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Preface

The evolution of economic agglomeration in cities is observed historically. The most likely and most accepted scenario is the evolution from an evenly spread population of farmers with evenly spread economic activities en route to agglomeration of economic activities in a few urban regions. Urbanization is indeed the cradle of economic development and prosperity. Nowadays, megalopolises prosper worldwide: Tokyo, Jakarta, New York, Seoul, Manila, Mumbai, São Paulo, Mexico City, Delhi, and Shanghai, to name a few. A question to be answered is "*How* and *where* is spatial agglomeration self-organized?"

The problem of self-organization involves several aspects of human activities and, accordingly, is of interest in various fields of studies. The self-organization of hexagonal agglomeration patterns of industrial regions was first predicted by central place theory in economic geography based on an empirical investigation of southern German cities. Self-organization of such distributions in two-dimensional economic agglomeration was envisaged by Krugman, who developed a new economic geographical model for spatial agglomeration. This model incorporated microeconomic mechanisms, including the following: monopolistic competition model of Dixit–Stiglitz, increasing returns at the level of firms, iceberg transport costs, factor mobility, and so on. Krugman noticed the vital role of bifurcation in the evolution of economic agglomeration. This motivated a thorough study of the mathematical mechanism of bifurcation engendering economic agglomeration presented in this book.

A two-dimensional space for economic activities is modeled, in this book, by a hexagonal lattice with periodic boundaries. Places (cities) for economic activities, such as consumption and production, are located on the nodes of this lattice and are connected by roads forming a regular-triangular mesh. By virtue of periodic boundaries, every place on the lattice enjoys equal competition. Microeconomic interactions among the places are expressed by core–periphery models in new economic geography. Population distribution on this lattice is obtained as a solution to the governing equation of these models. Manufactured goods are transported along these roads at a certain transport cost. When the transport cost is high, the uniform state, in which each place has the same population and is in the same economic state, is stable. Yet, when the cost is reduced to a certain level in association with the progress of technology, the uniform state is destabilized by bifurcations to produce prospering places with increasing population and decaying ones with decreasing population. Then certain patterns, the hexagonal ones being of particular interest, are self-organized.

In this book, after a brief introduction of central place theory and new economic geography, the missing link between them is discovered by elucidating the mechanism of the evolution of bifurcating hexagonal patterns. Pattern formation by such bifurcation is a well-studied topic in nonlinear mathematics, and group-theoretic bifurcation analysis is a well-developed theoretical tool to investigate possible bifurcating patterns. A finite hexagonal lattice is used in this book to express uniformly distributed places, and the symmetry of this lattice is expressed by a finite group. Several mathematical methodologies indispensable for tackling the present problem are gathered in a self-contained manner. The existence of hexagonal distributions is verified by group-theoretic bifurcation analysis, first by applying the so-called equivariant branching lemma and next by solving the bifurcation equation. This book consequently offers a complete guide for the application of group-theoretic bifurcation analysis to economic agglomeration on the hexagonal lattice.

As a main technical contribution of this book, a complete analysis of bifurcating solutions for hexagonal distributions from critical points of multiplicity 12 is conducted. Mathematically, the analysis of hexagonal distributions is carried out in a streamlined manner by means of fundamental theoretical tools for integer matrices, such as the Smith normal form and determinantal divisors. In particular, a solvability criterion for a system of linear equations in integer unknown variables that refers to determinantal divisors plays a significant role. Duality nature between the two methods, one by the equivariant branching lemma and the other by bifurcation equations, is made clear. The equivalence of the results obtained by these two methods is established through a theorem for integrality of solutions, the so-called integer analogue of the Farkas lemma.

Numerical bifurcation analysis of an economy on the hexagonal lattice with periodic boundaries is conducted to demonstrate the emergence of hexagonal distributions envisaged in central place theory, led by Christaller and Lösch. Moreover, as a step toward a connection with the real world, self-organization of central places is demonstrated for a domain with the shape of southern Germany without periodic boundaries. This is the birthplace of central place theory, to which Christaller's theory was first applied, and nowadays cities of several sizes are scattered on its relatively flat land to exhibit a hierarchy of central places. Unlike the hexagonal lattice with symmetry, no bifurcation for agglomeration would occur on this domain as it has no symmetry, but it has turned out that bifurcation serves as an underlying mechanism for the progress of the agglomeration. The hexagonal patterns, observed also for the hexagonal lattice, have appeared in the middle of the domain, just as regularly-arrayed hexagonal cells appear in the experiment of Bénard convection in fluid dynamics. This indicates the generality of the emergence of hexagonal patterns in spatial agglomeration, regardless of the shape and the boundary conditions of the domain. Thus the agglomeration behaviors on the hexagonal lattice capture much desired realism at the expense of idealization of periodic boundaries. This suffices to show the role and the importance of the theoretical and computational analysis on the hexagonal lattice presented in this book.

Hexagonal agglomeration in economic geography investigated in this book is a topic of interdisciplinary study of various fields, encompassing central place theory in economic geography, core–periphery models in new economic geography, and bifurcation theory in nonlinear mathematics. Naturally, several prerequisites in preparation for this study are contained in this book. The book can be read profitably by those who study applied mathematics as it presents related backgrounds of central place theory and geographical models in a self-contained manner. The book can also be addressed to professional researchers of economic geography and new economic geography. A complete guide of group-theoretic bifurcation analysis is provided encompassing introductory fundamental issues and an application to the economy on the hexagonal lattice. A complete classification of hexagonal distributions that can appear on the hexagonal lattice is obtained so as to assist the understanding of agglomeration behavior. The present methodology is endowed with extendibility to other problems with other symmetry groups. In addition, numerically obtained agglomeration patterns would contribute to gaining an intuitive understanding of two-dimensional agglomeration. In particular, agglomeration patterns of southern Germany points to a promising direction of a further study. Ample references are introduced to assist readers who are interested in further study.

