BASIC MATRIX ANALYSIS AND SYNTHESIS

With Applications to Electronic Engineering

BY

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PREFACE

MATRICES and determinants are well-known mathematical tools for handling simultaneous equations.*(1, 2) The impressive potentialities of matrix theory have been quite early appreciated for network analysis and synthesis.(3, 4, 5) The rapid evolution of this branch of applied mathematics has produced a number of outstanding texts which deal with a broad range of engineering problems in terms of matrices(6, 7, 8, 9). This book has the somewhat modest aim of a basic, yet rigorous review of a narrower field. It is concerned with the application of matrix methods to practical electronics problems. The organization of the text is slanted to the needs of practising electronics engineers as well as students with some grounding in mathematics and network theory.

Transistors and other solid-state devices have slowly but surely displaced electronic tubes from many areas in communication and control engineering. Therefore, throughout this book the author has put the main emphasis on transistor applications, though semiconductor physics is not touched upon. In all cases the transistors are reduced to a simplified but adequate twoport linear model. The subsequent analysis and synthesis of more elaborate electronic systems, such as feedback amplifiers and oscillators, are studied in terms of these mathematical models. The author hopes to show that numerous and sometimes complex network problems yield to the basic matrix methods. Much care has been devoted to the selection of rewarding "live" topics, right out of the design engineers' notebook. These are all treated in con-

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* Superscript figures are to References on p.65.

siderable depth. Because of the fundamental character of the subject matter and the generalized form of presentation, it is believed that the book is reasonably free of obsolesence. Readers who are familiar with the author's previous book *Basic Matrix Algebra and Transistor Circuits*(10) will find this present volume as a logical extension and complement to the first one.

The material has been divided into three parts and each can be treated as a set of self-contained monographs. Part I covers the basic matrix theory of twoports and will be quite frequently referred to in the later parts. The mechanism of matrix and determinant operations is shown and explained in detail with applications to the study of twoport networks, both passive and active. The transfer characteristics and terminal impedances of these are derived in terms of the various matrix domains. Next, the elements of matrix synthesis are introduced and it is shown how some elaborate systems can be synthesized from the simple mathematical models. In conclusion, conversion of matrices is reviewed and it is demonstrated how these techniques may most efficiently be exploited.

Part II is devoted to the concept of impedance transformation and image matching in the different matrix domains. The practical application of this part is underscored by the generalized solution of a typical RF amplifier to antenna matching problem.

Part III is concerned with the analysis and synthesis of active networks. The mathematical model concepts of transistors and vacuum tubes are freely applied to a broad range of problems with an emphasis on practical applications such as conventional amplifiers, single- and multi-stage transistor feedback amplifiers and oscillators. In order to gain a broader appreciation of the analytical process involved, a sort of "mathematical slow-motion camera" has been focused on essential algebraic manipulations. Some attention has also been devoted to the techniques of engineering approximations. A whole set of approximate formulas have been derived in this process and shown to be applicable to a large group of routine design problems. The important mathematical processing of matrix conversions have been carefully woven into the texture of this part of the book.

The student will find that, as a general rule, a step by step mathematical reasoning leads to the final definitions or equations. Furthermore, the essential summaries are collated in a number of tables and these will be found also in the body of the text.

In conclusion, there is a liberal selection of keyed problem sets in the Appendix.

Historically, this book has evolved from direct research and development work involving the author. A substantial portion of the text has been written during his tenure as a group leader with the Canadian Marconi Company in Montreal. Therefore the author is particularly indebted to K. C. M. Glegg, Chief Engineer, Commercial Products Division, for permission to publish material of latest vintage on transistor feedback amplifiers, originally prepared as a Canadian Marconi Technical Report. It is a tribute to the generosity of the De Havilland Aircraft of Canada Limited that the manuscript of this book could be completed. The writer is most grateful to Dr. P. A. Lapp, Chief Engineer, and A. C. Stonell, Chief System Analyst, Special Products and Applied Research Division. Finally, a word of appreciation is due to the publishers of *Electronic Engineering* and *J. Brit. IRE* (now *IERE*) for granting permission to use material from the author's recent papers in these publications.

Toronto G. ZELINGER

1. REVIEW OF THE BASIC MATHEMATICAL OPERATIONS WITH MATRICES AND DETERMINANTS

(a) MATRICES AND DETERMINANTS; FUNDAMENTAL SIMILARITIES AND DIFFERENCES

Determinants and matrices may be classified as types of mathematical shorthand. One may also broadly state that matrices have been created for the purpose of manipulating a system of simultaneous equations, while determinants are used to solve for any particular unknown. Both determinant and matrix methods make use of detached coefficients. The determinant is a function with a definite algebraic or numerical value which is not the case with the matrix. In the later parts of this section the particular and distinct characteristics of matrices will be studied in greater detail. However, some familiarity with elementary algebra of determinants is assumed. For those who wish a review, a selected list of references is included.*

The fundamental similarities and differences of determinants and matrices may be conveniently demonstrated by considering a set of simultaneous equations:

$$y_1 = a_{11}x_1 + a_{12}x_2, \tag{1.1}$$

$$y_2 = a_{21}x_1 + a_{22}x_2, \tag{1.2}$$

where the x and y terms represent variables and the a terms represent constants. It will be recalled that in the algebra of determinants the detached coefficients are written within

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* See page 65.

two parallel bars:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv \varDelta_a \equiv \text{determinant.} \tag{1.3}$$

By definition, the determinant has a definite value which is obtained by cross-multiplication and subtraction of the diagonal terms:

$$\begin{vmatrix} \searrow & & \swarrow \\ a_{11} & & a_{12} \\ & \times & \\ a_{21} & & a_{22} \\ \swarrow & & \searrow \end{vmatrix} \equiv (a_{11}a_{22} - a_{12}a_{21}) \equiv \varDelta_a . \tag{1.4}$$

When applying the elementary rules of the algebra of determinants, a numerical or algebraic solution of the simultaneous eqns. (1.1) and (1.2) is possible.

Solving for x_1,

$$x_1 = \frac{\begin{vmatrix} \searrow & & \swarrow \\ y_1 & & a_{12} \\ & \times & \\ y_2 & & a_{22} \\ \swarrow & & \searrow \end{vmatrix}}{\varDelta_a} = \frac{(y_1a_{22} - y_2a_{12})}{\varDelta_a} . \tag{1.5}$$

Similarly, solving for x_2,

$$x_2 = \frac{\begin{vmatrix} \searrow & & \swarrow \\ a_{11} & & y_1 \\ & \times & \\ a_{21} & & y_2 \\ \swarrow & & \searrow \end{vmatrix}}{\varDelta_a} = \frac{(y_2a_{11} - y_1a_{21})}{\varDelta_a} . \tag{1.6}$$

It is apparent from eqns. (1.5) and (1.6) that provided $\varDelta_a \neq 0$, the determinant has a definite value which is a function of its constituent elements.

Reverting to the system of eqns. (1.1) and (1.2), if the detached a coefficients are enclosed within square brackets, they will signify the matrix A:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A \equiv \text{matrix}. \tag{1.7}$$

Similarly, the detached variables x and y are also written in matrix notation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \equiv Y \tag{1.8a}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \equiv X. \tag{1.8b}$$

By the use of the notations of eqns. (1.7) to (1.8b) inclusive, eqns. (1.1) and (1.2) may be rewritten in matrix from:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X}. \tag{1.9}$$

Equation (1.9) may be put in still more compact form:

$$[Y] = [A][X]. \tag{1.10}$$

The matrices in eqns. (1.9) and (1.10) describe a system where the variables x and y are related by the a terms. Note, however, the distinct characteristic of matrices that they describe a system completely and no "value" need be attached to them.

Matrices in general obey the ordinary rules of algebra, but an important exception is the operation of multiplication. In this section those basic matrix operations will be introduced which are of importance in network problems within the scope of this text.

(b) ADDITION AND SUBTRACTION

Except for algebraic sign the addition and subtraction of matrices are essentially similar forms of operation. The sum of two matrices is found by adding their corresponding elements. As an illustration, consider the matrices $[A]$ and $[B]$:

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$[B] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Now, by definition the sum of $[A]+[B]$:

$$[A]+[B] = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) \end{bmatrix}. \qquad (1.11)$$

Next consider the subtraction of the matrices $[C]$ and $[D]$, where

$$[C] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$[D] = \begin{bmatrix} d_{11} - d_{12} \\ d_{21} - d_{22} \end{bmatrix}.$$

Here, by definition the corresponding elements of $[D]$ will be subtracted from $[C]$:

$$[C]-[D] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} - \begin{bmatrix} d_{11} - d_{12} \\ d_{21} - d_{22} \end{bmatrix} = \begin{bmatrix} (c_{11}-d_{11}) & (c_{12}+d_{12}) \\ (c_{21}-d_{21}) & (c_{22}+d_{22}) \end{bmatrix}. \qquad (1.12)$$

The above results are completely general and the matrix elements may represent real or complex quantities.

(c) MULTIPLICATION

The technique of matrix multiplication is perhaps the most important tool for systematic study and solution of circuit problems. This will be extensively demonstrated in the later section of this book.

It is fortunate that both passive and active network problems may be efficiently handled with the aid of the twoport network theory. In this domain, circuit characteristics may be completely described with a 2 × 2 parameter matrix. That is a matrix with only two rows and two columns. It will be shown that relatively simple operations will be required to solve the majority of practical network problems. Matrix multiplication will be frequently used. At first reading it may appear confusing, but matrix multiplication is not commutative. It will be shown that $[A][B] \neq [B][A]$. In general, the multiplication of two matrices requires multiplying the row elements of the first matrix with the corresponding column elements of the second matrix and summing up the products. The sum of such products forms an element in the resulting product matrix. It is then situated at the intersection of the corresponding row and column. As an illustration let us take the matrix product of $[A][B] = [C]$. Dealing with 2×2 matrices, by definition

$$[A][B] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}. \quad (1.13)$$

Note that in this example the matrix $[A]$ has as many columns as the number of rows in the matrix $[B]$. In fact this condition is prerequisite for matrix multiplication.

Let us start by multiplying the first row of the A matrix with the first column of the B matrix. The sum of the products must be equal to C_{11} as in eqn. (1.13).

Now perform the actual multiplications and we find that the complete C matrix is made up from the elements as follows:

$$c_{11} = (a_{11}b_{11} + a_{12}b_{21}), \quad (1.14)$$

$$c_{12} = (a_{11}b_{12} + a_{12}b_{22}), \quad (1.15)$$

$$c_{21} = (a_{21}b_{11} + a_{22}b_{21}), \quad (1.16)$$

$$c_{22} = (a_{21}b_{12} + a_{22}b_{22}). \quad (1.17)$$

Thus, from eqns. (1.13) to (1.17) inclusive,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \overbrace{(a_{11}b_{11} + a_{12}b_{21})}^{c_{21}} & \overbrace{(a_{11}b_{12} + a_{12}b_{22})}^{c_{22}} \\ \underbrace{(a_{21}b_{11} + a_{22}b_{21})}_{c_{21}} & \underbrace{(a_{21}b_{12} + a_{22}b_{22})}_{c_{22}} \end{bmatrix}. \quad (1.18)$$

Consider now the situation which is often encountered when, in the product matrix $[A][B]$, the terms b_{12} and b_{22} are absent; then from eqn. (1.13)

$$[A][B] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}. \tag{1.19}$$

The product matrix in respect of the right-hand part may be written down by inspection of eqn. (1.18). Note, however, that all terms which contain the b_{12} or b_{22} terms are now zero. Thus, from eqns. (1.18) and (1.19),

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} \overbrace{(a_{11}b_{11} + a_{12}b_{21})}^{c_{11}} \\ \underbrace{(a_{21}b_{11} + a_{22}b_{21})}_{c_{21}} \end{bmatrix}. \tag{1.20}$$

At the outset of this paragraph it has been stated that in general $[B][A] \neq [A][B]$, that is matrix multiplication is not a commutative process. This will be easily proved by reversing the left-hand terms in eqn. (1.18):

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (b_{11}a_{11} + b_{12}a_{21}) & (b_{11}a_{12} + b_{12}a_{22}) \\ (b_{21}a_{11} + b_{22}a_{21}) & (b_{21}a_{12} + b_{22}a_{22}) \end{bmatrix}. \tag{1.21}$$

By definition, two matrices are equal if each of their corresponding elements are equal. When comparing the corresponding elements on the right-hand side in eqns. (1.18) and (1.21), it will be apparent that they are not equal.

Though matrix multiplication is not commutative, the associative law holds. It is permitted to split a multiple product in any desired manner. Consider the triple matrix products

$$ABC = [AB]C \tag{1.22}$$

or

$$ABC = A[BC]. \tag{1.23}$$

Note that in eqn. (1.22) the product AB is "postmultiplied" by C and in eqn. (1.23) the product BC is "premultiplied" by A.

Multiplication by a constant. The multiplication of a matrix $[A]$ by a scalar or constant quantity K is defined as follows:

$$K[A] = K\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} Ka_{11} & Ka_{12} \\ Ka_{21} & Ka_{22} \end{bmatrix}. \tag{1.24}$$

Accordingly, a matrix is multiplied by a constant if each element of the matrix is multiplied by that constant. Note here the important difference between matrices and determinants. The determinant is multiplied by a constant if the elements of one row or one column are multiplied by that constant.

(d) INVERSION

Consider a set of equations:

$$[A][B] = [C]. \tag{1.25}$$

When eqn. (1.25) is written out in expanded form, the matrices are identified as follows:

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{1.26}$$

$$[B] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \tag{1.27}$$

$$[C] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}. \tag{1.28}$$

If it is required to solve eqn. (1.25) for $[B]$, then one would be inclined to divide it through by $[A]$, that is $[B] = [C]/[A]$, which is entirely wrong.

Note that the operation of division in matrix algebra just does not exist. Equation (1.25) may be solved for $[B]$ by "premultiplication" of both sides by the "inverse" of the matrix $[A]$:

$$[A]^{-1}[A][B] = [A]^{-1}[C]. \tag{1.29}$$

By definition

$$[A]^{-1}[A] = 1. \tag{1.30}$$

Thus, from eqn. (1.29),

$$[B] = [A]^{-1}[C]. \tag{1.31}$$

For further processing, eqn. (1.31) is written out in its expanded form:

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}^{-1} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}. \tag{1.32}$$

At this stage for a complete solution of eqn. (1.32) an interpretation of the inverse matrix $[A]^{-1}$ would be required. However, for the time being the reader is asked to accept in good faith a definition of the inverse matrix. Later, in Chapter 3, the matrix inversion techniques of twoport mathematical models will be studied in greater depth.

The inverse matrix of $[A]^{-1}$ is defined by the following identities:

$$[A]^{-1} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}^{-1} \equiv \begin{bmatrix} \dfrac{A_{11}}{\Delta} & \dfrac{A_{21}}{\Delta} \\ \dfrac{A_{12}}{\Delta} & \dfrac{A_{22}}{\Delta} \end{bmatrix} \equiv \frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \tag{1.33}$$

where

$$\Delta = (a_{11}a_{22} - a_{21}a_{12}), \tag{1.34}$$

$$\Delta = \text{determinant of matrix,}$$

and

A_{11}, A_{12}, A_{21} and A_{22} are "cofactors".

By definition, the cofactor, or signed "minor" of the determinant is the element, or group of elements which remain after the indicated row and column are deleted. Therefore the cofactors of eqn. (1.33) may be identified as follows:

$$A_{11} = a_{22} \qquad A_{21} = -a_{12}$$
$$A_{12} = -a_{21} \qquad A_{22} = a_{11}$$

Now the procedure of the inversion of the matrix may be summarized:

First: Replace each element by its cofactor.

Second: Interchange rows and columns formed from cofactors.

Third: Divide each element by the determinant of the matrix.

2. THE TERMINATED TWOPORT AND MATHEMATICAL MODELS

THE importance of correct mathematical formulation of electronic circuit problems is well appreciated. It is recognized as the cornerstone of every successful design effort. If one considers a moderately complex electronic system such as an impedance matching network, a filter, a feedback amplifier or a transistor oscillator, one may end up with a formidable array of simultaneous equations if applying the classical methods of mathematical analysis. A familiarity with matrix algebra will reduce the mathematical drudgery substantially. This is because we are generally interested in finding the voltage or current transfer ratios between some designated pair of input and output ports.

As our insight into the intricacies of matrix manipulations broadens, we will learn the techniques whereby complex electronic systems or circuitry can be broken down quite easily into elementary twoport structures. Because of this fundamental concept, the compact and rigorous matrix algebra is admirably suited for handling the mathematical work. Therefore the reader will find the study of this chapter highly rewarding. The most important feature of the twoport approach is, that the complete formulation of network problems can be reduced to the mathematical model of a 2×2 matrix. It will be shown that there are only a few systematic ways of interconnecting the elementary twoports. From the possible six, only five are of particular interest and these will be studied in depth.

If our main concern is restricted to the input and output terminal pairs, we can legitimately represent an arbitrary twoport network, passive or active, as a "black box". We may also assign to such a simple linear model a set of generalized internal matrix parameters. Finally, the system concept is satisfied by adding generator and load terminations. If we apply basic principles of matrix algebra to this simple linear model, we can easily derive the transfer characteristics, input and output impedances in terms of the chosen parameters matrices. By means of a rigorous step by step algebraic process it will be shown how to obtain the desired transfer characteristics in each of the *Z, Y, h, g* and *ABCD* parameters. The results will be tabulated in a handy form which lends itself to convenient reference.

(a) THE TERMINATED TWOPORT NETWORK AND THE *Z* MATRIX

We may look upon the most general form of a twoport network as a "black box" and define this simple linear model mathematically in terms of the *Z* parameter matrix. It can be stipulated further that this twoport model may contain active or passive elements. Figure 2.1 shows such a generalized linear model with the appropriate sign convention. The equilibrium equations for this twoport may be written in matrix form thus:

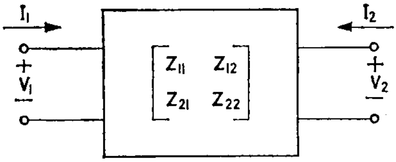


FIG. 2.1. Generalized twoport—unterminated.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{2.1}$$

If now generator and load are added, then our twoport model can be designated either as an amplifier, or a filter or an impedance transformer, depending on the nature of internal elements. A corresponding linear model will be of the form shown in the block diagram Fig. 2.2.

This block diagram will be utilized for the purpose of deriving generalized and meaningful operating characteristics of our system.

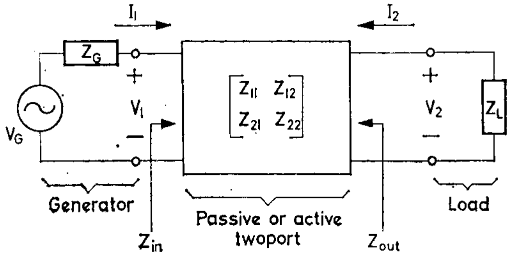


FIG. 2.2. The terminated twoport.

(i) Current transfer ratio

On expanding the matrix (2.1) we have a pair of simultaneous equations:

$$V_1 = Z_{11}I_1 + Z_{12}I_2, \tag{2.2}$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2, \tag{2.3}$$

where the open-circuit Z parameters are defined as follows. From eqn. (2.2)

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}; \quad \text{Input impedance.}$$

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}; \quad \text{Reverse transfer impedance.}$$

Similarly, from eqn. (2.3)

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}; \quad \text{Forward transfer impedance.}$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}; \quad \text{Output impedance.}$$

We are interested in obtaining an expression for the current I_2 through the load Z_L in relation to the input current I_1. From the block diagram in Fig. 2.2, we have the equality:

$$V_2 = -I_2 Z_L. \tag{2.4}$$

If we substitute for V_2 into eqn. (2.3), rearrange terms, then we obtain from eqns. (2.2) and (2.4) a new system matrix:

$$\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & (Z_{22}+Z_L) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{2.5}$$

Solving eqn. (2.5) for I_1 with determinants:

$$I_1 = \frac{\begin{vmatrix} V_1 & Z_{12} \\ 0 & (Z_{22}+Z_L) \end{vmatrix}}{\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & (Z_{22}+Z_L) \end{vmatrix}} = \frac{V_1(Z_{22}+Z_L)}{\underbrace{Z_{11}Z_{22}-Z_{12}Z_{21}}_{\Delta_Z}+Z_{11}Z_L}. \tag{2.6}$$

Note that Δ_Z is the determinant of the matrix of our original twoport in Fig. 2.1. Consequently, eqn. (2.6) may be put into the compact form

$$I_1 = \frac{V_1(Z_{22}+Z_L)}{\Delta_Z + Z_{11}Z_L}. \tag{2.7}$$

Similarly, solving eqn. (2.5) for I_2

$$I_2 = \frac{\begin{vmatrix} Z_{11} & V_1 \\ Z_{21} & 0 \end{vmatrix}}{\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & (Z_{22}+Z_L) \end{vmatrix}} = \frac{-V_1 Z_{21}}{\underbrace{Z_{11}Z_{22}-Z_{12}Z_{21}}_{\Delta_Z}+Z_{11}Z_L}, \tag{2.8}$$

$$I_2 = \frac{-V_1 Z_{21}}{\Delta_Z + Z_{11}Z_L}. \tag{2.9}$$

The current transfer ratio I_2/I_1 is obtained by dividing eqn. (2.9) by eqn. (2.7):

$$\frac{I_2}{I_1} = \left\{\frac{-V_1 Z_{21}}{\Delta_Z + Z_{11}Z_L}\right\}\left\{\frac{\Delta_Z + Z_{11}Z_L}{V_1(Z_{22}+Z_L)}\right\}, \tag{2.10}$$

$$\boxed{\frac{I_2}{I_1} = -\frac{Z_{21}}{Z_{22}+Z_L}} \tag{2.11}$$

(ii) Voltage transfer ratio

Reverting to eqn. (2.4), and by transposition, we find that

$$I_2 = -\frac{V_2}{Z_L}. \tag{2.12}$$

Substituting the right-hand part of Z_{12} for I_2 in eqn. (2.9),

$$-\frac{V_2}{Z_L} = -\frac{V_1 Z_{21}}{\Delta_Z + Z_{11} Z_L}, \tag{2.13}$$

$$\boxed{\frac{V_2}{V_1} = \frac{Z_{21} Z_L}{\Delta_Z + Z_{11} Z_L}} \tag{2.14}$$

(iii) Input impedance

From eqn. (2.7), by transposition, we obtain

$$\boxed{\frac{V_1}{I_1} = Z_{\mathrm{IN}} = \frac{\Delta_Z + Z_{11} Z_L}{Z_{22} + Z_L}} \tag{2.15}$$

(iv) Output impedance

Revert to the block diagram in Fig. 2.2 and note that

$$V_1 = -I_1 Z_G. \tag{2.16}$$

Now substitute for V_1 in eqn. (2.2) and obtain the system matrix which will contain the generator impedance Z_G in series with Z_{11}:

$$\begin{bmatrix} 0 \\ V_2 \end{bmatrix} = \begin{bmatrix} (Z_{11}+Z_G) & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{2.17}$$

Solving eqn. (2.17) for I_2 with determinants,

$$I_2 = \frac{\begin{vmatrix} (Z_{11}+Z_G) & 0 \\ Z_{21} & V_2 \end{vmatrix}}{\begin{vmatrix} (Z_{11}+Z_G) & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}}, \tag{2.18}$$

$$I_2 = \frac{V_2(Z_{11}+Z_G)}{\underbrace{Z_{11}Z_{22}-Z_{12}Z_{21}}_{\Delta_Z}+Z_{22}Z_G}, \tag{2.19}$$

$$I_2 = \frac{V_2(Z_{11}+Z_G)}{\Delta_Z+Z_{22}Z_G}. \tag{2.20}$$

By transposition

$$\boxed{\frac{V_2}{I_2} = Z_{\text{OUT}} = \frac{\Delta_Z+Z_{22}Z_G}{Z_{11}+Z_G}} \tag{2.21}$$

Note that the results are completely general and applicable to either passive or active linear twoports with arbitrary load and generator terminations.

(b) THE TERMINATED TWOPORT AND THE *Y* MATRIX

In the most general form we can represent the terminated twoport in terms of admittance parameters as shown in Fig. 2.3. We can define the unterminated centre portion of this model by the matrix equation as follows:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{2.22}$$

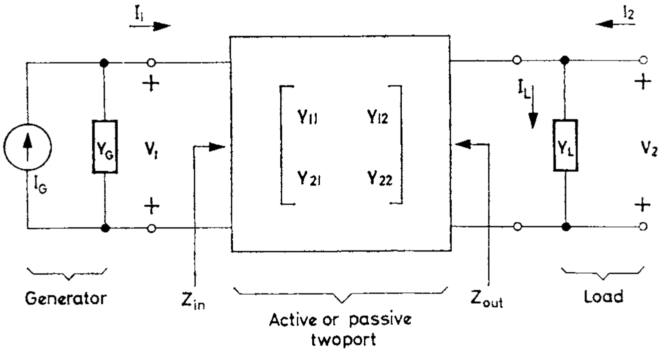


FIG. 2.3. The terminated twoport with *Y* parameters.

We can also easily find the physical interpretation of the short-circuit admittance parameters from eqn. (2.22):

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}; \quad \text{Input admittance.}$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}; \quad \text{Reverse transfer admittance.}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}; \quad \text{Forward transfer admittance.}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}; \quad \text{Output admittance.}$$

(ii) Current transfer ratio

With reference to Fig. 2.3, the current transfer ratio A_I is the ratio of the load and input currents. That is

$$A_I = \frac{I_L}{I_1}.$$

If the generator is the only current source feeding I_1 to the twoport network and associated load admittance, then the equilibrium equations of the system can be written down as

$$I_1 = Y_{11}V_1 + Y_{12}V_2, \tag{2.23}$$

$$0 = Y_{21}V_1 + (Y_{22} + Y_L)V_2. \tag{2.24}$$

Solve these equations simultaneously for V_2 by determinants:

$$V_2 = \frac{\begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & 0 \end{vmatrix}}{Y_{11}(Y_{22}+Y_L) - Y_{12}Y_{21}} = \frac{-Y_{21}I_1}{\underbrace{Y_{11}Y_{22} - Y_{12}Y_{21}}_{\Delta_Y} + Y_{11}Y_L}. \tag{2.25}$$

It will be recognized that the terms $(Y_{11}Y_{22} - Y_{12}Y_{21})$ represent the determinant of the matrix. Thus

$$(Y_{11}Y_{22} - Y_{12}Y_{21}) = \Delta_Y.$$

Therefore, by substituting into eqn. (2.25),

$$V_2 = \frac{-Y_{21}I_1}{\Delta_Y + Y_{11}Y_L}. \tag{2.26}$$

Again, with reference to the block diagram in Fig. 2.3,

$$V_2 = \frac{I_L}{Y_L} \tag{2.27}$$

where I_L is the current through the load admittance Y_L. Thus, from eqns. (2.26) and 2(.27), we obtain the current transfer ratio:

$$\boxed{A_I = \frac{I_L}{I_1} = -\frac{Y_{21}Y_L}{\Delta_Y + Y_{11}Y_L}} \tag{2.28}$$

Note, that the negative sign arises from the adopted sign convention.

(ii) Voltage transfer ratio

Reverting to the equilibrium equations (2.23) and (2.24), and solving them for V_1,

$$V_1 = \frac{\begin{vmatrix} I_1 & Y_{12} \\ 0 & (Y_{22}+Y_L) \end{vmatrix}}{Y_{11}(Y_{22}+Y_L) - Y_{12}Y_{21}} = \frac{I_1(Y_{22}+Y_L)}{\Delta_Y + Y_{11}Y_L}. \tag{2.29}$$

By definition of the voltage transfer ratio,

$$A_V = \frac{V_2}{V_1}. \tag{2.30}$$

Hence, the ratio of eqns. (2.26) and (2.29) is

$$\boxed{A_V = -\frac{Y_{21}}{Y_{22}+Y_L}} \tag{2.31}$$

(iii) Input impedance

Here again the operating condition corresponds to $I_2 = 0$, consequently the pair of equilibrium equations (2.23) and (2.24) still hold. Reverting to eqn. (2.29) and noting that $Z_{\text{IN}} = V_1/I_1$, by transposition

$$\boxed{Z_{\text{IN}} = \frac{Y_{22}+Y_L}{\Delta_Y+Y_{11}Y_L}} \tag{2.32}$$

(iv) Output impedance

With reference to the block diagram in Fig. 2.3 the condition applies when $I_1 = 0$. However, the source admittance Y_G remains operative. If Y_L is disconnected across the output terminals, then the equilibrium equations of the system may be written, by inspection,

$$0 = (Y_{11}+Y_G)V_1+Y_{12}V_2, \tag{2.33}$$

$$I_2 = Y_{21}V_1+Y_{22}V_2. \tag{2.34}$$

Solving simultaneously for V_2,

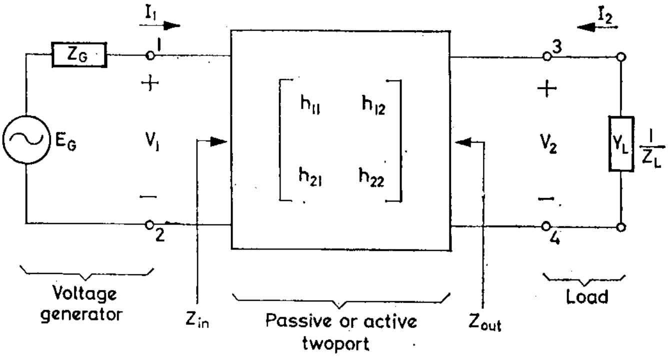
$$V_2 = \frac{\begin{vmatrix} (Y_{11}+Y_G) & 0 \\ Y_{21} & I_2 \end{vmatrix}}{(Y_{11}+Y_G)Y_{22}-Y_{12}Y_{21}} = \frac{(Y_{11}+Y_G)I_2}{Y_{11}Y_{22}-Y_{12}Y_{21}+Y_GY_{22}}. \tag{2.35}$$

By definition $Z_{\text{OUT}} = V_2/I_2$ thus, from eqn. (2.35),

$$\boxed{Z_{\text{OUT}} = \frac{Y_{11}+Y_G}{\Delta_Y+Y_GY_{22}}} \tag{2.36}$$

(c) THE TERMINATED TWOPORT IN TERMS OF THE *h* PARAMETER MATRIX

A particularly useful mathematical model for analytical work with transistors and feedback amplifiers is the *h* parameter matrix. A terminated twoport with *h* matrix notation



2.4. Fig. The terminated twoport with *h* parameters.

can be represented as shown in Fig. 2.4. If we take the unterminated "black box", the equilibrium equations are defined as follows:

$$V_1 = h_{11}I_1 + h_{12}V_2, \tag{2.37}$$

$$I_2 = h_{21}I_1 + h_{22}V_2. \tag{2.38}$$

The physical interpretation of the parameters are, from eqn. (2.37),

$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$; The dimensions of impedance.

$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$; The dimensionless constant of proportionality or the reverse- voltage transfer ratio.

Similarly, from eqn. (2.38),

$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$; The dimensionless constant of proportionality or the forward- current transfer ratio.

$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$; The dimensions of admittance.

By separating the coefficients, we can rearrange eqns. (2.37) and (2.38) into matrix form:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}. \tag{2.39}$$

(i) Voltage transfer ratio

Reverting again to Fig. 2.4, but now connecting a load admittance Y_L as shown across the output terminals 3–4, we can easily verify that,

$$I_2 = -V_2 Y_L. \tag{2.40}$$

Next, by rewriting the equilibrium equations (2.37) and (2.38), substituting eqn. (2.40) for I_2 and transposing, we get

$$V_1 = h_{11} I_1 + h_{12} V_2, \tag{2.41}$$

$$0 = h_{21} I_1 + (h_{22} + Y_L) V_2. \tag{2.42}$$

Solving these equations simultaneously for V_2 by using determinants,

$$V_2 = \frac{\begin{vmatrix} h_{11} & V_1 \\ h_{21} & 0 \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & (h_{22} + Y_L) \end{vmatrix}}, \tag{2.43}$$

$$V_2 = \frac{-h_{21} V_1}{h_{11} h_{22} - h_{12} h_{21} + h_{11} Y_L}. \tag{2.43a}$$

Note that $\Delta_h = (h_{11} h_{22} - h_{12} h_{21})$, where Δ_h is the determinant of the unterminated twoport h matrix.

Hence, we can put eqn. (2.43) into a more compact form,

$$V_2 = -V_1 \frac{h_{21}}{\Delta_h + h_{11} Y_L}. \tag{2.44}$$

By definition of the voltage transfer ratio A_V, then from eqn. (2.44)

$$\boxed{A_V = \frac{V_2}{V_1} = -\frac{h_{21}}{\Delta_h + h_{11} Y_L}} \tag{2.45}$$

(ii) Current transfer ratio

Reverting again to the terminated twoport in Fig. 2.4 and eqn. (2.40), we may replace Y_L by $1/Z_L$ and obtain

$$V_2 = -I_2 Z_L. \tag{2.46}$$

Substituting eqn. (2.46) for V_2 in eqn. (2.38),

$$I_2 = h_{21}I_1 - h_{22}Z_L I_2. \qquad (2.47)$$

By transposition:

$$I_2(1+h_{22}Z_L) = h_{21}I_1. \qquad (2.48)$$

The current transfer ratio A_I is defined as I_2/I_1, thus from eqn. (2.48),

$$\boxed{A_I = \frac{I_2}{I_1} = \frac{h_{21}}{1+h_{22}Z_L}} \qquad (2.49)$$

(iii) Input impedance

Again reverting to the equilibrium equations (2.41) and (2.42), as before, with load impedance Y_L connected and solving these equations simultaneously for I_1 by applying determinants,

$$I_1 = \frac{\begin{vmatrix} V_1 & h_{12} \\ 0 & (h_{22}+Y_L) \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & (h_{22}+Y_L) \end{vmatrix}}, \qquad (2.50)$$

$$I_1 = \frac{V_1(h_{22}+Y_L)}{h_{11}h_{22}-h_{12}h_{21}+h_{11}Y_L}. \qquad (2.50a)$$

Rewriting eqn. (2.50a) by putting $\Delta_h = (h_{11}h_{22}-h_{12}h_{21})$

$$I_1 = V_1\frac{h_{22}+Y_L}{\Delta_h+h_{11}Y_L}. \qquad (2.51)$$

We define the input impedance as $Z_{\text{in}} = V_1/I_1$. Therefore, from eqn. (2.51) by transposition and inversion,

$$\boxed{Z_{\text{in}} = \frac{V_1}{I_1} = \frac{\Delta_h+h_{11}Y_L}{h_{22}+Y_L}} \qquad (2.52)$$

(iv) Output impedance

We now connect the generator across the input terminals 1–2 and disconnect the load Y_L. We can rewrite the equilibrium equations (2.37) and (2.38) by substituting

$$V_1 = -I_1 Z_G, \tag{2.53}$$

hence

$$-I_1 Z_G = h_{11} I_1 + h_{12} V_2 \tag{2.54}$$

and

$$I_2 = h_{21} I_1 + h_{22} V_2. \tag{2.55}$$

By transposition of (2.54) the new set of equilibrium equations become

$$0 = (h_{11} + Z_G) I_1 + h_{12} V_2, \tag{2.56}$$

$$I_2 = h_{21} I_1 + h_{22} V_2. \tag{2.57}$$

Next we solve eqns. (2.56) and (2.57) simultaneously for V_2:

$$V_2 = \frac{\begin{vmatrix} (h_{11} + Z_G & 0 \\ h_{21} & I_2 \end{vmatrix}}{\begin{vmatrix} (h_{11} + Z_G) & h_{12} \\ h_{21} & h_{22} \end{vmatrix}}. \tag{2.58}$$

By cross-multiplication and subtraction,

$$V_2 = \frac{I_2 (h_{11} + Z_G)}{\Delta_h + h_{22} Z_G}. \tag{2.58a}$$

Note that by definition, the output impedance $Z_{\text{out}} = V_2/I_2$. Hence from eqn. (3.58a) by transposition;

$$\boxed{Z_{\text{out}} = \frac{V_2}{I_2} = \frac{(h_{11} + Z_G)}{\Delta_h + h_{22} Z_G}} \tag{2.59}$$

(d) THE TERMINATED TWOPORT IN TERMS OF THE g PARAMETER MATRIX

Although it will be shown later that the g matrix is the inverse of the h matrix, it is a profitable exercise to analyse the terminated twoport in terms of the g parameters. Figure

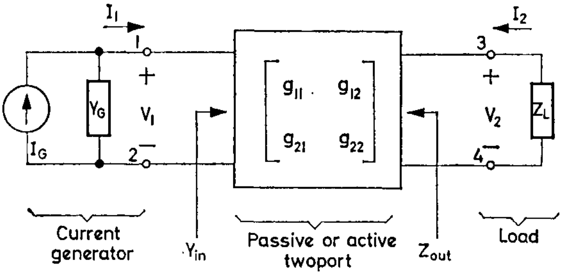


FIG. 2.5. Terminated twoport with g parameter matrices.

2.5 shows the corresponding linear model with appropriate sign convention.

We define the equilibrium equations for the unterminated "black box" as follows:

$$I_1 = g_{11}V_1 + g_{12}I_2, \tag{2.60}$$

$$V_2 = g_{21}V_1 + g_{22}I_2. \tag{2.61}$$

We can obtain the mathematical interpretation of the g parameters by simple algebraic process:

From eqn. (2.60)

$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0}$; The dimensions of admittance.

$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$; The dimensionless constant of proportionality, reverse- current transfer ratio.

Similarly, from eqn. (2.61)

$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$; The dimensionless constant of proportionality, the forward-voltage transfer ratio.

$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$; The dimensions of impedance.

(i) Current transfer ratio

With reference to Fig. 2.5, if load impedance Z_L is connected to the terminals 3–4 of our "black box", we have across the output port

$$V_2 = -I_2 Z_L. \tag{2.62}$$

Hence by substituting eqn. (2.62) into eqn. (2.61) for V_2 and transposing, we obtain a new set of equilibrium equations:

$$I_1 = g_{11}V_1 + g_{12}I_2. \tag{2.63}$$

$$0 = g_{21}V_1 + (g_{22} + Z_L)I_2. \tag{2.64}$$

Solving eqns. (2.63) and (2.64) simultaneously for I_2 by determinants,

$$I_2 = \frac{\begin{vmatrix} g_{11} & I_1 \\ g_{21} & 0 \end{vmatrix}}{\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & (g_{22}+Z_L) \end{vmatrix}} = \frac{-I_1 g_{21}}{\underbrace{(g_{11}g_{22} - g_{12}g_{21})}_{\Delta_g} + g_{11}Z_L}. \tag{2.65}$$

Rewriting and transposing eqn. (2.65) yields the desired expansion for current transfer ratio:

$$\boxed{A_I = \frac{I_2}{I_1} = -\frac{g_{21}}{\Delta_g + g_{11}Z_L}} \tag{2.66}$$

(ii) Voltage transfer ratio

Reverting to eqn. (2.61) and noting that for the terminated condition

$$I_2 = -\frac{V_2}{Z_L},$$

after substituting for I_2

$$V_2 = g_{21}V_1 - g_{22}\frac{V_2}{Z_L}. \tag{2.67}$$

By transposition

$$V_2\left\{1 + \frac{g_{22}}{Z_L}\right\} = g_{21}V_1 \tag{2.68}$$

and finally

$$\boxed{A_V = \frac{V_2}{V_1} = \frac{g_{21}}{1+(g_{22}/Z_L)}} \tag{2.69}$$

Since $Y_L = 1/Z_L$ from eqn. (2.69)

$$\boxed{A_V = \frac{V_2}{V_1} = \frac{g_{21}}{1+g_{22}Y_L}} \tag{2.70}$$

(iii) Input impedance

Consider again that the load impedance Z_L is connected across the output terminals 3–4. We again solve eqns. (2.63) and (2.64) simultaneously, but this time for V_1:

$$V_1 = \frac{\begin{vmatrix} I_1 & g_{12} \\ 0 & (g_{22}+Z_L) \end{vmatrix}}{\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & (g_{22}+Z_L) \end{vmatrix}}. \tag{2.71}$$

By cross-multiplication and subtraction,

$$V_1 = \frac{I_1(g_{22}+Z_L)}{\Delta_g+g_{11}Z_L}. \tag{2.71a}$$

Dividing by I_1 gives

$$\boxed{Z_{\text{in}} = \frac{V_1}{I_1} = \frac{g_{22}+Z_L}{\Delta_g+g_{11}Z_L}} \tag{2.72}$$

(iv) Output impedance

If in Fig. 2.5 we connect our generator across the input terminals 1–2 and remove the load from terminals 3–4, we have for the input current I_1,

$$I_1 = -V_1 Y_G. \tag{2.73}$$

Substitute now for I_1, into eqn. (2.60):

$$-V_1 Y_G = g_{11}V_1 + g_{12}I_2. \tag{2.74}$$

From eqns. (2.74) and (2.61) we obtain a new set of equilibrium equations:

$$0 = (g_{11}+Y_G)V_1 + g_{12}I_2, \tag{2.75}$$

$$V_2 = g_{21}V_1 + g_{22}I_2. \tag{2.76}$$

Solve eqns. (2.75) and (2.76) simultaneously for I_2:

$$I_2 = \frac{\begin{vmatrix} (g_{11}+Y_G) & 0 \\ g_{21} & V_2 \end{vmatrix}}{\begin{vmatrix} (g_{11}+Y_G) & g_{12} \\ g_{21} & g_{22} \end{vmatrix}} = \frac{V_2(g_{11}+Y_G)}{\Delta_g + g_{22}Y_G}. \tag{2.77}$$

By transposition and inversion of eqn. (2.77),

$$\boxed{Z_{\text{out}} = \frac{V_2}{I_2} = \frac{\Delta_g + g_{22}Y_G}{g_{11}+Y_G}} \tag{2.78}$$

(e) THE TERMINATED TWOPORT AND THE TRANSMISSION MATRIX

When manipulating our linear, terminated twoports in the preceeding chapters, we have, in fact practised implicitly some of the basic techniques of matrix synthesis. Our study of the transmission, or *ABCD* matrix, will offer even a better opportunity to gain skill in synthesizing mathematical models. Therefore, it will be convenient to study the terminal properties of a twoport system in different sequence.

(i) Input impedance

Let us consider separately our "black box" in terms of the *ABCD* matrix (as shown in Fig. 2.6).

The pertinent equilibrium equations are defined as follows:

$$V_1 = AV_2 + BI_2, \tag{2.79}$$

$$I_1 = CV_2 + DI_2. \tag{2.80}$$

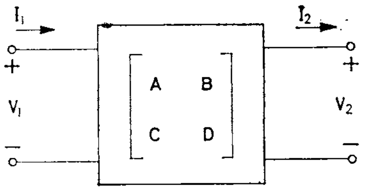


FIG. 2.6. Open-circuited twoport with *ABCD* parameter matrices.

The physical interpretation of the parameters are obtained from eqn. (2.79):

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0};$$ The dimensionless constant of proportionality, reverse- voltage transfer ratio.

$$B = \left.\frac{V_1}{I_2}\right|_{V_2=0};$$ The dimensions of impedance.

Similarly, from eqn. (2.80):

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0};$$ The dimensions of admittance.

$$D = \left.\frac{I_1}{I_2}\right|_{V_2=0};$$ The dimensionless constant of proportionality, the reverse-current transfer ratio.

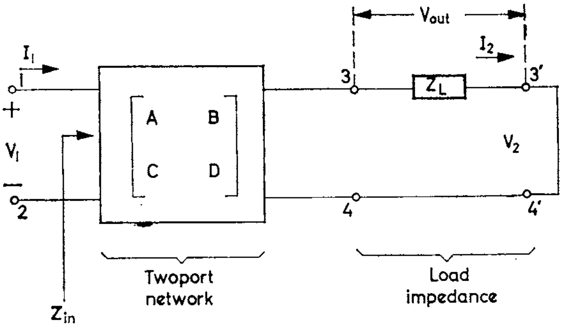


FIG. 2.7. The load-terminated twoport with *ABCD* parameter matrices.

Adding a load impedance Z_L by cascading as shown in Fig. 2.7, but leaving terminals 3′–4′ open-circuited, we can now relate the input and output variables by the matrix product:

$$\underbrace{\begin{bmatrix} V_1 \\ I_1 \end{bmatrix}}_{\text{Input}} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\substack{\text{Twoport} \\ \text{“black box”}}} \underbrace{\begin{bmatrix} 1 & Z_L \\ 0 & 1 \end{bmatrix}}_{\text{Load}} \underbrace{\begin{bmatrix} V_2 \\ I_2 \end{bmatrix}}_{\text{Output}}. \tag{2.81}$$

By multiplying out the *ABCD* twoport and load impedance matrices, we obtain from eqn. (2.81)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & (AZ_L + B) \\ C & (CZ_L + D) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{2.82}$$

Next, by performing the remaining multiplication on the right-hand side, we obtain a set of equilibrium equations with terminals 3′–4′ still open-circuited:

$$V_1 = AV_2 + (AZ_L + B)I_2, \tag{2.83}$$

$$I_1 = (V_2 + (CZ_L + D)I_2. \tag{2.84}$$

If we now short- circuit terminals 3′–4′; and put $V_2 = 0$, the ratio of eqns. (2.83) and (2.84) will yield an expression for the input impedance.

