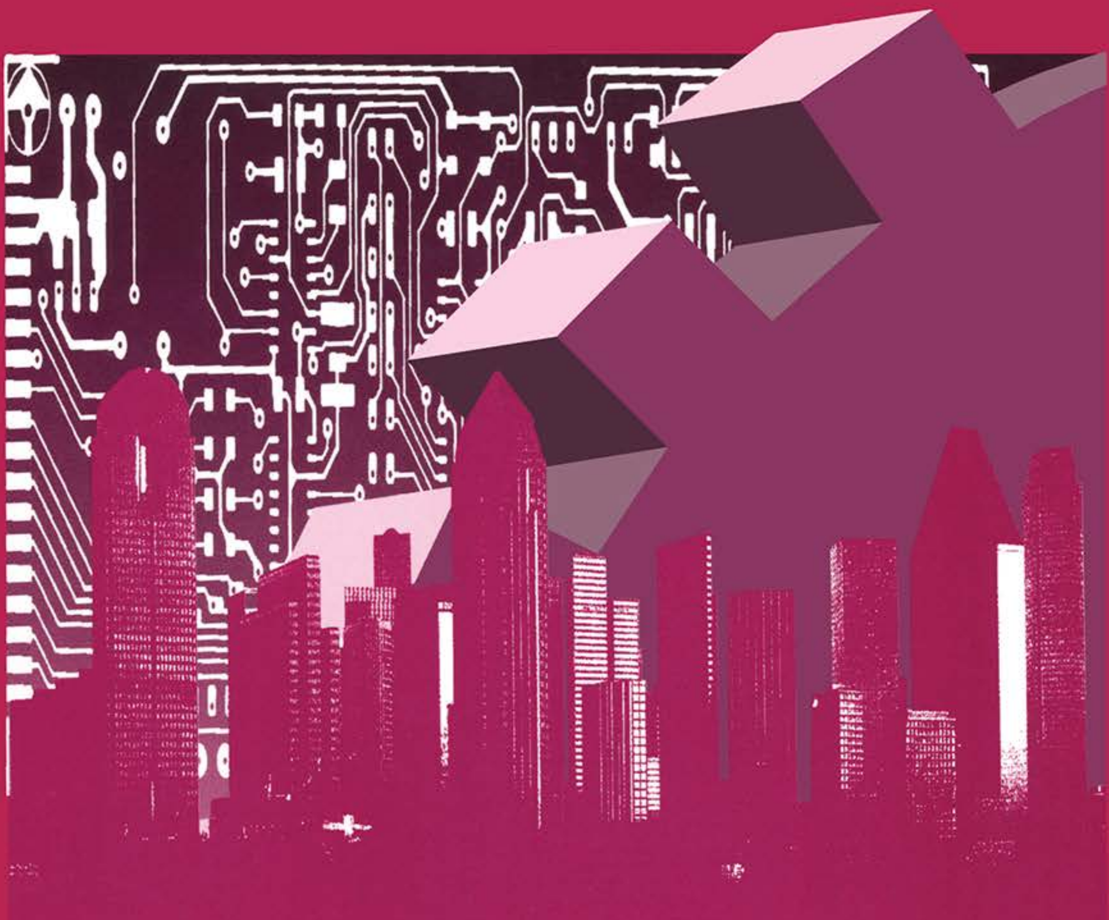
FOUNDATION SCIENCE FOR ENGINEERS



KEITH L. WATSON

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Keith L. Watson

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PREFACE

This book has been written for students without science A-levels who are entering an engineering degree or Higher National Diploma course via a foundation year. Very little scientific background is assumed and only an elementary knowledge of mathematics, which need extend no further than the simple properties of the right-angled triangle. Calculus is not required.

The book is divided into three parts, which may be taken either in series or in parallel. Emphasis has been placed on clarity and crispness of presentation, and on the provision of appropriate worked examples and practice questions. (The data supplied are approximate and are for illustrative purposes only.)

I have selected those topics which seem to me to provide the essential core material for any engineering foundation course. Practical work is not covered: the inclusion of instructions for safe and effective laboratory exercises over the whole range of topics would have lengthened the book considerably, and, furthermore, the needs of individual courses and the resources of individual institutions tend to determine their particular selection of specific exercises.

I am indebted to many colleagues here at Portsmouth for advice and comments on various parts of the manuscript; in particular, I should like to thank Dr Ray Batt, Professor Trevor Crabb, Michael Devane, Derek Hunter, Dr Tom Nevell, Bob Otter, Ron Parvin and Vic Riches — also Professor Brian Lee for his support. I am especially grateful to Kerry Lawrence of The Macmillan Press for her patience and encouragement, and to Professor John Wilson of the University of Northumbria at Newcastle, who reviewed the manuscript and made many helpful suggestions. Last, but not least, I thank my wife for her powers of endurance.

Portsmouth, 1992 KLW

Part 1

Force, Matter and Motion

TOPIC 1 QUANTITIES

COVERING:

- SI units;
- base and derived units;
- scalar and vector quantities;
- vector addition.

1.1 SI UNITS

Engineering quantities (pressure, temperature, power, and so on) need to be expressed in terms of an agreed system of units. SI units (Système International d'Unités) have been adopted in the UK and in many other countries, so we shall use them in this book. The system is founded on seven base units and two supplementary units from which all the others are derived.

The base units which are going to be of most interest to us are shown in Table 1.1 together with some of the derived units. We shall add to the list as we go along.

Derived units can be construed in terms of independent dimensions (such as length, mass and time) that are provided by the base units. Let us consider the unit of power as an illustration. Don't worry if you are unable to follow the scientific arguments too well at this stage. We shall go over them much more thoroughly later. The important thing to appreciate is that we can analyse the relationship between quantities in terms of their constituent base units.

First, velocity is a measure of change of position in unit time and its magnitude is given in metres per second ($m\ s^{-1}$). Acceleration is a measure of the rate at which velocity changes and its magnitude is given in metres per second per second ($m\ s^{-2}$).

The unit of force is called the *newton* (N). As we shall see later, it is defined as the force needed to give a mass of 1 kg an acceleration of 1 $m\ s^{-2}$. The relationship between these quantities is given by $F = ma$, where F represents force, m represents mass and a represents acceleration. The constituent base units of force are therefore those of mass times acceleration and

$$1\ N = 1\ kg\ m\ s^{-2}$$

Table 1.1

	Name	Symbol
Some base units:		
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
(For our purposes °C = K − 273: see Topic 16)		
Some derived units:		
Force	newton	N
Energy	joule	J
Power	watt	W
Pressure	pascal	Pa
Electric charge	coulomb	C

Taking this a step further, the *joule* (J) is the SI unit of energy. This is equal to the amount of work done when the point of application of a force of 1 newton moves through a distance of 1 metre in the direction of the force. If W is the work done and s is the distance moved in the direction of the force F, then $W = Fs$. The constituent base units of energy are therefore those of force times distance and

$$1 \text{ J} = 1 \text{ kg m}^2 \text{ s}^{-2}$$

Finally, power is the rate at which work is done or energy expended. The SI unit of power is called the *watt* (W), which corresponds to a rate of 1 joule per second. The base units of power are therefore those of energy divided by time and

$$1 \text{ W} = 1 \text{ kg m}^2 \text{ s}^{-3}$$

So, starting with the base units of mass, length and time, we have derived the unit of power.

Units often require prefixes to adjust them to an appropriate scale of magnitude for a particular measurement. For instance, we use *kilo*metres (km, a thousand metres) for large distances and *milli*metres (mm, a thousandth of a metre) for small ones. Table 1.2 shows the prefixes that we shall be using in this book.

1.2 SCALAR AND VECTOR QUANTITIES

Scalar quantities are those that are completely specified by their magnitude (e.g. length, speed and mass) and can be manipulated

Table 1.2

Prefix	Symbol	Factor	
giga-	G	$\times 10^9$	(a thousand million)
mega-	M	$\times 10^6$	(a million)
kilo-	k	$\times 10^3$	(a thousand)
deci-	d	$\times 10^{-1}$	(a tenth)
centi-	c	$\times 10^{-2}$	(a hundredth)
milli-	m	$\times 10^{-3}$	(a thousandth)
micro-	μ	$\times 10^{-6}$	(a millionth)
nano-	n	$\times 10^{-9}$	(a thousand millionth)
pico-	p	$\times 10^{-12}$	(a million millionth)

by simple arithmetic operations such as addition and multiplication. *Vector* quantities have both magnitude and direction (e.g. displacement, velocity and force). This makes them more complicated to handle, because angles are involved.

Let us start with displacement, i.e. change of position. This can be defined in terms of distance in a particular direction. But, as we can see from the example in Figure 1.1, there is another approach. Figure 1.1(a) shows a displacement of 5 units at an angle of 37° measured anticlockwise from the positive x-axis (i.e. from the 3 o'clock direction). Figure 1.1(b) shows the same displacement as the *resultant* of moving a distance of 4 units in the 0° direction, then 3 units in the 90° direction. Since $\tan 37° = 3/4$ and, from Pythagoras' theorem, $5 = \sqrt{4^2 + 3^2}$, we can see that both methods give the same result.

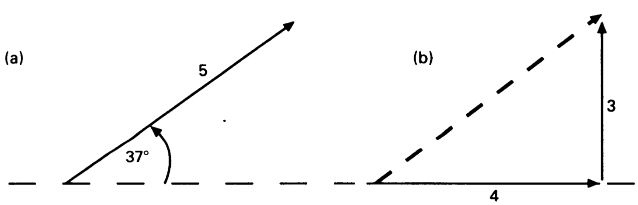

Figure 1.1

Vector quantities are often expressed in terms of distances and angles as in Figure 1.1(a), but sometimes we need to *resolve* them into perpendicular components as in Figure 1.1(b). From the figure we can see that the vertical component, equal to 3, is given by 5 sin 37 and the horizontal component, equal to 4, is given by 5 cos 37. This is useful when we need to find the resultant of two vector quantities.

For example, Figure 1.2(a) shows a displacement OA at an angle α followed by a second displacement AB at an angle β. Figure 1.2(b) shows the resultant OB in the direction θ. The total vertical displacement y is equal to (OA $\sin \alpha$ + AB $\sin \beta$) and the total horizontal displacement x is equal to (OA $\cos \alpha$ + AB $\cos \beta$). So, knowing the magnitude and direction of the two original displacements, we can

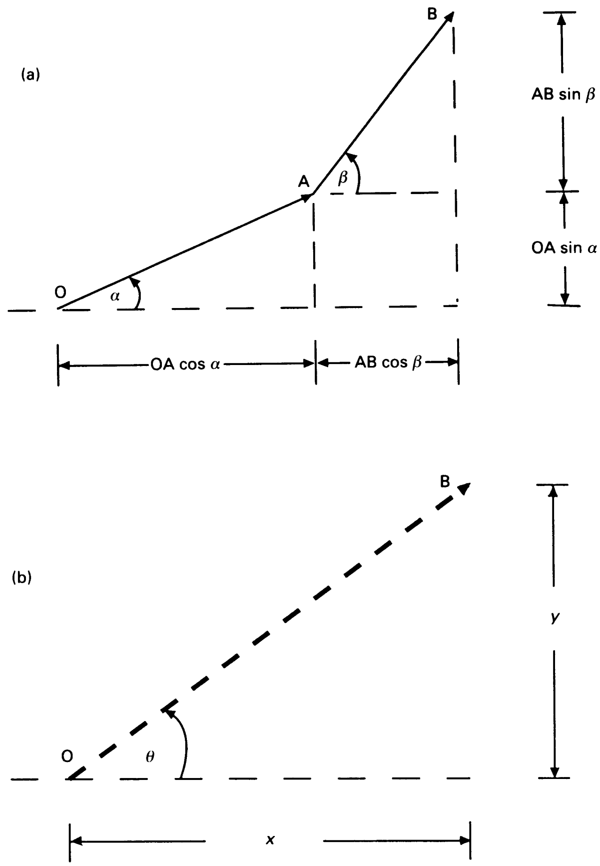

Figure 1.2

calculate y and x. We can then calculate the magnitude of OB from Pythagoras' theorem and its direction from $\tan \theta = y/x$.

Clearly we can use this method to obtain the resultant of as many displacements as we wish by plotting vectors head to tail. If the vectors form a closed loop, then the resultant (hence, the net displacement) is zero. Taking Figure 1.2 as an illustration, if we travelled from O to A, then from A to B, and finally from B back to O, the three displacement vectors would form a closed triangle and we would end up where we started. This is an obvious but important idea that we shall need to use later.

Note that velocity is a measure of displacement in unit time and that it is a vector quantity that can be manipulated in the same way as displacement. Later we shall use similar methods to handle forces.

Before tackling the questions at the end of the topic, make sure that you understand the following worked examples. And remember that, unless stated otherwise, angles are measured anticlockwise from the positive x-axis.

Worked Example 1.1

Find the magnitude and direction of the resultant of a displacement of 103 m at 62° followed by another of 59 m at 28°.

The displacements are shown in Figure 1.3.

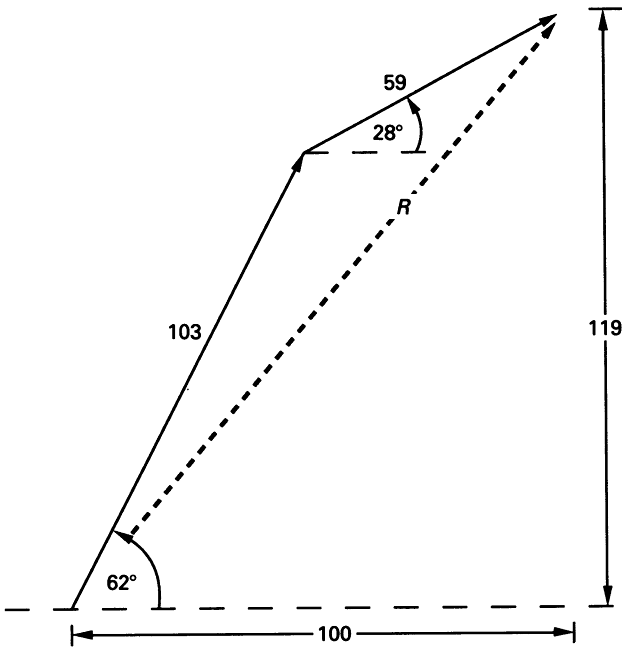

Figure 1.3

The total vertical displacement is equal to

$103 \sin 62 + 59 \sin 28 = 119$ m

The total horizontal displacement is equal to

$103 \cos 62 + 59 \cos 28 = 100$ m

The magnitude of the resultant R is equal to

$\sqrt{119^2 + 100^2} = 155$ m

The direction of the resultant R is equal to

$\tan^{-1} 119/100 = 50°$

Worked Example 1.2

Find the magnitude and direction of the resultant of successive displacements of 25 m at 90°, 30 m at 45° and 20 m at 300°.

The displacements are shown in Figure 1.4.

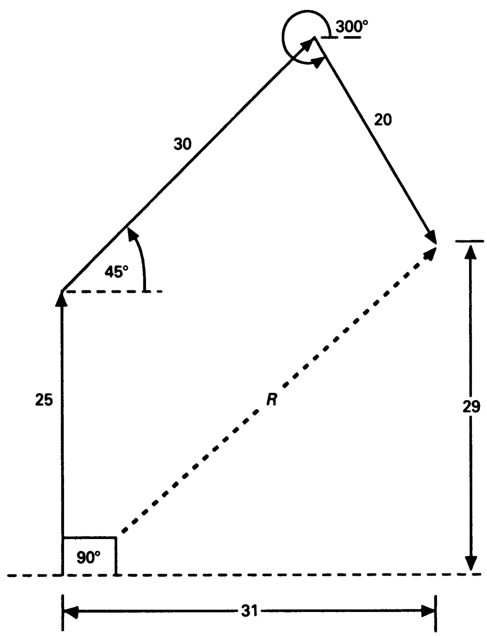

Figure 1.4

The total vertical displacement is equal to

$25 \sin 90 + 30 \sin 45 + 20 \sin 300 = 29$ m

The total horizontal displacement is equal to

$25 \cos 90 + 30 \cos 45 + 20 \cos 300 = 31$ m

The magnitude of the resultant R is equal to

$\sqrt{29^2 + 31^2} = 42$ m

The direction of the resultant R is equal to

$\tan^{-1} 29/31 = 43°$

Worked Example 1.3

If a boat is being rowed due north at 2 m s^{-1} and there is a current flowing due east at 1.5 m s^{-1}, what is the true velocity of the boat relative to the earth?

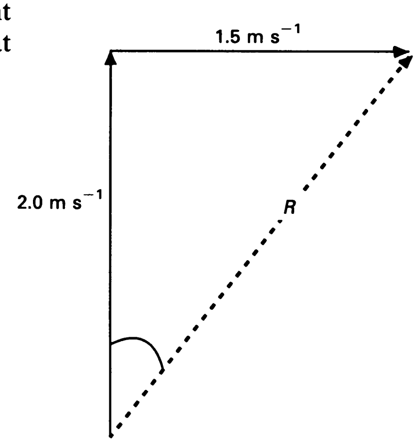

Figure 1.5

The velocity diagram is shown in Figure 1.5.
The magnitude of the resultant R is equal to

$$\sqrt{2^2 + 1.5^2} = 2.5 \text{ m s}^{-1}$$

The direction of the resultant R is equal to

$$\tan^{-1} \frac{1.5}{2} = 37° \text{ east of north}$$

i.e. 53° anticlockwise from the positive x-axis.

Questions

1. Find the magnitude and direction of the resultants of the following pairs of successive displacements:

   (a) 42 m at 31°, 53 m at 72°;
   (b) 117 m at 28°, 67 m at 147°;
   (c) 331 m at 47°, 158 m at 238°;
   (d) 97 m at 72°, 84 m at 163°.

2. A car climbs a hill at a road speed of 10 m s^{-1}. If the slope of the hill is 6° above the horizontal, what are the horizontal and vertical components of the car's velocity?
3. An object travels eastwards at 5 m s^{-1} for 7 s, then northwards at 7.5 m s^{-1} for 12 s, then westwards at 10 m s^{-1} for 15 s. Find how far and in what direction it must travel to return to its starting point.
4. A ship steers north at 5.0 m s^{-1} with a current flowing towards the south-east at 2.0 m s^{-1}. With what velocity must a man cross the deck to maintain a fixed position relative to the seabed?
5. An object travels 52 m in the 1 o'clock direction, then 71 m in the 5 o'clock direction, then 103 m in the 8 o'clock direction, then 43 m in the 11 o'clock direction. What is its resultant displacement?
6. A boat heads due north across a river 300 m wide which

flows from west to east at 1.5 m s^{-1}. If the boat moves at 5.0 m s^{-1} relative to the water:

(a) how long does it take to cross the river?
(b) how far downstream does it land?
(c) in what direction should it have headed to have crossed in the shortest possible distance, and
(d) how long would this have taken?

TOPIC 2 FORCES AND MATTER

COVERING:

- mass and gravitational force;
- internal forces and elastic behaviour;
- friction.

Isaac Newton (1642–1727) put forward three propositions concerning the relationship between the motion of a body and the forces acting upon it. These are known as *Newton's laws of motion*. For the moment we shall confine ourselves to the first. (We shall deal with the others in Topic 6.)

Newton's first law tells us that a stationary body remains at rest, or, if in motion, it moves in a straight line at constant speed, unless it is acted upon by a force (or the resultant of a number of forces). In mechanical terms, a force may therefore be regarded as an influence that tends to change a body's state of motion.

Scientists believe that there are only four fundamental types of force that operate in the universe. Two of these need not concern us, because they operate over extremely small distances and are of much more interest to nuclear physicists than to engineers. Of the remaining two, one stems from the gravitational interaction that arises between bodies because of their mass, and the other from the electromagnetic interaction due to the effects of electric charge.

2.1 MASS AND GRAVITATIONAL FORCE

The mass of a body is the quantity of matter that it contains. The SI unit of mass is the kilogram (kg).

Mass is formally defined in terms of inertia — that is to say, resistance to change of state of motion. We notice the inertia of our bodies when we are in a car that is changing speed or direction. We can feel the car forcing us to accelerate or decelerate against our natural tendency to remain at rest or move at constant speed in a straight line.

In practice, gravitation provides us with a much more convenient way of measuring mass. The law of universal gravitation, also named after Newton (who discovered it), is expressed by the equation

$$F = G\frac{m_1 m_2}{r^2} \tag{2.1}$$

where F is the gravitational force of attraction between two masses of magnitude m_1 and m_2 that are separated by a distance r. G is known as the *gravitational constant* and has the value 6.7×10^{-11} N m^2 kg^{-2}. Note that this law is an *inverse square law* — that is to say, the force is proportional to the reciprocal of the square of the distance between the masses. If the distance is doubled, the force is reduced to a quarter of its original magnitude; if the distance is halved, the force is quadrupled; and so on. Also note that we can verify the units of G by rearranging Equation (2.1) to give

$$G = \frac{Fr^2}{m_1 m_2}$$

which has the units of

$$\frac{\text{N m}^2}{\text{kg}^2}$$

From a gravitational point of view, we can assume that the earth behaves as though its mass is concentrated at its centre. If we substitute its mass (6.0×10^{24} kg) for m_1 and its radius (6.4×10^6 m) for r in Equation (2.1), and if we give m_2 the value 1.0 kg, then we obtain the gravitational force between the earth and a 1 kg mass at its surface as follows:

$$F = 6.7 \times 10^{-11}\frac{(6.0 \times 10^{24})(1.0)}{(6.4 \times 10^6)^2} = 9.8 \text{ N}$$

This means that there is a gravitational field of influence around the earth which causes any object at its surface to have a *weight* of 9.8 N per kilogram mass. (As a rough guide to magnitude, a medium-sized apple weighs about 1 N.)

9.8 N kg^{-1} is the *gravitational field strength* at the earth's surface and, if we give it the symbol g, we can write

$$W = mg \tag{2.2}$$

where W is the weight experienced by a mass m. (Engineers normally tend to interpret g in a different way, as we shall see in Topics 5 and 6.) Note that the weight of a given object will be different in places where the gravitational field strength is different. For example, a 1 kg object would weigh 1.6 N at the surface of the moon.

The gravitational interaction between objects on the human scale is very small. For instance, Equation (2.1) shows us that the force between two 1 kg masses 1 m apart is 6.7×10^{-11} N. Gravitational forces become large where astronomical masses (e.g. the earth) are involved, so that $m_1 \times m_2$ on the top line of Equation (2.1) becomes big enough to offset the tiny value of G.

Note that the centre of gravity of a body is the point at which its entire weight may be considered to act — for instance, the centre of a sphere or the mid-point of a ladder (assuming they are both uniform).

2.2 INTERNAL FORCES

A solid object will tend to resist being compressed or stretched because of opposing internal forces between its constituent atoms. These forces have electrical origins, which we shall discuss in some detail in Topic 15. For the time being, we can think of solid materials as consisting of atoms which are held together by *chemical bonds* that behave like tiny springs. A compressive force acting on a solid tends to push its constituent atoms together and a tensile force tends to pull them apart. In either case the deformation of the bonds results in the generation of an equal and opposite internal force that tends to restore the atoms to their original positions. If we remove the applied force, then the restoring force will return the solid to its original shape. This behaviour, called *elasticity*, is shared by all solids, even materials such as steel or concrete, although the deformation may not be obvious (for example, the deformation of a railway bridge supporting a train or a desk supporting a book.) Thus, a force can change the shape as well as the state of motion of a body.

Figure 2.1 represents the elastic behaviour of a steel wire. Let us imagine that the wire is suspended vertically from one end and that we stretch it by hanging a mass from the other. Figure 2.1 shows the relationship between the tensile force in the wire and the extension (increase in length) as the mass is increased. The chemical bonds in the wire stretch just far enough to provide an equal and opposite force to support the mass. The figure shows that if we double the mass so that the bonds have to provide twice the support, then the extension is doubled. If the force is trebled, the extension is trebled, and so on.

This is an example of Hooke's law, which tells us that elastic materials deform in proportion to the force causing their deformation. We shall consider this in more detail in Topic 20 (and we shall see that Hooke's law is not valid for all materials). In the meantime we should note that the argument can be broadened to state that, for the many materials that obey Hooke's law, *strain* is proportional to the *stress* which causes it. Strain ε is the amount of extension or compression per unit length and is given by $\varepsilon = \Delta l/l_o$ where Δl is the change in length and l_o is the original length. (Δ is a symbol that is used to represent a change in a quantity.) Note that strain is a dimensionless quantity, since it is obtained by dividing a

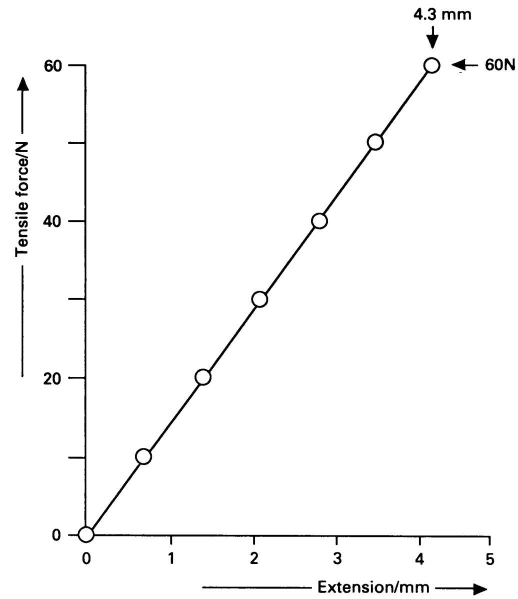

Figure 2.1

length by a length. It is sometimes expressed as a simple ratio and sometimes as a percentage.

Stress σ is the force per unit cross-sectional area and is given by $\sigma = F/A$, where F is the force (sometimes called the load) and A is the cross-sectional area. The unit of stress, N m^{-2}, is sometimes called the *pascal* (Pa).

The advantage of using stress and strain, as opposed to force and extension, is that we eliminate the effect of the dimensions of the specimen. (Under a given tensile load, the extension of a thick wire would be less than that of a thin one and the extension of a long wire would be greater than that of a short one.) A graph of stress against strain therefore represents the behaviour of the material alone and its slope E (= σ/ε), called the *Young's modulus*, gives a measure of the material's stiffness. We shall deal with stress/strain relationships in Topic 20. In the meantime, the following worked example uses the graph in Figure 2.1 to obtain Young's modulus directly.

Worked Example 2.1

Figure 2.1 (above) is a graph of load against extension for a wire of length 1.72 m and 0.40 mm diameter. Find the Young's modulus of the material.

From the discussion above we obtain the following expression

$$E = \frac{\sigma}{\varepsilon} = \frac{F}{A} \times \frac{l_o}{\Delta l} = \frac{l_o}{A} \times \frac{F}{\Delta l}$$

From the figure we see that the graph is a straight line which passes through both the origin and the point where $F = 60$ N and $\Delta l = 4.3$ mm. Its slope $(F/\Delta l)$ is therefore equal to

$$60/(4.3 \times 10^{-3})\ \text{N m}^{-1}$$

The radius r of the wire is equal to 0.2×10^{-3} m.

Substituting the values for l_o, A $(= \pi r^2)$ and $F/\Delta l$ in the expression for E that we obtained above,

$$E = \frac{1.72}{\pi(0.2 \times 10^{-3})^2} \times \frac{60}{(4.3 \times 10^{-3})} = 1.9 \times 10^{11}\ \text{N m}^{-2}$$

therefore,

$$E = 190\ \text{GPa}$$

If we unload the stretched wire represented in Figure 2.1, it will return to its original length. And we can repeat the loading/unloading cycle as many times as we like (within the limits imposed by metal fatigue, which we shall not be considering in this book). It is important to recognise that we must not use too great a load, otherwise the wire will deform *plastically* (i.e. stretch permanently) or even break. We shall discuss this in more detail in Topic 20. In the meantime, we shall confine our general discussion to stresses below the *elastic limit* (i.e. the stress level at which deformation ceases to be entirely elastic).

Note that our suspended mass is ultimately supported by the ground. If the wire is attached to the ceiling, then the mass and the wire are supported by an upward force generated by the slight bending of the joist to which the wire is fastened. (Bending of the joist is, in effect, a combination of tension and compression, where its bottom face tends to lengthen and become convex and its top face tends to shorten and become concave.) The joist is supported by an upward force generated by the compression of the wall which supports it. Finally, the wall is supported by an upward force generated by the deformation of the ground.

2.3 FRICTIONAL FORCE

Frictional forces arise through contact between objects and become apparent when we try to slide one surface over another.

Figure 2.2 shows a body lying on a horizontal plane. The weight W of the body is supported by the *normal* (i.e. perpendicular) force N exerted on it by the plane. Even the flattest surfaces are rough to varying degrees, so there will be high spots in the contact area

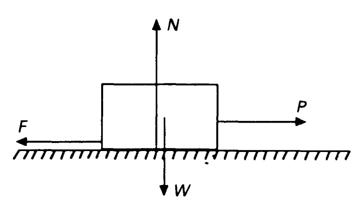

Figure 2.2

between the body and the plane which tend to interlock when they move relative to one another. The applied horizontal force P, acting towards the right, will therefore be opposed by a frictional resistance which will tend to prevent the body from sliding. In effect, P is opposed by a frictional force F which is the sum of all the tiny horizontal forces generated at the contact points where the surface irregularities push against one another. If we increase P, then, initially, F will increase to match it and the body will not move. Eventually P will reach a value just large enough to break down the interlocking between the surface irregularities. At this stage, when the body is just at the point of sliding, the frictional force F is given by the relationship

$$F = \mu_s N \tag{2.3}$$

The constant μ_s is called the *coefficient of static friction* and depends on the nature of the surfaces. The greater the friction between them then the greater will be the value of μ_s. Depending on conditions, the value for metals on certain plastics can be as low as 0.04, compared with around 1 for rubber on dry concrete.

As Equation (2.3) suggests, if we increase the normal force N by pressing the contact surfaces harder together, then the friction between them becomes proportionally greater.

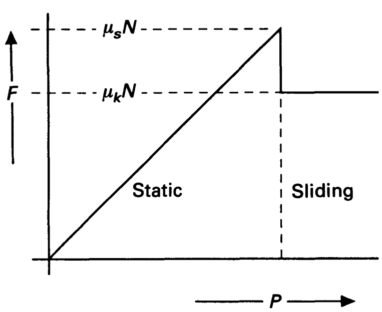

Figure 2.3

Once sliding begins, the frictional force F usually falls slightly, then tends to remain at a constant value that is independent of P. We therefore distinguish between the coefficients of static and kinetic (or dynamic) friction, μ_s and μ_k, respectively. μ_k (hence, F) tends to remain constant over a fairly wide range of sliding velocities and in many instances the values of μ_k and μ_s are not very different. The relationship between F and P is summarised in Figure 2.3.

From the figure we can see that:

$F\ (= P) < \mu_s N$ under static conditions

$F\ (= P) = \mu_s N$ at the point of sliding

$F = \mu_k N\ (< P)$ under sliding conditions

Equation (2.3) and its dynamic counterpart ($F = \mu_k N$) are approximate but, within their range of validity for a given system, they indicate that the frictional force is independent of the contact area.

Friction is a very complex phenomenon and it is impossible to devise a single physical model to cover all cases. Nevertheless it is worth making some general observations.

It is believed that, because of roughness, the points of real contact between two solid surfaces pressed together are usually few and far between. This means that the actual contact area is much smaller than the apparent area suggested by the overall dimensions of the surfaces involved. A normal force N will therefore be concentrated into regions of intense localised pressure at the contact points. It is believed that at these points some materials, particularly metals, tend to flow plastically, thereby causing an increase in the actual contact area (where the contact points are deformed) and even localised adhesion.

It appears that an increase in the value of N causes an increase in the actual contact area and, hence, an increase in frictional resistance. Also, for a given value of N, the sum of the areas of all the contact points appears to remain more or less the same whether they are spread out over a large apparent surface area or concentrated into a relatively small one. Furthermore, under sliding conditions, the areas of contact between the surfaces are being continually broken and remade. The fact that μ_k tends to be less than μ_s seems to suggest that, under sliding conditions, a proportion of the potential contact points across the interface are not connected at any given moment.

Another measure of frictional resistance is the *angle of friction*, which is illustrated in Figure 2.4. If the plane on which a body rests is tilted to an angle θ where the body is on the point of sliding downwards, then θ is the angle of friction, where

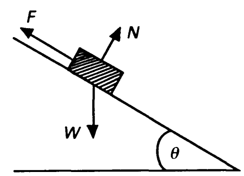

Figure 2.4

$$\mu_s = \tan\theta \tag{2.4}$$

Equation (2.4) is easy to verify. If the body is on the point of slipping, then its weight W is balanced by the normal force N supporting it perpendicular to the plane plus the frictional force F just stopping it from sliding down. N and F, in effect, are combined in an upward resultant force R (that is equal and opposite to W), as shown in Figure 2.5(a).

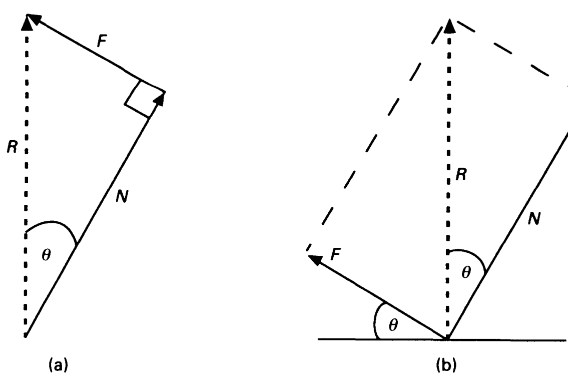

Figure 2.5

Note that, since they are vector quantities, we treat the forces N and F in the same way as the displacements in the last topic. We represent them by arrows indicating their direction, with length proportional to their magnitude. It is sometimes helpful to represent two forces acting at a point as the sides of a parallelogram (a rectangle in this particular case) where the diagonal from the point at which the forces meet represents the resultant (Figure 2.5b).

As Figure 2.5(b) shows, the angle between R and N must be θ. (F and N are mutually perpendicular, so they must make an angle θ with the horizontal and vertical respectively, as shown.) From the right-angled triangle in Figure 2.5(a) we can see that

$$F = R \sin \theta$$

and

$$N = R \cos \theta$$

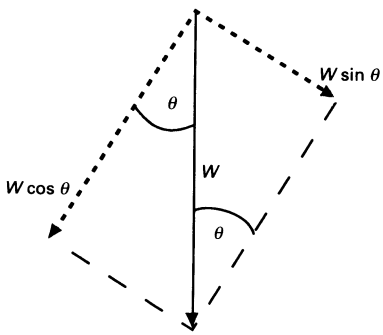

Figure 2.6

So, from Equation (2.3),

$$\mu_s = \frac{F}{N} = \frac{R \sin \theta}{R \cos \theta} = \tan \theta$$

as in Equation (2.4).

Rather than combine N and F to find the resultant force R, we could equally well tackle the problem by *resolving* the weight W into components parallel and perpendicular to the plane, as in Figure 2.6. These two components can then be equated to F and N, respectively, and

$$\mu_s = \frac{F}{N} = \frac{W \sin \theta}{W \cos \theta} = \tan \theta$$

Friction is sometimes reduced by lubrication. Putting oil between metal contact surfaces tends to keep them apart, so that frictional forces are reduced. Another approach is to mount things on wheels. This does not entirely solve the problem, because of *rolling friction* between the wheel and the surface over which it is running. The wheel tends to create a slight depression out of which it continuously tries to climb and, at the same time, it tends to flatten where it touches the surface.

Note that, in the present context, the term 'smooth' is sometimes used to describe a surface that is frictionless or that can be considered to be frictionless as an approximation.

Worked Example 2.2

(a) Assuming that $\mu_s = 0.55$, what is the minimum horizontal force needed to start a 50 kg box sliding across a horizontal floor? (b) What force is required if it is applied to the box by means of a rope inclined at 25° above the horizontal?

(a) The forces involved are shown in Figure 2.2 (page 15). The weight W of the box is supported by the normal force N, so that

$$N = W = 9.8 \times 50 = 490 \text{ newtons}$$

Therefore, from Figure 2.3 (page 16), to start the box sliding,

$$P = F = \mu_s N = 0.55 \times 490 = 270 \text{ newtons}$$

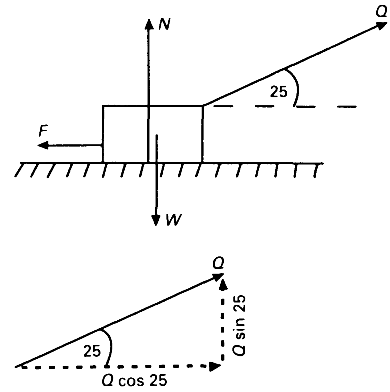

Figure 2.7

(b) Figure 2.7 shows that the applied force Q has a horizontal component Q cos 25 and a vertical component Q sin 25.

Considering the horizontal forces, we have

$$F = Q \cos 25$$

Considering the vertical forces, and remembering that $W = 490$ newtons,

$$N + Q \sin 25 = 490$$

Therefore,

$$N = 490 - Q \sin 25$$

When the box is on the point of sliding,

$$F = \mu_s N = 0.55 \times N$$

and, substituting for F and N,

$$Q \cos 25 = 0.55\,(490 - Q \sin 25)$$

which gives

$$Q = 237 \text{ newtons}$$

Questions

(Where necessary assume that $g = 9.8 \text{ N kg}^{-1}$.)

1. Estimate the mass of the moon, given that its diameter is 3500 km and that a 1.0 kg mass weighs 1.6 N (newtons) at its surface. ($G = 6.7 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$.)
2. A cable, 50 m long, experiences a strain of 0.05%. By how many mm has its original length increased?
3. A 0.4 mm diameter wire supports a mass of 2.5 kg.

(a) What is the tensile stress in the wire?
(b) What is the percentage strain if the value of Young's modulus is 2×10^{11} N m^{-2}?

4. A body of 25 kg mass is resting on a level floor. If $\mu_s = 0.4$, what is the frictional force?
5. A body of 25 kg mass lying on a level floor has a steadily increasing horizontal force applied to it. If $\mu_s = 0.4$, what is the maximum frictional force reached?
6. A body of 25 kg mass is sliding down a 45° slope. If $\mu_k = 0.6$, what is the frictional force?
7. For Figure 2.7, calculate the magnitude of Q if its direction is reversed (so that the box is being pushed by a force inclined at 25° below the horizontal).
8. A 50 kg box is placed on a ramp inclined at 25° to the horizontal.

   (a) Find the angle of friction; hence show that the box will not slide downwards.
   (b) Find the minimum force parallel to the ramp needed to start the box (i) sliding downwards and (ii) sliding upwards. (Assume $\mu_s = 0.55$.)

TOPIC 3 EQUILIBRIUM

COVERING:

- translational and rotational equilibrium;
- the components of forces;
- the moments of forces.

A body is in equilibrium when the forces acting on it balance each other so that it either remains at rest or, if it is moving, remains in a state of uniform motion (i.e. constant speed in a straight line). To understand equilibrium we need to recognise that forces can influence the motion of a body in two ways: they can affect its translational motion from one place to another (with all its parts moving in the same direction) and they can affect its rotation.

Two equal and opposite forces meeting at the same point in the body do not affect its state of motion at all, because they are in both translational and rotational equilibrium. If the two forces are separated so that their lines of action are parallel, as at the opposite ends of bicycle handlebars, then they are still in translational equilibrium, because they cancel each other out and their resultant is zero. However, they are no longer in rotational equilibrium, because they tend to cause the handlebars to turn.

We shall only consider coplanar forces here — that is to say, forces that act in the same plane. First we shall think about translational equilibrium in terms of the components of forces; then we shall move on to rotational equilibrium.

3.1 COMPONENTS OF FORCES

In the last topic we met simple cases of combining forces into a resultant and of resolving forces into components. We can very easily extend this to more complex systems, as we did with displacements. We can even plot force vectors head to tail on a scale drawing and find the resultant graphically, just as we can with displacements on a map.

If the vectors representing a system of forces form a closed loop, then the resultant is zero and the forces are in translational equilibrium. And if they are concurrent (i.e. if they meet at a point), then

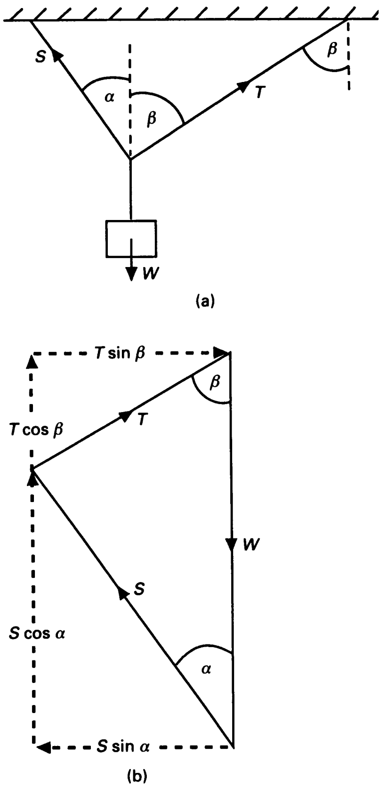

Figure 3.1

they are in rotational equilibrium too. For instance, Figure 3.1(a) shows a weight W being supported by two fixed strings which carry tensile forces S and T, respectively. Note that the values of the angles α and β are dictated by the lengths of the strings and the distance between their fixing points on the ceiling. The system is at rest and therefore in equilibrium.

How can the values of S and T be found? We could do it with a ruler and a protractor. In Figure 3.1(b) the length W is drawn to represent the weight. The angles α and β are then marked off at either end to give the directions of S and T. The point where the lines representing S and T intersect can then be found, and, hence, their lengths and the corresponding magnitudes.

Alternatively, since the forces are in equilibrium, the sum of all their horizontal components and of all their vertical components must separately be zero (since there can be no resultant); therefore,

$$T \sin \beta = S \sin \alpha$$

and

$$S \cos \alpha + T \cos \beta = W$$

Knowing the values of W, α and β (and assuming the strings have negligible mass), we can find S and T as in the following worked example.

Worked Example 3.1

Two strings (of negligible mass) support a weight of 80 N, as in Figure 3.1. If $\alpha = 35°$ and $\beta = 60°$, find the tension in each string.

Resolving horizontally,

$$S \sin 35 = T \sin 60$$

which gives

$$S = 1.5T$$

Resolving vertically,

$$80 = S \cos 35 + T \cos 60$$

and, since $S = 1.5T$, this gives

$$S = 69 \text{ N and } T = 46 \text{ N}$$

What happens if we disconnect the right-hand string from the ceiling and pull it horizontally, as in Figure 3.2, so that the left-hand string still makes the same angle α with the vertical as before? The triangle of forces is shown in Figure 3.2(b). T no longer has a vertical component. This means that S has to support all of W, which results in an increase in its vertical component and, hence, in its horizontal component. The increase in the latter is supported by T.

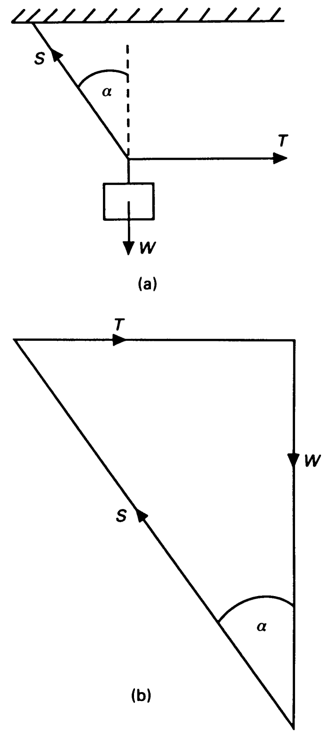

Figure 3.2

Worked Example 3.2

If, in Figure 3.2, $\alpha = 35°$ and $W = 80$ N, find the tension in each string.

From the triangle of forces in Figure 3.2(b),

$$S = W/\cos\alpha = 80/\cos 35 = 98 \text{ N}$$

and

$$T = W \tan\alpha = 80 \tan 35 = 56 \text{ N}$$

Finally, if we let go of the right-hand string completely (so that $T = 0$ and $\alpha = 0$), then $S = W$.

Figure 3.3 shows a variation on the same system. In Figure 3.3(a) we can see that the weight W is supported by two counterbalancing forces S and T supplied by masses suspended from the ceiling by pulleys. As long as there is no friction, the tension in each string can be assumed to be constant along its length, provided that its weight can be ignored. If we know S, T and W, then we can find α and β by constructing arcs of radius S and T, respectively from the ends of a line of length W, as in Figure 3.3(b). The point where the arcs intersect completes the triangle and enables us to measure α and β with a protractor. (Alternatively, these angles could be calculated by use of the cosine rule.)

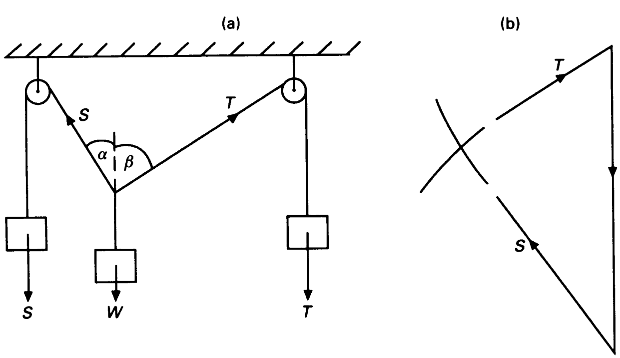

Figure 3.3

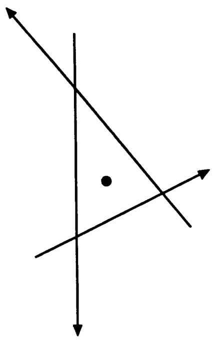

Figure 3.4

Note that in this case the values of S and T are fixed and the strings are free to move, so that the angles adjust themselves to bring the system to equilibrium. (In the fixed-string arrangement the geometry dictates the angles; the values of S and T therefore adjust themselves to bring the system to equilibrium.)

A system of forces will be in translational equilibrium provided that there is no net force in each of two directions (often taken to be mutually perpendicular for convenience). However, if the forces are not concurrent, they will tend to cause rotation. In the cases above, the lines of action of the forces S, T and W pass through the same point (where the strings join) and there is no tendency to rotation. In Figure 3.4 the three forces are not concurrent and the system will tend to rotate in an anticlockwise direction, even though it may be in translational equilibrium.

3.2 MOMENTS OF FORCES

The turning effect, or *moment*, of a force about a point (sometimes called *torque*) is defined as the product obtained by multiplying the force by the perpendicular distance of its line of action from the point. The units are newton metres, N m (not to be confused with joules, which are units of energy). The moment of the force F about the point O in Figure 3.5(a) is therefore $F \times s$ newton metres. This is consistent with practical experience (using a spanner, for example); a turning moment is made greater by increasing either the force F or its perpendicular distance s from the point, or both together. Note that if the line of action of the force passes through the point, then $s = 0$ and its moment is zero, so there will be no rotation.

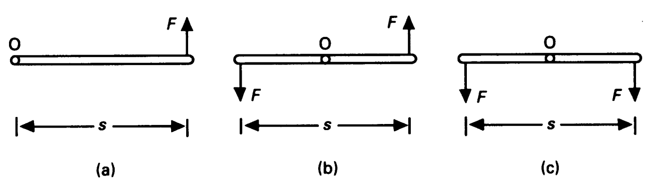

Figure 3.5

Figure 3.5(b) shows a *couple*, which consists of two equal and opposite parallel forces. This has a moment about O equal to the sum of the two individual moments. Each of these is equal to $F \times s/2$, giving a total moment of $F \times s$. Thus, the moment of a couple is equal to the product of one of the forces and the perpendicular distance between them. A couple clearly has no resultant, so it causes pure rotation, as in turning a key or a tap.

Figure 3.5(c) shows two equal parallel forces — for example, identical weights balanced at equal distances from the pivot of a see-saw. The anticlockwise moment of the left-hand force about O is equal to the clockwise moment of the right-hand force, so they balance each other — in other words, they are in rotational equilibrium. Two forces of different magnitude can be in rotational equilibrium provided that their respective values of $F \times s$ are equal and opposite — for instance, if one person twice the weight of the other is sitting half as far away from the pivot.

In fact we can have as many forces in rotational equilibrium as we like provided that the sum of the clockwise moments equals the sum of the anticlockwise moments. And we can take moments (i.e. find their sum) about any point we choose, although it is best to select one through which a force passes, or several forces in more complex cases; this makes their moments zero about the point and reduces the amount of calculation needed. It also enables us to eliminate an unwanted unknown force from a calculation.

Worked Example 3.3

A 3 m uniform beam of unknown mass, pivoted in the middle, supports a weight of 800 N at one end and another of 400 N at the other. Where must a further weight of 800 N act in order to bring the system to equilibrium?

Figure 3.6 shows the forces involved. The third weight must act between the 400 N weight and the pivot at some distance s from the pivot (which then supports a total weight of 2000 N plus the weight of

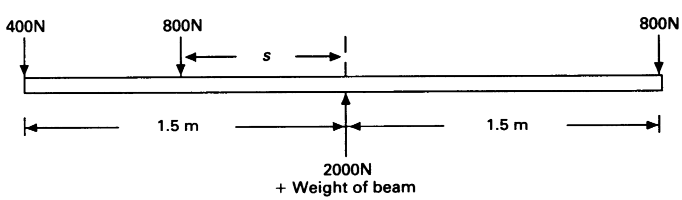

Figure 3.6

the beam). For equilibrium, the sum of the anticlockwise moments must equal the sum of the clockwise moments about any point. Taking moments about the pivot (to eliminate the unknown weight of the beam),

$$(400 \times 1.5) + (800 \times s) = (800 \times 1.5)$$

Therefore,

$$s = \frac{1200 - 600}{800} = 0.75 \text{ m}$$

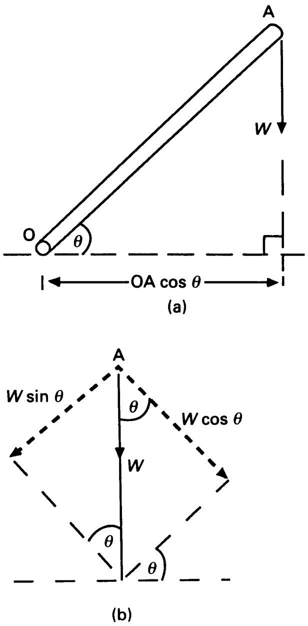

Figure 3.7

(In fact it would not have mattered if we had assumed s to be on the wrong side of the pivot — giving the third weight a clockwise moment. In this case we would have found that $s = -0.75$ m, the minus sign indicating a distance of 0.75 m from the pivot on the opposite side from the one we selected.)

Earlier we defined the moment of a force about a point as the product of the force and the perpendicular distance of its line of action from the point. In the examples we have considered so far the distance has been very obviously perpendicular to the line of action. Figure 3.7 shows a different case. In Figure 3.7(a), OA represents a lever arm pivoted at O with a body of weight W suspended from it at A. From our experience we would expect the moment of W about O to be at a minimum value (zero in fact) when OA is vertical and at a maximum when it is horizontal. As we can see from the figure, the perpendicular distance of the line of action of W from O is given by OA cos θ, where θ is the angle that OA makes with the horizontal. The moment of W about O is therefore $W \times$ OA cos θ. If OA is vertical, then $\theta = 90°$ or $270°$ and OA $\cos \theta = 0$ (i.e. the line of action passes through O, so W has no turning effect). If OA is horizontal, then $\theta = 0°$ or $180°$; therefore, $\cos \theta = 1$ or -1 and the perpendicular distance is OA, which is its maximum value and gives the maximum moment.

Figure 3.7(b) shows an alternative approach where W is resolved into components perpendicular to and along OA, i.e. $W \cos \theta$ and $W \sin \theta$, respectively. The moment of $W \sin \theta$ about O is zero, since its line of action passes through O, but the moment of the perpendicular component is OA $\times\ W \cos \theta$, which gives the same result as above.

3.3 EQUILIBRIUM CONDITIONS

For a body to be in equilibrium, the forces acting upon it must balance and their clockwise and anticlockwise moments about any point must be equal. For translational equilibrium, we usually resolve the forces into two mutually perpendicular directions, typically vertically and horizontally, or perpendicular and parallel to some convenient plane. And, as noted above, we usually take moments about a point through which at least one of the unknown forces passes.

Worked Example 3.4

A uniform ladder weighing 200 N rests against a smooth (frictionless) vertical wall and makes an angle of 70° with the ground, which is level. Find the force exerted on the bottom end of the ladder by the ground.

Figure 3.8 shows the forces acting on the ladder. A frictionless surface can only exert a force perpendicular to itself; therefore, the only force the ladder experiences where it touches the wall is the normal force P. The ladder is uniform, so its weight acts half-way along its length l. It is at rest, so there is sufficient frictional force F between its bottom end and the ground to prevent it from slipping. N is the upward force exerted on the ladder by the ground. The force we require is the resultant of F and N.

Vertically we find that

$$N = 200 \text{ newtons}$$

and horizontally

$$F = P$$

We can find P by taking moments about the bottom of the ladder (eliminating F and N), as follows

$$P \times \text{OB} = 200 \times \text{OA}$$

Therefore,

$$P \times l \sin 70 = 200 \times (l/2) \cos 70$$

which, on rearranging, gives

$$P = \frac{200}{2 \tan 70} = 36 \text{ N}$$

(Note that l cancels out, so we do not need to know the length of the ladder.)

The magnitude of the resultant is equal to

$$\sqrt{200^2 + 36^2} = 203 \text{ N}$$

The direction of the resultant is equal to

$$\tan^{-1} 200/36 = 80°$$

(Be careful not to confuse the direction of the resultant ($\tan^{-1} N/F$) with the angle the ladder makes with the ground.)

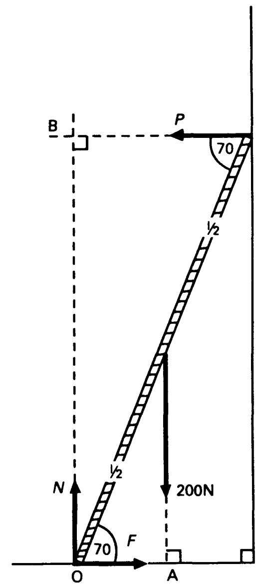

Figure 3.8

Questions

(Where necessary assume that $g = 9.8 \text{ N kg}^{-1}$.)

1. Determine whether each of the following coplanar systems of concurrent forces is in approximate equilibrium. If not, then find the magnitude and direction of the counterbalancing force required to bring the system

to equilibrium. (Angles are measured anticlockwise from the x-axis.)

(a) 5.0 N at 20°, 6.0 N at 60°, 10.3 N at 222°
(b) 28 N at 32°, 34 N at 214°, 56 N at 313°
(c) 125 N at 27°, 240 N at 137°, 530 N at 270°
(d) 35 N at 47°, 43 N at 116°, 61 N at 215°, 54 N at 327°
(e) 63 N at 28°, 47 N at 128°, 38 N at 139°, 34 N at 203°, 85 N at 293°
(f) 119 N at 47°, 116 N at 108°, 124 N at 146°, 196 N at 233°, 207 N at 328°

2. If, in Figure 3.3(a), $S = T = W = 1$ kg, what are the values of α and β?

3. A weightless beam, 10 m long and supported at either end, carries a 50 kg load 2.5 m from one end. Find the forces supporting the beam at either end.

4. A 150 kg steel bar, 2.0 m long and of uniform cross-section, is supported by two vertical wires, one fixed at one end of the bar and the second at a distance of 0.8 m from the other end. Find the tension in each wire.

5. A weightless horizontal cantilever beam projects 5 m from a vertical wall.

   (a) If a 15 kg mass is placed on the end of the beam furthest from the wall, find the moment of its weight about the point where the beam enters the wall.
   (b) If an additional two masses are placed on the beam, 10 kg at 1 m and 5 kg at 3 m from the wall, respectively, find the total moment about the point where the beam enters the wall.
   (c) If the three masses are combined into a single mass, find how far from the wall it must be placed to provide the same moment as in (b).

6. Seven coplanar forces act at the corners of a 1 m square as shown in Figure 3.9.

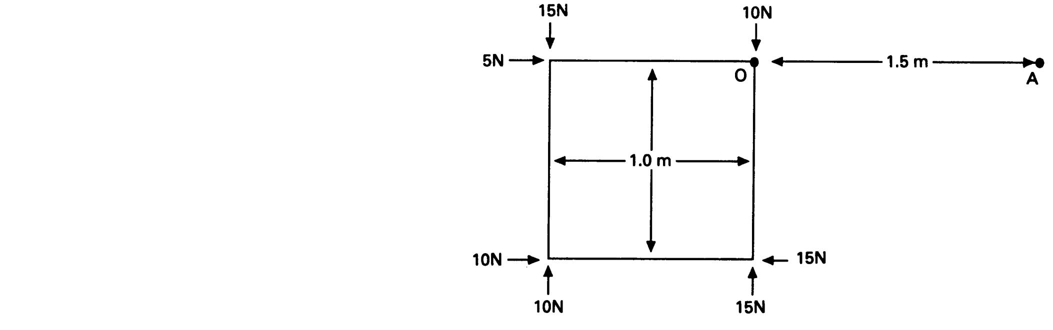

Figure 3.9

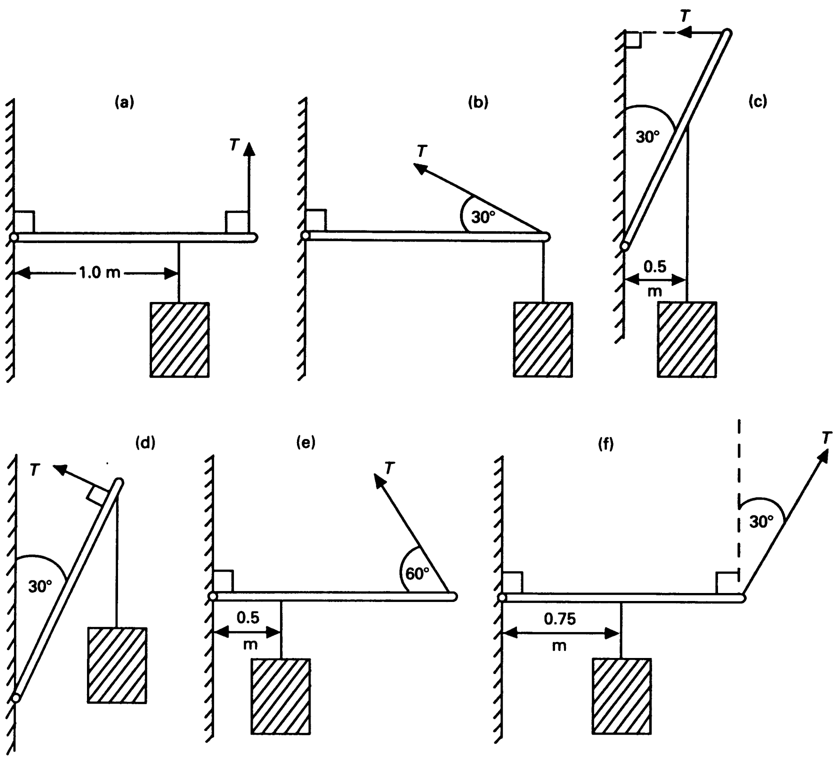

Figure 3.10

(a) Find the magnitude and direction of the resultant translational force.
(b) Find the net moment about O.
(c) Find the net moment about A.

7. Figure 3.10 shows a weightless rod, 1.5 m long, fastened to a vertical wall by a hinge at one end so that it can pivot vertically. A weightless wire is attached to the other end to enable it to support a 5.0 kg mass in the various arrangements shown. In each case (i) find the value of T, the tension in the wire, and (ii) find the vertical and horizontal components, and, hence, the magnitude and direction of the resultant force acting on the rod at the hinge.

8. If, in Worked Example 3.4, the ladder is on the point of slipping, what is the value of μ_s between the ladder and the ground?

9. If, in Worked Example 3.4, $\mu_s = 0.5$ between the ladder and the ground, what is the smallest angle the ladder could make with the ground (i.e. where it is on the point of slipping)?

TOPIC 4 PRESSURE AND UPTHRUST

COVERING:

- density and relative density;
- absolute pressure and gauge pressure;
- transmission of pressure;
- floating and sinking.

This topic is mainly concerned with forces that operate in liquids at rest, although some of it refers to gases as well.

4.1 DENSITY

First we need to recognise that the *density* of a substance is a measure of the mass it contains per unit volume. Expressing this in mathematical terms,

$$\rho = \frac{m}{V} \tag{4.1}$$

where m represents mass (kg), V volume (m^3) and ρ density (kg m^{-3}). Density is sometimes given in grams per cubic centimetre, g cm^{-3}. (Note that 1000 kg m^{-3} is equal to 1 g cm^{-3}.)

Relative density (formerly called *specific gravity*) is a dimensionless quantity which, for a liquid or a solid, is obtained by dividing its density by the density of water. The maximum density of water (1000 kg m^{-3}) is commonly used, in which case the relative density of a substance is obtained by dividing its density in kg m^{-3} by 1000. Although the density of water varies with temperature (its maximum occurring at 4 °C), the variation is small over the normal range of room temperatures.

In practice, the relative density of a liquid can be measured by dividing its mass by the mass of an equal volume of water. This can be done by weighing the liquids in a *relative density bottle*, which is a small flask with a special stopper that ensures a reproducible volume each time it is filled.

4.2 PRESSURE

Pressure is a measure of the normal (perpendicular) force acting per unit area on a surface. The SI unit of pressure is called the pascal (Pa) and 1 Pa = 1 N m^{-2}. (As we saw in Topic 2, the pascal can also be used as the unit of stress.)

Worked Example 4.1

A metal block, of relative density 8.90, measures 45 × 70 × 125 mm. How much does it weigh? What apparent pressure would each of its faces exert when resting on a level surface? (g = 9.8 N kg^{-1}.)

Substituting the data into Equation (4.1) ($m = V \times \rho$),

$$m = (0.045 \times 0.070 \times 0.125) \times 8.90 \times 10^3$$

which gives

$$m = 3.50 \text{ kg}$$

The block therefore weighs 3.50 × 9.8 = 34.3 N. Since pressure p = force/area, then, when the block stands on its end,

$$p = \frac{34.3}{0.045 \times 0.070} = 1.1 \times 10^4 \text{ Pa}$$

and when it lies on its side,

$$p = \frac{34.3}{0.045 \times 0.125} = 6.1 \times 10^3 \text{ Pa}$$

and when it lies flat,

$$p = \frac{34.3}{0.070 \times 0.125} = 3.9 \times 10^3 \text{ Pa}$$

Pressure exists at any point in the body of a liquid because of the weight of liquid above it. It therefore increases with depth. Figure 4.1 shows an imaginary horizontal area (= A) at a depth h below the surface of a liquid. In effect, the area A supports a column of liquid above it which has a volume $h \times A$. Since $m = V \times \rho$ (from Equation 4.1) the mass of liquid in the column will be $h \times A \times \rho$. Multiplying this by g (9.8 N kg^{-1}) gives its weight as $h \times A \times \rho \times g$. Since weight is

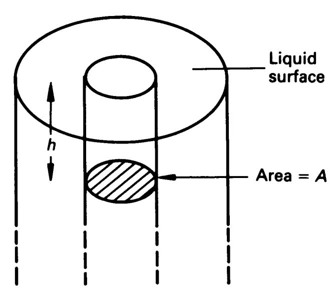

Figure 4.1

a force and the area over which it acts is A, the pressure p at A is given by

$$p = \frac{\text{force}}{\text{area}} = \frac{h \times A \times \rho \times g}{A} = \rho g h \qquad (4.2)$$

This means that, since g is constant, the pressure acting at any point in a liquid simply depends on its depth and the density of the liquid above it. In practice, liquids are normally subjected to atmospheric pressure. Taking this into account, the total pressure is given by

$$p = \rho g h + p_{\text{atm}}$$

where p_{atm} represents atmospheric pressure. This leads to a distinction between the *absolute pressure* relative to a perfect vacuum and the *gauge pressure* measured relative to atmospheric pressure. (Remember that gauge pressure is zero at atmospheric pressure.)

The *manometer*, which is essentially a U-tube containing a liquid, is a device which measures gauge pressure (Figure 4.2).

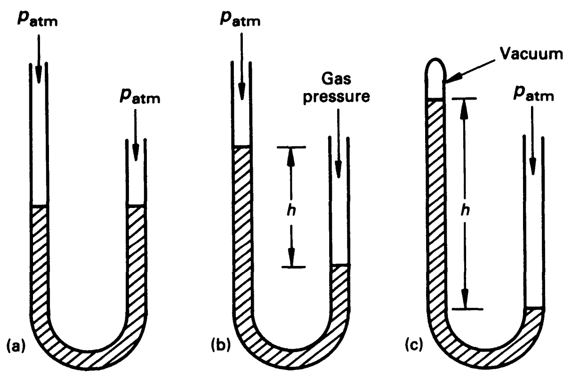

Figure 4.2

In Figure 4.2(a) both ends of the tube are open and the liquid surfaces on each side are level, because they are both subjected to atmospheric pressure. In Figure 4.2(b) the right-hand side is connected to a supply of gas under pressure. The liquid in the left-hand column will rise until the value of h is such that $\rho g h + p_{\text{atm}}$ balances the pressure of the gas supply. The difference in the height of the columns is therefore a measure of the gas pressure relative to the atmosphere — that is to say, $\rho g h$ gives the gauge pressure of the gas.

In Figure 4.2(c) the left-hand side has been sealed off with the space at the top under vacuum. The difference in the height of the columns now represents the absolute pressure at the open end compared with the vacuum. This is the basis of the mercury barometer used to measure atmospheric pressure. Atmospheric pressure fluctuates around 760 mmHg — that is to say, h equals 760 mm of mercury. (Hg is the chemical symbol for mercury.)

Worked Example 4.2

If the air pressure is 758 mmHg at the bottom of a mountain and 618 mmHg at the top, find the height of the mountain. (Assume that $\rho_{Hg} = 13.6 \times 10^3$ kg m^{-3} and that the average value of $\rho_{air} = 1.27$ kg m^{-3}.)

A column of air the same height as the mountain is equivalent to a mercury column of height equal to

$$0.758 - 0.618 = 0.140 \text{ m}$$

If ρgh for air $= \rho gh$ for mercury, then

$$1.27 \times g \times h_{air} = 13.6 \times 10^3 \times g \times 0.140$$

which gives

$$h_{air} = 1500 \text{ m}$$

Although the pascal is the SI unit of pressure, there are a number of others in use. 1 mmHg is a unit called the *torr*, which is often used in measuring low pressures. The *atmosphere* (atm), equivalent to 760 mmHg, is convenient for high pressures. If we take the density of mercury to be 13.6×10^3 kg m^{-3}, then Equation (4.2) tells us that 760 mmHg is equivalent to 101 kPa. (The accurate value for 1 atm is 101.325 kPa.) The *bar* is equal to 100 kPa and equivalent to 750 mmHg. Meteorologists use the *millibar*, which is 100 Pa. The pound per square inch (14.7 lb in^{-2} = 1 atm) was in common use at one time.

Pressure values are occasionally expressed in terms of the height of a column of water. Since the relative density of mercury is 13.6, a water column is much higher than the equivalent mercury column and this makes it easier to measure small pressures.

Liquids lack the rigidity of solids and have the ability to flow. This has important consequences when we think about applying pressure to a liquid at rest. First, the pressure at any point is the same in all directions. (Otherwise the liquid would not remain at rest.) Second, only normal forces can act between a liquid and a surface in contact with it. (Any parallel component would lead to relative movement between the two.) Third, a liquid will transmit an externally applied pressure uniformly throughout its volume, although there will still be the internal variation with depth.

This leads us on to the idea of amplifying forces by means of the

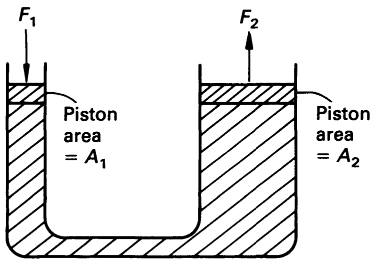

Figure 4.3

pressure in a liquid. Figure 4.3 shows a simplified version of a hydraulic press. It consists of two liquid-filled interconnected cylinders of different diameters, each fitted with a piston. If a downward force F_1, is applied to the smaller piston of area A_1, then the resulting pressure p transmitted by the liquid will be equal to F_1/A_1. The pressure will be converted to an upward force F_2 by the larger piston of area A_2, so that F_2/A_2 equals p. Since

$$\frac{F_2}{A_2} = p = \frac{F_1}{A_1}$$

$$\frac{F_2}{F_1} = \frac{A_2}{A_1}$$

The ratio between the output and the input force is therefore determined by the ratio between the areas of the output to the input piston.

4.3 UPTHRUST

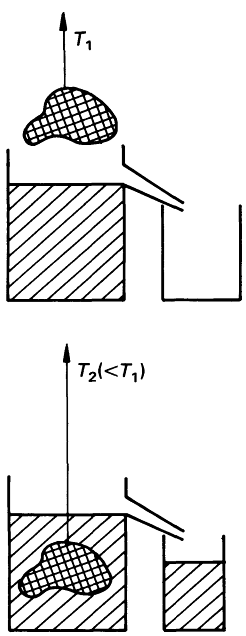

Figure 4.4

From experience we recognise that an object placed in a liquid seems to get lighter. *Archimedes' principle* tells us that an object that is either totally or partially immersed in a fluid (liquid or gas) experiences an *upthrust* equal to the weight of fluid it displaces.

An object totally immersed in a liquid will displace a volume of liquid equal to its own. Figure 4.4 shows how the displaced liquid can be collected via a side arm fitted to the container from which it is displaced. If the object is suspended from a spring balance, the upthrust it experiences corresponds to the apparent reduction in its weight ($T_1 - T_2$ in the figure) as it is immersed in the liquid. And, from Archimedes' principle, we find that the upthrust is equal to the weight of liquid displaced.

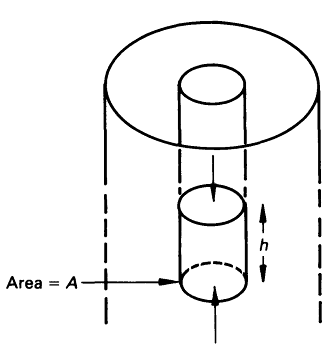

Figure 4.5

To help us understand this, Figure 4.5 shows a cylindrical solid object, of height h and cross-sectional area A, replacing part of the liquid column we considered earlier. The pressure acting on the top surface of the cylinder will be less than the pressure at the bottom, where the liquid is deeper. Since pressure acts perpendicularly to solid surfaces, the pressure at the top will act downwards and the pressure at the bottom will act upwards. The vertical pressure difference will be ρgh (where ρ is the density of the liquid), so the net upward force or upthrust acting on the cylinder will be $A \times \rho gh$, which is precisely the same as the weight of the liquid ($Ah \times \rho g$) that it replaces. (The horizontal pressure acting on the sides of the cylinder will have no vertical effect.)

If the density of the object is greater than the density of the liquid, its weight cannot be supported by the upthrust and it will sink. If its density is less than that of the liquid, the upthrust will exceed its weight and it will rise. If its density is the same, it will stay where it is.

Archimedes' principle provides us with a very simple method of finding the relative density of a solid by dividing its weight in air by its apparent loss in weight in water.

Worked Example 4.3

A cube of density 2500 kg m^{-3} weighs 24.5 N in air. Find its apparent weight if it is

(a) totally immersed in water ($\rho = 1000$ kg m^{-3});
(b) half immersed in water;
(c) totally immersed in liquid of density 800 kg m^{-3};
(d) totally immersed in mercury ($\rho = 13.6 \times 10^3$ kg m^{-3}).

($g = 9.8$ N kg^{-1}.)

The mass of the cube = 24.5/9.8 = 2.5 kg. The volume of the cube = 2.5/2500 = 1×10^{-3} m^3.

(a) From Equation (4.1) ($m = V \times \rho$), the mass of water displaced by the cube is equal to

$$1 \times 10^{-3} \times 1 \times 10^3 = 1 \text{ kg}$$

Therefore, the upthrust (weight of water displaced) is equal to

$$1 \times 9.8 = 9.8 \text{ N}$$

Therefore,

$$\text{apparent weight of cube} = 24.5 - 9.8 = 14.7 \text{ N}$$

(b) The upthrust on half the cube = 4.9 N; therefore,

$$\text{apparent weight of cube} = 24.5 - 4.9 = 19.6 \text{ N}$$

(c) The mass of liquid displaced by the cube is equal to

$$1 \times 10^{-3} \times 0.8 \times 10^3 = 0.8 \text{ kg}$$

Therefore,

$$\text{upthrust} = 0.8 \times 9.8 = 7.8 \text{ N}$$

Therefore,

$$\text{apparent weight of cube} = 24.5 - 7.8 = 16.7 \text{ N}$$

(d) The mass of mercury displaced by the cube is equal to

$$1 \times 10^{-3} \times 13.6 \times 10^3 = 13.6 \text{ kg}$$

Therefore,

$$\text{upthrust} = 13.6 \times 9.8 = 133.3 \text{ N}$$

Therefore,

$$\text{apparent weight of the cube} = 24.5 - 133.3 = -108.8 \text{ N}$$

That is to say, the upthrust exceeds the weight of the cube by 108.8 N, which is therefore the force that would be needed to hold the cube beneath the surface of the mercury.

