## Solutions to Problems

# ELECTRONIC AND ELECTRICAL ENGINEERING

## Principles and Practice

Lionel Warnes

# *Solutions to Problems*
# *Electronic and Electrical Engineering*

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## *Principles and Practice*

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# Chapter 1

**1** The equivalent circuit is shown in figure A1.1a, which reduces to that of figure A1.1b since the 8 Ω and 1 kΩ are in parallel and by the product-over-sum rule these combine to give a resistance of

$$R = \frac{8 \times 1000}{8 + 1000} = \frac{8000}{1008} = 7.9365\ \Omega$$

Then the total series resistance is $2 + 0.1 + 7.9365 = 10.0365\ \Omega$ and the current, $I$, is $9/10.0365 = 0.8967$ A. Then the voltage across the 7.9365Ω resistance is $0.8967 \times 7.9365 = 7.117$ V, which is the required voltmeter reading.

Placing the voltmeter across XY gives the equivalent circuit of figure A1.1c, which reduces to that of figure A1.1d by the product-over-sum rule: $8.1 \times 1000/1008.1 = 8.0349\ \Omega$. Then

$$I = 9/(2 + 8.0349) = 9/10.0349 = 0.89687 \text{ A}$$

The voltage drop across the 2Ω resistance is $2I = 1.79374$ V and the voltage across XY is $9 - 2I = 9 - 1.79374 = 7.206$ V, the required voltmeter reading.

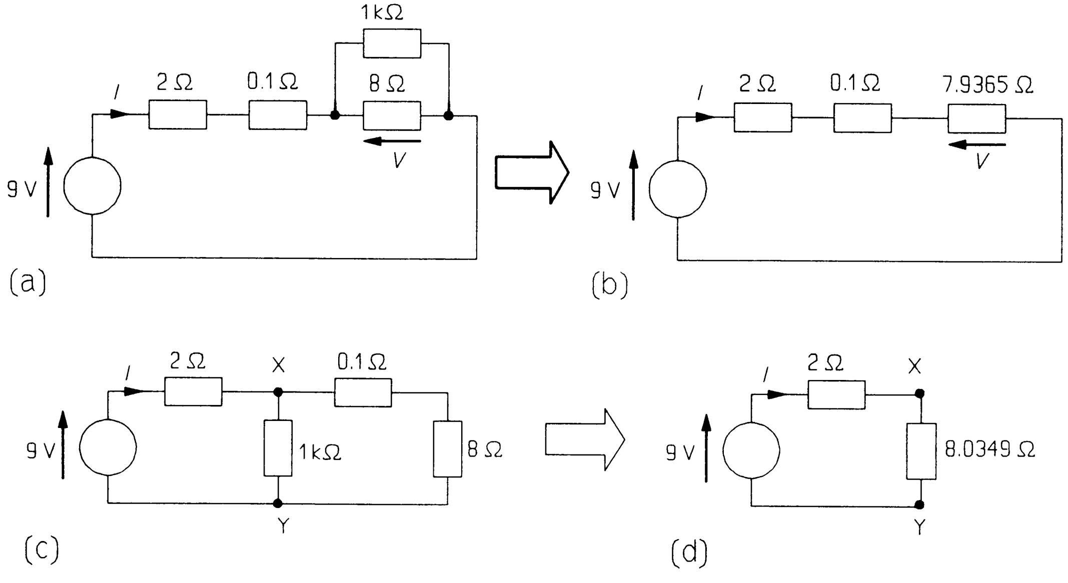


*Figure A1.1*

**2** The 2 and 1μF capacitances are in parallel and add up to 3 μF, which is in series with the 4μF capacitance to give a combined capacitance of $(3 \times 4)/7 = 12/7 = 1.714$ μF. This is in parallel with the 3μF capacitance and the combined capacitance is then 4.714 μF. Figure A1.2 shows the sequence.

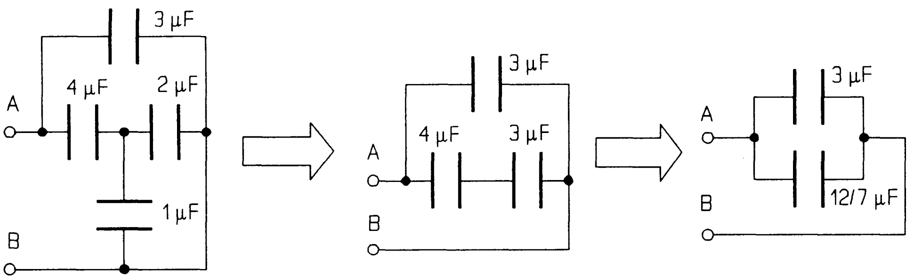


*Figure A1.2*

**3** The easiest method is to use the delta-star transformation on the upper delta network in figure A1.3a, treating inductances as resistances. The inductance attached to A in the star network of figure A1.3b is given by the product of the two inductances attached to A in the delta network divided by the sum of all three in the delta network:

$$L_A = \frac{15 \times 6}{15 + 6 + 9} = \frac{90}{30} = 3 \text{ mH}$$

And the same process applies to C and D, giving the circuit of figure A1.3b, which soon reduces to a single inductance of 6 mH.

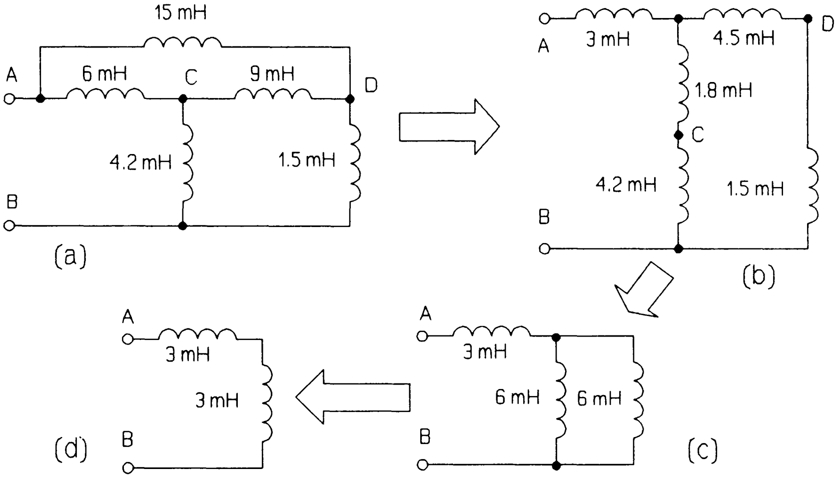


*Figure A1.3*

**4** The resultant capacitance of 2 μF in parallel with $C$ μF is $2 + C$ μF, which is in series with a 1μF capacitance and another 1μF capacitance as shown in figure A1.4. The overall capacitance is

$$\frac{1}{C} = \frac{1}{1} + \frac{1}{C+2} + \frac{1}{1} = 2 + \frac{1}{C+2} = \frac{2C+5}{C+2}$$

Hence $2C^2 + 5C = C + 2$, or $C^2 + 2C - 1 = 0$, so that

$$C = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2} = +0.414\ \mu\text{F}$$

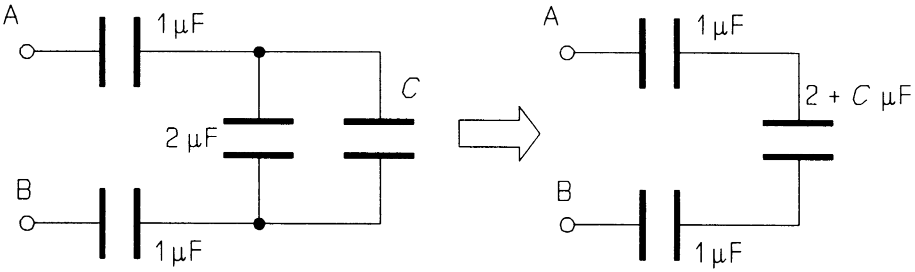


*Figure A1.4*

**5** The resistances must be arranged in parallel rows with different numbers of resistors in series. To take a concrete example, suppose we have 14 resistors, then we arrange them as shown in figure A1.5, which uses 10 resistors. If they are all $R$ ohms each then the resistance would be

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R} + \frac{1}{2R} + \frac{1}{3R} + \frac{1}{4R} = \frac{2.083}{R}$$

$$\Rightarrow \quad R_{\text{eq}} = 0.48R$$

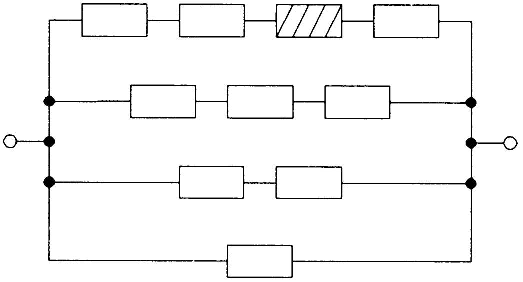


*Figure A1.5*

Now if one of these resistors is different the reading of the ohmmeter will depend on which row it is in and the row may therefore be identified, provided that $R$ is known. For example if it is in the 4th row and has resistance $R'$, the overall resistance is

$$R_{\text{eq}} = \frac{1}{1/R + 1/2R + 1/3R + 1/(3R + R')}$$

$$= \frac{3R + R'}{3 + r + 3/2 + r/2 + 1 + r/3 + 1} = \frac{3R + R'}{6.5 + 1.8333r}$$

where $r = R'/R$. If it is in the row of 4, then these 4 resistors are marked and each is placed in a different row of a fresh series-parallel combination. If the resistor is one of the 4 not used in making up the series-parallel combination, then the procedure is the same as if it had been in the row of 4. The procedure is repeated, the row containing the odd resistor is identified and hence the odd resistor.

**6** In the circuit of figure A1.6a, the voltage across the inductances in series is

$$v = v_1 + v_2 = L_1 \mathrm{d}i/\mathrm{d}t + L_2 \mathrm{d}i/\mathrm{d}t = (L_1 + L_2)\mathrm{d}i/\mathrm{d}t = L_{eq}\mathrm{d}i/\mathrm{d}t$$

hence $L_{eq} = L_1 + L_2$ and inductances in series are like resistances in series.

In the circuit of figure A1.6b by KCL

$$i = i_1 + i_2 = \frac{1}{L_1}\int v\mathrm{d}t + \frac{1}{L_2}\int v\mathrm{d}t = \left[\frac{1}{L_1} + \frac{1}{L_2}\right]\int v\mathrm{d}t = \frac{1}{L_{eq}}\int v\mathrm{d}t$$

Then $1/L_{eq} = 1/L_1 + 1/L_2$, inductances in parallel are like resistances in parallel.

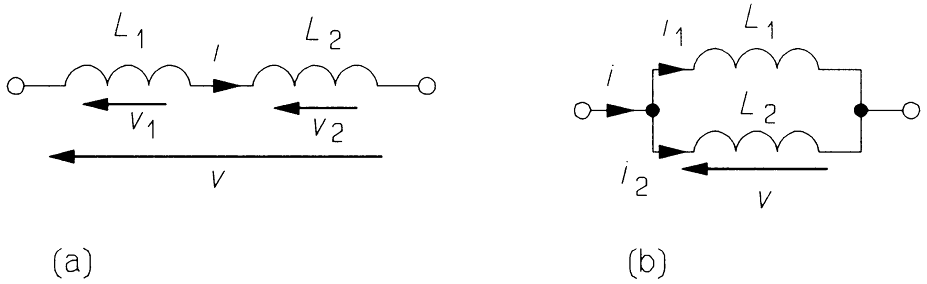


*Figure A1.6*

**7** We can see that in figure A1.7, the incoming current, $I$, must divide equally at the corner node because the cube is symmetrical: thus $I = 3I_1$.

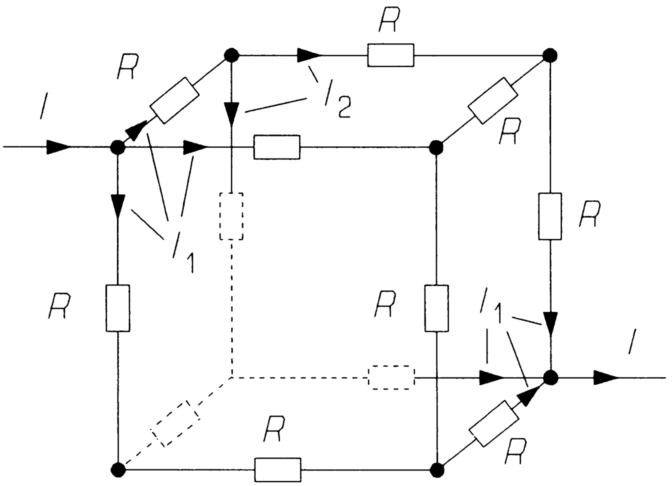


*Figure A1.7*

Similarly $I_1$ divides equally into two at the next corner and $I_1 = 2I_2$. The voltage across the cube diagonal by KVL is

$$V = I_1R + I_2R + I_1R = R(2I/3 + I/6) = 5IR/6$$
$$\Rightarrow \quad V/I = R_{eq} = 5R/6 = 5\ \Omega$$

**8** The original charge, $Q$, is divided between the capacitances and the final voltages on the capacitances are equal and opposite: $+V$ and $-V$, as in figure A1.8.

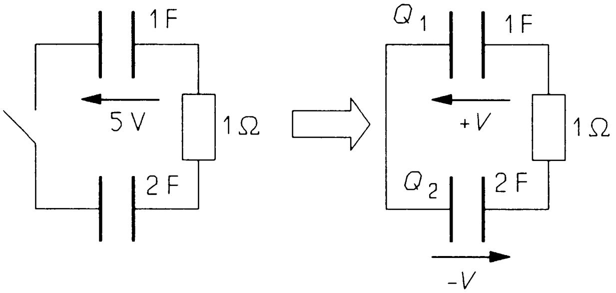


*Figure A1.8*

If the final charges are $Q_1$ and $Q_2$, then

$$Q = C_1V = 1 \times 5 = Q_1 + Q_2 = C_1V + C_2V = 3V$$

$$\Rightarrow \quad V = 5/3 = 1.67\ \text{V}$$

The energy stored at first is 0.5 × 1 × $5^2$ = 12.5 J. And the final energy stored is

$$0.5C_1V^2 + 0.5C_2V^2 = 0.5 \times 3 \times 1.67^2 = 4.17\ \text{J}$$

The energy lost in the resistance is 12.5 − 4.17 = 8.33 J.
[The current is $5\exp(-1.5t)$ A and then the energy lost in the resistance is given by

$$\int_0^\infty [5\exp(-1.5t)]^2 \mathrm{d}t = \int_0^\infty 25\exp(-3t)\mathrm{d}t = \left[\frac{-25}{3}\exp(-3t)\right]_0^\infty = 8.33\ \text{J}$$

which is the expected answer.]

When the resistance is zero, the calculation above cannot be made, but the same energy would still lost from the circuit by electromagnetic radiation from the accelerated charge. In practice there is always some resistance.

**9** In figure A1.9 the power in the load is

$$P_L = \frac{V_T^2 R_L}{(R_T + R_L)^2}$$

while the power from the source is

$$P_S = V_T I_L = V_T^2/(R_T + R_L)$$

The ratio of these is the efficiency:

$$\eta = \frac{P_L}{P_S} = \frac{V_T^2 R_L (R_T + R_L)}{(R_T + R_L)^2 V_T^2} = \frac{R_L}{R_T + R_L}$$

which is maximal when $R_T/R_L = 0$.

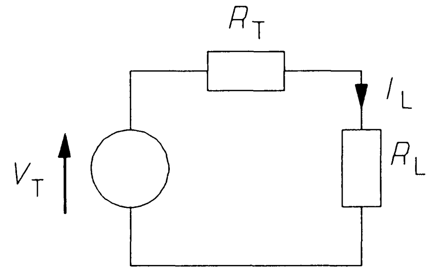


*Figure A1.9*

**10** We equate stored energies:

$$0.5CV^2 = 0.5LI^2 \Rightarrow I = V\sqrt{C/L} = 24\sqrt{32/0.5} = 192 \text{ kA}$$

**11** The stored energy is $\frac{1}{2}LI^2 = 5 \times 10^9 \times 3600$ J, so that

$$L = \frac{5 \times 10^9 \times 3600}{0.5 \times (2 \times 10^5)^2} = 900 \text{ H}$$

The terminal voltage is $L\mathrm{d}i/\mathrm{d}t = 5000$, so $(\mathrm{d}i/\mathrm{d}t)_{\max} = 5000/900 = 5.56$ A/s.

If this rate of discharge is sustained, the minimum time needed to reduce the current to zero is

$$T_{\min} = \frac{I}{(\mathrm{d}i/\mathrm{d}t)_{\max}} = \frac{2 \times 10^5}{5.56 \times 3600} = 10 \text{ hrs}$$

Then the average power delivered is $5 \times 10^9/10 = 500$ MW. The maximum output power is $VI_{\max} = 5000 \times 2 \times 10^5 = 1$ GW.

**12** The circuit reduction by source transformations is shown in figure A1.12a, with the final voltage source being replaced by short-circuit that leaves a 2.5Ω and a 2Ω resistance in parallel. Thus the Thévenin resistance is 2.5 Ω ‖ 2 Ω = 2.5 × 2/(2.5 + 2) = 5/4.5 = 1.11 Ω. The Thévenin voltage is found from the penultimate circuit of figure A1.12a using the voltage-divider rule:

$$V_T = \frac{2 \times 4.5}{2 + 2.5} = 2 \text{ V}$$

The Norton current is then $V_T/R_T = 2/1.11 = 1.8$ A and we find the maximum power transfer by connecting a 1.11Ω resistance across the Thévenin equivalent circuit of figure A1.12b. This give a load current of 2/2.22 = 0.9 A and a maximum load power of $0.9^2 \times 1.11 = 0.9$ W.

There is no safe way of calculating the power dissipated in the circuit except by finding the currents in all the resistive elements. Using the circuit of figure A1.12c, we can see that when no load is connected the voltage across the 2Ω resistance is $V_T$ or 2 V, so that $I_1 = 1$ A. The power developed is $1^2 \times 2 = 2$ W. Looking at node X, we see by KCL that $I_2 = 2$ A and the power developed in the 2.5Ω resistance is $2^2 \times 2.5 = 10$ W, for a total power of 12 W.

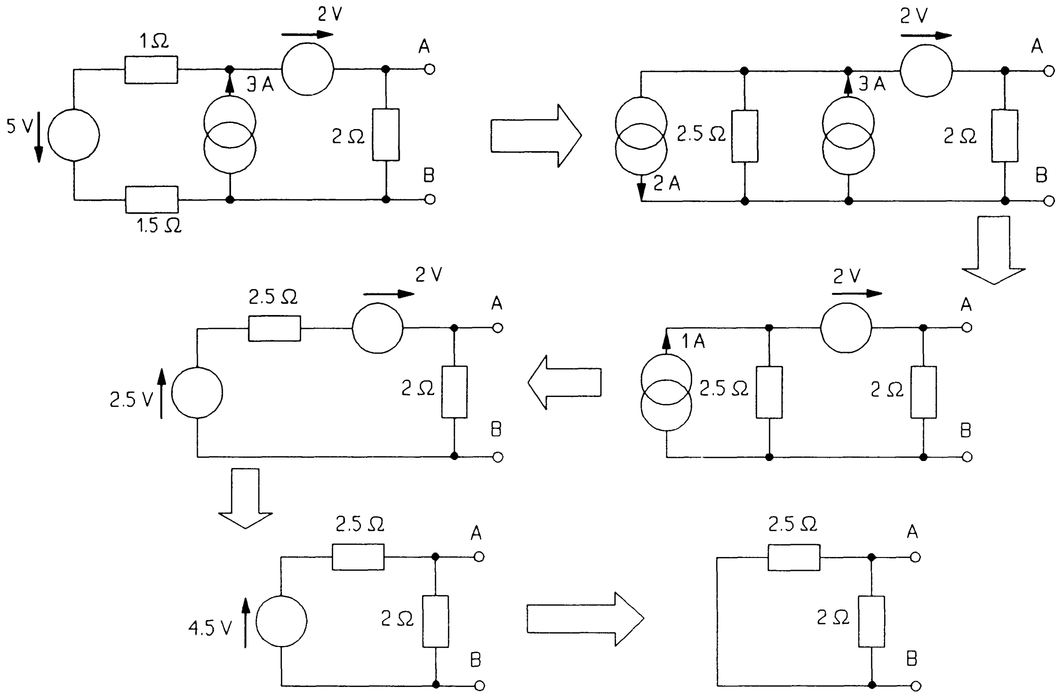
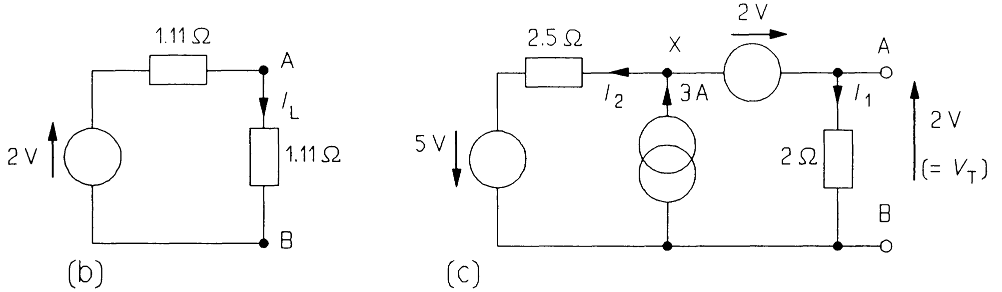


*Figure A1.12a*

When the circuit is delivering maximum power, the currents must be recalculated with the circuit of figure A1.12d, in which $I_1 = 1.11I_L/2 = 1.11 \times 0.9/2 = 0.5$ A. Then by KCL $I_2 = I_1 + I_L = 1.4$ A and at point X, $I_2 + I_3 = 3$ A, making $I_3 = 1.6$ A and the power developed in the 2Ω resistance $0.5^2 \times 2 = 0.5$ W, while the power in the 2.5Ω resistance is $1.6^2 \times 2.5 = 6.4$ W, for a total internal power dissipation of 6.9 W.



*Figures A1.12b and A1.12c*

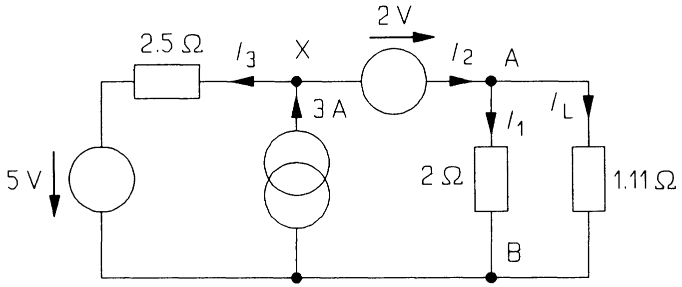


*Figure A1.12d*

**13** The star network can be transformed, using conductances, into the delta form of figure A1.13a, followed by two voltage-to-current source transformations in figure A1.13b. The resistances parallel to these current sources are combined in figure A1.13c and the current sources are then turned back into voltage sources in figure A1.13d, combined and transformed again to a current source as in figure A1.13e.

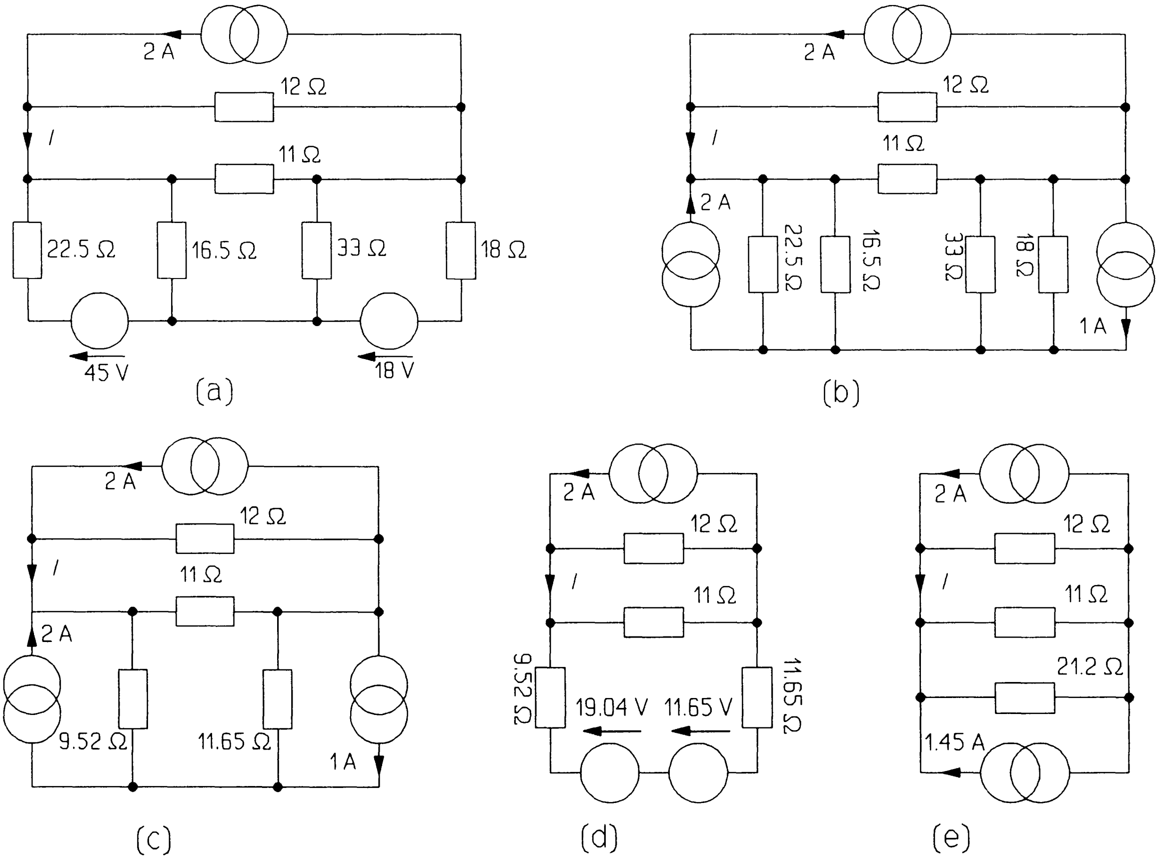


*Figure A1.13*

The final circuit preserves $I$, which is 2 A less the current through the 12Ω resistance. Since the total current is the sum of the two source currents, 3.45 A, the current in the 12Ω resistance is

$$I_{12\Omega} = \frac{1/12}{1/12 + 1/11 + 1/21.2} \times 3.45 = 1.3 \text{ A}$$

And then $I = 2 - 1.3 = 0.7$ A.

**14** The mesh currents are as in figure A1.14 and the mesh equations are

$$\begin{aligned} I_1 &= -2 \\ 12(I_2 - I_1) + 6(I_2 - I_4) + 3(I_2 - I_3) &= 0 \\ 22.5I_3 + 3(I_3 - I_2) + 9(I_3 - I_4) &= 45 \\ 9(I_4 - I_3) + 6(I_4 - I_2) + 18I_4 &= 18 \end{aligned}$$

These rearrange to

$$\begin{aligned} 1I_1 + 0I_2 + 0I_3 + 0I_4 &= -2 \\ -12I_1 + 21I_2 - 3I_3 - 9I_4 &= 0 \\ 0I_1 - 3I_2 + 34.5I_3 - 9I_4 &= 45 \\ 0I_1 - 6I_2 - 9I_3 + 33I_4 &= 18 \end{aligned}$$

Gaussian elimination leads to $I_4 = 0.815$ A. [This is the easiest way of solving the problem as the coefficients of the currents can be written down by inspection of the circuit. Students should write a program for Gaussian elimination in the general case.]

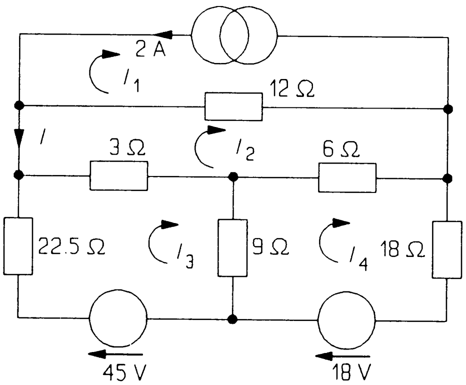


*Figure A1.14*

**15** The node voltages are shown in figure A1.15 and the nodal equations are:

$$\frac{V_1 - 45}{22.5} + \frac{V_1 - V_2}{3} - I = 0$$

$$\frac{V_2 - V_1}{3} + \frac{V_2}{9} + \frac{V_2 - V_3}{6} = 0$$

$$\frac{V_3 - V_2}{6} + \frac{V_3 + 18}{18} + I = 0$$

The extra variable, $I$, indicates a supernode, but $I$ is easily found from the additional equation for the currents at node 1:

$$\frac{V_1 - V_3}{12} - I - 2 = 0$$

Substituting for $I$ in the first and third equations leads to:

$$\begin{aligned} +0.461\,V_1 - 0.333\,V_2 - 0.0833\,V_3 &= 4 \\ -0.333\,V_1 + 0.611\,V_2 - 0.1667\,V_3 &= 0 \\ -0.0833\,V_1 - 0.1667\,V_2 + 0.306\,V_3 &= -3 \end{aligned}$$

Solving we find $V_1 = 12.23$, $V_2 = 5.75$ and $V_3 = -3.34$ V, so that the voltage across the current source is $V_1 - V_3 = 15.6$ V.

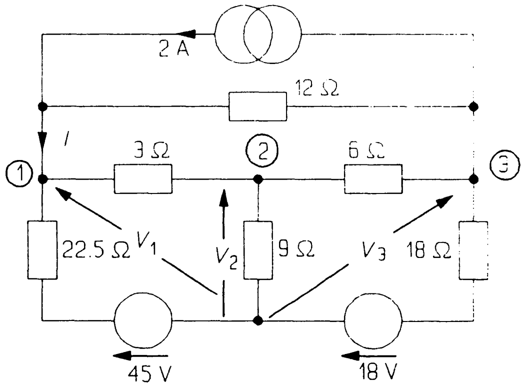


*Figure A1.15*

**16** The circuit is shown in figure A1.16, and applying KVL to loop 1 gives

$$12(2 - I) - 3(I - I) - 6I_3 = 0$$

But by KCL we can see that $I_3 = I - I_1 - I_2$, and the equation above becomes

$$-21I + 9I_1 + 6I_2 = -24$$

Then application of KVL to loop 2 gives

$$\begin{aligned} -22.5I_1 + 3(I - I_1) + 9I_2 &= 45 \\ \Rightarrow \quad 3I - 25.5I_1 + 9I_2 &= 45 \end{aligned}$$

KVL in loop 3 leads to

$$\begin{aligned} -9I_2 + 6I_3 + 18(I_3 - I) &= 18 \\ \Rightarrow \quad 6I - 24I_1 - 33I_2 &= 18 \end{aligned}$$

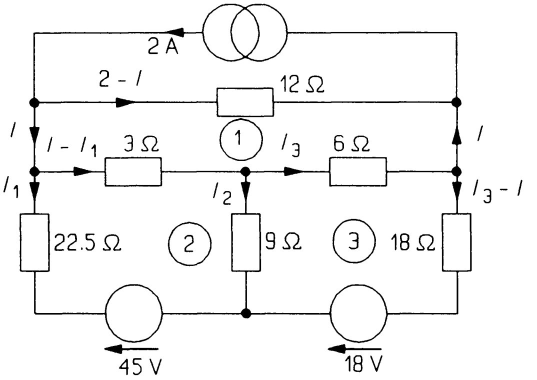


*Figure A1.16*

These three equations can be solved to give $I = 0.702$, $I_1 = -1.456$ and $I_2 = 0.641$ A. The current from the 18V source is $-I_1 - I_2 = -(-1.456) - 0.641 = 0.815$ A. The current in the 12Ω resistance is $2 - 0.7 = 1.3$ A and the voltage across it is therefore $12 \times 1.3 = 15.6$ V, as before.

**17** Figure A1.17 shows the transformations. The 12V source in series with 6 Ω is transformed into a 2A source in parallel with 6 Ω. The parallel current sources are then combined to give a $4 - 2 = 2$A source in parallel with 2 Ω (3 Ω ‖ 6 Ω = 2 Ω). This current source is next converted to a 4V source in series with 2 Ω, so that the series voltage sources may be combined into a single 8V source in series with 4 Ω. Thus $V_{AB} = -4$ V.

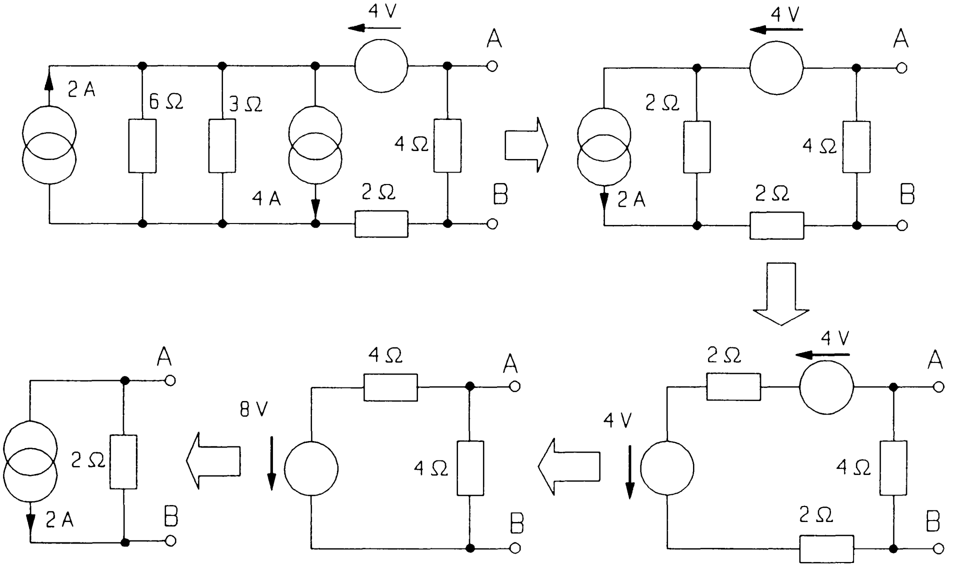


*Figure 1.17*

**18** If the network is infinite in extent then the current through $R_L$ is zero and it is immaterial what its value is; make it 1 Ω. The right-end parallel branches are then 2 Ω ‖ 2 Ω = 1 Ω. This can be added to the 1 Ω in series to make 2 Ω, and the whole process is repeated down to $R$, which must be 1 Ω for the resistance between AB to be 2 Ω.

**19** The circuit of figure P1.19 can be transformed into that of figure A1.19, from which we can find the load current, $I_L$:

$$I_L = \frac{G_L}{\Sigma G} \times \Sigma I$$

where $\Sigma G = G_1 + G_2 + G_3 + G_L$ and $\Sigma I = I_1 + I_2 + I_3$. The source currents are $I_1 = V_1 G_1$ etc. Then the load voltage is

$$V_L = \frac{I_L}{G_L} = \frac{\Sigma I}{\Sigma G} = \frac{V_1G_1 + V_2G_2 + V_3G_3}{\Sigma G}$$

Going back to figure P1.19, the currents from the sources are $(V_S - V_L)/R_S$ and the power delivered is $P_S = V_S(V_S - V_L)/R_S$.

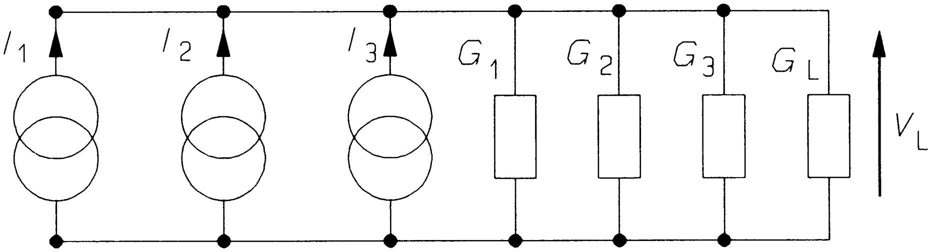


*Figure A1.19*

Putting in the numbers we find

$$V_L = \frac{1190 + 1200 + 1210}{30.1} = 119.60133 \text{ V}$$

and $P_1 = 121(121 - 119.60133)/0.1 = 1.692$ kW, $P_2 = 120(120 - 119.60133)/0.1 = 478$ W and $P_3 = 119(119 - 119.60133)/0.1 = -716$ W.

Changing the source resistances to 0.3, 0.2 and 0.1 Ω changes $V_L$ to

$$V_L = \frac{121 \times 3.333 + 120 \times 5 + 119 \times 10}{3.333 + 5 + 10 + 0.1} = 118.98731 \text{ V}$$

And the power delivered is $P_1 = 121(121 - 118.98731)/0.3 = 812$ W, and $P_2 = 120(120 - 118.98731)/0.2 = 608$ W, and $P_3 = 119(119 - 118.98731)/0.1 = 15$ W.

**20** We shall first consider the voltage source by itself, figure A1.20a. Here the current sources have been open-circuited as required. We see that $I_2$ and $I_3$ are zero as the terminals are open circuit. The current $I_1 = 8/6 = 1.333$ A, and we can also see that $V_1 + V_{4\Omega} = 0$. Now $V_{4\Omega} = 4I_1 = 5.333$ V, so that $V_1 = -5.333$ V. Next we consider the 2A source alone as in figure A1.20b, in which the voltage source has been short-circuited and the 1A source open-circuited.

The voltage, $V_2$, due to this current source is just the voltage across the 3Ω resistance. Since this carries all of the current from the source, $V_2 = -6$ V.

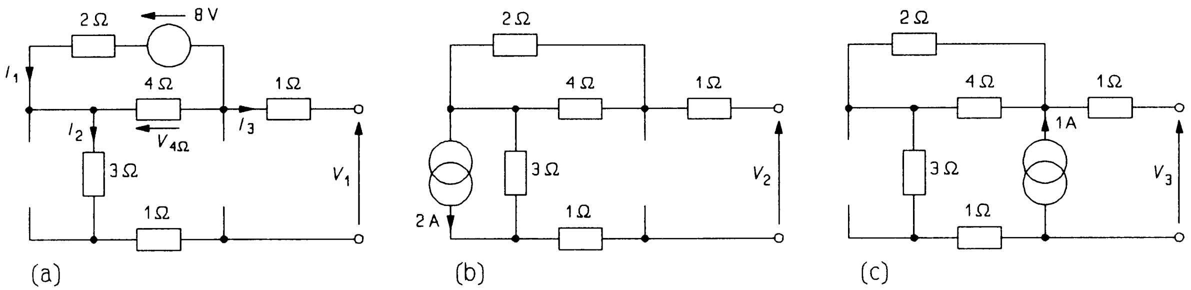


*Figure A1.20*

Finally we consider the 1A source by itself as in figure A1.20c, from which it can be seen that $V_3$ is the voltage across the 1A source, so we have to find the equivalent resistance of the path of the source current. The 4Ω and 2 Ω resistances are in parallel, constituting a resistance of $2 \times 4/(2 + 4) = 1.333$ Ω, which is in series with the 3Ω and 1Ω resistances, making 5.333 Ω in all. Thus, as the current is 1 A, $V_3 = 5.333$ V.

The Thévenin voltage, $V_T = V_1 + V_2 + V_3 = -5.333 - 6 + 5.333 = -6$ V.

**21** The voltage is found from

$$V = \frac{1}{C}\int i\mathrm{d}t = \frac{1}{200 \times 10^{-6}}\int (2 + 3\sin 2t)\mathrm{d}t \text{ mV}$$

$$= 5\int (2 + 3\sin 2t)\mathrm{d}t \text{ V} = 10 - 7.5\cos 2t \text{ V}$$

**22** The energy consumed by the resistance is given by

$$E = \int i^2R\mathrm{d}t = 583\int_0^{0.01} [0.63\exp(-100t)]^2\mathrm{d}t = 583\left[\frac{0.63^2\exp(-200t)}{-200}\right]_0^{0.01}$$

$$= \frac{583 \times 0.63^2}{200}[\exp(0) - \exp(-2)] = 1.157[1 - 0.1353] = 1 \text{ J}$$

The average power is $E/T = 1/0.01 = 100$ W and the r.m.s. current is

$$I_{\text{rms}}^2R = 100 \Rightarrow I_{\text{rms}} = \sqrt{100/583} = 0.414 \text{ A}$$

**23** The circuit of figure A1.23 shows the currents through the resistances.

By KVL in the left-hand loop

$$10000I_1 + 25000(I_1 + I_G) = 7$$

When $I_G = 0$ (the true balance point), then $I_1 = 0.2$ mA and when $I_G = 0.1$ μA, then $I_1 = 0.1999286$ mA (apparent balance point). Using KVL on the bottom right-hand loop yields

$$10000I_1 = 268(I_R + I_G)$$

When $I_G = 0$, this gives $I_R = 7.4627$ mA and when $I_G = 0.1$ $\mu$A, it gives $I_R = 7.4599$ mA.

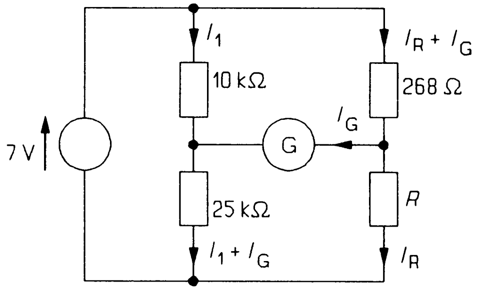


*Figure A1.23*

Using KVL on the outer loop gives

$$268(I_R + I_G) + I_RR = 7$$

When $I_G = 0$ this gives $R = 670$ Ω, and when $I_G = 0.1$ $\mu$A it gives $R = 670.35$ Ω, an error of 0.05%.

[The same result can more easily be obtained by neglecting the change in potential across $R$ due to $I_G$ and considering only that across the 10kΩ resistance. As the current, $I_1$, at true balance is 0.2 mA and $I_1 = 0.2001$ mA at the apparent balance, the value of $R$ is higher by $100(0.2001 - 0.2)/0.2 = 0.05\%$.]

**24** The problem requires us to find $I_R = I_C + I_L$ as $V_{AB} = I_RR + V_C$ (see figure A1.24).

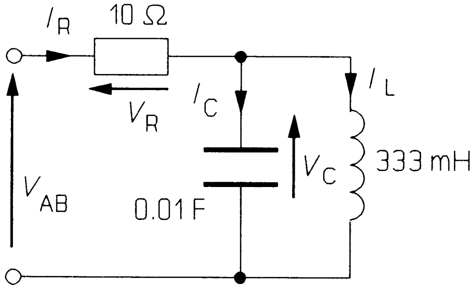


*Figure A1.24*

The capacitor voltage is

$$V_C = \frac{1}{C}\int I_C dt = 100\int 20t dt = 1000t^2 \text{ mV}$$

No constant of integration is required as the initial voltage is zero. This must also be the voltage across the inductance, $V_L$, which means that

$$I_L = \frac{1}{L}\int V_L \mathrm{d}t = 3\int 1000t^2 \mathrm{d}t = 1000t^3 \text{ mA}$$

Hence $I_R = 20t + 1000t^3$ mA and $V_R = 0.2t + 10t^3$ V. Then $V_R + V_C = 0.02t + t^2 + 10t^3$ V. Substituting $t = 0.3$ s leads to $V_{AB} = 0.06 + 0.09 + 0.27 = 0.42$ V.

**25** The secret is to short-circuit successive pairs of terminals. Short-circuiting terminals 1 and 3 in figures A1.25a and A1.25b and measuring the conductance between terminals 1 and 2 gives a conductance of $G_1 + G_2$ from figure A1.25b. This must be equal to the conductance measured in figure A1.25a, which is that of $G_1'$ in series with $G_2' \parallel G_3'$, that is

$$G_1 + G_2 = \frac{G_2' \times (G_1' + G_3')}{G_1' + (G_2' + G_3')} = \frac{G_2'(G_1' + G_3')}{\Sigma G'}$$

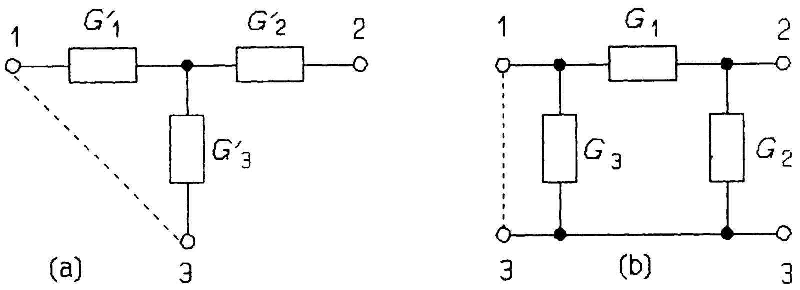


*Figure A1.25*

Then short-circuiting terminals 1 and 2 and measuring the conductance between terminals 1 and 3 leads to

$$G_2 + G_3 = \frac{G_3'(G_1' + G_2')}{\Sigma G'}$$

Finally we short-circuit terminals 2 and 3 and measure the conductance between terminals 1 and 3, which gives

$$G_1 + G_3 = \frac{G_1'(G_2' + G_3')}{\Sigma G'}$$

Subtracting the second equation from the first and adding the last to the result produces

$$G_1 + G_2 - G_2 - G_3 + G_1 + G_3 = \frac{G_2'(G_1' + G_3') - G_3'(G_1' + G_2') + G_1'(G_2' + G_3')}{\Sigma G'}$$

$$\Rightarrow \qquad 2G_1 = \frac{2G_1'G_2'}{\Sigma G'}$$

By similar means the other relationships of equation 1.50 may be found.

# Chapter 2

**1** Consider figure A2.1a, in which the waveform is triangular and symmetrical about the $t$-axis, so we need only consider the positive part.

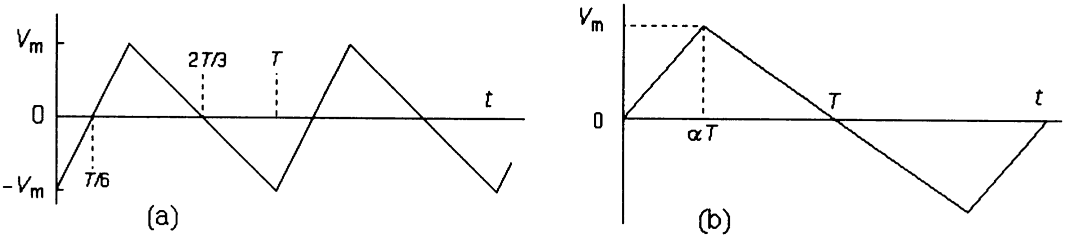


*Figure A2.1*

The shift along the $t$-axis is immaterial and we might as well start from the origin as in figure A2.1b. The equation of the straight line from the origin is $V_m t/\alpha T$ and the square of the first part of the waveform is

$$\int_0^{\alpha T} (V_m t/\alpha T)^2 \mathrm{d}t = \left[\frac{V_m^2 t^3}{3\alpha^2 T^2}\right]_0^{\alpha T} = \frac{V_m^2 \alpha T}{3}$$

The second part can be shifted to the origin so that the time span is from 0 to $(1 - \alpha)T$ and the square is

$$\int_0^{(1-\alpha)T} [V_m t/(1-\alpha)T]^2 \mathrm{d}t = \left[\frac{V_m^2 t^3}{3(1-\alpha)^2 T^2}\right]_0^{(1-\alpha)T} = \frac{V_m^2 (1-\alpha)T}{3}$$

Thus the r.m.s. value is

$$\sqrt{\frac{V_m^2 \alpha T + V_m^2 (1-\alpha)T}{3T}} = \frac{V_m}{\sqrt{3}}$$

The r.m.s. value is independent of $\alpha$. The rectified time-averaged value is

$$\frac{1}{T}\left[\int_0^{\alpha T} \frac{V_m t}{\alpha T}\mathrm{d}t + \int_0^{(1-\alpha)T} \frac{V_m t}{(1-\alpha)T}\mathrm{d}t\right]$$

$$= \frac{V_m}{2T^2}\left(\left[\frac{t^2}{\alpha}\right]_0^{\alpha T} + \left[\frac{t^2}{1-\alpha}\right]_0^{(1-\alpha)T}\right) = \frac{V_m}{2}[\alpha + (1-\alpha)] = \frac{V_m}{2}$$

So the FF is

$$\frac{V_m/\sqrt{3}}{V_m/2} = 2/\sqrt{3} = \frac{2\sqrt{3}}{3} = 1.155$$

2 The reactance of the inductance of 50 mH at 2000 rad/s is $j50 \times 10^{-3} \times 2000 = j100\ \Omega$. This is in series with 10 Ω for a impedance of $(10 + j100)\ \Omega$. The reactance of the 5 μF capacitance is $-j/(5 \times 10^{-6} \times 2000) = -j100\ \Omega$, which is in parallel with $(10 + j100)\ \Omega$, so that the combined impedance is

$$\mathbf{Z} = \frac{-j100 \times (10 + j100)}{-j100 + 10 + j100} = \frac{100\angle -90^\circ \times 100.5\angle 84.29^\circ}{10} = 1005\angle -5.71^\circ\ \Omega$$

In rectangular form this is $1000 - j100$ and is in series with 1 kΩ, for a combined impedance of $2000 - j100\ \Omega$ or in polar form $2002.5\angle -2.86^\circ\ \Omega$.

Since the r.m.s. voltage across this impedance is $24\surd 2/\surd 2 = 24$ V, the current, **I**, is

$$\mathbf{I} = \frac{24\angle 30^\circ}{2002.5\angle -2.86^\circ} = 11.985\angle 32.86^\circ \text{ mA}$$

Thus $\mathbf{V}_R = \mathbf{I}R = 11.985\angle 32.86^\circ$ V. Then $\mathbf{V}_C + \mathbf{V}_R = 24\angle 30^\circ$V, so that

$$\mathbf{V}_C = 24\angle 30^\circ - 11.985\angle 32.86^\circ = 20.785 + j12 - 10.067 - j6.503$$

$$= 10.718 + j5.497 = 12.045\angle 27.15^\circ \text{ V}$$

Then

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{jX_C} = \frac{12.045\angle 27.15^\circ}{100\angle -90^\circ} = 120.45\angle 117.15^\circ \text{ mA}$$

Next we find

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{10 + j100} = \frac{12.045\angle 27.15^\circ}{100.5\angle 84.29^\circ} = 119.85\angle 57.14^\circ \text{ mA}$$

The phasor diagram is shown in figure A2.2.

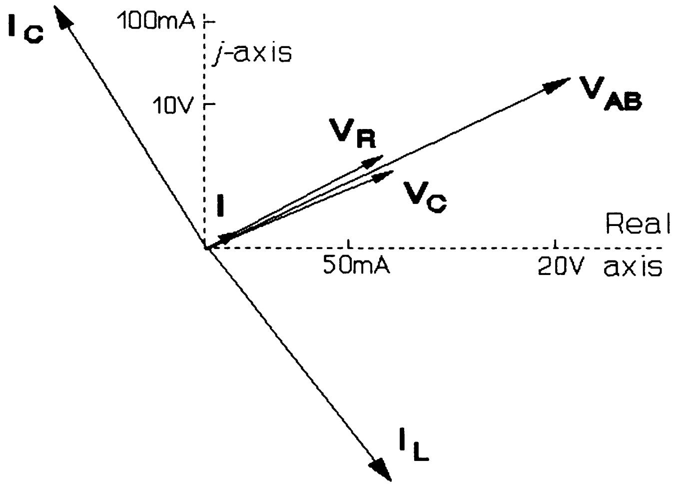


*Figure A2.2*

**3** The current in each branch is the same and is

$$\mathbf{I} = \mathbf{V}_{in}/(R + 1/j\omega C)$$

The voltage across $R$ is $\mathbf{V}_R = \mathbf{I}R$ and the voltage across $C$ is $\mathbf{V}_C = \mathbf{I}/j\omega C$ and

$$\mathbf{V}_o = \mathbf{V}_C - \mathbf{V}_R = \mathbf{I}/j\omega C - \mathbf{I}R = \mathbf{I}(1/j\omega C - R)$$

$$= \frac{\mathbf{V}_{in}(1/j\omega C - R)}{R + 1/j\omega C} \Rightarrow \frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1 - j\omega RC}{1 + j\omega RC}$$

The magnitude of $\mathbf{V}_o/\mathbf{V}_{in}$ is

$$\left|\frac{\mathbf{V}_o}{\mathbf{V}_{in}}\right| = \left|\frac{1 - j\omega RC}{1 + j\omega RC}\right| = \frac{\sqrt{1 + (\omega RC)^2}}{\sqrt{1 + (\omega RC)^2}} = 1$$

The phase shift is $\tan^{-1}(-\omega RC) - \tan^{-1}(\omega RC) = -2\tan^{-1}(\omega RC) = -90°$ when $\tan^{-1}(\omega RC) = 45°$, or when $\omega RC = 1$, or $f = \omega/2\pi = 1/2\pi RC$.

The phase shift is $-120°$ when $\tan^{-1}(\omega RC) = 60°$, or when $\omega RC = \sqrt{}3$, that is when $f = \sqrt{}3/2\pi RC$.

**4** At balance, let the current through the left-hand side of the bridge be **I** and let that through the right-hand side be $\mathbf{I}_1$. For balance the potentials at the central nodes of these branches are the same, that is

$$\mathbf{I}R = \mathbf{I}_1(R_1 + j\omega L_1) \text{ and } \mathbf{I}/j\omega C = \mathbf{I}_1(R_2 + 1/j\omega C_2)$$

Dividing equations gives

$$j\omega RC = \frac{R_1 + j\omega L_1}{R_2 + 1/j\omega C_2}$$

$$\Rightarrow \quad j\omega RR_2C + RC/C_2 = R_1 + j\omega L_1$$

Equating real parts gives $R_1 = RC/C_2$ and equating the imaginary, $L_1 = RR_2C$: $\omega$ cancels.

**5** The balance conditions are found as in problem 2.4 and are

$$\frac{R_1 + 1/j\omega C_1}{R_4} = \left[\frac{R_2/j\omega C_2}{R_2 + 1/j\omega C_2}\right] \div R_3$$

$$\Rightarrow \quad \frac{j\omega R_1R_3C_1 + R_3}{j\omega R_4C_1} = \frac{R_2}{j\omega R_2C_2 + 1}$$

$$\Rightarrow \quad j\omega R_2R_4C_1 = -\omega^2R_1R_2R_3C_1C_2 + R_3 + j\omega R_2R_3C_2 + j\omega R_1R_3C_1$$

Equating the real parts gives $0 = \omega^2 R_1R_2C_1C_2 - 1$, or $\omega = 1/\sqrt{(R_1R_2C_1C_2)}$. Equating the imaginary parts yields $\omega R_2R_4C_1 = \omega R_2R_3C_2 + \omega R_1R_3C_1$ or $C_1(R_2R_4 - R_1R_3) = C_2R_2R_3$, which is $C_2/C_1 = R_4/R_3 - R_1/R_2$.

**6** The current source transforms to a voltage source of $2\angle 30° \times 4\angle -90° = 8\angle -60°$ V, which can then be added to the other voltage source to give a voltage of

$$\mathbf{V}_S = 8\angle -60° - 12\angle -120° = 4 - j6.928 + 6 + j10.392$$

$$= 10 + j3.464 = 10.583\angle 19.11° \text{ V}$$

The impedance is series with this is $1 + j2 - j4 = 1 - j2 = 2.236\angle -63.43°\ \Omega$. The voltage source can then be transformed into a current source of value

$$\mathbf{I}_S = \frac{\mathbf{V}_S}{\mathbf{Z}_S} = \frac{10.583\angle 19.11°}{2.236\angle -63.43°} = 4.733\angle 82.54° \text{ A}$$

This in parallel with $2.236\angle -63.43°\ \Omega \parallel 2\ \Omega$ or

$$\mathbf{Z}_T = \frac{2.236\angle -63.43° \times 2}{2.236\angle -63.43° + 2} = \frac{4.472\angle -63.43°}{3 - j2}$$

$$= \frac{4.472\angle -63.43°}{3.606\angle -33.69°} = 1.24\angle 29.74°\ \Omega$$

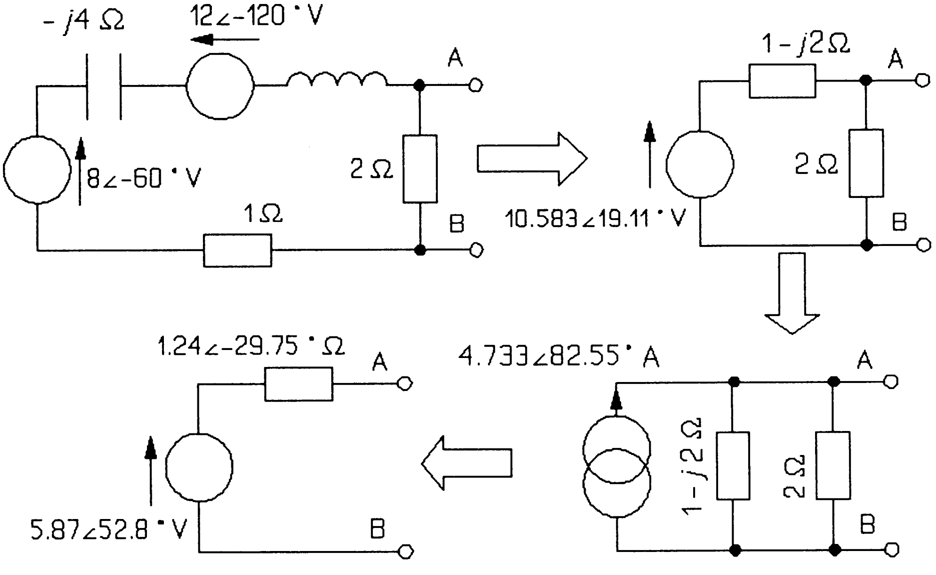


*Figure A2.6*

The current source becomes a Thévenin voltage of

$$\mathbf{V}_T = \mathbf{I}_S \times \mathbf{Z}_T = 4.733\angle 82.54° \times 1.24\angle -29.75 = 5.87\angle 52.8° \text{ V}$$

Figure A2.6 shows the stages in the circuit transformation.

The maximum power transfer theorem requires $\mathbf{Z}_L = \mathbf{Z}_T^*$, that is $\mathbf{Z}_L + \mathbf{Z}_T = 2\,\mathrm{Re}(\mathbf{Z}_T) = 2R_L = 2 \times 1.24\cos -29.75°\ \Omega$, or $2.153\ \Omega$, in which case

$$\mathbf{I}_L = \mathbf{V}_T/2.153 = 5.87\angle 52.8°/2.153 = 2.726\angle 52.8° \text{ A}$$

and $P_L = I_L^2R_L = 2.726^2 \times 1.077 = 8.00$ W. Then

$$Q_L = I_L^2X_L = I_L^2 \times Z_L\sin 29.75° = 2.726^2 \times 0.615 = 4.57 \text{ var (lagging)}$$

(The current lags the voltage: the load is inductive, $\mathbf{Z}_L$ has a positive phase angle).

7 The current source can be taken first, with the voltage source short-circuited as well as the 2Ω resistance across AB, as in figure A2.7a. The current from the source is

$$\mathbf{I}_1 = \frac{-j4}{1+j2-j4} \times 2\angle 30° = \frac{8\angle -60°}{2.236\angle -63.43°}$$

$$= 3.578\angle 3.43° \text{ A}$$

The voltage source is taken next with the current source open-circuited as in figure A2.7b, and the current, $\mathbf{I}_2$, is

$$\frac{-12\angle -120°}{1+j2-j4} = \frac{-12\angle -120°}{2.236\angle -63.43°} = -5.367\angle -56.57° \text{ A}$$

The sum of the two is the required current, **I**:

$$\mathbf{I} = 3.578\angle 3.43° - 5.367\angle -56.57° = 3.572 + j0.214 - 2.957 + j4.479$$

$$= 0.615 + j4.693 = 4.733\angle 83.53° \text{ A}$$

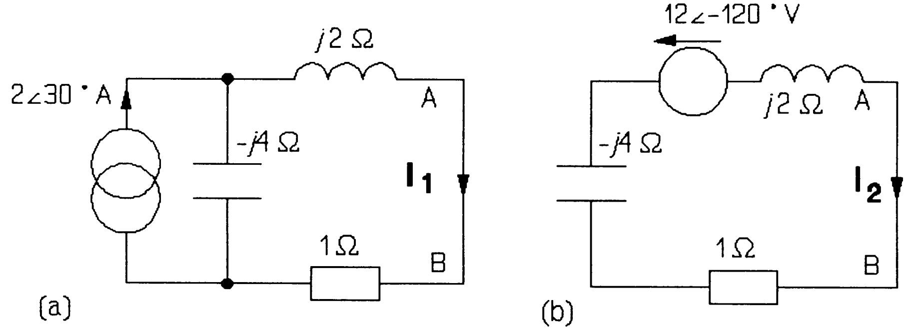


*Figure A2.7*

8 For the load to be matched, the Thévenin impedance of the circuit to the left of the 75Ω resistance, as in figure A2.8, must be wholly real and equal to 75 Ω.

The Thévenin impedance, looking into the terminals of figure A2.8, is

$$\mathbf{Z}_T = \frac{1}{j\omega C} + \frac{j2000\omega L}{2000 + j\omega L} = \frac{-j}{\omega C} + \frac{j2000\omega L(2000 - j\omega L)}{2000^2 + \omega^2 L^2}$$

$$= j\left[\frac{-1}{\omega C} + \frac{2000^2\omega L}{2000^2 + \omega^2 L^2}\right] + \frac{2000\omega^2 L^2}{2000^2 + \omega^2 L^2}$$

Equating the real part to 75 Ω gives

$$\frac{2000\omega^2 L^2}{2000^2 + \omega^2 L^2} = 75 \quad \Rightarrow \quad 2000^2\omega^2 L^2 = 75 \times 2000^2 + 75^2\omega^2 L^2$$

$$\therefore \quad L = \sqrt{\frac{75 \times 2000^2}{(2000-75)\omega^2}} = \frac{395}{2\pi \times 1.5 \times 10^6} = 41.9\ \mu\text{H}$$

Equating the imaginary part to zero gives

$$2000^2\omega^2 LC = 2000^2 + \omega^2 L^2$$

$$\Rightarrow \quad C = \frac{2000^2 + \omega^2 L^2}{2000^2\omega^2 L}$$

And then if $L = 41.9\ \mu$H, we find $C = 279$ pF.

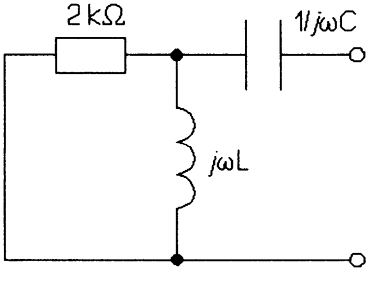


*Figure A2.8*

**9** The equivalent circuit is shown in figure A2.9, from which we can see that

$$\mathbf{Z}_1 = \frac{R_1/j\omega C_1}{R_1 + 1/j\omega C_1} = \frac{R_1}{1 + j\omega R_1 C_1}$$

where $C_1 = 110$ pF ∥ 20 pF = 130 pF and $R_1 = 1$ MΩ. Also $\mathbf{Z} = R/(1 + j\omega RC)$ and by the voltage-divider rule

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{\mathbf{Z}_1}{\mathbf{Z} + \mathbf{Z}_1} = \frac{R_1/(1 + j\omega R_1 C_1)}{R/(1 + j\omega RC) + R_1/(1 + j\omega R_1 C_1)} = \frac{R_1(1 + j\omega RC)}{R_1(1 + j\omega RC) + R(1 + j\omega R_1 C_1)}$$

Thus if $|\mathbf{V}_o/\mathbf{V}_{in}| = 0.1$, we have, taking magnitudes

$$\frac{1}{10} = \left|\frac{R_1 + j\omega RR_1C}{(R + R_1) + j\omega RR_1(C + C_1)}\right|$$

For this to hold for DC ($\omega = 0$), we require $10R_1 = R + R_1$, or $R = 9R_1 = 9\ \mathrm{M}\Omega$. For it to hold at any $\omega$, we require additionally that $10RR_1C = RR_1(C + C_1)$, or $10C = C + C_1$, that is $C = C_1/9 = 130/9 = 14.4$ pF.

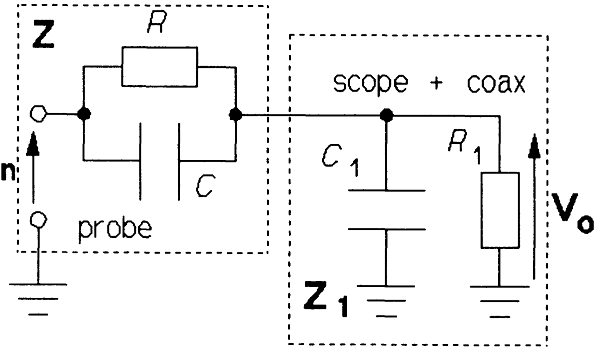


*Figure A2.9*

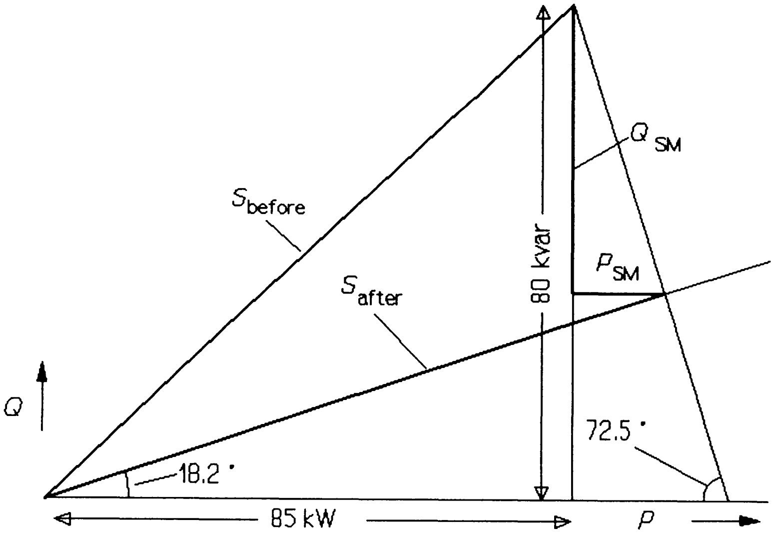


*Figure A2.10*

**10** The power diagram of figure A2.10 is constructed as follows: from the origin draw a horizontal line along the $P$-axis of length 60 + 25 = 85 kW (100 × 0.6 is the power delivered to the motor). Then from the 85kW point on the $P$-axis draw a line of length 80 kvar parallel to the $+Q$-axis to represent the kvar supplied to the motor: $100 \times \sqrt{[1 - 0.6^2]} = 80$. From the origin we then draw a line at an angle of $\cos^{-1}0.95$

= 18.2°, as the solution must lie on this line to have the correct final p.f. From the end of the 80kvar line we draw a line to intercept the $P$-axis at $\cos^{-1}0.3 = 72.5°$ to represent the kvar supplied by the synchronous motor and the kW supplied to it. Where this line intercepts the line from the origin at 18.2° is the point giving the total $P$ and $Q$ consumed by the circuit. Measuring the lengths indicated gives $P_{SM} = 14.7$ kW and $Q_{SM} = 47$ kvar.

It is easier to take an algebraic approach:

$$P = 85 + 0.3S_{SM}$$
$$Q = 80 - \sqrt{1 - 0.8^2}\,S_{SM}$$

And the ratio $Q/P = \tan\phi$, $\cos\phi = 0.95$; solving this gives the same answer.

The line current is proportional to the length of the hypotenuse, $S$. Before correction

$$S = \sqrt{80^2 + 85^2} = 116.7 \text{ kVA}$$

After correction the total power consumption is $P_T = 85 + 14.7 = 99.7$ kW and so $S_{after} = P_T/0.95 = 99.7/0.95 = 104.9$ kVA. The line current is then 94 × 104.9/116.7 = 84.5 A. Measuring the diagram gave 84.6 A.

**11** The reactance of 18 mH at 50 Hz is $100\pi \times 18 \times 10^{-3} = 5.655\ \Omega$ and the star-connected impedance can be converted to delta form using admittances:

$$Y_\Delta = Y_* /3 = \frac{1}{3(4 + j5.655)} = \frac{1}{20.78\angle 54.73°} = 48.12\angle -54.73° \text{ mS}$$

For a p.f. of 0.9, the phase angle of the admittance must be $\cos^{-1}0.9 = 25.84°$, so we must add to the $j$-part with capacitive susceptance while the real part stays the same at $48.12\cos(-54.73°) = 27.8$ mS. The imaginary part was $48.12\sin(-54.73°) = -39.3$ mS and must be $-27.8\tan 25.84° = -13.5$ mS after correction. The capacitive susceptance required is therefore $39.3 - 13.5 = 25.8$ mS, thus

$$C = \frac{25.8 \times 10^{-3}}{\omega} = \frac{25.8 \times 10^{-3}}{100\pi} = 82\ \mu\text{F}$$

Correction to unity p.f. requires

$$C = \frac{39.3 \times 10^{-3}}{100\pi} = 125\ \mu\text{F}$$

The current drawn is proportional to the magnitude of the admittance. After correction to a p.f. of 0.9 the admittance is

$$Y_1 = \sqrt{27.8^2 + 13.5^2} = 30.9 \text{ mS}$$

The ratio is 30.9/48.12 = 0.642, so the current drawn is 64.2 A. The admittance after correcting to unit p.f. is just 27.8 mS, so the current drawn is 100 × 27.8/48.12 = 57.7 A.

The reduction in current is 36% for 82 μF and 42% for 125 μF, the extra 6% reduction in current has required 50% more capacitance.

**12** We see at once that $Q = f_0/\Delta f = 300/10 = 30$, which is very large. At resonance the circuit is wholly real and the power dissipation is $V^2/R$, or $R = V^2/P = 200^2/100 = 400$ Ω. Now

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi \times 3 \times 10^5 L}{400} = 30$$

$$\Rightarrow \quad L = \frac{30 \times 400}{6\pi \times 10^5} = 6.37 \text{ mH}$$

The resonant frequency is given by $f_0 = 1/2\pi\surd(LC)$, so that $C = 1/(2\pi f_0)^2 L = 44.2$ pF.

The capacitor voltage at resonance is $QV_S = 30 \times 200 = 6$ kV.

The bandwidth is 10 kHz, so the lower half-power point is at 300 − 5 = 295 kHz, which means the power dissipated at 295 kHz is 0.5 × 100 = 50 W.

305 kHz is the other half-power frequency, so the power consumption is $I^2R = 50$ W and $I = \surd(50/400) = 0.354$ A.

**13** The circuit is that of figure A2.13. The admittance is

$$\mathbf{Y} = \frac{1}{R + j\omega L} + j\omega C = \frac{R}{R^2 + \omega^2 L^2} + j\left[\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right]$$

The resonant frequency is when **Y** is entirely real and this occurs when

$$\omega C = \omega L/(R^2 + \omega^2 L^2)$$

Substituting for $L$, $R$ and $\omega$ leads to $C = 5.99$ μF. Then the admittance at resonance is

$$\mathbf{Y}_0 = R/(R^2 + \omega_0^2 L^2)$$

The circuit's approximate $Q = \omega_0 L/R = 2\pi \times 2000 \times 10^{-3}/3 = 4.2$ and then its bandwidth is $f_0/Q = 2000/4.2 = 480$ Hz.

The half-power points occur when $V = V_0/\surd 2$, which is the same as when $Z = Z_0/\surd 2$, $Z_0$ being the impedance of the circuit at resonance:

$$\mathbf{Z}_0 = 1/\mathbf{Y}_0 = (R^2 + \omega_0^2 L^2)/R = (3^2 + 12.57^2)/3 = 55.64\ \Omega$$

We are therefore looking for the frequencies when $Z = 39.34$ Ω. Now **Z** is

$$\mathbf{Z} = \frac{(R + j\omega L)/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega RC}$$

Thus

$$|\mathbf{Z}|^2 = 39.34^2 = \frac{R^2 + (\omega L)^2}{(1 - \omega^2 LC)^2 + (\omega RC)^2} = 1548\ \Omega^2$$

This rearranges to

$$1548LC\omega^4 + \omega^2(1548RC - L^2 - 3096LC) + 1548 - R^2 = 0$$

Putting in the values for $R$, $C$ and $L$ gives

$$\omega^4 - 3.429 \times 10^8\omega^2 + 2.771 \times 10^{16} = 0$$

From which $\omega^2 = 2.125 \times 10^8$ or $1.304 \times 10^8$, or $f_1 = 1817$ Hz and $f_2 = 2320$ Hz.

At 2 kHz, the current through $C$ is $\mathbf{I}_C = j\omega C \times \mathbf{V} = 2\pi \times 2000 \times 5.99 \times 10^{-6} \angle -90^\circ \times 40 = 3.01 \angle -90^\circ$ A.

The current through the inductance at resonance is

$$\mathbf{I}_L = \frac{\mathbf{V}}{R + j\omega_0 L} = \frac{40\angle 0^\circ}{3 + j12.57} = \frac{40\angle 0^\circ}{12.92\angle 76.6^\circ} = 3.1\angle -76.6^\circ \text{ A}$$

The current from the source is then

$$\mathbf{I} = \mathbf{I}_C + \mathbf{I}_L = j3.01 + 0.72 - j3.01 = 0.72\angle 0^\circ \text{ A}$$

Alternatively, the circuit at resonance is a resistance of $(1 + Q^2)R = (1 + 4.2^2) \times 3 = 55.9\ \Omega$, so the current drawn is $40/55.9 = 0.72 \angle 0^\circ$ A.

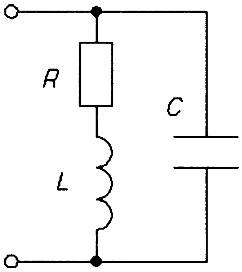


*Figure A2.13*

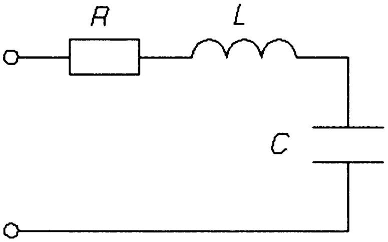


*Figure A2.14*

**14** The circuit is the series resonant circuit of figure A2.14, for which the admittance is

$$\mathbf{Y} = \frac{1}{R + j(\omega L - 1/\omega C)} = \frac{R}{R^2 + (\omega L - 1/\omega C)^2} - j\left[\frac{\omega L - 1/\omega C}{R^2 + (\omega L - 1/\omega C)^2}\right]$$

The conductance is maximum when $\omega = \omega_0 = 1/\surd(LC)$, and is then $1/R$, so that $R = 1/G_{\max} = 1/16.2$ mS $= 62\ \Omega$.

The susceptance is maximum or minimum when

$$\frac{dB}{d\omega} = 0 = \frac{(L + 1/\omega^2 C)(R^2 + [\omega L - 1/\omega C]^2) - 2(\omega L - 1/\omega C)(L + 1/\omega^2 C)(\omega L - 1/\omega C)}{D}$$

where $D$ is the denominator. From this we find $R^2 = (\omega L - 1/\omega C)^2$ and then we find

$$\pm R = \omega L - 1/\omega C \;\Rightarrow\; \omega^2 \pm \omega R/L - 1/LC = 0$$

Solving the quadratic, $\omega_{1,2} = \mp R/2L + \sqrt{R^2/4L^2 + 1/LC}$

These are the lower and upper half-power points. The bandwidth is $(\omega_2 - \omega_1)/2\pi = f_2 - f_1 = 3$ kHz $= R/2\pi L$. Thus $L = 62/2\pi \times 3000 = 3.29$ mH.

Now $f_0 = \sqrt{(f_1 f_2)} = 41.5$ kHz and $Q = f_0/\Delta f = 41.5/3 = 13.8$.

We find $C$ from $\omega_0 = 1/\sqrt{(LC)}$:

$$\Rightarrow \quad C = 1/\omega_0^2 L = [(2\pi \times 41500)^2 \times 3.29 \times 10^{-3}]^{-1} = 4.5 \text{ nF}$$

**15** The capacitances have reactances of $-j/2\pi \times 4974 \times 8 \times 10^{-6} = -j4\ \Omega$. The mesh currents are shown in figure A2.15, from which we have in mesh 1

$$(\mathbf{I}_1 - \mathbf{I}_2)(3 - j4) + 2\mathbf{I}_1 - j4(\mathbf{I}_1 - \mathbf{I}_3) = 0$$

which rearranges to

$$(5 - j8)\mathbf{I}_1 - (3 - j4)\mathbf{I}_2 + j4\mathbf{I}_3 = 0 \qquad (1)$$

Then in mesh 2: $6 = (3 - j4)(\mathbf{I}_2 - \mathbf{I}_1) + \mathbf{V}$

Mesh 3 gives

$$\mathbf{V} = -j4(\mathbf{I}_3 - \mathbf{I}_1) + j5\mathbf{I}_3 \quad \Rightarrow \quad \mathbf{V} = j4\mathbf{I}_1 + j\mathbf{I}_3$$

These last two equations may be combined, producing

$$6 = (3 - j4)(\mathbf{I}_2 - \mathbf{I}_1) - j4\mathbf{I}_1 + j\mathbf{I}_3$$

$$\Rightarrow \quad (3 - j8)\mathbf{I}_1 + (3 - j4)\mathbf{I}_2 + j\mathbf{I}_3 = 6 \qquad (2)$$

The remaining equation is

$$\mathbf{I}_2 - \mathbf{I}_3 = 10\angle -90° \quad \Rightarrow \quad \mathbf{I}_3 = \mathbf{I}_2 + j10$$

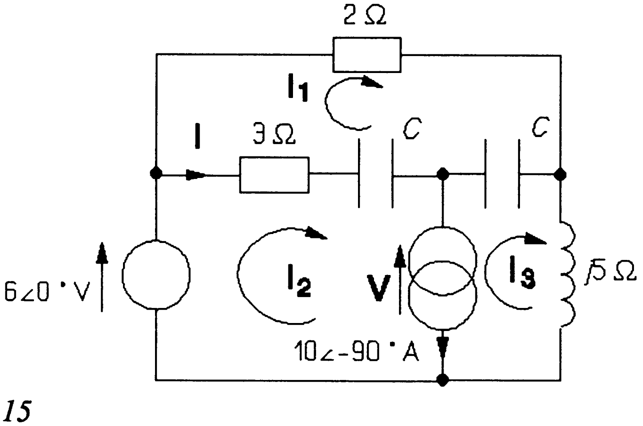


*Figure A2.15*

Substituting for $\mathbf{I}_3$ in the mesh 1 equation gives

$$(5-j8)\mathbf{I}_1 + (-3+j4)\mathbf{I}_2 + j4(\mathbf{I}_2 + j10) = 0$$

$$\Rightarrow \quad (5-j8)\mathbf{I}_1 + (-3+j8)\mathbf{I}_2 = 40 \qquad (3)$$

And substituting for $\mathbf{I}_3$ in the combined mesh 2 and 3 equations yields

$$(-3+j8)\mathbf{I}_1 + (3-j4)\mathbf{I}_2 + j1(\mathbf{I}_2 + j10) = 6$$

$$\Rightarrow \quad (-3+j8)\mathbf{I}_1 + (3-j3)\mathbf{I}_2 = 16 \qquad (4)$$

We can eliminate $\mathbf{I}_2$ from these by multiplying (4) by $(-3+j8)$ and (3) by $(3-j3)$:

$$(-3+j8)^2\mathbf{I}_1 + \alpha = 16(-3+j8)$$

$$\text{and} \quad (5-j8)(3-j3)\mathbf{I}_1 + \alpha = 40(3-j3)$$

where $\alpha$ is the term in $\mathbf{I}_2$. Subtracting these from each other gives

$$(8.544\angle 110.56°)^2\mathbf{I}_1 - (9.434\angle -57.99°)(4.243\angle -45°) = -48 + j128 - 120 + j120$$

$$\Rightarrow \quad (73\angle -138.88° - 40\angle -102.99°)\mathbf{I}_1 = -168 + j248$$

$$\Rightarrow \quad (-55 - j48 + 8.99 + j38.98)\mathbf{I}_1 = 299.55\angle -124.1°$$

$$\Rightarrow \quad (46.88\angle -168.9°)\mathbf{I}_1 = 299.55\angle -124.1°$$

$$\Rightarrow \quad \mathbf{I}_1 = 6.39\angle 293° = 6.39\angle -67° \text{ A}$$

Substituting for $\mathbf{I}_1$ in equation (4) gives

$$8.544\angle 110.56° \times 6.39\angle -67° + (4.234\angle -45°)\mathbf{I}_2 = 16$$

$$\Rightarrow \quad 54.6\angle 43.56° + (4.234\angle -45°)\mathbf{I}_2 = 16$$

$$\Rightarrow \quad \mathbf{I}_2 = \frac{16 - 39.57 - j37.63}{4.234\angle -45°} = \frac{44.4\angle -122.1°}{4.234\angle -45°} = 10.49\angle -77.1° \text{ A}$$

Then

$$\mathbf{I} = \mathbf{I}_2 - \mathbf{I}_1 = 10.49\angle -77.1° - 6.39\angle -67° = 2.34 - j10.23 - 2.5 + j5.88$$

$$= -0.16 - j4.35 = 4.35\angle -92° \text{ A}$$

And finally we find $\mathbf{V}$ by KVL:

$$6 = \mathbf{I}(3-j4) + \mathbf{V}$$

$$\Rightarrow \quad \mathbf{V} = 6 - 5\angle -53.13° \times 4.35\angle -92° = 6 - 21.75\angle -145.13° \text{ V}$$

$$\therefore \quad \mathbf{V} = 23.84 + j12.43 = 26.9\angle 27.5° \text{ V}$$

**16** For nodal analysis we use the circuit of figure A2.16, in which the three nodal voltages are $6\angle 0°$, $V_2$ and $V_3$. There are only two unknowns. At node 2 the nodal equation is

$$\frac{V_2 - 6}{3 - j4} + \frac{V_2 - V_3}{-j4} + 10\angle -90° = 0$$

$$\Rightarrow \quad \frac{V_2 - 6}{5\angle -53.13°} + \frac{V_2 - V_3}{4\angle -90°} + 10\angle -90° = 0$$

$$\Rightarrow \quad V_2\left[\frac{1}{5\angle -53.13°} + \frac{1}{4\angle -90°}\right] - \frac{V_3}{4\angle -90°} + 10\angle -90° + \frac{6}{5\angle -53.13°} = 0$$

$$\Rightarrow \quad V_2(0.2\angle 53.13° + 0.25\angle 90°) - 0.25V_3\angle 90° + 10\angle -90° - 1.2\angle 53.13° = 0$$

$$\Rightarrow \quad V_2(0.12 + j0.16 + j0.25) - 0.25V_3\angle 90° - j10 - 0.72 - j0.96 = 0$$

$$\Rightarrow \quad V_2(0.4272\angle 73.69°) - 0.25V_3\angle 90° = 0.72 + j10.96 = 10.98\angle 86.24° \qquad (1)$$

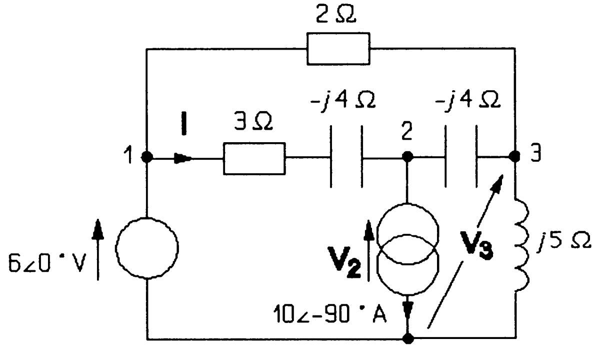


*Figure A2.16*

At node 3 the nodal equation is

$$\frac{V_3 - V_2}{-j4} + \frac{V_3}{j5} + \frac{V_2 - 6}{2} = 0$$

$$\Rightarrow \qquad 0.25V_2\angle -90° + 0.5025V_3\angle 5.71° = 3 \qquad (2)$$

We can eliminate $V_3$ from equations (1) and (2) by multiplying (1) by $0.5025\angle 5.71°$:

$$V_2(0.4272 \times 0.5025\angle(73.69° + 5.71°) - \alpha = 10.98 \times 0.5025\ \angle(86.24° + 5.71°)$$

$$\Rightarrow \quad V_2(0.2147\angle 79.4°) - \alpha = 5.518\angle 91.95° \qquad (3)$$

where $\alpha$ is the term to be eliminated.