Hence

$$\boxed{Z_{\text{in}} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{AZ_L + B}{CZ_L + D}} \tag{2.85}$$

(ii) Output impedance

We can synthesize a mathematical model by cascading the generator and the original *ABCD* twoport as shown in Fig. (2.8). Since we are now concerned with the output characteristics of our system, the assumed direction of current flow has been reversed. Consequently, the input and output quantities are now related by the inverse of the transmission

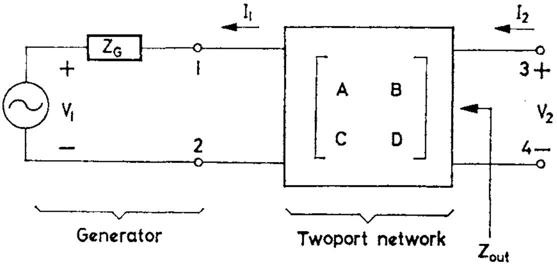


FIG. 2.8. Generator side- terminated twoport with reversed current flow.

matrix.

$$\underbrace{\begin{bmatrix} V_2 \\ I_2 \end{bmatrix}}_{\text{Output}} = \underbrace{\begin{bmatrix} \frac{D}{\Delta} & \frac{B}{\Delta} \\ \frac{C}{\Delta} & \frac{A}{\Delta} \end{bmatrix}}_{\substack{\text{Inverse} \\ \text{of } ABCD \\ \text{matrix}}} \underbrace{\begin{bmatrix} 1 & Z_G \\ 0 & 1 \end{bmatrix}}_{\substack{\text{Generator} \\ \text{impedance}}} \underbrace{\begin{bmatrix} V_1 \\ I_1 \end{bmatrix}}_{\text{Input}}, \tag{2.86}$$

where, by definition, we have

$\Delta = (AD - BC)$; The determinant of the unterminated twoport matrix.

Performing the triple matrix multiplication on the right-hand side of eqn. (2.86), we obtain a new pair of equilibrium equations namely

$$V_2 = \frac{D}{\Delta} V_1 + \frac{DZ_G + B}{\Delta} I_1 \tag{2.87}$$

and

$$I_2 = \frac{C}{\Delta} V_1 + \frac{CZ_G + A}{\Delta} I_1. \tag{2.88}$$

As a next step, we take the ratio of eqns. (2.87) and (2.88) and by putting $V_1 = 0$, the desired expression for the output impedance will be obtained:

$$\boxed{Z_{\text{out}} = \frac{V_2}{I_2}\bigg|_{V_1=0} = \frac{DZ_G + B}{CZ_G + A}} \tag{2.89}$$

(iii) Current transfer ratio

Reverting now to the linear model shown in Fig. 2.7 and short- circuiting the primed terminals 3′–4′, then $V_2 = 0$. Consequently, eqn. (2.84) yields directly the current transfer ratio:

$$\boxed{A_I = \frac{I_2}{I_1}\bigg|_{V_2=0} = \frac{1}{CZ_L+D}} \tag{2.90}$$

(iv) Voltage transfer ratio

In deriving an expression for the voltage transfer ratio, it is convenient to represent the linear twoport with an admittance Y_L as the load as shown in Fig. 2.9. We obtain the overall transmission matrix of the cascaded structures by multiplication of the parameter matrices:

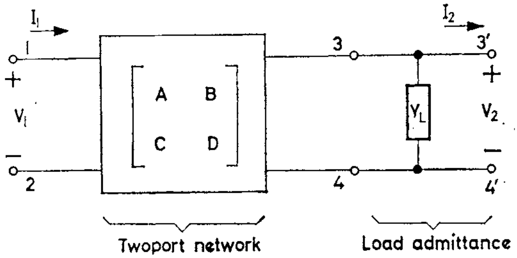


Fig. 2.9. The load admittance terminated twoport with *ABCD* parameter matrices.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_L & 1 \end{bmatrix} = \begin{bmatrix} (A+BY_L)+B \\ (C+DY_L)+D \end{bmatrix}. \tag{2.91}$$

With reference to Fig. 2.9 and the above matrix product, the system equilibrium equation may be written down as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} (A+BY_L) & B \\ (C+DY_L) & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{2.92}$$

Expanding the matrix product on the right- hand side, we obtain a new set of equilibrium equations:

$$V_1 = (A+BY_L)V_2 + BI_2, \tag{2.93}$$

$$I_1 = (C+DY_L)V_2 + DI_2. \tag{2.94}$$

If we now put $I_2 = 0$, then eqn. (2.93) will yield the desired expansion for voltage transfer ratio.

Thus from eqn. (2.93) by transposition,

$$\boxed{A_V = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{1}{A+BY_L}} \tag{2.95}$$

By definition $Y_L = 1/Z_L$, therefore eqn. (2.95) can be rewritten

$$\boxed{A_V = \frac{Z_L}{AZ_L+B}} \tag{2.96}$$

TABLE I. THE TERMINAL PROPERTIES IN TERMS OF THE PARAMETERS MATRIX

	A_v	A_i	Z_0	Z_i
z	$\frac{z_{21}Z_L}{\Delta_z+z_{11}Z_L}$	$\frac{z_{21}}{z_{22}+Z_L}$	$\frac{\Delta_z+z_{22}Z_g}{z_{11}+Z_g}$	$\frac{\Delta_z+z_{11}Z_L}{z_{22}+Z_L}$
y	$\frac{-y_{21}}{y_{22}+Y_L}$	$\frac{-y_{21}Y_L}{\Delta_y+y_{11}Y_L}$	$\frac{y_{11}+Y_g}{\Delta_y+y_{22}Y_g}$	$\frac{y_{22}+Y_L}{\Delta_y+y_{11}Y_L}$
h	$\frac{-h_{21}}{h_{11}Y_L+\Delta_h}$	$\frac{-h_{21}Y_L}{h_{22}+Y_L}$	$\frac{h_{11}+Z_g}{\Delta_h+h_{22}Z_g}$	$\frac{\Delta_h+h_{11}Y_L}{h_{22}+Y_L}$
g	$\frac{g_{21}}{1+g_{22}Y_L}$	$\frac{g_{21}}{\Delta_g+g_{11}Z_L}$	$\frac{\Delta_g+g_{22}Y_g}{g_{11}+Y_g}$	$\frac{g_{22}+Z_L}{\Delta_g+g_{11}Z_L}$
AB CD	$\frac{Z_L}{AZ_L+B}$	$\frac{1}{CZ_L+D}$	$\frac{DZ_G+B}{CZ_G+A}$	$\frac{AZ_L+B}{CZ_L+D}$

Definitions:

$$Z_G = \frac{1}{Y_G}; \qquad Z_L = \frac{1}{Y_L}.$$

Determinants of matrices:

$$\Delta_z = z_{11}z_{22}-z_{12}z_{21},$$
$$\Delta_y = y_{11}y_{22}-y_{12}y_{21},$$
$$\Delta_h = h_{11}h_{22}-h_{12}h_{21},$$
$$\Delta_g = g_{11}g_{22}-g_{12}g_{31}.$$

3. INTERRELATIONS AND CONVERSION OF MATRICES

We have seen in the preceding chapter that a twoport network can be describėd by at least five different systems of matrix parameters. These systems are generated essentially by permutating the relative positions of dependent and independent variables in the equilibrium equations. Intuitively, one would be inclined to believe that these different modes of network representations should be quite identical. If so, then one may ask the legitimate question as, to what practical use can thus be of such a multiplicity of systems. It will be shown that, indeed, we can express an arbitrary system of matrices in terms of any of the other four parameter matrices. As to the practical utility of the conversion, inversion or transformation of matrices gives the following results:

(a) The solution for the unknown quantity is obtained by algebraic inversion.
(b) Simplified mathematical models of complex electronic systems.
(c) Synthesis of complex electronic systems from elementary linear twoports.

All the above aspects of matrix interrelations will be studied here and in the chapters to follow. The inversion of the Z into Y matrix and the g into h matrix will be treated in detail. In both cases it will be shown that the inversion yields the solution of a pair of simultaneous equations for the unknown quantity. With similar detailed approach, the transformation of the Y and h matrices into the $ABCD$ domain will be derived.

We will be able to appreciate the full practical utility of these transformation techniques when the results applied to the analysis and synthesis of feedback amplifiers and oscillators in Part III are studied. A tabulated summary of matrix interrelations will be found in this chapter, the contents of which will be referred to quite frequently.

(a) INVERSION OF THE Z MATRIX INTO Y MATRIX

In Chapter 2 we have already defined the Z matrix of a twoport, which, for easy reference, is repeated here:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{3.1}$$

When expanding the matrix (3.1), we obtain a pair of equilibrium equations with I_1 and I_2 as the unknown quantities:

$$V_1 = Z_{11}I_1 + Z_{12}I_2, \tag{3.2}$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2. \tag{3.3}$$

We may wish to describe the same terminal conditions of a particular twoport, but in terms of I_1 and I_2 as the known variables. This can be easily accomplished by solving eqns. (3.2) and (3.3) simultaneously for I_1 and I_2.

Applying determinants and solving for I_1,

$$I_1 = \frac{\begin{vmatrix} V_1 & Z_{12} \\ V_2 & Z_{22} \end{vmatrix}}{\Delta_Z} = \frac{Z_{22}}{\Delta_Z} V_1 - \frac{Z_{12}}{\Delta_Z} V_2. \tag{3.4}$$

Similarly, solving eqns. (3.2) and (3.3) for I_2,

$$I_2 = \frac{\begin{vmatrix} Z_{11} & V_1 \\ Z_{21} & V_2 \end{vmatrix}}{\Delta_Z} = \frac{Z_{11}}{\Delta_Z} V_2 - \frac{Z_{21}}{\Delta_Z} V_1. \tag{3.5}$$

Note, that eqns. (3.4) and (3.5) are essentially containing all the coefficients with the dimensions of admittance.

Rearranging terms and putting these new equations into matrix form, we obtain

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \dfrac{Z_{22}}{\Delta_Z} & -\dfrac{Z_{12}}{\Delta_Z} \\ -\dfrac{Z_{21}}{\Delta_Z} & \dfrac{Z_{11}}{\Delta_Z} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{3.6}$$

We can now identify each term in this matrix as an admittance parameter in terms of impedances, as follows;

$$Y_{11} \equiv \frac{Z_{22}}{\Delta_Z},$$

$$Y_{12} \equiv -\frac{Z_{12}}{\Delta_Z},$$

$$Y_{21} \equiv -\frac{Z_{21}}{\Delta_Z},$$

$$Y_{22} \equiv \frac{Z_{11}}{\Delta_Z}.$$

Thus, eqn. (3.6) indeed represents a Y matrix in terms of the original Z parameters.

(b) INVERSION OF THE *g* MATRIX INTO *h* MATRIX

Consider the mathematical model of a twoport in terms of the g matrix parameters;

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}. \tag{3.7}$$

In expanding this matrix, we obtain a pair of equilibrium equations with I_1 and V_2 as the dependent variables;

$$I_1 = g_{11}V_1 + g_{12}I_2, \tag{3.8}$$

$$V_2 = g_{21}V_1 + g_{22}I_2. \tag{3.9}$$

The solution of these equations for V_1 and I_2 corresponds to the algebraic inversion of the g matrix in eqn. (3.7). By applying determinants, we proceed now to solve, simultaneously, eqns. (3.8) and (3.9):

$$V_1 = \frac{\begin{vmatrix} I_1 & g_{12} \\ V_2 & g_{22} \end{vmatrix}}{\Delta_g} = \frac{g_{22}}{\Delta_g} I_1 - \frac{g_{12}}{\Delta_g} V_2. \tag{3.10}$$

Similarly, the solution for I_2 is

$$I_2 = \frac{\begin{vmatrix} g_{11} & I_1 \\ g_{21} & V_2 \end{vmatrix}}{\Delta_g} = \frac{g_{11}}{\Delta_g} V_2 - \frac{g_{21}}{\Delta_g} I_1. \tag{3.11}$$

Rearranging eqns. (3.10) and (3.11) into matrix form:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{g_{22}}{\Delta_g} & -\frac{g_{12}}{\Delta_g} \\ -\frac{g_{21}}{\Delta_g} & \frac{g_{11}}{\Delta_g} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}. \tag{3.12}$$

Here we recognize from positions of the variables that the coefficients must have the dimensions of the h parameters expressed in terms of the original g matrix. Accordingly, the elements in the matrix (3.12) may be now identified as follows:

$$h_{11} \equiv \frac{g_{22}}{\Delta_g},$$

$$h_{12} \equiv -\frac{g_{12}}{\Delta_g},$$

$$h_{21} \equiv -\frac{g_{21}}{\Delta_g},$$

$$h_{22} \equiv \frac{g_{11}}{\Delta_g}.$$

From a closer look at the last two exercises of matrix inversion, it will be now quite apparent that the process of matrix inversion has been essentially a solution of a set of equilibrium equations. Therefore, for the mathematical model of a twoport, the algebraic manipulation of matrix inversion can be essentially simplified into the following three steps;

1. Interchange the Z_{11} and Z_{22} terms (g_{11} and g_{22} terms).
2. Change the algebraic signs of the remaining terms.
3. Divide each element with the determinant of the original matrix.

(c) CONVERSION OF THE *Y* MATRIX INTO *ABCD* MATRIX

The usefulness of this type of conversion is often apparent when one desires to cascade twoport networks which were originally defined by admittance parameters. In the usual twoport matrix notation and in terms of the *Y* parameters, we have then

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{3.13}$$

Multiplying out the right-hand matrices, we obtain a pair of equilibrium equations:

$$I_1 = Y_{11}V_1 + Y_{12}V_2, \tag{3.14}$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2. \tag{3.15}$$

In accordance with the constraints of the *ABCD* matrix, we wish to manipulate eqns. (3.14) and (3.15) in such a manner that the dependent variables V_1 and I_1 will appear on the left side only. This can be accomplished in a few algebraic steps as follows.

Solve eqns. (3.14) and (3.15) simultaneously, first for V_1:

$$V_1 = \frac{\begin{vmatrix} I_1 & Y_{12} \\ I_2 & Y_{22} \end{vmatrix}}{\Delta_Y} = \frac{Y_{22}}{\Delta_Y} I_1 - \frac{Y_{12}}{\Delta_Y} I_2. \tag{3.16}$$

Similarly, solve eqns. (3.14) and (3.15) for V_2:

$$V_2 = \frac{\begin{vmatrix} Y_{11} & I_1 \\ Y_{21} & I_2 \end{vmatrix}}{\Delta_Y} = \frac{Y_{11}}{\Delta_Y} I_2 - \frac{Y_{21}}{\Delta_Y} I_1. \tag{3.17}$$

Next, substitute eqn. (3.16) for V_1 into eqn. (3.14):

$$I_1 = Y_{11}\left\{\frac{Y_{22}}{\Delta_Y} I_1 - \frac{Y_{12}}{\Delta_Y} I_2\right\} + Y_{12}V_2. \tag{3.18}$$

Expand:

$$I_1 = \frac{Y_{11}Y_{22}}{\Delta_Y} I_1 - \frac{Y_{11}Y_{12}}{\Delta_Y} I_2 + Y_{12}V_2. \tag{3.19}$$

Transpose the I_1 terms to the left side:

$$I_1\left\{1 - \frac{Y_{11}Y_{22}}{\Delta_Y}\right\} = -\frac{Y_{11}Y_{12}}{\Delta_Y} I_2 + Y_{12}V_2. \tag{3.20}$$

Divide both sides by $\{1-(Y_{11}Y_{22})\Delta_Y\}$:

$$I_1 = -\frac{Y_{11}Y_{12}}{\Delta_Y}\left\{\frac{\Delta_Y}{\Delta_Y - Y_{11}Y_{22}}\right\} I_2 + \left\{\frac{\Delta_Y Y_{12}}{\Delta_Y - Y_{11}Y_{22}}\right\} V_2. \tag{3.21}$$

Expand separately the denominators in the brackets:

$$\Delta_Y - Y_{11}Y_{22} = Y_{11}Y_{22} - Y_{12}Y_{21} - Y_{11}Y_{22}. \tag{3.22}$$

Hence

$$\Delta_Y - Y_{11}Y_{22} = -Y_{12}Y_{21}. \tag{3.23}$$

Substitute now the right-hand part of eqn. (3.23) into the denominators of eqn. (3.21) and rearrange terms:

$$I_1 = -\frac{\Delta_Y Y_{12}}{Y_{12}Y_{21}} V_2 + \frac{Y_{11}Y_{12}}{Y_{12}Y_{21}} I_2. \tag{3.24}$$

Cancel where applicable:

$$\boxed{I_1 = -\frac{\Delta_Y}{Y_{21}} V_2 + \frac{Y_{11}}{Y_{21}} I_2} \tag{3.25}$$

Use now eqn. (3.25) and substitute for I_1 in eqn. (3.16):

$$V_1 = \frac{Y_{22}}{\Delta_Y}\left\{-\frac{\Delta_Y}{Y_{21}}V_2+\frac{Y_{11}}{Y_{21}}I_2\right\}-\frac{Y_{12}}{\Delta_Y}I_2. \qquad (3.26)$$

Expand and rearrange terms:

$$V_1 = -\frac{Y_{22}}{Y_{21}}V_2+\left\{\frac{Y_{11}Y_{22}}{\Delta_Y Y_{21}}-\frac{Y_{12}}{\Delta_Y}\right\}I_2. \qquad (3.27)$$

Multiply and divide the second term in the bracket by Y_{21}:

$$V_1 = -\frac{Y_{22}}{Y_{21}}V_2+\left\{\frac{Y_{11}Y_{22}-Y_{12}Y_{21}}{\Delta_Y Y_{21}}\right\}I_2. \qquad (3.28)$$

By definition, $\Delta_Y = (Y_{11}Y_{22}-Y_{12}Y_{21})$.

Hence eqn. (3.28) simplifies to the final form:

$$\boxed{V_1 = -\frac{Y_{22}}{Y_{21}}V_2+\frac{1}{Y_{21}}I_2} \qquad (3.29)$$

Now, from eqns. (3.25) and (3.29) we can construct the *ABCD* matrix in terms of the original *Y* parameters:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} -\frac{Y_{22}}{Y_{21}} & -\frac{1}{Y_{21}} \\ -\frac{\Delta_Y}{Y_{21}} & -\frac{Y_{11}}{Y_{21}} \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \qquad (3.30)$$

The elements in the matrix (3.30) may be identified as follows:

$$A \equiv -\frac{Y_{22}}{Y_{21}},$$

$$B \equiv -\frac{1}{Y_{21}},$$

$$C \equiv -\frac{\Delta_Y}{Y_{21}},$$

$$D \equiv -\frac{Y_{11}}{Y_{21}}.$$

(d) CONVERSION OF THE *h* MATRIX INTO *ABCD* MATRIX

Revert now to eqns. (2.37) and (2.38) for a twoport in terms of the h parameters. If we choose to reverse the assumed current flow of I_2, then we may rewrite the equilibrium equations with a change in sign of I_2 only;

$$V_1 = h_{11}I_1+h_{12}V_2, \tag{3.31}$$

$$-I_2 = h_{21}I_1+h_{22}V_2. \tag{3.32}$$

It is now further required that eqns. (3.31) and (3.32) be transformed. The variables will be then be related as dictated by the structure of the transmission matrix.

From eqn. (3.32) by algebraic transposition,

$$-h_{21}I_1 = I_2+h_{22}V_2, \tag{3.33}$$

$$I_1 = -\frac{1}{h_{21}}I_2-\frac{h_{22}}{h_{21}}V_2. \tag{3.34}$$

Substituting now from eqn. (3.34) for I_1 into eqn. (3.31),

$$V_1 = h_{11}\left(-\frac{1}{h_{21}}I_2-\frac{h_{22}}{h_{21}}V_2\right)+h_{12}V_2. \tag{3.35}$$

$$V_1 = -\frac{h_{11}}{h_{21}}I_2-\frac{h_{11}h_{22}}{h_{21}}V_2+h_{12}V_2. \tag{3.36}$$

Rearranging terms,

$$V_1 = \left(h_{12}-\frac{h_{11}h_{22}}{h_{21}}\right)V_2-\frac{h_{11}}{h_{21}}I_2, \tag{3.37}$$

$$V_1 = \frac{h_{12}h_{21}-h_{11}h_{22}}{h_{21}}V_2-\frac{h_{11}}{h_{21}}I_2. \tag{3.38}$$

Multiplying the numerator and denominator of the first term on the right-hand side by -1,

$$V_1 = -\left(\frac{h_{11}h_{22}-h_{12}h_{21}}{h_{21}}\right)V_2-\frac{h_{11}}{h_{21}}I_2. \tag{3.39}$$

It will now be recognized that the numerator of the first term represents the determinant of the h matrix, that is

$$(h_{11}h_{22}-h_{12}h_{21}) \equiv \Delta_h. \tag{3.40}$$

The identity of eqn. (3.40) enables us to rewrite eqn. (3.39) in a more compact form:

$$V_1 = -\frac{\Delta_h}{h_{21}} V_2 - \frac{h_{11}}{h_{21}} I_2. \tag{3.41}$$

Note that the coefficients in eqns. (3.34) and (3.41) correspond to the elements A, B, C and D in the transmission matrix. Rewriting eqns. (3.34) and (3.41) as a pair of equilibrium equations,

$$V_1 = \overbrace{-\frac{\Delta_h}{h_{21}}}^{A} V_2 - \overbrace{\frac{h_{11}}{h_{21}}}^{B} I_2, \tag{3.42}$$

$$I_1 = \underbrace{-\frac{h_{22}}{h_{21}}}_{C} V_2 - \underbrace{\frac{1}{h_{21}}}_{D} I_2. \tag{3.43}$$

Expressing eqns. (3.42) and (3.43) in matrix form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = -\frac{1}{h_{21}} \begin{bmatrix} \Delta_h & h_{11} \\ h_{22} & 1 \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{3.44}$$

From eqns. (3.42), (3.43) and (3.44), it is now evident that the *ABCD* matrix has been defined in terms of the hybrid or h parameters:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv -\frac{1}{h_{21}} \begin{bmatrix} \Delta_h & h_{11} \\ h_{22} & 1 \end{bmatrix}. \tag{3.45}$$

The elements in eqn. (3.45) are identified as follows:

$$A \equiv -\frac{\Delta_h}{h_{21}},$$

$$B \equiv -\frac{h_{11}}{h_{21}},$$

$$C \equiv -\frac{h_{22}}{h_{21}},$$

$$D \equiv -\frac{1}{h_{21}},$$

(e) TABLE OF MATRIX CONVERSIONS, INTRODUCTORY COMMENTS

We have seen that the conversion of matrices, although quite straight forward, is a somewhat lengthy algebraic operation. A systematic permutation of the five different systems so far studied will yield twenty matrix interrelations. These have been summarized in Table II. Most readers will be familiar with the use of tables for Laplace transformations or tables of trigonometric interrelations. Essentially, the use of tables for obtaining matrix interrelations is also quite simple and warmly recommended once the student has acquired the proper comprehension of algebraic manipulations leading to them. By referring to Table II, each of the intersections of a vertical column and a horizontal row yield the desired transform pair.

TABLE II. MATRIX INTERRELATIONS

From → To ↓	z	y	h	g	$\begin{matrix} A & B \\ C & D \end{matrix}$
z		$\begin{matrix} \frac{y_{22}}{\Delta_y} & \frac{-y_{12}}{\Delta_y} \\ \frac{-y_{21}}{\Delta_y} & \frac{y_{11}}{\Delta_y} \end{matrix}$	$\begin{matrix} \frac{\Delta_h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{-h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{11}} & \frac{-g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{\Delta_g}{g_{11}} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$
y	$\begin{matrix} \frac{z_{22}}{\Delta_z} & \frac{-z_{12}}{\Delta_z} \\ \frac{-z_{21}}{\Delta_z} & \frac{z_{11}}{\Delta_z} \end{matrix}$		$\begin{matrix} \frac{1}{h_{11}} & \frac{-h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_h}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_g}{g_{22}} & \frac{g_{12}}{g_{22}} \\ \frac{-g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{matrix}$	$\begin{matrix} \frac{D}{B} & \frac{-\Delta}{B} \\ \frac{-1}{B} & \frac{A}{B} \end{matrix}$
h	$\begin{matrix} \frac{\Delta_z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{-z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{y_{11}} & \frac{-y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta_y}{y_{11}} \end{matrix}$		$\begin{matrix} \frac{g_{22}}{\Delta_g} & \frac{-g_{12}}{\Delta_g} \\ \frac{-g_{21}}{\Delta_g} & \frac{g_{11}}{\Delta_g} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta}{D} \\ \frac{-1}{D} & \frac{C}{D} \end{matrix}$
g	$\begin{matrix} \frac{1}{z_{11}} & \frac{-z_{12}}{z_{11}} \\ \frac{z_{21}}{z_{11}} & \frac{\Delta_z}{z_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_y}{y_{22}} & \frac{y_{12}}{y_{22}} \\ \frac{-y_{21}}{y_{22}} & \frac{1}{y_{22}} \end{matrix}$	$\begin{matrix} \frac{h_{22}}{\Delta_h} & \frac{-h_{12}}{\Delta_h} \\ \frac{-h_{21}}{\Delta_h} & \frac{h_{11}}{\Delta_h} \end{matrix}$		$\begin{matrix} \frac{C}{A} & \frac{-\Delta}{A} \\ \frac{1}{A} & \frac{B}{A} \end{matrix}$
$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} \frac{z_{11}}{z_{21}} & \frac{\Delta_z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{matrix}$	$\begin{matrix} \frac{-y_{22}}{y_{21}} & \frac{-1}{y_{21}} \\ \frac{-\Delta_y}{y_{21}} & \frac{-y_{11}}{y_{21}} \end{matrix}$	$\begin{matrix} \frac{-\Delta_h}{h_{21}} & \frac{-h_{11}}{h_{21}} \\ \frac{-h_{22}}{h_{21}} & \frac{-1}{h_{21}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{21}} & \frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\Delta_g}{g_{21}} \end{matrix}$	

Definitions: Determinants of Matrices: $\Delta = AD - BC$, $\Delta_z = z_{11}z_{22} - z_{12}z_{21}$, $\Delta_y = y_{11}y_{22} - y_{12}y_{21}$, $\Delta_h = h_{11}h_{22} - h_{12}h_{21}$, $\Delta_g = g_{11}g_{22} - g_{12}g_{21}$.

4. ELEMENTS OF MATRIX SYNTHESIS AND NETWORK MODELS

IN THIS chapter some of the most useful aspects of matrix algebra will be studied, namely, the orderly process of matrix synthesis of complex networks and systems, which are made up from elementary twoport building blocks. The student will be introduced to this new field by the way of the simplest possible network configurations, such as a single series impedance and a single shunt admittance element. Next, the more elaborate two and three element models will be treated, such as the familiar L, T and π networks. Finally, with an emphasis on the system concept, some fundamental techniques of twoport interconnections will be demonstrated.

The step-by-step worked-out examples will cover the ground of typical synthesis topics in terms of all five matrix domains. This sort of mathematical processing will enable the student to appreciate that the synthesis of a particular type of network will be most conveniently handled by one particular matrix form. It will then become quite obvious why the series connection of impedances is handled with the Z parameter matrix, the parallel connections with the Y matrix and the cascading of twoports with the $ABCD$ or transmission matrix, etc.

(a) THE SINGLE ELEMENT TWOPORT

Let us consider the simplest possible building block of any electronic system, namely that of a single impedance or admittance element. We can represent these elementary twoports as shown in Figs. 4.1a and 4.1b. Bearing in mind the adopted sign convention, we can easily obtain meaningful

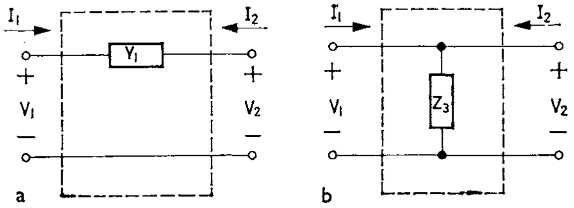


FIG. 4.1a,b. Single element twoports.

expressions for the equilibrium equations in terms of the Y and Z parameters.

From Fig. 4.1a, we can write down, by inspection, the nodal equations:

$$I_1 = Y_1 V_1 - Y_1 V_2, \tag{4.1}$$

$$I_2 = -Y_1 V_1 + Y_1 V_2. \tag{4.2}$$

From the detached coefficients we can now form the Y parameter matrix of the single element Y_1:

$$[Y] = \begin{bmatrix} Y_1 & -Y_1 \\ -Y_1 & Y_1 \end{bmatrix}, \tag{4.3}$$

where the element positions in the array are defined as follows:

$$Y_{11} = Y_{22} = Y_1$$

and $$Y_{12} = Y_{21} = -Y_1.$$

With similar reasoning for Fig. 4.1b, the equilibrium loop equations involving the impedance Z_3 are

$$V_1 = Z_3 I_1 + Z_3 I_2, \tag{4.4}$$

$$V_2 = Z_3 I_1 + Z_3 I_2. \tag{4.5}$$

Here the detached coefficients will yield the Z parameter matrix:

$$[Z] = \begin{bmatrix} Z_3 & Z_3 \\ Z_3 & Z_3 \end{bmatrix}, \tag{4.6}$$

where the elements in the array are identified as follows:

$$Z_{11} = Z_{12} = Z_{21} = Z_{22} = Z_3.$$

(b) THE TWO-ELEMENT L NETWORK IN TERMS OF Z AND Y PARAMETERS

Combining a series and a shunt impedance element will yield the L network as shown in Fig. 4.2. This simple linear model may represent a resistive voltage- divider, an impedance transformer or a low-pass filter. By inspection, we can easily write down a pair of loop equations in terms of the Z parameters:

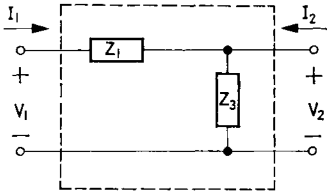


FIG. 4.2. The series input L network with Z parameters.

$$V_1 = (Z_1+Z_3)I_1+Z_3I_2, \tag{4.7}$$

$$V_2 = Z_3I_1+Z_3I_2. \tag{4.8}$$

From the detached coefficients, the Z parameter matrix can now be formed:

$$[Z] = \begin{bmatrix} (Z_1+Z_3) & Z_3 \\ Z_3 & Z_3 \end{bmatrix}, \tag{4.9}$$

where

$$Z_{11} = (Z_1+Z_3),$$

$$Z_{12} = Z_{21} = Z_{22} = Z_3.$$

If we choose to synthesize an L network with shunt input elements, the corresponding linear model is obtained as shown in Fig. 4.3. It will be easily seen that the appropriate

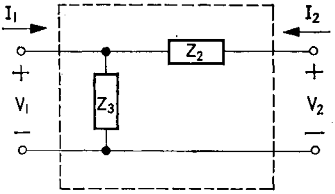


FIG. 4.3. The series output L network with Z parameters.

Z parameter matrix must be of the form

$$[Z]=\begin{bmatrix} Z_3 & Z_3 \\ Z_3 & (Z_2+Z_3) \end{bmatrix}, \tag{4.10}$$

where the element positions may be identified accordingly:

$$Z_{11}=Z_{12}=Z_{21}=Z_3$$

and
$$Z_{22}=(Z_2+Z_3).$$

Consider next the L networks in terms of the admittance or Y parameters as shown in Figs. 4.4a and 4.4b.

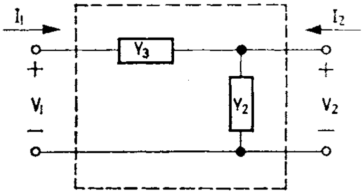


FIG. 4.4a. L network with shunt admittance output − Y parameter notation.

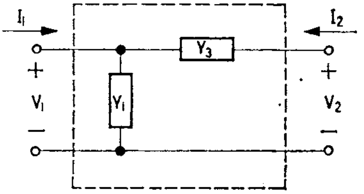


FIG. 4.4b. L network with shunt admittance input − Y parameter notation.

The nodal equilibrium equations with respect to the linear model in Fig. 4.4a are

$$I_1=Y_3V_1-Y_3V_2, \tag{4.11}$$

$$I_2=-Y_3V_1+(Y_2+Y_3)V_2. \tag{4.12}$$

The detached coefficients in eqns. (4.11) and (4.12) will make up the elements of the Y parameter matrix:

$$[Y]=\begin{bmatrix} Y_3 & -Y_3 \\ -Y_3 & (Y_2+Y_3) \end{bmatrix}, \tag{4.13}$$

where we can identify the elements as

$$Y_{11} = Y_3,$$
$$Y_{12} = Y_{21} = -Y_3,$$
$$Y_{22} = (Y_2 + Y_3).$$

With similar reasoning for the twoport model in Fig. 4.4b, we obtain the Y matrix:

$$[Y] = \begin{bmatrix} (Y_1 + Y_3) & -Y_3 \\ -Y_3 & Y_3 \end{bmatrix}, \qquad (4.14)$$

The elements here are identified as

$$Y_{11} = (Y_1 + Y_3),$$
$$Y_{12} = Y_{21} = -Y_3,$$
$$Y_{22} = Y_3.$$

(c) THE THREE-ELEMENT NETWORK IN TERMS OF Z AND Y PARAMETERS

It is well known that a large class of practical impedance transformers and filter networks can be generalized as a T or π structure. The appropriate linear models are shown in Figs. 4.5a and 4.5b respectively. If we now revert to Fig.

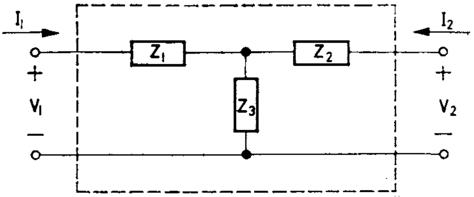


FIG. 4.5a. Twoport T network with impedance parameters.

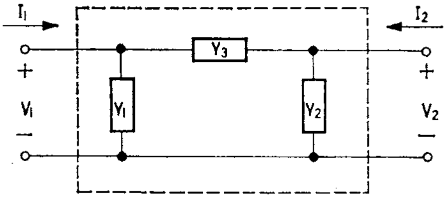


FIG. 4.5b. Twoport π network with admittance parameters.

4.3 and the matrix (4.10), one can easily synthesize the T structure by simply adding an impedance branch Z_1 to the element Z_{11}. Thus from eqn. (4.10) and Fig. 4.5a, the Z matrix of the T network is obtained:

$$[Z]_T = \begin{bmatrix} (Z_1+Z_3) & Z_3 \\ Z_3 & (Z_2+Z_3) \end{bmatrix}, \tag{4.15}$$

where, by definition of the elements in the array,

$$\begin{aligned} Z_{11} &= (Z_1+Z_3), \\ Z_{12} &= Z_{21} = Z_3, \\ Z_{22} &= (Z_2+Z_3). \end{aligned}$$

If a similar treatment is applied to the π network, we can obtain a mathematical model in terms of the Y parameters. This time we revert to Fig. 4.4b and the corresponding matrix (4.14). By adding the admittance element Y_2 to the Y_{22} position, we have in fact synthesized the mathematical model of our π network:

$$[Y]_\pi = \begin{bmatrix} (Y_1+Y_3) & -Y_3 \\ -Y_3 & (Y_2+Y_3) \end{bmatrix}. \tag{4.16}$$

The elements in the array are now identified as follows:

$$\begin{aligned} Y_{11} &= (Y_1+Y_3), \\ Y_{12} &= Y_{21} = -Y_3, \\ Y_{22} &= (Y_2+Y_3). \end{aligned}$$

(d) THE *ABCD* OR TRANSMISSION MATRIX APPLIED TO THE SYNTHESIS OF SIMPLE NETWORK MODELS

A closer look at the linear models of L, T or π networks will suggest the definition that these are synthesized by cascading of series impedances and shunt admittances. By way of rather complete mathematical proof, it has been shown that the transmission matrix of cascaded networks is obtained by the matrix product of the constituent *ABCD* matrices.(8, 10) Here we shall review the basic techniques of linear network manipulations with the *ABCD* matrix as the vehicle.

(i) The *L* network with series input element

Consider a simple low-pass filter section or a matching network which is made up from a series impedance Z_1 and a shunt admittance Y as shown in Fig. 4.6. Note that we are dealing with generalized parameters. The physical impedance may consist of several series- connected resistive and reactive components. Similarly, the admittance may represent the sum of several parallel- connected conductances and susceptances.

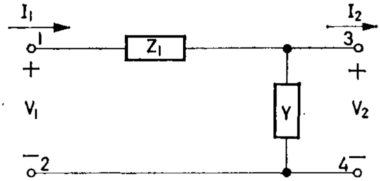


FIG. 4.6. Twoport network of cascaded Z and Y elements.

The behaviour of the composite network is completely defined by the matrix product of the transmission matrices of the series Z and parallel Y elements:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{ZY} = \underbrace{\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}}_{\substack{\text{Transmission} \\ \text{matrix of } Z_1}} \times \underbrace{\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}}_{\substack{\text{Transmission} \\ \text{matrix of } Y}}. \tag{4.17}$$

Substituting the appropriate parameter matrices into the right-hand part of eqn. (4.17)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{ZY} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}. \tag{4.18}$$

Note that the elementary transmission matrices are in the same order as the elements in the composite network.

Performing now the matrix multiplication on the right-hand side of eqn. (4.18),

$$\begin{bmatrix}1 & Z_1\\0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0\\Y & 1\end{bmatrix}=\begin{bmatrix}\overbrace{(1+Z_1Y)}^{A} & \overbrace{Z}^{B}\\ \underbrace{Y}_{C} & \underbrace{1}_{D}\end{bmatrix}. \qquad (4.19)$$

Using now the matrix (4.19), the equilibrium equation with respect of the complete cascaded network in Fig. 4.6 may be written down in a compact form:

$$\begin{bmatrix}V_1\\I_1\end{bmatrix}=\begin{bmatrix}\overbrace{(1+Z_1Y)}^{A} & \overbrace{Z}^{B}\\ \underbrace{Y}_{C} & \underbrace{1}_{D}\end{bmatrix}\times\begin{bmatrix}V_2\\I_2\end{bmatrix}. \qquad (4.20)$$

(ii) The *L* network with shunt input element

If the network in Fig. 4.6 is turned around, the shunt element will be transferred to the input side as shown in Fig. 4.7. With similar reasoning as with Fig. 4.6 the transmission matrix of this composite network will be also synthesized from the elementary transmission matrices.

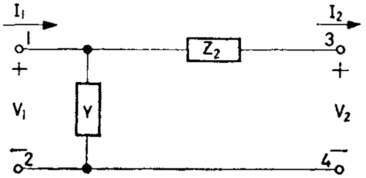


FIG. 4.7. Twoport network of cascaded *Y* and *Z* elements.

Again cascade the transmission matrices in the same order as the elements in the composite network shown in Fig. 4.7.

$$\begin{bmatrix}A & B\\C & D\end{bmatrix}_{YZ_2}=\begin{bmatrix}1 & 0\\Y & 1\end{bmatrix}\times\begin{bmatrix}1 & Z_2\\0 & 1\end{bmatrix}=\begin{bmatrix}\overbrace{1}^{A} & \overbrace{Z_2}^{B}\\ \underbrace{Y}_{C} & \underbrace{(1+YZ_2)}_{D}\end{bmatrix}. \qquad (4.21)$$

(iii) The T network

The transmission matrix of a T network as shown in Fig. 4.8 may be easily constructed from the elementary building blocks of series impedances Z_1 and Z_2 and a parallel admittance element Y.

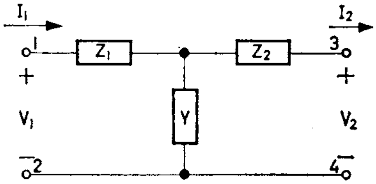


FIG. 4.8. The T network.

Cascading the elementary transmission matrices in the same order as the physical elements in the T network, the transmission matrix of the composite network will be obtained from the product of the elementary matrices:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{T\text{ network}} = \underbrace{\begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix}}_{\text{Series } Z_1} \times \underbrace{\begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}}_{\text{Parallel } Y} \times \underbrace{\begin{bmatrix} 1 & Z_2 \\ 0 & 1 \end{bmatrix}}_{\text{Series } Z_2}. \qquad (4.22)$$

In the previous paragraph we have seen that the matrix product of Y and Z_2 elements is of the form defined by eqn. (4.21). Therefore, computational effort may be saved by substituting the right-hand part of eqn. (4.21) for the YZ_2 product into eqn. (4.22):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{T\text{ network}} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & Z_2 \\ Y & (1+YZ_2) \end{bmatrix}. \qquad (4.23)$$

Performing now the remaining operation of multiplication on eqn. (4.23), the transmission matrix of the T network is obtained:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{T\text{network}} = \begin{bmatrix} \overbrace{(1+YZ_1)}^{A} & \overbrace{(Z_1+Z_2+YZ_1Z_2)}^{B} \\ \underbrace{Y}_{C} & \underbrace{(1+YZ_2)}_{D} \end{bmatrix}. \qquad (4.23a)$$

(iv) The π network

The transmission matrix of a π network is also derived from the elementary shunt and series elements. It is done in a similar fashion as it has been shown for the T network. Reverting to Fig. 4.2 and eqn. (4.19). When adding a second Y element across terminals 1–2, a π network will be formed as shown in Fig. 4.9. The transmission matrix of this network is found by successive multiplication of the component elementary matrices:

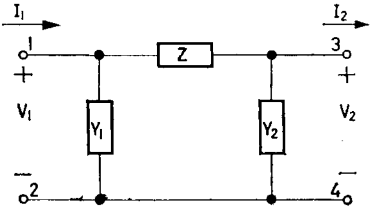


FIG. 4.9. The π network.

Thus from Fig. 4.9

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\pi\,\text{network}} = \begin{bmatrix} 1 & 0 \\ Y_1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ Y_2 & 1 \end{bmatrix}. \qquad (4.24)$$

Note, that, the product of the last two terms on the right-hand side of this equation has been already defined by eqn. (4.19).

Hence, by substituting eqn. (4.19) into eqn. (4.24),

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\pi\,\text{network}} = \begin{bmatrix} 1 & 0 \\ Y_1 & 1 \end{bmatrix} \times \begin{bmatrix} (1+ZY_2) & Z \\ Y_2 & 1 \end{bmatrix}. \qquad (4.25)$$

Performing now the remaining multiplication on the right-hand side of eqn. (4.25) the transmission matrix of the π network will be obtained:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\pi\,\text{network}} = \begin{bmatrix} \overbrace{(1+ZY_2)}^{A} & \overbrace{Z}^{B} \\ \underbrace{(Y_1+Y_2+ZY_1Y_2)}_{C} & \underbrace{(1+ZY_1)}_{D} \end{bmatrix}. \qquad (4.26)$$

(e) TWOPORT INTERCONNECTIONS AND SYNTHESIS OF GENERALIZED MATHEMATICAL MODELS

In progressing so far, the student has already acquired a fair amount of working experience in analysing and synthesizing some simple twoport models from elementary building blocks. There is, however, one more area to be explored before he could tackle with confidence practical electronic design problems. He requires the additional tool of a broader interpretation of arbitrary twoport interconnection techniques.

We have seen in the preceding chapter that the series connecting of two impedances, or the parallel connecting of two admittances is mathematically tantamount to simple algebraic summation of these elements. We may now expand this definition so that it should include the valid interconnection of arbitrary twoports. It will be shown that indeed, by observing a few simple rules in effecting the interconnections, the basic law of algebraic summation applies. The key to valid interconnection of twoports is shown to require the adherence to consistent matrix notations.

(i) The Z matrix interconnection

Consider a pair of arbitrary twoports which are mathematically defined by the Z parameter matrix. We can represent the twoports as "black boxes" and interconnected as shown in Fig. 4.10. The matrices $[Z]_A$ and $[Z]_B$ are completely general and may stand for, passive or active twoports. By definition for the series–series type of interconnection, consistency requires that the same current flows in and out of each terminal pair. On the other hand the sum of voltage drops must be equal to the applied terminal voltages. Consequently, the interconnected sytem matrix is obtained by the algebraic summation of the elements in the matrices $[Z]_A$ and $[Z]_B$. That is

$$[Z] = [Z]_A + [Z]_B, \qquad (4.27)$$

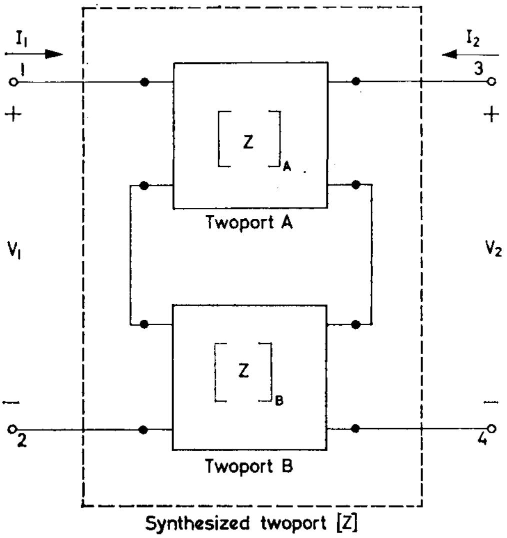


FIG. 4.10. Interconnection of twoports with Z parameter matrices.

where
$$[Z]_A = \begin{bmatrix} Z_{11a} & Z_{12a} \\ Z_{21a} & Z_{22a} \end{bmatrix}. \tag{4.28}$$

and
$$[Z]_B = \begin{bmatrix} Z_{11b} & Z_{12b} \\ Z_{21b} & Z_{22b} \end{bmatrix}. \tag{4.29}$$

By substituting for $[Z]_A$ and $[Z]_B$ in eqn. (4.27), we obtain the overall system matrix of the interconnected twoports:

$$[Z] = \begin{bmatrix} (Z_{11a}+Z_{11b}) & (Z_{12a}+Z_{12b}) \\ (Z_{21a}+Z_{21b}) & (Z_{22a}+Z_{22b}) \end{bmatrix}. \tag{4.30}$$

(ii) The Y matrix interconnection

For parallel connected "black boxes" in terms of the Y parameters Fig. 4.11 applies. Here the voltages across the terminal pairs are identical. Therefore the overall sytem matrix of this type of interconnection is obtained by the algebraic summation of the corresponding elements in the admittance matrices $[Y]_A$ and $[Y]_B$.

$$[Y] = [Y]_A + [Y]_B, \tag{4.31}$$

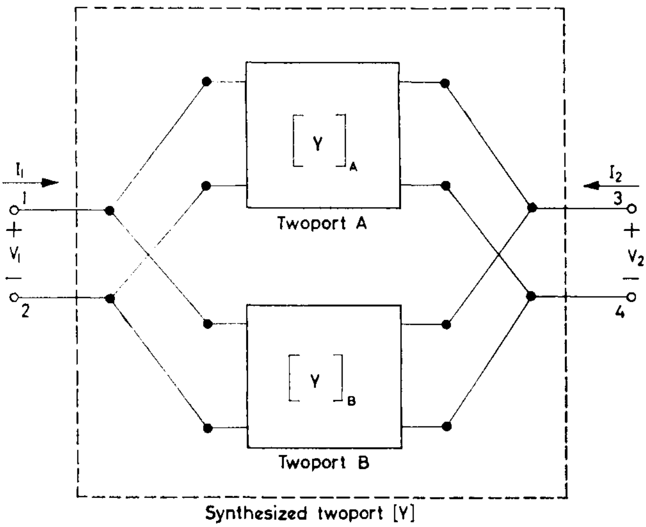


FIG. 4.11. Interconnection of twoports with Y parameter matrices.

where
$$[Y]_A = \begin{bmatrix} Y_{11a} & Y_{12a} \\ Y_{21a} & Y_{22a} \end{bmatrix} \tag{4.32}$$

and
$$[Y]_B = \begin{bmatrix} Y_{11b} & Y_{12b} \\ Y_{21b} & Y_{22b} \end{bmatrix}. \tag{4.33}$$

Substituting into eqn. (4.31) the expanded form for the matrix of the parallel connected twoports is

$$[Y] = \begin{bmatrix} (Y_{11a}+Y_{11b}) & (Y_{12a}+Y_{12b}) \\ (Y_{21a}+Y_{21b}) & (Y_{22a}+Y_{22b}) \end{bmatrix}. \tag{4.34}$$

So far we have assumed that each term of the above arrays has a definite physical meaning. In actual practice this may not be necessarily so. In fact we may choose to manipulate a mathematical model where some elements of the matrix are reduced to zero. For the sake of argument let us stipulate that our "black box" $[Y]_A$ represents an active twoport, a transistor or a vacuum tube. Assume further that in the operating range of interest, we can consider this active twoport as a unilateral device, that is, power can flow only in the forward direction. Such an idealized condition is tantamount to the mathematical constraint that $Y_{12a} = 0$. With similar reasoning we are at liberty to specify the "black box" designated by $[Y]_B$ as a package say, containing isolated input- and output- matching networks. Therefore, in mathematical language $Y_{12b} = Y_{21b} = 0$. If we now attempt to synthesize the mathematical model of our idealized amplifying system, the interconnections in Fig. 4.12 will apply.