If an object is floating, so that the upthrust supports its weight exactly, then it must displace its own weight of liquid. If it has the same density as the liquid, then it will be totally immersed. If it is of lower density, then it will only need to displace part of its volume, so only part of it will sink below the surface; and the higher the density of the liquid the less the object has to sink. A steel ship floats because it is hollow and its average density is lower than that of water. But the density of water varies with dissolved salts and with temperature; thus, the ship floats higher in sea-water than in fresh water and lower in warm water than in cold water. The *hydrometer*, used to measure the relative density of liquids (e.g. battery acid), works in a similar way.

Worked Example 4.4

A piece of wood, of density 650 kg m^{-3}, measures 400 × 300 × 50 mm thick. How much of its thickness is immersed when it is floating in water? By how much does it sink if a 1 kg mass is placed centrally on top of it?

$$m = V \times \rho$$

Therefore,

$$m = (0.4 \times 0.3 \times 0.05) \times 650 = 3.9 \text{ kg}$$

This will displace 3.9 kg water, which occupies

$$3.9/1000 = 3.9 \times 10^{-3} \text{ m}^3$$

If d is the thickness immersed, then

$$d \times 0.4 \times 0.3 = 3.9 \times 10^{-3}\ \text{m}^3$$

which gives

$$d = 33\ \text{mm}$$

The 1 kg mass will displace a further $1.0 \times 10^{-3}\ \text{m}^3$ of water, so the wood will sink a further distance equal to

$$\frac{1.0 \times 10^{-3}}{0.4 \times 0.3}\ \text{m} = 8\ \text{mm}$$

Questions

(Where necessary assume that $g = 9.8\ \text{N kg}^{-1}$ and that $\rho_{water} = 1000\ \text{kg m}^{-3}$.)

1. What is the mass of air in a room measuring $7 \times 3 \times 3$ m? (Assume $\rho_{air} = 1.3\ \text{kg m}^{-3}$.)
2. What is the volume of a sample of concrete that weighs 28.7 N and is known to have a density of $2.39 \times 10^3\ \text{kg m}^{-3}$?
3. Assuming the earth's mass to be 6.0×10^{24} kg and its radius to be 6.4×10^6 m, what is its average density?
4. A bottle weighs 95.7 g when it is empty and 308.4 g when it is full of water. If 207.3 g of sand is placed in the empty bottle, then 133.6 g of water is required to fill it. Find the density of the sand, giving your answer in kg m^{-3}.
5. The pistons of a hydraulic press have diameters of 16 mm and 32 mm, respectively. What input force should be applied to the smaller piston to produce an output force of 1000 N, and what would be the corresponding pressure?
6. Convert the following to Pa: (a) 3.2 lb in^{-2}, (b) 14 torr, (c) 998 mbar, (d) 765 mmHg, (e) 8.9 atm.
7. An open-ended U-tube of uniform bore is partially filled with mercury. If 38.9 cm^3 of liquid of density 800 kg m^{-3} is poured into one side, what volume of water must be poured into the other to keep the mercury levels equal?

8. An irregularly shaped metal object weighs 15.6 N in air, 13.6 N in water and 14.0 N in an unknown liquid.

   (a) Find the density of the metal.
   (b) Find the density of the unknown liquid.

9. A polythene block of mass 138 g is held under water by means of a thread fastened to the bottom of the container. Find the tension in the thread. ($\rho = 920$ kg m^{-3} for the polythene.)

TOPIC 5 DISPLACEMENT, VELOCITY AND ACCELERATION

COVERING:

- accelerated motion in a straight line;
- the equations of motion (kinematic equations);
- acceleration due to gravity;
- projectiles.

So far our discussion has been about systems that are either at rest or in a state of uniform motion. This topic introduces the idea of accelerated motion but, for the moment, without any reference to the forces that cause it.

There are two basic quantities that help us to define motion — namely length and time. The first quantity derived from these is velocity, which is the rate at which displacement changes with time.

Before going any further, it is important to make a careful distinction between speed and velocity. Speed is a scalar quantity measured in terms of magnitude alone — that is to say, it represents the rate of change of distance with time. Velocity is a vector quantity which has direction as well as magnitude. At any particular moment, the speed of a moving body represents the magnitude of its velocity. If a body travels along a curved path at constant speed, its velocity is changing because its direction is changing.

To start at the simplest level, we shall consider motion in a straight line. (In effect, we can then think of any change in displacement in terms of change in distance alone.) Note that because they are both directional, displacement and velocity can have negative values signifying movement opposite to some reference direction already taken as being positive. For example, if we think of an object thrown straight upwards as having a positive velocity, then once it starts to fall it will have a negative velocity.

Acceleration is the rate at which velocity changes. In terms of SI units, velocity is measured in metres per second (m s^{-1}) and acceleration in metres per second per second (m s^{-2}). When velocity is decreasing, acceleration has a negative value and is generally called either retardation or deceleration.

These definitions of velocity and acceleration lead to some useful equations that describe the motion of a body undergoing uniform acceleration in a straight line. (Uniform acceleration means that velocity is changing at a constant rate. For example, a body starting from rest with a uniform acceleration of 2 m s^{-2} will have a velocity of 0 m s^{-1} at 0 s, 2 m s^{-1} at 1 s, 4 m s^{-1} at 2 s, 6 m s^{-1} at 3 s, and so on.)

If a body accelerates uniformly from an initial velocity u to a final velocity v in a period of time t, then the acceleration a is given by the change in velocity per unit time, as follows:

$$a = \frac{v - u}{t}$$

which, on rearranging, becomes

$$v = u + at \tag{5.1}$$

Since velocity is defined as displacement per unit time, the displacement s of the body can be obtained by multiplying its average velocity, $(u + v)/2$, by the time t over which the acceleration takes place. Thus,

$$s = \left(\frac{u + v}{2}\right) t \tag{5.2}$$

By substituting $(u + at)$ for the final velocity v (from Equation 5.1), Equation (5.2) becomes

$$s = \left(\frac{u + u + at}{2}\right) t$$

Therefore,

$$s = \frac{2ut + at^2}{2}$$

and

$$s = ut + \frac{1}{2}at^2 \tag{5.3}$$

Finally, by substituting $2s/(u + v)$ for t (from Equation 5.2), Equation (5.1) becomes

$$v = u + a\,\frac{2s}{(u + v)}$$

and, on rearrangement,

$$2as = (v - u)(u + v)$$

Therefore,

$$2as = v^2 - u^2$$

and

$$v^2 = u^2 + 2as \qquad (5.4)$$

This gives us four equations involving the five quantities u, v, t, a and s, as summarised in Table 5.1. Careful inspection of the table shows that we can calculate the value of any two quantities provided that we know the values of the other three. For example, given u, a and s, we can calculate v from Equation (5.4) and, quite independently, t from Equation (5.3); given u, v and t, we can calculate a from Equation (5.1) and s from Equation (5.2) — and so on.

Table 5.1

Equation	Involves	Omits
(5.1) $v = u + at$	v, u, a, t	s
(5.2) $s = \frac{(u + v)}{2} t$	s, u, v, t	a
(5.3) $s = ut + \frac{1}{2} at^2$	s, u, t, a	v
(5.4) $v^2 = u^2 + 2as$	v, u, a, s	t

Worked Example 5.1

An object is uniformly accelerated from rest to 25 m s^{-1} over a period of 10 s. Find:

(a) the acceleration;
(b) the distance travelled;
(c) the extra time needed to reach 50 m s^{-1};
(d) the uniform acceleration needed to bring it to rest from 50 m s^{-1} in a distance of 250 m.

(a) $u = 0$, $v = 25$, $t = 10$, $a = ?$

From Equation (5.1)

$$25 = a \times 10$$

Therefore,

$$a = 2.5 \text{ m s}^{-2}$$

(b) $u = 0$, $v = 25$, $t = 10$, $s = ?$

From Equation (5.2)

$$s = \frac{(0 + 25)}{2} \times 10 = 125 \text{ m}$$

(c) $u = 25$, $v = 50$, $a = 2.5$, $t = ?$

From Equation (5.1)

$$50 = 25 + (2.5 \times t)$$

Therefore,

$$t = 10 \text{ s}$$

(d) $u = 50$, $v = 0$, $s = 250$, $a = ?$

From Equation (5.4) (where a is a retardation)

$$0 = 50^2 + 2 \times (-a) \times 250$$

Therefore,

$$a = 5 \text{ m s}^{-2}$$

Worked Example 5.2

An object at rest experiences a uniform acceleration of 5 m s^{-2} for 6 s. It maintains constant velocity for 14 s and is then brought to rest in 5 s by a uniform retardation. How far has it travelled?

Acceleration stage:

$u = 0$, $t = 6$, $a = 5$, $s = ?$, $v = ?$

From Equation (5.3)

$$s = \frac{1}{2} \times 5 \times 36 = 90 \text{ m}$$

From Equation (5.1)

$$v = 5 \times 6 = 30 \text{ m s}^{-1}$$

Constant velocity stage:

$$u = v = 30,\ t = 14,\ a = 0,\ s = ?$$

From Equation (5.3)

$$s = 30 \times 14 = 420 \text{ m}$$

Retardation stage:

$$u = 30,\ v = 0,\ t = 5,\ s = ?$$

From Equation (5.2)

$$s = \frac{(30 + 0)}{2} \times 5 = 75 \text{ m}$$

Therefore,

total distance travelled = 90 + 420 + 75 = 585 m

Note that all the four equations (5.1–5.4) contain u. If u is one of two unknowns, its value must be calculated first by using the appropriate equation. Then it can be substituted into any of the others to find the value of the second unknown. For example, if u and v are unknown, the value of u must be obtained from Equation (5.3); then the value of v can be obtained from any of the others.

Worked Example 5.3

A moving particle experienced an acceleration of 10 m s^{-2} over a period of 8 s, during which time it travelled 400 m. What were its initial and final velocities?

$$t = 8,\ a = 10,\ s = 400,\ u = ?,\ v = ?$$

From Equation (5.3)

$$400 = (u \times 8) + (\frac{1}{2} \times 10 \times 64)$$

Therefore,

$$u = 10 \text{ m s}^{-2}$$

From Equation (5.1)

$$v = 10 + (10 \times 8) = 90 \text{ m s}^{-1}$$

Equations (5.1)–(5.4), which are called the *equations of motion* or *kinematic equations*, can be used for calculations that involve bodies moving under the influence of gravity. We shall begin by considering this in terms of motion in a vertical straight line.

The first thing we need to recognise is that all objects allowed to fall freely close to the earth's surface experience the same downward acceleration. This fact may seem surprising and contrary to experience, because it tends to be obscured by the effects of air resistance and, for very light objects, the slight upthrust in air. A leaf or a feather seems to drift downwards at a more or less steady rate, and even a dense object reaches a steady *terminal velocity* when its weight is balanced by the effects of upthrust and air resistance (so that the resultant force is zero). But if an object of any mass or density is allowed to fall *freely* (i.e. with unimpeded motion, as in a vacuum), then it will accelerate uniformly. This *acceleration due to gravity*, sometimes called the *acceleration of free fall*, has the value 9.8 m s^{-2}. (There are slight variations over the earth's surface but we shall ignore them in this book.)

As we shall see in the next topic, it is no coincidence that the strength of the earth's gravitational field has the same numerical value as the acceleration due to gravity. In fact, engineers tend to regard g as an acceleration (9.8 m s^{-2}) rather than a field strength (9.8 N kg^{-1}). So, provided that any effect due to the air is small enough to be ignored, $a = g = 9.8$ m s^{-2} in the equations of motion when they are applied to objects moving vertically under the influence of gravity.

Sometimes the calculations are simplified by either u or v having the value zero. For example, if an object falls from rest (so that $u = 0$), then Equation (5.1) gives the relationship between its velocity and the time it has been falling (i.e. $v = gt$) and we can calculate the velocity after a given time or the time required to reach a given velocity. Similarly, Equations (5.3) and (5.4) give simple relationships between displacement and time and between final velocity and displacement, respectively, if the object falls from rest.

If an object is thrown vertically upwards with a known initial velocity u, then, by setting v to zero, we can calculate the maximum height s it reaches (Equation 5.4) and the time t it takes to get there

(Equation 5.1). Note that if we take the upward direction as positive, then u and s have positive values and $a = -g$ because it acts downwards.

Worked Example 5.4

An object is thrown vertically upwards with an initial velocity of 20 m s^{-1}. What height does it reach, and what is the total time it takes to return to its starting point?

$u = 20,\ v = 0,\ a = g = 9.8,\ s = ?,\ t = ?$

From Equation (5.4) (taking upwards as positive)

$$0 = 400 + (2 \times (-9.8) \times s)$$

which gives

$$s = 20.4 \text{ m}$$

From Equation (5.1)

$$0 = 20 + ((-9.8) \times t)$$

which gives

$$t = 2.04 \text{ s}$$

The object will take an equal time to return to its starting point, so the total time is 4.08 s.

Now let us add a second dimension, Figure 5.1 shows the path of an object thrown horizontally from a height s above the ground. Its velocity is resolved into horizontal and vertical components along the x- and y-axes, respectively. Since the object is thrown horizontally, its initial velocity u has no vertical component (i.e. $u_x = u$ and $u_y = 0$) but it immediately starts to fall, so at any subsequent moment its velocity v is the resultant of v_x and v_y. These components can be considered quite independently and the object takes the same length of time to hit the ground as if it had fallen from rest from the same height. Knowing the height s, we can calculate the time of flight t from Equation (5.3) (remembering that $u_y = 0$). If we neglect air resistance, the horizontal component of the velocity will remain constant and equal to u_x throughout the flight. This means that we

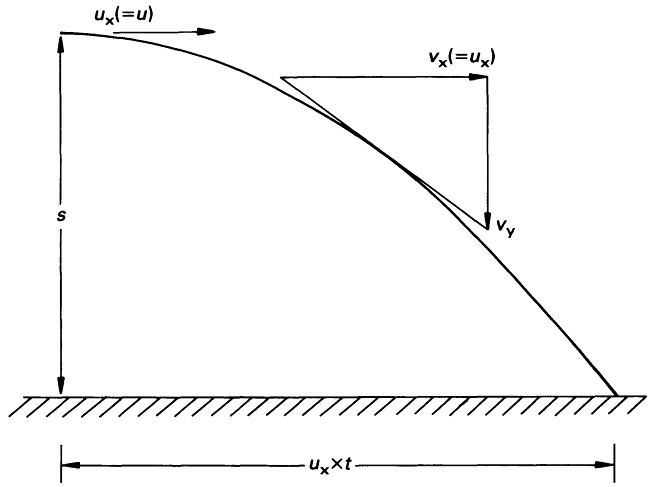

Figure 5.1

can calculate the horizontal distance the object travels by multiplying u_x by t.

We can also calculate the magnitude and direction of its velocity by finding the vertical component v_y at any time (Equation 5.1) or at any height (Equation 5.4) and combining it with the horizontal component v_x $(= u_x)$.

If the object had been projected from the ground with an equal and opposite velocity to that with which it had landed, then its path would have been the same as in Figure 5.1, but, of course, traversed in the opposite direction. (In this case the figure only shows half the path if the object goes on to complete its trajectory and return to the ground.)

Worked Example 5.5

An object is thrown horizontally at 6.0 m s^{-1} from a height of 3.3 m above level ground. How far does the object travel horizontally, and what is the magnitude and the direction of its velocity when it hits the ground?

Vertically

$$u_y = 0,\ a = g = 9.8,\ s = 3.3,\ t = ?,\ v_y = ?$$

From Equation (5.3) (taking downwards as positive)

$$3.3 = \frac{1}{2} \times 9.8 \times t^2$$

which gives

$$t = 0.82 \text{ s}$$

Therefore, the horizontal distance travelled is equal to

$$6 \times 0.82 = 4.9 \text{ m}$$

From Equation (5.4)

$$v_y^2 = 2 \times 9.8 \times 3.3$$

Therefore,

$$v_y = 8.0 \text{ m s}^{-1}$$

The magnitude of the final velocity is equal to

$$\sqrt{6.0^2 + 8.0^2} = 10.0 \text{ m s}^{-1}$$

The direction of the final velocity is equal to
$\tan^{-1} 8/6 = 53°$ below the horizontal

Worked Example 5.6

An object is projected at 10 m s^{-1} at an angle of 53° above the horizontal. What is the maximum height it reaches, and what horizontal distance does it cover?

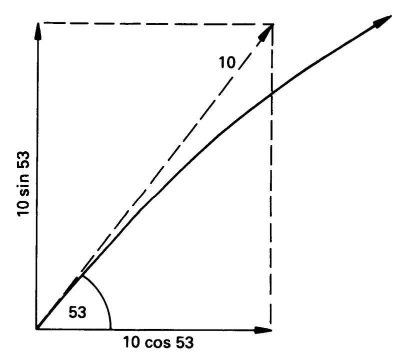

Figure 5.2

The horizontal and vertical components of the initial velocity are shown in Figure 5.2. First, note that 10 sin 53 = 8 and 10 cos 53 = 6.
Vertically:

$$u_y = 10 \sin 53,\ v_y = 0,\ a = g = 9.8,\ s = ?,\ t = ?$$

From Equation (5.4) (taking upwards as positive)

$$0 = (10 \sin 53)^2 + (2 \times (-9.8) \times s)$$

which gives

$$s = 3.3 \text{ m}$$

From Equation (5.1)

$$0 = 10 \sin 53 + ((-9.8) \times t)$$

which gives

$t = 0.82$ s

Horizontally:

$u_x = 10 \cos 53$, $t = 0.82$, $s = ?$

From Equation (5.3) $a = 0$ if we ignore wind resistance, and

$s = 10 \cos 53 \times 0.82 = 4.9$ m

The object travels a further 4.9 m in returning to the ground therefore total horizontal distance $= 2s = 9.8$ m

---

Questions

(Apart from Questions 9 and 10, assume motion in a straight line. Where necessary, assume $g = 9.8$ m s^{-2} and neglect air resistance and upthrust.)

1. An object is travelling at 12 m s^{-1}. For what period of time must it be accelerated at 4 m s^{-2} in order to reach 48 m s^{-1}?
2. What distance is taken for an object to reach 50 m s^{-1} from rest if it is accelerated at 2.5 m s^{-1}?
3. An object is projected vertically upwards from ground level at 49 m s^{-1}. Find its height after 1 s, 3 s, 5 s, 7 s and 9 s. What is its total time of flight?
4. An object is travelling at 12 m s^{-1}. It experiences a uniform acceleration and covers the next 36 m in 6 s. What is its final velocity and its acceleration?
5. An object initially at rest experiences an acceleration of 6 m s^{-1} for 4.5 s. What acceleration is required to return it to rest in 9 s?
6. An object that had fallen from a height of 128 m was found to have penetrated the ground to a depth of 40 mm. Estimate its average deceleration.
7. An object accelerates from 10 m s^{-1} to 50 m s^{-1} over a distance of 120 m. Calculate the acceleration and the time taken.
8. An object thrown vertically upwards reached a height of 28.8 m in 1.75 s. What maximum height did it reach and what time did it take to get there?

9. An object is dropped from an aircraft that is flying horizontally with a velocity of 100 m s^{-1} at a height of 250 m. What horizontal distance does the object cover before hitting the ground and what is the magnitude and direction of its impact velocity?

10. An object is projected with a velocity u at an angle θ above the horizontal. If g is the acceleration due to gravity, derive formulae giving (a) the maximum height to which it rises and (b) the time it takes to get there.

TOPIC 6 FORCE AND MOTION

COVERING:

- Newton's laws of motion;
- force and acceleration;
- action and reaction.

Newton's first law tells us that an object will remain at rest or in a state of uniform motion unless a force acts on it. In effect, this defines force as an influence which tends to change the velocity of an object. (Remember that the force in question might be the resultant of two or more others.)

Newton's second law tells us that, when an object is acted upon by a force, it will accelerate in accordance with the expression

$$F = ma \tag{6.1}$$

where F is the force (measured in N), m is the mass of the object (in kg) and a is its acceleration (in m s^{-2}).

As we might expect, the acceleration lies in the same direction as that of the force. The equation tells us that the larger the mass the larger the force needed to produce a given acceleration. Or, looking at it another way, the magnitude of a force can be found by measuring the acceleration that it gives to a known mass. Furthermore, if there is no force, then there can be no acceleration (which is another way of stating the first law). The equation also tells us that 1 N is the magnitude of the force that gives a 1 kg mass an acceleration of 1 m s^{-2}. (As we shall see in the next topic, the second law can also be stated in terms of momentum rather than acceleration.)

In Topic 2 we saw that the force due to gravity acting on a mass m (i.e. its weight W) provides a measure of the gravitational field strength g. At the earth's surface $g = W/m = 9.8$ N kg^{-1}. If the mass is allowed to fall freely, then Equation (6.1) tells us that, since $F = W$, W/m also gives its acceleration. So we can regard g either as a gravitational field strength or as an acceleration equal to 9.8 m s^{-2}. As we noted in the last topic, engineers generally regard it as an acceleration, so we shall do the same.

Note that Equation (6.1) does not tell us anything about the displacement or velocity of an object. If we need to know about these, and the force is constant, then we use the equations of motion.

Worked Example 6.1

A steady horizontal force of 12 N is applied to a 6 kg mass that is at rest on a smooth level surface. After 15 s (a) how far has the mass moved and (b) what is the magnitude of its velocity?

(a) $F = ma$

Therefore,

$$a = F/m = 12/6 = 2 \text{ m s}^{-2}$$

Now

$$u = 0, t = 15, a = 2, s = ?$$

so substituting in

$$s = ut + \frac{1}{2}at^2$$

we obtain

$$s = \frac{1}{2} \times 2 \times 15^2 = 225 \text{ m}$$

(b) $u = 0, \quad t = 15, \quad a = 2, v = ?$

Substituting in

$$v = u + at$$

we obtain

$$v = 2 \times 15 = 30 \text{ m s}^{-1}$$

Worked Example 6.2

A 15 kg mass is uniformly accelerated from 10 m s^{-1} to 20 m s^{-1} over a distance of 300 m. What force is being exerted on it?

$$u = 10, v = 20, s = 300, a = ?$$

Substituting in

$$v^2 = u^2 + 2as$$

we obtain

$$400 = 100 + (2 \times a \times 300)$$

which gives

$$a = 0.5 \text{ m s}^{-2}$$

Therefore,

$$F = ma = 15 \times 0.5 = 7.5 \text{ N}$$

Worked Example 6.3

A ball of 100 g mass fell from a height of 10 m and rebounded to a height of 5 m. Assuming that it remained in contact with the ground for 12 ms, what was the average force it exerted on the ground? ($g = 9.8 \text{ m s}^{-2}$.)

Considering the ball as it fell downwards,

$$u = 0,\ a = 9.8,\ s = 10,\ v = ?$$

Substituting in

$$v^2 = u^2 + 2as$$

we obtain

$$v^2 = 2 \times 9.8 \times 10$$

which gives

$v = 14.0 \text{ m s}^{-1}$ downwards at impact

Considering the ball as it rebounded upwards,

$$v = 0,\ a = -\,9.8,\ s = 5,\ u = ?$$

Substituting in

$$v^2 = u^2 + 2as$$

we obtain

$$0 = u^2 + (2 \times (-9.8) \times 5)$$

which gives

$$u = 9.9 \text{ m s}^{-1} \text{ upwards after impact}$$

The change in velocity due to the impact was therefore 23.9 m s^{-1} (i.e. from 14.0 m s^{-1} downwards to 9.9 m s^{-1} upwards). This took place over a period of 12×10^{-3} s and corresponds to an average acceleration of

$$\frac{23.9}{12 \times 10^{-3}} = 2.0 \times 10^3 \text{ m s}^{-2}$$

Therefore,

$$F = ma = 0.1 \times 2.0 \times 10^3 = 200 \text{ N}$$

Newton's third law states that action and reaction are equal and opposite. In other words, if one object exerts a force on another, then the second object exerts an equal but opposite force on the first. For instance, viewing Newton's third law in terms of his law of gravitation (Equation 2.1 on page 12), we can say that the earth exerts a gravitational force on an apple above its surface and the apple exerts an equal and opposite gravitational force on the earth. If the apple is released, the second law tells us that

$$\text{force} = m_{\text{apple}} \times a_{\text{apple}} = m_{\text{earth}} \times a_{\text{earth}}$$

That is to say, the apple will fall towards the earth and, at the same time, the earth will fall towards the apple though its acceleration (a_{earth} = force/m_{earth}) will be infinitesimal because its mass is so large.

The third law applies equally well to objects in contact. If the apple is lying on the ground, it presses downwards with a force equal to its weight and the ground reacts with an equal and opposite force on the apple. When a motor car tyre pushes backwards on the road, there is an equal and opposite reaction as the road pushes forward on the tyre — and the car moves forwards.

The laws of motion help us where vertical forces and acceleration are involved. Let us imagine an object of mass m resting on the floor of a lift. If the lift is stationary or moving with uniform velocity (i.e. $a= 0$), then there can be no net vertical force acting on the object, because $F = ma = 0$. The object exerts a downward force mg on the floor and there is an equal and opposite reaction as the floor pushes upwards on the object.

If the lift starts to move upwards with a uniform acceleration a, then not only does it have to support the object's weight mg, it also has to support the additional force $F = ma$ needed to make the object accelerate upwards. The floor of the lift therefore pushes upwards on the object with a total force $(mg + ma)$ and the object experiences an apparent increase in weight.

If the lift starts to move downwards with an acceleration a, then subsequent events depend on whether a is less than, equal to or greater than g, the acceleration due to gravity.

If the lift is propelled downwards so that a is greater than g, then the object will be left behind and, neglecting the effects of the air, will fall freely with acceleration g (until the ceiling of the lift catches up with it). If the cable breaks, then $a = g$ and the lift and the object will both fall freely together.

If a is less than g, then the resultant force ma accelerating the object downwards will be its weight mg less the support it receives from the floor (i.e. its apparent weight), so that

$$ma = mg - \text{apparent weight}$$

Therefore,

$$\text{apparent weight} = mg - ma$$

We shall consider further consequences of Newton's laws in the next topic.

Worked Example 6.4

An object of 30 kg mass is being propelled across a horizontal surface with a horizontal force of 202 N. If the coefficient of kinetic friction is 0.55, find the acceleration. (Assume $g = 9.8 \text{ m s}^{-2}$.)

$$\text{frictional force} = \mu_k N = 0.55 \times 30 \times 9.8 = 162 \text{ N}$$

Therefore,

$$\text{resultant force} = 202 - 162 = 40 \text{ N}$$

and

$$a = F/m = 40/30 = 1.3 \text{ m s}^{-2}$$

Worked Example 6.5

A 6 kg mass is connected to a 3 kg mass by a string that passes over a frictionless pulley. Assuming that the string and pulley have no effect on the motion of the system, find the tension in the string and the acceleration of the masses. (Assume $g = 9.8 \text{ m s}^{-2}$.)

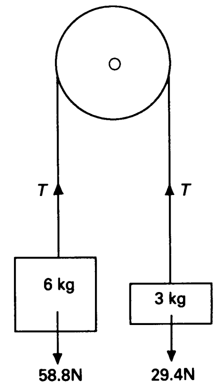

Figure 6.1

The weights experienced by the masses are 58.8 N and 29.4 N, as shown in Figure 6.1. Since the pulley is frictionless, the tension T in the string will be uniform along its length. Clearly the 6 kg mass will fall and the 3 kg mass will rise with the same acceleration. Taking this direction (with the pulley rotating anticlockwise) as positive, then the net force acting on the 6 kg mass is $(58.8 - T)$ and, from the second law,

$$a = \frac{F}{m} = \frac{(58.8 - T)}{6}$$

Similarly, for the 3 kg mass

$$a = \frac{F}{m} = \frac{(T - 29.4)}{3}$$

Since a is the same in both cases

$$\frac{(58.8 - T)}{6} = \frac{(T - 29.4)}{3}$$

which gives

$$T = 39.2 \text{ N}$$

and, from above,

$$a = \frac{(58.8 - T)}{6} = \frac{(58.8 - 39.2)}{6} = 3.3 \text{ m s}^{-2}$$

Questions

(Where necessary assume that $g = 9.8 \text{ m s}^{-2}$.)

1. If an apple weighing 1.0 N falls to the ground, what is the acceleration of the earth towards the apple? (Mass of the earth $= 6.0 \times 10^{24}$ kg.)

2. What is the acceleration of an 18 kg mass upon which a force of 27 N is being exerted? If it starts from rest, how long does it take to cover a distance of 50 m?
3. A mass of 8 kg is being pulled by a force of 34 N in the 3 o'clock direction and by a force of 18 N in the 9 o'clock direction. If its initial speed is 15 m s^{-1}, how long does it take for it to reach 30 m s^{-1}?
4. A horizontal force of 247 N is applied to an object of 25 kg mass resting on a level surface. Assuming $\mu_s = \mu_k = 0.60$, what distance does the object travel in 10 s?
5. A 22 kg mass is subjected to three forces: 12 N at 0°, 18 N at 45° and 24 N at 90°. Find its acceleration.
6. A 6 kg mass on a frictionless horizontal surface is subjected to two forces: 14 N at 30° and 17 N at 60°. If it starts at rest, what point will it reach after 12 s?
7. An object of mass 4.5 kg accelerates at a rate of 2.0 m s^{-2} when acted upon by a force of 35 N parallel to the horizontal surface which supports it. Find the coefficient of kinetic friction between the object and the surface.
8. A catapult is used to throw a 50 g stone horizontally with a velocity of 14 m s^{-1}. If the catapult sling had been drawn back through a distance of 550 mm, estimate the average force it exerted on the stone when it was released.
9. A 10 kg mass hangs from a string which can support a maximum load of 200 N. What is the maximum acceleration that can be used to raise the mass vertically by pulling the string without breaking it?
10. A 10 g projectile exerted an average force of 100 N on its target, which it penetrated to a depth of 0.5 m. Estimate its velocity on impact.
11. A passenger of 60 kg mass stands in a lift which accelerates upwards at 0.5 m s^{-2}. Find (a) the passenger's apparent weight, and (b) the tension in the lift cable. (The mass of the empty lift is 840 kg.)

TOPIC 7 MOMENTUM AND IMPULSE

COVERING:

- linear momentum;
- momentum, force and impulse;
- the principle of conservation of momentum.

Linear momentum is a physical quantity that provides another approach to the behaviour of objects in motion. It is a vector quantity obtained by multiplying the mass of an object by its velocity. It therefore has the unit kg m s^{-1}. Its direction is the same as that of the velocity of the object. The word *linear* is used to distinguish it from the angular momentum of a rotating body, which we shall meet later. The use of the word 'momentum' alone implies linear momentum, and that is the convention we shall adopt here.

Newton's second law of motion is often stated in a form telling us that the rate at which the momentum of an object changes with time is proportional to, and in the same direction as, the net force acting upon it. If the velocity of an object of mass m changes from u to v, then the change of momentum is given by $(mv - mu)$. If this change is brought about by a net force F acting on the object for a period of time t, then, using SI units, the second law can be expressed in the form

$$F = \frac{(mv - mu)}{t} \tag{7.1}$$

Note that, in the case of retardation, u will be greater than v and that, in any case, u and v may have positive or negative values, depending on the reference direction chosen. F may therefore have a negative value, which simply means that it is acting in the opposite direction to that originally chosen as being positive. Also note that $(v - u)/t$ gives the acceleration a, so Equation (7.1) can be reduced to the form we used in the last topic. Thus,

$$F = m\frac{(v - u)}{t} = ma$$

Rearranging Equation (7.1), we find that

$$Ft = (mv - mu)$$

Ft represents the *impulse* given to the body which changes its momentum. Impulse is a vector quantity like momentum and has the unit N s, which is, as the equation suggests, equivalent to momentum. (Remember that, by definition, 1 N = 1 kg m s^{-2}, so 1 N s = 1 kg m s^{-1}.) Note that a small force exerted over a long period can provide the same change in momentum as a large force over a short period. (It is better to fall onto a mattress than a concrete floor, because the same change in momentum is spread over a longer period and the average retarding force is correspondingly smaller.)

We now have three ways of viewing Equation (7.1):

force = mass × acceleration

force = change of momentum per unit time

impulse = force × time = change of momentum

The last of these leads us to a very important principle. Consider two objects which collide. According to Newton's third law, the force exerted by object A on object B will be equal and opposite to that exerted by B on A. Their respective impulses Ft must therefore be equal and opposite (since t is the same for both) and so must their respective changes in momentum. The total momentum of the system therefore remains constant. This principle, which is true for any system with any number of interacting objects, is known as the *principle of conservation of momentum*. It can be stated in the form that the total momentum of a system of interacting bodies remains unchanged as long as no external resultant force acts upon it.

This applies to all types of collisions, either where the objects involved move apart afterwards or where they stick together. If two objects of mass m_1 and m_2 collide with initial velocities u_1 and u_2, respectively, and then continue on their separate ways with velocities v_1 and v_2, their total momentum remains constant. Thus, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \tag{7.2}$$

If the objects stick together, then Equation (7.2) becomes

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v_c$$

where v_c is the velocity of the combined masses.

The principle can also be applied to a bullet fired from a gun. The momentum of the bullet is equal and opposite to the momentum of the gun as it recoils, so the total momentum remains zero:

$$m_{bullet}v_{bullet} - m_{gun}v_{gun} = 0$$

Similarly, the thrust of a rocket motor results from the momentum of its exhaust gases.

Worked Example 7.1

An object of mass 9.0 kg moves in a straight line at 5.0 m s^{-1} and collides with another object of mass 5.0 kg moving at 3.0 m s^{-1} in the same direction.

(a) If the 5.0 kg object moves on at 5.5 m s^{-1}, find the velocity of the 9.0 kg object.
(b) If the objects stick together, find the velocity of the combination.

(a) $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

Taking the common direction of motion of the objects as positive and substituting the given values,

$$(9.0 \times 5.0) + (5.0 \times 3.0) = (9.0 \times v_1) + (5.0 \times 5.5)$$

which gives

$$v_1 = 3.6 \text{ m s}^{-1} \text{ (in the same direction).}$$

(b) $m_1u_1 + m_2u_2 = (m_1 + m_2)v_c$

Taking the common direction of motion of the objects as positive and substituting the given values,

$$(9.0 \times 5.0) + (5.0 \times 3.0) = 14.0 \times v_c$$

which gives

$$v_c = 4.3 \text{ m s}^{-1} \text{ (in the same direction)}$$

Worked Example 7.2

An object of 5.0 kg mass moving in the 9 o'clock direction at 10.0 m s^{-1} collides with an object of 2.0 kg mass moving in the 3 o'clock direction at 7.5 m s^{-1}. What is their final velocity if they stick together?

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v_c$$

Taking the 3 o'clock direction as positive and substituting the given values,

$$(5.0 \times (-10.0)) + (2.0 \times 7.5) = 7.0 \times v_c$$

which gives

$$v_c = -5.0 \text{ m s}^{-1}$$

the minus sign indicating that v_c is in the opposite direction to that chosen as positive.

Worked Example 7.3

A ball of 100 g mass fell vertically to the ground with an impact velocity of 14 m s^{-1} and rebounded at 10 m s^{-1}. Assuming that it remained in contact with the ground for 12 ms, what was the average force it exerted on the ground?

$$F = m\frac{(v - u)}{t}$$

Taking the upward direction as positive and substituting the given values,

$$F = 0.1\frac{(10 - (-14))}{12 \times 10^{-3}} = 200 \text{ N}$$

Worked Example 7.4

If, in Worked Example 7.2, the 2.0 kg mass had been moving in the 12 o'clock direction at 7.5 m s^{-1}, what would the combined final velocity have been?

Since the momenta of the objects are vector quantities, their resultant can be found from the momentum diagram in Figure 7.1. From the diagram

$$R = \sqrt{50.0^2 + 15.0^2} = 52.2 \text{ kg m s}^{-1}$$

Therefore,

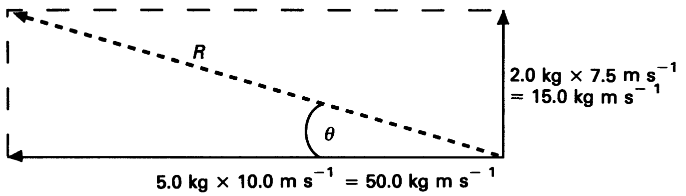

Figure 7.1

velocity = momentum/mass = 52.2/7.0 = 7.5 m s^{-1}

and

$\theta = \tan^{-1} 15.0/50.0 = 17°$

i.e. 163° anticlockwise from the positive *x*-axis.

Questions

(Where necessary assume that g = 9.8 m s^{-2}.)

1. What force is required to uniformly accelerate a 1000 kg vehicle from 40 km per hour to 60 km per hour in 10 s?
2. A projectile of mass 10 g exerts an average force of 20 N on its target as it is brought to a halt 0.05 s after its initial impact. Estimate the magnitude of its impact velocity.
3. A bullet of 15 g mass leaves the muzzle of a 5 kg rifle at 450 m s^{-1}. Find the recoil velocity of the rifle.
4. An object of 6000 kg mass, initially at rest, is subjected to a steady force of 1000 N for a period of 2 min. Assuming linear motion under frictionless conditions, calculate the object's final velocity. Compare its momentum with that of a 2000 kg mass after they have both been subjected to the same 1000 N force for a period of 3 min.
5. An object of 10 kg mass falls from a height of 10 m. Estimate the average retarding force acting upon it if it falls onto (a) a concrete floor which stops it in 0.01 s, and (b) a mattress which stops it in 0.06 s.
6. Water emerging from a 12 mm diameter horizontal pipe at 4.0 m s^{-1} strikes a vertical wall normal to its surface. What is the force exerted on the wall, assuming

that all the momentum of the water is lost on impact. ($\rho_{water} = 1 \times 10^3$ kg m^{-3}.)

7. A 3.00 kg object, at rest on a smooth frictionless horizontal surface, is struck by a bullet travelling horizontally at 370 m s^{-1}. The bullet is embedded in the object and they move off together at 4.27 m s^{-1}. Assuming motion in a straight line, find the mass of the bullet.

8. A 10 kg object falls vertically onto a 25 kg object travelling at 20 m s^{-1} in the 3 o'clock direction on a smooth horizontal frictionless surface. Find their combined velocity.

9. A 5 kg object is moving at 5 m s^{-1} in the 3 o'clock direction on a smooth horizontal surface. A 50 g bullet fired from the 6 o'clock direction strikes the object at 400 m s^{-1} and becomes embedded in it. What is their joint final velocity?

10. A rocket motor producing a thrust of 5.4 kN burns fuel at a rate of 3 kg s^{-1}. At what speed are the exhaust gases being ejected?

TOPIC 8 WORK, ENERGY AND POWER

COVERING:

- energy as the capacity to do work;
- potential energy (including strain energy);
- kinetic energy;
- the principle of conservation of energy;
- power and efficiency.

8.1 WORK

When an object moves under the influence of a force, then work is done according to the equation

$$W = F \times s \tag{8.1}$$

where W is the work done, F is the force (N) and s is the displacement (m) in the direction of the force. The SI unit of work is the *joule* (J), which is a scalar quantity defined as the work done when the point of application of a force of 1 N moves 1 m in the direction along which it is being applied. If the force is applied at an angle to the displacement, as in Figure 8.1, then we must use the magnitude of its component in the displacement direction, in which case Equation (8.1) becomes

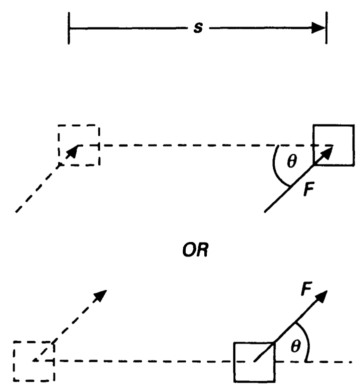

Figure 8.1

$$W = F \cos \theta \times s \tag{8.2}$$

where θ is the angle which the force makes with the displacement. If $\theta = 0°$, then $\cos \theta = 1$ and we have Equation (8.1). If $\theta = 90°$, $\cos \theta = 0$, so the force has no component and can do no work in the displacement direction.

Worked Example 8.1

A box is pulled a distance of 15 m across a level floor by a force of 237 N applied to it via a rope inclined at 25° above the horizontal (as in Figure 2.7 on page 19). How much work is done?

$$W = F \times \cos \theta \times s$$

and substituting the values given,

$$W = 237 \times \cos 25^\circ \times 15 = 3.2 \text{ kJ}$$

Note that there must be both force and displacement for work to be done. If an object is moving with uniform velocity with no net force acting upon it, then there is no work being done on it or by it. Nor is any work being done if the object remains stationary, no matter how large a force there might be acting on it.

8.2 ENERGY

Energy, which is also a scalar quantity measured in joules, is the capacity to do work. It exists in many forms (electrical energy, mechanical energy, thermal energy, and so on) and is transformed from one form to another when work is done. (There is also energy associated with mass in accordance with Einstein's theory of relativity but we shall not consider it here.)

In effect, we can regard energy as being stored by a system when work is done on it, or as being changed into another form, or forms, when work is done by a system. In the present context we shall consider potential energy and kinetic energy as representing mechanical work stored by an object by virtue of its position and motion, respectively.

8.3 POTENTIAL ENERGY

In the broadest sense, the potential energy of a system is derived from the relative position of its components. An apple hanging from a tree has potential energy. It has the capacity to do work by virtue of its position above the ground; it could, for example, be connected to a generator so that its potential energy is converted to electrical energy as it falls. In fact, the potential energy is possessed by the earth/apple system because of the gravitational force between them but, in view of the infinitesimal influence of the apple on the earth, it is more sensible to think in terms of the apple's potential energy in the earth's gravitational field.

The potential energy stored by an object by virtue of its height above the ground (or any other reference level) can be regarded as the work done in raising it against its weight mg through a vertical distance h. That is to say,

$$\text{potential energy } (= W = Fs) = mg \times h \qquad (8.3)$$

The potential energy of the object remains constant anywhere on an *equipotential surface* at any fixed height above a given reference level. It follows that, since horizontal movement has no effect on the potential energy, the route taken by an object to reach a given height, no matter how circuitous, has no effect on its final potential energy.

It is often helpful to be able to view work and energy in pictorial terms. The plot of force against displacement in Figure 8.2(a) gives a graphical description of how potential energy is stored in raising an object of weight mg vertically to a height h. Since the force required ($= mg$) is constant, the plot is a horizontal straight line of length h at a distance mg above the x-axis. The rectangle under the line therefore has an area $mg \times h$ which, as Equation (8.3) shows, represents the work done and, hence, the final potential energy of the object.

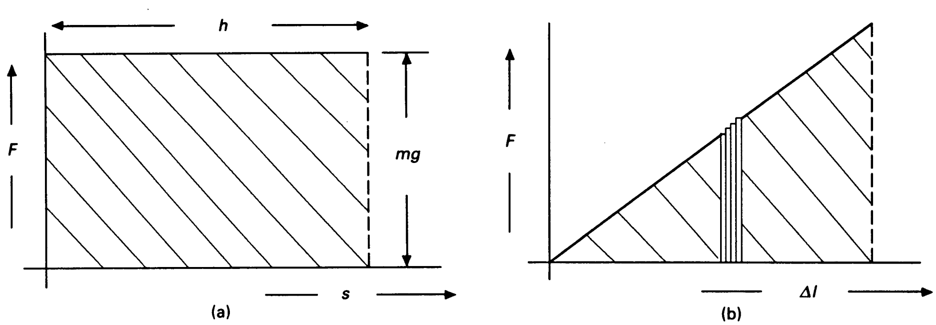

Figure 8.2

Strain energy, stored in a stretched wire, for example, can be treated in a similar way. Figure 8.2(b) represents the linear (i.e. proportional) relationship, following Hooke's law, between the stretching force F and the extension Δl (see Topic 2). (In effect, the extension is the displacement of the point of application of the force at the free end of the wire.) Like the potential energy example in Figure 8.2(a), the work done in stretching the wire is represented by the area under the graph.

We can think of this area as the sum of a very large number of extremely narrow strips (a few of which are shown in Figure 8.2b). Each strip is so narrow that it can be regarded as a rectangle where F is virtually constant. F, however, varies from one rectangle to the next, so we can regard the area under any force/displacement graph, no matter what its shape, as being made up of these narrow rectangles each representing a tiny amount of work done. In the case of a wire to which Hooke's law applies, the heights of the rectangles increase uniformly to give a triangle under the graph with an area equal to $\frac{1}{2}(F \times \Delta l)$ representing the work done and, hence, the strain energy stored in the wire.

Worked Example 8.2

From Figure 2.1 (page 14) find the energy stored in the wire when it is stretched by a force of 60 N.

Taking the extension to be 4.3 mm at 60 N,

$$\text{strain energy} = \frac{1}{2} \times F \times \Delta l = \frac{1}{2} \times 60 \times 0.0043 = 0.13 \text{ J}$$

8.4 KINETIC ENERGY

From Equation (5.4) (page 41) we know that if an object is accelerated from rest (i.e. $u = 0$) to a speed v, then $v^2 = 2as$, so the acceleration it experiences is given by $a = v^2/2s$. From Newton's second law of motion, the force required to produce this acceleration in a body of mass m is

$$F = ma = m \times \frac{v^2}{2s}$$

and, rearranging,

$$Fs = \frac{1}{2} mv^2 \tag{8.4}$$

where Fs is the work done in accelerating the object to velocity v. $\frac{1}{2}mv^2$ therefore represents the *kinetic energy* stored by an object by virtue of its motion. Since the kinetic energy of an object depends on v^2, then doubling v will increase its kinetic energy by a factor of 4 and trebling v will increase it by a factor of 9, and so on.

If the object had had an initial velocity u and, therefore, an initial kinetic energy $\frac{1}{2}mu^2$, then the work done to bring about the change in kinetic energy in accelerating or retarding it to velocity v would have been

$$Fs = \frac{1}{2} mv^2 - \frac{1}{2} mu^2 \tag{8.5}$$

Worked Example 8.3

An object of mass 5.0 kg falls to the ground from a height of 10.0 m. Find its total energy (a) initially, (b) half-way down and (c) immediately before impact. (Assume $g = 9.8$ m s^{-1} and that the air has no effect.)

(a) Initially, at a height of 10.0 m, the object possesses only potential energy and

$$mgh = 5.0 \times 9.8 \times 10.0 = 490 \text{ J}$$

(b) After falling 5.0 m, the velocity of the object (from Equation 5.4) is given by

$$v^2 = 2gs = 2 \times 9.8 \times 5.0 = 98$$

so its kinetic energy is

$$\frac{1}{2} mv^2 = \frac{1}{2} \times 5.0 \times 98 = 245 \text{ J}$$

Its potential energy (5.0 m above the ground) is given by

$$mgh = 5.0 \times 9.8 \times 5.0 = 245 \text{ J}$$

The total energy of the object is the sum of its kinetic energy and its potential energy, which is equal to

$$245 + 245 = 490 \text{ J}$$

(c) Immediately before impact (after falling 10.0 m) the potential energy of the object is zero and its velocity is given by

$$v^2 = 2gs = 2 \times 9.8 \times 10.0 = 196$$

so its total energy is entirely kinetic energy, given by

$$\frac{1}{2} mv^2 = \frac{1}{2} \times 5.0 \times 196 = 490 \text{ J}$$

(Note that the total energy of the object remains constant as it falls.)

Worked Example 8.4

If, in Worked Example 8.3, the object penetrates the ground to a depth of 50 mm after impact, estimate the average retarding force.

$v = 0$, $s = 0.05$ and, from Worked Example 8.3, $u = \sqrt{196}$.

$$Fs = \frac{1}{2} mv^2 - \frac{1}{2} mu^2$$

Therefore, taking downwards as the positive direction,

$$F = -\frac{mu^2}{2s} = -\frac{5.0 \times 196}{2 \times 0.05} = -9.8 \text{ kN}$$

(The minus sign indicates that the force acts upwards.)

8.5 CONSERVATION OF ENERGY

Although energy can be changed from one form to another, the *principle of conservation of energy* tells us that it cannot be created or destroyed. Worked Example 8.3 illustrates how the total energy of a falling object remains constant as its gravitational potential energy is traded for kinetic energy. Car engines change chemical energy (in petrol) into mechanical energy, electric generators change mechanical energy into electrical energy, and so on.

In practice, energy transformation always involves wastage. Much of the chemical energy stored in petrol is wasted in the form of heat as a by-product of providing a motor car with kinetic energy. Even a falling mass will not normally reach its theoretical velocity (hence, its theoretical kinetic energy) because of the air; and lifting the mass in the first place involves losses due to friction and raising moving parts of the lifting gear. Nevertheless none of the energy is truly lost, only changed into unwanted by-products.

Some energy changes are not so obvious. In the last topic we saw that momentum is conserved in collisions. But some, or even all, of the kinetic energy may be transformed. A collision between steel ball-bearings will be almost perfectly elastic; any kinetic energy converted to strain energy around the contact points will be recovered as the balls spring apart, although there might be slight losses in the form of heat or sound energy. At the other extreme, inelastic collisions can involve total kinetic energy loss. Two identical lumps of putty colliding with equal and opposite velocities (and momenta) along the same line will stick to each other and come to rest, so that all their kinetic energy is lost as heat and sound.

8.6 POWER

Power is a scalar quantity which gives the rate at which energy is transformed or work is done. The unit of power is the *watt* (W), which is equivalent to one joule per second, so

$$W \text{ (watts)} = \text{J s}^{-1} = \text{N m s}^{-1} = \text{N} \times \text{m s}^{-1}$$

$\text{N} \times \text{m s}^{-1}$ are the units of force times velocity. That is to say, if an object moves at a velocity v under the influence of a force F, then the mechanical power delivered to the object is $F \cos \theta \times v$, where θ is the angle between the line of action of the force and the direction of motion.

(Note that 1 horsepower is a unit of power that is equivalent to 746 W.)

8.7 EFFICIENCY

The efficiency of a machine or a process is given by the ratio between the useful energy output and the energy input, commonly expressed as a percentage. A motor that requires 1.0 kW of electrical power to provide 0.75 kW of mechanical power has an efficiency of

$$\frac{0.75}{1.00} \times 100 = 75\%$$

Questions

(Where necessary assume that $g = 9.8 \text{ m s}^{-2}$.)

1. 735 J is available to raise an object off the ground. What is the largest mass that can be lifted to a height of 15 m?
2. An object of 12.5 kg mass rests on a smooth (frictionless) horizontal surface. Calculate the work done if a 50 N force is applied to the object for a period of 10 s (a) parallel to the surface, (b) vertically upwards.
3. An object of 3 kg mass is travelling with a uniform velocity of 12 m s^{-1} across a smooth horizontal surface when a force of 38 N is suddenly applied in the displacement direction. What is the velocity of the object after it has travelled a further 30 m?
4. An object of 15 kg mass travels at 40 m s^{-1} across a smooth horizontal surface. Find the retarding force needed to bring it to a halt in a distance of 50 m.
5. An object 15 m above the ground has a potential energy of 735 J and a kinetic energy of 12.25 kJ. What is its speed?
6. What is the efficiency of a pump which uses 1.5 kW of electrical power in raising 85 litres of water through a height of 12 m in 10 s? (1 litre = 1000 cm^3 and $\rho_{water} = 1000 \text{ kg m}^{-3}$.)
7. A ball-bearing suspended from a weightless thread

swings to and fro so that its highest point is 94 mm and its lowest point is 32 mm above the table top. Find its maximum velocity.

8. An 80 kg man climbs a staircase 3 m high in 9.8 s. If he worked just as hard to drive a pedal-operated generator with an efficiency of 75%, how much electrical power would he produce?
9. An object of mass 2.5 kg, resting on a horizontal surface, is subjected to a horizontal force of 20 N. If $\mu_s = \mu_k = 0.49$, find the velocity of the object after it has travelled 10 m.
10. What is the power consumption when an object is pushed 16 m up a 26° slope in 12 s by a horizontal force of 50 N?

TOPIC 9 MOTION IN A CIRCLE

COVERING:

- angular displacement, velocity and acceleration;
- angular equations of motion;
- centripetal force and acceleration.

So far we have tended to think about objects that are either at rest or moving in straight lines. Now we need to consider circular motion and find angular equivalents of the linear parameters that we have already met.

9.1 ANGULAR DISPLACEMENT, VELOCITY AND ACCELERATION

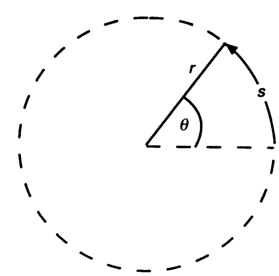

Figure 9.1

Let us begin by assuming that an object is moving round a circular track at a constant speed.

There are two easy ways of describing its displacement over a given period. As Figure 9.1 suggests, we can use the distance s it has moved round the circumference from its starting point. Alternatively we can use the *angular displacement* θ, i.e. the angle through which the radius has moved.

The SI unit for the measurement of angles is the radian (rad). In terms of Figure 9.1, the angle θ in radians is given by the length of the arc s divided by the radius r. Thus,

$$\theta = \frac{s}{r} \text{ rad} \qquad (9.1)$$

It follows that 1 rad is the angle where $s = r$ and that a linear (as opposed to angular) displacement round the circumference is given by $s = r\theta$. Note that for one complete lap of the circle $s = 2\pi r$ and

$$\theta = \frac{2\pi r}{r} = 2\pi = 6.28 \text{ rad}$$

which is equivalent to 360°; therefore, 1 rad = 57.3°. *Angular velocity* ω is the rate of angular displacement with time t and

$$\omega = \frac{\theta}{t} \text{ rad s}^{-1} \tag{9.2}$$

This can also be written in terms of the object's linear speed v ($= s/t$) around the circumference of the circle, since by combining Equations (9.1) and (9.2) to eliminate θ we have

$$\omega = \frac{s}{r \times t}$$

and because $v = s/t$

$$\omega = \frac{v}{r} \tag{9.3}$$

Note that the object's linear speed is related to its *period* T (the time for one complete revolution) by the expression $v = 2\pi r/T$, i.e. the circumference divided by the time taken to travel round it. Substituting ωr for v (Equation 9.3),

$$\omega r = \frac{2\pi r}{T}$$

Therefore,

$$T = \frac{2\pi}{\omega} \tag{9.4}$$

Now let us imagine that the angular velocity of the object is varying uniformly with time. If ω changes from ω_1 to ω_2 in time t, then the *angular acceleration* α is given by

$$\alpha = \frac{(\omega_2 - \omega_1)}{t} \text{ rad s}^{-2} \tag{9.5}$$

From Equation (9.3), $(\omega_2 - \omega_1) = (v_2 - v_1)/r$, so that

$$\alpha = \frac{(v_2 - v_1)}{rt}$$

and, since the linear acceleration a round the circumference of the circle is given by $(v_2 - v_1)/t$, then

$$\alpha = \frac{a}{r} \tag{9.6}$$

Thus, angular displacement, velocity and acceleration can all be obtained by dividing their linear counterparts by the radius r.

9.2 ANGULAR EQUATIONS OF MOTION

In Topic 5 we obtained four equations (5.1 – 5.4) giving relationships between displacement (s), velocity (u and v), uniform linear acceleration (a) and time (t). By using similar arguments (or by dividing each linear quantity by r) we can obtain four equivalent equations for motion in a circle where θ replaces s, ω_1 and ω_2 replace u and v, respectively, and α replaces a. Thus, we have

$$\omega_2 = \omega_1 + \alpha t \quad (9.7)$$

$$\theta = \frac{(\omega_1 + \omega_2)}{2} t \quad (9.8)$$

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2 \quad (9.9)$$

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta \quad (9.10)$$

Worked Example 9.1

A car with 500 mm diameter wheels moves in a straight line at 85 km per hour. How fast do the wheels turn (a) in rpm (revolutions per minute), and (b) in rad s^{-1}?

Converting km per hour to m s^{-1}, the linear speed of any point on the wheels' circumference is given by

$$85 \times \frac{1000}{3600} = 23.6 \text{ m s}^{-1}$$

and

$$\text{circumference} = 2\pi \times 0.25 = 1.57 \text{ m}$$

Therefore,

(a) the number of revolutions per minute is given by

$$\frac{\text{linear speed}}{\text{circumference}} \times 60 = \frac{23.6}{1.57} \times 60 = 900$$

and

(b) from Equation (9.3)

$$\omega = \frac{v}{r} = \frac{23.6}{0.25} = 94 \text{ rad s}^{-1}$$

Worked Example 9.2

A wheel initially at rest experiences a uniform angular acceleration of 5 rad s^{-2} for 6 s. It maintains a constant angular velocity for 14 s and is then brought to rest in 5 s by a uniform angular retardation. Find the total angular displacement.

Acceleration stage:

$\omega_1 = 0,\ t = 6,\ \alpha = 5,\ \theta = ?,\ \omega_2 = ?$

From Equation (9.9)

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2 = \frac{1}{2} \times 5 \times 36 = 90 \text{ rad}$$

From Equation (9.7)

$$\omega_2 = \omega_1 + \alpha t = 5 \times 6 = 30 \text{ rad s}^{-1}$$

Constant velocity stage:

$\omega_1 = \omega_2 = 30,\ t = 14,\ \alpha = 0,\ \theta = ?$

From Equation (9.9)

$$\theta = 30 \times 14 = 420 \text{ rad}$$

Retardation stage:

$\omega_1 = 30,\ \omega_2 = 0,\ t = 5,\ \theta = ?$

From Equation (9.8)

$$\theta = \frac{(\omega_1 + \omega_2)}{2} t = \frac{(30 + 0)}{2} \times 5 = 75 \text{ rad}$$

Therefore,

total angular displacement = 90 + 420 + 75 = 585 rad.

(Now compare this with Worked Example 5.2 on page 42.)

9.3 CENTRIPETAL ACCELERATION AND FORCE

For an object to move in a circle, there must be an inward force acting on it to overcome its natural tendency to follow a straight line. This *centripetal* (i.e. centre-seeking) force can be provided in many ways. For example, by gravitation, as in keeping the moon orbiting round the earth, or by the tension in a piece of string used to swing an object round in a circle. At any instant the object is moving in the direction given by the tangent to the circle at that point, so that, in the case of the second example, if the string breaks, the object will fly off along the tangent. The object is suddenly liberated from the centripetal force provided by the tension in the string, so it will continue in a straight line (or it would do in the absence of gravity). Centripetal force is also provided by friction between the tyres and the road when a car turns a corner.

An object moving in a circle experiences a change of velocity, even though its linear speed might be constant, because of the continuous change in its direction. Change of velocity means acceleration, and for acceleration we need a force such as the tension in a string continuously pulling an object away from a straight path. This centripetal acceleration must be distinguished from the angular acceleration α and its linear counterpart (which in any case would both be zero if ω is constant). Let us consider how it can be quantified.

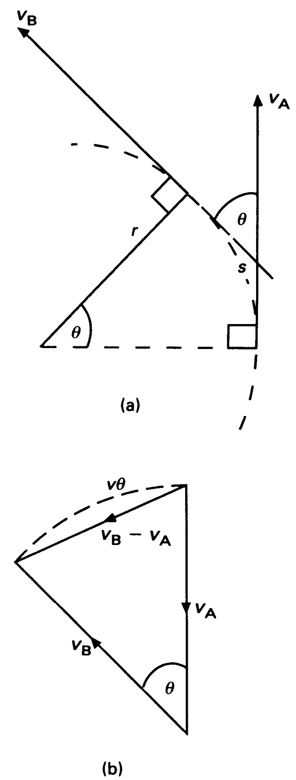

Figure 9.2

Figure 9.2(a) shows the path of an object moving round a circle of radius r at constant linear speed. In moving through an angle θ (equivalent to a linear distance s) its velocity changes from v_A to v_B in time t. Its change in velocity ($v_B - v_A$) is represented vectorially in Figure 9.2(b); note that the direction of v_A is reversed to make it negative, so that

$$v_B + (-v_A) = v_B - v_A$$

(The angle between v_A and v_B is θ, since this is the angle between the respective radii which they meet perpendicularly in Figure 9.2a.)

θ is shown as a large angle for clarity. If we make it very small, then the straight line representing the change of velocity ($v_B - v_A$) in Figure 9.2(b) will merge with the dotted curve which represents the arc of the circle of radius v (where v is the magnitude of v_A and v_B). Thus, if θ is in radians, then, to a very close approximation, Equation (9.1) gives us the change of velocity ($v_B - v_A$) as $v\theta$, which, since $\theta = s/r$ from Figure 9.2(a), is equal to vs/r. Since $s = vt$,

$$\text{change of velocity} = \frac{v^2 t}{r}$$

and the magnitude of the associated centripetal acceleration a_c is therefore given by

$$a_c = \frac{\text{change of velocity}}{t} = \frac{v^2 t}{r} \times \frac{1}{t}$$

Therefore,

$$a_c = \frac{v^2}{r} \tag{9.11}$$

Furthermore, since $v^2 = \omega^2 r^2$ (from Equation 9.3),

$$a_c = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r \tag{9.12}$$

If θ is made extremely small, so that v_A and v_B virtually overlap, then $(v_B - v_A)$ is perpendicular to the tangent at that point and is therefore directed towards the centre of the circle along the radius. The instantaneous acceleration therefore acts towards the centre of the circle.

If the mass of the object is m, then the centripetal force acting on it is given by Newton's second law as follows:

$$F\ (= ma_c) = m \times \frac{v^2}{r} \tag{9.13}$$

or

$$F = m\omega^2 r \tag{9.14}$$

where a_c is obtained from Equations (9.11) and (9.12), respectively.

As noted earlier, centripetal force can be provided in a number of ways. Figure 9.3 shows an object of mass m swinging round in a horizontal circle of radius r on the end of a string, fixed at the top, which makes an angle of φ with the vertical. At a given linear speed v

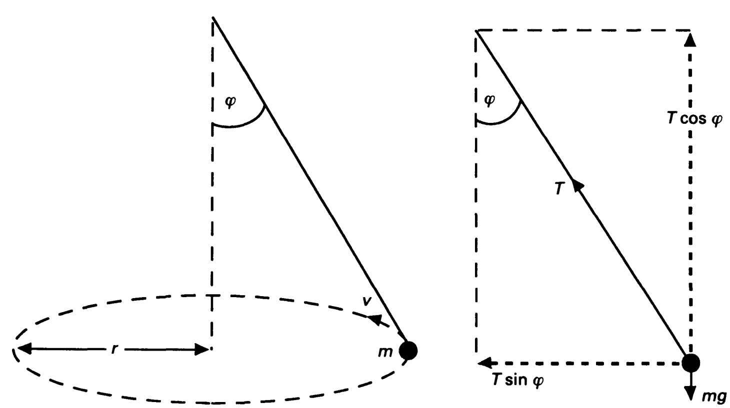

Figure 9.3

the horizontal component of the tension T in the string provides the centripetal force, so that

$$T \sin \varphi = \frac{mv^2}{r}$$

The vertical component of T supports the weight of the object, so that

$$T \cos \varphi = mg$$

Putting these together,

$$\tan \varphi \left(= \frac{T \sin \varphi}{T \cos \varphi} \right) = \frac{mv^2}{r} \times \frac{1}{mg}$$

Therefore,

$$\tan \varphi = \frac{v^2}{gr} \qquad (9.15)$$

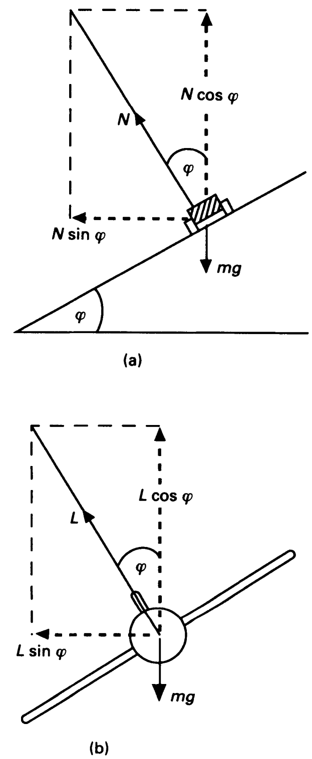

Figure 9.4

We can use a similar approach to banking a curved road to reduce the risk of skidding. Figure 9.4(a) shows the end view of a car of mass m travelling at a linear velocity v round a curve of radius r banked at an angle φ. The horizontal component of the normal force N acting on the car provides the centripetal force, the vertical component supports the car's weight, and putting them together as above again gives $\tan \varphi = v^2/gr$. So, for a given curve (where φ and r are fixed), there is a particular linear speed v, independent of the car's mass, where friction makes no contribution to the centripetal force. (Note that friction is still needed to prevent the car from slipping inwards at lower speeds and outwards at higher speeds.)

As Figure 9.4(b) suggests, the same equation applies to an aircraft making a banked turn; the vertical component of the lift force L experienced by the wings supports the aircraft's weight, while the horizontal component provides the centripetal force.

Going back to the car, if the road is horizontal ($\varphi = 0°$), then friction has to supply all the centripetal force. If the tyres are not slipping, then the contact area can be assumed to be instantaneously at rest. Therefore, the maximum possible centripetal force F corresponding to the maximum possible linear speed v is given by

$$F = \mu_s N = \mu_s mg = \frac{mv^2}{r}$$

so

$$\mu_s = \frac{mv^2}{r} \times \frac{1}{mg} = \frac{v^2}{gr}$$

and the maximum possible linear speed to negotiate the curve of radius r without skidding is given by $\sqrt{\mu_s gr}$.

Finally, let us briefly consider what happens when the object in Figure 9.3 moves through a vertical circle rather than a horizontal one. In this case the object, the string and its fixing point at the centre all lie within the plane of the circle. At the top of the circle the centripetal force is provided by the tension T in the string plus the object's weight, so that

$$\frac{mv^2}{r} = T + mg$$

and

$$T = \frac{mv^2}{r} - mg \tag{9.16}$$

Obviously there is a critical value of v where mg equals mv^2/r and the tension in the string is zero, so that the object only just manages to complete the circle. If v is even slightly below this value, the object will start to fall before it reaches the top. The same basic argument applies to swinging a bucket of water around vertically without spilling it.

At the bottom of the circle the weight of the object acts away from the centre, so T must provide the centripetal force and support the weight of the object, so that

$$T = \frac{mv^2}{r} + mg \tag{9.17}$$

In this case v must not be too large; otherwise the string may break.

Worked Example 9.3

Find the period of rotation of an object swinging round in a horizontal circle of 500 mm radius on the end of a string 1300 mm long (as in Figure 9.3). (g = 9.8 m s^{-2}.)

The vertical distance of the plane of rotation below the suspension point is given by

$$\sqrt{1.3^2 - 0.5^2} = 1.2$$

Therefore,

$$\tan\varphi = \frac{0.5}{1.2} = \frac{v^2}{gr} = \frac{v^2}{9.8 \times 0.5}$$

which gives

$$v = 1.43 \text{ m s}^{-1}$$

but, since $v = 2\pi r/T$,

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 0.5}{1.43} = 2.2 \text{ s}$$

Worked Example 9.4

(a) Find the minimum linear velocity for the object in Worked Example 9.3 to maintain continuous circular motion in a vertical plane. (b) If the mass of the object is 250 g, and if its linear velocity remains the same as in (a), then find its apparent weight at the bottom of the circle. ($g = 9.8$ m s^{-2}.)

(a) The minimum value of v is where the tension in the string is zero, so that, from Equation (9.16),

$$\frac{mv^2}{r} = mg$$

Therefore,

$$v = \sqrt{rg} = \sqrt{1.3 \times 9.8} = 3.6 \text{ m s}^{-1}$$

(b) Substituting $\sqrt{rg}$ for v in Equation (9.17),

$$T = \frac{mrg}{r} + mg = 2mg = 2 \times 0.25 \times 9.8 = 4.9 \text{ N}$$

which is the apparent weight of the object exerted on the string (i.e. twice its actual weight).

Questions

(Where necessary assume that $g = 9.8$ m s^{-2}.)

1. If the minute hand of a clock is 115 mm long, find the linear speed at its tip.
2. A record-player turntable takes 1.8 s to reach $33\frac{1}{3}$ rpm from rest. How many revolutions does this take?
3. The moon's mass is 7.3×10^{22} kg and its orbit round the earth has an average radius of 3.8×10^{8} m and a period of 27 days. Estimate the magnitude of the gravitational force between the earth and the moon.
4. A truck travels round a circular track of 80 m radius at a uniform linear speed of 10 m s^{-1}. Find the angle to which the track would need to be banked to eliminate the need for radial friction.
5. An object is turning through a vertical circle of 0.69 m radius on the end of a string. Find the minimum angular velocity required to maintain circular motion.
6. If the mass of the object in the previous question is 0.1 kg, estimate the minimum possible breaking strength of the string.
7. Estimate the angle at which an aircraft should be banked to make a horizontal turn of 3.5 km radius at a speed of 450 km per hour.
8. What is the tension in the string in Worked Example 9.3 if the mass of the object is 190 g?
9. A rotating shaft experienced an angular acceleration of 10 rad s^{-2} over a period of 8 s, during which time its angular displacement was 400 rad. What were its initial and final angular velocities?
10. An aircraft loops the loop with a radius of 750 m at a constant linear speed of 360 km h^{-1}. What is the magnitude of the force exerted by the seat on a 70 kg pilot (a) at the top, and (b) at the bottom of the loop?

TOPIC 10 ROTATION OF SOLIDS

COVERING:

- moment of inertia;
- angular momentum;
- rotational kinetic energy;
- torque, work and power.

In the previous topic we considered the translational motion of an object, treating it as a particle moving round a circular path. Now we shall consider a solid object, such as a shaft or a flywheel, rotating about an axis without necessarily moving from one place to another.

We have already met the idea that an object resists change in its state of translational motion because of its inertia. In an analogous way, an object resists change in its rotational state because of its *moment of inertia*.

10.1 MOMENT OF INERTIA

Figure 10.1 represents a solid object rotating about a fixed axis O perpendicular to the page. Let us focus on a single component particle of mass m rotating about O at a distance r. If we want to change the speed of the particle, then, considering it separately from all the others, we would have to apply a tangential force to it (in the same direction as its motion) in accordance with Newton's second law

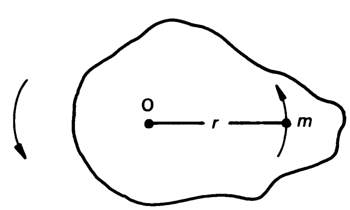

Figure 10.1

$$F = ma$$

but, since $a = \alpha r$ (Equation 9.6 on page 72), then

$$F = m\alpha r$$

where α is the angular acceleration.

In Topic 3 (Section 3.2) we saw that a *torque* about a point is found by multiplying the force producing it by the perpendicular distance of the line of action of the force from the point. The torque T needed to change the angular velocity of our particle is therefore given by

$$T = F \times r = m\alpha r \times r = mr^2\alpha$$

But the object consists of a large number of component particles, each with its own particular value of mr^2, so the total torque needed to change the angular velocity of the object as a whole is given by

$$T = \left(\sum mr^2\right) \alpha$$

where Σmr^2 is the sum of the individual mr^2 values of all the component particles about the axis. (Note that Σ is simply a mathematical symbol meaning 'the sum of'.) In fact, the quantity Σmr^2, which is given the symbol I and has units of kg m^2, represents the moment of inertia of the object, so

$$T \left(= \left(\sum mr^2\right) \alpha\right) = I\alpha \qquad (10.1)$$

This equation is the rotational version of Newton's second law where moment of inertia is analogous to mass. The greater the moment of inertia of a body the greater the torque needed to provide a given angular acceleration.

Since the value of mr^2 for each component particle in the object is the product of its mass and the square of its distance from the axis, it follows that the moment of inertia of a body depends on the way in which the total mass is distributed about the axis. Figure 10.2 shows some examples. If the total mass M is distributed in the form of a thin

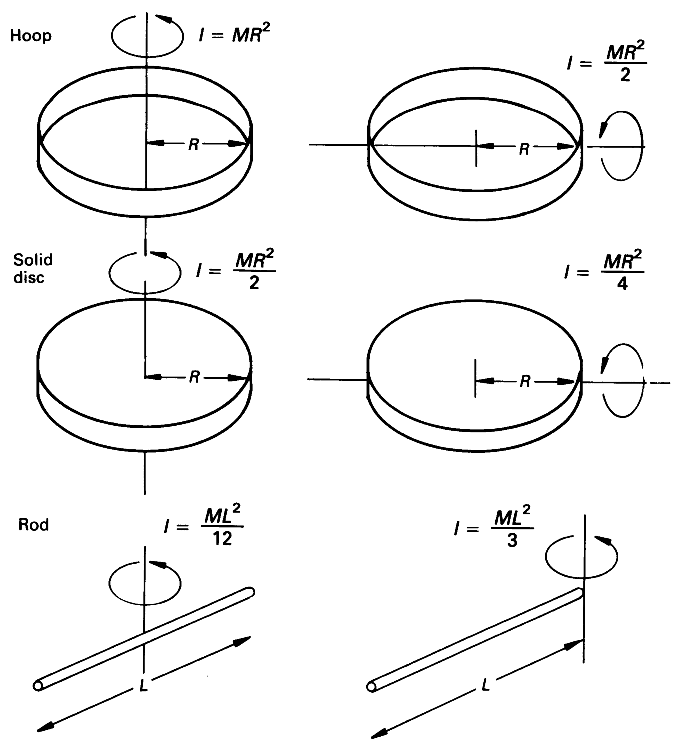

Figure 10.2

hoop of radius R, and the axis of rotation passes through the centre of the hoop normal to its plane, then $I = MR^2$, because all the component particles are at a distance R from the axis. For a given mass M, I increases as R^2; thus, the moments of inertia of a series of hoops of different sizes, but identical mass, increase rapidly with radius. For a solid disc rotating about the corresponding axis $I = MR^2/2$, which is less than for the hoop, because the mass is distributed closer to the axis. For this reason flywheels tend to have their mass concentrated at the rim, because this increases their moment of inertia. If the axis of rotation is changed, as on the right-hand side of Figure 10.2, then I changes, because the mass is distributed differently about it. For example, $I = MR^2/4$ for a thin solid disc with its axis of rotation along a diameter.