The book comprises two parts. Part I is devoted to the preparation of fundamental issues, whereas the hexagonal agglomeration in economic geography is revealed in light of bifurcation in Part II.

Part I is organized as follows. Chapter 1 introduces several fundamental and introductory issues. The hexagonal market areas of Christaller and Lösch's hexagons studied in central place theory are introduced. As a step toward a connection with the real world, self-organization of a hexagonal distribution called Christaller's $k = 3$ system is demonstrated for a domain with the shape of southern Germany with a microeconomic mechanism of Krugman's core–periphery model in new economic geography. Such a distribution is observed much clearer for an economy on the hexagonal lattice with periodic boundaries. This shows the suitability of such hexagonal lattice as a spatial platform for agglomeration in economic geography. The equilibrium equation of Krugman's core–periphery model is formulated by assembling several relations on economic concepts, and its stability is described. The history of the study of self-organization of cities is reviewed, encompassing works in economic geography, new economic geography, and physics. Chapter 2 presents fundamentals of group-theoretic bifurcation theory. Agglomeration of population in a two-place economy is advanced in order to demonstrate the predominant role of bifurcation in economic agglomeration. The mechanism of this bifurcation is elucidated by group-theoretic bifurcation analysis of a system with dihedral-group symmetry. Bifurcation equation, equivariant branching lemma, and block-diagonalization are introduced as mathematical tools used to tackle bifurcation of a symmetric system. Chapter 3 serves as a bridge to the study in Part II. The spatial agglomeration in a racetrack economy is investigated as an application of group-theoretic bifurcation analysis to a problem with a simple group, the dihedral group. Theoretically possible agglomeration (bifurcation) patterns of this economy are predicted by this analysis and the existence of these patterns is demonstrated by numerical bifurcation analysis. Such prediction and demonstration are conducted in Part II, in a more general setting, for a larger group expressing the symmetry of economy on a hexagonal lattice to clarify the existence of the hexagonal patterns of Christaller and Lösch.

In Part II, we would like to tackle the objective of this book: investigation of the mechanism of the hexagonal agglomeration in economic geography on a hexagonal lattice in light of bifurcation. Hexagonal population distributions of several sizes are shown to be self-organized from a uniformly inhabited state, which is modeled by a system of places (cities) on a hexagonal lattice. Microeconomic interactions among the places are expressed by core–periphery models in new economic geography. We search for hexagonal distributions of Christaller and Lösch using group-theoretic bifurcation theory. The symmetries of possible bifurcating solutions can be determined from the algebraic structure of the group that describes the symmetry of the system. Hence, the first step of the bifurcation analysis is to identify the underlying group and its algebraic structure. After an introduction of the hexagonal lattice as a two-dimensional spatial platform of economic agglomeration, the symmetry group of this lattice is presented (Chap. 5). In comparison with the dihedral group, which describes the symmetry of the racetrack economy (Chap. 3 in Part I), this symmetry group has a more complicated structure, thereby, entailing a far more complicated bifurcation mechanism. Such complexity, however, is untangled by group-theoretic (equivariant) bifurcation analysis in Chaps. 8 and 9.

Part II is organized as follows. Chapter 4, serving as a prelude of Part II, presents the equilibrium equation of core–periphery models and gives the proof of the equivariance of this equation. Theoretical results of Chaps. 5–9 that elucidate the mechanism of these bifurcations are previewed. Bifurcations on the hexagonal lattice engendering hexagonal distributions of interest are demonstrated by numerical bifurcation analysis. Chapter 5 introduces a hexagonal lattice as a two-dimensional discretized uniform space for economic agglomeration. Hexagonal distributions on this lattice, corresponding to those envisaged by Christaller and Lösch in central place theory (Sect. 1.2), are explained, parameterized, and classified. The symmetry group of the hexagonal lattice is presented. Chapter 6 gives a derivation of irreducible representations of this group according to a standard procedure known as the method of little groups in group representation theory. Chapter 7 presents matrix representations of the group for the hexagonal lattice. Among the irreducible representations of this group, those which are relevant to this lattice are identified. Chapters 8 and 9 present group-theoretic bifurcation analysis, respectively, by using the equivariant branching lemma and by solving the bifurcation equation.

Irreducible representations and the sizes of the hexagonal lattice that can engender hexagonal patterns of interest are set forth and classified. Asymptotic forms of bifurcating equilibrium paths and the directions of these paths are presented.

It would be our great pleasure if this book contributes to a better understanding of self-organization of economic agglomeration.

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Tokyo, Japan Kazuo Murota

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The authors met with the concept of self-organizing economy in the book:

P. Krugman, *The Self-organizing Economy*, Blackwell, Oxford, 1996.

This book opened the eyes of the authors to interdisciplinary studies encompassing social science and natural science.