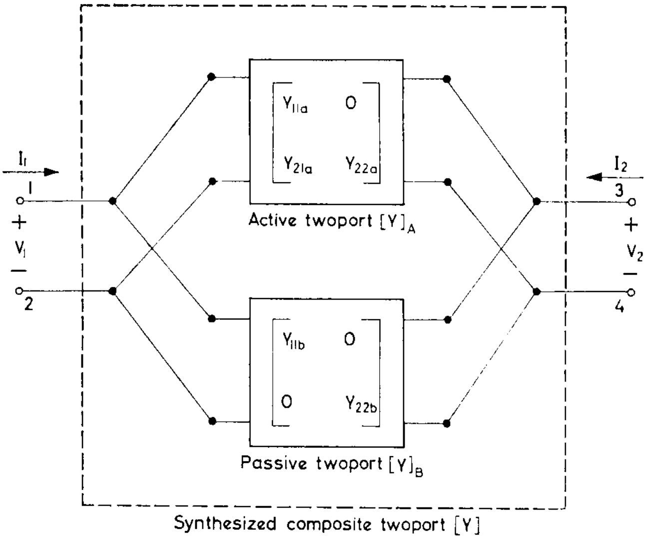


FIG. 4.12. Synthesis of an amplifier model from active and passive twoports.

The sytem matrix of this linear model is obtained as the algebraic sum of the constituent admittance matrices:

$$[Y] = [Y]_A + [Y]_B.$$

In expanded form

$$[Y] = \begin{bmatrix} \overbrace{(Y_{11a} + Y_{11b})}^{Y_{11}} & \overbrace{0}^{Y_{12}} \\ \underbrace{Y_{21a}}_{Y_{21}} & \underbrace{(Y_{22a} + Y_{22b})}_{Y_{22}} \end{bmatrix}. \tag{4.35}$$

Since the Y_{12} term is zero, we may therefore conclude that this mathematical model describes a unilateral amplifier, that is a system which will support power flow only in the forward direction.

(iii) The *h* matrix interconnection

Revert now to eqns. (2.37) and (2.38) for the fundamental definitions of the *h* parameters. It will be recalled, that as far as the input terminal pairs are concerned, the parameter h_{11} has the dimensions of an impedance. Similarly, with respect of the output port the parameter h_{22} has the dimensions of an admittance. Therefore, consistency demands the series connection of input ports and parallel connection of output ports when one wishes to interconnect twoports which are defined by *h* matrices. In this manner valid interconnection is obtained as shown in Fig. 4.13. Note that each twoport is

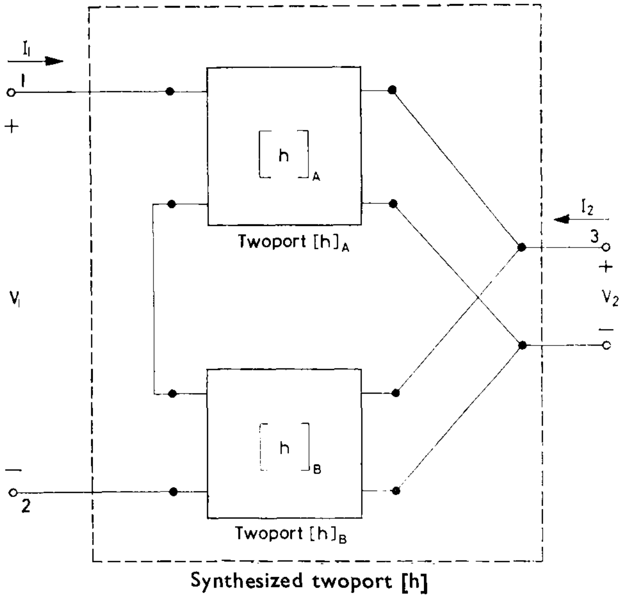


FIG. 4.13. Interconnection of twoports with *h* parameter matrices.

defined by an array of h parameter matrix, $[h]_A$ and $[h]_B$ respectively, where

$$[h]_A = \begin{bmatrix} h_{11a} & h_{12a} \\ h_{21a} & h_{22a} \end{bmatrix} \tag{4.36}$$

and

$$[h]_B = \begin{bmatrix} h_{11b} & h_{12b} \\ h_{21b} & h_{22b} \end{bmatrix}. \tag{4.37}$$

The composite system which has been synthesized by the process of interconnection, is again completely defined by the algebraic sum of the separate matrices, that is

$$[h] = [h]_A + [h]_B. \tag{4.38}$$

Writing in fully-expanded form,

$$[h] = \begin{bmatrix} (h_{11a} + h_{11b}) & (h_{12a} + h_{12b}) \\ (h_{21a} + h_{21b}) & (h_{22a} + h_{22b}) \end{bmatrix}. \tag{4.39}$$

The student may by now appreciate that the rules of consistent twoport interconnections are quite simple and can be utilized with profit for mathematical processing of a large array of electronic problems. As an introduction to a typical but simplified feedback system study, let us consider the linear model of a feedback amplifier as represented with the block diagram in Fig. 4.14. For the purpose of this exercise, it is quite legitimate to assume that the active twoport is a unilateral device, for example a high-frequency transistor, which is operated as an audio-frequency amplifier. Therefore, we may state that the reverse parameter $h_{12a} = 0$. Furthermore, by assuming the correct phase relations, negative feedback is established by the passive twoport $[h]_B$ with the impedance configurations as shown.

Since we have assigned physical impedance parameters to the feedback network, it will be necessary to express the matrix $[h]_B$ in terms of the impedances Z_1 and Z_2. This can be accomplished through some simple algebraic manipulations. Let us consider the twoport equilibrium equations in

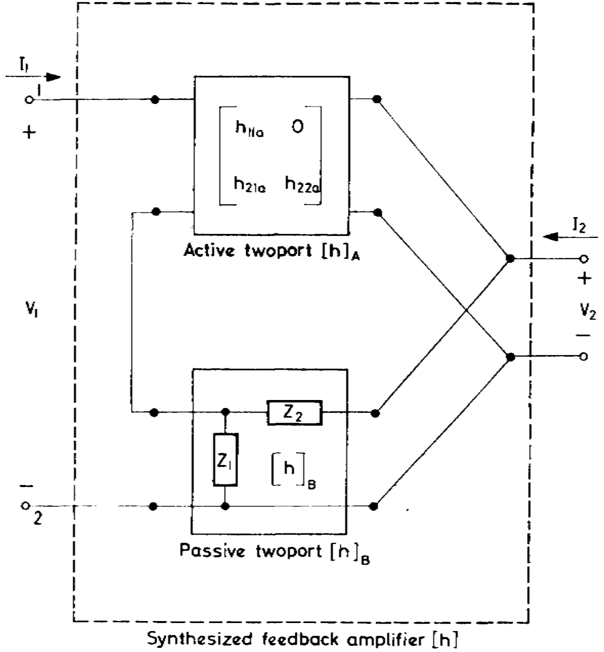


FIG. 4.14. Feedback amplifier synthesized from linear models $[h]_A$ and $[h]_B$.

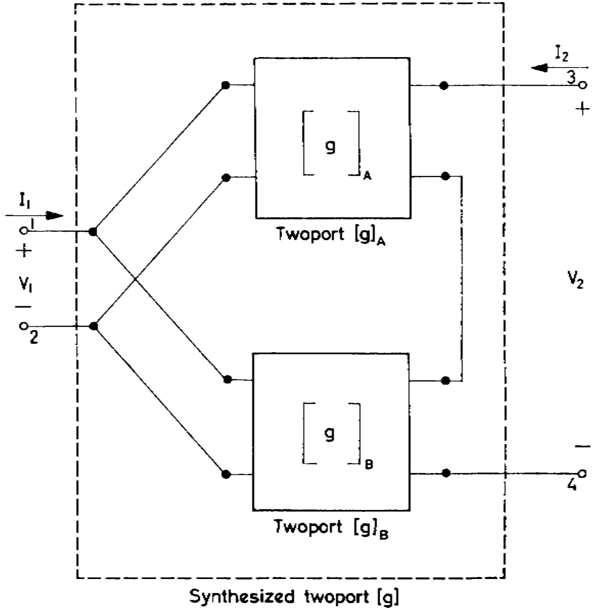


FIG. 4.15. Interconnection of twoports with g parameter matrices

terms of h parameters:

$$V_1 = h_{11}I_1 + h_{12}V_2, \tag{4.40}$$

$$I_2 = h_{21}I_1 + h_{22}V_2. \tag{4.41}$$

Apply eqns. (4.40) and (4.41) to the feedback network $[h]_B$ in Fig. 4.14, and ignore the active twoport $[h]_A$. By means of elementary algebra and network theory we obtain;

$$h_{11b} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{Z_1Z_2}{Z_1+Z_2}, \tag{4.42}$$

$$h_{12b} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{-Z_1}{Z_1+Z_2}, \tag{4.43}$$

$$h_{21b} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{Z_1}{Z_1+Z_2}, \tag{4.44}$$

$$h_{22b} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{Z_1+Z_2}. \tag{4.45}$$

Thus, using eqns. (4.42) to (4.45) inclusive, we can write down for the feedback network the expanded matrix

$$[h]_B = \begin{bmatrix} \dfrac{Z_1Z_2}{Z_1+Z_2} & -\dfrac{Z_1}{Z_1+Z_2} \\ \dfrac{Z_1}{Z_1+Z_2} & \dfrac{1}{Z_1+Z_2} \end{bmatrix}. \tag{4.46}$$

Finally, the h matrix of the complete feedback system is obtained by substituting into eqn. (4.38) the expanded matrices for $[h]_A$ and $[h]_B$ respectively.

$$[h] = \begin{bmatrix} \overbrace{\left(h_{11a}+\dfrac{Z_1Z_2}{Z_1+Z_2}\right)}^{h_{11}} & \overbrace{-\dfrac{Z_1}{Z_1+Z_2}}^{h_{12}} \\ \underbrace{\left(h_{21a}+\dfrac{Z_1}{Z_1+Z_2}\right)}_{h_{21}} & \underbrace{\left(h_{22a}+\dfrac{1}{Z_1+Z_2}\right)}_{h_{22}} \end{bmatrix}. \tag{4.47}$$

(iv) The *g* matrix interconnection

In Chapter 2 we have already dealt with the physical interpretation of the g parameters. The reader may easily verify that as far as terminal conditions are concerned, the element g_{11} has the dimensions of admittance and the element g_{22} that of an impedance. Therefore a valid interconnection of twoports which are defined in terms of the g matrix, will require the parallel connection of input and series connection of output terminals. Such a consistent interconnection is shown in the block diagram of Fig. 4.15. The mathematical model of the system formed by the interconnection will again be a 2×2 matrix, the elements of which are formed again by a term-by-term algebraic summation of the elements in $[g]_A$ and $[g]_B$ respectively.

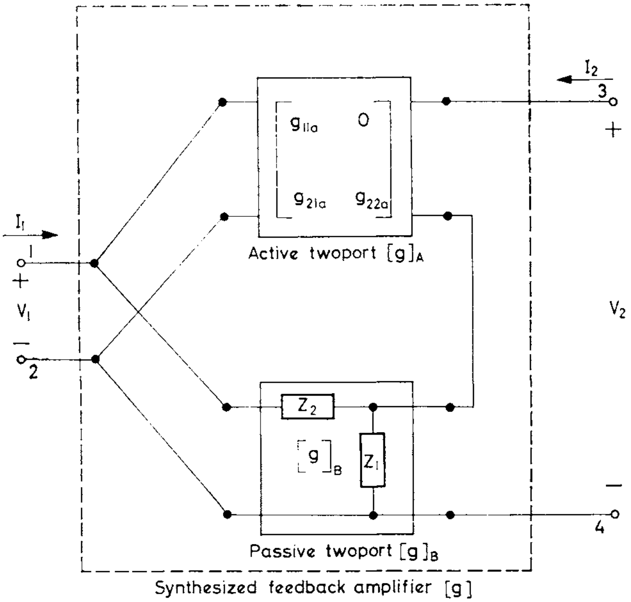


FIG. 4.16. Feedback amplifier synthesized from active and passive twoports in the g matrix domain.

Hence, by definition,

$$[g] = [g]_A + [g]_B. \qquad (4.48)$$

In expanded form

$$[g] = \begin{bmatrix} (g_{11a}+g_{11b}) & (g_{12a}+g_{12b}) \\ (g_{21a}+g_{21b}) & (g_{22a}+g_{22b}) \end{bmatrix}. \qquad (4.49)$$

This expression is also completely general and of fundamental importance to network and feedback system analysis.

As a useful exercise, let us assign the role of an active unilateral twoport to the "black box" $[g]_A$ and that of a passive feedback network to the box $[g]_B$ as shown in Fig. 4.16. It will be now requried to define the elements in the matrix $[g]_B$ in terms of the impedances Z_1 and Z_2. In reverting to Chapter 2, we find the pair of appropriate equilibrium equations of our immediate interest, namely

$$I_1 = g_{11}V_1 + g_{12}I_2, \qquad (4.50)$$

$$V_2 = g_{21}V_1 + g_{22}I_2. \qquad (4.51)$$

If we ignore the active twoport for the time being, then by a straightforward algebraic proces we can derive from eqns. (4.50) and (4.51) the elements in the feedback network matrix $[g]_B$ as follows:

$$g_{11b} = \frac{I_1}{V_1}\bigg|_{I_2=0} = \frac{1}{Z_1+Z_2}, \qquad (4.52)$$

$$g_{12b} = \frac{I_1}{I_2}\bigg|_{V_1=0} = \frac{Z_1}{Z_1+Z_2}, \qquad (4.53)$$

$$g_{21b} = \frac{V_2}{V_1}\bigg|_{I_2=0} = -\frac{Z_1}{Z_1+Z_2}, \qquad (4.54)$$

$$g_{22b} = \frac{V_2}{I_2}\bigg|_{V_1=0} = \frac{Z_1Z_2}{Z_1+Z_2}. \qquad (4.55)$$

Thus, from eqns. (4.52) to (4.55) inclusive, we may write down the g matrix of the feedback network $[g]_B$;

$$[g]_B = \begin{bmatrix} \dfrac{1}{Z_1+Z_2} & \dfrac{Z_1}{Z_1+Z_2} \\ -\dfrac{Z_1}{Z_1+Z_2} & \dfrac{Z_1Z_2}{Z_1+Z_2} \end{bmatrix}. \qquad (4.56)$$

Note that in this particular case the matrix (4.56) is the inverse of the matrix (4.46), derived earlier for the h matrix version of feedback twoport. Therefore, it would appear that we could also obtain $[g]_B$ by the inversion of $[h]_B$.

We can now synthesize the simplified mathematical model of our feedback system. This is carried out by substituting into eqn. (4.49) the elements from the matrix (4.56):

$$[g] = \begin{bmatrix} \overbrace{\left(g_{11a} + \frac{1}{Z_1 + Z_2}\right)}^{g_{11}} & \overbrace{\frac{Z_1}{Z_1 + Z_2}}^{g_{12}} \\ \underbrace{\left(g_{21a} - \frac{Z_1}{Z_1 + Z_2}\right)}_{g_{21}} & \underbrace{\left(g_{22a} + \frac{Z_1 Z_2}{Z_1 + Z_2}\right)}_{g_{22}} \end{bmatrix}. \tag{4.57}$$

As already pointed out, the g matrix interconnection is of basic importance in network analysis and feedback amplifier design. Some practical applications will be studied in greater depth in later parts of this text.

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1. IMPEDANCE TRANSFORMATION AND IMAGE MATCHING, GENERAL

As AN introduction to the mathematics of impedance transformation, a brief definition of the concept of image matching seems to be desirable. Therefore let us consider a somewhat idealized but sufficiently descriptive case which is illustrated in Fig. 1.1. We have there a generator with an internal impedance Z_G which is connected through some medium to a load Z_L. If the interconnecting medium between the terminal pairs 1–2 and 3–4 has negligible losses, then we can state that an optimum power transfer to the load is achieved if

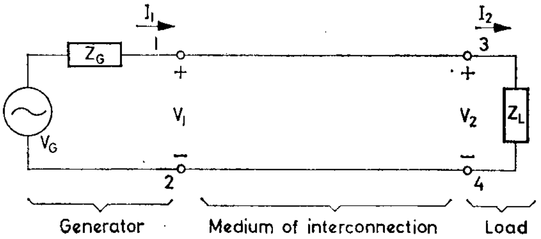

FIG. 1.1. Direct interconnection of generator and load.

$$Z_G = Z_L^*, \tag{1.1}$$

that is, generator and load have conjugate impedances.(1, 2, 3) From the theory of complex numbers, this equality exists only if

$$R_G = R_L \tag{1.2}$$

and

$$X_G = X_L, \tag{1.3}$$

where R_G and R_L are the resistive parts and X_G and X_L are the reactive parts of the generator and load impedances respectively.

We are concerned here primarily with impedance matching as applied to physical engineering problems. Therefore the above idealized situation can hardly be expected to be adequate. In reality the generator may be disguised as an electromechanical or electro-optical transducer, or the output terminals of a transistor, or a vacuum tube amplifier, etc. Similarly, for the load we may visualize a spectrum filter, or the input port of a transistor amplifier or a radiating antenna. Consequently, more often than not the medium of interconnections is by no means negligible. Therefore when we focus our attention on impedance matching techniques in the practical sense, we will endeavour to develop a generalized mathematical procedure of optimizing power transfer from a source or generator, to an arbitrary load. In doing so it will be convenient to represent our linear model as shown in Fig. 1.2, where, the box designated as impedance matching device, may also absorb the spurious reactances of the interconnecting medium.

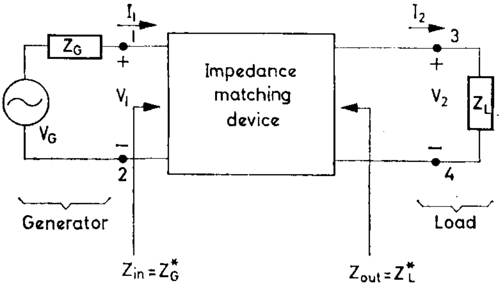

FIG. 1.2. Interconnection of generator and load with image impedance matching.

With reference to Fig. 1.2 generally, we can assume that $Z_G \neq Z_L^*$. However, by interposing a properly designed impedance matching device, we can easily accomplish that both generator and load simultaneously look into image matched terminations:

$$Z_G = Z_{\text{in}}^* \tag{1.4}$$

and

$$Z_L = Z^*_{\text{out}}. \tag{1.5}$$

These relations define the image matching and consequently satisfy the requirements of optimum power transfer. Since in this part of the text weare concerned only with image-matched terminal conditions, therefore, for the linear model in Fig. 1.2, we are at liberty to stipulate that

$$Z_G = Z_{\text{in}} \tag{1.6}$$

and

$$Z_L = Z_{\text{out}}. \tag{1.7}$$

Familiar with the basic matrix techniques in Part I, the student is now well equipped with the prerequisite tools for successfully tackling the important and practical aspects of impedance transformation. We can visualize impedance matching devices in many forms, passive and active. But as far as input and output terminal pairs are concerned, we can always manipulate them into a twoport system. Therefore, it seems to be logical that the mathematical model of a 2×2 matrix will completely define the transfer characteristics. In the following pages we hope to demonstrate that this is indeed so.

2. IMAGE IMPEDANCE MATCHING OF PASSIVE AND ACTIVE TWOPORTS

(a) IMAGE IMPEDANCE MATCHING IN TERMS OF THE *ABCD* MATRIX

In Part I we have already derived the precise expressions for the transfer characteristics and terminal impedances of twoports in terms of the general *ABCD* parameters. The pertinent equilibrium equations have been defined as

$$V_1 = AV_2 + BI_2, \tag{2.1}$$

$$I_1 = CV_2 + DI_2, \tag{2.2}$$

where the coefficients A, B, C and D have the following interpretations;

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \tag{2.3}$$

$$B = \left.\frac{V_1}{I_2}\right|_{V_2=0}, \tag{2.4}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}, \tag{2.5}$$

$$D = \left.\frac{I_1}{I_2}\right|_{V_2=0}. \tag{2.5a}$$

We have solved eqns. (2.1) and (2.2) for input and output impedances of the terminated twoport. The results are reproduced here;

$$Z_{\text{in}} = \frac{AZ_L + B}{CZ_L + D}, \tag{2.6}$$

$$Z_{\text{out}} = \frac{DZ_G + B}{CZ_G + A}. \tag{2.7}$$

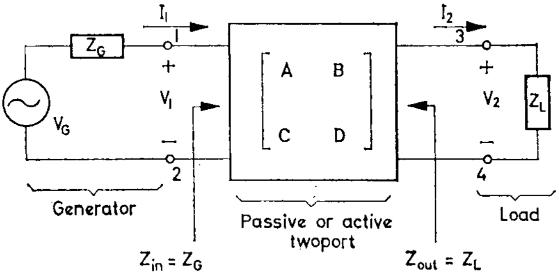

FIG. 2.1. Linear model of image-matched twoport with *ABCD* parameter matrices.

It will be convenient now to redraw the terminated twoport as shown in Fig. 2.1 and stipulate that both generator and load terminations are image matched. Therefore, by definition,

$$Z_{\text{in}} = Z_G$$

and

$$Z_{\text{out}} = Z_L.$$

Since the above conditions hold, we can redefine the left-hand part of eqns. (2.6) and (2.7):

$$Z_G = \frac{AZ_L + B}{CZ_L + D}, \tag{2.8}$$

$$Z_L = \frac{DZ_G + B}{CZ_G + A}. \tag{2.9}$$

We have to solve now these equations simultaneously for Z_G and Z_L respectively.

From eqn. (2.8) by cross-multiplication,.

$$CZ_LZ_G + DZ_G = AZ_L + B. \tag{2.10}$$

Similarly, cross-multiplying eqn. (2.9),

$$CZ_LZ_G + AZ_L = DZ_G + B. \tag{2.11}$$

From eqns. (2.10) and (2.11), by transposition and rearranging terms,

$$CZ_LZ_G + DZ_G - AZ_L - B = 0 \tag{2.12}$$

and

$$CZ_LZ_G - DZ_G + AZ_L - B = 0. \tag{2.13}$$

By addition of eqn. (2.12) to eqn. (2.13),

$$2CZ_LZ_G - 2B = 0, \tag{2.14}$$

$$CZ_LZ_G = B, \tag{2.15}$$

$$Z_LZ_G = \frac{B}{C}. \tag{2.16}$$

Next, subtract eqn. (2.13) from eqn. (2.12):

$$2DZ_G - 2AZ_L = 0, \tag{2.17}$$

$$DZ_G = AZ_L, \tag{2.18}$$

$$\frac{Z_G}{Z_L} = \frac{A}{D}. \tag{2.19}$$

Now multiply eqn. (2.16) by eqn. (2.19):

$$Z_LZ_G\left\{\frac{Z_G}{Z_L}\right\} = \left\{\frac{B}{C}\right\}\left\{\frac{A}{D}\right\} \tag{2.20}$$

$$Z_G^2 = \frac{BA}{CD} \tag{2.21}$$

$$\boxed{Z_G = \sqrt{\left(\frac{BA}{CD}\right)}} \tag{2.22}$$

These are the constraints for an image- matched generator. In order to obtain a corresponding form for the load termination, this time we divide eqn. (2.16) by eqn. (2.19):

$$Z_LZ_G\left\{\frac{Z_L}{Z_G}\right\} = \left\{\frac{B}{C}\right\}\left\{\frac{D}{A}\right\}, \tag{2.23}$$

$$Z_L^2 = \frac{BD}{CA}, \tag{2.24}$$

$$\boxed{Z_L = \sqrt{\left(\frac{BD}{CA}\right)}} \tag{2.25}$$

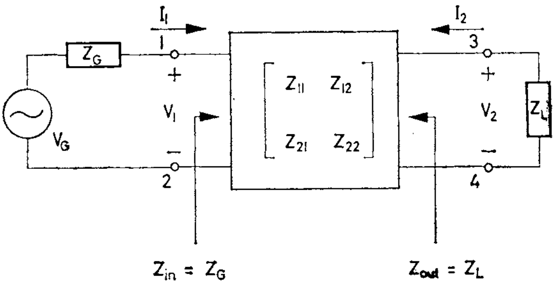

FIG. 2.2. Linear model of the image- matched twoport with Z parameter matrices.

Hence, for image matched conditions of the linear model in Fig. 2.1, eqns. (2.22) and (2.25) must be simultaneously satisfied.

(b) IMAGE IMPEDANCE MATCHING IN TERMS OF THE Z PARAMETER MATRIX

Consider a terminated twoport and stipulate that input and output terminal pairs are image matched in terms of the Z parameter matrix. We can readily construct the corresponding block diagram with the notations shown in Fig. 2.2.

By definition, we can now revert to eqns. (2.15) and (2.21) in Part I:

$$Z_G = \frac{\Delta_Z + Z_{11}Z_L}{Z_{22} + Z_L} \tag{2.26}$$

and

$$Z_L = \frac{\Delta_Z + Z_{22}Z_G}{Z_{11} + Z_G}\,. \tag{2.27}$$

Equations (2.26) and (2.27) have to be solved now simultaneously for Z_G and Z_L respectively.

From eqns. (2.26) and (2.27) by cross multiplication,

$$Z_{22}Z_G + Z_GZ_L = \Delta_Z + Z_LZ_{11} \tag{2.28}$$

and

$$Z_LZ_{11} + Z_GZ_L = \Delta_Z + Z_{22}Z_G\,. \tag{2.29}$$

Transpose and rearrange terms:

$$Z_G Z_L + Z_{22} Z_G - \varDelta_Z - Z_L Z_{11} = 0 \tag{2.30}$$

and

$$Z_G Z_L - Z_{22} Z_G - \varDelta_Z + Z_L Z_{11} = 0. \tag{2.31}$$

Add eqn. (2.30) to eqn. (2.31):

$$2Z_G Z_L - 2\varDelta_Z = 0, \tag{2.32}$$

$$Z_G Z_L = \varDelta_Z. \tag{2.33}$$

Next subtract eqn. (2.31) from eqn. (2.30):

$$2Z_{22} Z_G - 2Z_L Z_{11} = 0, \tag{2.34}$$

$$\frac{Z_G}{Z_L} = \frac{Z_{11}}{Z_{22}}. \tag{2.35}$$

Now multiply eqn. (2.35) by eqn. (2.33):

$$Z_G^2 = \varDelta_Z \frac{Z_{11}}{Z_{22}}, \tag{2.36}$$

$$\boxed{Z_G - \sqrt{\left(\varDelta_Z \frac{Z_{11}}{Z_{22}}\right)}} \tag{2.37}$$

Finally, divide eqn. (2.33) by eqn. (2.35):

$$Z_L^2 = \varDelta_Z \frac{Z_{22}}{Z_{11}}, \tag{2.38}$$

$$\boxed{Z_L = \sqrt{\left(\varDelta_Z \frac{Z_{22}}{Z_{11}}\right)}} \tag{2.39}$$

Hence, for image-matched conditions, eqns. (2.37) and (2.39) must be simultaneously satisfied.

(c) IMAGE IMPEDANCE MATCHING IN TERMS OF THE *h* PARAMETER MATRIX

Expressions for the input and output impedances in terms of the h parameter matrix have been derived in Part I. Therefore we may revert to eqns. (2.52) and (2.59). By stipulating image matched conditions, we can write

$$Z_{\text{in}} = Z_G = \frac{\Delta_h + h_{11} Y_L}{h_{22} + Y_L} \tag{2.40}$$

and

$$Z_{\text{out}} = Z_L = \frac{h_{11} + Z_G}{\Delta_h + h_{22} Z_G}. \tag{2.41}$$

Since in consistent notations the output port has the dimensions of admittance, therefore, by definition for the load, we have

$$Y_L = \frac{1}{Z_L}.$$

By inverting eqn. (2.41), we obtain

$$Y_L = \frac{\Delta_h + h_{22} Z_G}{h_{11} + Z_G}. \tag{2.42}$$

Now, eqns. (2.40) and (2.42) are a consistent pair as demanded by the structure of the h matrix. The corresponding block diagram is shown in Fig. 2.3.

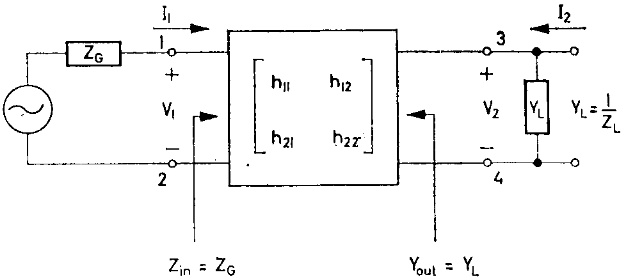

FIG. 2.3. Linear model of image-matched twoport with h parameter matrices.

The image matched conditions of the twoport are obtained if we solve simultaneously eqns. (2.40) and (2.42) for Z_G and Y_L. Therefore, by cross-multiplying eqns. (2.40) and (2.42),

$$h_{22}Z_G + Y_L Z_G = \Delta_h + h_{11}Y_L \tag{2.43}$$

and

$$h_{11}Y_L + Y_L Z_G = \Delta_h + h_{22}Z_G. \tag{2.44}$$

By transposing and rearranging terms of eqn. (2.43),

$$Y_L Z_G + h_{22}Z_G - \Delta_h - h_{11}Y_L = 0. \tag{2.45}$$

Similarly, from eqn. (2.44),

$$Y_L Z_G - h_{22}Z_G - \Delta_h + h_{11}Y_L = 0. \tag{2.46}$$

By adding eqn. (2.45) to eqn. (2.46),

$$2Y_L Z_G - 2\Delta_h = 0, \tag{2.47}$$

$$Y_L Z_G = \Delta_h, \tag{2.48}$$

Next, subtracting eqn. (2.46) from eqn. (2.45),

$$2h_{22}Z_G - 2h_{11}Y_L = 0, \tag{2.49}$$

$$\frac{Z_G}{Y_L} = \frac{h_{11}}{h_{22}}. \tag{2.50}$$

Now multiply eqn. (2.48) by eqn. (2.50):

$$Z_G^2 = \Delta_h \frac{h_{11}}{h_{22}}, \tag{2.51}$$

$$\boxed{Z_G = \sqrt{\left(\Delta_h \frac{h_{11}}{h_{22}}\right)}} \tag{2.52}$$

Finally, for the output port matching, divide eqn. (2.48) by eqn. (2.50):

$$Y_L^2 = \Delta_h \frac{h_{22}}{h_{11}}, \tag{2.53}$$

$$\boxed{Y_L = \sqrt{\left(\Delta_h \frac{h_{22}}{h_{11}}\right)}} \tag{2.54}$$

Equations (2.52) and 2.54) must be simultaneously satisfied for image matched conditions of the twoport in Fig. 2.3.

(d) IMAGE IMPEDANCE MATCHING IN TERMS OF ARBITRARY MATRIX PARAMETERS

In the course of our study of matrix transformations, Chapter 3 of Part I, we have found that the Y matrix is the inverse of the Z matrix. Similarly, the g matrix is the inverse of the h matrix. Therefore, by making use of the logical concepts of duality, from eqns. (2.37) and (2.39) we can write down the conditions of image matching in terms of the Y parameter matrix

$$\boxed{Y_G = \sqrt{\left(\Delta_Y \frac{Y_{11}}{Y_{22}}\right)}} \tag{2.55}$$

and for the load

$$\boxed{Y_L = \sqrt{\left(\Delta_Y \frac{Y_{22}}{Y_{11}}\right)}} \tag{2.56}$$

With similar reasoning as above, from eqns. (2.52) and (2.54), we obtain for image matching in terms of the g matrix parameters

$$\boxed{Y_G = \sqrt{\left(\Delta_g \frac{g_{11}}{g_{22}}\right)}} \tag{2.57}$$

and

$$\boxed{Z_L = \sqrt{\left(\Delta_g \frac{g_{22}}{g_{11}}\right)}} \tag{2.58}$$

A digression of this chapter will reveal that if consistent matrix notations are adhered to, then the form of defining equations of image impedance matching are essentially the same whether we chose Z, Y, h or g matrices.

Therefore if M stands for a generalized immitance matrix, we have, by definition of the conditions for image match of the generator,

$$\boxed{M_G = \sqrt{\left(\Delta_m \frac{m_{11}}{m_{22}}\right)}} \tag{2.59}$$

Similarly, the conditions for image match of the load:

$$\boxed{M_L = \sqrt{\left(\varDelta_m \frac{m_{22}}{m_{11}}\right)}} \tag{2.60}$$

where

$\varDelta_m = (m_{11}m_{22}-m_{12}m_{21})$; determinant of the matrix.

3. APPLICATION OF THE *Z* MATRIX TO IMAGE MATCHING OF A COMPLEX LOAD AND AN RF AMPLIFIER

(a) DEFINITION OF THE PROBLEM

The matching of a complex source impedance to a complex load is perhaps one of the most fundamental and classical problems in electronic engineering.(1, 2, 3) The special case of antenna impedance matching to radio-frequency (RF) amplifiers has been extensively treated in the published literature, which covers some excellent analytical and graphical techniques.(4, 5, 6) We intend to approach here the antenna matching problem from yet a different angle. It will be shown that matrix methods offer significant advantages in handling the mathematics and obtaining exact results in this process.

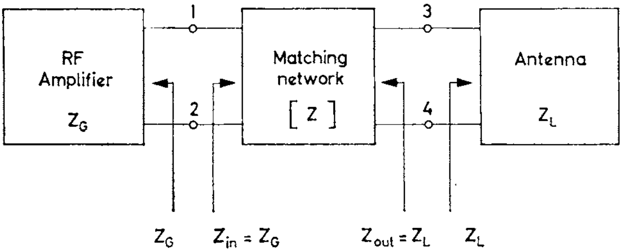

FIG. 3.1. Block diagram of the antenna matching problem.

A pictorial statement of the antenna matching problem is represented with the block diagram of Fig. 3.1. The "black boxes", stand for the RF amplifier and the antenna respectively. The physical arrangement of impedances or admittances inside the boxes may be of arbitrary configuration. However, as far as the accessible terminal pairs 1–2 and 3–4 are concerned, we stipulate that they look into conjugate impedances of Z_G and Z_L respectively. The matching network which appears to be "embedded" between Z_G and Z_L is therefore called upon to present an image match across input and output ports.

For the purpose of this particular problem it is convenient to treat both source and load as series impedances, although physically they may contain parallel elements. It can be shown that parallel connected impedances may be represented by an equivalent series combination and vice-versa. The algebraic process of series–parallel transformation is an extremely useful analytical tool for dealing with impedance matching problems.

Consider a parallel and a series form of complex generator as shown in Fig. 3.2a and b. We may wish to specify that, when looking into the terminals 1–2, the impedances and phase angles are identical, that is

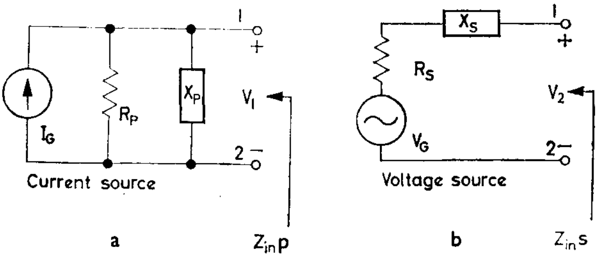

FIG. 3.2a,b. Equivalent complex generators—parallel and series representation.

$$Z_{\text{in},\text{p}} = Z_{\text{in},\text{s}}. \tag{3.1}$$

Consequently, we may write down the following equilibrium equation:

$$R_s + jX_s = \frac{jX_p R_p}{R_p + jX_p}, \tag{3.2}$$

where R_s and R_p are the series and parallel resistances, jX_s and jX_p are the series and parallel reactances respectively.

Multiplying through eqn. (3.2) with R_p+jX_p and equating separately the real and imaginary parts, we obtain by simple algebraic process the following set of transformations.

$$R_p = R_s\left\{1+\left(\frac{X_s}{R_s}\right)^2\right\}, \tag{3.3}$$

$$X_p = X_s\left\{1+\left(\frac{R_s}{X_s}\right)^2\right\}, \tag{3.4}$$

and

$$R_s = \frac{R_p}{\{1+(R_p/X_p)^2\}}, \tag{3.5}$$

$$X_s = \frac{X_p}{\{1+(X_p/R_p)^2\}}. \tag{3.6}$$

Having established precise mathematical relations between the parallel and series network configurations, we are now at liberty to redraw our block diagram in Fig. 3.1, to a somewhat more informative configuration as shown in Fig. 3.3. We have assumed a capacitive series reactance $-jX_L$ associated with the load, and series reactance $+jX_G$ associated with the generator. Naturally, other combinations are also

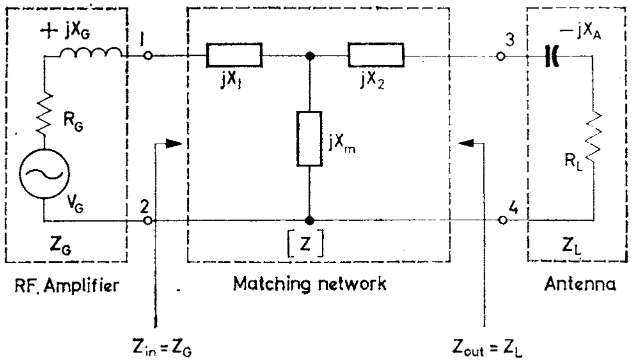

FIG. 3.3. Linear model of the antenna matching problem.

possible. The physical matching network is a T structure which is made up from yet unspecified reactances X_1, X_2, and X_m.

(b) SYNTHESIS OF MATHEMATICAL MODEL, CONDITIONS OF IMAGE MATCHING

From our linear model in Fig. 3.3 we can proceed to the synthesis of a suitable matrix form which may be utilized for deriving the optimum parameters of the T network. We recognize here that the series–series type of interconnection calls for the Z matrix. The reader is therefore referred to eqns. (2.37) and (2.39) in the preceding chapter, defining the condition of simultaneous image impedance matching in terms of the Z matrix parameters. In accordance with our definition of conjugate image matching, the twoport T structure in Fig. 3.3 must satisfy the following terminal conditions:

$$Z_{\text{in}} = R_G - jX_G \tag{3.7}$$

and

$$Z_{\text{out}} = R_L + jX_A. \tag{3.8}$$

It may be further assumed that the T network is made up from purely reactive elements, since resistive losses may be lumped together with R_G and R_L.

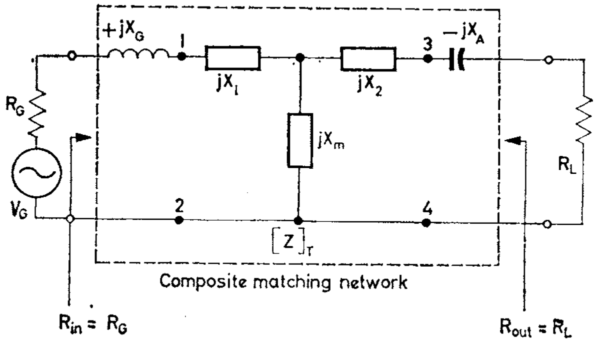

FIG. 3.4. Modified impedance matching problem, the T network absorbs generator and load reactances.

From the analytical point of view, it is quite legitimate to combine the series reactances of the generator and the load with the series arms of the T network. Accordingly, the modified linear model is shown in Fig. 3.4. It will be apparent that by this artifice our problem has been simplified to that of optimum power transfer between a resistive generator R_G and a resistive load R_L. Since the reactances jX_G and $-jX_A$ are now combined with the series arms of the T network, it will be possible to write down by inspection the Z matrix of this new mathematical model:

$$[Z]_T = \begin{bmatrix} \overbrace{j(X_G + X_1 + X_m)}^{Z_{11}} & \overbrace{jX_m}^{Z_{12}} \\ \underbrace{jX_m}_{Z_{21}} & \underbrace{j(-X_A + X_2 + X_m)}_{Z_{22}} \end{bmatrix}. \qquad (3.9)$$

We may now revert to eqns. (2.37) and (2.39). The conditions of conjugate impedance matching is satisfied if

$$R_G = \sqrt{\left(\Delta_Z \frac{Z_{11}}{Z_{22}}\right)} \qquad (3.10)$$

and

$$R_L = \sqrt{\left(\Delta_Z \frac{Z_{22}}{Z_{11}}\right)}, \qquad (3.11)$$

where Z_{11}, Z_{22} and Δ_Z are applicable to the terms of the matrix (3.9).

By simple algebraic manipulations of eqns. (3.10), (3.11) and the matrix (3.9), the parameters X_1, X_2 and X_m can be derived.

First, multiply eqn. (3.10) by eqn. (3.11):

$$R_G R_L = \left\{\sqrt{\left(\Delta_Z \frac{Z_{11}}{Z_{22}}\right)}\right\} \left\{\sqrt{\left(\Delta_Z \frac{Z_{22}}{Z_{11}}\right)}\right\} \qquad (3.12)$$

$$R_G R_L = \Delta_Z . \qquad (3.13)$$

Next, expand the determinant of matrix (3.9) and substitute the results into the right-hand part of eqn. (3.13):

$$R_G R_L = -(X_G + X_1 + X_m)(-X_A + X_2 + X_m) + X_m^2. \quad (3.14)$$

Recall, that according to the definition of conjugate impedance matching, the algebraic sum of reactances looking into the terminals 1–2 and 3–4 of the composite network in Fig. 3.4, must add up to zero. Or, in other words, the elements Z_{11} and Z_{22} of the matrix (3.9) should be simultaneously equal to zero.

Therefore we obtain from eqns. (3.9) or (3.14)

$$(X_G + X_1 + X_m) = 0, \quad (3.15)$$

$$(-X_A + X_2 + X_m) = 0. \quad (3.16)$$

If eqns. (3.15) and (3.16) are satisfied, then eqn. (3.14) reduces to

$$R_G R_L = X_m^2, \quad (3.17)$$

$$\boxed{X_m = \sqrt{(R_G R_L)}} \quad (3.17a)$$

Reverting to eqn. (3.15), by transposition,

$$\boxed{X_1 = -X_G - X_m} \quad (3.18)$$

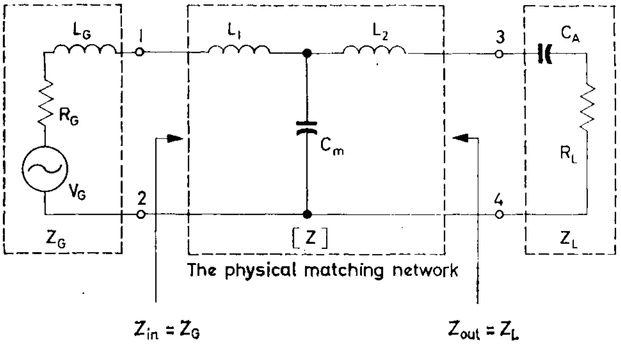

FIG. 3.5. Practical configuration of the antenna- matching *T* network.

Similarly, from eqn. (3.16),

$$\boxed{X_2 = X_A - X_m} \tag{3.19}$$

If we now assign the desirable low-pass structure to our matching network, we can redraw the linear model to its final and practical form as in Fig. 3.5. The parameters of L_1, L_2 and C_m are easily obtained from eqns. (3.17), (3.18) and (3.19) respectively.