10.2 ANGULAR MOMENTUM

Angular momentum is the rotational counterpart of linear momentum (mv), which we met in Topic 7. Moment of inertia and angular velocity are the rotational counterparts of mass and linear velocity and

$$\text{angular momentum} = I\omega$$

In Topic 7 we noted that Newton's second law can be expressed in terms of change of linear momentum. The same is true of angular momentum, since, from Equation (10.1),

$$T = I\alpha = I\frac{(\omega_2 - \omega_1)}{t} = \frac{I\omega_2 - I\omega_1}{t}$$

And, as with linear momentum, there is the *principle of conservation of angular momentum*, which states that the total angular momentum of a system remains constant if there is no net torque acting on it. Skaters make use of this principle to control the rate at which they spin. If they tuck their arms in close to their bodies, then their moment of inertia decreases, so their angular velocity increases in order to keep the value of $I\omega$ constant. Similarly, if they stretch their arms out, then their angular velocity decreases.

10.3 ENERGY, WORK AND POWER

A rotating object must possess kinetic energy, since its component parts are moving. As we might expect from its translational counterpart ($\frac{1}{2}mv^2$),

$$\text{rotational kinetic energy} = \frac{1}{2}I\omega^2$$

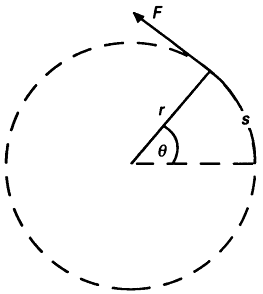

Figure 10.3

But this energy was originally provided in accelerating the object from an initial angular velocity of zero. Let us see how we can quantify the work done.

Figure 10.3 shows a tangential force F that, in moving through the distance s, has produced rotation through the angle θ. In this case the work done W is given by

$$W = F \times s$$

and, since $s = r\theta$,

$$W = Fr\theta$$

Therefore, since $Fr = T$ (torque), we have

$$W = T\theta$$

If this is the work done in accelerating the object from rest to an angular velocity ω, then

$$T\theta = \frac{1}{2} I\omega^2$$

Alternatively, $T\theta$ might represent the energy transmitted over a period of time t through a rotating shaft, say from a turbine to an electric generator, in which case the power P transmitted is given by

$$P = \frac{\text{work}}{\text{time}} = \frac{T\theta}{t}$$

and, since $\omega = \theta/t$,

$$P = T\omega$$

which is equivalent, in linear terms, to force times velocity.

Finally, we should note that an object may possess both rotational and translational kinetic energy, as in the case of a round object rolling along the ground.

10.4 SUMMARY

The angular quantities that we have met in this topic are listed below (with their linear counterparts in parentheses):

- moment of inertia I (mass m);
- torque $T = I\alpha$ (force $F = ma$);
- angular momentum $I\omega$ (mv);

- kinetic energy $\frac{1}{2}I\omega^2$ ($\frac{1}{2}mv^2$);
- work $T\theta$ (Fs);
- power $T\omega$ (Fv).

Worked Example 10.1

Find the total kinetic energy of a 50 kg disc, 400 mm in diameter, rolling across a level surface at 70 rpm.

Considering translational plus rotational motion, the total kinetic energy E_k is given by

$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

but since $v = r\omega$ (Equation 9.3 on page 72) and $I = \frac{1}{2}mr^2$ in this case

$$E_k = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\omega^2 = \frac{3}{4}mr^2\omega^2$$

then, on substituting,

$$E_k = \frac{3}{4} \times 50 \times (0.2)^2 \times \left(\frac{70}{60} \times 2\pi\right)^2 = 81 \text{ J}$$

Worked Example 10.2

If the disc in the previous example is mounted as a flywheel, find the torque required to raise it from rest to 200 rpm in 5 revolutions.

In this case $I = \frac{1}{2}mr^2$ and, on substituting,

$$I = \frac{50 \times (0.2)^2}{2} = 1 \text{ kg m}^2$$

$$200 \text{ rpm} = \frac{200}{60} \times 2\pi = 21 \text{ rad s}^{-1}$$

and

$$\omega_1 = 0, \ \omega_2 = 21, \ \theta = 5 \times 2\pi, \ \alpha = ?$$

so, from Equation (9.10) (page 73),

$$(21)^2 = 0 + (2 \times \alpha \times 5 \times 2\pi)$$

which gives

$$\alpha = 7 \text{ rad s}^{-2}$$

Therefore,

$$T\,(= I\alpha) = 1 \times 7 = 7 \text{ N m}.$$

Worked Example 10.3

The flywheel in the previous example, rotating freely (i.e. no external torque) at 21 rad s^{-1}, is connected via a friction clutch of zero I to a second flywheel, with $I = 2$ kg m^2, that is at rest. Find (a) the final combined angular velocity, and (b) the heat energy dissipated in the clutch.

(a) Angular momentum is conserved; therefore,

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\,\omega_{A+B}$$

and, substituting,

$$(1 \times 21) + 0 = (1 + 2)\,\omega_{A+B}$$

which gives

$$\omega_{A+B} = 7 \text{ rad s}^{-1}$$

(b) Kinetic energy before engaging clutch is equal to

$$\frac{1}{2}I_A\omega_A^2 + \frac{1}{2}I_B\omega_B^2 = \left(\frac{1}{2} \times 1 \times (21)^2\right) + 0 = 220.5 \text{ J}$$

Kinetic energy after engaging clutch is equal to

$$\frac{1}{2}\,I_{A+B}\omega_{A+B}^2 = \frac{1}{2} \times 3 \times 7^2 = 73.5 \text{ J}$$

Therefore, the heat dissipated is equal to

$$220.5 - 73.5 = 147 \text{ J}$$

Worked Example 10.4

A shaft is being driven at a constant 200 rpm by a steady torque of 7 N m. Find the power consumption.

$$\text{power} = T\omega = 7 \times \frac{200 \times 2\pi}{60} = 147 \text{ W}$$

Questions

1. A 5 kg disc, 200 mm in diameter, is mounted as a flywheel and rotates at 300 rpm. Find (a) its moment of inertia, (b) its kinetic energy, (c) its total kinetic energy, if it is allowed to roll across a level surface with the same angular velocity.

2. A 12 g coin 30 mm in diameter is tossed so that it spins at 300 rpm about an axis along its diameter. Estimate its rotational kinetic energy.

3. A steady torque of 12 N m is applied to a flywheel at rest that has a moment of inertia of 6 kg m^2. After 15 s what is (a) the angular displacement, (b) the angular velocity, (c) the work done, (d) the power consumption and (e) the kinetic energy of the flywheel?

4. Find the moment of inertia about the axis of rotation of an object, initially at rest, which is accelerated to an angular velocity of 180 rad s^{-1} in 15 s by a torque of 15 N m.

5. What is the angular acceleration of a flywheel with a moment of inertia of 18 kg m^2 upon which a torque of 27 N m is acting? If the flywheel starts from rest, how long does it take to turn through 50 rad?

6. A 200 mm diameter flywheel rotating at 200 rad s^{-1} is brought to rest in 125 revolutions by a braking torque of 0.26 N m. Find the mass of the flywheel, assuming that it is a disc.

7. A shaft, with $I = 5$ kg m^2, freely rotating clockwise at 50 rad s^{-1}, is connected via a friction clutch to a shaft with $I = 10$ kg m^2 that is freely rotating anticlockwise at 100 rad s^{-1}. Find (a) their combined angular velocity and (b) the kinetic energy lost.

8. A rod of negligible mass rotates horizontally about its centre with two identical masses attached, one at each

side, at equal distances from the axis of rotation. Assuming free rotation with no external torque applied, and an angular velocity of 40 rad s^{-1} with the masses 500 mm apart, find the new angular velocity if the masses move to a position 550 mm apart as they rotate.

9. Shaft A, driven at a constant 150 rad s^{-1}, is connected via a friction clutch to shaft B, which has a moment of inertia of 0.04 kg m^2. Shaft B accelerates uniformly to the same angular velocity as shaft A over a period of 0.75 s. By consideration of the kinetic energy acquired by shaft B and the torque acting upon it, find the heat energy dissipated in the clutch.

10. A round object of 2 kg mass and 750 mm diameter, starting from rest, takes 3.8 s to roll 15 m down a 25° slope. Find (a) the object's final total kinetic energy and (b) its moment of inertia about the axis of rotation; hence (c) establish whether it is a disc or a hoop.

TOPIC 11 SIMPLE HARMONIC MOTION

COVERING:

- a mathematical model;
- the simple pendulum;
- vertical oscillation of a mass hanging from a spring;
- damping.

Having considered linear, circular and rotational motion, we now move on to vibrational motion, or oscillation, such as that of a pendulum, where an object is displaced from some central equilibrium position, then released so that it oscillates backwards and forwards about it. Such behaviour can often be described in terms of *simple harmonic motion*, which is characterised by an acceleration towards the equilibrium position that has a magnitude proportional to the displacement from it.

11.1 A MATHEMATICAL MODEL

Figure 11.1 provides the basis of a mathematical model of simple harmonic motion.

The figure shows a point P moving round a circle of radius r with a constant linear speed v (= ωr). As P makes successive revolutions, the point X, i.e. the vertical projection from P onto the horizontal diameter, moves to and fro along it with simple harmonic motion about the centre.

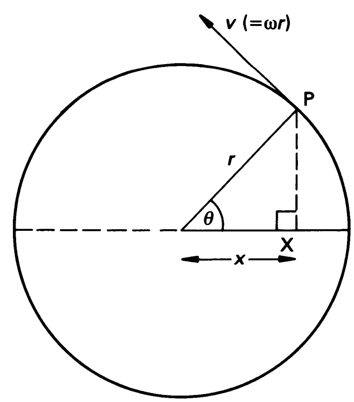

Figure 11.1

Before examining this idea more closely, note that the positive direction runs along the positive x-axis from the centre of the circle. The displacement x of the point X from the centre is given by $r\cos\theta$. When X is to the left of centre, then the value of x is negative, since $\cos\theta$ is always negative on that side. Since $x = r\cos\theta$ and $\theta = \omega t$ (Equation 9.2 on page 72),

$$x = r\cos\omega t \tag{11.1}$$

We have implied here that $\theta = 0$ at $t = 0$, which is not necessarily true. If there is already an angular displacement, say φ, at time $t = 0$, then $\theta = (\omega t + \varphi)$. Also note that the maximum displacement, $x = r$ or $-r$, is called the *amplitude*.

Figure 11.2 illustrates how displacement varies with time. T, the unit of time used in the figure, is the period $T = 2\pi/\omega$ that corresponds to one complete oscillation (Equation 9.4 on page 72). As Figures 11.1 and 11.2 both imply, X is moving at its fastest through the central position, then it slows down until it stops and reverses direction at the maximum displacement, then it accelerates towards the centre — and so on.

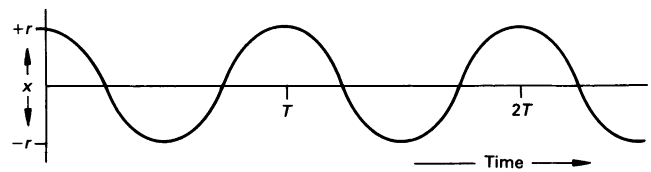

Figure 11.2

From Topic 9 we know that P must experience a centripetal acceleration ($= \omega^2 r$) inwards along the radius (Equation 9.12 on page 76). The acceleration a of X is given by the horizontal component of the centripetal acceleration as follows:

$$a = -\omega^2 r \cos\theta$$

and, since $x = r\cos\theta$,

$$a = -\omega^2 x \qquad (11.2)$$

This means that, since ω is constant, the acceleration of X is proportional to its displacement x. The negative sign indicates that the acceleration is directed towards the centre (i.e. in the negative direction when X is to the right of centre and x is positive, and in the positive direction when X is to the left of centre and x is negative). X therefore executes simple harmonic motion in accordance with the characteristics noted at the beginning of the topic.

Now let us consider the force involved where a mass is moving with simple harmonic motion. If an object of mass m experiences an acceleration $-\omega^2 x$, then, from Newton's second law, it will experience a force F, given by

$$F\ (= ma) = -m\omega^2 x$$

Since m and ω are both constants,

$$F = -kx \qquad (11.3)$$

where the constant k ($= m\omega^2$) represents the force per unit displacement.

This tells us that, in simple harmonic motion, the object experiences a restoring force acting towards the central equilibrium point, which, like the acceleration, is proportional to the object's displacement from it. Note that any oscillating mechanical system with a proportional relationship between the restoring force and the displacement will execute simple harmonic motion. Now since $\omega = \sqrt{k/m}$ (from above), the period of oscillation is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$$

and therefore

$$T = 2\pi\sqrt{m/k} \qquad (11.4)$$

The energy of the object is the sum of its potential and kinetic energies, which are continuously interchanging from 100% potential energy at the extreme positions to 100% kinetic energy at the centre. For example, Figure 11.3 shows how the total energy of a mass swinging to and fro at the end of a weightless string is its potential energy (mgh) at its extreme positions, where it is stationary, and its kinetic energy ($\frac{1}{2}mv^2$) at the central position, where it is at the lowest point of its travel.

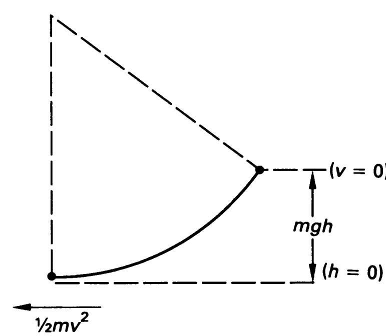

Figure 11.3

First let us consider the general case. The total energy of any object undergoing simple harmonic motion can be viewed in terms of the initial work done in displacing it from its central equilibrium position to one of the extreme positions prior to releasing it in the first instance. The work is done against the restoring force F, which, as we saw earlier, must increase proportionally to the displacement x. The potential energy stored by this process is therefore given by the area $\frac{1}{2}Fx$ under the plot of displacing force (of magnitude F) against x in Figure 11.4. When the object is at its maximum displacement, where $x = r$, then its total energy E is given by

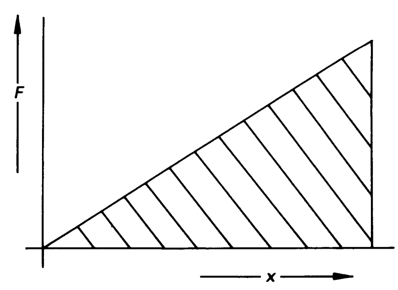

Figure 11.4

$$E = \frac{1}{2}Fr$$

and since the magnitude of F is given by $kr = m\omega^2 r$ (see Equation 11.3), then

$$E = \frac{1}{2}m\omega^2 r^2 \qquad (11.5)$$

When the object is released and starts to move, its potential energy is traded for kinetic energy. Its potential energy at any given displacement x is then given by $\frac{1}{2}m\omega^2x^2$, since this would have been the work needed to displace the object from rest at equilibrium to that position. The kinetic energy of the object at that point will be the difference between its total energy and its potential energy

$$\frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 r^2 - \frac{1}{2} m\omega^2 x^2$$

Therefore,

$$v^2 = \omega^2(r^2 - x^2)$$

so the magnitude of the velocity is given by

$$v = \omega \sqrt{r^2 - x^2} \tag{11.6}$$

Equation (11.6) confirms our earlier observations about the velocity, i.e. that v is zero where $x = r$ at the maximum displacement, and v has its greatest magnitude (ωr) at the centre. (In general, we shall try to avoid any possible confusion with the sign convention — positive to the right for our model — by using the magnitudes of the quantities involved and specifying their directions where necessary.)

Note that the rate at which a system oscillates is often expressed in terms of the frequency f, which is the reciprocal of the period T:

$$f = \frac{1}{T} \tag{11.7}$$

Frequencies are expressed in hertz (Hz), where 1 Hz is 1 cycle per second. Also note that, since $\omega = 2\pi/T$ (Equation 9.4 on page 72), then

$$\omega = 2\pi f \tag{11.8}$$

Worked Example 11.1

A 950 g mass moves in simple harmonic motion with a frequency of 20 Hz and an amplitude of 100 mm. Find

(a) its maximum and minimum speeds and where these occur;
(b) its maximum and minimum accelerations and where these occur;
(c) its speed and acceleration 30 mm from the extreme positions;
(d) the maximum restoring force acting upon it.

(a) Since $v = \omega \sqrt{r^2 - x^2}$ (Equation 11.6), the maximum speed occurs where $x = 0$ (at the central position); hence, $v = \omega r$ and, since $\omega = 2\pi f$ (Equation 11.8),

$$v = 2\pi fr$$

Substituting,

$$v = 2\pi \times 20 \times 0.1 = 12.6 \text{ m s}^{-1}$$

The minimum speed, $v = 0$, occurs where $x = r$ (at the extreme positions).

(b) Since $a = \omega^2 x$ towards the central point (Equation 11.2) and $\omega = 2\pi f$ (Equation 11.8), the maximum acceleration occurs where $x = r$ (at the extreme positions) and is given by

$$a = 4\pi^2 f^2 r$$

Substituting,

$$a = 4\pi^2 \times (20)^2 \times 0.1 = 1580 \text{ m s}^{-2} \text{ (inwards)}$$

At the central position $x = 0$, therefore, $a = 0$.

(c) From Equations (11.6) and (11.8)

$$v = 2\pi f \sqrt{r^2 - x^2}$$

Therefore, 30 mm from the extreme positions, where $x = 0.07$ m,

$$v = 2\pi \times 20 \times \sqrt{0.1^2 - 0.07^2} = 9 \text{ m s}^{-1}$$

Furthermore,

$$a = 4\pi^2 f^2 x$$

and, substituting,

$$a = 4\pi^2 \times (20)^2 \times 0.07 = 1105 \text{ m s}^{-2} \text{ (inwards)}$$

(d) Since the maximum acceleration occurs at the extreme positions, then so does the maximum restoring force, which is given by

$$F = ma = m \times 4\pi^2 f^2 r$$

and, substituting,

$$F = 0.95 \times 4\pi^2 \times (20)^2 \times 0.1 = 1500 \text{ N}$$

Now let us consider some practical examples of simple harmonic motion.

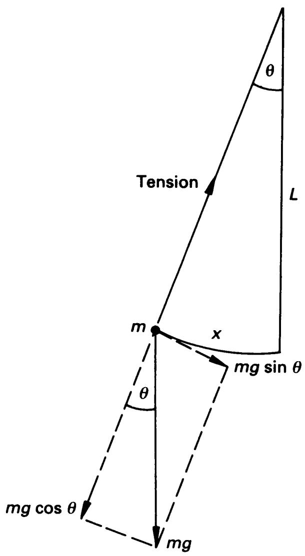

Figure 11.5

11.2 THE SIMPLE PENDULUM

The simple pendulum consists of a mass m that swings through a small angle on the end of a string of negligible mass hanging from a fixed point (Figure 11.5).

The figure shows the pendulum at a moment when the string makes an angle θ with the vertical and the mass is displaced a distance x along an arc of radius L (where L is the length of the string). The weight mg is resolved into two components: $mg \cos \theta$, supported by the tension in the string, and $mg \sin \theta$, which is the magnitude of the force F restoring the mass to the central position. If θ is small, then $\sin \theta$ is very close to θ in radians (for example, $10° = 0.1745$ rad and $\sin 10° = 0.1736$). Thus,

$$F = mg \sin \theta$$

and, if θ rad is small,

$$F = mg\theta$$

and since, from the figure, $\theta = x/L$ rad,

$$F = \frac{mgx}{L} = \frac{mg}{L} \times x$$

Therefore,

$$F = kx, \text{ where } k = \text{constant} = mg/L$$

So, provided that the amplitude is small, the restoring force is proportional to the displacement and the pendulum will execute simple harmonic motion. Furthermore, since $k = mg/L$ and therefore $m/k = L/g$ then, from Equation (11.4),

$$T \;(= 2\pi \sqrt{m/k}) = 2\pi \sqrt{L/g} \qquad (11.9)$$

which means that the period of oscillation of the pendulum depends only on its length, assuming that g is constant.

Worked Example 11.2

A 20 g bullet is fired at a 20 kg stationary target suspended by a rope. The bullet becomes embedded in the target, which subsequently swings to and fro in simple harmonic motion with period 4 s and amplitude 255 mm. Assuming the mass of the rope may be ignored, estimate the impact velocity of the bullet.

Equation (11.6) tells us that the magnitude of the velocity of the swinging target is at its maximum value $v = \omega r$ at the central position. Since $\omega = 2\pi f$ (Equation 11.8) and $f\ (= 1/T) = 1/4 = 0.25$ Hz, and $r = 0.255$ m, then, at the central position,

$$v = 2\pi f r = 2\pi \times 0.25 \times 0.255 = 0.40 \text{ m s}^{-1}$$

Assuming the system is undamped (see below), this value of v is the combined velocity following the impact of the bullet on the target. The impact velocity of the bullet may therefore be obtained by consideration of the conservation of momentum. Where the subscripts b, t and c refer to the bullet, the target and their combination, respectively,

$$m_b v_b + m_t v_t = m_c v_c$$

and, substituting,

$$(0.02 \times v_b) + 0 = (20 + 0.02) \times 0.40$$

which gives

$$v_b = 400 \text{ m s}^{-1}$$

11.3 A MASS HANGING FROM A SPRING

Let us assume that the spring in Figure 11.6(a) obeys Hooke's law and is of negligible mass. If an object of mass m is suspended from the lower end and extends the spring by a distance d, then, at the equilibrium position in Figure 11.6(b), the tension in the spring is given by

$$mg = kd$$

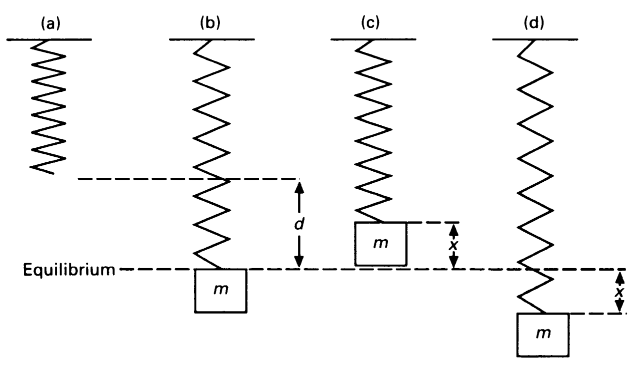

Figure 11.6

where k (= mg/d) is the *stiffness* of the spring (i.e. force per unit extension).

If the object is raised slightly, then released so that it oscillates vertically, then at any subsequent displacement x the restoring force towards the equilibrium position will be the resultant of the object's weight pulling it downwards and the tension in the spring pulling it upwards.

If the object is above the equilibrium position, as in Figure 11.6(c), then the tension in the spring is $k(d - x)$ and the downward restoring force R_d is given by

$$R_d = \text{weight of object} - \text{tension in spring}$$

Therefore,

$$R_d = mg - k(d - x)$$

and, since $mg = kd$ (from above),

$$R_d = kd - kd + kx = kx$$

If the object is below the equilibrium position, as in Figure 11.6(d), the tension in the spring is $k(d + x)$ and the upward restoring force R_u is given by

$$R_u = \text{tension in the spring} - \text{weight of object}$$

Therefore,

$$R_u = k(d + x) - mg$$

and, since $mg = kd$,

$$R_u = kd + kx - kd = kx$$

Thus, the restoring force acts towards the equilibrium position and is proportional to the displacement, so the object will execute simple harmonic motion.

Furthermore, since $m/k = d/g$ (from above), then Equation (11.4) gives

$$T\ (= 2\pi\sqrt{m/k}) = 2\pi\sqrt{d/g} \qquad (11.10)$$

Thus, the period of the oscillation depends on d, the extension at equilibrium, which can be varied by changing m.

11.4 DAMPING

There are many other examples of simple harmonic motion: for example, the oscillation of a vertical float of uniform cross-section in a liquid or of a liquid in a U-tube of uniform cross-section, or the rotational oscillation of a torsion pendulum (e.g. an object twisting about the vertical axis of a wire from which it is suspended).

Many practical systems are more complex than our discussion might seem to suggest. For instance, restoring forces may not be proportional to displacement, and *damping* may be an important factor.

So far we have assumed that the energy of the oscillating system remains constant and that we are dealing with *free* oscillations that continue indefinitely with constant amplitude. Damping causes the amplitude to decay, as in Figure 11.7, for example, because energy is lost to the surroundings. Even a simple pendulum will come to rest eventually because of the damping loss due to the frictional effect of air resistance. Mechanical energy is converted to heat energy and the temperature of the surrounding air increases. The pendulum in a clock loses energy by friction in the mechanism as well as to the air, and the damping losses are topped up by mechanical energy stored in a spring or a raised weight.

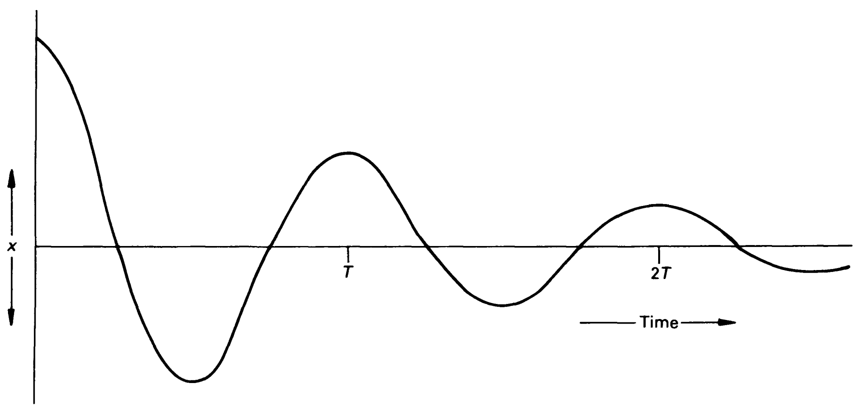

Figure 11.7

Damping is often deliberately introduced into mechanical systems. For instance, shock absorbers are used in a motor car to damp the suspension and minimise oscillation.

Questions

(Where necessary assume that $g = 9.8 \text{ m s}^{-2}$. Assume free (undamped) oscillations.)

1. Find the length of the simple pendulum with a period of (a) 1 s, (b) 2 s and (c) 4 s.

2. A spring is extended by 25 mm because of a load of 500 g suspended from it. If the load is increased to 1 kg and allowed to oscillate vertically, find the period of its oscillation.
3. If the length of a simple pendulum is reduced by 1.2 m with the result that its frequency is doubled, what was its original length?
4. A spring is extended by 20 mm because of a mass suspended from it. Find the frequency if the mass is allowed to oscillate vertically.
5. Find (a) the maximum speed and (b) the maximum acceleration of an object moving in simple harmonic motion with a period of 2 s and an amplitude of 95 mm.
6. If the period of vertical oscillation of a 0.80 kg mass suspended from a spring is 0.75 s, find the stiffness of the spring.
7. An object is placed on a horizontal surface which oscillates vertically with simple harmonic motion. When the frequency is increased to 7 Hz, the object is on the point of losing contact with the surface. At what point of the cycle does this occur and what is the amplitude of the oscillation?
8. A point that is moving with simple harmonic motion has a velocity of 5 m s^{-1} at a distance of 12 m from its central position, and 12 m s^{-1} at a distance of 5 m from it. Find the frequency of its oscillation.
9. A 2.5 kg mass moves with simple harmonic motion at a frequency of 15 Hz and with an amplitude of 50 mm. Find (a) its total energy, (b) its maximum speed and (c) the maximum restoring force.
10. The period of a torsion pendulum is given by the rotational analogy of Equation (11.4), where m is replaced by the moment of inertia and k by the torsional stiffness (i.e. torque per unit angular displacement, N m rad^{-1}).

    A rod, 1200 mm long, is suspended from its centre by a wire. A torque of 0.175 N m is required to turn the rod 10° about its suspension point. If the period of oscillation is 1.5 s, then, with reference to Figure 10.2 (page 82), estimate the mass of the rod.

TOPIC 12 MECHANICAL WAVES

COVERING:

- the description of mechanical waves;
- reflection, refraction, diffraction and interference;
- wave speed;
- standing waves;
- resonance.

In the previous topic we considered the continuous interchange of potential and kinetic energy in oscillating systems where the total energy remains trapped or would remain trapped in the absence of damping. Figure 11.2 (page 90) shows a wave-like relationship between displacement and time for an isolated oscillating system of this kind.

In this topic we shall consider wave motion via oscillations in a continuous medium which enables energy to be carried from one place to another.

12.1 THE NATURE OF WAVE MOTION

Figure 12.1 shows a rope, stretched horizontally, that is being forced to oscillate vertically at one end. The figure represents snapshots taken at intervals of a quarter of a period. The energy fed into one end of the rope is transferred from one part to the next in a *progressive* (i.e. travelling) wave that moves towards the other end.

The *amplitude* is half the total wave height from trough to crest. The *wavelength* is the distance between any two adjacent points along the wave train that are *in phase*, i.e. exactly 360° apart, such as from crest to crest or from trough to trough. If the frequency of the oscillation initiating the wave motion is f hertz, then the number of waves passing any particular point along their path is f per second. If the wavelength is λ metres, then the *wave speed* v (m s^{-1}) is given by

$$v = f\lambda \tag{12.1}$$

This is sometimes expressed in the form $v = \lambda/T$, where T is the period.

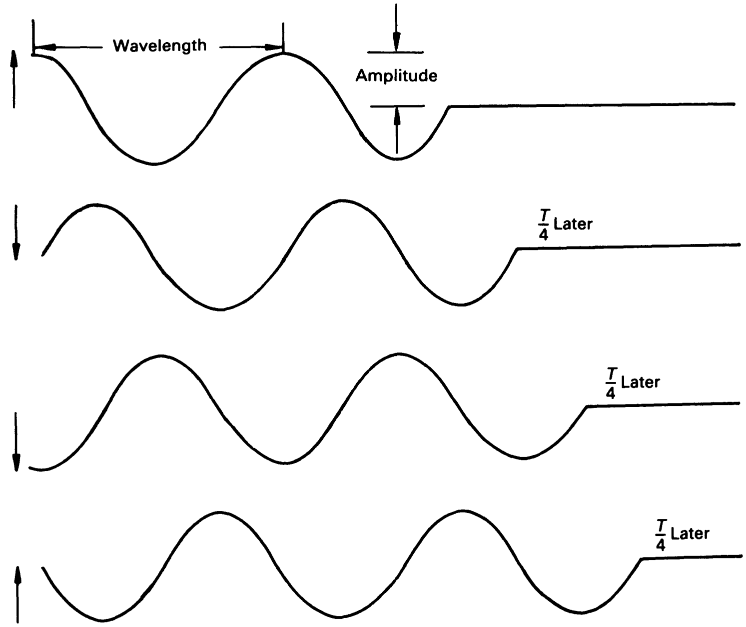

Figure 12.1

The type of wave in Figure 12.1 is described as *transverse*, because the oscillations are perpendicular to the direction in which the wave is travelling. By contrast, *longitudinal* waves involve oscillations which are parallel to the direction of travel, as shown in Figure 12.2. This figure might represent, say, a snapshot of successive pulses of tension and compression passing down the coils of a stretched spring as one end is forced to oscillate along its longitudinal axis.

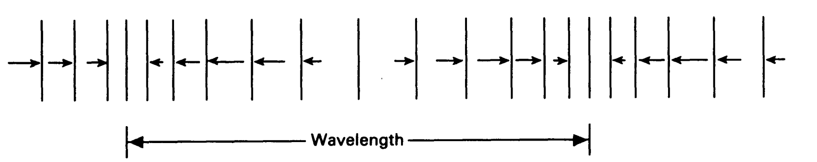

Figure 12.2

It is important to note that Figure 11.2 (page 90) represents displacement against time for a particular point, whereas Figures 12.1 and 12.2 represent displacement against distance at a particular time.

We shall use sound as one illustration of mechanical wave motion. Sound travels through air as longitudinal waves consisting of successive compressions and rarefactions moving at about 340 m s^{-1} at frequencies in the range 20–20 000 Hz. Corresponding wavelengths therefore range from about 17 m down to 17 mm. (Note that the pitch of musical notes is related to their frequency, based on 440 Hz for the A above middle C.)

Forgetting the sensitivity of a listener's hearing, the loudness of a sound is determined by its *intensity*. This is measured in terms of the amount of energy that sound waves would carry in 1 second through a 1 m^2 aperture perpendicular to their direction of propagation. The unit of intensity is therefore W m^{-2}. (As a rough guide to magnitude, the threshold of hearing is about 10^{-12} W m^{-2}, conversation is about 10^{-6} W m^{-2} and the threshold of pain about 1 W m^{-2}.)

Sound waves normally tend to spread out in all directions from their source. A point source in a uniform medium lies at the centre of a series of expanding spherical wavefronts (i.e. surfaces of constant phase) rather like the ripples spreading out when a stone is dropped into a pond. This means that the intensity progressively decreases as the surface area of the wavefronts increases with their expansion. At a distance r from a sound source of power P watts the energy is distributed over the area of a sphere of radius r, which is equal to $4\pi r^2$, so the intensity I at that distance is given by the inverse square relationship

$$I = \frac{P}{4\pi r^2} \tag{12.2}$$

Surface waves on water depend on a combination of transverse and longitudinal motion. Figure 12.3 shows how individual water molecules move in circular orbits, completing one lap for each wave that passes. The figure represents a snapshot of one complete wavelength. As well as moving vertically, each molecule moves forwards with the

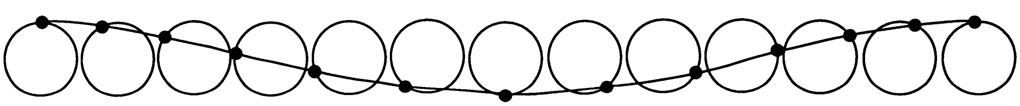

Figure 12.3

wave on the crest and backwards in the trough. Similar orbits occur below the surface, rapidly becoming smaller with depth. In shallow water these orbits interact with the bottom and the wave speed decreases as the depth decreases. If the waves continue to arrive with the same frequency, then the wavelength decreases in accordance with Equation (12.1); that is to say, if $f = v/\lambda$ and f remains constant, then λ must decrease if v decreases.

12.2 WAVE BEHAVIOUR

Wave behaviour is conveniently demonstrated in a device called a *ripple tank*, which has a transparent bottom and contains a shallow layer of water. Waves, or ripples, travelling across the surface of the water are viewed on a screen by passing light through the bottom of the tank. Figure 12.4(a) represents an overhead view of straight, parallel waves being generated by means of a horizontal bar, just dipping into the water, which is made to oscillate vertically. If a small sphere is used instead of the bar, as in Figure 12.4(b), then circular waves diverge from it like those from a stone dropped into a pond. The movement of the waves can be frozen with a stroboscope.

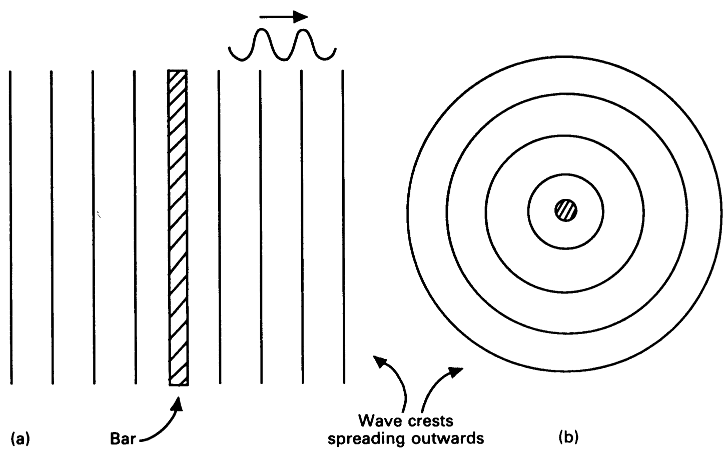

Figure 12.4

Reflection, refraction, diffraction and interference are important aspects of wave behaviour which can be demonstrated with the ripple tank.

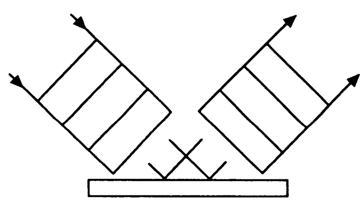

Figure 12.5

Figure 12.5 shows *reflection* demonstrated by placing a barrier diagonally across the path of a train of straight, parallel waves.

Echoes are simply reflections of sound waves. Sound is reflected from hard, flat surfaces in a similar way to light from a mirror (see Figure 13.3 on page 112); the angle of reflection is equal to the angle of incidence.

The reflection of ultrasonic waves (i.e. those above the frequency range of the human ear) is used at sea for echo-sounding and in engineering for non-destructive testing, e.g. for locating cracks. It is also used in medicine for forming images of the inside of the body.

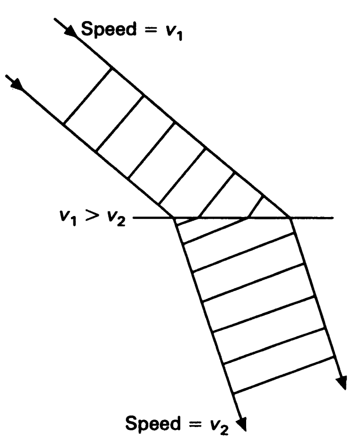

Figure 12.6

Figure 12.6 shows *refraction*, which is the change of direction a wavefront experiences when it passes obliquely through the interface between two media in which it has different speeds. This can be demonstrated in the ripple tank by placing a flat sheet of glass on the bottom so that the depth of water (hence, the wave speed and the wavelength) changes abruptly. It is important to remember that the frequency remains constant as long as the bar continues to oscillate at the same rate.

Refraction explains why waves tend to reach the beach parallel to the shoreline. When a wave approaches at an angle from deep water, then the part of the wavefront closest to the beach slows first as it encounters shallow water, while the parts further out are unaffected and tend to catch up. As these start to slow, the part closest to the shore slows even further and the process continues as progressive refraction along the wavefront tends to turn it parallel to the shore.

As we shall see in the next topic, refraction is an important aspect of the behaviour of light.

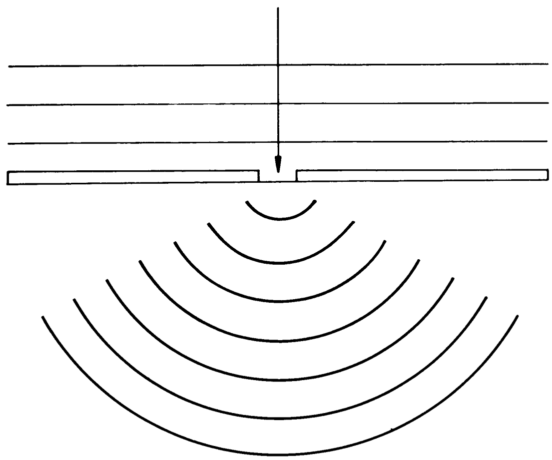

Figure 12.7

Figure 12.7 shows an example of *diffraction* in the ripple tank, where a straight, parallel wavefront passes through an aperture with a width that is similar to the wavelength. Waves passing through the aperture are diffracted — that is to say, they spread out from it with circular wavefronts just as though they had originated there. If the aperture is large compared with the wavelength, then the wavefronts pass through, straight and parallel, more or less unaltered apart from bending at the ends, where they are diffracted round the edges of the aperture. Because of their wavelength, sound waves are diffracted round the corners of buildings and by apertures such as doorways. The diffraction of light waves, which have wavelengths of less than a thousandth of a millimetre, operates on a much smaller scale, as we shall see in the next topic.

Before moving on to interference, we need to take note of the *principle of superposition*, which tells us that, where two or more waves meet, the total displacement at any given point is the sum of their individual displacements. This general idea applies to any waves of the same type but not necessarily of the same shape or wavelength. Figure 12.8(a) shows two waves, identical in this case, which are exactly in phase and superpose constructively to give increased displacement where they meet. On the other hand, Figure 12.8(b) shows that if the same two waves are exactly out of phase, they superpose destructively and cancel each other. Note that if two waves pass

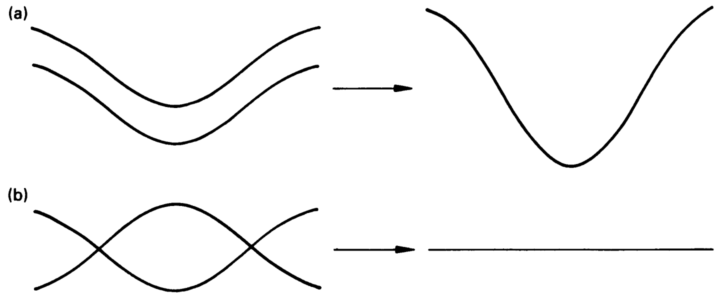

Figure 12.8

through each other, they will continue on their separate ways unaffected by their temporary superposition where they met.

Interference occurs as a result of the superposition of two or more coherent wave motions of comparable amplitude, such as those in Figure 12.8. (*Coherent* waves are those that have a constant phase relationship.)

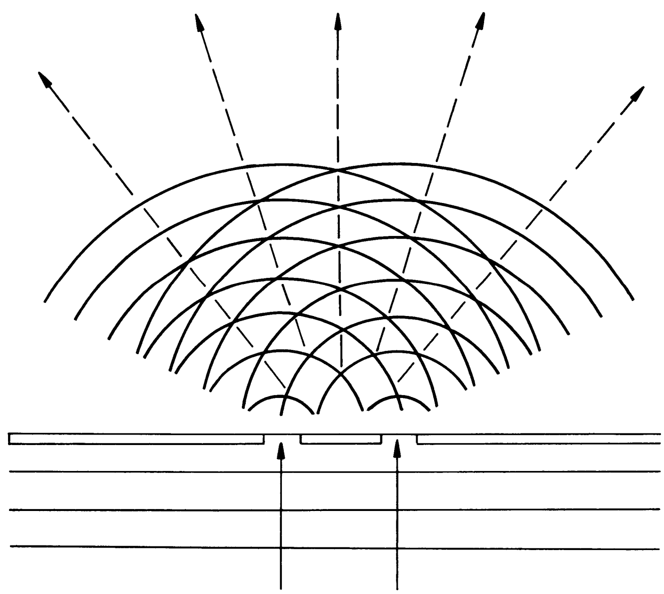

Figure 12.9

Figure 12.9 shows waves originating from two small apertures so that their wavefronts overlap.

Constructive interference occurs along the dotted lines where the two wave trains are exactly in phase; crest meets crest where the wavefronts cross and trough meets trough between the crests, to give correspondingly higher crests and deeper troughs, as in Figure 12.8(a).

Destructive interference occurs at those angles in between the dotted lines, where the waves are exactly out of phase, i.e. where troughs meet crests and the waves cancel each other out, as in Figure 12.8(b).

12.3 WAVE SPEED

Now we need to consider the speed of mechanical waves passing through matter. In the last topic we met the idea that mechanical oscillations involve the interconversion of potential and kinetic energy and depend on mass and on some kind of restoring force. These factors are also involved in the propagation of mechanical waves. For instance, the speed of a wave passing down a taut string is given by

$$v = \sqrt{\frac{F}{m/l}} \tag{12.3}$$

where v (m s^{-1}) is the wave speed, F (N) is the tension in the string and m/l (kg m^{-1}) is the mass per unit length of string. If the tension is increased, then the force restoring any displacement in the string will be greater and will tend to return the string to its equilibrium position more quickly. A heavier string, with a greater mass per unit length, has greater inertia and will therefore be returned more slowly.

Similar arguments apply to sound waves passing through matter. The speed of sound can generally be expressed in terms of the elastic behaviour and density of the medium as follows:

$$v = \sqrt{\frac{\text{elastic modulus}}{\text{density}}} \tag{12.4}$$

Young's modulus (Topic 2) is used for the elastic modulus where longitudinal waves pass down a solid rod of small diameter relative to the wavelength. The stiffer the chemical bonds in the material the more rapid its response to the displacement of its constituent atoms. The less dense the material the less its inertia and the faster its response to restoring forces.

In fluids, sound is transmitted by pressure changes, so the *bulk modulus* is used. Bulk modulus is a measure of the volume elasticity of a substance, as we shall see in Topic 20.

The speed of sound in air at 1 atm is about 331 m s^{-1} at 0 °C and 344 m s^{-1} at 20 °C and, by comparison, about 1500 m s^{-1} in water and about 5000 m s^{-1} in steel.

Interesting things happen when a sound source moves at a speed that is significant relative to the speed of sound. For example, the pitch of a car horn appears to drop as it passes a stationary observer.

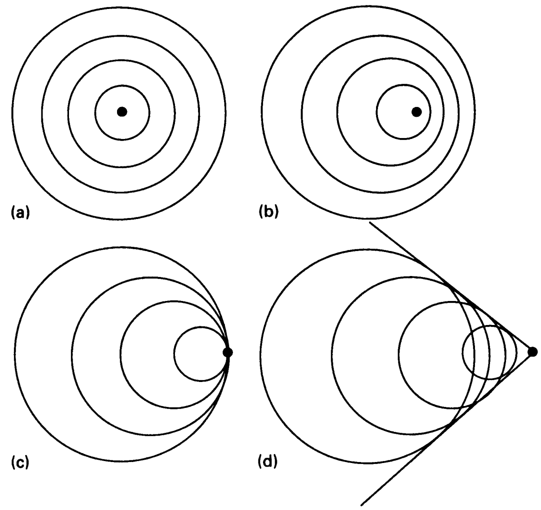

Figure 12.10

This is an example of the Doppler effect. Figure 12.10(a) shows that a stationary sound source remains at the centre of the spherical wavefronts that it produces, but in Figure 12.10(b), where the source is moving from left to right, it tends to catch up with the wavefronts ahead and leave behind those at the rear. Owing to the effective squashing and stretching of the wavelengths, a stationary observer hears a frequency that appears to be higher than the actual frequency as the source approaches and lower as it moves away. The Doppler effect also occurs if the source is stationary and the observer is moving.

As the velocity is further increased, the wavefronts crowd closer together until at the speed of sound (Figure 12.10c) they can no longer outrun the source and so form a barrier which the source must penetrate to achieve supersonic speeds. Above the speed of sound, as in Figure 12.10(d), the source outstrips the wavefronts, leaving a conical shock wave behind it that is defined by the tangential envelope enclosing the wavefronts. This is the source of the sonic boom from supersonic aircraft.

12.4 STANDING (STATIONARY) WAVES

Standing or stationary waves, as opposed to progressive or travelling waves, are so called because they do not appear to move. They occur, for example, when a progressive wave is reflected straight back along its own path, so that it interacts with the wave moving forwards in the opposite direction. Under the right conditions the superposition of two identical progressive waves moving in opposite directions will cause a standing wave to be set up.

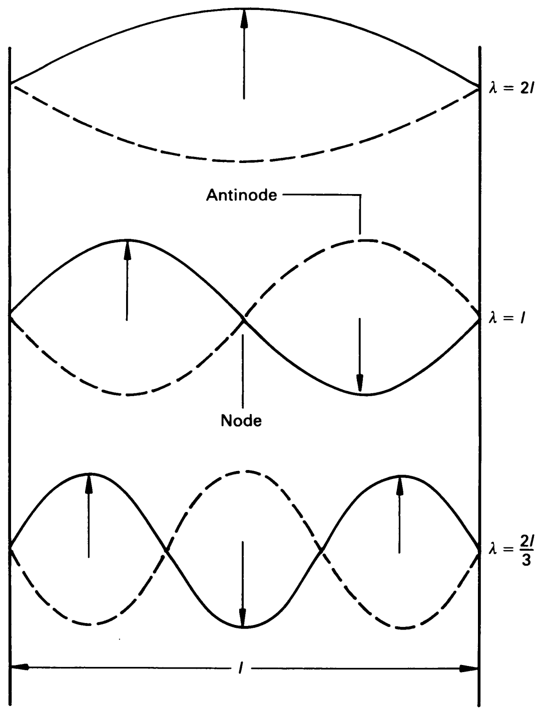

Figure 12.11

Standing waves are formed in taut strings where progressive transverse wave trains move in opposite directions and are continuously reflected at either end to give patterns such as those illustrated in Figure 12.11. The figure represents the times when the displacements are at their maxima. There are points of zero displacement called *nodes*, which never vibrate, and points of maximum amplitude called *antinodes*. The solid and dotted lines indicate how the amplitude varies in between. The simplest mode of vibration, called the *fundamental* mode, has one loop corresponding to half a wavelength, so that $\lambda = 2l$, where l is the length of the string. For the *first overtone*, with two loops, $\lambda = l$; for the *second overtone* $\lambda = 2l/3$; and so on. We can see that, in general, $\lambda = 2l/n$, where n is the number of loops. The corresponding frequencies are given by $f\ (= v/\lambda) = vn/2l$, where v is the speed of the transverse waves in the string.

These various frequencies, sometimes referred to as *harmonics*, are therefore simple multiples of the fundamental frequency; thus, the fundamental frequency is referred to as the *first harmonic* ($n = 1$), the first overtone ($n = 2$) as the *second harmonic*, the second overtone ($n = 3$) as the *third harmonic*, and so on.

Musical instruments produce mechanical standing waves (transverse in strings and longitudinal in air columns) which cause progressive longitudinal waves to spread through the surrounding air to the listener's ears. Although the predominant mode is usually the fundamental, overtones are produced as well — and the combination of different amplitudes of the various overtones give different instruments their characteristic sound quality.

From above, the fundamental frequency of a taut string is given by $f = v/2l$ (since $n = 1$), so, substituting for v from Equation (12.3),

$$f = \frac{1}{2l}\sqrt{\frac{F}{m/l}} \tag{12.5}$$

Thus, a string may be tuned to a particular fundamental frequency by varying its tension. The equation also suggests why long, heavy strings are used for low notes, and short, light strings for high ones.

12.5 RESONANCE

If an object is subjected to vibrations of a frequency that coincides with one of its own natural frequencies, then *resonance* occurs. Thus, a note sung near a piano will cause some of the strings to resonate or vibrate in sympathy.

Resonance is of great interest to engineers. Sometimes it merely causes irritating vibrations, but sometimes it can be destructive, particularly when the energy supplied by the source exceeds the damping losses, so that the amplitude builds up. (As we know from experience, the amplitude of a pendulum or a swing can be greatly increased by pushing it quite gently in time with its own natural frequency.) Powerful singers are said to be able to break wine glasses by hitting a resonant frequency, and marching soldiers break step crossing bridges to avoid the same basic type of problem. Machinery can cause resonance in the floor that supports it, and some readers will have seen the famous film of the Tacoma Narrows suspension bridge collapsing because of resonance effects due to the wind.

Questions

1. A thin card produces a musical note when it is held lightly against the spokes of a rotating wheel. If the wheel has 32 spokes, how quickly must it rotate, in revolutions per minute, in order to produce the A above middle C (i.e. 440 Hz)?
2. Assume the speed of sound in air to be 340 m s^{-1}.
   (a) A clap of thunder arrives 5 s after the lightning flash. Assuming that light travels at infinite speed, how close is the storm?
   (b) Find the wavelength in air of the musical note C with the frequency 262 Hz.

(c) How long does it take for the echo to reach a person who fires a gun while standing 68 m from the base of a cliff?

(d) The wavelength of sound ranges from 17 mm up to 17 m in air. Find the corresponding frequency range.

3. A person stands between two parallel cliffs and fires a gun. The first echo arrives after 1 s and the second after 2 s. How far apart are the cliffs? (Assume the speed of sound is 340 m s^{-1}.)

4. Two people are standing 85 m from the base of a straight cliff. One fires a gun and the other hears two reports, 0.75 s and 0.90 s after seeing the smoke. Find (a) the speed of sound and (b) the distance between the two people.

5. Find the power output of a sound source which produces a sound intensity of 1.4×10^{-4} W m^{-2} at a distance of 25 m.

6. An underwater sound source is operating at 256 Hz. Find the wavelength of the sound (a) under the water and (b) after it has passed through the surface into the air above. (Assume the speed of sound is 1460 m s^{-1} in the water and 340 m s^{-1} in the air.)

7. If the speed of transverse waves is 384 m s^{-1} along a taut string 0.75 m long, find the fundamental frequency of the string and the frequency of its second and third harmonics.

8. The length l of a wire under constant tension was adjusted by varying the distance between its two supports in order to tune it to various frequencies, as follows:

f/Hz	250	300	350	400	450	500
l/m	0.877	0.730	0.629	0.549	0.490	0.440

Manipulate the data to give a straight line relationship and, from a plot, read off the frequency corresponding to $l = 0.500$ m.

9. (a) If it takes 1.9 ms for a sound pulse to travel down a steel rod of length 9.5 m and density 7.8×10^3 kg m^{-3}, estimate the Young's modulus of the steel.

(b) If E represents Young's modulus and ρ represents density, show that the base units of $\sqrt{E/\rho}$ can be reduced to those of velocity.

10. A sound pulse enters one end of a lead rod and of an aluminium rod at the same moment. What must be the relative lengths of the rods if the pulse is to emerge from both simultaneously? (For lead $E = 1.6 \times 10^{10}$ N m^{-2} and $\rho = 1.1 \times 10^4$ kg m^{-3}, and for aluminium $E = 7.0 \times 10^{10}$ N m^{-2} and $\rho = 2.7 \times 10^3$ kg m^{-3}.)

TOPIC 13 ELECTROMAGNETIC WAVES

COVERING:

- the nature of electromagnetic waves;
- reflection and refraction;
- total internal reflection;
- diffraction and interference;
- polarisation.

An electromagnetic wave can be considered as a progressive transverse wave that consists of a fluctuating electric field coupled with a fluctuating magnetic field at right angles to it, as shown in Figure 13.1. Don't worry if this seems a difficult idea at this stage; it will become clearer when we discuss electric and magnetic fields in later topics. For the moment the important thing to remember is that, unlike mechanical waves, electromagnetic waves do not necessarily require a medium for their propagation. They travel through empty space (vacuum) at a speed of very nearly 3×10^8 m s^{-1}, commonly called the speed of light (symbol c), and their frequency can be obtained from their wavelength via Equation (12.1) ($v = f\lambda$). The speed of light in air is very slightly less than in vacuum but considerably less in some other materials, as we shall see later.

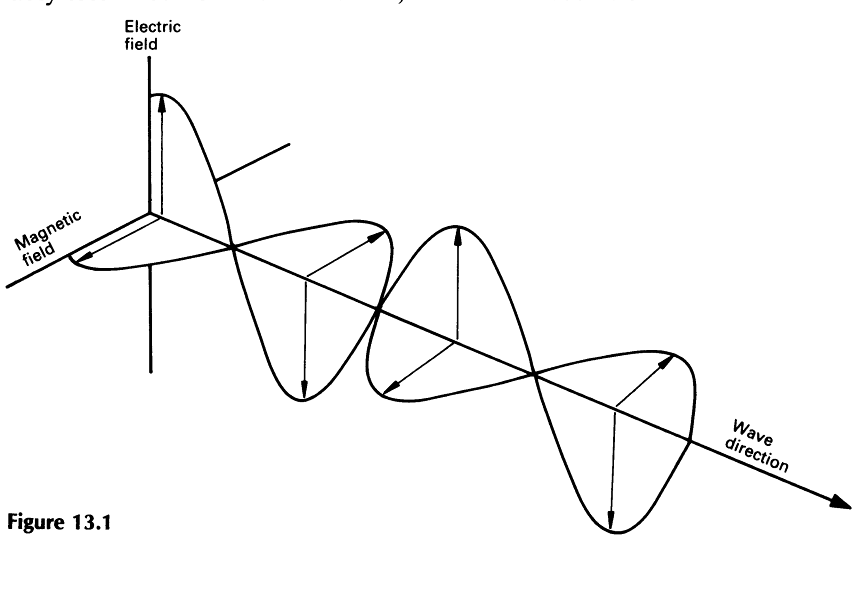

Figure 13.1

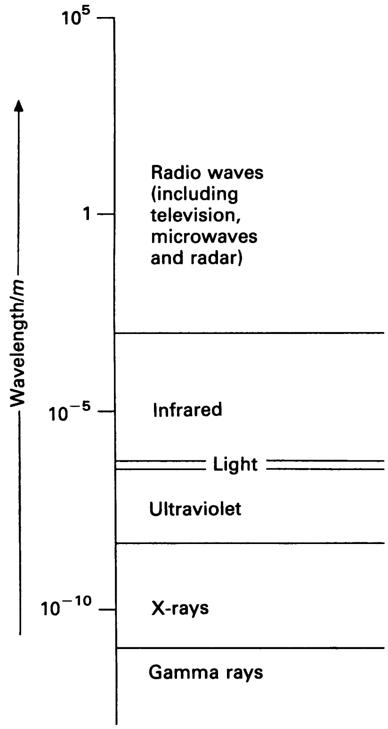

Figure 13.2

The *electromagnetic spectrum* (Figure 13.2) is divided into various types of waves (X-rays, light, radio waves, etc.) which we recognise from everyday experience. These have different names, because they are produced in different ways and have different effects, but the essential difference between them is their wavelength. The boundaries in the figure are approximate, since there are regions of overlap.

In this topic we shall use light to illustrate the nature of electromagnetic waves. Like mechanical waves, they exhibit reflection, refraction, diffraction and interference. And when they fall on the surface of an object they may be reflected by it, transmitted through it, absorbed by it, or some combination of the three. (They may also be scattered, but this is beyond the scope of our discussion.)

Light occupies a narrow band of wavelengths between infrared and ultraviolet. Note that, in a similar way to pitch and sound frequency, colour is a sensation attributable to different wavelengths of light ranging across the colours of the rainbow from about 700 nm (1 nm = 1×10^{-9} m) at the red end down to about 400 nm at the violet end. White light is a mixture of all these wavelengths. In white light a red object appears red because it reflects red light and absorbs the other colours; similarly, a red filter transmits red light. In white light white objects appear white because they reflect all the wavelengths and black objects appear black because they absorb them all. (The absorbed energy serves to increase the internal energy of the object, i.e. the kinetic and potential energy of its component particles (Topic 16), and this usually results in a temperature rise.)

13.1 REFLECTION

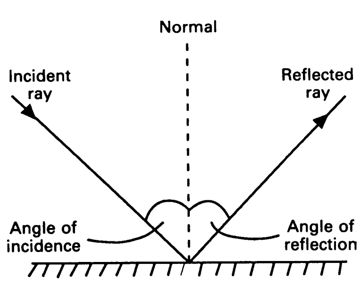

Figure 13.3

Objects are made visible by light reflected from their surface. If the surface is rough, then it will appear dull because the reflected light is diffuse, i.e. reflected in all directions by surface irregularities. On the other hand, a smooth, flat, shiny surface like a mirror will reflect a beam of light more or less intact.

Figure 13.3 shows a *ray* representing the direction in which the waves are travelling in a beam of light that is being reflected by a mirror. (Note that the direction of a ray is normal to the wavefronts which it represents.) There are two laws governing reflection. One tells us that the incident ray and the reflected ray lie in the same plane with the normal to the reflecting surface where they meet. The other tells us that the angle of reflection is equal to the angle of incidence relative to the normal.

Figure 13.4 illustrates how light rays diverging from an object form an image in a mirror which gives the impression that the object lies behind the reflecting surface. The direction of the rays changes where they are reflected by the mirror, but the observer interprets them as travelling in straight lines.

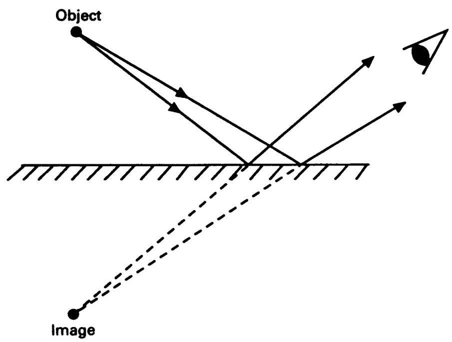

Figure 13.4

13.2 REFRACTION

Figure 13.5 shows how light is refracted when it passes obliquely through the interface between two media in which it has different speeds. The first thing to note is that the incident ray, the refracted ray and the normal all lie in the same plane but, unlike reflection, the angle of refraction r is different from the angle of incidence i.

The figure illustrates the case where the speed of light c is greater in medium 1 than in medium 2, i.e. $c_1 > c_2$. Figure 13.5(b) enables us to find the relationship between i, r, c_1 and c_2. This is less complicated than it might seem. Just follow the argument very carefully, a step at a time.

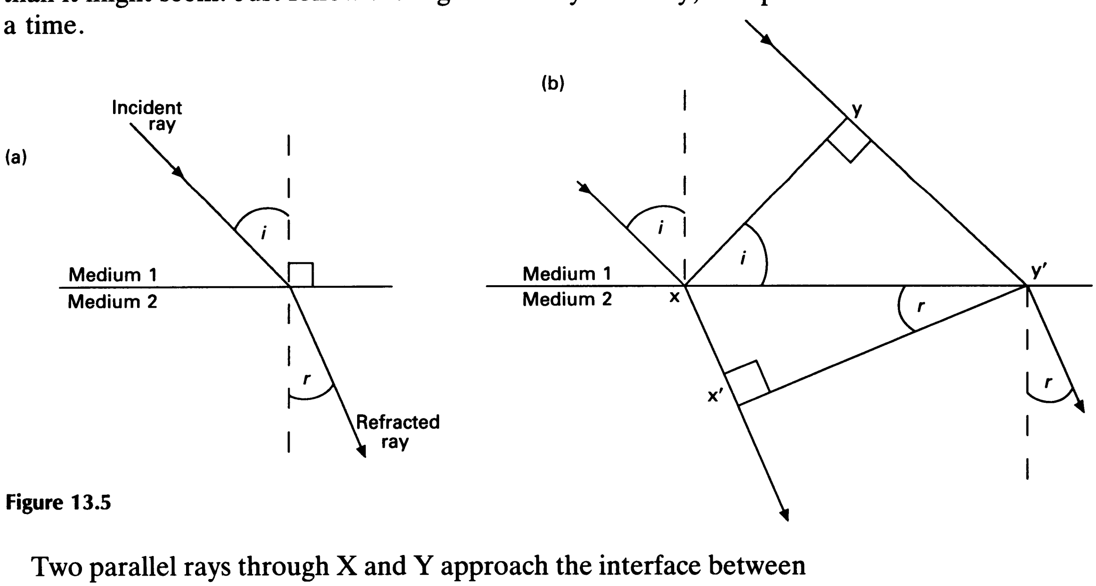

Figure 13.5

Two parallel rays through X and Y approach the interface between the media at an angle of incidence i. Since $c_1 > c_2$, any point on the wavefront XY slows down immediately it crosses the interface. If the wavefront travels from Y to Y′ in time t, then the distance YY′ equals

c_1t (i.e. speed × time). During the same interval the other end of the wavefront travels the correspondingly shorter distance XX′, equal to c_2t, through medium 2. The wavefront at X′Y′ has therefore changed direction and the refracted rays are bent inwards towards the normal so that the angle of refraction r is less than the angle of incidence i. The greater the difference between c_1 and c_2 the greater the change of direction.

Since the ray approaching X is perpendicular to the wavefront XY, and since the normal at X is perpendicular to the interface, then YXY′ = i. By a similar argument XY′X′ = r. It follows that

$$\mathrm{YY'} = \mathrm{XY'} \sin i = c_1 t$$

and

$$\mathrm{XX'} = \mathrm{XY'} \sin r = c_2 t$$

Therefore, dividing one equation by the other,

$$\frac{\sin i}{\sin r} = \frac{c_1}{c_2} \qquad (13.1)$$

Note that the rays in the figure are reversible and that light will travel along the same path in the opposite direction. In other words, Equation (13.1) still applies where $c_2 > c_1$, but in such cases $r > i$ and refracted rays are bent away from the normal.

13.3 REFRACTIVE INDEX

Equation (13.1) is the basis of *Snell's law*, which tells us that, for two given media, sin i/sin r is a constant. This constant (= c_1/c_2) is called the *relative refractive index* ${}_1n_2$ for waves passing from medium 1 to medium 2.

The *absolute refractive index* n of a particular medium is given by the ratio between the velocity of light in vacuum, c, and the velocity of light in the medium, c_m, so that $n = c/c_m$. Therefore, $c_1 = c/n_1$ and $c_2 = c/n_2$, so

$$\frac{c_1}{c_2} = \frac{c}{n_1} \times \frac{n_2}{c} = \frac{n_2}{n_1}$$

and, substituting n_2/n_1 for c_1/c_2 in Equation (13.1),

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1} \qquad (13.2)$$

At ordinary temperature and pressure the absolute refractive index of air has a value of 1.0003, so if medium 1 is air, then, to a very close approximation, Equation (13.2) reduces to $\sin i / \sin r = n_2$.

To take a few examples, the absolute refractive index of water is 1.33 and that of glass is typically about 1.5–1.7, depending on its type, while that of diamond is about 2.44.

Refractive index varies slightly with wavelength, so values are often quoted for monochromatic light (i.e. a single wavelength), commonly $\lambda = 589.3$ nm, which is in the yellow part of the spectrum.

Figure 13.6 illustrates how refraction accounts for the fact that objects submerged in water appear to be closer to the surface than they really are. Light rays from the object bend away from the normal where they leave the water but the observer interprets them as travelling in straight lines. This is also the reason why straight objects appear to bend upwards where they are partly under water.

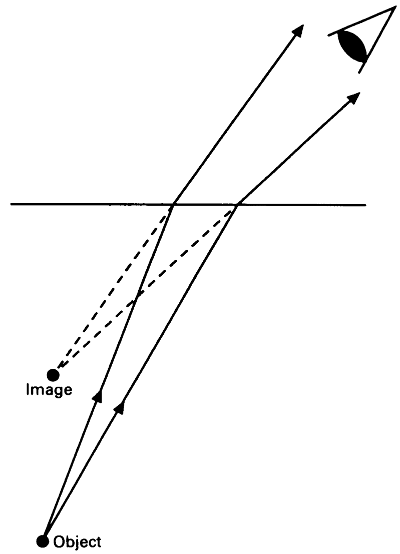

Figure 13.6

When light passing through one medium meets the interface with another of lower refractive index, then *total internal reflection* may occur. Figure 13.7 shows that at zero or relatively low angles of incidence ((a) and (b)) a light ray will pass through the interface and, except at $i = 0°$, will be refracted away from the normal. When the angle of incidence is increased to the *critical angle* i_c, as in (c), the refracted ray passes along the interface and $r = 90°$. When the angle of incidence is greater than i_c, as in (d), total internal reflection occurs, i.e. the ray is reflected from the interface. (Weak internal reflections may be seen for angles of incidence less than i_c; hence the term *total* internal reflection for greater angles.) Substituting i_c for i and 90° for r in Equation (13.2), we have

$$\sin i_c = \frac{n_2}{n_1} \qquad (13.3)$$

If medium 2 is air, then, to a close approximation, the absolute refractive index of medium 1 is given by $1/\sin i_c$.

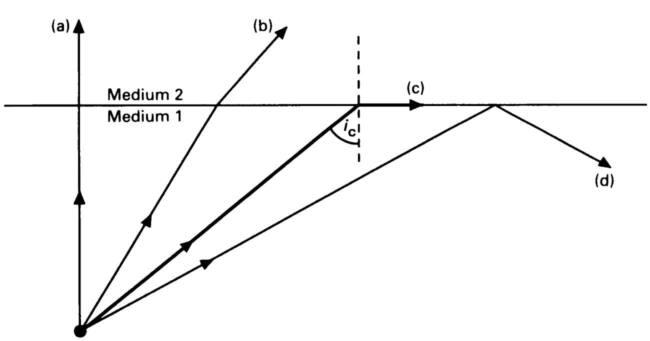

Figure 13.7

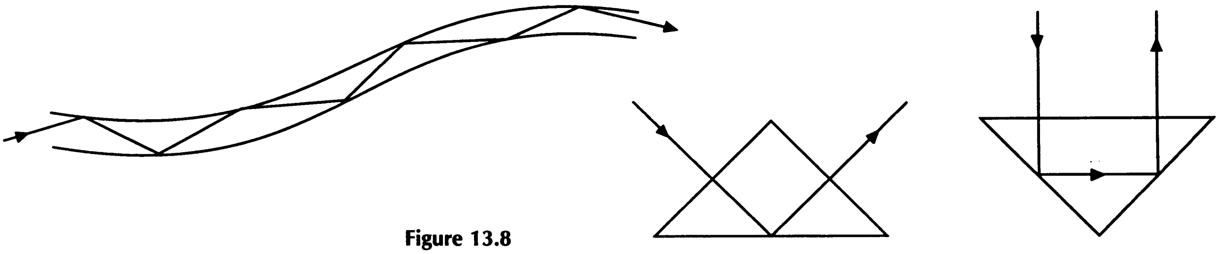

Figure 13.8

Figure 13.8 illustrates how total internal reflection traps light inside glass fibres so that it can be piped from one place to another. It also shows how prisms can be used for reflecting light in optical equipment such as periscopes and binoculars.

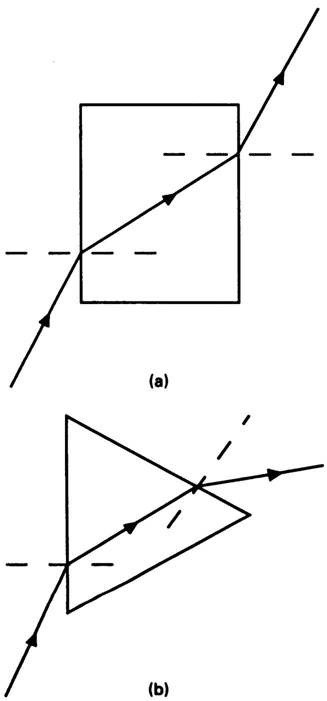

Figure 13.9

13.4 PRISMS AND LENSES

Figure 13.9 illustrates the deviation of light by refraction through a prism. Figure 13.9(a) shows a ray passing through a rectangular block of material. The path by which the light emerges from the block is parallel to the path by which it enters, because the deviation it experiences on entry is cancelled by the equal and opposite deviation on leaving. In the case of the prism in Figure 13.9(b), the angles are such that the ray is bent twice in the same direction and the deviation on leaving the prism is added to the deviation on entering it.

Figure 13.10 shows how the behaviour of a lens can be viewed in terms of a series of prisms. Convex lenses (thicker at the centre) are called *converging* lenses because they bend a parallel beam of light inwards so that it converges to a point called the *principal focus* (at F in Figure 13.10a). Concave lenses (thinner at the centre) are called *diverging* lenses because they bend a parallel beam of light outwards so that it seems to diverge from a principal focus (at F in Figure 13.10b).

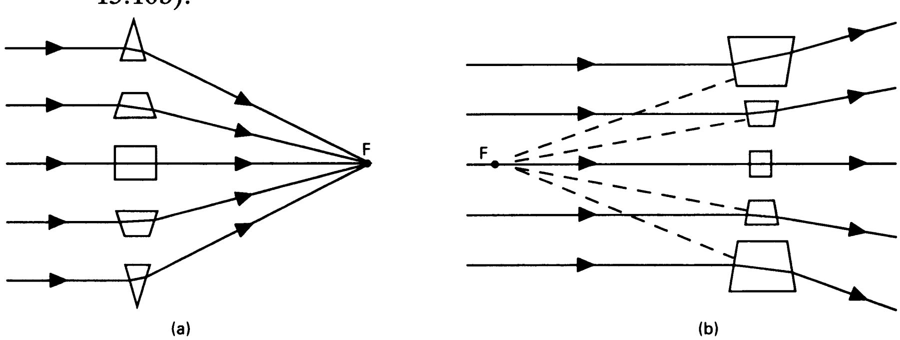

Figure 13.10

A prism will disperse a beam of light into a spectrum consisting of its component wavelengths. *Dispersion* is a consequence of the variation of refractive index with wavelength mentioned earlier. For many substances, refractive index increases with decreasing wavelength; thus, the violet end of the spectrum will be refracted through the greatest angle when white light is dispersed by a glass prism (Figure 13.11). Raindrops disperse sunlight to form rainbows, and simple lenses produce images with coloured fringes because they focus different wavelengths at slightly different positions.