The hexagonal patterns of economic agglomeration on the hexagonal lattice were found in May 2009 by a master course student of the first author. This finding triggered our serious group-theoretic analysis to sort out Christaller's hexagons of three kinds, which was reported in:

K. Ikeda, K. Murota, T. Akamatsu, T. Kono, Y. Takayama, G. Sobhaninejad, A. Shibasaki: Self-organizing hexagons in economic agglomeration: core-periphery models and central place theory, Technical Report METR 2010-28, Department of Mathematical Informatics, University of Tokyo, October 2010.

This report and a study thereafter that blossomed into this book were supported by researchers in various fields, encompassing economic geography, new economic geography, and nonlinear mathematics. The authors would like to acknowledge the contributions of the following: Professor Takashi Akamatsu provided us with knowledge on core–periphery models in new economic geography and Professor Tatsuhito Kono introduced us to fundamentals of economic geography. Yuki Takayama, Reza Sobhaninejad, Akira Shibasaki, and Naoki Kondo contributed to the numerical nonlinear bifurcation analysis of hexagonal patterns on the hexagonal lattice.

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Contents

Part I
Economic Agglomeration and Bifurcation: Introduction

Chapter 1
Hexagonal Distributions in Economic Geography and Krugman's Core–Periphery Model

Abstract At the beginning of this book, several fundamental concepts related to the study of hexagonal economic agglomerations are presented and importance of this study is demonstrated. Christaller's three hexagonal market areas associated with market, traffic, and administrative principles and Lösch's hexagons derived from geometrical consideration in central place theory are introduced. As a step toward a connection with the real world, self-organization of central places is demonstrated for a domain with the shape of southern Germany with a microeconomic mechanism of Krugman's core–periphery model in new economic geography. Such a distribution is observed much clearer for an economy on the hexagonal lattice with periodic boundaries to demonstrate the importance of the group-theoretic study on this lattice conducted in this book. Nonlinear equilibrium equations and the stability of Krugman's core–periphery model are introduced. History of the study of self-organization of cities is reviewed, encompassing works in economic geography, new economic geography, and physics.

Keywords Agglomeration of population • Bifurcation • Central place theory • Christaller's hexagonal market area • Core–periphery model • Economic agglomeration • Krugman model • Lösch's hexagons • Southern Germany • Spatial equilibrium • Stability

1.1 Introduction

Hierarchical urbanization of megalopolises, cities, towns, villages, and so on displays interesting scattering patterns that hint at the existence of an underlying mechanism. A first attempt to elucidate such a mechanism was conducted for southern Germany by Christaller, 1933 [8]. Figure 1.1 depicts a distribution of large cities in southern Germany, where cities of various sizes are distributed. In particular, Frankfurt, Stuttgart, Nuremberg, and Munich appear to be approximately equidistant. A question to be answered is "*How* and *where* are these cities

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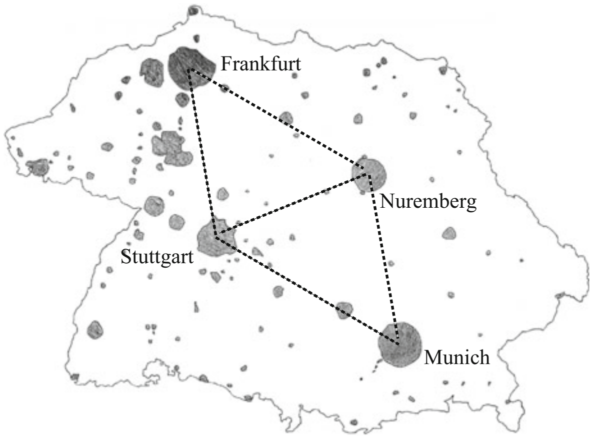

Fig. 1.1 Distribution of large cities in southern Germany

self-organized?" Although diversified studies have been conducted in social and natural sciences, to be described in Sect. 1.6, the elucidation of this mechanism remains difficult.

Successful simulation of self-organization is indeed a difficult task as it involves the modeling of various aspects: geometry of locations, microeconomic activities, interaction of places via transportation of goods, and so on. In addition, there are detailed factors, such as the location of houses and factories and the shipment of goods by trucks and trains. In this book, to avoid excessive complexity, we specifically examine the most likely and most generally accepted scenario: an evolution from an evenly spread population of farmers to an agglomeration of economic activities in a few urban regions. In accordance with this scenario, we introduce the modeling of several issues:

- For the modeling of locations, we refer to *central place theory*, which describes geometrically possible spatial patterns of urbanization that are self-organized from a uniform economic space, as explained in Sect. 1.2.
- For the modeling of economic activities, we utilize the *core–periphery model* in new economic geography that incorporates several microeconomic mechanisms, such as interactions occurring among production with increasing returns, transport costs, and factor mobility, as expounded in Sect. 1.5.

By virtue of this modeling, a nonlinear equilibrium equation can be formulated, in which the population migrating among places serves as an independent variable and the transport cost serves as a bifurcation parameter. As a solution to this equation, we can obtain loci of equilibria expressing the progress of agglomeration parameterized by the transport cost. The analysis of the equation is, however, complicated due to the multiplicity of solutions by bifurcations.

The book, accordingly, has become an interdisciplinary study of hexagonal patterns in central place theory in economic geography, core–periphery models for economic agglomeration in new economic geography, and group-theoretic bifurcation theory. At the beginning of this book, fundamentals of central place theory and core–periphery models are presented, whereas the history of economic geography and new economic geography is given in Sect. 1.6 and the group-theoretic bifurcation theory is shown in detail in Chap. 2.