From eqn. (3.17):

$$X_m = \frac{1}{\omega C_m} = \sqrt{(R_G R_L)}, \tag{3.20}$$

$$C_m = \frac{1}{\omega \sqrt{(R_G R_L)}}. \tag{3.21}$$

Similarly, from eqn. (3.18):

$$X_1 = j\omega L_1 = j\frac{1}{\omega C_m} - j\omega L_G, \tag{3.22}$$

$$L_1 = \frac{1}{\omega^2 C_m} - L_G. \tag{3.23}$$

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INTRODUCTORY REMARKS

Active network design, whether it involves vacuum tubes or transistors, may be looked upon essentially as an optimizing procedure. Once the active device has been chosen, we can reduce it to a reasonable approximation of a linear model, then assign the appropriate twoport matrix notations to the elements. Next, one generally decides on the type of most desirable input, output and feedback networks if any. As a rule, these are passive elements, therefore conveniently characterized also with twoport matrix parameters. Finally, by combining the consistent sets of matrices, we have in fact synthesized a new mathematical model. This new model, again, is a 2×2 array and obeying all rules of matrix algebra. In this manner one can reduce a complex electronic system to the familiar twoport linear model. By means of conventional algebraic process or applying the standards forms from the tables in Part I, we can now easily derive the pertinent transfer characteristics and terminal conditions.

The remaining parts of this text will be devoted to a representative range of active network topics. We start off with the modest but very fundamental single-stage amplifiers and then progress gradually to more complex feedback systems. The emphasis will be, of course, on transistor applications, but a substantial coverage of basic techniques on vacuum tube amplifiers and oscillators is also included.

The student will find that there is a clear division of topics. Each chapter has been moulded into the style of a self-contained monograph. By underscoring the "why and how" aspects of all mathematical manipulations, the author has slanted the discussions to the needs of the practising electronics engineer. Adequate attention has been given to the problems of engineering approximations. Matrix forms of exact and approximate design equations have been consolidated in a number of tables which are found in the main body of the text.

1. VACUUM TUBES, LINEAR MODELS AND MATRICES

(a) GENERAL DEFINITIONS

In order to handle efficiently vacuum-tube problems, we require a reasonably accurate linear model of this device. It will be demonstrated that the fundamental and universally accepted resistive model of the vacuum tube can be related to the floating admittance matrix. For an exhaustive discussion of this most elegant mathematical tool and the underlaying philosophy, references (1) to (4) may be consulted. Here we are concerned only with a basic, yet adequately rigorous study of fundamental vacuum- tube circuits. All three models of operation will be covered, that is grounded cathode, grounded grid and grounded plate or cathode follower.

In order to consolidate the general concepts of the floating admittance matrix, refer to the three-terminal network as shown in Fig. 1.1. Where the admittance parameters Y_1, Y_2 and Y_3 are completely general. That is, they can be passive or active, resistive or reactive. Note also, that there is no common reference node. We also know that Kirchoff's law states that the algebraic sum of currents entering and leaving a nodal point must add up to zero. Therefore, applying this law to our three-terminal model in Fig. 1.1, we can write down the equilibrium equation:

$$I_1 + I_2 + I_3 = 0. \qquad (1.1)$$

The above definition of the floating admittance matrix can be profitably related to the vacuum tube with the terminal notations in Fig. 1.2a. We have included an external grid-to- cathode conductance g_{gk} which is always existing in physical circuits. A corresponding unilateral resistive model is shown in Fig. 1.2b, where the admittance parameters are defined as follows:

$$g_{gk} = \frac{1}{r_g}\,; \quad \text{Grid- to- cathode conductance.}$$

$$g_{pk} = \frac{1}{r_p}\,; \quad \text{Plate- to- cathode conductance.}$$

$$g_m = \frac{\mu}{r_p}\,; \quad \text{Forward transconductance.}$$

$$\mu = \text{Amplification factor.}$$

We have assumed, quite legitimately, a unilateral and resistive model, which is sufficiently accurate for most engineer-

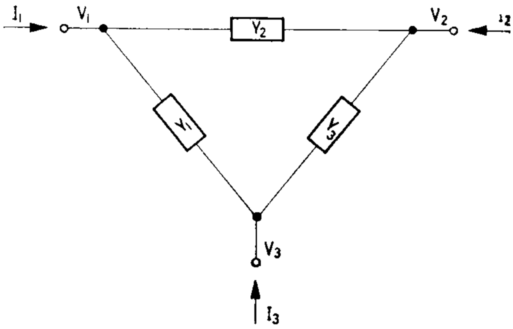

FIG. 1.1. The floating three-terminal admittance network.

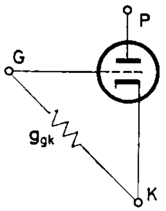

FIG. 1.2a.The floating triode, schematic.

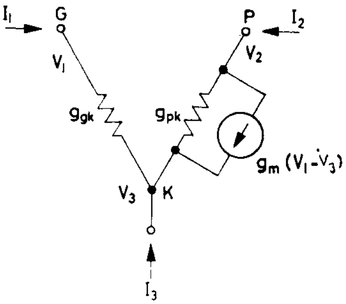

FIG. 1.2b. Linear resistive model of the floating triode.

ing applications. At high frequencies one may wish to include inter-electrode capacitances. Mathematically, this can be accounted for by the simple algebraic addition of admittances as explained in Chapter 4 of Part I.

The nodal eqn. (1.1) stipulates that the algebraic sum of currents entering and leaving any nodal point must be equal to zero. Therefore it would appear to be both logical and instructive to develop the admittance matrices separately for each of the three nodal junctions. The linear superposition theorem permits the synthesis of the complete mathematical model by the algebraic summation of these matrices. We also know, that the linear model with three floating nodal junctions calls for a 3 × 3 matrix. It will be shown, that once we have synthesized the floating admittance matrix, we can readily derive the appropriate matrices of the vacuum tube for any desired mode of ground reference. The algebraic process is deceivingly simple. Zeros are substituted for all elements in the selected row and column. Thus, producing a 2 × 2 matrix, the mathematical model will describe precisely active twoports.

(b) FLOATING ADMITTANCE MATRIX OF THE TRIODE

To begin with the synthesis of our mathematical model refer to Fig. 1.2b. When considering the conductance g_{gk}, the elements of Y_{11} and Y_{33} are generated in a 3 × 3 array:

$$Y_{11} = \frac{I_1}{V_1} = g_{gk}. \tag{1.2}$$

Similarly,

$$Y_{33} = \frac{I_3}{V_3} = g_{gk}. \tag{1.3}$$

Consistency demands that each row or column adds up to zero, which can be satisfied if

$$Y_{13} = Y_{31} = -\frac{I_3}{V_1} = -g_{gk}. \tag{1.4}$$

Therefore, we have the matrix involving g_{gk} only:

$$[Y]_{g_{gk}} = \begin{array}{c|c|c|c|c} & V_1 & V_2 & V_3 & \\ \hline I_1 & g_{gk} & 0 & -g_{gk} & G \\ \hline I_2 & 0 & 0 & 0 & P \\ \hline I_3 & -g_{gk} & 0 & g_{gk} & K \\ \hline & G & P & K & \end{array} \tag{1.5}$$

Next let us consider the plate conductance g_{pk}. We have the corresponding elements defined in a second 3×3 matrix:

$$Y_{22} = Y_{33} = g_{pk}. \tag{1.6}$$

Similarly, for the mutual negative elements,

$$Y_{23} = Y_{32} = -g_{pk}. \tag{1.7}$$

Entering these parameters in the array which relates to g_{pk} only:

$$[Y]_{g_{pk}} = \begin{array}{c|c|c|c|c} & V_1 & V_2 & V_3 & \\ \hline I_1 & 0 & 0 & 0 & G \\ \hline I_2 & 0 & g_{pk} & -g_{pk} & P \\ \hline I_3 & 0 & -g_{pk} & g_{pk} & K \\ \hline & G & P & K & \end{array} \tag{1.8}$$

Finally, for the active generator $g_m(V_1 - V_3)$, the array involves the parameter of g_m only:

$$Y_{21} = Y_{33} = g_m \tag{1.9}$$

and

$$Y_{23} = Y_{31} = -g_m. \tag{1.10}$$

$$[Y]_{g_m} = \begin{array}{c|c|c|c|c} & V_1 & V_2 & V_3 & \\ \hline I_1 & 0 & 0 & 0 & G \\ \hline I_2 & g_m & 0 & -g_m & P \\ \hline I_3 & -g_m & 0 & g_m & K \\ \hline & G & P & K & \end{array} \tag{1.11}$$

The complete floating admittance matrix is next obtained by the algebraic summation of matrices (1.5), (1.8) and (1.11) respectively.

$$[Y]_{\text{triode}} = \begin{array}{c|c|c|c|c} & V_1 & V_2 & V_3 & \\ \hline I_1 & g_{gk} & 0 & -g_{gk} & G \\ \hline I_2 & g_m & g_{pk} & -g_{pk}-g_m & P \\ \hline I_3 & -g_{gk}-g_m & -g_{pk} & g_{gk}+g_{pk}+g_m & K \\ \hline & G & P & K & \end{array} \tag{1.12}$$

Note that the sum of any row or column is zero, precisely, as Kirchoff's eqn. (1.1) stipulates. We can now proceed and derive from this matrix the mathematical models for any of the three operating modes of the triode.

The grounded cathode mode. The cathode is the No. 3 nodal junction. Therefore by collapsing the third row and column in the matrix (1.12), we obtain the twoport admittance matrix, appropriate for grounded cathode operation:

$$[Y]_{GK} = \begin{array}{c|c|c|c} & V_1 & V_2 & \\ \hline I_1 & g_{gk} & 0 & G \\ \hline I_2 & g_m & g_{pk} & P \\ \hline & G & P & \end{array} \tag{1.13}$$

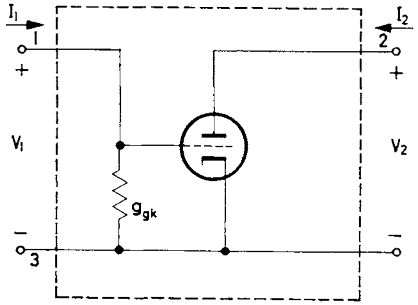

FIG. 1.3. Twoport notation of the grounded- cathode triode.

The corresponding linear model and terminal notation is shown in Fig. 1.3.

The grounded grid mode. Since the grid terminal is the No. 1 node, therefore, by collapsing the first row and column in the floating matrix (1.12), we obtain the twoport admittance matrix of the grounded grid configuration:

$$[Y]_{GG} = \begin{array}{c|c|c|c} & V_2 & V_3 & \\ \hline I_2 & g_{pk} & -g_{pk}-g_m & P \\ \hline I_3 & -g_{pk} & g_{pk}+g_{pk}+g_m & K \\ \hline & P & K & \end{array} \tag{1.14}$$

Note here that the input variables are V_3 and I_3. It is convenient to transpose them by simple operation of reversing the diagonals in the matrix (1.14):

$$[Y]_{GG} = \begin{array}{c|c|c|c} & V_3 & V_2 & \\ \hline I_3 & g_{gk}+g_{pk}+g_m & -g_{pk} & K \\ \hline I_2 & -g_{pk}-g_m & g_{pk} & P \\ \hline & K & P & \end{array} \tag{1.15}$$

The twoport connection corresponding to this matrix is shown in Fig. 1.4.

The grounded plate or cathode follower mode. Here the desired ground reference is the plate terminal and corresponds

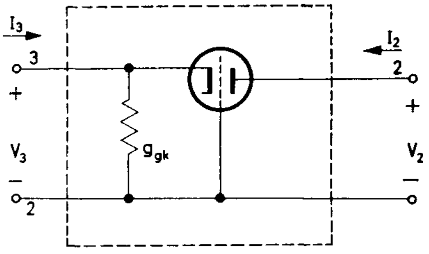

FIG. 1.4. Twoport notation of the grounded- grid triode.

to the nodal junction No. 2. Hence, by collapsing the second row and second column in the matrix (1.12), we obtain the admittance matrix of the cathode follower:

$$[Y]_{GP} = \begin{array}{c|c|c|c} & V_1 & V_3 & \\ \hline I_1 & g_{gk} & -g_{gk} & G \\ \hline I_3 & -g_{gk}-g_m & g_m+g_{gk}+g_{pk} & K \\ \hline & G & K & \end{array} \qquad (1.16)$$

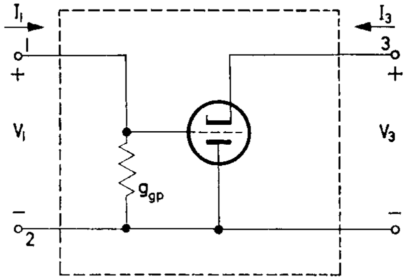

FIG. 1.5a. Twoport notation of the cathode follower.

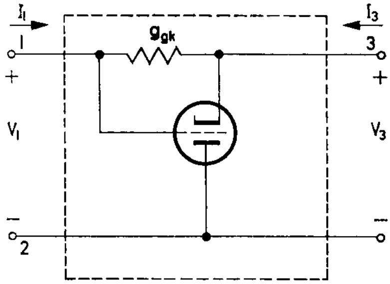

FIG. 1.5b. Practical form of cathode follower twoport notation.

The appropriate twoport connections are shown in Fig. 1.5a. Although this terminal arrangement is formally correct, yet a more practical model is obtained if we rearrange the physical connections of the grid adittance as shown in Fig. 1.5b. Note that we have put $g_{gk} = 0$ and added a new admittance element g_{gp} between grid and plate, which, of course, happens to be the ground reference. Thus g_{gp} will occupy the position of Y_{11} in a new mathematical model:

$$[Y]_{GP} = \begin{array}{c|c|c|c} & V_1 & V_3 & \\ \hline I_1 & g_{gp} & 0 & G \\ \hline I_3 & -g_m & g_m+g_{pk} & K \\ \hline & G & K & \end{array} \qquad (1.17)$$

Having derived the twoport admittance matrices from the 3×3 floating admittance matrix, we are now at liberty to discard the original terminal notations. Therefore we revert to the more convenient 2×2 matrix conventions, where the variables V_1 and I_1 are associated with the input terminals, and V_2 and I_2 with the output, respectively.

(c) TRANSFORMATION OF THE ADMITTANCE MATRIX OF THE TRIODE TO ARBITRARY MATRIX DOMAINS

A digression of Part I will indicate that we can substantially broaden the usefulness of matrix analysis with the powerful tool of transformation techniques. Therefore we are tempted to extend now the mathematical definition of the vacuum triode to the quite useful *Z*, *g*, *h* and *ABCD* matrix domains. The requisite matrix transformations can be carried out rapidly with the aid of tables in Part I. The actual algebraic manipulations involved in transforming the triode *Y* matrix parameters to the *Z* and *ABCD* domain will be worked out in detail. The relevant results are collated in Table III which contains the complete array of matrix forms for the triode.

(i) Transformation to the *Z* matrix form

Revert to Table II in Part I for the appropriate matrix interrelations. At the intersection of the *Y* column and the

Z row, we will find the desired transformation matrix:

$$[Z] = \begin{bmatrix} \frac{Y_{22}}{\Delta_Y} & -\frac{Y_{12}}{\Delta_Y} \\ -\frac{Y_{21}}{\Delta_Y} & \frac{Y_{11}}{\Delta_Y} \end{bmatrix}. \tag{1.18}$$

The grounded cathode operation. We have the Y parameters from the matrix (1.13) as follows:

$$Y_{11} = g_{gk},$$
$$Y_{12} = 0,$$
$$Y_{21} = g_m,$$
$$Y_{22} = g_{pk}.$$

By inspection, the determinant of the matrix (1.13):

$$\Delta_Y = g_{gk} g_{pk}. \tag{1.19}$$

Hence a direct substitution of the above parameters into the matrix (1.18) will yield

$$[Z]_{GK} = \begin{bmatrix} \frac{1}{g_{gk}} & 0 \\ -\frac{g_m}{g_{gk} g_{pk}} & \frac{1}{g_{pk}} \end{bmatrix}. \tag{1.20}$$

By the definition of the triode admittances, we have

$$\frac{1}{g_{gk}} = r_g,$$
$$\frac{1}{g_{pk}} = r_p,$$
$$g_m = \frac{\mu}{r_p}.$$

Hence the matrix (1.20) can be simplified and rewritten in the form

$$[Z]_{GK} = \begin{bmatrix} \overset{Z_{11}}{\overbrace{r_g}} & \overset{Z_{12}}{\overbrace{0}} \\ \underset{Z_{21}}{\underbrace{-\mu r_g}} & \underset{Z_{22}}{\underbrace{r_p}} \end{bmatrix}. \tag{1.21}$$

The grounded grid operation. We have the Y parameters from the matrix (1.15)

$$\begin{aligned} Y_{11} &= (g_{gk}+g_{pk}+g_m), \\ Y_{12} &= -g_{pk}, \\ Y_{21} &= -(g_{pk}+g_m). \\ Y_{22} &= g_{pk}. \end{aligned}$$

The determinant corresponding to these parameters:

$$\begin{aligned} \Delta_Y &= (g_{gk}+g_{pk}+g_m)g_{pk}-(-g_{pk}-g_m)(-g_{pk}) \\ \Delta_Y &= g_{gk}g_{pk}. \end{aligned} \tag{1.22}$$

By direct substitution into the matrix (1.18)

$$[Z]_{GG} = \begin{bmatrix} \dfrac{g_{pk}}{g_{gk}g_{pk}} & \dfrac{g_{pk}}{g_{gk}g_{pk}} \\ \left\{\dfrac{g_{pk}}{g_{gk}g_{pk}}+\dfrac{g_m}{g_{gk}g_{pk}}\right\} & \left\{\dfrac{g_{gk}}{g_{gk}g_{pp}}+\dfrac{g_{pk}}{g_{gk}g_{pk}}+\dfrac{g_m}{g_{gk}g_{pk}}\right\} \end{bmatrix}. \tag{1.23}$$

Rearranging terms and substituting $\mu = g_m/g_{pk}$,

$$[Z]_{GG} = \begin{bmatrix} \dfrac{1}{g_{gk}} & \dfrac{1}{g_{gk}} \\ \left\{\dfrac{1}{g_{gk}}+\dfrac{\mu}{g_{gk}}\right\} & \left\{\dfrac{1}{g_{pk}}+\dfrac{1}{g_{gk}}+\dfrac{\mu}{g_{gk}}\right\} \end{bmatrix}. \tag{1.24}$$

Next, change the admittances to resistances and note that in this particular configuration $1/g_{gk} = r_k$:

$$[Z]_{GG} = \begin{bmatrix} r_k & r_k \\ (r_k+\mu r_k) & (r_k+r_p+\mu r_k) \end{bmatrix}. \tag{1.25}$$

Simplifying and collecting terms will yield the final form for grounded- grid mode of operation:

$$[Z]_{GG} = \begin{bmatrix} \overbrace{r_k}^{Z_{11}} & \overbrace{r_k}^{Z_{12}} \\ \underbrace{(1+\mu)r_k}_{Z_{21}} & \underbrace{r_p+(1+\mu)r_k}_{Z_{22}} \end{bmatrix}. \tag{1.26}$$

The cathode follower operation. From the matrix (1.17) we have the elements identified as

$$\begin{aligned} Y_{11} &= g_{gp}, \\ Y_{12} &= 0, \\ Y_{21} &= -g_m. \\ Y_{22} &= g_{pk}+g_m. \end{aligned}$$

The determinant of the matrix, by inspection,

$$\Delta_Y = g_{gp}(g_{pk}+g_m). \tag{1.27}$$

Performing the appropriate substitutions into the transformation matrix (1.18), we obtain the Z matrix of the grounded plate or the cathode follower connection:

$$[Z]_{GP} = \begin{bmatrix} \dfrac{(g_{pk}+g_m)}{g_{gp}(g_{pk}+g_m)} & 0 \\ \dfrac{g_m}{g_{gp}(g_{pk}+g_m)} & \dfrac{g_{gp}}{g_{gp}(g_{pk}+g_m)} \end{bmatrix}. \tag{1.28}$$

Cancelling terms where applicable,

$$[Z]_{GP} = \begin{bmatrix} \dfrac{1}{g_{gp}} & 0 \\ \dfrac{g_m}{g_{gp}(g_{pk}+g_m)} & \dfrac{1}{(g_{pk}+g_m)} \end{bmatrix}. \tag{1.29}$$

Rewriting eqn. (1.29) by changing the admittances to resistive parameters,

$$[Z]_{GP} = \begin{bmatrix} r_g & 0 \\ \left\{\dfrac{\mu}{r_p}\right\} \dfrac{1}{(1/r_g r_p)+(\mu/r_g r_p)} & \dfrac{1}{(1/r_p)+(\mu/r_p)} \end{bmatrix} \tag{1.30}$$

Simplify by multiplying numerators and denominators by r_g in the position Z_{21} and by r_p in the position Z_{22}:

$$[Z]_{GP} = \begin{bmatrix} \overbrace{r_g}^{Z_{11}} & \overbrace{0}^{Z_{12}} \\ \underbrace{\dfrac{\mu r_g}{(1+\mu)}}_{Z_{21}} & \underbrace{\dfrac{r_p}{(1+\mu)}}_{Z_{22}} \end{bmatrix} \tag{1.31}$$

which is the final and practical form of the cathode follower's Z matrix.

(ii) Transformation to the *ABCD* matrix form

Revert again to Table II in Part I. At the intersection of the Y column and the $ABCD$ row, we locate the required transformation interrelations:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -\frac{Y_{22}}{Y_{21}} & -\frac{1}{Y_{21}} \\ -\frac{\Delta_Y}{Y_{21}} & -\frac{Y_{11}}{Y_{21}} \end{bmatrix}. \tag{1.32}$$

As before, we make the appropriate substitutions from the twoport admittance matrices into the transform matrix above.

The grounded cathode operation. Revert to eqn. (1.13) and substitute the specified admittance parameters into the transform matrix (1.32):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GK} = \begin{bmatrix} -\frac{g_{pk}}{g_m} & -\frac{1}{g_m} \\ -\frac{g_{gk}g_{pk}}{g_m} & -\frac{g_{gk}}{g_m} \end{bmatrix}. \tag{1.33}$$

By definition $g_m = \mu g_{pk}$, therefore- substituting this identity into the C term in the matrix (1.33) and changing admittances to resistances, we obtain the transformed twoport matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GK} = \begin{bmatrix} -\frac{1}{\mu} & -\frac{1}{g_m} \\ -\frac{1}{\mu r_g} & -\frac{1}{g_m r_g} \end{bmatrix}. \tag{1.34}$$

The grounded grid operation. Revert to eqn. (1.15) and substitute the appropriate elements into the transform matrix (1.32):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GG} = \begin{bmatrix} \dfrac{g_{pk}}{g_m+g_{pk}} & \dfrac{1}{g_m+g_{pk}} \\ \dfrac{g_{gk}g_{pk}}{g_m+g_{pk}} & \dfrac{g_{gk}+g_{pk}+g_m}{g_m+g_{pk}} \end{bmatrix}. \tag{1.35}$$

Simplify this expression by changing the admittances to resistances and noting that here $1/g_{gk} = r_k$ and $g_m = \mu/r_p$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GG} = \begin{bmatrix} \dfrac{1}{(1+\mu)} & \dfrac{r_p}{(1+\mu)} \\ \dfrac{1}{(1+\mu)r_k} & \dfrac{r_p/r_k+(1+\mu)}{(1+\mu)} \end{bmatrix}. \tag{1.36}$$

Further simplification is possible by multiplying both numerator and denominator of the D term by r_k:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GG} = \begin{bmatrix} \dfrac{1}{(1+\mu)} & \dfrac{r_p}{(1+\mu)} \\ \dfrac{1}{(1+\mu)r_k} & \dfrac{r_p+(1+\mu)r_k}{(1+\mu)r_k} \end{bmatrix}. \tag{1.37}$$

This is a convenient mathematical model of the grounded grid connected triode in the *ABCD* matrix domain.

The grounded plate or cathode follower operation. Use again the transform matrix (1.32) and substitute the appropriate elements from the admittance matrix (1.17):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GP} = \begin{bmatrix} \dfrac{g_m+g_{pk}}{g_m} & \dfrac{1}{g_m} \\ \dfrac{g_{gp}(g_m+g_{pk})}{g_m} & \dfrac{g_{gp}}{g_m} \end{bmatrix}. \tag{1.38}$$

Simplify with the identity $g_m \equiv \mu g_{pk}$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GP} = \begin{bmatrix} \dfrac{\mu g_{pk}+g_{pk}}{\mu g_{pk}} & \dfrac{1}{g_m} \\ g_{gp}+\dfrac{g_{gp}g_{pk}}{\mu g_{pk}} & \dfrac{g_{gp}}{g_m} \end{bmatrix}. \tag{1.39}$$

Rearrange terms and replace admittances with resistive parameters:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GP} = \begin{bmatrix} \dfrac{(1+\mu)}{\mu} & \dfrac{1}{g_m} \\ \dfrac{1}{r_g}+\dfrac{1}{\mu r_g} & \dfrac{1}{g_m r_g} \end{bmatrix}. \qquad (1.40)$$

Rearrange terms in position C:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{GP} = \begin{bmatrix} \dfrac{(1+\mu)}{\mu} & \dfrac{1}{g_m} \\ \dfrac{(1+\mu)}{\mu r_g} & \dfrac{1}{g_m r_g} \end{bmatrix} \qquad (1.41)$$

which is the transmission matrix of the cathode follower connected triode.

By means of similar algebraic processing as shown above, we obtain the mathematical models of the triode also in the g and h matrix domains. The results are summarized in Table III.

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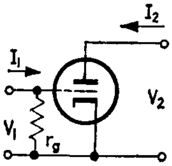
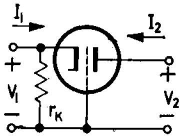
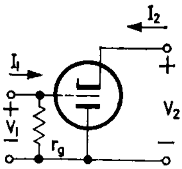
3. A. J. Cote jr. and J. B. Oakes, *Linear Vacuum Tube and Transistor Circuits*, McGraw-Hill, New York, 1961.
4. L. DePian, *Linear Active Network Theory*, Prentice-Hall, Englewood Cliffs, 1962.

TABLE III. TWOPORT MATRICES OF THE TRIODE

Mode of operation	[Z]	[Y]
 Grounded cathode	$\begin{matrix} r_g & 0 \\ -\mu r_g & r_p \end{matrix}$	$\begin{matrix} \frac{1}{r_g} & 0 \\ g_m & \frac{1}{r_p} \end{matrix}$
 Grounded grid	$\begin{matrix} r_k & r_k \\ (1+\mu)r_k & r_p+(1+\mu)r_k \end{matrix}$	$\begin{matrix} \frac{r_p+(1+\mu)r_k}{r_pr_k} & -\frac{1}{r_p} \\ -\frac{(1+\mu)}{r_p} & \frac{1}{r_p} \end{matrix}$
 Cathode follower	$\begin{matrix} r_g & 0 \\ \frac{\mu r_g}{(1+\mu)} & \frac{r_p}{(1+\mu)} \end{matrix}$	$\begin{matrix} \frac{1}{r_g} & 0 \\ -g_m & \frac{(1+\mu)}{r_p} \end{matrix}$

$[h]$	$[g]$	$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$
$\begin{matrix} r_g & 0 \\ g_m r_g & \frac{1}{r_p} \end{matrix}$	$\begin{matrix} \frac{1}{r_g} & 0 \\ \mu & r_p \end{matrix}$	$\begin{matrix} -\frac{1}{\mu} & -\frac{1}{g_m} \\ -\frac{1}{\mu r_g} & -\frac{1}{g_m r_g} \end{matrix}$
$\begin{matrix} \frac{r_p r_k}{r_p+(1+\mu)r_k} & \frac{r_k}{r_p+(1+\mu)t_k} \\ \frac{(1+\mu)r_k}{r_p+(1+\mu)r_k} & \frac{1}{r_p+(1+\mu)r_k} \end{matrix}$	$\begin{matrix} \frac{1}{r_k} & -1 \\ (1+\mu) & r_p \end{matrix}$	$\begin{matrix} \frac{1}{(1+\mu)} & \frac{r_p}{(1+\mu)} \\ \frac{1}{(1+\mu)r_k} & \frac{r_p+(1+\mu)r_k}{(1+\mu)r_k} \end{matrix}$
$\begin{matrix} r_g & 0 \\ -g_m r_g & \frac{(1+\mu)}{r_p} \end{matrix}$	$\begin{matrix} \frac{1}{r_g} & 0 \\ \frac{\mu}{(1+\mu)} & \frac{r_p}{(1+\mu)} \end{matrix}$	$\begin{matrix} \frac{(1+\mu)}{\mu} & \frac{1}{g_m} \\ \frac{(1+\mu)}{\mu r_g} & \frac{1}{g_m r_g} \end{matrix}$

2. SINGLE-STAGE VACUUM TUBE AMPLIFIERS, Z MATRIX APPLICATIONS

(a) GROUNDED CATHODE AMPLIFIER

In the preceding chapter we have derived a family of matrix relations for the vacuum tube. It will now be in order to consider the practical implementation of matrix analysis to some typical amplifier design topics. The student may find it profitable to compare the rigorous and compact matrix analysis with the established classical methods.(1, 2, 3)

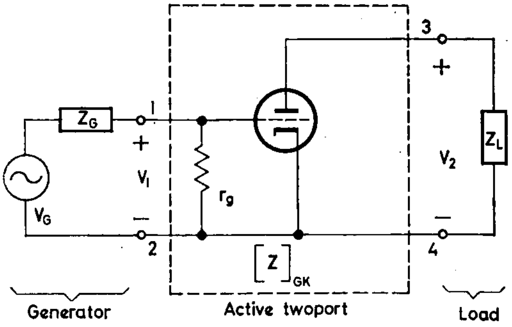


FIG. 2.1a. Building blocks of the triode amplifier.

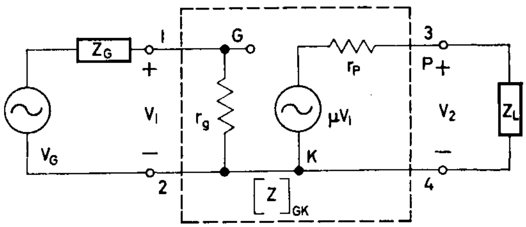


FIG. 2.1b. Linear model of the triode amplifier.

One may consider the simple amplifier in Fig. 2.1a as being synthesized from elementary building blocks, such as the driving generator, tube and load. The same physical amplifier can be represented as the linear model shown in Fig. 2.1b, where the familiar sign conventions apply. If we choose to describe the boxed active- twoport with the Z matrix parameters, then, by reverting to eqn. (1.21) or to Table III, the mathematical model of the triode is evidently

$$[Z]_{GK} = \begin{bmatrix} r_g & 0 \\ -\mu r_g & r_p \end{bmatrix}. \tag{2.1}$$

The gain and terminal impedances of the amplifier can be computed by applying the appropriate formulas from Table I in Part I.

We find the expression for the voltage gain A_v:

$$A_v = \frac{Z_{21}Z_L}{\Delta_Z + Z_{11}Z_L}. \tag{2.2}$$

The corresponding elements from the matrix (2.1) are evidently

$$Z_{21} = -\mu r_g,$$
$$Z_{11} = r_g,$$
$$\Delta_Z = r_g r_p.$$

Substituting these into eqn. (2.2),

$$A_v = \frac{-\mu r_g Z_L}{r_g r_p + r_g Z_L}. \tag{2.3}$$

Divide both numerator and denominator by r_g:

$$\boxed{A_v = \frac{-\mu Z_L}{r_p + Z_L}} \tag{2.4}$$

Equation (2.4) may be put into a different form by dividing both numerator and denominator by μ:

$$\boxed{A_v = \frac{-g_m Z_L}{1 + (Z_L/r_p)}} \tag{2.5}$$

If $r_p \gg Z_L$ which is generally true for tetrodes and pentodes, the eqn. (2.5) simplifies to

$$\boxed{A_v \approx -g_m Z_L} \tag{2.6}$$

Next, compute the input impedance Z_{in}. Again from Table I,

$$Z_{\text{in}} = \frac{\Delta_Z + Z_{11}Z_L}{Z_{22}+Z_L}. \tag{2.7}$$

Substitute for the Z parameters elements from the matrix (2.1):

$$Z_{\text{in}} = \frac{r_g r_p + r_g Z_L}{r_p + Z_L}. \tag{2.8}$$

Rearrange terms:

$$Z_{\text{in}} = r_g \frac{(r_p + Z_L)}{(r_p + Z_L)}, \tag{2.8a}$$

$$\boxed{Z_{\text{in}} = r_g} \tag{2.9}$$

which would be expected from physical reasoning.

Finally, for the output impedance, from Table I,

$$Z_{\text{out}} = \frac{\Delta_Z + Z_{22}Z_G}{Z_{11}+Z_G}. \tag{2.10}$$

Substitute into this expression the appropriate Z parameters from the matrix (2.1):

$$Z_{\text{out}} = \frac{r_g r_p + r_p Z_G}{r_g + Z_G} = r_p \frac{(r_g + Z_G)}{(r_g + Z_G)}, \tag{2.11}$$

$$\boxed{Z_{\text{out}} = r_p} \tag{2.12}$$

This result is also consistent with physical reasoning.

(b) NEGATIVE FEEDBACK AMPLIFIER

A simple but efficient feedback configuration is shown in Fig. 2.2. The cathode current through impedance Z_k gives rise to a negative feedback potential in series with the generator V_G. It will be practical to split up and redraw the boxed part $[Z]_A$ of the circuit diagram as shown in Fig. 2.3. We have now the familiar series–series type of twoport interconnection. The corresponding mathematical model can be synthesized by the algebraic summation of the constituent twoport matrix parameters.

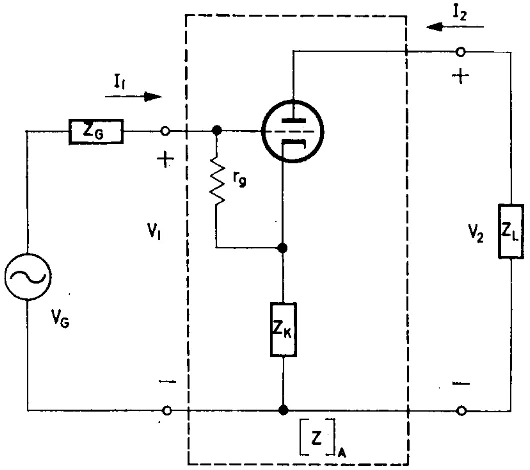


FIG. 2.2. Feedback amplifier with series negative feedback through Z_k.

By inspection, from Fig. 2.3, we can write down the matrix of the feedback impedance Z_k:

$$[Z]_k = \begin{bmatrix} Z_k & Z_k \\ Z_k & Z_k \end{bmatrix}. \tag{2.13}$$

The second matrix, for the active twoport $[Z]_{Gk}$ has already been specified and given by eqn. (2.1). Hence, the mathematical model of the feedback amplifier $[Z]_A$:

$$[Z]_A = [Z]_{Gk} + [Z]_k. \tag{2.14}$$

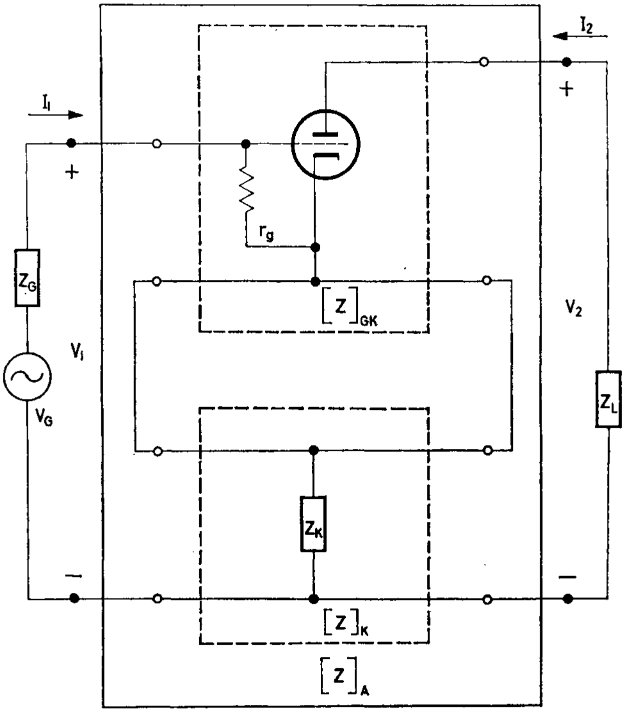


FIG. 2.3. Feedback amplifier $[Z]_A$ synthesized from active and passive twoports $[Z]_{GK}$ and $[Z]_K$.

Writing this in expanded form as the algebraic sum of eqns. (2.1) and (2.13) for the right-hand part:

$$[Z]_A = \begin{bmatrix} \overbrace{(r_g + Z_k)}^{Z_{11}} & \overbrace{Z_k}^{Z_{12}} \\ \underbrace{(-\mu r_g + Z_k)}_{Z_{21}} & \underbrace{(r_p + Z_k)}_{Z_{22}} \end{bmatrix}. \tag{2.15}$$

The gain and terminal impedances are next computed by simple algebraic manipulation. However, first we require to evaluate the determinant of the matrix (2.15);

$$\Delta_z = (r_g + Z_k)(r_p + Z_k) - (-\mu r_g + Z_k)Z_k, \tag{2.16}$$

$$\Delta_z = Z_k\{r_p + r_g(1+\mu)\} + r_g r_p. \tag{2.17}$$

Note that according to the adopted sign convention μ is a negative quantity. Furthermore, for practical vacuum tubes $\mu \gg 1$, and eqn. (2.17) simplifies to

$$\Delta_z = Z_k(r_p + \mu r_g) + r_p r_g. \tag{2.18}$$

We can now proceed to compute the voltage gain A_v by substituting into eqn. (2.2) the appropriate elements from the matrix (2.15):

$$\boxed{A_v = \frac{(-\mu r_g + Z_k)Z_L}{Z_k(r_p + \mu r_g) + r_p r_g + (r_g + Z_k)Z_L}} \tag{2.19}$$

Equation (2.19) may be simplified if $r_g \gg Z_k$. We proceed then by dividing both numerator and denominator by r_g and rearranging terms:

$$\boxed{A_v \approx \frac{-\mu Z_L}{\mu Z_k + r_p + Z_L}} \tag{2.20}$$

Compare now this expression with eqn. (2.7), derived for the grounded cathode amplifier and note that by adding a negative feedback impedance the voltage gain has been reduced by the factor μZ_k in the denominator.

Next, we can obtain the input impedance by substituting into eqn. (2.7) the appropriate elements from the matrix (2.15):

$$Z_{\text{in}} = \frac{Z_k(r_p + \mu r_g) + r_p r_g + (r_g + Z_k)Z_L}{r_p + Z_k + Z_L}. \tag{2.21}$$

By rearranging terms,

$$\boxed{Z_{\text{in}} = \frac{Z_k(\mu r_g + r_p + Z_L) + r_g(r_p + Z_L)}{r_p + Z_k + Z_L}} \tag{2.22}$$

If $Z_k \ll (r_p + Z_L)$ then eqn. (2.22) reduces to

$$\boxed{Z_{\text{in}} \approx r_g \left\{1 + \frac{\mu Z_k}{r_p + Z_L}\right\}} \tag{2.23}$$

Compare eqns. (2.9) and (2.23) and note that the negative series feedback resulted in an increase of input impedance by the amount indicated in the second term of eqn. (2.23).

Finally, for the output conditions, revert to eqn. (2.10) and substitute into it the elements from the matrix (2.15):

$$Z_{\text{out}} = \frac{Z_k(r_p+\mu r_g)+r_p r_g+(r_p+Z_k)Z_G}{r_g+Z_k+Z_G}. \qquad (2.24)$$

Rearrange terms:

$$\boxed{Z_{\text{out}} = \frac{Z_k(\mu r_g+r_p+Z_G)+r_p(r_g+Z_G)}{Z_k+r_g+Z_G}} \qquad (2.25)$$

If $Z_k \ll (r_g+Z_G)$ then eqn. (2.25) simplifies to

$$\boxed{Z_{\text{out}} \approx r_p+\frac{Z_k(\mu r_g+r_p+Z_G)}{(r_g+Z_G)}} \qquad (2.26)$$

In comparing eqns. (2.12) and (2.26), we find that the application of the series negative feedback resulted in the increase of output impedance by the amount of the second term in eqn. (2.26).

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6. A. J. Cote Jr. and J. B. Oakes, *Linear Vacuum Tube and Transistor Circuits*, McGraw-Hill, New York, 1961.
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3. THE SINGLE-STAGE TRANSISTOR AMPLIFIER DESIGN, APPLICATIONS OF THE *h* AND *ABCD* OR TRANSMISSION MATRIX PARAMETERS

(a) INTRODUCTION

This chapter deals with an amplifier design technique which utilizes the transistor hybrid *h* parameters, those which are frequently listed in manufacturer's data sheets or may be established by measurements. Matrix analysis will be used for manipulating the transistor and external circuit parameters as best suited to the problem at hand. The concept will be exploited whereby the transistor is considered as forming a link in the chain between generator and output networks. The passive four-terminal networks themselves are conveniently described by the general parameters of the transmission matrix. An analytical approach based on this philosophy, it is believed, yields a better "feel" of the transistor's linear model and its inherent limitations. Furthermore, the compact yet rigorous matrix analysis will contribute to a systematic and elegant mathematical procedure for solving the majority of amplifier design problems.

(b) GENERAL CONSIDERATION OF DESIGN TARGET

When designing a linear AF or RF amplifier one is usually concerned with the establishment of quantitative data in respect of:

1. Current gain with specified generator and load terminations.
2. Voltage gain with specified generator and load terminations.
3. Input impedance of the transistor with specified load terminations.
4. Output impedance of the transistor with specified generator source impedance.
5. Device dissipation and d.c. biasing conditions.

Here we are concerned with items 1 to 4 only and it is tacitly assumed that the correct d.c. bias conditions exist.

(c) INTERCONNECTING THE TRANSISTOR AND TERMINATING NETWORKS, DEFINITION OF PARAMETER MATRICES

It is well known that the transistor is a non-unilateral device. Therefore amplifier gain characteristics and terminal conditions are strongly interrelated. These will also be demonstrated later with mathematical reasoning.

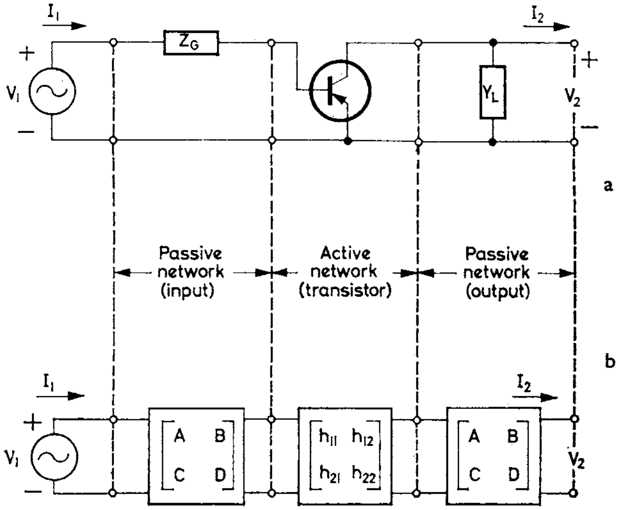

FIG. 3.1. The single-stage transistor amplifier, general configuration.

Consider a transistor amplifier in the most general form in Fig. 3.1a. A block diagram which conveys also the required basic information is shown in Fig. 3.1b. Of course, the transistor may operate in any of the conventional modes, that is common-emitter, common-base or common- collector. Note, furthermore, that the transistor and terminating networks have been designated in Fig. 3.1b. with internal parameter matrices. The transistor appears to be "embedded" between the two "black boxes" containing the terminating networks. It forms a link between these terminations.

If taken separately with due consideration of the customary sign convention, then either of the terminating twoport networks may be redrawn as shown in Fig. 3.2.

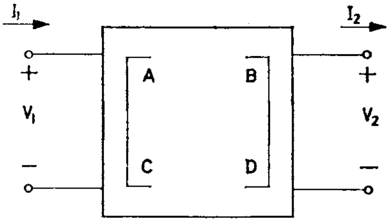

FIG. 3.2. The generalized twoport network.

The pair of equilibrium equations corresponding to this network model have been defined as follows:

$$V_1 = AV_2 + BI_2, \tag{3.1}$$

$$I_1 = CV_2 + DI_2. \tag{3.2}$$

Expressing eqns. (3.1) and (3.2) in matrix form,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{3.3}$$

Reverting now to the remaining link in the chain, the middle "black box" stands for the transistor with the hybrid *h* parameters. A block diagram of this network with the customary sign convention is shown in Fig. 3.3. Here the

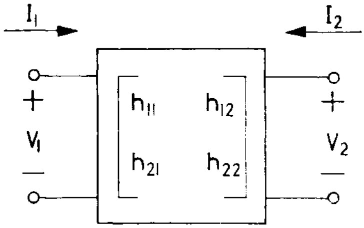

FIG. 3.3. Twoport network with hybrid h parameters.

dependent variables are the input voltage V_1 and output current I_2.

Writing down the corresponding equilibrium equations,

$$V_1 = h_{11}I_1 + h_{12}V_2, \tag{3.4}$$

$$I_2 = h_{21}I_1 + h_{22}V_2. \tag{3.5}$$

From these equations a two-generator linear model of the transistor may be synthesized as shown in Fig. 3.4.