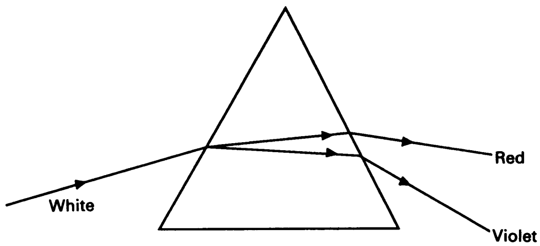

Figure 13.11

13.5 DIFFRACTION AND INTERFERENCE

If monochromatic light passes through a fine slit to illuminate two other fine slits, closely spaced and parallel to the first, then the double slit behaves as a pair of coherent sources which produce interference patterns, as represented in Figure 12.9 (page 104). The effect of constructive and destructive interference may be seen as alternate light and dark bands, parallel to the slits, that are known as *Young's fringes* (after Thomas Young, who demonstrated them in 1801).

A *diffraction grating* consists of many uniformly spaced parallel slits, normally up to about 1000 per mm. These are made by ruling extremely fine lines on suitable materials such as glass, or more commonly (and more cheaply) by casting plastic replicas. (*Reflection gratings* are opaque and work with reflected rather than transmitted light.)

Figure 13.12(a) represents light waves passing through a transmission grating with the diffracted wavefronts emerging from the slits. (Only some of the diffracted wavefronts are shown.) The dotted diagonal line represents the tangential envelope to a series of diffracted wavefronts in which each is exactly one wavelength out of step with its neighbours. All the wavelengths along this line are therefore exactly in phase and reinforce one another. This produces a beam of diffracted light which deviates from the direction of the original beam by an angle θ.

In Figure 13.12(b) d represents the distance between centres of adjacent slits. θ is the angle of deviation of the beam, which is found when the path difference between adjacent wavefronts is equal to λ, the wavelength of the light. Simple trigonometry gives $\lambda = d \sin \theta$.

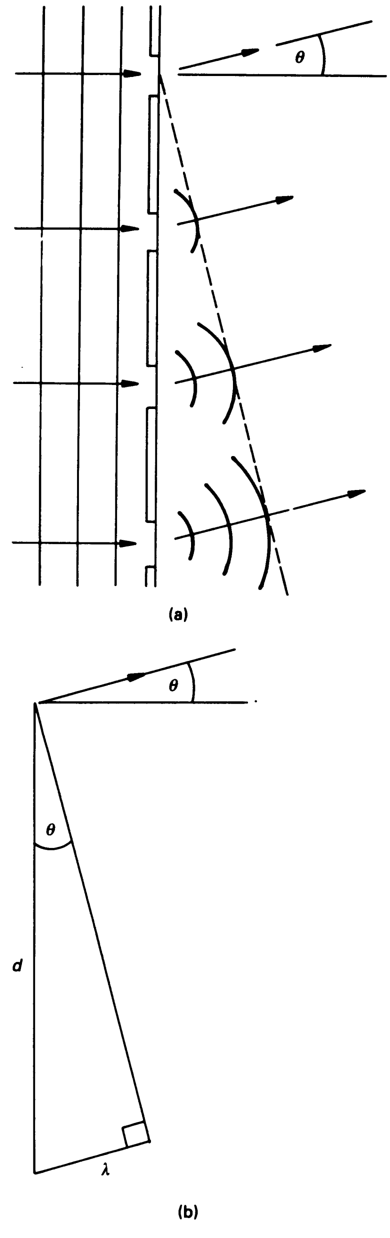

Figure 13.12

Hence, by knowing d and measuring θ, the wavelength can be calculated. If white light is used instead of monochromatic, then each component wavelength deviates to the angle given by the equation, with the result that a spectrum is obtained. As the equation tells us, the deviation will be greatest for the longer wavelengths towards the red end of the spectrum (as opposed to the case of the prism, where the shorter wavelengths suffer the greatest deviation).

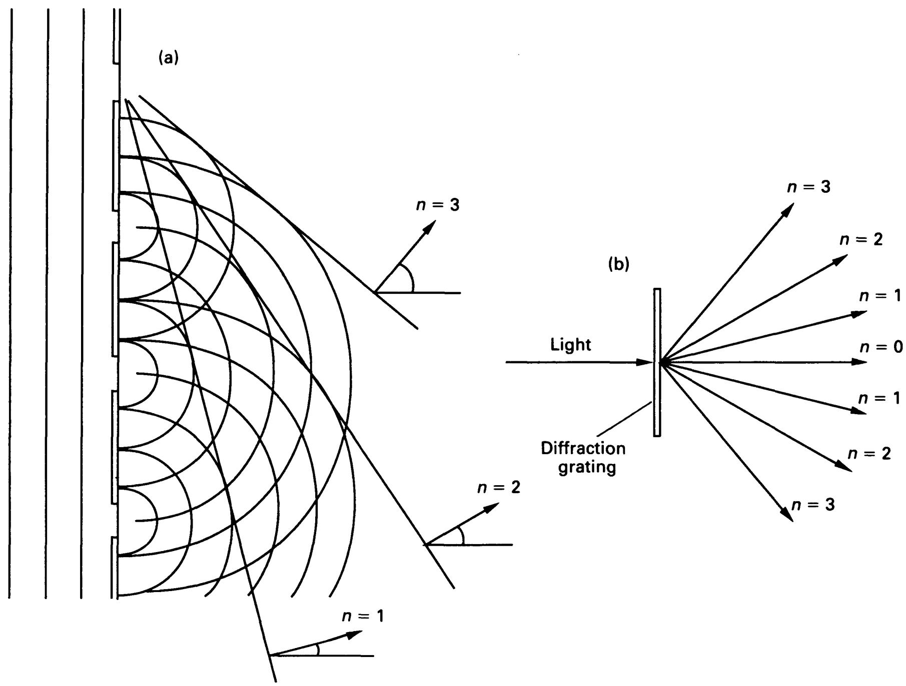

Figure 13.13

As Figure 13.13(a) indicates, there are other tangential envelopes corresponding to path differences of 2λ, 3λ, and so on. In general, therefore, we have

$$n\lambda = d \sin \theta \tag{13.4}$$

where n is the integral number of wavelengths that constitutes the path difference. This gives rise to first-, second- and third-order angles of diffraction where n equals 1, 2, 3, and so on, as indicated in Figure 13.13(b). Note that there will be a zero-order beam along the centre line where the path difference is zero and θ is therefore zero. On the other side of this there will be a second series of diffraction angles, symmetrical with the first, arising in precisely the same way but with the tangential envelopes facing the other direction, as shown. If white light is used, then corresponding orders of spectra will be seen (apart from a zero-order band of white light along the centre line, where $n = 0$ and $\theta = 0$).

Note that, since sin θ cannot exceed 1, $n\lambda$ cannot exceed d, which places a limit on the number of orders which can be obtained.

13.6 POLARISATION

The polarisation of light is best explained by a mechanical analogy using the rope in Figure 12.1 (page 100). Normally transverse waves of any orientation would travel along the rope, but if it is threaded through a vertical slot, then only vertical waves will be able to pass through. Similarly, a horizontal slot will only transmit horizontal waves. A vertical slot followed by a horizontal slot will stop any transverse waves but would have no effect on longitudinal waves (for example, those passing along a stretched spiral spring threaded through them).

Polaroid, the material used in certain types of sunglasses, acts as the optical equivalent of a slot, and two pieces of Polaroid crossed at 90° will stop light passing through. In a beam of unpolarised light the electric field vibrates at all angles within the plane perpendicular to the direction in which the light is travelling. In polarised light these vibrations are confined to one particular direction. As the nature of longitudinal waves suggests, they cannot be polarised.

Polarisation occurs on reflection from certain surfaces and Polaroid sunglasses can be used to cut down glare from the reflected light. Some transparent plastics will polarise light when they are under stress; these can be used to make models of engineering components which enable internal stress patterns to be made visible.

Questions

(Assume that $c = 3.00 \times 10^8$ m s^{-1}.)

1. If an electromagnetic wave has a period of 1.96×10^{-15} s, in what part of the spectrum does it lie?

2. Find the wavelengths transmitted by radio stations broadcasting on (a) 97.6 MHz and (b) 1215 kHz.

3. A source produces light of frequency 4.57×10^{14} Hz. What is the wavelength of the light (a) in air and (b) in water. (Assume $n_{water} = 1.33$.)

4. (a) If a light beam enters the surface of a smooth pond at 55° to the normal, by how much will it be deflected from its original path?
(b) If a light beam emerges from under the surface of a smooth pond at 55° to the normal, by how much has it been deflected from its original path? (Assume $n_{water} = 1.33$.)

5. What is the refractive index of a substance for which the critical angle in air is 49°?
6. The refractive index of diamond was found to be 2.44, using monochromatic light of frequency 5.09×10^{14} Hz. Find (a) the speed of light in diamond, (b) the wavelength of the monochromatic light in diamond, and the critical angle for internal reflection of diamond (c) in air and (d) in water. (Assume $n_{water} = 1.33$.)
7. A fish's eye view concentrates everything above the water surface into a circle of light which subtends an angle of 98° at the eye. (a) Explain this by assuming $n_{water} = 1.33$. (b) What happens outside the circle?
8. A source of monochromatic light gave a third-order angle of deviation of 48.6° with a particular diffraction grating. Find the first-order angle of deviation.
9. Find the wavelength of monochromatic light giving a first-order angle of deviation of 17.1° using a diffraction grating with 500 lines per mm.
10. What is the angular spread of (a) the first-order spectrum and (b) the third-order spectrum of visible light, from 390 nm to 740 nm, using a diffraction grating with 250 lines per mm?

Part 2

Structure and Properties of Matter

TOPIC 14 ATOMIC STRUCTURE AND THE ELEMENTS

COVERING:

- the electron, the proton and the neutron;
- the nucleus;
- the electronic structure of atoms;
- the elements and the periodic table;
- atomic mass.

So far our everyday general knowledge of gases, liquids and solids has provided us with sufficient background for our discussion. Now we have reached the point where we need to concern ourselves with the internal structure of matter. We shall start with atoms and see how differences in atomic structure lead to the various chemical elements such as hydrogen, carbon, oxygen, and so on.

14.1 THE CONSTITUENT PARTICLES OF ATOMS

Atoms are extremely small, with radii of about 10^{-10}m. We can view their structure in terms of three constituent fundamental particles — the *electron*, the *proton* and the *neutron*. These can be distinguished from one another by their mass and their charge (Table 14.1).

Table 14.1

Particle	Mass/kg	Mass/u	Charge/C
Electron	9.11×10^{-31}	5.5×10^{-4}	-1.60×10^{-19}
Proton	1.67×10^{-27}	1.0	$+1.60 \times 10^{-19}$
Neutron	1.67×10^{-27}	1.0	0

Because these particles are so small, it is sometimes convenient to express their mass in terms of *atomic mass units* (symbol u), where $1\ u = 1.66 \times 10^{-27}$ kg. (We shall see how we arrive at this value later in the topic.) Note that the mass of the electron can often be considered to be negligible compared with that of the proton and the neutron.

Most of us are aware of the fact that we can electrically charge certain objects such as plastic combs and pens by rubbing them on cloth so that they attract scraps of paper and hairs or even a thin stream of water running from a tap. There are two types of charge, positive and negative, and we need to remember that like charges repel each other, whereas opposite charges attract. The force between two charges obeys an inverse square law, called Coulomb's law, that is analogous to Newton's law of gravitation, which we met in Topic 2. In Topic 21 we shall see that this force, whether attractive or repulsive, is proportional to Q_1Q_2/r^2, where Q_1 and Q_2 represent the magnitudes of the charges and r the distance between them. The unit of charge, which we shall define in Topic 24, is the coulomb (symbol C).

As Table 14.1 shows, the charge on the electron is opposite but equal in magnitude to that on the proton. Atoms contain equal numbers of each and are therefore electrically neutral. (Neutrons are neutral, so they make no contribution to the balance of charge in the atom.)

Now we need to give some thought to how these particles are arranged.

14.2 THE NUCLEUS

At the centre of the atom lies the *nucleus*, which contains all the protons and neutrons and therefore all the positive charge and most of the mass. The number of protons characterises the nucleus as being that of a particular element and is called the *atomic number* (symbol Z). Obviously there must also be Z electrons in a neutral atom. To take a few examples, hydrogen has an atomic number of 1, carbon 6, chlorine 17 and iron 26. The protons and neutrons are held together by an extremely powerful nuclear force that we shall not consider here other than to note that it is very much stronger than the repulsive coulombic force between protons.

Light elements tend to have about equal numbers of neutrons and protons, but the neutron/proton ratio increases with atomic number to about one and a half for heavy elements. However, the number of neutrons varies for nuclei of the same element. This gives rise to *isotopes*, which are atoms with the same atomic number but which differ in the number of neutrons their nuclei contain. Thus, chlorine-37 has 17 protons and 20 neutrons, whereas chlorine-35 has 17 protons but only 18 neutrons. Note that the *mass number* (symbol A) is the total number of protons and neutrons in the nucleus (37 and 35 in the case of these two examples). It follows that the number of neutrons in the nucleus is given by $(A - Z)$. The convention for representing a particular atom, say of element X, is ^A_ZX. For example, the carbon-12 isotope is represented $^{12}_{6}C$, where C is the chemical symbol for carbon (see Table 14.2 on page 127).

Nuclei have radii of approximately 10^{-14} – 10^{-15} m and are therefore tiny compared with the overall size of the atoms which contain them. Since they contain most of the mass, they are of extremely high density.

14.3 THE ELECTRONIC STRUCTURE

The relative size of the nucleus means that the atom is mostly empty space. Early models of the atom suggested that the electrons revolve in orbits around the nucleus like the planets round the sun, the attractive force between the positive nucleus and the negative electrons providing the centripetal force. The total energy of the electron was viewed in terms of its potential energy due to its distance from the nucleus and its kinetic energy due to its motion. More recent developments have led to the idea of three-dimensional *orbitals* representing regions within the atom where the probability of finding an electron is high.

Rather than try to picture the atom, we shall simply think of the electrons as being arranged in energy levels. These are traditionally called *shells* (numbered 1, 2, 3, etc.) and are divided into *subshells*. The first and lowest energy shell has one subshell (labelled 1*s*), the second shell has two subshells (2*s* and 2*p*), the third has three (3*s*, 3*p* and 3*d*) and the fourth has four (4*s*, 4*p*, 4*d* and 4*f*). (The use of the letters *s*, *p*, *d* and *f* has an historical basis.) Figure 14.1 shows these and higher subshells arranged in columns according to which main shell they belong. They are plotted vertically in order of increasing energy. Note that, from the 4*s* upwards, the subshell energy levels overlap between the main shells.

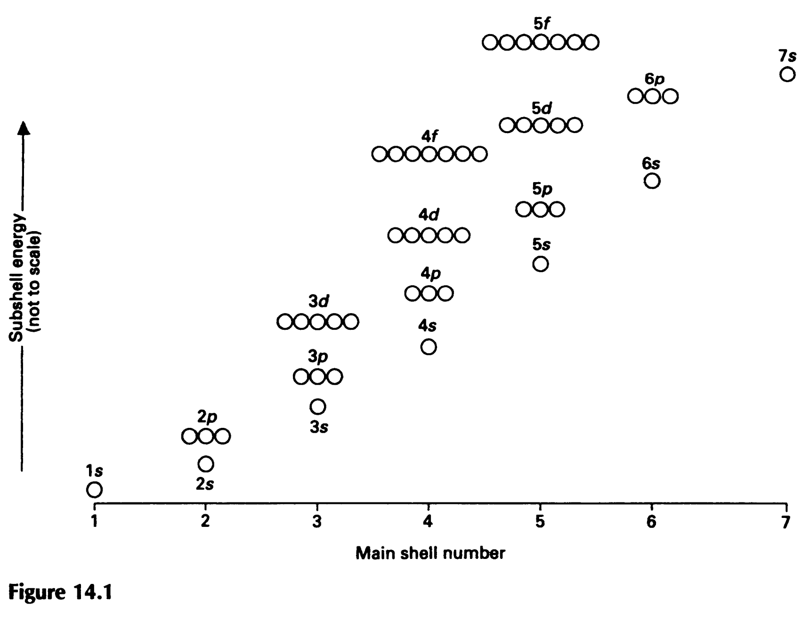

Figure 14.1

Each of the circles in the figure corresponds to one of the orbitals mentioned above and, in effect, represents an 'address' that can accommodate two electrons. Each *s* subshell can accommodate two electrons in one orbital, each *p* subshell can accommodate six in three orbitals, each *d* can accommodate ten in five orbitals and each *f* fourteen in seven. Note that two electrons occupying the same orbital must be of opposite *spin*. (In some respects an electron behaves as if it is spinning on its axis like a top; in terms of our simple model, it is as though the two electrons must spin in opposite directions, clockwise and anticlockwise, in order to coexist in the same orbital).

We shall now make use of Figure 14.1 as the basis of a paper exercise where we picture the chemical elements as a series, built up by successively adding electrons to the subshells. As we might expect, the lowest energy levels are filled first. (We should bear in mind that there must be an equal number of protons and the appropriate number of neutrons added to the nucleus at the same time.)

Table 14.2 shows the first thirty-six elements. Hydrogen has a single electron in the 1*s* subshell. The completion of this subshell with a second electron gives helium, whereupon the first main shell has its full complement of two electrons. The third electron enters the next lowest subshell, the 2*s*, to give lithium, and the fourth electron completes it, to give beryllium. The 2*p* subshell fills next to give the six elements from boron to neon. With neon the second main shell has its full complement of eight electrons.

Complications begin in the third shell. The 3*s* and 3*p* subshells fill first to give the elements from sodium to argon. But, as Figure 14.1 shows, the 4*s* subshell lies at a lower energy level than the 3*d* and therefore fills next to give potassium and calcium. The 3*d* follows then, at gallium, the 4*p*. This is as far as we need to go to get the general idea of viewing the elements in terms of filling subshells in order of increasing energy.

A very important consequence of the overlap between energy levels is that the outermost main shell of any atom cannot contain more than eight electrons; as Figure 14.1 shows, *d* and *f* subshells do not begin to fill until there are electrons present in a higher main shell.

The actual number of electrons in the outermost main shell is of enormous importance in determining the properties of an atom. For example, Figure 14.2 shows the *ionisation energy* of the first twenty elements in order of increasing atomic number. The ionisation energy is simply the amount of energy that would be required to completely remove an outermost electron from an atom against the attractive force due to the positive charge on the nucleus.

First, we can see that helium (He), neon (Ne) and argon (Ar) have high ionisation energies. This means that atoms of these elements are resistant to the removal of an outer electron and therefore have particularly stable electronic structures which make them extremely reluctant to combine chemically with other elements. This is reflected

Table 14.2

Element	Chemical Symbol	Atomic Number	1*s*	2*s*	2*p*	3*s*	3*p*	3*d*	4*s*	4*p*
Hydrogen	H	1	1							
Helium	He	2	2							
Lithium	Li	3	2	1						
Beryllium	Be	4	2	2						
Boron	B	5	2	2	1					
Carbon	C	6	2	2	2					
Nitrogen	N	7	2	2	3					
Oxygen	O	8	2	2	4					
Fluorine	F	9	2	2	5					
Neon	Ne	10	2	2	6					
Sodium	Na	11	2	2	6	1				
Magnesium	Mg	12	2	2	6	2				
Aluminium	Al	13	2	2	6	2	1			
Silicon	Si	14	2	2	6	2	2			
Phosphorus	P	15	2	2	6	2	3			
Sulphur	S	16	2	2	6	2	4			
Chlorine	Cl	17	2	2	6	2	5			
Argon	Ar	18	2	2	6	2	6			
Potassium	K	19	2	2	6	2	6		1	
Calcium	Ca	20	2	2	6	2	6		2	
Scandium	Sc	21	2	2	6	2	6	1	2	
Titanium	Ti	22	2	2	6	2	6	2	2	
Vanadium	V	23	2	2	6	2	6	3	2	
Chromium	Cr	24	2	2	6	2	6	5	1	
Manganese	Mn	25	2	2	6	2	6	5	2	
Iron	Fe	26	2	2	6	2	6	6	2	
Cobalt	Co	27	2	2	6	2	6	7	2	
Nickel	Ni	28	2	2	6	2	6	8	2	
Copper	Cu	29	2	2	6	2	6	10	1	
Zinc	Zn	30	2	2	6	2	6	10	2	
Gallium	Ga	31	2	2	6	2	6	10	2	1
Germanium	Ge	32	2	2	6	2	6	10	2	2
Arsenic	As	33	2	2	6	2	6	10	2	3
Selenium	Se	34	2	2	6	2	6	10	2	4
Bromine	Br	35	2	2	6	2	6	10	2	5
Krypton	Kr	36	2	2	6	2	6	10	2	6

in their name, the *inert gases*, sometimes called the noble gases. Table 14.2 shows that the outermost main shells of neon and argon contain the maximum number of eight electrons. Later we shall see that there are heavier elements with this outer octet which are members of the same family. Note that helium is also an inert gas but can only possess two outer electrons, because the first main shell has no *p* subshell.

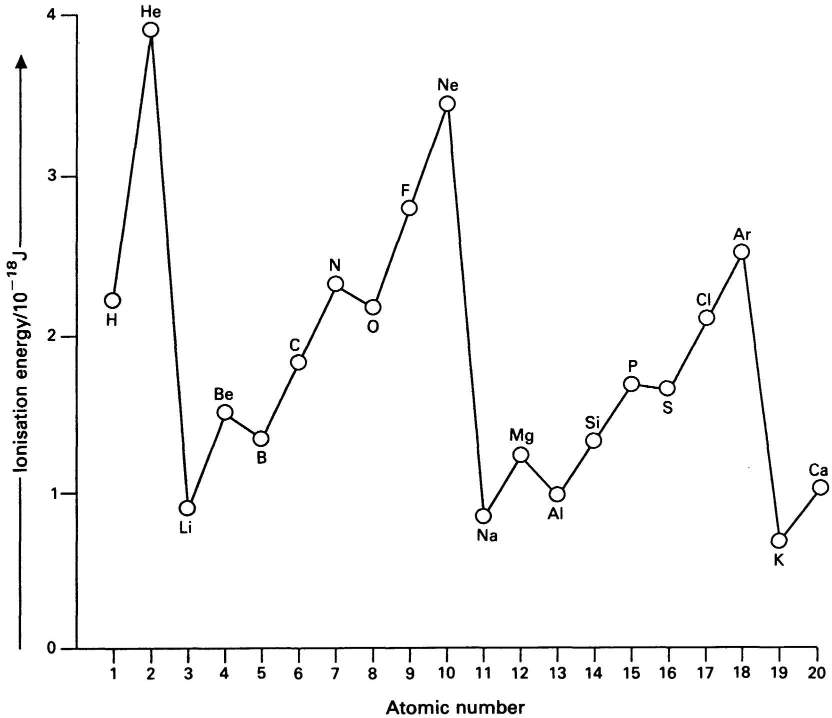

Figure 14.2

Lithium (Li), sodium (Na) and potassium (K) have low ionisation energies, which means that their outermost electrons are loosely held. Each of these elements has one more electron than the preceding inert gas, and Table 14.2 reminds us that the extra electron starts a new main shell. This makes these elements particularly susceptible to chemical combination with others. This family, of which these are the lighter members, is called the *alkali metals*.

Figure 14.2 shows that there is an increase in ionisation energy in building up from one alkali metal to the next inert gas. For example, it increases fourfold from lithium to neon, corresponding to atomic numbers from 3 to 10. This is hardly surprising, since the outer electron in lithium is attracted to the nucleus by the positive charge due to only three protons; the outer electrons in neon are attracted by ten protons, so we would expect them to be much more strongly held.

Note that there is an overall decrease in ionisation energy from the first 'octave' of elements (lithium to neon) to the second (sodium to argon). This is due to a larger 'screen' of inner electrons shielding the outer electrons from the attraction of the nucleus and making them easier to remove. We can regard the electrons in hydrogen and helium as being in direct sight of the nucleus (and helium has the highest ionisation energy of all the elements). From lithium to neon there are two screening electrons in the first main shell. From sodium to argon there are a total of ten, two in the first main shell and eight in the second, and there is a general reduction in ionisation energy.

The abrupt decrease in ionisation energy from an inert gas to the succeeding alkali metal is due to the sudden increase in the number of screening electrons accompanying an increase of only one proton in the nucleus.

14.4 THE PERIODIC TABLE

The cyclical, or periodic, variation in ionisation energy that we see in Figure 14.2 is reflected in the *periodic table*, which is used to classify the elements. Table 14.3 shows the first 36 elements arranged in *groups*, corresponding to the vertical columns, according to the number of outer electrons they possess. The group I elements have one outer electron, the group II elements have two, and so on. By convention, the inert gases in column 8 are given the group number 0.

The *s* and *p* blocks dividing the first four horizontal rows correspond to filling the *s* and *p* subshells in the first four main shells. Note

Table 14.3

GROUP NUMBER	I	II	III	IV	V	VI	VII	0
	s-BLOCK		*p*-BLOCK					
	H 1							He 2
	Li 3	Be 4	B 5	C 6	N 7	O 8	F 9	Ne 10
	Na 11	Mg 12	Al 13	Si 14	P 15	S 16	Cl 17	Ar 18
	K 19	Ca 20	Ga 31	Ge 32	As 33	Se 34	Br 35	Kr 36

TRANSITION ELEMENTS									
d-BLOCK									
Sc 21	Ti 22	V 23	Cr 24	Mn 25	Fe 26	Co 27	Ni 28	Cu 29	Zn 30

that there are no p subshells in the first main shell and, although helium can only have two electrons, it is properly regarded as an inert gas and therefore placed in column 8.

As we have already seen, following calcium (Z = 20), the $4p$ subshell does not fill until the $3d$ is complete. So, although the d-block elements from scandium to zinc (21 to 30) belong between calcium and gallium, they do not readily fit into the fourth row as it stands. Instead they form the first row of an inner series, called the transition elements, which fits in between groups II and III.

Table 14.4 shows the full periodic table, which includes the subshells above the $4p$. Further transition elements occur where the higher d subshells are being filled, and more complications begin with the lanthanide series, where the $4f$ subshell fills before the $5d$.

We needn't concern ourselves with the detailed structure of the full periodic table here other than to note that we can still view it in terms of filling subshells in order of increasing energy.

The elements below and to the left of the heavy line in Table 14.4 are metals. Be careful to note that this division is not precise, because some elements close to the line show behaviour that is partly metallic and partly non-metallic. Before we can appreciate the distinction, we need some basic understanding of the chemical bonds that occur between atoms. But first let us take a closer look at atomic mass.

Table 14.4

I	II											III	IV	V	VI	VII	0
H 1																	He 2
Li 3	Be 4	METALS										B 5	C 6	N 7	O 8	F 9	Ne 10
Na 11	Mg 12	TRANSITION ELEMENTS										Al 13	Si 14	P 15	S 16	Cl 17	Ar 18
K 19	Ca 20	Sc 21	Ti 22	V 23	Cr 24	Mn 25	Fe 26	Co 27	Ni 28	Cu 29	Zn 30	Ga 31	Ge 32	As 33	Se 34	Br 35	Kr 36
Rb 37	Sr 38	Y 39	Zr 40	Nb 41	Mo 42	Tc 43	Ru 44	Rh 45	Pd 46	Ag 47	Cd 48	In 49	Sn 50	Sb 51	Te 52	I 53	Xe 54
Cs 55	Ba 56	La 57	Hf 72	Ta 73	W 74	Re 75	Os 76	Ir 77	Pt 78	Au 79	Hg 80	Tl 81	Pb 82	Bi 83	Po 84	At 85	Rn 86
Fr 87	Ra 88	Ac 89															

LANTHANIDES

Ce 58	Pr 59	Nd 60	Pm 61	Sm 62	Eu 63	Gd 64	Tb 65	Dy 66	Ho 67	Er 68	Tm 69	Yb 70	Lu 71

ACTINIDES

Th 90	Pa 91	U 92	Np 93	Pu 94	Am 95	Cm 96	Bk 97	Cf 98	Es 99	Fm 100	Md 101	No 102	Lr 103

14.5 ATOMIC MASS

As we noted earlier, mass on the atomic scale is measured in atomic mass units. By convention 1 u is taken to be equal to one-twelfth of the mass of the carbon-12 atom, which gives it the value 1.66×10^{-27} kg. (For carbon-12, $A = 12$ and $Z = 6$, so this atom contains six neutrons and six protons in the nucleus and six electrons to balance the protons.)

The *relative atomic mass* (atomic weight) of an element is the average mass per atom expressed in atomic mass units; it is relative to one-twelfth of the mass of the carbon–12 atom and it is an average value, since elements occur naturally as mixtures of their isotopes. Table 14.5 gives some examples.

Table 14.5

Element	Symbol	Atomic no.	Relative atomic mass
Hydrogen	H	1	1.0
Carbon	C	6	12.0
Nitrogen	N	7	14.0
Oxygen	O	8	16.0
Sodium	Na	11	23.0
Aluminium	Al	13	27.0
Chlorine	Cl	17	35.5
Calcium	Ca	20	40.1
Iron	Fe	26	55.8
Copper	Cu	29	63.5
Zinc	Zn	30	65.4
Silver	Ag	47	107.9
Gold	Au	79	197.0

Although mass is measured in kg, the SI base unit for *amount of substance* is the mole (symbol mol), which is the amount containing a fixed number of the 'entities' that constitute the substance under consideration. This number, called the Avogadro constant, is 6.02×10^{23} mol^{-1} whatever the substance happens to be. The entities might be atoms, or ions or molecules (which we shall meet in the next topic). Thus, 1 mol of water contains 6.02×10^{23} water molecules and 1 mol of carbon contains 6.02×10^{23} carbon atoms. The mole is useful where the relative number of entities is important — for example, in studying a chemical reaction.

It is a simple matter to convert moles to mass. For example, carbon has a relative atomic mass of 12.0 and carbon atoms therefore have an average mass of $(12.0 \times 1.66 \times 10^{-27})$ kg each. The mass of 1 mol of carbon, i.e. 6.02×10^{23} carbon atoms, is therefore given by

$$6.02 \times 10^{23} \times (12.0 \times 1.66 \times 10^{-27}) \text{ kg}$$

which is equal to

$$0.012 \text{ kg} = 12 \text{ g}$$

1 mol is actually defined as the amount of substance that contains as many of the specified entities as there are atoms in 0.012 kg of carbon-12. For our purposes we can say that the mass of 1 mol of carbon or of any other element is equal to its relative atomic mass expressed in grams. Looking at this another way, 6.02×10^{23} atomic mass units have a mass of 1 g, because

$$6.02 \times 10^{23} \times 1.66 \times 10^{-27} = 0.001 \text{ kg} = 1 \text{ g}$$

Note that, although the SI unit of mass is the kilogram, scientists often work in grams (g). Remember that $1 \text{ g} = 1 \times 10^{-3} \text{ kg}$.

Questions

(Where necessary assume that $1 \text{ u} = 1.66 \times 10^{-27}$ kg *or* that the Avogadro constant $= 6.02 \times 10^{23} \text{ mol}^{-1}$.)

Making use of the information tabulated in this topic:

1. Identify by name the elements with the following atomic numbers and state which are metals: 3, 7, 10, 16, 19, 21, 25 and 35.
2. Find the number of neutrons contained in each of the following atoms: helium-4, nitrogen-14, carbon-14, oxygen-16, argon-40, potassium-40 and calcium-40.
3. Find the mass in kg of the following atoms: carbon, oxygen and iron.
4. Find the mass of silver that contains the same number of atoms as (a) 63.5 g of copper, (b) 3 g of carbon and (c) 4925 kg of gold.
5. Find the number of atoms in 1 kg of each of the following: hydrogen, copper and silver.
6. An 18 carat gold ring weighs 8.72 g. How many gold atoms does it contain? (18 carat gold contains 75% gold by weight.)
7. Find the number of atoms in a 1 mm cube of copper. ($\rho_{copper} = 8900 \text{ kg m}^{-3}$.)
8. The smallest entity identifiable as water is the water molecule, which contains two atoms of hydrogen and one of oxygen. If 180 g of water is completely decom-

posed into hydrogen and oxygen, what mass of hydrogen is produced?

9. What is the volume in mm^3 of a piece of copper containing 5×10^{22} atoms? ($\rho_{copper} = 8900\ kg\ m^{-3}$.)

10. In a certain silver/copper alloy 12.1% of the total number of atoms is copper. Find the percentage of silver by weight.

11. Assuming the value of the Avogadro constant and using the data in Table 14.5, (a) suggest what element it is that has atoms with an average mass of 9.27×10^{-26} kg; (b) find the mass of the oxygen atom; and (c) estimate the volume of the copper atom if $\rho_{copper} = 8900\ kg\ m^{-3}$.

TOPIC 15 CHEMICAL BONDING

COVERING:

- ionic, covalent and intermediate types of bond;
- metallic bond;
- intermolecular forces;
- relative molecular mass.

We need to have a basic understanding of chemical bonding, because it plays a central role in determining the behaviour of all substances, including engineering materials.

15.1 IONIC BONDING

In the last topic we saw that the inert gases have an outer octet of electrons, or a pair in the case of helium, that represents a particularly stable electronic structure. As we shall now see, other elements tend to behave in such a way that they achieve these stable configurations by losing or gaining electrons.

An atom of sodium (Na in group I) will tend to get rid of the single 3*s* electron in its outer shell, thereby achieving the neon configuration and becoming a positively charged sodium *ion* (Na^+) in the process. (Note that it does not become a neon atom, because the nucleus remains the same.) In chemical shorthand

$$Na = Na^+ + e^-$$

where e^- represents the electron. This tendency to form positive ions is characteristic of metallic elements.

By contrast, chlorine, a non-metal from the opposite side of the periodic table (Cl in group VII), is one electron short of the argon configuration. If it can gain an electron from elsewhere, it becomes a negatively charged chloride ion, thus

$$Cl + e^- = Cl^-$$

Both these tendencies are satisfied when sodium and chlorine are combined in sodium chloride, i.e. common salt, as follows:

$$Na + Cl = Na^+ + Cl^-$$

Sodium chloride is a crystalline solid held together by *ionic bonds*.

Ions formed from single atoms can be regarded as charged spheres. A sodium ion and a chloride ion will be drawn together by the attractive force between them due to their opposite charge. The closeness of their approach, however, is limited by their outer orbitals, which cannot interpenetrate, because they would then exceed their quota of two electrons where they occupy the same space. Furthermore, any interpenetration would result in reduced screening of the two nuclei, which would cause repulsion between them. The overall effect is that any attempt to squeeze the ions together results in a repulsive force and the outer shell of each ion behaves rather like a spherical elastic skin.

Figure 15.1 indicates how the attractive and repulsive forces vary with separation. The attractive force predominates at larger separations and the repulsive force only becomes important at smaller separations, where the outer shells approach closely. By summing the two relationships we obtain the net force/separation curve.

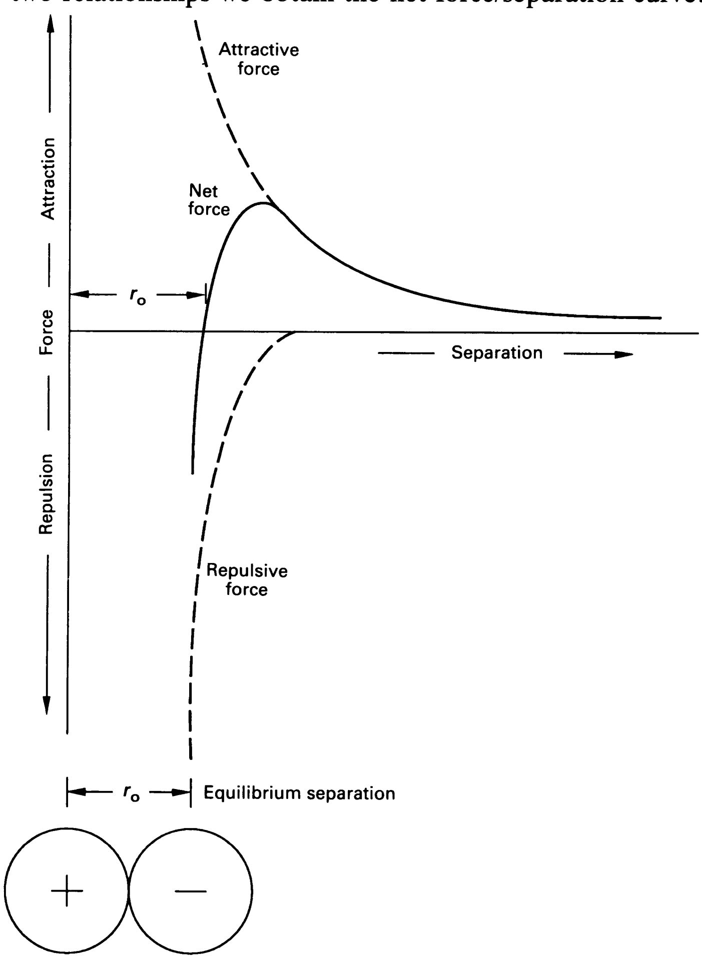

Figure 15.1

There is a point where the two component forces balance so that the net force is zero; this represents the *equilibrium separation* r_o, where the ions would naturally come to rest in the absence of any external forces. If we pull the ions apart from their equilibrium position, then the balance between the component forces is upset and an attractive force arises as we move up the net force/separation curve; the ions will move just far enough to generate an equal but opposite force resisting our effort in pulling them apart. Similarly, if we push the ions together, then the compressive force we apply to them will be opposed by an equal and opposite resistance due to the repulsive force that arises between them as we move down the net force/separation curve. So, as we noted in Topic 2, it is the deformation of chemical bonds that enables materials to resist tension and compression. The portion of the net force/separation curve close to the equilibrium position (corresponding to small deformations) is very nearly straight where it crosses the horizontal axis in Figure 15.1. This provides us with the fundamental basis of Hooke's law, namely that deformation is proportional to load and hence, Young's modulus (which is equal to the stress/strain ratio, as discussed in Topic 2). Furthermore, since the proportionality is maintained across the horizontal axis, Young's modulus applies to both tensile and compressive stresses in materials which follow this model.

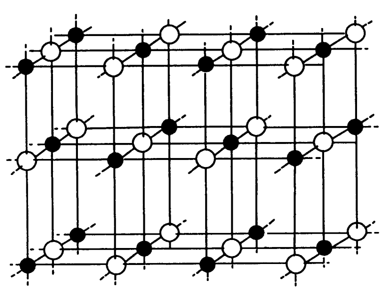

Figure 15.2

Figure 15.2 shows part of a simple ionic crystal structure in 'exploded' form for clarity. In general terms, ionic crystal structures are regular, extended lattice arrangements in which oppositely charged ions are drawn together but like-charged ions stay apart. In this particular case (named the *rocksalt* structure after naturally occurring sodium chloride) each positive ion is surrounded by six negative neighbours, and vice versa, so there are equal numbers of each overall.

Note that group II elements form ions by losing two electrons and group VI elements by gaining two; for example, magnesium oxide ($Mg^{2+}O^{2-}$) is an ionic substance (which, incidentally, also adopts the rocksalt structure.)

Many ionic structures are more complex. Electrical neutrality must be preserved; therefore, ions occur in different proportions where their charge magnitude differs. In calcium fluoride, for example, we need twice as many F^- ions as Ca^{2+} ions. The relative size of the ions is also an important factor in determining the way in which they pack together. Furthermore, there are many ions which contain more than one atom; for example, the sulphate ion (SO_4^{2-}) and the nitrate ion (NO_3^-).

15.2 COVALENT BONDING

In the absence of metallic elements, non-metals can achieve stable inert gas configurations by sharing electrons to form *covalent bonds*. For example, two hydrogen atoms can pool their single electrons to form a pair that is shared between them, effectively giving each the helium configuration. The two half-filled atomic orbitals combine to give a single molecular orbital, containing both electrons, which encloses both nuclei as indicated in Figure 15.3.

The electrons spend much of their time in between the nuclei and exert attractive forces on each which serve to tie them together.

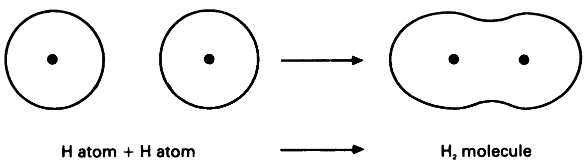

Figure 15.3

There is also an opposing repulsive force between the nuclei because of their like charges. The combination of these two forces gives a net force/separation curve of a similar form to that of the ionic bond. This means that the smallest entity of hydrogen that normally maintains an independent existence is the hydrogen *molecule*, H_2, in which two hydrogen atoms are joined by a covalent bond. Hydrogen molecules tend to remain separate under normal conditions and exist as a gas rather than coalesce to form a liquid or a solid.

Chlorine is also one electron short of its neighbouring inert gas configuration (argon), so, like hydrogen, two chlorine atoms share a pair of electrons to form a covalent bond, and the resulting Cl_2 molecules exist as a gas under normal conditions. (Note that electrons involved in forming chemical bonds are called *valence electrons*.)

Carbon is a particularly important element because it forms the basis of a vast number of covalently bonded molecules. As a member of group IV, each atom shares four pairs of electrons with other atoms and achieves the neon configuration. Since carbon forms four covalent bonds, we say it has a *valency* of 4. (The term 'valency' is also applied to the number of charges on an ion; thus, for example, Mg^{2+} has a valency of 2 and Cl^- has a valency of 1.)

Figure 15.4 shows some examples of carbon-based molecules that happen to be *hydrocarbons*, i.e. *compounds* containing only hydrogen and carbon. Compounds are substances which result from the chemical combination of elements. They should not be confused with *mixtures*, which contain more than one individual chemical substance and can be separated by physical methods.

Covalent bonds are often represented by straight lines between the atoms sharing the valence electrons. Figure 15.4(a) shows methane (CH_4), which is the main constituent of natural gas. (Natural gas is a mixture of gases, mostly hydrocarbons.) The four C–H bonds are identical and repel each other because of their identical charge distribution. They spread apart as far as possible, with the result that the methane *molecule* (i.e. the smallest entity of methane) has a tetrahedral form with equal angles between all the bonds. Figure 15.4(b)

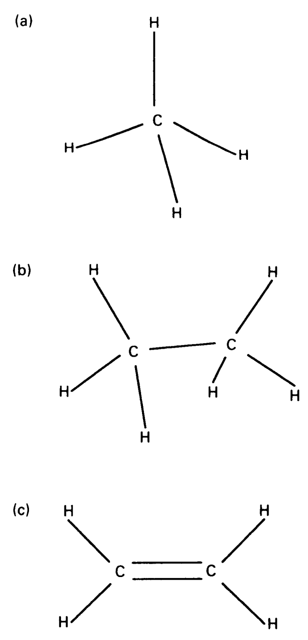

Figure 15.4

shows ethane (C_2H_6), which is also found in natural gas. Figure 15.4(c) shows ethene (C_2H_4), or ethylene, which is an important raw material in the chemical industry. Both ethane and ethene molecules contain two carbon atoms.

In ethane the bonds are distributed round each carbon atom in a tetrahedral configuration, as in methane, and one end of the molecule can rotate relative to the other, like a propeller, about the C–C axis. Ethane (CH_3–CH_3) is the smallest of a family of chain-like molecules which continues with propane (CH_3–CH_2–CH_3) and butane (CH_3–CH_2–CH_2–CH_3) and extends to higher members of great length. Ordinary rubber and many plastics consist of enormously long molecules based on covalently bonded carbon chains. (As we shall see in Topic 20, the elasticity of rubber depends on bond rotation about the axes of the –C–C– bonds along the length of the chains.)

In the ethene molecule the carbon atoms are joined by a *double bond*, where they share two pairs of electrons. This means that, because of its valency of 4, each carbon atom can only bond with two hydrogen atoms. The double bond pulls the carbon atoms closer together than the single bond; furthermore, it is stiffer and stronger, and requires more energy to break it. It also prevents rotation, and all the atoms in the ethene molecule lie in the same plane. (Triple bonds between carbon atoms are also possible but we shall not discuss them here.)

Now we can begin to see important differences between ionic and covalent bonding. The number of covalent bonds an atom can form is dictated by its valency and the bonds are formed specifically between those atoms sharing the valence electrons. This leads to the formation of individual molecules which have definite shapes and sizes and which are capable of existence as separate entities. On the other hand, ionic structures are simply formed from ions packed together, like charged spheres, to form extended crystal structures of no fixed size and in which individual molecules cannot be identified.

Having said this, covalent bonding can also lead to extended crystal structures. For instance, Figure 15.5 shows the structure of diamond, where carbon atoms are joined together so that each has four neighbours arranged around it in a tetrahedral configuration. Like an ionic crystal, this structure has no fixed size and it is not possible to identify an individual diamond molecule.

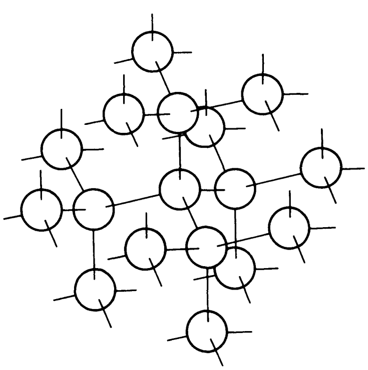

Figure 15.5

15.3 INTERMEDIATE BONDING (IONIC–COVALENT)

Compounds of the types we have been considering are seldom 100% ionic or 100% covalent — most lie somewhere in between. The best way of approaching this idea is to consider ionic bonds with covalent character and covalent bonds with ionic character.

Ions are of different sizes. Table 15.1 shows that there is a decrease in radius along the series from O^{2-} to Al^{3+}. All these ions have the

Table 15.1

Ion	Atomic no.	Radius/nm
O^{2-}	8	0.14
F^{-}	9	0.13
Na^{+}	11	0.10
Mg^{2+}	12	0.07
Al^{3+}	13	0.05

neon configuration and the decrease is associated with the increasing positive charge on the nucleus (from 8 to 13) as it holds the skin of outer electrons progressively more tightly. Thus, a highly charged positive ion tends to have a 'hard', compact skin. On the other hand, a highly charged negative ion tends to have a larger, 'soft', more deformable skin with the electrons less under the influence of the nucleus. As Figure 15.6 suggests, if we let two such ions come close together, then the negative ion will be distorted, or *polarised* by the positive ion, so that its centre of negative charge is displaced from its centre of positive charge at the nucleus. The outer electrons will therefore tend to spend more time between the two nuclei. In other words, the ionic bond will take on covalent character. The larger the size and the greater the charge on the negative ion the greater will be its susceptibility to polarisation. The smaller the size and the greater the charge on the positive ion the greater will be its polarising power.

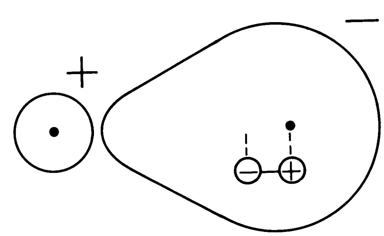

Figure 15.6

Polarisation can also occur in covalent bonds and this gives them ionic character. It occurs where the bonded atoms differ in their *electronegativity*, which is their tendency to attract electrons and to form negative ions. (As we might expect from the previous topic, electronegativity tends to increase from left to right across the periodic table and to decrease from top to bottom.)

If two different atoms are joined by a covalent bond, then the valence electrons will tend to be drawn towards the more electronegative atom. That end of the bond will therefore tend to acquire a negative bias, leaving the other end with a positive bias, thereby giving the bond partial ionic character. The greater the difference in electronegativity the greater the polarisation of the bond and the greater its ionic character. We shall meet an example of a polarised covalent bond below, when we discuss the water molecule.

15.4 METALLIC BONDING

Metallic elements are described as *electropositive* because they tend to form positive ions by losing electrons. In the absence of non-metallic elements with which to form ionic bonds, the valence electrons join forces to form a negative 'sea' or *electron gas*. This serves to bind the ions together to form a regular extended crystal structure, as suggested in Figure 15.7(a).

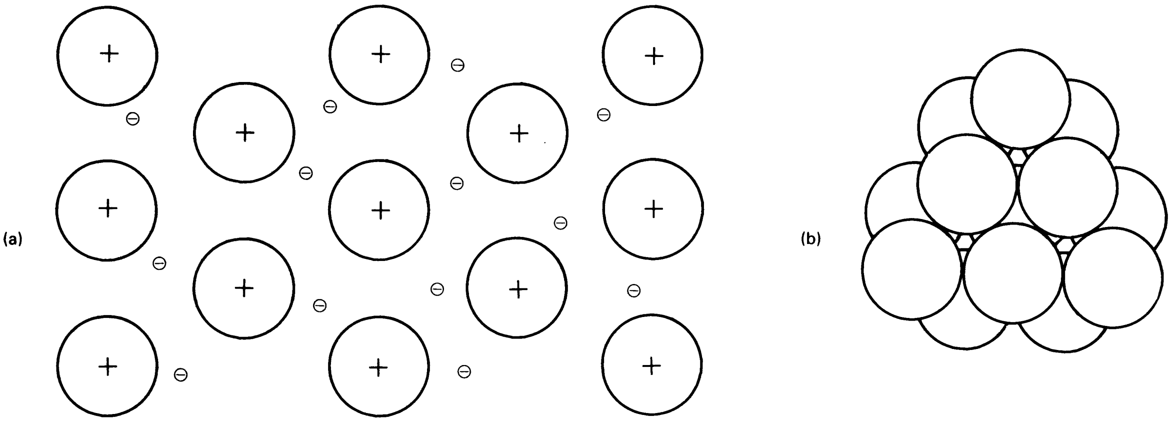

Figure 15.7

The electrons move randomly in between the positive ions, thus providing an attractive force that draws them together. Repulsion effects increase as the ions approach one another and, as before, we can view the bond in terms of a balance between attractive and repulsive forces. The separation between the nuclei of two neighbouring metal ions corresponds to the equilibrium position on the net force/separation curve in Figure 15.1 (page 135). This makes it possible to view metal atoms as elastic spheres and metal crystal structures in terms of three-dimensional geometrical patterns formed by packing spheres together, as in Figure 15.7(b), though without the same charge constraints that apply to ionic structures. (In fact, models of metal crystal structures can be constructed by glueing table tennis balls together.)

As this suggests, metal crystals are of no fixed size and individual molecules cannot be identified within them. Note that the metal ions do not all have to be of the same kind, so *alloys* can be formed in which different metals — copper and nickel, for example — are combined on the same crystal lattice.

As we shall see later, it is the freedom of movement of the valence electrons through the crystal lattice that makes metals good conductors of heat and electricity.

15.5 INTERMOLECULAR FORCES

Summarising very broadly, we have seen that ionic bonds are formed between metallic atoms and non-metallic atoms, covalent bonds are formed between non-metallic atoms, and metallic bonds are formed between metallic atoms. But this does not explain why molecules tend to stick together. For instance, water molecules form a coherent liquid or solid, depending on the temperature. To understand why they do this, we need to examine their structure.

Oxygen is a group VI element and an oxygen atom shares a pair of electrons with each of two hydrogen atoms. This gives oxygen the neon configuration and hydrogen the helium configuration. The resulting H_2O molecule is a stable entity capable of independent existence (Figure 15.8). The four outer electrons of the oxygen atom that are not involved in bonding form two so-called *lone pairs*. As far as the shape of the molecule is concerned, each lone pair occupies a non-bonding orbital of a similar form to that of the orbitals constituting the covalent bonds. The result is a tetrahedral molecule with an overall shape not unlike that of methane.

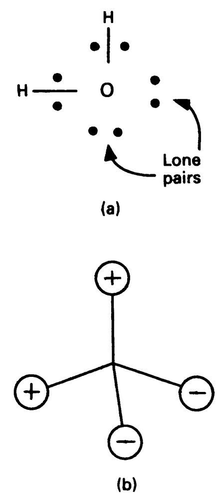

Figure 15.8

Figure 15.8(b) is a simplified representation of the charge distribution in the water molecule. The two lone pairs form negative regions in which there are no nuclei to balance their charge. The two bonding orbitals have a positive bias, because the central oxygen atom is electronegative and draws the valence electrons inwards from the hydrogen nuclei to leave a net positive bias at the outer ends of the bonds. Since there are no screening electrons round the hydrogen nuclei, the effect of this is particularly strong.

A charged plastic comb will attract a thin stream of water running from a tap, because the water molecules orientate themselves so that the orbitals that are oppositely charged to the comb tend to point towards it; the resulting attractive force will bend the stream of water. In a similar way, two water molecules that are close together and free to rotate will tend to orientate themselves so that oppositely charged orbitals point towards one another, with a resulting attractive force that holds the molecules together. This type of intermolecular force, involving hydrogen atoms covalently bonded to strongly electronegative atoms, is called *hydrogen bonding*. It provides a network of attractive forces within a collection of water molecules, and it is this that gives water its coherent nature. In fact, hydrogen bonding is a special case of the attractive forces that generally arise between any *polar* molecules (i.e. those that are permanently polarised).

But inert gases and non-polar molecules (e.g. H_2, Cl_2 and O_2, where the valence electrons are equally shared) will liquefy and solidify if the temperature is low enough. (Oxygen is a group VI element, so O_2 molecules are held together by a double bond O=O.) In such cases there can be no permanent polarisation. Figure 15.9(a), which represents a helium atom, suggests the reason.

The electrons are in perpetual motion, so that at any instant, unless they happen to be diametrically opposite to one another, the atom will be temporarily polarised. The magnitude and direction of the polarisation continuously changes as the displacement of the centre of negative charge fluctuates about the nucleus. The atom will tend to induce polarisation in its neighbours, as in Figure 15.9(b); in this case the nucleus of the left-hand atom is attracting the electrons in the right-hand atom or, looking at it the other way, the electrons in the right-hand atom are repelling the electrons in the left-hand atom.

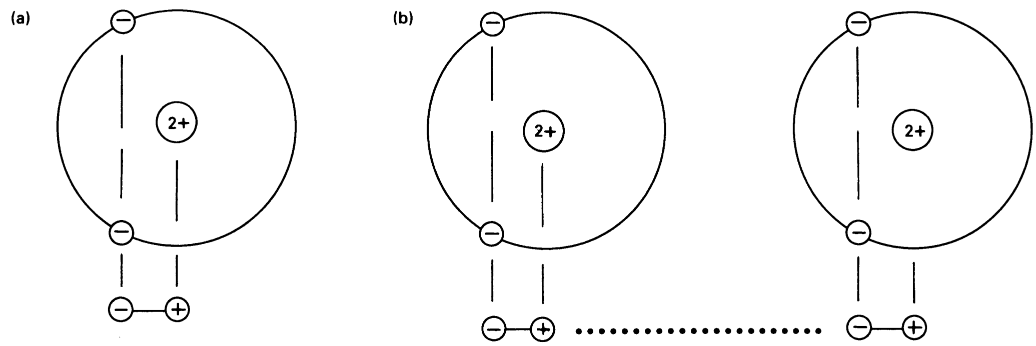

Figure 15.9

Thus, temporary polarisation in the two atoms will tend to fluctuate in sympathy, so that the dipoles are orientated similarly and an attractive force arises between them.

Such forces are called *van der Waals* forces. They arise wherever atoms and molecules are close together, in addition to any other types of bonding that may be operating. van der Waals forces are extremely weak in the case of helium but stronger with larger atoms and with molecules where there are more electrons involved. To take an extreme case, polythene consists of very large chain-like hydrocarbon molecules where attraction between the chains is provided by van der Waals forces.

As a broad generalisation, ionic and covalent bonds are the strongest, followed by metallic bonds, then hydrogen bonds, with van der Waals forces the weakest.

15.6 RELATIVE MOLECULAR MASS

The mass of a molecule is expressed in terms of its *relative molecular mass* (molecular weight). This is equal to the sum of the relative atomic masses of all the atoms it contains. For example, using the values from Table 14.5 (page 131), the relative molecular mass of water (H_2O) is 18.0 (i.e. $(2 \times 1.0) + 16.0$) and of ethane (C_2H_6) is 30.0 (i.e. $(2 \times 12.0) + (6 \times 1.0)$). And following on from the last section of the previous topic, the mass of 1 mol of water is 18.0 g and of ethane 30.0 g.

Although molecules cannot be identified in ionic structures, it is still helpful to use the chemical formulae of such compounds in a corresponding way. Thus, 1 mol of sodium chloride (NaCl) has a mass of 58.5 g (i.e. 23.0 g + 35.5 g).

Relative molecular mass values, or their ionic counterparts, enable us to quantify chemical reactions in terms of the masses of the substances involved. For example, the complete combustion of methane in oxygen yields carbon dioxide (CO_2) and water. In order to form a properly balanced chemical equation, the number of atoms of each element must be the same on each side. In this case the equation is balanced by having two oxygen molecules and two water molecules:

CH_4	+	$2O_2$	=	CO_2	+	$2H_2O$
16.0		2×32.0		44.0		2×18.0

The relative molecular masses underneath enable corresponding quantities to be calculated in whatever units are required. Thus, 160 kg of methane needs 640 kg of oxygen for complete combustion. 1.00 g of methane yields (36.0/16.0 =) 2.25 g of water. 22 mg of carbon dioxide is produced by 8 mg of methane, and so on.

Questions

(Use any previously tabulated data as required.)

1. Deduce the general type of chemical structure of the following: caesium chloride (CsCl), carbon tetrachloride (CCl_4), ice, solid argon, carbon dioxide, liquid ammonia (NH_3).

2. Find the relative molecular mass (or ionic counterpart) of each of the following: ethane, calcium carbonate ($CaCO_3$), ethanol (C_2H_5OH), silver nitrate ($AgNO_3$), chloroform ($CHCl_3$).

3. One compound contains 75% carbon and 25% hydrogen by weight and another contains 80% carbon and 20% hydrogen. Suggest what each might be.

4. Find the aluminium content of alumina (Al_2O_3) expressed as a weight percentage.

5. Find how much carbon dioxide results from the complete combustion of 4.5 g of propane (C_3H_8) if the reaction follows the equation

   $$C_3H_8 + 5O_2 = 3CO_2 + 4H_2O$$

6. The thermal decomposition of calcium carbonate ($CaCO_3$) yields CaO and CO_2. How much calcium carbonate would be needed to give 1 kg of carbon dioxide?

7. Write the balanced chemical equation for the complete combustion of ethyl alcohol (C_2H_5OH) to carbon dioxide and water, and hence find how much alcohol yields 14.35 g of carbon dioxide.

TOPIC 16 HEAT AND TEMPERATURE

COVERING:

- heat and internal energy;
- heat capacity and latent heat;
- thermal expansion and contraction;
- thermal stress.

Whether a particular substance exists as a solid, a liquid or a gas at a given temperature and pressure depends upon the strength of the forces of attraction between its constituent atoms, ions or molecules. Thus, at atmospheric pressure and room temperature the van der Waals forces between the oxygen molecules in air are not strong enough to make them stick together, nor are the hydrogen bonds between water molecules strong enough for them to form ice. On the other hand, nearly all metallic and ionic/covalent materials are solids. Furthermore, simple solids tend to turn to liquids, and liquids to gases, if they are heated. These observations seem to point to the idea that the cohesion due to the forces of attraction between atoms, ions and molecules is opposed by the effect of heat.

Heat is the energy that is transferred between two bodies as a result of a temperature difference between them. Heat will flow from the hotter to the colder body; therefore, *temperature* is the property which determines the direction in which the heat will flow.

First we must recognise that the heat absorbed by a body may increase its internal energy (associated with its temperature and physical state) and enable it to do external work on its surroundings, as, for example, in raising the temperature of a body of gas, enabling it to expand and drive a piston. It follows that the increase in the internal energy of a body equals the heat added to it less any external work that may be done by it. On the other hand, it is possible to increase the internal energy of a body simply by doing work on it without supplying any heat at all. For example, in braking a car, kinetic energy is transformed into an increase in the internal energy of the brakes, thereby raising their temperature. The increased temperature of the air compressed in a bicycle pump is a result of an increase in its internal energy due to work done on it. When a falling body hits the ground, its original potential energy is transformed into increased internal energy of the body and of the ground, and to some extent the air through which it fell, plus some sound energy.

These are all examples of the principle of conservation of energy which we met in Topic 8. They illustrate the need to view any change in the internal energy of a body as an exercise in accountancy: work and heat both represent energy in the process of being transferred either to or from the body. In this topic we shall consider the internal energy changes in a body as it absorbs or emits heat and, for the purposes of this discussion, we shall generally assume that conditions are such that a negligible amount of work is done.

First, we need to recognise that the atoms, ions or molecules that constitute a particular substance are in a state of continual thermal agitation which becomes more energetic if the temperature of the substance is increased. For simplicity we shall base our discussion on the general case of an unspecified model substance consisting of chemically bonded atoms.

In the solid state the atoms vibrate about fixed positions on the crystal lattice, where they are trapped between their neighbours. The thermal energy associated with their vibrations is insufficient to overcome the cohesion due to the forces of attraction between them. If the temperature is raised, the vibrations become more vigorous until, at the melting point, the atoms have sufficient energy to escape from their fixed positions and the substance changes from a solid to a liquid. Although the atoms still remain in contact with their neighbours, they now have the capacity for translational motion relative to one another — hence, liquids can flow. In the gaseous state the atoms have sufficient energy to overcome the forces of attraction and they fill the entire volume of the container they occupy irrespective of how large that might be. At normal pressures the atoms are widely separated and they move randomly and independently of one another, only interacting when they collide. Although the velocity of individual atoms will vary greatly, their mean velocity remains constant at a given temperature and increases as the temperature is raised.

Let us examine these ideas more closely.

16.1 A SIMPLE MODEL

We shall begin with the assumption that the general form of the net force/separation curve in Figure 15.1 (page 135) can be used to represent the behaviour of any of the types of bond that we met in the previous topic, including that between the atoms in our model substance. Figure 16.1 shows the way in which the corresponding potential energy varies. As before, the horizontal axis gives the position of the centre of one atom relative to the other. The equilibrium position, where the net force between the atoms is zero, corresponds to the potential energy minimum at the bottom of the trough. The shape of the trough reflects the effect of the attractive and repulsive

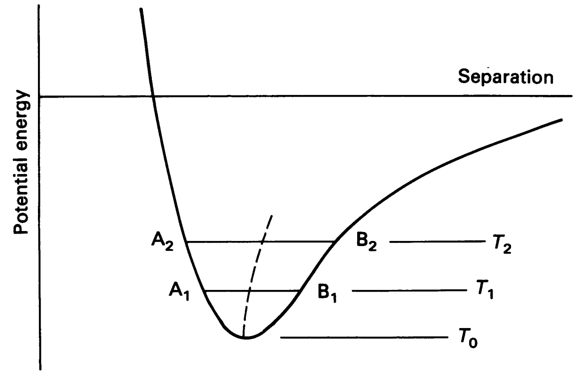

Figure 16.1

components of the net force/separation curve and is correspondingly asymmetric.

As we saw in the previous topic, any deformation of the bond results in an opposing force. The system will therefore tend to return to its equilibrium position in much the same way as the pendulum and the spring in Topic 11. The linear net force/separation relationship close to the equilibrium position suggests that if the atoms are displaced slightly and then released, they will oscillate with simple harmonic motion.

As we noted earlier, the atoms in a solid are normally in a continuous state of oscillation, and this provides us with a picture of the way in which the temperature of a body increases when it absorbs heat. There is a continuous interchange of kinetic and potential energy as the atoms oscillate to and fro between the points A_1 and B_1 in Figure 16.1 at a temperature T_1. The potential energy follows the curve throughout the cycle and is at a maximum at A_1 and B_1, where the kinetic energy is zero and the system is on the point of changing direction. From Topic 11 (see page 91) we would expect the sum of the potential energy and kinetic energy to remain constant throughout the cycle, in which case the horizontal line A_1B_1 represents the total energy. The kinetic energy at any point is therefore equal to the vertical distance between the horizontal line and the potential energy curve.

If heat is added to the system to raise the temperature to T_2, then the total energy increases to a higher level represented by the horizontal line A_2B_2. Conversely, if the system cools from T_2 to T_1, then the difference in internal energy is released to the surroundings as heat.

If the temperature is lowered to T_0, corresponding to the bottom of the trough, the system has no kinetic energy and the atoms come to rest; this represents *absolute zero*, which is the lowest temperature that is theoretically possible. The vertical distance of any one of the series of horizontal lines A_nB_n above the trough can be viewed in terms of the kinetic energy associated with the thermal motion of the system at a particular temperature and the potential energy associated with the interaction between the vibrating atoms. The quantity of heat which a body absorbs in increasing its temperature is known as its *heat capacity*. We shall define this more carefully very shortly. (Readers who know about quantum theory will have noticed that our simple model is valid for an 'average' oscillating atom but not for individuals, and that we have ignored the fact that the system must possess some residual kinetic energy at absolute zero.)

Let us assume that the mid-point of each horizontal line A_nB_n represents the average separation between the centres of the atoms. Because the potential energy curve is not symmetrical, this distance increases as the temperature is raised (following the dotted line in the figure). The substance of which the atoms form a representative part therefore undergoes *thermal expansion*. Conversely, it undergoes *thermal contraction* as it cools.

The addition or removal of heat does not always change the temperature of a substance. At the melting point the amplitude of the oscillations is large enough for the atoms to move past their neighbours and escape from their fixed positions on the crystal lattice. In effect, the breaking down of the crystal structure and the freedom of the atoms amounts to an increase in potential energy. The substance pays for this by absorbing heat from its surroundings at constant temperature until it has completely melted. This heat, called the *latent heat of fusion*, is returned to the surroundings if the substance resolidifies. Thus, ice must be supplied with latent heat for it to melt at 0 °C, and this heat is returned if the water refreezes at 0 °C. (The word 'latent' means 'concealed' in this context, because the heat transfer is not revealed as a temperature change.)

In the liquid state the forces of attraction are still able to maintain overall cohesion. The liquid still has a definite volume which is contained within its boundary surface (unlike a gas, which is diffuse and expands to fill its container). The liquid can flow, because its constituent atoms can readily change position relative to one another. If it is heated, the atoms move more vigorously, and if the temperature is raised to the boiling point, they will have enough energy to climb out of the potential energy trough and become independent of one another — thus, the forces of attraction are overcome and the substance changes from a liquid to a gas. During this change of state the temperature of the liquid remains constant as it absorbs its *latent heat of vaporisation*. Because the latent heat of vaporisation of a substance represents the energy required to completely separate the constituent atoms, it is generally considerably greater than the latent heat of fusion absorbed during the melting process.

Note that liquids tend to evaporate below their boiling points and some, such as alcohol and water, can become noticeably cooler in doing so. This is because molecules with sufficiently high kinetic energy are able to escape from the liquid surface. This has the effect of reducing the average kinetic energy of those remaining behind — hence, the liquid becomes cooler.

16.2 HEAT CAPACITY AND LATENT HEAT

Now we need to quantify the amount of heat involved as the temperature or physical state of a substance is changed. For example, how much heat must be supplied to cold water to raise its temperature to boiling point? And how much heat is required to change the boiling water to steam?

Figure 16.2 is an idealised plot of temperature against time which summarises the quantities involved as a substance is heated from the solid right through to the gaseous state.

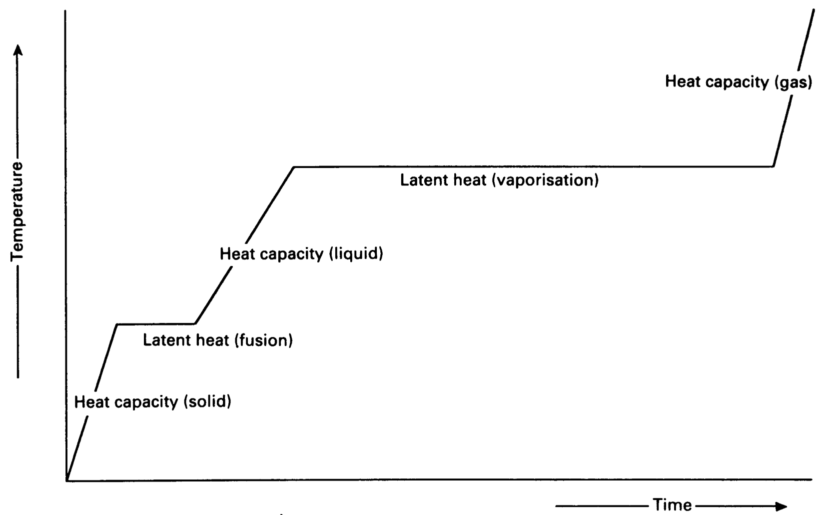

Figure 16.2

For measuring temperature we shall use the Celsius scale (formerly called the centigrade scale), where the values 0 °C and 100 °C are assigned to the freezing and boiling points of water at 1 atmosphere. (1 atmosphere is specified because the freezing and boiling points vary with pressure.) On this basis, absolute zero turns out to be −273.15 °C. The SI unit of temperature is the kelvin (symbol K) but this is much less commonly used than the degree Celsius. 0 K represents absolute zero and a temperature interval of 1 K is the same as 1 °C, so for our purposes °C = K − 273.

Different substances have different heat capacities, in other words, they require different quantities of heat to raise their temperature by the same amount. In general, provided that no change of state occurs,

$$Q = mc\theta \qquad (16.1)$$

where Q joules of heat are required to raise m kilograms of the substance through a temperature interval of θ K (or θ °C), c is called the *specific heat capacity* and is defined as the heat required to raise the temperature of 1 kg of the substance by 1 K (1 °C); its units are therefore J kg^{-1} K^{-1}. Note that c is often assumed to be constant, although it can vary quite considerably with temperature, depending on the substance and the temperature range involved. Table 16.1 shows some typical approximate mean values. Water has a particularly high value, making it a useful heat transfer medium for heating and cooling purposes.

Table 16.1

Substance	Specific heat capacity/J kg^{-1} K^{-1}
Aluminium	900
Brass	380
Ice	2100
Iron	460
Lead	130
Mild steel	480
Water	4200

Molar heat capacity (symbol C_m) is sometimes used instead of specific heat capacity. This is based on the mole rather than the kilogram and is obtained by multiplying the specific heat capacity by the mass of one mole (kg mol^{-1}) of the substance. The units are therefore

$$\text{J kg}^{-1}\text{ K}^{-1} \times \text{kg mol}^{-1} = \text{J mol}^{-1}\text{ K}^{-1}$$

For many simple solid substances the molar heat capacity is approximately 25 J mol^{-1} K^{-1} (Dulong and Petit's law). For gases the specific and molar heat capacities at constant volume are less than at constant pressure, where extra heat is needed to enable the gas to expand to keep its pressure constant. The specific heat capacity of dry air, for example, is about 0.7 kJ kg^{-1} K^{-1} at constant volume, compared with about 1 kJ kg^{-1} K^{-1} at constant pressure.

Used on its own, the term *heat capacity* (symbol C, with units of J K^{-1}) is applied to particular objects. In terms of Equation (16.1), $C = mc$ for an object of mass m made from a material with a specific heat capacity c.

The latent heat absorbed or emitted by a substance as it changes state at constant temperature is given by

$$Q = ml \tag{16.2}$$

where Q joules of heat are absorbed or emitted by m kilograms of the substance, and l is the *specific latent heat* in J kg^{-1}. (Note that latent

Table 16.2

Latent heat of	Specific latent heat/J kg^{-1}
Fusion (ice-water)	3.3×10^5
Vaporisation (water-steam)	2.3×10^6

heat is sometimes expressed as *molar latent heat*, i.e. J mol^{-1}.) Table 16.2 gives approximate values for the specific latent heats of fusion and vaporisation of water, which we shall need below.

Worked Example 16.1

Estimate the difference in water temperature between the top and the bottom of a waterfall 86 m high. (Assume $g = 9.8$ m s^{-2}.)

Assuming that the potential energy of the water at the top of the waterfall (mgh) is all used to raise its temperature, then, from Equation (16.1),

$$Q = mc\theta = mgh$$

Therefore, since $c = 4200$ (Table 16.1),

$$\theta = \frac{gh}{c} = \frac{9.8 \times 86}{4200} = 0.20\ ^\circ\text{C}$$

Worked Example 16.2

Calculate the power required to maintain the temperature of a house that is losing heat at a rate of 28.8 MJ per hour.

A heat loss of 28.8 MJ per hour is offset by a heat input equal to

$$\frac{28.8 \times 10^6}{3600} = 8 \times 10^3 \text{ J s}^{-1} = 8 \text{ kW}$$

Worked Example 16.3

Starting with 2 kg of ice at −5 °C, find the heat required at each stage to effect the following changes: (a) heat the ice to 0 °C; (b) change the ice to water at 0 °C; (c) heat the water to 100 °C; (d) change the water to steam at 100 ° C.

Using data from Tables 16.1 and 16.2:

(a) $Q = mc\theta = 2 \times 2100 \times 5 = 21$ kJ
(b) $Q = ml = 2 \times (3.3 \times 10^5) = 660$ kJ
(c) $Q = mc\theta = 2 \times 4200 \times 100 = 840$ kJ
(d) $Q = ml = 2 \times (2.3 \times 10^6) = 4600$ kJ

Worked Example 16.4

500 g of boiling water is poured into a 1.25 kg aluminium saucepan at 20 °C. Find the final temperature, assuming no heat losses.