This book offers a group-theoretic methodology to elucidate the mechanism of self-organization of hexagonal patterns in economic agglomeration. In this chapter, the occurrence of self-organization is demonstrated and explained on the basis of computational simulations of economic agglomeration in southern Germany and in a hexagonal lattice.

This chapter is organized as follows. Hexagonal market areas in central place theory in economic geography are introduced in Sect. 1.2. Economic agglomeration in a domain of the shape of southern Germany with regular-triangular meshes is simulated to demonstrate the emergence of hexagonal patterns in Sect. 1.3. Emergence of hexagonal patterns on a hexagonal lattice with regular-triangular meshes and periodic boundaries is demonstrated numerically in Sect. 1.4. Krugman's core–periphery model is presented in Sect. 1.5. History of the study of self-organization of cities is reviewed in Sect. 1.6.

1.2 Christaller's Hexagonal Market Areas and Lösch's Hexagons

Self-organization of hexagonal distributions has been studied by Christaller and Lösch in central place theory.[1] The concept of *flat earth* is introduced on the basis of several simplifying assumptions, such as

- The land surface is completely flat and homogeneous in every aspect. It is, in technical terms, an *isotropic plain*.
- Movement can occur in all directions with equal ease and there is only one type of transportation.
- The plain is limitless or unbounded, so that complications that tend to occur at boundaries do not need to be dealt with.
- The population is spread evenly over the plain.

1.2.1 Christaller's Hexagonal Distributions

Christaller, 1933 [8] considered a hierarchical structure of industries with different sizes of demand and showed that a nested set of *hexagonal market areas* of

[1]For central place theory, see, for example, Lösch, 1940 [23]; Lloyd and Dicken, 1972 [22]; Isard, 1975 [20]; and Beavon, 1977 [4].

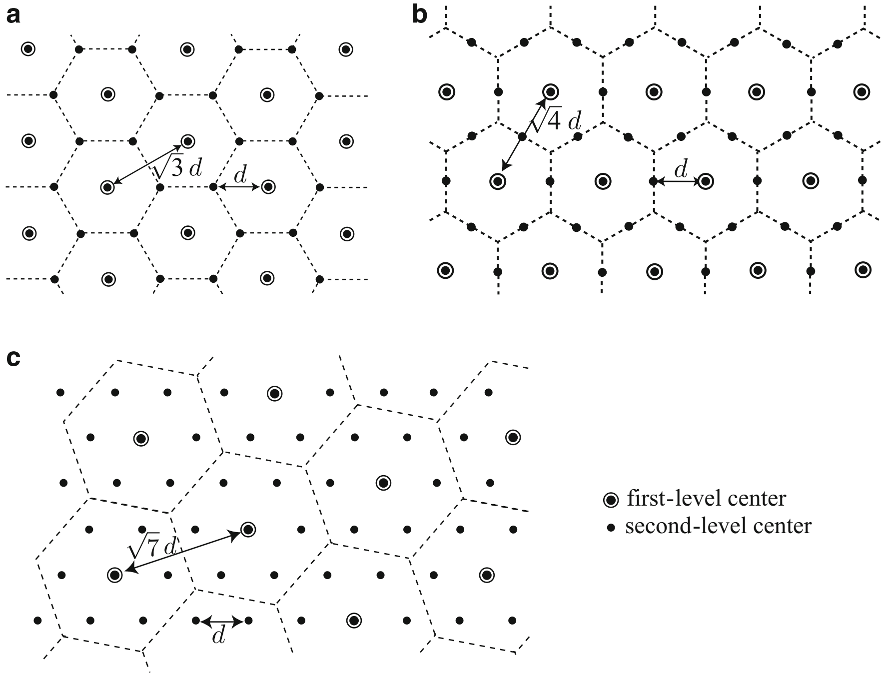

Fig. 1.2 Three systems predicted by Christaller. (**a**) Christaller's $k = 3$ system. (**b**) Christaller's $k = 4$ system. (**c**) Christaller's $k = 7$ system. The *dashed lines* denote hexagonal market areas

places, such as cities, towns, and villages, emerges. A hierarchy of places with different levels exists in each market area governed by the highest-level (first-level) center with the largest population, the second-level center with the second largest population, and so on.[2] Self-organization of hexagonal market areas of three kinds shown in Fig. 1.2 was advanced as a key concept.

Christaller introduced the so-called *k value* as an important index to characterize hexagonal market areas, as stated by Dicken and Lloyd, 1990 (p. 28) [11] as

> Christaller's model, then, implies a fixed relationship between each level in the hierarchy. This relationship is known as a k value (k meaning a constant) and indicates that each center dominates a discrete number of lower-order centers and market areas in addition to its own.

The k value has a geometrical implication in that it is proportional to the size (area) of the hexagonal market area. Its square root $\sqrt{k}$ is proportional to the *spatial period* L, the shortest distance between the first-level centers, which represents the radius of hexagons. The three smallest values, $k = 3$, 4, and 7, are associated with Christaller's systems (Fig. 1.2).

[2]Such a hierarchy is also called metropolis, city, town, village, hamlet; or A-level center, B-level center, C-level center, and so on.