Then we multiply equation (2) by $0.25\angle 90°$ to obtain

$$\mathbf{V}_2(0.25 \times 0.25\angle(-90° + 90°)) + \alpha = 3 \times 0.25\angle 90°$$

$$\Rightarrow \quad 0.0625\mathbf{V}_2 + \alpha = 0.75\angle 90° \tag{4}$$

Adding equations (3) and (4) gives

$$\mathbf{V}_2(0.2147\angle 79.4° + 0.0625) = 5.518\angle 91.95 + 0.75\angle 90°$$

$$\Rightarrow \quad \mathbf{V}_2(0.0395 + 0.0625 + j0.211) = -0.188 + j5.514 + j0.75$$

$$\Rightarrow \quad \mathbf{V}_2(0.2344\angle 64.2°) = 6.267\angle 91.7°$$

$$\Rightarrow \quad \mathbf{V}_2 = 26.74\angle 27.5° \text{ V}$$

Finally we find **I** by using KVL on the lower left-hand mesh:

$$6 = \mathbf{I}(3 - j4) + \mathbf{V}_2$$

$$\Rightarrow \quad \mathbf{I} = \frac{6 - \mathbf{V}_2}{3 - j4} = \frac{6 - 23.72 - j12.35}{3 - j4} = \frac{21.6\angle -145.1°}{5\angle -53.13} = 4.32\angle -92° \text{ A}$$

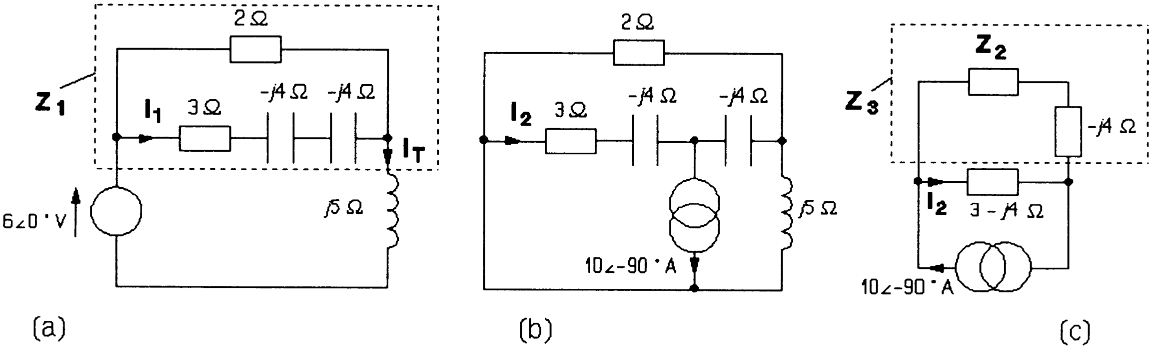


*Figure A2.17*

**17** Taking the voltage source first, the circuit is as in figure A2.17a, from which we can see that the 2Ω resistance is in parallel with an impedance of 3 − *j*8 Ω, giving an impedance of

$$\mathbf{Z}_1 = \frac{2(3 - j8)}{2 + (3 - j8)} = \frac{6 - j16}{5 - j8} \ \Omega$$

$\mathbf{Z}_1$ is in series with $j5$ Ω and $\mathbf{I}_T = 6/(\mathbf{Z}_1 + j5)$ A. Then $\mathbf{I}_1$ is found:

$$\mathbf{I}_1 = \frac{2}{2 + 3 - j8} \times \mathbf{I}_T = \frac{12}{(5 - j8)(\mathbf{Z}_1 + j5)}$$

$$= \frac{12}{(5-j8)[(6-j16)/(5-j8)+j5)]} = \frac{12}{6-j16+j5(5-j8)}$$

$$= \frac{12}{6-j16+j25+40} = \frac{12}{46+j9}\ \mathrm{A}$$

Taking the current source alone as in figure A2.17b we see that the $j5\ \Omega$ is in parallel with the $2\ \Omega$ giving an impedance of $\mathbf{Z}_2 = j10/(2+j5)\ \Omega$, which is in series with $-j4\ \Omega$. This impedance is

$$\mathbf{Z}_3 = \mathbf{Z}_2 - j4 = \frac{j10}{2+j5} - j4 = \frac{j10-j8-j^2 20}{2+j5} = \frac{20+j2}{2+j5}$$

$\mathbf{Z}_3$ is in parallel with $3-j4\ \Omega$ and the current source as in figure A2.17c, and then $\mathbf{I}_2$ is

$$\mathbf{I}_2 = \frac{\mathbf{Z}_3}{\mathbf{Z}_3+3-j4} \times 10\angle -90^\circ = \frac{(20+j2)10\angle -90^\circ}{20+j2+(2+j5)(3-j4)}$$

$$= \frac{(20.1\angle 5.71^\circ)(10\angle -90^\circ)}{(20+j2)+(6-j^2 20+j15-j8)} = \frac{201\angle -84.29^\circ}{46+j9}$$

Then we find $\mathbf{I}$ by adding $\mathbf{I}_1$ and $\mathbf{I}_2$:

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \frac{12}{46=j9} + \frac{201\angle -84.29^\circ}{46+j9} = \frac{12+20-j200}{46+j9}$$

$$= \frac{202.54\angle -80.91^\circ}{46.87\angle 11.07^\circ} = 4.32\angle -92^\circ\ \mathrm{A}$$

$\mathbf{V}$ is found as in problem 2.15.

**18** (a) The r.m.s. value is

$$\sqrt{\frac{1}{T}\left[\int_0^{T/2} 4^2 \mathrm{d}t + \int_{T/2}^{T}(-2)^2\mathrm{d}t\right]} = \sqrt{\frac{1}{T}\left[16\frac{T}{2} + 4\left[T - \frac{T}{2}\right]\right]} = \sqrt{10}\ \mathrm{V}$$

(b) This is just half a sinewave so the r.m.s. value must be $V_\mathrm{m}/2$, the formal proof is

$$V_\mathrm{rms} = \sqrt{\frac{\int_0^{T/2} V_\mathrm{m}^2 \sin^2\omega t\,\mathrm{d}t}{T}} = \sqrt{\frac{V_\mathrm{m}^2\int_0^{T/2} 0.5(1-\cos 2\omega t)\,\mathrm{d}t}{T}} = \sqrt{\frac{0.5V_\mathrm{m}^2[t - \sin 2\omega t]_0^{T/2}}{T}}$$

$$= \sqrt{\frac{0.5V_\mathrm{m}^2[T/2 - \sin\omega T]}{T}} = \sqrt{\frac{V_\mathrm{m}^2}{4}} = \frac{V_\mathrm{m}}{2}$$

(c) The r.m.s. value is

$$I_{\text{rms}} = \sqrt{\frac{\int_0^{T/3} 8^2 \mathrm{d}t + \int_{T/3}^{T} (-4)^2 \mathrm{d}t}{T}} = \sqrt{\frac{64T/3 + 16(T - T/3)}{T}} = \sqrt{32} = 4\sqrt{2} \text{ A}$$

(d) The r.m.s. value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\left[\int_0^{T/3} 6^2 \mathrm{d}t + \int_{T/3}^{T} (-6)^2 \mathrm{d}t\right]} = \sqrt{36/3 + 36 - 36/3} = 6 \text{ A}$$

A square wave can be treated as DC + AC for power purposes only if the time spent at the peak value is the same as the time spent at the trough. If the square wave is shifted by $\Delta$ from a peak-peak value of $\pm y$, its r.m.s. value is

$$y_{\text{rms}} = \sqrt{\frac{\int_0^{T/2} (y + \Delta)^2 \mathrm{d}t + \int_{T/2}^{T} (-y + \Delta)^2 \mathrm{d}t}{T}} = \sqrt{\frac{y^2 + 2y\Delta + \Delta^2 + y^2 - 2y\Delta + \Delta^2}{2}} = \sqrt{y^2 + \Delta^2}$$

Thus the r.m.s. value is the same as the square root of sum of the squares of the alternating and direct components. Making the high/low shift at any time but $T/2$ invalidates this as the product terms $\pm y\Delta$ do not cancel.

**19** The parallel part of the circuit can be made to resonate at 50 Hz, thereby having maximum impedance and blocking that frequency. We need to check that the coil, that is the 80mH inductance and series 5Ω resistance, has a high enough Q for this. At 50 Hz, $X_L = 2\pi \times 50 \times 80 \times 10^{-3} = 8\pi$ and $Q = X_L/R = 1.6\pi \approx 5$, which is high enough to approximate the resonant frequency with $\omega_0 = 1/\angle(LC) = 100\pi$

$$\Rightarrow \quad C = \frac{1}{\omega_0^2 L} = \frac{1}{(100\pi)^2 \times 0.08} = 127 \ \mu\text{F}$$

At 200 Hz the parallel branch is capacitive and we can make it resonate with the series inductance, $L$, at 200 Hz, when it will have minimum impedance. The circuit at 200 kHz is as in figure A2.19a, with the 80mH inductance having now a reactance $j100.5$ Ω and the 127μF capacitance a reactance of $-j6.27$ Ω. The parallel branch has an impedance of

$$\mathbf{Z}_p = \frac{(5 + j100.5)(-j6.27)}{5 + j100.5 - j6.27} = \frac{100.62\angle 87.15° \times 6.27\angle -90°}{94.36\angle 86.96°} = 6.69\angle -89.81°$$

which is nearly $-j6.69$ Ω. Thus we require $\omega L = 6.69$, or $L = 6.69/2\pi \times 200 = 5.3$ mH. The attenuation is the ratio of $|V_o/V_{\text{in}}|$ at the two frequencies. At 50 Hz the circuit is that of figure A2.19b, and we must find the impedance

$$\mathbf{Z}_1 = j1.67 + [(5 + j25.1) \parallel -j25.1]$$

Or, $$\mathbf{Z}_1 = j1.67 + \frac{(5+j25.1)(-j25.1)}{5} = j1.67 + \frac{630 - j125.5}{5} = 126 - j23.4\ \Omega$$

Then $$\left|\frac{\mathbf{V}_o}{\mathbf{V}_{in}}\right| = \left|\frac{10}{10 + 126 - j23.4}\right| = \sqrt{\frac{100}{136^2 + 23.4^2}} = 0.0725$$

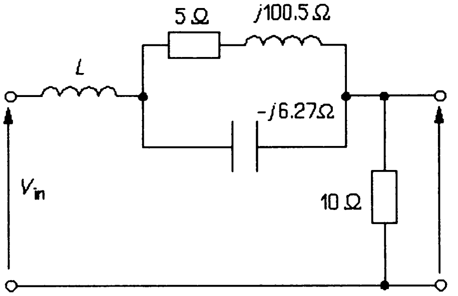


*Figure A2.19a*

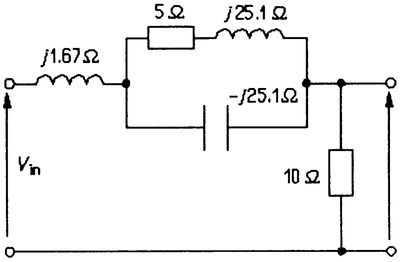


*Figure A2.19b*

At 200 Hz, the parallel branch has an impedance of $6.69\angle -89.81°\ \Omega$, the reactive part of which is cancelled by the reactance of the 5.3mH inductance, leaving just 0.022 Ω, in series with the 10 Ω across the output, so that

$$|\mathbf{V}_o/\mathbf{V}_{in}| = \frac{10}{10 + 0.022} = 0.9978$$

The ratio of the voltage ratios, which is the relative attenuation, is then 0.0725/0.9978 = 0.0727 or −22.8 dB.

The above analysis can be shortened considerably by taking $\mathbf{V}_o = \mathbf{V}_{in}$ at 200 Hz and taking the parallel branch to be $Q^2R = 5.03^2 \times 5 = 126\ \Omega$ at 50 Hz and ignoring the reactance of the 5.3mH inductance, so that $|\mathbf{V}_o/\mathbf{V}_{in}| = 10/136 = 0.0735$ and the ratio of the two responses is then $20\log 0.0735 = -22.7$ dB, practically the same.

**20** The two T networks can be transformed to $\pi$ form using admittances. Taking the upper T first the admittances are $j\omega C$, $j\omega C$ and $2/R$ and the admittance of the top part of the $\pi$-network is

$$\mathbf{Y}_1 = \frac{j\omega C \times j\omega C}{2/R + j2\omega C} = \frac{-\omega^2C^2R^2}{2R + j2\omega CR^2}$$

The lower T-network is transformed similarly and the upper part of the $\pi$-network has an admittance of

$$\mathbf{Y}_2 = \frac{1/R \times 1/R}{2/R + j2\omega C} = \frac{1}{2R + j2\omega CR^2}$$

Figure A2.20 shows the circuit after these transformations. The other admittances are inconsequential.

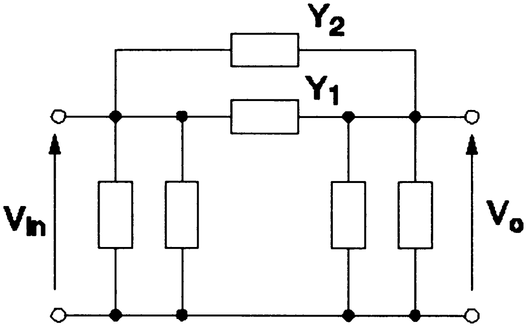


*Figure A2.20*

Thus the admittance of the transformed network between the upper input and output terminals is

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = \frac{1 - \omega^2R^2C^2}{2R + j2\omega CR^2}$$

If this is zero the input impedance will be infinite, that is when $\omega^2R^2C^2 = 1$ or $\omega = 1/RC$.

**21** If the source voltage is **V** then the load current is

$$\mathbf{I}_L = \frac{\mathbf{V}}{\mathbf{Z} + \mathbf{Z}_L}$$

The load power is

$$P_L = I_L{}^2R_L = \frac{V^2R_L}{|Z + Z_L|^2} = \frac{V^2R_L}{(R + R_L)^2 + (X + X_L)^2}$$

If $R_L$ alone can vary we can differentiate:

$$\frac{dP_L}{dR_L} = \frac{V^2[(R + R_L)^2 + (X + X_{L^2}] - 2(R + R_L)V^2R_L}{D^2}$$

where $D$ is the denominator. Setting this to zero for a maximum gives

$$\begin{aligned} V^2[(R + R_L)^2 + (X + X_L)^2] &= 2(R + R_L)V^2R_L \\ \Rightarrow \quad (R + R_L)^2 + (X + X_L)^2 &= 2(R + R_L)R_L \\ \Rightarrow \quad R^2 + 2RR_L + R_L{}^2 + (X + X_L)^2 &= 2RR_L + 2R_L{}^2 \\ \Rightarrow \quad R_L{}^2 &= R^2 + (X + X_L)^2 \end{aligned}$$

which is the required result.

**22** The expression for $P_L$ is the same as in problem 21. When $X_L$ is varied and $R_L$ is constant the differentiation goes

$$\frac{dP_L}{dX_L} = \frac{-2(X + X_L)V^2R_L}{D^2}$$

which is zero when $X = -X_L$.

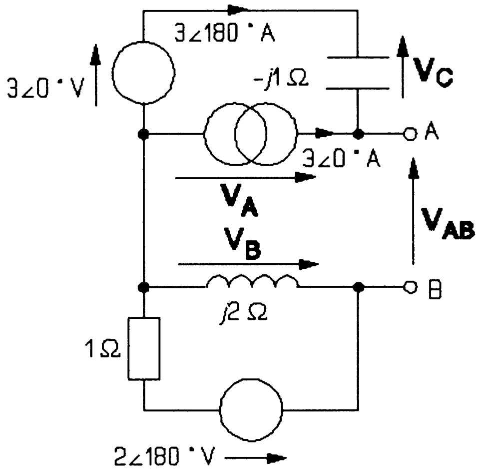


*Figure A2.22*

In the circuit of figure A2.22, we can find $\mathbf{V}_A$ and $\mathbf{V}_B$ and then $\mathbf{V}_{AB} = \mathbf{V}_T = \mathbf{V}_A - \mathbf{V}_B$.

In the top loop the clockwise current is $-3\angle 0°$, so $\mathbf{V}_C = 3\angle 180° \times 1\angle -90° = 3\angle 90°$ V. Then $\mathbf{V}_A = 3\angle 0° - \mathbf{V}_C = 3 - j3$ V.

In the bottom loop

$$\mathbf{V}_B = \frac{j2}{1 + j2} \times 2\angle 180° = \frac{2\angle 90° \times 2\angle 180°}{2.236\angle 63.43°}$$

$$= 1.789\angle 206.57° \text{ V}$$

$$\therefore \quad \mathbf{V}_T = \mathbf{V}_A - \mathbf{V}_B = 3 - j3 - 1.789\angle 206.57°$$

$$= 3 + 1.6 - j3 + j0.8 = 5.1\angle 25.56° \text{ V}$$

The Thévenin impedance looking into AB is just $-j1 + (j2 \parallel 1)$ Ω, or

$$\mathbf{Z}_T = -j1 + \frac{j2}{1 + j2} = \frac{-j1 + 2 + j2}{1 + j2} = \frac{2.236\angle 26.57°}{2.236\angle 63.43°}$$

$$= 1\angle -36.86° = 0.8 - j0.6 \ \Omega$$

We can therefore develop maximum power in a 2Ω load if we add a series inductance of reactance 0.6 Ω. The power developed is

$$P_L = \frac{V_T^2R_L}{(R_T + R_L)^2} = \frac{5.1^2 \times 2}{(0.8 + 2)^2} = 6.64 \text{ W}$$

# Chapter 3

**1** (a) In Figure A3.1a, node A is a virtual ground, so that $I_{in} = V_{in}/R_1$ and $Z_{in} = R_1$.
(b) In figure A3.1b, if the op amp is ideal, $I_{in} = 0$ and $Z_{in} = \infty$.
(c) In figure A3.1c, $I_{in} = (V_{in} - V_o)/R$. The voltage at node A is $V_{in}$ and then

$$I = (V_o - V_{in})/R = V_{in}/R \Rightarrow V_o = 2V_{in}$$

Thus $\quad I_{in} = (V_{in} - V_o)/R = -V_{in}/R \Rightarrow Z_{in} = V_{in}/I_{in} = -R$

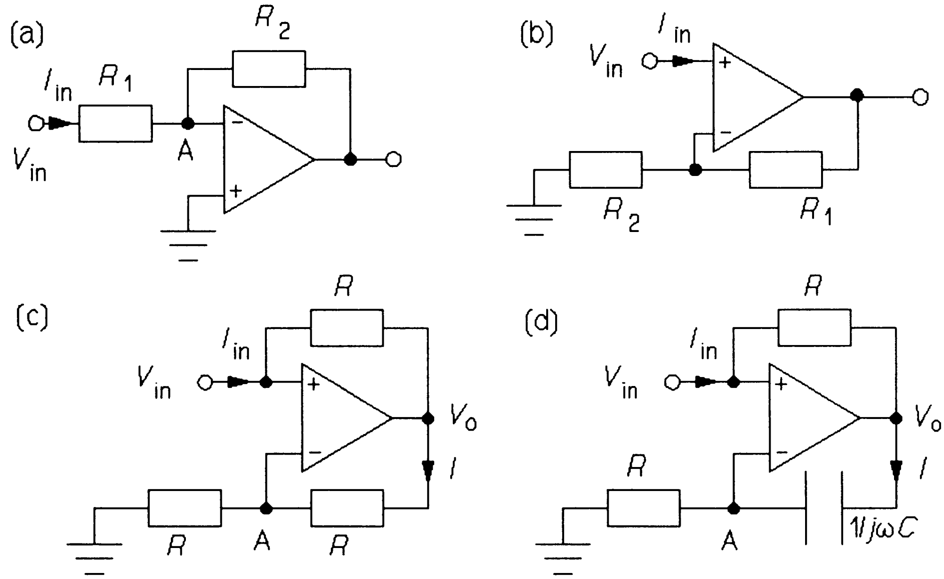


*Figure A3.1*

(d) In figure A3.1d, $\mathbf{I}_{in} = (\mathbf{V}_{in} - \mathbf{V}_o)/R$. The voltage at node A is $\mathbf{V}_{in}$ and then

$$\mathbf{I} = (\mathbf{V}_o - \mathbf{V}_{in})j\omega C = \mathbf{V}_{in}/R \Rightarrow \mathbf{V}_o = \mathbf{V}_{in}(1 + 1/j\omega CR)$$

Substituting for $\mathbf{V}_o$ in the equation for $\mathbf{V}_{in}$ gives

$$\mathbf{I}_{in} = \frac{\mathbf{V}_{in} - \mathbf{V}_{in}(1 + 1/j\omega CR)}{R} = \frac{-\mathbf{V}_{in}}{j\omega CR^2}$$

$$\therefore \quad \mathbf{Z}_{in} = \mathbf{V}_{in}/\mathbf{I}_{in} = -j\omega CR^2$$

**2** (a) The voltage at node A in figure A3.2a is $\mathbf{V}_o$ and therefore

$$\mathbf{I}_2 = (\mathbf{V}_{in} - \mathbf{V}_o)j\omega C = \mathbf{V}_o/R_2$$

$$\Rightarrow \quad \mathbf{V}_o(1/R_2 + j\omega C) = \mathbf{V}_{in} j\omega C \Rightarrow \mathbf{V}_o = \frac{\mathbf{V}_{in} j\omega CR_2}{1 + j\omega CR_2}$$

Next we find $\mathbf{I}_{in} = \mathbf{I}_1 + \mathbf{I}_2$:

$$\mathbf{I}_1 = \frac{\mathbf{V}_{in} - \mathbf{V}_o}{R_1} = \frac{\mathbf{V}_{in}}{R_1}\left[1 - \frac{j\omega CR_2}{1 + j\omega CR_2}\right] = \frac{\mathbf{V}_{in}}{R_1}\left[\frac{1}{1 + j\omega CR_2}\right]$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_o}{R_2} = \frac{\mathbf{V}_{in} j\omega CR_2}{R_2(1 + j\omega CR_2)} = \frac{\mathbf{V}_{in} j\omega C}{1 + j\omega CR_2}$$

So that

$$\mathbf{I}_{in} = \frac{\mathbf{V}_{in}}{R_1}\left[\frac{1}{1 + j\omega CR_2}\right] + \frac{\mathbf{V}_{in} j\omega C}{1 + j\omega CR_2} = \mathbf{V}_{in}\left[\frac{1/R_1 + j\omega C}{1 + j\omega CR_2}\right]$$

$$\therefore \quad \mathbf{Z}_{in} = \frac{\mathbf{V}_{in}}{\mathbf{I}_{in}} = \frac{R_1(1 + j\omega CR_2)}{1 + j\omega CR_1}$$

Then if $1/CR_2 << \omega << 1/CR_1$, or $\omega CR_2 >> 1$ and $\omega CR_1 << 1$; $\mathbf{Z}_{in} \approx j\omega CR_1R_2$. The circuit is an active inductor of inductance $L_{eq} = CR_1R_2 = 10^{-7} \times 500 \times 10^7 = 500$ H.

(b) The right-hand op amp circuit is a negative impedance converter (NIC) with an impedance of $-R$, so that the circuit is equivalent to that of figure A3.2b, another NIC, for which

$$\mathbf{Z}_{in} = -\{R + [(R + \mathbf{Z}) \parallel -R]\} = -\left[R + \left[\frac{-R^2 - R\mathbf{Z}}{\mathbf{Z}}\right]\right] = \frac{R^2}{\mathbf{Z}}$$

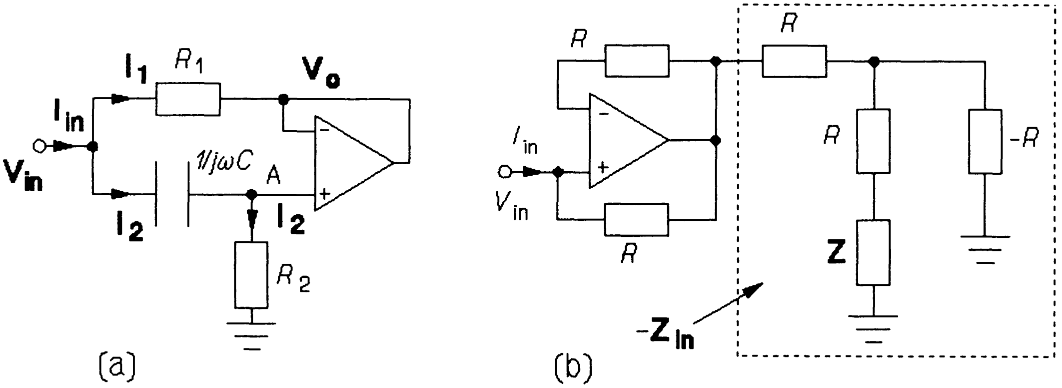


*Figure A3.2*

3 The input impedance, $\mathbf{Z}_1 = R_1 + 1/j\omega C_1$, while the feedback impedance, $\mathbf{Z}_2$, is $R_2 \parallel 1/j\omega C_2 = R_2/(1 + j\omega C_2R_2)$, giving

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{-\mathbf{Z}_2}{\mathbf{Z}_1} = \frac{-R_2/(1 + j\omega C_2R_2)}{R_1 + 1/(j\omega C_1)} = \frac{-j\omega C_1R_2}{(1 + j\omega C_2R_2)(1 + j\omega C_1R_1)}$$

Thus when $\omega$ is small $\omega C_1R_1 << 1$ and $\omega C_2R_2 << 1$, $\mathbf{V}_o/\mathbf{V}_{in} = -j\omega C_1R_2$. When $\omega$ is large and $\omega C_1R_1 >> 1$ and $\omega C_2R_2 >> 1$, then

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} \approx \frac{-j\omega C_1R_2}{j\omega C_2R_2 \times j\omega C_1R_1} = \frac{-1}{j\omega C_2R_1}$$

**4** At node B in figure A3.4, $\mathbf{V}_B = \mathbf{V}_o$ and summing currents out of the node gives

$$\mathbf{V}_o j\omega C + \frac{\mathbf{V}_o - \mathbf{V}_A}{R} = 0 \quad \Rightarrow \quad \mathbf{V}_A = \mathbf{V}_o(1 + j\omega CR)$$

At node A summing currents out gives

$$\frac{\mathbf{V}_A - \mathbf{V}_{in}}{R} + \frac{\mathbf{V}_A - \mathbf{V}_o}{R} + (\mathbf{V}_A - \mathbf{V}_o)j\omega C = 0$$

$$\Rightarrow \quad \mathbf{V}_A(2 + j\omega CR) - \mathbf{V}_o(1 + j\omega CR) = \mathbf{V}_{in}$$

$$\Rightarrow \quad \mathbf{V}_o[(2 + j\omega CR)(1 + j\omega CR) - (1 + j\omega CR)] = \mathbf{V}_{in}$$

$$\mathbf{V}_o(1 + j\omega CR)^2 = \mathbf{V}_{in}$$

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1}{(1 + j\omega CR)^2} = \frac{1}{1 - \omega^2C^2R^2 + j2\omega CR}$$

Taking magnitudes: $$\left|\frac{\mathbf{V}_o}{\mathbf{V}_{in}}\right| = \sqrt{\frac{1}{(1 - \omega^2C^2R^2)^2 + (2\omega CR)^2}}$$

$$= \sqrt{\frac{1}{1 - 2\omega^2C^2R^2 + \omega^4C^4R^4 + 4\omega^2C^2R^2}}$$

$$= \sqrt{\frac{1}{1 + 2\omega^2C^2R^2 + \omega^4C^4R^4}} = \frac{1}{1 + \omega^2C^2R^2}$$

Since $$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1}{1 - \omega^2C^2R^2 + j2\omega CR}$$

the phase shift is 90° when $\omega^2C^2R^2 = 1$ or $\omega = 1/CR$ and then

$$\mathbf{V}_o/\mathbf{V}_{in} = 1/j2\omega CR = 1/j2 = 0.5\angle -90°$$

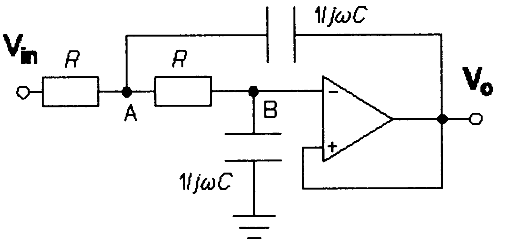


*Figure A3.4*

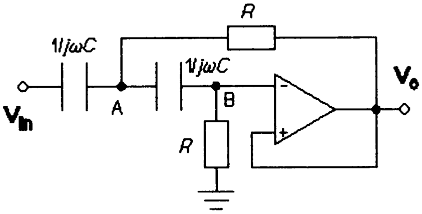


*Figure A3.5*

**5** At node B in figure A3.5, $\mathbf{V}_B = \mathbf{V}_o$ and the sum of the outgoing currents is

$$\mathbf{V}_o/R + (\mathbf{V}_o - \mathbf{V}_A)j\omega C = 0$$

$$\Rightarrow \quad \mathbf{V}_A = \mathbf{V}_o(1 + 1/j\omega CR)$$

Then at node A the sum of the outgoing currents is

$$(\mathbf{V}_A - \mathbf{V}_{in})j\omega C + (\mathbf{V}_A - \mathbf{V}_o)j\omega C + (\mathbf{V}_A - \mathbf{V}_o)/R = 0$$

$$\mathbf{V}_A(j2\omega C + 1/R) - \mathbf{V}_o(1/R + j\omega C) = \mathbf{V}_{in}j\omega C$$

Substituting for $\mathbf{V}_A$ leads to

$$\mathbf{V}_o[(1 + 1/j\omega CR)(2 + 1/j\omega CR) - (1 + 1/j\omega CR)] = \mathbf{V}_{in}$$

$$\Rightarrow \quad \frac{\mathbf{V}_O}{\mathbf{V}_{in}} = \frac{1}{(1 + 1/j\omega CR)^2} = \frac{1}{1 - 1/\omega^2C^2R^2 + 2/j\omega CR}$$

$$= \frac{\alpha^2}{\alpha^2 - 1 - j2\alpha}, \quad \text{where } \alpha \equiv \omega CR$$

When $\alpha = 1$

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1}{-j2\alpha} = \frac{1}{-j2} = 0.5\angle 90°$$

**6** At node A in figure A3.6, $V_A = 0$; at node B the sum of the outgoing currents is

$$\frac{V_B - V_o}{100} + \frac{V_B}{1} + \frac{V_B - V_A}{100} = 0$$

$$\Rightarrow \quad 1.02V_B = V_o/100 \quad \Rightarrow \quad V_B = V_o/102$$

Then summing currents into node A

$$V_{in}/100 + V_B/100 = 0 \Rightarrow V_B = -V_{in}$$

Thus $V_{in} = -V_o/102$ and if $V_{in} = 10$ mV, $V_o = -102 \times 10 \times 10^{-3} = -1.02$ V.

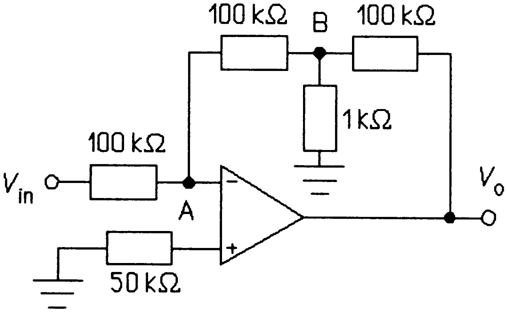


Figure A3.6

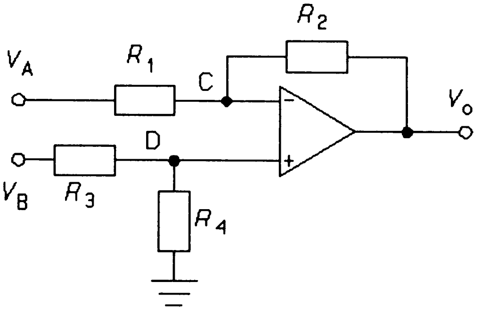


Figure A3.7

7 At node D in figure A3.7 the voltage is $V_D = V_B R_4/(R_3 + R_4) = V_C$. Then at node C we can sum the currents out:

$$\frac{V_C - V_A}{R_1} + \frac{V_C - V_o}{R_2} = 0$$

$$\Rightarrow \quad \frac{V_C(R_1 + R_2)}{R_1} - \frac{R_2 V_A}{R_1} = V_o$$

Substituting for $V_C$ gives

$$\frac{V_B R_4(R_1 + R_2)}{R_2(R_3 + R_4)} - \frac{R_2 V_A}{R_1} = V_o$$

So that when $R_1 = R_3$ and $R_2 = R_4$, $V_o = V_B - V_A$.

8 The output voltage in the absence of input is given by

$$V_o = \frac{-1}{RC}\int V_d \mathrm{d}t = \frac{-1}{RC}\int (V_{os} + I_b R)\mathrm{d}t$$

Where $V_{os}$ is the offset voltage at the input which is $1.2 \times 5 = 6$ μV for a 5K rise in temperature. The input bias current is $10 \times 5 = 50$ pA, so that $I_b R = 50 \times 10^{-12} \times 147 \times 10^3 = 7.35$ μV, for a total output drift of

$$V_o = \frac{-1}{147000 \times 10^{-6}}\int (6 + 7.35) \times 10^{-6}\mathrm{d}t = 91\ \mu\mathrm{V}/s$$

Putting a 147kΩ resistance from the + input to ground would eliminate the effect of $I_b$.

A drift of 1 mV on the output in 5 min is $1 \times 10^{-3}/300 = 3.33\ \mu V/s$, so with a temperature rise of 0.1 K, the input voltage drift is $1.2 \times 0.1 = 0.12\ \mu V$ and the max gain is $3.33/0.12 = 27.8$ only.

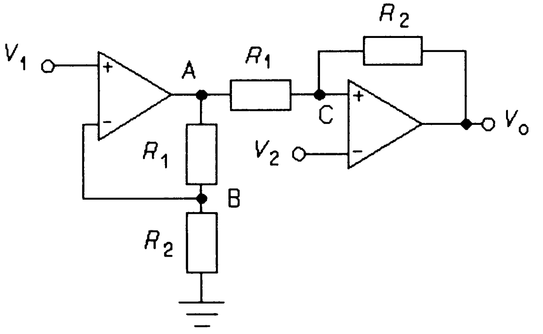


*Figure A3.9*

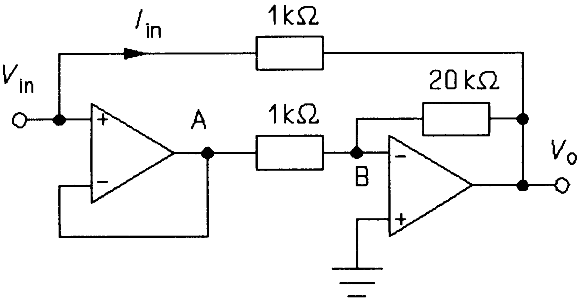


*Figure A3.10*

**9** At node B in figure A3.9, $V_B = V_1 = 0.2$ V and at node C, $V_C = V_2 = 0.65$ V. Summing currents out of node B yields

$$\frac{V_B - V_A}{10} + \frac{V_B}{90} = 0 \Rightarrow \quad V_A = V_B(1 + 0.111) = 0.222 \text{ V}$$

Summing currents out of node C yields

$$\frac{V_C - V_A}{10} + \frac{V_C - V_o}{90} = 0 \Rightarrow \quad \frac{0.65 - 0.222}{10} + \frac{0.65 - V_o}{90} = 0$$

From which $V_o = 4.5$ V.

**10** At node A in figure A3.10 we see that $V_A = V_{in}$ and at node B, that $V_B = 0$ V. Summing currents at node B:

$$(V_B - V_A)/1 + (V_B - V_o)/20 = 0$$

$$\Rightarrow \quad -V_{in} - V_o/20 = 0 \Rightarrow \quad V_o = -20V_{in}$$

Now $Z_{in} = V_{in}/I_{in}$ and

$$I_{in} = (V_{in} - V_o)/1000 = (V_{in} + 20V_{in})/1000 = 0.021V_{in}$$

Thus $Z_{in} = 1/0.021 = 48\ \Omega$.

**11** At node B in figure A3.11, $V_B = 0$ and summing currents out gives

$$(V_B - V_A)/1 + (V_B - V_o)/33 = 0 \Rightarrow \quad V_A = -V_o/33$$

Summing currents out of node A gives

$$(V_A - V_{in})/1 + (V_A - V_B)/1 + (V_A - V_o)/33 = 0$$

$$\Rightarrow \quad 2.03V_A - V_{in} - V_o/33 = 0$$

Substituting for $V_A$ produces

$$-V_o(2.03/33 + 1/33) = V_{in} \quad \Rightarrow \quad V_o/V_{in} = -33/3.03 = -10.89$$

Then

$$I_{in} = \frac{V_{in} - V_A}{1000} = \frac{V_{in} + V_o/33}{1000} = \frac{V_{in}(1 - 10.89/33)}{1000}$$

$$= 0.067V_{in} \quad \Rightarrow \quad Z_{in} = V_{in}/I_{in} = 1.493\ \text{k}\Omega$$

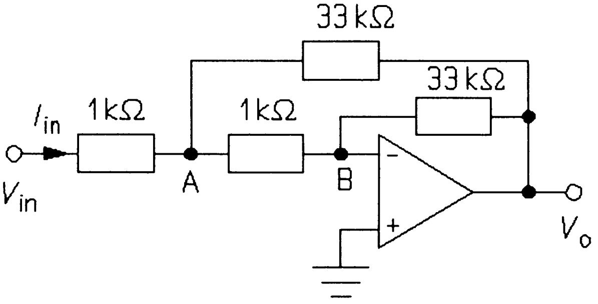


*Figure A3.11*

**12** We use equation 3.17 which is

$$V_{ref}(\text{upper}) = \frac{2GV_+}{2G + G_f} + \frac{G_fV_+}{G + G_f}$$

And as $\quad V_{ref}(\text{lower}) = \dfrac{2GV_+}{2G + G_f}, \quad$ we see that the hysteresis is

$$V_{ref}(\text{upper}) - V_{ref}(\text{lower}) = \frac{G_fV_+}{G + G_f} = \frac{5G_f}{G + G_f} = 1$$

$$\Rightarrow \quad 4G_f = G \Rightarrow 4/R_f = 1/20\text{k}$$

Then $R_f = 80$ kΩ.

The upper reference voltage is found from

$$V_{ref}(\text{upper}) = \frac{2GV_+}{2G + G_f} + \frac{G_fV_+}{G + G_f} = \frac{10G}{2G + 0.25G} + \frac{5 \times 0.25G}{G + 0.25G}$$

$$\therefore \quad V_{ref}(\text{upper}) = \frac{10}{2.25} + \frac{1.25}{1.25} = 4.44 + 1 = 5.44\ \text{V}$$

# Chapter 4

1 The transformed circuit is as in figure A4.1, from which we see that the impedance seen by the 12/$s$ source is

$$\tilde{\mathbf{Z}} = 500 + (500/s \parallel 100) = \frac{50000}{100s + 500} + 500$$

$$= \frac{500 + 500(s+5)}{s+5} = \frac{500s + 3000}{s+5}$$

Thus the current from the source is

$$\tilde{\mathbf{i}}_{\mathrm{s}} = 12/s \div \tilde{\mathbf{Z}} = \frac{12/s \times (s+5)}{500s + 3000} = \frac{12(s+5)}{500s^2 + 3000s}$$

By the current divider rule

$$\tilde{\mathbf{i}} = \frac{100\tilde{\mathbf{i}}_{\mathrm{s}}}{500/s + 100} = \frac{12(s+5)}{(5/s + 1)(500s^2 + 3000s)} = \frac{12(s+5)}{(s+5)(500s + 3000)}$$

$$= \frac{12}{500s + 3000} = \frac{0.024}{s+6} \Rightarrow i(t) = 24\exp(-6t)u(t) \text{ mA}$$

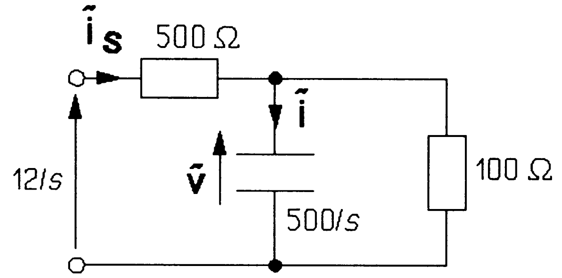


*Figure A4.1*

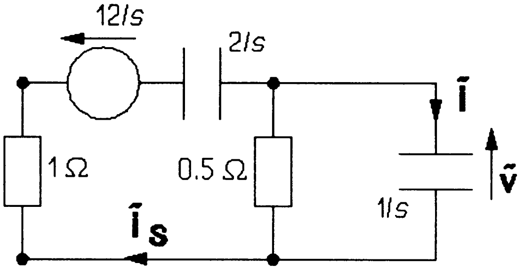


*Figure A4.2*

The transformed voltage is found from

$$\tilde{\mathbf{v}} = \tilde{\mathbf{i}}\tilde{\mathbf{X}}_{\mathrm{C}} = \frac{0.024}{s+6} \times \frac{500}{s} = \frac{12}{s(s+6)} \equiv \frac{A}{s} + \frac{B}{s+6}$$

$$\therefore \qquad A(s+6) + Bs \equiv 12$$

$$\Rightarrow \quad A + B = 0 \text{ (coeff. of } s\text{) and } 6A = 12 \text{ (constant term)}$$

And so $A = 2$ and $B = -2$, making $v(t) = [2 - 2\exp(-6t)]u(t)$ V.

**2** The 0.5F capacitance is charged to 12 V and then when S1 is opened and S2 closed, current will flow from it into the 1F capacitance via the 1Ω resistance. The charging current is thus −12A, as $i(t)$ is negative. After charging the 0.5F capacitance for a time, it will begin to discharge through the 0.5Ω resistance and eventually the current flow will cease when both $i(\infty)$ and $v(\infty)$ will be zero; $v(0) = 0$ of course, as the 1F capacitance was originally uncharged.

The transformed circuit is as in figure A4.2, from which we find the total impedance seen by the $12/s$ source to be

$$\tilde{\mathbf{Z}} = 2/s + 1 + (1/s \parallel 0.5) = (2+s)/s + 1 + 0.5/(0.5s+1)$$

$$= \frac{s+2}{s} + \frac{1}{s+2} = \frac{(s+2)^2+s}{s(s+2)} = \frac{s^2+5s+4}{s(s+2)}$$

The current from the source is then

$$\tilde{\mathbf{I}}_S = -12/s \div \tilde{\mathbf{Z}}$$

And so the transformed current through the 1F capacitance is

$$\tilde{\mathbf{I}} = \frac{0.5}{0.5+1/s} \times \tilde{\mathbf{I}} = \frac{s}{s+2} \times \frac{-12/s}{\tilde{\mathbf{Z}}} = \frac{-12}{(s+2)\tilde{\mathbf{Z}}}$$

$$= \frac{-12s}{s^2+5s+4} = \frac{-12s}{(s+4)(s+1)} \equiv \frac{A}{s+4} + \frac{B}{s+1}$$

Then we find

$$A(s+1) + B(s+4) \equiv -12s$$

Equating coefficients of $s$: $A + B = -12$
Equating constant terms: $A + 4B = 0$
From which we find $A = -16$ and $B = 4$, so that

$$i(t) = [4\exp(-t) - 16\exp(-4t)]u(t) \text{ A}$$

The transformed voltage is given by

$$\tilde{\mathbf{V}} = \tilde{\mathbf{I}} \times \tilde{\mathbf{X}}_C = \frac{-12s}{(s+4)(s+1)} \times \frac{1}{s} = \frac{-12}{(s+4)(s+1)} \equiv \frac{A}{s+4} + \frac{B}{s+1}$$

Proceeding as before we find

$$A(s+1) + B(s+4) \equiv -12 \Rightarrow A = 4 \text{ and } B = -4$$

$$\therefore \quad v(t) = 4[\exp(-t) - \exp(-4t)]u(t) \text{ V}$$

We differentiate this to find the time of maximum voltage

$$\mathrm{d}v/\mathrm{d}t = -4\exp(-t) + 16\exp(-4t) = 0$$

$$\Rightarrow\ 4\exp(-4t) = \exp(-t)$$

$$\Rightarrow\ \ln 4 - 4t = -t$$

$$\Rightarrow\ t = (\ln 4)/3 = 0.462\ \mathrm{s}$$

3 Initially there is no current through the inductance and so

$$i(0) = 60/(300 + 300) = 0.1\ \mathrm{A}$$

Eventually all the current flows through the inductance and therefore

$$i(\infty) = 60/300 = 0.2\ \mathrm{A}$$

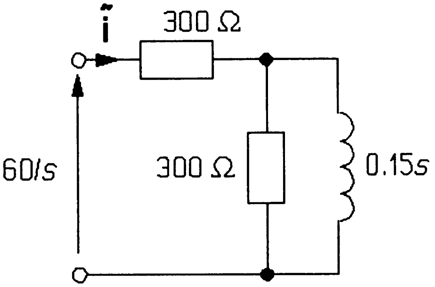


*Figure A4.3*

The transformed circuit is shown in figure A4.3 from which we see that

$$\tilde{\mathrm{i}} = \frac{60/s}{300 + (300 \parallel 0.15s)} = \frac{60/s}{300 + (300 \times 0.15s)/(300 + 0.15s)}$$

$$= \frac{60(300 + 0.15s)}{300s(300 + 0.15s) + 45s^2} = \frac{s + 2000}{10s^2 + 10000s}$$

Hence $\quad \tilde{\mathrm{i}} = \dfrac{s + 2000}{10s(s + 1000)} \equiv \dfrac{A}{s} + \dfrac{B}{s + 1000}$

$$\therefore \quad A(s + 1000) + Bs \equiv 0.1(s + 2000)$$

And thus $A = 0.2$, $B = -0.1$ and $i(t) = [0.2 - 0.1\exp(-1000t)]u(t)$ A.

The voltage across the inductance is the same as that across the parallel 300Ω resistance and can be found from the current flowing through it:

$$\tilde{\mathbf{i}}_R = \frac{0.15s}{300 + 0.15s} \times \tilde{\mathbf{i}} = \frac{0.15s(s + 2000)}{10(300 + 0.15s)(s^2 + 1000s)} = \frac{0.1}{s + 1000}$$

Thus $v_R = v_L = i_R R = 30\exp(-1000t)$ V, and after 10 ms this is $30e^{-10} = 1.362$ mV.

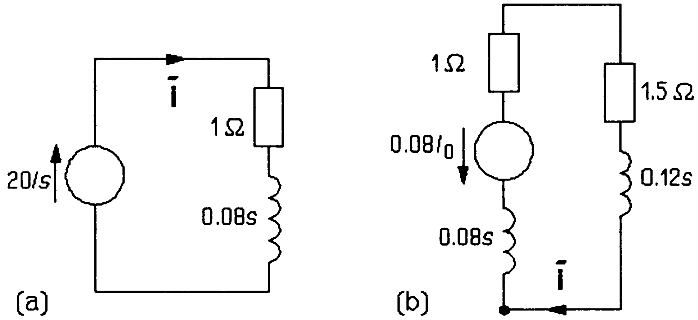


*Figure A4.4*

**4** We first find the current through the 80mH inductance at $t = t_1 = 60$ ms, for which the transformed circuit is shown in figure A4.4a, and we see that the current is

$$\tilde{\mathbf{i}} = \frac{20/s}{1 + 0.08s} = \frac{250}{s(s + 12.5)} \equiv \frac{A}{s} + \frac{B}{s + 12.5}$$

giving $A(s + 12.5) + Bs \equiv 250$, so $A = 20$ and $B = -20$, making

$$i(t) = [20 - 20\exp(-12.5t)]u(t) \text{ A.}$$

So after 60 ms the current is $20 - 20\exp(-0.75) = 10.553$ A.

This current is the initial current, $I_0$, in the 80mH inductance when S2 is closed and S1 opened to give figure A4.4b, the transformed circuit that can be solved. The time can be shifted by 60 ms by letting $t_1 = t - 0.06$. We find the transformed current as usual:

$$\tilde{\mathbf{i}} = \frac{-LI_0}{2.5 + 0.2s} = \frac{-0.08 \times 10.553}{0.2s + 2.5} = \frac{-4.22}{s + 12.5}$$

Hence $i(t_1) = -4.22\exp(-12.5t_1)u(t_1)$ A.

**5** The output voltage in the $s$-domain is

$$\tilde{\mathbf{v}}_o = \frac{2000}{2000 + 200000/s} \times \tilde{\mathbf{v}}_{in} = \frac{s}{s + 100} \times \frac{5}{s + 100}$$

$$= \frac{5(s + 100) - 500}{(s + 100)^2} = \frac{5}{s + 100} - \frac{500}{(s + 100)^2}$$

This detransforms to

$$v_0(t) = [5\exp(-100t) - 500t\exp(-100t)]u(t) \text{ V}$$

Differentiating to find the minimum gives

$$\mathrm{d}v_0/\mathrm{d}t = -500\exp(-100t) - 500\exp(-100t) - 100 \times -500t\exp(-100t)$$

$$= [50000t - 1000]\exp(-100t) = 0$$

From which we find $t = 0.02''$ and substituting this into the expression for $v_0(t)$ yields

$$v_0(\text{min}) = 5\exp(-100 \times 0.02) - 500 \times 0.02\exp(-100 \times 0.02)$$

$$= 5\exp(-2) - 10\exp(-2) = -0.677 \text{ V}$$

**6** When $v_{\text{in}}(t) = V\exp(-at)$ the transformed output voltage is

$$\tilde{\mathbf{v}}_o = \frac{V}{s + a} \times \frac{s}{s + 100} \equiv \frac{A}{s + a} + \frac{B}{s + 100}$$

Hence $A(s + 100) + B(s + a) \equiv Vs$

$\therefore$ $A + B = V$ (coeff. of $s$)

and $100A + aB = 0$ (constant term)

Thus $A = aV/(a - 100)$ and $B = -100V/(a - 100)$ and

$$v_0(t) = \frac{aV\exp(-at)}{a - 100} - \frac{500\exp(-100t)}{a - 100}$$

Substituting $V = 5$ V and $a = 100$ gives an indeterminate answer. Without going into too much mathematics, we can write $a = 100 - \alpha$ ($\alpha << 1$) and then

$$v_0(t) = \frac{(100 - \alpha)V\exp(-100 + \alpha)t}{-\alpha} - \frac{100V\exp(-100t)}{-\alpha}$$

$$= \frac{(100 - \alpha)V\exp(-100t)\exp(\alpha t)}{-\alpha} - \frac{100V\exp(-100t)}{-\alpha}$$

$$= \frac{(100 - \alpha)V\exp(-100t)(1 + \alpha t)}{-\alpha} - \frac{100V\exp(-100t)}{-\alpha}$$

$$= \frac{(100 + 100\alpha t - \alpha)V\exp(-100t) - 100V\exp(-100t)}{-\alpha}$$

$$= V\exp(-100t) - 100V\exp(-100t)$$

where terms in $\alpha^2$ and higher are ignored. This agrees with the previous problem's answer.

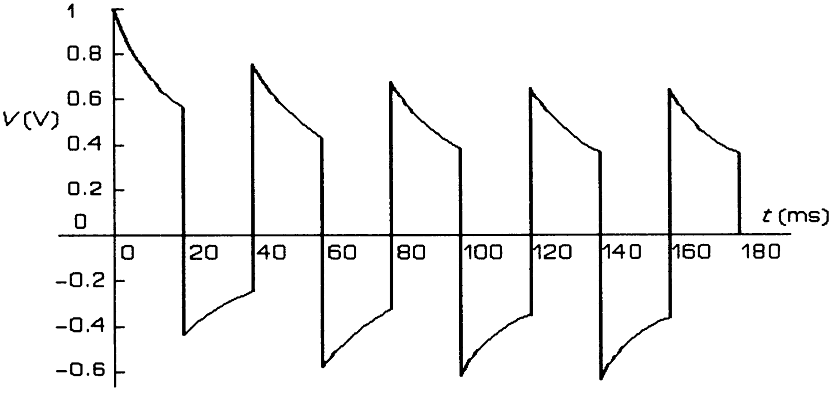


*Figure A4.7*

7 The capacitor first charges up according to the equation

$$v_C = 1 - \exp(-t/RC) = 1 - \exp(-t/35), \quad t \text{ in ms}$$

Thus at $t = 0$, $v_C = 0$ V and at $t = 20$ ms, $v_C = 0.4353$ V, while $v_R = 1 - v_C = 1$ V at $t = 0$ ms and 0.5647 V at $t = 20$ ms. At this point the voltage applied is zero and the capacitor discharges according to the equation $v_C = 0.4353\exp(-t/35)$, moving the time origin along 20 ms for convenience. Thus after 20 ms it reaches 0.2458 V. Meanwhile, since now $v_R + v_C = 0$, $v_R$ has jumped down to $-0.4353$ V at 20 ms and then risen to $-0.2458$ V. At $t = 40$ ms, the voltage applied is 1 V once more and the capacitor charges up according to the equation

$$v_C = 0.2458 + 0.7542(1 - \exp[-t/35]) = 1 - 0.7542\exp(-t/35) \text{ V}$$

$v_R = 1 - v_C = 0.7542\exp(-t/35)$ V (starting the clock again). Thus at $t = 60$ ms, $v_o = 0.4259$ V and when the applied voltage is zero it jumps down to $-(1 - 0.4259) = -0.5741$ V, and the cycle is repeated (see figure A4.7).

The settling time will be approximately 5 time constants, 175 ms, to within 1% of the steady-state, or 4.6 × 35 = 161 ms, to be more exact: the transition at 180 ms will be the first to be within 1%.

**8** In figure A4.8 we see that the transformed voltage at A is $\tilde{v}_{in}$ and the nodal equation is

$$\frac{\tilde{v}_{in}}{R_2} + \frac{\tilde{v}_{in} - \tilde{v}_o}{R_1 \| 1/Cs} = 0 \Rightarrow \frac{\tilde{v}_{in}}{R_2} + \frac{(\tilde{v}_{in} - \tilde{v}_o)(R_1Cs + 1)}{R_1} = 0$$

$$\Rightarrow \tilde{v}_o(1 + R_1Cs) = \tilde{v}_{in}(1 + R_1/R_2 + R_1Cs)$$

$$\tilde{v}_o = \frac{\tilde{v}_{in}(s + 1/R_1C + 1/R_2C)}{s + 1/R_1C}$$

Substituting $R_1 = 80\ \mathrm{k\Omega}$, $R_2 = 2\ \mathrm{k\Omega}$, $C = 4\ \mu\mathrm{F}$ and $\tilde{v}_{in} = 0.2/s$ gives

$$\tilde{v}_o = \frac{0.2}{s}\frac{(s + 10^3/320 + 10^3/8)}{(s + 10^3/320)} = \frac{0.2(s + 25.63)}{s(s + 3.125)} \equiv \frac{A}{s} + \frac{B}{s + 3.125}$$

Thus $A(s + 3.125) + Bs \equiv 0.2s + 25.63$, so that $A = 8.2$, $A + B = 0.2$ and $B = -8$. Detransforming leads to

$$v_o(t) = [8.2 - 8\exp(-3.125t)]u(t)\ \mathrm{V}$$

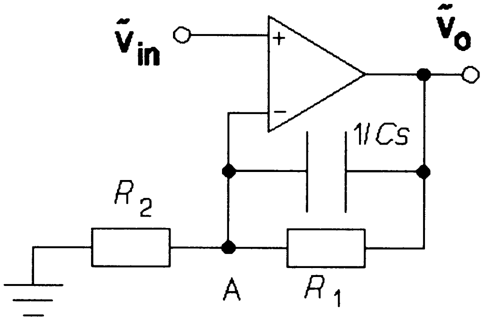


*Figure A4.8*

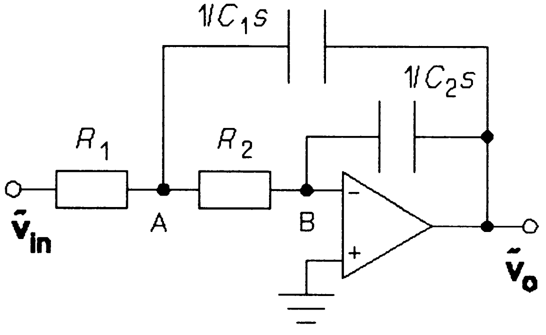


*Figure A4.9*

When $v_{in}(t) = 10t$ mV, we can replace $\tilde{v}_{in}$ by $10/s^2$ and working in mV find

$$\tilde{v}_o = \frac{10(s + 128.15)}{s^2(s + 3.125)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{D}{s + 3.125}$$

Whence

$$As(s + 3.125) + B(s + 3.125) + Ds^2 \equiv 10s + 1281.25$$

$$\Rightarrow \quad A + D = 0 \quad \text{(coeff. of } s^2\text{)}$$

$$3.125A + B = 10 \quad \text{(coeff. of } s\text{)}$$

$$B = 1281.25/3.125 = 410 \quad \text{(constant term)}$$

$$\text{and } A = -128, \ D = 128$$

The detransformed voltage is

$$v_o(t) = [-128 + 410t + 128\exp(-3.125t)]u(t) \text{ mV}$$

**9** At node B in figure A4.9, $\tilde{v}_B = 0$, while the nodal equation at A is

$$\frac{\tilde{v}_A - \tilde{v}_{in}}{R_1} + \frac{\tilde{v}_A - \tilde{v}_B}{R_2} + (\tilde{v}_A - \tilde{v}_o)C_1 s = 0$$

$$\Rightarrow \quad \tilde{v}_A(1 + R_1/R_2 + R_1C_1s) - \tilde{v}_o R_1 C_1 s = \tilde{v}_{in}$$

The nodal equation at B is

$$-\tilde{v}_A/R_2 - \tilde{v}_o C_2 s = 0 \Rightarrow \tilde{v}_A = -\tilde{v}_o R_2 C_2 s$$

Substituting this into the equation at node A gives

$$-\tilde{v}_o(R_2C_2s + R_1C_2s + R_1R_2C_1C_2s^2) - \tilde{v}_o R_1C_1s = \tilde{v}_{in}$$

$$\Rightarrow \quad -\tilde{v}_o s(R_1C_1 + R_1C_2 + R_2C_2 + R_1R_2C_1C_2s) = \tilde{v}_{in}$$

With $R_1C_1 = R_2C_2 = 1$ and $R_1C_2 = 0.01$, while $\tilde{v}_{in} = 10$, this equation becomes

$$-\tilde{v}_o s(s + 2.01) = 10 \tag{A4.9}$$

$$\Rightarrow \quad \tilde{v}_o = \frac{-10}{s(s + 2.01)} \equiv \frac{A}{s} + \frac{B}{s + 2.01}$$

$$\Rightarrow \quad A = -10/2.01 = -4.975 \quad B = -A = 4.975$$

The detransformed voltage is $v_o(t) = -4.975u(t) + 4.975\exp(-2.01t)u(t)$ V.

**10** When $v_{in}(t) = 2.5u(t)$ V, $\tilde{v}_{in} = 2.5/s$ and equation A4.9 becomes

$$-\tilde{v}_o s(s + 2.01) = 2.5/s$$

$$\Rightarrow \quad \tilde{v}_o = \frac{-2.5}{s^2(s + 2.01)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{D}{s + 2.01}$$

$$\Rightarrow \quad As(s + 2.01) + B(s + 2.01) + Ds^2 \equiv -2.5$$

From which we find $A + D = 0$ (coeff. of $s^2$), $2.01A + B = 0$ (coeff. of $s$) and $2.01B = -2.5$ (constant term), so $B = -1.244$, $A = -B/2.01 = 0.619$ and $D = -0.619$. The detransformed output voltage is

$$v_o(t) = [0.619 - 1.244t - 0.619\exp(-2.01t)]u(t) \text{ V}$$

If the supply voltages are ±15 V, then the time to saturate will be when

$$v_o(t) \approx -15 \approx 0.619 - 1.244t$$

$$\Rightarrow \quad t \approx -15.619/-1.244 = 12.6 \text{ s}$$

The exponential part is negligible.

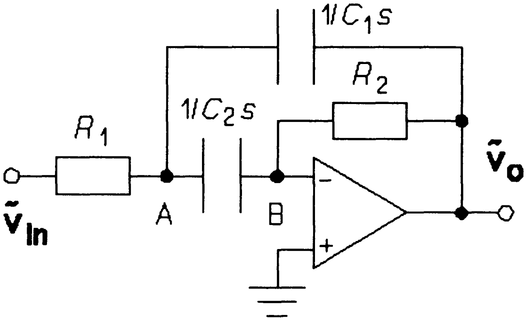


*Figure A4.11*

**11** At node B in figure A4.11, $\tilde{v}_B = 0$ and then by KCL

$$-\tilde{v}_A C_2 s - \tilde{v}_o/R_2 = 0 \Rightarrow \tilde{v}_A = -\tilde{v}_o/R_2C_2s$$

The nodal equation at A is

$$\frac{\tilde{v}_A - \tilde{v}_{in}}{R_1} + (\tilde{v}_A - \tilde{v}_o)C_1s + \tilde{v}_A C_2 s = 0$$

$$\tilde{v}_A(1 + R_1C_1s + R_1C_2s) - \tilde{v}_o R_1 C_1 s = \tilde{v}_{in}$$

Substituting for $\tilde{v}_A$ yields

$$-\tilde{v}_o\left[\frac{1}{R_2C_2s} + \frac{R_1(C_1 + C_2)}{R_2C_2}\right] - \tilde{v}_o R_1 C_1 s = \tilde{v}_{in} \qquad \text{(A4.11)}$$

Then with $R_1 = 250\ \Omega$, $R_2 = 12.5\ \text{k}\Omega$, $C_1 = C_2 = 100$ nF and $\tilde{v}_{in} = 1.4/s$, this becomes

$$-\tilde{v}_o(800/s + 0.04 + 2.5 \times 10^{-5}s) = 1.4/s$$

$$\Rightarrow \quad \tilde{v}_o = \frac{-1.4}{2.5 \times 10^{-5}s^2 + 0.04s + 800} = \frac{-56000}{s^2 + 1600s + 3.2 \times 10^7}$$

$$= \frac{-56000}{(s + 800)^2 + 3.2 \times 10^7 - 800^2} = \frac{-56000}{(s + 800)^2 + 5600^2}$$

which can be detransformed to $v_o(t) = -10\exp(-800t)\sin(5600t)u(t)$ V.

The circuit's Q-factor is $\omega_0/\beta = \omega_0/2a$, where $a = 800$ and $\omega_0 = \surd(32 \times 10^6) = 5657$ rad/s, making $Q = 5657/1600 = 3.54$, not 3.5 as would be obtained using $\omega_n = 5600$ rad/s, though the difference is insignificant.

**12** This is a matter of putting $R_1 = 1$ kΩ, $R_2 = 6.76$ kΩ, $C_1 = 100$ nF, $C_2 = 2.5$ μF and $\tilde{\mathbf{v}}_{in} = 1.4/s$ into equation A4.11 to obtain

$$-\tilde{\mathbf{v}}_o\left[\frac{1}{6760 \times 2.5 \times 10^{-6}s} + \frac{10^3(10^{-7} + 2.5 \times 10^{-6})}{6760 \times 2.5 \times 10^{-6}}\right] - \tilde{\mathbf{v}}_o \times 10^3 \times 10^{-7}s = \frac{1.4}{s}$$

$$\Rightarrow \quad \tilde{\mathbf{v}}_o = \frac{-1.4}{59.2 + 0.1538s + 10^{-4}s^2} = \frac{-14000}{s^2 + 1538s + 592000} = \frac{-14000}{(s + 769)^2}$$

which detransforms to $v_o(t) = -14000t\exp(-769t)u(t)$ V. This has maximum magnitude when

$$\frac{\mathrm{d}v_o(t)}{\mathrm{d}t} = -14000\exp(-769t) - 769 \times -14000t\exp(-769t) = 0$$

$$\Rightarrow \quad t = 1/769 = 1.3 \text{ ms}$$

Then substituting this value of $t$ in the expression for $v_o(t)$ gives

$$|v_o(\max)| = 14 \times 1.3 \times \exp(-769 \times 1.3 \times 10^{-3}) = 18.2\exp(-1) = 6.7 \text{ V}$$

**13** The impedance of the transformed series RLC circuit is

$$\tilde{\mathbf{Z}} = Ls + R + 1/Cs$$

And if the transformed input voltage is $V/s$, the current flowing is

$$\tilde{\mathbf{i}} = \frac{V/s}{\tilde{\mathbf{Z}}} = \frac{V}{Ls^2 + Rs + 1/C} = \frac{V/L}{s^2 + Rs/L + 1/LC}$$

Substituting $R = 0.01$ Ω, $L = 0.3$ μH, $C = 13$ μF and $V = 25$ kV gives

$$\tilde{\mathbf{i}} = \frac{8.33 \times 10^{10}}{s^2 + 33333s + 2.56 \times 10^{11}} = \frac{164600 \times 5.061 \times 10^4}{(s + 16667)^2 + (5.061 \times 10^4)^2}$$

which detransforms to $i(t) = 165\exp(-16667t)\sin(5.061 \times 10^5t)u(t)$ kA. A graph of this is shown in figure A4.13. The current is at a maximum when

$$i(t) = A\exp(-at)\sin\omega t$$

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} = A\omega\exp(-at)\cos\omega t - aA\exp(-at)\sin\omega t = 0$$

$$\Rightarrow \quad \omega\cos\omega t = a\sin\omega t \quad \Rightarrow \quad \tan\omega t = \omega / a$$

$$\Rightarrow \quad t = \omega^{-1}\tan^{-1}(\omega / a) = 1.98 \times 10^{-6}\tan^{-1}(5.061 \times 10^{5} / 16667)$$

$$= 1.98 \times 1.54 = 3.05\ \mu\text{s}$$

The maximum current is $i_{\max} = 165\exp(-0.051)\sin 1.54 = 157$ kA and so the maximum power is $157^2 \times 0.01$ MW = 246 MW.

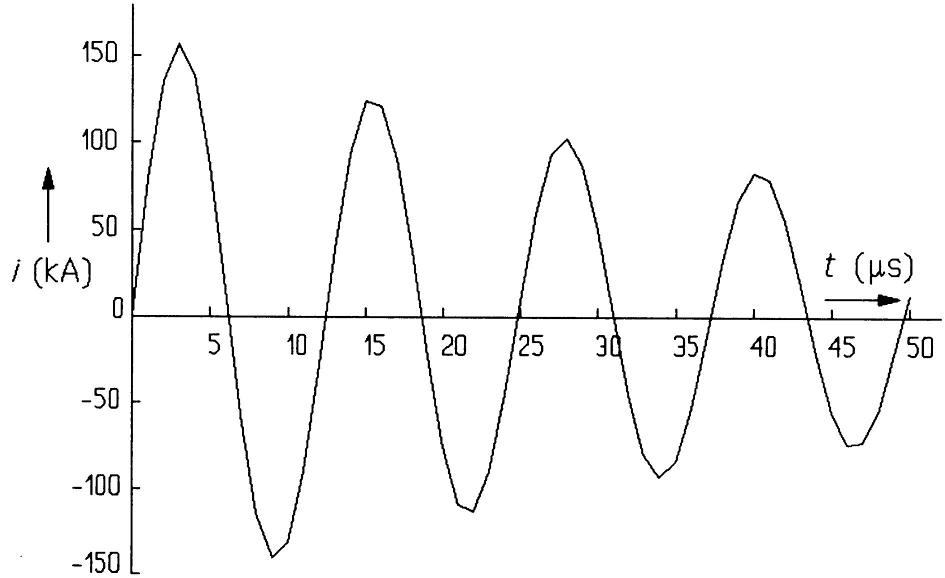


*Figure A4.13*

**14** The critical damping resistance is found from

$$R^2/4L^2 = 1/LC \quad \Rightarrow \quad R = 2\sqrt{L/C} = 2\sqrt{0.023} = 0.304\ \Omega$$

The critically-damped current is

$$\bar{\mathrm{i}} = \frac{V/L}{(s + 1/\sqrt{LC})^2} = \frac{8.33 \times 10^{10}}{(s + 5.064 \times 10^5)^2}$$

which detransforms to $i(t) = 83.3t\exp(-5.064 \times 10^5 t)$ GA, a graph of which is shown in figure A4.14. The current is a maximum when $t = (5.064 \times 10^5)^{-1} = 1.975\ \mu$s. At this time $i_{\max} = 60.5$ kA and the power is $60.5^2 \times 0.304 = 1\,110$ MW or 1.11 GW.

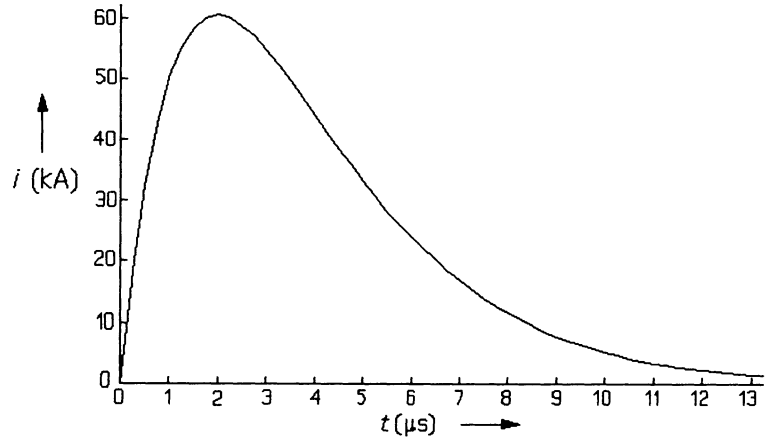


*Figure A4.14*

**15** The integration for $\mathscr{L}(\sin\omega t)$ goes

$$\int_0^\infty \sin\omega t\exp(-st)\mathrm{d}t = [-s^{-1}\exp(-st)\sin\omega t]_0^\infty - \int_0^\infty -s^{-1}\exp(-st)\omega\cos\omega t\,\mathrm{d}t$$

$$= \frac{\omega}{s}\int_0^\infty \cos\omega t\exp(-st)\mathrm{d}t = \frac{\omega}{s}[-s^{-1}\exp(-st)\cos\omega t]_0^\infty - \frac{\omega}{s}\int_0^\infty -s^{-1}\exp(-st)\omega(-\sin\omega t)\mathrm{d}t$$

$$\therefore \quad \mathscr{L}(\sin\omega t) = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\mathscr{L}(\sin\omega t)$$

$$\Rightarrow \quad (1 + \omega^2/s^2)\mathscr{L}(\sin\omega t) = \omega/s^2$$

$$\Rightarrow \quad \mathscr{L}(\sin\omega t) = \frac{\omega}{s^2 + \omega^2}$$

And we see from line two of this that $\mathscr{L}(\sin\omega t) = (\omega/s)\mathscr{L}(\cos\omega t)$, which means that

$$\mathscr{L}(\cos\omega t) = (s/\omega)\mathscr{L}(\sin\omega t) = \frac{s}{s^2 + \omega^2}$$

For $\mathscr{L}[f^n(t)]$ we have

$$\int_0^\infty \exp(-st)f^n(t)\mathrm{d}t = [\exp(-st)f^{n-1}]_0^\infty - \int_0^\infty -s\exp(-st)f^{n-1}\mathrm{d}t$$

$$= -f^{n-1}(0) + s\int_0^\infty \exp(-st)f^{n-1}dt$$

Then $$s\int_0^\infty \exp(-st)f^{n-1}dt = -s^2 f^{n-2}(0) + s^2\int_0^\infty \exp(-st)f^{n-2}\mathrm{d}t$$

Thus the transform is as given in table 4.1.

# Chapter 5

1 The circuit in figure P5.1 has a voltage response of

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{R_2 \parallel 1/j\omega C}{R_1 + (R_2 \parallel 1/j\omega C)} = \frac{R_2}{R_1(1 + j\omega CR_2) + R_2} = \frac{R_2/(R_1 + R_2)}{1 + j\omega R'}$$

where $R' \equiv R_1 \parallel R_2$. Then we can substitute $R_1 = 1$ kΩ, $R_2 = 9$ kΩ and $C = 111$ nF to obtain

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{0.9}{1 + j\omega \times 111 \times 10^{-9} \times 900} = \frac{0.9}{1 + j\omega/10^4}$$

Thus when $\omega << 10^4$ rad/s the response is constant at $20\log_{10}0.9 = -0.9$ dB.

When $\omega >> 10^4$ rad/s the response drops $-20$ dB per decade.

The phase response is $-\tan^{-1}(\omega/10^4)$, that is 0° when $\omega << 10^4$ rad/s and $-90°$ when $\omega >> 10^4$ rad/s. $\phi = -45°$ at $10^4$ rad/s, $-6°$ at $10^3$ rad/s and $-84°$ at $10^5$ rad/s. Figure A5.1 shows the Bode diagram.

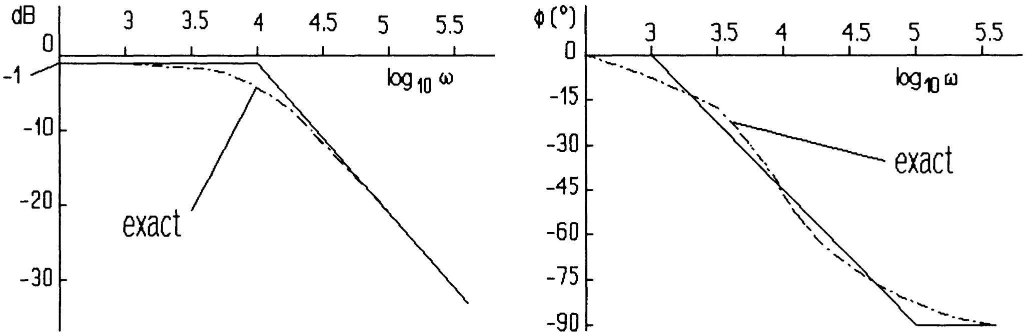


*Figure A5.1*

2 The response of the circuit of figure P5.2 is

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{j\omega RL}{R(R + j\omega RL) + j\omega RL} = \frac{j\omega/2}{R/2L + j\omega}$$

Substituting $R = 10$ kΩ and $L = 3.18$ mH gives

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{j\omega/2}{3.14 \times 10^6 + j\omega} = \frac{jf/2}{5 \times 10^5 + jf}$$

When $f << 500$ kHz the response increases +20 dB/decade. When $f >> 500$ kHz the response is constant at $20\log_{10}0.5 = -6$ dB.

The phase response is $\phi = 90° - \tan^{-1}(f/5 \times 10^5)$, which is +90° when $f << 500$ kHz and 0° when $f >> 500$ kHz. Also $\phi = 45°$ when $f = 500$ kHz, 84° when $f = 50$ kHz and 6° when $f = 5$ MHz.

Figure A5.2 shows the Bode diagram.

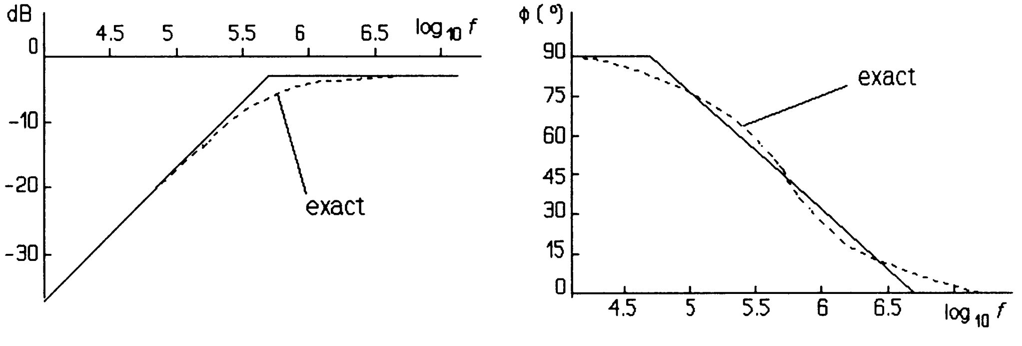


*Figure A5.2*

3 The circuit of figure P5.3 will approximate to $R_2$ across the output and $C_1$ at the input, so the low-frequency response will be approximately

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{R_2}{R_2 + 1/j\omega C_1} = \frac{j\omega C_1 R_2}{1 + j\omega C_1 R_2} \approx j\omega C_1 R_2$$

when $\omega << 1/C_1R_2$.

At high frequencies the circuit approximates to $C_2$ across the output and $R_1$ at the input, giving a high-frequency response of

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1/j\omega C_2}{R_1 + 1/j\omega C_2} = \frac{1}{1 + j\omega C_2 R_1} \approx \frac{1}{j\omega C_2 R_1}$$

when $\omega >> 1/C_2R_1$.