Let the final temperature = T. The heat gained by the saucepan is given by

$$mc\theta = 1.25 \times 900 \times (T - 20)$$

This is equal to the heat lost by the water, which is given by

$$mc\theta = 0.5 \times 4200 \times (100 - T)$$

Hence,

$$T = 72\ °\text{C}$$

16.3 EXPANSIVITY

Thermal expansion and contraction, if unrestrained, can generate large enough stresses to cause problems in engineering structures such as bridges and railway lines unless proper allowance is made to relieve or accommodate them.

We can calculate the length change of a material from its *linear expansivity*, α. This is the fractional length increase per unit temperature rise as given by

$$\alpha = \frac{L_2 - L_1}{L_1} \times \frac{1}{\theta} \tag{16.3}$$

where the length L_1 increases to L_2 as the result of a temperature rise of θ. This equation can be rearranged to give

$$L_2 = L_1 (1 + \alpha\theta) \tag{16.4}$$

There are corresponding expressions for area and volume expansion:

$$A_2 = A_1 (1 + \beta\theta) \tag{16.5}$$

$$V_2 = V_1 (1 + \gamma\theta) \tag{16.6}$$

where A and V represent area and volume, respectively, and β is the *area* or *superficial expansivity* and γ the *volume* or *cubic expansivity*. (As a reasonable approximation, $\beta = 2\alpha$ and $\gamma = 3\alpha$.) Note that the values of all three expansivities generally vary to some extent with temperature, although they are often taken to be constant unless accurate results are needed. Some typical approximate mean values of α at normal temperature are shown in Table 16.3.

Table 16.3

Substance	Linear expansivity/K^{-1}
Aluminium	24×10^{-6}
Brass	19×10^{-6}
Copper	17×10^{-6}
Glass	9×10^{-6}
Steel	12×10^{-6}

Obviously we cannot measure α or β for liquids, because they flow; instead we measure their volume expansivity in a container. But this, of course, expands too. The volume of the space within a solid container changes just as though it is made from the same material as the container itself (i.e. as though the container is solid throughout). Liquids generally expand more than solids; hence, they generally expand more than the space inside their containers. The apparent expansion of a liquid is therefore less than its absolute (i.e. real) expansion by an amount corresponding to the expansion of the space it occupies inside the container. The volume expansivities are related approximately as follows:

$$\gamma_{\text{absolute}} = \gamma_{\text{apparent}} + \gamma_{\text{container}}$$

The thermal expansion of a material obviously results in a decrease in its density. In the case of liquids and gases, which can flow, this leads to natural convection, which we shall discuss in more detail in the next topic.

Water behaves in a special way. Ice has a very open crystal structure, where each molecule is surrounded tetrahedrally by four others; each of its orbitals attracts an oppositely charged orbital from each of its four neighbours. When ice melts, the open, rigid crystal structure is disrupted and tends to collapse. This allows the molecules to move closer together, effectively increasing the average number of neighbours per molecule. Water therefore occupies less volume in the liquid state and is more dense than ice (hence, icebergs float and frozen water pipes burst). The density continues to increase as the temperature is raised above 0 °C but by 4 °C the normal thermal expansion starts to win and water expands from a maximum density value at 4 °C as its temperature is raised further. As we noted in Topic 4, the maximum density of water is 1000 kg m^{-3} (= 1 g cm^{-3}). To a very close approximation, this is equivalent to 1 kg per litre. Although the litre is not an SI unit, it is very widely used for volume measurement. (The former definition of the litre was the volume occupied by 1 kg of water at 4 °C but now it is 1 cubic decimetre (i.e. 1×10^{-3} m^3), which is not quite the same.)

Worked Example 16.5

What is the volume at 89 °C of a glass flask which has a capacity of 1.000 l (litre) at 15 °C?

For glass $\gamma = 3\alpha = 3 \times (9 \times 10^{-6}) = 27 \times 10^{-6}\ K^{-1}$. From Equation (16.6),

$$V_2 = V_1(1 + \gamma\theta)$$

Therefore,

$$V_2 = 1.000(1 + (27 \times 10^{-6} \times 74)) = 1.002\ l$$

16.4 THERMAL STRESS

Before leaving this topic, let us consider the stress that arises in a solid material, say a metal bar, if it is restrained from expanding when it undergoes a temperature increase θ. From Equation (16.3), the unrestrained expansion of the bar from L_1 to L_2, expressed as a fractional length increase, would have been $(L_2 - L_1)/L_1 = \alpha\theta$. In effect, the restraint acting on the bar compresses it from L_2 to L_1, thereby producing a strain $\varepsilon = (L_2 - L_1)/L_2$. To a close approximation $(L_2 - L_1)/L_1 = (L_2 - L_1)/L_2$, because L_1 and L_2 will generally have very similar values. It follows that the effective compressive strain ε is given by $\alpha\theta$. But we know from Topic 2 that $E = \sigma/\varepsilon$, where E represents Young's modulus and σ the stress. The thermal stress in the bar is therefore given by

$$\sigma\ (= E \times \varepsilon) = E\alpha\theta \qquad (16.7)$$

Thermal stress is the reason why an ordinary drinking-glass is liable to crack when hot water is poured into it; the inside of the glass tries to expand before the outside. (Heat-resistant glass with low expansivity is much less susceptible to thermal stress.)

Worked Example 16.6

A steel bar is restrained from expansion while its temperature is raised from 3 °C to 28 °C. (a) Find the resulting thermal stress, and (b) find the corresponding restraining force if the bar has a diameter of 70 mm. (Assume $E_{steel} = 2 \times 10^{11}$ N m^{-2}.)

(a) From Equation (16.7),

$$\sigma = E\alpha\theta$$

Therefore, since $\alpha = 12 \times 10^{-6}$ (Table 16.3),

$$\sigma = (2 \times 10^{11}) \times (12 \times 10^{-6}) \times (28 - 3) = 60 \text{ MN m}^{-2}$$

(b) force = stress × cross-sectional area, which is equal to

$$(60 \times 10^{6}) \times \pi(0.035)^2 = 0.23 \text{ MN}$$

Questions

(Use any previously tabulated data as required.)

1. In each case find the temperature rise if the heat produced by a 70 W heater over a period of 1 min is absorbed by 1 kg of the following: (a) water, (b) ice, (c) aluminium, (d) iron and (e) lead.
2. Find the heat evolved as 5 kg of water at each of the following temperatures is changed to ice at 0 °C; (a) 0 °C, (b) 5 °C, (c) 50 °C.
3. Estimate what the theoretical minimum velocity of a snowball at 0 °C would have to be for it to completely melt on impact.
4. A continuous-flow water heater is required to raise the temperature of a water supply from 5 °C to 25 °C at a flow rate of 1.25 litres per minute. Assuming no heat losses, what must the power output of the heater be?
5. A 2 kW electric kettle contains 1.5 kg of water at 10 °C. Assuming the heat capacity of the kettle is 200 J K^{-1}, and ignoring any heat losses, (a) find the time that would be required for the water to come to the boil, and (b) find the extra time required to change a third of the water into steam.

6. (a) How much water at 0 °C must be added to cool 2.5 kg of water from 40 °C to 20 °C?
   (b) How much ice at 0 °C would have had the same effect?

7. A copper tube is 0.500 m long at 20 °C. Its length increases by 0.68 mm when steam at 100 °C is passing through it. Find the linear expansivity of the copper.

8. A steel measuring tape, correctly calibrated at 10 °C, gives the distance between two points as 30.000 m at a temperature of 30 °C. Find the true distance.

9. At 17 °C a brass sphere has a diameter of 49.95 mm and a steel tube has an internal diameter of 50.00 mm. At what temperature is the sphere an exact fit inside the tube?

10. If a steel strut is 1.6 m long at −10 °C, to what temperature must it be heated to increase its length by 0.5 mm? If this expansion is restrained completely, what would be the magnitude of the compressive stress in the strut? (Assume $E_{\text{steel}} = 2 \times 10^{11}$ N m^{-2}.)

11. A steel drum, filled to the brim, holds 200 litres of liquid at 5 °C. If the liquid has a cubic expansivity of 8.7×10^{-4} K^{-1}, how much overflows if the temperature rises to 35 °C?

12. At 20 °C the difference in length between a steel rod and a brass rod is 175 mm. Find the length of each rod if this difference is to remain constant at any normal temperature.

TOPIC 17 HEAT TRANSFER

COVERING:

- conduction, convection and radiation.

There are three principal mechanisms involved in heat transfer. These are *conduction*, *convection* and *radiation*. (Note that other processes such as evaporation and condensation can also be significant.) Heat transfer is central to many areas of engineering, from domestic refrigerators to nuclear power stations, and very important in others.

Conduction involves heat transfer from a hotter to a colder part of a body — for example, through the base of a heated saucepan. *Natural convection* involves the movement of a fluid, such as air or water, which becomes less dense and tends to rise as it is heated. *Forced convection* involves an external agency such as a fan or a pump to move the fluid. Radiated heat, from the sun, for example, is carried by electromagnetic waves (Topic 13) and requires no transfer medium at all.

In practical situations all three processes may operate simultaneously but, to keep things simple, we shall consider them separately.

17.1 CONDUCTION

Heating one end of a solid bar will cause its constituent atoms to vibrate more vigorously about their fixed positions on the crystal lattice. Some of the vibrational energy will be passed on to neighbouring atoms via the chemical bonds and, as the process continues, heat is transferred along the rod from the hot end towards the cold.

Let us assume that a bar of length L has a cross-sectional area A and that its end faces are parallel and maintained at temperatures θ_1 and θ_2, respectively, where $\theta_1 < \theta_2$ as in Figure 17.1. Let us also assume that the bar is lagged along its length to prevent any heat loss from its sides and that the system has been given sufficient time to reach a steady state. ('Steady state' implies that, although the temperature differs at different points within the body, it remains constant at any given point.) The graph at the bottom of the figure

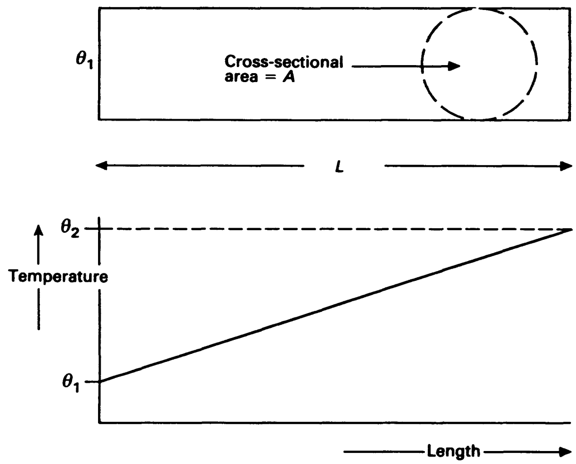

Figure 17.1

shows that the temperature gradient along the bar is $(\theta_2 - \theta_1)/L$. The rate at which heat flows through the bar (from right to left) will be proportional to its cross-sectional area and to the temperature gradient, so that

$$\frac{Q}{t} = kA\frac{(\theta_2 - \theta_1)}{L} \tag{17.1}$$

where Q/t is the rate of heat flow in J s^{-1} (or W). k, the coefficient of proportionality, is the *thermal conductivity* of the material and has the units W m^{-1} K^{-1} (i.e. J s^{-1} m^{-1} K^{-1}). Table 17.1 shows some typical values. They tend to vary with temperature and other factors such as moisture content. They may vary considerably for different varieties of the same material. For example, the value for concrete ranges from below 0.2 to above 3.5 W m^{-1} K^{-1}, depending on type and moisture content.

Equation (17.1) tells us that the heat flow is proportional to the temperature difference $(\theta_2 - \theta_1)$ and inversely proportional to L/kA. The quantity L/kA is called the *thermal resistance* of the material; as we shall see from Topic 25, it is the thermal analogy of electrical resistance.

Remember that, in practice, the surface temperature of a conductor may not be the same as that of its surroundings. For example, the inside surface temperature of a window pane may be significantly different from room temperature because of the insulating effect of the layer of air immediately adjacent to the glass. Such effects can reduce heat flow very considerably.

Metals generally seem to feel colder than other materials at the same temperature, because they are better at conducting heat away from the body. Their high thermal conductivity stems from the freedom of movement of the valence electrons that maintain the metallic

Table 17.1

Material	Thermal conductivity/W m^{-1} K^{-1}
Metals:	
Aluminium	210
Copper	400
Steel	50
Non-metals:	
Concrete	1.5
Brick	0.6
Glass	1
Ice	2.1
Plaster	0.13
Wood (parallel to grain)	0.38
Wood (perpendicular to grain)	0.15
Water	0.58
Thermal insulators:	
Air	0.02
Glass wool	0.04
Polystyrene (expanded)	0.03

bond throughout the metal crystal structure. The free electrons in the hot part of the metal gain kinetic energy and rapidly transfer it to the colder parts as they migrate there. Some heat is still transferred via the vibrations of the crystal lattice but much less than by the free electrons. There tend to be no free electrons in ionic and covalent substances, so these have to depend on the lattice vibrations; hence, they generally have low thermal conductivity values compared with metals.

Air has very low thermal conductivity, because its constituent molecules are widely separated and can only pass kinetic energy from one to the other during their relatively infrequent collisions. If convection currents can be prevented, then air is a very good thermal insulator. Glass wool contains air trapped between the fibres, which gives it its very low thermal conductivity. Expanded polystyrene relies on the same basic principle and the insulating properties of some of the non-metals in Table 17.1 benefit from small quantities of trapped air. (Note that a vacuum is an ideal insulator in that it allows no possibility of conduction or convection; however, it will not stop the transmission of radiant heat.)

Equation (17.1) tells us that the heat flow through a material is inversely proportional to its thickness. This leads us to an important point about thermal insulation. Let us assume that a 1 m^2 sheet of expanded polystyrene 10 mm thick has one face maintained at -10 °C and the other at 30 °C. The equation tells us that the heat flow through the sheet is 120 W (see Worked Example 17.1 below). If we successively double the thickness of the sheet to 20 mm, 40 mm and 80 mm, the heat flow falls to 60 W, 30 W and 15 W, respectively; in other words, the first increase of 10 mm saves 60 W, a further increase of 20 mm saves an extra 30 W, but the final increase of 40 mm saves only another 15 W. Obviously the law of diminishing returns is operating: there is a critical thickness beyond which the heat saved over a given period of time does not justify the cost of the extra polystyrene used and the space wasted.

In many practical situations it is necessary to consider heat passing through successive thicknesses of different materials — for example, the three layers in a double-glazed window (two of glass and one of air).

Let us consider a simple example where a composite sheet built up from two layers has a thermal gradient between its faces. Since sheets tend to be flat and thin, lateral heat loss to the surroundings through the edge is confined to a relatively small strip round the periphery. Away from the edge, a section through the thickness of the sheet can be regarded in the same way as a lagged bar, because it is surrounded by material with an identical thermal gradient and there is no tendency for lateral heat loss. Figure 17.2 represents such a section through a composite of two materials, A and B. Heat will flow through both materials at the same rate Q/t, so if the temperature at the interface between them is θ_{AB}, and if $\theta_1 < \theta_2$, then, from Equation (17.1),

$$\frac{Q}{t} = k_A A_A \frac{(\theta_{AB} - \theta_1)}{L_A} = k_B A_B \frac{(\theta_2 - \theta_{AB})}{L_B} \qquad (17.2)$$

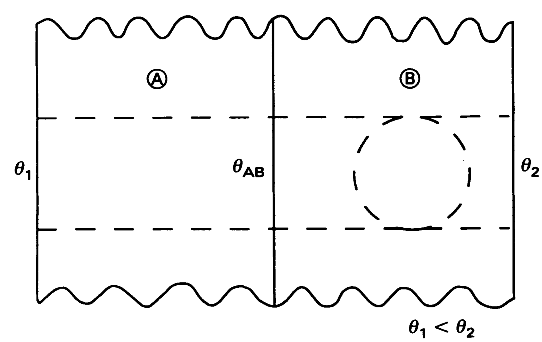

Figure 17.2

Worked Example 17.1

A sheet of expanded polystyrene, 1 m^2 in area, has one face maintained at −10 °C and the other at 30 °C. Find the heat flow through the sheet as its thickness is successively doubled from 10 mm to 20, 40 and finally 80 mm.

From Table 17.1, $k = 0.03$ W m^{-1} K^{-1}, and, substituting the data into Equation (17.1),

$$\frac{Q}{t} = \frac{0.03 \times 1 \times 40}{L}$$

which gives

120 W for $L = 0.01$ m,
60 W for $L = 0.02$ m,
30 W for $L = 0.04$ m, and
15 W for $L = 0.08$ m.

Worked Example 17.2

A composite sheet consists of a 30 mm thickness of material A and a 10 mm thickness of material B. (a) Find the heat flow through a 9 × 9 m panel of the composite when the exposed surface of A is maintained at 5 °C and the exposed surface of B at 25 °C. (b) Find the heat flow if the temperatures are reversed. (c) Sketch the temperature gradient in each case. (Assume $k_A = 0.1$ and $k_B = 0.3$ W m^{-1} K^{-1}.).

(a) Letting θ_{AB} represent the temperature at the interface, and substituting the data into Equation (17.2),

$$\frac{Q}{t} = 0.1 \times 3^2 \frac{(\theta_{AB} - 5)}{0.03} = 0.3 \times 3^2 \frac{(25 - \theta_{AB})}{0.01}$$

which gives

$$\theta_{AB} = 23\ °C$$

and, substituting for θ_{AB} above,

$$\frac{Q}{t} = 0.1 \times 3^2 \frac{(23 - 5)}{0.03} = 540\ W$$

(b) $$\frac{Q}{t} = 0.1 \times 3^2 \frac{(25 - \theta_{AB})}{0.03} = 0.3 \times 3^2 \frac{(\theta_{AB} - 5)}{0.01}$$

which gives

$\theta_{AB} = 7$ °C; hence, $Q/t = 540$ W

(c)

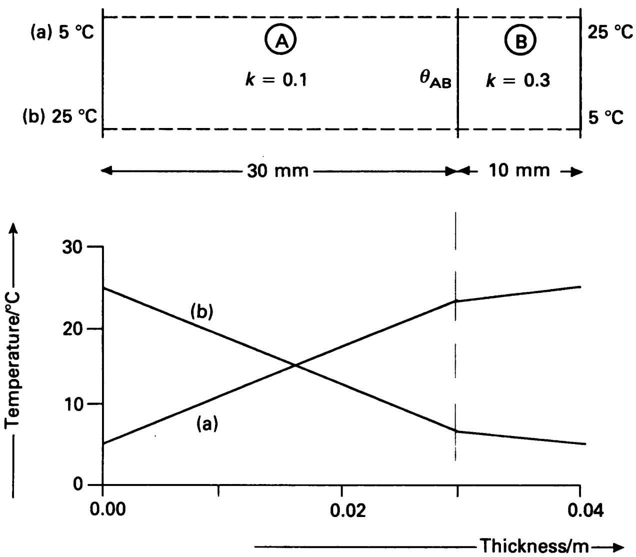

Figure 17.3

Worked Example 17.3

A lake is covered with a thickness of 20 mm of ice which is increasing by 6 mm per hour. Assuming that the water underneath the ice is at 0 °C, estimate the air temperature. (Assume $\rho_{ice} = 920$ kg m^{-3}.)

An increase in thickness of 6 mm over an area of 1 m^2 of ice is a volume increase of $1 \times 0.006 = 0.006$ m^3. The mass of 0.006 m^3 of ice is equal to

$$0.006 \text{ m}^3 \times 920 \text{ kg m}^{-3} = 5.52 \text{ kg}$$

To produce 5.52 kg of ice from water at 0 °C requires the removal of $Q = ml$ joules of heat, where $m = 5.52$ kg and, from Table 16.2 (page 150), $l = 3.3 \times 10^5$ J kg^{-1}. ml joules are removed in 1 h; therefore, the rate of removal is

$$\frac{Q}{t} = \frac{ml}{3600} = \frac{5.52 \times 3.3 \times 10^5}{3600} = 506 \text{ J s}^{-1}$$

From Table 17.1 (page 158) $k_{ice} = 2.1$ W m^{-1} K^{-1}, and, substituting in

Equation (17.1), where θ_1 is the air temperature,

$$\frac{Q}{t} = 506 = 2.1 \times 1^2 \frac{(0 - \theta_1)}{0.02}$$

which gives

$$\theta_1 = -4.8\ °C$$

17.2 CONVECTION

Newton's law of cooling is an empirical law telling us that the rate at which a body loses heat is proportional to its excess temperature over that of its surroundings. The discussion above suggests that the law should be valid for a hot body connected to cooler surroundings via a thermal conductor. However, it is normally viewed in the context of forced convection, although it works fairly well for natural convection if the excess temperature is small.

Let us assume that the temperature of a hot body is θ_2 and that of the surroundings is θ_1. The rate of heat loss Q/t from the body will be roughly proportional to its surface area A and, under conditions where Newton's law of cooling applies,

$$\frac{Q}{t} = k' A(\theta_2 - \theta_1) \qquad (17.3)$$

k', the constant of proportionality, depends on the shape, orientation and surface characteristics of the body, and on the nature and flow characteristics of the cooling fluid.

It is useful to remember that if a body loses Q joules of heat, then its temperature will fall by Q/mc °C, where m is its mass and c is the specific heat capacity of the substance from which it is made (see Equation 16.1, page 148). (This implies that the temperature will remain uniform throughout the body as it cools, but in practice there may well be significant temperature variation through a given cross-section at any moment.)

17.3 RADIATION

If a body is completely surrounded by vacuum, then it can only exchange energy with its surroundings by radiation. Thus, the earth receives radiated energy from the sun in the form of heat and light.

In Topic 13 we noted that the essential difference between the various types of electromagnetic radiation is their wavelength. We also noted that black objects appear black because they absorb all the visible wavelengths. Physicists talk about a hypothetical body called a *black body* which has no ability to reflect incident radiation but will absorb it all completely. As a corollary of this property, a black body is the best possible emitter of *thermal radiation*. This is the radiation emitted by all objects by virtue of their temperature and is of particular interest to us in the present context. If the temperature of a black body is increased, it will emit more thermal radiation and, at the same time, the wavelength of the most intense radiation will decrease. Obviously, a black body will cease to appear black when it is so hot that it emits light.

Stefan's law gives the total radiant energy emitted by a black body per unit time (i.e. total radiant power) per unit surface area, as follows:

$$\frac{P}{A} = \sigma T^4 \tag{17.4}$$

where P represents the radiant power emitted by a black body of surface area A, and T is the absolute temperature (K) of its surface. σ is a constant, called Stefan's constant, which has the value 5.67×10^{-8} W m^{-2} K^{-4}. The equation tells us that the body will emit radiation at all temperatures greater than absolute zero.

If the surroundings of the black body are at some lower temperature T_s, then it will receive radiant energy from them at a rate proportional to T_s^4 and its net rate of radiant energy loss P_{net} will be proportional to $(T^4 - T_s^4)$. If $(T - T_s)$ is small, it is a fairly simple mathematical exercise to show that $(T^4 - T_s^4)$ is approximately equal to $4\,T_s^3(T - T_s)$; thus, the net rate of radiant energy loss is approximately proportional to the temperature difference if this is small. So.

$$\frac{P_{net}}{A} = \sigma(T^4 - T_s^4) \tag{17.5}$$

and

$$\frac{P_{net}}{A} \approx \sigma \times 4\,T_s^3(T - T_s) \tag{17.6}$$

provided that $(T - T_s)$ is small.

Clearly there will be a net loss of energy from the black body until its temperature reaches that of its surroundings. It will still continue to radiate energy when this happens, but it will then be absorbing it from the surroundings at the same rate.

Real objects are not perfect absorbers and emitters like black bodies — in fact, some are rather poor. The *emissivity*, ε, of a non-black body can be defined as the power emitted per unit area expressed as a fraction of that radiated by a black body at the same temperature, ε ranges from 1 for a perfect emitter down to 0, and Stefan's law can be expressed in the form

$$\frac{P}{A} = \sigma\varepsilon T^4 \tag{17.7}$$

Worked Example 17.4

A 25 mm diameter solid metal sphere is cooling under conditions such that there are negligible heat losses by conduction and convection. If the sphere behaves as a black body, with a net rate of heat loss of 100 W where its surroundings are maintained at 15 °C, then estimate (a) its temperature and (b) the rate at which this is falling. (Assume that the density of the metal $\rho = 9 \times 10^3$ kg m^{-3} and that its specific heat capacity c = 400 J kg^{-1} K^{-1}. Assume that Stefan's constant = 5.67×10^{-8} W m^{-2} K^{-4}.)

(a) The surface area of a sphere of radius r is $4\pi r^2$. Substituting the data into Equation (17.5), where T (K) is the temperature of the sphere,

$$\frac{100}{4\pi(0.0125)^2} = 5.67 \times 10^{-8} (T^4 - 288^4)$$

which gives

$$T = 975 \text{ K} = 702 \text{ °C}$$

(b) The volume of a sphere of radius r is given by

$$\frac{4}{3}\pi r^3$$

Therefore, its mass m is given by

$$m = \frac{4}{3}\pi r^3 \times \rho$$

and its thermal capacity mc is given by

$$mc = \frac{4}{3}\pi r^3 \times \rho \times c$$

and, substituting the data,

$$mc = \frac{4}{3}\pi(0.0125)^3 \times (9 \times 10^3) \times 400 = 29.5 \text{ J K}^{-1}$$

An object with a thermal capacity of 29.5 J K^{-1} losing 100 J in 1 s will experience an average fall in temperature of

$$\frac{100 \text{ J s}^{-1}}{29.5 \text{ J K}^{-1}} = 3.4 \text{ K s}^{-1}$$

Questions

(Use any previously tabulated data as required. Assume that Stefan's constant = 5.67×10^{-8} W m^{-2} K^{-4}.)

1. For each of the following materials estimate the thickness that would provide equivalent thermal resistance to that of 12 mm of expanded polystyrene:

   (a) glass wool;
   (b) plaster;
   (c) solid glass;
   (d) steel.

2. Two cylinders of identical shape and size, one copper and the other steel, are joined end to end and lagged along their length. The exposed copper face is maintained at 0 °C and the exposed steel face at 100 °C. Under steady state conditions, (a) what is the temperature at the interface, and (b) what would the relative lengths of the cylinders have to be for an interface temperature of 50 °C?

3. A water tank, made from steel sheet 5 mm thick, has a layer of scale inside that is 0.6 mm thick. When the tank contains water at 90 °C, it loses heat at a rate of 35 kW m^{-2} through its sides. Estimate the temperature of the outside surface of the metal. (Assume that the thermal conductivity of the scale is 1.2 W m^{-1} K^{-1}.)

4. A sheet of aluminium 3 mm thick is sandwiched between two sheets of steel each 1 mm thick. What would be the heat flow per unit area through the thickness of the composite if one face is maintained at 0 °C and the other at 100 °C?

5. An object, hanging from an insulating thread, is cooling in a steady breeze at an air temperature of 15 °C. When it reaches 55 °C, it is losing heat at a rate of 30 W. Assuming negligible radiation losses, estimate the

rate at which it will lose heat when it reaches 35 °C.

6. The temperature of each of the following is falling at a rate of 0.75 °C per minute. In each case estimate the rate of heat loss:

   (a) a metal object with a heat capacity of 200 J K^{-1};
   (b) a metal object of mass 0.5 kg and specific heat capacity 400 J kg^{-1} K^{-1};
   (c) a 47.5 mm diameter solid metal sphere with a density of 8900 kg m^{-3} and a specific heat capacity of 400 J kg^{-1} K^{-1}.

7. The temperature of a mild steel can of mass 0.525 kg containing 0.240 kg of water was found to fall at a rate of 1 °C in 2 min 45 s. A sufficient quantity of hot water was added to the can to return it to its original temperature and, under the same cooling conditions, the temperature was found to fall at a rate of 1 °C in 8 min 15 s. Estimate how much water was added to the can.

8. Two ball-bearings, one twice the diameter of the other but otherwise identical, and both at the same temperature, are allowed to cool under identical conditions where Newton's law of cooling is valid. Find their relative initial rates of (a) loss of heat and (b) loss of temperature.

9. A black body, completely isolated in space, is cooling from 650 °C.

   (a) At what temperature does it radiate energy at half its initial rate?
   (b) At what rate does it radiate energy when it has cooled to 0 °C?

   (Assume space is at absolute zero.)

10. A 100 m diameter sphere, completely isolated in space, is radiating energy at a rate of 22.8 MW. If its surface temperature is 127 °C, estimate its emissivity. (Assume space is at absolute zero.)

TOPIC 18 GASES

COVERING:

- ideal gas;
- Boyle's law, the pressure law and Charles' law;
- the ideal gas equation;
- real gas and vapour.

In Topic 16 we noted that the constituent particles in a gas are free to move around independently of one another. The constituent particles of most ordinary gases are molecules. For instance, air consists of roughly 80% N_2 and 20% O_2 molecules with small quantities of CO_2 and other molecular substances. (The minor constituents also include the inert gases, which exist as single atoms because of their stable electronic configurations.)

At 0 °C and atmospheric pressure a litre of gas contains approximately 2.7×10^{22} molecules; this means that the average distance between two neighbours is about 3 nm (3×10^{-9} m), which is roughly ten times the size of a small molecule. Gas molecules move around at high speed in a random and disordered fashion, continually colliding with one another and the walls of their container. At ordinary temperatures the molecules in air travel at hundreds of metres per second, with a typical *mean free path* (average distance between collisions) of around 100 nm. The random nature of their motion and their frequent collisions means that at any instant the molecules have a wide range of speeds. However, their average kinetic energy is constant at a given temperature and proportional to the absolute temperature.

With this simple picture in mind, we can regard the addition of thermal energy to a gas as resulting in an increase in the translational kinetic energy (hence, the speed) of its constituent molecules.

18.1 IDEAL GAS

Scientists use the hypothetical concept of an *ideal gas* in which the individual molecules are assumed to have no volume and experience no intermolecular forces of the kind that we discussed in Topic 15.

Their collisions with one another and with the walls of their container are assumed to be perfectly elastic.

Air at ordinary temperatures and pressures behaves more or less like an ideal gas. Anyone who uses a bicycle pump is familiar with the 'springy' nature of air. This is quantified in Boyle's law, which relates the volume V and pressure p of a given mass of ideal gas by

$$V \propto 1/p$$

or

$$pV = \text{constant}$$

Thus, the pressure of the gas is inversely proportional to its volume. That is to say, if its volume is halved, its pressure is doubled, and if its volume is doubled, its pressure is halved, and so on. It is most important to remember that Boyle's law only holds at constant temperature and that it is only strictly valid for an ideal gas.

The pressure that a gas exerts on its container is due to the force arising from the change of momentum as the molecules bounce off the container walls. Because of the enormous numbers of molecules involved, these collisions result in a steady pressure (force per unit area).

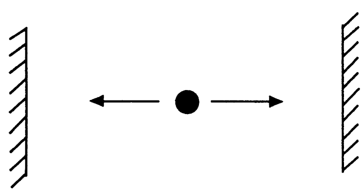

Figure 18.1

Figure 18.1 shows a simple one-dimensional model where the gas in its container is represented by a single molecule moving to and fro perpendicularly between two parallel walls. At a given temperature the molecule moves at a constant speed corresponding to the constant average kinetic energy of the actual gas molecules. If the distance between the walls is halved, the frequency of the collisions is doubled and therefore the pressure is doubled; and if the distance is doubled, the frequency is halved — hence, in one-dimensional terms, pV = constant.

If the temperature is raised, there will be a corresponding increase in the kinetic energy of the gas molecules. This is represented by an increase in the speed of the molecule in our model. There will be an increase in the force as it collides with the walls and, assuming the walls are fixed (constant volume), there will be an increase in the frequency of its collisions — hence, there will be a pressure increase. The relationship between the pressure and absolute temperature T of a given mass of ideal gas at constant volume is given by the *pressure law*

$$p \propto T$$

or

$$p/T = \text{constant}$$

In other words, the pressure of a given mass of an ideal gas at constant volume is proportional to its absolute temperature.

If the gas is allowed to remain at constant pressure, by letting it expand as its temperature is raised, then the relationship between its volume and absolute temperature is given by *Charles' law* as follows:

$$V \propto T$$

or

$$V/T = \text{constant}$$

That is to say, the volume of a given mass of ideal gas at constant pressure is proportional to its absolute temperature.

These three laws are combined in the *ideal gas equation*

$$pV = nRT \tag{18.1}$$

where V is in m^3 and p is in Pa. (Remember that the SI unit of pressure is the pascal (Topic 4) and that 1 Pa = 1 N m^{-2}.) T is the absolute temperature (K), n is the amount of gas in moles (mol) and R is the *universal molar gas constant* (often abbreviated to *gas constant*), which has the value 8.31 J mol^{-1} K^{-1}.

From Equation (18.1),

$$(nR =) \frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2} \tag{18.2}$$

where the subscripts 1 and 2 denote the pressure, volume and temperature of a given quantity of a gas under two different sets of conditions.

If $T_1 = T_2$, then Equation (18.2) reduces to Boyle's law:

$$p_1V_1 = p_2V_2$$

and if $V_1 = V_2$, then it reduces to the pressure law:

$$\frac{p_1}{T_1} = \frac{p_2}{T_2}$$

and if $p_1 = p_2$, it reduces to Charles' law:

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

Furthermore, a plot of p against $1/V$ at constant temperature will be a straight line of slope nRT, and so on.

Worked Example 18.1

An air bubble trebles in volume as it rises from the bottom of a lake to the surface. Estimate the depth of the lake. (Assume that $g = 9.8 \text{ m s}^{-2}$, atmospheric pressure is 101 kPa and the density of water $\rho_{water} = 1000 \text{ kg m}^{-3}$.)

If p_{atm} and V_{atm} are the pressure and volume of the bubble at atmospheric pressure at the surface of the lake and p_b and V_b are the corresponding values at the bottom then, from Boyle's law, assuming the water temperature is constant,

$$p_{atm} \times V_{atm} = p_b \times V_b$$

but

$$V_{atm} = 3V_b$$

Therefore,

$$p_{atm} \times 3V_b = p_b \times V_b$$

and

$$p_b = 3p_{atm}$$

But p_b results from the pressure due to the depth of water plus atmospheric pressure; therefore, the pressure due to the depth of water alone is equal to

$$p_b - p_{atm} = 3p_{atm} - p_{atm} = 2p_{atm}$$

and, from Equation (4.2) (page 32), this is equal to $\rho_{water}gh$, where h is the depth of the lake.

Rearranging $\rho_{water}gh = 2p_{atm}$ gives

$$h = \frac{2p_{atm}}{\rho_{water}g} = \frac{2 \times 101 \times 10^3}{1000 \times 9.8} = 20.6 \text{ m}$$

Worked Example 18.2

A vertical glass tube, sealed at its bottom end, contains a 144 mm column of air trapped beneath a 126 mm column of mercury. Find the atmospheric pressure if, when the tube is laid horizontally, the air column is 168 mm long.

Vertical position:
The pressure acting on the trapped air column is $(p_{atm} + 126)$ mmHg. Assuming, for simplicity, that the tube is of unit internal cross-sectional area, then

$$pV = (p_{atm} + 126) \times 144$$

Horizontal position:
The pressure acting on the trapped air column is p_{atm} mmHg. Therefore,

$$pV = p_{atm} \times 168$$

From Boyle's law

$$(p_{atm} + 126) \times 144 = p_{atm} \times 168$$

which gives

$$p_{atm} = 756 \text{ mmHg}$$

Worked Example 18.3

The following p–V data was obtained using 1.88×10^{-3} moles of gas at 26 °C;

Pressure/kPa	100	150	200	250	300
Volume/cm³	46.6	31.1	23.3	18.6	15.5

By non-graphical means (a) confirm that the gas obeys Boyle's law and (b) estimate the value of the universal molar gas constant.

(a) If Boyle's law is obeyed, then pV should be constant. For each pair of values,

$$pV = 4.66, 4.67, 4.66, 4.65 \text{ and } 4.65 \text{ (average 4.66)}$$

(Remember that 1 kPa = 1×10^3 Pa and 1 cm³ = 1×10^{-6} m³.)

(b) From Equation (18.1) (page 169)

$$R = \frac{pV}{nT} = \frac{4.66}{1.88 \times 10^{-3} \times 299} = 8.3 \text{ J mol}^{-1} \text{ K}^{-1}$$

Note that the units of R are easily derived as follows:

$$R = \frac{pV}{nT} = \frac{\text{N m}^{-2} \times \text{m}^3}{\text{mol} \times \text{K}} = \text{N m mol}^{-1}\ \text{K}^{-1} = \text{J mol}^{-1}\ \text{K}^{-1}$$

Figure 18.2 is a plot of p against V for a given mass of ideal gas at temperatures T_1 and T_2, where $T_2 > T_1$. Boyle's law is obeyed along each *isotherm* (constant temperature line). The vertical and horizontal arrows, at constant V and constant p, represent changes following the pressure law and Charles' law, respectively.

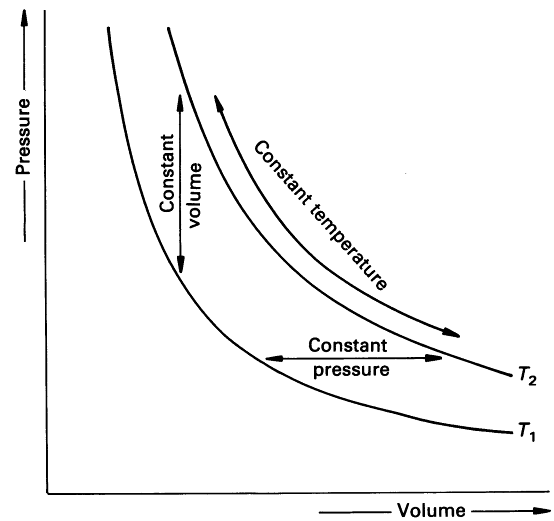

Figure 18.2

Charles' law can be expressed approximately by

$$V = V_0(1 + \theta/273) \tag{18.3}$$

where V_0 is the volume of gas at 0 °C and V is its volume at θ °C. (Compare this equation with Equation 16.6 (page 152), noting that, although the expansivity of different solids and different liquids can vary considerably, the volume expansivity for an ideal gas at 0 °C is approximately 1/273 K^{-1}.)

Figure 18.3 shows Equation (18.3) plotted as a graph. This again draws our attention to the significance of −273 °C as the lowest possible temperature (absolute zero). Of course, the atoms in a real gas would not have zero volume at absolute zero as the figure seems to suggest; they would form a solid.

The equivalent equation for the pressure law is

$$p = p_0(1 + \theta/273) \tag{18.4}$$

and this gives a corresponding graph of the same form as Figure 18.3.

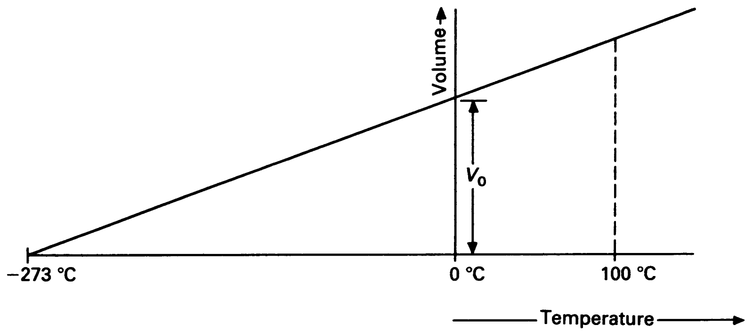

Figure 18.3

There are two further points that we should note about ideal gases. *Avogadro's hypothesis* states that equal volumes of ideal gases at the same temperature and pressure contain the same number of molecules. At *standard temperature and pressure* (S.T.P.) 1 mol of an ideal gas occupies about 22.4×10^{-3} m³, or 22.4 litres. S.T.P., which is fixed at 0 °C and 1 atm pressure (760 mmHg), is a standard condition that makes a useful baseline to which any quantity of ideal gas at any temperature and pressure can be reduced.

Dalton's law of partial pressures states that the total pressure of a mixture of gases equals the sum of the partial pressures of its components. (The *partial pressure* of each component is the pressure it would exert if it was the only occupant of the volume containing the mixture.)

Real gases deviate from ideal behaviour because real atoms and molecules have significant volume and, as we saw in Topic 15, they have forces operating between them.

18.2 REAL GASES

The behaviour of real gases was investigated last century by Andrews. Figure 18.4 gives a broad picture of the relationship between pressure and volume based on his results with carbon dioxide. We shall use this to illustrate our discussion.

At relatively high temperatures, represented by the uppermost isotherm in Figure 18.4(a), the cohesive effect of the attractive forces between the gas molecules is small compared with the disruptive effect of thermal energy, and the isotherm is similar to that of an ideal gas. As the temperature is lowered, the effects of intermolecular forces become more significant and the isotherm becomes correspondingly distorted. Before we consider the critical isotherm corresponding to the critical temperature T_c, let us see what happens as we compress the gas along the isotherm XYY′Z corresponding to some temperature below T_c.

From X to Y it behaves as a gas but at Y it starts to condense. From

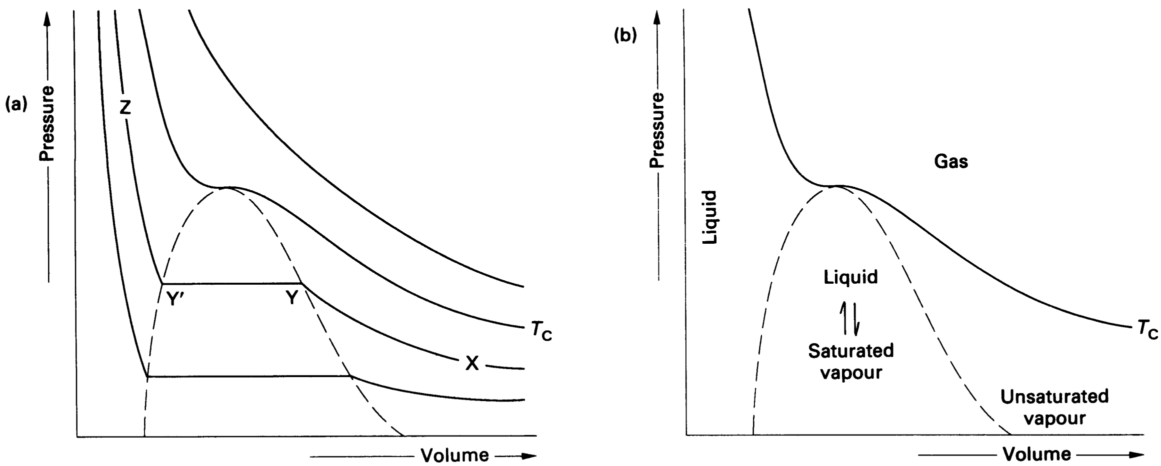

Figure 18.4

Y to Y′ the pressure remains constant while the volume is reduced, and the gas progressively condenses until at Y′ it has entirely liquefied. Y′ to Z represents the compression of the liquid; liquids are obviously much less compressible than gases, because the molecules are close together, and the graph is correspondingly steep.

The *critical temperature* T_c is the temperature above which a gas cannot be liquefied by increasing its pressure. The *critical isotherm* at T_c therefore represents a boundary above which the substance must exist in gaseous form irrespective of pressure, as shown in Figure 18.4(b). Of course, the gas molecules can still be forced very close together by using high pressure, but above T_c this produces a highly compressed gas, not a liquid.

As Figure 18.4(b) suggests, a *vapour* is a substance in the gaseous state but below its critical temperature; thus, it can be liquefied by pressure alone.

In Topic 16 we noted that liquids evaporate because molecules with sufficiently high kinetic energy escape from the surface. And, as we might expect, if the temperature of the liquid is raised, the number of sufficiently energetic molecules will increase and the evaporation rate will increase. If we enclose a liquid under a vacuum, the molecules which escape will form a vapour above the surface. A number of the vapour molecules will *condense* back into the liquid as they collide with its surface and an equilibrium will be established when the condensation rate is equal to the evaporation rate. Under these conditions the vapour is described as *saturated* and the *vapour pressure* (the pressure exerted by the vapour) is called the *saturated vapour pressure* (s.v.p.). The saturated vapour pressure increases with temperature as the kinetic energy of the molecules is increased; however, it does not depend on the volume of the space above the liquid, because if this is changed and the temperature remains the same (as between Y′ and Y in Figure 18.4), then an imbalance between the evaporation and condensation rates restores the vapour pressure to its original saturated value. If the volume is so large that the liquid evaporates completely before the saturated vapour pressure is reached, then the vapour is described as *unsaturated* (as between Y and X in the figure).

The dotted curve in the figure encloses an area within which liquid and saturated vapour coexist; it encloses the horizontal parts of all the isotherms below T_c. The pressure corresponding to each of these represents the saturated vapour pressure at that temperature. If the temperature is raised or lowered, the saturated vapour pressure will increase or decrease accordingly. If the temperature is raised to the point where the saturated vapour pressure is equal to the external pressure, then evaporation will occur throughout the body of the liquid; in other words, the liquid will boil. The boiling point normally quoted for a liquid is that at standard atmospheric pressure; it generally occurs somewhere about two-thirds of the critical temperature on the absolute scale.

If an unsaturated vapour is cooled to the point where it reaches its s.v.p., then it will normally start to condense, like moisture on a cold window, for example. *Relative humidity* is a measure of the extent to which air is saturated with water vapour. Thus, '60% relative humidity' means that the air contains 60% of the moisture that it would contain if it was saturated. The *dew point* is the temperature where the water vapour in the air is just saturated.

Questions

(Use any previously tabulated data as required. Assume ideal gas behaviour. Where necessary, assume atmospheric pressure = 101 kPa and R = 8.3 J mol^{-1} K^{-1}. Assume absolute pressure as opposed to gauge pressure (Topic 4) unless otherwise stated or implied. g = 9.8 m s^{-2}.)

1. The temperature of 2400 mm^3 of gas is increased from 27 °C to 57 °C. What volume change must be made to keep its pressure constant?

2. The pressure inside a 12 litre gas cylinder fell from 0.82 MPa to 0.21 MPa, owing to a leaky valve. Estimate the volume of escaped gas at atmospheric pressure.

3. A pressure gauge, used to measure tyre pressures before and after a journey, gave readings of 190 kPa and 210 kPa, respectively. If the initial temperature was 18 °C (and the volume of the tyres remained constant), estimate the air temperature in the tyres after the journey.

4. 7.5 litres of gas at 27 °C and 505 × 10^3 Pa is allowed to

expand to atmospheric pressure. Find its volume at 7 °C.

5. Calculate the volume of 1 mol of N_2 at standard temperature and pressure.
6. If a sample of gas occupies 1.20 m^3 at 27 °C and 1950 mmHg pressure, find (a) its volume at s.t.p. and (b) its amount in moles.
7. A cylinder with internal dimensions of 100 mm length and 50 mm diameter contains gas at 14 °C and a pressure of 2 atmospheres. Estimate how many gas molecules the cylinder contains.
(Avogadro constant = 6.02×10^{23} mol^{-1}.)
8. Assuming that air consists of 80% nitrogen (N_2) and 20% oxygen (O_2) by volume, estimate its density at s.t.p.
9. A device like that in Worked Example 18.2 contained a mercury column 120 mm long. When the tube was held vertically, the length of the air column was (a) 79 mm with the open end upwards and (b) 109 mm with the open end downwards. Calculate the atmospheric pressure at the time.
10. Use a graphical method to answer Worked Example 18.3.
11. A device like that in Worked Example 18.2 contained a column of air trapped by water (rather than mercury). With the tube horizontal and atmospheric pressure equal to 759 mmHg, the length of the air column is 235 mm at 10 °C and 340 mm at 60 °C. Assuming that the saturated vapour pressure of water is 9 mm at 10 °C (and that Dalton's law applies to the trapped air/water vapour mixture), estimate the saturated vapour pressure of water at 60 °C.

TOPIC 19 LIQUIDS

COVERING:

- non-viscous behaviour (Bernoulli's equation);
- viscous flow (Poiseuille's formula);
- motion in a viscous fluid (Stokes's law);
- surface tension.

We have seen that the forces of attraction operating in a liquid are able to withstand the disruptive effect of thermal energy to the extent that the constituent particles form a coherent mass but remain capable of movement relative to one another. A liquid therefore has a more or less fixed volume contained within its boundary surface and is capable of flowing.

In this book we shall confine ourselves to *laminar flow*. This is flow that can be viewed in terms of the movement of layers (laminae) of liquid where successive particles passing the same point follow the same path (as indicated by the arrows in Figure 19.1). By contrast, in *turbulent flow*, which tends to occur at higher velocities, the flow pattern is broken and irregular, and eddies are formed.

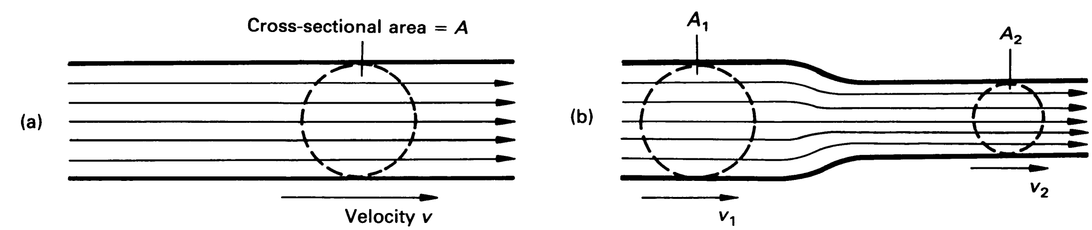

Figure 19.1

Reynolds demonstrated the difference between laminar and turbulent flow in a classic experiment towards the end of the last century. He introduced a 'thread' of liquid dye via a fine nozzle into water flowing through a glass tube. He found that as long as the velocity of the water remained low (where the flow is laminar), the thread of dye remained intact as it was carried along. At higher velocities (where the flow becomes turbulent) the thread broke up and the dye mixed with the water.

Figure 19.1(a) represents laminar flow through a uniform pipe. If the cross-sectional area of the pipe is A and liquid is flowing through it with an average velocity v, then the flow rate, expressed as the volume of liquid passing a given point in one second, will be Av. If the cross-sectional area of the pipe varies, as in Figure 19.1(b), then the velocity varies to maintain a constant flow rate and, assuming the liquid is incompressible,

$$A_1v_1 = A_2v_2 \tag{19.1}$$

This equation, called the *continuity equation*, tells us that, as the pipe narrows, the liquid velocity increases.

19.1 IDEAL (NON-VISCOUS) LIQUIDS

The next step is to consider *Bernoulli's equation*, which describes the behaviour of an ideal liquid. For the purposes of our discussion, an ideal liquid is incompressible and non-viscous and has flow properties that depend only on its density. (As we shall see in the next section, viscosity is a measure of the internal friction in a fluid that gives it resistance to flow.) To identify the parameters involved in Bernoulli's equation, let us consider the case of a liquid flowing through the curiously shaped pipe in Figure 19.2. At any position in the system, p represents the pressure, v represents the velocity of the liquid and h represents the height above some reference level. Let us assume that the liquid is flowing from position 1 to position 2 because of a pressure difference between them. The work done on the fluid by the net force due to the pressure difference results in an increase in its potential energy as it moves uphill and in its kinetic energy because of the increased velocity where the pipe narrows.

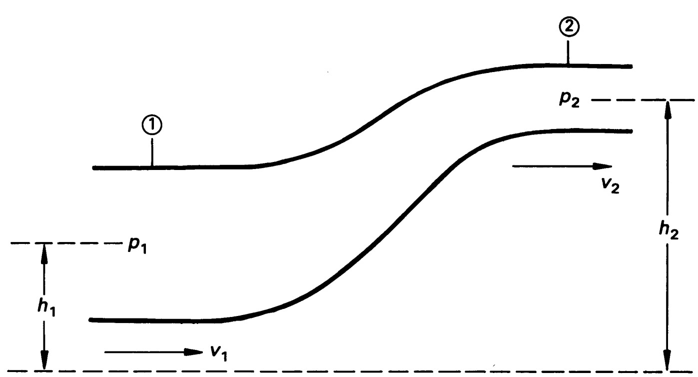

Figure 19.2

Bernoulli's equation tells us that for any small volume anywhere in an ideal liquid the sum of the three parameters involved (pressure, potential energy and kinetic energy) is a constant. In mathematical terms

$$p + \rho gh + \frac{1}{2}\rho v^2 = \text{constant} \quad (19.2)$$

and applying this to Figure 19.2

$$p_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2 \quad (19.3)$$

where ρ is the density of the ideal liquid and g is the acceleration due to gravity.

Bernoulli's equation is essentially a statement of the law of conservation of energy, where each term is expressed as energy per unit volume. If this idea is difficult to grasp, just think about what happens if we multiply both sides of Equation (19.3) by unit volume (1 m^3). The potential and kinetic energy terms contain density rather than mass, but

$$\text{density} \times m^3 = \text{kg m}^{-3} \times m^3 = \text{kg} = \text{mass}$$

which returns them to their more familiar form (mgh and $\frac{1}{2}mv^2$). Furthermore, multiplying pressure by unit volume gives units of energy

$$\text{pressure} \times m^3 = \text{N m}^{-2} \times m^3 = \text{N m} = \text{J} = \text{energy}$$

In theory Bernoulli's equation is valid for all fluids (liquids and gases), provided that they are incompressible and have zero viscosity. Real fluids are, of course, compressible, particularly gases. They are also viscous, particularly liquids, and the mechanical work done in overcoming the viscosity of a liquid appears as heat, with the result that there is a decrease in the quantity ($p + \rho gh + \frac{1}{2}\rho v^2$). Nevertheless Bernoulli's equation is extremely important and has wide applications.

In some cases one or other of the parameters may be eliminated. For instance, p_1 and p_2 are both equal to atmospheric pressure in Worked Example 19.1 (below). In this example we find that the theoretical velocity with which liquid escapes from a small hole in the side of a tank is given by $\sqrt{2gh}$, where the hole is a distance h below the liquid surface (Torricelli's theorem). This turns out to be the same velocity as that which a body reaches by falling freely from rest through the same distance. In Question 1 you are asked to use Bernoulli's equation to derive the expression for the pressure at a given depth below the surface of a stationary liquid (Equation 4.2 on page 32). Provided that the liquid is stationary, then $v_1 = v_2 = 0$.

The potential energy term is eliminated for the case shown in Figure 19.3, where a liquid encounters a constriction in a horizontal pipe through which it is flowing. The vertical tubes are simply mano-

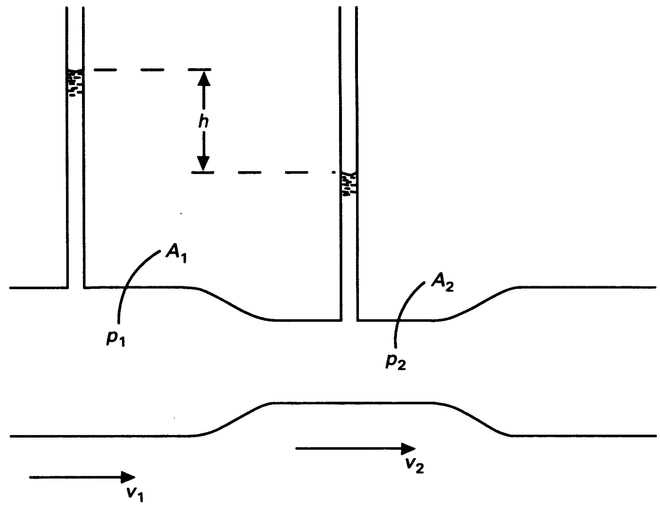

Figure 19.3

meters (see Topic 4) to indicate the gauge pressures p_1 in the pipe and p_2 in the constriction, where the velocity has increased from v_1 to v_2, owing to the reduction in cross-sectional area from A_1 to A_2. Since the pipe is horizontal, the potential energy terms cancel and Equation (19.3) becomes

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \tag{19.4}$$

and, on rearranging,

$$p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) \tag{19.5}$$

Contrary to what we might expect, the manometer levels in the figure show that the pressure falls where the liquid velocity increases in the constriction, and the equation tells us that the greater the velocity change the greater the pressure drop.

This effect has many applications. For instance, an aircraft wing is designed so that the air flow over the top surface is faster than that over the bottom and the resulting pressure difference provides a lift force. (This example departs from our simple model, because air is compressible.)

In Worked Example 19.2, Equation (19.5) is used as the theoretical basis for the Venturi meter, which is a device used for measuring flow rate through a pipe. Worked Example 19.3 uses Equation (19.4) for the Pitot tube, which is a device used for measuring flow velocity.

Worked Example 19.1

Find an expression for the theoretical velocity v with which liquid escapes from a small hole in the side of a large tank at a distance h below the liquid surface.

Let p_1, h_1 and v_1 be the pressure, height and velocity of the liquid at its upper surface, and p_2, h_2 and v_2 the corresponding values where it escapes from the hole at a distance h below (see Figure 19.4).

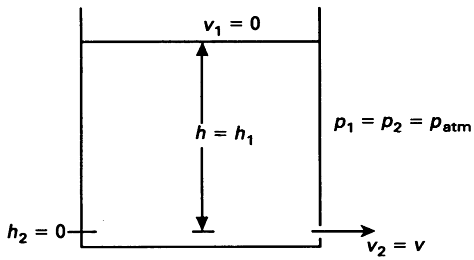

Figure 19.4

The liquid experiences atmospheric pressure p_{atm} at its upper surface and where it emerges from the hole; therefore, $p_1 = p_2 = p_{atm}$.

If we take the level of the hole to be the reference level, then $h_2 = 0$ and $h = h_1$.

If the cross-sectional area of the tank is large compared with that of the hole, then we can assume that the surface level falls at a negligible rate as the liquid escapes; therefore, $v_1 = 0$.

If the liquid density is ρ, then, substituting in Equation (19.3),

$$p_{atm} + \rho g h_1 + 0 = p_{atm} + 0 + \frac{1}{2}\rho v_2^2$$

Therefore,

$$v_2^2 = 2gh_1$$

and, since $v = v_2$ and $h = h_1$,

$$v = \sqrt{2gh}$$

(This is the same as the speed of a body of mass m that has fallen from rest at a height h where its potential energy was mgh; in this case $\frac{1}{2}mv^2 = mgh$.)

Worked Example 19.2

The device illustrated in Figure 19.3 can be used to measure flow rate. Find an expression giving flow rate in terms of the height difference h between the levels in the manometer tubes and the cross-sectional area of the pipe (A_1) and of the constriction (A_2).

From Equation (19.1)

$$A_1v_1 = A_2v_2$$

Therefore,

$$v_1^2 = \frac{A_2^2v_2^2}{A_1^2}$$

and, substituting for v_1^2 in Equation (19.5),

$$p_1 - p_2 = \frac{1}{2}\rho\left(v_2^2 - \frac{A_2^2v_2^2}{A_1^2}\right)$$

and

$$p_1 - p_2 = \frac{1}{2}\rho v_2^2\frac{(A_1^2 - A_2^2)}{A_1^2}$$

which rearranges to give

$$v_2 = A_1\sqrt{\frac{2(p_1 - p_2)}{\rho(A_1^2 - A_2^2)}}$$

Since the flow rate is equal to A_2v_2, then, substituting for v_2 from above,

$$\text{flow rate} = A_2 \times A_1\sqrt{\frac{2(p_1 - p_2)}{\rho(A_1^2 - A_2^2)}}$$

But if h is the height difference between the manometer levels, then

$$(p_1 - p_2) = \rho gh$$

and

$$\frac{(p_1 - p_2)}{\rho} = gh$$

Therefore,

$$\text{flow rate} = A_1A_2\sqrt{\frac{2gh}{(A_1^2 - A_2^2)}}$$

Note that we have assumed an ideal liquid. In practice this device, and the device in Worked Example 19.3 below, would be calibrated (Topic 26) to take into account the effect of friction and other complicating factors associated with real liquids.

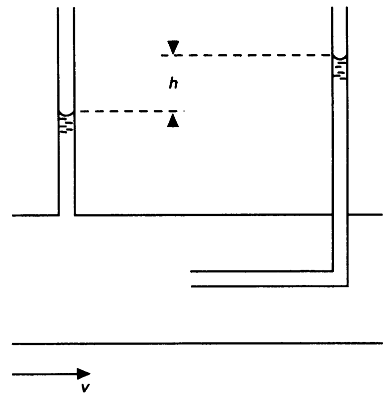

Figure 19.5

Worked Example 19.3

The device illustrated in Figure 19.5 can be used to measure flow velocity through a pipe. Find an expression giving flow velocity v in terms of the height difference h between the levels in the vertical tubes.

The left-hand tube indicates the pressure p_1, where the liquid flows through the pipe with velocity v_1.

The bottom end of the right-hand tube faces into the moving liquid, which is forced up to a height h above the level in the other tube. The liquid at the bottom end is stationary; therefore, $v_2 = 0$ with a corresponding pressure p_2.

Since the liquid flow through the pipe is horizontal, we can ignore the potential energy term in Bernoulli's equation and, from Equation (19.4),

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + 0$$

and

$$\frac{1}{2}\rho v_1^2 = p_2 - p_1$$

But

$$p_2 - p_1 = \rho g h$$

Therefore,

$$\frac{1}{2}\rho v_1^2 = \rho g h$$

and

$$v = v_1 = \sqrt{2gh}$$

19.2 REAL (VISCOUS) LIQUIDS

Until now, the only property of a fluid that we have considered is its density. Now we have reached the point where we need to take viscosity into account.

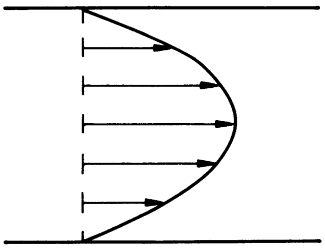

Figure 19.6

Viscosity is the frictionlike resistance to flow that arises from the cohesive forces between the constituent particles in a liquid. If the temperature is raised, the thermal energy of the constituent particles is increased and relative movement between them becomes easier. Thus, the viscosity of most liquids decreases with temperature.

In the absence of viscosity the velocity of a liquid flowing through a pipe would be uniform over the entire cross-section. For a real viscous liquid the velocity profile looks something like that in Figure 19.6, where the length of the arrows represents speed. The liquid molecules in immediate contact with the pipe adhere to its surface and tend to remain stationary. The stationary layer provides resistance to the movement of the layer of molecules adjacent to it and this in turn provides resistance to the movement of the next layer, and so on, giving a velocity profile similar to that in the figure. (Note that the work done in overcoming viscosity ultimately raises the total kinetic energy of the molecules and therefore appears as heat.)

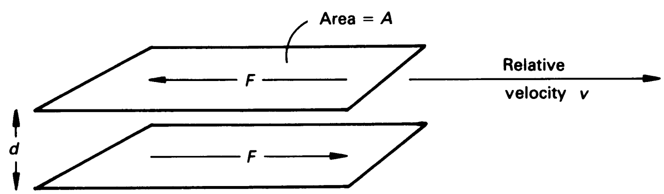

Figure 19.7

Viscosity can be quantified in terms of the friction-like force acting between adjacent layers of liquid moving relative to one another. In Figure 19.7 both layers are moving from left to right but the upper layer moves faster, so that their relative velocity v is equal to $(v_{\text{upper}} - v_{\text{lower}})$. The lower layer therefore exerts a retarding force F on the upper layer and experiences an equal and opposite reaction force in the forward direction. F is an example of a shear force and the resulting shear stress is given by F/A, where A is the area of contact between the layers. (We shall look at shear in more detail in the next topic; for the time being, just think of shear forces as acting parallel to a plane, unlike tensile and compressive forces, which act perpendicularly to it.) If d is the distance between the two layers, then for the purposes of our discussion we can say that, for many liquids, F is given by

$$F = \frac{\eta A v}{d} \tag{19.6}$$

where η is the *coefficient of dynamic viscosity*, often simply called *viscosity*. The SI unit of viscosity is pascal seconds, Pa s (i.e. N s m^{-2}). (Viscosity is sometimes measured in non-SI units called poises, where

10 poises = 1 Pa s.) Some approximate values of η at 20 °C are given in Table 19.1.

Table 19.1

Substance	Viscosity/Pa s at 20 °C
Air	1.8×10^{-5}
Water	1.0×10^{-3}
Machine oil	$1 - 6 \times 10^{-1}$

From Equation (19.6) we see that, for the liquids to which it applies, the shear stress F/A is proportional to the velocity gradient v/d and η is the constant of proportionality; thus, a large shear stress coupled with a small velocity gradient indicates a liquid of high viscosity (for instance, machine oil compared with water).

There are two important laws governing viscous flow. *Poiseuille's formula* gives the flow rate through a cylindrical pipe. If the pipe has an internal radius r and a length l, and the pressure difference between its ends is $(p_2 - p_1)$, then the volume flow rate V of a fluid of viscosity η is given by

$$V = \frac{\pi r^4}{8\eta}\left(\frac{p_2 - p_1}{l}\right) \tag{19.7}$$

Thus, flow rate is proportional to the pressure gradient $(p_2 - p_1)/l$ and inversely proportional to the viscosity of the fluid. Somewhat more surprisingly, it is also proportional to r^4, so, all other things being equal, doubling the radius of the pipe increases the flow rate sixteenfold! Another way of looking at the equation is to say that flow rate is proportional to the pressure difference $(p_2 - p_1)$ and inversely proportional to the resistance to flow as given by $8\eta l/\pi r^4$.

Before moving on to the other important law, we should note that the type of flow under given conditions can be predicted with the aid of the *Reynolds number*, *Re*. This is a dimensionless quantity given by

$$Re = \frac{v\rho l}{\eta} \tag{19.8}$$

where v, ρ and η are the velocity, density and viscosity of the fluid and l is a linear dimension that is characteristic of the system. If, for a straight uniform pipe, we take v as equal to the volume flow rate divided by the cross-sectional area and l as equal to the internal diameter, then we can normally expect laminar flow if the value of *Re* is below about 2000 and turbulent flow if it is above about 4000. Either may be possible between 2000 and 4000. Thus, Equation (19.8) tells us that laminar flow is favoured by low values of v, ρ and l and by high viscosity.

The second important law is *Stokes's law*. This gives the viscous resistive force F that acts on a sphere of radius r moving with velocity v through a fluid of viscosity η as follows:

$$F = 6\pi\eta r v \qquad (19.9)$$

Thus, the resistance experienced by the sphere is proportional to η, r and v. Worked Example 19.4 (below) shows how Stokes's law can be used to determine the viscosity of a liquid by measuring the terminal velocity of a sphere falling through it under gravity (e.g. a small ball bearing falling through oil); note that for Stokes's law to agree closely with experimental results Re should be less than 0.1, where l is taken to be the diameter of the sphere and v its terminal velocity. Furthermore, the liquid container needs to be large, so that the walls and the bottom have an insignificant effect on the velocity of the sphere.

Worked Example 19.4

Assuming Stokes's law, find an expression whereby the viscosity η of a fluid of density ρ_f may be obtained from the terminal velocity v_t of a sphere of radius r and density ρ_s falling through it under gravity.

The downward force acting on the sphere (i.e. its weight) is opposed by the upthrust due to the fluid it displaces and by the viscous resistive force it experiences. When the sphere has reached its terminal velocity v_t, where its acceleration is zero (Topic 5), then the net downward force acting on it is zero (because $F = ma = m \times 0$). Now

$$\text{mass of sphere} = \text{volume} \times \text{density} = \frac{4}{3}\pi r^3 \times \rho_s$$

and

$$\text{mass of displaced fluid} = \frac{4}{3}\pi r^3 \times \rho_f$$

and, from Equation (19.9),

$$\text{viscous resistive force} = 6\pi\eta r v$$

At terminal velocity v_t the net downward force is equal to

$$\text{weight of sphere} - \text{upthrust} - \text{resistive force} = 0$$

Therefore,

$$\frac{4}{3}\pi r^3 \rho_s g - \frac{4}{3}\pi r^3 \rho_f g - 6\pi\eta r v_t = 0$$

which rearranges to

$$\eta = \frac{2r^2 (\rho_s - \rho_f)g}{9v_t}$$

19.3 SURFACE TENSION

Surface tension is the property that makes a liquid behave as though it has an elastic skin. It is the reason why water forms drops and why its surface can support small objects such as sewing needles whose density would otherwise cause them to sink.

Surface tension has a molecular basis. A molecule in the body of a liquid is completely surrounded by neighbours and therefore experiences attractive forces more or less uniformly in all directions. But a surface molecule experiences a net attractive force into the body of the liquid, since it only has neighbours on that side (apart from a few in the surrounding vapour). Because the surface molecules tend to be pulled inwards, the liquid will tend to adjust its shape to minimise its surface area. Since the sphere is the geometric shape with the smallest surface area for a given volume, liquids tend to form spherical drops, although factors such as gravity and the effect of other surfaces normally prevent this.

Surface tension is given the symbol γ and can be defined as the force in the liquid surface acting perpendicularly to a line of unit length lying in the same plane. The units are therefore N m^{-1} (and for water at ordinary temperatures γ is about 0.073 N m^{-1}). To make this idea clearer, Figure 19.8 shows a film of liquid, such as a soap solution, stretched across a wire frame which has one side, of length l, that can be moved without any frictional resistance. Because the film tries to retract in order to reduce its surface area, a force F must be applied to the movable side to hold it stationary. By our definition above $F = 2l \times \gamma$. (The factor 2 is necessary because the film has two surfaces, an upper and a lower, both of which are pulling on the frame.)

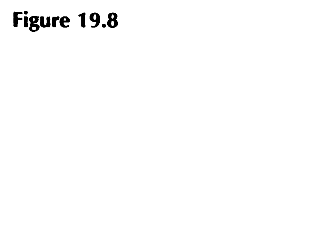

Figure 19.8

If we stretch the film by pulling the movable side against the surface tension through a distance d, then the work done, equal to $2l\gamma \times d$, is stored as *surface energy*. Surface energy is given the symbol σ and has units of J m^{-2}. In terms of Figure 19.8, the area of the newly created surface is $2ld$; therefore,

$$\sigma = \frac{2l\gamma d}{2ld} = \gamma$$

so γ and σ have the same numerical value. (Note that σ = J m^{-2} = N m m^{-2} = N m^{-1} = γ.)

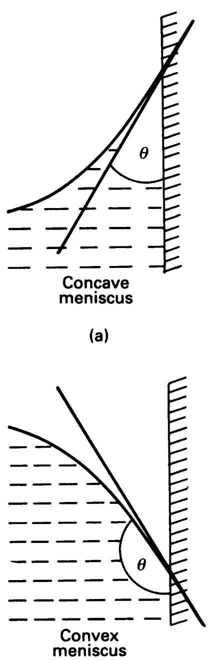

Figure 19.9

When a liquid comes into contact with a solid, forces of attraction between the liquid molecules and the solid surface cause adhesion between the two. A detailed discussion is beyond the scope of this book, but we need a broad picture of some of the ideas involved.

The adhesion between water and clean glass is much greater than the cohesion between water molecules themselves. Water therefore tends to form as large an interface with glass as possible, and spreads out over its surface and wets it. If the glass is vertical, then the water will tend to climb up it and form a concave meniscus similar to that in Figure 19.9(a). By contrast, the adhesion between mercury and clean glass is smaller than the cohesion in liquid mercury. Mercury therefore forms discrete drops on a horizontal glass surface, and a convex meniscus on a vertical one, as in Figure 19.9(b), in order to minimise the area of its interface with the glass.

The *angle of contact*, θ in the figure, is the angle measured through the liquid between the liquid and solid surfaces where they meet. The smaller the value of θ the greater the tendency of the liquid to wet the solid surface. In fact, θ is zero for water on clean glass and about 140° for mercury on clean glass.

Wetting agents are used to reduce θ for many purposes: for example, fluxes are used to promote the wetting of metal surfaces with molten solder. Surfaces are sometimes waterproofed by treating them with substances to increase their contact angle with water. Capillarity, which causes liquids to rise in narrow tubes where $\theta < 90°$, is due to surface tension.

Questions

(Use any previously tabulated data as required. $\rho_{water} = 1000$ kg m^{-3}. $g = 9.8$ m s^{-2}.)

1. The gauge pressure p at a depth h below the surface of a stationary liquid of density ρ is given by the expression $p = \rho gh$, where g is the acceleration due to gravity. Derive this expression from Bernoulli's equation.

2. A large steel tank floating in water has a 24 mm diameter hole sealed with a plug 250 mm below the water line. Estimate the initial flow rate into the tank if the plug is removed.

3. Express the flow rate from the previous question in terms of molecules per minute.
   (Avogadro constant = 6.02×10^{23} mol^{-1}.)

4. By consideration of the units on their respective right-hand sides confirm that (a) Equation (19.7) gives a volume flow rate, (b) Equation (19.8) gives a dimensionless number and (c) Equation (19.9) gives a force.

5. An oil drop of density 900 kg m^{-3} and of radius 2.5×10^{-6} m fell a distance of 10.0 mm through air in 14.7 s. Estimate the viscosity of the air, assuming that its density may be ignored.

6. An open-ended horizontal tube 500 mm long with an internal diameter of 2 mm was sealed into the bottom of a water tank. 139 cm^3 of water flowed from the tube over the course of 1 min. (a) Check whether the water would have experienced laminar flow through the tube and (b) estimate the depth of water in the tank.

7. A workman accidentally drilled a horizontal hole in a water pipe 3.29 m above ground level. The escaping water travelled a horizontal distance 4.92 m before hitting the ground. Find the gauge pressure in the pipe.