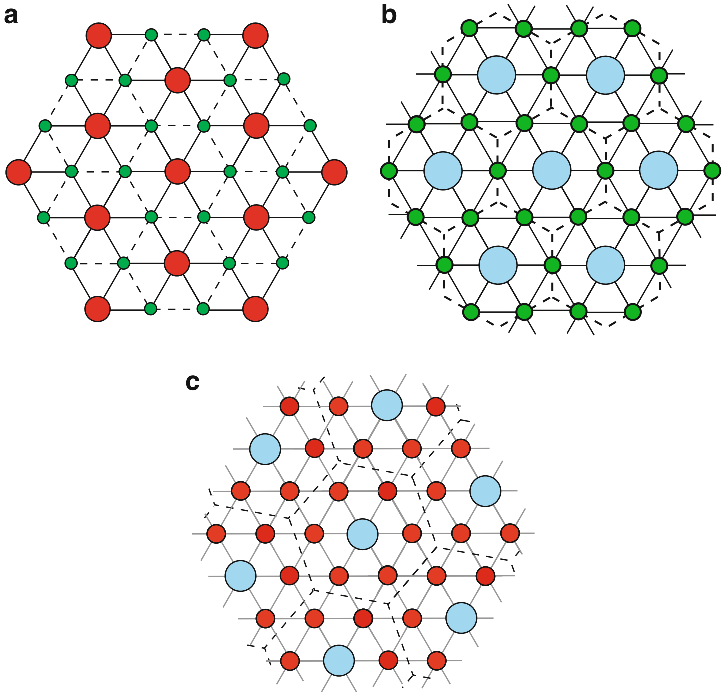

Fig. 1.3 Hexagonal distributions of Christaller on the hexagonal lattice. (**a**) Christaller's $k = 3$ system. (**b**) Christaller's $k = 4$ system. (**c**) Christaller's $k = 7$ system. The *larger circles* represent the first-level centers and the *smaller ones* represent the second-level centers; the *dashed lines* denote hexagonal market areas

Christaller's three systems are explained by market, traffic, and administrative principles, respectively (Christaller, 1933 [8]; Dicken and Lloyd, 1990, Chap. 1 [11]). These principles are explained below with reference to Fig. 1.3, which displays Christaller's hexagonal distributions on a hexagonal lattice.

- In the $k = 3$ *system* in Fig. 1.3a, neighboring first-level centers (larger circles) are connected by two kinked roads, each of which passes a second-level center (smaller circle) at the kink. This system is explained by Christaller's *market principle* of supplying the maximum number of evenly distributed consumers from the minimum number of central places.
- In the $k = 4$ *system* in Fig. 1.3b, neighboring first-level centers are connected by a straight road that passes a second-level center. This system is explained by Christaller's *traffic principle* of achieving efficient transportation. Christaller wrote: "The traffic principle states that the distribution of central places is most favorable when as many important places as possible lie on one traffic route between two important towns, the route being as straightly and as cheaply as possible."

- The distribution in the $k = 7$ *system* in Fig. 1.3c agrees with Christaller's *administrative principle* of avoiding the sharing of a satellite place by two first-level centers to prevent administrative conflict. Christaller stated: "The ideal of such a spatial community has the nucleus as the capital (a central place of a higher rank), around it, a wreath of satellite places of lesser importance, and toward the edge of the region a thinning population density—and even uninhabited areas."

The number N_j of the jth level centers dominated by the first-level center is given by the recurrence formula[3]

$$N_1 = 1, \quad N_j = k^{j-1} - k^{j-2}, \quad j = 2, 3, \ldots; \ k = 3, 4, 7. \tag{1.1}$$

For example, we have

$$N_1 : N_2 : N_3 : \cdots = \begin{cases} 1 : 2 : 6 : 18 : 54 : 162 : \cdots & \text{for } k = 3 \text{ system,} \\ 1 : 3 : 12 : 48 : 192 : \cdots & \text{for } k = 4 \text{ system.} \end{cases}$$

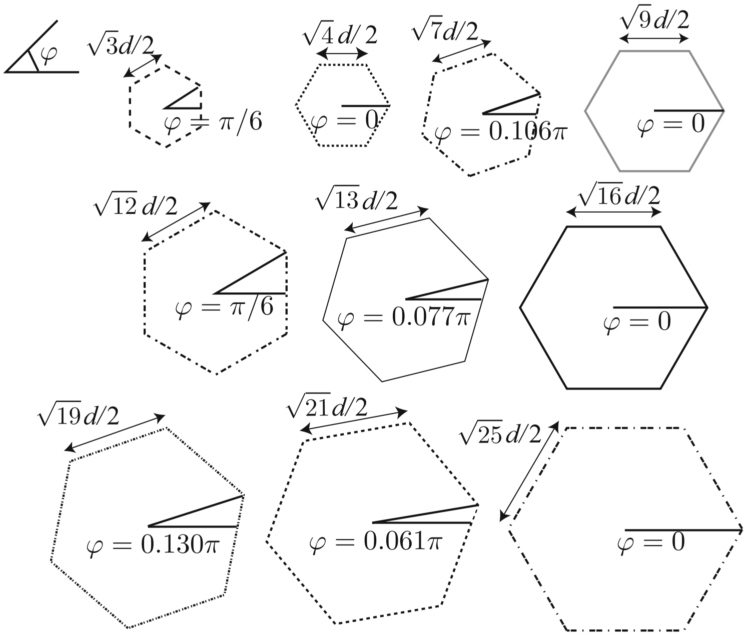

Fig. 1.4 Lösch's ten smallest hexagons

[3]See Christaller, 1933 (1966, p. 67) [8] for the case of $k = 3$ and Dicken and Lloyd, 1990, Chap. 1 [11] for the case of $k = 4$.