Thus the low-frequency response increases at +20 dB/decade for $\omega < 1/C_1R_2$, while the high-frequency response falls at −20 dB/decade when $\omega > 1/C_2R_1$. The phase response is +90° at low frequencies and −90° at high frequencies. The actual response is

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{R_2 \parallel 1/j\omega C_2}{R_1 + 1/j\omega C_1 + (R_2 \parallel 1/j\omega C_2)} = \frac{R_2/(1 + j\omega C_2 R_2)}{R_1 + 1/j\omega C_1 + R_2/(R_2 + j\omega C_2 R_2)}$$

$$= \frac{j\omega C_1R_2}{(1+j\omega C_1R_1)(1+j\omega C_2R_2)+j\omega C_1R_2} = \frac{j\omega C_1R_2}{(1-\omega^2C_1C_2R_1R_2)+j\omega(R_1C_1+R_2C_2+R_1C_2)}$$

Substituting $R_1 = 1$ kΩ, $R_2 = 10$ kΩ, $C_1 = 100$ nF and $C_2 = 1$ μF leads to

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{j\omega 10^{-3}}{(1-\omega^2 10^{-6})+j\omega 0.0111}$$

Note that $1/R_1C_2 = 1$ krad/s $= 1/R_2C_1$ and at this frequency the denominator is just $j11.1$, giving a response of $-20\log_{10}11.1 = -20.9$ dB and $\phi = 0°$. We can factorise the denominator to find the corner frequencies:

$$(1-\omega^2 10^{-6}) + j\omega 0.0111 = (1+j\omega T_1)(1+j\omega T_2)$$

$$\Rightarrow \quad T_1T_2 = 10^{-6} \text{ and } T_1+T_2 = 0.0111$$

$$\Rightarrow \quad T_1(0.0111-T_1) - 10^{-6} = 0$$

$$\Rightarrow \quad T_1^2 - 0.0111T_1 + 10^{-6} = 0$$

$$\Rightarrow \quad T_1 = 0.011009 \text{ and } T_2 = 9.0833\times 10^{-5}$$

The break points are $\omega_1 = 1/T_1 = 90.8$ rad/s and $\omega_2 = 1/T_2 = 11.009$ krad/s. The response is

$$H(j\omega) = \frac{j\omega/1000}{(1+j\omega/90.8)(1+j\omega/11009)}$$

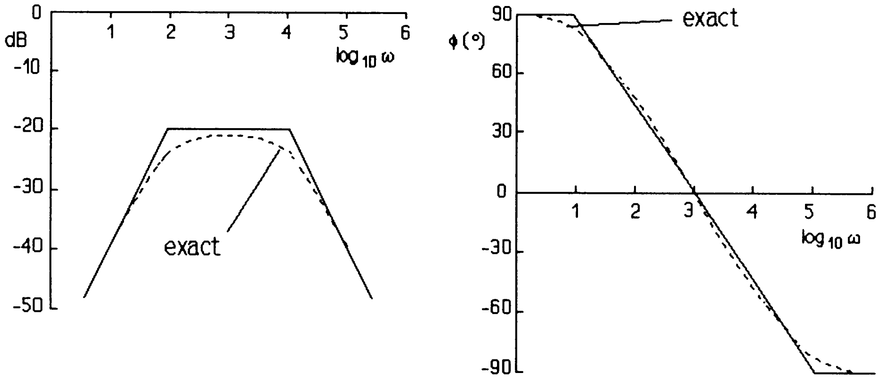


Figure A5.3

Below 90.8 rad/s the amplitude response rises at 20 dB/decade. At $\omega = \omega_1$ the response will be 3 dB down from the maximum at 1 krad/s. Between 90.8 rad/s and 11 krad/s the response will be flat. It will then fall at 20 dB/decade above 11 krad/s.

At 9 rad/s and below the phase response will be 90°. At 90.8 rad/s the phase response will be $90° - 45° = 45°$. At 1 krad/s the phase response is $90° - 90° - 0° = 0°$. At 11 krad/s the phase response is $90° - 90° - 45° = -45°$. Finally, at 110 krad/s and above the phase response is $90° - 90° - 90° = -90°$. Figure A5.3 shows the Bode plot.

4 The feedback loop has an impedance of $R_2 \parallel 1/j\omega C = R_2/(1 + j\omega C_2)$ and so

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{-R_2/R_1}{1 + j\omega CR_2}$$

Thus the corner frequency is $\omega_c = 1/CR_2 = (20 \times 10^{-9} \times 500 \times 10^3)^{-1} = 100$ rad/s and the gain is $R_2/R_1 = 100$ or $20\log_{10}100 = 40$ dB. The amplitude response is flat up to 100 rad/s and declines at 20 dB/decade thereafter as shown in figure A5.4, which is the Bode plot. The exact response is 3 dB down at the corner frequency as usual.

The phase response is $180° - \tan^{-1}\omega CR_2$ or 180° at $\omega < 0.1\omega_c$, 135° at $\omega = \omega_c$ and 90° for $\omega > 10\omega_c$ as shown in figure A5.4.

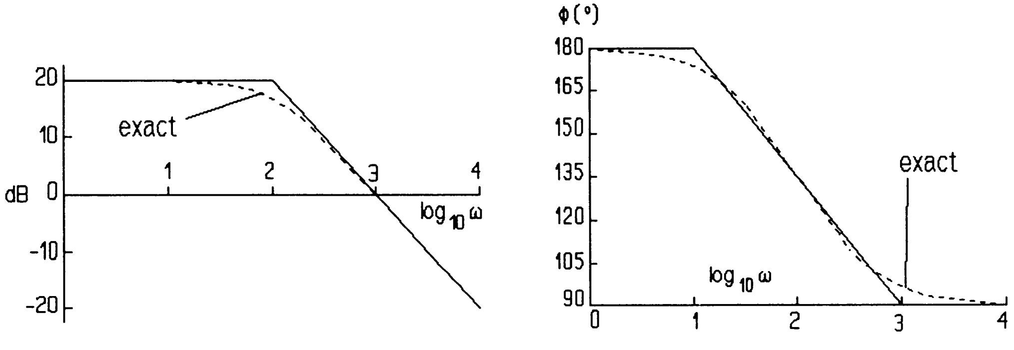


*Figure A5.4*

5 At low frequencies $1/j\omega C >> R$, so that at node B in figure A5.5a, $\mathbf{V}_B = \mathbf{V}_o$ and at node A, $\mathbf{V}_A = \mathbf{V}_B\ \mathbf{V}_o$ and no current flows in the feedback loop, nor from the input: $\mathbf{V}_o = \mathbf{V}_{in}$, the amplitude response is 0 dB at low frequencies and the phase response is 0°. At high frequencies $R >> 1/j\omega C$, so that $\mathbf{V}_B = \mathbf{V}_A = 0$ - the capacitor is a short-circuit to ground. The circuit is then a feedback amplifier with unity gain and 180° phase shift: $\mathbf{V}_o = -\mathbf{V}_{in}$. We note then that $\mathbf{V}_A = \mathbf{V}_B$ and that

$$\mathbf{V}_B = \frac{\mathbf{V}_o(1/j\omega C)}{R + 1/j\omega C} = \frac{\mathbf{V}_o}{1 + j\omega CR}$$

At node A $$\frac{\mathbf{V}_A - \mathbf{V}_{in}}{R} + \frac{\mathbf{V}_A - \mathbf{V}_o}{R} = 0$$

Or, $$2\mathbf{V}_A - \mathbf{V}_o = \mathbf{V}_{in}$$

Substituting for $\mathbf{V}_A$ yields

$$\frac{2\mathbf{V}_o}{1 + j\omega CR} - \mathbf{V}_o = \mathbf{V}_{in} \Rightarrow \frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1 + j\omega CR}{1 - j\omega CR}$$

The amplitude response is

$$A_{dB} = 20\log_{10}\left[\frac{1 + (\omega CR)^2}{1 + (\omega CR)^2}\right] = 20\log_{10}1 = 0$$

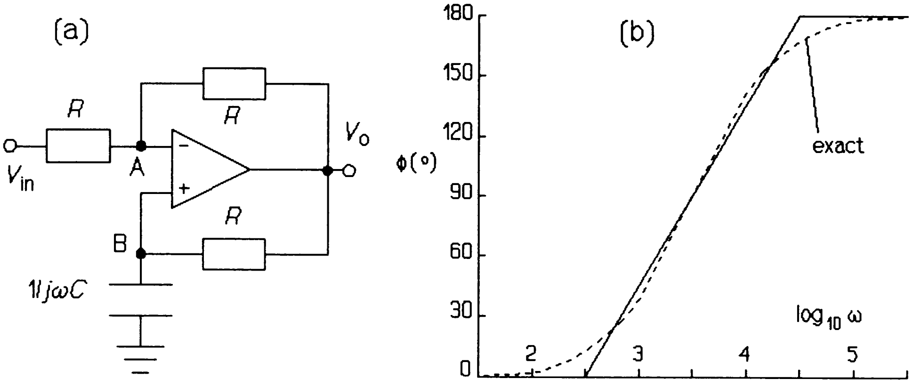


*Figure A5.5*

The phase response is $\tan^{-1}(\omega CR) - \tan^{-1}(-\omega CR) = 2\tan^{-1}(\omega CR)$. The corner frequency is $1/CR = 3.333$ krad/s and the exact phase response is $2 \times \tan^{-1}0.1 = 11.4°$ at 333 rad/s and $180 - 11.4° = 168.6°$ at 33.33 krad/s. Figure A5.5b shows the phase response.

6 In figure A5.6a we can see that at low frequencies $1/j\omega C_2 >> R_2$, making $R_2$ the only important component in the feedback path. Similarly $1/j\omega C_1 >> R_1$ and $R_1$ can be neglected, which means the circuit is just a simple amplifier with $R_2$ in the feedback loop and $1/j\omega C_1$ at the input and

$$\mathbf{V}_o/\mathbf{V}_{in} = -j\omega C_1R_2$$

At high frequencies the inequalities are reversed and so the feedback loop has only $1/j\omega C_2$ in it and the input impedance is $R_1$ and the response will be

$$\mathbf{V}_o/\mathbf{V}_{in} = -1/j\omega C_2R_1$$

We note that at node B, $\mathbf{V}_B = 0$ and then

$$\mathbf{V}_A/R_1 = -\mathbf{V}_o j\omega C_2 \quad \Rightarrow \quad \mathbf{V}_A = -\mathbf{V}_o j\omega C_2 R_1$$

At node A $$(\mathbf{V}_A - \mathbf{V}_{in}) j\omega C_1 + \frac{\mathbf{V}_A - \mathbf{V}_o}{R_2} + \frac{\mathbf{V}_A}{R_1} = 0$$

$$\Rightarrow \quad \mathbf{V}_A(1/R_1 + 1/R_2 + j\omega C_1) - \mathbf{V}_o/R_2 = \mathbf{V}_{in} j\omega C_1$$

Substituting for $\mathbf{V}_A$ gives

$$\mathbf{V}_o j\omega C_2 R_1 (1/R_1 + 1/R_2 + j\omega R_2 C_1) - \mathbf{V}_o/R_2 = -\mathbf{V}_{in} j\omega C_1$$

$$\Rightarrow \quad \mathbf{V}_o(j\omega C_2 R_2 + j\omega C_2 R_1 - \omega^2 C_1 C_2 R_1 R_2 + 1) = -\mathbf{V}_{in} j\omega C_1 R_2$$

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{-j\omega C_1 R_2}{1 - \omega^2 C_1 C_2 R_1 R_2 + j\omega C_2(R_1 + R_2)}$$

At low frequencies the denominator → 1 and the response is $-j\omega C_1 R_2$ and at high frequencies the denominator → $-\omega^2 C_1 C_2 R_1 R_2$, so the response is then

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{-j\omega C_1 R_2}{-\omega^2 C_1 C_2 R_1 R_2} = \frac{j}{\omega C_2 R_1} = \frac{-1}{j\omega C_2 R_1}$$

Putting $R_1 = R_2 = 33\ \text{k}\Omega$, $C_1 = 30$ nF and $C_2 = 1$ nF into the general response equation gives

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{-j\omega \times 9.9 \times 10^{-3}}{1 - 3.267 \times 10^{-8}\omega^2 + j\omega \times 6.6 \times 10^{-5}}$$

The denominator will not factorise into terms like $1 + j\omega T$: the circuit is underdamped and a modification of equation 5.20 applies

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{j\Omega\alpha}{1 - \Omega^2 + j\Omega/Q}$$

where $\Omega = \omega/\omega_0$ and $Q = \omega_0/\beta$, while $\omega_0 = 1/\surd(3.267 \times 10^{-8}) = 5.533$ krad/s, $\alpha = 9.9 \times 10^{-3}\omega_0 = 54.8$ and

$$\beta = \frac{6.6 \times 10^{-5}}{3.267 \times 10^{-8}} = 2.02 \text{ krad/s}$$

$$\Rightarrow \quad Q = \omega_0/\beta = 2.74 \quad \text{and} \quad \alpha = 54.8$$

Though not 'large', this value of $Q$ is enough to increase the response at $\omega_0$ by $20\log_{10} Q = 8.75$ dB above the linear approximation which is +20 dB/decade below $\omega_0$ and −20 dB/decade above $\omega_0$. The −3dB points are roughly at $\omega_0(1 \pm 1/2Q) \approx 4.52$ and 6.54 krad/s while at $\omega = 0.5\omega_0$, $A_{dB} = 31$ dB (2 dB above the approximate value) and at $1.5\omega_0$, $A_{dB} = 35.5$ dB (4.3 dB above the linear approximation). Figure A5.6b shows the amplitude response.

The phase response is −90° at low frequencies and −90° − 180° = −270° at high frequencies, because the phase of the response's denominator is +180° when $\omega >> \omega_0$. To see this consider when $\omega = 10\omega_0$, the denominator is $-99 + j3.65$, in the second quadrat of the Argand diagram, a phase angle of +178°.

At $\omega = \omega_0$ the phase angle is −90° − 90° = −180°. With $Q = 2.74$, the phase response is closer to a step function than a ramp function: approximately −90° up to 5.533 krad/s and −270° after. The approximate half-power points are at $\omega_0(1 \pm 1/2Q) = 4.523$ and 6.543 krad/s, where the phases are −135° and −225° respectively. We can make additional corrections at $\omega = 10^{3.5} = 3.162$ krad/s ($\phi = -107°$) and $\omega = 10^4$ rad/s ($\phi = -254°$) to give the 'exact' phase plot of figure A5.6c.

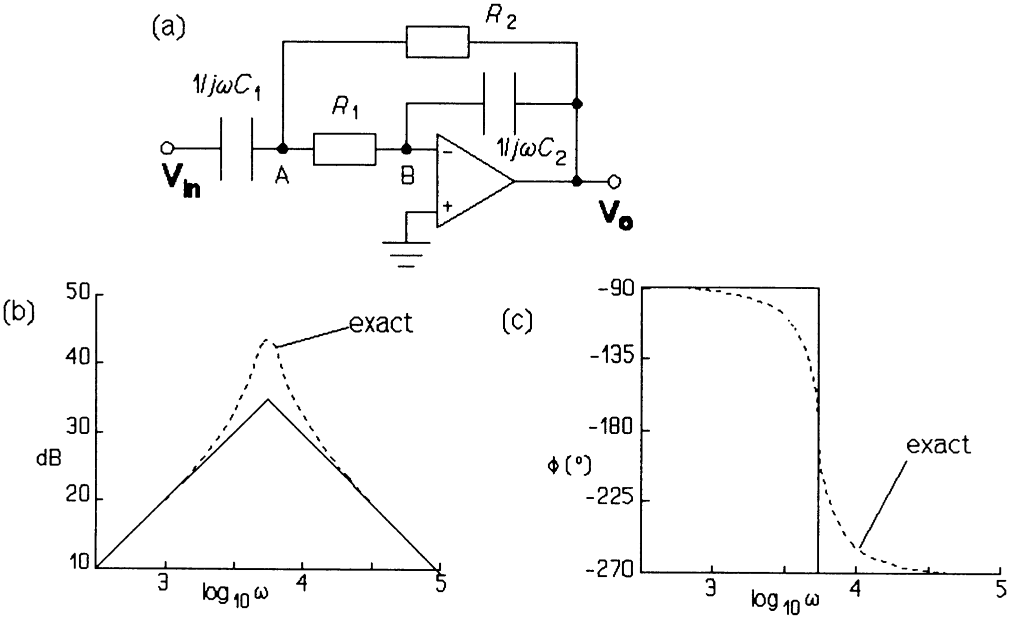


*Figure A5.6*

7 At low frequencies the circuit in figure A5.7a is just two capacitors as $1/j\omega C >> R$, and then $V_o = V_{in}$. At high frequencies the left-hand capacitor is a short-circuit to ground and the voltage across it is $V_{in}/j\omega CR$ and $V_o$ is $1/j\omega CR$ times this or $-1/(\omega CR)^2$.

The nodal equation at B in figure A5.7a is

$$\frac{\mathbf{V}_o - \mathbf{V}_1}{R} + \mathbf{V}_o j\omega C = 0 \Rightarrow \mathbf{V}_1 = \mathbf{V}_o(1 + j\omega CR)$$

And at A: $$\frac{\mathbf{V}_1 - \mathbf{V}_{in}}{R} + \frac{\mathbf{V}_1 - \mathbf{V}_o}{R} + \mathbf{V}_1 j\omega C = 0$$

$$\Rightarrow \mathbf{V}_1(2 + j\omega CR) - \mathbf{V}_o = \mathbf{V}_{in}$$

Substituting for $\mathbf{V}_1$ gives

$$\mathbf{V}_o(1 + j\omega CR)(2 + j\omega CR) - \mathbf{V}_o = \mathbf{V}_{in}$$

$$\Rightarrow \mathbf{V}_o(2 + j3\omega CR - \omega^2C^2R^2 - 1) = \mathbf{V}_{in}$$

$$\Rightarrow \frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1}{1 - \omega^2C^2R^2 + j3\omega CR}$$

Thus at low frequencies the response is 1 and at high frequencies it is $-1/(\omega CR)^2$ as expected.

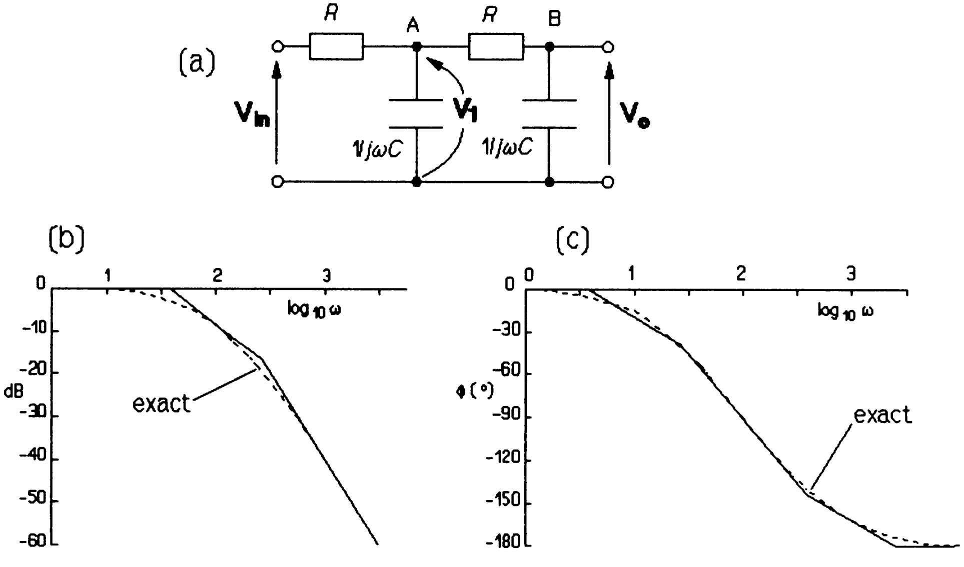


*Figure A5.7*

When $R$ = 22 kΩ, $C$ = 455 nF, $RC$ = 0.01 and the response is

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{1}{1 - 10^{-4}\omega^2 + j\omega 0.03} = \frac{1}{(1 + j\omega 0.0262)(1 + j\omega 3.82 \times 10^{-3})}$$

And so up to $\omega = 1/0.0262 = 38.2$ rad/s, the amplitude response is 0 dB, falling at 20 dB/decade thereafter until $\omega = 1/3.82 \times 10^{-3} = 262$ rad/s when it falls at 40 dB/decade. The exact amplitude plot differs by 3 dB at the most, at the corner frequencies. Figure A5.7b shows the amplitude part of the Bode diagram.

The phase response is zero up to 3.82 rad/s, then falls linearly at 45°/decade. The second term in the denominator comes into play at 26.2 rad/s and contributes $-45°$ at 262 rad/s and $-90°$ at 2.62 krad/s (see phase plot of figure A5.7c). Thus the phase is changing at $-90°$ per decade from 26.2 rad/s to 382 rad/s and at $-45°$/decade from 382 rad/s to 2.62 krad/s when it reaches $-180°$. The corrected plot is most different from the approximate plot at 3.82 rad/s and 2.62 krad/s, when the error is $\tan^{-1}0.1 \approx 6°$.

8 In the circuit of figure A5.8a we can see that at low frequencies $\mathbf{V}_o = \mathbf{V}_{in}$ since the capacitors' reactances are much greater than $R$. At high frequencies $\mathbf{V}_o = \mathbf{V}_A/j\omega CR$ and $\mathbf{V}_A = \mathbf{V}_{in}/j\omega CR$, so that $\mathbf{V}_o = \mathbf{V}_{in}/(j\omega CR)^2 = -\mathbf{V}_{in}/\omega^2C^2R^2$.

On the output side we see that

$$\mathbf{V}_o = \frac{1/j\omega C\mathbf{V}_A}{1/j\omega C + R} = \frac{\mathbf{V}_A}{1 + j\omega CR}$$

while on the input side we have

$$\mathbf{V}_A = \frac{\mathbf{V}_{in}}{1 + j\omega CR}$$

since $\mathbf{V}_+ = \mathbf{V}_- = \mathbf{V}_A$. Substituting for $\mathbf{V}_A$ leads to

$$\mathbf{V}_o/\mathbf{V}_{in} = 1/(1 + j\omega CR)^2$$

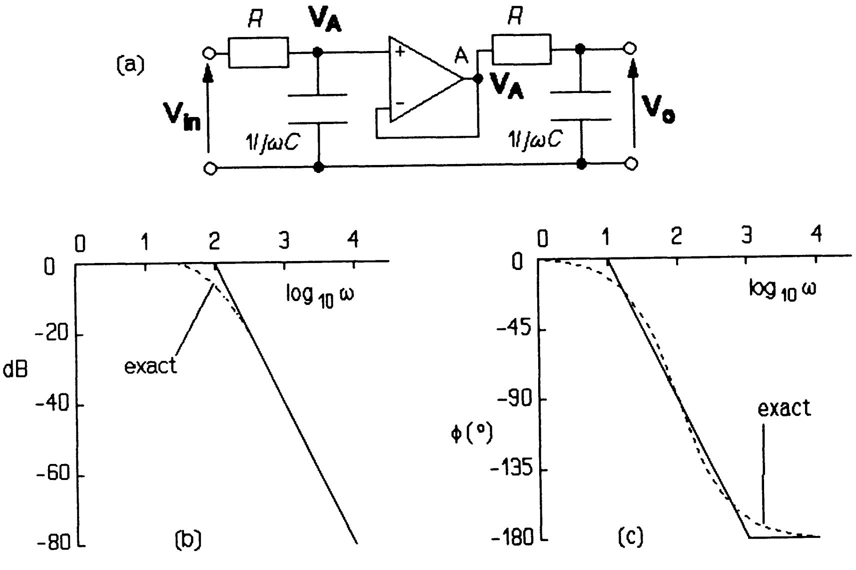


*Figure A5.8*

When $RC = 0.01$ the amplitude response is 0 dB up to 100 rad/s and falls at 40 dB/decade thereafter. The correction is $-6$ dB at 100 rad/s as figure A5.8b shows.

The phase response is 0° up to 10 rad/s and then $-90°$ per decade up to 1 krad/s, when it reaches $-180°$. The correction is $2\tan^{-1}0.1 \approx -11°$ at 10 rad/s and $+11°$ at 1 krad/s, as shown in figure A5.8c, which is the phase plot.

9 At low frequencies the circuit is $1/j\omega C$ in series with $j\omega L$, so that

$$\mathbf{V}_o/\mathbf{V}_{in} = \frac{j\omega L}{1/j\omega C} = -\omega^2 LC$$

At high frequencies the circuit is $1/j\omega C$ in series with $R$, but $1/j\omega C \rightarrow 0$, making $\mathbf{V}_o = \mathbf{V}_{in}$. The impedance of the inductance and resistance in parallel is

$$\mathbf{Z} = \frac{j\omega LR}{R + j\omega L}$$

And then $$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{\mathbf{Z}}{\mathbf{Z} + 1/j\omega C} = \frac{j\omega RL/(R + j\omega L)}{j\omega RL/(R + j\omega L) + 1/j\omega C}$$

$$= \frac{j\omega RL \times j\omega C}{j\omega RL \times j\omega C + R + j\omega L} = \frac{-\omega^2 LC}{1 - \omega^2 LC + j\omega L/R}$$

When $R = 10\ \text{k}\Omega$, $L = 3$ mH and $C = 3$ nF, we have

$$\mathbf{V}_o/\mathbf{V}_{in} = \frac{-9 \times 10^{-12}\omega^2}{1 - 9 \times 10^{-12}\omega^2 + j3 \times 10^{-7}\omega} = \frac{-\Omega^2}{1 - \Omega^2 + j\Omega/Q}$$

where $\Omega \equiv \omega/\omega_0$, $\omega_0 = 1/\sqrt{(3 \times 10^{-12})} = 333$ krad/s and $Q = (3 \times 10^{-7}\omega_0)^{-1} = 10$.

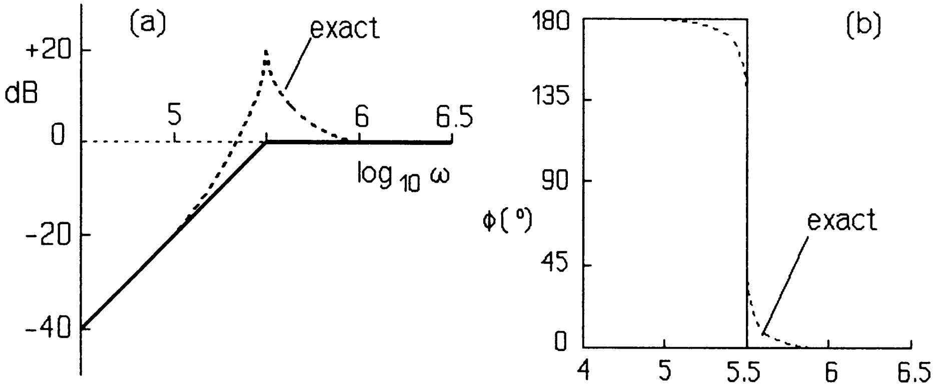


Figure A5.9

The amplitude response rises from low frequencies at 40 dB/decade up to the corner frequency, 333 krad/s, where it is 0 dB. It is flat thereafter, as shown by the approximate response line in figure A5.9b, but the $Q$ is 10 (large), so at 333 krad/s the exact response is $20\log_{10} 10 = 20$ dB. The 3dB-correction frequencies are at $0.54 \times 333 = 180$ krad/s and $1.31 \times 333 = 436$ krad/s as shown in the 'exact' plot (figure A5.9a).

At low frequencies the phase response is 180°, at $\omega_0$ it is $180° - 90° = 90°$, and at high frequencies it is $180° - 180° = 0°$. The phase change occurs almost in step fashion at 333 krad/s as $Q$ is large. The phase is 135° at $0.95\omega_0$ (the lower half-power point) and 45° at $1.05\omega_0$ (the upper half-power point). Figure A5.9b shows the phase response.

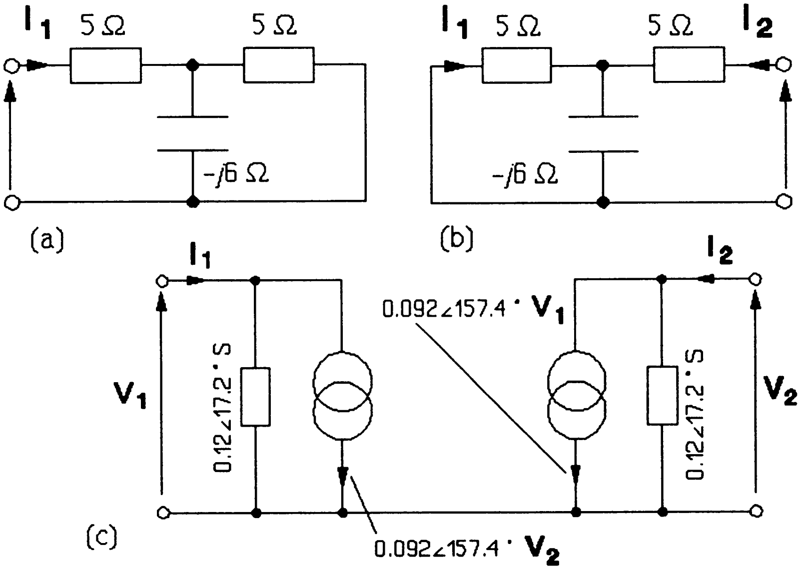


*Figure A5.10*

**10** The $y$-parameters are found from $\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2$ and $\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2$, so that from the first of these we have

$$\mathbf{y}_{11} = [\mathbf{I}_1/\mathbf{V}_1]_{\mathbf{V}_2 = 0}$$

that is as in figure A5.10a, where port 2 is short-circuited. $\mathbf{I}_1$ is then given by

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{5 + (5 \parallel -j6)} = \frac{\mathbf{V}_1}{5 - j30/(5 - j6)}$$

$$\Rightarrow \quad \frac{\mathbf{I}_1}{\mathbf{V}_1} = \mathbf{y}_{11} = \frac{5 - j6}{25 - j60} = \frac{7.81\angle -50.2°}{65\angle -67.4°} = 120\angle 17.2° \text{ mS}$$

Or one can say $\mathbf{y}_{11} = 0.2$ S in series with $(0.2 + j0.1667)$ S, which is

$$y_{11} = \frac{0.2 \times (0.2 + j0.1667)}{0.2 + (0.2 + j0.1667)} = \frac{0.052\angle 39.8^\circ}{0.433\angle 22.6^\circ} = 120\angle 17.2^\circ \text{ mS}$$

Next we find $y_{12}$ from

$$y_{12} = [I_1/V_2]_{V_1 = 0}$$

that is as in figure A5.10b, when port 1 is short-circuited. Now

$$I_1 = -I_2 \times \frac{-j6}{5 - j6} = I_2 \times \frac{6\angle 90^\circ}{7.81\angle -50.2^\circ} = I_2 \times 0.768\angle 140.2^\circ$$

while $I_2 = V_2 y_{22} = V_2 y_{11}$, as the network is symmetrical. Then

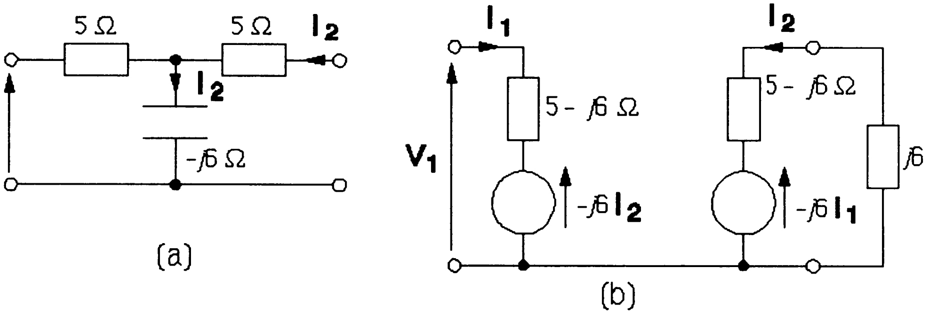
$$I_1 = I_2 \times 0.768\angle 140.2^\circ = V_2 \times 0.12\angle 17.2^\circ \times 0.768\angle 140.2^\circ = V_2 \times 0.092\angle 157.4^\circ$$

$$\Rightarrow \quad I_1/V_2 = y_{12} = 92\angle 157.4^\circ \text{ mS}$$

The network's symmetry also means that $y_{12} = y_{21}$ and the *y*-equivalent circuit is as figure A5.10c, from which we see that

$$V_2 = \frac{-0.092\angle 157.4^\circ \times V_1}{0.12\angle 17.2^\circ} = \frac{-0.092\angle 157.4^\circ \times 15\angle -60^\circ}{0.12\angle 17.2^\circ}$$

$$= -11.5\angle 80.2^\circ = 11.5\angle -99.8^\circ \text{ V}$$



(a) (b)

*Figure A5.11*

**11** The *z*-parameters are defined by $V_1 = z_{11}I_1 + z_{12}I_2$ and $V_2 = z_{21}I_1 + z_{22}I_2$, from which

$$z_{11} = z_{22} = \left[\frac{V_1}{I_1}\right]_{I_2 = 0}$$

The symmetry of the network means that $\mathbf{z}_{11} = \mathbf{z}_{22}$ and $\mathbf{z}_{12} = \mathbf{z}_{21}$, and we see that port 2 must be open circuit and immediately gives $\mathbf{z}_{11} = 5 - j6\ \Omega$.

$\mathbf{z}_{12}$ is found from

$$\mathbf{z}_{12} = \left[\frac{\mathbf{V}_1}{\mathbf{I}_2}\right]_{\mathbf{I}_1 = 0}$$

This situation is shown in figure A5.11a, from which it is seen that if $\mathbf{I}_1 = 0$, then $\mathbf{V}_1$ is the voltage across the capacitance, the current in which is $\mathbf{I}_2$. Thus $\mathbf{V}_1/\mathbf{I}_2 = \mathbf{z}_{12} = -j6\ \Omega$.

The $z$-parameter equivalent circuit, with $j6\ \Omega$ across port 2, is that of figure A5.11b, from which we deduce

$$\mathbf{I}_2 = \frac{-(-j6\mathbf{I}_1)}{5 - j6 + j6} = j1.2\mathbf{I}_1$$

And also

$$\mathbf{I}_1 = \frac{\mathbf{V}_1 - (-j6\mathbf{I}_2)}{5 - j6} = \frac{\mathbf{V}_1 + j6\mathbf{I}_2}{5 - j6} = \frac{120\angle 60° - 7.2\mathbf{I}_1}{5 - j6}$$

$$\Rightarrow \quad (5 - j6)\mathbf{I}_1 = 120\angle 60° - 7.2\mathbf{I}_1$$

$$\Rightarrow \quad \mathbf{I}_1 = \frac{120\angle 60°}{12.2 - j6} = \frac{120\angle 60°}{13.6\angle -26.2°} = 8.82\angle 86.2°\ \text{A}$$

And so $\mathbf{I}_2 = j1.2\mathbf{I}_1 = 10.6\angle 176.2°$ A.

**12** The reciprocity theorem enables us to write down the short-circuit current through AB directly

$$\mathbf{I}_{AB} = \mathbf{I}_{XY} \times \frac{\mathbf{V}_{XY}}{\mathbf{V}_{AB}} = 0.662\angle 70.7° \times \frac{1.34\angle -58°}{10\angle 0°} = 88.7\angle 12.7°\ \text{mA}$$

Then the current through the inductance is

$$\mathbf{I}_L = \frac{5}{5 + j4} \times \mathbf{I}_{AB} = \frac{5 \times 88.7\angle 12.7°}{6.4\angle 38.7°} = 69\angle -26°\ \text{mA}$$

**13** The $h$-parameter equivalent circuit with a 2kΩ (500μS) load is shown in figure A5.13 and can be derived from the $h$-parameter defining equations.

From this we see that the conductance across the current source is 100 + 500 = 600 μS, so that

$$V_2 = -100I_1/600 \times 10^{-6} = -167 \times 10^3 I_2$$

Then looking at the input side we have

$$V_1 = 10^3 I_1 + 10^{-3} V_2 = 10^3 I_1 - 167 I_1 = 833 I_1$$

The voltage gain is therefore

$$V_2/V_1 = -167 \times 10^3 I_1/833 I_1 = -200$$

Now $I_2$ is the current through the 500$\mu$S load which is

$$I_2 = -500 \times 100 \times I_1/600 = -83 I_1$$

The current gain is −83.

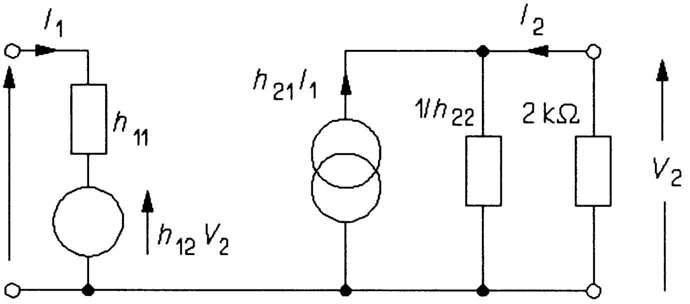


*Figure A5.13*

**14** Table 5.1 gives the *a*-parameters in terms of the *h*-parameters as

$$a_{11} = -\Delta h/h_{21} = -(h_{11}h_{22} - h_{12}h_{21})/h_{21} = -(10^3 \times 10^{-4} - 10^{-3} \times 10^2)/h_{21} = 0$$

$$a_{12} = -h_{11}/h_{21} = -10^3/10 = -10$$

$$a_{21} = -h_{22}/h_{21} = -10^{-4}/10 = 10^{-5}$$

$$a_{22} = -1/h_{21} = -1/10 = -0.1$$

The *a*-parameters of the combination are given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ -10^{-6} & -10^{-2} \end{bmatrix} \begin{bmatrix} 0 & -10 \\ -10^{-6} & -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 + 10^{-7} & 0 + 0.1 \\ 0 + 10^{-8} & 10^{-5} + 10^{-4} \end{bmatrix}$$

Therefore
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 10^{-7} & 0.1 \\ 10^{-8} & 1.1 \times 10^{-4} \end{bmatrix}$$

When the output of this combination is open circuit, the *a*-parameter definitions give

$$[V_2/V_1]_{I_2=0} = 1/a_{11} = 10^5$$

To find the voltage gain with load we can return to the *h*-parameter equivalent circuit. The *h*-parameters of the combination are given by

$$h_{11} = a_{12}/a_{22} = 0.1/1.1 \times 10^{-4} = 909\ \Omega$$

$$h_{12} = \Delta a/a_{22} = (10^{-5} \times 1.1 \times 10^{-4} - 0.1 \times 10^{-8})/1.1 \times 10^{-4} = 9.09 \times 10^{-7}$$

$$h_{21} = -1/a_{22} = -1/1.1 \times 10^{-4} = -9090$$

$$h_{22} = a_{21}/a_{22} = 10^{-8}/1.1 \times 10^{-4} = 90.9\ \mu\text{S}$$

With no load $I_2 = 0$, and the defining equation for $I_2$ gives

$$V_2 = -h_{21}I_1/h_{22} = 10^8 I_1$$

And so on the input side

$$V_1 = h_{11}I_1 + h_{12}V_2 = 909I_1 - 9.09 \times 10^{-7} \times 10^8 I_1 = 818I_1$$

Thus the no-load voltage gain, $V_2/V_1 = 10^8/818 = 1.22 \times 10^5$.

With a 10kΩ load, the *h*-parameter equivalent circuit is as in figure A5.14, which gives

$$V_2 = -h_{21}I_1Z_o = 9090I_1 \times 10 \times 11 \times 10^3/21 = 47.6 \times 10^6 I_1$$

where $Z_o = 10\ \text{k}\Omega \parallel 11\ \text{k}\Omega$. Then on the input side

$$V_1 = h_{11}I_1 + h_{12}V_2 = 909I_1 + 9.09 \times 10^{-7} \times 47.6 \times 10^6 I_1 = 10^3 I_1$$

Finally

$$V_2/V_1 = 47.6 \times 10^6 I_1/10^3 I_1 = 47.6 \times 10^3$$

Note that $V_2$ is in phase with $V_1$ and so the ratio is positive, not negative.

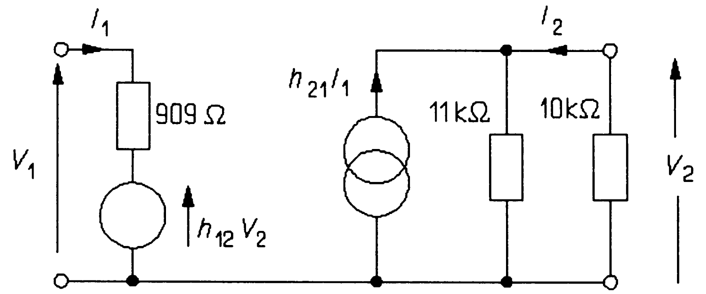


*Figure A5.14*

# Chapter 6

**1** (a) The resistance is given by $R = l/\sigma A$, so

$$l = R\sigma A = 10^3 \times 60 \times 10^6 \times \pi(0.5 \times 10^{-3})^2 = 47.1 \text{ km}$$

(b) We first calculate the conductivity from

$$\sigma = nq\mu = 10^{20} \times 1.6 \times 10^{-19} \times 0.1 = 1.6 \text{ S/m}$$

and then

$$l = R\sigma A = 10^3 \times 1.6 \times \pi(0.5 \times 10^{-3})^2 = 1.26 \text{ mm}$$

(c) If $\sigma = nq\mu$

$$\mu_{\text{Cu}} = \frac{\sigma}{nq} = \frac{60 \times 10^6}{8.5 \times 10^{28} \times 1.6 \times 10^{-19}} = 0.0044 \text{ m}^2\text{V}^{-1}\text{s}^{-1}$$

The ratio of the mobilities is 0.1/0.0044 = 22.7

**2** Equation 6.2 is

$$n_{\text{i}} = N\exp(-E_{\text{g}}/2kT)$$

Thus the conductivity is

$$\sigma = nq\mu = Nq\mu\exp(-E_{\text{g}}/2kT)$$

$$\Rightarrow \quad \ln\sigma = \ln(Nq\mu) - E_{\text{g}}/2kT$$

$$\Rightarrow \quad \frac{1}{\sigma}\frac{\text{d}\sigma}{\text{d}T} = \frac{E_{\text{g}}}{2kT^2}$$

and we find

$$\frac{\Delta\sigma}{\sigma} = \frac{E_{\text{g}}\Delta T}{2kT^2} = \frac{qE_{\text{g}}\Delta T}{2kT^2} \quad (E_{\text{g}} \text{ in eV})$$

Substituting $\Delta T = 2$ K, $T = 300$ K, $q = 1.6 \times 10^{-19}$ C and $k = 1.38 \times 10^{-23}$ J/K gives

$$\Delta\sigma/\sigma = 0.129E_{\text{g}}$$

Then for Ge, $\Delta\sigma/\sigma = 0.129 \times 0.66 = 0.085$ or 8.5%; for Si, $\Delta\sigma/\sigma = 14.2\%$; and for GaAs, $\Delta\sigma/\sigma = 18.2\%$.

**3** The conductivity is given by

$$\sigma = nq\mu = nqAT^{-1.5}$$

where $A$ is a constant. Taking natural logarithms,

$$\ln\sigma = \ln(nqA) - 1.5\ln T$$

$$\Rightarrow \quad \frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}T} = \frac{-1.5}{T}$$

When $\Delta T = 2$ K and $T = 300$ K, we find

$$\frac{\Delta\sigma}{\sigma} = \frac{-1.5\Delta T}{T} = \frac{-3}{300} = -0.01$$

The conductivity has gone *down* by 1%, and has not increased as in the case of intrinsic material.

**4** Once more we use $\sigma = nq\mu$, and this time

$$\sigma = N\exp(-E_c/2kT) \times q \times AT^{-1.5} = qNAT^{-1.5}\exp(-E_c/2kT)$$

$$\Rightarrow \quad \ln\sigma = \ln(qNA) - 1.5\ln T - E_c/2kT$$

Differentiating with respect to $T$

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}T} = \frac{-1.5}{T} + \frac{E_c}{2kT^2}$$

This is zero when

$$T = \frac{E_c}{3k} = \frac{0.05 \times 1.6 \times 10^{-19}}{3 \times 1.38 \times 10^{-23}} = 193 \text{ K}$$

Thus the temperature coefficient of the conductivity of a semiconductor can be positive, negative or zero.

**5** The question assumes that when doped the majority carriers are the only significant contributors to the conductivity. Then

$$\sigma = nq\mu \Rightarrow \quad \mu = \sigma/nq = 10/10^{22} \times 1.6 \times 10^{-19} = 6.25 \times 10^{-3} \text{ m}^2\text{V}^{-1}\text{s}^{-1}$$

The hole mobility will be a fifth of this or $1.25 \times 10^{-3}$ m$^2$V$^{-1}$s$^{-1}$.
The conductivity of a p-type sample doped with $10^{21}$ atoms/m$^3$ is

$$\sigma = pq\mu_p = 10^{21} \times 1.6 \times 10^{-19} \times 1.25 \times 10^{-3} = 0.2 \text{ S/m}$$

**6** The conductivity in a near-intrinsic semiconductor is due to both types of carrier, so that

$$\sigma = n_e q \mu_e + n_h q \mu_h$$

The subscripts *e* and *h* refer to electrons and holes respectively. Now the product $n_e n_h$ is constant, at constant temperature, and we can write $n_e n_h = A$, $n_e = A/n_h$, leading to

$$\sigma = \frac{Aq\mu_e}{n_h} + n_h q \mu_h = \frac{Aq\alpha\mu_h}{n_h} + n_h q \mu_h$$

Differentiating this with respect to $n_h$ leads to

$$\frac{d\sigma}{dn_h} = \frac{-A\alpha q \mu_h}{n_h^2} + q\mu_h$$

Setting this to zero for minimum conductivity gives $n_h = \sqrt{(A\alpha)}$. But $A$ is the product of the carrier concentrations, including the intrinsic carrier concentrations and so $A = np$, where $n$ = intrinsic electron concentration and $p$ = intrinsic hole concentration and $n = p$ in intrinsic material, so that $A = p^2$ and the minimum conductivity occurs when $n_h = \sqrt{(A\alpha)} = \sqrt{(p^2\alpha)} = p\sqrt{\alpha}$. The number of electrons/unit volume is then $n_e = A/n_h = p^2/p\sqrt{\alpha} = p/\sqrt{\alpha}$. Substituting these value for $n_e$ and $n_h$ into the conductivity equation gives

$$\sigma = \frac{pq\mu_e}{\sqrt{\alpha}} + \sqrt{\alpha}pq\mu_h = \frac{pq\alpha\mu_h}{\sqrt{\alpha}} + \alpha pq\mu_h = 2pq\mu_h\sqrt{\alpha}$$

Substituting $\mu_h = 0.009\ \mathrm{m^2V^{-1}s^{-1}}$, $\alpha = 3$ and $p = 10^{16}\ \mathrm{m^{-3}}$ leads to $\sigma_{min} = 50\ \mu\mathrm{S/m}$, and the electron concentration is then $10^{16}/\sqrt{3} = 5.8 \times 10^{15}\ \mathrm{m^{-3}}$ while the hole concentration is $10^{16} \times \sqrt{3} = 1.73 \times 10^{16}\ \mathrm{m^{-3}}$.

# Chapter 7

1 The problem has hidden difficulties, since $I_s$ depends on the intrinsic carrier concentration and that is temperature-dependent too. We can write

$$I = I_s\exp(qV/1.8kT) \Rightarrow I_s = I\exp(-qV/1.8kT)$$

$$I_s = 5 \times 10^{-3}\exp(-1.6 \times 10^{-19} \times 0.7/1.8 \times 1.38 \times 10^{-23} \times 300) = 1.485 \text{ nA}$$

Now the intrinsic carrier concentration is proportional to $\exp(-E_g/2kT)$, so that

$$I_s = A\exp(-E_g/2kT)$$

where $A$ is a constant that can be found if we substitute $E_g = 1.1$ eV, for Si:

$$1.485 \times 10^{-9} = A\exp\left[\frac{-1.1 \times 1.6 \times 10^{-19}}{2 \times 1.38 \times 10^{-23} \times 300}\right] \Rightarrow A = 2.53$$

Thus at 350 K

$$I_s = 2.53\exp\left[\frac{1.1 \times 1.6 \times 10^{-19}}{2 \times 1.38 \times 10^{-23} \times 350}\right] = 30.9 \text{ nA}$$

Then we can find $V$ from

$$I = 5 \times 10^{-3} = I_s\exp\left[\frac{qV}{1.8kT}\right] = 30.9 \times 10^{-9}\exp\left[\frac{1.6 \times 10^{-19}V}{1.8 \times 1.38 \times 10^{-23} \times 350}\right]$$

$$162 \times 10^3 = \exp(18.4V) \Rightarrow V = \frac{\ln(162 \times 10^3)}{18.4} = 0.65 \text{ V}$$

2 In figure A7.2 the nodal equation at A is

$$\frac{15 - 1 - V_A}{10} + \frac{-2 - 1 - V_A}{1} = \frac{V_A - 1 - (-15)}{20} + \frac{V_A - 1}{5}$$

$$\Rightarrow \quad 1.4 - 3 - 0.7 + 0.2 = V_A(0.1 + 1 + 0.05 + 0.2)$$

$$\Rightarrow \quad 1.35V_A = -2.1 \Rightarrow V_A = -1.56 \text{ V}$$

implying that D3 and D4 are reverse biased, so we must recalculate with $I_3$ and $I_4$ both zero:

$$\frac{15 - 1 - V_A}{10} = \frac{V_A - 1 - (-15)}{20} \Rightarrow 0.15V_A = 0.7 \Rightarrow V_A = +4.67 \text{ V}$$

which makes D4 forward biased, so reworking the nodal equation again gives

$$\frac{15 - 1 - V_A}{10} = \frac{V_A - 1 - (-15)}{20} + \frac{V_A - 1}{5}$$

$$\Rightarrow 1.4 - 0.7 + 0.2 = V_A(0.1 + 0.05 + 0.2) \Rightarrow V_A = 0.9/0.35 = 2.571 \text{ V}$$

This is now consistent with D3 being reverse biased and the rest forward biased, so that

$$I_1 = \frac{15 - 1 - 2.571}{10} = 1.143 \text{ A} \; ; \; I_2 = \frac{2.571 - 1 - (-15)}{20} = 0.829 \text{ A}$$

$$I_3 = 0 \quad ; \; I_4 = \frac{2.571 - 1}{5} = 0.314 \text{ A}$$

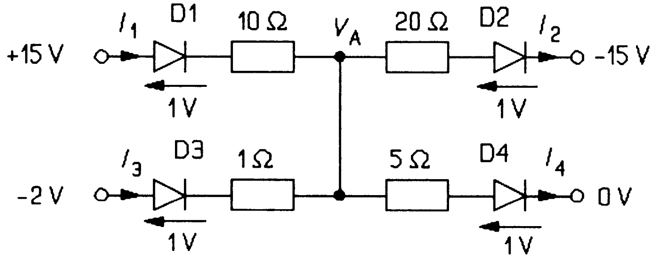


*Figure A7.2*

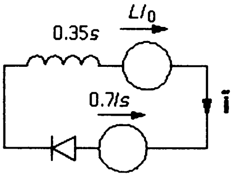


*Figure A7.3*

3 The circuit in the $s$-domain after the switch is opened is as shown in figure A7.3, in which the initial current through the inductor, $I_0 = 75/3 = 25$ A. Then

$$\bar{i} = \frac{0.35I_0 - 0.7/s}{0.35s} = \frac{I_0}{s} - \frac{2}{s^2} = \frac{25}{s} - \frac{2}{s^2}$$

Detransforming, $i(t) = (25 - 2t)u(t)$ A, but this becomes zero at $t = 12.5''$, when the current ceases, so we must add on a term $2(t - 12.5)u(t - 12.5)$ to ensure $i = 0$ for $t > 12.5''$.

The power consumption is $V_D i(t) = (17.5 - 1.4t)u(t) + 1.4(t - 12.5)u(t - 12.5)$ W. We can find the average power from $E = 0.5LI_0^2 = 109.4$ J, divided by the time, 12.5″, giving 8.75 W. This answer can be checked by integrating $p(t)$, and dividing by 12.5:

$$P = \frac{\int_0^{12.5}(17.5 - 1.4t)\mathrm{d}t}{12.5} = \frac{\left[17.5t - 0.7t^2\right]_0^{12.5}}{12.5} = \frac{17.5 \times 12.5 - 0.7 \times 12.5^2}{12.5} = 8.75\text{ W}$$

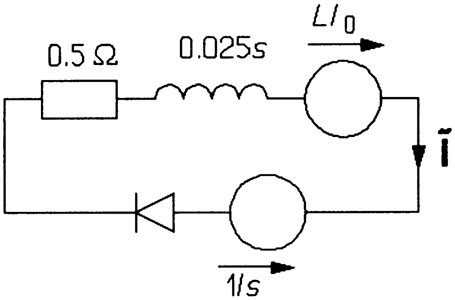


*Figure A7.4*

**4** The circuit in the $s$-domain after the switch has been opened is as in figure A7.4, from which we find

$$\tilde{\mathrm{I}} = \frac{LI_0 - 1/s}{0.5 + 0.025s}$$

The initial current, $I_0 = 3/0.5 = 6$ A and then $LI_0 = 0.025 \times 6 = 0.15$, making

$$\tilde{\mathrm{I}} = \frac{0.15 - 1/s}{0.025s + 0.5} = \frac{6s - 40}{s(s + 20)} \equiv \frac{A}{s} + \frac{B}{s + 20}$$

$$\Rightarrow \quad A(s + 20) + Bs \equiv 6s - 40 \quad \Rightarrow \quad A = -2, \quad B = 8$$

And detransformation gives

$$i(t) = -2 + 8\exp(-20t) \text{ A}$$

This only holds until the current becomes zero at $t = t_0$ and for $t \geq 0$.
We must find $t_0$ from

$$i(t_0) = 0 = -2 + 8\exp(-20t_0) \quad \Rightarrow \quad \exp(-20t_0) = 0.25$$

$$-20t_0 = \ln 0.25 \quad \Rightarrow \quad t_0 = 69.3 \text{ ms}$$

**5** The supply that back-biases the diode controls the average power developed in the load of course, and this can only be found by integrating the instantaneous power between appropriate limits and dividing by the time. The voltage across the 12Ω resistance is shown in figure A7.5, and is $V_m \sin \omega t - V_b - 0.7$ V. The part of the waveform above the $t$-axis must be found, squared and integrated.

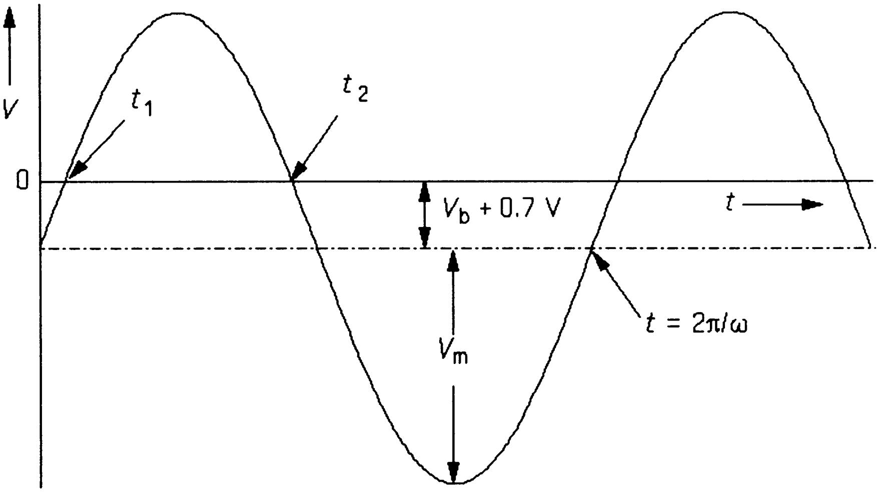


*Figure A7.5*

When $V_b = 1$ V, and $V_m = 6$ V, the conducting phase angle is $\sin^{-1}(1.7/6) = 16.5°$, and the extinction phase angle is $180° - 16.5° = 163.5°$. The energy for one cycle is thus

$$E = \int p dt = \frac{1}{R}\int_{t_1}^{t_2}(V_m \sin\omega t - 1.7)^2 \mathrm{d}t$$

where $\omega t_1 = \theta_1 = 16.5\pi/180$ radians and $\omega t_2 = \theta_2 = 163.5\pi/180$ radians. Substituting $\theta$ for $\omega t$ and putting $R = 12\ \Omega$ and $V_m = 6$ V gives

$$E = \frac{1}{12\omega}\int_{\theta_1}^{\theta_2}(6\sin\theta - 1.7)^2 \mathrm{d}\theta = \frac{1}{12\omega}\int_{\theta_1}^{\theta_2}(36\sin^2\theta - 20.4\sin\theta + 2.89)\mathrm{d}\theta$$

$$= \frac{1}{12\omega}\int_{\theta_1}^{\theta_2}(18 - 18\cos 2\theta - 20.4\sin\theta + 2.89)\mathrm{d}\theta = \frac{[20.89\theta - 9\sin 2\theta + 20.4\cos\theta]_{\theta_1}^{\theta_2}}{12\omega}$$

But this must be averaged over one cycle which is from $t = 0$ to $t = 2\pi/\omega$, so that the average power is

$$P = \frac{E}{2\pi/\omega} = \frac{\omega E}{2\pi} = \frac{20.89(\theta_2 - \theta_1) - 9(\sin 2\theta_2 - \sin 2\theta_1) + 20.4(\cos\theta_2 - \cos\theta_1)}{24\pi}$$

$$= \frac{\frac{20.89\pi}{180}(163.5° - 16.5°) - 9(\sin 327° - \sin 33°) + 20.4(\cos 163.5° - \cos 16.5°)}{24\pi}$$

$$= \frac{53.6 + 9.8 - 39.1}{24\pi} = 0.322 \text{ W}$$

When the bias voltage is increased to 2 V, the integration goes

$$P = \frac{\int_{\theta_1}^{\theta_2} (6\sin\theta - 2.7)^2 d\theta}{24\pi}$$

where $\theta_1 = \sin^{-1}(2.7/6) = 26.7°$ and $\theta_2 = 153.3°$, giving

$$P = \frac{25.29(\theta_2 - \theta_1) - 9(\sin 2\theta_2 - \sin 2\theta_1) + 32.4(\cos\theta_2 - \cos\theta_1)}{24\pi}$$

$$= \frac{\dfrac{25.29\pi \times 126.5}{180} + 9 \times 1.607 - 32.4 \times 1.786}{24\pi} = 0.165 \text{ W}$$

When $V_b = 4$ V, $\theta_1 = \sin^{-1}(4.7/6) = 51.6°$, $\theta_2 = 128.4°$ and the integration produces

$$P = \frac{40.09(\theta_2 - \theta_1) - 9(\sin 2\theta_2 - \sin 2\theta_1) + 56.4(\cos\theta_2 - \cos\theta_1)}{24\pi} = 15.9 \text{ mW}$$

**6** Here the voltage across the resistance is $V_m \sin\omega t + V_b - 0.7$ and the conduction angle is $\sin^{-1}[(0.7 - V_b)/6]$ or $-2.9°$, $-12.5°$ and $-33.4°$ for $V_b = 1$, 2 and 4 V respectively. The first integration, with $\theta_1 = -2.9°$ and $\theta_2 = 182.9°$, yields

$$P = \frac{\int_{\theta_1}^{\theta_2}(6\sin\theta + 0.3)^2 d\theta}{24\pi} = \frac{18.09(\theta_2 - \theta_1) - 9(\sin 2\theta_2 - \sin 2\theta_1) - 3.6(\cos\theta_2 - \cos\theta_1)}{24\pi}$$

$$= \frac{\dfrac{18.09\pi(182.9 + 2.9)}{180} - 9 \times 0.202 + 3.6 \times 1.997}{24\pi} = 0.849 \text{ W}$$

And when $V_b = 2$ V, $\theta_1 = \sin^{-1}(-1.3/6) = 12.5°$ and $\theta_2 = 192.5°$. The integration goes

$$P = \frac{\dfrac{19.69\pi \times 205}{180} - 9 \times 0.845 + 15.6 \times 1.953}{24\pi} = 1.24 \text{ W}$$

Finally, when $V_b = 4$ V the power is

$$P = \frac{\dfrac{28.89\pi \times 246.8}{180} - 9 \times 1.838 + 39.6 \times 1.67}{24\pi} = 2.31 \text{ W}$$

7 We can write down from figure A7.7 using KVL that

$$V_s + V_b = IR + V_D + I_D r_d$$

$$\Rightarrow \quad 0.2\sin\omega t + 1.2 = 10(I_D + I_L) + 0.6 + 0.9I_D$$

as $I = I_D + I_L$. Also by KVL

$$V_D + I_D r_d = 50I_L \quad \Rightarrow \quad 0.6 + 0.9I_D = 50I_L$$

Then $10I_L = 0.12 + 0.18I_D$, which can be substituted in the previous equation to yield

$$0.2\sin\omega t + 1.2 = 10I_D + 0.12 + 0.18I_D + 0.6 + 0.9I_D$$

$$\Rightarrow \quad 0.2\sin\omega t + 0.48 = 11.08I_D$$

Thus $I_D = 18\sin\omega t + 43.3$ mA.

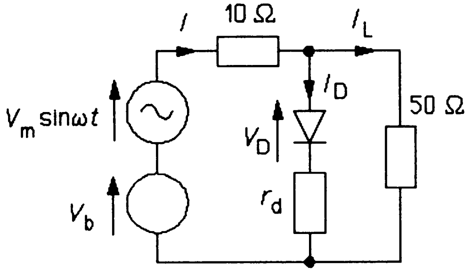


Figure A7.7

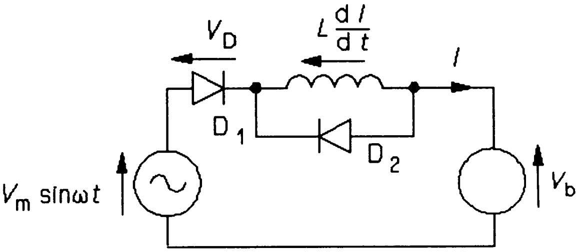


Figure A7.8

8 In figure A7.8, $D_1$ conducts during the positive half cycle at an angle of $\theta_c = \sin^{-1}[(V_b + V_D)/V_m]$ and ceases to conduct at an angle of 180° − $\theta_c$. The current reaches a maximum when the diode ceases to conduct. $D_2$ is assumed to have reduced the inductor current to zero before the next positive half-cycle commences. The KVL equation from figure A7.8 is

$$V_m\sin\omega t = L\mathrm{d}i/\mathrm{d}t + V_D + V_b \quad \Rightarrow \quad I_{max} = \frac{1}{L}\int_{t_1}^{t_2}(V_m\sin\omega t - V_D - V_b)\mathrm{d}t$$

$$= \frac{1}{\omega L}\int_{\theta_c}^{\pi - \theta_c}(V_m \sin\theta - V_D - V_b)d\theta = \frac{[-V_m\cos\theta - V_D\theta - V_b\theta]_{\theta_c}^{\pi - \theta_c}}{\omega L}$$

$$= \frac{2V_m\cos\theta_c - (V_D + V_b)(\pi - 2\theta_c)}{\omega L}$$

where $t_1 = \theta_c/\omega$ and $t_2 = (\pi - \theta_c)/\omega$. Then when $V_m = 10$ V, $V_b = 5$ V, $V_D = 0.7$ V, $L = 15$ mH and $\omega = 2\pi f = 100\pi$, we find $\theta_c = 0.6065$ radians and $I_{max} = 1.154$ A.

When we are given $I_{max}$ finding $V_b$ is more tricky, since we have also to find the conduction angle. An analytical solution is impossible and it is best to resort to guesswork.

The equation for $I_{max}$ can be expressed as

$$I_{max}\omega L = 2V_m\cos\theta_c - (V_b + V_D)(\pi - 2\theta_c)$$

$$\Rightarrow \quad V_b = \frac{2V_m\cos\theta_c - I_{max}\omega L}{\pi - 2\theta_c} - V_D = \frac{20\cos\theta_c - 2.356}{\pi - 2\theta_c} - 0.7$$

Let us guess a value for $\theta_c$, which will be somewhere between 0 and $\pi/2$ radians, let us say $\pi/4$ radians, and then

$$V_b = \frac{14.142 - 2.356}{1.571} - 0.7 = 6.8 \text{ V}$$

This gives $\theta_c = \sin^{-1}(7.5/10) = 0.848$ radians, so the refined value of $V_b$ is

$$V_b = \frac{13.23 - 2.356}{1.446} - 0.7 = 6.82 \text{ V}$$

**9** The capacitance is given by $C = 10^{-10}/\sqrt{V}$ F, which is 100 pF for $V = 1$ V and 22.4 pF when $V = 20$ V, giving

$$f = \frac{1}{2\pi\sqrt{10 \times 10^{-6} \times 100 \times 10^{-12}}} = 5.03 \text{ MHz and } f = \frac{1}{2\pi\sqrt{10^{-5} \times 22.4 \times 10^{-12}}} = 10.6 \text{ MHz}$$

The frequency in terms of voltage is

$$f = \frac{1}{2\pi\sqrt{10^{-5} \times 10^{-10}V^{-1/2}}} = 5 \times 10^6 V^{1/4} \text{ Hz}$$

$$\Rightarrow \quad \ln f = \ln(5 \times 10^6) + 0.25\ln V$$

Differentiating with respect to $V$ gives

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}V} = \frac{1}{4V} \Rightarrow \frac{\Delta f}{f} = \frac{1}{4}\frac{\Delta V}{V}$$

Thus if $\Delta V/V = 0.01$, $\Delta f/f = 0.0025$, so if $f = 7$ MHz, $\Delta f = 7 \times 10^6 \times 0.0025 = 17.5$ kHz.

**10** In the circuit of figure A7.10, we can see that

$$V_s = 300I + V_{Z0} + r_z I_Z = 300(I_L + I_Z) + V_{Z0} + r_z I_Z$$

where $V_{Z0}$ is the zener voltage at zero current, that is $V_{Z0} = 10 - 15 \times 10 \times 10^{-3} = 9.85$ V, since the rated voltage is 10 V at 10 mA and the dynamic resistance, $r_z = 15\ \Omega$.

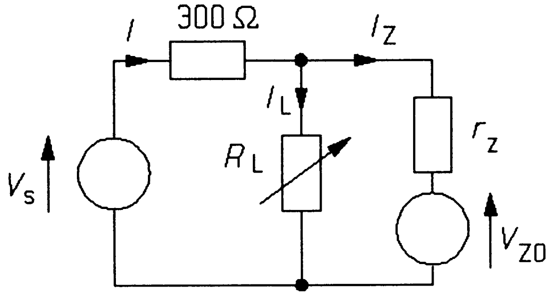


*Figure A7.10*

With these values substituted we have

$$V_s = 300(I_L + I_Z) + 9.85 + 15I_Z \Rightarrow I_Z = \frac{V_s - 9.85 - 300I_L}{315}$$

We can see that $I_Z$ will be least when $V_s$ is least (20 V) and $I_L$ is maximal (25 mA), giving

$$I_{Z\min} = \frac{20 - 9.85 - 300 \times 25 \times 10^{-3}}{315} = 8.41 \text{ mA}$$

The corresponding zener voltage is

$$V_{Z\min} = V_{Z0} + I_Z r_z = 9.85 + 8.41 \times 10^{-3} \times 15 = 9.976 \text{ V}$$

And $I_{Z\max}$ occurs when $V_s = 25$ V and $I_L = 5$ mA, being

$$I_{Z\max} = \frac{25 - 9.85 - 300 \times 5 \times 10^{-3}}{315} = 43.3 \text{ mA}$$

The corresponding zener voltage is

$$V_{Z\max} = 9.85 + 43.3 \times 10^{-3} \times 15 = 10.5 \text{ V}$$

The maximum power dissipation of the zener is $V_{Z\max} I_{Z\max} = 10.5 \times 43.3 \times 10^{-3} = 455$ mW.

**11** The photon energy is

$$E = \frac{1.24}{\lambda} \Rightarrow \frac{dE}{d\lambda} = \frac{-1.24}{\lambda^2} \Rightarrow \Delta E = \frac{-1.24\Delta\lambda}{\lambda^2}$$

But $\Delta E = 3kT$, so that

$$3kT = \frac{-1.24\Delta\lambda}{\lambda^2} \Rightarrow \Delta\lambda = \frac{-3kT\lambda^2}{1.24} \approx -2.4kT\lambda^2$$

The minus sign indicates only that the linewidth decreases as the photon energy goes up (or as the wavelength goes down).

The bandgap is related to wavelength by

$$E_g = hf = hc/\lambda \Rightarrow \lambda = hc/E_g$$

where $h$ = Planck's constant = $6.627 \times 10^{-34}$ Js and $c = 3 \times 10^8$ m/s. Thus if $E_g$ = 1.42 eV = $1.42 \times 1.6 \times 10^{-19}$ J, the photon's wavelength is

$$\lambda = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{1.42 \times 1.6 \times 10^{-19}} = 0.875 \ \mu\text{m}$$

Then the linewidth at 77 K is

$$\Delta\lambda = \frac{2.4 \times 1.38 \times 10^{-23} \times 77 \times 0.875^2}{1.6 \times 10^{-19}} = 0.0122 \ \mu\text{m}$$

as $kT$ has to be in eV we divide by $q$, the electronic charge. Thus at 77 K the $Q$ is

$$Q = \lambda/\Delta\lambda = 0.875/0.0122 = 72$$

while at 300 K the linewidth will be

$$\Delta\lambda = \frac{2.4 \times 1.38 \times 10^{-23} \times 300 \times 0.875^2}{1.6 \times 10^{-19}} = 0.0475 \ \mu\text{m}$$

and then $Q = 0.875/0.0475 = 18.4$

# Chapter 8

1 In figure A8.1 using KVL down the right-hand side gives

$$V_{CC} = I_C R_C + V_{CEsat} + (I_C + I_B)R_E$$

Substituting $I_C/\beta$ for $I_E$ gives

$$V_{CC} = I_C R_C + V_{CEsat} + (I_C + I_C/\beta)R_E$$

$$\Rightarrow \quad I_C = \frac{V_{cc} - V_{CEsat}}{R_C + R_E + R_E/\beta} = \frac{8.8}{2000 + 470 + 470/150} = 3.56 \text{ mA}$$

Using KVL on the base side gives

$$V_{CC} = I_B R_B + V_{BE} + I_E R_E = I_B R_B + V_{BE} + (\beta + 1)I_B R_E$$

$$\Rightarrow \quad R_B = \frac{V_{CC} - V_{BE} - (\beta + 1)I_B R_E}{I_B} = \frac{\beta(V_{CC} - V_{BE})}{I_C} - (\beta + 1)R_E$$

Substituting $\beta = 150$, $I_C = 3.56$ mA, $R_C = 2$ kΩ and $R_E = 470$ Ω gives $R_B = 279$ kΩ.

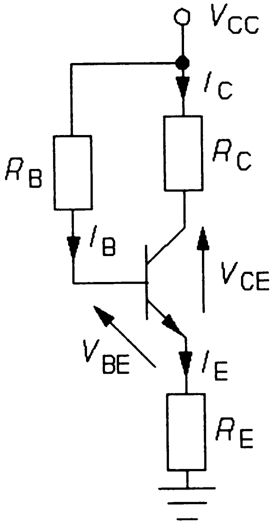


Figure A8.1

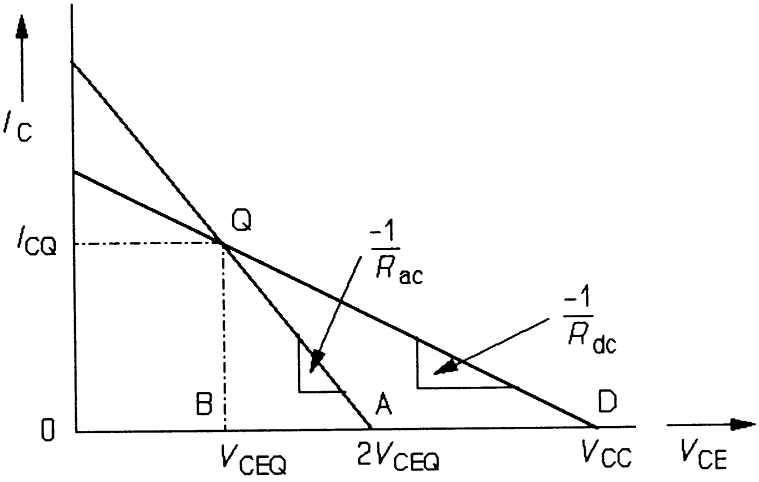


Figure A8.2

If the transistor is to be saturated the collector current must be maintained at 3.56 mA when $50 \leq \beta \leq 300$. Thus the maximum base current will be when $\beta = 50$. Substituting into the previous equation this value of $\beta$ gives

$$R_B = \frac{50(9 - 0.7)}{3.56 \times 10^{-3}} - 51 \times 470 = 82.6 \text{ k}\Omega$$

The collector current is very nearly independent of base current when $\beta \geq 50$.

2 The optimal Q-point is shown in figure A8.2, and it is placed so that the intercept of the AC load line with the $V_{CE}$ axis is at $2V_{CEQ}$, which ensures that the output voltage will be nearly $\pm V_{CEQ}$, assuming $V_{CEsat}$ is negligible.

Then in triangle BDQ we see that the slope of the load line, QD is $-1/R_{dc}$ and this is QB/QD, thus

$$\frac{QB}{QD} = \frac{1}{R_{dc}} = \frac{I_{CQ}}{V_{cc} - V_{CEQ}} \Rightarrow V_{CEQ} = V_{CC} - I_{CQ}R_{dc}$$

Since QB $= I_{CQ}$ and QD $= V_{CC} - V_{CEQ}$.

Then in triangle ABQ, the slope of the load line, QA is $-1/R_{ac}$ and this is QB/AB, so that

$$\frac{QB}{AB} = \frac{1}{R_{ac}} = \frac{I_{CQ}}{V_{CEQ}} \Rightarrow I_{CQ} = \frac{V_{CEQ}}{R_{ac}}$$

But $V_{CEQ} = V_{CC} - I_{CQ}R_{dc}$, so that

$$I_{CQ} = \frac{V_{CC}}{R_{ac}} - \frac{I_{CQ}R_{dc}}{R_{ac}} \Rightarrow I_{CQ}(R_{ac} + R_{dc}) = V_{CC}$$

Hence
$$I_{CQ} = \frac{V_{CC}}{R_{dc} + R_{ac}}$$

3 Looking at figure A8.3, we can estimate $h_{fe}$ at the Q-point:

$$h_{fe} = \left[\frac{\delta I_b}{\delta I_C}\right]_Q = \frac{(55 - 35) \times 10^{-6}}{2 \times 10^{-3}} = 100$$

Estimation of $h_{oe}$ requires measuring the slope of the two lines marked '$I_B = 35$ $\mu$A' and '$I_B = 55$ $\mu$A' and averaging, which is not very accurate as the slopes are nearly zero. Taking the upper line and measuring at $V_{CE} = 16$ V we find $I_C = 5.1$ mA, and at $V_{CE} =$ 4 V we find $I_C = 4.95$ mA, making a slope of 0.15 mA ÷ 12 V = 12.5 $\mu$S. Measuring the lower line at the same voltages yields $\delta I_C = 3.13 - 3.00 = 0.13$ mA, for a slope of 10.8 $\mu$S, giving an average slope of about 12 $\mu$S for $h_{oe}$, which can safely be neglected.

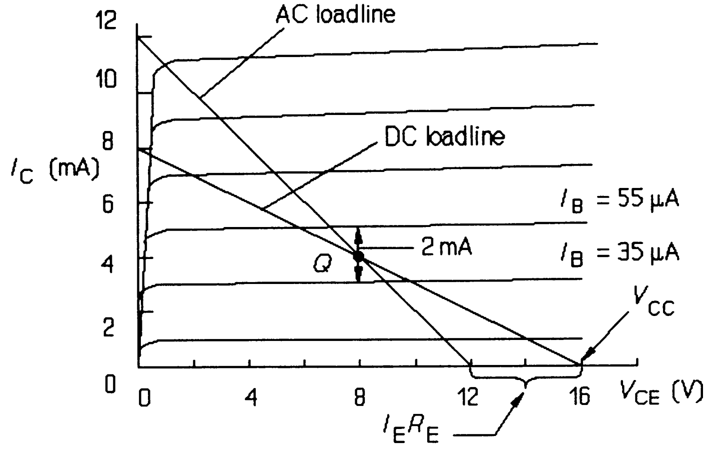


*Figure A8.3*

We can estimate $h_{ie}$ from

$$h_{ie} = \frac{V_T}{I_{BQ}} = \frac{kT/q}{I_{BQ}} = \frac{0.026}{45 \times 10^{-6}} \approx 600\ \Omega$$

The slope of the DC load line is $-8 \times 10^{-3} \div 16 = -5 \times 10^{-4}$ S $= -1/R_{dc}$, which makes $R_{dc} = R_C + R_E = 2$ kΩ. The slope of the AC load line is $-12 \times 10^{-3} \div 12 = -10^{-3}$ S $= -1/R_{ac}$, making $R_{ac} = R_C = 1$ kΩ and then $R_E = 1$ kΩ.

The Q-point is placed so that the transistor runs into cut off at $V_{CE} = 12$ V long before it runs into saturation, so the output voltage swing is $\pm(V_{CE}[\text{cut-off}] - V_{CEQ}) = \pm(12 - 8) = \pm 4$ V, or 8 V in all. There will be some distortion. The relatively undistorted output voltage swing will be from when $I_B = 15\ \mu$A to $I_B = 75\ \mu$A, that is from $V_{CE} = 11$ V to $V_{CE} = 5$ V, a swing of 6 V.

When the load is 2 kΩ, $R_{ac} = R_L \parallel R_C = 667\ \Omega$, while $R_{dc} = 2$ kΩ, thus the optimal value of $I_{CQ}$ is

$$I_{CQ} = \frac{V_{CC}}{R_{dc} + R_{ac}} = \frac{16}{2.667 \times 10^3} = 6 \text{ mA}$$

The slope of the DC load line is $5 \times 10^{-4}$, so that

$$V_{CEQ} = V_{CC} - \frac{I_{CQ}}{\text{dc slope}} = 16 - \frac{6 \times 10^{-3}}{5 \times 10^{-4}} = 4 \text{ V}$$

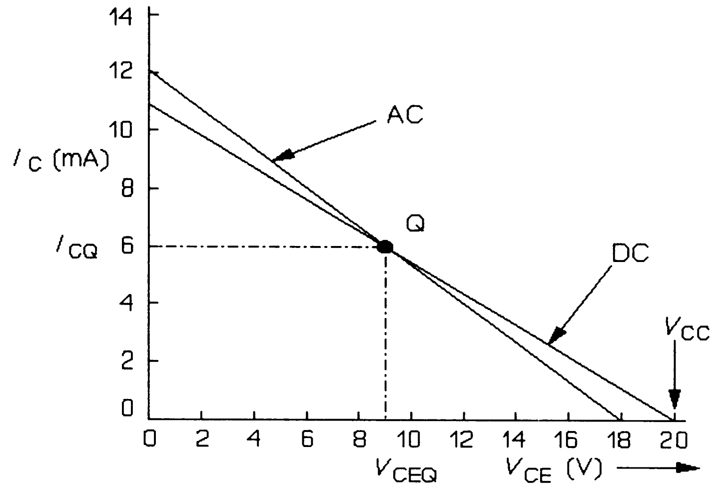


*Figure A8.4*

**4** $R_{ac} = R_C = 1.5$ kΩ while $R_{dc} = R_C + R_E = 1.83$ kΩ, so that the optimal value of $I_{CQ}$ is

$$I_{CQ} = \frac{V_{CC}}{R_{ac} + R_{dc}} = \frac{20}{(1.5 + 1.83) \times 10^{-3}} = 6 \text{ mA}$$

and $$V_{CEQ}(\text{opt}) = V_{CC} - I_{CQ}R_{dc} = 20 - 6 \times 1.83 = 9 \text{ V}$$

Figure A8.4 shows AC and DC load lines for the optimally-biased transistor with no load.

If $I_1 = 10I_{BQ}$ and $I_{BQ} = I_C/\beta = 6 \times 10^{-3}/100 = 60\ \mu$A, then $I_1 = 0.6$ mA, so the current through $R_2 = I_1 - I_B = 0.54$ mA. By KVL $I_2R_2 = V_{BE} + I_ER_E = 0.7 + 1.01 \times 6 \times 10^{-3} \times 330 = 2.7$ V, therefore $R_2 = 2.7/0.54$ kΩ $= 5$ kΩ.

Also by KVL on the input side,

$$V_{CC} = 20 = I_1R_1 + I_2R_2 = 0.6R_1 + 2.7$$

$$\Rightarrow \quad R_1 = 17.3/0.6 \text{ k}\Omega = 28.8 \text{ k}\Omega$$

**5** We can use the result of problem 8.2 again, with $R_{ac} = 2$ kΩ ‖ 3 kΩ = 1.2 kΩ and $R_{dc} = 2$ kΩ, so that

$$I_{CQ}(\text{opt}) = \frac{V_{CC}}{R_{ac} + R_{dc}} = \frac{12}{1.2 + 2} \text{ mA} = 3.75 \text{ mA}$$

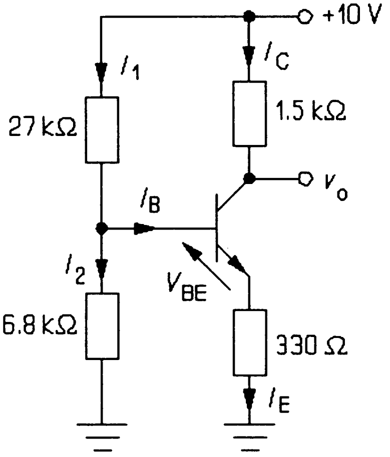
and

$$V_{CEQ}(\text{opt}) = V_{CC} - I_{CQ}R_{dc} = 12 - 3.75 \times 2 = 4.5 \text{ V}$$

Then in this case, we require $I_{BQ} = I_{CQ}/\beta = 3.75 \times 10^{-3}/180 = 20.83\ \mu$A and so by KVL

$$V_{CC} = 12 = I_BR_B + V_{BE} = 20.8 \times 10^{-6}R_B + 0.7$$

$$\Rightarrow \quad R_B = 11.3/20.83 \times 10^{-6} = 542\ \text{k}\Omega$$



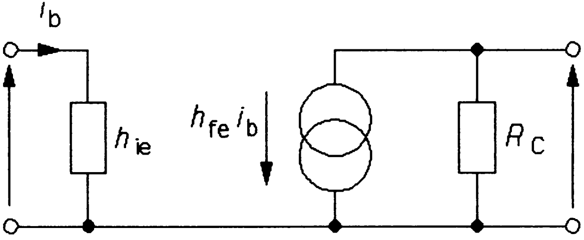


*Figure A8.6b*

*Figure A8.6a*

6 Using KVL on the lower half of the circuit of figure A8.6a gives

$$R_2I_2 = R_2(I_1 - I_B) = V_{BE} + I_ER_E = V_{BE} + (\beta + 1)I_BR_E$$

$$\Rightarrow \quad I_1 = \frac{V_{BE} + I_B[(\beta + 1)R_E + R_2]}{R_2} = 0.1029 + 6.872I_B$$

(working in mA and kΩ). Then also by KVL

$$I_1R_1 + V_{BE} + I_ER_E = V_{CC}$$

$$\Rightarrow I_1 = \frac{V_{CC} - V_{BE} - (\beta + 1)I_BR_E}{R_1} = 0.3444 - 1.479I_B$$

Then equating the two expression for $I_1$ leads to

$$0.1029 + 6.872I_B = 0.3444 - 1.479I_B \quad \Rightarrow \quad 8.351I_B = 0.2415\ \text{mA}$$

giving $I_B = 29\ \mu$A and hence $h_{ie} = 0.02/29 \times 10^{-6} = 690\ \Omega$. Taking $I_C = I_E$ makes almost no difference to this result.