8. A clean rectangular glass plate, measuring 75 × 16 mm and 1.9 mm thick, is suspended so that its long edges are horizontal, its faces are vertical and it just makes contact with a horizontal water surface. Assuming that $\gamma_{water} = 0.073$ N m^{-1}, estimate the force due to surface tension that must be overcome to separate the plate from the water.

9. Estimate the energy required to divide a 2 mm diameter raindrop into ten million identical droplets. (For a sphere of radius r, surface area = $4\pi r^2$ and volume = $4\pi r^3/3$. Assume $\gamma_{water} = 0.073$ N m^{-1}.)

TOPIC 20 SOLIDS

COVERING:

- elastic deformation and modulus of elasticity;
- stress/strain relationships;
- plastic deformation;
- brittle behaviour.

We have already seen that the constituent atoms, ions or molecules in a solid material are trapped between their neighbours because they have insufficient thermal energy to escape. They are confined to fixed positions on a crystal lattice, held by a network of cohesive forces that tends to oppose any attempt to deform it, and it is this that gives crystalline solids their characteristic rigidity. We shall begin by considering elasticity, which is the property of a solid that tends to return it to its original dimensions when it has been deformed.

20.1 ELASTIC DEFORMATION

We shall start with a model solid in which the constituent atoms are held together by chemical bonds represented by the net force/separation curve in Figure 20.1. In the absence of external forces the atoms will adopt the equilibrium separation r_0, where the attractive and repulsive components in the bond are balanced (see Topic 15). If we apply an external force to pull the atoms apart or push them together, then an opposing force of equal magnitude will be generated within the bond by the imbalance between the attractive and repulsive components as we move up or down the net force/separation curve. At small displacements the net force/separation curve is virtually linear; hence, it provides the basis of Hooke's law, as we saw in Topic 15 (see page 136). If the external force is removed, then the atoms will return to their equilibrium separation r_0.

Figure 20.2 represents the mechanical deformation of a model material. In tension and compression equal and opposite forces act along the same line and the length of the specimen changes accordingly. In shear (to the right of Figure 20.2) the forces are out of alignment and this results in a twisted deformation of the specimen.

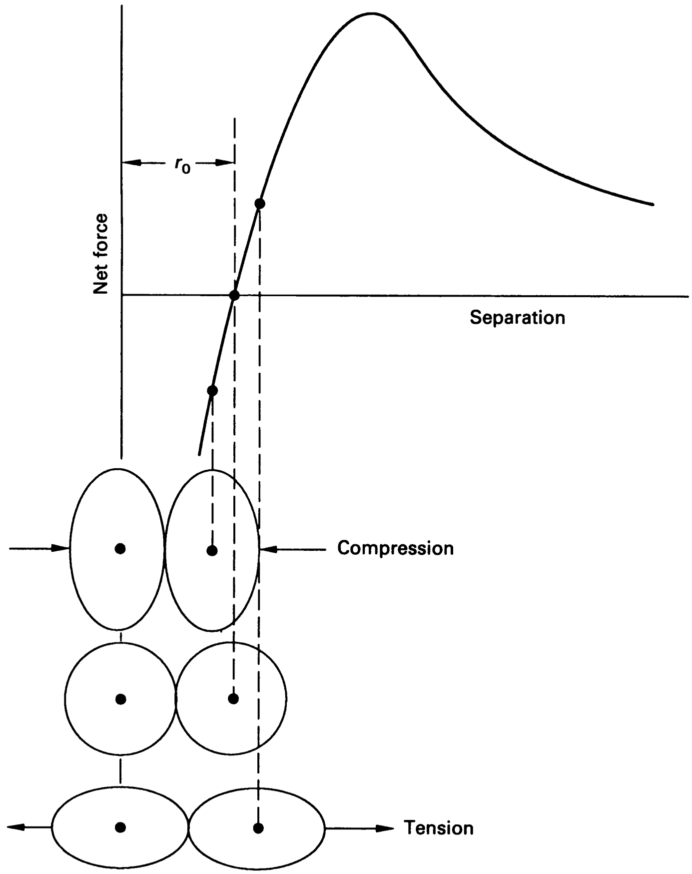

Figure 20.1

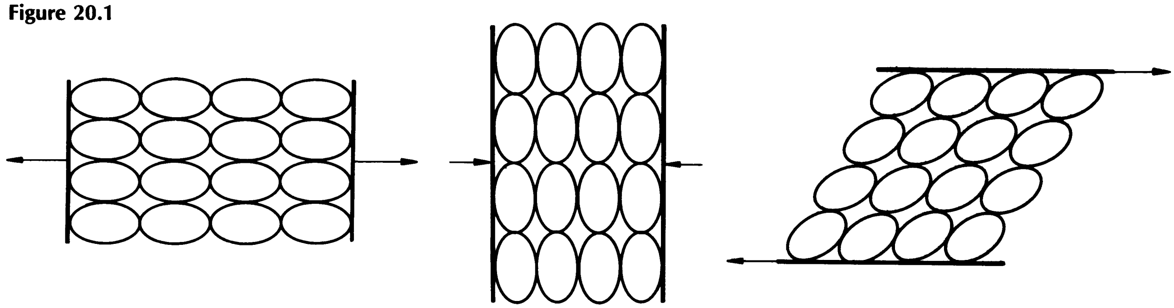

Figure 20.2

(Note that in this case, as the figure stands, additional forces that are needed to prevent rotation have been omitted for simplicity.)

In Topic 2 we found the *Young's modulus* of a material as follows:

$$E = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta l/l_o} = \frac{\sigma}{\varepsilon} \tag{20.1}$$

where Δl represents the extension and l_o the undeformed length, as indicated in Figure 20.3(a). Some approximate Young's modulus values are given in Table 20.1. (Remember that 1 GPa = 1×10^9 N m^{-2}.)

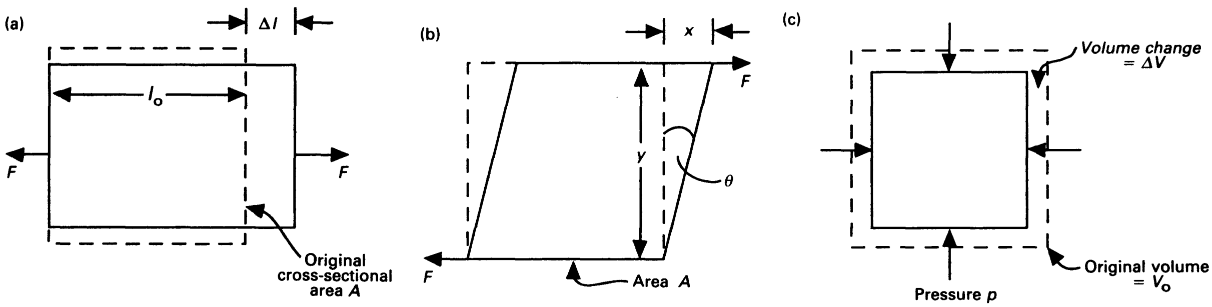

Figure 20.3

Table 20.1

Material	Young's modulus/GPa
Aluminium	70
Brass	110
Copper	110
Glass	70
Nylon	3
Steel	200

As Figure 20.3(a) suggests, materials stretched under tension tend to become thinner (fatter under compression) and in most cases there is a change in volume. The transverse strain ε_t accompanying the longitudinal strain ε_l is given by the Poisson's ratio of the material, μ, where

$$\mu = -\frac{\varepsilon_t}{\varepsilon_l} \tag{20.2}$$

The negative sign is needed because a positive longitudinal strain (extension) gives a negative transverse strain (contraction), and vice versa.

Now let us consider the equivalent of Young's modulus in shear. In Figure 20.3(b) the shear stress (symbol τ) is given by F/A, where F is the tangential force along the plane of area A over which it is applied. The angle of shear θ is a measure of the shear strain and $x/y = \tan\theta = \theta$ radians if θ is small. Thus, at small strains the *shear modulus G* is given by

$$G = \frac{\tau}{\theta} \tag{20.3}$$

(If we think of G as F/A divided by x/y, then we have a parallel with viscosity (see Equation 19.6 on page 184).)

The third type of deformation, shown in Figure 20.3(c), is the change in volume ΔV which occurs when an object is subjected to hydrostatic pressure — for example, at the bottom of the sea. In this case the stress is the pressure p. (Remember that pressure is the normal force per unit area acting on a surface (Topic 4).) The volume strain is the fractional volume change $\Delta V/V_o$, where V_o is the original volume. In this case the modulus is the *bulk modulus* K, given by

$$K = -\frac{p}{\Delta V/V_o} \tag{20.4}$$

The negative sign is needed because a positive pressure change gives a negative volume change, and vice versa.

Note that strain is a dimensionless quantity, so all three moduli have the units of stress, N m^{-2} (or Pa).

For isotropic materials (i.e. those with uniform properties in all directions) the elastic moduli and Poisson's ratio are related by

$$G = \frac{E}{2(1 + \mu)} \tag{20.5}$$

and

$$K = \frac{E}{3(1 - 2\mu)} \tag{20.6}$$

Many solids have a value for Poisson's ratio of between about $\frac{1}{3}$ and $\frac{1}{4}$, which, when substituted in the equations above, gives values for G of about $0.4E$ and for K between about $0.7E$ and $1.0E$.

Note that most solids behave elastically at only very low strains, generally less than 1%, above which they either break in a brittle fashion or deform plastically before fracture. Rubber is exceptional in that it remains elastic up to very large strains, sometimes several hundred per cent, although the stress/strain relationship is not linear, as we shall see later.

Figure 2.1 (page 14) shows the load/extension plot for a particular wire that happened to be 1.72 m long and 0.40 mm in diameter. A more fundamental approach would be to present the stress/strain plot for the material from which the wire was made. In the worked example below, the original load/extension data for the wire is reprocessed and stress is plotted against strain. (Note that strain can be represented as a straightforward ratio, but it is often expressed as a percentage by multiplying it by 100. Since strains in real structures are often very small, engineers sometimes prefer to work in *microstrain*, which is strain multiplied by a million. Thus, an extension of

1 mm in 1 m can be expressed as 0.001 strain, 0.1% strain or 1000 microstrain.)

Worked Example 20.1

The following load/extension data was obtained for a wire 1.72 m long and 0.40 mm in diameter. Convert the data into stress/strain form and plot it to obtain the Young's modulus for the material from which the wire was made.

Load/N	10	20	30	40	50	60
Extension/mm	0.7	1.5	2.1	2.9	3.6	4.3

Table 20.2

Load/N	Extension/m $\times 10^{-3}$	Stress/MPa	Strain/$\times 10^{-3}$
10	0.7	80	0.41
20	1.5	159	0.87
30	2.1	239	1.22
40	2.9	318	1.69
50	3.6	398	2.09
60	4.3	477	2.50

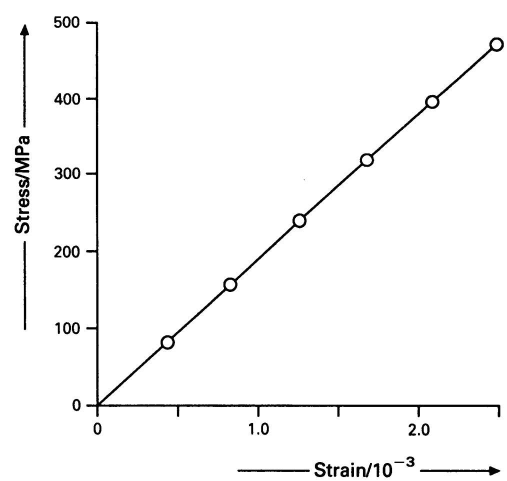

Figure 20.4

Table 20.2 shows load divided by the original cross-sectional area ($\pi \times 0.0002^2$) to give tensile stress, and extension divided by the original length to give strain. Figure 20.4 shows stress plotted against strain. Since the graph is a straight line and passes through the origin, then, from the figure, estimating the stress to be 480 MPa at a strain of 2.5×10^{-3},

$$E = \frac{480 \times 10^6}{2.5 \times 10^{-3}} = 190 \text{ GPa}$$

Figure 20.5 shows a plot that was obtained in basically the same way as Figure 20.4 but, in this case, an ordinary rubber band was used instead of a metal wire. It is clear that it does not obey Hooke's law. Furthermore, we can see from the scales on the two figures (and, of course, we know from experience) that a rubber band stretches very much more easily than a metal wire. Clearly the mechanism of rubber-like elasticity is fundamentally different.

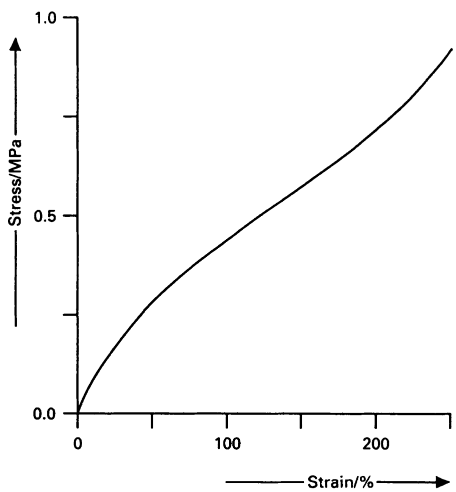

Figure 20.5

20.2 RUBBER-LIKE ELASTICITY

Rubber consists of enormously long chain-like molecules which are randomly tangled together in a disordered mass rather like a plate of spaghetti. At normal temperatures there is sufficient thermal energy for rotation to occur about the covalent bonds along the length of the chains (see Topic 15). The result of this is that the chains are in a constant state of random wriggling motion, continuously changing their shape. On application of an external force to stretch the rubber, the chains tend to straighten by untwisting through bond rotation so that they are brought into partial alignment. On releasing the external force, the chains wriggle back into their disorganised and more compact configurations so that the rubber as a whole retracts. Deformation by bond rotation processes of this kind is much greater, and much smaller forces are involved, than in the bond distortion mechanism we discussed earlier (Figure 20.1).

20.3 PLASTIC DEFORMATION

All solid materials subjected to a continuously increasing tensile stress break sooner or later. But before they do, many of them undergo a significant amount of *plastic deformation*, which is a permanent deformation that results from internal structural changes. For instance, many metals stretch elastically, so that, on unloading, they return to their original lengths, but if they have been stretched too far, they are left with a permanent length increase. In other words, they have an elastic limit corresponding to a stress level above which some of the strain is permanent.

The detailed mechanisms of plastic deformation for different materials are complex. Nevertheless, in view of its importance in engineering, we shall look at it briefly in terms of a very simple atomic model of a metal. This is based on the fact that adjacent planes of atoms in a metallic crystal structure can slip over one another under the influence of stress. This is illustrated in idealised form in Figure 20.6(a). Provided that the stress is large enough to move each atom over its neighbour in the adjacent plane (Figure 20.6b), then there is nothing in the nature of the metallic bond to prevent this process from continuing step by step. (The existence of such planes is implied in Figure 15.7 on page 140.) Bonding between the planes is not interrupted, so the force required is less than that needed to pull them apart completely. Our simple model therefore suggests that metals tend to deform plastically in this way rather then snap in a brittle fashion like glass. We cannot, of course, continue stretching the specimen indefinitely. Sooner or later, depending on the material, it will break, although the fracture processes involved are complex and beyond the scope of our discussion.

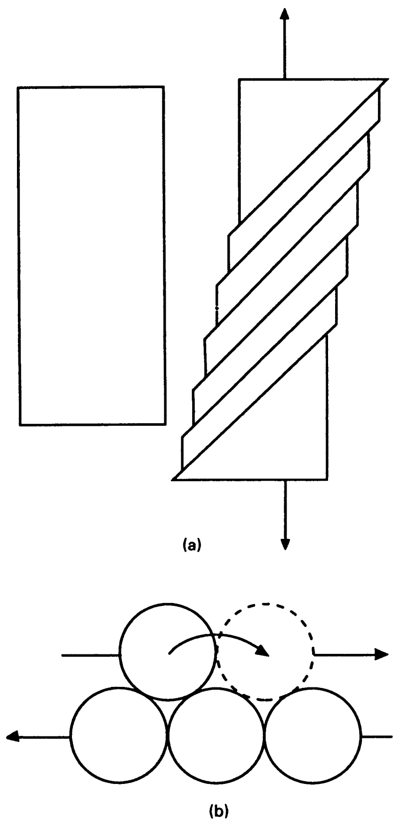

Figure 20.6

20.4 BRITTLE FRACTURE

Brittle materials, like glass, are characterised by their inability to undergo plastic deformation. If a brittle material is fractured, the broken pieces can often be glued back together, so the repaired object is more or less restored to its original dimensions.

This is consistent with our simple two-atom model of elasticity. If we apply stress to a material, it responds by deforming elastically. If the stress is increased, the chemical bonds are progressively deformed until those that are the most highly stressed cannot support the load and the material breaks. The bonds within the broken pieces are no longer subjected to external stress and they return to their equilibrium positions. We can therefore reconstruct the original object by glueing the pieces back together.

Ceramics are non-metallic inorganic materials that include, for example, glass and fired clay products. They often contain compounds of metals and non-metals and they are inclined to be brittle because the ionic–covalent bonding on which they generally depend tends to resist the slip processes that readily occur in metals. At one end of the scale the covalent bond is rigid, directional and specific between the bonded atoms. And at the other end, as Figure 20.7 indicates, relative movement between the planes in an ionic structure as shown would bring positive ions into contact with positive ions and negative with negative. This would lead to strong repulsive forces causing the planes to separate rather than slip over one another.

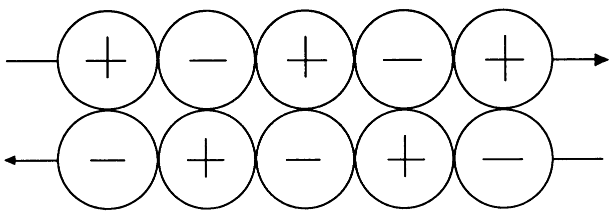

Figure 20.7

Glasses are *amorphous solids* — that is to say, they are non-crystalline, as suggested in Figure 20.8. Like a liquid, their structure has no long-range regularity and such materials are sometimes called *supercooled liquids*.

Rubber only exhibits its special elastic behaviour above its *glass transition temperature*. Below this, there is insufficient thermal energy for bond rotation and the chains are frozen into fixed configurations. The material loses its rubbery characteristics and behaves in a similar way to glass. A number of hard, brittle plastics are based on chain-like molecular structures which have their glass transition temperatures above room temperature.

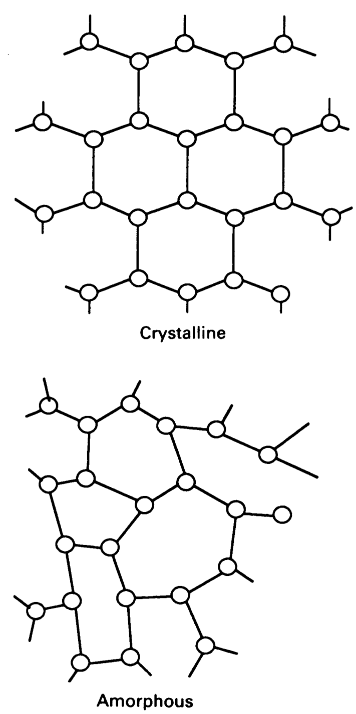

Figure 20.8

Questions

(Use any previously tabulated data as required and assume that Hooke's law is obeyed. Assume that g = 9.8 m s^{-2} and that Poisson's ratio for steel is 0.3.)

1. A 1.5 mm diameter wire supports a 9 kg load. A column of 1.4 m square cross-section supports a 10×10^6 kg load. Find the longitudinal stress in each case.
2. An 8 kg load is suspended by a steel wire originally 1 mm in diameter and 5 m in length before loading.

Find (a) the tensile stress in the wire and (b) by how much the load has stretched it.

3. Find the tensile stress in a steel rod originally 2 m long that has been stretched by 1 mm.
4. A 5.0 kg mass is to be suspended from a 2.0 mm diameter wire or thread, 3.0 m long, made from either (a) brass, (b) steel or (c) nylon. Predict the extension in mm in each case.
5. A copper wire has a diameter of 2 mm and a length of 6 m. An aluminium wire has a rectangular cross-section 1.5 $\times$ 2.5 mm and a length of 7.5 m. Find the total extension if the wires are joined end to end and used to suspend a 10 kg load.
6. A copper wire 2.2 m long and an aluminium wire 1.4 m long, both 2 mm in diameter, are joined end to end. What tensile load would produce a total extension of 1 mm?
7. An unstressed steel strip is 100.00 mm in width. Find the width of the strip when it is subjected to a longitudinal tensile stress of 400 MPa.
8. Find the percentage volume change of a steel cylinder after it has been subjected to a longitudinal tensile stress of 400 MPa.
9. A solid steel object sinks to a depth of 5 km below the surface of the sea. Estimate the percentage volume change it experiences, assuming the density of sea-water is 1030 kg m^{-3}.
10. A bar made from an unidentified material is initially 100.000 mm in diameter and 1000.00 mm long. When subjected to a longitudinal tensile force of 3.927 MN, its diameter becomes 99.925 mm and its length 1002.50 mm. Estimate (a) the shear modulus and (b) the bulk modulus of the material.

Part 3
Electricity and Magnetism

TOPIC 21 ELECTRIC CHARGE

COVERING:

- the nature of electric charge;
- force between electric charges (Coulomb's law);
- electrostatic induction.

Electricity and magnetism both stem from electric charge.

Charge, like mass, is a fundamental concept that lies at the limit of our absolute understanding of the physical world. We do not know precisely what it is, but we can describe its properties in terms of the effects it produces.

We can deliberately charge certain objects, such as plastic combs or pens, by rubbing them with a cloth so that they attract scraps of paper or even a thin stream of water running from a tap. This demonstrates a fundamental similarity between charge and mass in that they both give rise to forces.

In Topic 14 we saw that the electron and the proton carry equal but opposite charges (negative and positive, respectively) of 1.60×10^{-19} C. It is helpful to bear in mind that, no matter how we choose to define charge, these two particles form its natural units. Matter normally contains electrons and protons in more or less equal numbers and is therefore generally electrically neutral. Charge only becomes apparent when electrons and protons become separated from one another — by friction, for example.

If we rub certain objects with a cloth, then they become charged, either positively or negatively, depending on the materials involved. This can be explained in terms of the transfer of electrons from the cloth to the object, or vice versa, when they are rubbed together. The loss of electrons from one to the other leaves a positive charge due to the unbalanced protons left behind.

It is important to note the *principle of conservation of charge*. Although positive and negative charges may be separated as above, or indeed combined to neutralise each other, the net charge within any particular system remains the same.

We should also note that electric current is simply a flow of charge and that it is the freedom of movement of valence electrons through metals (see Topic 15) that makes them good electrical conductors. By contrast, ideal electrical insulators do not conduct electricity, because they possess no free charge-carriers. For example, the valence electrons in solid covalent and ionic substances are localised within their respective bonds or ions and are not free to carry current. It should be noted, however, that *electrolytes* contain ionic substances, either molten or in solution, that enable them to conduct electricity because of the freedom of movement of the ions. We shall see later (Topic 24) that *semiconductors* are intermediate between insulators and conductors in their electrical conductivity. (In practice, real insulators do allow some very slight movement of charge.)

In this topic we shall confine our discussion to *electrostatics* — that is to say, the study of electric charge at rest.

21.1 COULOMB'S LAW

The force between electric charges was investigated in the eighteenth century by the French scientist Coulomb. In Topic 14 we briefly met the law named after him. This is expressed by the equation

$$F = \frac{1}{4\pi\,\varepsilon_0} \times \frac{Q_1Q_2}{r^2} \tag{21.1}$$

F is the magnitude of the force in newtons between two charges and is repulsive or attractive depending on whether they are of like or opposite sign. Q_1 and Q_2 are the magnitudes of the charges in coulombs, r is the distance in metres between them and ε_0 is a constant with the value $8.85 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$. (The units for ε_0 are obtained here by consideration of Equation 21.1 but it is more usual to express them rather differently, as we shall see in Topic 23.) ε_0 is called the *absolute permittivity of free space* or the *electric constant*.

Equation (21.1) only applies under vacuum conditions; the presence of a *dielectric* or non-conducting substance between the charges reduces the magnitude of the force and the equation becomes

$$F = \frac{1}{4\pi\,\varepsilon} \times \frac{Q_1\,Q_2}{r^2} \tag{21.2}$$

where ε is the *absolute permittivity* of the substance (often simply called the *permittivity*). The *relative permittivity* ε_r, sometimes called the *dielectric constant*, is given by the ratio between the absolute permittivity of the substance and that of free space, so that $\varepsilon_r = \varepsilon/\varepsilon_0$. Table 21.1 gives some typical approximate relative permittivity values. For many practical purposes, air may be considered to have the same permittivity as that of free space. We shall discuss dielectrics in more detail in Topic 23.

Coulomb's law is an inverse square law, like Newton's law of gravitation. The electric force acts along the straight line between the two charges and is a vector quantity like gravitational force. How-

Table 21.1

Substance	Relative permittivity
Air	1.00
Polythene	2.3
Glass	4–7
Water	80

ever, there is an important difference in that gravitational force is always attractive, whereas electric force can be either attractive or repulsive.

Before examining electric forces any further, let us put the magnitude of the coulomb into perspective. At the atomic level we know that an electron carries a charge of 1.60×10^{-19} C. This means that we need 6.25×10^{18} electrons (i.e. about 1×10^{-5} mol) to provide a total negative charge of one coulomb. In due course we shall see that the ampere is the basic unit used to measure electric current (Topic 24). One coulomb is the amount of charge that is transported past any point in a conductor in one second by a current of one ampere. In everyday terms, one ampere would be the current drawn by a 240 volt lighting circuit with four 60 watt light bulbs. On this basis a coulomb seems to be a fairly modest amount of charge.

Now let us consider the repulsive force between two like charges of one coulomb each if, for instance, we placed them 25 mm apart in air. Letting $Q_1 = Q_2 = 1$ C and $r = 0.025$ m in Equation (21.1),

$$F = \frac{1}{4\pi \times 8.85 \times 10^{-12}} \times \frac{1 \times 1}{(0.025)^2} = 1.44 \times 10^{13}\ \text{N}$$

A medium-sized apple weighs about 1 N, so the force between the charges is equivalent to the weight of over fourteen million million apples. This extraordinary figure tells us that, although the coulomb represents a modest charge in terms of electric current, it is a very large charge in terms of electrostatics. In practice, the magnitude of ordinary electrostatic charges is rarely more than a tiny fraction of a coulomb, although thousands of volts may be involved.

You will find that Question 3 makes the interesting point that, on an atomic scale, the gravitational force between electrons and protons is negligible compared with the electrical force. Gravitational forces become important where very large masses, such as bodies on an astronomical scale, are involved.

21.2 ELECTROSTATIC INDUCTION

So far we have tended to think in terms of point charges of negligible volume, as opposed to charged objects on the human scale. The

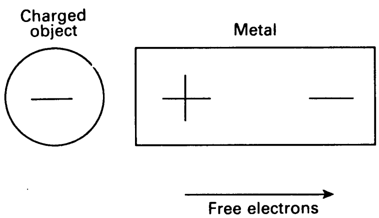

Figure 21.1

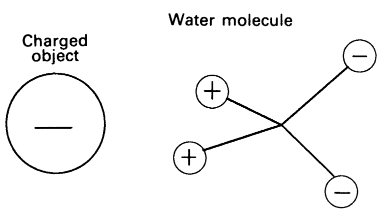

Figure 21.2

charge on an insulator rubbed with a cloth would tend to be distributed over the area covered by the cloth.

In the case of metals, with their free valence electrons, it is possible to 'induce' a temporary unevenness in the charge distribution. If a piece of metal is brought close to a charged object, then an attractive force will arise between them. The explanation is simple. If the object is negatively charged, then it will repel the free electrons towards the far side of the metal, as in Figure 21.1. This will leave an equivalent surplus of protons behind. Thus, the metal nearest the charged object has a net positive charge, resulting in an attractive force between the two. If the object had been positively charged, then a surplus of delocalised electrons would have been attracted towards it, making the nearer side of the metal negative — again leading to an attractive force.

Electrostatic induction, as this way of inducing charge separation is called, is also responsible for the attractive force that pulls a thin stream of water towards a charged object. The water molecule is polar, as we saw in Topic 15, with two positively and two negatively biased orbitals. Any charged object nearby will tend to attract the orbitals of opposite charge, and repel those of like charge, so that the molecules will tend to orientate themselves accordingly, as in Figure 21.2. This, in turn, will give rise to an attractive force between the charged object and the water. Similar electrostatic induction effects cause scraps of paper and hair to be attracted by charged objects.

Questions

(Use any previously tabulated data as required. Assume that $\varepsilon_0 = 8.85 \times 10^{-12}$ C^2 N^{-1} m^{-2}.)

1. (What happens to the force between two separate charges if the distance between them is (a) halved, (b) trebled?
2. Two positive charges of equal magnitude experience a repulsive force of 0.133 N between them when they are separated by a distance of 13 mm in air.
   (a) What is the magnitude of each charge?
   (b) What is the magnitude of the force if the charges are moved to a distance of 65 mm apart?
   (c) What is the magnitude of the force if the charges are separated by a 14 mm thickness of polythene?
3. Assuming that the hydrogen atom consists of an electron orbiting around a proton at a distance of 5.29×10^{-11} m, find (a) the electric force and (b) the gravitational force between them. (See Table 14.1 on page 123 and Equation 2.1 on page 12, and assume that $G = 6.7 \times 10^{-11}$ N m^2 kg^{-2}.)

4. Two identical charges, each of 15 g mass and negligible volume, are both suspended from the same point by non-conducting weightless strings 1500 mm in length. Find the magnitude of each charge if the angle between the strings is 5°. (g = 9.8 m s^{-2}.)

TOPIC 22 ELECTRIC FIELD

COVERING:

- electric field strength;
- the analogy with gravitational field;
- field lines;
- uniform and non-uniform fields;
- potential and potential difference.

In the previous topic we saw that forces exist between electric charges. It follows that a charge must in some way influence the space around itself. This property can be described in terms of an *electric field* surrounding the charge. Where two or more charges are involved, their fields interact with one another to produce forces.

The idea of an electric field is nothing more than an imaginary device to help us picture how charges behave. The same idea is used where other types of force operate at a distance — for instance, gravitational force and, as we shall see later, magnetic force. Nuclear scientists think in terms of fields within the nucleus which are responsible for the forces that hold protons and neutrons together. Thus, the effects of field forces range in scale from the nuclear to the astronomical.

22.1 FIELD STRENGTH

Newton's law of gravitation (Topic 2) tells us that there is an attractive force between any two objects by virtue of their mass — between the earth and an apple, for example. For convenience we tend to think of the earth as having a gravitational field which causes any other mass close to its surface to experience a weight of 9.8 N kg^{-1}. We can therefore say that the earth has a gravitational field strength of 9.8 N kg^{-1} close to its surface. By contrast, the moon has a corresponding gravitational field strength of about 1.6 N kg^{-1}. As this suggests, we could determine the strength of an unknown gravitational field by measuring the force acting on a test mass.

In a similar way we can determine the strength of an electric field by measuring the force acting on a test charge. The electric force F is given by

$$F = qE \tag{22.1}$$

where E represents the electric field strength and q represents the magnitude of the charge used to measure it. (Note the close analogy with the gravitational force $F = mg$.) The electric field strength can be expressed as force per unit charge, N C^{-1} (since $E = F/q$), in just the same way that gravitational field strength can be expressed as force per unit mass. (The magnitude of our test charge would have to be very small to avoid the possibility of inducing charge separation in nearby objects, and, hence, altering the field that we are trying to measure.)

In the previous topic we noted that electric force is a vector quantity. By convention, the field direction is taken to be that of the force acting on a positive charge within the field.

Now let us use Coulomb's law to find an expression for the strength of the field due to a point charge of magnitude Q. We shall use a very small positive test charge q and place it at a distance r from Q. The force acting between the charges is given by Equation (21.1) (page 202) as follows:

$$F = \frac{1}{4\pi\varepsilon_0} \times \frac{Qq}{r^2}$$

If we substitute the right-hand side of this expression for F in Equation (22.1), we get

$$E = \frac{F}{q} = \frac{Q}{4\pi\varepsilon_0 r^2} \tag{22.2}$$

Remember that, since q is positive, the direction of the field is towards Q if Q is negative, and away if it is positive. Also remember that the force acting on a negative charge will be in the opposite direction to that of the field.

22.2 FIELD LINES

At this stage it is useful to introduce another imaginary thinking aid — namely the *electric field line*. This helps us to picture the strength and direction of an electric field. A field line represents the direction of the force acting on a positive test charge at any given point; hence, it represents the field direction there. Figure 22.1 shows field lines around (a) a negative charge, (b) a positive charge and (c) a stronger positive charge. Field strength is indicated by the concentration of field lines; where they are closer together, the field is stronger. The greater the magnitude of the charge the greater the number of field lines entering it or emerging from it. And as the distance from the

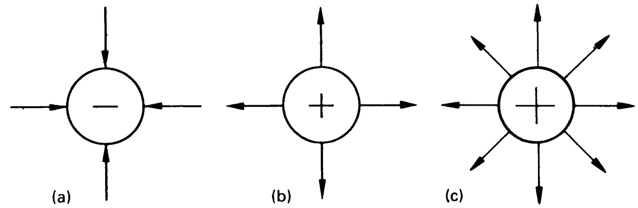

Figure 22.1

charge increases, the field lines diverge and the field becomes weaker.

Figure 22.2(a) represents the pattern of field lines around two equal and opposite charges. Some readers will recognise the similarity with the pattern that iron filings make in the magnetic field around a bar magnet (see Figure 27.1 on page 262). If the charges are alike and are of equal magnitude, as in Figure 22.2(b), then their overall pattern is similar to that around a single charge, particularly when they are close together.

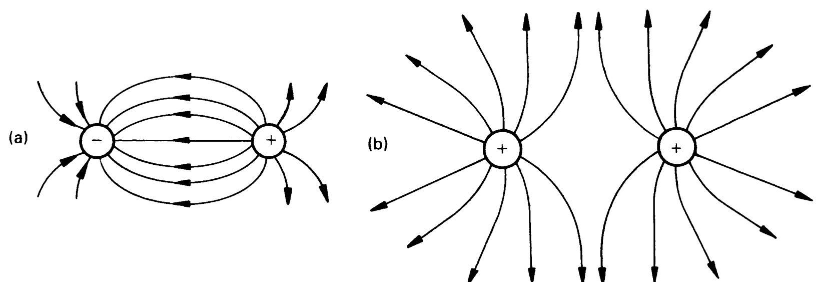

Figure 22.2

Since field lines reflect the force acting on a test charge, they are sometimes called lines of force. The nineteenth century scientist Michael Faraday pictured field lines as representing a state of strain, imagining them to be in tension (hence, tending to shorten), while at the same time repelling each other laterally. Thought of in those terms, the patterns in Figure 22.2 help us to picture the way in which force fields arise between charged objects.

The fields that we have considered so far vary in magnitude and direction from point to point, and are therefore described as being non-uniform. By contrast, a uniform field has constant strength and direction throughout. Figure 22.3 represents the uniform electric field between two parallel, oppositely charged metal plates. The field lines are parallel and uniformly spaced, apart from at the edges, where they tend to escape to some extent and bulge outwards. We shall ignore the edge effect for the purposes of our discussion.

Returning to the gravitational analogy, because the earth is so large the gravitational field over a small area of its surface is virtually uniform. Locally, the gravitational field lines, along which test masses would fall under gravity, can be regarded as parallel for most practical purposes.

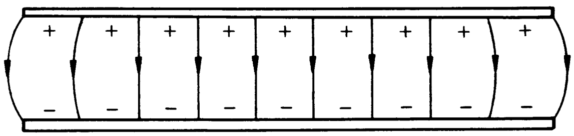

Figure 22.3

22.3 POTENTIAL AND POTENTIAL DIFFERENCE

Another aspect of the analogy between gravitational and electric fields is potential energy. In Topic 8 we saw that the work W needed to raise a mass m to a height h above the ground is equal to mgh joules (Equation 8.3 on page 64). This work is stored as mgh joules worth of potential energy so long as the mass remains at that height. Thus, the *potential* at any point in a gravitational field can be expressed in terms of $(mgh)/m = gh$ joules of potential energy per kilogram mass. Furthermore, we can define an *equipotential* as a surface over which the potential has a constant value — for example, at a particular height above ground level. Obviously an object can move anywhere across an equipotential surface without its potential energy changing. (No component of the force due to the field can lie within the plane of the equipotential; therefore, any field line must cut an equipotential perpendicularly to its surface.)

For practical purposes, we are often interested in the *potential difference* between two levels, as in lifting a mass from one height to another. It is important to remember that the potential difference between two levels corresponds to the distance between them in the field direction (e.g. vertically at the surface of the earth), irrespective of the actual route taken.

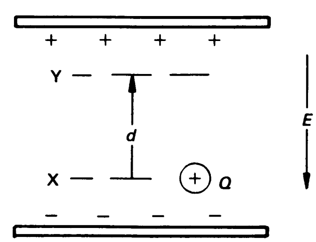

Figure 22.4

Similar considerations apply to a charge in an electric field. Figure 22.4 shows two oppositely charged parallel plates. A positive charge Q placed between the plates will be repelled by the positive plate and attracted towards the negative. To move the charge towards the positive plate, say from the equipotential at X to that at Y, we would need to apply a force F of QE newtons (Equation 22.1) in the opposite direction to that of the field. To move it a distance d in that direction, we would have to do QEd joules of work (just as we have to do mgh joules of work to lift a mass through a height h). Thus, we can express electric potential and define electric equipotentials in joules per coulomb (J C^{-1}), just as we can express their gravitational counterparts in joules per kilogram. (Remember that electric potential decreases in the field direction.)

Because of its size, it is convenient to regard the earth as representing the practical zero of electrical potential, since it acts as an enormous reservoir of charge whose level is virtually constant. By *earthing* a charged object any excess or deficiency of electrons would flow to or from the earth to give the object zero potential relative to the earth.

Later we shall find it necessary to consider potential differences in electrical circuits. For example, we shall regard the function of a battery as raising a charge through a potential difference in much the same way as a hoist raises a mass.

22.4 THE VOLT

Potential difference V is defined as the energy change involved in raising or lowering unit charge from one point to another; obviously, it could be expressed as joules per coulomb but potential difference and potential are given a special unit of their own — namely the *volt* (V). If the transfer of 1 coulomb of charge between two points involves an energy change of 1 joule, then the potential difference between them is 1 volt. For instance, a 1.5 volt battery will give 1.5 joules of energy to each coulomb of charge that passes through it.

Figure 22.4 helps to make this clear. If the work W required to move the charge Q from X to Y is given by $W = QEd$ joules, then V, the potential difference between X and Y (i.e. the energy change per coulomb moved), is given by

$$V = W/Q = (QEd)/Q = Ed \text{(V or J C}^{-1}\text{)}$$

This provides us with two useful relationships:

$$W = QV \qquad (22.3)$$

and

$$V = Ed \qquad (22.4)$$

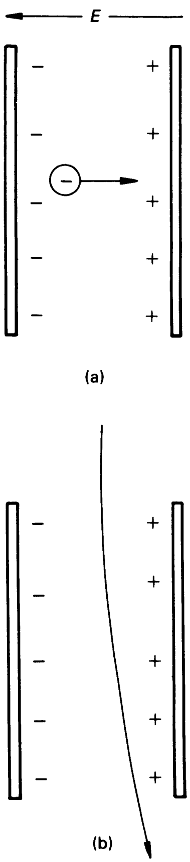

Figure 22.5

Equation (22.3) gives us the energy change involved when a charge moves through a potential difference without our needing to know the field strength or the route taken by the charge. Later on we shall see that this equation can be put to use in various contexts — for example, in converting an electric current to heat, or in storing energy in an electric field. For the time being, as an illustration, let us use it to consider what happens if we release an electron in an electric field between two parallel plates, as in Figure 22.5(a).

The electron will experience a force (hence, an acceleration) in a similar way to a mass in a gravitational field. (We assume that there is a vacuum between the plates; otherwise air molecules will get in the way of the electron and interfere with its acceleration.) As the electron falls through a given potential difference, it trades potential energy for kinetic energy, like a mass falling in a gravitational field. (Remember that an electron 'falls' in the opposite direction to a positive charge.) The kinetic energy ($\frac{1}{2}mv^2$) acquired by the electron is equal to the work done on it (QV) by the electrostatic force due to the field. (Note that if a moving electron enters the field between the plates, then its path will be deflected accordingly, as in Figure 22.5b, for example.)

The *electronvolt* (eV) is a non-SI unit of energy which is very convenient for dealing with events on an atomic scale. It can be defined as the kinetic energy an electron acquires as it falls freely through a potential difference of 1 volt. Its value in joules can be obtained from Equation (22.3) as follows:

$$W = QV = 1.6 \times 10^{-19} \times 1.0 = 1.6 \times 10^{-19} \text{ J}$$

(Since 1 V is equivalent to 1 J C^{-1}, the units of $Q \times V$ are C × J C^{-1} = J.)

Let us now consider Equation (22.4) ($V = Ed$). This relates the potential difference between two points in a uniform electric field to the distance between them. As an illustration, Figure 22.6 shows a uniform field (neglecting edge effects) between two parallel plates that are 0.25 m apart with a potential difference of 500 V between them. Equipotentials are drawn at 100 V (i.e. 0.05 m) intervals, and potential is plotted against distance in the lower half of the figure.

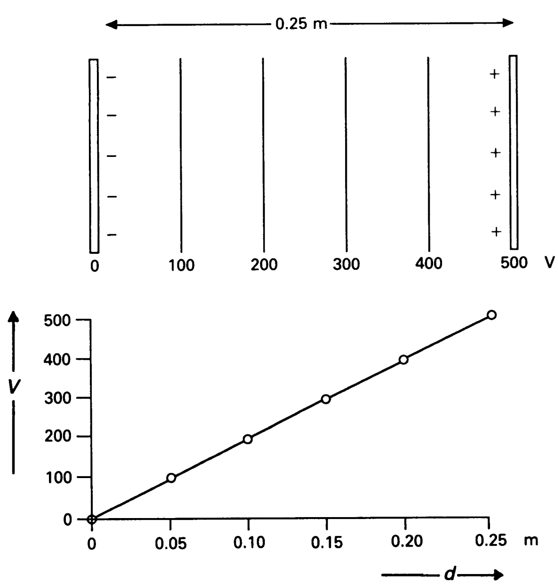

Figure 22.6

The plot is a straight line and the slope represents the electric field strength; thus,

$$E = \frac{V}{d} = \frac{500}{0.25} = 2000 \text{ V m}^{-1}$$

We now have two ways of regarding electric field strength: in terms of either force (N C^{-1}) or energy (V m^{-1}). We can readily show that N C^{-1} and V m^{-1} are equivalent to one another, as follows:

$$1 \text{ V} = 1 \text{ J C}^{-1}$$

and because 1 J = 1 N m,

$$1 \text{ V} = 1 \text{ N m C}^{-1}$$

Dividing both sides by m,

$$1 \text{ V m}^{-1} = 1 \text{ N m C}^{-1} \text{ m}^{-1} = 1 \text{ N C}^{-1}$$

The following worked example compares both viewpoints.

Worked Example 22.1

If an electron is released at the inner surface of the left-hand plate in Figure 22.6, find its velocity when it reaches the other plate (a) by force considerations and (b) by energy considerations.

(a) The electron would experience a force of magnitude F towards the right-hand plate, given by

$$F = QE = 1.6 \times 10^{-19} \times 2000 = 3.2 \times 10^{-16} \text{ N}$$

The resulting acceleration a is, from $F = ma$ (Equation 6.1 on page 50), given by

$$a = \frac{F}{m} = \frac{3.2 \times 10^{-16}}{9.1 \times 10^{-31}} = 3.5 \times 10^{14} \text{ m s}^{-2}$$

(The acceleration due to this tiny force is enormous because of the extremely low mass of the electron.)

The final velocity v of the electron is given by Equation (5.4) (page 41):

$$v^2 = u^2 + 2as$$

Since u (the initial velocity) is zero and s (the distance between the plates) is 0.25 m, the magnitude of v is given by

$$v = \sqrt{2as} = \sqrt{2 \times 3.5 \times 10^{14} \times 0.25} = 13 \times 10^6 \text{ m s}^{-1}$$

(b) If we assume that all the work done on the electron (QV) is converted to kinetic energy ($\frac{1}{2}mv^2$), then

$$\frac{1}{2}mv^2 = QV$$

and

$$v = \sqrt{\frac{2QV}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 500}{9.1 \times 10^{-31}}} = 13 \times 10^6 \text{ m s}^{-1}$$

(Note that we have to be careful with classical dynamic calculations like this, since the mass of an object increases when it is moving close to the speed of light. This is a consequence of the theory of relativity, which we shall not consider here. In this particular case the speed is less than 5% of that of light and the mass increase is less than 0.1% of the electron's mass at rest.)

Questions

(Use any previously tabulated data as required. Where appropriate, assume vacuum conditions unless otherwise stated. Assume that $g = 9.8 \text{ m s}^{-2}$ and that the speeds involved are too low to have a significant effect upon mass.)

1. Figure 22.7 shows two parallel metal plates 60 mm apart with a potential difference of 300 V between them.
   (a) What is the direction of the electric field between the plates?
   (b) If a proton at rest is released from the mid-point between the plates, in which direction will it move?
   (c) What is the field strength between the plates (i) in N C^{-1} and (ii) in V m^{-1}?
   (d) Find the energy required to move a proton from (i) O to X, (ii) O to Y and (iii) O to Z.
   (e) Find the energy required to move an electron from X to O.
   (f) If a proton is released from X, find its speed when it reaches O.
   (g) If an electron is released from O, find its speed when it reaches X.

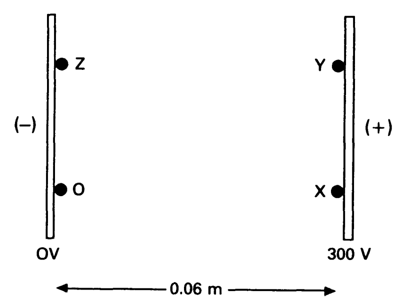

Figure 22.7

2. A potential difference of 200 V exists between two parallel plates that are 50 mm apart. Find (a) the force and (b) the acceleration experienced by an electron in the electric field between the plates.
3. What strength of electric field would just support the weight of a Ca^{2+} ion?
4. A negatively charged droplet of 4.9×10^{-15} kg mass is suspended in the electric field between two parallel horizontal plates 6 mm apart with a potential difference of 450 V between them. Find the number of surplus electrons carried by the droplet.
5. A 10 nC charge is moved 70 mm perpendicularly and then 50 mm parallel to the direction of a uniform electric field of 2000 V m^{-1} strength. Find the overall change in potential energy of the charge.
6. An electron, initially at rest, is released in a uniform electric field 500 V m^{-1} in strength. Estimate (a) its speed after it has travelled 400 mm and (b) how long it takes to travel this distance.
7. Repeat Question 6, replacing the electron with a proton.
8. An electrically charged droplet of 4×10^{-15} kg mass falls vertically at a steady speed through air between two vertical parallel plates set 10 mm apart. A potential difference of 500 V is applied to the plates, whereupon the droplet falls at an angle of 31.5° to the vertical. Estimate the magnitude of the charge carried by the droplet.
9. An electron, initially at rest, is accelerated through a potential difference of 285 V. It then passes midway between two parallel plates providing a uniform electric field perpendicular to the direction in which it is travelling. The plates are 50 mm long and 25 mm apart and there is a potential difference of 71 V between them. Find (a) the speed of the electron after its initial acceleration and (b) the transverse deflection experienced by the electron as it emerges from between the plates.

TOPIC 23 CAPACITANCE

COVERING:

- the parallel-plate capacitor;
- dielectrics;
- energy stored in a capacitor;
- capacitors in parallel and in series.

The creation of an electric field involves separating positive and negative charge. Work has to be done which is then stored in the field as potential energy. Figure 23.1 illustrates this in terms of an electric cell or battery connected across two parallel metal plates. The figure shows the conventional symbol for an electric cell (although the signs are usually omitted). A battery is simply a number of cells connected together to form a single unit.

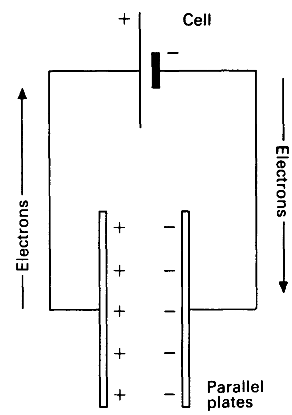

Figure 23.1

Electrons will flow towards the positive terminal of the cell from the plate to which it is connected. At the same time electrons will flow from the negative terminal of the cell onto the other plate. In effect, the cell transfers electrons from one plate to the other. The build-up of charge on the plates increasingly opposes the flow of electrons, which therefore decreases and eventually stops when the potential difference across the plates is equal to the voltage of the cell. If the cell is disconnected and the plates connected directly to one another, then electrons will flow from the negative plate to the positive until the potential difference between them is zero.

A device such as this, capable of storing electric charge, is called a *capacitor*. Its ability to store charge is measured in terms of its *capacitance* C, given by

$$C = Q/V, \text{ or } Q = CV \qquad (23.1)$$

where Q is the charge stored on either plate and V is the potential difference between them. The unit of capacitance is called the *farad* (symbol F) but, as it stands, this is much too large a quantity for many purposes, so the microfarad (1 μF = 1×10^{-6} F) and the picofarad (1 pF = 1×10^{-12} F) are in common use.

In this topic we shall confine ourselves to the so-called *parallel-plate capacitor* of the type shown in Figure 23.1.

Note that capacitance is dependent on the area of the plates and on the separation between them. Assuming that they are each of area A and separated by a distance d, and that the non-uniformity of the field at the edges can be ignored, then C is proportional to A/d. However, as we shall now see, capacitance is also increased by the presence of a *dielectric* material between the plates.

Practical capacitors are normally of the basic parallel-plate type, although a variety of geometrical arrangements are used. For example, metal foil and thin sheets of dielectric material are stacked in layers or rolled up together like a swiss roll to provide high capacitance in a small volume. Some capacitors have movable plates with adjustable overlap so that their effective area and, hence, their capacitance, can be varied.

23.1 DIELECTRICS

A dielectric is a non-conductor of electricity and can be used simply as an insulator. For this purpose it must have adequate *dielectric strength* — that is to say, it must not break down electrically and lose its insulating properties. Dielectric strength is generally expressed in terms of the maximum electric field the material will withstand. (Note that air normally has a dielectric strength of about 3 kV mm^{-1}. Some ceramics and polymers used as insulators have values well in excess of 10 kV mm^{-1}.)

Although dielectrics are insulators, they do respond to electric fields. In Topic 21 we saw that the force between two charges is reduced by the presence of a dielectric between them. Equation (21.2) (page 202) shows that this force is inversely proportional to the permittivity ε of the dielectric. (Remember that $\varepsilon = \varepsilon_r\varepsilon_0$, where ε_r is the relative permittivity, or dielectric constant, and ε_0 is the permittivity of free space.)

The presence of a dielectric between the plates of a capacitor will increase its capacitance C in accordance with the expression

$$C = \varepsilon_r C_0 \tag{23.2}$$

where C_0 is the capacitance when there is a vacuum between the plates. From this it follows that relative permittivity can be defined as the ratio C/C_0.

Now let us consider what actually happens when a dielectric is inserted into the space between a pair of charged plates separated by vacuum. If the capacitor is electrically isolated, then the charge on the plates remains fixed and the potential difference between them will fall. This is what we would expect from Equation (23.1). Since $V = Q/C$, and Q is fixed, then an increase in C by a factor of ε_r (from Equation 23.2) will lead to a corresponding reduction in V. If the capacitor is connected to a cell or a battery, as in Figure 23.1, then the insertion of a dielectric will cause a further flow of electrons, so that the potential difference across the plates remains equal to the voltage of the cell. In this case V is fixed, where $Q = CV$, and the presence of the dielectric raises C by a factor of ε_r, so the charge stored by the capacitor is increased accordingly. These effects suggest that the presence of the dielectric reduces the electric field between the plates, so that more charge is required to restore the potential difference between them to its original value. The reason for this is polarisation.

As Figure 23.2 suggests, the nucleus of an atom in an electric field tends to be shifted in the field direction, whereas the outer electrons tend to be shifted in the opposite direction. The centres of positive and negative charge distributions within the atom are thus displaced from one another, producing a dipole orientated in opposition to the field. In effect, this gives a resultant field smaller than the original. Where a substance consists of polar molecules with permanently polarised structures (Topic 15), their tendency to orientation can lead to marked dielectric behaviour. For example, as we saw in Topic 21, water is a highly polar molecule which, in the liquid state, has considerable freedom to align itself in an electric field. Water therefore has a high relative permittivity value (see Table 21.1 on page 203). The bending and stretching of polar bonds can also contribute to polarisation.

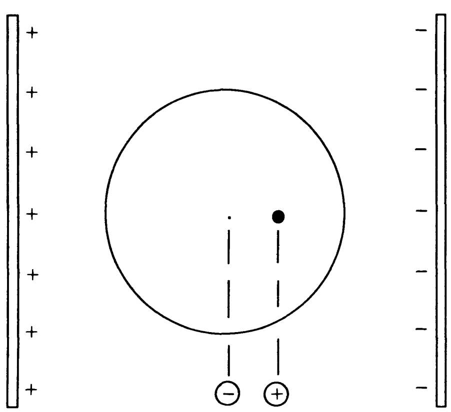

Figure 23.2

Earlier we noted that capacitance C is proportional to A/d, where A is the area of each plate and d is the distance between them. We can now make this relationship into an equation, because the permittivity ε of the dielectric (where $\varepsilon = \varepsilon_r\varepsilon_0$) gives us the constant of proportionality as follows:

$$C\left(=\frac{\varepsilon_r\varepsilon_0 A}{d}\right)=\frac{\varepsilon A}{d} \qquad (23.3)$$

Note that the normal unit of permittivity (given by rearrangement of this equation) is farads per metre, F m^{-1}. Also note that ε_r is a dimensionless quantity, since it is given by the ratio $\varepsilon/\varepsilon_0$.

23.2 ENERGY STORED IN A CHARGED CAPACITOR

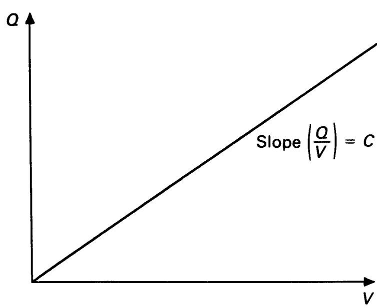

Figure 23.3

The energy stored in a charged capacitor is the total work done in transferring the electrons from one plate to the other. The work done in transferring the first electron is virtually zero, because the initial potential difference between the plates is zero. Since the relationship between Q and V is a straight line (see Figure 23.3), the work done in transferring successive electrons increases linearly as the charge builds up on the plates. If, at a given moment, the total charge transferred is Q and the potential difference is V, then we can say that the electrons have been transferred through an average potential difference equal to $(0 + V)/2$. Thus, W, the total work done, is given by the total charge multiplied by the average potential difference, as follows:

$$W = Q \times \frac{(0 + V)}{2} = \frac{1}{2} QV \qquad (23.4)$$

(We can regard the energy stored in a charged capacitor as analogous to the strain energy stored in a stretched wire. Strain energy is given by the area of the triangle under the force/extension line (Figure 8.2(b) on page 65). Similarly, the energy stored in the capacitor is given by the area of the triangle under the Q/V line.)

Note that, since $Q = CV$, W can also be expressed in the forms

$$W\left(= \frac{1}{2} QV \right) = \frac{1}{2} CV^2 \qquad (23.5)$$

and

$$W = \frac{1}{2} \frac{Q^2}{C} \qquad (23.6)$$

Worked Example 23.1

A capacitor of unknown value was charged using a 10 V battery. It was then disconnected from the battery and discharged through a small electric motor, which raised a 100 g mass to a height of 1 m. Estimate the unknown capacitance.

Assuming that all the stored electrical energy was converted to gravitational potential energy, then

$$\frac{1}{2}CV^2 = mgh$$

which can be rearranged to

$$C = \frac{2mgh}{V^2}$$

Substituting the values given above, and assuming that $g = 9.8 \text{ m s}^{-2}$,

$$C = \frac{2mgh}{V^2} = \frac{2 \times 0.1 \times 9.8 \times 1}{10^2} = 0.02 \text{ F}$$

23.3 CAPACITORS COMBINED IN PARALLEL AND IN SERIES

Where a number of capacitors are combined in an electrical circuit, their resultant capacitance can be found by considering them as either *parallel* or *series* combinations.

Figure 23.4 shows a parallel combination of two capacitors with capacitances C_1 and C_2, respectively. (The figure shows the conventional symbol for a capacitor.) Both capacitors are connected across the same battery, so the potential difference V across each is the same. The charge carried by each is therefore given by $Q_1 = C_1V$ and $Q_2 = C_2V$, respectively (from Equation 23.1). The combined charge Q is given by

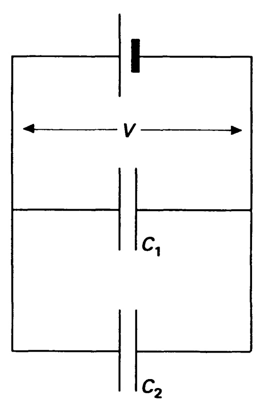

Figure 23.4

$$Q = Q_1 + Q_2 = V(C_1 + C_2)$$

Therefore, the combined capacitance C is

$$C = C_1 + C_2$$

This argument can be extended to give the general equation for any number of capacitances in parallel, as follows:

$$C = C_1 + C_2 + C_3 + \cdots C_n \qquad (23.7)$$

Figure 23.5 shows a series combination. In this case the battery produces a charge of $+Q$ on plate D and of $-Q$ on plate G. Provided that the plates in each capacitor are sufficiently large and closely

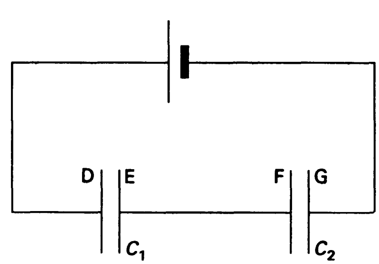

Figure 23.5

spaced together, E and F carry charges of $-Q$ and $+Q$ induced by their proximity to plates D and G, respectively. In this case the charge stored by each capacitor is equal and the potential difference across each is given by $V_1 = Q/C_1$ and $V_2 = Q/C_2$, respectively. The overall potential difference V, governed by the battery, is divided across the capacitors, so that

$$V = (V_1 + V_2) = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

The combined capacitance C is therefore given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

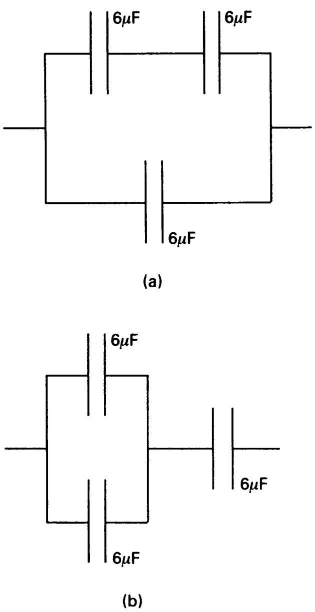

Figure 23.6

Again the argument can be extended to give a general equation for any number of capacitances in series, as follows:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \frac{1}{C_n} \tag{23.8}$$

Worked Example 23.2 illustrates more complex networks involving both parallel and series combinations.

Worked Example 23.2

Find the resultant capacitance of each of the combinations shown in Figure 23.6.

(a) First, considering the pair in series,

$$\frac{1}{C} = \frac{1}{6} + \frac{1}{6}$$

which gives $C = 3\ \mu F$.

Then, considering the parallel combination of this with the third capacitor,

$$C = 3 + 6 = 9\ \mu F$$

(b) First, considering the pair in parallel.

$$C = 6 + 6 = 12\ \mu F$$

Then, considering the series combination of this with the third capacitor,

$$\frac{1}{C}=\frac{1}{12}+\frac{1}{6}$$

which gives $C = 4\ \mu F$.

Worked Example 23.3

A 3 μF capacitor is charged to 60 V and a 6 μF capacitor is charged to 120 V. The capacitors are then connected with their like-charged plates together. Find the total stored energy before and after connection.

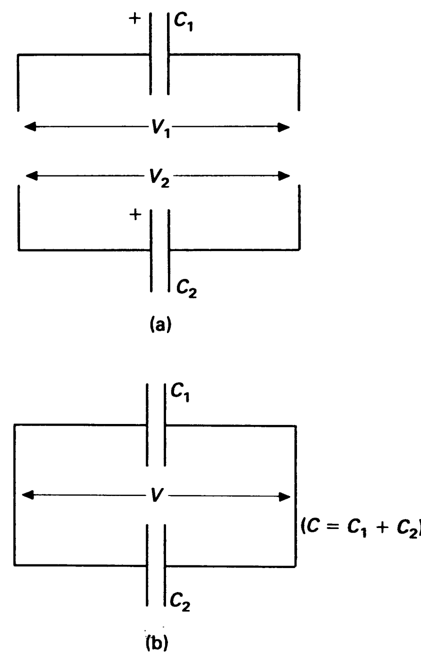

Figure 23.7

Figure 23.7 shows the capacitors (a) before, and (b) after connection.

(a) Before connection:

The total stored energy is given by

$$\frac{1}{2}C_1V_1^2+\frac{1}{2}C_2V_2^2$$

and, substituting the given values,

$$\left(\frac{1}{2}\times 3\times 10^{-6}\times 60^2\right)+\left(\frac{1}{2}\times 6\times 10^{-6}\times 120^2\right)$$

$$= 0.0486\ \text{J}$$

The total stored charge is given by

$$C_1V_1 = C_2V_2$$

and, substituting the given values,

$$(3\times 10^{-6}\times 60)+(6\times 10^{-6}\times 120)$$

$$= 9\times 10^{-4}\ \text{C}$$

(b) After connection:

Initially the capacitors have different potentials. On connection, there will be a redistribution of the total charge Q to equalise the potential difference across the parallel combination, which has a combined capacitance ($C_1 + C_2 =$) 9 μF. From Equation (23.6), the stored energy is given by

$$\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}\frac{(9\times 10^{-4})^2}{9\times 10^{-6}} = 0.045\ \text{J}$$

Note that the stored energy lost on connection (i.e. 0.0486 − 0.045 = 0.0036 J) is converted to heat in the connecting wires.

Also note that, after connection, the potential difference V is the same across both capacitors and can be found from

$$V = \frac{Q}{C} = \frac{9 \times 10^{-4}}{9 \times 10^{-6}} = 100 \text{ V}$$

Using this approach, the stored energy after connection can be obtained from

$$\frac{1}{2}QV = \frac{1}{2} \times 9 \times 10^{-4} \times 100 = 0.045 \text{ J}$$

or from

$$\frac{1}{2}CV^2 = \frac{1}{2} \times 9 \times 10^{-6} \times 100^2 = 0.045 \text{ J}$$

Questions

(Use any previously tabulated data as required.)

1. Find the charge on a 5 μF capacitor with a potential difference of 80 V between its plates.
2. Two parallel plates, each 0.2 m square, are positioned 2 mm apart in air. How many electrons must be transferred from one plate to the other to give a potential difference of 190 V between them?
3. A parallel-plate capacitor, with air between its plates, is charged by using a 60 V battery. Find the potential difference between the plates if, after disconnecting the battery, the air between them is replaced with polythene.
4. What is the total charge stored by one 2 μF, one 3 μF, one 5 μF and two 10 μF capacitors all connected in parallel across a 24 V battery?
5. Find the various capacitances obtainable from three 3 μF capacitors in combination.
6. Two parallel-plate capacitors are identical except that the plates of one are spaced twice as far apart as those of the other. If the capacitors are connected in series across a 12 V battery, estimate the potential difference across each.

7. A capacitor holds a charge of 0.1 mC after being charged to 50 V. Find (a) its capacitance, and (b) the amount of energy stored.

8. A 20 V battery is used to charge a 0.021 F capacitor. After disconnection from the battery, the energy stored in the capacitor is used to heat 10 g of water that is initially at a temperature of 15.0 °C. Find the final temperature of the water.

9. A 150 V battery is connected across a 4 μF and an 8 μF capacitor combined (a) in parallel, (b) in series. Find the charge stored by each capacitor in both cases.

10. A 3 μF capacitor, previously charged to 120 V, is connected in parallel with an uncharged 5 μF capacitor. Find (a) the final voltage across the capacitors and (b) the energy lost in heating the connecting wires.

TOPIC 24 ELECTRIC CURRENT

COVERING:

- electric current as a flow of charge;
- current in metal conductors;
- conventional current;
- power;
- current in semiconductors.

Electric current provides a very convenient means of transporting energy from place to place. Metal conductors are used to carry it, and all sorts of electrical devices are available to convert it into heat, light or whatever other form of energy is required.

As we noted in Topic 21, electric current is simply a flow of electric charge. It may be the flow of electrons through a vacuum, or ions through a gas or an electrolyte; more importantly from our point of view, it may be the flow of electrons through a metal or, as we shall see later in this topic, electrons and positive holes through semiconducting materials.

The magnitude of an electric current could be defined as the rate of flow of charge — say the number of coulombs passing a given point in a conductor in one second. But electric current (like mass, length and time) is one of the seven base SI units (see Topic 1) and, as we shall see later, the base unit of current, the ampere (A), is defined in quite a different way. The coulomb is formally derived from the ampere, rather than the other way round, and is defined as the quantity of charge which flows in one second past a point in a conductor which is carrying a steady current of one ampere. Thus,

$$Q = It, \text{ or } I = \frac{Q}{t} \qquad (24.1)$$

where I amperes is the steady current that flows when a total charge Q coulombs passes a given point at a uniform rate over a period of t seconds. To take a very simple example, the uniform flow of a total of 135 C over a period of 45 s is equivalent to a current of 3 A.

24.1 ELECTRIC CURRENT IN METAL CONDUCTORS

First let us estimate the velocity of the free electrons which constitute an electric current through a metal conductor. Figure 24.1 shows a conductor with a cross-sectional area of A square metres. If the average velocity of the free electrons flowing along the length of the conductor is v metres per second, and if X and Y are v metres apart, then all the free electrons between X and Y at a given moment will pass Y in 1 s. The volume of metal between X and Y is given by $v \times A$ (i.e. length $\times$ cross-sectional area), so, if the metal contains n free electrons per unit volume, then nvA electrons will pass Y in 1 s. This corresponds to a total charge of $nvAe$, where e represents the charge on each electron. Putting this into the form of an equation, the current I is given by

$$I = nvAe \qquad (24.2)$$

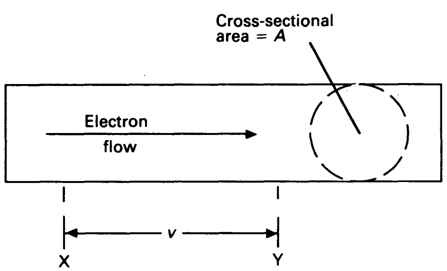

Figure 24.1

To estimate the velocity of the electrons, let us consider a current of 1 A flowing in a copper wire of 1 mm^2 cross-sectional area. In Question 7 at the end of Topic 14 you should have found that a 1 mm cube of copper contains 8.4×10^{19} atoms. A copper atom has one outer electron (Table 14.2 on page 127), so it is not unreasonable to assume a value for n of 8.4×10^{28} electrons per m^3. Rearranging Equation (24.2) and substituting these values,

$$v = \frac{I}{nAe}$$

Therefore,

$$v = \frac{1}{8.4 \times 10^{28} \times 1 \times 10^{-6} \times 1.6 \times 10^{-19}}$$

$$= 0.074 \times 10^{-3}\ \text{m s}^{-1}$$

This is equivalent to about a quarter of a metre per hour, which bears no comparison with the speeds of electrons in vacuum that we met in Topic 22. Clearly the electrons carrying a current through a metal conductor encounter considerable resistance to their movement.

To understand the reason for this, we need to remember that our atomic model of a metal is an ordered crystal structure of positive ions within which the valence electrons are free to move. If the crystal was perfectly regular and the ions stationary, then the electrons would be able to move through the 'corridors' between the rows of ions without any resistance. However, because of their thermal energy, the ions continuously vibrate about their mean positions on the crystal lattice, partially blocking the corridors as they move to and fro, so that the electrons collide with them from time to time. The collision frequency is expressed in terms of the *mean free path*, which is the average length of free flight between collisions. In the case of copper at ordinary temperatures, the mean free path is about 40×10^{-9} m (40 nm). This is equivalent to a row of about 160 copper atoms. At higher temperatures, thermal vibration is more vigorous and the collisions more frequent; hence, the mean free path becomes correspondingly shorter.

Normally the free valence electrons in a metal move at very high speeds in an entirely random fashion, rather like gas molecules. They have wide ranges of velocity and kinetic energy, and each electron continually changes speed and direction as it collides with metal ions in the crystal lattice. However, because there are so many electrons moving at random, their overall distribution is uniform and there is no net flow of charge in any particular direction.

If a potential difference is applied across the ends of a metal conductor, the electrons will tend to move towards the positive end. Their movement will still be essentially random, but during each flight between collisions they will respond to the force due to the applied electric field. Electrons that are moving towards the positive end will be accelerated, and those moving away from it will be decelerated. The result will be an overall drift superimposed on their random motion.

The drifting electrons experience a gain in kinetic energy associated with their net acceleration towards the positive end of the conductor. They share this energy with the metal ions as they collide with them, then they move on, providing a continual transfer of energy to the ions which ultimately appears as a temperature rise in the metal. By its very nature, the temperature rise will itself cause increased resistance to the passage of electrons which, in turn, will lead to a further increase in temperature — and so on. Sooner or later an equilibrium may be established where the conductor loses heat to its surroundings at the same rate as that at which it gains heat from the passage of the electrons. Obviously, the conversion of electricity to heat is wasteful and potentially dangerous in an electric cable but essential in a heating element (or in a piece of fuse wire, which melts when the current exceeds a particular value). Temperature rise is therefore an important factor in the design of a conductor.

Apart from temperature, the ease with which electrons pass through a metal depends upon structural features. For instance, the electrical conductivity of pure copper is roughly halved by substituting 10% of the atoms with zinc. Because of their different size, the substituent atoms distort the copper crystal lattice and, hence, the electron pathways between the ions. This will have the effect of reducing the mean free path and increasing the frequency of collisions. The mass of a substituent atom is also important, because this will affect its response to a collision and the extent to which it vibrates afterwards, and, hence, the transfer of heat into the metal structure. The charge on the ion also has an effect: zinc ions carry a double charge, in contrast to the single charge on copper, and this will create electrical irregularities in the electron pathways.

So now we can see that metals show an almost friction-like resistance to the passage of electrons which is due to structural irregularities of one kind or another.

24.2 CONVENTIONAL CURRENT DIRECTION

Before we go any further, there is a very important convention that we must remember. In practice, the direction of an electric current is taken to be that in which positive charge carriers would move (i.e. opposite to that of the electron drift). This *conventional current* direction was established before the discovery of the electron's role as the charge carrier in metals, and it remains to this day. For many practical purposes it makes no difference whether we think about negative charge moving one way or positive charge moving the other.

24.3 ENERGY CONVERSION AND POWER

Figure 24.2 represents a lamp powered by a battery. We shall assume that the wires connecting the bulb to the battery offer negligible resistance to the current. The filament in the bulb, however, offers so much resistance that it becomes hot enough to emit light.

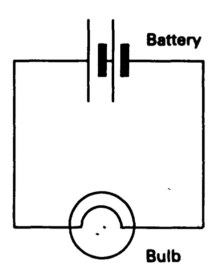

Figure 24.2

Thinking in terms of conventional current, the positive terminal of the battery represents a point of high potential and the negative terminal a point of low potential. Our notional positive charge will leave the positive terminal and pass through the filament, providing energy to raise its temperature. The charge, depleted of its energy, will return to the battery via the negative terminal. The potential difference across the filament, and the total energy per coulomb converted by it (to heat and light), are related by Equation (22.3) ($V = W/Q$). Thus, a potential difference of 6 V means that 6 J are extracted from each coulomb passing through the filament, 12 J from 2 C, 18 J from 3 C, and so on. This argument applies generally to devices which convert electrical energy to other forms — for example, electric motors which convert it to mechanical energy, and batteries on charge which convert it to chemical energy. We should note that most devices convert some of the electrical energy into unwanted by-products, such as heat, which reduce their efficiency.

In Topic 8 we noted that power is the quantity used to measure the rate at which energy is converted from one form to another. Thus, the power of an electrical device is the rate at which it converts electrical energy into other forms. If W joules of electrical energy are converted in t seconds, then the power P is given by W/t. But $W = QV$; therefore, W/t is equal to QV/t. Since $Q = It$ (Equation 24.1),

$$P\left(=\frac{W}{t}=\frac{QV}{t}\right)=\frac{ItV}{t}$$

and

$$P = IV \tag{24.3}$$

where P is in watts (W). This equation gives the rate of conversion to all forms of energy — for example, light plus heat in a light bulb, or mechanical energy plus heat in an electric motor.