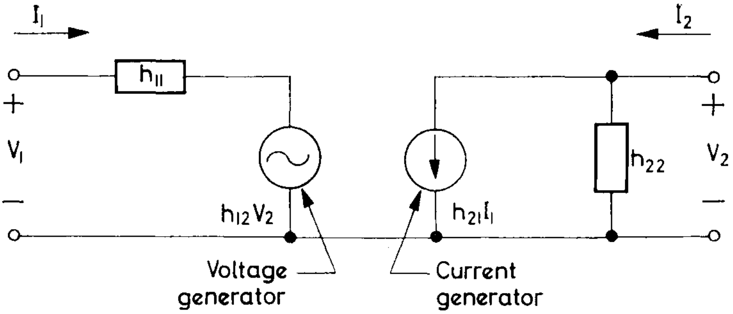

FIG. 3.4. Linear model of transistor with hybrid h parameters.

(d) THE MATHEMATICAL MODEL OF THE TRANSISTOR AS *ABCD* MATRIX IN TERMS OF HYBRID *h* PARAMETERS

Equations (3.4) and (3.5) are not particularly attractive for the type of interconnection shown in Fig. 3.1. We have represented the transistor as an active twoport, linking the input and output terminations. They may be looked upon as three twoports in cascade. Therefore, in order to establish the correct mathematical description of our system, we require to specify them in terms of the *ABCD* matrix. In fact we need to be concerned only with the transformation of the transistor *h* parameters into *ABCD* format. By reverting to Chapter 3 in Part I we find there the derivation of the appropriate transformation:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -\dfrac{\Delta_h}{h_{21}} & -\dfrac{h_{11}}{h_{21}} \\ -\dfrac{h_{22}}{h_{21}} & -\dfrac{1}{h_{21}} \end{bmatrix}. \qquad (3.6)$$

This matrix will be now used to compute our amplifier transfer characteristics and terminal conditions. The linear model corresponding to matrix (3.6) may be drawn as shown in Fig. 3.5 which is essentially identical to Fig. 3.4 except for the reversal of the assumed direction of the output current I_2.

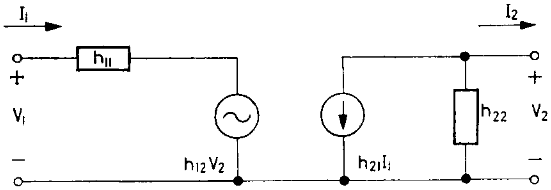

FIG. 3.5. Linear model of transistor with hybrid parameters after direction of output current I_2 reversed.

(e) INPUT IMPEDANCE

In the simplest form the transistor and the cascaded load termination may be represented either as in Fig. 3.6 or as in Fig. 3.7.

Consider first Fig. 3.6; the transmission matrix of this composite network is formed from the product of the individual transmission matrices.

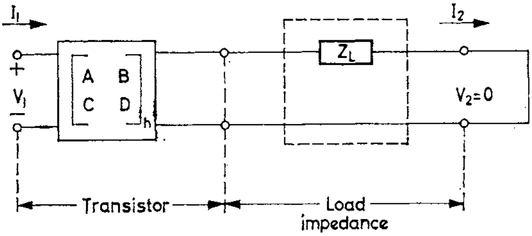

FIG. 3.6. Transistor amplifier with generalized load impedance.

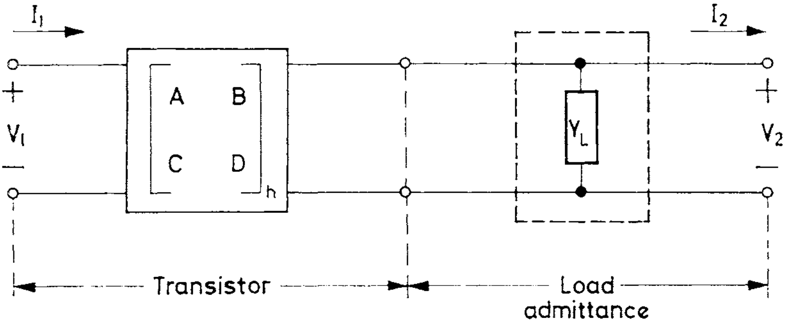

FIG. 3.7. Transistor amplifier with generalized load admittance.

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}_h}_{\text{Transistor}} \times \underbrace{\begin{bmatrix} 1 & Z_L \\ 0 & 1 \end{bmatrix}}_{\text{Load impedance}} = \underbrace{\begin{bmatrix} A & (AZ_L+B) \\ C & (CZ_L+D) \end{bmatrix}}_{\text{Matrix product}}. \tag{3.7}$$

The equilibrium equation corresponding to the network configuration in Fig. 3.6, may be written down at once:

$$V_1 = AV_2 + (AZ_L + B)I_2, \tag{3.8}$$

$$I_1 = CV_2 + (CZ_L + D)I_2. \tag{3.9}$$

Note that these expressions are identical with eqns. (2.83) and (2.84) of Part I.

From the ratio of eqns. (3.8) and (3.9) the input impedance may be readily obtained:

$$Z_{\text{in}} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{AZ_L + B}{CZ_L + D}. \tag{3.10}$$

Substituting now from the transistor matrix (3.6) the applicable h parameters into eqn. (3.10):

$$\boxed{Z_{\text{in}} = \frac{\Delta_h Z_L + h_{11}}{h_{22} Z_L + 1}} \tag{3.11}$$

It is often convenient to consider the output load network as an admittance parameter. This will yield the type of block diagram shown in Fig. 3.7.

The transmission matrix of this cascaded network is similarly obtained from the product of the trasnmission matrices of transistor and load admittance Y_L:

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}_h}_{\text{Transistor}} \times \underbrace{\begin{bmatrix} 1 & 0 \\ Y_L & 1 \end{bmatrix}}_{\substack{\text{Load}\\ \text{admittance}}} = \underbrace{\begin{bmatrix} (A+BY_L) & 1 \\ (C+DY_L) & D \end{bmatrix}}_{\text{Matrix product}}. \tag{3.12}$$

From the matrix product (3.12) a pair of equilibrium equations are readily obtained:

$$V_1 = (A+BY_L)V_2 + I_2, \tag{3.13}$$

$$I_1 = (C+DY_L)V_2 + DI_2. \tag{3.14}$$

The input impedance is defined as the ratio of eqns. (3.13) and (3.14):

$$Z_{\text{in}} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{A+BY_L}{C+DY_L}\,. \tag{3.15}$$

Substituting into eqn. (3.15) the appropriate parameters from the transistor matrix (3.6),

$$\boxed{Z_{\text{in}} = \frac{\Delta_h + h_{11} Y_L}{h_{22} + Y_L}} \tag{3.16}$$

(f) OUTPUT IMPEDANCE

It has been pointed out already that the transistor is a nonunilateral device. Consequently, the output impedance will also depend on the input termination. Consider the cascaded network structure in Fig. 3.8, consisting of a generator V_1 with an internal source impedance of Z_G and the transistor:

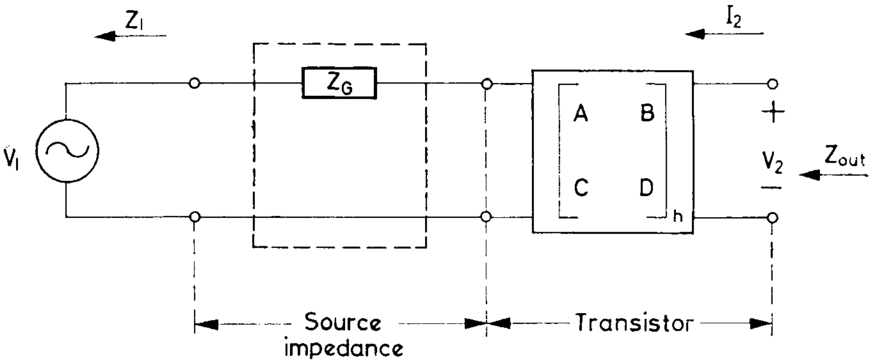

FIG. 3.8. Reversed current flow for the output impedance derivation.

By definition, the output impedance is equal to the ratio of V_2/I_2. Therefore, when setting up the equilibrium equations for this configuration, it will be necessary to choose V_2 and I_2 as dependent variables. It will be also required that the conventional direction of current flow be reversed as shown in Fig 2.8. In terms of matrices; the input and output quantities are now related by the inverse of the transmission matrix of the transistor. Thus from eqn. (3.3) and Fig. 3.8,

$$\underbrace{\begin{bmatrix} V_2 \\ I_2 \end{bmatrix}}_{\text{Output}} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}_h^{-1}}_{\substack{\text{Inverse of}\\ \text{transistor}\\ \text{matrix}}} \times \underbrace{\begin{bmatrix} 1 & Z_G \\ 0 & 1 \end{bmatrix}}_{\substack{\text{Source}\\ \text{impedance}}} \times \underbrace{\begin{bmatrix} V_1 \\ I_1 \end{bmatrix}}_{\text{input}}. \tag{3.17}$$

Performing the inversion of the transistor matrix, from eqn. (3.17),

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} D & B \\ C & A \end{bmatrix} \times \begin{bmatrix} 1 & Z_G \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}, \tag{3.18}$$

where $\Delta \equiv (AD-BC) =$ Determinant of the transmission matrix.

When multiplying out the triple matrix product, the desired pair of equilibrium equations will be obtained:

$$V_2 = \frac{D}{\Delta} V_1 + \frac{DZ_G + B}{\Delta} I_1, \tag{3.19}$$

$$I_2 = \frac{C}{\Delta} V_1 + \frac{CZ_G + A}{\Delta} I_1. \tag{3.20}$$

Note here also that these expressions are identical with eqns. (2.87) and (2.88) in Part I. Hence taking the ratios

$$Z_{\text{out}} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = \frac{DZ_G + B}{CZ_G + A}. \tag{3.21}$$

It is now an easy matter to substitute into eqn. (3.21) the applicable h parameters from the transistor matrix (3.6):

$$\boxed{Z_{\text{out}} = \frac{Z_G + h_{11}}{h_{22}Z_G + \Delta_h}} \tag{3.22}$$

If $h_{22}Z_G \ll \Delta$, then for the purpose of engineering approximations eqn. (3.22) may be simplified:

$$\boxed{Z_{\text{out}} \approx \frac{Z_G + h_{11}}{\Delta_h}} \tag{3.23}$$

(g) CURRENT GAIN

For the network model of the transistor amplifier in Fig. 3.6. the equilibrium equations (3.8) and (3.9) hold. Taking the ratio of I_2/I_1, from eqn. (3.9),

$$\left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{1}{CZ_L + D}. \tag{3.24}$$

Substituting into this equation the applicable h parameters from the transistor matrix (3.6),

$$\boxed{\frac{I_2}{I_1} = -\frac{h_{21}}{h_{22}Z_L+1}} \tag{3.25}$$

The negative sign in eqn. (3.25) is due to the direction assumed for the current flow. If $h_{22}Z_L \ll 1$, then for the purpose of engineering approximations eqn. (3.25) simplifies to

$$\frac{I_2}{I_1} \approx -h_{21}, \tag{3.26}$$

which is, by earlier definitions, the short-circuit current gain.

(h) VOLTAGE GAIN

For derivation of the voltage gain formula it is convenient to revert to Fig 3.7 and the related equilibrium equation (3.13). Note that in this case the load is represented by the admittance parameter Y_L:

From eqn. (3.13) by algebraic transposition,

$$\left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{1}{A+BY_L}. \tag{3.27}$$

Repeat here again the routine of substituting into eqn. (3.27) the appropriate h parameters from the transmission matrix (3.6) of the transistor:

$$\boxed{\frac{V_2}{V_1} = -\frac{h_{21}}{\Delta_h+h_{11}Y_L}} \tag{3.28}$$

The negative sign is attributed again to the adopted sign convention.

Equation (3.28) may be put into a different form by remembering that

$$Y_L = 1/Z_L \tag{3.29}$$

$$\frac{V_2}{V_1} = -\frac{h_{21}}{\Delta_h + h_{11}\dfrac{1}{Z_L}}. \tag{3.30}$$

Multiplying numerator and denominator by Z_L,

$$\boxed{\frac{V_2}{V_1} = -\frac{h_{21}Z_L}{\Delta_h Z_L + h_{11}}} \tag{3.31}$$

If $\Delta_h Z_L \ll h_{11}$, then for the purpose of engineering approximations eqn. (3.31) may be simplified to

$$\boxed{\frac{V_2}{V_1} \approx -\frac{h_{21}}{h_{11}} Z_L = -h_{21}\frac{Z_L}{h_{11}}} \tag{3.32}$$

(i) POWER GAIN

The operating power gain PG of an amplifier is defined as:

$$\mathrm{PG} = I_G V_G, \tag{3.33}$$

where I_G = Current gain as defined in eqn. (3.25),
V_G = Voltage gain as defined by eqns. (3.28) or (3.31),

hence the product of eqns. (3.25) and (3.28) will satisfy eqn˙ (3.33):

$$\mathrm{PG} = \left[-\frac{h_{21}}{h_{22}Z_L + 1}\right]\left[-\frac{h_{21}}{\Delta_h + h_{11}Y_L}\right], \tag{3.34}$$

$$\boxed{\mathrm{PG} = \frac{h_{21}^2}{(h_{22}Z_L + 1)(\Delta_h + h_{11}Y_L)}} \tag{3.35}$$

Alternatively, the power gain is obtained from the product of eqns. (3.25) and (3.31):

$$\mathrm{PG} = \left[-\frac{h_{21}}{h_{22}Z_L + 1}\right]\left[-\frac{h_{21}Z_L}{\Delta_h Z_L + h_{11}}\right], \tag{3.36}$$

$$\boxed{\mathrm{PG} = \frac{h_{21}^2 Z_L}{(h_{22}Z_L + 1)(\Delta_h Z_L + h_{11})}} \tag{3.37}$$

If, for a practical transistor amplifier one substitutes numerical constants into eqn. (3.37), then one will find that generally $h_{11} \gg \Delta_h Z_L$. Therefore, for the majority of applications, eqn. (3.37) may be simplified to

$$\boxed{\mathrm{PG} \approx \frac{h_{21}^2 Z_L}{h_{11}(1+h_{22}Z_L)}} \tag{3.38}$$

Further simplification will be possible if $h_{22}Z_L \ll 1$. In such cases eqn. (3.38) reduces to

$$\boxed{\mathrm{PG} \approx h_{21}^2 \frac{Z_L}{h_{11}}} \tag{3.39}$$

In the above analysis, transistor and terminating networks have been considered in a generalized form. Therefore if they are complex, then in the power gain equations (3.35) to (3.39), inclusive, the real parts must be taken.

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4. THE SINGLE-STAGE COMPOSITE FEEDBACK TRANSISTOR AMPLIFIER DESIGN, *Y* AND *ABCD* MATRIX APPLICATIONS

(a) INTRODUCTORY REMARKS

It is well known that manufacturing tolerances of transistors are extremely wide. It is not uncommon that the most significant parameter, the forward current gain β, varies as much as 50–100% among transistors of the same type. This state of affairs prompted amplifier designers to seek relief by the exploitation of negative feedback stabilization techniques. An impressive array of literature exists already and deals with several aspects of gain stabilization and linearization. Some of the outstanding and most readable contributions towards a clear mathematical formulation and design procedure of specific transistor feedback amplifiers will be found in references (1) to (8), inclusive.

At high-frequency operation, in addition to stabilization problems due to the wide variation of intrinsic current gain, various other factors appear, which also require attention; notably, the external stray capacitances and lead inductances. These parasitic elements may become very annoying with printed circuitry and high-density packaging techniques. In fact they could completely mask the intrinsic reverse and output admittances of the transistor.

With the intent of predicting more confidently the performance of any type of transistor feedback amplifier, we will attempt to tackle here the amplifier design problem in a broad, general fashion. To this end a linear model of the transistor amplifier is constructed, featuring simultaneous series and shunt feedback paths. A rigorous mathematical expression of this model is then derived in terms of the resistive transistor parameters, external feedback, load and generator admittance matrices. The complete transistor amplifier performance, including gain as well as input and output impedances, is derived with matrix analysis as the vehicle. Meaningful results are obtained in terms of realizable impedance ratios, thus clearly identifying the controlling mechanism and interplay of the individual feedback paths. By simple manipulation of the impedance ratios, the gain, stability, linearity and terminal impedances of the amplifier may be virtually tailored to the requirements of the designer. The physical transistor parameters will enter as second order quantities only.

The problem of approximations has also been adequately dealt with. Furthermore, the limiting conditions are considered in detail when either shunt or series feedback is the dominating factor, resulting in further significant simplification of the basic design equations; thus yielding in this process a whole range of approximate formulas, which are all applicable to practical engineering design work.

(b) FORMULATION OF THE AMPLIFIER DESIGN TARGET

The assignment of a linear transistor- amplifier design usually carries a set of tight specifications as to gain, bandwidth, stability, and input and output impedance levels. Unfortunately, more often than not, the transistor itself is the least predictable link in the whole system. Established design procedures take advantage of negative feedback stabilization techniques, both a.c. and d.c. types [references (1) to (8) inclusive]. Figure 4.1 shows schematically the simplest amplifier configuration with series, or emitter feedback, produced by the impedance element Z_e. It is assumed that the correct base and collector bias conditions exist. The bias network, of course, can be absorbed in the generalized source and load impedances Z_s and Z_L respectively.

The shunt feedback variety of stabilization is illustrated schematically in Fig. 4.2, where Z_R is the generalized collector to base feedback impedance. Here again the d.c. biasing network is absorbed in the impedances Z_R, Z_S and Z_L respectively.

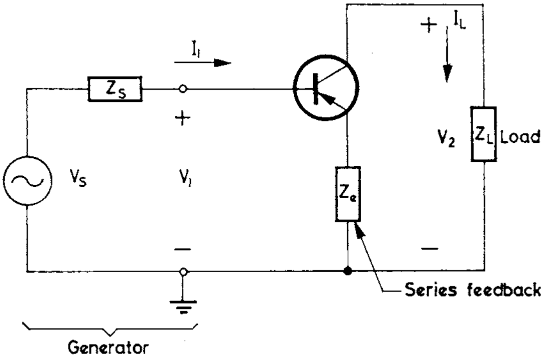


FIG. 4.1. Amplifier with series feedback.

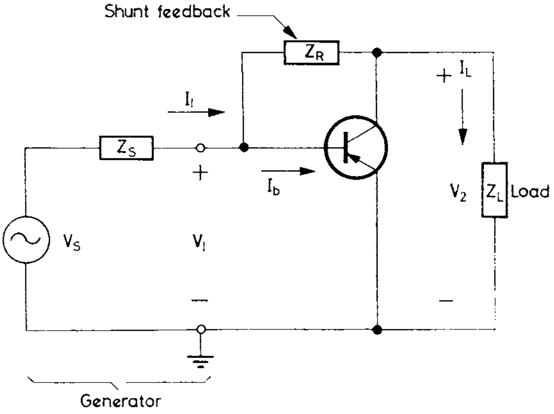


FIG. 4.2. Amplifier with shunt feedback.

It is well understood that at sufficiently high operating frequencies, generally both feedback types may simultaneously exist; firstly through the internal elements of the transistor itself and next via the external collector- to- base stray capacitances and emitter lead reactances. Therefore, it would appear to be logical to include these strays into the general system of simultaneously acting series and shunt feedback paths as shown in Fig. 4.3.

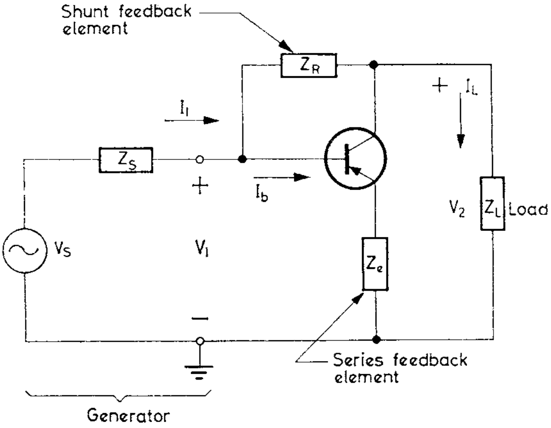


FIG. 4.3. The single-stage transistor amplifier with composite feedback.

As the next step, assume that unique, yet simple, mathematical relations could be established between feedback elements, terminating impedances and dominant transistor parameters. It will be demonstrated by means of some elementary matrix and algebraic manipulations that it is indeed so. The student will find that once the desired mathematical forms have been obtained, then feedback amplifiers can be confidently designed with respect to the predicted gain and terminal impedances. In the final results, the physical parameters of the transistor will enter only as second order quantities or even be negligible.

(c) SYNTHESIS OF THE COMPOSITE FEEDBACK AMPLIFIER

From the large assortment of possible linear twoport representation of the transistor [references (9), (10), (12) and (13)], the resistive *T* model has been selected for reasons of its simplicity and adequacy for mathematical processing.

Reverting now to Fig. 4.3 and redrawing this schematically to suit our purpose, a linear model can thus be synthesized as shown in detail in Fig. 4.4. Inside the large boxed area, we have our active twoport, consisting of the intrinsic transistor T network, with series feedback impedance Z_E and shunt feedback admittance Y_R. Note also that the generator and load terminations are now designated as admittances, since

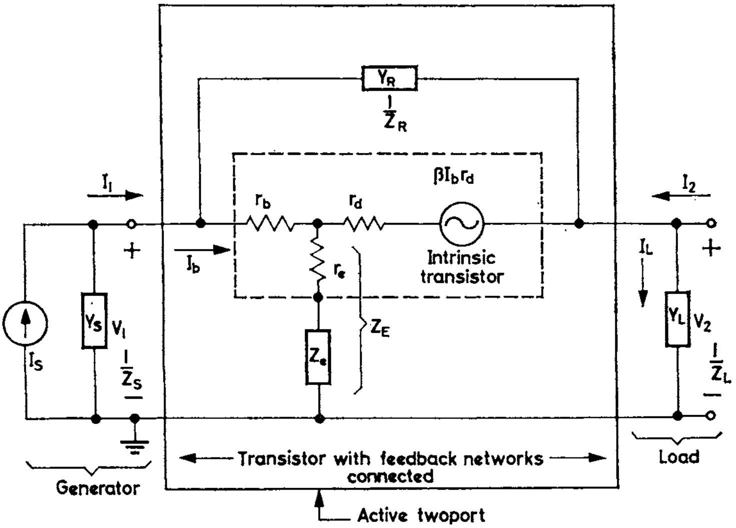


FIG. 4.4. Linear model of transistor amplifier with composite feedback.

$$Y_S = \frac{1}{Z_S} \quad \text{and} \quad Y_L = \frac{1}{Z_L}.$$

This linear model will eventually yield the basic and comprehensive mathematical description of our system, and which will be suitable for further processing.

(d) DETAILED MATRIX ANALYSIS

Having synthesized a composite feedback amplifier by the interconnection of the individual building blocks, it will be next required to define the system in mathematical terms. The topology of the network model in Fig. 4.4 calls for the application of the admittance matrix.

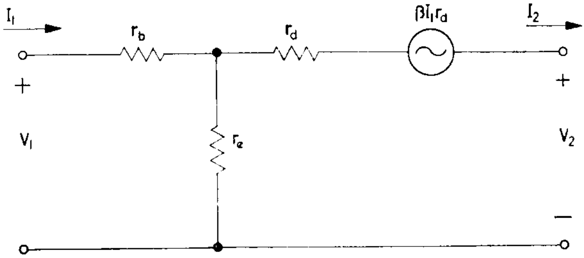


FIG. 4.5. Active twoport as resistive model of the common-emitter transistor.

It is well known that the transistors, as with any other linear twoport structures, can be described mathematically in terms of various matrix parameters. Because of the simplest physical interpretation, the one generator resistive T configuration has been selected in this study. Figure 4.5 shows this elementary but adequate network model with appropriate parameter and sign notation. The transmission matrix of this structure has been derived elsewhere(12) and defined:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{CE} = \frac{1}{r_e+\beta r_d}\begin{bmatrix} (r_e+r_b) & r_b(r_e+r_d)+r_e r_d(1+\beta) \\ 1 & (r_e+r_d) \end{bmatrix}, \tag{4.1}$$

$$r_d = (1-\alpha)r_c \approx \frac{r_c}{\beta}.$$

As a first step towards sophistication of this simple transistor model, consider the addition of a series emitter feedback impedance Z_e as shown in Fig. 4.6. Let $Z_E = (Z_e+r_e)$. The general validity of the transmission matrix (4.1) will still

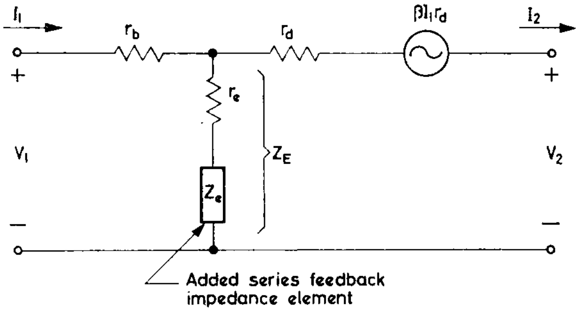


FIG. 4.6. Active twoport resistive model with emitter feedback impedance element added.

hold if we substitute Z_E for r_e. Thus for our active twoport model with emitter degeneration, we may write down the transmission matrix

$$\underset{\substack{\text{CE with}\\\text{series}\\\text{feedback}}}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} = \frac{1}{Z_E+\beta r_d}\begin{bmatrix} (Z_E+r_b) & r_b(Z_E+r_d)+Z_E r_d(1+\beta) \\ 1 & (Z_E+r_d) \end{bmatrix}. \tag{4.2}$$

For an amplifier configuration which features shunt feedback in addition to the series element Z_e, it is convenient to manipulate the mathematics in terms of admittances. Therefore we will proceed to transform the transmission matrix (4.2) into a Y matrix. By applying the elementary rules of matrix transformation or using tables, in Part I,

$$\underset{\substack{\text{transistor}\\\text{CE with}\\\text{series}\\\text{feedback}}}{[Y]} = \begin{bmatrix} Y_i & Y_r \\ Y_f & Y_0 \end{bmatrix} = \begin{bmatrix} \dfrac{D}{B} & -\dfrac{\Delta}{B} \\ -\dfrac{1}{B} & \dfrac{A}{B} \end{bmatrix}, \tag{4.3}$$

where the admittance parameters are defined as

Y_i = input admittance,
Y_r = reverse transfer admittance,
Y_f = forward transfer admittance,
Y_o = output admittance.

To evaluate the admittance parameters, substitute the approprite elements from the transmission matrix (4.2) into the right-hand part of eqn. (4.3), thus obtaining

$$\varDelta = \frac{Z_E}{Z_E + \beta r_d} : \text{ Determinant of the transmission matrix} \tag{4.4}$$

$$Y_i = \frac{D}{B} = \frac{(Z_E + r_d)}{r_b(Z_E + r_d) + Z_E r_d(1 + \beta)}, \tag{4.5}$$

$$Y_r = -\frac{\varDelta}{B} = -\frac{Z_E}{r_b(Z_E + r_d) + Z_E r_d(1 + \beta)}, \tag{4.6}$$

$$Y_f = -\frac{1}{B} = -\frac{(Z_E + \beta r_d)}{r_b(Z_E + r_d) + Z_E r_d(1 + \beta)}, \tag{4.7}$$

$$Y_0 = \frac{A}{B} = \frac{(Z_E + r_b)}{r_b(Z_E + r_d) + Z_E r_d(1 + \beta)}. \tag{4.8}$$

It is now a good time to stop and consider the magnitudes involved in the above expressions. One will find in any physical transistor amplifier that

$$Z_E \ll r_d \quad \text{and} \quad \beta \gg 1$$

Therefore eqns. (4.5) to (4.8) will substantially simplify. Yet the parameters are still sufficiently accurate for all engineering applications.

Hence the final forms of the twoport parameters are

$$Y_i = \frac{1}{r_b + \beta Z_E}, \tag{4.9}$$

$$Y_r = -\frac{Z_E}{r_d(r_b + \beta Z_E)}, \tag{4.10}$$

$$Y_f = \frac{\beta}{(r_b + \beta Z_E)}, \tag{4.11}$$

$$Y_0 = \frac{(Z_E + r_b)}{r_d(r_b + \beta Z_E)}. \tag{4.12}$$

Finally substitute eqns. (4.9) to (4.12) into the matrix (4.3):

$$\underset{\substack{\text{CE with}\\\text{series}\\\text{feedback}}}{[Y]} = \begin{bmatrix} \dfrac{1}{(r_b + \beta Z_E)} & -\dfrac{Z_E}{r_d(r_b + \beta Z_E)} \\ \dfrac{\beta}{(r_b + \beta Z_E)} & \dfrac{(Z_E + r_b)}{r_d(r_b + \beta Z_E)} \end{bmatrix}. \tag{4.13}$$

We have now in matrix (4.13) a satisfactory mathematical model of the transistor with emitter impedance degeneration.

Next, the admittance matrix of the shunt feedback element Y_R must be specified which is also inside the boxed area of our active twoport. By definition, the admittance matrix of Y_R is

$$[Y_R] = \begin{bmatrix} Y_R & -Y_R \\ -Y_R & Y_R \end{bmatrix}. \tag{4.14}$$

It is now evident that the sum of matrices (4.13) and (4.14) yield the admittance matrix of the linear model of our composite, active twoport.

$$\underset{\substack{\text{CE with}\\ \text{composite}\\ \text{feedback}}}{[Y]} = \begin{array}{cc} Y_{11} & Y_{12} \\ \left[\begin{array}{cc} \dfrac{1}{(r_b+\beta Z_E)}+Y_R & -\dfrac{Z_E}{r_d(r_b+\beta Z_E)}-Y_R \\ \dfrac{\beta}{(r_b+\beta Z_E)}-Y_R & \dfrac{(r_b+Z_E)}{r_d(r_b+\beta Z_E)}+Y_R \end{array}\right. & \left.\right] \\ Y_{21} & Y_{22} \end{array}. \tag{4.15}$$

It remains to specify the admittance matrices of the generator and load terminations. Using again elementary rules of matrix algebra,

$$[Y_S] = \begin{bmatrix} Y_S & 0 \\ 0 & 0 \end{bmatrix}; \quad \text{Source admittance matrix.} \tag{4.16}$$

$$[Y_L] = \begin{bmatrix} 0 & 0 \\ 0 & Y_L \end{bmatrix}; \quad \text{Load admittance matrix.} \tag{4.17}$$

Finally, a complete mathematical description of the amplifier is obtained by the summing up of the admittance matrices (4.15), (4.16) and ((4.17):

$$\underset{\substack{\text{CE}\\ \text{feedback}\\ \text{amplifier}}}{[Y]} = \begin{bmatrix} \dfrac{1}{(r_b+\beta Z_E)}+Y_R+Y_S & -\dfrac{Z_E}{r_d(r_b+\beta Z_E)}-Y_R \\ \dfrac{\beta}{(r_b+\beta Z_E)}-Y_R & \dfrac{r_b+Z_E}{r_d(r_b+\beta Z_E)}+Y_R+Y_L \end{bmatrix}. \tag{4.18}$$

This appears to be a rather bulky expression. Fortunately, substantial simplifications are possible, for the following reasons. If one considers the range of magnitudes encountered in a practical transistor feedback amplifier, one will find that for any physical amplifier configuration,

$$\frac{(r_b + Z_E)}{r_d(r_b + \beta Z_E)} \ll (Y_R + Y_L). \tag{4.19}$$

Similarly, when combining reactive reverse parameters of the transistor with the external element Y_R,

$$\frac{Z_E}{r_d(r_b + \beta Z_E)} \ll Y_R. \tag{4.20}$$

Because of the validity of inequalities (4.19) and (4.20) it is perfectly legitimate to omit the first terms in the elements Y_{12} and Y_{22} of the matrix (4.18). Thus the admittance matrix of the complete feedback amplifier simplifies to

$$\underset{\substack{\text{CE}\\ \text{feedback}\\ \text{amplifier}}}{[Y]} = \begin{bmatrix} \dfrac{1}{(r_b + \beta Z_E)} + Y_R + Y_S & -Y_R \\ \dfrac{\beta}{(r_b + \beta Z_E)} - Y_R & Y_R + Y_L \end{bmatrix}. \tag{4.21}$$

With similar reasoning, the parameter matrix of the active twoport can also be simplified.

Thus eqn. (4.15) reduces to the very compact form

$$\underset{\substack{\text{CE with}\\ \text{series and}\\ \text{shunt}\\ \text{feedback}}}{[Y]} = \begin{bmatrix} \overbrace{\dfrac{1}{(r_b + \beta Z_E)} + Y_R}^{Y_{11}} & \overbrace{-Y_R}^{Y_{12}} \\ \underbrace{\dfrac{\beta}{(r_b + \beta Z_E)} - Y_R}_{Y_{21}} & \underbrace{Y_R}_{Y_{22}} \end{bmatrix}. \tag{4.22}$$

Using now matrix (4.22) to describe the active twoport in our feedback amplifier, a new and simplified linear model can be synthesized as shown in Fig. 4.7. Compare this model with the block diagram Fig. 2.3 of Chapter 2 in Part I. The symbolic parameter equivalence will be at once apparent. Therefore the gain, input and output impedance formulas of the Y parameter twoport are directly applicable to our model of a physical transistor feedback amplifier.

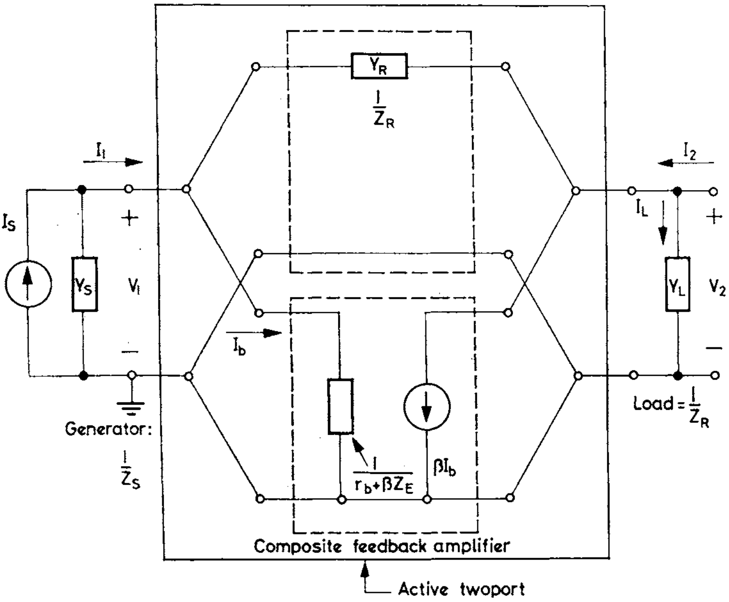


FIG. 4.7. Linear model of the composite feedback amplifier with generator and load terminations in terms of admittance parameters.

By reverting to Table I in Part I, with a systematic substitution into the standard forms of the appropriate elements from the matrix (4.22), we can derive a whole range of meaningful design equations.

(e) CURRENT GAIN

From Table I in Part I the current gain in terms of admittance parameters is

$$A_i = -\frac{Y_{21}Y_L}{\Delta_Y + Y_{11}Y_L}, \tag{4.23}$$

where Δ_Y = Determinant of the matrix (4.22):

$$\Delta_Y = \left(\frac{1}{r_b+\beta Z_E} + Y_R\right)Y_R - \left(\frac{\beta}{r_b+\beta Z_E} - Y_R\right)(-Y_R), \tag{4.24}$$

$$\Delta_Y = \frac{Y_R}{r_b+\beta Z_E}(1+\beta). \tag{4.25}$$

Since $\beta \gg 1$, eqn. (4.25) simplifies to

$$\Delta_Y = \frac{\beta Y_R}{r_b+\beta Z_E}. \tag{4.26}$$

Now by simple algebraic substitution into eqn. (4.23) from eqns. (4.22) and (4.26),

$$A_i = -\frac{\left(\dfrac{\beta}{r_b+\beta Z_E} - Y_R\right)Y_L}{\dfrac{\beta Y_R}{r_b+\beta Z_E} + \left(\dfrac{1}{r_b+\beta Z_E} + Y_R\right)Y_L}. \tag{4.27}$$

Divide numerator and denominator by Y_L, then multiply numerator and denominator by $(r_b+\beta Z_E)$:

$$A_i = -\frac{\beta - (r_b+\beta Z_E)Y_R}{\beta\dfrac{Y_R}{Y_L} + 1 + (r_b+\beta Z_E)Y_R}. \tag{4.28}$$

Express eqn. (4.28) in terms of impedance ratios and rearrange. Remember that $Y_R = (1/Z_R)$ and $Y_L = (1/Z_L)$.

$$\boxed{A_i = -\frac{\beta\left(1-\dfrac{Z_E}{Z_R}\right) - \dfrac{r_b}{Z_R}}{1+\beta\left(\dfrac{Z_L}{Z_R}+\dfrac{Z_E}{Z_R}\right)+\dfrac{r_b}{Z_R}}} \tag{4.29}$$

For all engineering applications eqn. (4.29) can be designated as an exact expression. Generally, the ratio of r_b/Z_R is a second or third order quantity and, consequently, for the broadest applications, eqn. (4.29) reduces to

$$\boxed{A_i = -\frac{\beta\left(1-\frac{Z_E}{Z_R}\right)}{1+\beta\left(\frac{Z_L}{Z_R}+\frac{Z_E}{Z_R}\right)}} \tag{4.30}$$

Under conditions where either series or shunt feedback is absent, eqn. (4.30) reduces to very simple form:

With shunt feedback only, that is $Z_E \to 0$,

1st approximation: $$A_i \approx -\frac{\beta}{1+\beta(Z_L/Z_R)}. \tag{4.31}$$

2nd approximation: $$A_i \approx -\frac{Z_R}{Z_L}. \tag{4.32}$$

With series feedback only, that is $Z_R \to \infty$,

1st approximation: $$A_i \approx -\beta. \tag{4.33}$$

Note that with series feedback only, the current gain is practically independent of the feedback impedance.

(f) VOLTAGE GAIN

From Table I in Part I,

$$A_v = -\frac{Y_{21}}{Y_{22}+Y_L}. \tag{4.34}$$

Substitute now for Y_{21} and Y_{22} the appropriate elements from the parameter matrix (4.22):

$$A_v = -\frac{\beta/(r_b+\beta Z_E)-Y_R}{Y_R+Y_L}. \tag{4.35}$$

Multiply numerator and denominator by $(r_b+\beta Z_E)$:

$$A_v = -\frac{\beta-(r_b+\beta Z_E)Y_R}{(r_b+\beta Z_E)(Y_R+Y_L)}. \tag{4.36}$$

Expand the products and rearrange terms as impedance ratios:

$$\boxed{A_v = -\frac{\beta\left(1-\frac{Z_E}{Z_R}\right)-\frac{r_b}{Z_R}}{\beta\left(\frac{Z_E}{Z_R}+\frac{Z_E}{Z_L}\right)+\frac{r_b}{Z_R}+\frac{r_b}{Z_L}}} \tag{4.37}$$

For a composite feedback amplifier the ratio of r_b/Z_R is generally a second or third order quantity, hence for practical engineering design purposes eqn. (4.37) reduces to the simpler form

$$\boxed{A_v = -\frac{\beta\left(1-\frac{Z_E}{Z_R}\right)}{\beta\left(\frac{Z_E}{Z_R}+\frac{Z_E}{Z_L}\right)+\frac{r_b}{Z_L}}} \tag{4.38}$$

For those applications where either the series or shunt feedback dominates, eqn. (4.37) simplifies:

With shunt feedback only, that is $Z_E \to 0$,

1st approximation: $$A_v = -\frac{\beta Z_L}{r_e(1+\beta(Z_L/Z_R))}. \tag{4.39}$$

With series feedback only, that is $Z_R \to \infty$,

1st approximation: $$A_v \approx -\frac{Z_L}{Z_E+(r_b/\beta)}. \tag{4.40}$$

2nd approximation: $$A_v \approx -\frac{Z_L}{Z_E}. \tag{4.40a}$$

(g) INPUT IMPEDANCE

Reverting to Table I in Part for the expression of input impedance in terms of Y parameters, we find

$$Z_{\text{in}} = \frac{Y_{22}+Y_L}{\Delta_Y+Y_{11}Y_L}. \tag{4.41}$$

Substituting again eqn. (4.26) into eqn. (4.41) for Δ_Y and the appropriate elements from the parameter matrix (4.22),

$$Z_{\text{in}} = \frac{Y_R + Y_L}{\beta Y_R/(r_b + \beta Z_E) + [1/(r_b + \beta Z_E) + Y_R] Y_L}. \tag{4.42}$$

Multiply both denominator and numberator by $(r_b + \beta Z_E)$:

$$Z_{\text{in}} = \frac{(r_b + \beta Z_E)(Y_R + Y_L)}{\beta Y_R + Y_L + (r_b + \beta Z_E) Y_R Y_L}. \tag{4.43}$$

Expand the products:

$$Z_{\text{in}} = \frac{r_b Y_R + \beta Z_E Y_R + r_b Y_L + \beta Z_E Y_L}{\beta Y_R + Y_L + r_b Y_R Y_L + \beta Z_E Y_R Y_L}. \tag{4.44}$$

Divide numerator and denominator by Y_L, then rearrange terms and finally express admittances with the reciprocal impedances.

$$\boxed{Z_{\text{in}} = \frac{\beta Z_E \left(1 + \frac{Z_L}{Z_R}\right) + r_b \left(1 + \frac{Z_L}{Z_R}\right)}{1 + \beta \left(\frac{Z_L}{Z_R} + \frac{Z_E}{Z_R}\right) + \frac{r_b}{Z_R}}} \tag{4.45}$$

This is an exact expression which contains a number of second order terms, such as Z_E/Z_R in the denominator and Z_L/Z_R the last term in the numerator. Therefore, for the majority of engineering applications eqn. (4.45) simplifies to

$$\boxed{Z_{\text{in}} = \frac{\beta Z_E \left(1 + \frac{Z_L}{Z_R}\right) + r_b}{1 + \beta \frac{Z_L}{Z_R} + \frac{r_b}{Z_R}}} \tag{4.46}$$

If either series or shunt feedback is negligible, then eqn. (4.46) further simplifies:

With shunt feedback dominating. This corresponds to the condition when $Z_e \to 0$, Consequently, $Z_E = r_e$.

1st approximation: $$Z_{\text{in}} \approx \frac{\dfrac{Z_R}{Z_L}\left(r_e + \dfrac{r_b}{\beta}\right) + r_e}{1 + \dfrac{Z_R}{\beta Z_L}}. \tag{4.47}$$

2nd approximation $$Z_{\text{in}} \approx \frac{Z_R}{Z_L}\left(r_e + \frac{r_b}{\beta}\right). \tag{4.48}$$

With series feedback only, that is $Z_R \to \infty$,

1st approximation: $$Z_{\text{in}} \approx \beta Z_E + r_b. \tag{4.49}$$

2nd approximation: $$Z_{\text{in}} \approx \beta Z_E. \tag{4.50}$$

(h) OUTPUT IMPEDANCE

From Table I in Part I, we can write down the expression for output impedance in terms of our linear model in Fig. 4.5:

$$Z_{\text{out}} = \frac{Y_{11} + Y_S}{\Delta_Y + Y_S Y_{22}}. \tag{4.51}$$

Use again the substitutions from the parameter matrix (4.22) and eqn. (4.26):

$$Z_{\text{out}} = \frac{\dfrac{1}{(r_b + \beta Z_E)} + Y_R + Y_S}{\dfrac{\beta Y_R}{(r_b + \beta Z_E)} + Y_S Y_R}. \tag{4.52}$$

Multiply first numerator and denominator by $(r_b + \beta Z_E)$, then divide by Y_S:

$$Z_{\text{out}} = \frac{\dfrac{1}{Y_S} + r_b \dfrac{Y_R}{Y_S} + \beta Z_E \dfrac{Y_R}{Y_S} + r_b + \beta Z_E}{\beta \dfrac{Y_R}{Y_S} + r_b Y_R + \beta Z_E Y_R}. \tag{4.53}$$

Express the admittances in terms of reciprocal impedance parameters and rearrange terms:

$$\boxed{Z_{\text{out}} = \frac{Z_S + \beta Z_E\left(1 + \dfrac{Z_S}{Z_R}\right) + r_b\left(1 + \dfrac{Z_S}{Z_R}\right)}{\beta\left(\dfrac{Z_S}{Z_R} + \dfrac{Z_E}{Z_R}\right) + \dfrac{r_b}{Z_R}}} \tag{4.54}$$

This is again an exact expression which contains a number of second and third order terms, such as the terms r_b/Z_R in the denominator and Z_S/Z_R in the last bracket of the numerator. Hence for all engineering applications, eqn. (4.54) simplifies to

$$\boxed{Z_{\text{out}} = \frac{Z_S + \beta Z_E\left(1 + \frac{Z_S}{Z_R}\right) + r_b}{\beta\left(\frac{Z_S}{Z_R} + \frac{Z_E}{Z_R}\right)}} \qquad (4.55)$$

This equation again further simplifies if only one kind of feedback is acting.

With shunt feedback dominating.

Under these conditions $Z_e \overset{\rightarrow}{\equiv} 0$.

Consequently $Z_E = r_e$. From eqn. (4.55)

1st approximation: $$Z_{\text{out}} \approx \frac{Z_R}{\beta} + \frac{Z_R}{Z_S}\left(r_e + \frac{r_b}{\beta}\right). \qquad (4.56)$$

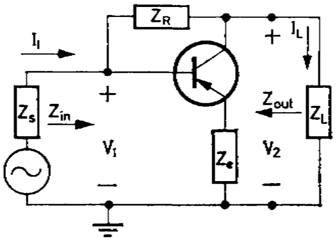
2nd approximation: $$Z_{\text{out}} \approx \frac{Z_R}{\beta} + r_e\frac{Z_R}{Z_S}. \qquad (4.57)$$

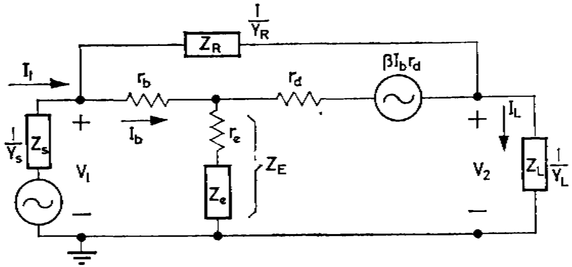
(i) CONCLUDING REMARKS

The above analysis has been carried out in terms of generalized parameters. Therefore the results are valid for any type of physical network configuration. Thus the individual blocks or elements may be resistive, reactive or complex. If the dominating parameters in the operating region are resistive, then a single-stage amplifier can be considered as an absolutely stable twoport. However, if the terminating elements have reactive components, then certain combinations will yield negative input impedance, thus the system becomes unstable. An extensive treatment of the stability problem will be found in references (10) and (13).