The approximate *h*-parameter circuit is shown in figure A8.6b, from which one deduces that $v_o = -h_{fe}i_bR_C$ and as $i_b = v_s/h_{ie}$, we have

$$\frac{v_o}{v_s} = \frac{-h_{fe}R_C}{h_{ie}} = \frac{-120 \times 1500}{690} = -261$$

A capacitively-coupled load of 2 kΩ will be in parallel with $R_C$, so that

$$R_o = R_C \parallel R_L = \frac{2 \times 1.5}{2 + 1.5} = 0.857 \text{ k}\Omega$$

And then

$$A_{vL} = \frac{-h_{fe}R_o}{h_{ie}} = \frac{-120 \times 857}{690} = -149$$

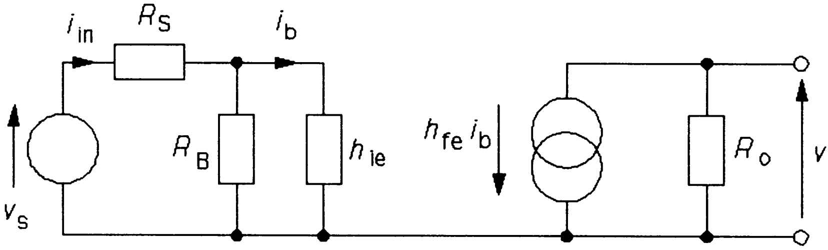


*Figure A8.6c*

When the signal source has internal resistance, the approximate $h$-parameter circuit is as in figure A8.6c and in this case we must find $i_b$:

$$i_b = \frac{R_B i_{in}}{R_B + h_{ie}}$$

where

$$R_B = R_1 \parallel R_2 = \frac{27 \times 6.8}{27 = 6.8} = 5.43 \text{ k}\Omega$$

And

$$i_{in} = \frac{v_s}{R_S + R_{in}} = \frac{v_s}{R_S + R_B \parallel h_{ie}} = \frac{v_s}{2000 + \frac{5430 \times 690}{5430 + 690}}$$

$$= \frac{v_s}{2000 + 612} = \frac{v_s}{2612}$$

Thus

$$i_b = \frac{R_B i_{in}}{R_B + h_{ie}} = \frac{R_B v_s}{2612(R_B + h_{ie})} = \frac{5430 v_s}{2612(5430 + 690)} = \frac{v_s}{2944}$$

And then

$$v_o = -h_{fe}i_bR_o = \frac{-120 \times 857 \times v_s}{2944} = -35v_s$$

That is $A_{vL} = -35$.

Including $h_{oe}$ in the equivalent circuit places an additional resistance of $1/h_{oe} = 20$ kΩ in parallel with $R_C$ and $R_L$, making

$$R_o = (1/20 + 1/2 + 1/1.5)^{-1} = 0.822 \text{ k}\Omega$$

instead of 0.857 kΩ, reducing $A_{vL}$ to

$$A_{vL} = \frac{0.822 \times -35}{0.857} = -33.5$$

7 The no-load voltage gain is given by

$$A_{v0} = -h_{fe}i_bR_C$$

if $h_{re}$ and $h_{oe}$ are both zero. When loaded the voltage gain is

$$A_{vL} = h_{fe}i_bR_o$$

where $R_o = R_C \parallel R_L$. The ratio of the two is

$$\frac{A_{v0}}{A_{vL}} = \frac{R_C}{R_CR_L/(R_C + R_L)} = 1 + \frac{R_C}{R_L} = 1.5$$

so that $R_C = 0.5R_L = 1.5$ kΩ.

By KVL, $V_{CC} = I_{CQ}R_C + V_{CEQ}$, so that $-12 = I_{CQ}R_C - 6$, leading to $I_{CQ}R_C = -6$ and $I_{CQ} = -4$ mA (the negative sign indicates the current flows into the supply and out of ground).

The quiescent base current is found from

$$V_{CC} = I_{BQ}R_B + V_{BEQ}$$

Substituting $-12$ V for $V_{CC}$, $-0.7$ V for $V_{BEQ}$ and 400 kΩ for $R_B$ leads to $I_{BQ} = -28.25$ μA (the negative sign indicating current flowing out of the base) and thus $h_{fe} \approx h_{FE} = I_{CQ}/I_{BQ} = 4 \times 10^{-3}/28.25 \times 10^{-6} = 142$.

To find $h_{ie}$ we use the equation for $A_{v0}$:

$$A_{v0} = -h_{fe}R_C/h_{ie} \quad \Rightarrow \quad h_{ie} = h_{fe}R_C/A_{v0} = 142 \times 1500/150 = 1420 \ \Omega$$

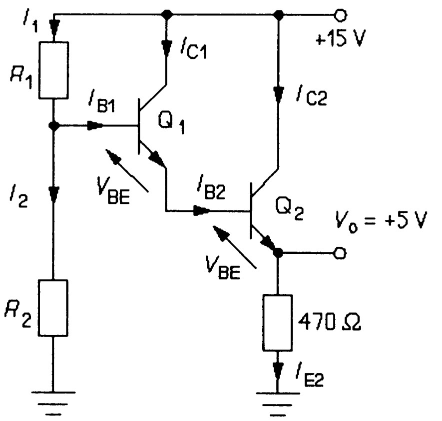


Figure A8.8

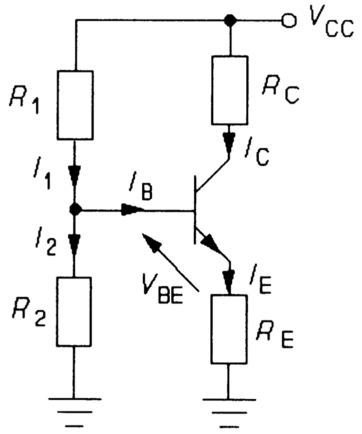


Figure A8.9

**8** In figure A8.8 we use KVL on the input side and find

$$I_1R_1 + I_2R_2 = V_{CC} \tag{A8.1}$$

But $R_1 = R_2$ and $I_1 = I_2 + I_{B1}$. Now $I_{E2}R_{E2} = 470I_{E2} = V_o = 5$, making $I_{E2} = 10.64$ mA. Thus $I_{E1} = I_{B2} = I_{E2}/(\beta + 1) = 10.64/31 = 0.343$ mA. Therefore $I_{B1} = I_{E1}/(\beta + 1) = 0.343/31 = 11.06\ \mu$A.

Looking at the lower part of the circuit we see by KVL that

$$I_2R_2 = 2V_{BE} + 5 = 6.4 \text{ V}$$

Equation A8.1 then becomes

$$I_1R_1 + I_2R_2 = (I_2 + I_{B1})R_1 + I_2R_1 = 2I_2R_2 + I_{B1}R_1 = 15$$

$$\Rightarrow \quad I_{B1}R_1 = 15 - 2 \times 6.4 = 2.2$$

and $R_1 = 2.2/11.06 \times 10^{-6} = 199$ kΩ.

**9** The circuit of figure A8.9, the fixed bias CE amplifier, by KVL on the input side gives

$$I_1R_1 + I_2R_2 = I_1R_1 + (I_1 - I_{BQ})R_2 = V_{CC}$$

$$\Rightarrow \quad I_1 = \frac{V_{CC} + I_{BQ}R_2}{R_1 + R_2} \tag{A9.1}$$

And using KVL on the lower half of the circuit gives

$$I_2R_2 = (I_1 - I_{BQ})R_2 = V_{BEQ} + I_{EQ}R_E$$

$$\Rightarrow \quad I_1 = \frac{V_{BEQ} + I_{EQ}R_E + I_{BQ}R_2}{R_2}$$

Substituting this into equation A9.1 gives

$$\frac{V_{BEQ} + I_{EQ}R_E + I_{BQ}R_2}{R_2} = \frac{V_{CC} + I_{BQ}R_2}{R_1 + R_2}$$

Whence

$$\frac{I_{EQ}R_E}{R_2} + I_{BQ} - \frac{I_{BQ}R_2}{R_1 + R_2} = \frac{V_{CC}}{R_1 + R_2} - \frac{V_{BEQ}}{R_2}$$

Replacing $I_{EQ}$ by $I_{CQ}(\beta + 1)/\beta$ and $I_{BQ}$ by $I_{CQ}/\beta$ gives

$$\frac{I_{CQ}(\beta + 1)R_E}{\beta} + \frac{I_{CQ}}{\beta} - \frac{I_{CQ}R_2}{\beta(R_1 + R_2)} = \frac{V_{CC}}{R_1 + R_2} - \frac{V_{BEQ}}{R_2}$$

which can be rearranged to

$$I_{CQ}\left[\frac{(\beta + 1)R_E}{R_2} + \frac{R_1}{R_1 + R_2}\right] = \beta\left[\frac{V_{CC}}{R_1 + R_2} - \frac{V_{BEQ}}{R_2}\right]$$

Multiplying by $R_2$ and replacing $R_1R_2/(R_1 + R_2)$ by $R_B$ gives

$$I_{CQ}[(\beta + 1)R_E + R_B] = \beta\left[\frac{V_{CC}R_2}{R_1 + R_2} - V_{BEQ}\right] = \beta\,[V_{CC}R_B/R_1 - V_{BEQ}]$$

which finally produces

$$I_{CQ} = \frac{\beta(V_{CC}R_B/R_1 - V_{BEQ})}{R_B + (\beta + 1)R_E} \qquad \text{(A9.2)}$$

Thus the fixed-bias circuit of figure P8.2, where $R_2 = \infty$ and $R_B = R_1$ would yield from the above equation,

$$I_{CQ} = \frac{\beta(V_{CC} - V_{BEQ})}{R_B + (\beta + 1)R_E}$$

Either result gives the *same* variation of $I_{CQ}$ with $\beta$, the difference lies in the relative magnitudes of $R_B$. Taking logs of equation A9.2 gives

$$\ln I_{CQ} = \ln\beta + \ln(V_{CC}R_B/R_1 - V_{BEQ}) - \ln[R_B + (\beta + 1)R_E]$$

Then differentiating with respect to $\beta$ gives

$$\frac{1}{I_{CQ}}\frac{dI_{CQ}}{d\beta} = \frac{1}{\beta} - \frac{R_E}{R_B + (\beta + 1)R_E} = \frac{R_B + R_E}{\beta[R_B + (\beta + 1)R_E]}$$

Since $R_B >> R_E$ this approximates to

$$\frac{\Delta I_{CQ}}{I_{CQ}} = \frac{\Delta\beta}{\beta[1 + (\beta + 1)R_E/R_B]}$$

In the case of the fixed-bias circuit, $R_B >> (\beta + 1)R_E$, while for the voltage-divider bias circuit $R_B << (\beta + 1)R_E$. In the former case the above expression becomes

$$\frac{\Delta I_{CQ}}{I_{CQ}} \approx \frac{\Delta\beta}{\beta}$$

And in the latter case it becomes

$$\frac{\Delta I_{CQ}}{I_{CQ}} \approx \frac{\Delta\beta}{\beta(\beta + 1)R_E/R_B}$$

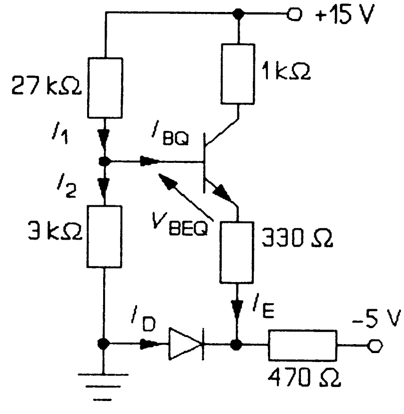


*Figure A8.10*

**10** In the circuit of figure A8.10, by applying KVL to the lower left loop we find

$$I_2R_2 + V_D = V_{BEQ} + I_{EQ}R_E$$

If $V_D = V_{BEQ}$, then $I_2R_2 = I_{EQ}R_E$, and $V_{BEQ}$ drops out.

Using KVL on the left-hand side gives

$$I_1R_1 + I_2R_2 = (I_2 + I_{BQ})R_1 + I_2R_2 = V_{CC}$$

Now $I_2 = I_{EQ}R_E/R_2$, so the above equation becomes

$$(I_{EQ}R_E/R_2 + I_{BQ})R_1 + I_{EQ}R_E = V_{CC}$$

And as $I_{EQ} = (\beta + 1)I_{BQ}$, this in turn becomes

$$I_{BQ}\left[\frac{(\beta + 1)R_E + R_2}{R_2}\right]R_1 + I_{BQ}(\beta + 1)R_E = I_{BQ}\left[\frac{(\beta + 1)(R_1 + R_2)R_E + R_1R_2}{R_2}\right] = V_{CC}$$

Rearranging this gives

$$I_{BQ} = \frac{R_2V_{CC}/(R_1 + R_2)}{(\beta + 1)R_E + R_B} = \frac{V_{CC}R_B/R_1}{(\beta + 1)R_E + R_B}$$

where $R_B = R_1 \parallel R_2$. Again, this is independent of $V_{BEQ}$, as it should be.

Substituting $V_{CC} = 15$ V, $R_1 = 27$ kΩ, $R_2 = 3$ Ω, $R_B = 2.7$ kΩ, $R_E = 330$ Ω and $\beta = 100$ gives $I_{BQ} = 41.63$ μA, so that $I_{CQ} = \beta I_{BQ} = 100 \times 0.04163 = 4.163$ mA.

$I_D$ can be found by applying KVL to the bottom branch:

$$-V_D - (I_{EQ} + I_D)R_D = -V_{EE}$$

$$\Rightarrow \quad I_D = \frac{-V_D + V_{EE}}{R_D} - I_{EQ} = \frac{-0.7 + 5}{0.47} - (101 \times 0.04163) = 4.944 \text{ mA}$$

Ideally we require $I_D = I_E$ for the same voltage drop in the diode and base-emitter junction which means that

$$2I_{EQ} = \frac{V_{EE} - V_D}{R_D} \Rightarrow R_D = \frac{V_{EE} - V_D}{2(\beta + 1)I_{BQ}} = \frac{4.3}{2 \times 101 \times 0.04163} = 0.511 \text{ k}\Omega$$

In practice, the change would have negligible effect.

**11** In the circuit of figure A8.11, KVL on the right-hand side gives

$$I_{CQ}R_C + V_{CEQ} + I_{EQ}R_E = V_{CC}$$

$$\Rightarrow \quad \beta I_{BQ}R_C + V_{CEQ} + (\beta + 1)I_{BQ}R_E = V_{CC}$$

$$\Rightarrow \quad I_{BQ} = \frac{V_{CC} - V_{CEQ}}{\beta R_C + (\beta + 1)R_E} = \frac{24 - 12}{120 \times 2000 + 121 \times 470} = 40.4\ \mu A$$

And also by KVL

$$I_{CQ}R_C + I_{BQ}R_B + V_{BEQ} + I_{EQ}R_E = V_{CC} \qquad \text{(A11.1)}$$

$$\Rightarrow \quad R_B = \frac{V_{CC} - V_{BEQ}}{I_{BQ}} - \beta R_C - (\beta + 1)R_E$$

Substituting for $I_{BQ}$ and the other values gives $R_B = 280\ k\Omega$.

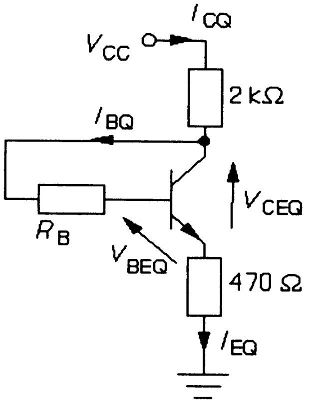


*Figure A8.11*

Going back to equation A11.1, which is

$$I_{CQ}R_C + I_{BQ}R_E + V_{BEQ} + I_{EQ}R_E = V_{CC}$$

$$\Rightarrow \quad I_{CQ}R_C + I_{CQ}R_B/\beta + (\beta + 1)I_{CQ}R_E/\beta = V_{CC} - V_{BEQ} \qquad \text{(A11.2)}$$

$$\Rightarrow \quad I_{CQ} = \frac{\beta(V_{CEQ} - V_{BEQ})}{\beta R_C + R_B + (\beta + 1)R_E}$$

Taking logarithms and differentiating with respect to $\beta$ gives

$$\ln I_{CQ} = \ln\beta + \ln(V_{CC} - V_{BEQ}) - \ln[R_B + \beta R_C + (\beta + 1)R_E]$$

$$\frac{1}{I_{CQ}}\frac{dI_{CQ}}{d\beta} = \frac{1}{\beta} - \frac{R_C + R_E}{R_B + \beta R_C + (\beta + 1)R_E} = \frac{R_B + R_E}{\beta[R_B + \beta R_C + (\beta + 1)R_E]}$$

$$\Rightarrow \quad \frac{\Delta I_{CQ}}{I_{CQ}} = \frac{\Delta\beta}{\beta}\frac{R_B + R_E}{R_B + \beta R_C + (\beta + 1)R_E} = \frac{\Delta\beta}{\beta}\frac{280.47}{576.87} = 0.486\frac{\Delta\beta}{\beta}$$

Then a 10% increase in $\beta$ causes a 4.9% increase in $I_{CQ}$. Substituting $\beta = 120$ and then $\beta = 132$ into equation A11.2 will show that $I_{CQ}$ increases by 4.6%.

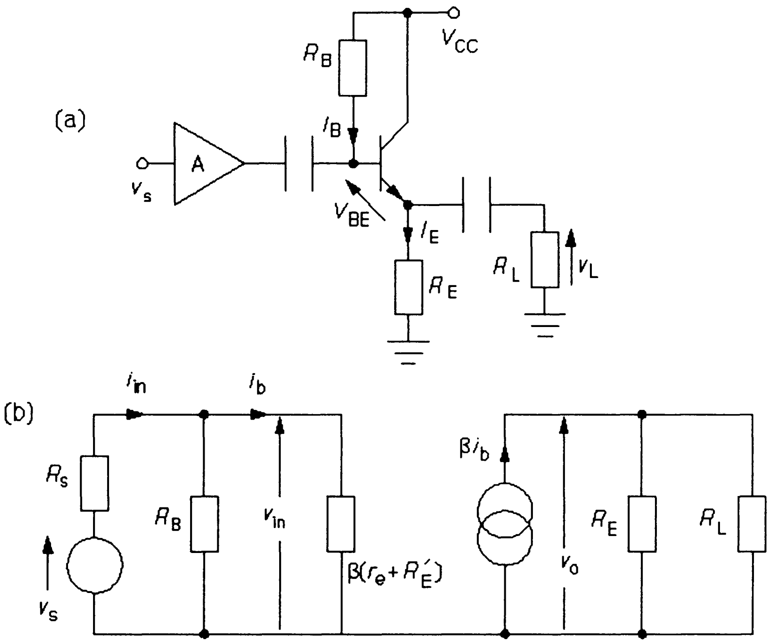


*Figure A8.12*

**12** We use the equivalent circuit of figure A8.12b, and since the load resistance is small we need $r_e$, which is found from $r_e = 26/I_C$ (assuming the base-emitter junction temperature is 300 K), with $r_e$ in Ω and $I_C$ in mA. Looking at figure A8.12a, KVL gives

$$V_{CC} = I_B R_B + V_{BE} + I_E R_E = I_B R_B + V_{BE} + (\beta + 1)I_B R_E$$

$$\Rightarrow \quad I_B = \frac{V_{CC} - V_{BE}}{R_B + (\beta + 1)R_E} = \frac{30 - 0.7}{350 + 151 \times 2.2} = 0.043 \text{ mA}$$

Thus $I_C = \beta I_B = 150 \times 0.043 = 6.45$ mA and in figure A8.12b, $r_e = 26/6.45 = 4.0$ Ω. Now $R_E' = R_E \parallel R_L = 2200 \parallel 20 = 19.8$ Ω, so that $r_e + R_E' = 4 + 19.8 = 23.8$ Ω and $\beta(r_e + R_E') = 150 \times 23.8 = 3.57$ kΩ.

The amplifier's output resistance, $R_s$ = 2 kΩ, so the input voltage to the transistor's base, $v_{in}$, is

$$v_{in} = v_s \times \frac{R_{in}}{R_{in} + R_s} = \frac{3.53 \times 100v_s}{3.53 + 2} = 63.8v_s$$

where $R_{in} = R_B \parallel \beta(r_e + R_E') = 350\ k\Omega \parallel 3.57\ k\Omega = 3.53\ k\Omega$ and the no-load voltage gain of the amplifier is 100.

Thus $i_b = v_{in}/\beta(r_e + R_E') = 63.8v_s/3570 = 0.0179v_s$ and the voltage across the load is

$$v_L = \beta i_b R_E' = 150 \times 0.0179v_s \times 19.8 = 53v_s$$

Making the simplifying approximations that $R_E' = R_L = 20\ \Omega$ and that $R_B$ can be neglected leads to $r_e + R_L = 24\ \Omega$, $\beta(r_e + R_L) = 3.6\ k\Omega$. Then $v_{in} = 3.57 \times 100v_s/5.57 = 64v_s$, while $i_b = v_{in}/\beta(r_e + R_L) = 64v_s/3500 = 0.0178v_s$. Thus $v_L/v_s = \beta i_b R_L/v_s = 150 \times 0.0178 \times 20 = 53$, which is the same as the 'exact' solution.

Making the customary assumption that $r_e$ can be neglected leads to $\beta R_L = 3\ k\Omega$ and $v_{in} = 60v_s$ and $i_b = 60v_s/3000 = 0.02v_s$, so that $v_L/v_s = 150 \times 0.02 \times 20 = 60$. In all probability this is sufficiently accurate for any practical purpose.

Finding the current gain requires us to find $i_L$, that is $v_L/R_L = 53v_s/R_L = 2.65v_s$. And $i_s = v_s/R_A = v_s/1000$, as $R_A$, the input resistance of the amplifier, is 1 kΩ. Hence

$$\frac{i_L}{i_s} = \frac{2.65v_s}{v_s/1000} = 2650$$

# Chapter 9

1 To find $R_S$ and $R_D$ we first use KVL on the output side of figure A9.1 to obtain

$$I_{DQ}(R_D + R_S) + V_{DSQ} = V_{DD}$$

$$\Rightarrow \quad R_D + R_S = \frac{V_{DD} - V_{DSQ}}{I_{DQ}} = \frac{15 - 6}{3} = 3\ \mathrm{k\Omega}$$

And $R_S$ can be found from $V_{GSQ}$ using KVL on the lower half of the circuit of figure A9.1

$$V_{GSQ} + I_{DQ}R_S = 0 \quad \Rightarrow \quad R_S = -V_{GSQ}/I_D$$

Finally $V_{GSQ}$ is found from

$$I_{DQ} = I_{DSS}(1 - |V_{GSQ}/V_P|)^2 \quad \Rightarrow \quad V_{GSQ} = V_P\left(1 - \sqrt{1 - I_{DQ}/I_{DSS}}\right)$$

Substituting $V_P = -4$ V, $I_{DSS} = 10$ mA and $I_{DQ} = 3$ mA leads to $V_{GSQ} = -1.809$, then

$$R_S = -V_{GSQ}/I_{DQ} = 1.809/3 \times 10^{-3} = 603\ \Omega$$

making $R_D = 3 - 0.603 = 2.597\ \mathrm{k\Omega}$.

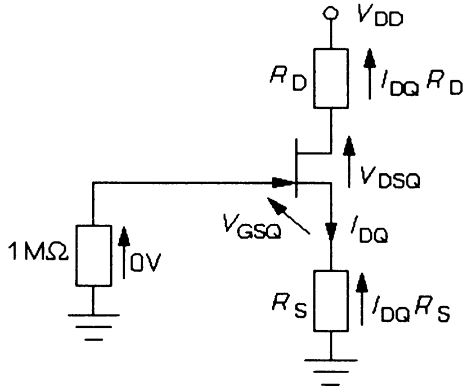


*Figure A9.1*

2 Again we use the fact that $V_{GSQ} = -I_{DQ}R_S = -0.56I_{DQ}$, working in V, mA and kΩ. Then

$$I_{DSQ} = I_{DSS}(1 - |V_{GSQ}/V_P|)^2 = 10(1 - |0.56I_{DQ}/-4|)^2 = 10(1 - 0.14I_{DQ})^2$$

We could solve the quadratic equation, but it is easier to use the Newton-Raphson approximation

$$\delta I_{DQ} = -\epsilon/f'$$

where $\delta I_{DQ}$ is the correction to be added to the value of $I_{DQ}$, $f' = df/dI_{DQ}$ and

$$f = I_{DQ} - 10(1 - 0.14I_{DQ})^2$$

while $\epsilon$ is the error. In this case we guess first that $I_{DQ} = 3$ mA and substitute into $f$. Differentiating $f$ gives

$$f = I_{DQ} - 10(1 - 0.14I_{DQ})^2$$

$$\Rightarrow \ f' = 1 + 2.8(1 - 0.14I_{DQ})$$

Substituting $I_{DQ} = 3$ mA into $f$ gives

$$\epsilon = f(3) = 3 - 10(1 - 0.14 \times 3)^2 = -0.364$$

And substituting the same into $f'$ gives

$$f'(3) = 1 + 2.8(1 - 0.14 \times 3)^2 = 2.624$$

Thus

$$\delta I_{DQ} = -\epsilon/f' = -(-0.364)/2.624 = +0.14 \text{ mA}$$

Then $V_{GSQ} = -I_{DQ}R_S = -3.14 \times 0.56 = -1.76$ V and

$$V_{DSQ} = V_{DD} - I_{DQ}(R_D + R_S) = 15 - 3.14 \times 2.76 = 6.33 \text{ V}$$

3 As before $I_{DQ}R_S = -V_{GSQ}$ and

$$I_{DQ} = I_{DSS}\left(1 - \left|\frac{V_{GSQ}}{V_P}\right|\right)^2 = I_{DSS}\left(1 - \left|\frac{I_{DQ}R_S}{V_P}\right|\right)^2 = 8(1 - 0.2I_{DQ})^2$$

Then taking a first guess at $I_{DQ} = 3$ mA, we find

$$\epsilon_1 = 3 - 8(1 - 0.2 \times 3)^2 = 1.72$$

And

$$f' = 1 - 2 \times 8(1 - 0.2I_{DQ})(-0.2) = 1 + 3.2(1 - 0.2I_{DQ})$$

$$\Rightarrow \ f'(3) = 1 + 3.2(1 - 0.2 \times 3) = 2.28$$

Then

$$\delta I_{DQ} = -\epsilon_1/f'(3) = -1.72/2.28 = 0.75$$

And our corrected value of $I_{DQ} = 3 - 0.74 = 2.26$ mA. We must go round again for another correction:

$$\epsilon_2 = 2.25 - 8(1 - 0.2 \times 2.25)^2 = -0.17$$

While

$$f'(2.25) = 1 + 3.2(1 - 0.2 \times 2.25) = 2.76$$

and so

$$\delta I_{DQ} = -\epsilon_2/f'(2.25) = 0.17/2.76 = 0.06$$

making $I_{DQ} = 2.31$ mA. Solving the quadratic gives the answer more quickly.
From this, $I_{DQ}(R_D + R_S) = 2.31 \times 2.76 = 6.38$ V and $V_{DSQ} = 15 - 6.38 = 8.62$ V.

**4** Since $I_{DQ}$ is given we find $V_{GSQ}$ from

$$I_{DQ} = -3 = I_{DSS}\left(1 - \left|\frac{V_{GSQ}}{V_P}\right|\right)^2 = -6(1 - |V_{GSQ}/3|)^2$$

$$\Rightarrow \quad 1 - |V_{GSQ}/3| = \sqrt{3/6} = 0.707$$

$$\Rightarrow \quad V_{GSQ} = 3 \times 0.293 = 0.879 \text{ V}$$

Since there is no current flowing into the gate, the gate bias resistor network is a pure voltage divider and then

$$V_{1M\Omega} = \frac{1}{1+8} \times -12 = -1.333 \text{ V}$$

In figure A9.4, KVL on the lower loop gives

$$V_{1M\Omega} = V_{GSQ} + V_{RS} = V_{GSQ} + I_{DQ}R_S = 0.879 + 3R_S = -1.333 \text{ V}$$

$$\therefore \quad R_S = (-1.333 - 0.879)/3 = 0.737 \text{ k}\Omega$$

Then as

$$V_{DD} = I_{DQ}(R_D + R_S) + V_{DSQ}$$

$$\Rightarrow \quad V_{DSQ} = V_{DD} - I_{DQ}(R_D + R_S) = -12 - (-3)(R_D + R_S) = -5 \text{ V}$$

$$\Rightarrow \quad R_D + R_S = 7/3 = 2.333 \text{ k}\Omega$$

$$\Rightarrow \quad R_D = 2.333 - R_S = 2.333 - 0.737 = 1.596 \text{ k}\Omega$$

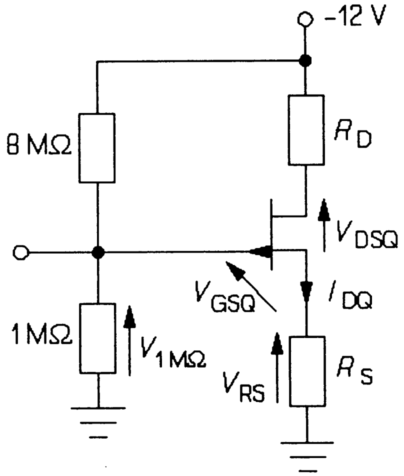


*Figure A9.4*

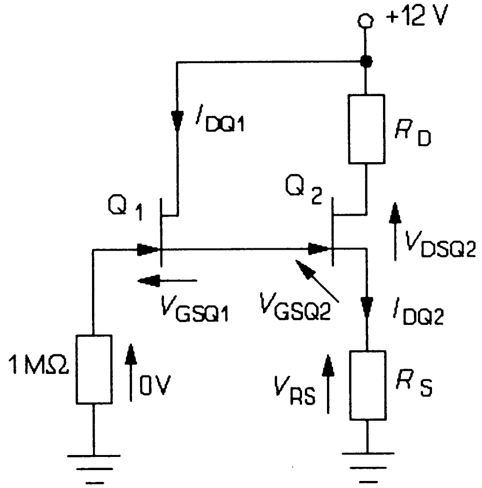


*Figure A9.5*

**5** The first thing to note is that the source of Q1 is connected to the gate of Q2, which draws no current making $I_{DQ1} = 0$, which means that $V_{GSQ1} = V_P = -2$ V. Using KVL on the lower loop of figure A9.5 yields

$$0 = V_{GSQ1} + V_{GSQ2} + I_{DQ2}R_S$$

From this we see that

$$V_{GSQ2} = -V_{GSQ1} - I_{DQ2}R_S = 2 - I_{DQ2}$$

as $R_S = R_D = 1$ kΩ. Therefore

$$I_{DQ2} = I_{DSS}\left(1 - \left|\frac{V_{GSQ2}}{V_P}\right|\right)^2 = 8\left(1 - \left|\frac{2 - I_{DQ2}}{2}\right|\right)^2$$

It is best to guess $I_{DQ2} = 3$ mA and then letting

$$f = I_{DSQ2} - 8(1 - |1 - 0.5I_{DSQ2}|)^2$$

so that $f(3) = 1$ and as

$$f' = \mathrm{d}f/\mathrm{d}I_{DQ2} = 1 - 2 \times 8(1 - |1 - 0.5I_{DQ2}|)(-0.5) = 1 + 8(1 - |1 - 0.5I_{DQ2}|)$$

we find $f'(3) = 5$ and the correction to $I_D$ is

$$\delta I_{DQ2} = -f(3)/f'(3) = -1/5 = -0.2$$

making the corrected value for $I_{DQ2} = 3 - 0.2 = 2.8$ mA. Going round again gives $f(2.8) = -0.08, f'(2.8) = 5.8$, and $\delta I_{DQ2} = +0.08/5.8 = 0.014$ mA, giving $I_{DQ2} = 2.814$ mA.

Now $V_{DD} = I_{DQ2}(R_D + R_S) + V_{DSQ2}$, from which

$$V_{DSQ2} = V_{DD} - I_{DQ2}(R_D + R_S) = 12 - 2.814 \times 2 = 6.37 \text{ V}$$

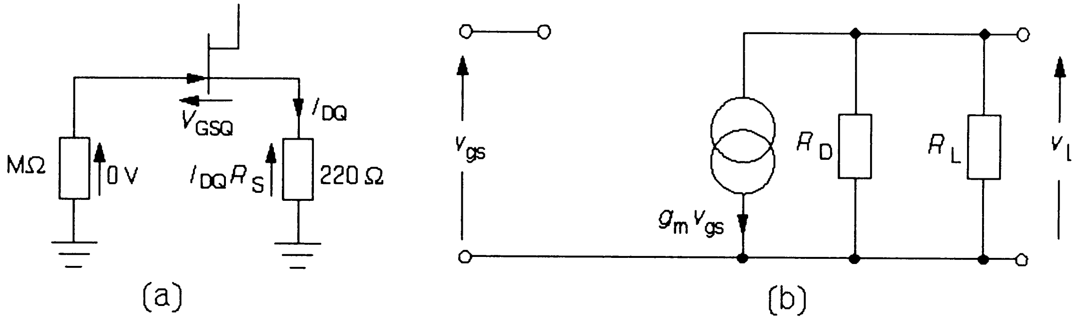


*Figure A9.6*

6 With the gate grounded in figure A9.6a, we have $V_{GSQ} = -I_{DQ}R_S = 0.22I_{DQ}$ and then

$$I_{DQ} = I_{DSS}(1 - |V_{GSQ}/V_P|)^2 = 7(1 - |0.22I_{DQ}/3.3|)^2$$

This produces the quadratic equation

$$I_{DQ}^2 - 62.14I_{DQ} + 225 = 0 \Rightarrow I_{DQ} = \frac{62.14 \pm \sqrt{62.14^2 - 4 \times 225}}{2} = 3.86 \text{ mA}$$

The other solution is discarded as physically impossible. Then $V_{GSQ} = -I_{DQ}R_S = 0.22 \times 3.86 = 0.85$ V and

$$g_m = \frac{2\sqrt{I_{DSS}I_{DQ}}}{|V_P|} = \frac{2 \times 5.2}{3.3} = 3.15 \text{ mS}$$

The equivalent circuit is as shown in figure A9.6b which shows that

$$R_o = R_D \parallel R_L = (1/2.2 + 1/3.3)^{-1} = 1.32 \text{ k}\Omega$$

and so

$$v_L = g_m v_{gs} R_o = 3.15 \times 10^{-3} \times 0.1 \times 1.32 \times 10^3 = 0.416 \text{ V}$$

as $v_{gs} = v_g = 0.1$ V.

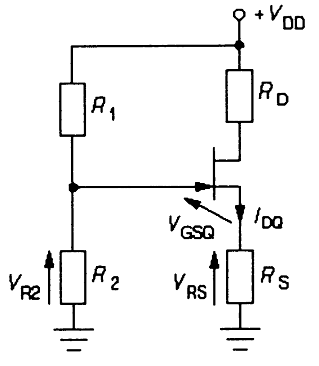


*Figure A9.7*

7 From figure A9.7 the gate bias voltage is

$$V_{R2} = \frac{R_2}{R_1 + R_2} \times V_{DD} = \frac{1 \times 24}{6 + 1} = 3.43 \text{ V}$$

Using KVL on the lower half of the circuit gives

$$V_{R2} = V_{GSQ} + V_{RS} \Rightarrow V_{GSQ} = V_{R2} - V_{RS} = 3.43 - 1.8I_{DQ}$$

Then $$I_{DQ} = I_{DSS}(1 - |V_{GSQ}/V_P|)^2 = 8\left(1 - \left|\frac{3.43 - 1.8I_{DQ}}{5}\right|\right)^2$$

$$= 8(1 - |0.686 - 3.6I_{DQ}|)^2$$

Let $$f = I_{DQ} - 8(1 - |0.686 - 0.36I_{DQ}|)^2$$

Then $$f' = 1 + 5.76(1 - |0.686 - 0.36I_{DQ}|)$$

Guessing $I_{DQ} = 3$ mA leads to

$$f(3) = 3 - 8(1 - 0.394)^2 = 0.0621$$

and so $$f'(3) = 1 + 5.76 \times 0.606 = 4.49$$

Thence $$\delta I_{DQ} = -f(3)/f'(3) = -0.0621/4.49 = -0.014 \text{ mA}$$

and the corrected value of $I_{DQ} = 2.986$ mA.

The transconductance is found from

$$g_m = \frac{2\sqrt{I_{DQ}I_{DSS}}}{|V_P|} = \frac{2\sqrt{2.986 \times 8}}{5} = 1.955 \text{ mS}$$

And the output voltage is $v_L = -g_m v_{gs} R_o$ as shown by figure A9.6b, where $R_o = R_D \parallel R_L = (1/1.8 + 1/5)^{-1} = 1.324$ kΩ. Thus as $v_{gs} = v_g$, $v_L / v_g = -g_m R_o = -1.955 \times 1.324 = -2.588$.

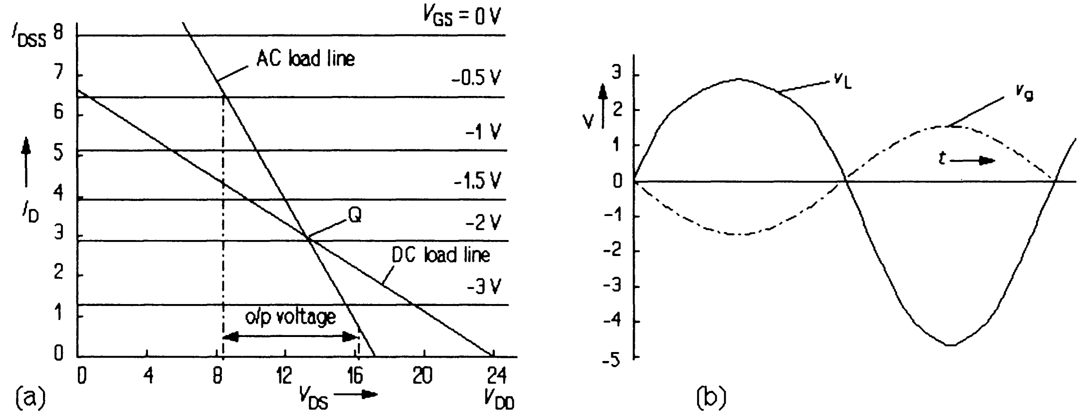


*Figure A9.8*

8 Figure A9.8a shows the load lines and the drain characteristics. The Q-point is at $I_{DQ} = 2.986$ mA, and $V_{DSQ} = V_{DD} - I_{DQ}(R_D + R_S) = 24 - 2.986 \times 3.6 = 13.25$ V. The slope of the DC load line is

$$\frac{\Delta I_D}{\Delta V_{DS}} = \frac{-1}{R_D + R_S} = \frac{-1}{3.6} = -0.278 \text{ mS}$$

The AC load line has a gradient of $-1/R_o$ and passes through the Q-point. Its intercept on the $V_{DS}$ axis is

$$V_{DSQ} - (I_{DQ} \div \text{AC slope}) = V_{DSQ} + I_{DQ}R_o$$

$$= 13.25 + 2.986 \times 1.324 = 17.2 \text{ V}$$

The drain characteristics are found from

$$I_D = I_{DSS}(1 - |V_{GS}/V_P|)^2 = 8(1 - |0.2V_{GS}|)^2$$

and are marked for $V_{GS} = 0, -0.5, -1$ etc. The quiescent gate-source voltage from the diagram is just over −2 V (calculated value −1.944 V).

The no-load output voltage without a bypass capacitor will be along the DC load line and will reach a minimum of about 0.1 V and a maximum of about 21.2 V.

The load voltage can only take on values corresponding to intercepts of $V_{GS}$ values with the AC load line. Thus when $v_g = +1.5$ V, $V_{GS} = V_{GSQ} + 1.5 = -2 + 1.5 = -0.5$ V, approximately, and the corresponding value of $V_{DS} = 8.6$ V, approximately. And when $v_g = -1.5$ V, $V_{GS} = -2 - 1.5 = -3.5$ V, approximately, with a corresponding value of $V_{DS} = 16.2$ V, approximately, for a load voltage swing of $16.2 - 8.6 = 7.6$ V. The load-voltage waveform can be constructed from the intercepts on the load line and then $v_L = V_{DS} - V_{DSQ}$, as in the table below:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $v_g$ | 1.5 | 1.0 | 0.5 | 0 | −0.5 | −1.0 | −1.5 |
| $V_{GS}$ | −0.5 | −1.0 | −1.5 | −2.0 | −2.5 | −3.0 | −3.5 |
| $V_{DS}$ | 8.6 | 10.3 | 11.9 | 13.4 | 14.6 | 15.5 | 16.2 |
| $v_L$ | −4.7 | −3.0 | −1.4 | +0.1 | +1.3 | +2.2 | +2.9 |
| $\omega t$ | 90° | 42° | 19° | 0° | −19° | −42° | −90° |

Figure A9.8b shows the load-voltage waveform.

The ratio $v_L/v_g$ is $-4.7/1.5 = -3.1$ when $v_g$ is maximum and $-2.9/1.5 = -1.9$ when $v_g$ is minimum, but taking the peak-to-peak voltage the ratio is $(-4.7 - 2.9)/3 = -2.5$ (the calculated result is $-2.588$), as suggested by problem 9.7.

9 Figure A9.9a shows the load lines and drain characteristics, which are the same as figure A9.8a, but now the peak gate voltages are +2 V and −2 V. When $v_g$ exceeds 1.944 V, the drain current reaches $I_{DSS}$, 8 mA and further increases in $v_g$ have no effect.

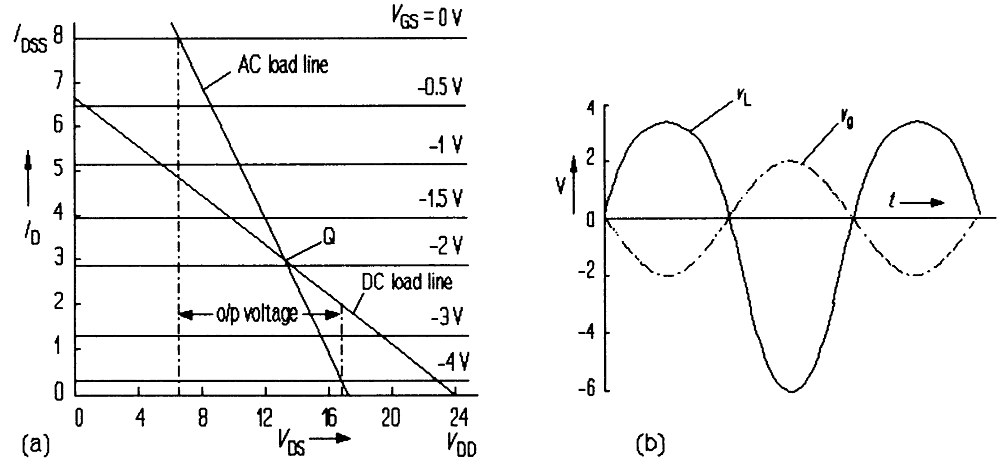


*Figure A9.9*

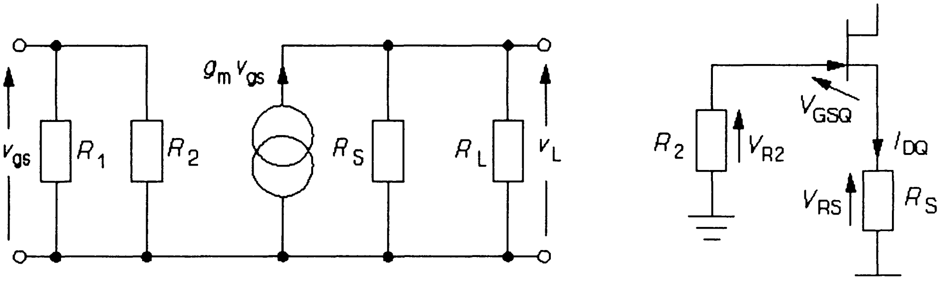
The table has the values below:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $v_g$ | +2.0 | +1.5 | +1.0 | +0.5 | 0 | −0.5 | −1.0 | −1.5 | −2.0 |
| $V_{GS}$ | 0.0 | −0.5 | −1.0 | −1.5 | −2.0 | −2.5 | −3.0 | −3.5 | −4.0 |
| $V_{DS}$ | 7.2 | 8.6 | 10.3 | 11.9 | 13.4 | 14.6 | 15.5 | 16.2 | 16.7 |
| $v_L$ | −6.1 | −4.7 | −3.0 | −1.4 | +0.1 | +1.3 | +2.2 | +2.9 | +3.4 |
| $\omega t$ | 90° | 49° | 30° | 14° | 0° | −14° | −30° | −49° | −90° |

Figure A9.9b shows the load-voltage waveform.

When the FET is not loaded and $R_S$ is not bypassed the output voltage is constrained to the DC load line and $V_{DS}(\text{min}) = 0$ V (assuming $R_{DS}(\text{on}) = 0$), while $V_{DS}(\text{max})$ is 22.7 V.

$v_L/v_g = -6.1/2 = -3.05$ when $v_g$ is maximum, $= -3.4/2 = -1.7$ when $v_g$ is minimum and the peak-to-peak ratio is $(-6.1 - 3.4)/4 = -9.5/4 = -2.4$, that is its magnitude has declined from the value of 2.59 it had previously, because of the restriction in $V_{GS}$.



(a) (b)

*Figure A9.10*

**10** Figure A9.10a shows the equivalent circuit. The load voltage is $g_m v_{gs} R_o$ so that the voltage gain is

$$\frac{v_L}{v_g} = \frac{g_m v_{gs} R_o}{v_g} = \frac{g_m(v_g - v_L)R_o}{v_g}$$

$$\Rightarrow \quad (v_L/v_g)(1 + g_m R_o) = g_m R_o$$

$$\Rightarrow \quad A_{vL} = v_L/v_g = \frac{g_m R_o}{1 + g_m R_o}$$

The usual FET drain-current equation applies

$$I_{DQ} = I_{DSS}\left(1 - \left|\frac{V_{GSQ}}{V_P}\right|\right)^2 \quad \Rightarrow \quad 1 - \left|\frac{V_{GSQ}}{V_P}\right| = \sqrt{\frac{I_{DQ}}{I_{DSS}}} = \sqrt{\frac{1}{3}} = 0.577$$

Thus $$|V_{GSQ}/V_P| = 1 - 0.5773 = 0.4227$$

$$\Rightarrow \quad V_{GSQ} = -3 \times 0.4227 = -1.268 \text{ V}$$

But figure A9.10b shows that $V_{GSQ} + V_{RS} = V_{R2}$ where $V_{RS} = I_{DQ}R_S = 4$ V, so that $V_{R2} = -1.268 + 4 = 2.732$ V. But the voltage across $R_2$ is

$$V_{R2} = 2.732 = \frac{R_2 V_{DD}}{R_1 + R_2} = \frac{9}{R_1 + 1}$$

whence $R_1 = 2.294$ MΩ.

Now the transconductance is given by

$$g_m = \frac{2}{|V_P|}\sqrt{I_{DQ} I_{DSS}} = \frac{2}{3}\sqrt{4 \times 12} = 4.62 \text{ mS}$$

While $R_o = R_S \parallel R_L = 1$ kΩ $\parallel$ 5 Ω = 0.833 kΩ, so that $g_m R_o = 3.85$ and the voltage gain is

$$A_{vL} = \frac{3.85}{1 + 3.85} = 0.794$$

The input resistance, from figure A9.10a, is $R_{in} = R_1 \parallel R_2 = 1$ MΩ $\parallel$ 2.294 MΩ = 0.696 MΩ.

**11** As there is no source resistor, the gate-source voltage, $V_{GS}$, is zero and so then the drain current, $I_D = I_{DSS} = 12$ mA. Finally $V_{DS} + I_D R_D = V_{DD}$, so that

$$V_{DS} = V_{DD} - I_D R_D = 15 - 12 \times 0.68 = 6.84 \text{ V}$$

**12** Equation 9.1 is

$$I_D = I_{DSS}\left(1 - \left|\frac{V_{GS}}{V_P}\right|\right)^2 \Rightarrow 1 - \left|\frac{V_{GS}}{V_P}\right| = \sqrt{\frac{I_D}{I_{DSS}}}$$

Equation 9.3 is

$$g_m = \frac{2I_{DSS}}{|V_P|}\left(1 - \left|\frac{V_{GS}}{V_P}\right|\right)$$

so that substitution for $1 - |V_{GS}/V_P|$ gives

$$g_m = \frac{2I_{DSS}}{|V_P|}\sqrt{\frac{I_D}{I_{DSS}}} = \frac{2\sqrt{I_{DSS} I_D}}{|V_P|}$$

which is equation 9.4.

# Chapter 10

1 In figure A10.1 we can see that

$$V_{BE1} = V_{BE2} + I_2R$$

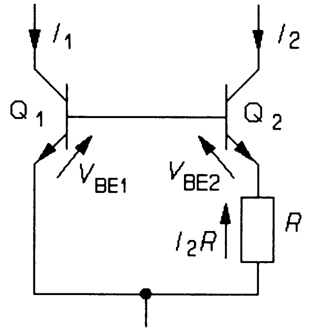


*Figure A10.1*

We are told to use equation 7.1, which is

$$I = I_s[\exp(qV/kT) - 1]$$

where $q = 1.6 \times 10^{-19}$ C, $k = 1.38 \times 10^{-23}$ J/K, $T = 300$ K and for Q1, $V = V_{BE1} =$ 0.7 V, so that $qV/kT = 27.05$ and therefore

$$I_1 = 3 \times 10^{-3} = I_s \exp(27.05) = 5.6 \times 10^{11} I_s$$

$$\Rightarrow \quad I_s = \frac{3 \times 10^{-3}}{5.6 \times 10^{11}} = 5.36 \times 10^{-15} \text{ A}$$

Since $I_2 = 5$ $\mu$A we have

$$I_2 = I_s \exp(38.65 V_{BE2})$$

$$\Rightarrow \quad V_{BE2} = \frac{\ln(I_2/I_s)}{38.65} = 0.0259 \ln\left[\frac{5 \times 10^{-6}}{5.36 \times 10^{-15}}\right] = 0.535 \text{ V}$$

Therefore as $I_2R = V_{BE1} - V_{BE2} = 0.7 - 0.535 = 0.165$ V, $R = 0.165/I_2 = 33$ kΩ.

Assuming that $V_{BE1}$ is 0.7 V when the temperature is increased to 350 K (it will actually by less and so will the increase in $I_2$, but this will serve as a worst case analysis) means that we can take $qV/kT$ to be 27.05 × 300/350 = 23.2 and then

$$I_1 = I_s \exp(23.2) = 3 \times 10^{-3} \Rightarrow I_s = 2.52 \times 10^{-13} \text{ A}$$

This value can be substituted into the equation for $I_2$:

$$I_2 = I_s\exp(33.13V_{BE2}) = 2.52 \times 10^{-13}\exp(33.13V_{BE2})$$

$$\Rightarrow \quad V_{BE2} = \frac{\ln I_2 - \ln(2.52 \times 10^{-13})}{33.13} = 0.0302\ln I_2 + 0.876 \text{ V}$$

But since $I_2R + V_{BE2} = V_{BE1} = 0.7$, we can equate the expressions for $V_{BE2}$,

$$V_{BE2} = 0.7 - I_2R = 0.7 - 33000I_2 = 0.0302\ln I_2 + 0.876$$

which leads to

$$f(I_2) = 33000I_2 + 0.0302\ln I_2 + 0.176 = 0$$

This is best solved by Newton-Raphson, with an initial guess that $I_2 = 5$ μA, making

$$f(5 \times 10^{-6}) = 33000 \times 5 \times 10^{-6} + 0.0302\ln(5 \times 10^{-6}) + 0.176$$

$$= 0.165 - 0.369 + 0.176 = -0.028$$

Then differentiating $f(I_2)$ gives

$$f'(I_2) = 33000 + 0.0302/I_2 \quad \Rightarrow \quad f'(5 \times 10^{-6}) = 39040$$

The correction to be added to $I_2$ is

$$\Delta I_2 = \frac{-f(I_2)}{f'(I_2)} = \frac{-(-0.028)}{39040} = 7 \times 10^{-7} \text{ A}$$

making $I_2 = 5.7$ μA. In practice the increase in temperature would affect $V_{BE1}$ and $V_{BE2}$ similarly and the change in $I_2$ would be less than this.

2 When the area of a single die or chip is 35 mm$^2$, the yield is

$$y = 100\exp(-35/50) = 49.7\%$$

The area of the wafer is $\pi r^2 = \pi(75)^2 = 17670$ mm$^2$, so that the maximum number of chips is 17670/35 ≈ 500 and the number of good chips is 500 × 0.496 = 248. Thus the cost per chip is £500/248 = £2.02, though in practice the number of chips will be rather less and the cost per chip rather more.

If the cost per good chip is 10p, then the number of good chips, $n$, is 500/0.1 = 5000. If the wafer area is $A_W$, the total number, $N$, of chips, good and bad is

$$N = A_W/A = \pi r^2/A$$

And as $n = yN$, where $y$ is the yield, we have

$$n = yN = \exp(-A/D) \times \frac{\pi r^2}{A}$$

$$\Rightarrow \quad 5000 = \frac{\pi 75^2}{A} \times \exp(-A/50)$$

We can solve this with Newton-Raphson's method; let

$$f(A) = 5000 - \frac{\pi 75^2}{A}\exp\left[\frac{-A}{50}\right]$$

and $$\frac{\mathrm{d}f(A)}{\mathrm{d}A} = f'(A) = \frac{\pi 75^2}{A^2}\exp\left[\frac{-A}{50}\right] + \frac{\pi 75^2}{50A}\exp\left[\frac{-A}{50}\right] = \frac{\pi 75^2}{A^2}\left[1 + \frac{A}{50}\right]\exp\left[\frac{-A}{50}\right]$$

Then as $A$ is small, $\exp(-A/50)$ is $\approx 1$ and $\pi 75^2/A \approx 5000$, making $A \approx 3.5$ mm$^2$. Substituting in $f(A)$ and $f'(A)$ gives

$$f(3.5) = 5000 - \frac{\pi 75^2}{3.5}\exp\left[\frac{-3.5}{50}\right] = 292$$

$$f'(3.5) = \frac{\pi 75^2}{3.5^2}\left[1 + \frac{3.5}{50}\right]\exp\left[\frac{-3.5}{50}\right] = 1439$$

The area correction is now

$$\Delta A = \frac{-f(3.5)}{f'(3.5)} = -\frac{292}{1439} = -0.2$$

making $A = 3.3$ mm$^2$.

With one good chip/wafer

$$n = 1 = Ny = \frac{\pi 75^2}{A}\exp\left[\frac{-A}{50}\right]$$

As before

$$f(A) = 1 - \frac{\pi 75^2}{A}\exp\left[\frac{-A}{50}\right]$$

$$f'(A) = \frac{\pi 75^2}{A^2}\left[1 + \frac{A}{50}\right]\exp\left[\frac{-A}{50}\right]$$

This time $A$ is large and $\exp(-A/50)$ is small. As a guess we can try $A = 200$ which gives

$$f(200) = -0.62 \quad \text{and} \quad f'(200) = 0.0405$$

The correction to $A$ is $+0.62/0.0405 = +15$; then with $A = 215$ the new correction is

$$\Delta A = \frac{-f(215)}{f'(215)} = \frac{0.115}{0.0275} = 4.2$$

making $A = 219\ \text{mm}^2$. There is little point in further accuracy and in practice $200\ \text{mm}^2$ is likely to be as close as necessary.

When $A = 400\ \text{mm}^2$ and $n = 1$ we have

$$n = 1 = Ny = \frac{\pi 75^2}{400}\exp\left[\frac{-400}{D}\right]$$

$$\Rightarrow \quad \frac{-400}{D} = \ln\left[\frac{400}{\pi 75^2}\right] \quad \Rightarrow \quad D = 105.6\ \text{mm}^2$$

3 Rearranging the formula $R = sl/w$ into $s = Rw/l = 100 \times 8 \times 10^{-6}/100 \times 10^{-3} = 8\ \text{m}\Omega/\square$.

Assuming the laser cut to be infinitely thin, the conductor is split into two and a cut of length $y$ results in additional conductor of width 10 $\mu$m and length $2y$. If $y$ is in $\mu$m, the increase in resistance is $8 \times 2y/10 = 1.6y\ \text{m}\Omega$. Now to trim a 100Ω resistor to 0.1% requires an additional resistance of 0.1 Ω or 100 mΩ, so that $1.6y = 100$ and $y = 62.5$ $\mu$m.

The maximum increase in resistance is when $y = 0.5$ mm or 500 $\mu$m, which makes $R = 1.6 \times 500 = 800\ \text{m}\Omega$. The trimming range is from 99.2 Ω up. It is clearly necessary to have coarse trimming too.

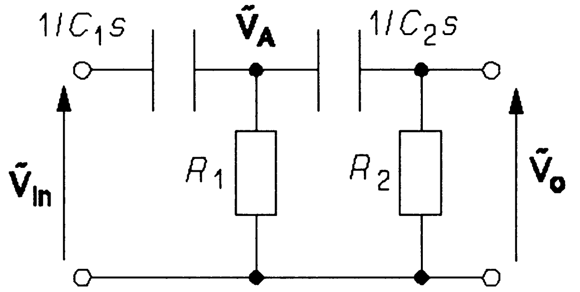


*Figure A10.4*

4 Looking at figure A10.4 and using nodal analysis, we find from the output node that

$$\tilde{\mathbf{V}}_o/R_2 + (\tilde{\mathbf{V}}_o - \tilde{\mathbf{V}}_A)C_2s = 0 \Rightarrow \tilde{\mathbf{V}}_A = \tilde{\mathbf{V}}_o(1 + 1/R_2C_2s)$$

And at node A
$$(\tilde{\mathbf{V}}_A - \tilde{V}_{in})C_1s + \frac{\tilde{\mathbf{V}}_A}{R_1} + (\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_o)C_2s = 0$$

which rearranges to

$$\tilde{\mathbf{V}}_A(1/R_1 + C_1s + C_2s) - \tilde{\mathbf{V}}_o C_2 s = \tilde{\mathbf{V}}_{in} C_1 s$$

Substituting for $\tilde{\mathbf{V}}_A$ gives

$$\tilde{\mathbf{V}}_o(1 + 1/R_2C_2s)(C_1s + C_2s + 1/R_1) - \tilde{\mathbf{V}}_o C_2 s = \tilde{\mathbf{V}}_{in} C_1 s$$

$$\Rightarrow \quad \tilde{\mathbf{V}}_o(C_1s + 1/R_1 + C_1/R_2C_2 + 1/R_2 + 1/R_1R_2C_2s) = \tilde{\mathbf{V}}_{in} C_1 s$$

Thus the voltage response is

$$\mathrm{H}(s) = \frac{\tilde{\mathbf{V}}_o}{\tilde{\mathbf{V}}_{in}} = \frac{C_1 s}{C_1s + 1/R_1 + C_1/R_2C_2 + 1/R_2 + 1/R_1R_2C_2s}$$

$$= \frac{R_1R_2C_1C_2s^2}{R_1R_2C_1C_2s^2 + 1 + R_1C_1s + R_2C_2s + R_1C_2s}$$

Replacing $s$ by $j\omega$ and letting $\omega_c^2 = 1/R_1R_2C_1C_2$, we obtain

$$H(j\omega) = \frac{-\omega^2/\omega_c^2}{1 - \omega^2/\omega_c^2 + j\omega(R_1C_1 + R_2C_2 + R_1C_2)}$$

$$= \frac{-\Omega^2}{1 - \Omega^2 + j\Omega\omega_c(R_1C_1 + R_2C_2 + R_1C_2)}$$

$$= \frac{-\Omega^2}{1 - \Omega^2 + j\Omega/Q}$$

where $\Omega \equiv \omega/\omega_c$ and $Q \equiv 1/\omega_c(R_1C_1 + R_2C_2 + R_1C_2)$.

To find the maximum value for $Q$, put it in the form

$$Q = \frac{1}{\omega_c(R_1C_1 + R_2C_2 + R_1C_2)} = \frac{\sqrt{R_1C_1R_2C_2}}{R_1C_1 + R_2C_2 + R_1C_2} = \frac{\sqrt{T_1T_2}}{T_1 + T_2 + T_2R_1/R_2}$$

where $T_1 \equiv R_1C_1$ and $T_2 \equiv R_2C_2$. This can be partially differentiated with respect to $T_1$ to give

$$\frac{\partial Q}{\partial T_1} = \frac{0.5\sqrt{T_2/T_1}\,(T_1 + T_2 + T_2R_1/R_2) - \sqrt{T_1T_2}}{\text{DENOMINATOR}}$$

Equating the numerator to zero for a maximum in $Q$ leads to

$$0.5\sqrt{T_2/T_1}\,(T_1 + T_2 + T_2R_1/R_2) = \sqrt{T_1T_2}$$

$$\Rightarrow \quad 0.5(T_1 + T_2 + T_2R_1/R_2) = T_1$$

$$\Rightarrow \quad T_1 = T_2(1 + R_1/R_2)$$

Substitution of this value for $T_1$ into the expression for $Q$ produces

$$Q_{\max} = \frac{T_2\sqrt{1 + R_1/R_2}}{2T_2(1 + R_1/R_2)} = \frac{0.5}{\sqrt{1 + R_1/R_2}}$$

Thus making $R_1 << R_2$ results in $Q_{\max} = 1/2$ and at the same time, since $T_1 = T_2(1 + R_1/R_2)$, $T_1 = T_2$. When $Q = 1/2$, the voltage response is

$$H(j\Omega) = -\Omega^2/(1 - \Omega^2 + j\Omega/Q)$$

And the magnitude of this is

$$H(j\Omega) = \frac{-\Omega^2}{1 - \Omega^2 + j2\Omega}$$

$$\Rightarrow \quad |H(j\Omega)| = \frac{\Omega^2}{\sqrt{(1 - \Omega^2)^2 + 4\Omega^2}} = \frac{\Omega^2}{\sqrt{1 + 2\Omega^2 + \Omega^4}} = \frac{\Omega^2}{1 + \Omega^2}$$

Therefore when $\omega = \omega_c$, $\Omega = 1$ and $|H(j\omega)| = 1/2$ which is $-6$ dB. When $Q < Q_{\max}$, the response at the corner frequency is less than this.

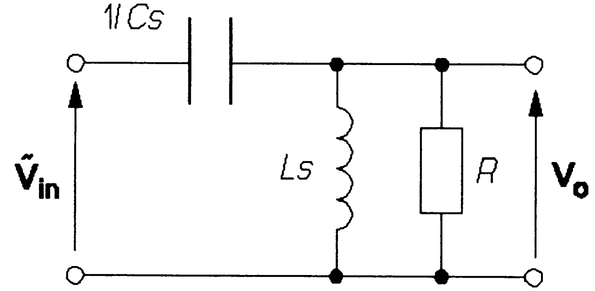


*Figure A10.5*

**5** The circuit of figure A10.4 is a high-pass filter and so the RLC circuit with a similar response is that of figure A10.5 (which is the same as that of figure 10.10a). Then $\omega_c = \omega_0 = 1/\surd(LC)$, $\beta = R/L$ and $Q = Q_p = R/\omega_0 L = 1/2$. Thus

$$L = \frac{R}{\omega_0 Q_p} = \frac{10^3}{2\pi \times 10^3 \times 0.5} = \frac{1}{\pi} = 0.318 \text{ mH}$$

As $$\omega_0 = \frac{1}{\sqrt{LC}}, \quad C = \frac{1}{\omega_0^2 L} = \frac{1}{4\pi \times 10^6} = 79.6 \text{ nF}$$

The response at 300 Hz or $\Omega = 0.3$ is

$$|H(j\Omega)| = \frac{\Omega^2}{1 + \Omega^2} = \frac{0.3^2}{1 + 0.3^2} = 0.0826$$

And $20\log_{10}|H(j\Omega)| = -21.7$ dB. The approximate response is $40\log_{10}\Omega^2 = 40\log_{10}0.09 = -20.9$ dB.

**6** The analysis of the circuit of figure A10.6 follows the standard nodal method. Noting that at node B, $\tilde{\mathbf{V}}_B = \tilde{\mathbf{V}}_o$, and then

$$(\tilde{\mathbf{V}}_o - \tilde{\mathbf{V}}_A)C_2 s + \tilde{\mathbf{V}}_o/R_1 = 0$$

$$\Rightarrow \qquad \tilde{\mathbf{V}}_A = \tilde{\mathbf{V}}_o(1 + 1/R_1 C_2 s)$$

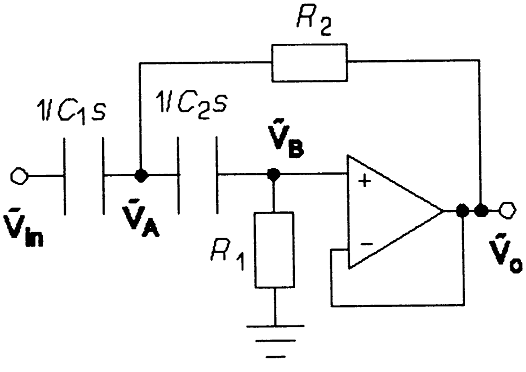


*Figure A10.6*

And at node A,

$$(\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_{in})C_1 s + \frac{\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_o}{R_2} + (\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_o)C_2 s = 0$$

Collecting terms we have

$$\tilde{\mathbf{V}}_A(C_1s + C_2s + 1/R_2) - \tilde{\mathbf{V}}_o(C_2s + 1/R_2) = \tilde{\mathbf{V}}_{in}C_1s$$

Substituting for $\tilde{\mathbf{V}}_A$ gives

$$\tilde{\mathbf{V}}_o(1 + 1/R_1C_2s)(C_1s + C_2s + 1/R_2) - \tilde{\mathbf{V}}_o(C_2s + 1/R_2) = \tilde{\mathbf{V}}_{in}C_1s$$

$$\Rightarrow \quad \tilde{\mathbf{V}}_o(C_1s + C_1/R_1C_2 + 1/R_1 + 1/R_1R_2C_2s) = \tilde{\mathbf{V}}_{in}C_1s$$

which produces

$$H(s) = \frac{\tilde{\mathbf{V}}_o}{\tilde{\mathbf{V}}_{in}} = \frac{C_1s}{C_1s + C_1/R_1C_2 + 1/R_1 + 1/R_1R_2C_2s} = \frac{R_1R_2C_1C_2s^2}{1 + R_1R_2C_1C_2s^2 + C_1R_2s + C_2R_2s}$$

And replacing $s$ by $j\omega$ gives

$$H(j\omega) = \frac{-R_1R_2C_1C_2\omega^2}{1 - R_1R_2C_1C_2\omega^2 + j\omega R_2(C_1 + C_2)}$$

Then if we replace $\omega^2R_1R_2C_1C_2$ by $\Omega^2$ we obtain

$$H(j\Omega) = \frac{-\Omega^2}{1 - \Omega^2 + j\Omega/Q}$$

provided that $\Omega/Q = \omega R_2(C_1 + C_2)$, or $Q = \Omega/\omega R_2(C_1 + C_2) = \omega\surd(R_1R_2C_1C_2)/\omega R_2(C_1 + C_2)$. Hence

$$Q = \frac{\sqrt{R_1R_2C_1C_2}}{(C_1 + C_2)R_2}$$

7 For a 1kHz cut-off $\omega_c = 2\pi \times 10^3 = 1/\surd(R_1C_1R_2C_2)$, and we are told that $R_1C_1 = R_2C_2$, thus $\omega_c = 1/R_2C_2$ and so

$$C_2 = \frac{1}{\omega_cR_2} = \frac{1}{2\pi \times 10^3 \times 10 \times 10^3} = 15.9 \text{ nF}$$

The $Q$ is to be $1/\surd 2$, so that

$$Q = \frac{1}{\sqrt{2}} = \frac{\sqrt{R_1R_2C_1C_2}}{(C_1 + C_2)R_2} = \frac{R_2C_2}{R_2(C_1 + C_2)} = \frac{C_2}{C_1 + C_2}$$

Thus $\surd C_2 = C_1 + C_2$, or $C_1 = C_2(\surd - 1) = 15.9 \times 0.414 = 6.59$ nF.
Because $R_1C_1 = R_2C_2$, $R_1 = R_2C_2/C_1 = 10/(\surd 2 - 1) = 24.1$ kHz.

The cut-off is at 1 kHz, and there $\Omega = 1$, making the response

$$|H(j\Omega)| = \frac{\Omega^2}{\sqrt{(1-\Omega^2)^2 + (\Omega/Q)^2}} = \frac{1}{1/Q} = Q$$

And $20\log_{10} Q = 20\log_{10} \sqrt{2} = -3$ dB.

At 50 Hz, $\Omega = 0.05$, so the response is

$$|H(j\Omega)| \approx \Omega^2 = 2.5 \times 10^{-3}$$

which is $-52$ dB. The 'approximation' is very, very nearly exact. If the 50Hz component at the input is 20 mV, it will be $2.5 \times 10^{-3} \times 20 \times 10^{-3} = 50\ \mu$V at the output.

**8** Consider the circuit of figure A10.8 (which is the same as figure 10.13a). The voltage at the inverting $(-)$ input is $\tilde{V}_o/K$ by the voltage divider rule. Thus $\tilde{V}_B = \tilde{V}_o/K$ since the voltages at the inputs must be the same. Considering node B and using KCL as usual we obtain

$$\tilde{V}_B Cs + \frac{\tilde{V}_B - \tilde{V}_A}{R} = 0 \Rightarrow \tilde{V}_A = \tilde{V}_B(RCs + 1) = \frac{\tilde{V}_o(RCs + 1)}{K}$$

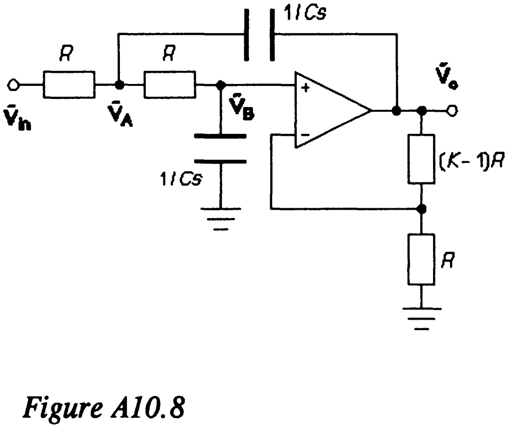


*Figure A10.8*

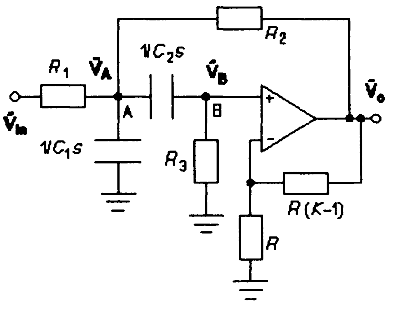


*Figure A10.9*

At node A the nodal current equation is

$$\frac{\tilde{V}_A - \tilde{V}_{in}}{R} + (\tilde{V}_A - \tilde{V}_o)Cs + \frac{\tilde{V}_A - \tilde{V}_B}{R} = 0$$

Collecting terms gives

$$\tilde{\mathbf{V}}_A(2/R + Cs) - \tilde{\mathbf{V}}_o Cs - \tilde{\mathbf{V}}_B/R = \tilde{\mathbf{V}}_{in}/R$$

Multiplying by $R$ and substituting for $\tilde{\mathbf{V}}_A$ and $\tilde{\mathbf{V}}_B$ leads to

$$\frac{\tilde{\mathbf{V}}_o(1 + RCs)(2 + RCs)}{K} - \tilde{\mathbf{V}}_o RCs - \frac{\tilde{\mathbf{V}}_o}{K} = \tilde{\mathbf{V}}_{in}$$

$$\Rightarrow \quad \tilde{\mathbf{V}}_o[1 + (3 - K)RCs + R^2C^2s^2] = \tilde{\mathbf{V}}_{in}K$$

Thus the frequency-domain response is

$$H(j\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{K}{1 - \omega^2R^2C^2 + j\omega RC(3 - K)}$$

Letting $\Omega = \omega RC$ gives

$$H(j\Omega) = \frac{K}{1 - \Omega^2 + j\Omega(3 - K)}$$

The amplitude response is

$$|H(\Omega)| = \frac{K}{\sqrt{(1 - \Omega)^2 + \Omega^2(3 - K)^2}} = \frac{K}{\sqrt{1 + ([3 - K]^2 - 2)\Omega^2 + \Omega^4}}$$

Equation 10.25 with $n = 2$ (that is a one-stage filter) is

$$|H(\Omega)| = \frac{1}{\sqrt{1 + \Omega^4}}$$

To make the denominators equal requires that $(3 - K)^2 - 2 = 0$, or $K = 3 - \sqrt{2}$. The stage gain is then $K = 1.586$, rather than unity.

**9** The circuit of figure A10.9 is analysed as usual by nodal analysis. The voltage at the inverting (−) input is $\tilde{\mathbf{V}}_o/K$ and this must also be $\tilde{\mathbf{V}}_B$.

At node B

$$\tilde{\mathbf{V}}_B/R_3 + (\tilde{\mathbf{V}}_B - \tilde{\mathbf{V}}_A)C_2s = 0 \Rightarrow \tilde{\mathbf{V}}_A = \tilde{\mathbf{V}}_B(1 + 1/R_3C_2s)$$

At node A

$$\frac{\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_{in}}{R_1} + \frac{\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_o}{R_2} + (\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_B)C_2s + \tilde{\mathbf{V}}_AC_1s = 0$$

Collecting terms gives

$$\tilde{\mathbf{V}}_A(1/R_1 + 1/R_2 + C_1s + C_2s) - \tilde{\mathbf{V}}_o/R_2 - \tilde{\mathbf{V}}_BC_2s = \tilde{\mathbf{V}}_{in}/R_1$$

Then substituting for $\tilde{\mathbf{V}}_A$ and $\tilde{\mathbf{V}}_B$ leads to

$$\frac{\tilde{\mathbf{V}}_o}{K}\left[1 + \frac{1}{R_3C_2s}\right]\left[\frac{1}{R_1} + \frac{1}{R_2} + C_1s + C_2s\right] - \frac{\tilde{\mathbf{V}}_o}{R_2} - \frac{\tilde{\mathbf{V}}_oC_2s}{K} = \frac{\tilde{\mathbf{V}}_{in}}{R_1}$$

$$\Rightarrow \tilde{\mathbf{V}}_o\left[\frac{1}{R_1} + \frac{1-K}{R_2} + C_1s + \frac{1}{R_1R_3C_2s} + \frac{1}{R_3R_2C_2s} + \frac{C_1}{R_3C_2} + \frac{1}{R_3}\right] = \frac{\tilde{\mathbf{V}}_{in}K}{R_1}$$

Dividing by $C_1s$ produces

$$\tilde{\mathbf{V}}_o\left[\frac{s}{R_1C_1} + \frac{s(1-K)}{R_2C_1} + s^2 + \frac{1}{R_1R_3C_1C_2} + \frac{1}{R_2R_3C_1C_2} + \frac{s}{R_3C_2} + \frac{s}{R_3C_1}\right] = \frac{\tilde{\mathbf{V}}_{in}Ks}{R_1C_1}$$

$$\Rightarrow \frac{\tilde{\mathbf{V}}_o}{\tilde{\mathbf{V}}_{in}} = \frac{Ks/R_1C_1}{\dfrac{R_1 + R_2}{R_1R_2R_3C_1C_2} + s^2 + s\left[\dfrac{1}{R_1C_1} + \dfrac{1}{R_3C_2} + \dfrac{1}{R_3C_1} + \dfrac{1-K}{R_2C_1}\right]}$$

And replacing $s$ by $j\omega$ gives $H(j\omega)$.

**10** Figure A10.10a shows the feedback path for the Colpitts oscillator, whose full circuit is shown in figure A10.10b. The feedback ratio is $X_{C2}/X_{C1} = C_1/C_2 = 0.05$, while the equivalent capacitance is

$$C_{eq} = \frac{C_1C_2}{C_1 + C_2} = \frac{0.05C_2^2}{1.05C_2} = 0.0476C_2$$

Thus if $f_0 = 100$ kHz,

$$\omega_0 = 2\pi f_0 = 2\pi \times 10^5 = \frac{1}{\sqrt{LC_{eq}}} = \frac{1}{\sqrt{0.0476LC_2}}$$

$$\Rightarrow \quad C_2 = \frac{1}{0.0476\omega_0^2 L} = \frac{1}{0.0476 \times 4\pi^2 \times 10^{10} \times 100 \times 10^{-6}} = 0.532\ \mu\text{F}$$

And so $C_1 = 0.05C_2 = 26.6$ nF.

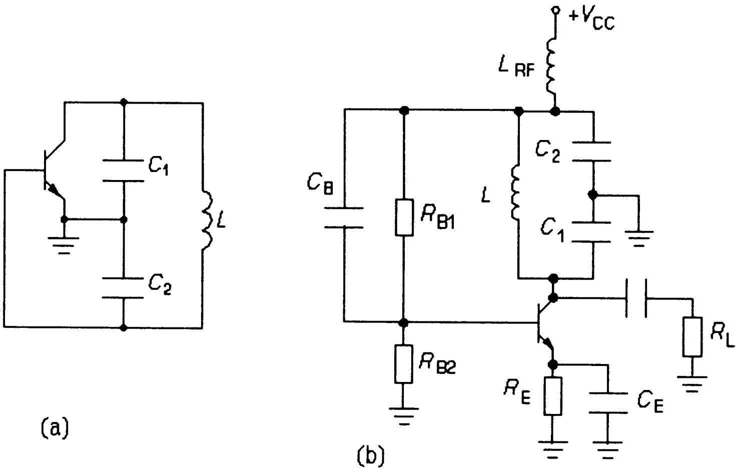


*Figure A10.10*

The voltage gain is

$$A_{vL} = \frac{h_{fe}R_L}{h_{ie}}$$

assuming that the bias resistances are large compared to $h_{ie}$ and that $h_{re}$ and $h_{oe}$ are negligible. Now $h_{ie} = 0.04/I_B = 0.04h_{FE}/I_C \approx 0.04h_{fe}/I_C$, making the voltage gain

$$A_{vL} = \frac{h_{fe}R_L}{h_{ie}} = \frac{h_{fe}R_L}{0.04h_{fe}/I_C} = 25I_CR_L$$

But for the circuit to oscillate the voltage gain must be greater than 20, the feedback ratio, thus

$$A_{vL} = 25I_CR_L > 20 \Rightarrow \quad I_C > \frac{20}{25R_L} = \frac{20}{25 \times 50} = 16\ \text{mA}$$

The collector current must therefore be at least 16 mA.