24.4 ELECTRIC CURRENT IN SEMICONDUCTORS

In view of the great technological importance of semiconductors, we need to have some understanding of how they work. As their name implies, they are neither good conductors nor good insulators but lie somewhere in between. There are many semiconducting materials. Silicon and germanium are well-known examples which we shall use as the basis of our discussion.

As their position in group IV of the periodic table suggests, atoms of silicon and germanium form four covalent bonds with their neighbours in a similar way to carbon atoms in diamond. At absolute zero both silicon and germanium behave as insulators. However, their valence electrons are rather loosely held, so that, at higher temperatures, some of them have sufficient thermal energy to become detached from the bonds and turn into conduction electrons that are able to transport charge.

To detach a valence electron in diamond involves overcoming an energy barrier of about 6 eV, whereas it only requires about 1.1 eV and 0.7 eV for silicon and germanium, respectively. The effect of this is that, at room temperature, silicon and germanium possess enough conduction electrons to make them semiconductors, whereas diamond remains an insulator. Diamond becomes a semiconductor if its temperature is raised sufficiently and, in general, the conductivity of semiconductors increases with increasing temperature as more electrons have sufficient thermal energy to jump the barrier.

When a valence electron becomes detached, it leaves behind it a positively charged vacant site called a *hole*. A second valence electron from a nearby covalent bond may then move into the hole, in which case it will leave a new hole behind it. In effect, this electron and the original hole will exchange places, as indicated by the arrows in Figure 24.3, the electron moving one way and the positive hole in the opposite direction. This means that holes can act as charge carriers which contribute towards the electrical conductivity of the semiconductor.

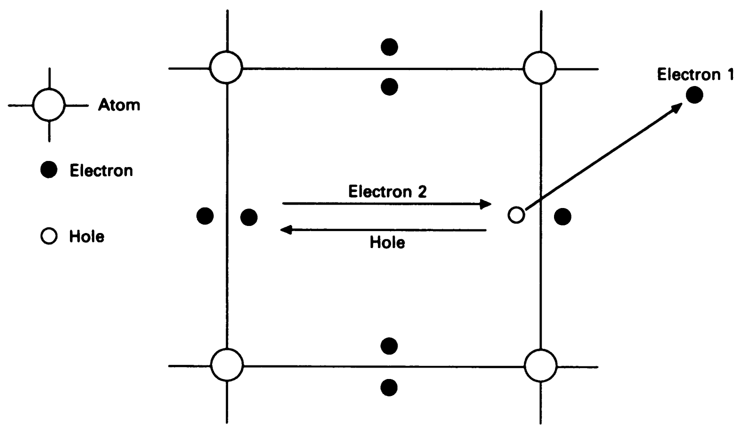

Figure 24.3

Intrinsic semiconductors are so called because their semiconductivity is an inherent property of the material. Intrinsic semiconductors may be *doped* with minute traces of impurities in order to control the flow of current by introducing additional charge carriers. These doped materials are called *extrinsic semiconductors*. They are classified as *n-type* if the additional charge carriers are electrons and *p-type* if they are holes. (p- and n- signify positive and negative charge carriers, respectively.)

n-Type semiconductors can be made from silicon and germanium by doping them with group V elements (i.e. elements with five outer electrons) such as phosphorus, arsenic and antimony. Each impurity atom takes up a position on the crystal lattice, forming four covalent bonds with its neighbours, leaving the fifth electron free to act as a charge carrier. Electrons are the *majority carriers* (i.e. the predominant type), although intrinsic *minority carriers* (holes) will still be present. In this case the impurity atoms donate electrons and are known as *donors*. It is important to note that the crystal as a whole remains electrically neutral, because all of its constituent atoms are neutral.

If silicon or germanium is doped with a group III element (such as boron, aluminium or gallium), with three outer electrons, the impurity atoms form three covalent bonds with their neighbours, leaving a hole which can accept electrons. In this case the majority charge carriers are holes and we have a p-type semiconductor with the impurity atoms described as *acceptors*.

24.5 p–n JUNCTIONS

Many semiconductor devices make use of *p–n junctions* specially formed between p- and n-type materials.

A single junction forms the basis of the *semiconductor diode*, which has the property of allowing conventional current to flow from the p side to the n side but not the other way. Figure 24.4 illustrates how this works in principle.

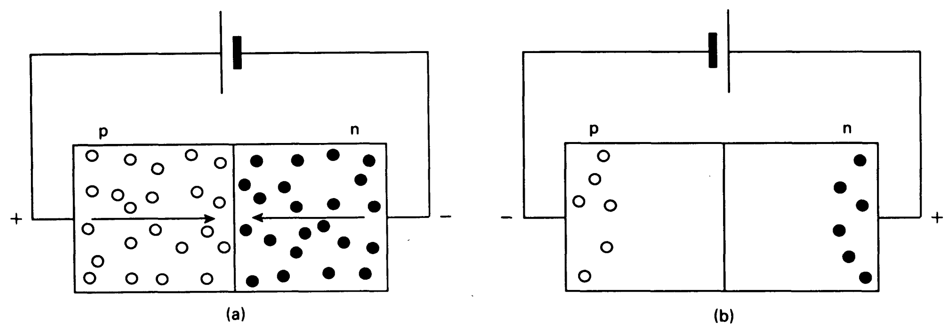

Figure 24.4

Figure 24.4(a) shows a cell connected across the junction so that the p end is positive and the n end is negative. The majority carriers on each side (holes and electrons, respectively) move towards the junction, where they meet and combine. Fresh electrons enter the n-type material from the negative terminal of the cell and fresh holes are formed at the positive terminal where electrons are withdrawn. Current will flow as these processes continue and the junction is said to be *forward-biased*.

The recombination of electrons and holes at a forward-biased p–n junction is accompanied by the liberation of energy, usually in the form of heat. Some types of junction emit useful amounts of light and can be used as the basis of *light-emitting diodes*, such as those used for digital displays.

If the applied voltage is reversed, as in Figure 24.4(b), the electrons and holes tend to be drawn away from the junction and no current flows (apart from a very small *leakage current* due to the thermal formation of a few intrinsic carriers). In this case the junction is said to be *reverse-biased*.

Questions

(Use any previously tabulated data as required.)

1. In 1 min how many electrons pass a given point in a wire carrying a current of 0.2 A?
2. What is the power consumed by an electrical device carrying a current of 500 mA and across which there is a potential difference of 10 V?
3. Find the energy lost by an electron in passing through the filament of a torch bulb across which there is a potential difference of 2.5 V.
4. In 1 min a charge of 3 C passes through an electrical device across which there is a potential difference of 120 V. Find the power consumed by the device.
5. An electrical device consumes a total of 120 J of electrical energy when 8 C passes through it at a steady rate over a period of 20 s. Find (a) the power consumption of the device, (b) the potential difference across it, and (c) the magnitude of the current passing through it.
6. Assuming that the current and duration of a lightning flash are of the order of 1×10^4 A and 1×10^{-3} s, respectively, estimate the quantity of charge involved. Assuming a potential difference of 200 MV between the cloud and the ground, estimate the energy dissipated.
7. Find the potential difference across an electrical device which consumes 1200 J of electrical energy when a steady current of 2 A flows through it for a period of 10 s.
8. Water at 10 °C passes into a continuous-flow electric water heater at a rate of 100 g per minute and emerges at 15 °C. Assuming there are no heat losses, find the current in the heater element if the potential difference across it is 14 V.
9. An electrical machine operates with an efficiency of 80% while drawing a current of 6 A at 160 V. How much power is wasted?

TOPIC 25 RESISTANCE

COVERING:

- simple measurement of resistance;
- resistance and resistivity;
- I–V characteristics;
- resistors in series and in parallel;
- e.m.f.;
- internal resistance;
- power.

In the previous topic we saw how metal conductors tend to resist the flow of electric charge that constitutes an electric current. We have now reached the point where we need to be able to quantify this electrical resistance.

We already know that a potential difference is needed to make a current flow through a conductor. Extending this idea, a bad conductor offers greater electrical resistance than a good one; it therefore requires a correspondingly greater potential difference to give the same current under the same conditions. Resistance R is defined as the ratio between the potential difference V across the conductor and the current I that is passing through it. Thus,

$$R = \frac{V}{I} \tag{25.1}$$

The unit of resistance is called the *ohm* (symbol Ω).

To take a simple example, if a bulb operating at 3 V draws a current of 0.25 A, then it has a resistance R, given by

$$R = V/I = 3/0.25 = 12\ \Omega$$

Looking at this another way ($I = V/R$), a conductor with a small resistance will allow a large current to pass for a given potential difference. To illustrate this, let us compare the currents drawn by two different bulbs, say with resistances of 12 Ω and 15 Ω, respectively, both operating at 3 V.

For the 12 Ω bulb,

$I = V/R = 3/12 = 0.25$ A

and for the 15 Ω bulb,

$I = V/R = 3/15 = 0.2$ A

25.1 MEASURING RESISTANCE

Figure 25.1 shows a simple way of measuring the resistance of a bulb, or indeed any other electrical component, using Equation (25.1).

The voltmeter is connected in parallel with the bulb in order to measure the potential difference across it. The current will divide at the point X, some passing through the bulb and the rest through the voltmeter; then it will recombine at the point Y. Ideally a voltmeter should have very high resistance so that it diverts an insignificant proportion of the current from the component across which it is connected. (In terms of $I = V/R$, the larger the value of R the smaller the value of I.) The voltmeter will then cause very little disturbance to the conditions in the main circuit.

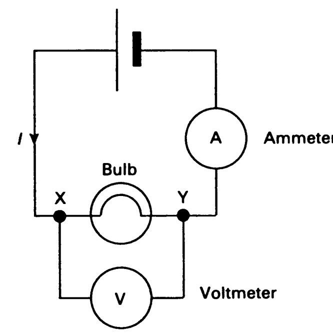

Figure 25.1

The ammeter measures the current and is connected in series with the bulb, so that the same current flows through them both (assuming a negligible current through the voltmeter). Ideally, an ammeter should have very low resistance so that it disturbs the conditions in the circuit as little as possible.

The resistance of the component is then calculated from the voltmeter and ammeter readings by use of Equation (25.1).

We shall look at another method of measuring resistance in the next topic.

25.2 RESISTANCE AND RESISTIVITY

The resistance of a conductor (e.g. a wire) depends on its dimensions. If we double its cross-sectional area without changing anything else, then twice as much charge can pass and the current will be doubled. (The same effect would be achieved by connecting a second identical conductor in parallel with the first.) On the other hand, if we double the length (in effect, adding a second identical conductor in series with the first), then the current has to travel twice as far and will therefore experience twice the resistance. If nothing else is changed, the current will be halved (because $I = V/R$).

These observations are summarised by the equation

$$R = \rho \times \frac{L}{A} \qquad (25.2)$$

where L and A are the length and cross-sectional area of the conductor in m and m^2, respectively. ρ, the constant of proportionality, represents the *resistivity* of the material from which the conductor is made. Rearrangement of Equation (25.2) tells us that the unit of ρ is Ω m. Resistivity is a property of enormous variation, ranging, for example, from 1.6×10^{-8} Ω m for silver to around 1×10^{18} Ω m for silica glass. (The reciprocal of resistivity ($1/\rho$) is called *conductivity* (σ). This is measured in Ω^{-1} m^{-1} or, in SI units, siemens per metre, where siemens (S) is the SI unit of electrical *conductance*. 1 S = 1 Ω^{-1}; thus, conductance is the reciprocal of resistance.)

Table 25.1

Substance	Resistivity/Ω m
Silver	1.6×10^{-8}
Copper	1.7×10^{-8}
Aluminium	2.8×10^{-8}
Tungsten	5.5×10^{-8}
Nichrome (Ni/Cr alloy)	100×10^{-8}
Germanium	0.5
Silicon	2.5×10^{3}
Silica glass	1×10^{18}

Table 25.1 shows the approximate resistivities of various materials at ordinary temperatures.

Copper has slightly higher resistivity than silver but is much more widely used as an electrical conductor, because of the cost factor. Aluminium is used in overhead power lines, because, although its resistivity is around one and a half times that of copper, its density is about three times less. Thus, on a weight basis (important for suspended cables), aluminium is an appropriate choice. Tungsten is used for light bulb filaments, which get very hot, because it has a very high melting point (about 3400 °C). Nichrome, which is a nickel/chromium alloy with fairly high resistivity, is used for making heating elements.

Knowing the dimensions of a conductor, Equation (25.2) enables us to calculate its resistance from its resistivity, or vice versa. It also suggests that, if we are simply interested in transporting electricity, we should make conductors as thick and as short as possible in order to minimise their resistance. If the resistance is high, then electrical energy will be wasted in heating the conductor and there will be a voltage drop along its length (given by $V = IR$).

25.3 *I–V* CHARACTERISTICS OF A METALLIC CONDUCTOR

We would normally expect the current flowing through a metallic conductor to increase if we increase the potential difference across its

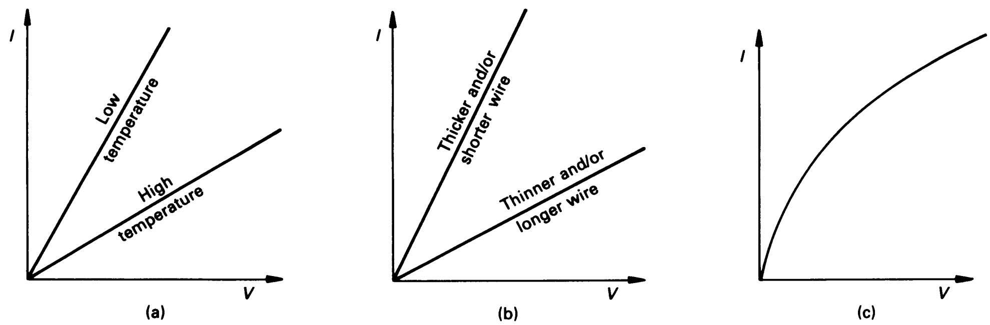

Figure 25.2

ends. Figure 25.2 illustrates several general points about the relationship between current and voltage.

Figure 25.2(a) shows the I–V characteristics of an ordinary metal conductor determined at two different temperatures. As we saw in the previous topic, the electrical conductivity of a metal decreases with increasing temperature. This is reflected in Figure 25.2(a), which shows that the conductor allows less current to pass at high temperature. The figure also shows that, at constant temperature, current is proportional to potential difference. In other words, the I–V characteristic is linear. From this we can say that the resistance of the metal is constant, because the ratio V/I ($= R$) is the same at any point on the line. As Figure 25.2(a) stands, the steeper the line the lower the resistance, since the slope I/V (the reciprocal of the resistance) represents the conductance.

The proportionality between current and potential difference is described by Ohm's law (named after Georg Ohm, who discovered it early in the nineteenth century). This may be expressed in the form

$$\frac{V}{I} = \text{constant}$$

It is most important to recognise that this equation is not a restatement of Equation (25.1) ($R = V/I$), which merely defines resistance under particular current and voltage conditions. The resistance of many materials changes with voltage. Ohm's law describes the relationship between I and V only for those whose resistance remains constant.

Figure 25.2(b) simply follows from Equation (25.2) and shows the effect of changing the dimensions of a conductor, say a wire, made from a given material. A thicker and/or shorter wire will allow more current to pass than a thinner and/or longer wire under the same conditions.

Under constant conditions, metals generally obey Ohm's law. They tend to deviate from linear behaviour if their temperature is not kept constant. For example, Figure 25.2(c) shows the I–V characteristic of a bulb filament. The slope of the curve progressively decreases as the potential difference is increased. This is because the resistance of the filament increases as it becomes hotter.

25.4 RESISTORS

Although we have discussed resistance in terms of conductors and light bulb filaments, there are electronic components called *resistors* which are specifically designed to provide resistance in electrical circuits. Resistors are made from a variety of materials — for example, from wire wound into a coil, or from metal oxides or carbon. There are two main types — namely fixed and variable. Fixed resistors, with a fixed value, are simply provided with a connection at either end. Variable resistors consist of a track, sometimes a wire-wound coil, with a sliding contact that can be moved along its length. The resistance over the total track length is, of course, fixed (and there are normally connections at either end), but intermediate values can be tapped off by using the sliding contact to vary the length of wire or track through which current has to pass.

Resistors commonly have values of thousands or even millions of ohms, in which case the convenient units to use are kilohms (kΩ) or megohms (MΩ), respectively.

Where a number of resistors are interconnected, we can use Ohm's law to evaluate their combined effect in terms of a single resistance value.

25.5 RESISTORS IN SERIES

Figure 25.3 shows three resistances (R_1, R_2 and R_3) connected in series, so the same current I will pass through them all (as can readily be demonstrated by connecting an ammeter into any part of the circuit). Between entering R_1 and leaving R_3, each coulomb will have lost energy, as heat, equivalent to the total drop in potential V across

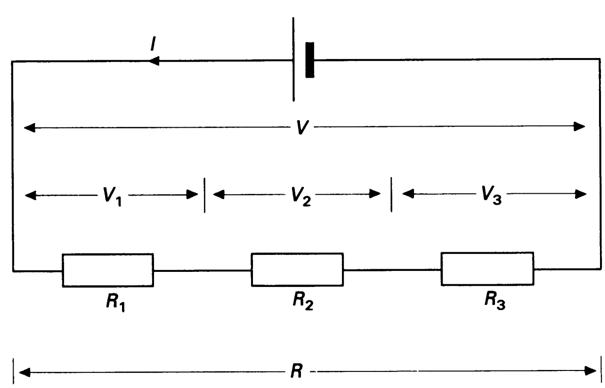

Figure 25.3

the resistors. The total potential drop will be the sum of the separate drops across the individual resistors (as can readily be demonstrated with a voltmeter). In general,

$$V = V_1 + V_2 + V_3 + \cdots V_n$$

and, since $V = IR$,

$$IR = IR_1 + IR_2 + IR_3 + \cdots IR_n$$

Dividing both sides by I, which is constant throughout,

$$R = R_1 + R_2 + R_3 + \cdots R_n \qquad (25.3)$$

Thus, the total combined resistance is the sum of the individual values.

Worked Example 25.1

Find the current that flows when a potential difference of 12 V is maintained across a 2 Ω resistor and a 4 Ω resistor connected in series. Hence find the potential difference across each resistor.

The circuit is shown in Figure 25.4.

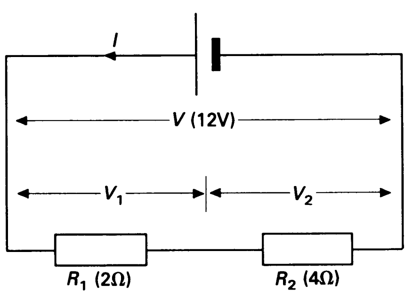

Figure 25.4

From Equation (25.3), the resistors have a combined value of 2 Ω + 4 Ω = 6 Ω. A potential difference of 12 V across a resistance of 6 Ω will result in a current I, given by

$$I = V/R = 12/6 = 2 \text{ A}$$

The potential difference V_1 and V_2 across each resistor is therefore given by

$$V_1 = IR_1 = 2 \times 2 = 4 \text{ V}$$

and

$$V_2 = IR_2 = 2 \times 4 = 8 \text{ V}$$

The example above shows that two resistors in series divide the total voltage across them in the ratio of their respective resistance values. The general case is easy to prove. The current is the same throughout the circuit; therefore,

$$I = \frac{V_1}{R_1} = \frac{V_2}{R_2}$$

so

$$\frac{V_1}{V_2} = \frac{R_1}{R_2} \tag{25.4}$$

Furthermore, if V is the total voltage across the series combination, then

$$I = \frac{V}{R_1 + R_2}$$

and, since $V_1 = IR_1$,

$$V_1 = \frac{V}{R_1 + R_2} \times R_1 \tag{25.5}$$

and similarly for V_2.

This is the basis of the *potential divider*, which is a device used to divide a voltage into given fractions.

25.6 RESISTORS IN PARALLEL

Figure 25.5 shows three resistances (R_1, R_2 and R_3) connected in parallel. In this case it is the potential difference V across them that is fixed, while the current is divided between them. The total current I is given by

$$I = I_1 + I_2 + I_3 + \cdots I_n$$

and, since $I = V/R$

$$\frac{V}{R} = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} + \cdots \frac{V}{R_n}$$

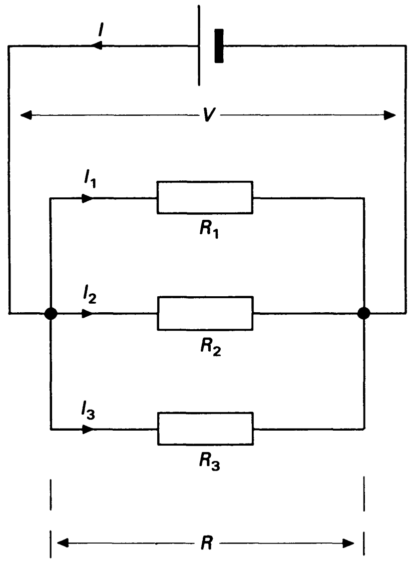

Figure 25.5

Dividing both sides by V, which is constant,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \frac{1}{R_n} \qquad (25.6)$$

Thus, resistances in parallel have a combined value less than any of the individual values.

(Be very careful not to confuse these equations with the corresponding equations for capacitors. Remember that, for series combinations,

$$R = R_1 + R_2 + R_3 + \cdots R_n$$

and

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \frac{1}{C_n}$$

and for parallel combinations,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \frac{1}{R_n}$$

and

$$C = C_1 + C_2 + C_3 + \cdots C_n)$$

Worked Example 25.2

Find the current that flows when a 2 Ω resistor and a 4 Ω resistor are connected in parallel across a 12 V battery.

The circuit is shown in Figure 25.6.

From Equation (25.6), the resistors have a combined value given by

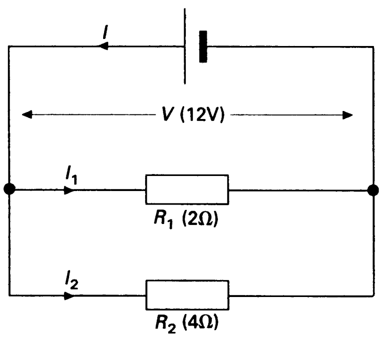

Figure 25.6

$$\frac{1}{R} = \frac{1}{2} + \frac{1}{4} = 0.75$$

Therefore, $R = 1/0.75 = 1.33\ \Omega$.

A potential difference of 12 V across a resistance of 1.33 Ω will result in a current I, given by

$$I = V/R = 12/1.33 = 9\ \text{A}$$

(Alternatively, the currents I_1 and I_2 flowing through each resistor are given by

$$I_1 = V/R_1 = 12/2 = 6\ \text{A}$$

and

$$I_2 = V/R_2 = 12/4 = 3\ \text{A}$$

Therefore,

$$I = I_1 + I_2 = 9\ \text{A})$$

Worked Example 25.3

For the circuit in Figure 25.7, (a) find the potential difference between A and B and between B and C, and (b) find the current passing through the 20 Ω resistor and through the 30 Ω resistor.

The resistance R_{BC} of the parallel combination between B and C is given by

$$\frac{1}{R_{BC}} = \frac{1}{20} + \frac{1}{30}$$

which gives $R_{BC} = 12\ \Omega$.

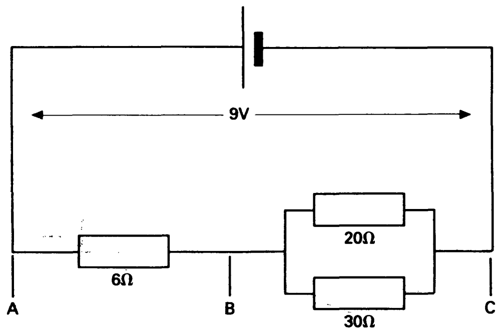

Figure 25.7

The total resistance R_{AC} is 6 Ω + 12 Ω = 18 Ω. Since the potential difference across AC is 9 V, the current drawn by the total combination is given by

$$I = V/R = 9/18 = 0.5\ \mathrm{A}$$

(a) The potential difference V_{AB} across AB is given by

$$V_{AB} = I \times R_{AB} = 0.5 \times 6 = 3\ \mathrm{V}$$

and the potential difference V_{BC} across BC is given by

$$V_{BC} = I \times R_{BC} = 0.5 \times 12 = 6\ \mathrm{V}$$

(b) Since the potential difference across BC is 6 V, the current passing through the 20 Ω resistor is given by

$$I = V/R = 6/20 = 0.3\ \mathrm{A}$$

and through the 30 Ω resistor by

$$I = V/R = 6/30 = 0.2\ \mathrm{A}$$

25.7 E.M.F. AND INTERNAL RESISTANCE

The basic function of an electric cell or battery is to use chemical energy to raise charge through a potential difference. Equation (22.3) ($W = QV$) tells us, for example, that a 3.0 V cell gives 3.0 J of energy to each coulomb passing through it.

When a cell is not being used, the voltage across its terminals is called the *electromotive force* or *e.m.f.* (symbol E), as indicated in

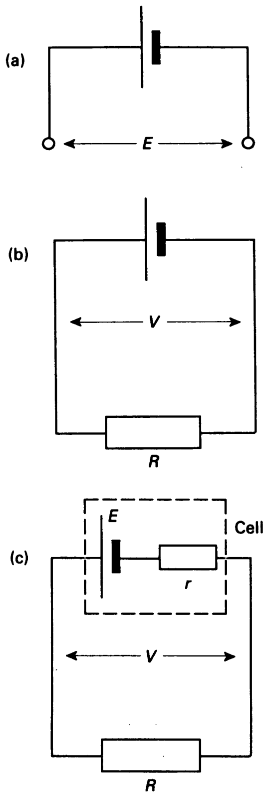

Figure 25.8

Figure 25.8(a). The e.m.f. is the electrical energy supplied by the cell per coulomb. If the cell is being used to maintain a current, say through the resistance R in Figure 25.8(b), then the potential difference V across the resistor, and across the cell, is found to be less than the e.m.f. The reason for this is that the cell has its own *internal resistance* as charge moves through its interior. This means that, when a current flows around the circuit, electrical energy is consumed by the cell as well as by the resistor. It can be helpful to view internal resistance as a component of the circuit into which the cell is connected. Figure 25.8(b) is therefore redrawn in Figure 25.8(c) to show the circuit as a cell of e.m.f. E and internal resistance r connected across the resistance R.

R is in series with r, so the total resistance in the circuit is $(R + r)$. From Ohm's law, the current I flowing through the circuit will be

$$I = \frac{E}{R + r} \tag{25.7}$$

which, on rearranging, gives

$$E = IR + Ir = V + Ir$$

where V $(= IR)$ represents the potential difference across the cell terminals. Ir is the inaccessible voltage lost across the resistance of the cell which produces heat and which only becomes evident when a current I flows. On rearranging,

$$V = E - Ir \tag{25.8}$$

That is to say, when current passes through the circuit, each coulomb of charge passing through the cell gains E joules but also loses Ir joules to the internal resistance. This leaves V joules for external use. Because the magnitude of Ir increases with I, the usable voltage V of the cell decreases in proportion to the current drawn from it. Equation (25.8) enables us to calculate the internal resistance of a cell or other source of e.m.f. simply by subtracting the potential difference across its terminals from the e.m.f. and dividing the result by the current.

25.8 BATTERIES

As we noted in Topic 23, a battery is simply a number of cells connected together to form a single unit. Batteries may have cells connected in series or in parallel.

If the cells are in series with positive terminals connected to negative, as in Figure 25.9(a), then their combined e.m.f. and their combined internal resistance are both obtained by adding the individual values together as follows:

$$E = E_1 + E_2 + E_3 + \ldots E_n \tag{25.9}$$

and

$$r = r_1 + r_2 + r_3 + \ldots r_n \tag{25.10}$$

For example, six 2.0 V lead–acid cells are used in series in 12 V car batteries. (Lead-acid cells have low internal resistance, which allows high currents to be drawn briefly for starting engines.)

If identical cells are connected in parallel, as in Figure 25.9(b), then their combined e.m.f. is the same as a single cell but their *capacity* is correspondingly increased. Capacity is measured in ampere-hours, 1 ampere-hour being the charge passing a point in 1 h in a conductor carrying a steady current of 1 A. (In practice, the actual capacity generally depends on the rate at which a cell is discharged.) As we might expect from Equation (25.6), the combined resistance is given by r/n, where r is the internal resistance of each of the n identical parallel cells.

The parallel combination of different types of cell is more complex and we shall consider it in the next topic.

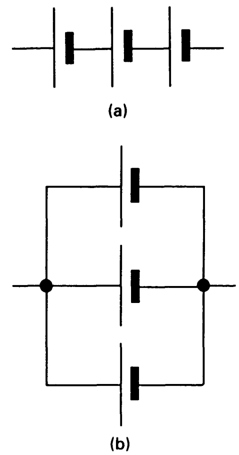

Figure 25.9

25.9 POWER

Equation (24.3) ($P = IV$) applies to any device converting electrical energy to another form. If the device is a resistor that obeys Ohm's law and converts the electrical energy entirely to heat, then we can modify the equation by introducing R and eliminating I or V as required. Thus,

$$P = IV$$

and, since $I = V/R$,

$$P = \frac{V^2}{R} \tag{25.11}$$

Also, since $V = IR$,

$$P = I^2R \tag{25.12}$$

Worked Example 25.4

A potential difference of 12 V is maintained across an 8 Ω and a 16 Ω resistor connected (a) in series, and (b) in parallel. Find the electrical power consumed in each case by each resistor.

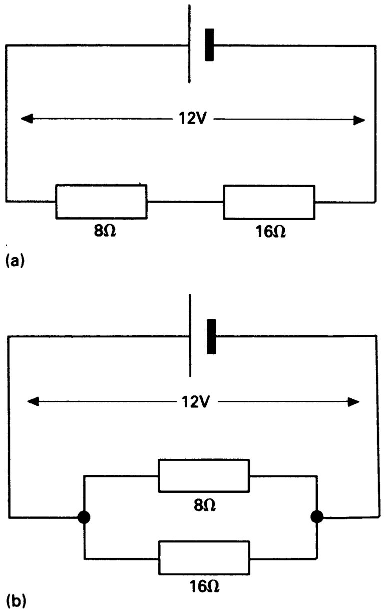

Figure 25.10

The circuits are shown in Figure 25.10.

(a) The total resistance of the series combination in Figure 25.10(a) is (8 Ω + 16 Ω =) 24 Ω; therefore, the current I flowing round the circuit is given by

$$I = \frac{V}{R} = \frac{12}{24} = 0.5 \text{ A}$$

The power consumed by the 8 Ω resistor is given by

$$P = I^2 \times R = 0.5^2 \times 8 = 2 \text{ W}$$

and by the 16 Ω resistor by

$$P = I^2 \times R = 0.5^2 \times 16 = 4 \text{ W}$$

(b) The power consumed by the 8 Ω resistor is given by

$$P = \frac{V^2}{R} = \frac{12^2}{8} = 18 \text{ W}$$

and by the 16 Ω resistor by

$$P = \frac{V^2}{R} = \frac{12^2}{16} = 9 \text{ W}$$

Questions

(Use any previously tabulated data as required.)

1. Find the resistance of a 100 m length of copper wire 1.2 mm in diameter.
2. A copper wire, 10 m long and 1.7 mm² in cross-sectional area, carries a current of 15 A. Find the potential drop along its length.
3. The individual resistors shown in Figure 25.11 all have the value of 3 Ω. Find the combined resistance across AB in each case.
4. What value resistor must be connected with a 20 Ω resistor to give a combined value of 12 Ω? Should it be connected in series or in parallel?
5. (a) Show that the combined resistance R of two resistors R_1 and R_2 connected in parallel is given by

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

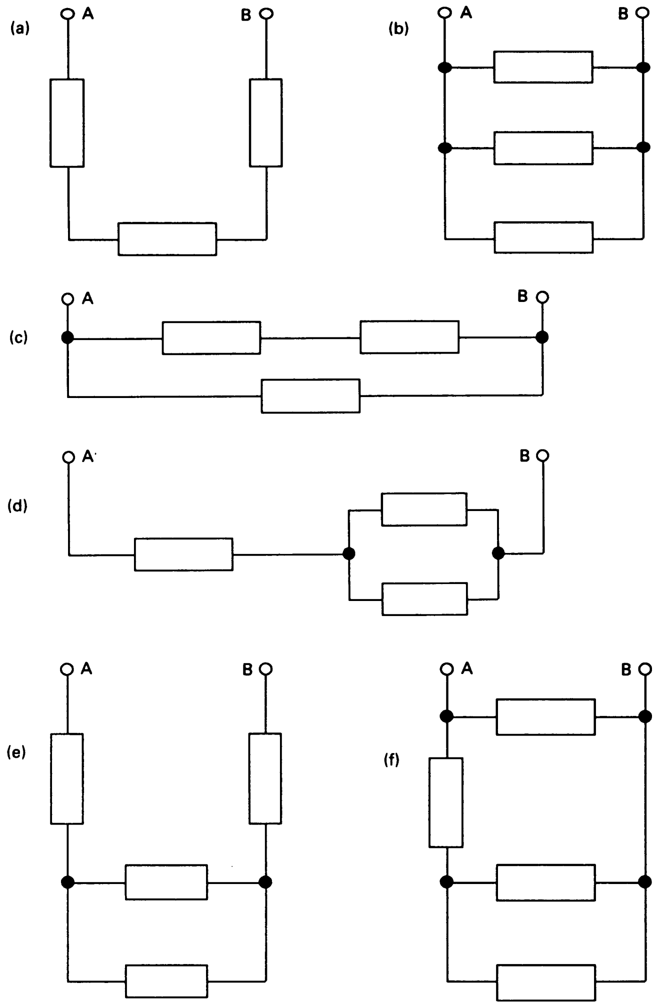

Figure 25.11

(b) Two resistances connected in parallel have a combined value of 1.2 Ω. When connected in series, they have a combined value of 5 Ω. Find their individual values.

6. For the circuit shown in Figure 25.12, find (a) the potential difference across the 20 Ω resistor, (b) the

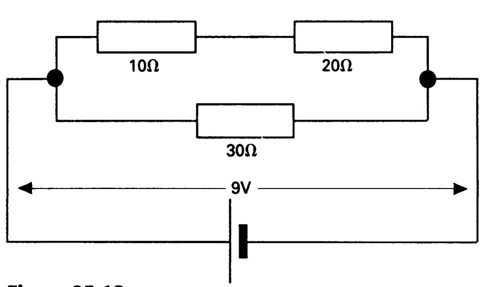

Figure 25.12

combined resistance of the three resistors, and (c) the current through the 10 Ω resistor.

7. Two resistors are connected in series across a 3 V supply. One of the resistors, with a value of 200 Ω, has a potential difference of 0.1 V across it. Find the value of the other resistance.

8. (a) A battery of 4.5 V e.m.f. with an internal resistance of 1.25 Ω is used to maintain a current of 0.2 A through a resistor. Find the potential difference across the resistor.
   (b) Find the potential difference if the resistor is changed for one that draws a current of 0.8 A.
   (c) Estimate the current which flows if the terminals of the battery are momentarily short-circuited with a very low resistance conductor.

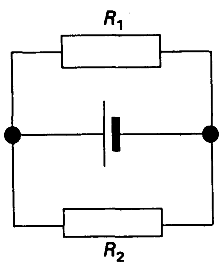

Figure 25.13

9. In the circuit shown in Figure 25.13 the cell has an e.m.f. of 6 V and an internal resistance of 0.5 Ω. R_1 and R_2 are resistors of 3 Ω and 6 Ω, respectively. Find the current through (a) the cell, (b) R_1 and (c) R_2.

10. Two identical cells, each with an e.m.f. of 4.5 V and an internal resistance of 1.5 Ω, are connected (a) positive terminal to negative in series and (b) positive terminal to positive in parallel across a 3 Ω resistance. Find the current through the resistance in each case.

11. A current of 1.2 mA flows through a resistor across which there is a potential difference of 6 V. Find the resistance and the power consumption of the resistor.

12. The maximum power rating of a particular 100 Ω resistor was given as 2.25 W. Find the maximum current that it would safely carry.

13. By considering the units on the right-hand sides of Equation (24.3) (page 228) and Equations (25.11) and (25.12), confirm that the quantity obtained in each case is power.

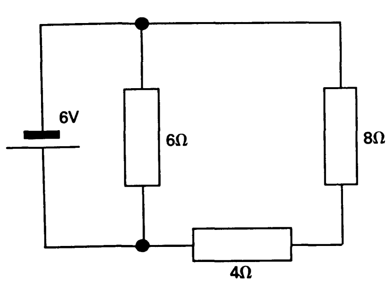

Figure 25.14

14. For the circuit in Figure 25.14 find (a) the power supplied by the battery and (b) the power dissipated in the 8 Ω resistor. (Assume that the internal resistance of the battery is negligible.)

TOPIC 26 SOME SIMPLE CIRCUITS

COVERING:

- shunts and multipliers (ammeters and voltmeters);
- potential dividers;
- the potentiometer;
- the Wheatstone bridge;
- Kirchhoff's laws.

The purpose of this topic is to develop some of the ideas that we have already met and to broaden our discussion of electrical circuits.

26.1 SHUNTS AND MULTIPLIERS

In this section we shall consider ammeters and voltmeters based on the moving-coil galvanometer. Nowadays these are tending to be superseded by digital electronic instruments; nevertheless they provide a good basis for our discussion.

The moving-coil galvanometer is an instrument that measures small electric currents. Essentially it consists of a coil mounted between the poles of a permanent magnet in such a way that the coil rotates when a current is passed through it. (We shall consider this in more detail in the next topic.) Rotation is resisted by a spring, so that the angle through which the coil moves gives a measure of the current. This is usually indicated by the position of a needle against a calibrated scale. As far as its behaviour as a circuit element is concerned, the instrument can generally be treated as a resistor because of the resistance of its internal wiring, particularly that of the coil.

Moving-coil galvanometers usually need to be modified for use as ammeters, because they are generally too sensitive for normal currents and the needle tends to go off scale. The range of the instrument needs to be adjustable, so that the current corresponding to full-scale deflection of the needle can be varied as required. This is very easily done using a *shunt*, which is a resistance connected in parallel with the galvanometer which allows a fixed proportion of the total current to bypass the instrument itself, as in Figure 26.1.

R_g is the resistance of the galvanometer G. By adjusting the shunt resistance R_s, we can vary the proportion in which the total current

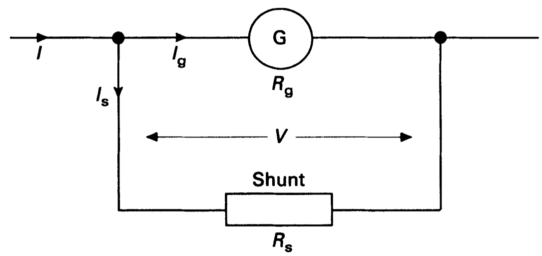

Figure 26.1

I is divided into I_g and I_s, the parallel currents through the galvanometer and through the shunt, respectively. Since the potential difference V across both resistances is the same, then, from Ohm's law,

$$V = I_s R_s = I_g R_g$$

Therefore,

$$R_s = \frac{I_g R_g}{I_s}$$

and, since $I_s = I - I_g$,

$$R_s = \frac{I_g R_g}{I - I_g} \tag{26.1}$$

By adjusting R_s we can adjust the range of the instrument to suit the magnitude of the current being measured. Obviously, I_g must not exceed the full-scale deflection current; otherwise the needle will go off scale.

Worked Example 26.1

A moving-coil galvanometer shows full-scale deflection with a potential difference of 75 mV across the terminals and a current of 15 mA flowing through the coil. How can the instrument be adapted to measure currents up to 2.5 A?

The resistance of the galvanometer is obtained from Equation (25.1) (on page 232) as follows:

$$R = \frac{V}{I} = \frac{75 \times 10^{-3}}{15 \times 10^{-3}} = 5\ \Omega$$

If, in Figure 26.1, I_g is not to exceed 0.015 A, then a shunt is required which allows (2.5 − 0.015) A to bypass the galvanometer when the total current I is 2.5 A. The shunt resistance may therefore be obtained from Equation (26.1) as follows:

$$R_s = \frac{I_g R_g}{I - I_g} = \frac{15 \times 10^{-3} \times 5}{(2.5 - 0.015)} = 0.03\ \Omega$$

Since a potential difference applied across the terminals of a galvanometer will cause a current to flow, we can use it as a voltmeter. If the wiring inside the galvanometer obeys Ohm's law, then there will be a linear relationship between V and I and it can be calibrated to measure either. But, as with the ammeter, we need to be able to vary the sensitivity of the instrument. In this case we use a *multiplier*, as in Figure 26.2.

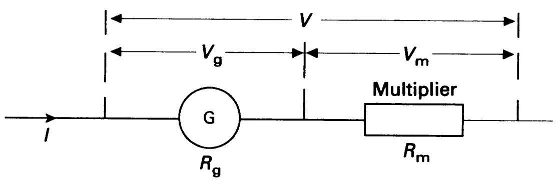

Figure 26.2

A multiplier is simply a series resistance, R_m in the figure. Together with the galvanometer resistance R_g, it divides the potential difference V in accordance with Equation (25.4) (page 238). Since the current I is the same through both resistances, because they are in series, then

$$I = \frac{V_g}{R_g} = \frac{V_m}{R_m}$$

Therefore,

$$R_m = \frac{R_g V_m}{V_g}$$

and, since $V_m = V - V_g$,

$$R_m = \frac{R_g (V - V_g)}{V_g} \qquad (26.2)$$

Thus, we can adjust the range of the voltmeter by varying the multiplier resistance, remembering that V_g must not exceed the full-scale deflection voltage.

Worked Example 26.2

How can the moving-coil galvanometer in Worked Example 26.1 be adapted to measure voltages up to 12 V?

If, in Figure 26.2, V_g is not to exceed 0.075 V, then a multiplier is required across which there is a potential difference of (12 – 0.075) V when the total potential difference is 12·V. The multiplier resistance may therefore be obtained from Equation (26.2) as follows

$$R_m = \frac{R_g (V - V_g)}{V_g} = \frac{5(12 - 0.075)}{0.075} = 795\ \Omega$$

The ideal voltmeter has infinite resistance and the ideal ammeter has zero resistance, so that they have no effect on the current flowing through the circuit where they are being used. In practice, real moving-coil instruments have finite resistance, which can affect the measurements significantly. For example, let us consider the measurement of resistance by the so-called *ammeter–voltmeter method* which we met in the last topic (Figure 25.1 on page 233). The two arrangements in Figure 26.3 incorporate a *rheostat* to control the current through the circuit. (A rheostat is a variable resistor connected so that current flows between one of the end connections and the sliding contact.)

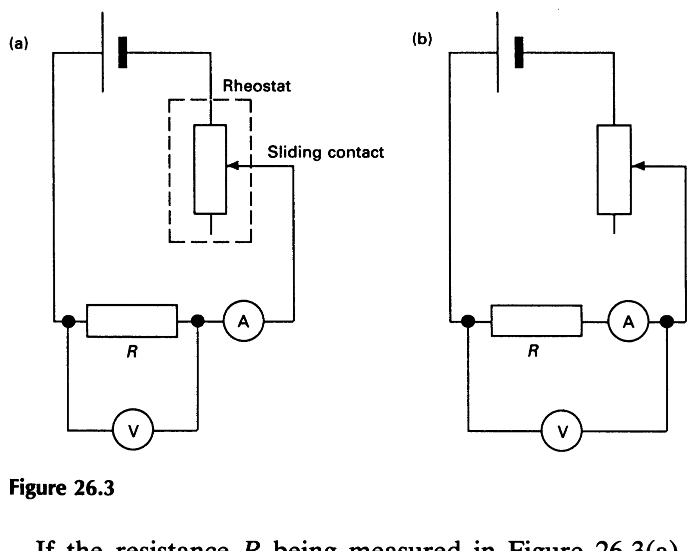

Figure 26.3

If the resistance R being measured in Figure 26.3(a) has a sufficiently high value, then the current flowing through it may be low enough for the current through the voltmeter V to be significant. This problem can be overcome by connecting the voltmeter across both the resistance and the ammeter A, as in Figure 26.3(b), so that the ammeter measures the true current through the resistance. Note that this arrangement will only give a satisfactory measurement where the resistance of the ammeter is negligible compared with the component under test; otherwise there will be a significant voltage drop across the ammeter.

26.2 POTENTIAL DIVIDERS

In the last topic we met a potential divider in the form of two resistors in series which enable a fraction of the total voltage across them to be tapped off according to Equation (25.5) (page 238). Potential dividers take a variety of different forms, including a chain of any number of series resistors dividing the voltage across it into as many portions as there are individual resistances.

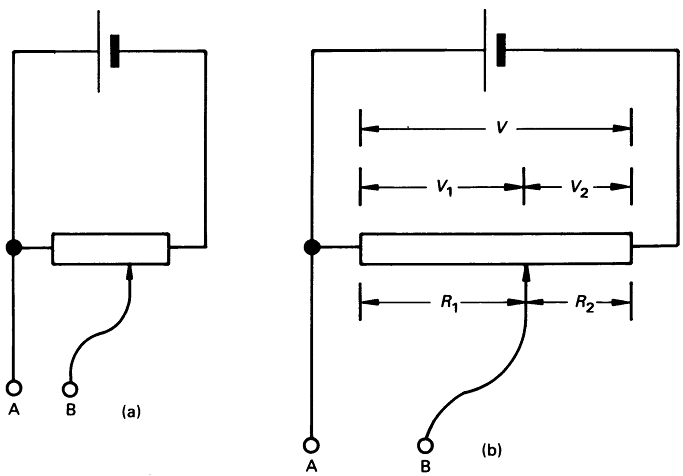

Figure 26.4

Figure 26.4(a) shows a potential divider based on a variable resistor with both ends connected across a cell. The sliding contact divides the track into two parts, corresponding to R_1 and R_2 in Equation (25.5), as shown in Figure 26.4(b). By adjusting the position of the sliding contact, the potential difference across the output terminals AB may be varied from zero up to the maximum potential difference V across the ends of the variable resistor. As the following worked example demonstrates, this voltage will fall when an external current is drawn from the output terminals.

Worked Example 26.3

For the potential divider shown in Figure 26.4, $R_1 = 200\ \Omega$, $R_2 = 100\ \Omega$ and $V = 12$ V. Assuming the battery has negligible internal resistance, find the potential difference across the output terminals AB

(a) when no external current is being drawn from them, and (b) when an external resistance of 200 Ω is connected across them.

(a) From Equation (25.5),

$$V_1 = \frac{R_1}{R_1 + R_2} \times V = \frac{200}{200 + 100} \times 12 = 8 \text{ V}$$

(and $V_2 = 4$ V).
(b) With the external resistance R_e connected, the combined resistance R_{AB} across AB is given by

$$\frac{1}{R_{AB}} = \frac{1}{R_1} + \frac{1}{R_e} = \frac{1}{200} + \frac{1}{200}$$

Hence, $R_{AB} = 100\ \Omega$.
Since $R_{AB} = R_2 = 100\ \Omega$, the voltage V is divided into equal halves (i.e. $V_1 = V_2 = 6$ V). The potential difference across AB is therefore reduced to 6 V by the effect of the external resistance. (Note that the greater the external resistance the less will be the current drawn by it and the less the reduction in the potential difference across AB.)

The *potentiometer* (see Figure 26.5(a)) is a very simple form of potential divider. It consists of a resistance wire of uniform cross-section which has a constant potential difference maintained across its ends by means of a so-called driver cell. A sliding contact is used to tap off any fraction of the total potential difference. A scale is fixed parallel to the wire so that the lengths l_1 and l_2 (corresponding to R_1 and R_2 in Figure 26.4) can be measured. The galvanometer G has a central zero position so that it can detect current flowing in either direction when other circuit elements are connected across AB.

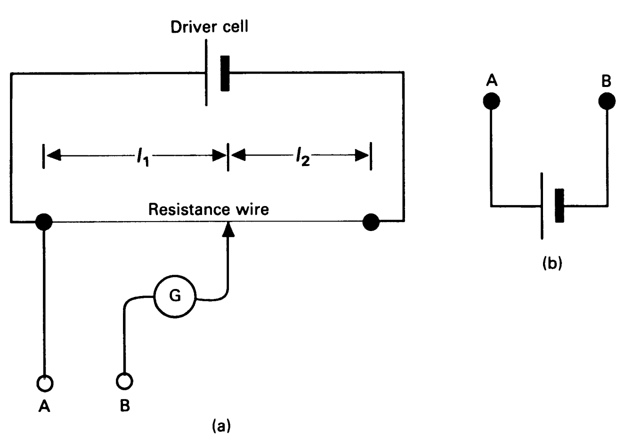

Figure 26.5

The potentiometer measures potential difference and can be used to find the true e.m.f. of a cell. The cell is connected across AB so that it is in opposition to the driver (as indicated in Figure 26.5(b)). Provided that the cell e.m.f. is less than the potential difference between the ends of the potentiometer, then a length of wire l_1 can be found where the potential difference across it exactly balances the e.m.f. of the cell. Under these conditions, no current flows through the galvanometer and the potentiometer is said to be *balanced*. Since no current is flowing through the cell under test, the potential difference across its terminals is equal to its e.m.f. (because $Ir = 0$ in Equation 25.8 on page 242). Thus, the balanced length l_1 is proportional to the e.m.f. of the cell. If this procedure is repeated with another cell, then

$$\frac{E_A}{E_B} = \frac{l_A}{l_B} \tag{26.3}$$

where E_A and E_B are the e.m.f. values and l_A and l_B are the respective values of the balance length l_1 for the two cells. The e.m.f. of an unknown cell can therefore be obtained by comparing it with a cell of accurately known e.m.f.

The potentiometer needs modification for measuring very small e.m.f. values, because the balance length will be very small and subject to large errors in its measurement. The problem could be solved by using a very long potentiometer wire but this would generally be impracticable. However, the same effect can be achieved by connecting a large resistor in series with the wire, as in Figure 26.6(a). The total voltage across the series pair is then divided according to Equation (25.5) (page 238), where R_1 is the resistance of the whole length of the wire and R_2 is the added series resistance.

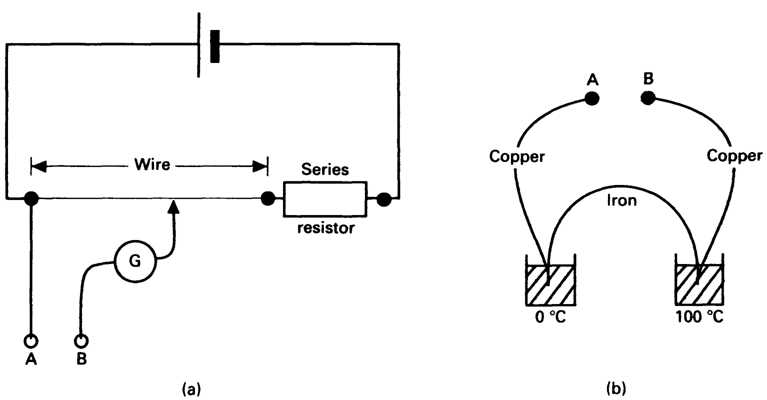

Figure 26.6

This arrangement can be used to measure the small e.m.f. produced by a *thermocouple*. Thermocouples are devices that are used for measuring temperature. For example, if a piece of iron wire is connected to the potentiometer via two copper wires and the two iron–copper junctions are at different temperatures, say 0 °C and 100 °C as in Figure 26.6(b), then a small e.m.f. can be measured. This e.m.f. varies with the temperature difference and the device can therefore be used as a thermometer by keeping one of the junctions at a known reference temperature.

Since the potentiometer draws no external current, it behaves, in effect, like a voltmeter of infinitely high resistance and therefore does not disturb any circuit into which it is connected. It can be used to measure potential difference accurately, because the balance length can be measured accurately. Being a *null method* (i.e. a balance method), it does not rely on the accuracy of the galvanometer. It therefore has a number of advantages over ordinary voltmeters and can even be used to calibrate them. (*Calibration* is the determination of the true values corresponding to the actual readings given by any type of instrument.)

A variable potential difference can be applied across the terminals of an ordinary voltmeter by using a variable resistor as a potential divider (as in Figure 26.4). For any given reading on the voltmeter, the true potential difference across its terminals can be measured by using the potentiometer. The voltmeter can therefore be calibrated over its whole range by varying the potential difference.

The potentiometer can be used to measure current by finding the potential difference across a known resistance through which the current is passing and then applying Ohm's law. This may involve inserting into the circuit an accurately known resistance of sufficiently low value not to disturb the current. This principle can be used to calibrate an ammeter by measuring the true current passing through it for any given reading.

The value of an unknown resistance can be obtained by connecting a known resistance in series with it and using the potentiometer to measure the potential difference across each resistance in turn when they are both carrying the same current. If V_A and V_B are the potential differences corresponding to the balance lengths l_A and l_B for the resistances R_A and R_B, respectively, then

$$\frac{l_A}{l_B}\left(=\frac{V_A}{V_B}=\frac{IR_A}{IR_B}\right)=\frac{R_A}{R_B} \tag{26.4}$$

(Before we leave this section, note that variable resistors used in electronic circuits are sometimes referred to as potentiometers.)

26.3 THE WHEATSTONE BRIDGE

The Wheatstone bridge is a circuit that can be used for the accurate measurement of resistance. It consists of a network of four resistances R_1, R_2, R_3 and R_4, connected as shown in Figure 26.7. For the purposes of our discussion let us assume that the voltages across them are V_1, V_2, V_3 and V_4, respectively.

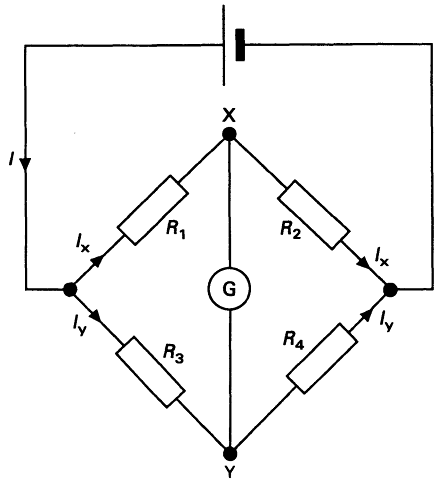

Figure 26.7

If the values of the resistances are such that no current flows through the galvanometer, then the bridge is said to be *balanced*. The potential difference across XY must then be zero and therefore $V_1 = V_3$ and $V_2 = V_4$. Furthermore, the current I entering the network divides into I_x through R_1 and R_2, and I_y through R_3 and R_4 (remembering that none flows through the galvanometer).

If $V_1 = V_3$, then $I_xR_1 = I_yR_3$ (since $V = IR$), and if $V_2 = V_4$, then $I_xR_2 = I_yR_4$.

Dividing one equation by the other,

$$\frac{I_xR_1}{I_xR_2} = \frac{I_yR_3}{I_yR_4}$$

Therefore,

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \qquad (26.5)$$

Knowing the values of three of the resistances, the fourth may be found.

The *metre bridge*, shown in Figure 26.8, is a simple practical version of the Wheatstone bridge. It consists of a resistance wire 1 m

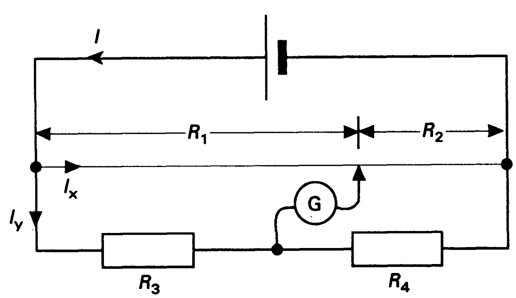

Figure 26.8

long which enables the ratio R_1/R_2 to be continuously varied with a sliding contact. Knowing this ratio and the value of either R_3 or R_4, the value of the fourth resistance can be found from Equation (26.5).

26.4 KIRCHHOFF'S LAWS

Kirchhoff's laws are useful when it comes to considering steady currents flowing through circuits which are too complicated to be treated as series and parallel combinations of resistances and e.m.f.s. In such cases we consider circuits in terms of *junctions* (where three or more conductors meet) and closed *loops*. We have already met the basic ideas involved. Now we shall formalise them.

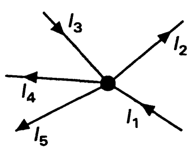

Figure 26.9

The first law states that the sum of the currents entering a junction in a circuit is equal to the sum of the currents leaving it. In mathematical terms $\Sigma I = 0$ (where Σ means 'the sum of'). Currents arriving are normally treated as positive and those leaving as negative. In terms of Figure 26.9, which shows two currents arriving at a junction and three leaving it,

$$I_1 - I_2 + I_3 - I_4 - I_5 = 0$$

or

$$I_1 + I_3 = I_2 + I_4 + I_5$$

This is the statement of a special case of the principle of conservation of charge. It also expresses the idea that charge does not accumulate at any point under steady state conditions.

Kirchhoff's second law tells us that, in any closed loop in a circuit, the sum of the e.m.f.s is equal to the sum of the potential differences across the resistances in the loop. In mathematical terms, $\Sigma E = \Sigma IR$. This is, of course, the statement of a special case of the principle of conservation of energy. As we shall see, we have to think carefully about the signs in applying the second law. If we follow the conventional current flowing round a simple circuit, then the e.m.f. of a cell (or any source of e.m.f.) represents a rise in potential and the potential difference across a resistance represents a drop in potential. Figure 26.10 shows a more awkward case where two cells are connected in parallel to form a closed loop which is part of a larger circuit. E_1 and E_2 are the e.m.f.s of the cells and r_1 and r_2 are their respective internal resistances.

Applying the first law to the junctions at either end of the loop simply tells us that the sum of the currents I_1 and I_2 through the two branches of the loop is equal to the total current I_3.

We can apply the second law by travelling either clockwise or anticlockwise round the loop formed by the two branches. In either case we equate the sum of the e.m.f.s (i.e. the potential rises) to the

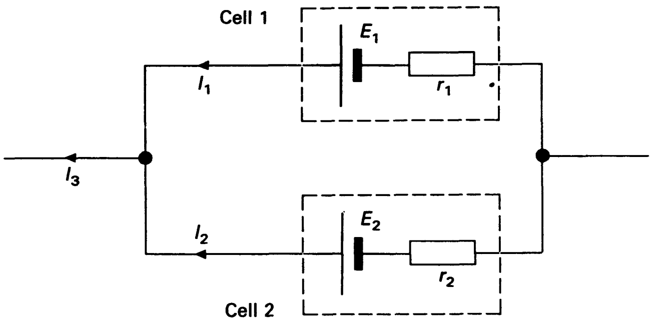

Figure 26.10

sum of the potential drops, taking into account our direction of travel. We take an e.m.f. as positive if we enter the cell via the negative terminal and exit via the positive; we take a potential drop as positive if we travel through the resistance in the same direction as the current. Thus, if we choose to travel clockwise round the loop in Figure 26.10, then E_2 and I_2r_2 are positive, while E_1 and I_1r_1 are negative. We have

$$\Sigma E = E_2 - E_1$$

and

$$\Sigma IR = I_2r_2 - I_1r_1$$

Therefore,

$$E_2 - E_1 = I_2r_2 - I_1r_1$$

The following worked example illustrates a simple application of Kirchhoff's laws. First, we identify the current passing through each branch. (In some cases it may be difficult to decide the current direction, but if the wrong assumption is made, then the calculated value will simply be negative.) Next we apply the first law to the junctions, then the second law to the closed loops.

Worked Example 26.4

A 10 Ω resistance is connected in parallel with two cells simultaneously, one of 3 V e.m.f. and 1 Ω internal resistance and the other of 6 V e.m.f. and 2 Ω internal resistance, with their positive terminals together. Find the current in each branch of the circuit.

The circuit is shown in Figure 26.11.

Let us choose I_1 and I_2 as the currents in the upper and lower branches in the directions shown.

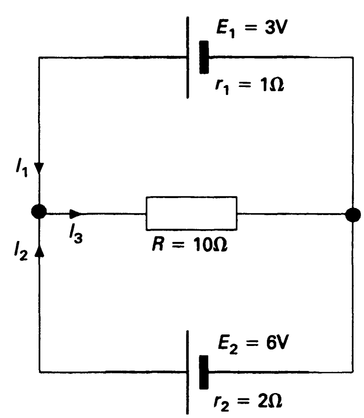

Figure 26.11

Applying the first law to the left-hand junction, I_1 and I_2 combine to give I_3, which is the current through the resistance in the middle branch.

Applying the second law to the upper loop (containing cell 1 and the resistance R), then, in the anticlockwise direction,

$$E_1 = I_1r_1 + I_3R$$

Therefore,

$$E_1 = I_1r_1 + (I_1 + I_2)R$$

and, substituting the given values,

$$3 = 11I_1 + 10I_2 \qquad \text{(a)}$$

Applying the second law to the lower loop (containing cell 2 and the resistance R), then, in the clockwise direction,

$$E_2 = I_2r_2 + I_3R$$

Therefore,

$$E_2 = I_2r_2 + (I_1 + I_2)R$$

and, substituting the given values,

$$6 = 10I_1 + 12I_2 \qquad \text{(b)}$$

Combining (a) and (b) to eliminate I_2 gives

$$I_1 = -0.75 \text{ A}$$

and substituting -0.75 A for I_1 in either (a) or (b) gives

$$I_2 = 1.125 \text{ A}$$

Therefore,

$$I_3 = I_1 + I_2 = 0.375 \text{ A}$$

We can check the values of I_1 and I_2 by applying the second law to the outer loop containing just the cells (but not the resistance R). Taking the clockwise direction,

$$\Sigma E = E_2 - E_1 = 6 - 3 = 3 \text{ V}$$

and

$$\Sigma IR = I_2r_2 - I_1r_1 = (1.125 \times 2) - (-0.75 \times 1) = 3 \text{ V}$$

(As noted above, the negative value found for I_1 simply tells us that its direction is opposite to that initially chosen and shown in Figure 26.11.)

Questions

1. A battery of 12 V e.m.f. and 15 Ω internal resistance is connected across a voltmeter of (a) 500 Ω resistance (b) 5000 Ω resistance. In each case find the potential difference across the voltmeter.

2. A 9.5 Ω resistance and an ammeter of 0.1 Ω resistance are connected in series with a cell of 1.5 V e.m.f. and 0.4 Ω internal resistance. Find (a) the current passing through the ammeter and (b) the potential difference across the cell.

3. Two 1000 Ω resistors are connected in series across a battery of 6 V e.m.f. and negligible internal resistance.
   (a) Find the potential difference across a voltmeter of 2000 Ω resistance connected in parallel across one of the resistors.
   (b) What would the potential difference be if the voltmeter had infinite resistance?
   (c) What would the resistance of the voltmeter have to be for a potential difference of 2.9 V?

4. (a) In Worked Example 26.3 (page 251), find the potential difference across AB if the value of the external resistance had been 66.7 Ω.
   (b) Find the current passing through R_2 (i) in this arrangement and (ii) in both arrangements in the worked example.

5. Twelve identical pieces of wire, each of 12 Ω resistance, are connected together to form the edges of a cube. A current of 2 A enters this network at one corner of the cube and leaves it by the corner diagonally opposite. (a) Find the current in each wire. Find (b) the potential difference between the points where the current enters and leaves the network and (c) the total resistance between these points.

6. A current of 0.2 A flows through the 50 Ω resistance in the circuit shown in Figure 26.12. Using Kirchhoff's

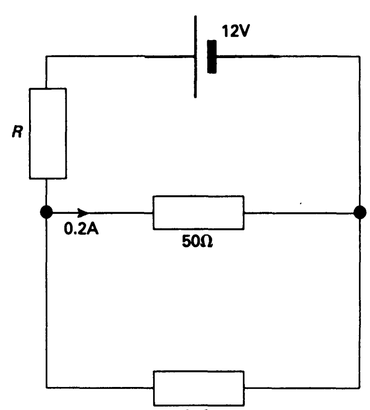

Figure 26.12

laws, find the value of the resistance R, assuming that the battery has negligible internal resistance.

7. In the Wheatstone bridge circuit shown in Figure 26.7 (page 255) $R_1 = 10\ \Omega$, $R_2 = R_3 = 20\ \Omega$ and $R_4 = 40\ \Omega$. The battery has an e.m.f. of 6 V and an internal resistance of 2 Ω. Using Kirchhoff's laws, find the current drawn from the battery.

TOPIC 27 MAGNETIC FIELDS

COVERING:

- permanent magnets;
- fields around current-carrying conductors;
- force on a current-carrying conductor in a magnetic field;
- force on a moving charge in a magnetic field;
- torque on a current-carrying coil in a magnetic field;
- force between current-carrying conductors.

We have already seen that an electric charge gives rise to an electric field. In this topic we shall see that if an electric charge is in motion, it will produce a magnetic field as well.

27.1 PERMANENT MAGNETS

The magnetic field surrounding a permanent magnet is associated with the motion of the electrons within its constituent atoms. The earth behaves like a permanent magnet for reasons that are not fully understood but are believed to have their origins in electric currents which circulate in its molten core.

Permanent magnets have equal and opposite north-seeking and south-seeking *poles*, normally called north and south poles, corresponding to the way in which a freely suspended bar magnet aligns itself in the earth's magnetic field. The opposite poles of two magnets attract one another and their like poles repel because of the forces arising from the interaction of their magnetic fields. From this it follows that the earth's north pole is actually a magnetic south pole, because it attracts the north-seeking pole of a compass. Similarly, its south pole is a magnetic north pole. (A compass is simply a small magnet, pivoted to allow it to align itself in a magnetic field.)

Magnetic fields, like electric fields, can be represented by field lines whose concentration indicates the field strength. The field direction at any point is taken to be the direction of the force acting on a north pole placed there. The field pattern round a permanent magnet can be plotted with a small compass, but it can be revealed much more quickly by covering the magnet with a sheet of stiff paper and sprinkling iron filings on top. On gently tapping the paper, the iron filings

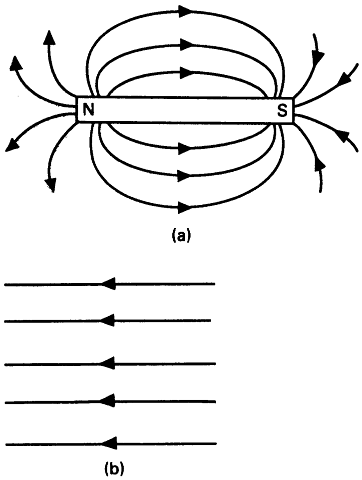

Figure 27.1

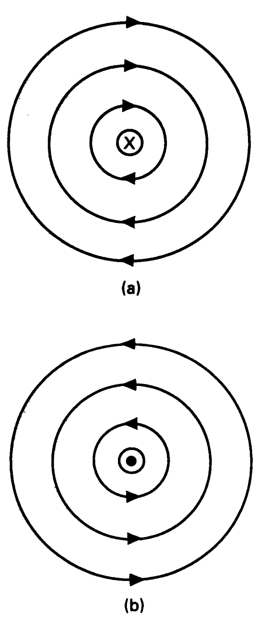

Figure 27.2

align themselves in the field. Figure 27.1(a) represents the way in which the field varies around a bar magnet. Such fields are described as non-uniform. By contrast, Figure 27.1(b) shows straight, parallel, equally spaced field lines that are characteristic of uniform fields.

The earth behaves rather as though it contains a bar magnet along its magnetic axis. A compass needle free to rotate in any direction would set horizontally at the magnetic equator and vertically at the magnetic poles, with intermediate angles elsewhere. In the UK it would point downwards at an angle of roughly 70° below the horizontal. (This angle is called the *angle of dip* or the *inclination*.)

27.2 MAGNETIC FIELDS AROUND CONDUCTORS

Since electric current is a flow of charge, we find magnetic fields associated with current-carrying conductors such as wires. In this section we shall consider the field patterns associated with a straight wire, a flat circular coil and a solenoid (i.e. a long cylindrical coil).

Figure 27.2(a) represents the field pattern around a long straight wire running perpendicularly through the page with the current passing downwards into the paper. The cross represents the tail of a departing arrow indicating the current direction. In Figure 27.2(b) the dot represents the tip of an approaching arrow indicating that the current direction is upwards out of the paper. If the current is sufficiently large, the field will be strong enough for interference from the earth's magnetic field to be insignificant and the field lines will form concentric circles around the wire. The field direction, given by the so-called *corkscrew rule*, is the direction of rotation of a right-hand screw thread advancing in the conventional current direction. (Just think of the cross in Figure 27.2(a) as the head of an ordinary screw.)

Figure 27.3 represents the field through a flat circular coil viewed from above, with the plane of the coil set at right angles to the plane of the paper. The cross and the dot indicate the current direction through the opposite sides. Very close to the wire the field pattern takes the form of more or less concentric circles which become progressively distorted further away as the field due to current in other parts of the coil becomes more significant. The direction of the field lines in the figure is still consistent with the corkscrew rule.

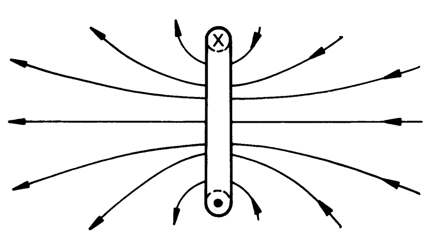

Figure 27.3

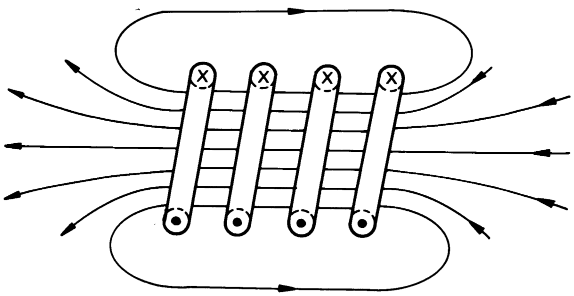

Figure 27.4

Figure 27.4 represents the field associated with a solenoid (somewhat simplified for our purposes). As we can see, there is a region of uniform magnetic field inside. The corkscrew rule still applies and the pattern is essentially an extended version of that associated with a flat coil. It is also similar to that of the permanent bar magnet in Figure 27.1(a). In fact, the solenoid will align itself in a magnetic field in just the same way as a bar magnet, and this leads us on to consider the force experienced by a current-carrying conductor in a magnetic field.

27.3 FORCE ON A CONDUCTOR IN A MAGNETIC FIELD

When a current flows through a wire suspended vertically between the poles of a U-shaped permanent magnet, the wire experiences a force and will move as shown in Figure 27.5. The direction in which it moves is given by *Fleming's left-hand rule*, which is a mnemonic involving the first and second fingers and the thumb of the left hand mutually arranged at right angles. If the *F*irst finger points in the *F*ield direction and the se*C*ond finger points in the *C*urrent direction then the thu*M*b gives the direction of *M*otion.

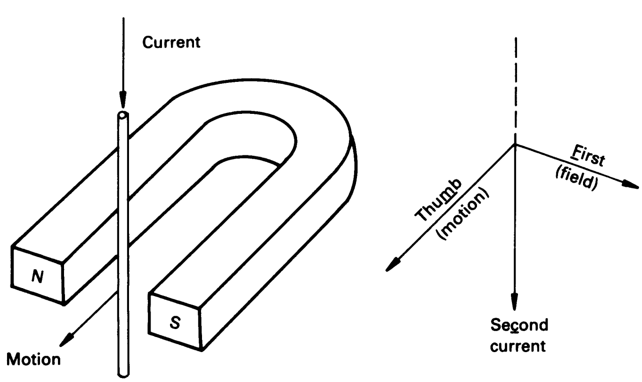

Figure 27.5

The magnitude of the force acting on the wire is proportional to the strength of the field expressed in terms of its *magnetic flux density* or *magnetic induction* (symbol B). The force F, also proportional to the current I and to the length of wire l in the field, is given by

$$F = BIl \qquad (27.1)$$

Rearrangement of this equation ($B = F/Il$) tells us that we could measure flux density by finding the force acting on a metre length of wire carrying a current of 1 A at right angles to the field. The unit of B is N A^{-1} m^{-1} and is called the *tesla* (symbol T).

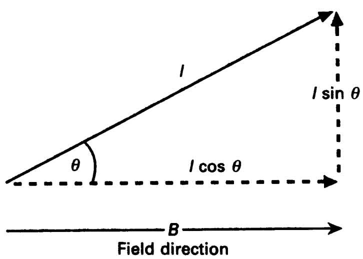

Figure 27.6

Equation (27.1) only applies when the current direction is perpendicular to the field direction. If the two directions are parallel, then the wire will experience no force at all. If the wire makes an angle θ with the field direction, as in Figure 27.6, then the force acting on the parallel component of the current ($I \cos \theta$) will be zero and the force on the perpendicular component will be proportional to $I \sin \theta$. Equation (27.1) should then be written

$$F = BIl \sin \theta \qquad (27.2)$$

When $\theta = 90°$, then $\sin \theta = 1$ and $F = BIl$. When $\theta = 0°$, then $\sin \theta = 0$ and $F = 0$. Remember that the direction of the force is perpendicular to the plane containing the current and field directions.