1.2.2 Lösch's Hexagons

Lösch, 1940 [23] demonstrated, for a single industry, that market areas must be hexagonal in order to minimize transport costs for a given density of central places. The *ten smallest hexagons* shown in Fig. 1.4 were presented as fundamental sizes of market areas. Based on the geometry of the hexagonal lattice, the *normalized spatial period* L/d was shown to take some specific values, such as

$$\frac{L}{d} = \sqrt{D} = 1, \sqrt{3}, 2, \sqrt{7}, 3, \sqrt{12}, \sqrt{13}, 4, \sqrt{19}, \sqrt{21}, 5, \dots, \tag{1.2}$$

where d is the distance between two neighboring places and D is an important parameter for the characterization of Lösch's hexagons.

It should be emphasized that central place theory relies on a normative and geometrical approach and does not reveal the microeconomic or mathematical mechanism of self-organization of hexagonal patterns. This book underpins this theory by mathematical study of the geometry of the hexagonal lattice (Chap. 5) and elucidates the mechanism of self-organization in light of bifurcation theory (Chaps. 6–9).

1.3 Agglomeration in Southern Germany: Realistic Spatial Platform

As a step toward a connection with the real world, self-organization of central places is demonstrated here by economic agglomeration analysis[4] of the domain in Fig. 1.5. The shape of this domain was chosen to mimic the shape of southern Germany.[5]

In comparison with the hexagonal lattice to be studied in Sect. 1.4, this domain has the following two characteristics.

- An irregular shape without any symmetry.
- Nonperiodic boundaries.

No bifurcation for agglomeration would occur on this domain as it has no symmetry, but it will turn out that bifurcation serves as an underlying mechanism for the

[4] This domain comprises 404 places connected by a set of triangular meshes and the nominal mesh size $d = 1/32$. For microeconomic modeling, the core–periphery model with (4.6) in Sect. 4.5 was used with the parameter values of $(\sigma, \mu, \theta) = (5.0, 0.4, 10000)$. See also Sect. 1.5 for fundamental issues of this model.

[5] Southern Germany is the birthplace of central place theory to which Christaller's theory was first applied (Christaller, 1933 [8]), and nowadays cities of several sizes are scattered on its relatively flat land to exhibit a hierarchy of central places.

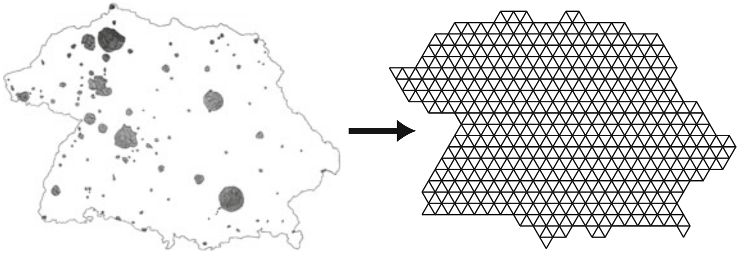

Fig. 1.5 Grid model of southern Germany. Places are located on the nodes of the grid; a regular-triangular mesh of roads is assumed to exist even at the location where the grid is absent

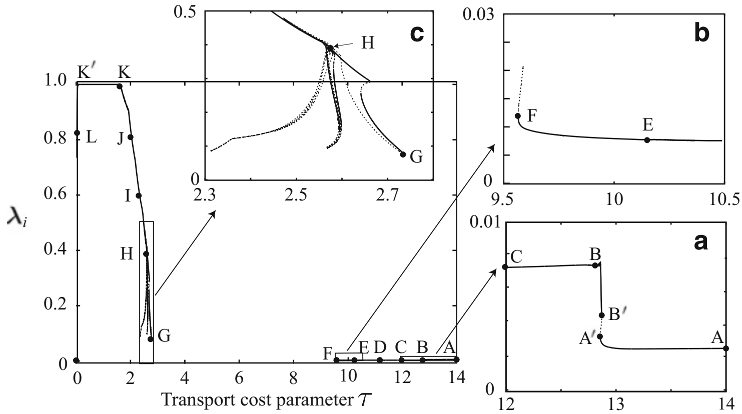

Fig. 1.6 Change of the population size λ_i at a place in the middle of the domain plotted against the transport cost parameter τ. *Solid curves* mean stable equilibria; *dotted curves*, unstable equilibria

progress of the agglomeration. The major objective of this book is to develop a theoretical framework to explain the mechanism of this agglomeration behavior.

The population λ_i at a place in the middle is plotted[6] against the transport cost parameter τ in Fig. 1.6 ($0 < \tau < 14.0$) with enlarged views (a)–(c), and in Fig. 1.7 population distributions are shown by expressing population size by the area of *black circle*. The curves in Fig. 1.6c are very complicated forming several *loops* that are a mixture of stable and unstable equilibria (see Sect. 1.5.4 for the definition of stability). Several characteristic states of the progress of agglomeration as τ decreases (in association with the development of technology) were observed as explained below.

[6]The population versus the transport cost parameter curve in Fig. 1.6 was obtained only in the ranges of $\tau > 9.57$ and $\tau < 2.73$ because the curve formed a plethora of loops and was too complicated in the range of $2.73 < \tau < 9.57$ between points G and F.