**11** The voltage at the inverting (−) input in figure A10.11 is $\tilde{\mathbf{V}}_o/K$, and this is also the voltage at the inverting (+) input, $\tilde{\mathbf{V}}_A$. At node A the nodal equation is

$$\tilde{\mathbf{V}}_A C_1 s + \frac{\tilde{\mathbf{V}}_A}{R_1} + \frac{\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_o}{R_2 + 1/C_2 s} = 0$$

Collecting terms

$$\tilde{\mathbf{V}}_A \left[ C_1 s + \frac{1}{R_1} + \frac{1}{R_2 + 1/C_2 s} \right] = \frac{\tilde{\mathbf{V}}_o}{R_2 + 1/C_2 s}$$

Substituting for $\tilde{\mathbf{V}}_A$ yields

$$\frac{1}{K} \left[ C_1 s + \frac{1}{R_1} + \frac{1}{R_2 + 1/C_2 s} \right] = \frac{1}{R_2 + 1/C_2 s}$$

Multiplying both sides by $K(R_2 + 1/C_2 s)$ gives

$$(C_1 s + 1/R_1)(R_2 + 1/C_2 s) + 1 = K$$

$$\Rightarrow \quad C_1 R_2 s + C_1/C_2 + R_2/R_1 + 1/R_1 C_2 s + 1 = K$$

Replacing $s$ by $j\omega$ leads to

$$j\omega C_1 R_2 + C_1/C_2 + R_2/R_1 + 1/j\omega R_1 C_2 + 1 = K$$

$$\Rightarrow \quad j\left(\omega C_1 R_2 - 1/\omega R_1 C_2\right) + 1 + C_1/C_2 + R_1/R_2 = K$$

And for this to be satisfied with $K$ real requires

$$\omega C_1 R_2 = 1/\omega R_1 C_2 \quad \Rightarrow \quad \omega = (R_1 R_2 C_1 C_2)^{-1/2}$$

and then $$K = 1 + \frac{C_1}{C_2} + \frac{R_2}{R_1}$$

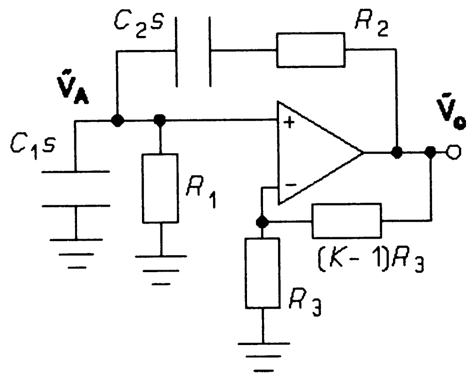


*Figure A10.11*

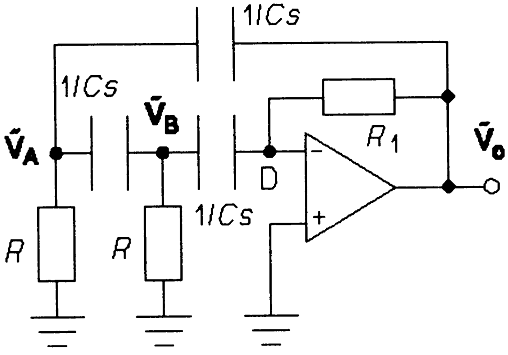


*Figure A10.12*

**12** There are three nodes in the circuit of figure A10.12, but node D is at the inverting ($-$) input where the voltage is zero as the non-inverting ($+$) input is grounded. Thus at node D

$$\frac{-\tilde{\mathbf{V}}_o}{R_1} = \tilde{\mathbf{V}}_B Cs \quad \Rightarrow \quad \tilde{\mathbf{V}}_B = \frac{-\tilde{\mathbf{V}}_o}{R_1 Cs}$$

At node B

$$\tilde{\mathbf{V}}_B Cs + \frac{\tilde{\mathbf{V}}_B}{R} + (\tilde{\mathbf{V}}_B - \tilde{\mathbf{V}}_A)Cs = 0$$

$$\Rightarrow \quad \tilde{\mathbf{V}}_A = \tilde{\mathbf{V}}_B \left[2 + \frac{1}{RCs}\right] = \frac{-\tilde{\mathbf{V}}_o \left[2 + \frac{1}{RCs}\right]}{R_1 Cs} = -\tilde{\mathbf{V}}_o \left[\frac{2}{R_1 Cs} + \frac{1}{RR_1 C^2 s^2}\right]$$

At node A

$$\frac{\tilde{\mathbf{V}}_A}{R} + (\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_B)Cs + (\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_o)Cs = 0$$

$$\tilde{\mathbf{V}}_A \left[2 + \frac{1}{RCs}\right] - \tilde{\mathbf{V}}_B - \tilde{\mathbf{V}}_o = 0$$

Substituting for $\tilde{\mathbf{V}}_A$ and $\tilde{\mathbf{V}}_B$ gives

$$\frac{-\tilde{\mathbf{V}}_o}{R_1 Cs} \left[2 + \frac{1}{RCs}\right]^2 + \frac{\tilde{\mathbf{V}}_o}{R_1 Cs} - \tilde{\mathbf{V}}_o = 0$$

$$\Rightarrow \quad -\tilde{\mathbf{V}}_o \left[\frac{3}{R_1 Cs} + \frac{4}{RR_1 C^2 s^2} + \frac{1}{R_1 R^2 C^3 s^3} + 1\right] = 0$$

Replacing $s$ by $j\omega$ leads to

$$-\mathbf{V}_o \left[\frac{3}{j\omega R_1 C} - \frac{1}{j\omega^3 R_1 R^2} + 1 - \frac{4}{\omega^2 R_1 R C^2}\right] = 0$$

If the circuit is to have any output, $\mathbf{V}_o \neq 0$ and the term in brackets must be zero, which can only be so if the real and the imaginary parts are both zero, that is

$$\omega^2 R_1 R C^2 = 4 \quad \text{and} \quad 3\omega^2 R^2 C^2 = 1$$

From the latter condition $\omega = 1/(RC\sqrt{3})$ and $\omega^2 RC^2 = 1/3R$, which can be substituted into the former condition to give $R_1 = 12R$.

# Chapter 11

1 Figure A11.1 shows the AC load line for an ideal transformer-coupled class-A amplifier operating at maximum efficiency, with the quiescent point at $I_{CQ}$, $V_{CC}$. It shows that the output voltage of the amplifier is $v_{ce} = V_{CC}(1 + \sin\omega t)$, as the output is sinusoidal and goes along the AC load line from 0 to $2V_{CC}$. The current likewise is sinusoidal and goes from 0 to $2I_{CQ}$. Since $i_c = 0$ when $v_{ce} = 2V_{CC}$ and $i_c = 2I_{CQ}$ when $v_{ce} = 0$, then $i_c = I_{CQ}(1 - \sin\omega t)$. The instantaneous power dissipated in the transistor is therefore

$$v_{ce}i_c = V_{CC}I_{CQ}(1 + \sin\omega t)(1 - \sin\omega t)$$

$$= V_{CC}I_{CQ}(1 - \sin^2\omega t) = 0.5V_{CC}I_{CQ}(1 + \cos 2\omega t)$$

The average power dissipated in the transistor is then

$$P = \frac{0.5V_{CC}I_{CQ}\int_0^T(1 + \cos 2\omega t)\,dt}{T} = \frac{0.5V_{CC}I_{CQ}\left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T}{T}$$

$$= \frac{0.5V_{CC}I_{CQ}[T + \sin 4\pi]}{T} = 0.5V_{CC}I_{CQ}$$

as $T = 2\pi/\omega$. This is the same as the power dissipated in the load, so 50% of the input power, ideally, appears as heat in the transistor.

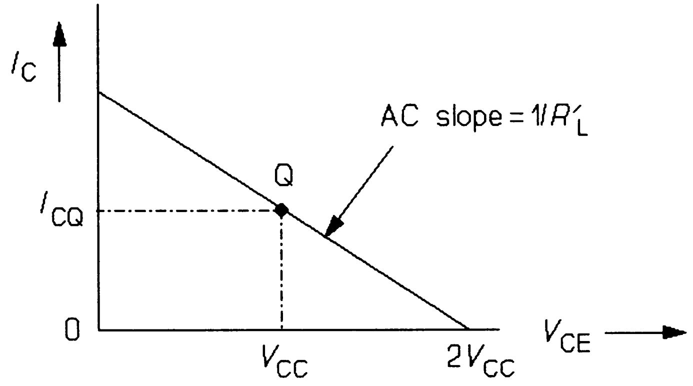


*Figure A11.1*

2 The output resistance of the CE amplifier is $1/h_{oe} \parallel R_C$ and so maximum power is transferred when $R_L = 1/h_{oe} \parallel R_C = R_C/(1 + h_{oe}R_C)$. Now the efficiency of the capacitively-coupled CE amplifier is given by equation 11.8 which is

$$\eta = \frac{R_C R_L}{2(R_C + R_L)(R_C + 2R_L)}$$

Substituting $R_C/(1 + h_{oe}R_C)$ for $R_L$ gives

$$\eta = \frac{R_C^2/(1 + h_{oe}R_C)}{2[R_C + R_C/(1 + h_{oe}R_C)][R_C + 2R_C/(1 + h_{oe}R_C)]}$$

Multiplying both numerator and denominator by $(1 + h_{oe}R_C)/R_C^2$ gives

$$\eta = \frac{1 + h_{oe}R_C}{2(2 + h_{oe}R_C)(3 + h_{oe}R_C)} \approx \frac{1 + h_{oe}R_C}{12 + 10h_{oe}R_C} \approx \frac{1}{12}$$

since $h_{oe}R_C << 1$, usually; the efficiency with maximum power transfer is 1/12 or 8.33%, and if $1/h_{oe} = 10R_C$, the efficiency is only slightly increased to 8.46%.

3 The load must be transformed into one of $V_{CC}/I_{1max}$ in the primary in order to obtain maximum load power. Since $V_{CC} = 12$ V and $I_{1max} = 2$ A, the load had to be 6 Ω when referred to the primary. Since

$$R_1/R_2 = n^2 = 6/50000 \Rightarrow n = \sqrt{6/50000} = 0.011$$

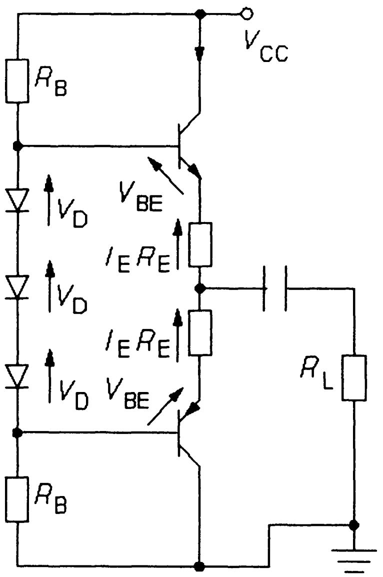


*Figure A11.4*

4 Figure A11.4 shows the class-B BJT push-pull amplifier. The load power for the class-B amplifier is given by

$$P_L = \tfrac{1}{2}I_{mL}{}^2R_L \Rightarrow I_{mL} = \sqrt{2P_L/R_L} = \sqrt{200/10} = \sqrt{20}\ \text{A}$$

But the maximum load current is

$$I_{mL} = \frac{\frac{1}{2}V_{CC}}{R_E + R_L}$$

$$\Rightarrow \quad V_{CC} = 2(R_E + R_L)I_{mL} = 2 \times 12 \times \sqrt{20} = 107.3\ \text{V}$$

while the input power is the product of $V_{CC}$ and the time average of the current. The time-averaged current supplied is $I_{mL}/\pi$, so $P_{in} = V_{CC}I_{mL}/\pi$ and the efficiency is then

$$\eta = \frac{P_L}{V_{CC}I_{mL}/\pi} = \frac{100}{107.3 \times \sqrt{20}/\pi} = 0.654$$

or 65.4%.

The power lost in the transistors is found from

$$P_{TX} = P_{in} - P_L - I_L^2R_E$$

Though there are two emitter resistors, each conducts during one half cycle only, so that we can consider the load current to flow for the whole time through one. But $I_L = I_{mL}/\surd 2$, making $I_E^2R_E = ½I_{mL}^2R_E = 0.5 \times 20 \times 2 = 20$ W. We are assuming negligible base current so that the losses in $R_B$ and the diodes can be neglected. The power lost in the transistors is therefore 152.7 − 100 − 20 = 32.7 W, or 16.35 W in each.

The voltage drop across $R_E$ during standby is $½(3V_D - 2V_{BE}) = ½V_D = 0.35$ V, so that the quiescent emitter current is $0.35/R_E = 0.175$ A, and the power supplied during standby must be $V_{CC}I_C \approx V_{CC}I_E = 107.3 \times 0.175 = 18.8$ W.

**5** If the supply voltage is 200 V and the maximum current is 8 A, with a load power of 250 W, then we can calculate the conduction angle, $\theta_c$ from

$$P_L = 0.5r_1V_{CC}I_m$$

and $$r_1 = 0.2026\theta_c - 0.01386\theta_c^2$$

The first equation gives

$$r_1 = \frac{2P_L}{V_{CC}I_m} = \frac{2 \times 250}{200 \times 8} = 0.3125$$

while the second yields

$$0.3125 = 0.2026\theta_c - 0.01386\theta_c^2$$

$$\Rightarrow \quad \theta_c^2 - 14.62\theta_c + 22.55 = 0$$

$$\theta_c = 7.31 \pm \sqrt{7.31^2 - 22.55} = 12.87 \text{ or } 1.7525 \text{ rad}$$

Taking the latter value as the former makes no physical sense, we find the conduction angle to be 1.7525 rad or 100.4°, and from this we can calculate the efficiency using

$$\eta = 100 - 6.84\theta_c = 100 - 6.84 \times 1.7525 = 88\%$$

The power drawn from the supply is

$$P_S = r_0 V_{CC} I_m = (\theta_c/\pi^2) \times V_{CC} I_m = (1.7525/\pi^2) \times 200 \times 8 = 284 \text{ W}$$

The amplifier must dissipate

$$P_A = P_S - P_L = 284 - 250 = 34 \text{ W}$$

The base bias is found from

$$\theta_c = 2\cos^{-1}\left[\frac{|V_{BB}| + V_{BE}}{V_m}\right]$$

$$\Rightarrow \quad |V_{BB}| = V_m \cos(\theta_c/2) - V_{BE} = 5\cos 50.2° - 0.7 = 2.5 \text{ V}$$

Thus $V_{BB} = -2.5$ V, since the bias voltage is negative.

6 The conduction angle is given by

$$\theta_c = 2\cos^{-1}\left[\frac{|V_{BB}| + V_{BE}}{V_m}\right] = 2\cos^{-1}\left[\frac{10 + 0.7}{15}\right] = 89°$$

In radians, $\theta_c = 89°/180 = 1.553$ rad. Then $r_0 = \theta_c/\pi^2 = 0.1574$ and the power supplied is

$$P_S = r_0 V_{CC} I_m = 0.1574 \times 60 \times 3.2 = 30.2 \text{ W}$$

$r_1$ is given by

$$r_1 = 0.2026\theta_c - 0.01386\theta_c^2 = 0.2026 \times 1.553 - 0.10386 \times 1.553^2 = 0.281$$

From which we find the load power

$$P_L = 0.5 r_1 V_{CC} I_m = 27 \text{ W}$$

And the efficiency is

$$\eta = 100 - 6.84\theta_c = 89.4\%$$

As a check, $P_L/P_S = 27/30.2 = 0.894$ or 89.4%.

**7** The output power is given by

which rearranges to

$$\delta = \frac{2P_L}{\eta V_{in} I_{Cmax}} = \frac{2 \times 225}{0.85 \times 160 \times 10} = 0.33$$

And $V_{CEmax} = 2V_{in} = 320$ V. The load voltage is $V_L = \surd(P_L R_L) = \surd(225 \times 500) = 335$ V.
The ripple is given by

$$\Delta V = \frac{V_L}{RCf} \Rightarrow C = \frac{1}{Rf\Delta V/V_L} = \frac{1}{500 \times 10000 \times 0.02} = 10\ \mu\text{F}$$

**8** If the power is equally distributed (a big 'if'), then each transistor dissipates 50/4 = 12.5 W, so that the temperature difference from junction to case is

$$\Delta T = P\theta_{JC} = 12.5 \times 2.8 = 35°\text{C}$$

And if the junction is at 150°C, the case must be at 150 − 35 = 115°C. The temperature difference between case and heat sink is

$$\Delta T = P\theta_{paste} = 12.5 \times 0.2 = 2.5°\text{C}$$

Thus the temperature of the heat sink is 115 − 2.5 = 112.5°C.

The temperature difference between the heat sink and ambient thus becomes 112.5 − 50 = 62.5°C and so

$$\theta_{HA} = \Delta T/P = 62.5/50 = 1.25°\text{C/W}$$

**9** The heat sink's temperature with fan cooling would be

$$T_{HS} = T_A + P\theta_{HA} = 50 + 50 \times 0.7 = 85°\text{C}$$

Then the case temperature is

$$T_C = T_{HS} + \tfrac{1}{4}P\theta_{paste} = 85 + 0.25 \times 50 \times 0.2 = 85 + 2.5 = 87.5°\text{C}$$

And the junction temperature is

$$T_J = T_C + \tfrac{1}{4}P\theta_{JC} = 87.5 + 0.25 \times 50 \times 2.8 = 122.5°\text{C}$$

If the junction temperature is 150°C then the temperature of the heat sink is

$$T_{HS} = T_J + \tfrac{1}{4}P(\theta_{JC} + \theta_{paste}) = T_J - 0.25P \times 3 = 150 - 0.75P$$

But the heat sink has to remove $P$ watts to ambient at 50°C, and its temperature is therefore

$$T_{HS} = T_A + P\theta_{HA} = 50 + 0.7P$$

Equating the two expressions for $T_{HS}$ gives

$$150 - 0.75P = 50 + 0.7P$$

$$\Rightarrow \quad 1.45P = 100 \quad \Rightarrow \quad P = 69 \text{ W}$$

which represents an increase in power dissipation of 38%.

# Chapter 12

**1** Consider a small element of the circumference of the coil, as shown in figure A12.1, whose length is $\delta l$, and which is inclined at an angle, $\phi$, to the vertical. The Lorentz force on this element is

$$\delta\mathbf{F} = i\delta\mathbf{l} \times \mathbf{B}$$

where $\delta\mathbf{l}$ is directed along the tangent to the circumference in the direction of $i$. The force's direction is given by the right-hand corkscrew rule. When the plane of the coil makes no angle ($\theta = 0°$) with $\mathbf{B}$, the angle between $\delta\mathbf{l}$ and $\mathbf{B}$ is $(90° - \phi)$ and the force's magnitude is

$$\delta F = Bi\delta l\sin(90° - \phi) = Bi\delta l\cos\phi$$

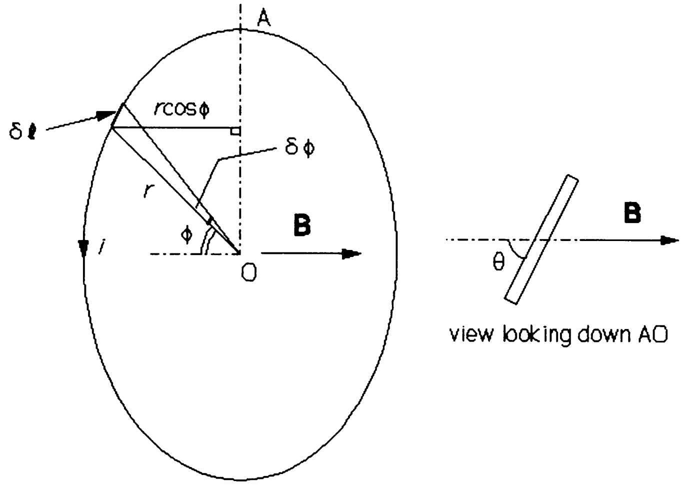


*Figure A12.1*

The moment of this force and the corresponding element on the other side of the circumference is

$$\delta M = 2r\cos\phi\delta F = 2Bir\cos^2\phi\,\delta l$$

since the separation between elements is $2r\cos\phi$. But $\delta l = r\delta\phi$ making

$$\delta M = Bir^2\cos^2\phi\,\delta\phi$$

And when the plane of the coil is inclines at $\theta$ to $\mathbf{B}$, the moment of the element is $\delta\mathbf{M}\cos\theta$. Integrating $\delta\mathbf{M}$ from $\phi = -\pi/2$ to $\phi = \pi/2$ gives

$$M = \int \mathrm{d}M\cos\theta = \int_{-\pi/2}^{\pi/2} 2Bir^2\cos\theta\cos^2\phi\,\mathrm{d}\phi$$

Now $$\int_{-\pi/2}^{\pi/2} 2\cos^2\phi \, \mathrm{d}\phi = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\phi)\mathrm{d}\phi = \left[\phi + \frac{\sin 2\phi}{2}\right]_{-\pi/2}^{\pi/2} = \pi$$

Thus $$M = \pi Bir^2 \cos\theta = \pi BAi \cos\theta$$

2 The induced e.m.f. is

$$E = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = B\frac{\mathrm{d}A}{\mathrm{d}t} = Bl\frac{\mathrm{d}x}{\mathrm{d}t} = Blv$$

And in air $B = \mu_0 H = 4\pi \times 10^{-7} \times 40 = 50.3\ \mu\mathrm{T}$, so that

$$E = Blv = 50.3 \times 10^{-6} \times 2.65 \times 112000/3600 = 4.14 \text{ mV}$$

The power dissipated is $V^2/R$, where

$$R = \frac{l}{\sigma A} = \frac{2.65}{50 \times 10^6 \times 10^{-5}} = 5.3 \text{ m}\Omega$$

$$\therefore \quad P = (4.14 \times 10^{-3})^2/5.3 \times 10^{-3} = 3.23 \text{ mW}$$

The force on the conductor is given by

$$\mathbf{F} = i\mathbf{l} \times \mathbf{B}$$

and this has a magnitude of

$$F = ilB\sin\theta = ilB\sin 90^\circ = ilB$$

where $i = E/R$ and $R = l/\sigma A$, so that

$$F = E\sigma AB = 4.14 \times 10^{-3} \times 50 \times 10^6 \times 10^{-5} \times 50.3 \times 10^{-6} = 104\ \mu\mathrm{N}$$

3 This is an exercise in using Fleming's right-hand (generator) rule, with the thumb indicating the motion of the conductor (the sea) into the N Sea and the first finger indicating the direction of the earth's magnetic field (downwards) as in figure A12.3, making the current flow from France to England as indicated by the second finger. If the current flows from France to England, the channel acts as a generator and thus England is at the higher potential.

The magnitude of the e.m.f. is given by

$$E = \mathrm{d}\Phi/\mathrm{d}t = B\mathrm{d}A/\mathrm{d}t = Bl\mathrm{d}x/\mathrm{d}t = Blv$$

where $l$ is the width of the Channel in m, $v$ is the speed of its flow in m/s and $B = \mu_0 H$

$= 4\pi \times 10^{-7} \times 25 = 31.4\ \mu T$. The speed is 4 km/h, which is $4000/3600 = 1.11$ m/s, making

$$E = 31.4 \times 10^{-6} \times 35000 \times 1.11 = 1.22\ \text{V}$$

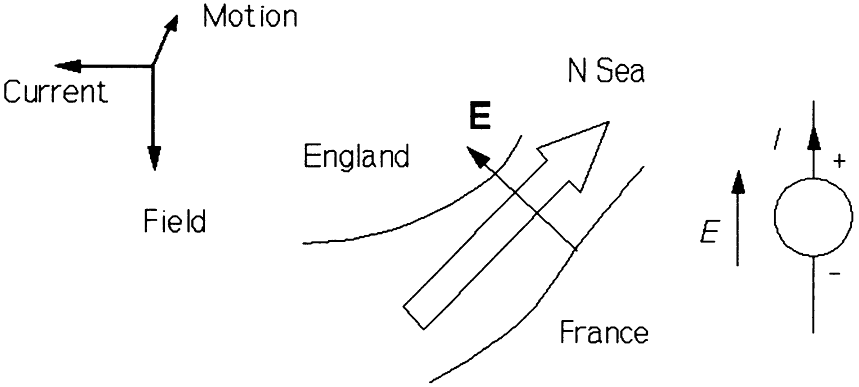


*Figure A12.3*

**4** The equation used is

$$B = \mu_0(H + M)$$

where $H = NI/l = 330 \times 1.2/0.5 = 792$ A/m and $M = M_{sat} = 2 \times 10^5$ A/m, making

$$B = \mu_0(792 + 2 \times 10^5) = 4\pi \times 10^{-7} \times 200792 = 0.252\ \text{T}$$

The flux is then $\Phi = BA = 0.252 \times 4 \times 10^{-4} = 0.101$ mWb.

The relative permeability is

$$\mu_r = \frac{B}{\mu_0 H} = \frac{0.252}{4\pi \times 10^{-7} \times 792} = 253$$

**5** First we must find the reluctance of the air gap from $\mathbb{R}_g = l_g/\mu_0 A_g$, where $l_g$ is the length of the gap along the direction of **B** and $A_g = A_c$ = cross-sectional area of the core normal to **B**. Because there are two gaps each of length 0.05 mm as shown in figure A12.5, $l_g = 0.1$ mm $= 10^{-4}$ m, and

$$\mathbf{R}_g = \frac{10^{-4}}{4 \times 10^{-4}\mu_0} = \frac{0.25}{\mu_0}$$

The reluctance of the magnetic core is

$$\mathbf{R}_c = \frac{l_c}{\mu_0\mu_r A_c} = \frac{0.5}{253 \times 4 \times 10^{-4}\mu_0} = \frac{4.94}{\mu_0}$$

The combined circuit reluctance is

$$\mathbf{R} = \mathbf{R}_g + \mathbf{R}_c = (0.25 + 4.94)/\mu_0 = 5.19/\mu_0$$

Then we can find the flux from

$$NI = \Phi\mathbf{R} \Rightarrow \Phi = \frac{NI}{\mathbf{R}} = \frac{330 \times 1.2}{5.19/\mu_0} = 95.9\ \mu\text{Wb}$$

And the flux density is

$$B = \frac{\Phi}{A_c} = \frac{95.6 \times 10^{-6}}{4 \times 10^{-4}} = 0.24\ \text{T}$$

The relative permeability is

$$\mu_r = \frac{B}{\mu_0 H} = \frac{0.24}{4\pi \times 10^{-7} \times 792} = 241$$

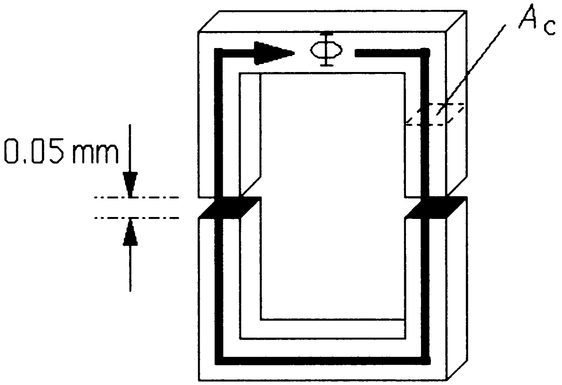


*Figure A12.5*

Increasing the gap to 2 mm makes $l_g$ = 4 mm and

$$\mathbf{R}_g = \frac{4 \times 10^{-3}}{4 \times 10^{-4}\mu_0} = \frac{10}{\mu_0}$$

Since $\mathbb{R}_c$ remains unchanged we have a circuit reluctance of $14.94/\mu_0$ and the flux is

$$\Phi = \frac{NI}{\mathbf{R}} = \frac{330 \times 1.2}{14.94/\mu_0} = 33.3\ \mu\text{Wb}$$

making

$$B = \frac{\Phi}{A_c} = \frac{33.3 \times 10^{-6}}{4 \times 10^{-4}} = 0.0833\ \text{T}$$

and

$$\mu_r = \frac{B}{\mu_0 H} = \frac{0.0833}{792\mu_0} = 83.7$$

**6** The loop area is $4B_rH_c$ and this is the hysteresis loss per cycle per m$^3$, that is

$$P_h = 4\Lambda B_r H_c f = 4\Lambda B_m H_c f$$

$$= 4 \times 3 \times 10^{-3} \times 0.25 \times 5 \times 24000 = 360 \text{ W}$$

$\Lambda = 3 \times 10^{-3}$ m$^3$, the volume of the core.

**7** The eddy-current losses are

$$P_e = \tfrac{1}{6}(\pi b f B_m)^2 \sigma \Lambda$$

$$= \tfrac{1}{6}(\pi \times 0.3 \times 10^{-3} \times 50 \times 0.68)^2 \times 0.7 \times 10^6 \times 4 \times 10^{-4} = 0.048 \text{ W}$$

where $b = 0.3$ mm, $f = 50$ Hz, $B_m = 0.68$ T, $\sigma = 0.7$ MS/m and $\Lambda = 4 \times 10^{-4}$ m$^3$

Since the losses go as $f^2$, at 2 kHz, they will be $(2000/50)^2 = 1600$ times greater, or 76.6 W.

The eddy-current density at 50 Hz is given by

$$J = \sqrt{\frac{\sigma P_e}{\Lambda}} = \sqrt{\frac{0.7 \times 10^6 \times 0.048}{4 \times 10^{-4}}} = 9.16 \text{ kA/m}^2$$

And at 2 kHz it will be $2000/50 = 40$ times greater, or 366 kA/m$^2$.

**8** The coil inductance is given by

$$L = \mu_r \mu_0 \Lambda n^2$$

where $\Lambda$ is the core volume and $n$ is the number of turns/$l$. Now $\Lambda = A_c l$, $A_c$ being the cross-sectional area of the core, $8 \times 10^{-5}$ m$^2$, and $l = 2\pi r = 2\pi \times 20 \times 10^{-3} = 0.1257$ m. From the equation above we find

$$n = \sqrt{\frac{L}{\mu_r \mu_0 \Lambda}} = \sqrt{\frac{60 \times 10^{-3}}{325 \times 4\pi \times 10^{-7} \times 8 \times 10^{-5} \times 0.1257}} = 3822 \text{ turns/m}$$

Thus $N = nl = 3822 \times 0.1257 = 480$ turns.

The inductance is proportional to $N^2$, so that

$$\ln L = 2N + K$$

where $K$ is independent of $N$ and therefore differentiating $L$ with respect to $N$ yields

$$\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}N} = 2 \Rightarrow \frac{\delta L}{L} = 2\frac{\delta N}{N}$$

If $\delta N = 0.5$ and $N = 480$, $\delta N/N = 1.04 \times 10^{-3}$, and then $\delta L/L = 2.08 \times 10^{-3}$, making

$$\delta L = 2.08 \times 10^{-3}L = 2.08 \times 10^{-3} \times 60 = 0.125 \text{ mH}$$

**9** The hysteresis losses/cycle will be

$$P_h = 4B_rH_c = 4 \times 0.8 \times 40 = 128 \text{ J/m}^3$$

which is $128f$ W/m$^3$, where $f$ is the frequency.

The eddy-current losses/m$^3$ will be

$$P_e = \tfrac{1}{6}(\pi b f B_m)^2\sigma$$

where $b$ = lamination thickness, 0.25 mm, $f$ is the frequency, $B_m = B_r = 0.8$ T and $\sigma$, the conductivity is 1.5 MS/m. Substituting these values in the formula above produces $P_e = 0.0987f^2$ W/m$^3$. Equating $P_h$ and $P_e$ gives

$$128f = 0.0987f^2 \Rightarrow f = 1296 \text{ Hz}$$

At 5 kHz the hysteresis losses are 128 × 5000 = 640 kW/m$^3$, while the eddy-current losses are

$$P_e = \tfrac{1}{6}(\pi b \times 5000 \times 0.8)^2 \times 1.5 \times 10^6 = 3.96 \times 10^{13}b^2 \text{ W/m}^3$$

Equating the losses gives

$$3.95 \times 10^{13}b^2 = 640 \times 10^3 \Rightarrow b = 0.127 \text{ mm}$$

# Chapter 13

1 The e.m.f. is given by

$$e = \frac{Z\Phi_p n}{60} \Rightarrow Z = \frac{60e}{\Phi_p n} = \frac{60 \times 450}{0.03 \times 1000} = 900$$

Since there are three pole pairs, if the e.m.f. and the flux/pole are unchanged in a wave-wound machine, the number of conductors must be reduced by a factor of three to 300.

The excitation is constant and so if the speed is reduced to 825 r/min from 1000 r/min, the generated e.m.f. must be reduced in proportion, that is

$$E = 825 \times 450/1000 = 371 \text{ V}$$

Then if the current supplied is 200 A, the armature voltage drop is $I_a R_a = 200 \times 0.05 = 10$ V making the terminal voltage

$$V = E - I_a R_a = 371 - 200 \times 0.05 = 361 \text{ V}$$

2 The flux/pole for a lap-wound machine is given by

$$\Phi_p = \frac{60e}{Zn} = \frac{60 \times 660}{300 \times 2 \times 720} = 91.7 \text{ mWb}$$

where $Z$, the number of conductors, is $300 \times 2$.

A wave-wound generator has an e.m.f. given by

$$e = Z\Phi_p np/60$$

where $p$ is the number of pole pairs, in this case 6. Thus for the same number of conductors and generated e.m.f. the flux/pole is six times less or $91.7/6 = 15.3$ mWb.

3 We must use the equivalent circuit of figure A13.3, from which we can see that $I_f = V/R_f = 660/240 = 2.75$ A. Also we can see that $I_a = I_L + I_f$ and

$$I_L = P_L/V = 2 \times 10^5/660 = 303 \text{ A}$$

so that $I_a = 303 + 2.75 = 305.75$ A. Now the armature voltage drop is

$$I_a R_a = E - V = 690 - 660 = 30 \text{ V}$$

which gives

$$R_a = \frac{E - V}{I_a} = \frac{30}{305.75} = 0.0981\ \Omega$$

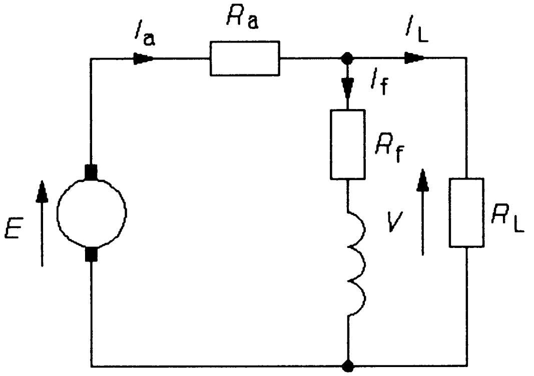


*Figure A13.3*

Using the same equivalent circuit and the same equations we can find the terminal voltage when the generated e.m.f. is 690 V as follows. The load current is

$$I_L = P_L / V = 10^5 / V$$

And the field current must remain at 2.75 A for the e.m.f. to stay constant, as the speed will be constant. Thus the armature current is

$$I_a = I_L + I_f = 10^5 / V + 2.75$$

Then the armature voltage drop is

$$I_a R_a = (10^5 / V + 2.75) \times 0.0981$$

But this is $E - V = 690 - V$ giving

$$690 - V = 0.0981(10^5 / V + 2.75)$$

which rearranges to the quadratic equation

$$V^2 - 689.73V + 9810 = 0$$

$$\Rightarrow \quad V = \frac{689.73 \pm \sqrt{689.73^2 - 4 \times 9810}}{2} = 675.2\ \text{V}$$

(The other solution is $V = 14.5$ V, which is impossible.)

**4** The graph of open-circuit e.m.f. against field current is as in figure A13.4a, wherein by drawing a horizontal line at 160 V we can see that the corresponding field current is 1.7 A making the field resistance 160/1.7 = 94 Ω. The initial slope of the curve is clearly 38/0.25 = 152 Ω and this is the critical field resistance.

If a load is attached and supplied with 20 A with the same generator speed of 900 r/min and field resistance, the terminal voltage will fall because of the armature voltage drop and this will lead to a decrease in generated e.m.f. as the field coil is in parallel with the load. The total armature current is $I_L + I_f \approx 20 + 1.7 = 21.7$ A, making the armature voltage drop, $I_aR_a$, $21.7 \times 0.46 = 10$ V. Since the terminal voltage must lie on the field load line and be 10 V below the generated (≈ open-circuit) e.m.f., we can find the solution as shown in the expanded inset, which gives $E = 154$ V and $V = 144$ V.

The increase of speed to 1 200 r/min from 900 r/min will cause the e.m.f. to increase by 1 200/900 for the same field current. We can therefore draw another $E$ - $I_f$ curve from the data given by increasing the voltages by a factor of 4/3 to give the curve in figure A13.4b; and drawing a field load line with a slope of 110 Ω as shown gives the open-circuit e.m.f. as 228 V.

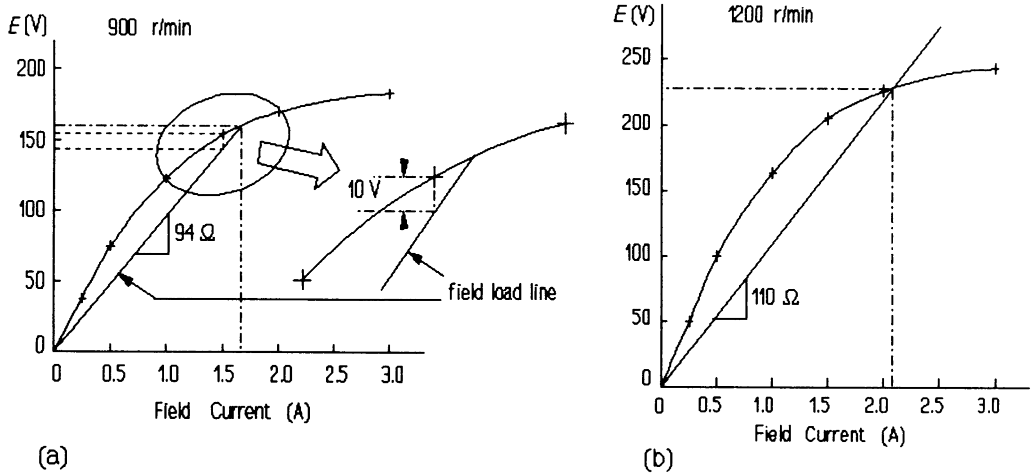


*Figure A13.4*

**5** The amp-turns/pole on full load for the shunt winding are $0.278 \times 1800 = 500.4$, and on no load they are $0.25 \times 1800 = 450$, the difference being 50 amp-turns, which must be supplied by the series field winding on full load. The full load current is

$$I_{FL} = P_{FL}/V_T = 3000/120 = 25 \text{ A}$$

Thus the series coil requires 50/25 = 2 turns/pole.

If the number of turns/pole in the shunt winding were to be 1 400 with the same no-load and full-load field currents (0.25 and 0.278 A), then the series winding would overcompensate, since the amp-turns on no load would be 1400 × 0.25 = 350, while on full load they would be 1400 × 0.278 = 389.2, a difference of 39.2 amp-turns, while the series field coil would supply 2 × 25 = 50 amp-turns as before. The series field current must be reduced to

$$I_s = \frac{39.2}{50} \times 25 = 19.6 \text{ A}$$

This can be achieved with a parallel resistance, $R_p$, as in figure A13.5, and the voltage across this is $I_sR_s$ = 19.6 × 0.02 = 0.392 V. The total current through $R_s$ and $R_p$ remains 25 A, so that the current through $R_p$ is $I_{FL} - I_s = 25 - 19.6 = 5.4$ A, making $R_p = 0.392/5.4 = 0.0726\ \Omega$.

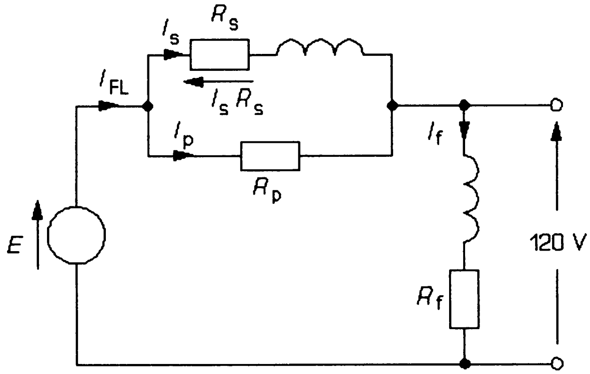


*Figure A13.5*

**6** If the armature current is unchanged, the back e.m.f. is unchanged also, since

$$E = V - I_aR_a$$

and the supply voltage, $V$ is constant. But

$$E \propto n\Phi \propto nI_f = nV/R_f \propto n/R_f$$

If, therefore, the speed increases, the field resistance must increase in proportion, giving

$$R_{f2} = R_{f1}n_2/n_1 = 480 \times 675/600 = 540\ \Omega$$

The armature voltage drop is $I_aR_a = 95 \times 0.32 = 30.4$ V, so that the back e.m.f. is

$$E = V - I_aR_a = 440 - 30.4 = 409.6 \text{ V}$$

When the armature current is 50 A, the back e.m.f. is

$$E = V - I_aR_a = 440 - 50 \times 0.32 = 424 \text{ V}$$

The speed, for constant field current, is proportional to the back e.m.f., so that

$$n_2 = E_2 n_1 / E_1 = 424 \times 600/409.4 = 621 \text{ r/min}$$

The speed alters little for a near halving of armature current.

The power output is $EI_a$, which gives 409.4 × 95 = 38.9 kW at 600 r/min and 424 × 50 = 21.2 kW at 621 r/min.

7 The power output is

$$P = EI_a = (V - I_a R_a) I_a \Rightarrow I_a^2 - \frac{VI_a}{R_a} + \frac{P}{R_a} = 0$$

Substituting for $V = 400$ V, $P = 32$ kW and $R_a = 0.32\ \Omega$ leads to

$$I_a^2 - 1250 I_a + 10^5 = 0 \Rightarrow I_a = 625 \pm \sqrt{625^2 - 10^5} = 85.9 \text{ A}$$

The other solution is 1.164 kA, which we reject as it is far too large.

The back e.m.f. is proportional to $\Phi n$, and as $\Phi \propto I_f$ we have

$$E = knI_f = knV/R_f$$

In the previous problem we found $E = 409.6$ V when $n = 600$ r/min, $R_f = 480\ \Omega$ and $V = 440$ V. Substituting these in the above equation leads to

$$k = \frac{ER_f}{nV} = \frac{409.6 \times 480}{600 \times 440} = 0.7447$$

The back e.m.f. is $V - I_a R_a = 400 - 85.9 \times 0.32 = 372.5$ V, so when $R_f = 480\ \Omega$ the speed is

$$n = \frac{ER_f}{kV} = \frac{372.5 \times 480}{0.7447 \times 400} = 600 \text{ r/min}$$

And if the speed is 660 r/min the field resistance is

$$R_f = \frac{knV}{E} = \frac{0.7447 \times 660 \times 400}{372.5} = 528\ \Omega$$

8 We first find the back e.m.f. from

$$E = V - I_a R_a = 200 - 18 \times 0.73 = 187 \text{ V}$$

The back e.m.f. is proportional to $\Phi n \propto n/R_f$, or

$$E = kn/R_f \Rightarrow k = ER_f/n = 187 \times 400/1320 = 56.7$$

Then when the motor is not loaded $E = V$, the supply voltage, very nearly, and

$$V = kn_0/R_f \Rightarrow R_f = kn_0/V = 56.7 \times 1320/200 = 374\ \Omega$$

The no-load speed is given by

$$n_0 = VR_f/k = 200 \times 510/56.7 = 1800\ \text{r/min}$$

with $R_f = 510\ \Omega$.

9 The armature voltage drop is $I_aR_a = 20.5 \times 0.96 = 19.7$ V. Then

$$E = V - I_aR_a = 240 - 19.7 = 220.3\ \text{V}$$

Now the field current, $I_f = V/R_f = 240/900 = 0.2667$ A, from which we find

$$\Phi = 60I_a - 50I_a^2 = 60 \times 0.2667 - 50 \times 0.2667^2 = 12.44\ \text{mWb}$$

And $E = k\Phi n$, so that

$$k = \frac{E}{\Phi n} = \frac{220.3}{12.44 \times 1222} = 0.01449\ \text{V/mWb/rpm}$$

The no-load speed, $n_0$, occurs when $E = V$ and then

$$V = k\Phi n_0 \Rightarrow \Phi = \frac{V}{kn_0} = \frac{240}{0.01449 \times 12.44} = 1331\ \text{r/min}$$

When $n_0 = 1100$ r/min

$$\Phi = \frac{V}{kn_0} = \frac{240}{0.01449 \times 1100} = 15.06\ \text{mWb}$$

We can then find $I_f$ from

$$\Phi = 60I_f - 50I_f^2 \Rightarrow I_f^2 - 1.2I_f + \Phi/60 = 0$$

$$I_f^2 - 1.2I_f + 0.3012 = 0$$

$$\Rightarrow I_f = 0.6 \pm \sqrt{0.6^2 - 0.3012} = 0.3575\ \text{A}$$

The other solution lies outside the range of the equation's applicability. Then $R_f = V/I_f = 240/0.3575 = 671\ \Omega$.

When $R_f = 1\ \text{k}\Omega$, $I_f = V/R_f = 240/1000 = 0.24$ A, which gives

$$\Phi = 60I_f - 50I_f^2 = 60 \times 0.24 - 50 \times 0.24^2 = 11.52\ \text{mWb}$$

As the armature current is the same as at first, $E = 220.3$ V also and as $E = \mathrm{k}\Phi n$

$$n = \frac{e}{k\Phi} = \frac{220.3}{0.01449 \times 11.52} = 1320 \text{ r/min}$$

And when $R_f = 500\ \Omega$, $I_f = 240/500 = 0.48$ A, from which we find the flux:

$$\Phi = 60I_f - 50I_f^2 = 60 \times 0.48 - 50 \times 0.48^2 = 17.28 \text{ mWb}$$

Hence

$$E = k\Phi n = 0.01449 \times 17.28 \times 900 = 225.3 \text{ V}$$

Therefore $$I_aR_a = V - E = 240 - 225.3 = 14.7 \text{ V}$$

$$\Rightarrow \quad I_a = \frac{V - E}{R_a} = \frac{14.7}{0.96} = 15.3 \text{ A}$$

**10** The armature voltage drop is $I_aR_a = 200 \times 0.125 = 25$ V, which makes the back e.m.f.

$$E = V - I_aR_a = 600 - 25 = 575 \text{ V}$$

The mechanical power is

$$P_m = EI_a = 575 \times 200 = 115 \text{ kW}$$

Hence the torque is

$$T = \frac{60P}{2\pi n} = \frac{60 \times 115}{2\pi \times 740} = 1.484 \text{ kNm}$$

The copper (armature joule) losses are

$$I_a^2R_a = 200^2 \times 0.125 = 5 \text{ kW}$$

And so the fixed losses are also 5 kW for a total loss of 10 kW. The power supplied is $P_S = VI_a = 600 \times 200 = 120$ kW, so the efficiency is

$$\eta = 1 - \frac{\text{losses}}{P_S} = 1 - \frac{10}{120} = 0.917 = 91.7\%$$

If the armature current is 90 A then the copper losses are

$$P_{Cu} = I_a^2R_a = 90^21 \times 0.,125 = 1012.5 \text{ W}$$

while the fixed losses remain at 5 kW, for total losses of 6 kW. Now the input power is $VI_a = 600 \times 90 = 54$ kW, making the efficiency

$$\eta = 1 - \frac{\text{losses}}{P_s} = 1 - \frac{6}{54} = 0.889 = 88.9\%$$

When $I_a = 250$ A the input power is $VI_a = 600 \times 250 = 150$ kW. The copper losses are

$$P_{Cu} = I_a^2 R_a = 250^2 \times 0.125 = 7.8125 \text{ kW}$$

The total losses are then $5 + 7.8125 = 12.8125$ kW and the efficiency is

$$\eta = 1 - \frac{\text{losses}}{P_s} = 1 - \frac{12.8125}{150} = 0.915 = 91.5\%$$

**11** When $T = 70$°C, $R_T = 0.816\ \Omega$ and then we find $R_0$ in the equation

$$R_T = R_0(1 + 0.005T) \Rightarrow R_0 = \frac{R_T}{1 + 0.005T} = \frac{0.816}{1 + 0.005 \times 70} = 0.604\ \Omega$$

Now the mechanical power developed is $EI_a$, and if this is halved when the field excitation and the speed are unchanged - implying that $E$ is also unchanged - then $I_a$ must be halved also to 7 A.

The armature losses are $I_a^2R_a$ and the temperature rise in the armature is proportional to them:

$$\Delta T \propto I_a^2 R_a = kI_a^2 R_a \Rightarrow k = \frac{\Delta T}{I_a^2 R_a}$$

When $I_a = 14$ A, $\Delta T = 70 - 20 = 50$°C, so that

$$k = \frac{\Delta T}{I_a^2 R_a} = \frac{50}{14^2 \times 0.816} = 0.3125\ °\text{C/W}$$

With $I_a = 7$ A, the armature losses are $7^2R_a = 49R_a$, and the armature temperature rise is

$$\Delta T = kI_a^2 R_a = 0.3125 \times 49R_a = 15.31R_a$$

Thus we can solve the equation for $R_a$, which is

$$R_a = R_0(1 + 0.005T) = 0.604(1 + 0.005[20 + \Delta T])$$

$$= 0.604(1.1 + 0.005\Delta T) = 0.604(1.1 + 0.005 \times 15.31R_a)$$

$$= 0.6644 + 0.0462R_a \Rightarrow R_a = \frac{0.6644}{0.9538} = 0.697\ \Omega$$

The armature power dissipation is thus $I_a^2R_a = 49 \times 0.697 = 34.1$ W, and its temperature rise is

$$\Delta T = kI_a^2R_a = 0.3125 \times 34.1 = 10.7°\text{C}$$

that is the armature temperature is 20 + 10.7 = 30.7°C.

When delivering rated power, $I_a$ = 14 A and the armature's temperature rise is

$$\Delta T = kI_a^2R_a = 0.3125 \times 14^2R_a = 61.25R_a$$

Hence

$$\begin{aligned} R_a &= R_0(1 + 0.005[40 + \Delta T]) \\ &= 0.604(1.2 + 0.005 \times 61.25R_a) \\ &= 0.7248 + 0.185R_a \end{aligned}$$

$$\Rightarrow \quad R_a = \frac{0.7248}{0.815} = 0.89\ \Omega$$

And then $\Delta T = 61.25R_a = 61.25 \times 0.89$ = 54.5°C, making $T$ = 40 + 54.5 = 94.5°C.

**12** First find the back e.m.f. from

$$E = V - I_aR_a = 650 - 466 \times 0.05 = 626.7\ \text{V}$$

The power output of the motor is $P_1 = E_1I_{a1} = 626.7 \times 466 = 292$ kW.

Next we can find the proportionality constant, $k$, in

$$E = kI_an \quad \Rightarrow \quad k = \frac{E}{I_an} = \frac{626.7}{466 \times 550} = 2.445\ \text{mV/A/rpm}$$

If the torque is constant while the speed is reduced to 400 r/min then the power developed is

$$P_2 = n_2P_1/n_1 = 400 \times 292/550 = 212.4\ \text{kW}$$

Thus the back e.m.f. must be

$$E_2 = kn_2I_{a2} = 2.445 \times 10^{-3} \times 400I_a = 0.978I_a$$

and then the power developed is

$$P_2 = E_2I_a2 = 0.978I_{a2}^2 = 212.4\ \text{kW}$$

$$\Rightarrow \quad I_{a2} = \sqrt{212400/0.978} = 466\ \text{A}$$

Thus $E_2 = 0.978I_{a2} = 455.76$ V. Then

$$I_{a2}R_{a2} = V - E_2 = 650 - 455.76 = 194.24 \text{ V}$$

$$\Rightarrow \quad R_a = 194.24/466 = 0.417 \ \Omega$$

The extra resistance is 0.417 − 0.05 = 0.367 Ω.

When the torque is 75% of its previous value the power developed is

$$P_3 = 0.75 \times \frac{n_1 P_1}{n_2} = 0.75 \times \frac{400P_1}{550} = 0.5455P_1 = 159.3 \text{ kW}$$

The back e.m.f. is

$$E_3 = kn_3I_{a3} = 2.445 \times 10^{-3} \times 400I_{a3} = 0.978I_{a3}$$

Therefore

$$P_3 = E_3I_{a3} = 0.978I_{a3}^{\,2} = 159300 \text{ W}$$

$$\Rightarrow \quad I_{a3} = \sqrt{159300/0.978} = 403.6 \text{ A}$$

And $E_3 = 0.978I_{a3} = 0.978 \times 403.6 = 394.7$ V, giving

$$R_{a3} = \frac{V - E_3}{I_{a3}} = \frac{650 - 394.7}{403.6} = 0.633 \ \Omega$$

Hence the extra resistance required is 0.63 − 0.05 = 0.58 Ω.

**13** Using the equation

$$E = knI_a$$

where we found in the previous problem that $k = 2.445 \times 10^{-3}$, with $I_a = 1$ kA and $n = 200$ r/min we find $E = 489$ V, and then

$$P_m = \frac{2\pi nT}{60} = EI_a$$

$$\Rightarrow \quad T = \frac{60EI_a}{2\pi n} = \frac{60 \times 489 \times 1000}{2\pi \times 200} = 23.35 \text{ kNm}$$

Also

$$R_a = \frac{V - E}{I_a} = \frac{650 - 489}{1000} = 0.161 \ \Omega$$

The power input to the motor is $P_S VI_a = 650 \times 1000 = 650$ kW, while the armature losses are $I_a^2R_a = 1000^2 \times 0.161 = 161$ kW, making the efficiency

$$\eta = 1 - \frac{\text{losses}}{P_S} = 1 - \frac{161}{650} = 75.2\%$$

# Chapter 14

1 Taking $\mathbf{E}_{RN}$ as the reference phasor in figure A14.1, the line voltage being 415 V,

$$\mathbf{E}_{RN} = \frac{415\angle 0^\circ}{\sqrt{3}} = 240\angle 0^\circ \text{ V}$$

Then the star-to-neutral voltage is

$$\mathbf{V}_{SN} = \frac{\mathbf{E}_{RN}\mathbf{Y}_R + \mathbf{E}_{YN}\mathbf{Y}_Y + \mathbf{E}_{BN}\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B}$$

where $\mathbf{Y}_R = \dfrac{1}{-j1000} = 1\angle 90^\circ$ mS and $\mathbf{Y}_Y = \mathbf{Y}_B = \dfrac{1}{1000} = 1\angle 0^\circ$ mS

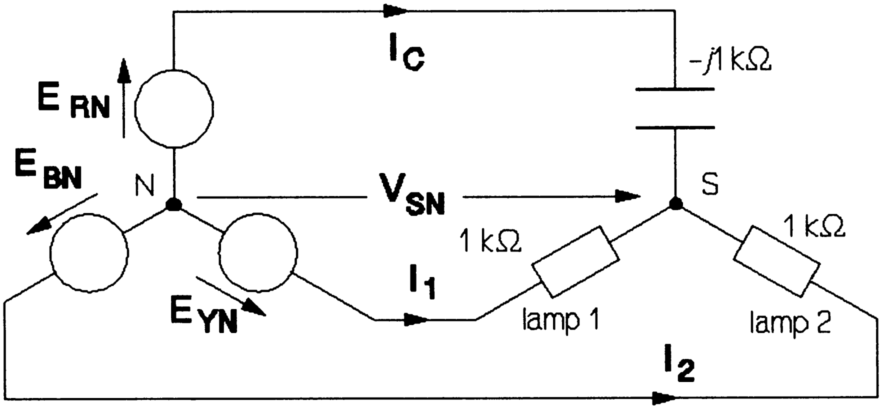


*Figure A14.1*

Thus
$$\mathbf{V}_{SN} = \frac{240\angle 0^\circ \times 1\angle 90^\circ + 240\angle -120^\circ \times 1\angle 0^\circ + 240\angle 120^\circ \times 1\angle 0^\circ}{1\angle 90^\circ + 1\angle 0^\circ + 1\angle 0^\circ}$$

$$= \frac{j240 - 120 - j208 - 120 + j208}{j1 + 1 + 1} = \frac{-240 + j240}{2 + j1}$$

$$= \frac{339\angle 135^\circ}{2.236\angle 26.57^\circ} = 152\angle 108.43^\circ \text{ V}$$

$\mathbf{I}_1$ is therefore given by

$$\mathbf{I}_1 = \frac{\mathbf{E}_{YN} - \mathbf{V}_{SN}}{1000} = \frac{240\angle -120° - 152\angle 108.43°}{1000}\ \text{A}$$

$$= \frac{-120 - j208 + 48 - j144}{1000} = \frac{-72 - j352}{1000} = 0.359\angle -101.6°\ \text{A}$$

and $\mathbf{I}_2$ by

$$\mathbf{I}_2 = \frac{\mathbf{E}_{BN} - \mathbf{V}_{SN}}{1000} = \frac{240\angle 120° - 152\angle 108.43°}{1000} = \frac{-120 + j208 + 48 - j144}{1000}$$

$$= \frac{-72 + j64}{1000} = 0.0963\angle 138.4°\ \text{A}$$

These are as obtained in section 14.3.2 apart from a phase shift of 30° caused by choosing $\mathbf{E}_{RN}$ as the reference phasor instead of $\mathbf{E}_{RY}$ ($\mathbf{E}_{RY} = \sqrt{3}\mathbf{E}_{RN}\angle 30°$).

2 The phase angle, $\phi$, is 35° and is given by

$$\tan\phi = \frac{\sqrt{3}(P_2 - P_1)}{P_1 + P_2} = \tan 35° = 0.7$$

so that, if $P_1 = 30.9$ kW, we have

$$\frac{\sqrt{3}(P_2 - P_1)}{P_1 + P_2} = \frac{\sqrt{3}(P_2 - 30.9)}{30.9 + P_2} = 0.7 \Rightarrow \sqrt{3}(P_2 - 30.9) = 21.63 + 0.7P_2$$

$$\Rightarrow P_2(\sqrt{3} - 0.7) = 30.9\sqrt{3} + 21.63 \qquad \Rightarrow P_2 = \frac{75.15}{1.032} = 72.8\ \text{kW}$$

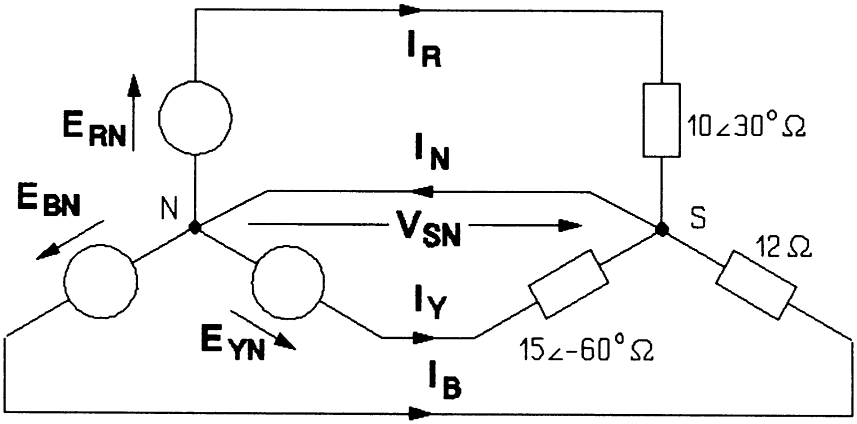


*Figure A14.3*

**3** Figure A14.3 shows the circuit. The star-neutral voltage is given by

$$\mathbf{V}_{SN} = \frac{\mathbf{E}_{RN}\mathbf{Y}_R + \mathbf{E}_{YN}\mathbf{Y}_Y + \mathbf{E}_{BN}\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B + \mathbf{Y}_N}$$

$\mathbf{E}_{RN} = 240\angle 0°$ V and is the reference phasor, while $\mathbf{Y}_R = 1/\mathbf{Z}_R$ etc. and then

$$\mathbf{V}_{SN} = \frac{\dfrac{240\angle 0°}{10\angle 30°} + \dfrac{240\angle -120°}{15\angle -60°} + \dfrac{240\angle 120°}{12}}{\dfrac{1}{10\angle 30°} + \dfrac{1}{15\angle -60°} + \dfrac{1}{12} + \dfrac{1}{0.1}} = \frac{24\angle -30° + 16\angle -60° + 20\angle 120°}{0.1\angle 30° + 0.067\angle 60° + 0.083 + 10}$$

$$= \frac{20.78 - j12 + 8 - j13.86 - 10 + j17.32}{0.087 - j0.05 + 0.033 + j0.058 + 10.083} = \frac{18.78 - j8.54}{10.2 + j0.008}$$

$$= \frac{20.63\angle -24.45°}{10.2\angle 0.04°} = 2.022\angle -24.5° \text{ V}$$

Having found $\mathbf{V}_{SN}$ we next find the line currents, starting with $\mathbf{I}_R$

$$\mathbf{I}_R = \frac{\mathbf{E}_{RN} - \mathbf{V}_{SN}}{\mathbf{Z}_R} = \frac{240\angle 0° - 2.022\angle -24.5°}{10\angle 30°} = \frac{240 - 1.84 + j0.84}{10\angle 30°}$$

$$= \frac{238.2\angle 0.2°}{10\angle 30°} = 23.83\angle -29.8° \text{ A}$$

And
$$\mathbf{I}_Y = \frac{\mathbf{E}_{YN} - \mathbf{V}_{SN}}{\mathbf{Z}_Y} = \frac{240\angle -120° - 2.022\angle -24.5°}{15\angle -60°}$$

$$= \frac{-120 - j207.85 - 1.84 + j0.84}{15\angle -60°} = \frac{-121.84 - j207.01}{15\angle -60°}$$

$$= \frac{240.2\angle -120.5°}{15\angle -60°} = 16.01\angle -60.5° \text{ A}$$

While $\mathbf{I}_B$ is

$$\mathbf{I}_B = \frac{\mathbf{E}_{BN} - \mathbf{V}_{SN}}{\mathbf{Z}_B} = \frac{240\angle 120° - 2.022\angle -24.5°}{12} = \frac{-120 + j207.85 - 1.84 + j0.84}{12}$$

$$= \frac{-121.84 + j208.69}{12} = \frac{241.65\angle 120.3°}{12} = 20.14\angle 120.3° \text{ A}$$

Finally, $$\mathbf{I}_N = \frac{\mathbf{V}_{SN}}{R_N} = \frac{2.022\angle -24.5°}{0.1} = 20.22\angle -24.5° \text{ A}$$

Checking that the line currents add up to $\mathbf{I}_N$ we have

$$\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B = 23.82\angle -29.8° + 16.01\angle -60.5° + 20.14\angle 120.3°$$

$$= 20.67 - j11.84 + 7.88 - j13.93 - 10.16 + j17.39$$

$$= 18.39 - j8.38 = 20.21\angle -24.5° \text{ A}$$

The power in the phases is

$$P_R = I_R^2 Z_R \cos\phi_R = 23.82^2 \times 10\cos 30° = 4.914 \text{ kW}$$

$$P_Y = I_Y^2 Z_Y \cos\phi_Y = 16.01^2 \times 15\cos 60° = 1.922 \text{ kW}$$

$$P_B = I_B^2 Z_B \cos\phi_B = 20.14^2 \times 12 = 4.867 \text{ kW}$$

$$P_N = I_N^2 R_N = 20.22^2 \times 0.1 = 40.9 \text{ W}$$

**4** The star-neutral voltage is found as before, but since there is no connection between the star point of the load and the neutral of the generator, $\mathbf{Y}_N = 0$ and is omitted from the denominator. The numerator of the expression for $\mathbf{V}_{SN}$ is the same as in the previous problem, $20.63\angle -26.63°$ A, but the denominator must be worked out again using more decimal places:

$$\mathbf{V}_{SN} = \frac{\mathbf{E}_{RN}\mathbf{Y}_R + \mathbf{E}_{YN}\mathbf{Y}_Y + \mathbf{E}_{BN}\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B} = \frac{20.63\angle -24.45°}{0.1\angle -30° + 0.0667\angle 60° + 0.08333}$$

$$= \frac{20.63\angle -24.45°}{0.0866 - j0.05 + 0.03333 + j0.05774 + 0.08333} = \frac{20.63\angle -24.45°}{0.2033 + j0.00774}$$

$$= \frac{20.63\angle -24.45°}{0.2034\angle 2.18°} = 101.4\angle -26.63° \text{ V}$$

As before we find the line currents:

$$\mathbf{I}_R = \frac{\mathbf{E}_{RN} - \mathbf{V}_{SN}}{\mathbf{Z}_R} = \frac{240\angle 0° - 101.4\angle -26.63°}{10\angle 30°} = \frac{240 - 90.64 - j45.45}{10\angle 30°}$$

$$= \frac{156.1\angle 16.92°}{10\angle 30°} = 15.61\angle -13.1° \text{ A}$$

$$\mathbf{I}_Y = \frac{240\angle -120° - 101.4\angle -26.63°}{15\angle -60°} = \frac{-120 - j207.85 - 90.64 + j45.45}{15\angle -60°}$$

$$= \frac{266\angle -142.4°}{15\angle -60°} = 17.73\angle -82.4°\ \text{A}$$

$$\mathbf{I}_B = \frac{240\angle 120° - 101.4\angle -26.63°}{12} = \frac{-120 + j207.85 - 90.64 + j45.45}{12}$$

$$= 27.45\angle 129.7°\ \text{A}$$

Then the power is:

$$P_R = I_R^2 Z_R \cos\phi_R = 15.61^2 \times 10\cos 30° = 2.11\ \text{kW}$$

$$P_Y = I_Y^2 Z_Y \cos\phi_Y = 17.73^2 \times 15\cos 60° = 2.358\ \text{kW}$$

$$P_B = I_B^2 Z_B \cos\phi_B = 27.45^2 \times 12 = 9.042\ \text{kW}$$

5 The line voltage is 415 V and since the load is delta-connected, $E_P = E_L = 415$ V. The phase current is

$$I_P = E_P/Z_P = 415/30 = 13.83\ \text{A}$$

The power developed in a balanced three-phase load is

$$P = P_1 + P_2 = 3E_P I_P \cos\phi = 3 \times 415 \times 13.83\cos 36° = 13.93\ \text{kW}$$

Then

$$\tan\phi = \frac{\sqrt{3}(P_1 - P_2)}{P_1 + P_2} = \tan 36° = 0.7265$$

$$\Rightarrow \quad \sqrt{3}(P_1 - P_2) = 0.7265(P_1 + P_2) = 0.7265 \times 13.93 = 10.12$$

$$\Rightarrow \quad P_1 - P_2 = 5.84 \quad \text{and as } P_1 + P_2 = 13.93,\ 2P_1 = 19.77$$

Hence $P_1 = 9.885$ kW and $P_2 = 4.045$ kW.

One wattmeter will read zero if either $\cos(\phi + 30°) = 0$ or if $\cos(30° - \phi) = 0$, that is if $\phi = +60°$ (inductive load) or $\phi = -60°$ (capacitive load). Here the load is inductive and the phase impedance is

$$\mathbf{Z}_P = 30\cos 36° + j\sin 36° = 24.27 + j17.63\ \Omega$$

With the resistive part held constant at 24.27 Ω, we require an inductive reactance to make the load's phase angle 60°, that is $X_L = 24.27\tan 60° = 42.04\ \Omega$.

In this case the phase impedance of the load is

$$\mathbf{Z}_P = \sqrt{24.27^2 + 42.04^2}\angle 60° = 48.54\angle 60° \ \Omega$$

and the magnitude of the phase current is

$$I_P = 415/48.54 = 8.55 \text{ A}$$

Thus as $P_2 = 0$,

$$P_1 + P_2 = P_1 = 3E_P I_P \cos\phi = 3 \times 415 \times 8.55\cos 60° = 5.32 \text{ kW}$$

Keeping the phase reactance of the load constant at 17.63 Ω and then making the impedance's phase angle 60° by varying $R$ requires that

$$R = 17.63/\tan 60° = 10.18 \ \Omega$$

And then the phase impedance is

$$\mathbf{Z}_P = 10.18 + j17.63 = 20.36\angle 60°$$

The phase current is therefore

$$I_P = E_P/Z_P = 415/20.36 = 20.38 \text{ A}$$

Thus $P_1 = 3E_P I_P \cos\phi = 3 \times 415 \times 20.38\cos 60° = 12.69$ kW.

**6** Figure A14.6 shows the symmetrical motor phases, the additional 3kW loads across the RB and BY phases and the connections of the two wattmeters. We can work out the line voltages from the given phase sequence, BYR, taking $\mathbf{E}_{RN}$ as the reference phase to start with:

$$\mathbf{E}_{RN} = 240\angle 0° \text{ V} \quad \mathbf{E}_{BN} = 240\angle -120° \text{ V} \quad \mathbf{E}_{YN} = 240\angle 120° \text{ V}$$

Then the line voltages are

$$\mathbf{E}_{RY} = \mathbf{E}_{RN} - \mathbf{E}_{YN} = 240\angle 0° - 240\angle 120° = 240 + 120 - j207.8 = 415\angle -30° \text{ V}$$

$$\mathbf{E}_{RB} = \mathbf{E}_{RN} - \mathbf{E}_{BN} = 240\angle 0° - 240\angle -120° = 240 + 120 + j207.8 = 415\angle 30° \text{ V}$$

$$\mathbf{E}_{BY} = \mathbf{E}_{BN} - \mathbf{E}_{YN} = 240\angle -120° - 240\angle 120° = -120 - j207.8 + 120 - j207.8$$

$$= 415\angle -90° \text{ V}$$

Adding 30° to these to make $\mathbf{E}_{RY}$ the reference phasor produces the line voltages

$$\mathbf{E}_{RY} = 415\angle 0°\ \text{V} \quad \mathbf{E}_{RB} = 415\angle 60°\ \text{V} \quad \mathbf{E}_{BY} = 415\angle -60°\ \text{V}$$

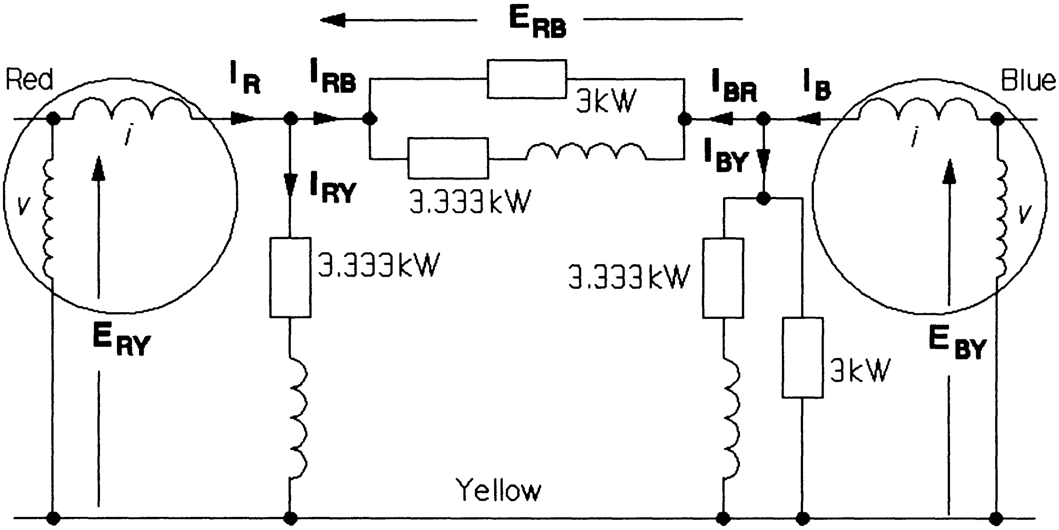


*Figure A14.6*

The next step is to work out the phase currents and hence the line currents, from which the wattmeter readings can be found. The motor consumes 10/3 = 3.333 kW per phase with a lagging phase angle of 30°, which means that $Q$, the reactive power, is

$$Q = P\tan\phi = 3.333\tan 30° = 1.924\ \text{kvar}$$

Only the motor's phase is connected between R and Y so that in this phase the apparent power, $S$ is

$$S = \sqrt{P^2 + Q^2} = \sqrt{3.333^2 + 1.924^2} = 3.848\ \text{kVA}$$

And so $I_{RY} = S/E_P = 3\,848/415 = 9.27$ A, lagging the line voltage by 30°. Since $\mathbf{E}_{RY}$ is the reference, $\mathbf{I}_{RY} = 9.27\angle -30°$ A.

In the other two phases is an extra resistance in parallel with the motor that consumes 3 kW, so that the total power for the phase is 6.333 kW and $Q$ is still 1.924 kvar, making the apparent power

$$S = \sqrt{P^2 + Q^2} = \sqrt{6.333^2 + 1.924^2} = 6.619\ \text{kVA}$$

Thus the phase current is $I_P = 6619/415 = 15.95$ A. This part of the load has a phase angle of $\tan^{-1}(Q/P) = \tan^{-1}0.3038 = 16.9°$ (lagging).

Therefore the current in the phase between the red and blue lines is $15.95\angle -16.9°$ plus the phase angle of the line, and $\mathbf{E}_{RB} = 415\angle 60°$ V, giving $\mathbf{I}_{RB} = 15.95\angle 43.1°$ A and the current in the phase between the blue and yellow lines will be $15.95\angle -76.9°$ A.

The line currents are found from these phase currents

$$\mathbf{I}_{\mathbf{R}} = \mathbf{I}_{\mathbf{RB}} + \mathbf{I}_{\mathbf{RY}} = 15.95\angle 43.1° + 9.27\angle -30° = 11.65 + j10.9 + 8.03 - j4.635$$

$$= 19.68 + j6.265 = 20.65\angle 17.66° \text{ A}$$

Then the power measured by the wattmeter whose current coil is in the red line is

$$P_{\text{R}} = E_{\text{RY}}I_{\text{R}}\cos\phi_{\text{R}} = 415 \times 20.65\cos 17.66° = 8.165 \text{ kW}$$

And the blue line current is

$$\mathbf{I}_{\mathbf{B}} = \mathbf{I}_{\mathbf{BR}} + \mathbf{I}_{\mathbf{BY}} = -\mathbf{I}_{\mathbf{RB}} + \mathbf{I}_{\mathbf{BY}} = -15.95\angle 43.1° + 15.95\angle -76.9°$$

$$= -11.65 - j10.9 + 3.615 - j15.535 = 27.63\angle -106.9° \text{ A}$$

so that the wattmeter whose current coil is in the blue line reads

$$P_{\text{B}} = E_{\text{BY}}I_{\text{B}}\cos\phi_{\text{B}} = 415 \times 27.63\cos 46.9° = 7.835 \text{ kW}$$

Note that $\phi_{\text{B}}$ is the angle between $\mathbf{E}_{\mathbf{BY}}$ and $\mathbf{I}_{\mathbf{B}}$.

# Chapter 15

1 Using the e.m.f. equation for the side with the most turns (the secondary) we find

$$E_2 = 2200 = 4.44B_mAfN_2 = 4.44 \times 0.55 \times 0.015^2 \times 15000N_2$$

Whence

$$N_2 = \frac{2200}{4.44 \times 0.55 \times 0.015^2 \times 15000} = 267 \text{ turns}$$

And the primary turns are then

$$N_1 = \frac{E_1N_2}{E_2} = \frac{500 \times 267}{2200} = 60.7 \text{ turns}$$

The nearest whole number is 61.

2 The step-up ratio is 4:1 and the primary voltage is 240 V, so the secondary voltage is to be 4 × 240 = 960 V, and the turns required are

$$N_2 = \frac{E_2}{4.44B_mAf} = \frac{960}{4.44 \times 0.7 \times 0.0082 \times 50} = 753 \text{ turns}$$

The primary turns are then 753/4 = 188, to the nearest turn.

3 The equivalent circuit is as in figure A15.3.

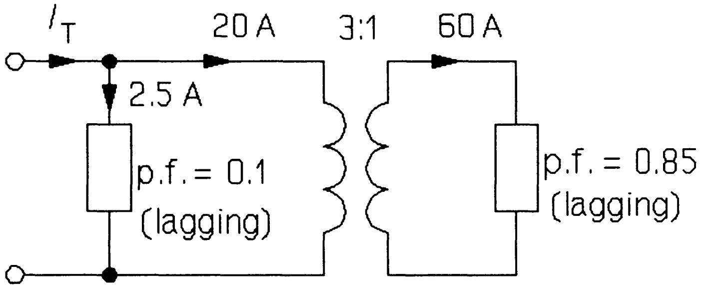


*Figure A15.3*

If the secondary current is 60 A, the primary current on 3:1 step-down transformer must be 60/3 = 20 A, at the same lagging p.f. of 0.85. We must then add to this the no-load current of 2.5 A at a lagging p.f. of 0.1 to obtain the primary current on load. If $\cos\phi = 0.85$, $\phi = 31.79°$ and $\sin\phi = 0.5268$, so that the load current referred to the primary is

$$\mathbf{I}_1 = 20\cos 31.79° + j20\sin 31.79° = 17 + j10.54 \text{ A}$$

while the no-load primary current's phase angle is $\cos^{-1}0.1 = 84.26°$ and this current is

$$\mathbf{I}_0 = 2.5\cos 84.26° + j2.5\sin 84.26° = 0.25 + j2.487 \text{ A}$$

The sum of the two currents is

$$\mathbf{I}_T = \mathbf{I}_1 + \mathbf{I}_0 = 17 + j10.54 + 0.25 + j2.487 = 21.6\angle 37.07° \text{ A}$$

The p.f. is then $\cos 37.07° = 0.798$ (lagging).

4 The circuit is shown in figure A15.4.

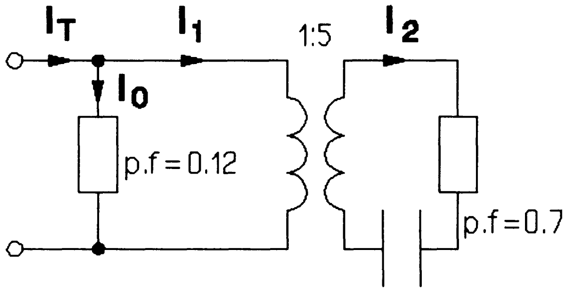


*Figure A15.4*

The secondary current is 8 A, which means that the primary current, $I_1$, is $8 \times 5 = 40$ A, since this is a step-up transformer with a secondary-to-primary voltage ratio of $V_2/V_1 = 5$. The load is capacitive with a leading p.f. of 0.7, making $\phi = -\cos^{-1}0.7 = -45.57°$ and the primary current due to the load is

$$\mathbf{I}_1 = 40\cos(-45.57°) + j40\sin(-45.57°) = 28 - j28.57 \text{ A}$$

The no-load current in the primary has a lagging p.f. of 0.12, so that $\phi = \cos^{-1}0.12 = 83.11°$ and the current is

$$\mathbf{I}_0 = 3\cos 83.11° + j\sin 83.11° = 0.36 + j2.978 \text{ A}$$

Therefore the total primary current is

$$\mathbf{I}_1 + \mathbf{I}_0 = 28 - j28.57 + 0.36 + j2.957 = 28.36 - j25.613 = 38.2\angle 42.1° \text{ A}$$

And thus the p.f. is $\cos 42.1° = 0.742$, leading.

5 The circuit is shown in figure A15.5.

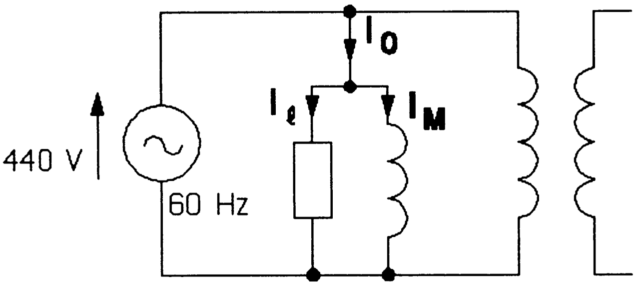


*Figure A15.5*

The no-load p.f. is 0.18 and thus $\phi = \cos^{-1}0.18 = 79.63°$. If the no-load current's magnitude is 1.4 A, then its phasor form is

$$\mathbf{I}_0 = 1.4\cos 79.63° + j1.4\sin 79.63° = 0.252 + j1.377 \text{ A} = I_l + jI_M$$

Thus the magnetising current, $I_M$, is 1.377 A. Since the primary voltage is 440 V, the power loss, $V_1I_0\cos\phi = V_1I_l = 440 \times 0.252 = 111$ W.