The following Table IV summarizes the pertinent terminal characteristics and design equations of the composite feedback transistor amplifier.

TABLE IV. THE COMPOSITE FEEDBACK AMPLIFIER TABLE OF TERMINAL PROPERTIES





	A_i Current gain	A_v Voltage gain	Z_{in} Input impedance	Z_{out} Output impedance
Generalized basic matrix formula	$-\frac{Y_{21}Y_L}{\Delta_Y+Y_{11}Y_L}$	$-\frac{Y_{21}}{Y_{22}+Y_L}$	$\frac{Y_{22}+Y_L}{\Delta_L+Y_{11}Y_L}$	$\frac{Y_{11}+Y_S}{\Delta_Y+Y_{22}Y_S}$
Fully expanded expression	$-\frac{\beta\left(1-\frac{Z_E}{Z_R}\right)-\frac{r_b}{Z_R}}{1+\beta\left(\frac{Z_L}{Z_R}+\frac{Z_E}{Z_R}\right)+\frac{r_b}{Z_R}}$	$-\frac{\beta\left(1-\frac{Z_E}{Z_R}\right)-\frac{r_b}{Z_R}}{\beta\left(\frac{Z_E}{Z_R}+\frac{Z_E}{Z_L}\right)+\frac{r_b}{Z_R}+\frac{r_b}{Z_L}}$	$\frac{\beta Z_E\left(1+\frac{Z_L}{Z_R}\right)+r_b\left(1+\frac{Z_L}{Z_R}\right)}{1+\beta\left(\frac{Z_L}{Z_R}+\frac{Z_E}{Z_R}\right)+\frac{r_b}{Z_R}}$	$\frac{Z_S+\beta Z_E\left(1+\frac{Z_S}{Z_R}\right)+r_b\left(1+\frac{Z_S}{Z_R}\right)}{\beta\left(\frac{Z_S}{Z_R}+\frac{Z_E}{Z_R}\right)+\frac{r_b}{Z_R}}$

Full practical equation	$-\dfrac{\beta\left(1-\dfrac{Z_E}{Z_R}\right)}{1+\beta\left(\dfrac{Z_L}{Z_R}+\dfrac{Z_E}{Z_R}\right)}$	$-\dfrac{\beta\left(1-\dfrac{Z_E}{Z_R}\right)}{\beta\left(\dfrac{Z_E}{Z_R}+\dfrac{Z_E}{Z_L}\right)+\dfrac{r_b}{Z_L}}$	$\dfrac{\beta Z_E\left(1+\dfrac{Z_L}{Z_R}\right)+r_b}{1+\beta\dfrac{Z_L}{Z_R}+\dfrac{r_b}{Z_R}}$	$\dfrac{Z_S+\beta Z_E\left(1+\dfrac{Z_S}{Z_R}\right)+r_b}{\beta\left(\dfrac{Z_S}{Z_R}+\dfrac{Z_E}{Z_R}\right)}$
Shunt feedback only				
1st approximation	$-\dfrac{\beta}{1+\beta\dfrac{Z_L}{Z_R}}$	$-\dfrac{\beta Z_L}{r_e\left(1+\beta\dfrac{Z_L}{Z_R}\right)}$	$\dfrac{\dfrac{Z_R}{Z_L}\left(r_e+\dfrac{r_b}{\beta}\right)+r_e}{1+\dfrac{Z_R}{\beta Z_L}}$	$\dfrac{Z_R}{\beta}+\dfrac{Z_R}{Z_S}\left(r_e+\dfrac{r_b}{\beta}\right)$
2nd approximation	$-\dfrac{Z_R}{Z_L}$		$\dfrac{Z_R}{Z_L}\left(r_e+\dfrac{r_b}{\beta}\right)$	$\dfrac{Z_R}{\beta}+r_e\dfrac{Z_R}{Z_S}$
Series feedback only				
1st approximation	$-\beta$	$-\dfrac{Z_L}{Z_E+\dfrac{r_b}{\beta}}$	βZ_E+r_b	$Z_d\dfrac{\beta Z_E+Z_S}{Z_E+Z_S}$
2nd approximation		$-\dfrac{Z_L}{Z_E}$	βZ_E	$\rightarrow\infty$

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5. THE TRANSISTOR VOLTAGE FEEDBACK PAIR, *h* MATRIX APPLICATIONS

(a) SYNTHESIS OF THE MATHEMATICAL MODEL

An amplifier configuration which is particularly useful when high input and low output impedances are desired is the series–parallel feedback type as shown in Fig. 5.1. In accordance with the adopted building block concept, we can consider that this amplifier circuit has been synthesized from a unilateral amplifier and a bilateral passive feedback twoport. When cut apart, these building blocks are of the form as shown in Fig. 5.2a and Fig. 5.2b.

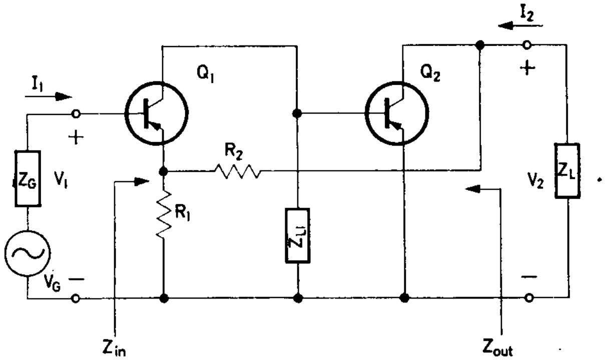

FIG. 5.1. The voltage feedback pair, simplified schematic.

With regard to the amplifier twoport we can make the valid assumption that the reactive components of the input and output impedances are absorbed by the external generator and load impedances. As the next step, we can replace the cascaded transistor pair with an equivalent single resistive model as shown in Fig. 5.3, where the composite current

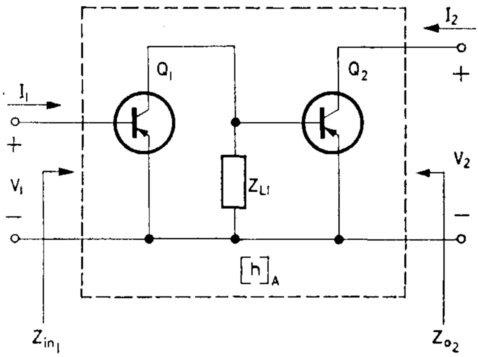

FIG. 5.2a. The amplifier section and twoport notations.

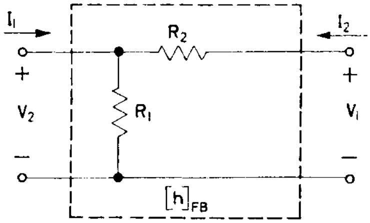

FIG. 5.2b. The passive feedback twoport.

FIG. 5.2. The voltage feedback pair split up into active and passive twoports.

gain β_0 is by definition the product of the short-circuit current gains of transistors Q_1 and Q_2 respectively. That is

$$A_{I0} = \beta_1\beta_2 = \beta_0. \tag{5.1}$$

If we conveniently stipulate that the load impedance Z_{L1} of the first transistor is the input impedance of the second stage, then the voltage gain A_{V0} in the absence of negative feedback is very nearly

$$A_{V0} \approx \beta_1\beta_2 \left\{\frac{Z_{\text{in}\,2}}{Z_{\text{in}\,1}}\right\} \left\{\frac{Z_L}{Z_{\text{in}\,2}}\right\}, \tag{5.2}$$

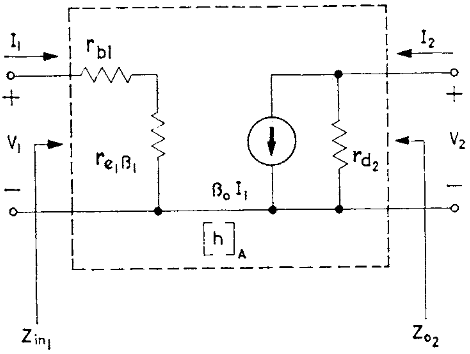

FIG. 5.3. Linear resistive model of the active twoport.

which can be written as

$$A_{V0} \approx \beta_0 \frac{Z_L}{Z_{\text{in}\,1}}, \tag{5.3}$$

where $Z_{\text{in}\,1}$ is the input impedance of the first stage and Z_L is the load impedance of the second stage. In terms of resistive transistor parameters

$$Z_{\text{in}\,1} = r_{b1} + r_{e1}\beta_1, \tag{5.4}$$

where, by definition r_{b1} and r_{e1} are the base and emitter resistances of transistor Q_1.

We restore now the connections between the amplifier and feedback network and redraw the composite circuit as shown in Fig. 5.4. Observe, that this is now a series–parallel type of twoport interconnection. From our studies in Part I we remember that such an interconnection calls for a mathematical model in the h matrix domain. Therefore, the equilibrium equation of the feedback amplifier in matrix form is

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}, \tag{5.5}$$

or in more compact notation

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = [h] \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}, \tag{5.6}$$

where the matrix h, of course, stands for the mathematical model of the feedback amplifier. Actually, this matrix is made up from the algebraic sum of two matrices:

$$[h] = [h]_A + [h]_{FB}\,. \tag{5.6a}$$

The matrices $[h]_A$ and $[h]_{FB}$ stand for the active and passive twoports in Figs. 5.2a and 5.2b respectively.

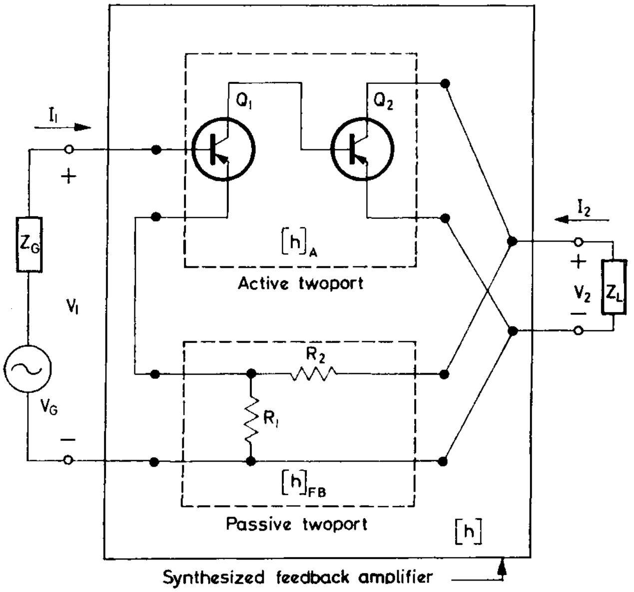

FIG. 5.4. Twoport representation of the voltage feedback pair.

In applying the h parameter constraints to the linear model of the amplifier twoport in Fig. 5.3, we obtain for $[h]_A$

$$[h]_A = \begin{bmatrix} \overbrace{Z_{\text{in}\,1}}^{h_{11a}} & \overbrace{0}^{h_{12a}} \\ \underbrace{\beta_0}_{h_{21a}} & \underbrace{\dfrac{1}{Z_{02}}}_{h_{22a}} \end{bmatrix}. \tag{5.7}$$

With a similar algebraic process applied to the feedback twoport in Fig. 5.2b,

$$[h]_{FB} = \begin{bmatrix} \overbrace{\dfrac{R_1 R_2}{R_1+R_2}}^{h_{11b}} & \overbrace{-\dfrac{R_1}{R_1+R_2}}^{h_{12b}} \\ \underbrace{\dfrac{R_1}{R_1+R_2}}_{h_{21b}} & \underbrace{\dfrac{1}{R_1+R_2}}_{h_{22b}} \end{bmatrix}. \qquad (5.8)$$

The complete mathematical model of the feedback pair is now easily synthesized by algebraic summation of matrices (5.7) and (5.8):

$$[h] = \begin{bmatrix} \overbrace{Z_{\text{in}\,1} + \dfrac{R_1 R_2}{R_1+R_2}}^{h_{11}} & \overbrace{-\dfrac{R_1}{R_1+R_2}}^{h_{12}} \\ \underbrace{\beta_0 + \dfrac{R_1}{R_1+R_2}}_{h_{21}} & \underbrace{\dfrac{1}{Z_{02}} + \dfrac{1}{R_1+R_2}}_{h_{22}} \end{bmatrix}. \qquad (5.9)$$

This mathematical model completely defines the internal characteristics of the feedback amplifier as shown within the boxed area in Fig. 5.4. The gain and terminal impedances with arbitrary generator and load terminations are easily derived in terms of these parameters.

(b) VOLTAGE GAIN

In reverting to Table I we find the desired equation for voltage gain of the terminated twoport in terms of h matrix parameters:

$$A_v = \frac{-h_{21}}{\Delta_h + h_{11}\dfrac{1}{Z_L}}, \qquad (5.10)$$

where h_{21} and h_{11} stand now for the corresponding elements in the matrix (5.9). However, before we can make use of this expression, it will be necessary to evaluate the determinant

Δ_h of the matrix (5.9). Hence, by cross-multiplication and subtraction of the diagonal terms,

$$\Delta_h = \beta_0 \frac{R_1}{R_1+R_2} + \left\{\frac{R_1}{R_1+R_2}\right\}^2 + \frac{Z_{\text{in}\,1}}{Z_{02}} + \frac{R_1R_2}{(R_1+R_2)Z_{02}} + $$

$$+ \frac{Z_{\text{in}\,1}}{R_1+R_2} + \frac{R_1R_2}{(R_1+R_2)^2}\,. \tag{5.11}$$

After rearranging terms,

$$\Delta_h = \beta_0 \frac{R_1}{R_1+R_2} + \frac{R_1R_2}{R_1+R_2}\left\{\frac{R_1}{(R_1+R_2)R_2} + \frac{1}{Z_{02}} + \frac{1}{R_1+R} + \frac{Z_{\text{in}\,1}}{R_1R_2}\right\}. \tag{5.12}$$

This expression looks rather formidable, but fortunately we can substantially simplify it, because in any physical feedback amplifier the following inequality holds:

$$\beta_0 \frac{R_1}{R_1+R_2} \gg 1. \tag{5.13}$$

Therefore in eqn. (5.12) the first term dominates. Hence, for the purpose of all engineering applications, eqn. (5.12) may be simplified to

$$\boxed{\Delta_h = \beta_0 \frac{R_1}{R_1+R_2}} \tag{5.14}$$

We can now proceed in evaluating the voltage gain of the feedback pair. Revert to eqn. (5.10) and substitute from the matrix (5.9) the appropriate elements and eqn. (5.14) for Δ_h:

$$A_V = -\frac{\beta_0 + \dfrac{R_1}{R_1+R_2}}{\beta_0 \dfrac{R_1}{R_1+R_2} + \left\{Z_{\text{in}\,1} + \dfrac{R_1R_2}{R_1+R_2}\right\}\dfrac{1}{Z_L}}\,. \tag{5.15}$$

Multiply through both numerator and denominator by Z_L:

$$A_V = -\frac{\beta_0 Z_L + \dfrac{R_1}{R_1+R_2} Z_L}{\beta_0 Z_L \dfrac{R_1}{R_1+R_2} + Z_{\text{in}\,1} + \dfrac{R_1R_2}{R_1+R_2}}\,. \tag{5.16}$$

Note that the second term in the numerator is negligible in relation to the first term. Therefore, by omitting it and rearranging the numerator,

$$\boxed{A_V = -\frac{\beta_0 Z_L}{[R_1/(R_1+R_2)]\,(\beta_0 Z_L + R_2) + Z_{\text{in}\,1}}} \qquad (5.17)$$

This expression is substantially accurate for practical feedback amplifiers. However, for the purpose of engineering approximations, it can be further simplified, since generally

$$\beta_0 Z_L \gg R_2 .$$

Therefore, from eqn. (5.17),

$$\boxed{A_V = -\frac{\beta_0 Z_L}{\beta_0 Z_L \dfrac{R_1}{R_1+R_2} + Z_{\text{in}\,1}}} \qquad (5.18)$$

If the first term in the denominator is very large, eqn. (5.18) reduces to

$$\boxed{A_V \approx -\left\{\frac{R_2}{R_1}+1\right\}} \qquad (5.19)$$

Note that the voltage gain is essentially independent of the intrinsic transistor characteristics.

(c) CURRENT GAIN

From Table I, we obtain the expression for current gain:

$$A_I = -\frac{h_{21}}{1+h_{22}Z_L}. \qquad (5.20)$$

By simple algebraic substitutions from the matrix (5.9) for the elements h_{21} and h_{22} in eqn. (5.20),

$$A_I = -\frac{\left\{\beta_0 + \dfrac{R_1}{R_1+R_2}\right\}}{1+\left\{\dfrac{1}{Z_{02}}+\dfrac{1}{R_1+R_2}\right\} Z_L}. \qquad (5.20a)$$

Multiply through numerator and denominator by (R_1+R_2):

$$A_I = -\frac{\beta_0(R_1+R_2)+R_1}{(R_1+R_2)+(R_1+R_2)\dfrac{Z_L}{Z_{02}}+Z_L}. \tag{5.21}$$

Note that the last term in the numerator is now quite negligible, hence it can be omitted. We can put eqn. (5.21) into a more convenient form by dividing both numerator and denominator again by the factor (R_1+R_2) and we get

$$A_I = -\frac{\beta_0}{1+Z_L/Z_{02}+Z_L/(R_1+R_2)}. \tag{5.22}$$

Rearrange terms:

$$\boxed{A_I = -\frac{\beta_0}{1+Z_L\left\{\dfrac{1}{Z_{02}}+\dfrac{1}{R_1+R_2}\right\}}} \tag{5.23}$$

For practical feedback amplifiers,

$$\frac{Z_L}{Z_{02}} \ll 1 \gg \frac{Z_L}{(R_1+R_2)}.$$

Hence eqn. (5.23) simplifies to the approximate form

$$\boxed{A_I \approx -\beta_0} \tag{5.24}$$

Note that the series–parallel feedback does not stabilize the current gain against transistor parameter variations.

(d) INPUT IMPEDANCE

We find from Table I the expression for the input impedance:

$$Z_{\text{in}} = \frac{\Delta_h + h_{11}\dfrac{1}{Z_L}}{h_{22}+\dfrac{1}{Z_L}}. \tag{5.25}$$

Substituting from the matrix (5.9) for the elements h_{11} and h_{22} and eqn. (5.14) for $\varDelta_h$,

$$Z_{\text{in}} = \frac{\beta_0 \dfrac{R_1}{R_1+R_2} + \left\{Z_{\text{in}\,1} + \dfrac{R_1R_2}{R_1+R_2}\right\}\dfrac{1}{Z_L}}{\dfrac{1}{Z_{02}} + \dfrac{1}{R_1+R_2} + \dfrac{1}{Z_L}}. \tag{5.26}$$

Multiply through both numerator and denominator by Z_L:

$$Z_{\text{in}} = \frac{\beta_0 Z_L \dfrac{R_1}{R_1+R_2} + Z_{\text{in}\,1} + \dfrac{R_1R_2}{R_1+R_2}}{\dfrac{Z_L}{Z_{02}} + \dfrac{Z_L}{R_1+R_2} + 1}. \tag{5.27}$$

Rearrange terms:

$$\boxed{Z_{\text{in}} = \frac{\dfrac{R_1}{R_1+R_2}(\beta_0 Z_L + R_2) + Z_{\text{in}\,1}}{1 + \dfrac{Z_L}{Z_{02}} + \dfrac{Z_L}{R_1+R_2}}} \tag{5.28}$$

This last expression is quite accurate for all design applications. Note, however, that the term $Z_{\text{in}1}$ in the numerator is small in comparison with the first group of terms. Similarly, the fractional terms in the denominator are generally very much less than unity. Therefore, for routine engineering approximations, eqn. (5.28) simplifies to

$$\boxed{Z_{\text{in}} \approx \frac{R_1}{R_1+R_2}(\beta_0 Z_L + R_2)} \tag{5.29}$$

Note that the input impedance of the *h* type feedback amplifier is very high.

(e) OUTPUT IMPEDANCE

Reverting again to Table I for the formula of the output impedance,

$$Z_{\text{out}} = \frac{h_{11} + Z_G}{\varDelta_h + h_{22} Z_G}. \tag{5.30}$$

Substitute here for the elements h_{11}, h_{22} and Δ_h as before:

$$Z_{\text{out}} = \frac{Z_{\text{in}1} + \dfrac{R_1 R_2}{R_1+R_2} + Z_G}{\beta_0 \dfrac{R_1}{R_1+R_2} + \left\{\dfrac{1}{Z_{02}} + \dfrac{1}{R_1+R_2}\right\} Z_G}. \tag{5.31}$$

Multiply numerator and denominator by (R_1+R_2):

$$Z_{\text{out}} = \frac{Z_{\text{in}1}(R_1+R_2) + R_1 R_2 + Z_G(R_1+R_2)}{\beta_0 R_1 + \dfrac{Z_G}{Z_{02}}(R_1+R_2) + Z_G}. \tag{5.32}$$

Rearrange terms:

$$Z_{\text{out}} = \frac{(R_1+R_2)(Z_{\text{in}1}+Z_G) + R_1 R_2}{\beta_0 R_1 + Z_G\left\{1 + \dfrac{R_1+R_2}{Z_{02}}\right\}}. \tag{5.33}$$

Divide through numerator and denominator by R_1:

$$Z_{\text{out}} = \frac{\left\{\dfrac{R_2}{R_1} + 1\right\}(Z_{\text{in}1}+Z_G) + R_2}{\beta_0 + \dfrac{Z_G}{R_1} + \dfrac{Z_G}{Z_{02}}\left\{\dfrac{R_2}{R_1} + 1\right\}}. \tag{5.34}$$

In practical feedback amplifiers $R_2 \gg R_1$; therefore, eqn. (5.34) can be simplified to

$$\boxed{Z_{\text{out}} = \frac{\dfrac{R_2}{R_1}(Z_{\text{in}\,1}+Z_G) + R_2}{\beta_0 + \dfrac{Z_G}{R_1} + \dfrac{Z_G}{Z_{02}}\left\{\dfrac{R_2}{R_1} + 1\right\}}} \tag{5.35}$$

For engineering approximations this expression can be further simplified. First by rearranging denominator and omitting the unity term

$$Z_{\text{out}} = \frac{\dfrac{R_2}{R_1}(Z_{\text{in}\,1}+Z_G) + R_2}{\beta_0 + \dfrac{Z_G}{R_1}\left\{1 + \dfrac{R_2}{Z_0}\right\}}, \tag{5.36}$$

and furthermore, if $Z_{02} \gg R_2$, then eqn. (5.36) will reduce to the simple form

$$\boxed{Z_{\text{out}} \approx \frac{R_2(Z_{\text{in}\,1} + Z_G + R_1)}{\beta_0 R_1 + Z_G}} \tag{5.37}$$

We note in conclusion, that the output impedance of the *h* type feedback amplifier is very low.

(f) SUMMARY OF TERMINAL PROPERTIES, (TABLE V)

The important analytical results pertaining to the *h* type transistor feedback pair is summarized in Table V. Circuit parameter definitions are related to the functional block diagrams which are self-explanatory.

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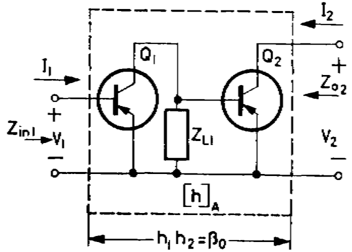
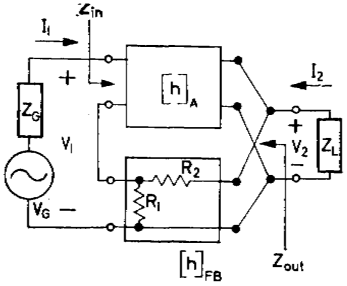
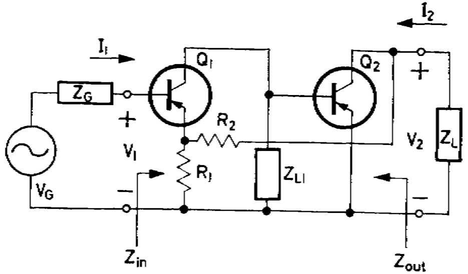
TABLE V. THE TRANSISTOR VOLTAGE FEEDBACK PAIR, SUMMARY OF TERMINAL PROPERTIES







	A_V Voltage gain	A_I Current gain	Z_{in} Input impedance	Z_{out} Output impedance
Generalized twoport matrix formula	$-\dfrac{h_{21}Z_L}{\Delta_h Z_L+h_{11}}$	$-\dfrac{h_{21}}{1+h_{22}Z_L}$	$\dfrac{\Delta_h Z_L+h_{11}}{1+h_{22}Z_L}$	$\dfrac{h_{11}+Z_G}{\Delta_h+h_{22}Z_G}$
Exact expression	$-\dfrac{\beta_0 Z_L}{\left(\dfrac{R_1}{R_1+R_2}\right)(\beta_0 Z_L+R_2)+Z_{\text{in}\,1}}$	$\dfrac{\beta_0}{1+\dfrac{Z_L}{Z_{02}}+\dfrac{Z_L}{R_1+R_2}}$	$\dfrac{\left(\dfrac{R_1}{R_1+R_3}\right)(\beta_0 Z_L+R_2)+Z_{\text{in}\,1}}{1+\dfrac{Z_L}{Z_{02}}+\dfrac{Z_L}{R_1+R_2}}$	$\dfrac{\dfrac{R_2}{R_1}(Z_{\text{in}\,1}+Z_G)+R_2}{\beta_0+\dfrac{Z_G}{R_1}\left(1+\dfrac{R_2}{Z_{02}}\right)}$
1st Approximation	$-\left(\dfrac{R_2}{R_1}+1\right)$	β_0	$\dfrac{R_1}{R_1+R_2}(\beta_0 Z_L+R_2)$	$\dfrac{R_2(Z_{\text{in}\,1}+Z_G+R_1)}{\beta_0 R_1+Z_G}$
Remarks	Stabilized voltage gain	Current gain is not stabilized by this type of feedback	High input impedance	Low output impedance

6. THE TRANSISTOR CURRENT FEEDBACK PAIR, g MATRIX APPLICATIONS

(a) SYNTHESIS OF THE MATHEMATICAL MODEL

The current feedback pair features low input impedance and high output impedance. These will be quite evident in the course of our study. The basic circuit diagram is shown in Fig. 6.1. It is taken for granted that the correct d.c. biasing conditions exist. We can look upon this amplifier configuration as having been synthesized from a unilateral two-stage transistor amplifier and a bilateral feedback two-port of resistors R_f and R_e. Furthermore, we can legitimately assume that the reactive components of each transistor, if any, are lumped into the impedances Z_G and Z_L respectively. As a convenient design simplification, we also assume that the interstage load impedance Z_{L1} includes, and is

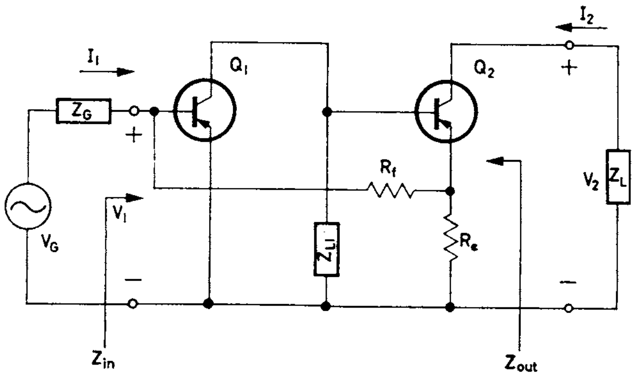


FIG. 6.1. The current feedback pair, simplified schematic.

dominated by, the input impedance of the second transistor Q_2. If we remove now the feedback elements R_f and R_e, we are left with an active twoport in the form of the conventional amplifier as shown in Fig. 6.2a. The resistive feedback

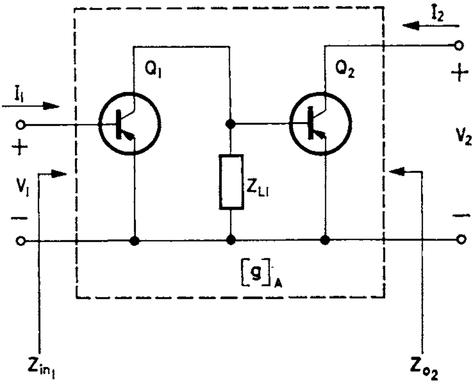


FIG. 6.2a. The amplifier section and twoport notations.

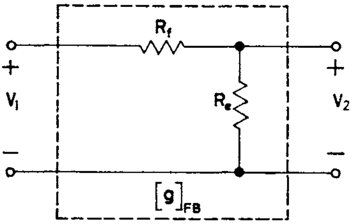


FIG. 6.2b. The passive feedback twoport.

FIG. 6.2. The current feedback pair split up into active and passive feedback twoport.

network can be redrawn separately as a passive twoport and is shown in Fig. 6.2b.

Having bisected the physical amplifier into independent active and passive twoports, it should now be instructive to re-combine them in accordance with the already familiar twoport notations. If we do so, the circuit block diagram is obtained as shown in Fig. 6.3. We note that this is a parallel–

series type of interconnection. Therefore, it calls for a mathematical model in the g matrix domain.

We have since stipulated that for the operating range of interest, our active twoport model is unilateral. Consequently,

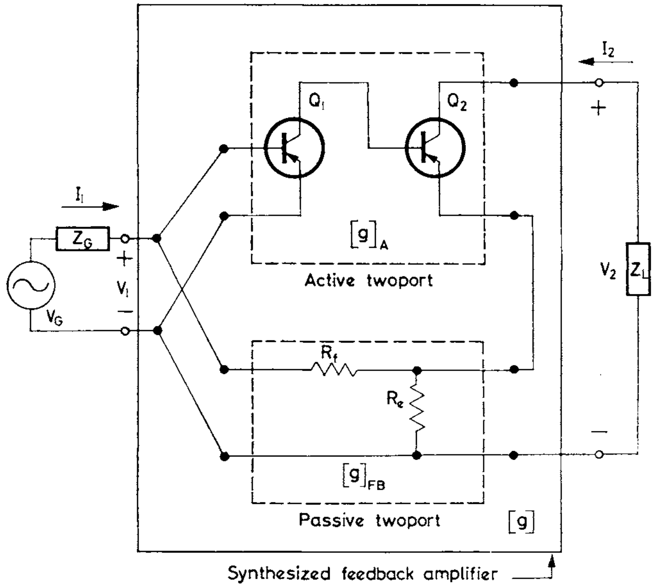


FIG. 6.3. Twoport representation of the current feedback pair.

in reverting to Fig. 6.2a, we can state that the short-circuit current gain of the cascaded stages can be expressed with good accuracy as the product of the individual current gains. That is

$$A_{I0} = \beta_1\beta_2 = \beta_0. \tag{6.1}$$

Similarly, the overall voltage gain is very nearly

$$A_{V0} \approx \beta_1\beta_2 \left\{\frac{Z_{\text{in}\,2}}{Z_{\text{in}\,1}}\right\} \left\{\frac{Z_L}{Z_{\text{in}\,2}}\right\}, \tag{6.2}$$

$$A_{V0} \approx \beta_0 \frac{Z_L}{Z_{\text{in}1}}, \tag{6.3}$$

where $Z_{\text{in }1}$ and $Z_{\text{in }2}$ are the input impedances of the first and second stages respectively. In terms of transistor constants

$$Z_{\text{in }1} = r_{b1} + r_{e1}\beta_1 \tag{6.4}$$

and

$$Z_{\text{in }2} = r_{b2} + r_{e2}\beta_2 = Z_{L1}, \tag{6.4a}$$

where r_b and r_e are the base and emitter resistances respectively.

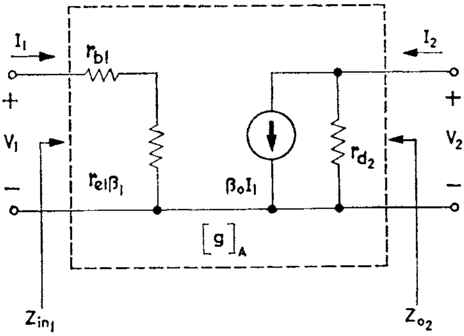


FIG. 6.4. Linear resistive model of the active twoport.

Using eqns. (6.1) and (6.4), we can construct a resistive model of the composite amplifier as shown in Fig. 6.4. The output impedance is that of the second transistor Q_2, which is by definition

$$Z_{02} = r_{d2} = \frac{r_c}{\beta_2}.$$

For the complete feedback amplifier in Fig 6.3, the equilibrium conditions are defined by the matrix equation

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} \tag{6.5}$$

or

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = [g] \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}, \tag{6.6}$$

where the matrix $[g]$ stands for the mathematical model of the amplifier. In other words it is the algebraic sum of the separate matrices of the active and passive twoports:

$$[g] = [g]_A + [g]_{FB}. \tag{6.7}$$

It will be now required to define the elements of the matrices $[g]_A$ and $[g]_{FB}$. By direct application of the basic rules as derived in Part I, from Fig. 6.4 we can write down directly the amplifier matrix $[g]_A$.

$$[g]_A = \begin{bmatrix} \overbrace{\dfrac{1}{Z_{\text{in}\,1}}}^{g_{11a}} & \overbrace{0}^{g_{12a}} \\ \underbrace{-\beta_0 \dfrac{Z_{02}}{Z_{\text{in}\,1}}}_{g_{21a}} & \underbrace{Z_{02}}_{g_{22a}} \end{bmatrix}. \tag{6.8}$$

Similarly, from Fig. 6.2b, for the passive feedback network $[g]_{FB}$ have

$$[g]_{FB} = \begin{bmatrix} \overbrace{\dfrac{1}{R_e+R_f}}^{g_{11b}} & \overbrace{\dfrac{R_e}{R_e+R_f}}^{g_{12b}} \\ \underbrace{-\dfrac{R_e}{R_e+R_f}}_{g_{21b}} & \underbrace{\dfrac{R_eR_f}{R_e+R_f}}_{g_{22b}} \end{bmatrix}. \tag{6.9}$$

By algebraic summation of these matrices, the process of synthesis of the mathematical model will be completed:

$$[g] = \begin{bmatrix} \overbrace{\left\{\dfrac{1}{Z_{\text{in}\,1}} + \dfrac{1}{R_e+R_f}\right\}}^{g_{11}} & \overbrace{\dfrac{R_e}{R_e+R_f}}^{g_{12}} \\ \underbrace{-\left\{\beta_0 \dfrac{Z_{02}}{Z_{\text{in}\,1}} + \dfrac{R_e}{R_e+R_f}\right\}}_{g_{21}} & \underbrace{\left\{Z_{02} + \dfrac{R_eR_f}{R_e+R_f}\right\}}_{g_{22}} \end{bmatrix} \tag{6.10}$$

which completely describes the boxed part of the feedback amplifier in Fig. 6.3.

(b) VOLTAGE GAIN

The gain and terminal conditionsof the amplifier can be now computed by reverting to Table I and substituting the elements from the matrix (6.10) into the appropriate g parameter equations. Accordingly, for the voltage gain, we have, if $Y_L \doteq 1/Z_L$,

$$A_v = \frac{g_{21}Z_L}{g_{22}+Z_L}. \tag{6.11}$$

Substitute into eqn. (6.11) for the elements g_{21} and g_{22} from the matrix (6.10):

$$A_v = -\frac{\left\{\beta_0 \frac{Z_{02}}{Z_{\text{in}1}} + \frac{R_e}{R_e+R_f}\right\} Z_L}{Z_{02}+\frac{R_eR_f}{R_e+R_f}+Z_L}. \tag{6.12}$$

Multiply through numerator and denominator by (R_e+R_f):

$$A_v = -\frac{\beta_0 Z_L Z_{02}\left\{\frac{R_e+R_f}{Z_{\text{in}1}}\right\}+R_eZ_L}{Z_{02}(R_e+R_f)+R_eR_f+Z_L(R_e+R_f)}. \tag{6.13}$$

Divide through by Z_{02} both numerator and denominator:

$$A_v = -\frac{\beta_0 \frac{Z_L}{Z_{\text{in}1}}(R_e+R_f)+R_e\frac{Z_L}{Z_{02}}}{(R_e+R_f)\left(1+\frac{Z_L}{Z_{02}}\right)+\frac{R_eR_f}{Z_{02}}}. \tag{6.14}$$

Divide both numerator and denominator next by R_e:

$$\boxed{A_v = -\frac{\beta_0 \frac{Z_L}{Z_{\text{in}1}}\left\{\frac{R_f}{R_e}+1\right\}+\frac{Z_L}{Z_{02}}}{\left\{\frac{R_f}{R_e}+1\right\}\left\{1+\frac{Z_L}{Z_{02}}\right\}+\frac{R_f}{Z_{02}}}} \tag{6.15}$$

We can neglect the last term in the numerator as it is negligible. Therefore, from eqn. (6.15),

$$\boxed{A_v = -\frac{\beta_0 \dfrac{Z_L}{Z_{\text{in}\,1}}\left\{\dfrac{R_f}{R_e}+1\right\}}{\left\{\dfrac{R_f}{R_e}+1\right\}\left\{1+\dfrac{Z_L}{Z_{02}}\right\}+\dfrac{R_f}{Z_{02}}}} \qquad (6.16)$$

If $R_f \ll Z_{02}$, then eqn. (6.16) simplifies to

$$A_v \approx -\frac{\beta_0 \dfrac{Z_L}{Z_{\text{in}\,1}}}{1+\dfrac{Z_L}{Z_{02}}}. \qquad (6.17)$$

In physical transistor amplifiers generally $Z_L \ll Z_{02}$, therefore eqn. (6.17) reduces to the approximate expression

$$\boxed{A_v \approx -\beta_0 \frac{Z_L}{Z_{\text{in}\,1}}} \qquad (6.18)$$

We note that this type of feedback interconnection does not stabilize the voltage gain.

(c) CURRENT GAIN

From Table I, we have the formula for current gain A_I;

$$A_I = \frac{g_{21}}{\Delta_g + g_{11}Z_L}. \qquad (6.19)$$

Observe, that before we could make effective use of this equation, we would require to know the determinant Δ_g of the matrix (6.10). Therefore we proceed to evaluate the determinant of (6.10). By cross-multiplication and subtraction of the cross-products,

$$\Delta_g = \left\{\frac{1}{Z_{\text{in}\,1}}+\frac{1}{R_e+R_f}\right\}\left\{Z_{02}+\frac{R_eR_f}{R_e+R_f}\right\}+$$
$$+\left\{\beta_0 \frac{Z_{02}}{Z_{\text{in}\,1}}\left(\frac{R_e}{R_e+R_f}\right)+\left(\frac{R_e}{R_e+R_f}\right)^2\right\}. \qquad (6.20)$$

Expand the products and rearrange terms:

$$\Delta_g = \frac{R_e}{R_e+R_f}\left\{\beta_0\frac{Z_{02}}{Z_{\text{in}1}}+\frac{R_f}{Z_{\text{in}1}}+\frac{R_f}{R_e+R_f}+\frac{R_e}{R_e+R_f}+\frac{Z_{02}}{R_e}\right\}+ \\ +\frac{Z_{02}}{Z_{\text{in}1}}. \tag{6.21}$$

Since

$$\frac{R_f}{R_e+R_f}+\frac{R_e}{R_e+R_f} = 1, \tag{6.22}$$

this term is negligible in comparison with the other terms inside the bracket. Therefore by dropping this negligible term and rearranging eqn. (6.21),

$$\Delta_g = \frac{R_e}{R_e+R_f}\beta_0\frac{Z_{02}}{Z_{\text{in}1}}+\frac{Z_{02}}{Z_{\text{in}1}}+\frac{R_e}{R_e+R_f}\left\{\frac{R_f}{Z_{\text{in}1}}+\frac{Z_{02}}{R_e}\right\}. \tag{6.23}$$

Rearrange the first three terms and we obtain

$$\Delta_g = \frac{Z_{02}}{Z_{\text{in}1}}\left\{\beta_0\frac{R_e}{R_e+R_f}+1\right\}+\frac{R_e}{R_e+R_f}\left\{\frac{R_f}{Z_{\text{in}1}}+\frac{Z_{02}}{R_e}\right\}. \tag{6.24}$$

Since in practical feedback amplifiers

$$\beta_0\frac{R_e}{R_e+R_f} \gg 1,$$

eqn. (6.24) will further simplify to

$$\Delta_g = \beta_0\frac{Z_{02}}{Z_{\text{in}1}}\left\{\frac{R_e}{R_e+R_f}\right\}+\frac{R_e}{R_e+R_f}\left\{\frac{R_f}{Z_{\text{in}1}}+\frac{Z_{02}}{R_e}\right\}. \tag{6.25}$$

The right-hand part of this equation now contains two terms, or rather groups of terms. The first one is dominating, the second one is negligibly small. Retaining the dominating term only, eqn. (6.25) reduces to

$$\boxed{\Delta_g \approx \beta_0\frac{Z_{02}}{Z_{\text{in}1}}\left\{\frac{R_e}{R_e+R_f}\right\}} \tag{6.26}$$

We can now substitute this expression for A_g and the elements g_{11} and g_{21} from the matrix in eqn. (6.19):

$$A_I = \frac{-\left\{\beta_0 \dfrac{Z_{02}}{Z_{\text{in}1}} + \dfrac{R_e}{R_e+R_f}\right\}}{\beta_0 \dfrac{Z_{02}}{Z_{\text{in}1}} \left\{\dfrac{R_e}{R_e+R_f}\right\} + \left\{\dfrac{1}{Z_{\text{in}1}} + \dfrac{1}{R_\epsilon + R_f}\right\} Z_L}. \tag{6.27}$$

Multiply through by (R_e+R_f) both numerator and denominator:

$$A_I = \frac{-\left\{\beta_0 \dfrac{Z_{02}}{Z_{\text{in}1}} (R_e+R_f) + R_e\right\}}{\beta_0 \dfrac{Z_{02}}{Z_{\text{in}1}} R_e + \left\{\dfrac{R_e+R_f}{Z_{\text{in}1}} + 1\right\} Z_L}. \tag{6.28}$$

The last term R_e in the numerator is negligible as is the unity terms in the denominator. By omitting these, then multiplying both numerator and denominator by $Z_{\text{in}1}$, we obtain

$$A_I = -\frac{\beta_0 Z_{02}(R_e+R_f)}{\beta_0 Z_{02} R_e + (R_e+R_f)Z_L}. \tag{6.29}$$

Again dividing through both numerator and denominator, this time $\beta_0 Z_{02}$,

$$A_I = -\frac{(R_e+R_f)}{R_e + \left\{\dfrac{R_e+R_f}{\beta_0 Z_{02}}\right\} Z_L}. \tag{6.30}$$

Rearrange terms:

$$A_I = -\left\{\frac{R_f}{R_e} + 1\right\} \frac{1}{1 + \left\{\dfrac{R_e+R_f}{R_e \beta_0 Z_{02}}\right\} Z_L}. \tag{6.31}$$

A further rearranging of terms yields

$$\boxed{A_I = -\left\{\frac{R_f}{R_e} + 1\right\} \frac{1}{1 + \left\{\dfrac{R_f}{R_e} + 1\right\} \dfrac{Z_L}{\beta_0 Z_{02}}}} \tag{6.32}$$

In practical transistor smplifiers, generally the $(R_f/R_e) \gg 1$, hence eqn. (6.32) simplifies to

$$\boxed{A_I = -\left\{\frac{R_f}{R_e}+1\right\}\frac{1}{1+\dfrac{R_f}{\beta_0 R_e}\left(\dfrac{Z_L}{Z_{02}}\right)}} \qquad (6.33)$$

This expression may be considered as sufficiently accurate for the majority of engineering design work. However, if $\beta_0 = 1000$, then eqn. (6.33) can be simplified to

$$\boxed{A_I \cong -\left\{\frac{R_f}{R_e}+1\right\}} \qquad (6.34)$$

Note that this type of feedback interconnection stabilizes the current gain. If the open-loop current gain is sufficiently large, then the closed-loop current gain is effectively controlled by the ratio of the feedback resistors R_f and R_e respectively. In other words, the amplifier current gain is practically independent of the transistor intrinsic characteristics.