It is sometimes useful to resolve a magnetic field into components. For example, we noted that, in the UK, the earth's magnetic field dips at an angle of roughly 70° below the horizontal; the vertical and horizontal components of the earth's flux density, B_v and B_h, are therefore related by $B_v/B_h \simeq \tan 70°$.

27.4 FORCE ON A MOVING CHARGE IN A MAGNETIC FIELD

Equation (27.2) can readily be adapted to obtain the force acting on a single electron in a wire carrying an electric current in a magnetic field. Equation (24.2) (page 225) gives the magnitude of the current in a metal conductor as $I = nvAe$, where n is the number of free electrons per unit volume, v and e are their drift velocity and charge, respectively, and A is the cross-sectional area of the conductor. Substituting this for I in Equation (27.2) gives

$$F = BnvAel \sin \theta$$

But the number of free electrons in l metres of wire is nAl; therefore, the force acting on just one electron is given by

$$\frac{F}{nAl} = \frac{BnvAel \sin \theta}{nAl} = Bve \sin \theta$$

In general, a charge q moving with a speed v at an angle θ relative to the direction of a magnetic field B will experience a force F, given by

$$F = Bvq \sin \theta \qquad (27.3)$$

If the charge is moving perpendicularly to the field, then $F = Bvq$ because $\sin 90° = 1$.

From this it follows that a magnetic field will deflect a beam of charged particles passing through it (unless the beam is parallel to the field direction, in which case $\sin 0° = 0$ and $F = 0$). The force acting on the particles will not change their speed, because it always acts at right angles to their path. (Remember that, when applying Fleming's left-hand rule to a beam of negative particles, they travel in the opposite direction to a conventional current.)

The force due to a magnetic field can be cancelled out by the force due to a superimposed electric field. For example, a beam of electrons will be undeflected when it passes through superimposed magnetic and electric fields if they provide equal and opposite forces acting at right angles to the beam. Since the force due to an electric field of strength E is given by $F = qE$ (Equation 22.1 on page 207) then, when both the fields and the beam direction are mutually perpendicular, as in Figure 27.7, electrons of velocity v and charge e will be undeflected if

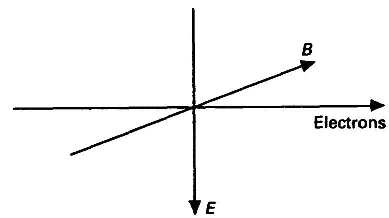

Figure 27.7

$$F = Bve = eE$$

and

$$v = \frac{E}{B} \qquad (27.4)$$

Note the field directions in Figure 27.7. In the absence of an electric field, Fleming's left-hand rule tells us that the electron beam would be deflected downwards (remembering that the equivalent conventional current flows in the opposite direction). The direction of the electric field must therefore be downwards in order to deflect the beam upwards (remembering that the direction of an electric field is that of the force acting on a positive charge).

27.5 TORQUE ON A COIL IN A MAGNETIC FIELD

In the last topic we noted that the moving-coil galvanometer relies on the principle that a current passing through a coil in a magnetic field can be used to produce rotation. The same principle applies to electric motors. We shall not go into the design of galvanometers or electric motors, but we need to understand the principle.

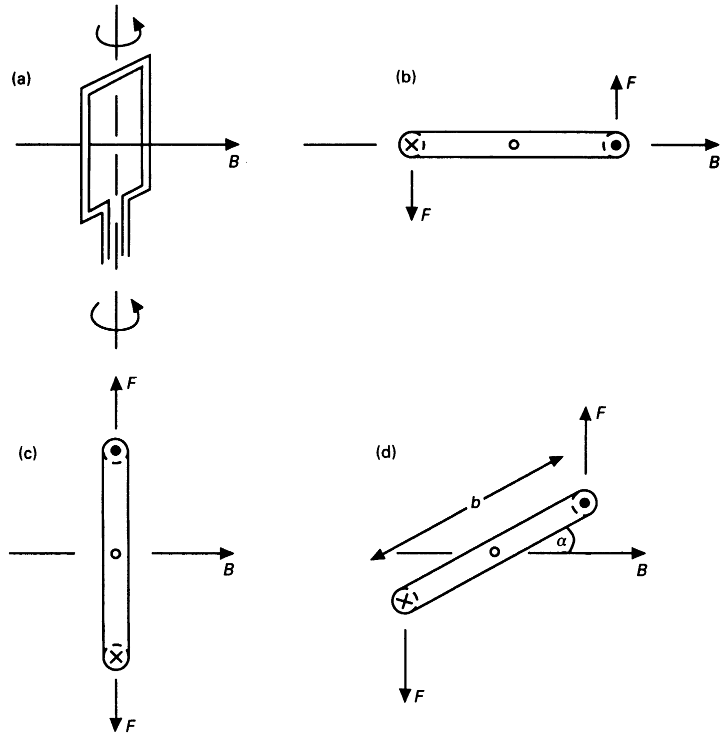

Figure 27.8

Figure 27.8(a) shows a rectangular coil that is free to rotate about its central vertical axis. The axis is at right angles to the direction of a uniform magnetic field of flux density B.

Figure 27.8(b) shows the top view of the coil when its plane is parallel to the field. The cross and dot show the current direction through the vertical sides of the coil. The direction of the force F acting on each vertical side is drawn in accordance with Fleming's left-hand rule. The current through the horizontal sides of the coil, at the top and bottom, is parallel to the field direction and therefore the magnetic force acting on them is zero. The forces on the vertical sides constitute a couple and the coil rotates until its plane is perpendicular to the field direction, as shown in Figure 27.8(c). The forces acting on the vertical sides are still the same as before but their lines of action both pass through the vertical axis, so there is no further tendency to rotate. Fleming's left-hand rule shows that there are now forces acting on the horizontal sides, but they are vertically opposed to one another and have no effect on the rotation of the coil.

Figure 27.8(d) shows the coil when it is inclined at an angle α to the field direction. From our discussion of the moments of forces in Topic 3 we can deduce that the torque T due to the couple about the axis of rotation is given by

$$T = F \times b \cos \alpha$$

where b is the width of the coil and $b \cos \alpha$ is the perpendicular distance between the lines of action of the forces F.

The vertical sides of the coil remain perpendicular to the direction of the field, whatever the value of α; therefore, F always has the value BIl (Equation 27.1), where l is the length of the vertical sides. If the coil has N turns, then, in effect, l is multiplied by N and $F = BINl$. Substituting this in the equation for T above,

$$T\ (= Fb \cos \alpha) = BINlb \cos \alpha$$

But lb is equal to the area A of the coil face; therefore,

$$T = BINA \cos \alpha \qquad (27.5)$$

If $\alpha = 0°$, then $\cos \alpha = 1$ and $T = BINA$, and if $\alpha = 90°$, then $\cos \alpha = 0$ and $T = 0$, as in Figures 27.8(b) and (c), respectively.

27.6 FORCES BETWEEN PARALLEL CONDUCTORS

Two straight current-carrying conductors placed parallel to one another each experience a force because of the magnetic field due to the other. The forces are attractive if the currents are flowing in the same direction and repulsive if they are in opposite directions.

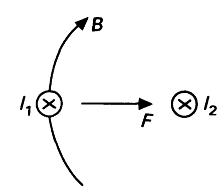

Figure 27.9

Figure 27.9 shows two parallel conductors running perpendicularly through the page, each carrying a current downwards into the paper. The current I_2 in the right-hand conductor produces a field B at the left-hand conductor (as in Figure 27.2a on page 262). The resulting force acting on the left-hand conductor pulls it towards the right (Fleming's left-hand rule). Similarly, the right-hand conductor experiences a force to the left because of the magnetic field due to the current I_1 in the left-hand conductor. Similar arguments show that the forces between the conductors are repulsive if the currents flow in opposite directions.

The ampere is defined as the steady current in each of two straight, parallel conductors of infinite length and negligible cross-sectional area, 1 metre apart in vacuum, that produces a force between them of 2×10^{-7} N per metre length.

Questions

(Use any previously tabulated data as required.)

1. A straight wire 100 mm long experiences a force of 15×10^{-3} N while it carries a current of 3 A perpendicular to a uniform magnetic field.

   (a) Find the flux density of the field.
   (b) Find the magnitude of the force acting on the wire if the wire makes an angle of 60° with the field.

2. (a) Find the direction of the force acting on an electron travelling horizontally in the 9 o'clock direction when it enters a horizontal magnetic field in the 12 o'clock direction.
   (b) Find the direction of the force acting on a conventional current flowing in the same direction under the same circumstances.

3. What angle should a current-carrying conductor make with a magnetic field so that the force acting on it is half its maximum possible value?

4. An electron is travelling in a straight line at 5×10^6 m s^{-1} towards a magnetic field of 0.025 T superimposed on an electric field of 125×10^3 V m^{-1}. If the electron path and the two fields are mutually perpendicular as in Figure 27.7, find the deflection experienced by the electron 0.2 μs after entering the fields.

5. An electron is travelling in a straight line at 10×10^6 m s^{-1} midway between two parallel plates providing a uniform electric field perpendicular to the direction in which it is travelling. The plates are 10 mm apart and there is a potential difference of 1000 V between them. Find the magnitude of the magnetic field which is required to maintain the straight path of the electron.

6. A square coil with 50 mm sides is made from 125 turns of wire which has a resistance of 4 Ω per metre. Estimate the torque acting on the coil when it is connected to a 12 V supply while it is suspended from the centre of one of its sides with its plane parallel to a uniform magnetic field of 0.04 T.

7. A 12 m length of metal wire, of 8900 kg m^{-3} density and 1.7×10^{-8} Ω m resistivity, is aligned horizontally in an west–east direction. A potential difference of 890 V across the ends of the wire provides just enough support for its weight. Estimate the magnitude of the earth's magnetic field acting horizontally at that point. (Assume g = 9.8 m s^{-2}.)

TOPIC 28 ELECTRO-MAGNETIC INDUCTION

COVERING:

- e.m.f. and current induced in a moving conductor;
- magnetic flux;
- e.m.f. induced in a rotating coil;
- inductance;
- the transformer.

In the last topic we saw how electric current produces motion in a magnetic field. In this topic we shall see how motion in a magnetic field *induces* electric current.

28.1 E.M.F. AND CURRENT INDUCED IN A MOVING CONDUCTOR

Figure 28.1(a) is to remind us of Fleming's left-hand rule. This is sometimes called the *motor rule*, because it concerns the motion produced by passing a current through a conductor in a magnetic field. We shall now move on to consider *Fleming's right-hand rule*, sometimes called the *generator rule*, because it concerns the current induced in a conductor that is being propelled through a magnetic field.

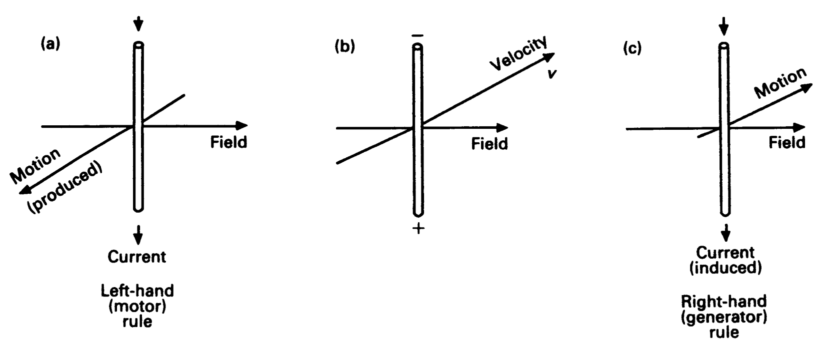

Figure 28.1

Figure 28.1(b) shows a vertical conductor, isolated from any external circuit, that is being propelled at a steady horizontal velocity v at right angles to a horizontal magnetic field. The charge carriers in the conductor are therefore being transported across the field at velocity v and, in effect, constitute a current in a similar way to charged particles in a beam. Consequently, they will experience a vertical force Bvq in accordance with Equation (27.3) (page 265) and in the direction given by Fleming's left-hand rule. If the conductor is metallic, then the charge carriers are electrons and will be displaced upwards (in the opposite direction to a conventional current). Since the positive metal ions cannot move, there will be a separation of charge, so that the top end of the conductor becomes negative while the bottom end is left positive. As charge separation continues, it creates an electric field of growing magnitude in the conductor which increasingly opposes further movement of electrons towards the top.

The electric force acting on each electron is equal to eE (Equation 22.1 on page 207), where E is the electric field strength and e the charge on the electron. Eventually this force will grow large enough to balance the magnetic force Bve (where $q = e$). An equilibrium will then be established in which

$$eE = Bve$$

and

$$E = Bv$$

(remembering that v is the velocity of the conductor).

Thus, a potential difference V is created between the ends of the conductor. If the length of the conductor is l, then, from Equation (22.4) (page 210),

$$E = V/l$$

and, substituting for E in $E = Bv$, we obtain

$$V = Bvl \tag{28.1}$$

If the ends of the conductor are connected to an external circuit and a steady velocity v is maintained, then it will act as a generator. (In the next section we shall treat V in Equation 28.1 as an e.m.f. E. Note that e.m.f. and electric field strength have the same symbol, so be very careful not to confuse these quantities.) The direction of the induced current is given by Fleming's right-hand rule, which is illustrated in Figure 28.1(c). The field and induced current directions are represented by the first and second fingers of the right hand. These are mutually perpendicular to the thumb, which represents the direction in which the conductor is being moved.

Lenz's law tells us that the direction of an induced current is always such that it opposes the change that is causing it. This is in agreement with Figure 28.1(c), where moving the conductor as shown produces a conventional current flowing downwards. This current is just like any other, so, according to Fleming's left-hand rule, its interaction with the magnetic field produces a force which acts in the opposite direction to the movement of the conductor, as in Figure 28.1(a).

We can regard this as an example of the principle of conservation of energy. Moving the conductor induces a current which creates an opposing force that requires work to be done to overcome it. Mechanical energy is therefore absorbed by the system and electrical energy is produced.

28.2 MAGNETIC FLUX

It is helpful to think of an induced e.m.f. as the result of a moving conductor cutting through magnetic field lines or, as we shall see later, magnetic field lines sweeping across a stationary conductor.

To develop this approach, we make use of a quantity called *magnetic flux* (symbol Φ), which we shall take to represent the number of field lines. The magnetic flux through a plane is obtained by multiplying the area A of the plane by the magnetic flux density B normal to its surface. If the field direction is perpendicular to the plane, then the three quantities are related by the expression $\Phi = BA$.

The unit of magnetic flux is the *weber* (Wb) and, since $B = \Phi/A$, 1 T is equivalent to 1 Wb m^{-2}. (Now we can see why B is called the flux density.)

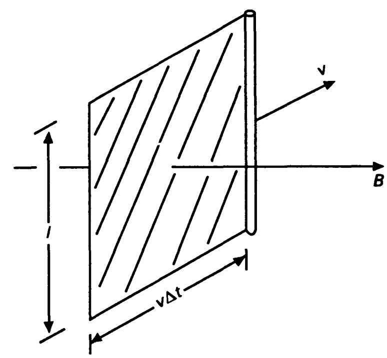

Figure 28.2

Let us imagine that a straight conductor of length l is moving at a steady velocity v through a magnetic field of flux density B, where the conductor, the field and the velocity are mutually perpendicular (see Figure 28.2). During a time interval Δt the conductor will have moved a distance $v\Delta t$ and swept out an area $lv\Delta t$. The flux cut by the conductor is therefore given by

$$\Delta\Phi = Blv\Delta t$$

This can be rearranged to give

$$Blv = \Delta\Phi/\Delta t$$

But, from Equation (28.1),

$$E = Bvl$$

where E is the induced e.m.f.; therefore,

$$E = \Delta\Phi/\Delta t \qquad (28.2)$$

Thus, the induced e.m.f. is equal to the rate of flux change which the conductor experiences in cutting through field lines.

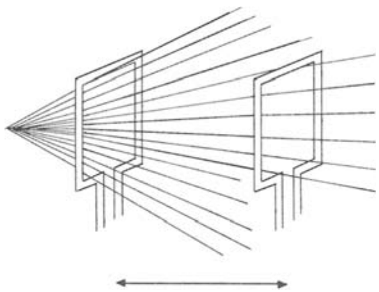

Figure 28.3

Bearing in mind that $\Phi = BA$, a flux change can also result from a change in B over a fixed area A. For example, a coil will experience change in the flux *linking* it (i.e. passing through it) if it is placed in a varying magnetic field.

Figure 28.3 illustrates a case which we can interpret either way. If the coil is moved through a non-uniform field, then it will cut field lines in the process. At the same time we can see that there will be a change in the number of field lines linking the coil. Either way an e.m.f. will be induced in the coil and for our purposes it is reasonable to assume that flux cutting or a change in flux linking have equivalent effects. (Note that the weber is actually defined as the flux which, when linking a coil of one turn, and when uniformly reduced to zero in one second, induces an e.m.f. of one volt in the coil.)

These ideas are embodied in *Faraday's laws of electromagnetic induction*, which tell us that the induced e.m.f. is proportional to the rate of cutting flux or the rate of change of flux linking. Note that if a coil has N turns, then the so-called *flux-linkage* is given by $N\Phi$ and any induced e.m.f. is correspondingly increased. Putting all these ideas together gives the *Faraday–Neumann law*, which can be expressed in the form

$$E = -N\frac{\Delta\Phi}{\Delta t} \qquad (28.3)$$

The minus sign, which is in accordance with Lenz's law, tells us that the direction of the e.m.f. is such that it opposes the change causing it.

28.3 E.M.F. INDUCED IN A ROTATING COIL

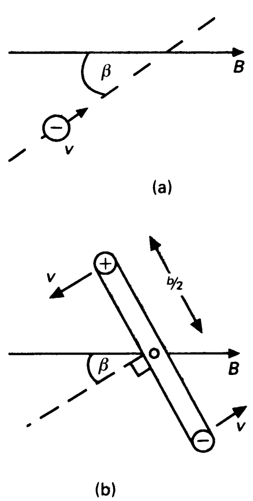

Figure 28.4

We have seen that if we push a conductor across a magnetic field at right angles to the field direction, then the induced e.m.f. is given by $E = Bvl$.

Figure 28.4(a) is looking down on top of a vertical conductor that is being pushed across a magnetic field at an angle β relative to the field direction. In this case the effective velocity of the conductor relative to the field is reduced to the perpendicular component $v \sin \beta$. The induced e.m.f. is therefore reduced to $Bvl \sin \beta$ (which becomes zero if the conductor moves parallel to the field.) The negative sign at the top of the conductor in the figure indicates the charge there due to the electrons moving upwards, as in Figures 28.1(b) and (c).

Figure 28.4(b) shows the conductor as one of a pair which we shall treat as the vertical sides of a rectangular coil of width b (as in Figure 27.8 on page 266). The coil is being rotated at a constant angular velocity about its central vertical axis, so that the vertical sides cut the magnetic flux. As the positive and negative signs indicate, the e.m.f.s in the two sides act in the same direction around the coil and reinforce each other, so that the total induced e.m.f. (between the ends of the coil) is obtained by adding them together. Therefore, at any instant when the normal to the plane of the coil makes an angle β with the field direction,

$$E = 2 \times Bvl \sin \beta$$

From Equation (9.2) (page 72) the angular velocity of the coil is given by $\omega = \beta/t$ rad s^{-1}, where it rotates through β radians in t seconds and, if $\beta = 0$ at $t = 0$, then $\beta = \omega t$. Furthermore, if v is the linear speed of the conductors around the circumference of a circle of radius $b/2$, then, from Equation (9.3) (page 72) $v = \omega b/2$. Substituting for β and v in the equation for E above, we get

$$E = 2B \frac{\omega b}{2} l \sin \omega t$$

and since the area A of the coil is equal to bl, then

$$E = BA\omega \sin \omega t$$

Finally, if the coil has N turns, then A is, in effect, multiplied by N, so that

$$E = BAN\omega \sin \omega t \qquad (28.4)$$

If required, this can be rewritten in terms of frequency of rotation f (Hz = s^{-1}), since $\omega = 2\pi f$.

Equation (28.4) tells us that if the coil rotates at a steady rate, then the induced e.m.f. across its ends varies sinusoidally with time — that is to say, it follows the pattern of a sine wave. Figure 28.5 shows how the e.m.f. between the ends of the coil alternates between positive and negative as each conductor changes direction relative to the field when the coil passes the point where its plane is perpendicular to the field direction. At this point the sides of the coil are travelling parallel to the field direction and $E = 0$.

When the plane of the coil is parallel to the field, its vertical sides are travelling at right angles to the field direction and the e.m.f. is at a maximum. At this point the normal to the plane of the coil is perpendicular to the field direction, so that ωt is either $\pi/2$ or $3\pi/2$. (Remember that ωt is in radians.) Sin ωt is therefore either 1 or -1. The e.m.f. is therefore either $+BAN\omega$ or $-BAN\omega$, where $BAN\omega$ is the amplitude of the sinusoidally alternating e.m.f.

If a resistance is connected across the coil, the alternating e.m.f. will produce an alternating current (a.c.) which periodically reverses its direction. (By contrast, direct current (d.c.) flows in one direction only.)

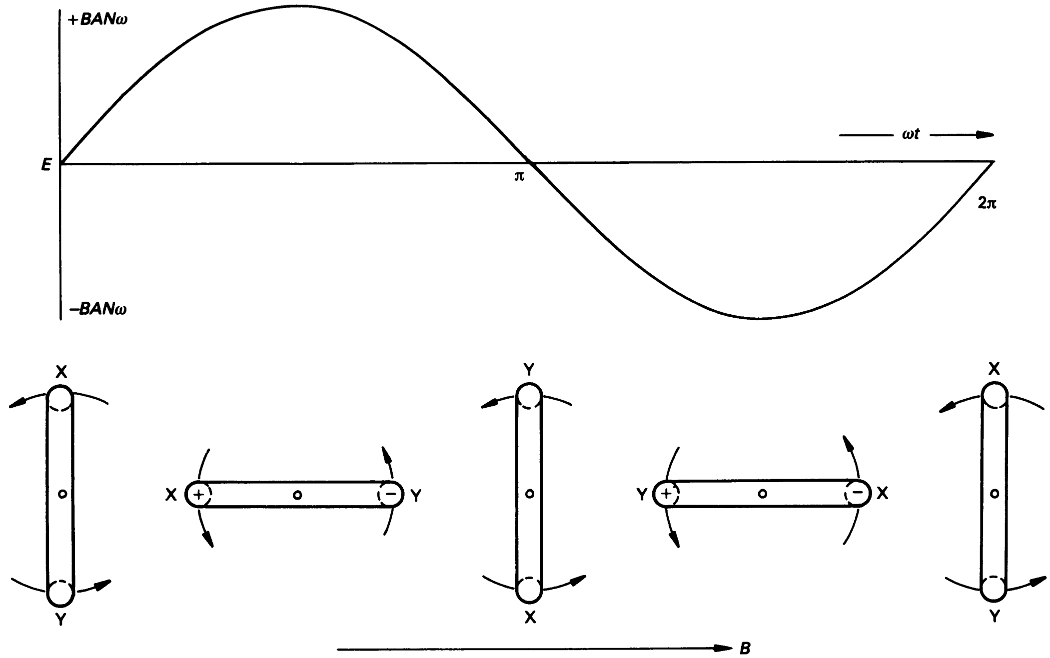

Figure 28.5

A major advantage of alternating current is that voltages can be stepped up and down very efficiently with *transformers*. However, before we can discuss transformers we must first consider *inductance*.

28.4 INDUCTANCE

Let us try to imagine what happens when a current is passed through a coil. Initially, before the current is switched on, there is no magnetic field, but as the current starts to flow, the field begins to develop. This provides a changing magnetic flux linking the coil or, if it is easier to picture them, field lines growing out from each loop of the coil which cut through their neighbouring loops. Either way, an e.m.f. is induced in the coil itself. According to Lenz's law, this e.m.f. opposes the change that is causing it, with the result that the rise in current is delayed, as indicated in Figure 28.6.

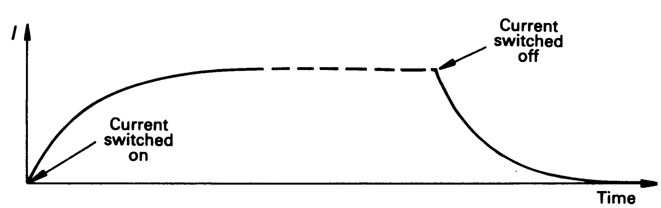

Figure 28.6

Where the curve levels off, the opposing e.m.f. is zero and the current has reached a steady value that depends on its source and the resistance of the circuit. The work done in raising the current to a steady value against the opposing e.m.f. is now stored in the magnetic field in an analogous way to the energy stored in the electric field of a capacitor.

If the current is switched off, the magnetic field collapses, thereby inducing an e.m.f. which opposes the decay of the current, as indicated in Figure 28.6. (On opening the switch contacts, this e.m.f. may be large enough to produce a visible spark as the energy stored in the magnetic field is dissipated.)

The unit of *inductance* is called the *henry* (symbol H). A coil or other *inductor* has an inductance of 1 henry if a current passing through it, while changing at the rate of 1 ampere per second, induces an e.m.f. of 1 volt. A given current change will induce a large e.m.f. in a coil with a large inductance and a small e.m.f. in one with a small inductance. Putting this into the form of an equation,

$$E = -L\frac{\Delta I}{\Delta t} \qquad (28.5)$$

where L is the inductance. The equation tells us that 1 H = 1 V s A^{-1}. The minus sign reminds us that the induced e.m.f. acts in opposition to the current change that is causing it (Lenz's law). Since the e.m.f. is induced in the same circuit through which the current is changing, L is often called *self-inductance*.

By contrast, the term *mutual inductance* applies when the changing current in one coil or circuit (called the primary) causes an e.m.f. to be induced in another (called the secondary) because of the changing flux linkage between them. (Or, if you prefer, because the expanding field lines from the primary cut across the secondary.)

A *mutual inductance* of 1 henry exists if a current passing through the primary, while changing at the rate of 1 ampere per second, induces an e.m.f. of 1 volt in the secondary. The e.m.f. E_s in the secondary is given by

$$E_s = -M\frac{\Delta I_p}{\Delta t} \qquad (28.6)$$

where M is the mutual inductance and $\Delta I_p/\Delta t$ is the rate of current change in the primary.

If an a.c. supply is connected across the primary, then an alternating e.m.f. will be induced in the secondary. This is what happens in transformers.

28.5 THE TRANSFORMER

Figure 28.7 shows a transformer with primary and secondary *windings* side by side on a *core*. (In practice, one set of windings is often wound on top of the other.) The core, made of iron, for example, provides a very efficient magnetic linkage between the primary and secondary windings. Iron is much better than air at conveying magnetic flux (just as copper is much better at conveying electric current), so practically all the flux remains within the core.

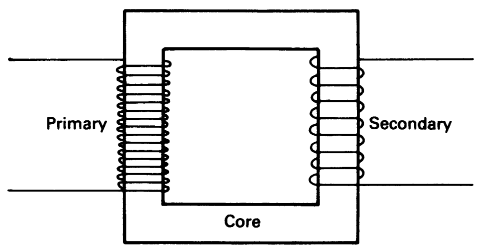

Figure 28.7

(Note that the changing flux induces *eddy currents* in the core material itself. These lead to the dissipation of electrical energy as heat. Eddy current losses can be minimised by constructing the core from laminations which are insulated from one another to interrupt the current pathways.)

The theory of transformers is beyond the scope of this book, so we shall confine ourselves to an outline discussion. An alternating voltage applied to the primary gives an alternating flux in the core which induces an alternating voltage in the secondary. Since the flux is the same through both, then the voltage V across each is proportional to the number of turns N and

$$\frac{V_s}{V_p} = \frac{N_s}{N_p} \tag{28.7}$$

where the subscripts identify the primary and secondary windings. If $N_s > N_p$, we have a step-up transformer, which increases the supply voltage; and if $N_s < N_p$, we have a step-down transformer, which reduces it.

Many transformers transfer power from the primary to the secondary with nearly 100% efficiency, in which case, using Equation (24.3) (page 228), we can write $I_sV_s = I_pV_p$, which gives

$$\frac{V_s}{V_p} = \frac{I_p}{I_s} \tag{28.8}$$

Questions

1. A straight conductor, 200 mm long and of negligible resistance, is moved at 20 m s^{-1} through a field of magnetic flux density of 5×10^{-3} T. Assuming that the conductor, the field and the direction of motion are mutually perpendicular, calculate the current if the conductor is connected across a 2.5 Ω resistor.

2. Calculate the induced e.m.f. across the ends of a straight conductor 4.0 m long under each of the following circumstances:

   (a) After having fallen freely through a distance of 10 m from a horizontal position at right angles to the magnetic north–south direction.
   (b) As in (a) but parallel to the magnetic north–south direction.
   (c) As it travels parallel to the ground at 250 km per hour in a direction at right angles to its longitudinal axis, which is orientated as in (b).

   (Assume $g = 9.8$ m s^{-2} and that the horizontal component of the earth's magnetic field is 1.9×10^{-5} T, while the angle of dip is 70°.)

3. An e.m.f. of 0.21 mV is induced across the ends of a straight, horizontal conductor 2.0 m long as it is moved vertically at 5.5 m s^{-1} at right angles to the magnetic north–south direction. The induced e.m.f. is 0.48 mV when the conductor is moved parallel to the ground at the same speed in a direction at right angles to its longitudinal axis. Calculate the flux density and the angle of dip of the magnetic field in that area.

4. A rectangular coil of 100 turns 60 mm wide and 100 mm long is rotated at 500 revolutions per minute about its central longitudinal axis, which is at right angles to a magnetic field of flux density 32 mT. Calculate the instantaneous value of the e.m.f. in the coil when its plane is at an angle of (a) 0°, (b) 60° and (c) 90° to the field direction.

5. If the coil in Question 4 is stationary and the flux density falls uniformly from 32 mT to zero in 48 s, calculate the e.m.f. induced in the coil when its plane is at an angle of (a) 90°, (b) 30° and (c) 0° to the field direction.

TOPIC 29 MAGNETIC BEHAVIOUR OF MATERIALS

COVERING:

- diamagnetism, paramagnetism and ferromagnetism;
- hysteresis;
- soft and hard magnets;
- magnetic circuits.

As Figure 27.4 (page 263) indicates, there is a region of more or less uniform magnetic field inside a solenoid when it carries an electric current. The flux density can be varied by filling the solenoid with a core of material. So-called *diamagnetic* materials slightly reduce the flux density and *paramagnetic* materials slightly increase it. On the other hand, *ferromagnetic* materials increase it greatly, some by a factor of many thousands.

Let us represent the uniform flux density in the solenoid by B_0 under vacuum conditions and by B when it is filled with different materials. From above, we can write $B < B_0$ for diamagnetic materials, $B > B_0$ for paramagnetic materials and $B \gg B_0$ for ferromagnetic materials. The ratio B/B_0, which is a dimensionless quantity, gives the *relative permeability* μ_r of a material. For air μ_r is very close to 1.

The magnetic behaviour of a material can be attributed to the orbital motion and spin of electrons in its constituent atoms. The electrons can be regarded as behaving like tiny circulating currents with associated magnetic fields.

29.1 DIAMAGNETISM AND PARAMAGNETISM

Diamagnetism is a very weak effect that is exhibited by all materials but is often swamped by the effects of paramagnetism and ferromagnetism. It results from changes in orbital motion which, in keeping with Lenz's law, tend to oppose an applied magnetic field, thereby decreasing the field in the material. Purely diamagnetic materials generally have complete electron pairing in their atomic and molecular structures.

Materials with structures containing unpaired electrons show *paramagnetic* behaviour. The unpaired electrons have associated magnetic fields which tend to become aligned in an applied field, thereby increasing the field in the material. Some metals show paramagnetic behaviour due to the spin of conduction electrons. Paramagnetism is opposed by the randomising effect of thermal agitation.

29.2 FERROMAGNETISM

As their name suggests, *ferromagnetic* materials are epitomised by iron. They are of great importance in electrical engineering and we shall concentrate on them in this topic.

In ferromagnetic materials the effect of unpaired electron spin in incomplete inner orbitals is great enough to cause such strong interactions between neighbouring atoms that they tend to become mutually aligned. Below a certain temperature, called the *Curie point* or *Curie temperature*, the magnetic axes of neighbouring atoms are able to remain aligned and the material exhibits ferromagnetic behaviour. Above the Curie point (about 760 °C in the case of iron) there is sufficient thermal energy to destroy the alignment and the material becomes paramagnetic.

Regions of uniform alignment are called *domains*. These are typically fractions of a millimetre in size and are, in effect, small permanent magnets. In a piece of unmagnetised ferromagnetic material the domains are orientated in different directions and therefore cancel each other out. If the material is subjected to an increasing external magnetic field, then domains that are aligned in more or less the same direction as the field will tend to grow at the expense of the others. Furthermore, the magnetic axes of domains that are not aligned in the field direction may rotate if the field is strong enough.

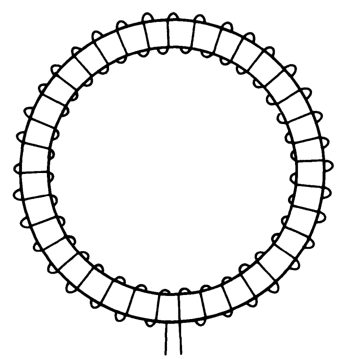

Figure 29.1

Saturation occurs when the magnetic axes of all the domains are aligned with the external field. If the field is then reduced to zero, the material will remain magnetised. To see how this happens, let us consider the relationship between B and B_0 for a ferromagnetic core in a solenoid. To avoid complications due to end effects, it is better to consider a *toroid* (Figure 29.1), which is an endless solenoid made in the form of a ring. (B_0 is readily controlled, since it varies proportionally with the current through the toroid.)

Figure 29.2 shows the typical form of the relationship between B and B_0 for an initially unmagnetised ferromagnetic material. (Generally, $B \gg B_0$, so the vertical and horizontal scales would normally be different for a real material.) As B_0 is increased from zero at O, the material becomes magnetised, as discussed above. B increases progressively less rapidly with B_0 as it reaches its saturation value at P. If B_0 is now reduced to zero, the material retains a remanent (i.e. residual) flux density (at Q) which is called the *remanence* or *retentivity*. The value of this indicates the degree of residual distortion of the domain structure. The remanent flux density can be reduced to zero (at R) by increasing B_0 in the reverse direction. The value of B_0

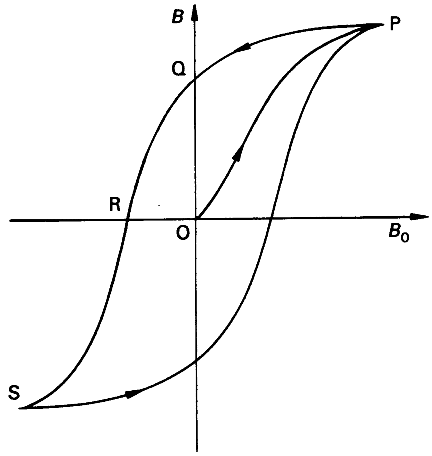

Figure 29.2

required to do this is proportional to the *coercive force* or *coercivity*, which is a measure of the difficulty of neutralising the residual distortion of the domain structure.

On further increasing the reversed field, the material becomes saturated in the reverse direction (at S). The whole process from P to S can then be repeated in the opposite direction, from S to P, to form a closed loop called the hysteresis loop. (The word *hysteresis* describes the lagging of an effect behind its cause, in this case the lagging of B behind B_0.)

It is evident from Figure 29.2 that the relative permeability μ_r ($= B/B_0$) is not constant for ferromagnetic materials. The maximum relative permeability, based on the largest value of B/B_0 on the initial magnetisation curve, is sometimes used to characterise a material, a high value indicating that the material is readily magnetised.

29.3 SOFT AND HARD MAGNETS

Ferromagnetic materials are described as *soft* or *hard*, depending on whether they readily lose their magnetism or tend to retain it. The use of these words is well exemplified by iron and steel, whose mechanical softness and hardness, respectively, is reflected in their magnetic properties. (The reasons for this are beyond the scope of our discussion.) Figure 29.3 shows hysteresis loops corresponding to an example of each type.

The loop for the soft magnetic material indicates low coercivity and remanence and a generally high relative permeability. This suggests a material that would be suitable for making *electromagnets*. An electromagnet is a temporary magnet, essentially a solenoid with a soft magnetic core giving a strong field, that can be controlled by varying the current. This principle is used in such diverse applications as

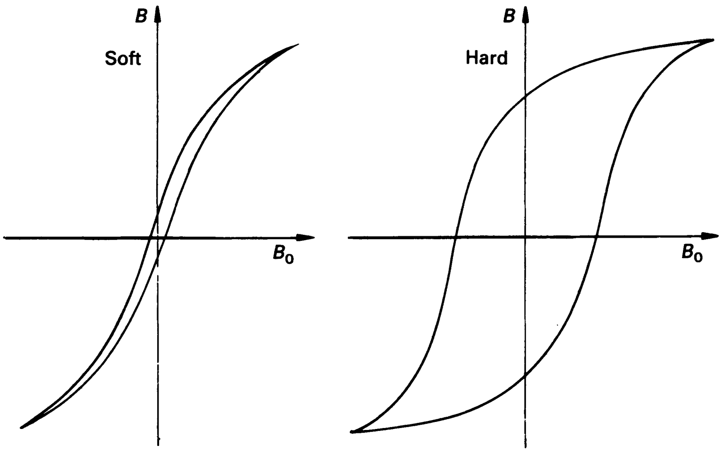

Figure 29.3

electric bells, relays, telephone receivers and for lifting heavy objects made from iron and steel.

Materials used for transformer cores must be magnetically soft in order to respond to alternating current. Ferromagnetic materials consume energy as their magnetisation direction is continually changed. This energy, which is proportional to the area enclosed by the hysteresis loop, is dissipated as heat. Thus, the hysteresis loop should be narrow.

Magnetically hard materials are used to make permanent magnets. Such materials generally have high remanence — that is to say, they retain a high remanent flux density when the magnetising field has been removed. They also tend to have high coercivity, so that they are not readily demagnetised. The shape of the hysteresis loop reflects their high resistance to alteration of the domain structure.

29.4 MAGNETIC CIRCUITS

There is an interesting analogy between electric circuits and the magnetic circuit formed by the toroid in Figure 29.1. Magnetic flux Φ is driven round the toroid by a *magnetomotive force* (*m.m.f.*) in an analogous way to a current being driven round an electric circuit by an e.m.f. The magnitude of the m.m.f. is given by NI (measured in ampere-turns) where N is the number of turns and I is the current. Thus, Φ is proportional to NI.

Furthermore, the *reluctance* R_m, which is the 'magnetic resistance' of the circuit, is given by the ratio m.m.f./Φ (analogous to $R = V/I$), thus,

$$\Phi = \frac{NI}{R_m} \tag{29.1}$$

The analogy extends further because

$$R_m = \frac{l}{\mu A} \tag{29.2}$$

where l is the length (i.e. mean circumference) and A is the cross-sectional area of the toroid. μ is the *absolute permeability* of the core material and is analogous to the electrical conductivity of a conductor (see Section 25.2).

Absolute permeability μ and relative permeability μ_r are related by $\mu = \mu_r\mu_0$ where μ_0, called the *magnetic constant*, is the absolute permeability of free space and has the value $4\pi \times 10^{-7}$ H m^{-1}. In Topic 21 we met the parallel relationship $\varepsilon = \varepsilon_r\varepsilon_0$ for the permittivity of a material in an electric field.

(The electromagnetic theory of James Clerk Maxwell, the nineteenth century scientist, showed that the speed c of electromagnetic waves in free space depends only upon ε_0 and μ_0 as follows:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Although this is really beyond the scope of our present discussion, it serves to illustrate the fundamental interrelationship between electricity and magnetism which has become evident from the previous two topics.)

Question

1. An iron ring, of 420 mm mean diameter and 1.6×10^{-3} m^2 cross-sectional area, has 1000 turns of wire wound uniformly around it (as in Figure 29.1). If a current of 1.4 A in the wire produces a magnetic flux of 4×10^{-3} Wb in the iron core, estimate the relative permeability of the iron.
($\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}.)

TOPIC 30 ALTERNATING CURRENT

COVERING:

- inductive reactance;
- capacitive reactance;
- impedance;
- phase angle;
- power dissipation.

The transmission of electrical power is more efficient when high voltages are used. To take an example, 100 kW of power is carried by a 1000 A current at 100 V, and by a 100 A current at 1000 V (remembering that 1 W = 1 A × 1 V). Assuming that identical cables of resistance R are used for both, the power loss P will be 100 times greater in the first case than the second because the current I is ten times greater and $P = I^2R$ (Equation 25.12 on page 243). Since transformers provide a very efficient means of stepping the voltage up or down, it makes good sense to use alternating current to transmit electrical power over long distances. In practice, enormous voltages are used for this purpose, sometimes as high as 400 kV.

Alternating current is extensively used for lighting, heating and driving machinery, and it can readily be *rectified* to direct current if a particular application demands it.

As far as we are concerned, the major difficulty with alternating current is its mathematical treatment, which is more complicated than for direct current. We have already seen that a steady direct current is opposed by the resistance of the circuit through which it flows. Alternating current is opposed not only by resistance, but also by any capacitance or inductance that the circuit may possess. But, before we move on to this, we must first understand how to quantify alternating current and voltage.

30.1 ALTERNATING CURRENT AND VOLTAGE

All we need to describe a steady direct current is its magnitude and direction. It is more difficult to describe an alternating current, because its magnitude and direction vary periodically with time. We shall confine our discussion to sinusoidal variation, although there are other types of waveform (e.g. square and sawtooth).

The *frequency* of an alternating current is measured in hertz (Hz), where 1 Hz is equal to one complete cycle per second. The ordinary mains supply in many countries is 50 Hz.

The *amplitude* or *peak value* is the maximum value, positive or negative. From our discussion in Section 28.3 we can say that the instantaneous value V of an alternating voltage at any time t is given by

$$V = V_0 \sin 2\pi ft \tag{30.1}$$

where V_0 is the peak voltage and f is the frequency. Similarly, the instantaneous value of an alternating current is given by

$$I = I_0 \sin 2\pi ft \tag{30.2}$$

For many practical purposes, we need average values that we can use in simple calculations. But alternating voltage and current have average values of zero over a complete cycle because they are positive for one half and negative for the other. We can get round the problem by considering the heating effect of a current, because this is the same in both directions. We can define the *effective* value I_{eff} of an alternating current in terms of the equivalent direct current that produces the same power dissipation in a given resistor. Although we shall not go into the reasons here, this turns out to be

$$I_{eff} = \frac{I_0}{\sqrt{2}} = 0.707\ I_0 \tag{30.3}$$

Similarly, the effective voltage is given by

$$V_{eff} = \frac{V_0}{\sqrt{2}} = 0.707\ E_0 \tag{30.4}$$

The 240 V quoted for the ordinary domestic mains supply in Britain is the effective value. Equation (30.4) gives the corresponding peak value as 339 V.

Note that, unless otherwise stated, any further reference to current I and voltage V in this topic implies their effective values.

30.2 REACTANCE

We are now in a position to consider the factors which oppose the reciprocating flow of alternating current through a circuit. As with direct current, alternating current is opposed by the electrical resistance of the materials of which the circuit is made. But, because it is continuously changing, alternating current is also opposed by the effects of inductance and capacitance. These effects are called *reactance* and are due to the occurrence of voltages that arise in inductors and capacitors which oppose the current.

We shall begin by considering *inductive reactance* X_L, which is the quantity that is used to measure the effect of an inductor such as a coil. The relationship of X_L to an inductor is parallel to that of resistance R to a resistor, and we can write

$$X_L = \frac{V}{I} \tag{30.5}$$

where V and I are the effective values of voltage and current. This expression is parallel to the definition of resistance ($R = V/I$) and, as we shall see below, reactance is measured in ohms.

A detailed mathematical analysis of reactance is beyond the scope of this book; nevertheless we need to understand it in semi-quantitative terms. As we saw in Topic 28, switching a direct current on or off through a coil (or other inductor) causes a self-induced e.m.f. which opposes the current and tends to maintain the status quo, thereby delaying current growth or decay. However, once a steady state has been reached, this e.m.f. disappears. In the case of alternating current, which is continuously changing, the opposing e.m.f. will also be continuous.

Inductive reactance increases with increasing frequency f and increasing inductance L and is given by

$$X_L = 2\pi fL \tag{30.6}$$

Since the unit of f is s^{-1} and the unit of L is V s A^{-1} (from Equation 28.5), the unit of $2\pi fL$ is V A^{-1}. This is the same as the unit of resistance R ($= V/I$), and reactance is measured in ohms. (Note that $f = 0$ for a steady direct current, in which case $X_L = 0$ and the current is impeded solely by the resistance of the coil.)

In a purely resistive circuit, with no reactance at all, the voltage and current are in phase. That is to say, their peaks and troughs coincide as indicated in Figure 30.1. The alternating current and voltage are represented by rotating vectors, called *phasors*, which turn anticlockwise with frequency f and whose length is equal to the amplitude. Their vertical component (the projection onto the vertical axis) represents the instantaneous value.

Figure 30.2 shows that, in the case of a pure inductor, the voltage *leads* the current by a quarter of a cycle. (The opposing e.m.f., which balances the applied voltage, is at a maximum where the rate of current change is at a maximum and $I = 0$.)

Capacitive reactance X_C is the quantity that is used to measure the

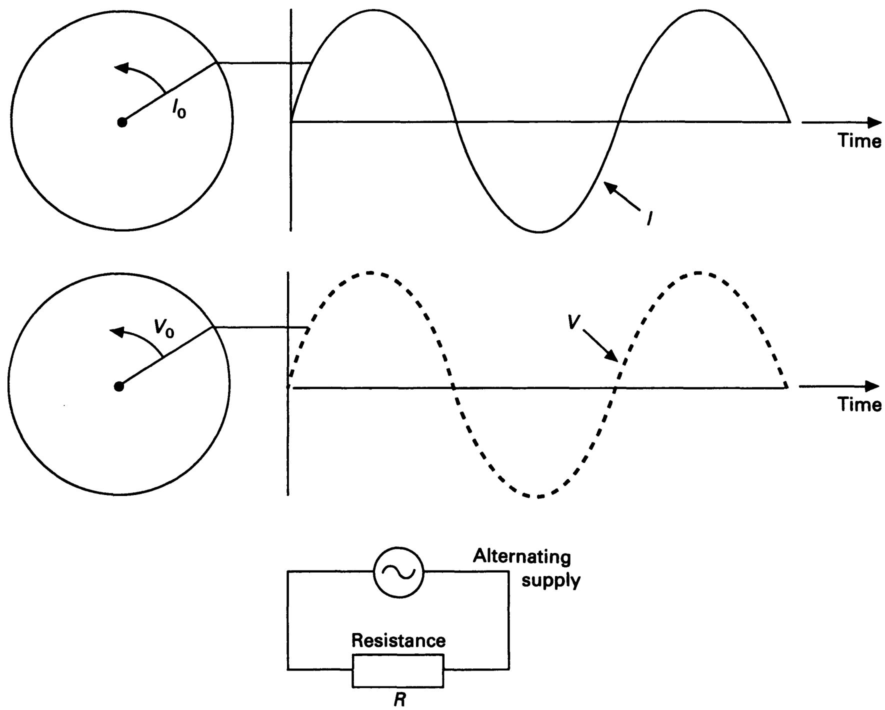

Figure 30.1

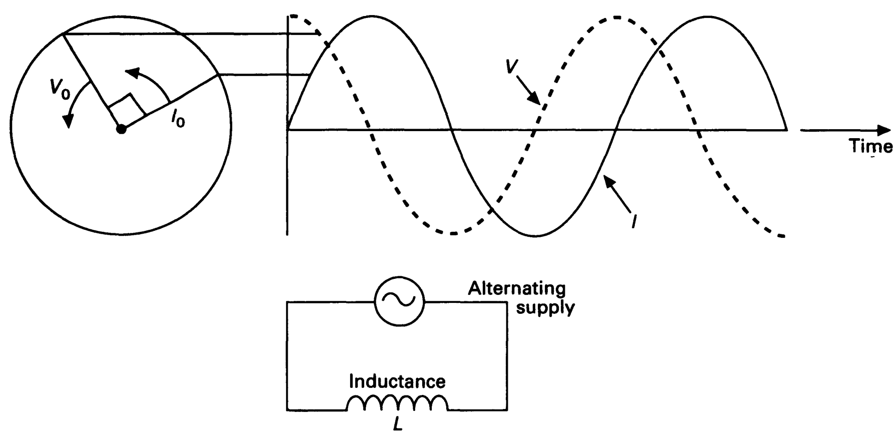

Figure 30.2

opposition of a capacitor to the flow of alternating current. In a similar way to inductive reactance,

$$X_C = \frac{V}{I} \tag{30.7}$$

and, similarly, the units are ohms.

Again, we need a semi-quantitative description. In Topic 23 we saw that direct current cannot flow through a capacitor, because of the insulating gap between its plates. If it is connected across a battery, then current flows as charge builds up on the plates. But the growing potential difference across the plates increasingly opposes and eventually stops the current flow.

A capacitor does not stop alternating current, however, because charge can flow backwards and forwards from plate to plate around an external circuit without actually crossing the gap between them. However, the build-up of charge, and the resultant opposing potential difference every half-cycle, will still tend to impede the current. Capacitive reactance decreases with increasing frequency f and increasing capacitance C and is given by

$$X_C = \frac{1}{2\pi f C} \tag{30.8}$$

Remember that the unit of capacitance can be expressed as C V^{-1} (Equation 23.1 on page 215) and that 1 C = 1 A s (Equation 24.1 on page 224). The unit of X_C is therefore given by

$$\frac{1}{s^{-1} \times A\,s\,V^{-1}} = \frac{V}{A}$$

Hence, X_C is measured in ohms. (Note that for direct current $f = 0$; therefore, X_C is infinite and current cannot flow.)

Figure 30.3 shows how the voltage in a purely capacitive circuit lags behind the current by a quarter of a cycle. (The capacitor has maximum charge when the current is zero and on the point of reversing.)

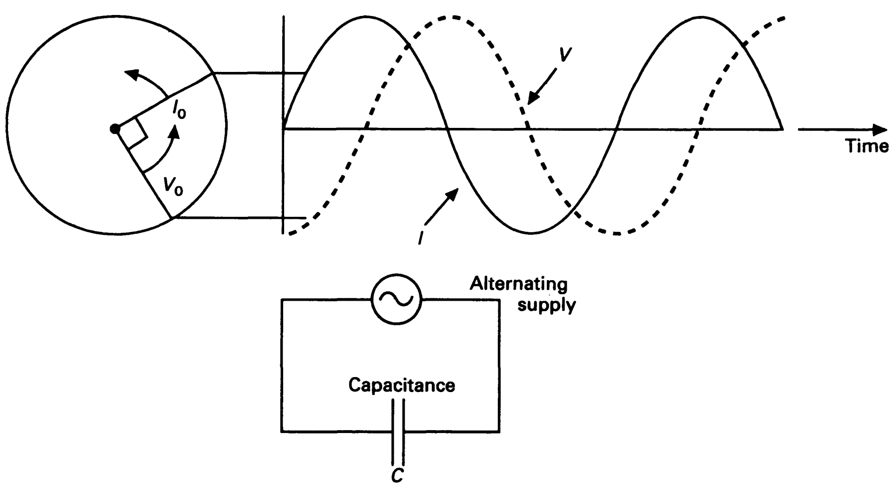

Figure 30.3

The word *CIVIL* is a useful mnemonic for phase relationships. *CIV* reminds us that, for capacitance *C*, *I* leads *V*. *VIL* reminds us that *V* leads *I* for inductance *L*.

30.3 IMPEDANCE AND PHASE ANGLE

Many circuits have inductance, capacitance and resistance. To take a very simple example, a coil will have inductance and it will have resistance due to the wire from which it is made. Figure 30.4(a) represents a simple series circuit containing all three and indicates the effective voltage across each. From the previous section we know that V_R is in phase with the current I, V_L is a quarter of a cycle ahead of it and V_C is a quarter of a cycle behind. These quantities are therefore represented as vectors, as in Figure 30.4(b), where the vector sum equals the applied voltage V as given by

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} \tag{30.9}$$

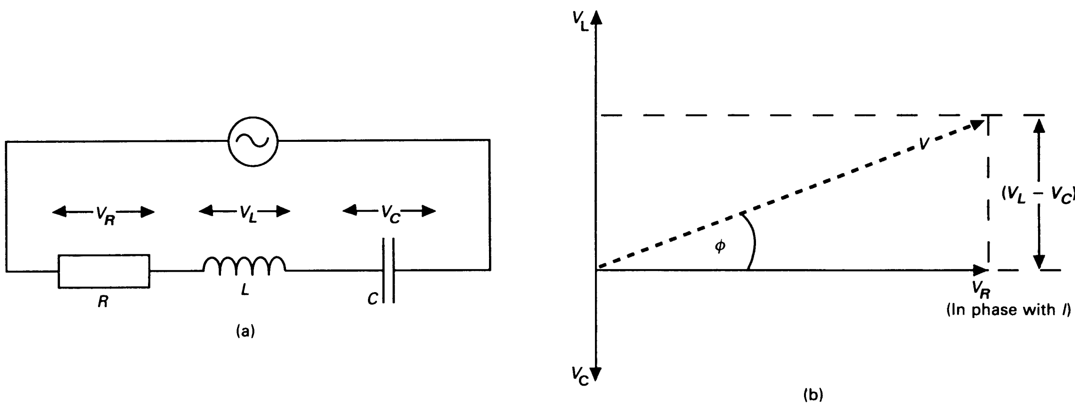

Figure 30.4

The circuit elements are in series; therefore, the current I is the same through each. From Equations (30.5) and (30.7) we know that $V_L = IX_L$ and $V_C = IX_C$; furthermore, $V_R = IR$. Therefore, from Equation (30.9), we can write

$$V = \sqrt{I^2R^2 + (IX_L - IX_C)^2}$$

and

$$V = I\sqrt{R^2 + (X_L - X_C)^2} \tag{30.10}$$

The total opposition to alternating current, called the *impedance Z*, is given by

$$Z = \frac{V}{I} = \sqrt{R^2 + (X_L - X_C)^2} \tag{30.11}$$

where Z is measured in ohms. (Remember that X_L and X_C depend on frequency; therefore, Z depends on frequency too.)

As the relationship between impedence and voltage suggests, we can draw a vector diagram in terms of impedance, as in Figure 30.5.

The *phase angle* Φ is the angle between the applied voltage and the current. Bearing in mind that V_R is in phase with the current, then, from Figure 30.4(b)

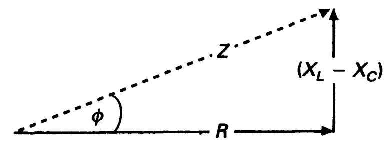

Figure 30.5

$$\tan \Phi = \frac{V_L - V_C}{V_R} \qquad (30.12)$$

or, from Figure 30.5,

$$\tan \Phi = \frac{X_L - X_C}{R} \qquad (30.13)$$

If $V_L < V_C$ (and $X_L < X_C$), then Φ will be negative, indicating that the current leads the voltage.

Although the above equations apply to series 'RLC' circuits they can be applied to series RL and RC circuits, where the respective C and L terms are zero; thus,

$$V = \sqrt{V_R^2 + V_L^2} \qquad (30.14)$$

$$V = \sqrt{V_R^2 + V_C^2} \qquad (30.15)$$

and so on.

30.4 POWER

Pure inductors and capacitors do not consume power, since energy is only temporarily stored in their respective magnetic and electric fields. On the other hand, power is dissipated by resistance and its average value P is given by I^2R, where I is the effective current in a circuit containing a resistance R.

If V leads I by Φ, as in Figure 30.4(b), V_R is the component of V across the resistance in the circuit and we can write

$$V_R = V \cos \Phi$$

Hence,

$$P\ (= IV_R) = IV \cos \Phi \qquad (30.16)$$

Cos Φ is called the *power factor* and, as we can see from Figure

30.5, it is equal to R/Z. If the circuit is purely resistive, then $\cos \Phi = 1$ and $P = IV$. If the circuit is purely inductive, then $\cos \Phi = 0$ and $P = 0$.

Questions

(Assume effective values for current and voltage.)

1. Cables with a total resistance of 12 Ω are used to carry current from a 25 kW supply. Find the power loss and voltage drop if the supply voltage is (a) 5 kV and (b) 10 kV.
2. What is the effective value of an alternating supply of 205 V amplitude?
3. A coil has a reactance at 50 Hz that is fifty times greater than its resistance. At what frequency would you expect its reactance to be a hundred times greater than its resistance?
4. A purely inductive circuit operating at 50 Hz has an inductance of 350 mH.

   (a) Find the reactance of the circuit.
   (b) Find the current if the voltage is 110 V.
5. A purely capacitive circuit operating at 70 Hz has a capacitance of 65 μF.

   (a) Find the reactance of the circuit.
   (b) Find the voltage if the current is 2 A.
6. What is the reactance at 35 Hz of a capacitor that has a reactance of 35 Ω at 70 Hz?
7. State the phase relationship between current and voltage for a pure (a) capacitance, (b) resistance and (c) inductance.
8. Find the impedance in a series circuit where

   (a) $R = 12.5\ \Omega$, $X_L = 25\ \Omega$ and $X_C = 15\ \Omega$
   (b) $R = 12.5\ \Omega$, $X_L = 15\ \Omega$ and $X_C = 25\ \Omega$
   (c) $R = 3\ \Omega$, $X_L = 6\ \Omega$ and $X_C = 2\ \Omega$.
9. Find the impedance of a series circuit in which $R = 10\ \Omega$, $L = 13.64$ mH and $C = 23.32\ \mu$F at 350 Hz.
10. Find the current when a 26 V alternating supply is connected across a circuit with an inductive reactance of 6 Ω in series with a resistance of 2.5 Ω.
11. A 15 V, 2 kHz supply is connected across the combination of a 3 Ω resistor in series with a 20 μF

capacitor. Find the voltage drop across (a) the resistor and (b) the capacitor.

12. (a) Find the voltage of an alternating supply connected in series with a resistor, an inductor and a capacitor across which there are potential differences of 3 V, 8 V and 4 V, respectively.
    (b) Find the phase angle.

13. Find the impedance and phase angle where $R = 5\ \Omega$, $X_L = 14\ \Omega$ and $X_C = 2\ \Omega$. If the applied voltage is 26 V then find the voltage drops across the resistance, the inductance and the capacitance, respectively.

14. Find the impedance of a circuit in which the resistance is 25 Ω and the phase angle is 60°.

15. Find the power consumed by the circuits which, when connected to a 240 V supply, carry currents of 1.5 A which lag behind the voltage by (a) 0°, (b) 41.4° and (c) 90°.

16. Find the phase angle in a series circuit in which $R = 3\ \Omega$, $X_L = 6\ \Omega$ and $X_C = 2\ \Omega$, and find the power dissipated when a current of 1.5 A is flowing through it.

ANSWERS TO QUESTIONS

Topic 1

1. (a) 89 m at 54°; (b) 102 m at 63°; (c) 178 m at 37°; (d) 127 m at 113°.
2. 9.9 m s^{-1} horizontally and 1.0 m s^{-1} vertically.
3. 146 m at 322°.
4. 3.9 m s^{-1} at 249°.
5. 58 m at 212° (approximately 8 o'clock).
6. (a) 60 s; (b) 90 m; (c) 17° west of north; (d) 62.6 s.

Topic 2

1. 7.3×10^{22} kg.
2. 25 mm.
3. (a) 195×10^{6} N m^{-2}; (b) 0.0975%.
4. 0 N.
5. 98 N.
6. 104 N.
7. 400 N.
8. (a) 29°; (b)(i) 37 N, (ii) 451 N.

Topic 3

1. (a) Yes; (b) 56 N at 127°; (c) 317 N at 78°; (d) yes; (e) yes; (f) yes
2. $\alpha = \beta = 60°$.
3. 123 N, 368 N.
4. 245 N, 1225 N.
5. (a) 735 N m; (b) 980 N m; (c) 3.3 m.
6. The forces are in equilibrium.
7. (a) 33 N, 16 N at 90°; (b) 98 N, 85 N at 0°; (c) 19 N, 53 N at 69°; (d) 25 N, 43 N at 60°; (e) 19 N, 34 N at 73°; (f) 28 N, 29 N at 119°.
8. 0.18.
9. 45°.

Topic 4

1. 82 kg.
2. 1.23×10^{-3} m^{3}.
3. 5500 kg m^{-3}.
4. 2620 kg m^{-3}.

5. 250 N; 1.2×10^6 Pa.
6. (a) 2.2×10^4 Pa; (b) 1.9×10^3 Pa; (c) 9.98×10^4 Pa; (d) 1.02×10^5 Pa; (e) 9.0×10^5 Pa.
7. 31.1 cm^3.
8. (a) 7.8×10^3 kg m^{-3}; (b) 0.80×10^3 kg m^{-3}.
9. 0.12 N.

Topic 5

1. 9 s.
2. 500 m.
3. 44 m, 103 m, 123 m, 103 m, 44 m; 10 s.
4. 0 m s^{-1}; −2 m s^{-2}.
5. −3 m s^{-2}.
6. 3.1×10^4 m s^{-2}.
7. 10 m s^{-2}; 4 s.
8. 32.0 m; 2.55 s.
9. 714 m; 122 m s^{-1} at 35° below the horizontal.
10. (a) $s = \dfrac{u^2 \sin^2 \theta}{2g}$; (b) $t = \dfrac{u \sin \theta}{g}$.

Topic 6

1. 1.7×10^{-25} m s^{-2}.
2. 1.5 m s^{-2}; 8.2 s.
3. 7.5 s.
4. 200 m.
5. 2.0 m s^{-2} in the 56° direction.
6. 360 m in the 46° direction.
7. 0.59.
8. 8.9 N.
9. 10.2 m s^{-2}.
10. 100 m s^{-1}.
11. (a) 618 N; (b) 9270 N.

Topic 7

1. 556 N.
2. 100 m s^{-1}.
3. 1.35 m s^{-1} in the opposite direction to the bullet.
4. 20 m s^{-1}; 1.8×10^5 kg m s^{-1} in both cases.
5. (a) 14 kN; (b) 2.3 kN.
6. 1.8 N.
7. 35 g.
8. 14.3 m s^{-1} in the 3 o'clock direction.
9. 6.3 m s^{-1} at 39° anticlockwise from 3 o'clock.
10. 1800 m s^{-1}.

Topic 8

1. 5 kg.
2. (a) 10 kJ; (b) 0 J.
3. 30 m s^{-1}.
4. 240 N.
5. 70 m s^{-1}.
6. 67%.
7. 1.1 m s^{-1} horizontally.
8. 180 W.
9. 8 m s^{-1}.
10. 60 W.

Topic 9

1. 2×10^{-4} m s^{-1}.
2. 0.5.
3. 2.0×10^{20} N.
4. 7.3°.
5. 3.8 rad s^{-1}.
6. 1.96 N.
7. 24.5°.
8. 2.0 N.
9. 10 rad s^{-1}; 90 rad s^{-1}.
10. (a) 250 N; (b) 1620 N.

Topic 10

1. (a) 0.025 kg m^2; (b) 12.3 J; (c) 37.0 J.
2. 3.3×10^{-4} J.
3. (a) 225 rad; (b) 30 rad s^{-1}; (c) 2700 J; (d) 360 W; (e) 2700 J.
4. 1.25 kg m^2.
5. 1.5 rad s^{-1}; 8.2 s.
6. 2.0 kg.
7. (a) 50 rad s^{-1}; (b) 37.5 kJ.
8. 33 rad s^{-1}.
9. 450 J.
10. (a) 124 J; (b) 0.28 kg m^2; (c) hoop.

Topic 11

1. (a) 0.25 m; (b) 0.99 m; (c) 3.97 m.
2. 0.45 s.
3. 1.6 m.
4. 3.5 Hz.
5. (a) 0.30 m s^{-1}; (b) 0.94 m s^{-2}.
6. 56 N m^{-1}.
7. At the top; 5 mm.
8. 0.16 Hz.

9. (a) 28 J; (b) 4.7 m s^{-1}; (c) 1.1 kN.
10. 476 g.

Topic 12

1. 825 rpm.
2. (a) 1.7 km; (b) 1.3 m; (c) 0.4 s; (d) 20 000 – 20 Hz.
3. 510 m.
4. (a) 340 m s^{-1}; (b) 255 m.
5. 1.1 W.
6. (a) 5.70 m; (b) 1.33 m.
7. 256 Hz; 512 Hz; 768 Hz.
8. 440 Hz.
9. (a) 2.0×10^{11} N m^{-2}.
10. The aluminium rod must be 4.2 times longer than the lead rod.

Topic 13

1. Light (because λ = 588 nm).
2. (a) 3.07 m; (b) 247 m.
3. (a) 656 nm; (b) 494 nm.
4. (a) 17°; (b) 17° (the light path is reversible).
5. 1.33.
6. (a) 1.23×10^8 m s^{-1}; (b) 242 nm; (c) 24.2°; (d) 33.0°.
7. (a) $2 \times i_c = 98°$; (b) total internal reflection.
8. 14.5°.
9. 588 nm.
10. (a) 5.6° – 10.7°; (b) 17.0° – 33.7°.

Topic 14

1. Lithium (metal); nitrogen; neon; sulphur; potassium (metal); scandium (metal); manganese (metal); bromine.
2. 2; 7; 8; 8; 22; 21; 20.
3. 1.99×10^{-26} kg; 2.66×10^{-26} kg; 9.26×10^{-26} kg.
4. (a) 107.9 g; (b) 27.0 g; (c) 2698 kg.
5. 6.0×10^{26}; 9.5×10^{24}; 5.6×10^{24}.
6. 2.0×10^{22}.
7. 8.4×10^{19}.
8. 20 g.
9. 590 mm^3.
10. 92.5%.
11. (a) Iron; (b) 2.66×10^{-26} kg; (c) 1.2×10^{-29} m^{-3}.

Topic 15

1. Ionic crystal; covalent molecule; covalent molecules joined by hydrogen bonds; atoms joined by van der Waals forces; covalent

molecule; covalent molecules joined by hydrogen bonds.
2. 30.0; 100.1; 46.0; 169.9; 119.5.
3. Methane; ethane.
4. 52.9%.
5. 13.5 g.
6. 2.275 kg.
7. $C_2H_5OH + 3O_2 = 2CO_2 + 3H_2O$; 7.5 g.

Topic 16

1. (a) 1 °C; (b) 2 °C; (c) 4.7 °C; (d) 9.1 °C; (e) 32.3 °C.
2. (a) 1650 kJ; (b) 1755 kJ; (c) 2700 kJ.
3. 812 m s^{-1}.
4. 1.75 kW.
5. (a) 293 s; (b) 575 s.
6. (a) 2.5 kg; (b) 0.5 kg.
7. 17×10^{-6} K^{-1}.
8. 30.007 m.
9. 160 °C.
10. 16 °C; 62.5 MN m^{-2}.
11. 5.15 l.
12. Steel rod 475 mm; brass rod 300 mm.

Topic 17

1. (a) 16 mm; (b) 52 mm; (c) 400 mm; (d) 20 m.
2. (a) 11.1 °C; (b) copper/steel = 8/1.
3. 69 °C.
4. 1.84 MW m^{-2}.
5. 15 W.
6. (a) 2.5 W; (b) 2.5 W; (c) 2.5 W.
7. 0.6 kg.
8. (a) Larger/smaller = 4/1; (b) larger/smaller = 1/2.
9. (a) 503 °C; (b) 315 W m^{-2}.
10. 0.5.

Topic 18

1. An increase of 240 mm^3.
2. 72.5 l.
3. 38 °C.
4. 35 l.
5. 22.4×10^{-3} m^3.
6. (a) 2.80 m^3; (b) 125 mol.
7. 1.0×10^{22}.
8. 1.3 kg m^{-3}.
9. 752 mm.
11. 149 mm.

Topic 19

2. 1.00 l s^{-1}.
3. 2.0×10^{27} molecules min^{-1}.
5. 1.8×10^{-5} Pa s.
6. (a) Laminar ($Re = 1475$); (b) 0.30 m.
7. 18 kPa.
8. 11×10^{-3} N.
9. 2×10^{-4} J.

Topic 20

1. 50 MPa in both cases.
2. (a) 100 MPa; (b) 2.5 mm.
3. 100 MPa.
4. (a) 0.43 mm; (b) 0.23 mm; (c) 15.6 mm.
5. 4.5 mm.
6. 8 kg.
7. 99.94 mm.
8. + 0.08%.
9. − 0.03%.
10. (a) 77 GPa; (b) 167 GPa.

Topic 21

1. (a) × 4; (b) × 1/9.
2. (a) 50 nC; (b) 5.32 mN; (c) 50 mN.
3. (a) 8.2×10^{-8} N; (b) 3.6×10^{-47} N.
4. 0.11 μC.

Topic 22

1. (a) Right to left; (b) right to left; (c)(i) 5000 N C^{-1}, (ii) 5000 V m^{-1}; (d)(i) 4.8×10^{-17} J, (ii) 4.8×10^{-17} J, (iii) 0 J; (e) 4.8×10^{-17} J; (f) 2.4×10^{5} m s^{-1}; (g) 1.0×10^{7} m s^{-1}.
2. (a) 6.4×10^{-16} N; (b) 7.0×10^{14} m s^{-2}.
3. 2.0×10^{-6} V m^{-1}.
4. 4.
5. 1 μJ.
6. (a) 8.4×10^{6} m s^{-1}; (b) 9.5×10^{-8} s.
7. (a) 2.0×10^{5} m s^{-1}; (b) 4.1×10^{-6} s.
8. 4.8×10^{-19} C.
9. (a) 1.0×10^{7} m s^{-1}; (b) 6.2 mm.

Topic 23

1. 400 μC.
2. 2.1×10^{11}.

3. 26 V.
4. 7.2×10^{-4} C.
5. 1 μF; 1.5 μF; 2 μF; 3 μF; 4.5 μF; 6 μF; 9 μF.
6. 8 V (wider-spaced plates); 4 V.
7. (a) 2 μF; (b) 2.5 mJ.
8. 15.1 °C.
9. (a) 600 μC, 1200 μC; (b) 400 μC, 400 μC.
10. (a) 45 V; (b) 13.5 mJ.

Topic 24

1. 7.5×10^{19}.
2. 5 W.
3. 4×10^{-19} J.
4. 6 W.
5. (a) 6 W; (b) 15 V; (c) 0.4 A.
6. 10 C; 2 GJ.
7. 60 V.
8. 2.5 A.
9. 192 W.

Topic 25

1. 1.5 Ω.
2. 1.5 V.
3. (a) 9 Ω; (b) 1 Ω; (c) 2 Ω; (d) 4.5 Ω; (e) 7.5 Ω; (f) 1.8 Ω.
4. 30 Ω in parallel.
5. 2 Ω; 3 Ω.
6. (a) 6 V; (b) 15 Ω; (c) 0.3 A.
7. 5800 Ω.
8. (a) 4.25 V; (b) 3.5 V; (c) 3.6 A.
9. (a) 2.4 A: (b) 1.6 A; (c) 0.8 A.
10. (a) 1.5 A; (b) 1.2 A.
11. 5 kΩ; 7.2 mW.
12. 0.15 A.
14. (a) 9 W; (b) 2 W.

Topic 26

1. (a) 11.65 V; (b) 11.96 V.
2. (a) 0.15 A; (b) 1.44 V.
3. (a) 2.4 V; (b) 3.0 V; (c) 14.5 kΩ.
4. (a) 4 V; (b)(i) 0.08 A, (ii) 0.04 A, 0.06 A.
5. (a) 0.67 A in the wires connected to the corners where the current enters or leaves the network and 0.33 A in the other wires; (b) 20 V; (c) 10 Ω.
6. 3.3 Ω.
7. 0.273 A.

Topic 27

1. (a) 0.05 T; (b) 13×10^{-3} N.
2. (a) Vertically upwards; (b) vertically downwards.
3. 30°.
4. Nil.
5. 0.01 T.
6. 1.5×10^{-3} N m.
7. 2×10^{-5} T.

Topic 28

1. 8 mA.
2. (a) 1.06 mV; (b) 0 V; (c) 14.5 mV.
3. 4.8×10^{-5} T; 66°.
4. (a) 1.0 V; (b) 0.5 V; (c) 0 V.
5. (a) 0.4 mV; (b) 0.2 mV; (c) 0 V.

Topic 29

1. 1875.

Topic 30

1. (a) 300 W, 60 V; (b) 75 W, 30 V.
2. 145 V.
3. 100 Hz.
4. (a) 110 Ω; (b) 1 A.
5. (a) 35 Ω; (b) 70 V.
6. 70 Ω.
7. (a) I leads V by 90°; (b) I and V are in phase; (c) V leads I by 90°.
8. (a) 16 Ω; (b) 16 Ω; (c) 5 Ω.
9. 14.50 Ω.
10. 4 A.
11. (a) 9 V; (b) 12 V.
12. (a) 5 V; (b) 53.1°.
13. 13 Ω; 67.4°; 10 V, 28 V, 4 V.
14. 50 Ω.
15. (a) 360 W; (b) 270 W; (c) 0 W.
16. 53.1°; 6.75 W.

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