The maximum core flux is

$$B_mA = \frac{E_1}{4.44N_1f} = \frac{440}{4.44 \times 135 \times 60} = 0.0122 \text{ mWb}$$

6 Figure A15.6 shows the equivalent circuit.

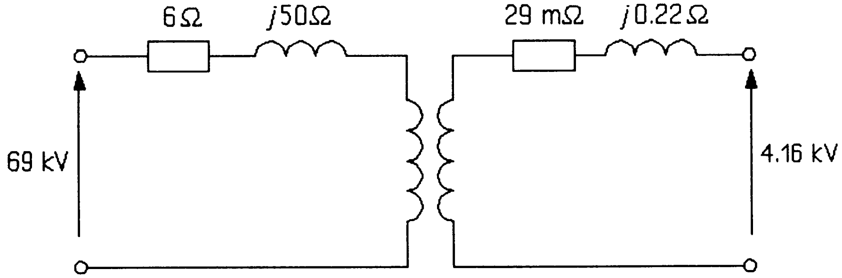


*Figure 15.6*

The turns ratio, $n$, from the 'nameplate' data is 69/4.16 = 16.59, so that secondary impedances are multiplied by $n^2 = 275$, when transformed into the primary. Hence the secondary has a resistance of 0.029 Ω and a leakage reactance of 0.22 Ω, which become $0.029 \times 275 = 7.975$ Ω and $0.22 \times 275 = 60.5$ Ω, in series with the primary resistance of 6 Ω and a leakage reactance of 50 Ω. Adding these to find the equivalent primary impedance gives

$$\mathbf{Z}_e = 6 + 7.975 + j(50 + 60.5) = 13.975 + j110.5 = 111.4\angle 82.79^\circ\ \Omega$$

(a) In supplying a resistive load of 2 MW, neglecting the no-load current means that the primary current is

$$\mathbf{I}_1 = \frac{2 \times 10^6}{69 \times 10^3}\angle 0^\circ = 29\angle 0^\circ\ \text{A}$$

The percentage regulation is

$$\frac{100(V_{2\text{NL}} - V_2)}{V_2} = \frac{100(V_{1\text{NL}} - E_1)}{V_{1\text{NL}}}$$

We find $E_1$ from

$$\mathbf{E}_1 = \mathbf{V}_1 - \mathbf{I}_1\mathbf{Z}_e$$

where $\mathbf{I}_1 = 29\angle 0^\circ$ A and $\mathbf{Z}_e = 111.4\angle 82.79^\circ\ \Omega$. Thus $\mathbf{I}_1\mathbf{Z}_e = 29 \times 111.4\angle 82.79^\circ = 3.231\angle 82.79^\circ$ kV and

$$\mathbf{E}_1 = 69\angle 0^\circ - 3.231\angle 82.79^\circ = 69 - 0.406 - j3.205$$

$$= 68.594 - j3.205 = 68.67\angle -2.68^\circ\ \text{kV}$$

Only the magnitude of $\mathbf{E}_1$ matters for regulation, so that

$$\%\text{Reg} = \frac{100 \times (69 - 68.67)}{69} = 0.48\%$$

(b) When the load is 3 MVA with a lagging p.f. of 0.9 (or $\phi = \cos^{-1}0.9 = 25.84^\circ$), the primary current is

$$\mathbf{I}_1 = \frac{3 \times 10^6}{69 \times 10^3}\angle -25.84^\circ = 43.48\angle -25.84^\circ\ \text{A}$$

Then

$$\mathbf{I}_1\mathbf{Z}_e = 43.48\angle -25.84^\circ \times 111.4\angle 82.79^\circ = 4.844\angle 56.95^\circ\ \text{kV}$$

$\mathbf{E}_1$ is found from

$$\mathbf{E}_1 = \mathbf{V}_1 - \mathbf{I}_1\mathbf{Z}_e = 69\angle 0^\circ - 4.884\angle 56.95^\circ = 69 - 2.664 - j4.094$$

$$= 66.336 - j4.094 = 66.46\angle -3.53^\circ\ \text{kV}$$

The regulation is therefore

$$\%\text{Reg} = \frac{100(69 - 66.46)}{69} = 3.7\%$$

The phase angle does not matter as far as regulation goes.

(c) When the load is 3 MVA at a leading p.f. of 0.95 (or $\phi = -\cos^{-1}0.95 = -18.19°$), the primary current is

$$\mathbf{I}_1 = \frac{3 \times 10^6}{69 \times 10^3} \angle 18.19° = 43.48 \angle 18.19° \text{ A}$$

giving

$$\mathbf{I}_1\mathbf{Z}_e = 43.48 \angle 18.19° \times 111.4 \angle 82.79° = 4.844 \angle 100.98° \text{ kV}$$

Hence $\mathbf{E}_1$ is

$$\mathbf{E}_1 = \mathbf{V}_1 - \mathbf{I}_1\mathbf{Z}_e = 69 \angle 0° - 4.884 \angle 100.98°$$

$$= 69 + 0.93 - 4.795 = 70.09 \angle -3.92° \text{ kV}$$

Now the regulation is

$$\%\text{Reg} = \frac{100(69 - 70.09)}{69} = -1.6\%$$

7 The circuit resembles that of figure A15.6. Working in per-unit values, both primary and secondary series (coil) resistances and leakage reactances are $0.002 + j0.05$ and referring the secondary to the primary will just double the values, so that the equivalent series impedance in the primary, $\mathbf{z}_e$, is

$$\mathbf{z}_e = 0.004 + j0.1 = 0.1 \angle 87.71° \text{ p.u.}$$

(a) The rated primary current is 1 p.u. and the voltage drop p.u. across $\mathbf{z}_e$ is

$$\mathbf{i}_1\mathbf{z}_e = 1 \angle 0° \times 0.1 \angle 87.71° = 0.1 \angle 87.71° \text{ p.u.}$$

The primary p.u. e.m.f., $\mathbf{e}_1$, is

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{i}_1\mathbf{z}_e = 1 \angle 0° - 0.1 \angle 87.71° = 1 - 0.004 - j0.1 = 1.001 \angle 5.73° \text{ p.u.}$$

And therefore

$$\%\text{Reg} = \frac{100(1 - 1.001)}{1} = -0.1\%$$

There is no need to convert the per-unit values to calculate the regulation.

(b) When rated power is supplied at a lagging p.f. of 0.8 ($\phi = \cos^{-1}0.8 = 36.87°$), the p.u. primary current is $\mathbf{i}_1 = 1 \angle -36.87°$, and the voltage drop across $\mathbf{z}_e$ is

$$\mathbf{i}_1\mathbf{z}_e = 1 \angle -36.87° \times 0.1 \angle 87.71° = 0.1 \angle 50.84° \text{ p.u.}$$

Then $\mathbf{e}_1$ is

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{i}_1\mathbf{z}_e = 1\angle 0° - 0.1\angle 50.84° = 1 - 0.0631 - j0.0775$$

$$= 0.94\angle -4.73° \text{ p.u.}$$

The regulation becomes

$$\%\text{Reg} = 100(1 - 0.94) = 6\%$$

(c) If the regulation with a lagging p.f. must be 1% or less then

$$\%\text{Reg} = 100(1 - e_1) = 1 \Rightarrow \quad e_1 = 0.99$$

But

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{i}_1\mathbf{z}_e = 1\angle 0° - 1\angle -\phi \times 0.1\angle 87.71° = 1 - 0.1\angle \theta$$

where $\theta = 87.71° - \phi$. ($\phi$ is positive as we have explicitly made the current lagging with a minus sign before $\phi$.) Putting this in rectangular form give

$$\mathbf{e}_1 = 1 - 0.1\angle \theta = 1 - 0.1\cos\theta - j0.1\sin\theta$$

Therefore the magnitude of $\mathbf{e}_1$ is

$$e_1 = \sqrt{(1 - 0.1\cos\theta)^2 + (0.1\sin\theta)^2} = \sqrt{1 + 0.01\cos^2\theta + 0.01\sin^2\theta - 0.2\cos\theta}$$

$$= \sqrt{1.01 - 0.2\cos\theta} = 0.99$$

Whence

$$1.01 - 0.2\cos\theta = 0.99^2 \Rightarrow \quad \cos\theta = 0.1495 \Rightarrow \theta = 81.4°$$

Thus $\phi = 87.71° - \theta = 87.71° - 81.4° = 6.31°$ and the smallest lagging p.f. that will permit 1% regulation is $\cos 6.31° = 0.994$.

8 The circuit resembles that of figure A15.5. The per-unit magnetising current is

$$\mathbf{i}_M = \frac{\mathbf{v}_1}{jx_M} = \frac{1\angle 0°}{50\angle 90°} = 0.02\angle -90°$$

And the p.u. core-loss current is

$$\mathbf{i}_1 = \frac{\mathbf{v}_1}{r_1} = \frac{1\angle 0°}{100\angle 0°} = 0.01\angle 0°$$

The p.u. standby current then becomes

$$\mathbf{i}_0 = \mathbf{i}_M + \mathbf{i}_l = 0.02\angle -90° + 0.01\angle 0° = 0.02236\angle -63.43° \text{ p.u.}$$

The base primary current is

$$I_{1B} = \frac{S}{V_1} = \frac{10^6}{33000} = 30.3 \text{ A}$$

making the standby current

$$\mathbf{I}_0 = I_{1B} \times \mathbf{i}_0 = 30.3 \times 0.02236\angle -63.43° = 0.678\angle -63.43° \text{ A}$$

The p.u. standby power loss is

$$p_0 = \frac{v_1^2}{r_1} = \frac{1^2}{100} = 0.01 \text{ p.u.}$$

And as the base power is 1 MVA, the standby power loss is

$$P_0 = P_B \times p_0 = Sp_0 = 10^6 \times 0.01 = 10 \text{ kW}$$

9 The impedance of 8% is made up of the primary leakage reactance plus the secondary leakage reactance referred to the primary, since the series resistance is negligible. If the p.u. values of these are the same then the primary leakage reactance is half of the impedance, or 4%. In the primary the base impedance is

$$Z_{1B} = \frac{V_1^2}{S} = \frac{(225 \times 10^3)^2}{60 \times 10^6} = 843.75 \ \Omega$$

making the primary leakage reactance 0.04 × 843.75 = 33.75 Ω.

In the secondary the base impedance is

$$Z_{2B} = \frac{V_2^2}{S} = \frac{(26.4 \times 10^3)^2}{60 \times 10^6} = 11.6 \ \Omega$$

and so the leakage reactance is 0.04 × 11.6 = 0.464 Ω.

The p.u. full-load copper losses are

$$p_{Cu} = i_1^2 r_1 + i_2^2 r_2 = 1^2 \times 0.002 + 1^2 \times 0.002 = 0.004 \text{ p.u.}$$

and since the base power is 60 MVA, this comes to

$$P_{Cu} = Sp_{Cu} = 60 \times 10^6 \times 0.004 = 240 \text{ kW}$$

If the efficiency is maximal on full load the copper losses equal the core (standby) losses, so that the total losses are twice the copper losses or 0.008 p.u., then the efficiency is

$$\eta = 1 - \text{p.u. losses} = 1 - 0.008 = 0.992$$

In calculating efficiency there is no need to convert the p.u. power.

**10** The power loss on no-load is

$$P_0 = V_1 I_0 \cos\phi = 330 \times 0.55 \times 0.3 = 54.45 \text{ kW}$$

Thus the losses on full load are twice these, 108.9 kW in all. The efficiency is then

$$\eta = 1 - \frac{\text{losses}}{P_{\text{input}}} = 1 - \frac{108.9 \times 10^3}{10 \times 10^6} = 1 - 0.011 = 0.989$$

If the core losses are 54.45 kW and 2 W/kg, the core must have a mass of

$$M = 54450/2 = 27225 \text{ kg} = 27.225 \text{ tonne}$$

And therefore the volume, $\Lambda$, is

$$\Lambda = M/\rho = 27.225/7.7 = 3.54 \text{ m}^3$$

where $\rho$ is the density.

The magnetising reactance is

$$X_M = \frac{V_1}{I_0 \sin\phi} = \frac{330 \times 10^3}{0.55 \sin 72.54°} = 629 \text{ k}\Omega$$

If the copper losses on full load are equal in primary and secondary, they are

$$P_{\text{Cu1}} = P_{\text{Cu2}} = \frac{54.45}{2} = 27.225 \text{ kW} = I_2^2 R_2$$

If the power rating is 10 MVA and $V_2$ = 22 kV, then $I_2 = 10^7/22000 = 455$ A, making

$$R_2 = P_{\text{Cu2}}/I_2^2 = 27225/455^2 = 0.132 \ \Omega$$

and $R_1 = 27225/30.3^2 = 29.7 \ \Omega$.

**11** On full load the copper losses equal the core losses and are half the total losses. The total losses are 100 − 99 = 1%, so the core losses are 0.5%. The core losses will be constant while the copper losses will go as the square of the current supplied.

If the transformer supplies full load for 4 h/day the energy output is

$$E_{FL} = P_{FL}t_{FL} = 4P_{FL} \text{ Wh}$$

where $P_{FL}$ = full-load power in W and $t_{FL}$ = time at full load in h.

During this time 1% of the input energy is lost, so the losses are

$$E_{lost} = 0.01P_{FL}t_{FL} = 0.04P_{FL} \text{ Wh}$$

On half load the current supplied is half and so the copper losses are 1/4 of the full-load losses, or 0.125%. The core losses, however, remain constant at 0.5% of the full-load power. The total losses are therefore 0.125% + 0.5% = 0.615%.

The output energy is

$$P_{HL} = 0.5P_{FL}t_{HL} = 0.5 \times 8P_{FL} = 4P_{FL} \text{ Wh}$$

While the losses are

$$E_{lost} = 0.00625 \times P_{FL}t_{HL} = 0.00625 \times 8P_{FL} = 0.05P_{FL} \text{ Wh}$$

On no load the output is zero and the losses are 0.5% of full-load losses. The lost energy is

$$E_{lost} = 0.005P_{FL}T_{NL} = 0.005 \times 12P_{FL} = 0.06P_{FL}$$

The total energy output is therefore $4P_{FL} + 4P_{FL} = 8P_{FL}$ Wh. The total energy lost is

$$\Sigma E_{lost} = 0.04P_{FL} + 0.05P_{FL} + 0.06P_{FL} = 0.15P_{FL}$$

The total input energy is $8.15P_{FL}$ Wh and the all-day efficiency is

$$\eta = \frac{\text{energy out}}{\text{energy in}} = \frac{8}{8.15} = 0.9816$$

**12** Let the full-load power output be $P$. Let the maximum efficiency, $\eta_{max} = 1 - 2\epsilon$, then at a power output of $\alpha P$, the fixed power losses are $\epsilon\alpha P$ and the copper power losses are also $\epsilon\alpha P$. Now at full power the output current is $1/\alpha$ times the current at an output power of $\alpha P$, so that the copper losses are $1/\alpha^2$ times those at an output power of $\alpha P$, that is

$$P_{Cu}(\text{f.p.}) = \frac{1}{\alpha^2} \times \epsilon\alpha P = \frac{\epsilon P}{\alpha}$$

If full power is maintained $x$ h/day, then the energy lost per diem in the copper is

$$E_{Cu}(\text{f.p.}) = \epsilon Px/\alpha$$

Similarly, at half power, the current is $1/2\alpha$ times that at $\alpha P$ and the copper losses are therefore $1/4\alpha^2$ times those at $\alpha P$, that is

$$P_{\mathrm{Cu}}(\text{h.p.}) = \frac{1}{4\alpha^2} \times \epsilon\alpha P = \frac{\epsilon P}{4\alpha}$$

If half power is maintained for $y$ h/day, the energy lost per diem in the copper is

$$E_{\mathrm{Cu}}(\text{h.p.}) = \epsilon Py/4\alpha$$

Thus the total energy lost in the copper per diem is

$$E_{\mathrm{Cu}} = \frac{\epsilon Px}{\alpha} + \frac{\epsilon Py}{4\alpha}$$

But the fixed losses of $\epsilon\alpha P$ continue 24 h/day, for an energy loss per diem of $24\epsilon\alpha P$. The total of the copper and fixed losses per diem is

$$E_{\mathrm{T}} = \frac{\epsilon Px}{\alpha} + \frac{\epsilon Py}{4\alpha} + 24\epsilon\alpha P$$

The only variable here is $\alpha$ and we can maximise efficiency by minimising these losses. Differentiating with respect to $\alpha$ and setting equal to zero gives

$$\frac{dE_{\mathrm{T}}}{d\alpha} = -\frac{\epsilon Px}{\alpha^2} - \frac{\epsilon Py}{4\alpha^2} + 24\epsilon P = 0$$

$$\Rightarrow \quad \alpha^2 = \frac{x + y/4}{24} \quad \Rightarrow \quad \alpha = \sqrt{\frac{x + y/4}{24}}$$

Applying this formula to problem 15.11 where $x = 4$ h, $y = 8$ h and $\eta_{\max} = 99\%$, we find

$$\alpha = \sqrt{\frac{x + y/4}{24}} = \sqrt{\frac{4 + 8/4}{24}} = 0.5$$

And

$$2\eta_{\max} = 1 - 2\epsilon \Rightarrow \epsilon = 0.5(1 - \eta_{\max}) = 0.5(1 - 0.99) = 0.005$$

The energy lost in the copper at full power is

$$E_{\mathrm{Cu}}(\text{f.p.}) = \epsilon Py/\alpha = 0.005 \times 4P/0.5 = 0.04P \text{ Wh}$$

And at half power it is

$$E_{\mathrm{Cu}}(\text{h.p.}) = \epsilon Py/4\alpha = 0.005 \times 8P/(4 \times 0.5) = 0.02P \text{ Wh}$$

While the fixed losses are

$$E_{\text{fixed}} = 24\epsilon\alpha P = 24 \times 0.005 \times 0.5P = 0.06P \text{ Wh}$$

We see that the copper and the fixed losses in a day are indeed equal, so that the total energy lost is 0.12*P*. The energy delivered in 24 h is

$$E_{\text{out}} = 4P + 8 \times 0.5P = 8P \text{ Wh}$$

This gives an all-day efficiency of

$$\eta_{\text{all-day}} = 1 - \frac{E_{\text{lost}}}{E_{\text{out}} + E_{\text{lost}}} = 1 - \frac{0.12}{8.12} = 0.9852$$

which is an improvement of 0.36% on that obtained when maximum efficiency was at rated output power.

**13** If the power rating is 500 kVA, then the output energy when operated for 4 h/day at rated power and 8 h/day at half power and unity p.f. is

$$E_{\text{out}} = 500 \times 4 + 0.5 \times 500 \times 8 = 4000 \text{ kWh/day}$$

Thus if a kWh costs 5p, the annual cost of this output is

$$C = 365 \times 5 \times 4000 = 7.3 \times 10^6 \text{ p}$$

If the all-day efficiency is 98.16%, then the cost of the losses is 1.84% of this sum or £1343 per annum. Improving the all-day efficiency to 98.52% means that the losses would be reduced to 1.48% or £1080.40, a saving of £262.60, a modest sum in view of the cost of installation, let alone the transformer itself.

**14** The equivalent circuit is shown in figure A15.14, in which the standby current, $I_0$, is 1.2 A and the standby power consumption is 65 W. The standby power consumption is given by

$$P_0 = 65 = V_1^2/R_0 = 240^2/R_0 \Rightarrow R_0 = 240^2/65 = 886 \ \Omega$$

The standby current is made up of the magnetising current, $\mathbf{I}_{\text{M}}$, and the core-loss current, $\mathbf{I}_l$, and the latter is

$$I_l = V_1/R_0 = 240/886 = 0.271 \text{ A}$$

so that, since $\mathbf{I}_{\text{M}}$ is in quadrature to $\mathbf{I}_l$

$$I_0^2 = 1.2^2 = I_M^{\,2} + I_l^2$$

From which

$$I_M = \sqrt{I_0^2 - I_l^2} = \sqrt{1.2^2 - 0.271^2} = 1.169 \text{ A}$$

Then

$$X_M = V_1/I_M = 240/1.169 = 205 \ \Omega$$

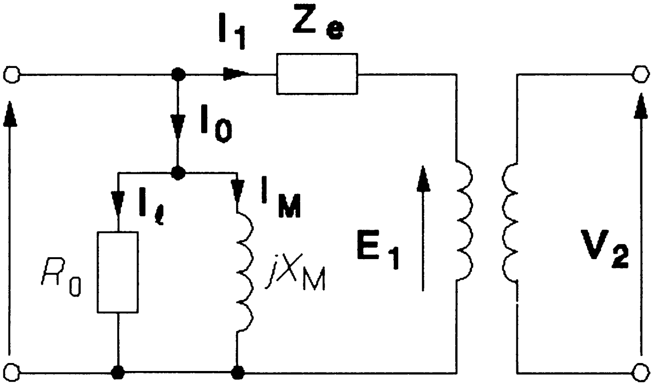


*Figure A15.14*

The rated primary current is $S/V_1 = 5000/240 = 20.83$ A. And since this is produced with 9.8 V on the primary, the effective impedance in the primary must be

$$Z_e = V_1/I_1 = 9.8/20.83 = 0.47 \ \Omega$$

The short-circuit test produces a power consumption of 77 W with rated current, which implies that

$$P_{SC} = I_1^2 R_e \ \Rightarrow \ R_e = P_{SC}/I_1^2 = 77/20.83^2 = 0.177 \ \Omega$$

Therefore

$$\mathbf{Z}_e = Z_e\cos\phi + jZ_e\sin\phi \ \Rightarrow \ Z_e\cos\phi = R_e$$

$$\Rightarrow \ \phi = \cos^{-1}(R_e/Z_e) = \cos^{-1}(0.177/0.47) = 67.8°$$

and $\mathbf{Z}_e = 0.47\angle 67.8° \ \Omega$.

When supplying full load at a p.f. of 0.9 the power output is

$$P_{out} = S\cos\phi = 5 \times 0.9 = 4.5 \text{ kW}$$

while the losses will be the standby losses of 65 W plus the short-circuit losses of 77 W, for a total of 142 W and then the efficiency is

$$\eta = \frac{P_{\text{out}}}{P_{\text{out}} + P_{\text{lost}}} = \frac{4500}{4500 + 142} = 0.9694$$

The load phase angle is $\cos^{-1}0.9 = 25.84°$ lagging, and so the input current is $20.83\angle -25.84°$ A. Thus the voltage drop across $\mathbf{Z}_e$ is

$$\mathbf{I}_1\mathbf{Z}_e = 20.83\angle\text{-}25.84° \times 0.47\angle 67.8° = 9.79\angle 41.96° \text{ V}$$

Hence the primary e.m.f. is

$$\mathbf{E} = \mathbf{V}_1 - \mathbf{I}_1\mathbf{Z}_e = 240\angle 0° - 9.79\angle 41.96° = 240 - 7.28 - j6.55 = 232.8\angle 1.61° \text{ V}$$

and the secondary terminal voltage is then 0.1 × 232.8 = 23.3 V.

# Chapter 16

1 The synchronous speed is given by

$$n_s = \frac{60f_s}{p} = \frac{60 \times 60}{(18/2)/3} = \frac{3600}{3} = 1200 \text{ r/min}$$

where $p$ is the number of pole pairs (= 18/2) per phase (= 3).
(a) The slip when the rotor speed is 1150 r/min is

$$s = \frac{n_s - n}{n_s} = \frac{1200 - 1150}{1200} = 0.0417 = 4.17\%$$

This is positive and < 1, so the machine is motoring.
(b) Above synchronous rotor speed, the machine is acting as a generator and the slip is

$$s = \frac{n_s - n}{n_s} = \frac{1200 - 1500}{1200} = -0.25 = -25\%$$

(c) The rotor speed is

$$n = 60f = 60\omega/2\pi = 60 \times -16/2\pi = -152.8 \text{ r/min}$$

Therefore the slip is

$$s = \frac{n_s - n}{n_s} = \frac{1200 + 152.8}{1200} = 1.127$$

The slip is positive and > 1, so the machine is braking.

2 At standstill $n = 0$ and the slip is 1. The machine may operate as a brake or a motor, depending on its state immediately prior to standstill.

If the slip is −33%, or −0.33, then the speed is

$$n = n_s - sn_s = 2000 - (-0.33) \times 2000 = 1.33 \times 2000 = 2660 \text{ r/min}$$

Since the slip is negative the machine is a generator.

If $s = 110\% = 1.1$, then

$$n = n_s - sn_s = 2000 - 1.1 \times 2000 = -200 \text{ r/min}$$

The slip is positive and > 1, so the machine is braking.

**3** The frequency of the rotor e.m.fs. is

$$f_r = sf_s = 60s$$

where $f_s$ is the supply frequency, 60 Hz. Thus the rotor e.m.f. frequencies are:

(a) $60 \times 0.0417 = 2.5$ Hz

(b) $60 \times (-0.25) = -15$ Hz

(c) $60 \times 1.127 = 67.6$ Hz

**4** The torque-slip equation is

$$T = \frac{K\alpha s}{s^2 + \alpha^2}$$

Differentiating with respect to $s$ yields

$$\frac{dT}{ds} = \frac{K\alpha(s^2 + \alpha^2) - 2s(K\alpha s)}{(s^2 + \alpha^2)^2}$$

which is zero when

$$K\alpha(s^2 + \alpha^2) = 2K\alpha s^2 \quad \Rightarrow \quad s^2 = \alpha^2$$

When $s = +\alpha$, the solution gives a maximum for $T$, and when $s = -\alpha$, it gives a minimum. Substituting $s = -\alpha$ leads to

$$T = \frac{-K\alpha^2}{\alpha^2 + \alpha^2} = \frac{-K}{2} = \frac{-V^2}{2\omega_s X_0}$$

**5** Equation 16.24 shows that the ratio of standstill torque to breakdown torque is

$$\frac{T_0}{T_m} = \frac{2\alpha}{1 + \alpha^2} = \frac{300}{800} = \frac{3}{8}$$

whence we find a quadratic equation for $\alpha$:

$$\alpha^2 - \frac{16\alpha}{3} + 1 = 0 \quad \Rightarrow \quad \alpha = \frac{8}{3} \pm \sqrt{\left[\frac{8}{3}\right]^2 - 1}$$

which leads to $\alpha = 5.14$ or $\alpha = 0.1946$, from which we take the latter value as $\alpha = R_2/X_0$ and is always less than 1.

This value of $\alpha$ can be substituted into equation 16.22 to produce

$$T = \frac{Ks\alpha}{s^2 + \alpha^2} = \frac{2T_m s\alpha}{s^2 + \alpha^2} = \frac{1.6 \times 0.1946s}{s^2 + 0.1946^2} = \frac{0.3114s}{s^2 + 0.0379}$$

since equation 16.23 gives $T_m = K/2$.

Then when $s = 0.03$,

$$T = \frac{0.3114 \times 0.03}{0.03^2 + 0.0379} = 0.241 \text{ Nm}$$

When $s = -0.5$, we find

$$T = \frac{0.3114 \times (-0.5)}{(-0.5)^2 + 0.0379} = -0.541 \text{ Nm}$$

And when $s = 1.2$ we obtain

$$T = \frac{0.3114 \times 1.2}{1.2^2 + 0.0379} = 0.253 \text{ Nm}$$

The standstill torque is 0.3 Nm, and then

$$T_0 = 0.3 = \frac{0.3114s}{s^2 + 0.0379} \Rightarrow s^2 - 1.038s + 0.0379 = 0$$

Solving the quadratic gives

$$s = 0.519 \pm \sqrt{0.519^2 - 0.0379} = 1 \text{ or } 0.0379$$

When the slip is 0.0379, the torque is equal to the standstill torque.

It is easily shown that this value is $\alpha^2$, since

$$T_0 = \frac{K\alpha s}{s^2 + \alpha^2} = \frac{K\alpha}{1 + \alpha^2} \Rightarrow s(1 + \alpha^2) = s^2 + \alpha^2$$

$$\Rightarrow s^2 - (1 + \alpha^2)s + \alpha^2 = 0 \Rightarrow s = 1 \text{ or } \alpha^2$$

**6** The per-phase equivalent circuit is shown in figure A16.6, in which $R_1$ is the stator series resistance, which must be half the resistance between two stator terminals if the connection is star; thus $R_1 = 0.5 \times 0.69 = 0.345\ \Omega$.

When not loaded $s \approx 0$, so the branch with $R_2/s$ in it is effectively open circuit, and the only components that draw current are $X_M$ and $R_0$. The line voltage of 415 V means a phase voltage of 240 V; the apparent power per phase is

$$S = V_P I_0 = 240 \times 0.9 = 216 \text{ VA}$$

The power consumed per phase is 106/3 = 35.3 W, and this is

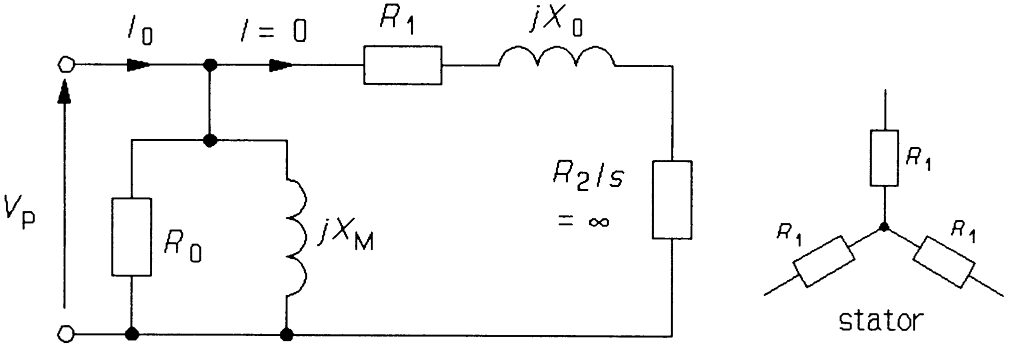


*Figure A16.6*

$$P = V_P^2/R_0 \quad \Rightarrow \quad R_0 = V_P^2/P = 240^2/35.3 = 1.63 \text{ k}\Omega$$

The reactive power per phase is

$$Q = \sqrt{S^2 - P^2} = \sqrt{216^2 - 35.3^2} = 213 \text{ var}$$

which is given by

$$Q = \frac{V_P^2}{X_M} \quad \Rightarrow \quad X_M = \frac{V_P^2}{Q} = \frac{240^2}{213} = 270 \ \Omega$$

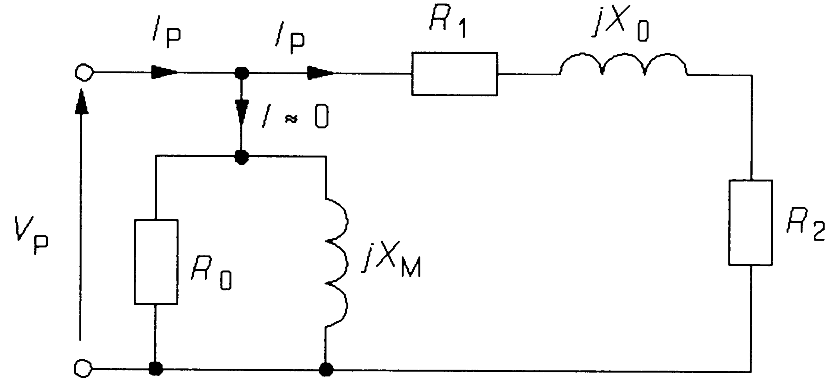


*Figure A16.7*

7 In the locked-rotor test, $s = 1$, and the equivalent circuit of figure A16.6 becomes that of figure A16.7, in which the current drawn by $X_M$ and $R_0$ is neglected ($I_0 \approx 0$). The line voltage is 258.5 V, making the phase voltage $258.5/\angle 3$ V. The apparent power per phase is

$$S = V_P I_P = 258.5 \times 15/\sqrt{3} = 2.239 \text{ kVA}$$

Now the total power consumption is 1.2 kW or 400 W/phase, and this is given by

$$P = I_P^2(R_1 + R_2) \Rightarrow R_1 + R_2 = P/I_P^2 = 400/15^2 = 1.778\ \Omega$$

And as $R_1 = 0.345\ \Omega$, $R_2 = 1.778 - 0.345 = 1.433\ \Omega$.

We find the value of $X_0$ from the reactive power as before:

$$Q = \sqrt{S^2 - P^2} = \sqrt{2239^2 - 400^2} = 2.203 \text{ kvar}$$

But the reactive power is given by

$$Q = I_P^2 X_0 \Rightarrow X_0 = Q/I_P^2 = 2203/15^2 = 9.79\ \Omega$$

8 The maximum torque is given by

$$T_m = \frac{V_P^2}{2\omega_s X_0}$$

where $\omega_s$ is

$$\omega_s = \frac{2\pi n_s}{60} = \frac{2\pi \times 3000}{60} = 314.6 \text{ rad/s}$$

and the phase voltage, $V_P$, is $415/\sqrt{3}$ V. Substituting into the equation for maximum torque yields

$$T_m = \frac{415^2}{3 \times 2 \times 314.6 \times 9.79} = 9.32 \text{ Nm}$$

This per phase, making the total torque 3 × 9.32 = 28 Nm.

At maximum torque the slip is $\alpha = R_2/X_0 = 1.433/9.79 = 0.146$ or 14.6%.

The maximum power in the rotor is

$$P_r = \omega_s T_m = 314.6 \times 28 = 8.8 \text{ kW}$$

The phase current is found from

$$\frac{P_r}{3} = \frac{I_P^2 R_2}{s} \Rightarrow I_P = \sqrt{\frac{sP_r}{3R_2}} = \sqrt{\frac{0.146 \times 8800}{3 \times 1.433}} = 17.3 \text{ A}$$

since the power per phase is $P_r/3$. The phase current is equal to the line current in a star-connected stator.

Finally, the copper losses in the rotor are given by

$$P_{rj} = 3I_P^2 R_2 = 3 \times 17.3^2 \times 1.433 = 1.29 \text{ kW}$$

The losses per phase are $I_P^2R_2$.

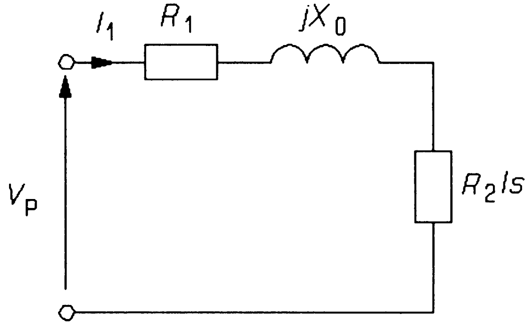


*Figure A16.9*

**9** The current drawn is reduced when we include $R_1$, the stator resistance. The equivalent circuit of figure A16.9 is used, from which we see that the current in the stator, $I_1$, is

$$\mathbf{I}_1 = \frac{V_P}{R_1 + R_2/s + jX_0} = \frac{240}{0.345 + 1.433/s + j9.79}$$

This current flows through the rotor-equivalent resistance $R_2/s$, developing rotor power of

$$P_r = I_1^2R_2/s$$

But $$I_1^2 = \frac{240^2}{(0.345 + 1.433/s)^2 + 9.79^2}$$

so that the rotor power is

$$P_r = \frac{240^2 \times 1.433/s}{(0.345 + 1.433/s)^2 + 9.79^2} = \frac{as}{(bs + c)^2 + es^2}$$

where $a = 240^2 \times 1.433 = 82540$, $b = 0.345$, $c = 1.433$ and $e = 9.79^2 = 95.84$. We can find when $P_r$ is a maximum by differentiating with respect to $s$

$$\frac{dP_r}{ds} = \frac{a[(bs + c)^2 + es^2] - [2b(bs + c) + 2es]as}{D^2} = 0$$

where $D$ stands for the denominator in the expression for $P_r$. This leads to

$$b^2s^2 + 2bcs + c^2 + es^2 = 2b^2s^2 + 2bcs + 2es^2$$

$$\Rightarrow \quad s = \sqrt{\frac{c^2}{b^2 + e}} = \sqrt{\frac{1.433^2}{0.345^2 + 95.84}} = 0.146$$

The slip is the same as before because $R_1^2 << X_0^2$.

Substituting 0.146 for $s$ in the expression for $P_r$ produces

$$P_r = \frac{82540/0.146}{(0.345 + 1.433/0.146)^2 + 9.79^2} = 2.84 \text{ kW/phase}$$

The total power is three times this, 8.52 kW, which is equal to $T_m\omega_s = 2\pi T_m n_s/60$, giving

$$T_m = \frac{60P_r}{2\pi n_s} = \frac{60 \times 8520}{2\pi \times 3000} = 27 \text{ Nm}$$

The current, $I_1$, is

$$\mathbf{I}_1 = \frac{240}{(0.345 + 1.433/0.146) + j9.79} = \frac{240}{10.16 + j9.79}$$

$$\Rightarrow \quad I_1 = \frac{240}{\sqrt{10.16^2 + 9.79^2}} = 17.0 \text{ A}$$

The stator's copper losses are $3I_1^2R_1 = 3 \times 17^2 \times 0.345 = 299$ W.

**10** Since the maximum power is developed at 3% slip we can find $X_0$ from

$$s = \alpha = R_2/X_0 = 0.08/X_0 \quad \Rightarrow \quad X_0 = 2.667\ \Omega$$

The maximum torque is given by

$$T_m = \frac{V_P^2}{2\omega_s X_0} = \frac{V_L^2/3}{4\pi n_s X_0/60} = \frac{6900^2/3}{4\pi \times 10 \times 2.667} = 47.35 \text{ kNm/phase}$$

making 3 × 47.35 = 142 kNm in total, and the rotor power is

$$P_r = \omega_s T_m = 2\pi n_s T_m/60 = 8.93 \text{ MW}$$

Ignoring the stator's copper losses and the no-load losses, the efficiency is approximately

$$\eta \approx \frac{P_m}{P_r} = \frac{(1 - s)R_2I_2^2/s}{I_2^2R_2/s} = 1 - s = 97\%$$

Including the stator resistance, $R_1$, means that the current, $I_1$, is given by

$$\mathbf{I}_1 = \frac{\mathbf{V}_P}{(R_1 + R_2/s) + jX_0} \quad \Rightarrow \quad I_1 = \frac{V_P}{\sqrt{(R_1 + R_2/s)^2 + X_0^2}}$$

Putting the values of $V_P$, $R_1$, $R_2$, $X_0$ and $s$ into this equation leads to

$$I_1 = \sqrt{\frac{6900^2/3}{(0.1 + 2.667)^2 + 2.667^2}} = 1.037 \text{ kA}$$

Then the rotor power is

$$P_r = I_1^2 R_2/s = 1.037^2 \times 10^6 \times 2.667 = 2.868 \text{ MW/phase}$$

or 3 × 2.868 = 8.6 MW in total.

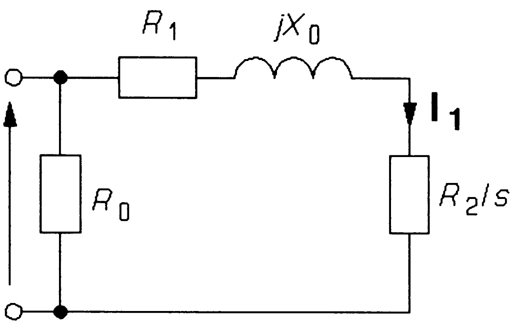


*Figure A16.11*

**11** If the parallel loss resistance is included in the equivalent circuit, then it looks like that of figure A16.11, from which we deduce that the parallel losses per phase are

$$P_0 = \frac{V_P^2}{R_0} = \frac{V_L^2}{3R_0} = \frac{6900^2}{3 \times 590} = 26.9 \text{ kW}$$

These losses must equal the copper losses in rotor and stator for maximum efficiency to be achieved. The copper losses are

$$P_{Cu} = I_1^2(R_1 + R_2) = I_1^2(0.1 + 0.08) = 26.9 \text{ kW}$$

$$\Rightarrow \quad I_1^2 = \frac{26900}{0.18} \quad \Rightarrow \quad I_1 = \sqrt{149444} = 386.6 \text{ A}$$

But from figure A16.11 we can see that

$$\mathbf{I}_1 = \frac{V_P}{(R_1 + R_2/s) + jX_0} = \frac{6900/\sqrt{3}}{(0.1 + 0.08/s) + j2.667}$$

Thus the magnitude of $\mathbf{I}_1$ is

$$I_1 = \frac{6900/\sqrt{3}}{\sqrt{(0.1 + 0.08/s)^2 + 2.667^2}} \Rightarrow \frac{6900^2/3}{(0.1 + 0.08/s)^2 + 7.113} = 386.6^2$$

Whence $$(0.1 + 0.08/s)^2 + 7.113 = \left[\frac{6900/\sqrt{3}}{386.6}\right]^2 = 106.2$$

$$\Rightarrow \qquad 0.1 + 0.08/s = \sqrt{99.07} = 9.953$$

$$s = \frac{0.08}{9.853} = 0.00812$$

At this slip the rotor power/phase is

$$P_r = I_1^2 R_2/s = 386.6^2 \times 0.08/0.00812 = 1.4725 \text{ MW}$$

The power lost is $2P_0 = 53.8$ kW/phase and thus the maximum efficiency is

$$\eta_{\max} = 1 - \frac{2P_0}{P_r} = 1 - \frac{53.8 \times 10^3}{1.4725 \times 10^6} = 0.963$$

At this efficiency the stator current is 386.6 A per phase, so the stator's apparent power is

$$S = V_P I_P = \frac{6900}{\sqrt{3}} \times 386.6 = 1.54 \text{ MVA}$$

for a total motor power of 3 × 1.54 = 4.62 MVA.

**12** The motor is at standstill when starting, so that $s = 1$ and the impedance per phase of the motor is

$$\mathbf{Z} = R_1 + R_2/s + jX_0 = R_1 + R_2 + jX_0 = 3 + j6.6 \ \Omega$$

The stator current is

$$\mathbf{I}_1 = \mathbf{V}_P/\mathbf{Z} = \frac{415/\sqrt{3}}{3 + j6.6} \Rightarrow I_1 = \frac{415/\sqrt{3}}{\sqrt{3^3 + 6.6^2}} = 33.05 \text{ A}$$

The rotor power/phase is

$$P_r = I_1^2R_2/s = 33.05^2 \times 1.6 = 1.748 \text{ kW}$$

which totals 3 × 1.748 = 5.244 kW. But the rotor power is given by

$$P_r = T\omega_s = 2\pi n_s T_0/60 \Rightarrow T_0 = \frac{60P_r}{2\pi n_s} = \frac{60 \times 5244}{2\pi \times 1500} = 33.4 \text{ Nm}$$

The maximum torque can be found from the maximum rotor power, which in turn is found from

$$P_r = I_1^2R_2/s = \frac{V_P{}^2R_2}{s[(R_1 + R_2/s)^2 + X_0^2]} \qquad \text{(A12.1)}$$

$$P_r = \frac{As}{(R_1s + R_2)^2 + s^2X_0^2} = \frac{As}{(1.4s + 1.6)^2 + 43.56s^2}$$

where $A \equiv V_P{}^2R_2 = (415^2/3) \times 1.6 = 91\,853$. Differentiating with respect to $s$ gives

$$\frac{dP_r}{ds} = \frac{A[(1.4s + 1.6)^2 + 43.56s^2] - As[2.8(1.4s + 1.6) + 87.52s]}{D^2}$$

where $D$ stands for the denominator. This expression is zero when

$$(1.4s + 1.6)^2 + 43.52s^2 = 2 \times 1.4s(1.4s + 1.6) + 87.12s^2 \qquad \text{(A12.2)}$$

$$\Rightarrow \quad 45.52s^2 = 2.56 \quad \Rightarrow \quad s = 0.237$$

(taking the positive square root) giving a maximum power/phase of

$$P_r(\text{max}) = \frac{91\,853 \times 0.237}{(1.4 \times 0.237 + 1.6)^2 + (0.237 \times 6.6)^2} = 3.523 \text{ kW}$$

This must equal $T\omega_s$:

$$T_m = \frac{P_r}{\omega_s} = \frac{60P_r}{2\pi n_s} = \frac{60 \times 3523}{2\pi \times 1500} = 22.43 \text{ Nm}$$

which is the maximum torque/phase so the total maximum torque is 3 × 22.43 = 67.3 Nm.

When a resistance of 5 Ω is placed in series with the stator, it effectively increases $R_1$ by 5 Ω to 6.4 Ω and the stator current at startup ($s = 1$) is

$$\mathbf{I}_1 = \frac{V_P}{(R_1 + R_2/s) + jX_0} = \frac{415/\sqrt{3}}{(6.4 + 1.6) + j6.6}$$

$$\Rightarrow \qquad I_1 = \frac{415/\sqrt{3}}{\sqrt{8^2 + 6.6^2}} = 23.1 \text{ A}$$

The rotor power/phase is then

$$P_r = I_1^2 R_2 = 23.1^2 \times 1.6 = 854 \text{ W}$$

And the per-phase torque is

$$T_0 = \frac{60 P_r}{2\pi n_s} = \frac{60 \times 854}{2\pi \times 1500} = 5.44 \text{ Nm}$$

giving a total start-up torque of 16.3 Nm.

A reactance of $j5$ Ω in series with the stator changes the start-up current to

$$I_1 = \frac{415/\sqrt{3}}{\sqrt{3^3 + 11.6^2}} = 20 \text{ A}$$

and the start-up rotor power/phase is then

$$P_r = I_1^2 R_2 = 20^2 \times 1.6 = 640 \text{ W}$$

giving a start-up torque of 4.07 Nm/phase and a total start-up torque of 12.2 Nm.

The maximum torque when a resistance is placed in series with the stator is found in the same way as before, but with $R_1 = 6.4$ Ω instead of 1.4 Ω. Substituting 6.4 for 1.4 in equation A12.2 gives

$$(6.4s + 1.6)^2 + 43.52s^2 = 2 \times 6.4s(1.4s + 1.6) + 87.12s^2$$

$$\Rightarrow \quad 84.48s^2 = 2.56 \quad \Rightarrow \quad s = 0.174$$

Hence the stator current is

$$I_1 = \frac{V_P}{(R_1 + R_2/s)^2 + X_0^2} = \frac{415/\sqrt{3}}{\sqrt{(6.4 + 1.6/0.174)^2 + 6.6^2}} = 14.15 \text{ A}$$

giving a maximum rotor power/phase of

$$P_r = I_1^2 R_2/s = 14.15^2 \times 1.6/0.174 = 1.841 \text{ kW}$$

which leads to a maximum torque/phase of

$$T_m = \frac{60P_r}{2\pi n_s} = \frac{60 \times 1841}{2\pi \times 1500} = 11.72 \text{ Nm}$$

and a total maximum torque of 35.2 Nm.

Placing a reactance of $j5$ Ω in series with the stator means that we must replace $X_0 = 6.6$ Ω in equation A12.1 with 11.6, which produces

$$P_r = I_1^2 R_2/s = \frac{V_P^2 R_2}{s[(R_1 + R_2/s)^2 + X_0^2]}$$

$$= \frac{As}{(R_1 s + R_2)^2 + s^2 X_0^2} = \frac{As}{(1.4s + 1.6)^2 + 134.56s^2}$$

And differentiating this with respect to $s$ results in

$$\frac{dP_r}{ds} = \frac{A[(1.4s + 1.6)^2 + 134.56s^2] - As[2.8(1.4s + 1.6) + 269.12s]}{D^2}$$

Setting the numerator equal to zero, we find

$$136.52s^2 = 2.56 \Rightarrow s = 0.137$$

Then the stator current becomes

$$I_1 = \frac{V_P}{(R_1 + R_2/s)^2 + X_0^2} = \frac{415/\sqrt{3}}{\sqrt{(1.4 + 1.6/0.137)^2 + 11.6^2}} = 13.71 \text{ A}$$

leading to a maximum rotor power/phase of

$$P_r = I_1^2 R_2/s = 13.71^2 \times 1.6/0.137 = 2.195 \text{ kW}$$

The maximum torque per phase is

$$T_m = \frac{60P_r}{2\pi n_s} = \frac{60 \times 2195}{2\pi \times 1500} = 13.97 \text{ Nm}$$

for a total for three phases of 41.9 Nm.

13 The rotor power of 12 kW implies 4 kW per phase. The maximum torque is given by

$$T_m = \frac{V_P^2}{2\omega_s X_0} \Rightarrow X_0 = \frac{V_P^2}{2\omega_s T_m} = \frac{V_P^2}{2P_r(\max)} = \frac{415^2}{2 \times 4000} = 21.5 \ \Omega$$

At maximum torque, the slip is given by

$$s = \alpha = R_2/X_0 \quad \Rightarrow \quad R_2 = sX_0 = 0.18 \times 21.5 = 3.875\ \Omega$$

The maximum torque is

$$T_m = P_r/\omega_s = \frac{60P_r}{2\pi n_s} = \frac{60 \times 12000}{2\pi \times 1800} = 63.7\ \text{Nm}$$

When a series resistance, $R$, is included in the equivalent circuit, this is as in figure A16.9, with $R_1$ replaced by $R$. The stator current is then

$$\mathbf{I}_1 = \frac{V_P}{(R + R_2/s) + jX_0} \quad \Rightarrow \quad I_1 = \frac{415}{\sqrt{(R + 3.875/s)^2 + 21.5^2}}$$

And the rotor power/phase is

$$P_r = I_1^2R_2/s = \frac{415^2 \times 3.875/s}{(R + 3.875/s)^2 + 21.5^2}$$

Differentiating this with respect to $s$ leads to

$$\frac{dP_r}{ds} = \frac{(-A/s^2)[(R + 3.875/s)^2 + 21.5^2] - 2(R + 3.875/s)(-3.975/s^2)(A/s)}{D^2}$$

where $D$ stands for the denominator. Setting the numerator equal to zero to obtain the condition for $P_r$ to be a maximum gives

$$(R + 3.875/s)^2 + 21.5^2 = 2(R + 3.875/s)(3.875/s)$$

$$\Rightarrow \quad R^2 + 21.5^2 = (3.875/s)^2$$

Thus when $s = 0.1$, this condition becomes

$$R^2 = (3.875/0.1)^2 - 21.5^2 \quad \Rightarrow \quad R = \sqrt{38.75^2 - 21.5^2} = 32.2\ \Omega$$

In this case the rotor power/phase is

$$P_r = I_1^2R_2/s = \frac{V_P^2R_2/s}{(R + R_2/s)^2 + X_0^2} = \frac{415^2 \times 38.75}{(32.2 + 38.75)^2 + 21.5^2} = 1.214\ \text{kW}$$

which is 3.64 kW in all. The torque is

$$T_m = 60P_r(\text{max})/2\pi n_s = 60 \times 3640/2\pi \times 1800 = 19.3\ \text{Nm}$$

**14** Maximum torque is obtained when the rotor power is maximum, so we must find the condition for this. Using the equivalent circuit of figure A16.9, the stator current is

$$I_1 = \frac{V_P}{\sqrt{(R_1 + R_2/s)^2 + X_0^2}}$$

and thus the rotor power is

$$P_r = I_1^2 R_2/s = \frac{V_P^2 R_2/s}{(R_1 + R_2/s)^2 + X_0^2} = \frac{V_P^2 R_2 s}{(R_1 s + R_2)^2 + s^2 X_0^2}$$

Differentiating with respect to $s$ leads to

$$\frac{dP_r}{ds} = \frac{V_P^2 R_2[(R_1 s + R_2)^2 + s^2 X_0^2] - V_P^2 R_2 s[2R_1(R_1 s + R_2) + 2sX_0^2]}{D^2}$$

where $D$ is the denominator. Setting the numerator equal to zero for a maximum in $P_r$ gives

$$(R_1 s + R_2)^2 + s^2 X_0^2 = 2R_1 s(R_1 s + R_2) + 2s^2 X_0^2$$

$$\Rightarrow \quad R_2^2 = (R_1^2 + X_0^2)s^2 \quad \Rightarrow \quad s = \frac{R_2}{\sqrt{R_1^2 + X_0^2}} = s_0$$

Letting $\beta \equiv \sqrt{(R_1^2 + X_0^2)}$, so that $s_0 = R_2/\beta$ for maximum power, we substitute for $s$ in the expression for the rotor power/phase and obtain

$$P_r(\max) = \frac{V_P^2 R_2/s_0}{(R_1 + R_2/s_0)^2 + X_0^2} = \frac{V_P^2 \beta}{(R_1 + \beta)^2 + X_0^2}$$

Since $\beta = \sqrt{(R_1^2 + X_0^2)}$, this means that $P_r(\max)$ is independent of $R_2$. The expression can be simplified by substituting for $\beta$ as follows:

$$P_r(\max) = \frac{V_P^2 \beta}{R_1^2 + 2R_1\beta + \beta^2 + X_0^2} = \frac{V_P^2 \beta}{2R_1\beta + 2\beta^2} = \frac{V_P^2}{2(R_1 + \beta)}$$

With three phases the total power is three times this.

# Chapter 17

1 Figure A17.1a shows the equivalent circuit of one phase of the star-connected generator, in which the line voltage of 6.9 kV has been converted to the phase voltage of 6.9/√3 kV. Attached to this phase of the generator is an 8Ω load ($R_{LS}$), which is the star-equivalent of the 24Ω delta-connected load.

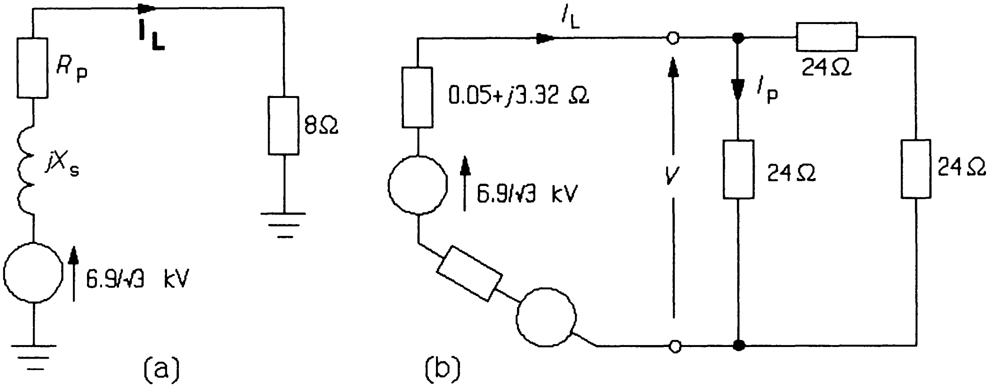


*Figure A17.1*

Neglecting the generator's phase resistance, $R_P$, means that the short-circuit current is

$$I_{SC} = 1.2 = \frac{6.9/\sqrt{3}}{X_s} \Rightarrow X_s = \frac{6.9/\sqrt{3}}{1.2} = 3.32\ \Omega$$

When the load is attached, the line current flowing is

$$\mathbf{I}_L = \frac{E_P}{R_P + jX_s + R_{LS}} = \frac{6.9/\sqrt{3}}{0.05 + j3.32 + 8} = \frac{6.9/\sqrt{3}}{8.05 + j3.32}$$

Taking magnitudes gives

$$I_L = \frac{6.9/\sqrt{3}}{\sqrt{8.05^2 + 3.32^2}} = 0.4575\ \text{kA}$$

The power dissipated in one phase of the load is $I_L^2 R_{LS}$, making the total load power

$$P = 3I_L^2 R_{LP} = 3 \times 0.4575^2 \times 8 = 5.02\ \text{MW}$$

The power dissipated in the generator per phase is $I_L^2 R_P$, giving a total of

$$P_G = 3I_L^2 R_P = 3 \times 457.5^2 \times 0.05 = 31400\ \text{W}$$

The regulation is defined by

$$\text{Reg} = \frac{E - V}{V}$$

where $E$ is the generator's open-circuit line e.m.f (6.9 kV) and $V$ is the terminal voltage on load, in this case that is the voltage across one phase of the load, as in figure A17.1b, which shows the load in delta form.

For a balanced, delta-connected load the phase current, $I_P$, is $I_L/\surd 3$, so that the terminal voltage on load is

$$V = I_P R_{LD} = I_L R_{LD}/\sqrt{3} = 457.5 \times 24/\sqrt{3} = 6.339 \text{ kV}$$

$R_{LD}$ is the delta-equivalent load resistance per phase. Thus the regulation is

$$\frac{E - V}{V} = \frac{6.9 - 6.339}{6.339} = 0.0885$$

2 The short-circuit ratio is 188/95 = 1.98 and the per-unit synchronous reactance is

$$x_s = \frac{1}{\text{SCR}} = \frac{1}{1.98} = 0.505$$

Assuming a star-connected generator, the base impedance per phase is

$$Z_B = \frac{E_P}{I_L} = \frac{11/\sqrt{3}}{1.575} = 4.03 \ \Omega$$

And thus the synchronous reactance is

$$X_s = Z_B x_s = 4.03 \times 0.505 = 2.035 \ \Omega$$

3 The circuit of figure A17.3 shows that the generator's per-unit e.m.f., $\mathbf{e} = \mathbf{v} + jx_s\mathbf{i}$, where $\mathbf{v}$, the reference phasor, is $1 \angle 0°$ p.u.
(a) Since $\mathbf{i}$ is rated current at unity p.f., $\mathbf{i}$ is also $1 \angle 0°$ p.u. Then

$$\mathbf{e} = \mathbf{v} + jx_s\mathbf{i} = 1 + j0.33 \times 1 = 1 + j0.33 \text{ p.u.}$$

To find the regulation we need only the magnitude of $\mathbf{e}$, that is

$$|\mathbf{e}| = e = \sqrt{1^2 + 0.33^2} = 1.053 \text{ p.u.}$$

And the regulation becomes

$$\frac{e - v}{v} = \frac{1.053 - 1}{1} = 0.053$$

In p.u. terms the regulation is just $e - 1$.

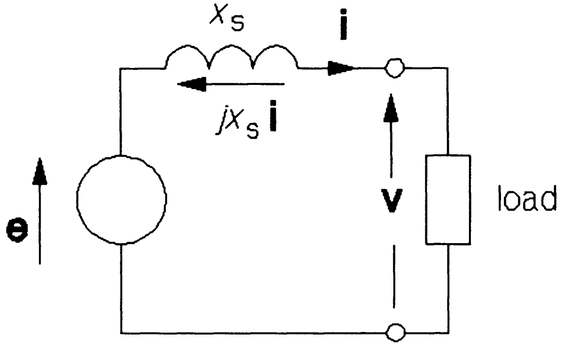


*Figure A17.3*

(b) When the load has a lagging p.f. of 0.8, the current is $1\angle -\cos^{-1}0.8 = 1\angle -36.87°$ p.u. and then the generator's p.u. e.m.f. is

$$\mathbf{e} = \mathbf{v} + jx_s\mathbf{i} = 1 + j0.33 \times 1\angle -36.87° = 1 + 0.33\angle(90° - 33.87°)$$

$$= 1 + 0.33\angle 53.13° = 1.198 + j0.264 \text{ p.u.}$$

Then the magnitude of **e** is $\surd(1.198^2 + 0.264^2) = 1.227$, and the regulation is 0.227 or 22.7%.

(c) With a leading p.f. of 0.8 and rated load, the current is $1\angle 36.87°$ p.u. This gives a generator e.m.f. of

$$\mathbf{e} = 1 + j0.33 \times 1\angle 36.87° = 1 + 0.33\angle 126.87°$$

$$= 1 - 0.198 + j0.264 = 0.802 + j0.264 \text{ p.u.}$$

Then $e = \surd(0.802^2 + 0.264^2) = 0.844$ and the regulation is $0.844 - 1 = -15.6\%$.

Including the per-phase resistance of the generator has an appreciable effect on the regulation, though it will have almost no effect on the short-circuit ratio, since the per-phase impedance of the generator is only slightly larger (0.3338 p.u.) than the per-unit synchronous reactance.

The calculations are exactly as before, but we use $\mathbf{z}_s$ $(= 0.05 + j0.33 = 0.3338 \angle 81.38°$ p.u.) in place of $jx_s$.

(a) The generator e.m.f. is given by

$$\mathbf{e} = \mathbf{v} + \mathbf{iz}_s = 1 + 1\angle 0° \times 0.3338\angle 81.38° = 1.05 + j0.33 \text{ p.u.}$$

thus $e = \surd(1.05^2 = 0.33^2) = 1.101$ and so the regulation is $1.101 - 1 = 10.1\%$.

(b) The current is $1\angle -36.87°$ p.u. and the e.m.f. is

$$\mathbf{e} = 1\angle 0° + 1\angle -36.87° \times 0.3338\angle 81.38° = 1\angle 0° + 0.3338\angle 44.51°$$

$$= 1.238 + j0.234 \text{ p.u.}$$

and therefore $e = \surd(1.238^2 + 0.234^2) = 1.26$, giving a regulation of $1.26 - 1 = 26\%$.

(c) This time the current is $1\angle 36.87°$ p.u. and the e.m.f. is

$$\mathbf{e} = 1\angle 0° + 1\angle 36.87° \times 0.338\angle 81.38° = 1 + 0.3338\angle 118.25°$$

$$= 1 - 0.158 + j0.294 = 0.842 + j0.294 \text{ p.u}$$

giving a regulation of $\surd(0.842^2 + 0.294^2) = 0.892 - 1 = -10.8\%$.

**4** The torque angle is that between **e** and **v** and since we have chosen **v** as the reference phasor, the torque angle is just the phase angle of **e**, the per-unit e.m.f. of the generator. A glance at the answers to the previous problem will show that neglecting the phase resistance of the generator, the e.m.fs. are:

(a) $\mathbf{e} = 1 + j0.33 = 1.053\angle 18.26°$ p.u., $\delta = 18.26°$.
(b) $\mathbf{e} = 1.198 + j0.264 = 1.227\angle 12.43°$ p.u., $\delta = 12.43°$.
(c) $\mathbf{e} = 0.802 + j0.264 = 0.844\angle 18.2°$ p.u., $\delta = 18.2°$.

And when the generator's phase resistance is included they are:

(a) $\mathbf{e} = 1.05 + j0.33 = 1.101\angle 17.45°$ p.u., $\delta = 17.45°$.
(b) $\mathbf{e} = 1.238 + j0.234 = 1.26\angle 10.7°$ p.u., $\delta = 10.7°$.
(c) $\mathbf{e} = 0.842 + j0.294 = 0.892\angle 19.25°$ p.u., $\delta = 19.25°$.

**5** The circuit is as in figure A17.3, from which it can be seen that

$$\mathbf{e} = \mathbf{v} + jx_s\mathbf{i}$$

But the current is to be rated current with a phase angle of $\phi$, that is $\mathbf{i} = 1\angle\phi$ p.u. The synchronous reactance is $jx_s = x_s\angle 90°$ p.u. and the reference p.u. phasor is $\mathbf{v} = 1\angle 0°$.

$$\therefore \quad \mathbf{e} = \mathbf{v} + x_s\angle 90° \times 1\angle\phi = 1\angle 0° + x_s\angle(90° + \phi)$$

$$= 1 + x_s\cos(90° + \phi) + jx_s\sin(90° + \phi)$$

$$= 1 + x_s(\cos 90°\cos\phi - \sin 90°\sin\phi) + jx_s(\sin 90°\cos\phi + \cos 90°\sin\phi)$$

$$= (1 - x_s\sin\phi) + jx_s\cos\phi$$

The magnitude of **e** is

$$e = \sqrt{(1 - x_s\sin\phi)^2 + (x_s\cos\phi)^2} = \sqrt{1 - 2x_s\sin\phi + x_s^2(\cos^2\phi + \sin^2\phi)}$$

$$= \sqrt{1 - 2x_s\sin\phi + x_s^2}$$

The regulation is zero when $e = v = 1$, that is when

$$\sqrt{1 - 2x_s\sin\phi + x_s^2} = 1 \Rightarrow \sin\phi = \tfrac{1}{2}x_s$$

Or when $$\text{p.f} = \cos\phi = \sqrt{1 - \tfrac{1}{4}x_s^2}$$

The phase angle of the current, $\phi$, must be positive (leading load p.f.) or $e > 1$.

The per-unit synchronous reactance of the generator in problem 17.3 is 0.33 and therefore the regulation will be zero when

$$\cos\phi = \sqrt{1 - 0.25 \times 0.33^2} = 0.9863 \Rightarrow \phi = 9.5°$$

**6** The generator has a synchronous reactance of

$$x_s = 1/\text{SCR} = 1/2.5 = 0.4 \text{ p.u.}$$

When it supplies its rated power then $v = 1$ and $i = 1$ and if the p.f. of the power supplied is 0.7 leading it means that the current's phase angle is $\cos^{-1}0.7 = +45.57°$, that is $\mathbf{i} = 1\angle 45.5°$ p.u. The circuit of figure A17.3 pertains and the generated e.m.f. must be

$$\mathbf{e} = \mathbf{v} + jx_s\mathbf{i} = 1\angle 0° + 0.4\angle(90° + 45.57°) = 1\angle 0° + 0.4\angle 135.57°$$

$$= 1 - 0.286 + j0.28 = 0.714 + j0.28 = 0.767\angle 21.4° \text{ p.u.}$$

Thus the actual e.m.f. is 0.767 times the line-neutral voltage, $33/\surd 3$ kV, which is 14.61 kV. The torque angle is the phase angle of the e.m.f. as **v** is the reference phasor, that is $\delta = 21.4°$.

The pull-out torque occurs when $\delta = 90°$ and then the power is

$$p = w_s t = 1 \times t = \frac{ev\sin\delta}{x_s} = \frac{0.767}{0.4} = 1.92 \text{ p.u.}$$

In per-unit terms the power, $p = t$, the per-unit torque, as $\omega_s = 1$ p.u.

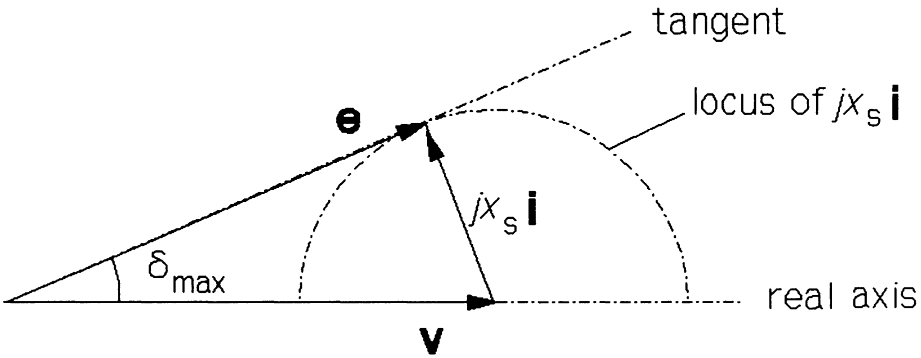


*Figure A17.7*

7 The phasor diagram for the generator is shown in figure A17.7, in which the locus of the $jx_s\mathbf{i}$ phasor is shown as a dotted semicircle of radius 0.4 p.u. (as $i = 1$ p.u. and $x_s = 0.4$ p.u.) centred on the end of **v**, which is the reference phasor and of length 1. Since $\mathbf{e} = \mathbf{v} + jx_s\mathbf{i}$, we find **e** by drawing a line from the origin to the end of the $jx_s\mathbf{i}$ phasor, from which we deduce that the maximum angle between **v** and **e** (which is the torque angle, $\delta$) occurs when **e** is a tangent to the semicircle, that is when **e** and $jx_s\mathbf{i}$ are at right angles. Therefore, by Pythagoras's theorem

$$e^2 + ix_s^2 = v^2 \Rightarrow e^2 + 0.4^2 = 1^2 \Rightarrow e = \sqrt{0.84} = 0.9165 \text{ p.u}$$

Then the actual e.m.f. is $0.9165 \times 33/\sqrt{3} = 17.46$ kV.

The torque angle is

$$\sin\delta = ix_s/v = 0.4 \Rightarrow \delta = 23.6°$$

The angle between the $jx_s\mathbf{i}$ phasor and **v** is $90° + \delta$, implying that the angle between **i** and **v** is 90° less, that is $\delta$ or 23.6°. The p.f. is therefore cos23.6° = 0.916.

8 The circuit diagram for one phase of the synchronous motor is as shown in figure 17.8a, being identical to that of the generator, but with the direction of the current reversed.

(a) The line voltage of 415 V becomes a phase voltage of $415/\sqrt{3} = 239.6$ V and thus $\mathbf{V} = 239.6\angle 0°$ V, being the reference phasor. The power consumption is 50 kW or

16.67 kW/phase, and thus the line current is

$$I_L = \frac{S}{V} = \frac{P}{V} = \frac{16667}{239.6} = 69.56 \text{ A} \Rightarrow \quad \mathbf{I}_L = 69.56\angle 0° \text{ A}$$

$P = S$ when the p.f. is unity and the current's phase angle is zero. Then the voltage across the synchronous reactance is

$$jX_s\mathbf{I}_L = j0.7 \times 69.56 = j48.69 \text{ V}$$

And therefore the generator e.m.f. becomes

$$\mathbf{E} = \mathbf{V} - jX_s\mathbf{I}_L = 239.6\angle 0° - j48.69 = 244.5\angle -11.5° \text{ V}$$

So if $E = 250$ V when $I_{ex} = 10$ A,

$$I_{ex} = \frac{244.5}{250} \times 10 = 9.78 \text{ A}$$

The torque angle can be found from

$$P = \left[\frac{EV}{X_s}\right] \sin\delta \Rightarrow \quad \sin\delta = \frac{PX_s}{EV} = \frac{16667 \times 0.7}{244.5 \times 239.6} = 0.199$$

which gives $\delta = 11.5°$.

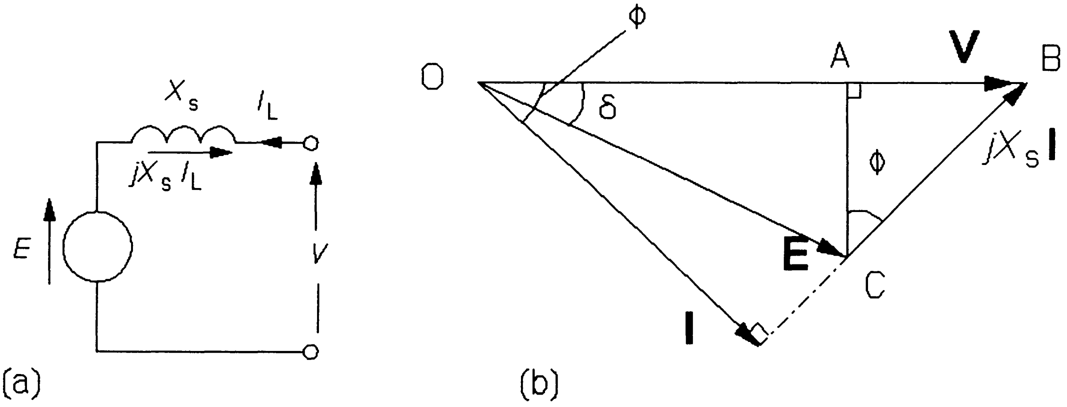


Figure A17.8

(b) When only the phase angle of the line current is given we must find **E** from

$$\mathbf{E} + jX_s\mathbf{I} = \mathbf{V} \;\Rightarrow\; \mathbf{I} = \frac{\mathbf{E} - \mathbf{V}}{jX_s}$$

Taking **V** as the reference phasor makes $\mathbf{E} = E\angle -\delta$, as in a motor **E** lags **V** by the torque angle. Then the equation above for **I** becomes

$$\mathbf{I} = \frac{\mathbf{V} - \mathbf{E}}{jX_s} = \frac{V - E\cos\delta + jE\sin\delta}{jX_s} = \frac{E\sin\delta}{X_s} + \frac{j(E\cos\delta - V)}{X_s}$$

Thus the phase angle of the current, $\phi = -\cos^{-1}0.9 = -25.84°$ (minus sign, since it is lagging), is given by

$$\tan\phi = \left[\frac{E\cos\delta - V}{E\sin\delta}\right] \qquad \text{(A8.1)}$$

But $E\sin\delta$ can be found from the power equation

$$P = \left[\frac{EV}{X_s}\right]\sin\delta \;\Rightarrow\; E\sin\delta = \frac{PX_s}{V} = \frac{16667 \times 0.7}{239.6} = 48.69 \text{ V}$$

And so from equation (A8.1) we find

$$E\cos\delta - V = E\sin\delta\tan\phi = 48.69\tan(-25.84°) = -23.6 \text{ V}$$

$$E\cos\delta = V - 23.6 = 239.6 - 23.6 = 216 \text{ V}$$

We have therefore found the two components of **E** and from them the magnitude of **E** is

$$E = \sqrt{E^2\sin^2\delta + E^2\cos^2\delta} = \sqrt{48.69^2 + 216^2} = 221.4 \text{ V}$$

The excitation current is 221.4 × 10/250 = 8.86 A and the torque angle is

$$\delta = \sin^{-1}\left[\frac{PX_s}{EV}\right] = \sin^{-1}\left[\frac{16667 \times 0.7}{221.4 \times 239.6}\right] = 12.7°$$

An alternative method using much less algebra employs the phasor diagram of figure A17.8b, from which we can see that OA = $E\cos\delta$, so that AB = OB − OA = $V - E\cos\delta$. Also AC = $E\sin\delta$ and $\angle$ABC = 90° − $\phi$, making $\angle$ACB = $\phi$ and hence

$$\tan\angle ACB = \tan\phi = \frac{\text{AB}}{\text{AC}} = \frac{V - E\cos\delta}{E\sin\delta}$$

The rest of the solution follows as before.

(c) When $\phi = \cos^{-1}0.8 = 36.87°$ (leading), we use the same equation as in part (b) and obtain

$$\tan\phi = \frac{E\cos\delta - V}{E\sin\delta} \Rightarrow E\cos\delta = V + E\sin\delta\tan\phi$$

$$E\cos\delta = V + (PX_s/V)\tan\phi = 239.6 + 48.69\tan 36.87° = 276.1 \text{ V}$$

Hence $$E = \sqrt{E^2\sin^2\delta + E^2\cos^2\delta} = \sqrt{276.1^2 + 48.69^2} = 280.4 \text{ V}$$

And then $I_{ex} = 280.4 \times 10/250 = 11.2$ A, while the torque angle is

$$\delta = \sin^{-1}\left[\frac{PX_s}{EV}\right] = \sin^{-1}\left[\frac{16667 \times 0.7}{280.4 \times 239.6}\right] = 10°$$

**9** It is easiest to use per-unit values from the outset and convert at the end. A power consumption of 16 kW when the rating is 20 kVA means that $p$, the per-unit power, is $16/20 = 0.8$ p.u. We find the torque angle from

$$p = \frac{ev\sin\delta}{x_s} = \frac{1 \times 1 \times \sin\delta}{0.8} = 0.8$$

$$\Rightarrow \sin\delta = 0.64 \Rightarrow \delta = 39.8°$$

since $v = 1$ p.u. by definition.

The power equation in p.u. terms is

$$\mathbf{vi}^* = \mathbf{v} \times \left[\frac{\mathbf{v} - \mathbf{e}}{jx_s}\right]^* = \left[\frac{1 - \mathbf{e}}{jx_s}\right]^* = \left[\frac{1 - e\cos\delta - je\sin\delta}{j0.8}\right]^*$$

$$= (-1.25e\sin\delta + j1.25[e\cos\delta - 1])^*$$

$$= -1.25e\sin\delta + j1.25(1 - e\cos\delta) = p + jq$$

where $\mathbf{i}^*$ etc. are the complex conjugates (that is the phase angle's sign is changed) and $\mathbf{v} = 1\angle 0°$, so that $\mathbf{v}^* = 1$. Then

$$q = 1.25e(1 - \cos\delta) \tag{A9.1}$$

substituting 1 for $e$ and $-39.8°$ for $\delta$ as $\mathbf{e}$ lags $\mathbf{v}$ by the torque angle: $\mathbf{e} = 1\angle -39.8°$ p.u. The reactive power is therefore

$$Q = Sq = 20 \times 0.29 = 5.8 \text{ kvar}$$

The pull-out torque occurs when $\sin\delta = 1$ and then $p = t$ as $\omega_s = 1$ p.u., which leads to

$$p = t = \frac{ev\sin\delta}{x_s} = \frac{ev}{x_s} = \frac{1 \times 1}{0.8} = 1.25 \text{ p.u.}$$

If $P = 20$ kW, $p = P/S = 20/20 = 1$ p.u. and then if $e = 1.85$ p.u., the power equation gives

$$p = \frac{ev\sin\delta}{x_s} = \frac{1.85 \times 1 \times \sin\delta}{0.8} = 1$$

$$\Rightarrow \quad \sin\delta = \frac{0.8}{1.85} \quad \Rightarrow \quad \delta = 25.62°$$

Then this value of $\delta$ is put into equation (A9.1) to give

$$q = \frac{1 - e\cos\delta}{x_s} = \frac{1 - 1.85\cos 25.62°}{0.8} = -0.835 \text{ p.u.}$$

Thus $Q = qS = -0.835 \times 20 = -16.7$ kvar. The negative sign indicates that the motor is supplying reactive power to the bus.

If $Q = -20$ kvar (supplied), $q = Q/S = -1$ and if $P = 20$ kW or 1 p.u., then using equation (2) once more leads to

$$q = \frac{1 - e\cos\delta}{x_s} \quad \Rightarrow \quad e\cos\delta = 1 - qx_s = 1 - (-1) \times 0.8 = 1.8 \text{ p.u.}$$

And the power equation is

$$e\sin\delta = \frac{px_s}{v} = \frac{1 \times 0.8}{1} = 0.8 \text{ p.u.}$$

Therefore

$$e = \sqrt{e^2\sin^2\delta + e^2\cos^2\delta} = \sqrt{1.8^2 + 0.8^2} = 1.97 \text{ p.u.}$$

**10** The p.f. is 0.1, so that the current's phase angle is $\cos^{-1}0.1 = 84.26°$, leading as the motor is a synchronous capacitor and supplies vars, that is $\mathbf{i} = 1\angle 84.26°$ p.u. since it operates at rated current. The complex power is

$$p = \mathbf{vi}^* = \cos\phi - j\sin\phi = p + jq$$

since $\mathbf{v} = 1\angle 0°$, $\mathbf{i} = 1\angle\phi$ and $\mathbf{i}^* = 1\angle -\phi$. Thus $q = -\sin\phi = -\sin 84.26° = -0.995$ and

$$Q = qS = -0.995 \times 100 = -99.5 \text{ Mvar}$$

The e.m.f. is given by

$$\mathbf{e} = \mathbf{v} - jx_s\mathbf{i} = 1\angle 0° - 0.8\angle 90° \times 1\angle 84.26°$$

$$= 1 - 0.8\angle 174.26° = 1.798\angle -2.55° \text{ p.u.}$$

The actual e.m.f. will be $eV_B$, where $V_B = 22/\surd 3 = 12.7$ kV is the base voltage. Thus $E = 1.798 \times 12.7 = 22.84$ kV.

The torque angle is the phase angle of **e**, 2.55°.

Reducing the e.m.f. to 1 p.u. with the same power consumption means that $e = v = 1$. The power consumption was $vi\cos\phi = 1 \times 1 \times 0.1 = 0.1$ p.u., so that the torque angle is

$$p = \frac{ev\sin\delta}{x_s} \Rightarrow \sin\delta = \frac{px_s}{ev} = \frac{0.1 \times 0.8}{1 \times 1} = 0.08$$

and therefore $\delta = 4.6°$.

The reactive power is given by

$$q = \frac{1 - e\cos\delta}{x_s} = \frac{1 - \cos 4.6°}{0.8} = 0.004 \text{ p.u.}$$

Hence $Q = qS = +0.4$ Mvar: the motor is acting as an inductor and is consuming vars, not supplying them.

# Chapter 18

1 The power consumption is

$$P = E_{DC}{}^2/R = 1.35E_L{}^2/R = 415^2/9.5 = 33 \text{ kW}$$

The diodes are conducting a third of the times each, so that at any one time there are two conducting, with a combined voltage drop of 1.4 V. The current flowing in the diodes is the load current, $I_{DC}$, which is $E_{DC}/R$ or $1.35 \times 415/9.5 = 58.97$ A, and then the power consumed by the diodes is $1.4 \times 58.97 = 82.6$ W. As a fraction of the power output thus is $82.6/33\,000 = 0.0025$ or 0.25%, a negligible amount.

2 The load voltage is $1.35E_L$ or 560 V and the load current is $560/9.5 = 59$ A. The ripple is found from equation 18.5:

$$L \approx \frac{(0.25 - 3.5r)R}{12fr} \Rightarrow 12Lfr + 3.5Rr \approx 0.25R$$

$$r \approx \frac{0.25R}{12Lf + 3.5R} = \frac{0.25 \times 9.5}{12 \times 0.03 \times 50 + 3.5 \times 9.5} = 0.046$$

where $r = \Delta I_{DC}/I_{DC}$, so that $\Delta I_{DC}$ = peak-peak ripple $= rI_{DC} = 0.046 \times 59 = 2.7$ A.

3 The r.m.s. line current is $0.816I_{DC}$ (equation 18.6), which is $0.816 \times 59 = 48.1$ A.

The p.f. is $P/S$, where $P$ is the power consumed in W and $S$ is the apparent power supplied in VA. The apparent power is $\surd 3E_LI_{rms} = 1.732 \times 415 \times 48.1 = 34.57$ kW, while the power in the load is 33 kW, giving a p.f. of $33/34.57 = 0.955$.

The reactive power supplied is

$$Q^2 = S^2 - P^2 = 34.57^2 - 33^2 \Rightarrow Q = 10.3 \text{ kvar}$$

4 For a six-pulse converter the DC voltage is

$$E_{DC} = 1.35E_L\cos\alpha = 1.35 \times 415\cos 30° = 485 \text{ V}$$

where $E_L$ is the line voltage, 415 V, and $\alpha$ is the delay angle, 30°. The load power is

$$P_L = E_{DC}{}^2/R_L = 485^2/9.5 = 24.8 \text{ kW}$$

The reactive power drawn from the supply is $Q = P\tan\alpha = 24.8\tan 24.8° = 14.3$ kvar.

For a power factor of 0.95, the effective phase angle, $\phi$, is $\cos^{-1}0.95 = 18.2°$, and this is given by

$$\tan\phi = Q/P \Rightarrow Q = P\tan\phi = 24.8\tan 18.2° = 8.15 \text{ kvar}$$

Thus the correction capacitors must supply a total reactive power, $Q_{total} = 14.3 - 8.15 = 6.15$ kvar.

The capacitors are placed across the lines, so that the reactive power in each is

$$Q = E_L^2/X_C \Rightarrow Q_{total} = 3E_L^2/X_C = 3E_L^2 \times 2\pi fC$$

From this expression we find

$$C = \frac{Q_{total}}{6\pi fE_L^2} = \frac{6150}{6\pi \times 50 \times 415^2} = 38\ \mu\text{F}$$

**5** When the power in the load is reduced to 10 kW, the DC voltage must be reduced also, to a value given by

$$P_L = E_{DC}^2/R_L \Rightarrow E_{DC} = \sqrt{P_L R_L} = \sqrt{10000 \times 9.5} = 308\ \text{V}$$

And as $E_{DC} = 1.35\cos\alpha$, where $\alpha$ is the delay angle, we find

$$\cos\alpha = \frac{E_{DC}}{1.35E} = \frac{308}{1.35 \times 415} = 0.55$$

Then $\alpha = 57°$.