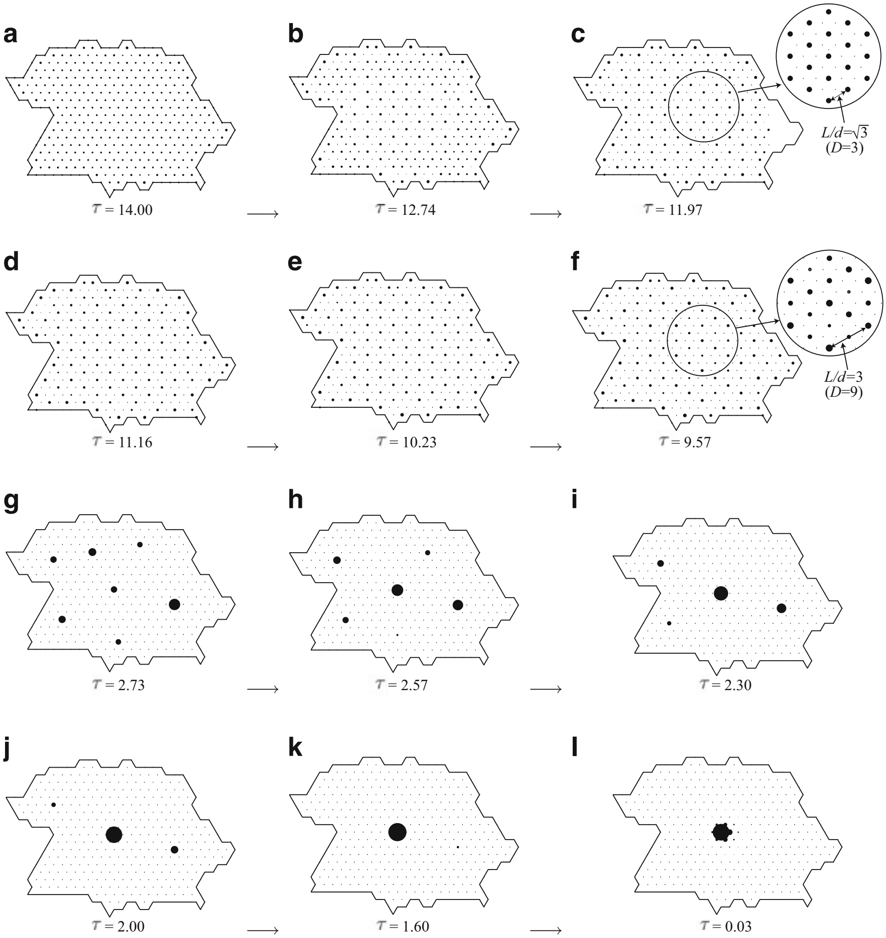

Fig. 1.7 Progress of agglomeration in association with the decrease of transport cost parameter τ observed at points A–L in Fig. 1.6. The area of *black circle* indicates population size. (**a**) Point A. (**b**) Point B. (**c**) Point C. (**d**) Point D. (**e**) Point E. (**f**) Point F. (**g**) Point G. (**h**) Point H. (**i**) Point I. (**j**) Point J. (**k**) Point K. (**l**) Point L

Emergence of Christaller's $k = 3$ System

In an early state ($14.0 > \tau > 12.86$), the population curve in Fig. 1.6 remained almost flat and the population of each place changed slowly, and the population was distributed almost uniformly (see Fig. 1.7a for $\tau = 14.0$).

A rapid increase of population λ_i started at $\tau = 12.86$ (point A′ in the enlarged view in Fig. 1.6a). There was a short unstable curve A′B′ subject to a snap back (shown by the dotted curve). The stability was lost at the minimal point A′ of τ and recovered at the maximal point B′ of τ. In association with monotonic reduction of the value of the transport cost parameter τ, we encounter the stable path AA′, a *dynamical shift* between A′ and B (bypassing B′), and another stable path BC in this sequence.

Along the curve AA′B′B, some places lost their population, other places gained population to grow into first-level centers, and, in turn, to self-organize a spatial pattern. At point B with $\tau = 12.74$ in Fig. 1.6, a zone containing sparsely and regularly distributed central places appeared in the middle of the domain (Fig. 1.7b). Since the distance between these places is equal to $L/d = \sqrt{D} = \sqrt{3}$, this demonstrates a self-organization of Christaller's $k = 3$ system with $D = 3$. This zone grew stably thereafter (Figs. 1.7c–e). At $\tau = 10.23$ at point E in Fig. 1.6, the sparsely- and regularly-distributed zone covered the domain away from the boundary (Fig. 1.7e). The curve AA′B′BC had a step-like shape with a snap back followed by a plateau. Such a shape of an equilibrium path is characteristic in the emergence of hexagonal patterns, as we will see in Sect. 1.4.