(d) INPUT IMPEDANCE

From Table I we have the definition for the input impedance:

$$Z_{\text{in}} = \frac{g_{22}+Z_L}{\Delta_g + g_{11}Z_L}. \qquad (6.35)$$

By direct substitution from the matrix (6.10) for the parameters g_{11} and g_{22} and eqn. (6.26) for the determinant Δ_g, we obtain

$$Z_{\text{in}} = \frac{Z_{02}+\dfrac{R_e R_f}{R_e+R_f}+Z_L}{\beta_0\dfrac{Z_{02}}{Z_{\text{in}1}}\left\{\dfrac{R_e}{R_e+R_f}\right\}+\left\{\dfrac{1}{Z_{\text{in}1}}+\dfrac{1}{R_e+R_f}\right\}Z_L}. \qquad (6.36)$$

Multiply both numerator and denominator by (R_e+R_f) and then rearrange terms:

$$Z_{\text{in}} = \frac{(R_e+R_f)(Z_{02}+Z_L)+R_eR_f}{\beta_0 R_e \dfrac{Z_{02}}{Z_{\text{in}1}} + (R_e+R_f)\dfrac{Z_L}{Z_{\text{in}1}} + Z_L}, \tag{6.37}$$

Note that the last terms in both numerator and denominator are negligible quantities. Hence eqn. (6.37) simplifies to

$$Z_{\text{in}} = \frac{(R_e+R_f)(Z_{02}+Z_L)}{\beta_0 R_e \dfrac{Z_{02}}{Z_{\text{in}1}} + (R_e+R_f)\dfrac{Z_L}{Z_{\text{in}1}}}, \tag{6.38}$$

Divide both numerator and denominator by (R_e+R_f):

$$Z_{\text{in}} = \frac{Z_{02}+Z_L}{\dfrac{\beta_0 Z_{02} R_e}{(R_e+R_f)Z_{\text{in}1}} + \dfrac{Z_L}{Z_{\text{in}1}}}. \tag{6.39}$$

Divide again numerator and denominator, this time by Z_{02}:

$$Z_{\text{in}} = \frac{1+\dfrac{Z_L}{Z_{02}}}{\dfrac{\beta_0}{Z_{\text{in}1}}\left\{\dfrac{R_e}{R_e+R_f}\right\} + \dfrac{Z_L}{Z_{02}Z_{\text{in}1}}}. \tag{6.40}$$

We note that the last term in the denominator is now negligible, hence eqn. (6.40) further simplifies to

$$Z_{\text{in}} = \frac{1+\dfrac{Z_L}{Z_{02}}}{\dfrac{\beta_0}{Z_{\text{in}1}}\left\{\dfrac{R_e}{R_e+R_f}\right\}}. \tag{6.41}$$

We can put this expression into a more meaningful form by multiplying through numerator and denominator by $Z_{\text{in}1}$:

$$\boxed{Z_{\text{in}} = \frac{Z_{\text{in}1}\left\{1+\dfrac{Z_L}{Z_{02}}\right\}}{\beta_0 \dfrac{R_e}{R_e+R_f}}} \tag{6.42}$$

If $(Z_L/Z_{02}) \ll 1 \gg (R_f/R_e)$, then eqn. (6.42) reduces to

$$\boxed{Z_{\text{in}} \approx \frac{Z_{\text{in1}}}{\beta_0 \dfrac{R_e}{R_f}}} \tag{6.43}$$

Thus the parallel–series type feedback will cause the input impedance of the amplifier to become very low.

(e) OUTPUT IMPEDANCE

Revert to Table I for the expression of the output impedance. If we replace Y_G by $1/Z_G$, then we obtain,

$$Z_{\text{out}} = \frac{\Delta_g Z_G + g_{22}}{g_{11} Z_G + 1}. \tag{6.44}$$

We may proceed now with the appropriate substitutions into eqn. (6.44). For the elements g_{11} and g_{22} from the matrix (6.10) and eqn. (6.26) for Δ_g:

$$Z_{\text{out}} = \frac{\beta_0 \left\{\dfrac{Z_{02}}{Z_{\text{in1}}}\right\} \left\{\dfrac{R_e}{R_e+R_f}\right\} Z_G + Z_{02} + \dfrac{R_e R_f}{R_e+R_f}}{\left\{\dfrac{1}{Z_{\text{in1}}} + \dfrac{1}{R_e+R_f}\right\} Z_G + 1}. \tag{6.45}$$

Multiply both numerator and denominator by (R_e+R_f) and rearrange terms:

$$Z_{\text{out}} = \frac{R_e \left\{\beta_0 \dfrac{Z_G}{Z_{\text{in1}}} Z_{02} + Z_{02} + R_f\right\} + Z_{02} R_f}{\left\{\dfrac{Z_G}{Z_{\text{in1}}}\right\} (R_e+R_f) + Z_G + (R_e+R_f)}. \tag{6.46}$$

Equation (6.46) will substantially simplify, since in most physical transistor amplifiers $\beta_0 \gg 1$ and $\beta_0 Z_{02} \gg R_f$. Similarly $(R_e+R_f) \gg Z_G$. Therefore, from eqn. (6.46),

$$Z_{\text{out}} = \frac{Z_{02}R_e\beta_0\dfrac{Z_G}{Z_{\text{in1}}} + Z_{02}R_f}{(R_e+R_f)\left\{\dfrac{Z_G}{Z_{\text{in1}}}+1\right\}}. \tag{6.47}$$

We can put this expression into a more meaningful form if divide both numerator and denominator by R_e and rearrange terms:

$$\boxed{Z_{\text{out}} = \frac{Z_{02}\left\{\beta_0\dfrac{Z_G}{Z_{\text{in1}}}+\dfrac{R_f}{R_e}\right\}}{\left\{\dfrac{R_f}{R_e}+1\right\}\left\{\dfrac{Z_G}{Z_{\text{in}}}+1\right\}}} \tag{6.48}$$

This expression is sufficiently accurate for all engineering applications. However, it may be further simplified if $(R_f/R_e) \gg 1$; If such condition holds, then from eqn. (6.48),

$$\boxed{Z_{\text{out}} \approx \left\{\frac{R_e}{R_f}\right\}\frac{\beta_0 Z_{02}}{\dfrac{Z_{\text{in1}}}{Z_G}+1}} \tag{6.49}$$

Note that the output impedance of the g-type feedback pair is generally very high. On the other hand, eqn. (6.43) would indicate a correspondingly low input impedance.

(f) SUMMARY OF TERMINAL RROPERTIES, (TABLE VI)

The results of the foregoing analysis are summarized in Table VI for convenient design reference.

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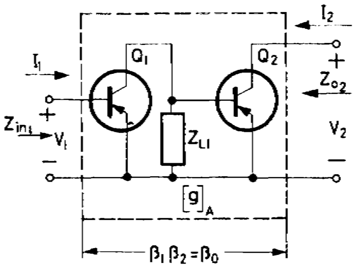
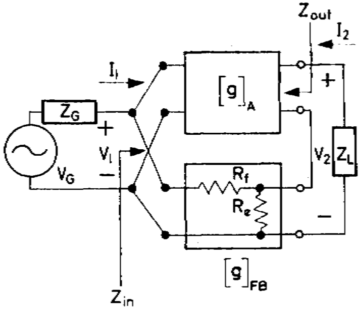
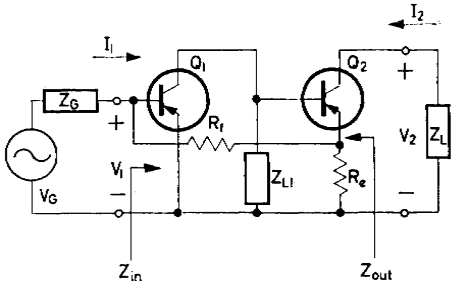
TABLE VI. THE TRANSISTOR CURRENT FEEDBACK PAIR, SUMMARY OF TERMINAL PROPERTIES







	A_V Voltage gain	A_I Current gain	Z_{in} Input impedance	Z_{out} Output impedance
Generalized twoport matrix formula	$\dfrac{g_{21}Z_L}{g_{22}+Z_L}$	$\dfrac{g_{21}}{\Delta_g+g_{11}Z_L}$	$\dfrac{g_{22}+Z_L}{\Delta_g+g_{11}Z_L}$	$\dfrac{\Delta_g Z_G+g_{22}}{g_{11}Z_G+1}$
Exact expression	$-\dfrac{\left\{\beta_0\dfrac{Z_L}{Z_{\text{in1}}}\right\}\left\{\dfrac{R_f}{R_e}+1\right\}}{\left\{\dfrac{R_f}{R_e}+1\right\}\left\{1+\dfrac{Z_L}{Z_{02}}\right\}+\dfrac{R_f}{Z_{02}}}$	$-\left\{\dfrac{R_f}{R_e}+1\right\}\dfrac{1}{1+\dfrac{R_f}{\beta_0 R_e}\left\{\dfrac{Z_L}{Z_{02}}\right\}}$	$\dfrac{Z_{\text{in1}}\left\{1+\dfrac{Z_L}{Z_{02}}\right\}}{\beta_0\left\{\dfrac{R_e}{R_e+R_f}\right\}}$	$\dfrac{Z_{02}\left\{\beta_0\dfrac{Z_G}{Z_{\text{in1}}}+R_f\right\}}{\left\{\dfrac{R_f}{R_e}+1\right\}\left\{\dfrac{Z_G}{Z_{\text{in1}}}+1\right\}}$
1st approximation	$-\beta_0\dfrac{Z_L}{Z_{\text{in1}}}$	$-\left\{\dfrac{R_f}{R_e}+1\right\}$	$\dfrac{Z_{\text{in1}}}{\beta_0\dfrac{R_e}{R_f}}$	$\left\{\dfrac{R_e}{R_f}\right\}\dfrac{\beta_0 Z_{02}}{\dfrac{Z_{\text{in1}}}{Z_G}+1}$
Remarks	Voltage gain is not stabilized by this type of feedback	Stabilized current gain	Low input impedance	High output impedance

7. THE FEEDBACK TRIPLET TRANSISTOR AMPLIFIER, APPLICATIONS OF THE *h* AND *Z* MATRICES

(a) GENERAL CONSIDERATIONS

In electronic design problems one is generally concerned with the efficient power transfer and amplification between some source and a specific load. The signal flow path may traverse several passive and active elements, such as impedance transformers, filters and amplifiers. With transistors as the active elements and at power levels of the order of 10 W, one is faced with uncomfortably low, input and output impedances. There are some areas of use where one would be happier if terminal impedances and transfer characteristics could be modified. It is well known that the application of negative feedback can profoundly modify the gain, linearity as well as the terminal impedances of an amplifier.(1, 2, 3) Transistor feedback amplifier design, particularly at low power levels, has been well covered in the literature.(4, 5, 6, 7) This chapter will be devoted to the more compact and elegant matrix methods of analysis and design of a three-stage transistor power amplifier.

Our study will commence with the recapitulation of the definitions for a single-stage non-feedback amplifier. Using this as a building block, a three-stage model will be then constructed. The approximate gain and terminal conditions of this elementary unilateral model will be related to the *h* and *Z* twoport parameters. In the next step a negativ feedback amplifier is a synthesized by the addition of a series feedback impedance.

A mathematical definition of the feedback system is obtained by combining algabraically the twoport Z parameters of the unilateral amplifier and the feedback impedance. The system matrix so formed will contain all pertinent characteristics of the feedback triplet. By simple algebraic manipulation of the system matrix, expressions are obtained for voltage gain, current gain, input and output impedances. The results are completely general and valid for series–series type of feedback amplifiers of arbitrary interstage complexity and load configuration.

Since transistors are subject to wide manufacturing tolerances, the problem of approximations has also been covered. A whole set of first and second order approximations have been derived which are usable for practical design work. The results are tabulated to show the mechanism of interplay between the controlling parameters. Thus the desired amplifier characteristics can be optimized with a minimum amount of computation.

A digression may be in order to Chapter 2, Part I, concerning twoport networks in terms of generalized Z parameters.

(b) DEFINITIONS OF THE NON-FEEDBACK SINGLE-STAGE AMPLIFIER MODEL

Before commencing the study of multistage feedback amplifiers, it will be instructive to recapitulate some of the well-established definitions pertaining to the basic building block, the elementary non feedback amplifier. Consider therefore, a single-stage transistor amplifier shown in Fig. 7.1.

The standard sign conventions apply and the current and voltage transfer ratios are defined to a good degree of accuracy as follows. Current gain A_I:

$$A_I = \frac{I_2}{I_1} = \frac{\beta}{1 + \frac{Z_L}{Z_{\text{out}}}}. \tag{7.1}$$

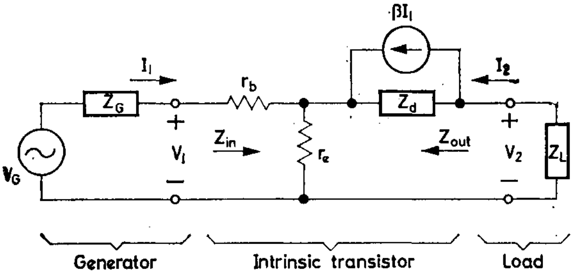


FIG. 7.1. Elementary single-stage amplifier.

Similarly, for voltage gain A_v:

$$A_V = \frac{V_2}{V_1} = -\beta\frac{Z_L}{Z_{\text{in}}}\left[\frac{1}{1+\frac{Z_L}{Z_{\text{out}}}}\right]. \tag{7.2}$$

If: $Z_L \ll Z_{\text{out}}$, then eqns. (7.1) and (7.2) reduce to

$$A_I \approx \beta, \tag{7.3}$$

$$A_V \approx -\beta\frac{Z_L}{Z_{\text{in}}}. \tag{7.4}$$

(c) DEFINITIONS OF THE THREE-STAGE NON-FEEDBACK MODEL

If we cascade three elementary amplifier stages of the type shown in Fig. 7.1, we obtain our three-stage non-feedback amplifier as in Fig. 7.2.

For individual stages, it is assumed that eqns. (7.3) and (7.4) are valid. The current and voltage gain between output and input terminal pairs may be considered as the product of individual gains as shown. It is further understood that as far as signal currents are concerned, the load impedance Z_{L1} of the first stage also includes the effective input impedance, and biasing resistors, if any, of the second stage. Similarly, Z_{L2} combines the corresponding components of the circuitry involving transistors Q_2 and Q_3.

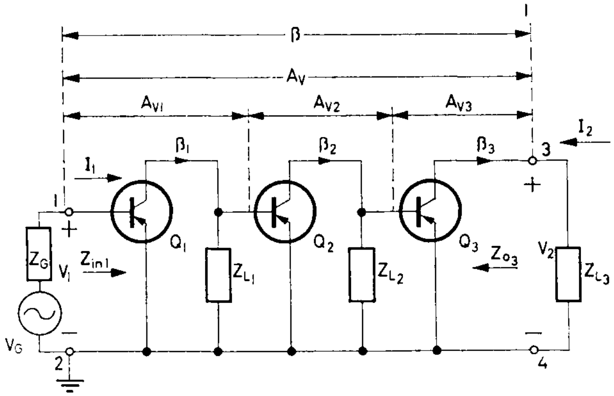


Fig. 7.2. The three-stage feedback amplifier.

Applying now eqns. (7.3) and (7.4) to the amplifier chain, the overall current and voltage gains are obtained.

From eqn. (7.3), and Fig. 7.2, the overall current gain is

$$\boxed{A_I = \frac{I_2}{I_1} = \beta_1 \times \beta_2 \times \beta_3 = \beta} \quad \text{(without feedback)} \qquad (7.5)$$

Similarly, from eqn. (7.4) and Fig. 7.2 for overall voltage gain.

$$A_V = \frac{V_2}{V_1} = A_{V1} \times A_{V2} \times A_{V3}, \qquad (7.6)$$

$$A_V = -\underbrace{\beta_1\beta_2\beta_3}_{\beta}\left(\frac{Z_{L1}}{Z_{\text{in}1}}\right)\left(\frac{Z_{L2}}{Z_{L1}}\right)\left(\frac{Z_{L3}}{Z_{L2}}\right), \qquad (7.7)$$

$$\boxed{A_V = -\beta\frac{Z_{L3}}{Z_{\text{in}1}}} \quad \text{(without feedback)} \qquad (7.8)$$

The linear model of one multistage amplifier has characteristics of an active unilateral twoport. The input and output ports are designated by the terminals 1–2 and 3–4 respectively. The amplifier as a system can be conveniently described in mathematical terms by a 2×2 parameter matrix. We prefer howewer, the familiar h parameter matrix.

$$[h]_{\text{amplifier}} = \begin{bmatrix} h_{11A} & h_{12A} \\ h_{21A} & h_{22A} \end{bmatrix}, \tag{7.9}$$

where, by definition and with reference to Fig. 7.2, the h parameters yield the following identities:

$$h_{11A} \equiv r_{b1} + \beta_1 r_{e1} \equiv Z_{\text{in}1}, \tag{7.10}$$

$$h_{21A} \equiv \beta_1\beta_2\beta_3 \equiv \beta, \tag{7.11}$$

$$h_{22A} \equiv \frac{1}{Z_{03}}. \tag{7.12}$$

Since our linear model in its elementary form represents a unilateral amplifier, the parameter h_{12} must therefore be zero. Hence in mathematical terms,

$$h_{12A} = 0. \tag{7.13}$$

With the definitions of the h parameters from eqns. (7.10) to (7.13) inclusive, the matrix (7.9) can be now rewritten in a form which represents physical conditions:

$$[h]_A = \begin{bmatrix} \overbrace{Z_{\text{in}1}}^{h_{11}} & \overbrace{0}^{h_{12}} \\ \underbrace{\beta}_{h_{21}} & \underbrace{\frac{1}{Z_{03}}}_{h_{22}} \end{bmatrix}. \tag{7.14}$$

(d) THE THREE-STAGE FEEDBACK AMPLIFIER AND THE Z PARAMETER MATRIX

It is well known that feedback modifies the gain, terminal impedances and frequency response of an amplifier. The stability criterion of feedback amplifiers will not be discussed here since the subject is well covered in the literature.(1, 5, 9)

RF power transistors exhibit notoriously low input and output impedances. We have therefore chosen the series–series feedback configuration in order to correct these undesirably low impedances and at the same time improve amplifier linearity. The basic method of feedback is shown in Fig. 7.3a and the physical arrangement in Fig. 7.3b. The feedback impedance Z_{fb} can be looked upon as a passive twoport and the amplifier as an active unilateral twoport network. It will be shown that the product βZ_{fb} has a profound effect on overall amplifier characteristics.

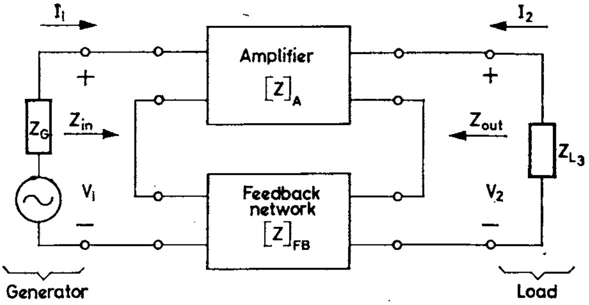


FIG. 7.3a. The basic series—series type of twoport interconnection.

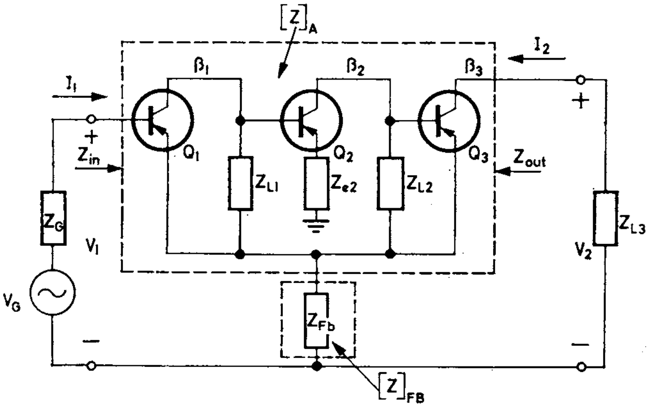


FIG. 7.3b. The physical interconnnection of amplifier and feedback twoports.

The physical interconnection of the unilateral amplifier with the passive feedback impedance will yield a new system matrix, formed by the algebraic summation of the Z matrices of the active and passive twoport parameters. Since the active twoport has been defined in terms of h parameters, it will be required to convert the matrix (7.14) into Z matrix form. Using algebraic process or tables, we will find that

$$[Z]_A = \begin{bmatrix} \overbrace{\dfrac{\Delta_{hA}}{h_{22A}}}^{Z_{11}} & \overbrace{\dfrac{h_{12A}}{h_{22A}}}^{Z_{12}} \\ \underbrace{-\dfrac{h_{21A}}{h_{22A}}}_{Z_{21}} & \underbrace{\dfrac{1}{h_{22A}}}_{Z_{22}} \end{bmatrix}, \qquad (7.15)$$

where Δ_h = Determinant of the amplifier's h matrix.

From eqn. (7.14) we obtain

$$\Delta_{hA} = \frac{Z_{\text{in}1}}{Z_{03}}. \qquad (7.16)$$

Substituting now the appropriate parameters from the matrix (7.14) into (7.15),

$$[Z]_A = \begin{bmatrix} \left(\dfrac{Z_{\text{in}1}}{Z_{03}}\right)(Z_{03}) & 0 \\ -\beta Z_{03} & Z_{03} \end{bmatrix}, \qquad (7.17)$$

$$[Z]_A = \begin{bmatrix} Z_{\text{in}1} & 0 \\ -\beta Z_{03} & Z_{03} \end{bmatrix}. \qquad (7.18)$$

Next, with reference to Fig. 7.3, we can write down by inspection the Z matrix of the feedback impedance Z_{fb}:

$$[Z]_{FB} = \begin{bmatrix} \overbrace{Z_{fb}}^{Z_{11}} & \overbrace{Z_{fb}}^{Z_{12}} \\ \underbrace{Z_{fb}}_{Z_{21}} & \underbrace{Z_{fb}}_{Z_{22}} \end{bmatrix}. \qquad (7.19)$$

Now, the complete feedback amplifier can be defined mathematically by the sum of the matrices (7.18) and (7.19):

$$[Z]_{\substack{\text{Feedback}\\ \text{amplifier}}} = [Z]_A + [Z]_{FB}. \tag{7.20}$$

Or in expanded form,

$$[Z]_{\substack{\text{Feedback}\\ \text{amplifier}}} = \begin{bmatrix} \overbrace{(Z_{\text{in}1}+Z_{fb})}^{Z_{11}} & \overbrace{Z_{fb}}^{Z_{12}} \\ \underbrace{(-\beta Z_{03}+Z_{fb})}_{Z_{21}} & \underbrace{(Z_{03}+Z_{fb})}_{Z_{22}} \end{bmatrix}. \tag{7.21}$$

This matrix now contains all information characterizing the feedback triplet in Fig. 7.3. Consequently, gain and terminal conditions can be obtained through simple algebraic manipulation. It is only required to apply substitution into the generalized twoport equations.

Before proceeding further, it will be necessary to evaluate the determinant of the matrix (7.21):

By cross-multiplication and subtraction,

$$\Delta_z = (Z_{\text{in}1}+Z_{fb})(Z_{03}+Z_{fb}) + \beta Z_{03} Z_{fb} - Z_{fb}^2. \tag{7.22}$$

Expanding the products and cancelling equal terms with opposite algebraic signs:

$$\Delta_z = Z_{\text{in}1} Z_{03} + Z_{\text{in}} Z_{fb} + Z_{fb} Z_{03} + Z_{fb}^2 + \beta Z_{03} Z_{fb} - Z_{fb}^2. \tag{7.23}$$

Rearranging terms:

$$\Delta_z = Z_{03} Z_{fb}(1+\beta) + Z_{\text{in}1}(Z_{03}+Z_{fb}). \tag{7.24}$$

In general for a multistage amplifier, $\beta \gg 1$, consequently eqn. (7.24) will simplify:

$$\Delta_z = \beta Z_{03} Z_{fb} + Z_{\text{in}1}(Z_{03}+Z_{fb}). \tag{7.25}$$

Rearranging terms:

$$\boxed{\Delta_z = Z_{fb}(\beta Z_{03} + Z_{\text{in}1}) + Z_{\text{in}1} Z_{03}} \tag{7.26}$$

(e) VOLTAGE GAIN

Revert to Fig. 7.3. and Table I in Part I. By substitution from the matrix (7.21) and eqn. (7.26),

$$A_V = \frac{Z_{21}Z_{L3}}{\Delta_z + Z_{11}Z_{L3}} =$$

$$= \frac{(-\beta Z_{03} + Z_{fb})Z_{L3}}{Z_{fb}(\beta Z_{03} + Z_{\text{in}1}) + Z_{\text{in}1}Z_{03} + (Z_{\text{in}1} + Z_{fb})Z_{L3}}. \qquad (7.27)$$

Expanding the denominator:

$$\mathrm{A}_V = \frac{-(\beta Z_{03} - Z_{fb})Z_{L3}}{Z_{fb}\beta Z_{03} + Z_{fb}Z_{\text{in}1} + Z_{\text{in}1}Z_{03} + Z_{\text{in}1}Z_{L3} + Z_{fb}Z_{L3}}. \qquad (7.28)$$

Rearranging terms:

$$A_V = -\frac{Z_{L3}(\beta Z_{03} - Z_{fb})}{Z_{fb}(\beta Z_{03} + Z_{\text{in}1} + Z_{L3}) + Z_{\text{in}1}(Z_{03} + Z_{L3})}. \qquad (7.29)$$

Since in any physical transistor feedback amplifier $\beta Z_{03} \gg Z_{fb}$, eqn. (7.23) will simplify to

$$A_V = -\frac{\beta Z_{L3}Z_{03}}{Z_{fb}(\beta Z_{03} + Z_{\text{in}1} + Z_{L3}) + Z_{\text{in}1}(Z_{03} + Z_{L3})}. \qquad (7.30)$$

Equation (7.30) now, can be put into a more convenient form by dividing both numerator and denominator by Z_{03}:

$$\boxed{A_V = -\frac{\beta Z_{L3}}{Z_{fb}\left[\beta + \frac{Z_{\text{in}1}}{Z_{03}} + \frac{Z_{L3}}{Z_{03}}\right] + Z_{\text{in}1}\left[1 + \frac{Z_{L3}}{Z_{03}}\right]}}$$

(7.31)

As a good approximation, eqn. (7.31) simplifies if $Z_{L3} \ll Z_{03}$:

$$\boxed{A_v \approx -\frac{\beta Z_{L3}}{Z_{fb}\left[\beta + \frac{Z_{\text{in}1}}{Z_{03}}\right] + Z_{\text{in}1}}} \qquad (7.32)$$

A further simplification is permissible if $Z_{\text{in}\,1} \ll Z_{fb}\beta$:

$$\boxed{A_V \approx -\frac{\beta Z_{L3}}{Z_{fb}\left[\beta+\dfrac{Z_{\text{in}\,1}}{Z_{03}}\right]}} \tag{7.33}$$

Reverting now to eqn. (7.32), dividing both numerator and denominator by $Z_{\text{in}\,1}$

$$A_V \approx -\frac{\beta\dfrac{Z_{L3}}{Z_{\text{in}1}}}{1+\dfrac{Z_{fb}}{Z_{\text{in}1}}\left[\beta+\dfrac{Z_{\text{in}1}}{Z_{03}}\right]}. \tag{7.34}$$

Note that this expression is of the form which is familiar from the feedback theory.(1, 2, 3) That is

$$A_V = \frac{G}{1+GB} \equiv \text{closed- loop gain,} \tag{7.34a}$$

where G = amplifier gain in the absence of feedback, B = complex feedback ratio, and GB = feedback loop gain.

When relating these definitions to eqn. (7.34), we find that

$$G = \beta\frac{Z_{L3}}{Z_{\text{in}\,1}} = \text{Amplifier gain in the absence of feedback,} \tag{7.34b}$$

$$GB = \frac{Z_{fb}}{Z_{\text{in}1}}\left[\beta+\frac{Z_{\text{in}\,1}}{Z_{03}}\right] = \text{Feedback loop gain,} \tag{7.34c}$$

$$B = \frac{Z_{fb}}{Z_{L3}}\left[1+\frac{Z_{\text{in}1}}{\beta Z_{03}}\right] = \text{Complex feedback ratio.} \tag{7.34d}$$

(f) CURRENT GAIN

Reverting to matrix (7.21) and to Table I in Part I and substituting, we can obtain the current gain of our amplifier:

$$A_I = \frac{Z_{21}}{Z_{22}+Z_L} = \frac{\beta Z_{03}+Z_{fb}}{Z_{03}+Z_{fb}+Z_{L3}}. \tag{7.35}$$

It has been earlier established that $Z_{fb} \ll \beta Z_{03}$, therefore eqn. (7.35) will simplify to

$$A_I = \frac{\beta Z_{03}}{Z_{03}+Z_{fb}+Z_{L3}}. \tag{7.36}$$

By dividing both numerator and denominator with Z_{03},

$$\boxed{A_I = \frac{\beta}{1+(Z_{fb}/Z_{03})+(Z_{L3}/Z_{03})}} \tag{7.37}$$

Further simplification is permissible if $Z_{fb} \ll Z_{03}$:

$$\boxed{A_I \approx \frac{\beta}{1+(Z_{L3}/Z_{03})}} \tag{7.38}$$

Note that the feedback has negligible effect on current gain. Furthermore, if $Z_{L3} \ll Z_{03}$, the eqn. (7.38) reduces to

$$\boxed{A_I \approx \beta} \tag{7.38a}$$

(g) INPUT IMPEDANCE

Revert again to Table I in Part I and substitute the appropriate parameters from the matrix (7.21) and eqn. (7.26) for Δ_z:

$$Z_{\text{in}} = \frac{\Delta_z + Z_{11}Z_{L3}}{Z_{22}+Z_{L3}} =$$

$$= \frac{Z_{fb}(\beta Z_{03}+Z_{\text{in}1})+Z_{\text{in}1}Z_{03}+(Z_{\text{in}1}+Z_{fb})Z_{L3}}{Z_{03}+Z_{fb}+Z_{L3}}. \tag{7.39}$$

Expand the numerator:

$$Z_{\text{in}} = \frac{Z_{fb}\beta Z_{03}+Z_{fb}Z_{\text{in}1}+Z_{\text{in}1}Z_{03}+Z_{\text{in}1}Z_{L3}+Z_{fb}Z_{L3}}{Z_{03}+Z_{fb}+Z_{L3}}. \tag{7.40}$$

Rearrange terms:

$$Z_{\text{in}} = Z_{\text{in}1}\frac{(Z_{03}+Z_{fb}+Z_{L3})}{(Z_{03}+Z_{fb}+Z_{L3})}+\frac{Z_{fb}(\beta Z_{03}+Z_{L3})}{(Z_{03}+Z_{fb}+Z_{L3})}, \quad (7.41)$$

$$\boxed{Z_{\text{in}} = Z_{\text{in}1}+Z_{fb}\frac{(\beta Z_{03}+Z_{L3})}{(Z_{03}+Z_{fb}+Z_{L3})}} \quad (7.42)$$

Simplification is possible, since generally $Z_{L3} \ll \beta Z_{03}$. Hence from eqn. (7.42)

$$Z_{\text{in}} = Z_{\text{in}1}+\frac{Z_{fb}\beta Z_{03}}{Z_{03}+Z_{fb}+Z_{L3}}. \quad (7.43)$$

Divide both numerator and denominator by Z_{03}:

$$\boxed{Z_{\text{in}} = Z_{\text{in}1}+\frac{\beta Z_{fb}}{1+(Z_{fb}/Z_{03})+(Z_{L3}/Z_{03})}} \quad (7.44)$$

A further simplification for engineering approximation is permissible if

$$\frac{Z_{L3}}{Z_{03}} > \frac{Z_{fb}}{Z_{03}} \ll 1$$

If this inequality holds, eqn. (7.44) reduces to

$$\boxed{Z_{\text{in}} \approx Z_{\text{in}1}+\frac{\beta Z_{fb}}{1+(Z_{L3}/Z_{03})}} \quad (7.45)$$

Note again that the product βZ_{fb} is the dominant controlling parameter of the input impedance. If $Z_{L3} \ll Z_{03}$ then eqn. (7.45) reduces to

$$\boxed{Z_{\text{in}} \approx Z_{\text{in}1}+\beta Z_{fb}} \quad (7.45a)$$

(h) OUTPUT IMPEDANCE

Reverting again to the matrix (7.21), to the determinant (7.26) and to Table I in Part I, by substituting the appropriate elements from matrix (7.21), we have the output impedance of our feedback amplifier:

$$Z_{\text{out}} = \frac{\Delta_Z + Z_{22}Z_G}{Z_{11}+Z_G} = \frac{Z_{fb}(\beta Z_{03}+Z_{\text{in}1}) + Z_{\text{in}1}Z_{03} + (Z_{03}+Z_{fb})Z_G}{Z_{\text{in}1}+Z_{fb}+Z_G}. \tag{7.46}$$

Expand the products in the numerator:

$$Z_{\text{out}} = \frac{Z_{fb}\beta Z_{03} + Z_{fb}Z_{\text{in}1} + Z_{\text{in}1}Z_{03} + Z_{03}Z_G + Z_{fb}Z_G}{Z_{\text{in}1}+Z_{fb}+Z_G}. \tag{7.47}$$

Rearrange terms:

$$Z_{\text{out}} = \frac{Z_{fb}(\beta Z_{03}+Z_{\text{in}1}+Z_G) + Z_{03}(Z_{\text{in}1}+Z_G)}{(Z_{\text{in}1}+Z_{fb}+Z_G)}. \tag{7.48}$$

Generally, in any physical transistor feedback, amplifier, $Z_{fb} \ll (Z_{\text{in}1}+Z_G)$. Consequently eqn. (7.48) simplifies to

$$Z_{\text{out}} = Z_{03}\frac{(Z_{\text{in}1}+Z_G)}{(Z_{\text{in}1}+Z_G)} + Z_{fb}\frac{(\beta Z_{03}+Z_{\text{in}1}+Z_G)}{Z_{\text{in}1}+Z_G}. \tag{7.49}$$

Further rearrange terms:

$$Z_{\text{out}} = Z_{03} + \frac{\beta Z_{fb}Z_{03}}{(Z_{\text{in}1}+Z_G)} + Z_{fb}\frac{(Z_{\text{in}1}+Z_G)}{(Z_{\text{in}1}+Z_G)}, \tag{7.50}$$

$$\boxed{Z_{\text{out}} = Z_{03} + Z_{fb} + \frac{\beta Z_{fb}Z_{03}}{(Z_{\text{in}1}+Z_G)}} \tag{7.51}$$

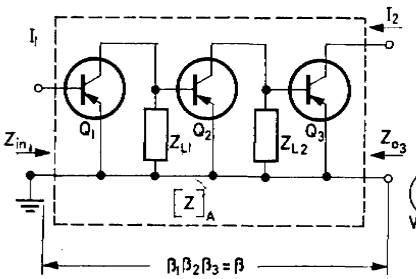
This expression simplifies if $Z_{fb} \ll Z_{03}$:

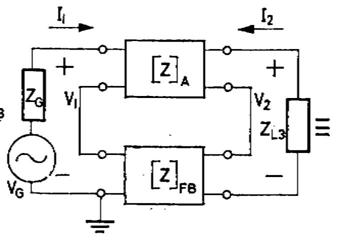
$$\boxed{Z_{\text{out}} \approx Z_{03} + \beta Z_{fb}\frac{Z_{03}}{(Z_{\text{in}1}+Z_G)}} \tag{7.52}$$

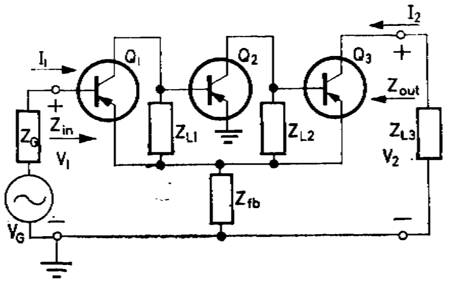
(i) CONCLUDING REMARKS

The foregoing analysis of the three-stage transistor amplifier has been carried out in a rather general form. We have assumed stable amplifier operation. Table VII contains a

TABLE VII. DEFINITIONS AND TERMINAL PROPERTIES OF THE FEEDBACK TRIPLET







	A_V Voltage gain	A_I Current gain	Z_{in} Input impedance	Z_{out} Output impedance
Generalized twoport matrix formula	$\frac{Z_{21}Z_{L3}}{\Delta_z+Z_{11}Z_{L3}}$	$\frac{Z_{21}}{Z_{22}+Z_{L3}}$	$\frac{\Delta_z+Z_{11}Z_{L3}}{Z_{22}+Z_{L3}}$	$\frac{\Delta_z+Z_{22}Z_G}{Z_{11}+Z_G}$
Fully expanded expression	$-\frac{\beta Z_{L3}}{Z_{fb}\left[\beta+\frac{Z_{\text{in1}}}{Z_{03}}+\frac{Z_{L3}}{Z_{03}}\right]+Z_{\text{in1}}\left[1+\frac{Z_{L3}}{Z_{03}}\right]}$	$\frac{\beta}{1+\frac{Z_{fb}}{Z_{03}}+\frac{Z_{L3}}{Z_{03}}}$	$Z_{\text{in1}}+\frac{\beta Z_{L3}}{1+\frac{Z_{fb}}{Z_{03}}+\frac{Z_{L3}}{Z_{03}}}$	$Z_{03}+Z_{fb}+\frac{\beta Z_{fb}Z_{03}}{Z_{\text{in1}}+Z_G}$
1st Approximation $Z_{fb} \ll Z_{03}$	$-\frac{\beta Z_{L3}}{Z_{fb}\left[\beta+\frac{Z_{\text{in1}}}{Z_{03}}\right]+Z_{\text{in1}}}$	$\frac{\beta}{1+\frac{Z_{L3}}{Z_{03}}}$	$Z_{\text{in1}}+\frac{\beta Z_{fb}}{1+\frac{Z_{L3}}{Z_{03}}}$	$Z_{03}+\frac{\beta Z_{fb}Z_{03}}{Z_{\text{in}}+Z_G}$
2nd Approximation $Z_{L3} \ll Z_{03} \gg Z_{fb}$	$-\frac{\beta Z_{L3}}{Z_{fb}\left[\beta+\frac{Z_{\text{in1}}}{Z_{03}}\right]}$	β	$Z_{\text{in1}}+\beta Z_{fb}$	

comprehensive summary of both exact and approximate design equations. For a review of the stability problem, references (1) to (6) inclusive may be consulted with profit.

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8. OSCILLATORY CIRCUITS, APPLICATIONS OF THE *Y* AND *h* PARAMETER MATRICES

(a) INTRODUCTORY REMARKS

It is well known that the mechanism of automatic amplitude regulation in vacuum tube and transistor oscillatory systems exploit the inherent nonlinear characteristics of the device. Therefore, matrix analysis of feedback oscillators will require a somewhat liberal interpretation of operating constraints. A logical extension of the linear twoport matrix theory is applicable if we consider the operating range of interest within the linear transfer characteristics of the active device. This is a perfectly legitimate assumption. Thus we can now formulate equilibrium equations. In fact, it will be shown that an oscillatorys ystem can be represented by a pair of suitably interconnected active and passive twoports. Oscillators are generally classified according to the mode of these interconnections.(1, 2) Some of the most useful types such as the *Colpitts* and *Hartley* may be reduced to a network model of parallel-connected active and passive twoports. In the linear mode of operation, matrix methods are admirably suited to the analysis and study of such systems.(1, 3, 4, 5, 10, 11)

There are several good examples in the published literature which deal with the various aspects of oscillator design.(6, 7, 8, 9) In the pages to follow we will attempt a comprehensive matrix analysis of the *Y* type Colpitts' oscillator. Typical transistor and tube variety of oscillator circuits will be studied. The basic circuits will be broken down into linear models which considerably simplify subsequent analytical work. Mathematically, these models are then described by the admittance matrix of the constituent twoports. A consistent step by step reasoning will be followed in deriving the frequency of oscillations and conditions of oscillations in terms of the pertinent circuit parameters.

(b) THE OSCILLATORY CIRCUITS, GENERAL

If d.c. biasing sources are ignored, the grounded plate or cathode follower Colpitts' oscillator is of the configuration as shown in Fig. 8.1. It is convenient to include the grid resistance R_G, which may be a physical element and part of the bias circuit or represent the input conductance of the tube at sufficiently high frequencies. The cathode resistor R_K is generally associated with the external load and as such it may also absorb the resistive losses of the inductance L.

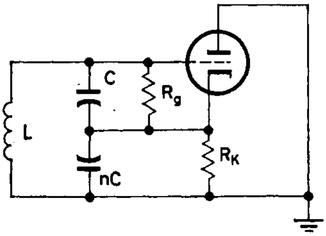

FIG. 8.1. The grounded-plate Colpitts' oscillator, simplified diagram.

The transistor oscillator of the grounded emitter variety is shown in Fig. 8.2. There again the d.c. biasing is ignored.

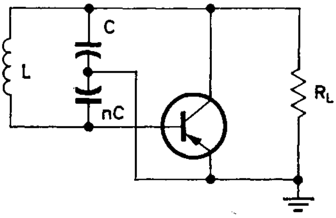

FIG. 8.2. The grounded-emitter Colpitts' oscillator, simplified diagram.

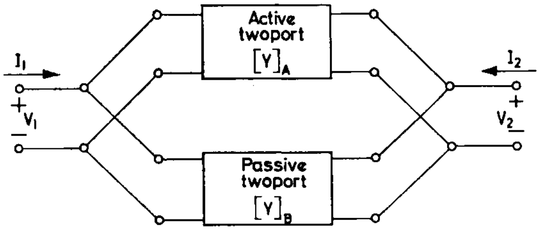

FIG. 8.3. Block diagram of the Colpitts' oscillator.

Both of these fundamental circuits may be generalized with the block diagram shown in Fig. 8.3.

By definition, the equilibrium equation which describes these parallel connected twoports is

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [Y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \tag{8.1}$$

where

$$[Y] = [Y]_A + [Y]_B. \tag{8.2}$$

The symbols $[Y]_A$ and $[Y]_B$ stand for the admittance parameter matrices of the active and passive twoports respectively. These parameter matrices will be obtained from the original circuit configurations in Figs. 8.1 and 8.2 by utilizing some basic principles of matrix algebra.

(c) THE GROUNDED-PLATE COLPITTS' OSCILLATOR

(i) Admittance matrix of the active twoport

As far as signal currents are concerned, we can redraw the oscillating triode as a twoport as shown in Fig. 8.4. The

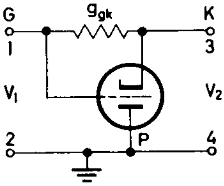

FIG. 8.4. Grounded-plate connection of the triode.

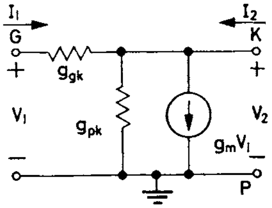

FIG. 8.5. Linear resistive model of the grounded-plate triode.

corresponding resistive linear model is represented in Fig. 8.5. In reverting to Chapter 1, we find that the grounded plate matrix form can be readily obtained from the floating admittance matrix. For convenience we repeat here the basic form:

$$[Y]_{\text{TRIODE}} = \begin{array}{c|c|c|c|c} & V_1 & V_2 & V_3 & \\ \hline I_1 & g_{gk} & 0 & -g_{gk} & G \\ \hline I_2 & g_m & g_{pk} & -g_m-g_{pk} & P \\ \hline I_3 & -g_{gk}-g_m & -g_{pk} & g_m+g_{pk}+g_{gk} & K \\ \hline & G & P & K & \end{array} \qquad (8.3)$$

From this 3×3 floating admittance matrix the twoport parameters may be obtained for any of the three possible grounded electrode configurations. For the purpose of this study we are interested in the grounded plate operating mode. The appropriate twoport parameter matrix is obtained by collapsing the second row and second column, leaving a 2×2 matrix:

$$[Y]_A = \begin{bmatrix} g_{gk} & -g_{gk} \\ -g_m-g_{gk} & g_m+g_{gk}+g_{pk} \end{bmatrix}. \qquad (8.4)$$

From this matrix, the resistive model of the grounded plate oscillator configuration can be synthesized and terminal conditions redefined as shown in Figs. 8.4 and 8.5.

(ii) Admittance matrix of the external feedback twoport

If we revert to Fig. 8.1, it is possible to redraw the passive circuit made up from L, C, nC and R_K in the form of a π network as shown in Fig. 8.6.

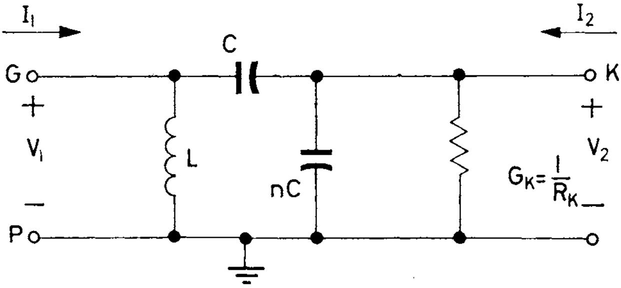

FIG. 8.6. Network model of the feedback circuit.

This network model can be described mathematically by the admittance parameter matrix.

From Fig. 8.6. by inspection,

$$[Y]_B = \begin{bmatrix} \frac{1}{pL}+pC & -pC \\ -pC & pC+pnC+G_K \end{bmatrix}. \tag{8.5}$$

(iii) Synthesis of the oscillating system

It is now quite feasible to synthesize the complete oscillating system from the constituent active and passive twoports. This is done simply by inserting into the block diagram in Fig. 8.3, the appropriate linear models from Figs. 8.5 and 8.6. The resulting oscillatory system is shown in Figs. 8.7a and 8.7b.

The new compound linear model in Fig. 8.7b, is mathematically completely described by the sum of the admittance matrices (8.4) and (8.5) as the "building blocks". Therefore the matrix of the oscillating system is

$$[Y]_{\text{osc.}} = [Y]_A + [Y]_B. \tag{8.6}$$

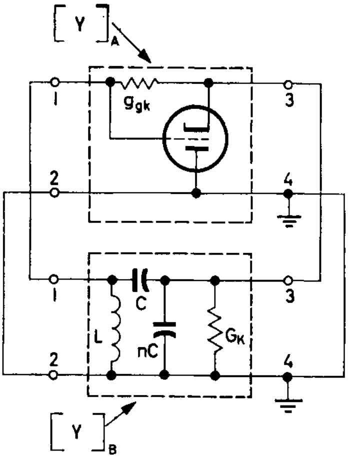

FIG. 8.7a. Oscillator circuit redrawn as parallel connected active and passive twoports.