The reactive power increases because of the larger delay angle:

$$Q = P\tan\alpha = 10\tan 57° = 15.2\ \text{kvar}$$

**6** The battery takes $V_B IT = V_B I/f$ J/cycle, where $f$ is the chopping frequency. This energy comes from the power supply, so that

$$V_B I/f = V_s IT_{on} \Rightarrow T_{on} = \frac{V_B}{V_s f} = \frac{12}{40 \times 500} = 0.6\ \text{ms}$$

The ripple current is the difference between maximum and minimum currents through the inductor, and when the GTO fires the voltage across the inductor is

$$V_s - V_B = L\frac{di}{dt} = L\frac{\Delta I}{T_{on}}$$

$$\Rightarrow \quad \Delta I = \frac{(V_s - V_B)T_{on}}{L} = \frac{(40 - 12) \times 0.6 \times 10^{-3}}{45 \times 10^{-3}} = 0.373\ \text{A}$$

In percentage terms that is $100 \times 0.373/15 = 2.5\%$.

7 The maximum voltage across the thyristor is

$$V_{\text{AK}}(\max) = V_{\text{m}}\sin\theta = 240\sqrt{2}\sin 45° = 240 \text{ V}$$

And so if the thyristor's firing current, $I_{\text{GT}}$, is 100 mA

$$I_{\text{GT}}R_{\text{G}} = V_{\text{AK}}(\max) \Rightarrow R_{\text{G}} = V_{\text{AK}}(\max)/I_{\text{GT}} = 240/0.1 = 2.4 \text{ k}\Omega$$

The power developed in the load resistance is given by equation 18.16,

$$P_{\text{R}} = \frac{V^2(2\pi - 2\theta + \sin 2\theta)}{4\pi R}$$

where $\theta = 45\pi/180$ radians. Thus, substituting 200 Ω for $R$ and 240 V for the supply r.m.s. voltage, we find

$$P_{\text{R}} = \frac{240^2(2\pi - \pi/2 + \sin 90°)}{4\pi \times 200} = 131 \text{ W}$$

The load r.m.s. voltage is then found from

$$V_{\text{L}} = \sqrt{P_{\text{R}}R} = \sqrt{131 \times 200} = 162 \text{ V}$$

And the p.f. is found from

$$\text{p.f.} = P_{\text{R}}/S = V_{\text{L}}I/VI = V_{\text{L}}/V = 162/240 = 0.675$$

And therefore the effective phase angle is $\phi = \cos^{-1}0.675 = 47.5°$ and the reactive power is

$$Q = P\tan\phi = 131\tan 47.5° = 143 \text{ var}$$

8 The current for this circuit (figure 18.11a) is given by equation 18.28

$$i(\theta) = \frac{1}{\omega L}\int_{\theta_{\text{on}}}^{\theta}(V_{\text{m}}\sin x - V_{\text{DC}}x)\,\text{d}x = \frac{\left[-V_{\text{m}}\cos x - V_{\text{DC}}x\right]_{\theta_{\text{on}}}^{\theta}}{\omega L}$$

Substituting $2\pi/3$ radians for $\theta$, $\pi/4$ radians for $\theta_{\text{on}}$, 339 V for $V_{\text{m}}$, $100\pi$ for $\omega$ and 0.1 H for $L$ gives

$$i = \frac{[-339\cos 2\pi/3 + 339\cos\pi/4 - 200\pi/3 + 100\pi/4]}{10\pi} = 8.9 \text{ A}$$

To find when the current is a maximum we can differentiate $i(\theta)$ with respect to $\theta$, which yields

$$\frac{\mathrm{d}i(\theta)}{\mathrm{d}\theta} = \frac{1}{\omega L}\left[V_{\mathrm{m}}\sin x - V_{\mathrm{DC}}\right]$$

and then set $\mathrm{d}i(\theta)/\mathrm{d}\theta$ to zero to obtain

$$V_{\mathrm{m}}\sin x = V_{\mathrm{DC}} \Rightarrow x = \sin^{-1}(V_{\mathrm{DC}}/V_{\mathrm{m}}) = \sin^{-1}(100/339) = 17.2°$$

This cannot be right as the thyristor has not been fired, and we must take the next value for $\theta$ which has $\sin\theta = 100/339 = 0.295$, that is 180° − 17.2° = 162.8°. At this angle the current is

$$i(\theta) = \frac{-339\cos 162.8° + 339\cos 45° - 100\pi \times 162.8/180 + 100\pi/4}{10\pi} = 11.4 \text{ A}$$

When $\theta = 135°$ the current is

$$i(\theta) = \frac{-339\cos 135° + 339\cos 45° - 100 \times 135\pi/180 + 100\pi/4}{\omega L}$$

$$= \frac{322}{100\pi L} = 15 \text{ A} \Rightarrow L = \frac{322}{100\pi \times 15} = 68 \text{ mH}$$

**9** The load voltage waveform is shown in figure A18.9, from which we can deduce that the r.m.s. voltage is

$$V_{\mathrm{rms}}^{\,2} = \frac{\int_\alpha^\pi V_{\mathrm{m}}^{\,2}\sin^2\theta\,\mathrm{d}\theta + \int_\pi^{2\pi} V_{\mathrm{m}}^{\,2}\sin^2\theta\,\mathrm{d}\theta}{2\pi} = \frac{V_{\mathrm{m}}^{\,2}\left[\int_\alpha^\pi (1 - \cos 2\theta)\,\mathrm{d}\theta + \int_\pi^{2\pi}(1 - \cos 2\theta)\,\mathrm{d}\theta\right]}{4\pi}$$

$$= \frac{V_{\mathrm{m}}^{\,2}\left\{\left[1 - 0.5\sin 2\theta\right]_\alpha^\pi + \left[1 - 0.5\sin 2\theta\right]_\pi^{2\pi}\right\}}{4\pi} = \frac{V_{\mathrm{m}}^{\,2}\left[\pi - \alpha + 0.5\sin\alpha + 2\pi - \pi\right]}{4\pi}$$

$$= \frac{V_{\mathrm{m}}^{\,2}\left[2\pi - \alpha + 0.5\sin 2\alpha\right]}{4\pi} \Rightarrow V_{\mathrm{rms}} = V_{\mathrm{m}}\sqrt{\frac{2\pi - \alpha}{4\pi} + \frac{\sin 2\alpha}{8\pi}}$$

When the thyristor is replaced by a diode all the source voltage (we are ignoring diode voltage drops) appears across the load and the r.m.s. voltage is $V_{\mathrm{m}}/\surd 2$. The value of $\alpha$ is effectively 0° for a diode, and then the r.m.s. voltage is

$$V_{\mathrm{rms}} = V_{\mathrm{m}}\sqrt{\frac{2\pi - 0}{4\pi} + \frac{0}{8\pi}} = \frac{V_{\mathrm{m}}}{\sqrt{2}}$$

which is correct.

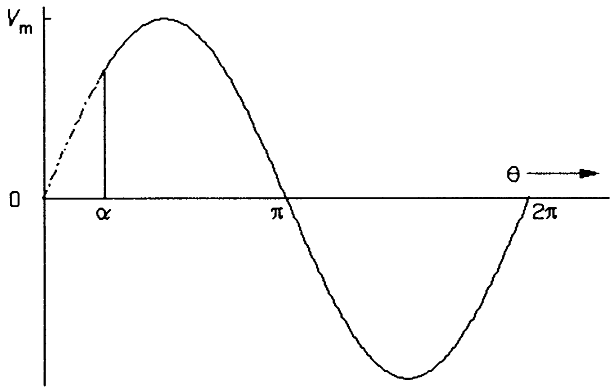


*Figure A18.9*

If the thyristor is not fired, we have a half-wave rectifier and then $V_{rms} = V_m/2$. In this case $\alpha = 180° = \pi$ radians and the r.m.s. voltage is given by

$$V_{rms} = V_m\sqrt{\frac{2\pi - \pi}{4\pi} + \frac{0}{8\pi}} = V_m\sqrt{\frac{1}{4}} = \frac{V_m}{2}$$

which is also as it should be.

**10** Substituting $\alpha = \pi/3$ radians (or 60°) into the formula for $V_{rms}$ yields

$$V_{rms} = V_m\sqrt{\frac{2\pi - \pi/3}{4\pi} + \frac{\sin 120°}{8\pi}} = 339\sqrt{0.4167 + 0.0345} = 228 \text{ V}$$

The current through the load is the same as the current through the supply so the p.f. is

$$\text{p.f.} = \frac{V_{rms}I}{VI} = \frac{V_{rms}}{V} = \frac{228}{339/sqrt2} = 0.95$$

where $V$ is the r.m.s. supply voltage, $V_m/\surd 2$.

If the load resistance is 8 Ω, the load power is

$$P_L = V_{rms}{}^2/R_L = 228^2/8 = 6.5 \text{ kW}$$

When the p.f. is 0.95, $\phi = \cos^{-1}0.95 = 18.2°$ and the reactive power is found from

$$Q = P_L\tan\phi = 6.5\tan 18.2° = 6.14 \text{ kvar}$$

# Chapter 19

**1** $A_{16} = 1010_2$, $1_{16} = 1_2 = 0001_2$, $F_{16} = 1111_2$ and $8_{16} = 1000_2$, so that

$$A1F8_{16} = 1010'0001'1111'1000_2 \text{ or } 1010000111111000_2$$

**2** The binary numbers are both first shifted left by two places to get rid of the binary point. The subtrahend is then 1011001111, which is 0100110000 in 1's complement form and by adding 1 we get the 2's complement form of 0100110001, which is added to 1100011010:

$$\begin{array}{r} 11'0001'1010 \\ +\ \underline{01'0011'0001} \\ 100'0100'1011 \end{array}$$

Suppressing the leading 1 from this result gives 1001011, which is then given back its binary point two places from the right: 10010.11 - the correct answer.

**3** Here we add the numbers two by two:

$$\begin{array}{r} 100'0111 \\ +\ \underline{1100'1100} \\ 1'0001'0011 \end{array}$$

Then the third number is added:

$$\begin{array}{r} 1'0001'0011 \\ +\ \underline{1110'1101} \\ 10'0000'0000 \end{array}$$

**4** BCD numbers are divided into groups of four and then these groups are converted to decimal form so that 1001'0110.0101 becomes $96.5_{10}$.

**5** The conversion from decimal to binary requires division of the integer part by two, and multiplication of the fractional part by two until no fractional part remains or it repeats, as in this case

$24 \div 2 = 12$ remainder 0 (LSB)
$12 \div 2 = 6$ remainder 0
$6 \div 2 = 3$ remainder 0
$3 \div 2 = 1$ remainder 1
$1 \div 2 = 0$ remainder 1 (MSB)

The integer part is thus $11000_2$. The fractional part converts as follows:

$0.12 \times 2 = 0.24$ integral part 0 (MSB)
$0.24 \times 2 = 0.48$ integral part 0
$0.48 \times 2 = 0.96$ integral part 0
$0.96 \times 2 = 1.92$ integral part 1
$0.92 \times 2 = 1.84$ integral part 1
$0.84 \times 2 = 1.68$ integral part 1
$0.68 \times 2 = 1.36$ integral part 1
$0.36 \times 2 = 0.72$ integral part 0
$0.72 \times 2 = 1.44$ integral part 1
$0.44 \times 2 = 0.88$ integral part 0
$0.88 \times 2 = 1.76$ integral part 1
$0.76 \times 2 = 1.52$ integral part 1
$0.52 \times 2 = 1.04$ integral part 1
$0.04 \times 2 = 0.08$ integral part 0
$0.08 \times 2 = 0.16$ integral part 0
$0.16 \times 2 = 0.32$ integral part 0
$0.32 \times 2 = 0.64$ integral part 0
$0.64 \times 2 = 1.28$ integral part 1
$0.28 \times 2 = 0.56$ integral part 0
$0.56 \times 2 = 1.12$ integral part 1 (LSB)

At this point the fraction recurs, so that $24.12_{10} = 11000.00011110101110000101_2$.

**6** The expression can be simplified by using boolean algebra, de Morgan's law or by drawing up a truth table. Boolean algebra yields

$$A + B + A'B' = A(B + B') + B + A'B' = AB + AB' + B + A'B'$$

$$= AB + B + B'(A + A') = B(A + 1) + B' = B + B' = 1$$

The use of de Morgan's law gives

$$A'B' = (A + B)'$$

And then replacing $(A + B)$ by $Q$ gives

$$F = A + B + A'B' = A + B + (A + B)' = Q + Q' = 1$$

This is readily seen with a truth table also:

| A | B | A′ | B′ | A′B′ | F |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 |

7 We can simplify the expression for $Y'$ by grouping terms such as the first two

$$ABC' + AB'C' = AC'(B + B') = AC'$$

and the last two

$$ABC + AB'C = AC(B + B') = AC$$

so that we find

$$Y' = AC' + A'B'C + AC = A(C + C') + A'B'C = A + A'B'C = A + B'C$$

The latter simplification is because

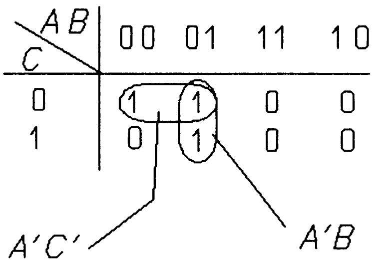
$$A + A'B'C = A(1 + B'C) + A'B'C = A + B'C(A + A') = A + B'C$$

Then we take the complement and use de Morgan's law:

$$Y = Y'' = \overline{A + B'C} = A'\overline{(B'C)} = A'(B'' + C') = A'B + A'C'$$

The truth table and Karnaugh map for $Y$ are

| A | B | C | Y' | Y |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 0 |



And we see that $Y = A'C' + A'B$.

The all-NAND form for $Y$ can be derived by using de Morgan's law as follows:

$$Y = A'(B + C') = A'\overline{\overline{(B + C')}} = A'\overline{B'C''} = A'\overline{B'C} = \overline{\overline{A'\overline{B'C}}}$$

Thus the circuit in 2-input NAND form is as shown in figure A19.7.

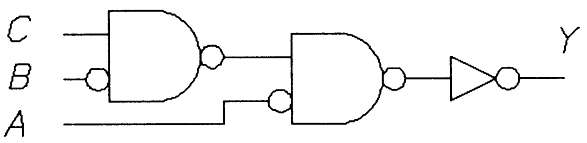


*Figure A19.7*

**8** The SOP expression for $Q$ is

$$Q = A'B'C + AB'C' + AB'C + ABC' + ABC = A'B'C + AB' + AB$$

$$= A + A'B'C = A + B'C$$

using the usual simplifications that $A(B + B') = A$ and $A + A'B = A + B$.

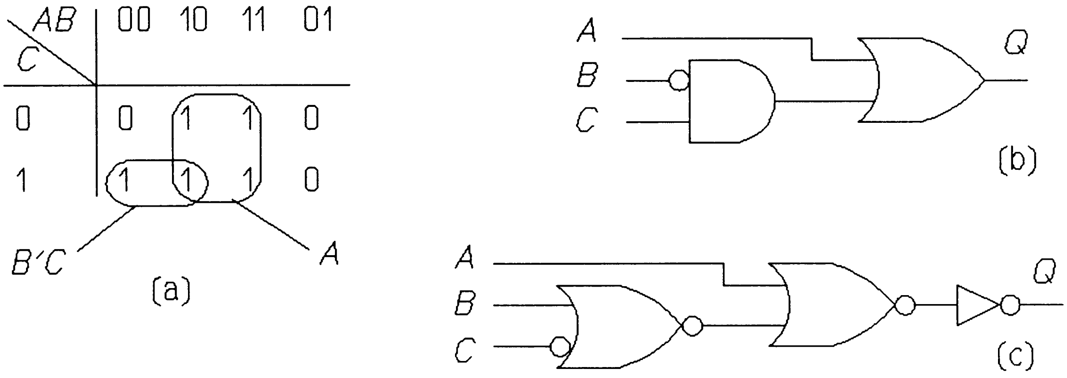


*Figure A19.8*

The Karnaugh map is shown in figure A19.8a, from which the minimal expression for $Q$ is found, leading to the logic diagram of figure A19.8b.

Using de Morgan's law leads to

$$Q = A + B'C = A + \overline{\overline{B'C}} = A + \overline{B + C'} = \overline{\overline{A + \overline{B + C'}}}$$

which is the all-NOR form and gives the circuit of figure A19.8c. The same result can be obtained using the circuit equivalents of figure A19.8d.

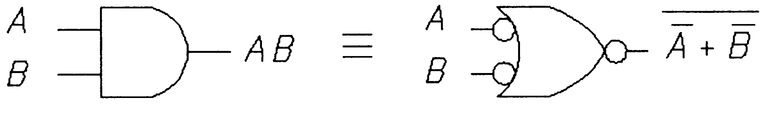
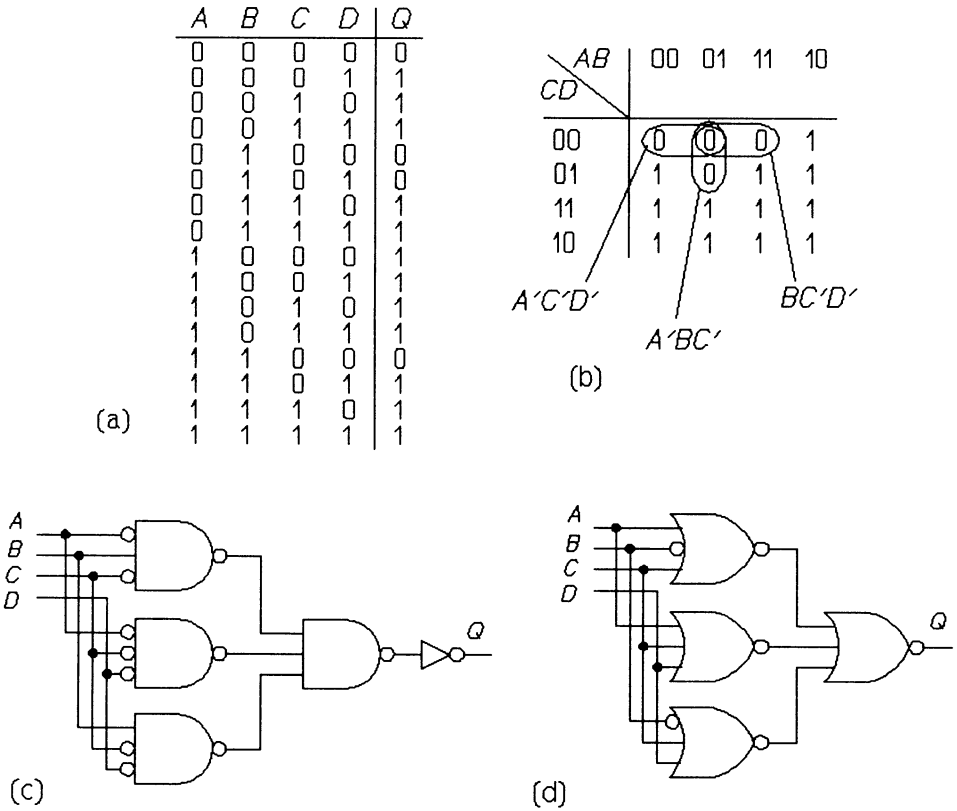


*Figure A19.8d*

**9** The truth table for the circuit of figure P19.9 is shown in figure A19.9a, and leads to the Karnaugh map of figure A19.9b, from which is found the minimal expression for $Q$:

$$Q' = A'BC' + A'C'D' + BC'D'$$



(a)

| A | B | C | D | Q |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

(b)

| CD \ AB | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 1 |
| 01 | 1 | 0 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 |

*Figure A19.9*

The minimal expression for $Q$ can be put into all-NAND form using de Morgan's law:

$$Q = Q'' = \overline{A'BC' + A'C'D' + BC'D'}$$

$$= \overline{A'BC'}\cdot\overline{A'C'D'}\cdot\overline{BC'D'} = \overline{\overline{\overline{A'BC'}\cdot\overline{A'C'D'}\cdot\overline{BC'D'}}}$$

leading to the circuit of figure A19.9c.

The all-NOR form is

$$Q' = \overline{\overline{A'BC'}} + \overline{\overline{A'C'D'}} + \overline{\overline{BC'D'}} = \overline{A + B' + C} + \overline{A + C + D} + \overline{B' + C + D}$$

$$\Rightarrow \quad Q = Q'' = \overline{\overline{A + B' + C} + \overline{A + C + D} + \overline{B' + C + D}}$$

which gives the circuit of figure A19.9d.

**10** The XOR function is defined by

$$A \oplus B = AB' + A'B$$

$$\begin{aligned}
\therefore \qquad (A\oplus B)\oplus C &= (AB' + A'B)C' + \overline{(AB' + A'B)}C \\
&= AB'C' + A'BC' + (\overline{AB'} \cdot \overline{A'B})C \\
&= AB'C' + A'BC' + (A' + B)(A + B')C \\
&= AB'C' + A'BC' + (A'A + A'B' + AB + BB')C \\
&= AB'C' + A'BC' + A'B'C + ABC
\end{aligned}$$

While

$$\begin{aligned}
A\oplus(B\oplus C) &= A\overline{(BC' + B'C)} + A'(BC' + B'C) \\
&= A(\overline{BC'} \cdot \overline{B'C}) + A'BC' + AB'C \\
&= A(B' + C)(B + C') + A'BC' + AB'C \\
&= A(B'B + B'C' + BC + CC') + A'BC' + AB'C \\
&= AB'C' + ABC + A'BC' + AB'C'
\end{aligned}$$

Then we can evaluate

$$\begin{aligned}
(A\oplus C)\oplus B &= (AC' + A'C)B' + \overline{(AC' + A'C)}B \\
&= AB'C' + A'B'C + (\overline{AC'} \cdot \overline{A'C})B \\
&= AB'C' + A'B'C + (A' + C)(A + C')B \\
&= AB'C' + A'B'C + (A'A + A'C' + AC + CC')B \\
&= AB'C' + A'B'C + A'BC' + ABC
\end{aligned}$$

And so the XOR function is associative.

**11** The truth table is shown in figure A19.11a and the Karnaugh map in figure A19.11b, leading to the burgling function

$$B = DH' + H'NW = H'(D + NW)$$

This can be arranged in all-NOR form using de Morgan's law as follows

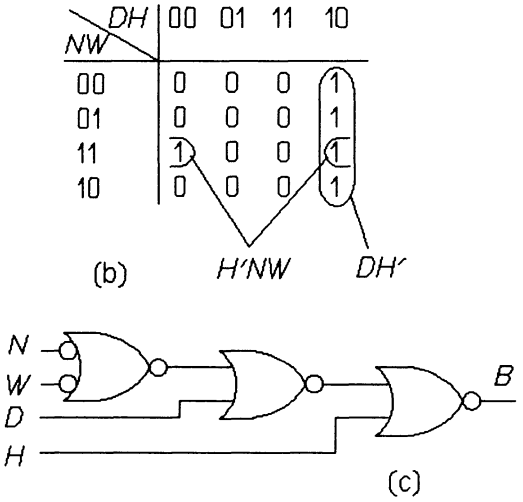
$$B = H'(D + NW) = H'(D + \overline{\overline{NW}}) = H'(D + \overline{N' + W'})$$

$$= \overline{\overline{H'(D + \overline{N' + W'})}} = \overline{H + \overline{D + \overline{N' + W'}}}$$

which leads to the circuit of figure A19.11c.

| D | H | N | W | B |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 |

(a)



| NW \ DH | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 1 |
| 01 | 0 | 0 | 0 | 1 |
| 11 | 1 | 0 | 0 | 1 |
| 10 | 0 | 0 | 0 | 1 |

(b) (c)

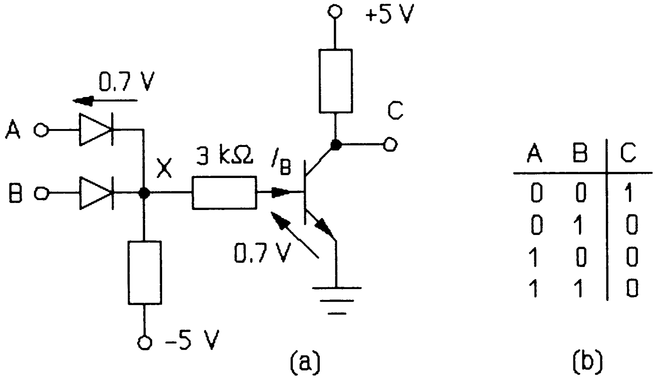
*Figure A19.11*

**12** In figure A19.12a we can see that the transistor will turn on if the voltage at X is positive by more than the base-emitter voltage drop of 0.7 V. Thus if either A or B is +5 V, the base current will be

$$I_B = \frac{V_X - V_{BE}}{3} = \frac{5 - 0.7 - 0.7}{3} = 1.2 \text{ mA}$$

The transistor will therefore turn fully on and be in saturation and the output voltage at C will be $V_{CEsat} \approx 0.2$ V, that is logical zero. If the potential at both A and B is zero the voltage at X will be −0.7V, and so the base current will be zero. The transistor is then turned fully off, in which case the collector current is zero and C will be at +5 V, logical

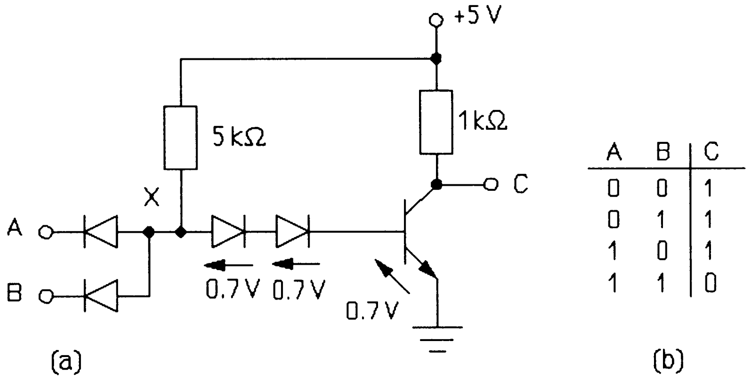
1. The truth table for the circuit is as in figure A19.12b and the device is seen to be a NOR gate.



| A | B | C |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

*Figure A19.12*

**13** In the circuit of figure A19.13a it can be seen that the potential at X must be at least +2.1 V (two diode drops plus $V_{BE}$) before the transistor can turn on. Thus when either of A or B is at 0 V (logical 0), the potential at X is limited to +0.7 V, the transistor is off and C is at +5 V (logical 1). If on the other hand BOTH of A and B are at +5 V (logical 1) the potential at X can rise above +2.1 V to turn the transistor on. In this case the transistor goes into saturation at C is at $V_{CEsat}$ or +0.2 V (logical 0). The truth table is as in figure A19.13b, from which it is seen that the circuit is a NAND gate.



| A | B | C |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

*Figure A19.13*

**14** The segment *d* is lit when the decimal number displayed is 0, 2, 3, 5, 6 or 8, that is binary 0000, 0010, 0011, 0101, 0110 and 0100. The truth table is as in figure A19.14a, giving rise to the Karnaugh map of figure A19.14b, in which the 'don't cares' (the binary numbers from 1100 to 1111) are indicated by an 'x'. The simplest groupings are those shown for $d = 0$, from which we deduce that the simplest logical expression for not lighting *d* is

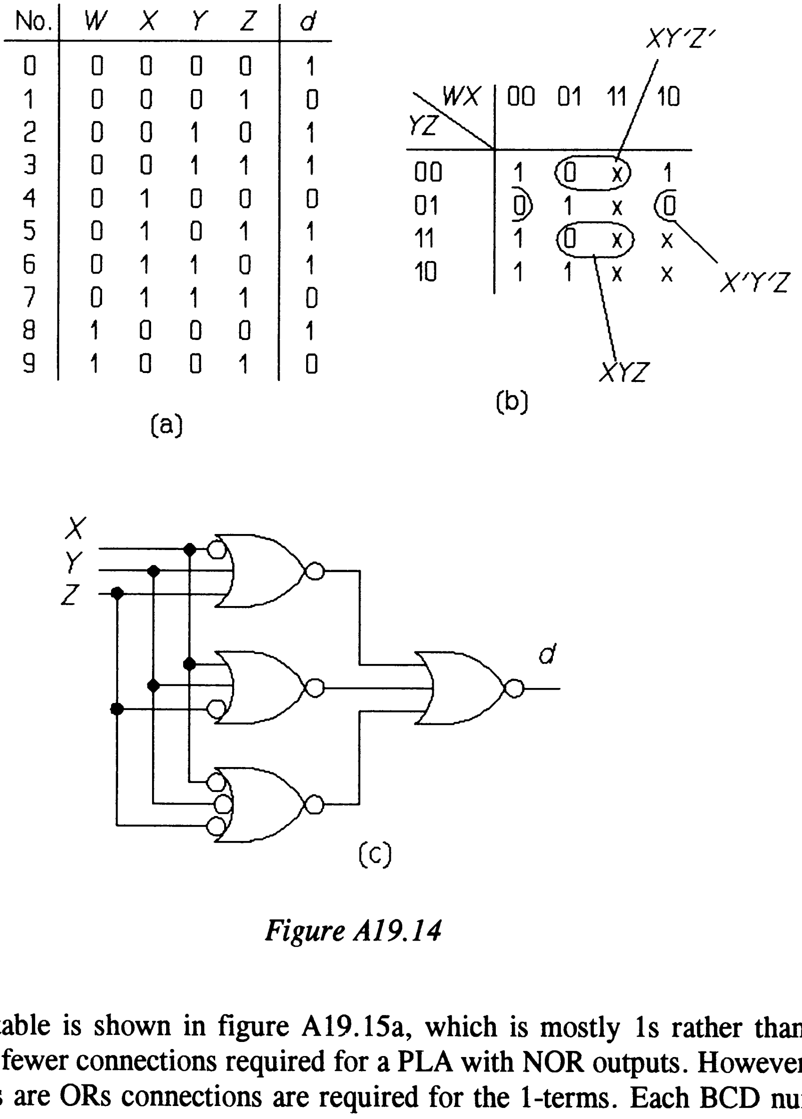
$$d' = XY'Z' + X'Y'Z + XYZ$$

From this we find

$$d = d'' = \overline{XY'Z' + X'Y'Z + XYZ} = \overline{\overline{\overline{XY'Z'}} + \overline{\overline{X'Y'Z}} + \overline{\overline{XYZ}}}$$

$$= \overline{\overline{\overline{X' + Y + Z} + \overline{X + Y + Z'} + \overline{X' + Y' + Z'}}}$$

utilising de Morgan's law. This expression is in all-NOR form and leads to the circuit of figure A19.14c.



| No. | W | X | Y | Z | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 | 1 |
| 3 | 0 | 0 | 1 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 |
| 5 | 0 | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 1 | 0 | 1 |
| 7 | 0 | 1 | 1 | 1 | 0 |
| 8 | 1 | 0 | 0 | 0 | 1 |
| 9 | 1 | 0 | 0 | 1 | 0 |

(a)

| YZ \ WX | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 0 | x | 1 |
| 01 | 0 | 1 | x | 0 |
| 11 | 1 | 0 | x | x |
| 10 | 1 | 1 | x | x |

(b)

(c)

*Figure A19.14*

**15** The truth table is shown in figure A19.15a, which is mostly 1s rather than 0s, so there would be fewer connections required for a PLA with NOR outputs. However, given that the outputs are ORs connections are required for the 1-terms. Each BCD number is given a 4-input AND gate and these are then connected to the appropriate 9-input OR which drives the segment required. For example, the BCD number 0110, decimal 6, is *W′XYZ′* and these inputs are connected to the AND gate labelled 6. Similarly, the segment, *c*, is required by the decimal numbers 0, 1, 3, 4, 5, 6, 7, 8 and 9 and so these

AND gates must be connected to the OR labelled *c*. The connections are shown in figure A19.15b.

| No. | *W* | *X* | *Y* | *Z* | *a* | *b* | *c* | *d* | *e* | *f* | *g* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 3 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 5 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 7 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |

*Figure A19.15a*

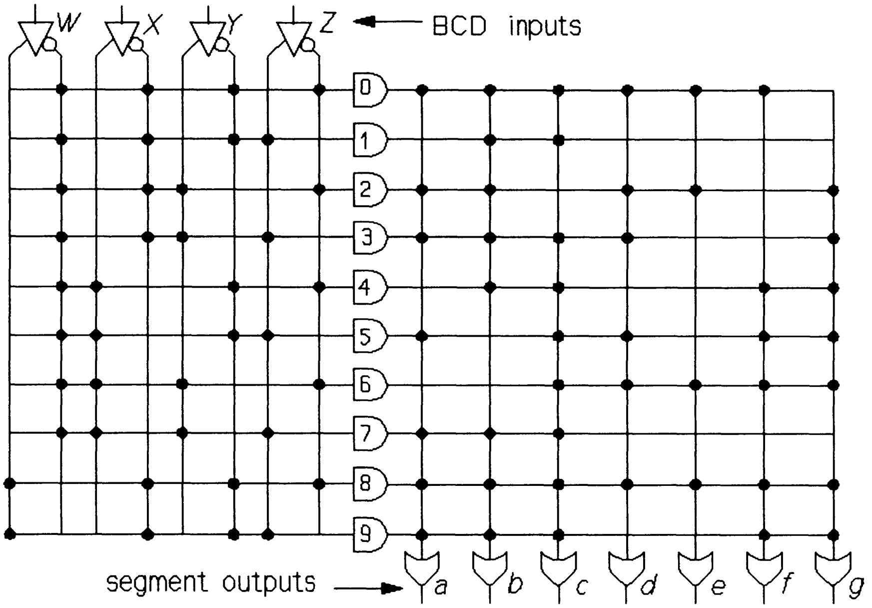


*Figure A19.15b*

# Chapter 20

1 We note that at all times $D_2 = Q_1$ and that $D_1 = (Q_1Q_2)'$ and that the device is positive-edge triggered. The table of figure A20.1a gives the successive states of the circuit when the initial state is $Q_1 = Q_2 = 1$ and the corresponding timing diagram is shown in figure A20.1b. When the initial state is $Q_1 = Q_2 = 0$, the successive states are shown in figure A20.1c, from which we see that the former sequence repeats.

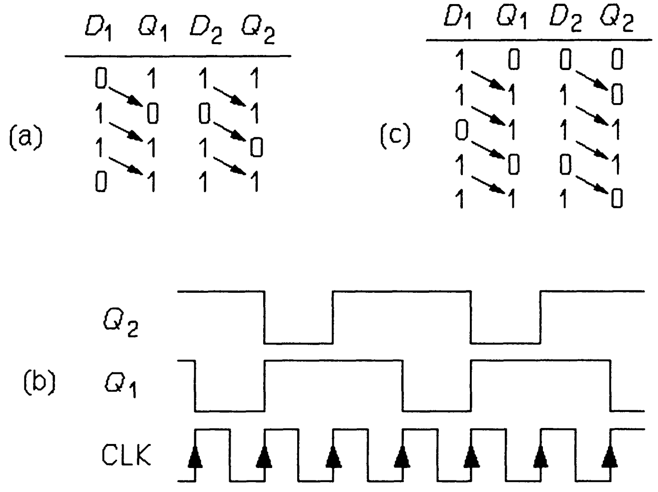


*Figure A20.1*

2 The counter sequences are shown in figure A20.2a, from which we can see that $D_1 = Q_1'$. We can also extract the truth tables for $D_2$ and $D_3$ as shown in figures A20.2b and A20.2c, leading to the Karnaugh maps of figures A20.2d and 20.2e. The Karnaugh maps indicate that

$$D_2 = Q_1Q_3 + Q_1'Q_3' \quad \text{and} \quad D_3 = Q_2Q_3 + Q_2'Q_3'$$

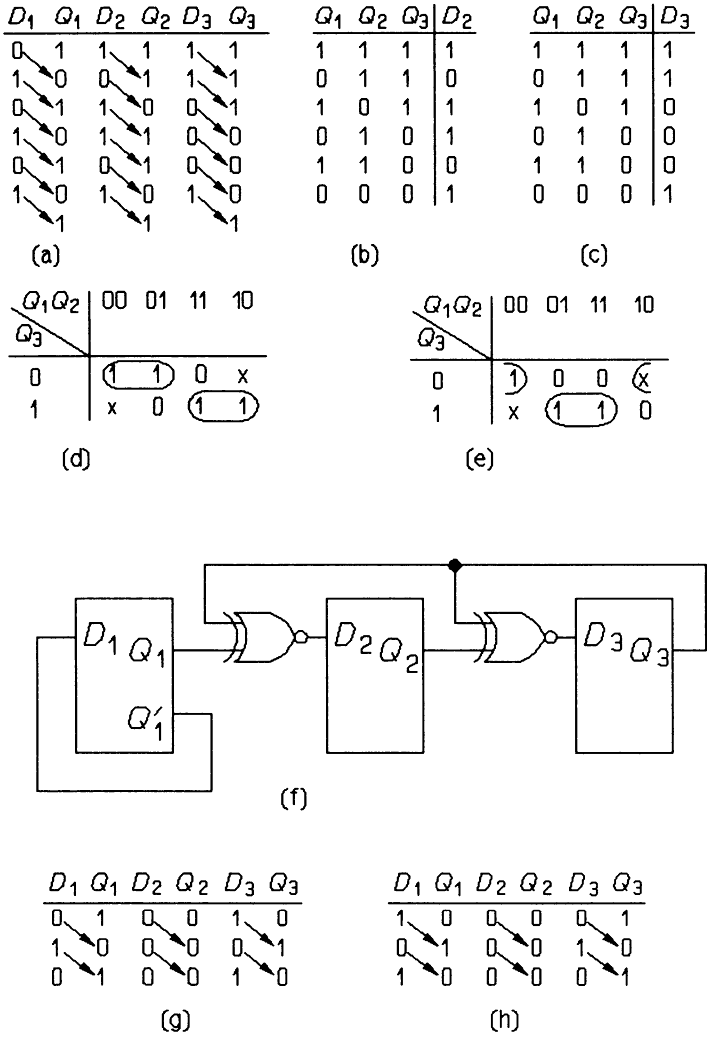
These are both XNOR functions and lead readily to the circuit of figure A20.2f.

To find out what happens when 100 is loaded into $Q_1$, $Q_2$ and $Q_3$, we must deduce the values of the $D$s and then load these into the $Q$s to find the next set of values as in the table of figure A20.2g. Now $D_1 = Q_1' = 0$, and we find $D_2$ from

$$D_2 = Q_1Q_3 + Q_1'Q_3' = 1.0 + 0.1 = 0$$

while
$$D_3 = Q_2Q_3 + Q_2'Q_3' = 0.0 + 1.1 = 1$$

And so on down the table. In the same way we can construct the sequences of the table in figure 20.2h, where the starting point is 001. As can be seen, in neither case does the desired sequence occur; the circuit alternates between 001 and 100.



| $Q_1$ | $Q_2$ | $Q_3$ | $D_2$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 |

(b)

| $Q_1$ | $Q_2$ | $Q_3$ | $D_3$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 |

(c)

| $Q_3$ \ $Q_1Q_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | x |
| 1 | x | 0 | 1 | 1 |

(d)

| $Q_3$ \ $Q_1Q_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | x |
| 1 | x | 1 | 1 | 0 |

(e)

*Figure A20.2*

3 The JK flip-flop's operational table is shown in figure A20.3a and we can use it to construct the table of successive counter states shown in figure A20.3b as follows. First enter 1s under $Q_1$, $Q_2$, $Q_3$ and $Q_4$. Examining the circuit diagram of figure A20.3c shows

$$J_1 = Q_4 = 1 \; ; \; J_2 = Q_1 = 1 \; ; \; J_3 = Q_2 = 1 \; ; \; J_4 = Q_3 = 1$$

$$K_1 = J_1 = 1 \; ; \; K_2 = Q_1' = J_2' = 0 \; ; \; K_3 = Q_2' = J_3' = 0 \; ; \; K_4 = Q_3' = J_4' = 0$$

Thus we can fill in the blanks in the first row of the table. On receipt of a clock pulse, the outputs can change state, depending on the states of the inputs. Thus as $J_1 = K_1 = 1$ (in the first row), $Q_1 = Q_p' = 0$ in row 2. And as $J_2 = 1$ while $K_2 = 0$ in row 1 means that $Q_2 = 1$ in row 2. The same reasoning gives $Q_3 = 1$ and $Q_4 = 1$ in row 2, and this last result means that $J_1 = K_1 = 1$. Having filled in row 2 we can fill in row 3 and so on until the sequence repeats with row 16 being the same as row 1.

(a)

| $J$ | $K$ | $Q$ |
|---|---|---|
| 0 | 0 | $Q_p$ |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | $\overline{Q}_p$ |

(b)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ | $J_3$ | $K_3$ | $Q_3$ | $J_4$ | $K_4$ | $Q_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |

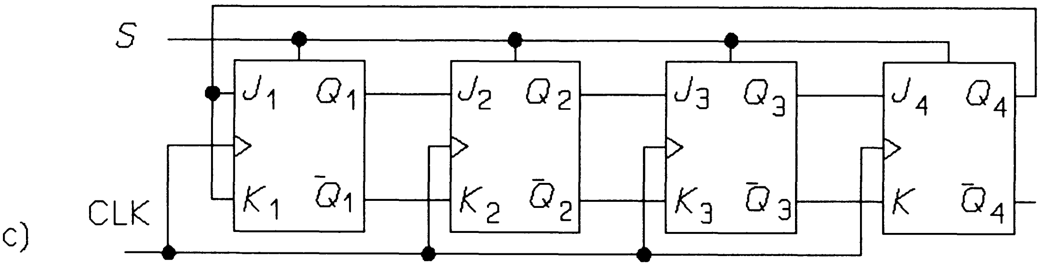


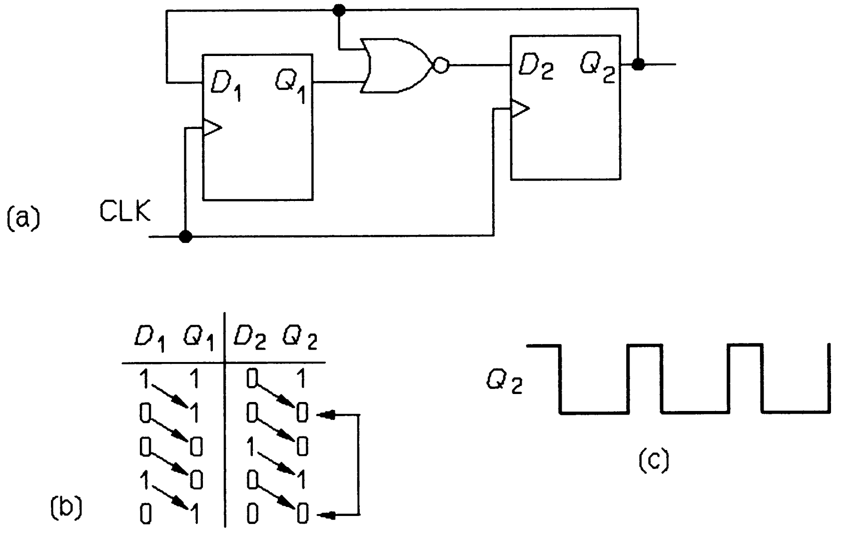
*Figure A20.3*

**4** Inspection of the circuit of figure A20.4a shows that $D_1 = Q_2$ and

$$D_2 = \overline{Q_1 + Q_2}$$

Thus we start off with $Q_1 = Q_2 = 1$, making $D_1 = 1$ and $D_2 = 0$ as shown in the first row of the table of figure A20.4b. On clocking, the inputs states are transferred to the outputs and the sequence of states is that in figure A20.4b, leading to the waveform for $Q_2$ shown in figure A20.4c. The second row of the table repeats at row 5 and the original state is never recovered.

**5** During discharge, $Q'$ is HIGH and the transistor is turned on, while $Q$ is LOW and $V_o = 0$. At the start of discharge, the capacitor voltage is ${}^2/_3V_{CC}$, and at the end of it



| $D_1$ | $Q_1$ | $D_2$ | $Q_2$ |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 |

*Figure A20.4*

${}^1/_3V_{CC}$, as set by the 1kΩ resistor network and the trigger and threshold comparators. The capacitor discharges through $R_B$, so that the capacitor voltage as a function of time is

$$v_C = \tfrac{2}{3}V_{CC}\exp(-t/R_BC)$$

If the time to discharge is $T_2$, then

$$\tfrac{1}{3}V_{CC} = \tfrac{2}{3}V_{CC}\exp(-T_2/R_BC) \;\Rightarrow\; T_2/R_BC = \ln 2 \;\Rightarrow\; T_2 \approx 0.7R_BC$$

When the capacitor voltage reaches ${}^1/_3V_{CC}$, the flip-flop is set, making $Q$ HIGH, $V_o = V_{CC}$ and $Q'$ LOW, thus turning of the transistor and allowing the capacitor to charge up to ${}^2/_3V_{CC}$ via the series resistors $R_A$ and $R_B$. The capacitor voltage as a function of time is therefore

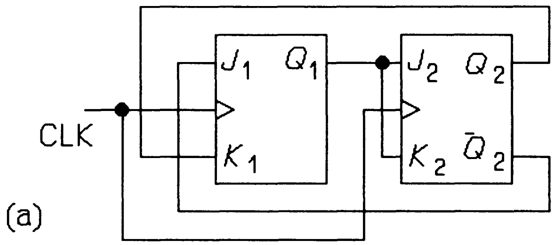
$$v_C = A + B\exp[-t/(R_A + R_B)C]$$

where $A$ and $B$ are constants determined from the conditions that at $t = 0$, $v_C = {}^1/_3V_{CC}$ and at $t = \infty$, $v_C = V_{CC}$. The former condition leads to $A + B = {}^2/_3V_{CC}$, and the latter to $A = V_{CC}$, making $B = -{}^2/_3V_{CC}$. Hence if the time taken to charge is $T_1$,

$$\tfrac{2}{3}V_{CC} = V_{CC} - \tfrac{2}{3}V_{CC}\exp[-T_1/(R_A + R_B)C]$$

$$\Rightarrow \quad \exp[-T_1/(R_A + R_B)C) = 1/2 \;\Rightarrow\; T_1 \approx 0.7(R_A + R_B)C$$

**6** From the circuit of figure A20.6a we can see that $J_2 = K_2 = Q_1$, $K_1 = Q_2$ and $J_1 = Q_2'$. Starting from $Q_1 = Q_2 = 1$ and using the operating table for JK flip-flops of figure A20.6b, we can construct the sequence in the table of figure A20.6c and the timing diagram of figure A20.6d, showing that the circuit divides the clock frequency by three and produces mark/space ratios of 2:1 ($Q_1$) and 1:2 ($Q_2$).



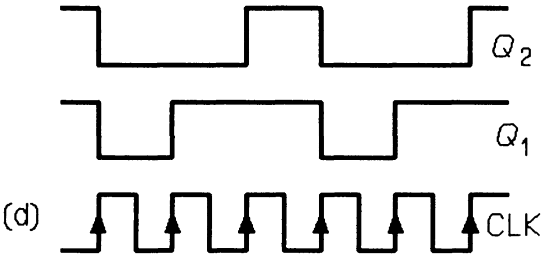
(a)

| J | K | Q |
|---|---|---|
| 0 | 0 | $Q_p$ |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | $\bar{Q}_p$ |

(b)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 |

(c)



(d)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 |

(e)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 |

(f)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |

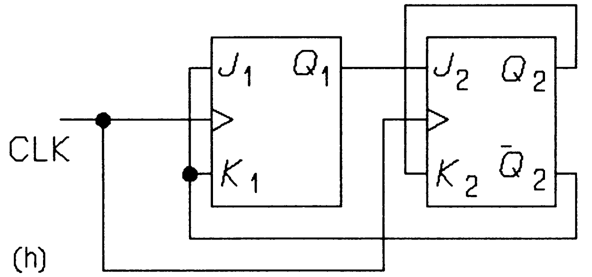
(g)

*Figures A20.6a to A20.6g*

If the initial state is set to $Q_1 = Q_2 = 0$, the sequence of figure A20.6e is produced, which gives the same waveforms as before. When the initial state is $Q_1 = 1$ and $Q_2 = 0$, the sequence of figure A20.6f results: again the same waveforms. But when the initial state is $Q_1 = 0$ and $Q_2 = 1$, the counter gets stuck as shown by figure A20.6g.

If the circuit is changed to that of A20.6h, in which $J_1 = K_1 = Q_2'$, $J_2 = Q_1$ and $K_2 = Q_2$, we find that starting from $Q_1 = Q_2 = 1$, we get the same sequence as before eventually as in figure A20.6j. However, no matter which combination we start from, this circuit does not get stuck and always gives the same output, as can be seen in the tables of figures A20.6k, A20.6m and A20.6n.

7 The tables for counting up and counting down are as in figure A20.7a and A20.7b, respectively. It can be seen that the table for counting down is just the inverse of that for counting up, so the counter circuit is that of the up-counter in figure 20.11, with counting down displayed by switching from the $Q$ outputs to the $Q'$ outputs as in figure A20.7c.



| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 |

(j)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 |

(k)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 |

(m)

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 |

(n)

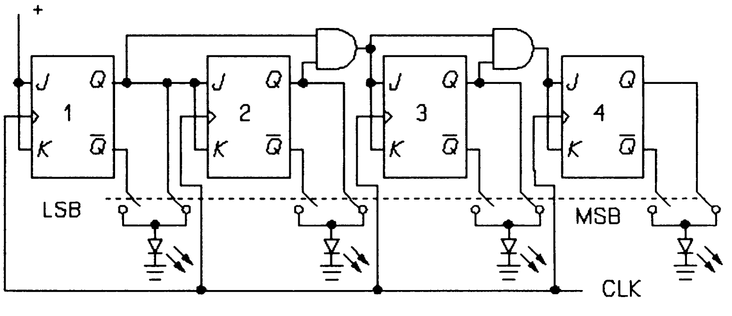
*Figures A20.6h to A20.6n*

| $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |

(a)

| $\bar{Q}_1$ | $\bar{Q}_2$ | $\bar{Q}_3$ | $\bar{Q}_4$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

(b)



(c)

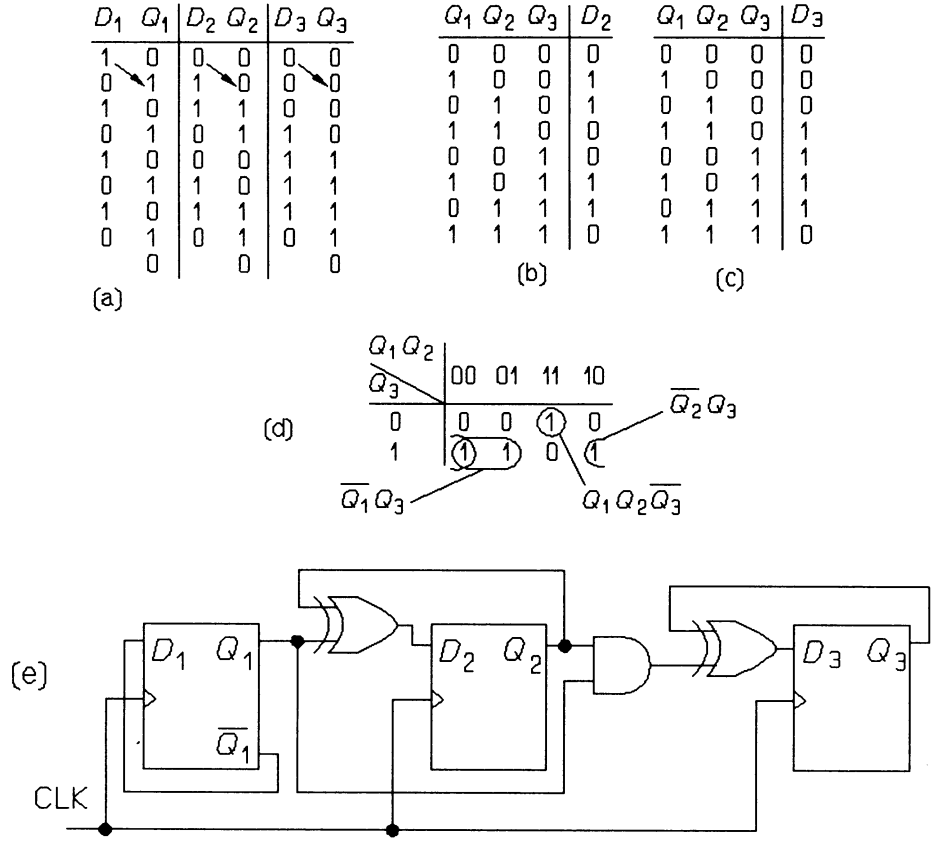
*Figure A20.7*

**8** The table for a 3-bit up-counter made from D-type flip-flops is shown in figure A20.8a, which is constructed by writing down the $Q$-values and then the $D$-values required which are the $Q$-values of the next clocked state. From this table we note first that $D_1 = Q_1'$ and then we construct the truth tables for $D_2$ and $D_3$, the tables of figures A20.8b and A20.8c. Table A20.8b indicates that $D_2$ is $Q_1 \oplus Q_2$, but we require a Karnaugh map to simplify the table of figure A20.8c, and this is shown in figure A20.8d. The logical expression for $D_3$ obtained from the Karnaugh map is

$$D_3 = Q_1Q_2Q_3' + Q_2'Q_3 + Q_1'Q_3 = (Q_1Q_2)Q_3' + (Q_1' + Q_2')Q_3$$

$$= (Q_1Q_2)Q_3' + (\overline{Q_1Q_2})Q_3 = (Q_1Q_2)\oplus Q_3$$

where we have used de Morgan's law to change ($Q_1' + Q_2'$) to $(Q_1Q_2)'$, leading to an XOR function at the end. The circuit is shown in figure A20.8e



| $D_1$ | $Q_1$ | $D_2$ | $Q_2$ | $D_3$ | $Q_3$ |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 |
|  | 0 |  | 0 |  | 0 |

(a)

| $Q_1$ | $Q_2$ | $Q_3$ | $D_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 |

(b)

| $Q_1$ | $Q_2$ | $Q_3$ | $D_3$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 |

(c)

| $Q_3 \backslash Q_1Q_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 |

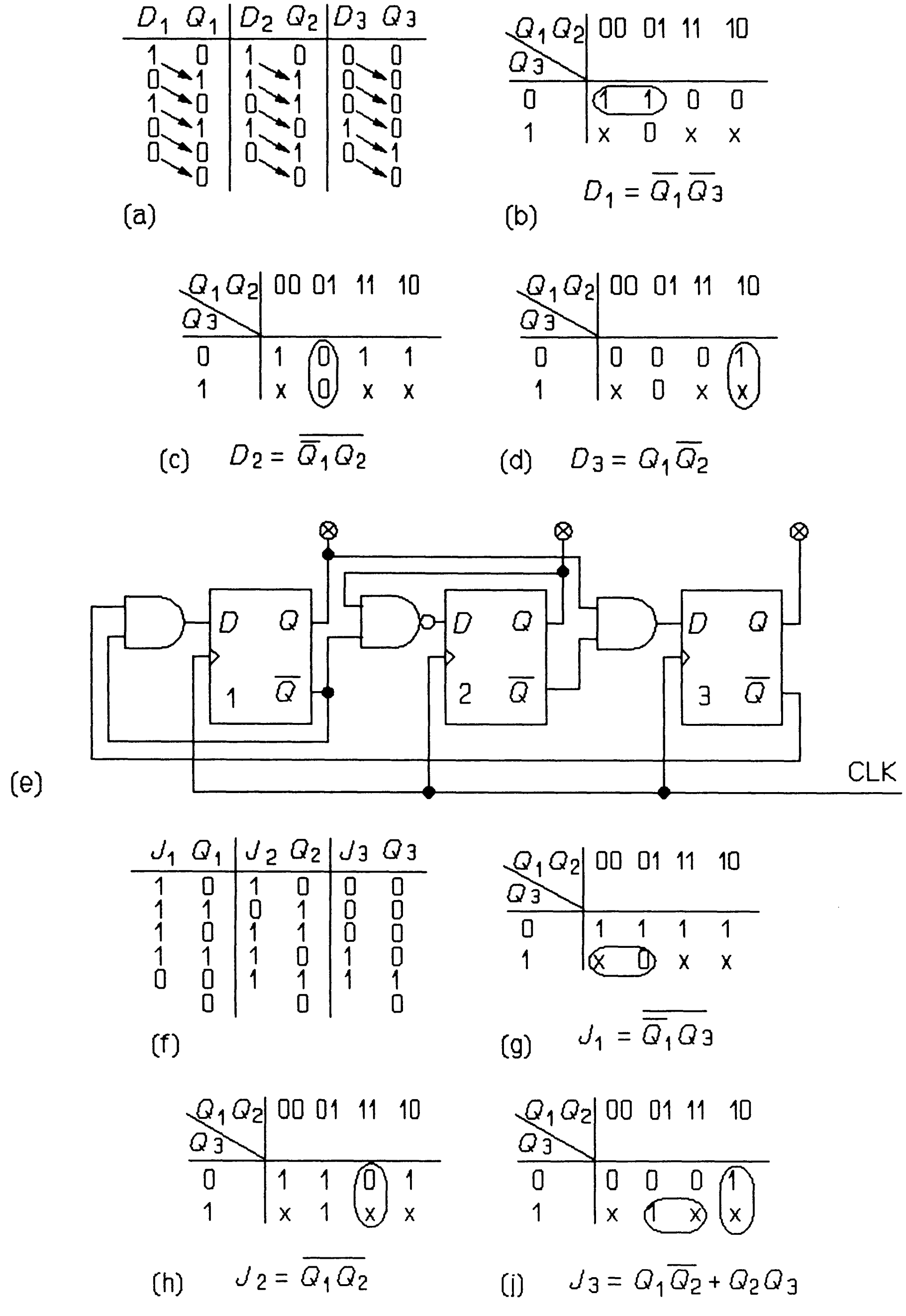
(d)

*Figure A20.8*

**9** A divide-by-five counter requires three flip-flops as there are 5 clock sequences. If the output, $Q_3$, is to have the state sequence 000010.., then we can assign different, but otherwise arbitrary, sequences to $Q_1$ and $Q_2$ and then deduce the logic for the prior input $D$-states. The table of figure A20.9a shows one such sequence starting from a reset, that is $Q_1 = Q_2 = Q_3 = 0$. From this table we see that the Karnaugh maps for $D_1$, $D_2$ and $D_3$ are as in figures 20.9b, 20.9c and 20.9d, leading to the circuit of figure 20.9e. The don't cares, that is combinations of the $Q$s not appearing in the table of figure 20.9a, are indicated by x in the Karnaugh maps. It may be that starting from one of these sequences the counter will hang up without achieving the desired division by five.

The same result can be obtained with JK flip-flops in the toggle mode, that is with $J = K$, though this is not essential. If the same sequence of states is employed for the outputs, then the required values of $J$ (and hence $K$) are as shown in the table of figure

A20.09f, leading to the Karnaugh maps of figures 20.9g,h and j and the circuit of figure 20.9k. Employing JKs in the non-toggle mode gives greater flexibility to the designer and leads to a circuit with fewer logic gates such as that of figure A20.9m, which is but one of several possible circuits.



*Figures A20.9a to A20.9j*

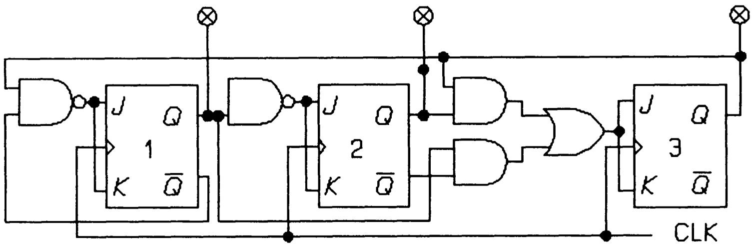


*Figure A20.9k*

| $J_1$ | $K_1$ | $Q_1$ | $J_2$ | $K_2$ | $Q_2$ | $J_3$ | $K_3$ | $Q_3$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |

$J_1 = J_2 = \overline{Q_3}$

$K_1 = Q_1$

$K_2 = \overline{Q_1}$

$J_3 = Q_1\overline{Q_2} = \overline{K_3}$

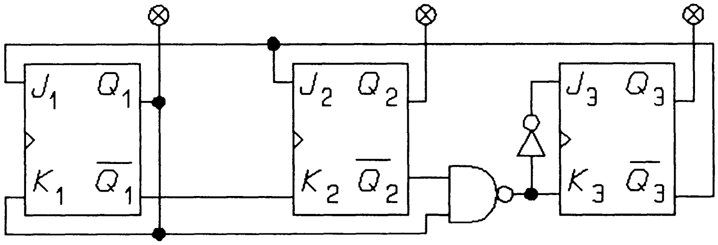
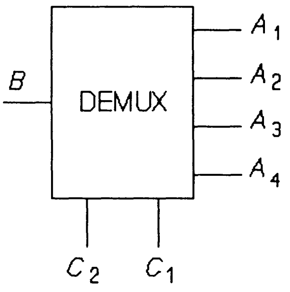


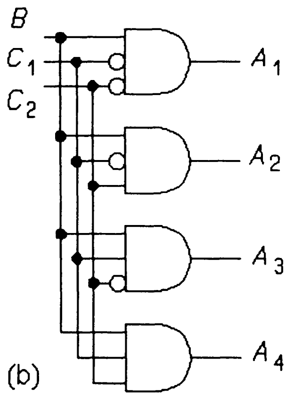
*Figure A20.9m*

**10** The demultiplexer must produce the *A*s from *B* when the control signals are as in the table of figure A20.10a (which is the same as that of figure 20.20b). Thus the ANDing of *B* by the *C*s will give the desired *A*s, which is shown in figure A20.10b.



| $C_1$ | $C_2$ | $B$ |
|---|---|---|
| 0 | 0 | $A_1$ |
| 0 | 1 | $A_2$ |
| 1 | 0 | $A_3$ |
| 1 | 1 | $A_4$ |

(a)



(b)

*Figure A20.10*

# Chapter 23

1 The modulation index is given by

$$m = \sqrt{2(r^2 - 1)}$$

where $r$ is the current or voltage ratio in the transmitting (or receiving antenna). Thus $r = 13.6/12 = 1.133$ and the modulation index is

$$m = \sqrt{2(1.133^2 - 1)} = 0.754$$

The power ratio is

$$\frac{P_m}{P_c} = \frac{P_c + P_s}{P_c} = 1 + \frac{1}{2}m^2 = 1 + 0.5 \times 0.754^2 = 1.2843$$

Thus $P_s = 0.2843P_c$, where the subscripts $m$, $c$ and $s$ refer to modulated, carrier and signal respectively and $P_c = I_c^2R_a = 12^2 \times 75 = 10.8$ kW, so that $P_s = 0.2843 \times 10.8 = 3.07$ kW.

The total power is

$$P_T = P_c + P_s = 10.8 + 3.07 = 13.87 \text{ kW}$$

so that the fractional power in the signal is

$$\frac{P_s}{P_T} = \frac{3.07}{13.87} = 0.221$$

The equation of an AM voltage wave of carrier power 1 W is

$$e = \sqrt{2}(1 + 0.5m\sin\omega_m t)\sin\omega_c t$$

In this case the r.m.s. carrier voltage is $I_cR_a = 12 \times 75 = 900$ V, so the peak voltage is $900\sqrt{2}$ V, while the carrier's angular frequency is $1.273 \times 2\pi = 8$ Mrad/s, and the modulation's angular frequency is $15.6 \times 2\pi = 98$ krad/s, making the equation of the voltage

$$e = 1.273[1 + 0.754\sin(98 \times 10^3 t)]\sin(8 \times 10^6 t) \text{ kV}$$

2 'Just over-modulated' means that $m = 1$, and substitution of this in the formula

$$m = \sqrt{2(r^2 - 1)} = 1 \Rightarrow r^2 = 1.5 \Rightarrow r = 1.225$$

yields $r = 1.225$ and thus $I_m = 1.225I_c = 1.225 \times 12 = 14.7$ A.

The total power is $P_m = I_m^2R_a = 14.7^2 \times 75 = 16.2$ kW, and as the carrier power is still 10.8 kW, the signal power is $16.2 - 10.8 = 5.4$ kW and the ratio of signal to carrier power is $5.4/10.8 = 1{:}2$.

3 Carson's rule gives a bandwidth of $2(f_m + \delta f) = 2(2.4 + 6) = 16.8$ kHz. The modulation index for FM is $m_f = \delta f/f_m = 6/2.4 = 2.5$, for which the coefficients are greater than 0.1 in table 23.1 up to $n = 3$ or 4, so that the bandwidth is effectively $2nf_m = 2 \times 3.5 \times 2.4 = 16.8$ kHz, taking an average for the order, $n$, of 3.5.

Doubling the modulating frequency's amplitude results in a doubling of the deviation, $\delta f$, to 12 kHz, making the modulation index 5, for which table 23.1 gives $n = 6$, and the bandwidth is $2nf_m = 2 \times 6 \times 2.4 = 28.8$ kHz. Using Carson's rule gives a bandwidth of $2(2.4 + 12) = 28.8$ kHz.

4 The modulation index is

$$m_f = \delta f/f_m = 20/5 = 4$$

The amplitude of the carrier frequency (order, $n = 0$) is $-0.4$ when $m_f = 4$, according to table 23.1, hence the carrier power is $(-0.4)^2 \times 10 = 1.6$ mW. The power residing in the sidebands must be $10 - 1.6 = 8.4$ mW.

The receiver's limited bandwidth of 50 kHz means that the higher order sidebands are lost in the receiver. With a bandwidth, $B$, of 50 kHz, the highest order of sideband receivable is

$$n = B/2f_m = 50/2 \times 5 = 5$$

The power residing in the sidebands up to and including the fifth is

$$P = 10[(-0.4)^2 + 2(0.07^2 + 0.36^2 + 0.43^2 + 0.28^2 + 0.13^2)] = 9.894 \text{ mW}$$

remembering to count the carrier once and the sidebands twice. Thus the power lost in the receiver is $10 - 9.894 = 0.106$ mW.

When the modulating frequency is doubled and the amplitude, power and maximum deviation are unchanged, the modulation index is halved to 2. The receiver's bandwidth is still 50 kHz, so the highest order of sideband receivable is $n = \Delta f/2f_m = 50/20 = 2.5$, or 2 since we deal in whole numbers. Table 23.1 reveals that the amplitude of the carrier is 0.22 for $m_f = 2$, so the carrier power at the receiver is $0.22^2 \times 10 = 0.48$ mW, making the total power in the sidebands $10 - 0.48 = 9.52$ mW.

The power in the carrier plus the receivable side bands is

$$P = 10[0.22^2 + 2(0.58^2 + 0.35^2)] = 9.66 \text{ mW}$$

Thus the power lost in the receiver must be $10 - 9.66 = 0.34$ mW.

Halving the amplitude means that the maximum deviation is reduced from 20 kHz to 10 kHz, and if the modulating frequency is 20 kHz, $m_f = \delta f/f_m = 10/20 = 0.5$, and virtually all of the received power is in the carrier, since its amplitude is 0.94 according to table 23.1, and so its power is $0.94^2 \times 10 = 8.84$ mW, if the received power is still 10 mW.

The power in the sidebands is therefore $10 - 8.84 = 1.16$ mW. The maximum order of receivable sideband is $50/(2 \times 20) = 1$ and the receivable power is

$$P = 10(0.94^2 + 2 \times 0.24^2) = 9.988 \text{ mW}$$

Then the power lost in the receiver is $10 - 9.988 = 0.012$ mW. Taking the amplitude of the sideband for which $n = 2$, that is 0.03, gives a lost power of $2 \times 0.03^2 \times 10 = 0.018$ mW. The two answers do not agree because the values in the table are rounded to 2 places of decimals.

5 The diode capacitance is

$$C_\mathrm{D} = KV^n = KV^{-0.4}$$

Then as $C_\mathrm{D} = 20$ pF when $V = 5$ V ($V$ is the reverse-bias voltage), we find

$$K = 20 \times 5^{0.4} = 38$$

in units of pF and V.

The modulation frequency and the modulation index are given, from which we can calculate the maximum deviation, as follows

$$\delta f = m_\mathrm{f} f_\mathrm{m} = 7.5 \times 20 = 150 \text{ kHz}$$

And the modulating voltage is to be a maximum of 25 mV, so we can use equation 23.33 as below

$$\frac{\mathrm{d}\omega}{\mathrm{d}V} = \frac{-n\omega}{2V(1 + c)} \Rightarrow \frac{\mathrm{d}\omega}{\omega} = \frac{\mathrm{d}f}{f} = \frac{-n}{2(1 + c)}\frac{\mathrm{d}V}{V}$$

where $f$ is the carrier frequency, 120 MHz. Replacing $\mathrm{d}f$ by $\delta f$ (=150 kHz) and $\mathrm{d}V$ by 25 mV leads to

$$\frac{150 \times 10^3}{120 \times 10^6} = 1.25 \times 10^{-3} = \frac{0.4}{2(1 + c)}\frac{25 \times 10^{-3}}{V}$$

$$\Rightarrow V(1 + c) = 4$$

But $c$ is the capacitance ratio, $C_\mathrm{D}/C$, which is

$$C_\mathrm{D}/C = KV^n/C = 38V^{-0.4}/180 = 0.211V^{-0.4}$$

$C$ being the series capacitance, 180 pF. This value for $c$ is substituted in the equation above to give

$$V(1 + c) = 4 \Rightarrow V(1 + 0.211V^{-0.4}) = V + 0.211V^{0.6} = 4$$

Then we can solve this using Newton-Raphson:

$$f(V) = V + 0.211V^{0.6} - 4 = 0 \Rightarrow f'(V) = \mathrm{d}f(V)/\mathrm{d}V = 1 + 0.1266V^{-0.4}$$

Making our first guess for $V$ as $V_1 = 4$ V, we find

$$f(4) = 4 + 0.211 \times 4^{0.6} - 4 = 0.485$$

and $$f'(4) = 1 + 0.1266 \times 4^{-0.4} = 1.073$$

Then the next approximation to $V$ is

$$V_2 = V_1 - \frac{f(V_1)}{f'(V_1)} = 4 - \frac{0.485}{1.073} = 3.55 \text{ V}$$

This is as near as makes no difference.

**6** The white noise can be taken to be Johnson noise whose voltage is

$$E_\mathrm{n} = \sqrt{4RkTB}$$

$$\Rightarrow \quad R = \frac{E_\mathrm{n}^2}{4kTB} = \frac{(0.6 \times 10^{-3})^2}{4 \times 1.38 \times 10^{-23} \times 300 \times 20 \times 10^6} = 1.09 \text{ M}\Omega$$

**7** There are 6 amplifier combinations to consider: 123, 132, 213, 231, 312 and 321, where the first number is the input amplifier and the last number denotes the output amplifier. It is almost a certainty that the amplifier with the lowest noise temperature must be placed first in the cascade for the lowest noise temperature overall, and that with the highest noise temperature should be placed first for the highest noise temperature overall, but it is best to evaluate all the combinations.

We have $G_1 = 6$ dB $= 4$, $T_1 = 50$ K; $G_2 = 30$ dB $= 1000$, $T_2 = 100$ K; $G_3 = 20$ dB $= 100$, $T_3 = 600$ K. The overall noise temperature of a cascaded series of amplifiers is

$$T_\mathrm{eq} = T_A + T_B/G_A + T_C/G_BG_A$$

where the amplifier order in the cascade is $ABC$, $A$ being the input and $C$ the output. And thus the six combinations give

$$T_\mathrm{eq}(123) = 50 + 100/4 + 600/(4 \times 1000) = 75.15$$

$$T_\mathrm{eq}(132) = 50 + 600/4 + 100/(4 \times 100) = 200.25$$

$$T_\mathrm{eq}(213) = 100 + 50/1000 + 600/(1000 \times 4) = 100.2$$

$$T_{eq}(231) = 100 + 600/1000 + 50/(1000 \times 100) = 100.6005$$

$$T_{eq}(312) = 600 + 50/100 + 100/(100 \times 4) = 600.75$$

$$T_{eq}(321) = 600 + 100/100 + 50/(100 \times 1000) = 601.0005$$

The lowest noise temperature overall is that of the 123 combination, 75.15 K and the highest by its reverse, 321, which is 601 K.

The noise factor is given by

$$T_{eq} = (F - 1)T_0$$

where $T_0$ is the ambient temperature, 300 K in this case. Thus the noise factors are

$$F_1 = 1 + T_1/T_0 = 1 + 50/300 = 1.1667$$

$$F_2 = 1 + T_2/T_0 = 1 + 100/300 = 1.3333$$

$$F_3 = 1 + T_3/T_0 = 1 + 600/300 = 3$$

The overall noise factor of the combination 123 is

$$F_{min} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} = 1.1667 + \frac{0.3333}{4} + \frac{2}{4 \times 1000} = 1.2505$$

which is found more simply by

$$F_{min} = \frac{T_{eq}(\text{min})}{T_0} + 1 = \frac{75.15}{300} + 1 = 1.2505$$

And that of the combination 321 is

$$F_{max} = F_3 + \frac{F_2 - 1}{G_3} + \frac{F_1 - 1}{G_3G_2} = 3 + \frac{0.3333}{100} + \frac{0.1667}{100 \times 1000} = 3.0033$$

Or, $$F_{max} = \frac{T_{eq}(\text{max})}{T_0} + 1 = \frac{601}{300} + 1 = 3.0033$$

**8** The thermal-noise voltage is

$$E_n = \sqrt{4kTBR} = \sqrt{4 \times 1.38 \times 10^{-23} \times 450 \times 2 \times 10^5 \times 10^4} = 7.05\ \mu\text{V}$$

The probability that the instantaneous voltage exceeds $E$ is given by

$$P(\geq E) = 0.5\,\text{erfc}(E/\sqrt{2}E_n)$$

Thus if $E = 6\ \mu\text{V}$, the probability of its being exceeded is

$$P(\geq 6\ \mu\text{V}) = 0.5\,\text{erfc}(6/\sqrt{2}/7.05) = 0.5\,\text{erfc}(0.6)$$

Looking up this probability in table 24.1 gives $P(\geq 6\ \mu\text{V}) = 0.198$, but as we are not interested in the sign of the voltage we must multiply by two, because the probability that an instantaneous voltage is less than $-6\ \mu\text{V}$ is also 0.198; thus $P(|E| \geq 6\mu\text{V}) = 0.396$.

If the probability is 50% that the instantaneous voltage exceeds $E$, say, then it must be 25% that it is greater than $+E$ and 25% that it is less than $-E$. We thus look for the value of $x$ corresponding to 0.25 in table 24.1, and we find that when $x = 0.5$, the probability is 0.2398 and when $x = 0.4$, the probability is 0.2858. The interpolation is

$$\Delta P = 0.2858 - 0.2398 = 0.046 \quad \text{and} \quad \Delta x = -0.1 \quad \Rightarrow \quad \frac{\Delta P}{\Delta x} = -0.46$$

The required $P$-value of 0.25 is 0.0102 away from 0.2398, so the correction to the $x$-value is -0.0102/0.46 = −0.022, giving $x = 0.4778$ as the required argument. The argument is given by

$$x = \frac{E}{\sqrt{2}E_n} \quad \Rightarrow \quad E = \sqrt{2}E_n x = \sqrt{2} \times 7.05 \times 0.4778 = 4.76\ \mu\text{V}$$

There is a 50% chance that the instantaneous voltage exceeds 4.76 $\mu$V in magnitude.

The probability that an instantaneous voltage exceeds $-5\ \mu\text{V}$ is 1 minus the probability that it is less than $-5\ \mu\text{V}$, which is easily found by calculating the value of the argument as

$$x = \frac{E}{\sqrt{2}E_n} = \frac{5}{\sqrt{2} \times 7.05} = 0.5015$$

The corresponding probability from table 24.1 is 0.2392 and subtracting this from 1 gives the required probability as 0.7608, which we square to get the probability of successive measurements exceeding $-5\ \mu\text{V}$ to be 0.579.

The probability that the voltage is less than $-3$ V is found by looking up the value in table 24.1 corresponding to

$$x = \frac{3}{\sqrt{2} \times 7.05} = 0.3$$

which is 0.3357.

The other argument corresponding to $+5\ \mu\text{V}$ is

$$x = \frac{5}{\sqrt{2} \times 7.05} = 0.5$$

for which we find $P = 0.2398$, which is the probability that the voltage lies above +5 $\mu$V. Hence the probability that an instantaneous voltage reading lies above $-3\ \mu\text{V}$ *and* below $+5\ \mu\text{V}$. Thus the probability that a voltage lies *outside* of the range is the sum of

0.3357 and 0.2398, which is 0.5755, and therefore the probability that it lies *inside* the range is 1 minus the probability that it lies outside = 1 − 0.5755 = 0.4245.
These answers can be understood more easily by referring to the bell-shaped curve (the gaussian probability distribution function) in figure A23.8, remembering that the area under the curve is the probability.

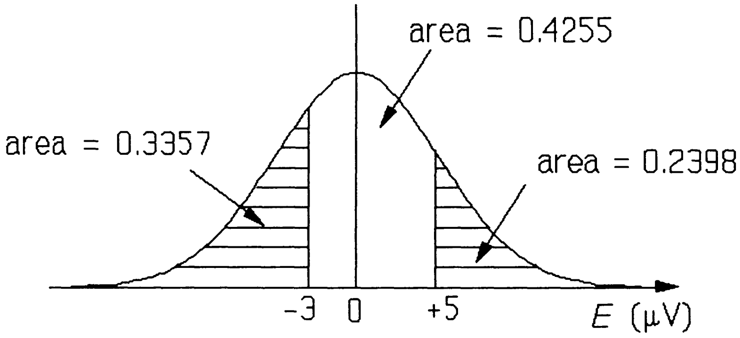


*Figure A23.8*

9 The noise in a FET is thermal noise of equivalent noise resistance $1/g_m$, or

$$R_{eq} = 1/g_m = 1/1.3 \times 10^{-3} = 769\ \Omega$$

Then the noise voltage is found from the Johnson noise formula

$$E_n = \sqrt{4kTBR_{eq}} = \sqrt{4 \times 1.38 \times 10^{-23} \times 350 \times 10^6 \times 769} = 3.85\ \mu\text{V}$$

The transconductance of a FET (see chapter 9) is given by

$$g_m = \frac{2\sqrt{I_{DSS}I_D}}{|V_P|} \propto \sqrt{I_D}$$

as $I_{DSS}$ and $V_P$ are constant for a given FET. Thus if $g_m = 1$ mS at $I_D = 1$ mA, it will be

$$g_m = \sqrt{\frac{100 \times 10^{-6}}{10^{-3}}} \times 1 = 0.316\ \text{mS}$$

and the equivalent noise resistance is $1/g_m = 3.16$ kΩ. The thermal noise at 400 K now becomes

$$E_n = \sqrt{4 \times 1.38 \times 10^{-23} \times 400 \times 10^6 \times 3160} = 8.35\ \mu\text{V}$$

The thermal noise in a gate resistance of 1 MΩ at 300 K is

$$E_n = \sqrt{4 \times 1.38 \times 10^{-23} \times 300 \times 10^6 \times 10^6} = 0.129 \text{ mV}$$

Then if the input signal, $E_s$, is 0.8 mV, the SNR is

$$SNR = 20\log_{10}(E_s/E_n) = 20\log_{10}(0.8/0.129) = 15.85 \text{ dB}$$

**10** The noise due to the spreading resistance of the base at 390 K is

$$E_{nb}^2 = 4kTr_b = 4 \times 1.38 \times 10^{-23} \times 390 \times 75 = 1.615 \times 10^{-18} \text{ V}^2/\text{Hz}$$

To this must be added the shot noise

$$E_{ns}^2 = \frac{2k^2T^2}{qI_E} = \frac{2 \times (1.38 \times 10^{-23} \times 390)^2}{1.6 \times 10^{-19} \times 0.1 \times 10^{-3}} = 3.62 \times 10^{-18} \text{ V}^2/\text{Hz}$$

The total noise voltage squared is the sum of the squares:

$$E_n^2 = E_{nb}^2 + E_{ns}^2 = 1.615 + 3.62 = 5.235 \times 10^{-18} \text{ V}^2/\text{Hz}$$

In a bandwidth of 3 MHz, this becomes

$$E_n = \sqrt{5.235 \times 10^{-18} \times 3 \times 10^6} = 3.96 \ \mu\text{V}$$

If the thermal and shot noise are equal

$$4kTr_b = 2k^2T^2/qI_E \Rightarrow I_E = kT/2qr_b$$

$$= 1.38 \times 10^{-23} \times 500/2 \times 1.6 \times 10^{-19} \times 75 = 0.2875 \text{ mA}$$

**11** If the noise is gaussian the probability that the r.m.s. noise voltage is exceeded is given by

$$P(>E_n) = 0.5\text{erfc}(E_n/\sqrt{2}E_n) = 0.5\text{erfc}(0.707)$$

Table 24.1 gives 0.5erfc(0.7) = 0.1611 and 0.5erfc(0.8) = 0.1289, so 0.5erfc(0.707) = 0.1588 and we therefore expect 15.88% of values to lie above $+E_n$ and 15.88% to lie below $-E_n$, or a total of 31.76% outside the lines drawn at $\pm E_n$. There are 59 point in figure 23.17, so we expect 0.1588 × 59 = 9 point to lie above $+E_n$ or below $-E_n$. A quick count of points in figure 23.17 shows that 10 lie below $-E_n$ and 9 lie above $+E_n$ with 38 in between and 2 on the lines. The proportion lying outside the lines is 19/59 = 0.322 or 32.2%, almost exactly the expected result for a gaussian. If, however, we include the two points on the lines in with the 19 lying outside the lines, the proportion is 21/59 = 0.356, or 35.6%, which is a little further away from the expected proportion, but not significantly so. Points on the line should not be used to prove the case we want to prove, but in this case do not matter much. One could take one point as being inside and one outside and then the number lying outside the lines would be 20 - also very close to the expected 19.

A rapid test to see if the mean is zero is simply to count the numbers above $+E_n$ and those below $-E_n$ and see if they are about equal. For large numbers of random values this is fairly reliable. For small numbers one could simply count the number lying above zero and see if it was about equal to the number below. In this case the former test leads us to conclude the mean is zero, but the latter gives 33 points above zero and only 26 below, but this is still not a sufficient imbalance that we can conclude the mean is not zero.

**12** The lower cut off is at

$$f_1 = \frac{1}{2\pi\tau_1} = \frac{1}{2\pi \times 0.5 \times 10^{-3}} = 318.3 \text{ Hz}$$

and the upper, $f_2 = 3.183$ kHz. In the circuit of figure A23.12a, the response is

$$\frac{\mathbf{V}_o}{\mathbf{V}_{in}} = \frac{R_2}{R_2 + \dfrac{R_1}{1 + j\omega CR_1}}$$

$$= \frac{R_2(1 + j\omega CR_1)}{R_2(1 + j\omega CR_1) + R_1} = \frac{1/CR_1 + j\omega}{\dfrac{R_1 + R_2}{CR_1R_2} + j\omega}$$

Therefore the lower cut-off frequency is $\omega_1 = 1/CR_1$, and so $CR_1 = 0.5$ ms. The upper cut-off frequency is at $\omega_2 = (R_1 + R_2)/CR_1R_2$, making $CR_1R_2/(R_1 + R_2) = 50\ \mu$s. The ratio is

$$\frac{\omega_2}{\omega_1} = 10 = \frac{R_1 + R_2}{R_2} \Rightarrow R_1 = 9R_2$$

The final equation comes from the requirement that $Z_{in} = 1$ kΩ at the lower cut off. From figure A23.12a we can see that the input impedance is

$$\mathbf{Z}_{in} = \frac{R_1}{1 + j\omega CR_1} + R_2 = \frac{R_1 + R_2 + j\omega CR_1R_2}{1 + j\omega CR_1}$$

The magnitude of $\mathbf{Z}_{in}$ is

$$Z_{in} = \sqrt{\frac{(R_1 + R_2)^2 + (\omega CR_1R_2)^2}{1 + (\omega CR_1)^2}}$$

And this is to be 1 kΩ when $\omega = \omega_1$, and $\omega_1 CR_1 = 1$. Substitution of these values in the equation for $Z_{in}$ gives

$$1000 = \sqrt{\frac{(R_1 + R_2)^2 + R_2^2}{2}} = \sqrt{\frac{(10R_2)^2 + R_2^2}{2}}$$

$$\Rightarrow \quad 101R_2^2 = 2 \times 10^6 \quad \Rightarrow \quad R_2 = 140.7\ \Omega$$

and hence $R_1 = 9R_2 = 1.266$ kΩ, while $C = \tau_1/R_1 = 0.5 \times 10^{-3}/1266 = 0.395\ \mu$F.

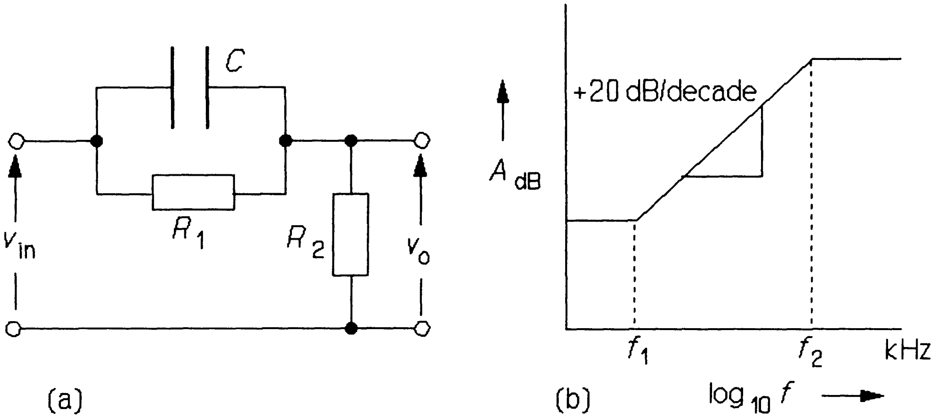


*Figure A23.12*

**13** For all the power to be in the sidebands and not in the carrier, we require that $m_f$ takes on one of its eigenvalues, 2.4, 5.5, 8.7 etc. Since $\delta f < 30$ kHz,

$$m_f = \delta f / f_m < 30/5 = 6$$

The eigenvalue $m_f = 5.5$ is suitable and makes $\delta f = m_f f_m = 5.5 \times 5 = 27.5$ kHz. By Carson's rule the bandwidth is

$$B = 2(f_m + \delta f) = 2(5 + 27.5) = 65 \text{ kHz}$$

**14** The flicker noise for a metal-film resistor is 0.1 μV/V/decade. If the 330kΩ resistor carries a current of 0.1 mA, then the voltage across it is 33 V and the flicker noise is 3.3 μV/decade. The noise lies in the band from 30 Hz to 100 kHz, which is

$$\log_{10}\left[\frac{100 \times 10^3}{30}\right] = 3.52 \text{ decades}$$

and so $E_{nf} = 3.3 \times 3.52 = 11.6\ \mu$V.

The thermal noise is

$$E_{nt} = \sqrt{4kTBR} = \sqrt{4 \times 1.38 \times 10^{-23} \times 410 \times 10^5 \times 330 \times 10^3} = 27.3\ \mu V$$

And the total noise is

$$E_n = \sqrt{E_{nf}^{\ 2} + E_{nt}^{\ 2}} = \sqrt{11.6^2 + 27.3^2} = 29.7\ \mu V$$

# Chapter 24

1 A BER of $10^{-5}$ means that the argument, $x$, of $0.5\,\mathrm{erfc}\,x$ in table 24.1 is 3.02. Using OOK and *average* power means that we can take the probability of error to be the same as with FSK that is

$$P_e = 0.5\,\mathrm{erfc}(A/2\sigma)$$

where $A$ is the amplitude of the signal and $\sigma$ the r.m.s. noise amplitude. Thus

$$\frac{A}{2\sigma} = 3.02 \;\Rightarrow\; \frac{A^2}{4\sigma^2} = 3.02^2 = \frac{P_s}{4P_n}$$

where $P_s$ is the average signal power and $P_n$ is the noise power, 2 nW. Hence

$$P_s = 4 \times 3.02^2 P_n = 4 \times 3.02^2 \times 2 = 73\text{ nW}$$

Since the average transmitted power is 3 kW, the signal is attenuated by

$$10\log_{10}\left[\frac{P_{\mathrm{TX}}}{P_s}\right] = 10\log_{10}\left[\frac{3000}{73 \times 10^{-9}}\right] = 106\text{ dB}$$

But the losses are 6 dB/km, so the range must be 106/6 = 17.7 km.

At 18 km the losses are 18 × 6 = 108 dB, so the signal power at the receiver is

$$P_s = \frac{3000}{10^{10.8}} = 47.55\text{ nW}$$

The noise power is 2 nW, and therefore

$$\frac{P_s}{P_n} = \frac{A^2}{\sigma^2} = \frac{47.55}{2} = 23.77$$

Hence

$$\frac{A}{2\sigma} = \frac{\sqrt{23.77}}{2} = 2.438$$

Looking up the corresponding error probability in table 24.1, we find $P_e = 2.91 \times 10^{-4}$, which means an actual bit error of $2.91 \times 10^{-4} \times 10 \times 10^6 = 2.91$ kbit/s.

An error rate of 1 bit/s gives an error probability of $10^{-7}$ and the corresponding value of $x$ in table 24.1 is 3.68, that is

$$x = A/2\sigma = 3.68 \Rightarrow A^2/\sigma^2 = P_s/P_n = 4 \times 3.68^2 = 54.2$$

Therefore the average signal power at the receiver, $P_s = 54.2P_n = 108.4$ nW. The attenuation is

$$10\log_{10}\left[\frac{P_{TX}}{P_s}\right] = 10\log_{10}\left[\frac{3000}{108.4 \times 10^{-9}}\right] = 104.4 \text{ dB}$$

Given that the attenuation is 6 dB/km, the range must be 104.4/6 = 17.4 km.

2 Spherical spreading means that the signal strength will decline as the square of the distance from the source, and so at $10^8$ km from the source the signal will be $10^{16}$ times less than at 1 km, that is

$$P = 195 \times 10^{-6} \times 10^{-16} = 1.95 \times 10^{-20} \text{ W/m}^2$$

The antenna has an area of 38 m$^2$, which means the power at the antenna must be

$$P_a = 1.95 \times 10^{-20} \times 38 = 7.41 \times 10^{-19} \text{ W}$$

The antenna has a gain of 76 dB (this gain is due to antenna geometry), so the signal power at the receiver is

$$P_s = 10^{7.6}P_a = 3.981 \times 10^7 \times 7.41 \times 10^{-19} = 29.5 \text{ pW}$$

The SNR is therefore 29.5/4.9 = 6.02 = $A^2/\sigma^2$. If OOK is used then

$$A/2\sigma = \sqrt{6.02}/2 = 1.227$$

Table 24.1 gives $P_e = 0.042$ when $x = 1.227$.

With PSK the argument of erfc is $A/\surd\sigma$ instead of $A/2\sigma$, so $x = 1.227 \times \surd 2 = 1.735$ and then $P_e$ according to table 24.1 is $7.2 \times 10^{-3}$. Cooling down to 77 K from 300 K will reduce the noise power by a factor of 77/300 and accordingly improve the signal to noise power ratio by 300/77, thus

$$A^2/\sigma^2 = 6.02 \times 300/77 = 23.45 \Rightarrow A/\sigma = 4.843$$

and $A/\surd\sigma = 3.425$, leading to $P_e$(PSK) = $6.6 \times 10^{-7}$.

For a BER of $10^{-9}$, the argument of erfc is 4.244, thus for PSK

$$\frac{A}{\sqrt{2}\,\sigma^2} = 4.244 \Rightarrow \frac{A^2}{\sigma^2} = 36.02$$

The ratio of temperatures is then 6/36 = 1/6 and the required temperature is 300/6 = 50 K.

3 A BER of $10^{-3}$ means that (from table 24.1) $x = 2.19$ and now since we are considering pulses, we use equation 24.7, which shows that

$$x = \frac{A}{2\sqrt{2}\,\sigma} = 2.19 \Rightarrow \frac{A^2}{\sigma^2} = 8 \times 2.19^2 = 38.4$$

This is the signal-to-noise within-pulse power ratio needed at the receiver amplifier, where the noise power is 0.1 pW, so the required signal pulse power is 3.84 pW. The receiving antenna has an area of 100 m$^2$, making the signal pulse power/m$^2$ $p_A = 0.0384$ pW/m$^2$. If the transmitted signal were subject to spherical spreading alone, the pulse power required would be

$$P = 4\pi r^2 p_A = 4\pi \times (4.5 \times 10^{12})^2 \times 0.0384 \times 10^{-12} = 9.77 \times 10^{12} \text{ W}$$

since the range, $r$, is $4.5 \times 10^{12}$ m. This power is reduced by the overall gain of the transmitter and receiver antennas, which is 80 dB, and the required transmitter pulse power becomes

$$P = 9.77 \times 10^{12} \times 10^{-8} = 97.7 \text{ kW}$$

Now the average available transmitter power is 0.85 kW, so the duty cycle for transmission can be a maximum of $0.85/97.7 = 8.7 \times 10^{-3}$. Since the minimum pulse length is 0.1 $\mu$s, the maximum bit rate is $10^7$ with a duty cycle of 1. But with OOK the duty cycle can be at most 0.5 (half the time, on average, 0s are transmitted and half the time 1s) and the maximum bit rate is therefore $2 \times 8.7 \times 10^{-3} \times 10^7 = 174$ kbit/s.

The television picture contains $625 \times 400 = 250 \times 10^3$ dots/frame, each dot contains three colours, so the total bits required (assuming no electron-beam intensity modulation in the television picture tube) is 750 kbit/frame, giving a maximum frame rate of 174/750 = 0.232 Hz. Using delta modulation would require only $0.12 \times 750 = 90$ kbits/frame, and the frame rate is then $174/90 = 1.93$ Hz.

4 Using PSK instead of OOK improves the pulse SNR by a factor of 4 and hence the transmitted pulse power required is reduced fourfold to 97.7/4 = 24.4 kW. However, the bit rate is only improved by a factor of two because PSK requires a pulse to be transmitted for both 0s and 1s; thus the bit rate becomes $174 \times 2 = 348$ kbit/s. The frame rates are likewise improved by a factor of 2 to 0.464 Hz and 3.86 Hz. Reducing the BER to $10^{-5}$ requires $x = 3.02$ instead of 2.19 and this means that the pulse power has to be increased to

$$P = 24.4 \times \left[\frac{3.02}{2.19}\right]^2 = 46.4 \text{ kW}$$

5 The receiver's noise power is

$$P_n = kT_{eq}B = 1.38 \times 10^{-23} \times 2500 \times 10^7 = 0.345 \text{ pW}$$

A BER of $10^{-6}$ corresponds to $x = 3.37$ in table 24.1 and therefore with PSK

$$x = \frac{A}{\sqrt{2}\,\sigma} = 3.37 \quad \Rightarrow \quad \frac{A^2}{\sigma^2} = 2 \times 3.37^2 = 22.7$$

Then

$$\frac{P_s}{P_n} = \frac{A^2}{\sigma^2} = 22.7 \quad \Rightarrow \quad P_s = 22.7P_n = 22.7 \times 0.345 = 7.83 \text{ pW}$$

The transmitted signal power is 176 W making the attenuation

$$A_{\mathrm{dB}} = 10\log_{10}\left[\frac{P_{\mathrm{TX}}}{P_s}\right] = 10\log\left[\frac{176}{7.83 \times 10^{-12}}\right] = 133.5 \text{ dB}$$

Of this 6 dB comes from the channel's fixed losses leaving 127.5 dB due to the cable's 0.13dB/m loss, thus the cable length must be

$$L = 127.5/0.13 = 981 \text{ m}$$

The reduction of the equivalent noise temperature to 300 K from 2500 K results in a gain of $10\log_{10}(2500/300) = 9.2$ dB, which is an extra length of $9.2/0.13 = 71$ m of cable, giving a total length of $981 + 71 = 1052$ m. This is, as usual, a very modest gain for a considerable improvement in noise factor. Only by improving the cable losses/m will commensurate gains in distance be achieved.

6 The value of $x^2$ corresponding to a $P_e = 10^{-4}$ from figure 24.3 is 7 and for OOK and an optimal detection process

$$E/8n_0 = x^2 = 7$$

where $E$ is the energy in the received pulse and $n_0$ is the spectrum noise level in W/Hz. We are given $n_0 = -135$ dB relative to 1 W/Hz, which is $3.16 \times 10^{-14}$ W/Hz and so

$$E_{\mathrm{RX}} = 2.636^2 \times 8n_0 = 7 \times 8 \times 3.16 \times 10^{-14} = 1.77 \times 10^{-12} \text{ J}$$

However, with an overall gain of 78 dB the required energy/pulse at the receiver is reduced by $10^{7.8}$ to $2.8 \times 10^{-20}$ J.

Now the pulse energy is

$$E_{\mathrm{TX}} = P_{\mathrm{TX}}\tau = 20 \times 10^3 \times 100 \times 10^{-6} = 2 \text{ J}$$

If the receiver is at a range, $r$, from the transmitter the received pulse energy would be

$$E_{\mathrm{RX}} = \frac{E_{\mathrm{TX}}A_{\mathrm{RX}}}{4\pi r^2}$$

if spherical spreading were the sole cause of attenuation. In addition there is an absorption loss of 8 dB/km or 0.008 dB/m, or a factor of $10^{-0.0008r}$ for $r$ m, and the received pulse power becomes

$$E_{RX} = SA = \frac{E_{TX}A_{RX}}{4\pi r^2} \times 10^{-0.0008r} = \frac{2 \times 0.2}{4\pi r^2} \times 10^{-0.0008r} = 2.8 \times 10^{-20}\ \mathrm{J}$$

where $S$ is the intensity loss due to spherical spreading and $A$ the intensity loss caused by absorption. It is instructive to enter some values for $r$ into the left hand side of this equation and see what results:

| $r$ (m) | $S$ | $A$ | $SA$ (J) |
|---|---|---|---|
| 1000 | $3.2 \times 10^{-8}$ | $1.6 \times 10^{-1}$ | $5.1 \times 10^{-9}$ |
| 5000 | $1.3 \times 10^{-9}$ | $1.0 \times 10^{-4}$ | $1.3 \times 10^{-13}$ |
| 10000 | $3.2 \times 10^{-10}$ | $1.0 \times 10^{-8}$ | $3.2 \times 10^{-18}$ |
| 12300 | $2.1 \times 10^{-10}$ | $1.4 \times 10^{-10}$ | $2.9 \times 10^{-20}$ |
| 13000 | $1.9 \times 10^{-10}$ | $4.0 \times 10^{-11}$ | $7.6 \times 10^{-21}$ |

The contribution of spherical spreading to the signal's attenuation is at first overwhelmingly predominant, but eventually it is overtaken and then swamped by the seemingly-modest part due to absorption. The exponential form of the latter causes the by-now-familiar threshold effect, and increasing the transmitted power hardly affects the range achievable.

7 The noise level in a bandwidth of 80 MHz is given by

$$N = B\sigma_0^2 = 8 \times 10^7 \times (0.8 \times 10^{-6})^2 = 5.12 \times 10^{-5}\ \mathrm{V}^2$$

The signal level is $S = (50 \times 10^{-3})^2 = 2.5 \times 10^{-3}\ \mathrm{V}^2$, giving a SNR of

$$S/N = 2.5 \times 10^{-3}/5.12 \times 10^{-5} = 48.83$$

If PAM is used with a BER of $10^{-5}$, the channel capacity is given by equation 24.29 as

$$C = B\log_2(1 + S/6N) = 3.322B\log_{10}(1 + S/6N)$$

$$= 3.322 \times 8 \times 10^7 \times \log_{10}(1 + 48.83/6) = 255\ \mathrm{Mbit/s}$$

When the BER is reduced to $10^{-7}$, table 24.1 gives $x = a/2\sqrt{2}\sigma = 3.68$, so that $h^2 = a^2/\sigma^2 = 8 \times 3.68^2 = 108$ and then

$$C = B\log_2(1 + 12S/h^2N) = 3.322B\log_{10}(1 + 12S/108N) = 3.322B\log_{10}(1 + S/9N)$$

Substituting $B = 80$ MHz and $S/N = 48.83$ into this gives $C = 215$ Mbit/s.

Shannon's formula gives

$$C = 3.322B\log_{10}(1 + S/N) = 3.322 \times 8 \times 10^7 \times \log_{10}(49.93) = 451 \text{ Mbit/s}$$

When the SNR is 4 and the bandwidth is unlimited

$$C_\infty = \frac{S}{2\sigma_0^2} = \frac{(50 \times 10^{-3})^2}{2 \times (0.8 \times 10^{-6})^2} = 1.95 \text{ Gbit/s}$$

**8** A SNR of 20 dB is the same as saying $S/N = 100$. Then if the bandwidth is 2 MHz, the Shannon formula gives

$$C = 3.322B\log_{10}(1 + S/N) = 3.322 \times 2 \times 10^6 \log_{10}(101) = 13.3 \text{ Mbit/s}$$

If PAM is used with a BER of $10^{-4}$, table 24.1 indicates that $x = a/2\surd\sigma = 2.636$, so that $a^2/\sigma^2 = h^2 = 8 \times 2.636^2 = 55.6$ and the channel capacity is

$$C = 3.322B\log_{10}(1 + 12S/h^2N) = 3.322B\log_{10}(1 + 12 \times 100/h^2)$$

$$= 3.322 \times 2 \times 10^6 \times \log_{10}(1 + 1200/55.6) = 9 \text{ Mbit/s}$$

Tripling the bandwidth will cause the SNR to be reduced by a factor of three if the spectrum noise level is unchanged, that is $S/N = 100/3 = 33.3$. Then the maximum information rate is (at the same BER as before)

$$C = 3.322 \times 3B\log_{10}(1 + S/N) = 3.322 \times 3 \times 10^6 \times \log_{10}(1 + 33.3) = 30.6 \text{ Mbit/s}$$

Using PAM and the same BER we find

$$C = 3.322 \times 3B\log_{10}(1 + 12 \times 33.3/55.6)$$

$$= 3.322 \times 6 \times 10^6 \times \log_{10}(8.19) = 18.2 \text{ Mbit/s}$$

**9** The quantisation SNR is given by

$$(S/N_q) \text{ dB} = 10\log_{10}(2^{2b} - 1) = 30 \Rightarrow 2^{2b} - 1 = 1000$$

$$\Rightarrow \quad 2b = \log_2 1001 = 3.322\log_{10}1001 \Rightarrow b = 4.98$$

The same result could have been obtained more quickly from

$$(S/N_q)_{\text{dB}} = 30 \approx 6b \Rightarrow b \approx 5$$

If the SNR at the receiver is 25 dB apart from quantisation noise, then $S/N = 10^{2.5} = 316$. Taking $S$ to be 1 W (any arbitrary value will do) then gives $N = 1/316 = 3.16$ mW.

For the quantisation noise we have $S/N_q = 10^3 = 1000$, so with $S = 1$ W, $N_q = 1$ mW. The total noise power is then 4.16 mW and

$$\frac{S}{N_{\text{Tot}}} = \frac{1}{4.16 \times 10^{-3}} = 240 \Rightarrow 10\log_{10}\left[\frac{S}{N_{\text{Tot}}}\right] = 23.8 \text{ dB}$$

The PCM signal is essentially OOK and the total noise calculated above is the noise which affects the receipt of a 1 or a 0, exactly as for OOK. Bearing in mind that the SNR refers to average signal power and not pulse power, the error probability is the same as for FSK, equation 24.8. Thus with a SNR of 23.8 dB, that is

$$SNR = \frac{A^2}{\sigma^2} = 10^{2.38} = 240 \Rightarrow \frac{A}{2\sigma} = x = \frac{\sqrt{240}}{2} = 7.746$$

This is a large value of $x$ so that

$$P_e = 0.5\text{erfc}(x) \approx \frac{\exp(-x^2)}{2\sqrt{\pi}x}$$

Substituting $x = 7.746$ into this leads to $P_e = 3.2 \times 10^{-28}$.

The theoretical minimum bandwidth required for a 5-bit code with a sampling frequency of 6 kHz is

$$B = bf_{\text{max}} = 0.5bf_s = 0.5 \times 5 \times 6 = 15 \text{ kHz}$$

where the sampling frequency is the Nyquist frequency. In practice the bandwidth required would be twice this or 30 kHz.

The information rate is $bf_s = 5 \times 6 = 30$ kbit/s.

Shannon's formula gives a maximum information rate of

$$C = 3.322B\log_{10}(1 + S/N) = 3.322 \times 15 \times 10^3 \times \log_{10}(1 + 240) = 119 \text{ kbit/s}$$

Equation 24.29 gives

$$C = 3.322B\log_{10}(1 + S/6N) = 3.322 \times 15 \times 10^3 \times \log_{10}(1 + 240/6) = 80 \text{ kbit/s}$$

The low information rate first calculated is based on OOK with a single amplitude. The calculated rate of 80 kbit/s is based on PAM with $2^b = 32$ amplitude levels and $P_e = 10^{-5}$. For this method of encoding $B$ is not 15 kHz as suggested but half the sampling frequency of 6 kHz, that is $B = 3$ kHz and $C = B\log_2 m^2 = 3 \times 3.322\log_{10}32^2 = 30$ kbit/s.

# Chapter 25

1 If $Q = \lambda/\Delta\lambda$ and $\lambda = 1.24/E_g$, then

$$\ln\lambda = \ln 1.24 - \ln E_g$$

Differentiating this gives

$$\frac{1}{\lambda}\frac{d\lambda}{dE_g} = \frac{-1}{E_g} \Rightarrow Q = \frac{\lambda}{\Delta\lambda} = \frac{E_g}{-\Delta E_g}$$

The minus sign merely indicates that $\lambda$ goes down when $E_g$ goes up, and may be ignored. Thus if the bandgap of GaAs is 1.43 eV and $Q = 1000$, the bandgap variation, $\Delta E_g = E_g/1000 = 1.43/1000 = 1.43$ meV.

If the bandgap variation in $In_xGa_{1-x}As$ varies linearly with $x$ and $E_g = 0.36$ eV for InAs and 1.43 eV for GaAs, then

$$E_g = 1.43 - 1.07x$$

Differentiating this yields

$$\frac{dE_g}{dx} = -1.07 \Rightarrow \frac{\Delta E_g}{E_g} = \frac{-1.07\Delta x}{E_g} = \frac{-1.07\Delta x}{1.43 - 1.07x}$$

When $x = 0.6$ this produces

$$\frac{\Delta E_g}{E_g} = \frac{1}{Q} = \frac{-1.07\Delta x}{1.43 - 1.07 \times 0.6} = -1.36\Delta x \Rightarrow \Delta x = \frac{1}{1.36 \times 1000} = 7.4 \times 10^{-4}$$

The minus sign is ignored for the reason previously given. The compositional variation, $\Delta x$, is $7.4 \times 10^{-4}$ or $\pm 3.7 \times 10^{-4}$, which is very little.

When $x = 0$ and $Q = 1000$, then $E_g = 1.43$ eV and $\lambda = 1.24/1.43 = 0.867$ $\mu$m. And as $Q = \lambda/\Delta\lambda$, $\Delta\lambda = \lambda/Q = 0.867/1000 = 0.867 \times 10^{-3}$ $\mu$m $= 0.867$ nm.

2 The current is given by

$$I = nq$$

where $n$ = number of electrons per second and $q$ is the electronic charge, $1.6 \times 10^{-19}$ C. If the efficiency is 100% all these electrons produce an outgoing photon and then $n$ is also the number of photons per second, but if the efficiency is $\eta$, $n$ electrons/s will only produce $\eta n$ photons/s, thus at a bit rate of 100 Mbit/s and an average of $10^7$ photons/bit

$$\eta n = 100 \times 10^6 \times 10^7 = 10^{15} \Rightarrow n = 10^{15}/\eta = 2 \times 10^{16}$$

When the electrode area is 100 × 20 $\mu m^2$, the current density is

$$J = \frac{I}{A} = \frac{nq}{A} = \frac{2 \times 10^{16} \times 1.6 \times 10^{-19}}{100 \times 20 \times 10^{-12}} = 1.6 \text{ MA/m}^2$$

or, in unapproved units, 1.6 $\text{A/mm}^2$.

If the wavelength of the emitted light is 0.8 $\mu$m, the energy of a single photon is

$$E_p = hf = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{0.8 \times 10^{-6}} = 2.5 \times 10^{-19} \text{ J}$$

Then the power in the light emitted is

$$P_o = n_p E_p = 10^{15} \times 2.5 \times 10^{-19} = 0.25 \text{ mW}$$

where $n_p$ is the number of photons emitted/s. If the efficiency is 5% the input power is 20 times this or 5 mW.

If there are $10^7$ photons/bit and half these are ones and half zeros on average with OOK, we should find there are 2 × $10^7$ photons in a 1 and no photons in a 0.

3 The acceptance angle is given by

$$\sin\alpha = \sqrt{n_1^2 - n_2^2} \quad \Rightarrow \quad n_1 = \sqrt{n_2^2 + \sin^2\alpha} = \sqrt{1.52^2 + \sin^2 12°} = 1.534$$

where $n_1$ is the RI of the core and $n_2$ that of the cladding.

The critical angle at the core/cladding interface is

$$\theta_c = \sin^{-1}(n_2/n_1) = \sin^{-1}(1.52/1.534) = 82.2°$$

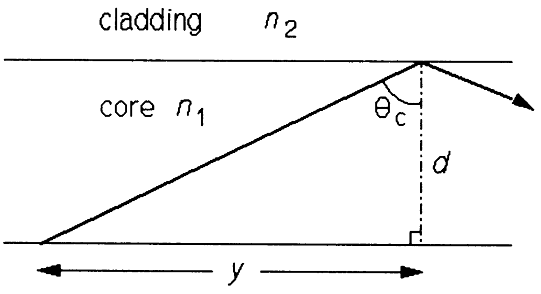


Figure A25.3

The geometry for a critically-angle ray travelling the maximum distance in the core is as shown in figure A25.3, where it can be seen that

$$y = d\tan\theta_c = 25\tan 82.2° = 182.5 \ \mu m$$

The number of total internal reflections/m is the reciprocal of this

$$N = 1/y = 1/182.5 \times 10^{-6} = 5480 \text{ per m}$$

If the attenuation at each reflection is $10^{-4}$ dB, then the attenuation of the critical ray per metre is $5480 \times 10^{-4} = 0.548$ dB/m.

**4** The number of modes is approximately

$$N \approx 0.5(\pi d/\lambda)^2(n_1^2 - n_2^2) = 0.5(\pi \times 25/0.5)^2(1.534^2 - 1.52^2) = 527$$

For $N$ to be 1 in this formula, the fibre diameter must be

$$d = \frac{\lambda}{\pi}\sqrt{\frac{2}{n_1^2 - n_2^2}} = \frac{0.5 \times 10^{-6}}{\pi}\sqrt{\frac{2}{1.534^2 - 1.52^2}} = 1.1\ \mu\text{m}$$

The 'exact' formula gives

$$d = \frac{0.77\lambda}{\sqrt{n_1^2 - n_2^2}} = \frac{0.77 \times 0.5 \times 10^{-6}}{\sqrt{1.534^2 - 1.52^2}} = 1.86\ \mu\text{m}$$

**5** The intermodal (or between-mode) dispersion is approximately

$$\Delta\tau = (n_1 - n_2)L/c$$

Substituting $L = 1000$ m, the length of the fibre, whose cladding RI is 1.53 and whose core RI is 1.52, with $c = 3 \times 10^8$ m/s we find $\Delta\tau_1 = 33.3$ ns. If the fibre is 0.25 km long then

$$\Delta\tau = \Delta\tau_1 L = 33.3 \times 0.25 = 8.33 \text{ ns}$$

leading to a maximum bit rate of approximately

$$R = 1/4\Delta\tau = (4 \times 8.33 \times 10^{-9})^{-1} = 30 \text{ Mbit/s}$$

When the fibre is 2.5 km long, it is a 'long' fibre so the dispersion is

$$\Delta\tau = \Delta\tau_1\sqrt{L} = 33.3 \times \sqrt{2.5} = 53 \text{ ns}$$

and the maximum bit rate becomes

$$R = 1/4\Delta\tau = (4 \times 53 \times 10^{-9})^{-1} = 4.7 \text{ Mbit/s}$$

6 The optimal grading index is given by $\alpha_{opt} = 2(1 - \Delta)$, where

$$\Delta = \frac{n_1 - n_2}{n_1} = \frac{1.54 - 1.51}{1.54} = 0.0195$$

$$\therefore \qquad \alpha_{opt} = 2(1 - 0.0195) = 1.961$$

If the actual grading index is 2, then

$$\frac{\Delta\tau}{L} = \frac{(n_1 - n_2)(\alpha - \alpha_{opt})}{c(\alpha + 2)} = \frac{0.03(2 - 1.961)}{4 \times 3 \times 10^8} = 9.75 \times 10^{-13} \text{ s/m} = 0.975 \text{ ps/km}$$

The dispersion of an optimally-graded fibre is

$$\frac{\Delta\tau}{L} = \frac{n_1\Delta^2}{8c} = \frac{1.54 \times 0.0195^2}{8 \times 3 \times 10^8} = 2.44 \times 10^{-13} \text{ s/m} = 0.244 \text{ ps/km}$$

The ratio of the two dispersions is 4:1.

7 The material dispersion in silica fibres between 0.8 $\mu$m and 1.6 $\mu$m is about

$$D_{mat} = \frac{\Delta\tau}{L\Delta\lambda} \approx -0.2\lambda + 260 \text{ ps/nm/km} = -0.2 \times 1000 + 260 = 60 \text{ ps/km}$$

if $\lambda = 1\ \mu\text{m} = 1000$ nm. Now a Q of 600 implies that

$$Q = \frac{\lambda}{\Delta\lambda} \Rightarrow \Delta\lambda = \frac{\lambda}{Q} = \frac{1000}{600} = 1.667 \text{ nm}$$

Hence the material dispersion is

$$\sigma_{mat} = \Delta\tau/L = D_{mat}\Delta\lambda = 60 \times 1.667 = 100 \text{ ps/km} = 0.1 \text{ ns/km}$$

When the intermodal dispersion, $\sigma_{mod}$, is 0.2 ns/km, the total dispersion is

$$\sigma_{tot} = \sqrt{\sigma_{mod}^2 + \sigma_{mat}^2} = \sqrt{0.2^2 + 0.1^2} = 0.224 \text{ ns/km}$$

In a fibre 10 km long (a 'long' fibre) the intermodal dispersion is

$$\Delta\tau_{mod} = \sigma_{mod}\sqrt{L} = 0.2 \times \sqrt{10} = 0.632 \text{ ns}$$

while the material dispersion is

$$\Delta\tau_{\mathrm{mat}} = \sigma_{\mathrm{mat}}L = 0.1 \times 10 = 1 \text{ ns}$$

And so the total dispersion is

$$\Delta\tau_{\mathrm{tot}} = \sqrt{\Delta\tau_{\mathrm{mod}}^2 + \Delta\tau_{\mathrm{mat}}^2} = \sqrt{1^2 + 0.632^2} = 1.18 \text{ ns}$$

The maximum bit rate for 1 km of fibre is

$$R = 1/4\Delta\tau_{\mathrm{tot}} = (4 \times 0.224 \times 10^{-9})^{-1} = 1.1 \text{ Gbit/s}$$

And for 10 km the maximum bit rate is

$$R = 1/4\Delta\tau_{\mathrm{tot}} = (4 \times 1.18 \times 10^{-9})^{-1} = 221 \text{ Mbit/s}$$

**8** The probability of error in the quantum limit is

$$P_{\mathrm{e}} = 0.5\exp(-m) = 10^{-6} \Rightarrow \quad m = -\ln(2 \times 10^{-6}) = 13.12$$

where $m$ is the average number of photons in a 1. Assuming OOK will produce the same number of 0s as 1s, the average number of photons per bit is $m/2 = 6.56$, that is 7 photons/bit since we require a whole number of photons. This is the minimum possible number of photons per bit.

In a system operating at 200 Mbit/s, there will have to be $200 \times 10^6 \times 7 = 1.4 \times 10^9$ photons/s in the quantum limit with a BER of $10^{-6}$. When the wavelength is 0.8 $\mu$m the energy in a photon is

$$E = hf = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{0.8 \times 10^{-6}} = 2.485 \times 10^{-19} \text{ J}$$

Thus the power in $1.4 \times 10^9$ photons is

$$P = 2.485 \times 10^{-19} \times 1.4 \times 10^9 = 0.348 \text{ nW}$$

For a BER of $10^{-6}$ we find from table 24.1 that the value of $x$ required is 3.37, and with OOK the average amplitude of signal current to noise current is given by

$$I_{\mathrm{s}}/2I_{\mathrm{n}} = A/2\sigma = x = 3.37 \Rightarrow \quad I_{\mathrm{s}}/I_{\mathrm{n}} = 2x = 6.74$$

Thus the average signal current is 67.4 nA. This requires

$$n_{\mathrm{s}} = \frac{I_{\mathrm{s}}}{q} = \frac{67.4 \times 10^{-9}}{1.6 \times 10^{-19}} = 4.2 \times 10^{11} \text{ electrons/s}$$

40% of the incident light is, however, reflected and of the remaining 60% only 10% is converted into photocurrent, that is

$$n_s = 0.1 \times 0.6 \times n_p = 0.06n_p$$

$$\Rightarrow \quad n_p = 16.7n_s = 16.7 \times 4.2 \times 10^{11} = 7 \times 10^{12} \text{ photons/s}$$

The energy of a photon is still $2.485 \times 10^{-19}$ J, so the power in the light signal is

$$P = 2.485 \times 10^{-19}n_p = 2.485 \times 10^{-19} \times 7 \times 10^{12} = 1.74 \ \mu\text{W}$$

The number of photons/bit is

$$n_b = \frac{n_p}{R} = \frac{7 \times 10^{12}}{200 \times 10^6} = 3.5 \times 10^4 \text{ photons/bit}$$

9 The electric field across the active layer is

$$\mathscr{E} = V/x = 30/30 \times 10^{-6} = 0.857 \text{ MV/m}$$

Thus the drift velocity of the electrons is

$$v_d = \mu\mathscr{E} = 0.2 \times 0.857 \times 10^6 = 171 \text{ km/s}$$

The transit time is therefore

$$\tau = x/v_d = 35 \times 10^{-6}/171 \times 10^3 = 0.2 \text{ ns}$$

If the number of incident photons/s is $n_i$ and 20% are reflected, then the number entering the active layer is $0.8n_i$ and the number absorbed is to be $0.5n_i$, so that the final number reaching the far side of the active layer is $0.3n_i$. If the absorption coefficient is $\alpha$ then

$$0.3n_p = 0.8n_p\exp(-\alpha x) \Rightarrow \alpha x = -\ln(0.3/0.8) = 0.98$$

$$\Rightarrow \alpha = 0.98/x = 0.98/35 \times 10^{-6} = 2.8 \times 10^4 \text{ m}^{-1}$$

The dopant density is found from

$$d = \sqrt{\frac{2\epsilon_0\epsilon_r(V_b + V_r)}{qN_D}} \quad \Rightarrow \quad N_D = \frac{2\epsilon_0\epsilon_r(V_b + V_r)}{qd^2}$$

$$\therefore \quad N_D = \frac{2 \times 8.85 \times 10^{-12} \times 2.5^2(1.5 + 30)}{1.6 \times 10^{-19} \times (35 \times 10^{-6})^2} = 1.78 \times 10^{19} \text{ m}^{-3}$$

# Chapter 26

**1** Strictly, the signal does not become compressed at all since $|v_{in}|$ is always less than or equal to $v_o$, but the time when the second part of the A-law operates is usually termed the compression phase. This occurs when $|v_{in}| = 1/A = 1/87.6$, that is when $\sin 2000t = 1/87.6$, and since 1/87.6 is small

$$\sin(1/87.6) = 1/87.6 \Rightarrow 2000t = 1/87.6 \Rightarrow t = 5.7\ \mu s$$

At this moment

$$v_o = \frac{A|v_{in}|}{1 + \ln A} = \frac{87.6 \times 1/87.6}{1 + \ln 87.6} = 0.1827\ \text{V}$$

When $t = 1\ \mu s$, $v_{in} = \sin(0.002) = 0.002$ and

$$v_o = \frac{87.6 \times 0.002}{1 + \ln 87.6} = 0.032\ \text{V}$$

Then when $t = 10\ \mu s$, $v_{in} = \sin(0.01) = 0.01$ and

$$v_o = \frac{1 + \ln(87.6 \times 0.01)}{1 + \ln 87.6} = 0.1585\ \text{V}$$

Figure A26.1 shows the graph of $v_o$ and $v_{in}$ for $0 \le t \le 785\ \mu s$.

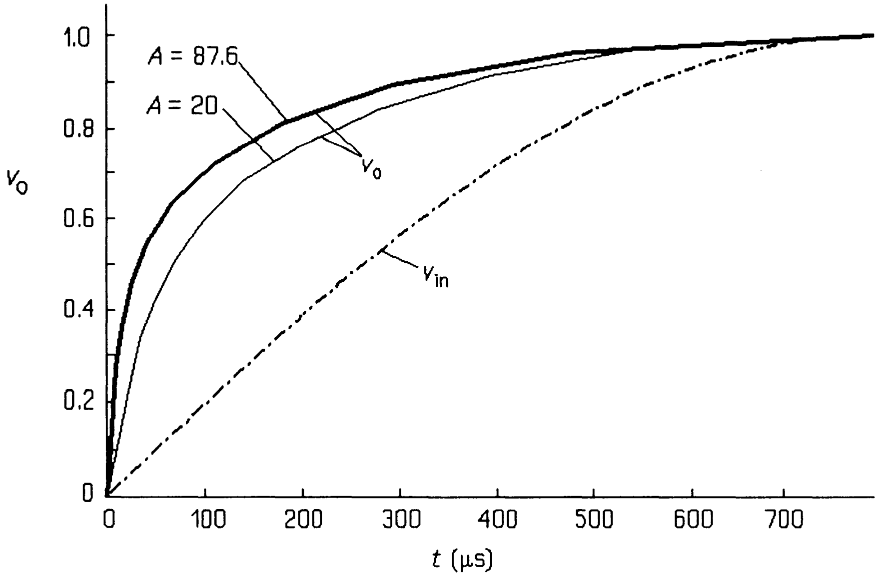


*Figure A26.1*

The first part of the A-law is a linear amplification of $v_{in}$ by a factor, $G$, where

$$G = \frac{A}{1 + \ln A} = \frac{87.6}{1 + \ln 87.6} = 16$$

The choice of $A = 87.6$ makes $G = 16$, a convenient integer. Making $A$ smaller has the effect of reducing $G$ almost in proportion and expands the range of signals which are amplified by $G$, since the changeover point is $v_{in} = 1/A$. For example when $A = 20$, $G = 5$ and the changeover point is at $v_{in} = 0.05$; the curve for this value of $A$ is shown in figure A26.1 also. Making $A = 202$, gives $G = 32$ and the changeover occurs at $v_{in} = 0.00495$.

2 The exchange traffic, $A = 2000 \times 0.03 = 60$ E, and table 26.2 gives $M = 75$ to give at least a 1% GOS, while for 0.2% GOS, 81 lines are required. The concentration ratios are therefore 2000/75 = 26.7:1 and 2000/81 = 24.7:1. Increasing the traffic by 20% to 72 E will reduce the GOS to 2.97% when $M = 81$ and 6.5% when $M = 75$, both unacceptable.

A BASIC program to compute GOS from $A$ and $M$ using equation 26.4 is as follows:

```
10 INPUT "A = ?"A
20 INPUT "M = ?"M
30 SUM = 1: REM: SUM = 1 when N = 0
40 TERM = 1: REM: TERM = 1 when N = 0
50 FOR N = 1 TO M
60   TERM = TERM*A/N: REM: TERM = A^N/N!
70   SUM = SUM + TERM
80 NEXT N
90 GOS = TERM/SUM: REM: TERM = last TERM = A^M/M!
100 PRINT GOS
110 END
```

3 With $M = 27$ and $A = 17.8$ we find GOS $= B = 1\%$, so the lost traffic is $AB = 17.8 \times 0.01 = 0.178$ E. When $M = 22$ and $A = 17.8$, we find $B = 6.173\%$, so the overflow traffic is $AB = 17.8 \times 0.06173 = 1.099$. Then this overflow traffic goes into the next group of 5 servers and we find, with $M = 5$ and $A$ = overflow = 1.099, that $B = 0.4456\%$. The lost traffic is now $AB = 1.099 \times 0.004456 = 0.0049$ E.

4 The traffic from C to T is 3 E and if the line between them is inoperable, all this traffic will have to pass through D and thence to T, making the traffic between C and D 3 + 2 = 5 E and that between D and T, 3 + 1 = 4 E. With 8 lines serving the traffic from C and D, the GOS on that route will be

$$B = \frac{A^M/M!}{\sum_{n=0}^{M} A^n/n!}$$

where $B$ = lost traffic = GOS, with $A = 5$ E and $M = 8$ lines. From this formula we find $B_1 = 0.07$, and with $A = 4$ E and $M = 8$ lines, we find $B_2 = 0.1172$.

The probability of a call getting through from C to D is $1 - B_1$, and the probability of a call getting through from D to T is $1 - B_2$, so the overall probability of a call getting through from C to T is $(1 - B_1)(1 - B_2)$ and the probability of blocking is 1 minus this or

$$GOS = 1 - (1 - B_1)(1 - B_2) = 1 - 0.93 \times 0.8828 = 0.179$$

The lost traffic for calls originating at C and going through T is 0.179 × 3 = 0.537 E, since the traffic affected is 3 E and its GOS is 0.179.

**5** If the lines between A and B are out of service and all the traffic is then routed through B and T, the total traffic through B and T and A and T is 2 + 4 = 6 E. If the lines between B and T can still offer a good GOS (we are not told how many there are and so must make this assumption), then the only constraint on the traffic is the number of lines between A and T, that is 8. Thus we find $B = 0.122$ from the Erlang formula of the previous problem with $A = 6$ E and $M = 8$.

With a traffic of 8 E, a 1% GOS requires 15 lines (according to table 26.2), a further 13 lines if 2 are already provided.

**6** The traffic between A and T must be the same as the sum of the traffic between A and B, A and C and A and D, that is 3 + 1 + 2 = 6 E. The traffic between B and T is likewise 3 + 3 + 4 = 10 E, and between C and T is 1 + 3 + 5 = 9 E and between D and T is 2 + 4 + 5 = 11 E. Let the coordinates of T be $(x,y)$, then the 'centre-of-mass' calculation for $x$ is

$$6[x - (-1)] + 10[x - 6] + 9[x - 4] + 11[x - 0] = 0$$

whence $x = 2.5$. For $y$

$$6[y - 6] + 10[y - 4] + 9[y - (-1)] + 11[y - 0] = 0$$

whence $y = 1.86$ and the coordinates of T are (2.5,1.86). The original value of $\Sigma Ad$ is

$$2\sqrt{1^2 + 6^2} + 3\sqrt{7^2 + 2^2} + 1\sqrt{5^2 + 7^2} + 3\sqrt{2^2 + 5^2} + 4\sqrt{6^2 + 4^2} + 5\sqrt{4^2 + 1^2} = 108.22$$

(in units of 10 Erlang-km). The value for the star network is

$$6\sqrt{3.5^2 + 4.14^2} + 10\sqrt{3.5^2 + 2.14^2} + 9\sqrt{1.5^2 + 2.86^2} + 11\sqrt{2.5^2 + 1.86^2} = 136.89$$

The difference is 28.67 E-km - an increase of 26.5%. The reason the star network increases $\Sigma Ad$ instead of reducing it is that the original network has too few centres (only four) so that only 4 × 3/2 = 6 interconnections suffice for a fully-connected network, while 4 are still required by the star network. If the number of centres had been 5, the number of interconnections required would have been 5 × 4/2 (= 10) for the fully-connected network and only 5 for the star network. In this case we might expect the two networks to have, in general, similar values of $\Sigma Ad$. In practice there would be some compromise between the two network extremes.

# Chapter 27

1 The cut-off frequency is

$$f_c = \frac{c}{2L} = \frac{3 \times 10^8}{2 \times 10^{-3}} = 150 \text{ GHz}$$

where $L$ = slot length = 1 mm. This is the frequency at which most radiation penetrates the braid. The minimum frequency which can penetrate the slot is given by $L = 0.01\lambda = 0.01c/f_{\min}$, so that $f_{\min} = 0.01c/L = 0.01 \times 3 \times 10^8/10^{-3} = 3$ GHz. At $0.1f_c = 15$ GHz, the waveguide attenuation is given by

$$S \approx 30t/L = 30 \times 0.4/1 = 12 \text{ dB}$$

Increasing the slot length to 3 mm will reduce the frequencies to a third of these values, namely to $f_c = 50$ GHz and $f_{\min} = 1$ GHz. Similarly the attenuation at 5 GHz will be reduced to 12/3 = 4 dB. The slot width's being increased to 1 mm has no effect on these figures. At 15 GHz, the previous value of $0.1f_c$, the waveguide attenuation is effectively now zero.

2 The skin depth formula is

$$\delta = \sqrt{2/\mu\sigma\omega}$$

which rearranges to

$$\omega = \frac{2}{\mu\sigma\delta^2} = \frac{2}{4\pi \times 10^{-7} \times 58 \times 10^6 \times (0.4 \times 10^{-3})^2} = 171.5 \text{ krad/s}$$

$$\Rightarrow \quad f = \omega/2\pi = 27.3 \text{ kHz}$$

where for Cu, $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m and $\sigma = 58$ MS/m.

At this frequency $t = \delta$ and so the absorption is $A = 8.7t/\delta = 8.7$ dB.

For copper, the reflection losses are $R = 168 - 10\log_{10}f = 124$ dB.

Now for steel, $\mu = \mu_r\mu_0 = 300 \times 4\pi \times 10^{-7}$ H/m and $\sigma = 2$ MS/m, so that

$$\omega = \frac{2}{300 \times 4\pi \times 10^{-7} \times 2 \times 10^6 \times (0.4 \times 10^{-3})^2} = 16.6 \text{ krad/s}$$

or $f = 2.64$ kHz.

At this frequency $t = \delta$ and $A = 8.7$ dB. We calculate the reflection losses from

$$R = 39.5 - 10\log_{10}(\omega\mu/\sigma)$$

$$\therefore \quad R = 39.5 - 10\log_{10}\left[\frac{16.6 \times 10^3 \times 300 \times 4\pi \times 10^{-7}}{2 \times 10^6}\right] = 94.5 \text{ dB}$$

3 The capacitance between two wires is given by

$$C = \frac{\pi\epsilon_r\epsilon_0 l}{\ln(2x/d)} = \frac{\pi \times 2.3 \times 8.85 \times 10^{-12} \times 0.25}{\ln 6} = 8.92 \text{ pF}$$

since $x$, the separation of the wires, is 3 mm (2 layers of 1mm thick insulation plus 1 conductor diameter, $d$, of 1 mm).

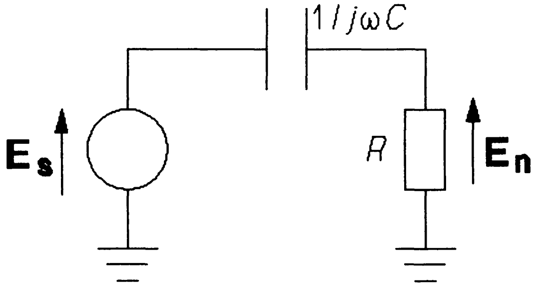


*Figure A27.3*

The equivalent circuit of the wires, neglecting capacitances to ground, is that of figure A27.3, from which we can see that the noise voltage is

$$\mathbf{E}_n/\mathbf{E}_s = \frac{R}{R + 1/j\omega C}$$

Hence,

$$\frac{1}{2.5} = \left|\frac{1}{1 + 1/j\omega CR}\right| = \frac{1}{\sqrt{1 + (\omega CR)^{-2}}}$$

$$\Rightarrow \quad 1 + (\omega CR)^{-2} = 2.5^2 \Rightarrow \omega CR = \frac{1}{\sqrt{2.5^2 - 1}}$$

$$\Rightarrow \quad \omega = \frac{0.4364}{CR} = \frac{0.4364}{8.92 \times 10^{-12} \times 10^4} = 4.89 \text{ Mrad/s}$$

or $f$ = 779 kHz.

If the capacitances to ground of the wires are taken into account, equation 27.1 applies, that is

$$\mathbf{E}_n = \frac{\omega C_{12} R_2 \mathbf{E}_s}{R_2\omega(C_{12} + C_2) - j}$$

$$\therefore \quad f_c = \frac{1}{2\pi R_2(C_{12} + C_2)} = \frac{1}{2\pi \times 10^4 \times 10.92 \times 10^{-12}} = 1.46 \text{ MHz}$$

The corner frequency, $f_c$, is such that 100 kHz is a 'low' frequency and we use the approximation of equation 27.3

$$E_n = \omega R_2 C_{12} E_s = 2\pi \times 10^5 \times 10^4 \times 8.92 \times 10^{-12} \times 2.5 = 0.14 \text{ V}$$

Conversely, 10 MHz is a 'high' frequency and we can use the approximation that is equation 27.4, giving

$$E_n = \frac{C_{12}E_s}{C_{12} + C_2} = \frac{8.92 \times 2.5}{8.92 + 2} = 2.04 \text{ V}$$

**4** The screened length, $l$, is 90% of 0.25 m or 0.225 m and the capacitance between screen and inner wire is

$$C_{2S} = \frac{2\pi\epsilon_r\epsilon_0 l}{\ln(d_2/d_1)} = \frac{2\pi \times 2.3 \times 8.85 \times 10^{-12} \times 0.225}{\ln 3} = 26.2 \text{ pF}$$

where $d_2$ = screen diameter = 3 mm (two thicknesses of 1mm insulation plus $d_1$ = 1 mm = wire diameter). The capacitance between the wires is then just that between the unscreened length, 0.025 m, which must be a 0.892 pF, a tenth of its previous value.

The maximum noise voltage is that at high frequencies when

$$E_n = \frac{C'_{12}E_s}{C'_{12} + C_{2S} + C'_1} \approx \frac{C'_{12}E_s}{C_{2S}} = \frac{0.892 \times 2.5}{26.2} = 85 \text{ mV}$$

The noise power is half the maximum noise power when $\omega = \omega_c$, the cut-off frequency, which is

$$\omega_c = \frac{1}{R_2(C'_{12} + C_{2S} + C'_2)} \approx \frac{1}{R_2C_{2S}} = \frac{1}{10^4 \times 26.2 \times 10^{-12}} = 3.82 \text{ Mrad/s}$$

or $f_c$ = 607 kHz. There is little practical point in calculating the exact frequency as it is only 4% less.

Again we can call 100 kHz a 'low' frequency so that by equation 27.5

$$E_n = \omega R_2 C'_{12}E_s = 2\pi \times 10^5 \times 10^4 \times 0.892 \times 10^{-12} \times 2.5 = 14 \text{ mV}$$

And 10 MHz is a high enough frequency to use equation 27.6, giving

$$E_n \approx \frac{C'_{12}E_s}{C_{2S}} = \frac{0.892 \times 2.5}{26.2} = 85 \text{ mV}$$

which is the maximum noise voltage.

**5** The attenuation losses are given by

$$A = 8.7t/\delta \text{ dB}$$

where $t$ is the shield thickness and $\delta$ is the skin depth, given by

$$\delta = \sqrt{2/\mu\sigma\omega}$$

so that the attenuation is

$$A = \frac{8.7t\sqrt{\mu\sigma\omega}}{\sqrt{2}}$$

The reflection losses are given by

$$R = 39.5 - 10\log_{10}(\omega\mu/\sigma) = 39.5 - 4.35\ln(\omega\mu/\sigma) \text{ dB}$$

The total losses are thus

$$S = A + R = \frac{8.7t\sqrt{\mu\sigma\omega}}{\sqrt{2}} + 39.5 - 4.35\ln(\omega\mu/\sigma)$$

Differentiating with respect to $\omega$ and setting equal to zero for a minimum, we find

$$\frac{\mathrm{d}S}{\mathrm{d}\omega} = \frac{8.7t\sqrt{\mu\sigma}}{2\sqrt{2}\sqrt{\omega}} - \frac{4.35}{\omega} = 0$$

$$\Rightarrow \quad \frac{t\sqrt{\mu\sigma}}{\sqrt{2}} = \frac{1}{\sqrt{\omega}}, \quad \therefore \; t = \sqrt{2/\mu\sigma\omega} = \delta$$

6 For round parallel wires the characteristic impedance is

$$Z_0 = 120\epsilon_r^{-1/2} = 120/\sqrt{2} = 85 \ \Omega$$

Thus the noise voltage is $Z_0\delta I$ = 85 mV.

For thin flat conductors the characteristic impedance is given by

$$Z_0 = \frac{377x}{b\sqrt{\epsilon_r}} = \frac{377 \times 0.5}{5 \times \sqrt{5}} = 17 \ \Omega$$

where $x$ = thickness of PVC = 0.5 mm, $b$ = conductor width = 5 mm and $\epsilon_r = 5$ for PVC. Thus an instantaneous change in current of 1 mA will produce a noise voltage of 17 mV.

When these rectangular conductors are placed side by side the characteristic impedance is given by

$$Z_0 = \frac{120\ln[\pi(x/b + 1)]}{\sqrt{\epsilon_r}} = \frac{120\ln(2\pi)}{\sqrt{4}} = 110 \ \Omega$$

where $x$ = 5 mm = gap between conductors, $b$ = width of conductors and $\epsilon_r = 4$ for PCB material. Thus the noise voltage is 110 mV when the line current changes by 1 mA instantaneously.

7 The area of the power line is $A = 5 \times 100 \times 10^{-6} = 5 \times 10^{-4}$ m$^2$, so that the capacitance of the line is given by

$$C = \frac{\epsilon_r \epsilon_0 A}{t} = \frac{4 \times 8.85 \times 10^{-12} \times 5 \times 10^{-4}}{1.3 \times 10^{-3}} = 13.6 \text{ pF}$$

Then the self-resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1 \times 10^{-6} \times 13.6 \times 10^{-12}}} = 857 \text{ Mrad/s}$$

or $f_0 = 136.4$ MHz.

The cross-sectional area of the power-line conductor is $0.5 \times 5 \times 10^{-6}$ m$^2$, and its DC resistance is

$$R_{DC} = \frac{l}{\sigma A_c} = \frac{100 \times 10^{-3}}{58 \times 10^6 \times 0.5 \times 5 \times 10^{-6}} = 0.69 \text{ m}\Omega$$

The skin depth is

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2}{4 \times 10^{-7} \times 58 \times 10^6 \times 857 \times 10^6}} = 5.66\ \mu\text{m}$$

Since this is much less than the conductor's thickness we can take

$$R_{AC} = \frac{R_{DC} t}{2\delta} = \frac{0.69 \times 0.5 \times 10^{-3}}{2 \times 5.66 \times 10^{-6}} = 30.5 \text{ m}\Omega$$

Then the Q-factor is given by

$$Q = \frac{\omega_0 L}{R_{AC}} = \frac{857 \times 10^6 \times 0.1\ 10^6}{30.5 \times 10^{-3}} = 2810$$

And thus the noise voltage at the self-resonant frequency is $QE_n = 0.562$ V.

8 The skin depth for copper at 50 Hz is

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 58 \times 10^6 \times 2\pi \times 50}} = 9.35 \text{ mm}$$

A shield of thickness 1 mm is approximately $0.1\delta$ thick and should be considered 'thin', as EMI design should err on the safe side. Thus the reflection loss given by equation 27.21 as

$$R_M(\text{Cu}) = 14.6 + 10\log_{10} r^2 f = 14.6 + 10\log_{10}(0.2^2 \times 50) = 17.6 \text{ dB}$$

should be reduced by 15 dB to 2.6 dB. In this case the induced voltage in the component is

$$E_n = 10^{-2.6/20} \times 34 = 25 \text{ mV}$$

When the shield is made from steel the skin depth becomes

$$\delta = \sqrt{\frac{2}{\mu_r \mu_0 \sigma \omega}} = \sqrt{\frac{2}{400 \times 4\pi \times 10^{-7} \times 6 \times 10^6 \times 100\pi}} = 1.45 \text{ mm}$$

This is very close to the shield thickness of 1 mm, and we must consider absorption losses, which are

$$A = 8.7t/\delta = 8.7 \times 1/1.45 = 6 \text{ dB}$$

Equation 27.20 gives the reflection loss as

$$R_M(\text{Fe}) = 20\log_{10}(0.25r\sqrt{\omega\mu_0\sigma/\mu_r})$$

$$\therefore \quad R_M(\text{Fe}) = 20\log_{10}(0.25 \times 0.2\sqrt{100\pi \times 4\pi \times 10^{-7} \times 6 \times 10^6/400)} = -18 \text{ dB}$$

The negative sign shows that the equation is inapplicable (the shield can hardly increase the EMI!) and that the reflection loss is probably zero. We therefore have only absorption losses of 6 dB to attenuate the EMI and the induced voltage is

$$E_n = 10^{-6/20} \times 34 = 17 \text{ mV}$$

To reduce the induced voltage to 1 mV requires total shield losses of $20\log_{10}34$ = 30.6 dB, of which the reflection losses with copper remain at 17.6 dB, leaving 13 dB to be absorbed. The absorption losses are

$$A = 8.7t/\delta = 8.7t/9.35 = 13 \Rightarrow t = 13 \times 9.35/8.7 = 14 \text{ mm}$$

With a steel shield we still have reflection losses of zero and all the losses have to come from absorption. The absorption losses are

$$A = 8.7t/\delta = 8.7t/1.45 = 30.6 \Rightarrow t = 30.6 \times 1.45/8.7 = 5.1 \text{ mm}$$

# Chapter 28

1 The voltage 6.31 ± 0.3% can be written 6.31 ± 0.02 V, so that adding it to 3.63 ± 0.03 V produces 9.94 ± 0.05 V, making the percentage error 0.05 × 100/9.94 = 0.5%. This is a conservative way of estimating the error.

We could combine the errors using equation 28.6, giving the most likely error, in which case we have

$$V = V_1 + V_2 \Rightarrow f_1 = \partial V/\partial V_1 = 1 = \partial V/\partial V_2 = f_2$$

Thus

$$s_V = \sqrt{s_{V1}^2 + s_{V2}^2} = \sqrt{0.03^2 + 0.02^2} = 0.036 \text{ V}$$

The percentage error in the sum is then 0.36%.

Subtraction yields 6.31 − 3.63 = 2.68 ± 0.05 V, or an error in percentage terms of

$$0.05 \times 100/2.68 = 1.9\%$$

Taking the most probable error as 0.036 V, the percentage error in the difference is

$$0.036 \times 100/2.68 = 1.34\%$$

2 The sum of the readings is 2390.5 μV and there are 12 readings, so the mean is 2390.5/12 = 199.2 μV. The mean is the best estimate of the true value.

The standard deviation, $\sigma_n$, is

$$\sqrt{\frac{\sum_{i=1}^{i=12} (x_i - \bar{x})^2}{n}} = \sqrt{\frac{(198.4 - 199.2)^2 + (198.9 - 199.2)^2 + \ldots}{12}} = 0.51 \ \mu\text{V}$$

The standard error of the mean is

$$s = \frac{\sigma_n}{\sqrt{n-1}} = \frac{0.51}{\sqrt{11}} = 0.15 \ \mu\text{V}$$

To reduce the standard error of the mean to 0.1 μV with a standard deviation of 0.51 μV requires

$$s = \frac{\sigma_n}{\sqrt{n-1}} = \frac{0.51}{\sqrt{n-1}} = 0.1 \Rightarrow n - 1 = (0.51/0.1)^2 = 26$$

Thus a further 14 readings are needed.

A reading of 202.1 μV is 2.9 μV from the mean, or nearly 6 standard deviations. This is a *very* improbable event, but whether it is included or not depends on the apparatus used, the way in which the reading was obtained and, above all, on the judgement of the observer. Note that including it in the analysis produces a mean of 199.4 μV and a standard deviation of 0.91 μV, while the standard error goes up to 0.26 μV. The *ideal* solution is to take more readings - though a great many more would be required to reduce $s$ to 0.15 μV - but that may not be possible. The *safest* course is to include it, especially if no other information is available beyond the readings themselves.

3 $V_x$ can be expressed as

$$V_x = VR_1/V_T$$

This expression is of the form

$$F(a,b,c) = a^u b^v c^w$$

with $a = V$, $b = R_1$ and $c = R_T$, and $u = 1$, $v = 1$ and $w = -1$. Thus from equation 28.8 we find

$$s_{Vx} = \sqrt{s_V^2 + s_{R1}^2 + s_{RT}^2} = \sqrt{0.2^2 + 0.1^2 + 0.1^2} = 0.245\%$$

taking the most probable error in the denominator as 0.1% since each resistance is in error by 0.1%. This is the most probable error.

Unless the magnitudes of the resistances are known we cannot use equation 28.6 to find the error in the denominator. If in the worst case all three are the same then the error in $R_T$ is $\sqrt{}0.03 = 0.173\%$, and the overall error becomes

$$s_{Vx} = \sqrt{0.2^2 + 0.1^2 + 0.173^2} = 0.283\%$$

which is a more conservative estimate. Adding the percentage errors is the most conservative way of all - therefore the safest - and gives an error estimate of 0.4%.

4 In this problem the magnitudes of the resistances are known and we can use equation 28.6 to find

$$f_V = \partial V_x/\partial V = R_1/R_T = 5.562/100 = 0.05562$$

with $R_T = R_2 + R_3 + R_4 = 100$ kΩ. The other partial derivatives are

$$f_{R1} = V/R_T = 3.573/10^5 = 3.573 \times 10^{-5} \text{ A}$$

$$f_{R2} = f_{R3} = f_{R4} = -VR_1/R_T^2 = -3.573 \times 5.562 \times 10^3/10^{10} = -1.99 \times 10^{-6} \text{ A}$$

Thus the overall error is

$$s_{\text{Vx}} = \sqrt{(f_{\text{V}}s_{\text{V}})^2 + (f_{\text{R1}}s_{\text{R1}})^2 + 3(f_{\text{R2}}s_{\text{R2}})^2}$$

$$= \sqrt{(0.05562 \times 0.005)^2 + (3.573 \times 10^{-5} \times 10)^2 + 3(1.99 \times 10^{-6} \times 10)^2} = 0.454 \text{ m}V$$

And therefore $V_x = 198.73 \pm 0.45$ mV.

5 Rearranging equation P28.5 to give $T$ leads to

$$T = \frac{R_T/R_0 - 1}{\alpha}$$

from which the partial derivatives are

$$f_{\text{RT}} = \partial T/\partial R_{\text{T}} = 1/R_0\alpha = (100 \times 0.00388)^{-1} = 2.5773 \text{ K}/\Omega$$

$$f_{\text{R0}} = \partial T/\partial R_0 = -R_{\text{T}}/\alpha R_0^2 = -115.33/0.0388 \times 10^4 = -2.9724 \text{ K}/\Omega$$

$$f_\alpha = \partial T/\partial \alpha = \frac{-(R_{\text{T}}/R_0 - 1)}{\alpha^2} = \frac{-(1.1533 - 1)}{0.00388^2} = -10183 \text{ K}^2$$

Then the contributions to the error in $T$ are $s_{\text{RT}}f_{\text{RT}} = 0.05 \times 2.5773 = 0.128865$ K, from $R_{\text{T}}$; $s_{\text{R0}}f_{\text{R0}} = 0.02 \times 2.9724 = 0.0594$ K from $R_0$; and $s_\alpha f_\alpha = 10^{-5} \times 10183 = 0.10183$ K from $\alpha$ (ignoring minus signs in the $f$-values). Thus we see that the largest source of error is that in $R_{\text{T}}$. From these we find the overall error

$$s_{\text{T}} = \sqrt{(s_{\text{RT}}f_{\text{RT}})^2 + (s_{\text{R0}}f_{\text{R0}})^2 + (s_\alpha f_\alpha)^2}$$

$$= \sqrt{0.128865^2 + 0.0594^2 + 0.10183^2} = 0.1747 \text{ K}$$

The temperature is

$$T = \frac{R_{\text{T}}/R_0 - 1}{\alpha} = \frac{0.1533}{0.00388} = 39.51 \pm 0.17°\text{C}$$

6 Again we rearrange equation P28.6 to give the temperature

$$T = \frac{B}{\ln(R_{\text{T}}/R_\infty)} = \frac{3333}{\ln(5238/0.5652)} = 291.43 \text{ K}$$

Then the partial derivative of $B$ is given by

$$f_{\text{B}} = \partial T/\partial B = \frac{1}{\ln(R_{\text{T}}/R_\infty)} = \frac{1}{\ln(5238/0.05652)} = 0.087$$

The error in $B$, $s_{\text{B}} = 0.1$ K, so that $s_{\text{B}}f_{\text{B}} = 0.008744$ K.

The partial derivative for $R_{\text{T}}$ is given by

$$f_{\mathrm{RT}} = \frac{-B/R_{\mathrm{T}}}{\left[\ln(R_{\mathrm{T}}/R_{\infty})\right]^2} = \frac{-3333/5238}{11.437^2} = -0.0049\ \mathrm{K}/\Omega$$

And as $s_{\mathrm{RT}} = 1\ \Omega$, $s_{\mathrm{RT}}f_{\mathrm{RT}} = -0.0049$ K.

Finally the partial derivative for $R_\infty$ is

$$f_{R\infty} = \frac{B/R_{\infty}}{\left[\ln(R_{\mathrm{T}}/R_{\infty})\right]^2} = \frac{3333/0.05652}{11.44^2} = 450\ \mathrm{K}/\Omega$$

And as $s_{\mathrm{R}\infty} = 10^{-5}\ \Omega$, $s_{\mathrm{R}\infty}f_{\mathrm{R}\infty} = 0.0045$ K.

The overall error in $T$ is

$$s_{\mathrm{T}} = \sqrt{(s_{\mathrm{B}}f_{\mathrm{B}})^2 + (s_{\mathrm{RT}}f_{\mathrm{RT}})^2 + (s_{\mathrm{R}\infty}f_{\mathrm{R}\infty})^2}$$

$$= \sqrt{0.0087^2 + 0.0049^2 + 0.0045^2} = 0.01\ \mathrm{K}$$

If the errors in $R_\infty$ and $B$ are negligible, only the error in $R_{\mathrm{T}}$ need be considered. At 0°C or 273.15 K, $R_{\mathrm{T}}$ is

$$R_{\mathrm{T}} = R_{\infty}\exp(B/T) = 0.05652\exp(3333/273.15) = 11.26\ \mathrm{k}\Omega$$

The partial derivative at 0°C is

$$f_{\mathrm{RT}} = \frac{\partial T}{\partial R_{\mathrm{T}}} = \frac{-B/R_{\mathrm{T}}}{\left[\ln(R_{\mathrm{T}}/R_{\infty})\right]^2} = \frac{-3333/11259}{\left[\ln(11259/0.05652)\right]^2} = -0.002\ \mathrm{K}/\Omega$$

Thus as the error in $T$ is $s_{\mathrm{RT}}f_{\mathrm{RT}} = 0.001$ K we require $s_{\mathrm{RT}} = 0.001/f_{\mathrm{RT}} = 0.001/0.002 = 0.5\ \Omega$, which is

$$s_{\mathrm{T}} = \frac{0.5 \times 100}{11259} = 0.0044\%$$

7 The deflection of a moving-coil meter is proportional to the current, $I_{\mathrm{M}}$, which is given by equation 28.18, that is

$$I_{\mathrm{M}} = \frac{[(1 + r)R_{\mathrm{s}} + R_{\mathrm{M}}]I_{\mathrm{F}}}{(1 + r)(R_{\mathrm{x}} + R_{\mathrm{s}}) + R_{\mathrm{M}}} = \frac{AI_{\mathrm{F}}}{(1 + r)R_{\mathrm{x}} + A}$$

where $A \equiv [(1 + r)R_{\mathrm{s}} + R_{\mathrm{M}}]$ is a meter constant, $r$ is also constant and $I_{\mathrm{F}}$ is the current giving full-scale deflection. Rearranging this equation to give $R_{\mathrm{x}}$ we find

$$[(1 + r)R_{\mathrm{x}} + A]I_{\mathrm{M}} = AI_{\mathrm{F}} \Rightarrow R_{\mathrm{x}} = \frac{AI_{\mathrm{F}}/I_{\mathrm{M}} - A}{1 + r} = B(I_{\mathrm{F}}/I_{\mathrm{M}} - 1)$$

where $B \equiv A/(1 + r)$, another constant. Taking natural logs and then differentiating the expression for $R_{\mathrm{x}}$ with respect to $I_{\mathrm{M}}$ leads to

$$\ln R_{\mathrm{x}} = \ln(I_{\mathrm{F}}/I_{\mathrm{M}} - 1) \Rightarrow \frac{1}{R_{\mathrm{x}}}\frac{\mathrm{d}R_{\mathrm{x}}}{\mathrm{d}I_{\mathrm{M}}} = \frac{1}{(I_{\mathrm{F}}/I_{\mathrm{M}} - 1)}\frac{-I_{\mathrm{F}}}{I_{\mathrm{M}}^2}$$

Replacing the differentials by $\delta I_{\mathrm{M}}$ etc. leads to

$$\frac{\delta R_{\mathrm{x}}}{R_{\mathrm{x}}} = \frac{-I_{\mathrm{F}}\delta I_{\mathrm{M}}}{(I_{\mathrm{F}}/I_{\mathrm{M}} - 1)I_{\mathrm{M}}^2} = \frac{I_{\mathrm{F}}^2\delta I_{\mathrm{M}}/I_{\mathrm{F}}}{(1 - I_{\mathrm{F}}/I_{\mathrm{M}})I_{\mathrm{M}}^2} = \frac{\delta I_{\mathrm{M}}/I_{\mathrm{F}}}{(1 - I_{\mathrm{F}}/I_{\mathrm{M}})(I_{\mathrm{M}}/I_{\mathrm{F}})^2}$$

But $\delta I_{\mathrm{M}}/I_{\mathrm{F}}$ (= $s$) is the error in $I_{\mathrm{M}}$ as a fraction of full-scale deflection, $I_{\mathrm{F}}$, while $I_{\mathrm{M}}/I_{\mathrm{F}} = \alpha$, the needle deflection as a fraction of full-scale deflection. Thus

$$\frac{\delta R_{\mathrm{x}}}{R_{\mathrm{x}}} = \frac{\delta I_{\mathrm{M}}/I_{\mathrm{F}}}{(1 - I_{\mathrm{F}}/I_{\mathrm{M}})(I_{\mathrm{M}}/I_{\mathrm{F}})^2} = \frac{s}{(1 - 1/\alpha)\alpha^2} = \frac{-s}{\alpha(1 - \alpha)}$$

The minus sign merely indicates that the deflection goes down when $R_{\mathrm{x}}$ goes up and can be ignored.

By differentiating this expression for $\delta R_{\mathrm{x}}/R_{\mathrm{x}}$ (= $\epsilon$) and setting it equal to zero, we can find when the error is least. Thus

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha} = \frac{s(1 - 2\alpha)}{\alpha^2(1 - \alpha)^2} = 0$$

And we see that $\alpha = 0.5$: half the full-scale deflection will give the least error.

At 20% f.s. deflection, $\alpha = 0.2$ and substituting this value of $\alpha$ and $s = 1\%$ into the expression for $\delta R_{\mathrm{x}}/R_{\mathrm{x}}$ leads to $\delta R_{\mathrm{x}}/R_{\mathrm{x}} = 1/(0.2 \times 0.8) = 6.25\%$. When $\alpha = 0.5$ we find the error in $R_{\mathrm{x}}$ is $1/(0.5 \times 0.5) = 4\%$. And $\alpha = 0.8$ gives the same error as $\alpha = 0.2$, obviously.

**8** The crest factor is the ratio of the peak to the r.m.s. value of a waveform.

(a) The r.m.s. value of a 'square' wave alternating between $+V_{\mathrm{m}}$ and $-V_{\mathrm{m}}$ is $V_{\mathrm{m}}$ and the peak value is also $V_{\mathrm{m}}$, so that the crest factor is 1. The square wave can have any value of $\alpha$, as shown in figure A28.8a, since the r.m.s. value is

$$V = \sqrt{\frac{\int_0^{\alpha T} V_{\mathrm{m}}^2\,\mathrm{d}t + \int_{\alpha T}^{T}(-V_{\mathrm{m}})^2\,\mathrm{d}t}{T}} = \sqrt{\frac{V_{\mathrm{m}}^2(\alpha T + T - \alpha T)}{T}} = V_{\mathrm{m}}$$

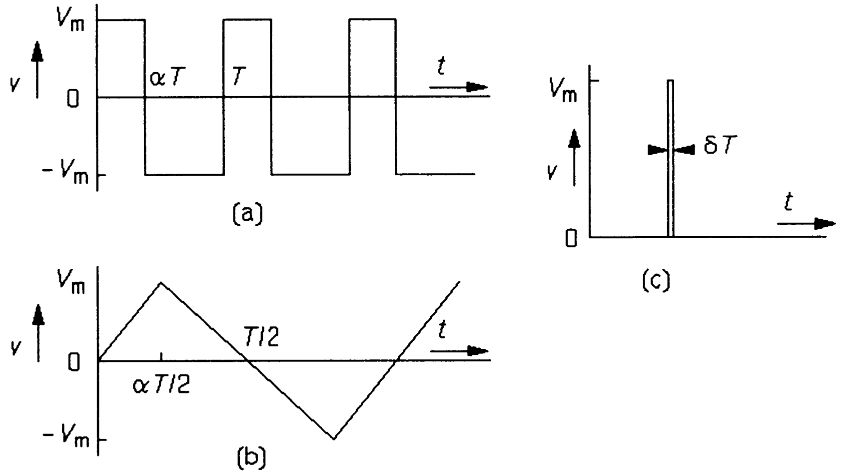


*Figure A28.8*

which is a long-winded way of stating the 'obvious'.

(b) A 'triangular' waveform, such as that shown in figure A20.8b, has an r.m.s. value of $V_m/\surd 3$ (see the solution to problem 2.1) and a peak value of $V_m$, and thus a crest factor of $\surd 3$.

(c) Any sinewave has an r.m.s. value of $V_m/\surd 2$ (see chapter 2, p 41) and therefore a crest factor of $\surd 2$.

(d) A voltage spike as shown in figure A28.8c has an r.m.s. value of

$$V = \sqrt{\frac{\int_T^{T+\delta T} V_m^2 \mathrm{d}t}{T}} = V_m \sqrt{\frac{\delta T}{T}}$$

The crest factor is then

$$CF = \frac{V_m}{V_m\sqrt{\delta T/T}} = \sqrt{\frac{T}{\delta T}} \rightarrow \infty \quad \text{as } \delta T \rightarrow 0$$

**9** A 5½-digit voltmeter will read up to 1.99999 V, with a variable scale factor, so that the accuracy is 0.00001 in 1.99999 or 1 in 2 × $10^5$. The number of bits required is

$$b \geq \frac{\log(2 \times 10^5)}{\log 2} = 17.6$$

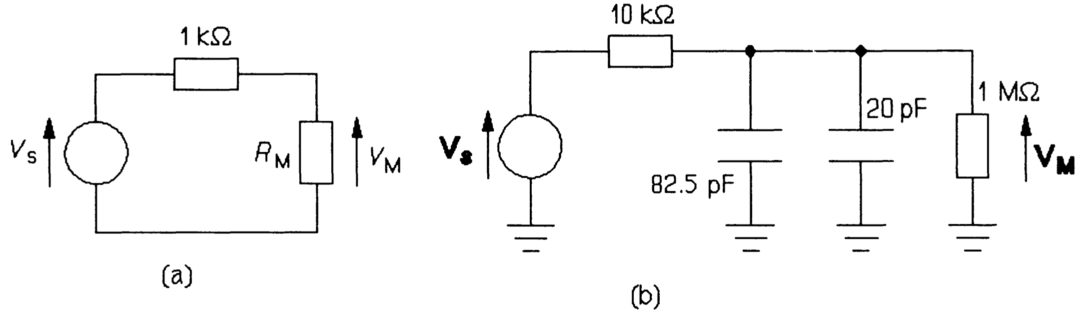
which means that at least 18 bits are required in the A/D converter.

A 16-bit A/D has an accuracy at best of 1 in $2^{16}$ or 1 in 65536, which is 0.0015%.

**10** The circuit is shown in figure A28.10a, from which we see that

$$\frac{V_M}{V_s} = \frac{R_M}{R_s + R_M} = \frac{R_M}{1 + R_M} = 0.998$$

$$\Rightarrow \quad 0.002R_M = 0.998\ \text{k}\Omega \quad \Rightarrow \quad R_M = 499\ \text{k}\Omega$$



(a)

(b)

*Figure A28.10*

The circuit for the AC source and oscilloscope is shown in figure 28.10b, in which the 82.5 pF is the cable capacitance (0.75 × 100 pF) and this is in parallel with the CRO's input impedance of 1 MΩ || 20 pF. The total capacitance then is 102.5 pF in parallel with 1 MΩ, which gives an impedance of

$$\mathbf{Z}_M = \frac{R_M / j\omega C_M}{R_M + 1/j\omega C_M} = \frac{R_M}{1 + j\omega C_M R_M}$$

where $C_M = 102.5$ pF and $R_M = 1$ MΩ. The voltage read by the CRO is the voltage across this combination in series with the source resistance, $R_s$ (= 10 kΩ), thus

$$\frac{\mathbf{V}_M}{\mathbf{V}_s} = \frac{\mathbf{Z}_M}{\mathbf{Z}_M + R_s} = \frac{R_M}{R_M + R_s(1 + j\omega C_M R_M)}$$

$$= \frac{1000}{1000 + 10(1 + j\omega 102.5 \times 10^{-6})} = \frac{100}{101 + j\omega 102.5 \times 10^{-6}}$$

Now if the CRO's reading is down by 0.5 dB from the source voltage, this means

$$|\mathbf{V}_M/\mathbf{V}_s| = 10^{-0.5/20} = 0.944 = \frac{100}{\sqrt{101^2 + (\omega 102.5 \times 10^{-6})^2}}$$

$$\Rightarrow \quad 101^2 + (\omega 102.5 \times 10^{-6})^2 = (100/0.944)^2 \quad \Rightarrow \quad \omega = 312\ \text{krad/s}$$

whence $f = 49.6$ kHz.