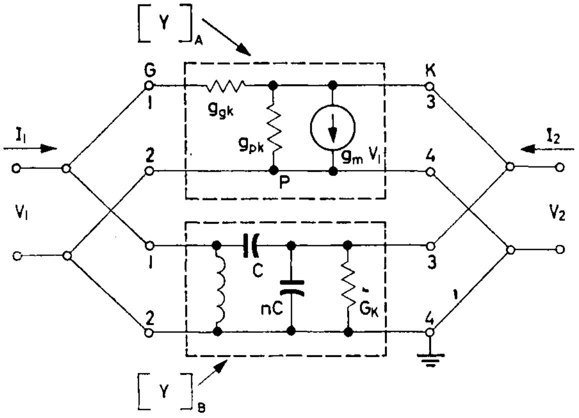

FIG. 8.7b. Linear model of the oscillating system.

$$[Y]_{\text{osc.}} = \begin{bmatrix} \overbrace{g_{gk} + \frac{1}{pL} + pC}^{Y_{11}} & \overbrace{-g_{gk} - pC}^{Y_{12}} \\ \underbrace{-g_m - g_{gk} - pC}_{Y_{21}} & \underbrace{g_m + g_{gk} + g_{pk} + G_k + pC + pnC}_{Y_{22}} \end{bmatrix} \quad .(8.7)$$

By definition, in an oscillating system the excitation function is zero. Therefore the equilibrium equation (8.1) for

$I_1 =' I_2 = 0$, simplifies to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = [Y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{8.8}$$

The nontrivial solution of eqn. (8.8) requires that the determinant of the Y matrix is zero. Hence for the condition of oscillation

$$[Y] = 0. \tag{8.9}$$

From eqn. (8.7),

$$[Y]_{\text{osc.}} = \begin{bmatrix} \overbrace{g_{gk} + \frac{1}{pL} + pC}^{Y_{11}} & \overbrace{-g_{gk} - pC}^{Y_{12}} \\ \underbrace{-g_m - g_{gk} - pC}_{Y_{21}} & \underbrace{g_m + (g_{gk} + g_{pk} + G_K) + pC(n+1)}_{Y_{22}} \end{bmatrix} = 0. \tag{8.10}$$

The Y_{22} of eqn. (8.10) can be written in a more compact form by putting

$$(g_{gk} + g_{pk} + G_K) = G_S, \tag{8.11}$$

where G_S = the sum of the passive conductance parameters. Hence, when using identity (8.11) the matrix (8.10) may be put into slightly more compact form:

$$[Y]_{\text{osc.}} = \begin{bmatrix} g_{gk} + \frac{1}{pL} + pC & -g_{gk} - pC \\ -g_m - g_{gk} - pC & g_m + G_S + pC(n+1) \end{bmatrix} = 0. \tag{8.12}$$

The difference of the cross-products will yield the determinant of this matrix:

$$|Y| = \left(g_{gk} + \frac{1}{pL} + pC\right)\{g_m + G_s + pC(n+1)\} - \\ - (g_m + g_{gk} + pC)(g_{gk} + pC) = 0. \tag{8.13}$$

Expanding the products:

$$\begin{aligned}|Y| = g_m g_{gk} + \frac{1}{pL} g_m + pCg_m + G_s g_{gk} + \frac{1}{pL} G_s + pCG_s \\ + npCg_{gk} + pCg_{gk} + \frac{C}{L}(n+1) + np^2C^2 + p^2C^2 \quad (8.14) \\ - g_m g_{gk} - g_{gk}^2 - g_{gk}pC - g_m pC - g_{gk}pC - p^2C^2 = 0.\end{aligned}$$

After cancellation of identical terms with opposing algebraic signs, eqn. (8.14) may now be rearranged:

$$\begin{aligned}|Y| = g_m \frac{1}{pL} + G_s g_{gk} + G_s \frac{1}{pL} + G_s pC + \\ + npCg_{gk} + \frac{C}{L}(n+1) + np^2C^2 - g_{gk}^2 - pCg_{gk} = 0.\end{aligned} \quad (8.15)$$

Now substitute $j\omega = p$ and separate the real and imaginary terms. From the theory of complex numbers, the real and imaginary parts will also be separately equal to zero.

Hence, the real part of eqn. (8.15) is

$$|Y|_{\text{Real}} = g_{gk}(G_s - g_{gk}) + \frac{C}{L}(n+1) - n\omega^2C^2 = 0. \quad (8.16)$$

Similarly, the imaginary part of eqn. (8.15) is

$$|Y|_{\text{Imaginary}} = \frac{1}{j\omega L}(g_m + G_s) + j\omega C\left\{G_s + g_{gk}(n-1)\right\} = 0. \quad (8.17)$$

Now, by definition, eqn. (8.16) will give the frequency of oscillations and eqn. (8.17) defines the criterion of oscillations.

(iv) The frequency of oscillations

Reverting to eqn. (8.16) and by transposition

$$n\omega^2C^2 = \frac{C}{L}(n+1) + G_s g_{gk} - g_{gk}^2, \quad (8.18)$$

$$\omega^2 = \frac{n+1}{LCn} + g_{gk}\frac{(G_s - g_{gk})}{C^2 n}, \qquad (8.19)$$

$$\boxed{\omega = \sqrt{\left[\frac{n+1}{LCn} + \frac{g_{gk}(G_s - g_{gk})}{C^2 n}\right]}} \qquad (8.20)$$

Equation (8.20) is an exact expression. From eqn. (8.11) note the identity

$$G_s \equiv (g_{gk} + g_{pk} + G_k).$$

In practical oscillator circuits the second term in eqn. (8.20) is a negligible quantity. Therefore for the purpose of engineering approximations, eqn. (8.20) reduces to

$$\boxed{\omega \approx \sqrt{\left(\frac{n+1}{LCn}\right)}} \qquad (8.21)$$

(v) The conditions of oscillation

Reverting to eqn. (8.17) and multiplying through by $j\omega C$.

$$\frac{C}{L}(g_m + G_s) - \omega^2 C^2 \{G_s + g_{gk}(n-1)\} = 0. \qquad (8.22)$$

By transposition

$$(g_m + G_s) = \omega^2 LC\{G_s + g_{gk}(n-1)\} \qquad (8.23)$$

$$\omega^2 LC = \frac{g_m + G_s}{G_s + g_{gk}(n-1)}. \qquad (8.24)$$

From eqn. (8.21) by squaring and transposition

$$\omega^2 LC = \frac{n+1}{n}. \qquad (8.25)$$

Now expressions (8.24) and (8.25) can be equated:

$$\frac{n+1}{n} = \frac{g_m + G_s}{G_s + ng_{gk} - g_{gk}}. \qquad (8.26)$$

By cross-multiplication:

$$G_s+nG_s+ng_{gk}+n^2g_{gk}-g_{gk}-ng_{gk} = ng_m+nG_s. \quad (8.27)$$

Rearrange terms and cancel where applicable:

$$n^2g_{gk}+ng_m+(G_s-g_{gk}) = 0. \quad (8.28)$$

Solve this quadratic for n by the standard form:

$$n_1, n_2 = \frac{-b\pm\sqrt{(b^2-4ac)}}{2a}, \quad (8.29)$$

where $a = g_{gk}$, $b = -g_m$, $c = G_s-g_{gk} = G_k+g_{pk}$.

$$\boxed{n_1, n_2 = \frac{g_m\pm\sqrt{[g_m^2-4g_{gk}(G_k+g_{pk})]}}{2g_{gk}}} \quad (8.30)$$

For a physical oscillatory circuit this expression can be again considerably simplified, since in general $g_{gk} \ll G_s$. If such conditions hold, eqn. (8.22) reduces to

$$\frac{C}{L}(g_m+G_s) = \omega^2C^2G_s, \quad (8.31)$$

$$\omega^2CL = \frac{g_m+G_s}{G_s}. \quad (8.32)$$

Substituting into eqn. (8.32) eqn. (8.25) for the term ω^2CL,

$$\frac{1+n}{n} = \frac{g_m+G_s}{G_s}, \quad (8.33)$$

$$\frac{1}{n} = \frac{g_m+G_s}{G_s}-1, \quad (8.34)$$

$$\frac{1}{n} = \frac{g_m+G_s-G_s}{G_s}. \quad (8.35)$$

By inversion from eqn. (8.35),

$$n = \frac{G_s}{g_m}. \quad (8.36)$$

Substituting, finally, eqn. (8.11) for G_s into eqn. (8.36):

$$\boxed{n = \frac{g_{gk}+g_{pk}+G_k}{g_m}} \tag{8.37}$$

It is evident from eqn. (8.37) that for sustained oscillations the limiting value of transconductance is

$$\boxed{g_m \geq \frac{g_{gk}+g_{pk}+G_k}{n}} \tag{8.38}$$

(d) THE GROUNDED EMITTER COLPITTS' OSCILLATOR

(i) Admittance matrix of the common-emitter transistor in terms of hybrid parameters

It is convenient to use the readily available hybrid parae meters of the transistor and then express the admittanc. matrix of the active twoport in terms of the h parameters- Here again the resistive and forward parameters are of interest. The reverse and reactive components will be combined with the external passive feedback network.

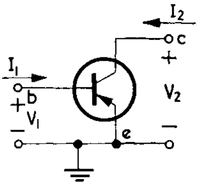

FIG. 8.8a. The grounded-emitter transistor.

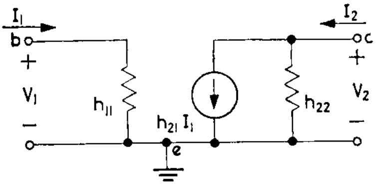

FIG. 8.8b. Linear resistive model of the grounded- emitter transistor if $h_{12} = 0$.

The grounded- emitter- connected transistor and the corresponding resistive model are shown in Fig. 8.8a and 8.8b. The equilibrium equations for the unilateral resistive model shown in Fig. 8.8b are defined as

$$V_1 = h_{11}I_1 + 0, \tag{8.39}$$

$$I_2 = h_{21}I_1 + h_{22}V_2. \tag{8.40}$$

By simple algebraic manipulation, the Y parameters can be obtained. From eqn. (8.39), by transposition,

$$I_1 = \frac{1}{h_{11}} V_1 + 0. \tag{8.41}$$

Substituting eqn. (8.41) into eqn. (8.40) for I_1,

$$I_2 = \frac{h_{21}}{h_{11}} V_1 + h_{22}V_2. \tag{8.42}$$

Now from eqns. (8.41) and (8.42) the Y matrix of the transistor may be formed in terms of the h parameters:

$$[Y]_{\text{Transistor}} = \begin{bmatrix} \frac{1}{h_{11}} & 0 \\ \frac{h_{21}}{h_{11}} & h_{22} \end{bmatrix}. \tag{8.43}$$

(ii) Admittance matrix of the external feedback twoport

Reverting to Fig. 8.2, the physical circuit of the oscillator, it is easy to see that the elements L, C, nC and R_L may be redrawn as a π network shown in Fig. 8.9

With reference to this network model, it will be assumed that the capacitors nC and C absorb the input and output capacitances of the transistor. The admittance matrix of the

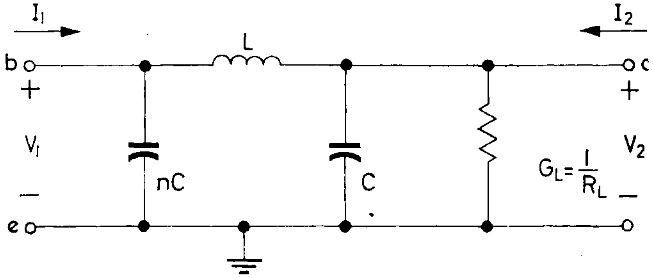

FIG. 8.9. π network equivalent of the feedback network.

twoport model in Fig. 8.9 may be written down by inspection:

$$[Y]_{\text{network}} = \begin{bmatrix} npC+\dfrac{1}{pL} & -\dfrac{1}{pL} \\ -\dfrac{1}{pL} & G_L+pC+\dfrac{1}{pL} \end{bmatrix}. \tag{8.44}$$

(iii) Synthesis of the oscillating system

By a process similar to the preceding vacuum tube example, the oscillating system will be synthesized by the parallel connection of active and passive twoports. The circuit configuration so obtained is shown in Fig. 8.10. An appropriate linear model may be drawn as in Fig. 8.11. Here again,

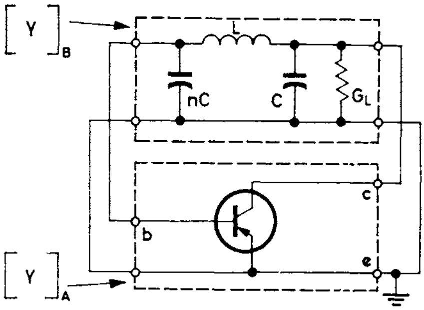

FIG. 8.10. The transistor grounded- emitter Colpitts' oscillator.

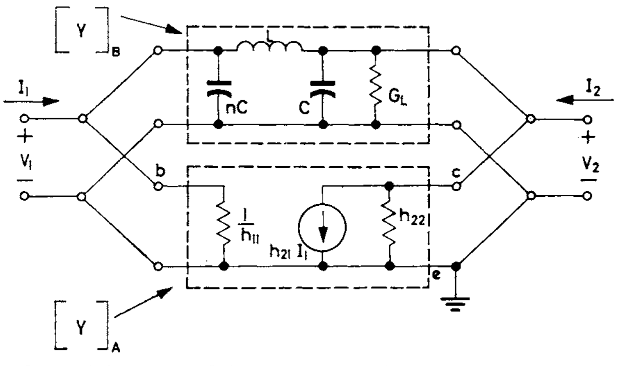

FIG. 8.11. Linear model of the transistor Colpitts' oscillator.

mathematically, the oscillating system is completely described by the sum of the admittance matrices of the constituent twoports.

Hence, adding eqn. (8.43) to eqn. (8.44),

$$[Y]_{\substack{\text{Oscillator}\\ \text{ce}}} = \begin{bmatrix} npC+\dfrac{1}{pL}+\dfrac{1}{h_{11}} & -\dfrac{1}{pL} \\ -\dfrac{1}{pL}+\dfrac{h_{21}}{h_{11}} & pC+\dfrac{1}{pL}+G_L+h_{22} \end{bmatrix}. \tag{8.45}$$

It is convenient to put in the Y_{22} term $G_0 \equiv G_L+h_{22}$, that is combine the load conductance with the transistor output conductance. With this slight simplification, the admittance matrix (8.45) may be rewritten:

$$[Y]_{\substack{\text{Oscillator}\\ \text{ce}}} = \begin{bmatrix} \overbrace{npC+\dfrac{1}{pL}+\dfrac{1}{h_{11}}}^{Y_{11}} & \overbrace{-\dfrac{1}{pL}}^{Y_{12}} \\ \underbrace{-\dfrac{1}{pL}+\dfrac{h_{21}}{h_{11}}}_{Y_{21}} & \underbrace{G_0+\dfrac{1}{pL}+pC}_{Y_{22}} \end{bmatrix}. \tag{8.46}$$

By definition, for the condition of oscillations the determinant of the system matrix (8.46) must be equated to zero.

Thus from the matrix (8.46),

$$|Y| = \left(npC+\frac{1}{pL}+\frac{1}{h_{11}}\right)\left(G_0+pC+\frac{1}{pL}\right)-\left(\frac{1}{p^2L^2}-\frac{h_{21}}{h_{11}pL}\right) = 0. \tag{8.47}$$

Expanding the products,

$$|Y| = G_0npC+G_0\frac{1}{pL}+G_0\frac{1}{h_{11}}+np^2C^2+\frac{C}{L}+pC\frac{1}{h_{11}}+$$

$$+n\frac{C}{L}+\frac{1}{p^2L^2}+\frac{1}{h_{11}pL}-\frac{1}{p^2L^2}+\frac{h_{21}}{h_{11}pL} = 0. \tag{8.48}$$

Rearranging eqn. (8.48) and cancelling identical terms with opposing algebraic signs:

$$|Y| = np^2C^2+\frac{C}{L}(n+1)+\frac{G_0}{h_{11}}+pC\left(\frac{1}{h_{11}}+nG_0\right)+$$
$$+\frac{1}{pL}\left(G_0+\frac{1}{h_{11}}+\frac{h_{21}}{h_{11}}\right) = 0. \qquad (8.49)$$

Put $p = j\omega$ and then equate the real and imaginary parts of eqn. (8.49) separately to zero.

$$|Y|_{\text{Real}} = -n\omega^2C^2+\frac{C}{L}(n+1)+\frac{G_0}{h_{11}} = 0, \qquad (8.50)$$

$$|Y|_{\text{Imaginary}} = j\omega C\left(\frac{1}{h_{11}}+nG_0\right)+\frac{1}{j\omega L}\left(G_0+\frac{1}{h_{11}}+\frac{h_{21}}{h_{11}}\right) = 0. \qquad (8.51)$$

(iv) Frequency of oscillations

Using eqn. (8.50) by transposition,

$$n\omega^2C^2 = \frac{C}{L}(n+1)+\frac{G_0}{h_{11}} \qquad (8.52)$$

$$\omega^2 = \frac{n+1}{LCn}+\frac{G_0}{h_{11}nC^2}, \qquad (8.53)$$

$$\boxed{\omega = \sqrt{\left(\frac{n+1}{LCn}+\frac{G_0}{h_{11}nC^2}\right)}} \qquad (8.54)$$

For a physical transistor oscillator the second term in eqn. (8.54) is generally negligible. Therefore for the purpose of engineering approximations, eqn. (8.54) reduces to

$$\boxed{\omega = \sqrt{\left(\frac{n+1}{LCn}\right)}} \qquad (8.55)$$

(v) Conditions of oscillation

Reverting to eqn. (8.51) and multiplying through by $j\omega C$,

$$Y|_{\text{Imaginary}} = -\omega^2C^2\left(\frac{1}{h_{11}}+nG_0\right)+\frac{C}{L}\left(G_0+\frac{1}{h_{11}}+\frac{h_{21}}{h_{11}}\right) = 0. \tag{8.56}$$

By transposition,

$$\omega^2LC = \frac{\left(G_0+\dfrac{1}{h_{11}}+\dfrac{h_{21}}{h_{11}}\right)}{\dfrac{1}{h_{11}}+nG_0}. \tag{8.57}$$

From eqn. (8.55)

$$\omega^2LC = \frac{n+1}{n}. \tag{8.58}$$

Substituting now eqn. (8.58) for ω^2LC into eqn. (8.57),

$$\frac{n+1}{n} = \frac{\left(G_0+\dfrac{1}{h_{11}}+\dfrac{h_{21}}{h_{11}}\right)}{\dfrac{1}{h_{11}}+nG_0}. \tag{8.59}$$

Multiply through the numerator and denominator of the right-hand part by h_{11}:

$$\frac{n+1}{n} = \frac{h_{11}G_0+1+h_{21}}{1+h_{11}nG_0}. \tag{8.60}$$

Cross-multiply:

$$(n+1)(1+h_{11}nG_0) = nh_{11}G_0+n+nh_{21}, \tag{8.61}$$

$$n+1+n^2h_{11}G_0+h_{11}nG_0 = nh_{11}G_0+n+nh_{21}. \tag{8.62}$$

Cancel similar terms and rearrange:

$$n^2h_{11}G_0-nh_{21}+1 = 0. \tag{8.63}$$

Solve eqn. (8.63) for n by the quadratic formula:

$$n_1,\, n_2, = \frac{-b\pm\sqrt{(b^2-4ac)}}{2a}, \tag{8.64}$$

where $a = h_{11}G_0$, $b = -h_{21}$, $c = 1$.

$$\boxed{n_1,\ n_2, = \frac{h_{21} \pm \sqrt{(h_{21}^2 - 4h_{11}G_0)}}{2h_{11}G_0}} \qquad (8.65)$$

Note that only positive values of n will yield meaningful solution. For the physical grounded- emitter oscillators, generally $h_{21}^2 \gg 4h_{11}G_0$.

Consequently, for the purpose of engineering design, eqn. (8.65) simplifies to

$$n = \frac{h_{21}+h_{21}}{2h_{11}G_0} = \frac{h_{21}}{h_{11}G_0}. \qquad (8.66)$$

By definitions,

$$h_{21} \equiv \beta = \text{Forward current gain},$$

$$G_0 \equiv h_{22}+G_L = h_{22}+\frac{1}{R_L}.$$

Therefore when substituting these identities into eqn. (8.66) the limiting value of n is

$$\boxed{n = \frac{\beta}{h_{11}\left(h_{22}+\frac{1}{R_L}\right)}} \qquad (8.67)$$

It is evident from eqn. (8.67) that for sustained oscillations the limiting magnitude of β is

$$\boxed{\beta \geqq nh_{11}\left(h_{22}+\frac{1}{R_L}\right)} \qquad (8.68)$$

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APPENDIX

Problems

Problem sets are keyed to the main body of the text, i.e. I-2d stands for problem d which is related to Part I, Chapter 2.

PART I

I-1a. Find the algebraic sum of the matrices:

$$\text{(i)}\quad \begin{bmatrix} ab & cd \\ kf & zw \end{bmatrix} + \begin{bmatrix} 3s & 2s \\ \frac{1}{s} & \frac{2}{s} \end{bmatrix}.$$

$$\text{(ii)}\quad \begin{bmatrix} (a+\mathrm{j}X) & (b-\mathrm{j}Y) \\ (c+\mathrm{j}KX) & (d+\mathrm{j}Y) \end{bmatrix} + B\begin{bmatrix} 2a & -b \\ 3c & -2d \end{bmatrix}.$$

I-1b. Perform the indicated multiplications:

$$\text{(i)}\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad \text{(ii)}\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\text{(iii)}\quad [a \quad b \quad c \quad d]\begin{bmatrix} K \\ L \\ M \\ N \end{bmatrix}.$$

$$\text{(iv)}\quad \begin{bmatrix} 1 & 0 \\ KY_1 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{Z}{F} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ MY_2 & 1 \end{bmatrix}.$$

I-1c. Expand the matrix products and form a set of simultaneous equations:

$$\begin{bmatrix} sL_1 & -\frac{1}{sC_1} & \frac{2}{sK_1} \\ s3L_2 & 0 & \frac{1}{sK_2} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

I-1d. Put into matrix form the equations:

(i) $Y = e_1x_1 + e_2x_2 - e_3x_3.$

(ii) $$\left.\begin{aligned} sI_1 + 10sI_2 &= v_1, \\ -0{\cdot}5sI_1 + +4sI_2 &= v_2. \end{aligned}\right\}$$

(iii) $$\left.\begin{aligned} (a+b)x_1 + (c+d)x_2 &= 3, \\ ex_1 + fx_2 \quad &= -6. \end{aligned}\right\}.$$

I-1e. Given the simultaneous equation, solve for the unknown variables V_1 and V_2, using the method of determinants.

$$\left.\begin{aligned} 3sV_1 + 9sV_2 &= I_1, \\ 4sV_2 - 2sV_1 &= I_2. \end{aligned}\right\}$$

I-1f. Put into matrix form the simultaneous equations and solve for x_1 and x_2:

$$\begin{aligned} F &= x_1 \sin A + x_2 \cos A, \\ W &= -x_1 \cos B + x_2 \cos B. \end{aligned}$$

I-1g. Find the inverse of the following matrices:

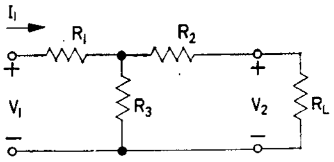
(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (ii) $K\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (iii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

I-1h. Derive the inverse matrices of the following expressions:

(i) $\begin{bmatrix} 3a & 4b \\ 2c & 7d \end{bmatrix}$. (ii) $\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$. (iii) $\begin{bmatrix} h_{11} & -h_{12} \\ h_{21} & h_{22} \end{bmatrix}$.

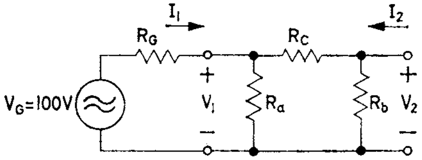
I-2a. Find the h matrix for the resistive network and then compute the input impedance and voltage transfer ratio if:

$R_1 = 1000$ ohm,
$R_2 = 200$ ohm,
$R_3 = 500$ ohm,
$R_L = 300$ ohm.

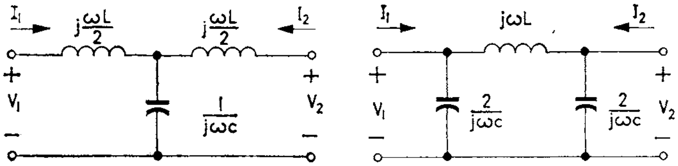


I-2b. Find the h matrix of the π network, derive expressions for output impedance and open-circuit current transfer ratio if:

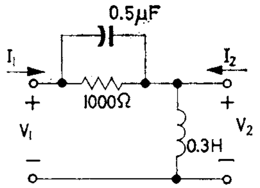
$R_a = 25{,}000$ ohm,
$R_b = 10{,}000$ ohm,
$R_c = 50{,}000$ ohm,
$R_G = 30{,}000$ ohm.



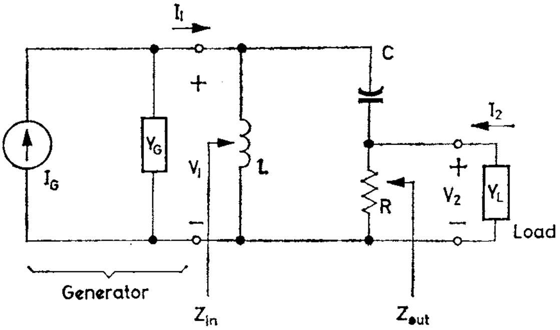
I-2c. Derive the $ABCD$ matrix for the reactive networks as shown



I-2d. Derive the transmission ($ABCD$) matrix for the equalizer network and compute the input impedance and open-circuit voltage transfer ratios at 100 c/s, 1 kc/s and 10 kc/s.

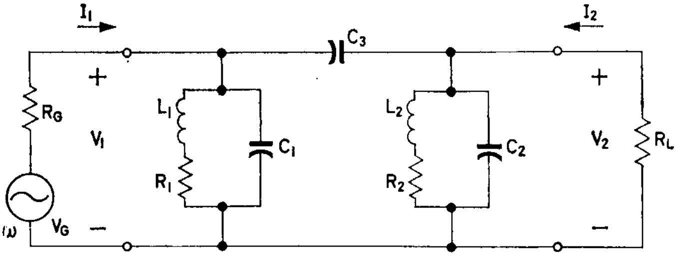


I-2e. Find the Y matrix, input impedance, output impedance, voltage and current transfer ratio for the terminated LCR network. The



generator is a current source with terminal admittance of Y_G and angular frequency of $\omega = 2\pi f$ cycles.

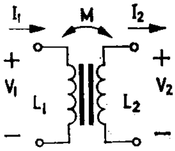
I-2f. Consider the capacitively coupled symmetrical coupled circuit. Derive an expression for voltage and current transfer ratio with resistive generator and load. Assume that $C_1 = C_2$, $L_1 = L_2$ and $R_1 = R_2$. (*Hint:* Try admittance parameters.)



I-3a. If the transmission matrix of the ideal transformer is given:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \dfrac{L_1}{M} & 0 \\ 0 & \dfrac{L_2}{M} \end{bmatrix}$$

then by using tables or algebraic methods, obtain the h and g matrices.



I-3b. Using tables, transform the h matrices into the g and Y domains:

$$\text{(i)}\quad [h] = \begin{bmatrix} \dfrac{ab}{a+b} & \dfrac{-a}{a+b} \\ \dfrac{a}{a+b} & \dfrac{1}{a+b} \end{bmatrix}.$$

$$\text{(ii)}\quad [h] = \begin{bmatrix} \dfrac{1}{j\omega C} & -1 \\ 1 & \dfrac{1}{R} \end{bmatrix}.$$

I-3c. Using algebraic methods or tables, transform the Y matrix into the Z, h, g and $ABCD$ domains:

$$[Y] = \begin{bmatrix} Y_1 & -\frac{Y_1}{K} \\ BY_1 & -\frac{B}{K}(Y_1+Y_2) \end{bmatrix}.$$

I-3d. A transistor in the common-base connection can be defined by the $ABCD$ matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{cb} = \frac{1}{r_b+\alpha r_c}\begin{bmatrix} (r_e+r_b) & r_e(r_b+r_c)+r_b r_c(1-\alpha) \\ 1 & (r_b+r_c) \end{bmatrix}.$$

By using tables, transform this mathematical model into the Z, Y, h and g domains. Show the simplifications which are feasible if the current gain $\alpha \to 1$.

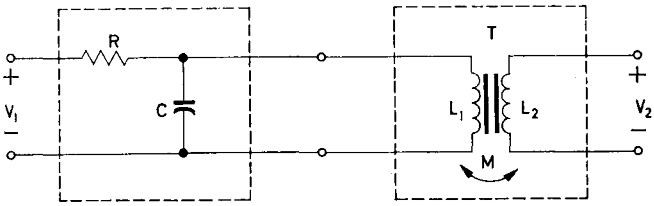
I-3e. For low-frequency applications the common-emitter mode operated transistor may be approximated by the h matrix:

$$[h]_{ce} = \begin{bmatrix} (r_b+\beta r_e) & 0 \\ \beta & \frac{\beta}{r_c} \end{bmatrix}.$$

Obtain a mathematical model of this transistor in the Y, Z, g and $ABCD$ matrix domains by using tables and/or algebraic process.

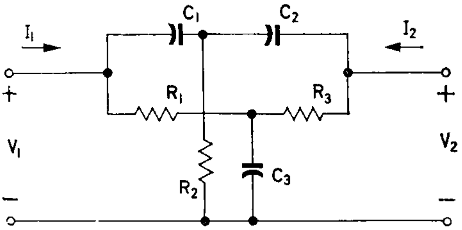
I-4a. Find the h matrix of the interconnected low-pass filter and ideal transformer.

(*Hint:* Kefer to Problem I-3a for the transmission matrix of the ideal transformer.)



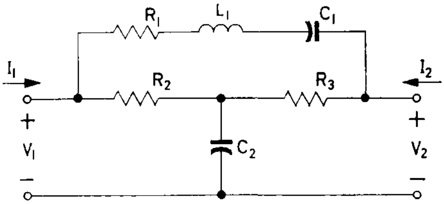
I-4b. Obtain the mathematical model in the Y matrix domain of the twin T filter network. Assume that $C_1 = C_2$ and $R_1 = R_2$.

(*Hint:* Split the network into two parallel T structures and find the Y matrices.)



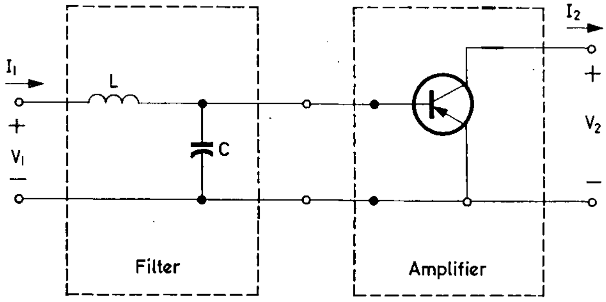
I-4c. The linear model of a frequency equalizing network is given, find the corresponding g matrix.

(*Hint:* Split the top and bottom parts.)



I-4d. A low-pass filter is terminated by a transistor amplifier in the common-emitter connection. Find the $ABCD$ matrix of the cascaded active and passive twoports.

(*Hint:* The h matrix of the transistor already defined in Problem I-3e)



I-4e. Given the Darlington- connected emitter follower pair. Find the Y matrix of the interconnection if the transistor parameters are defined in the h matrix domain:

$$[h]_a = \begin{bmatrix} h_{11a} & h_{12a} \\ h_{21a} & h_{22a} \end{bmatrix} \qquad [h]_b = \begin{bmatrix} h_{11b} & h_{12b} \\ h_{21b} & h_{22b} \end{bmatrix}.$$

Find the current gain and input impedance of the configuration if the resistive parameters are as follows:

$h_{11a} = 1000$ ohm, $h_{11b} = 100$ ohm,

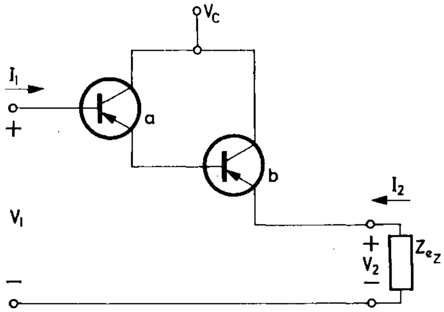
$h_{12a} = 0$, $h_{12b} = 10^{-3}$,

$h_{21a} = 50$, $h_{21b} = 30$,

$h_{22a} = 5 \times 10^{-5}$ mho, $h_{22b} = 10^{-4}$ mho,

$Z_{e2} = 1000$ ohm.

(*Hint:* Transform the h parameters into the *ABCD* domain.)



I-4f. A cascode amplifier with transistors 1 and 2, defined in terms of the *ABCD* matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_1 = \begin{bmatrix} \dfrac{r_e + r_b}{\beta r_d} & r_e \\ \dfrac{1}{\beta r_d} & \dfrac{1}{\beta} \end{bmatrix},$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_2 = \begin{bmatrix} \dfrac{r_e + r_b}{\alpha r_c} & \dfrac{r_e(r_b + r_c)}{\alpha r_c} \\ \dfrac{1}{\alpha r_c} & \dfrac{1}{\alpha} \end{bmatrix}.$$

The transistors are assumed to be identical with the resistive parameters of:

$r_e = 25$ ohm,

$r_b = 100$ ohm,

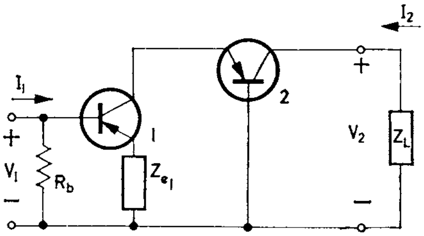
$\alpha = 0{\cdot}98$,

$\beta = 50$.

The base resistor $R_b = 50{,}000\,\Omega$ and it is required for biasing only. Find generalized expressions for:

(i) Current gain, voltage gain and input impedance.

(ii) If the load $Z_L = 2000$ ohm, what external impedance must be used for Z_{el} to obtain an amplifier input impedance of 10,000 ohm?



PART II

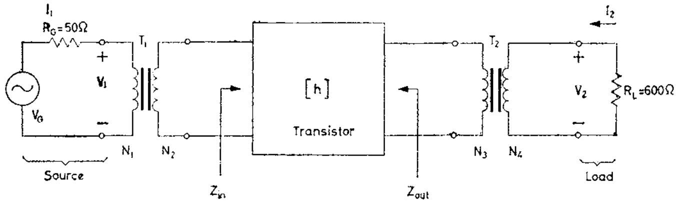
II-2a. Given a transistor power amplifier. It is required to provide optimum gain and power transfer between a 50 ohm source and a 600 ohm load as shown. Assume ideal transformers for T_1 and T_2.

The transistor is operating in the common-emitter mode and the intrinsic parameters are related to the h matrix:

$$[h]_{ce} = \begin{bmatrix} 500 & 0 \\ 75 & \frac{1}{5000} \end{bmatrix}.$$

(i) Find the turns ratio of the transformers for image matched source and load.

(ii) Compute the power gain under matched conditions.



PART III

III-1a. Find the optimum, image matched, input and output coupling network parameters for a grounded-grid RF amplifier stage V_2. At the operating frequency of 15 Mc/s satisfy that $R_{\mathrm{out}\,1} = R_{\mathrm{in}\,2}$ and

$R_{\text{out}\,2} = R_L$. Assume that the correct cathode, grid and plate bias conditions exist. Furthermore, that input and output capacitances of the tube V_2 can be considered as summed together with C_2 and C_3 respectively.

The parameters of V_2: $g_m = 20{,}000\ \mu\text{mho}$,

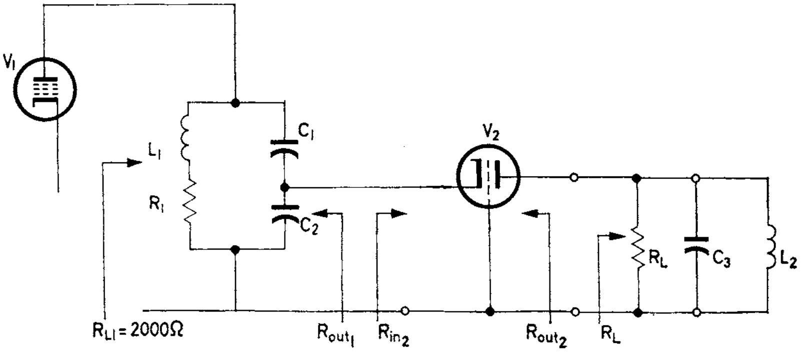
$$\mu = 60.$$

The desired selectivity of the tuned coupling networks when fully loaded, call for:

$$Q_1 = \frac{\omega L_1}{R_1} = 20$$

and

$$Q_2 = \frac{R_L}{\omega L_2} = 12.$$



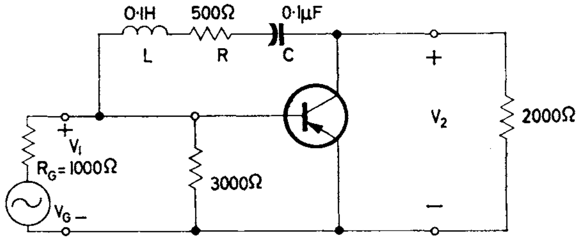
III-3a. Given an active filter in the form of a single-stage shunt feedback amplifier. The transistor is considered as sufficiently described with the resistive parameters:

$r_e = 30$ ohm,

$r = 100$ ohm,

$\beta = 50$,

$r_d = 20{,}000$ ohm.



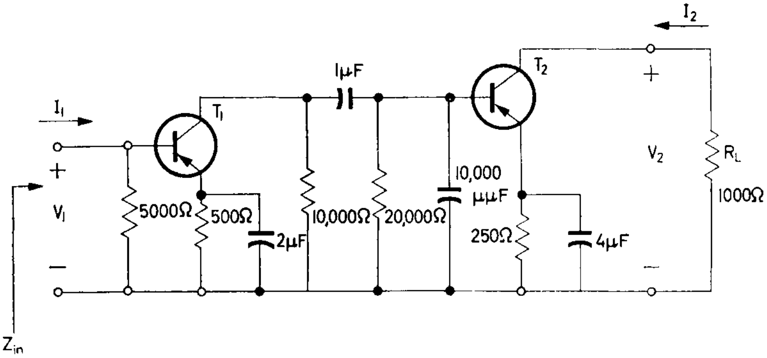
Find the voltage gain, current gain, input and output impedances with the external passive elements as shown:

(i) At the operating frequency where the LCR feedback network is series resonant.
(ii) At a frequency of one-tenth of resonance.
(iii) At a frequency of ten times resonance.

III-3b. Using matrix methods calculate the voltage gain, current gain and input impedance of the two-stage transistor audio amplifier at 10 c/s, 100 c/s, 1 kc/s and 10 kc/s. The two transistors have identical characteristics and in the operating range of interest consider them as purely resistive:

h_{11} = 1000 ohm, h_{12} = 0,

h_{21} = 50 ohm, h_{22} = 5×10^{-5} mho.



III-5a. Given a practical audio-frequency transistor voltage feedback pair. The operating range of interest extends from 100 c/s to 12 kc/s. The transistors are identical types, though there is a substantial spread of the manufacturing tolerances from transistor to transistor. At 1 mA collector current the following resistive parameters apply:

r_e = 25 ohm,

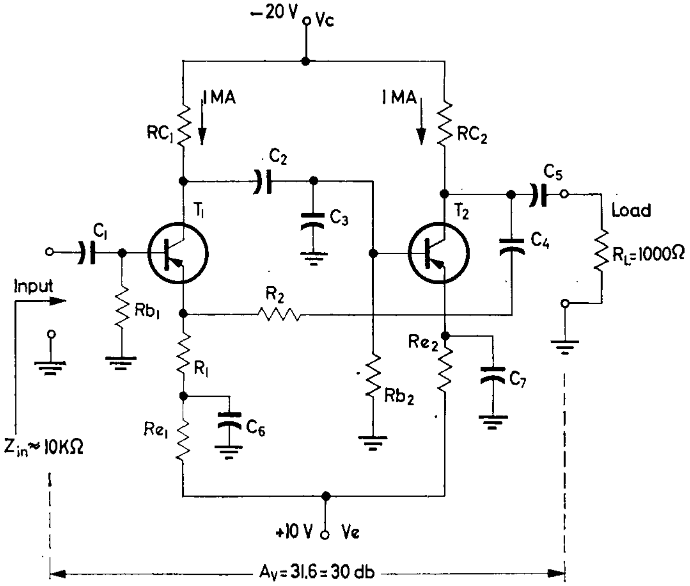
r_b = 50 ohm,

β = $40_{\min}$ to $250_{\max}$.

Assume further that h_{12} = 0 and h_{22} = 5×10^{-4} mho at β = 40. Compute the essential circuit parameters for 30 db voltage gain and 10,000 ohm input impedance. Ensure that the feedback network dissipates less than 10 per cent of the output power. What is the output impedance of the amplifier if both transistors have a current gain:

(i) $\beta_{\min}$.
(ii) $\beta_{\max}$.
(iii) If the first transistor has $\beta_{\max}$ and the second $\beta_{\min}$.

Allow mathematical simplifications whenever possible, provided that the error is less than 3 per cent.



III-6a. It is required to design and optimize the circuit parameters of a transistor video feedback amplifier. The operating frequency band extends from 30 c/s to 1·2 Mc/s. A stabilized current gain of 30 db is desired with an amplifier input impedance of 100 ohm or less. The manufacturing tolerances of transistors in a 2:1 range should introduce negligible change in the bandwidth and gain characteristics. Gain variation should not exceed 5% under the worst conditions.

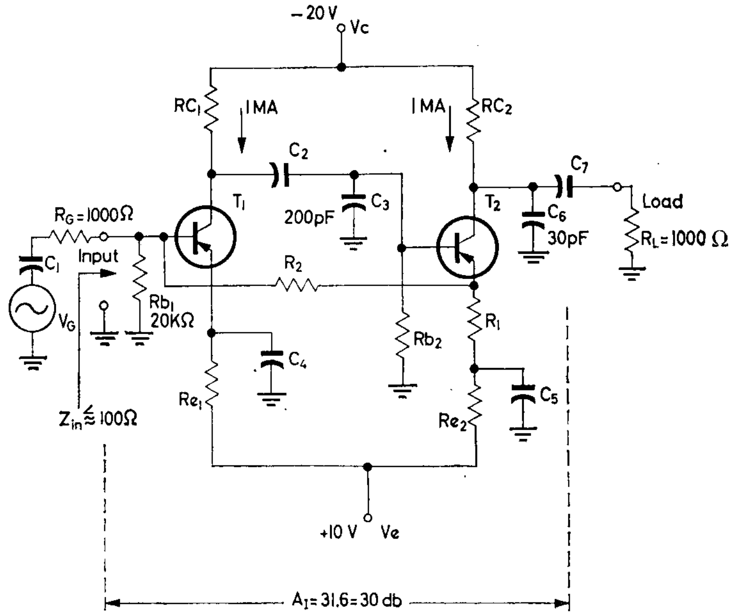
Assume that stray capacitances and transistor interelectrode capacitances can be shown by the capacitors C_3 and C_6 as above. The load is a 1000 ohm resistor. Both transistors are of the same type and in the operating range of interest they are defined by the resistive h parameters:

$$h_{11} \cong 800 \text{ ohm minimum,}$$
$$h_{12} = 0,$$
$$h_{22} \cong 2 \times 10^{-5} \text{ mho.}$$

If neglecting the intrinsic transistor base resistances:

(i) Find the minimum magnitude of h_{21} which will satisfy design objectives.

(ii) What is the current gain and input impedance of the amplifier if R_2 disconnected and R_1 shorted-out.



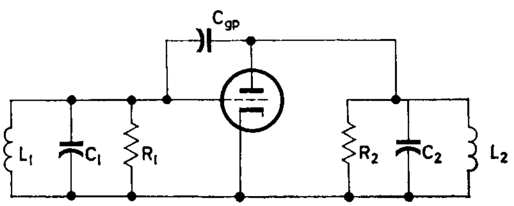
III-8a. Obtain the linear model and a corresponding mathematical model of the tuned-plate–tuned-grid oscillator. Derive generalized expressions for the conditions of oscillation and the frequency of oscillation. Assume that tube interelectrode capacitances are negligible. Compute the desired passive elements for operation at 1 kc/s and 1 Mc/s respectively if the following constraints apply:

$$g_m = 8000\ \mu\text{mho}, \quad r_p = 10{,}000\ \text{ohm},$$

$$Q_1 = Q_2 = \frac{R_1}{\omega L_1} = \frac{R_2}{\omega L_2} = 40,$$

$$C_1 = C_2.$$

(*Hint:* Try the Y matrices.)



III-8b. Given a Pierce type quartz crystal oscillator. The quartz is assumed to be inductive and series resonating at 1·7 Mc/s. The applicable constants of the quartz:

$$r_x = 50 \text{ ohm}, \qquad c_x < 0.1 \text{ pF},$$

$$Q_x = \frac{j\omega L_x}{r_x} = 20{,}000.$$

For the transistor in the h matrix domain:

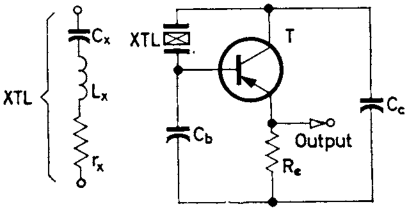
$$h_{11} = r_b + \beta r_e, \quad h_{12} = 0,$$

$$h_{21} = \beta, \qquad h_{22} = \frac{r}{\beta}$$

The corresponding numerical constants are:

$r_e = 25$ ohm, $r_b = 100$ ohm, $r_c = 2$ Megohm.

The output is taken across $R_e = 100$ ohm. Allowing for normal junction capacitances of the transistor, estimate the limiting value of β and optimum C_b and C_c.



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