Transition via the Hexagon with $D = 9$

After a relatively stable and calm era of the dominance of the $k = 3$ system with the spatial period $L/d = \sqrt{D} = \sqrt{3}$ during $12.74 > \tau > 10.23$, there appeared a transient state in Fig. 1.7f at $\tau = 9.57$, in which the spatial period among growing places elongated approximately to $L/d = \sqrt{D} = \sqrt{9} = 3$. Thus we encountered a cascade of bifurcations, in which the spatial period became $\sqrt{3}$ times repeatedly as

$$\frac{L}{d} = \sqrt{D}: \ 1 \to \sqrt{3} \to 3. \tag{1.3}$$

Such elongation progressed gradually without undergoing bifurcation and the initiation of the elongation was not clear.[7]

Agglomeration and Redispersion State

In the agglomeration and redispersion state ($2.73 > \tau > 0.0$), as shown in Fig. 1.6, the population λ_i in the middle of the domain grew rapidly between points G–K, hit the plateau between K–K′, and then decreased between K′–L. As shown in Fig. 1.7g, at point G with $\tau = 2.73$, the population is agglomerated at seven places denoted by *black circle*, which are approximately equidistant. Although these places do not have the same population and do not display a regular-hexagonal distribution due to the irregular shape of the domain, each place keeps sufficient and almost

[7] The cascade of spatial period elongation in (1.3) was observed more clearly for the 9×9 hexagonal lattice with periodic boundaries, for which this cascade was entailed by a cascade of bifurcations (Sects. 1.4.2 and 4.5.1).

the same distances from neighboring places to maintain its own market area. Such distribution agrees with Christaller's administrative principle of avoiding the sharing of a satellite place so as to prevent administrative conflict (Sect. 1.2).

The number of agglomerated places decreased in association with the decrease of τ (Figs. 1.7g–j), until the emergence of a megalopolis with a completely agglomerated population (Fig. 1.7k). Thereafter, the redispersion took place (Fig. 1.7l).

1.4 Hexagons on Hexagonal Lattice: Idealized Spatial Platform

As we have seen in Sect. 1.3, an agglomeration analysis for hexagonal patterns on the irregular shaped domain of southern Germany involved too complicated solution curves with a number of loops. In this book, we advance the hexagonal lattice as an idealized platform for the analysis of spatial agglomeration. The theoretical analysis and the numerical analysis of this lattice are previewed in this section. It is demonstrated that the agglomeration behavior on this lattice can capture essential characteristics of agglomeration in southern Germany at the expense of several idealizations.

1.4.1 Hexagonal Lattice and Possible Hexagonal Distributions

A completely homogeneous infinite two-dimensional land surface, the *flat earth* in central place theory (Sect. 1.2), needs to be expressed compatibly with the discretized analysis of core–periphery models. For this purpose, an $n \times n$ finite hexagonal lattice with periodic boundaries is used in this book.[8] Nodes on this lattice represent uniformly spread places of economic activities. These places are connected by roads of the same length d forming a regular-triangular mesh (see Fig. 1.8 for an example of $n = 4$), and goods are transported along these roads. In comparison with the domain with the shape of southern Germany studied in Sect. 1.3, the hexagonal lattice is endowed with uniformity, the same geometrical environment for every place, owing to periodic boundaries. Agglomeration would occur on this domain by way of bifurcations.

By a theoretical consideration, pertinent lattice sizes n that would engender hexagonal distributions of interest are given in Chap. 5. For example, Christaller's distributions (Fig. 1.3) are shown to be compatible with the lattice sizes

[8] The term of *hexagonal lattice* is commonly used in many fields of mathematical sciences, although it is also called *regular-triangular lattice*.

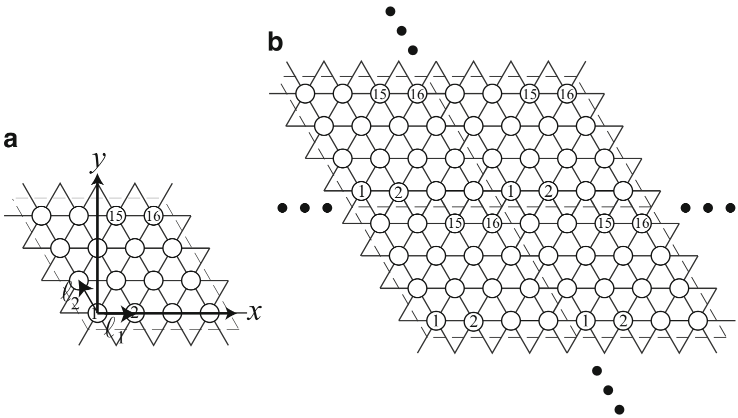

Fig. 1.8 A system of places on the 4×4 hexagonal lattice with periodic boundaries. (**a**) 4×4 hexagonal lattice. (**b**) Periodically repeated 4×4 hexagonal lattice

$$n = \begin{cases} 3m \text{ for Christaller's } k = 3 \text{ system,} \\ 2m \text{ for Christaller's } k = 4 \text{ system,} \\ 7m \text{ for Christaller's } k = 7 \text{ system} \end{cases} \tag{1.4}$$

$(m = 1, 2, \ldots)$.

Microeconomic structure of agglomeration is provided by the core–periphery model, which will be advanced in Sect. 4.2 as a variant of the general setting given in Sect. 1.5. Then the uniform population distribution, which we call the *flat earth equilibrium*, serves as the pre-bifurcation solution to the governing equation for this core–periphery model on the hexagonal lattice. Possible bifurcating hexagonal distributions from this equilibrium are investigated by group-theoretic bifurcation analysis (Chaps. 8 and 9).

1.4.2 Agglomeration Analysis for a Hexagonal Lattice

Figure 1.9 depicts equilibrium paths (the maximum population $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_{81})$ versus the transport cost parameter τ curves) for the 9×9 hexagonal lattice,[9] with hexagonal windows expressing hexagonal distributions observed on these paths.[10]

[9]The lattice size $n = 9$ was chosen in view of the theoretical result in (1.4).

[10]Each hexagonal window was obtained as a part of the spatially repeated infinite lattice (Fig. 1.8b).

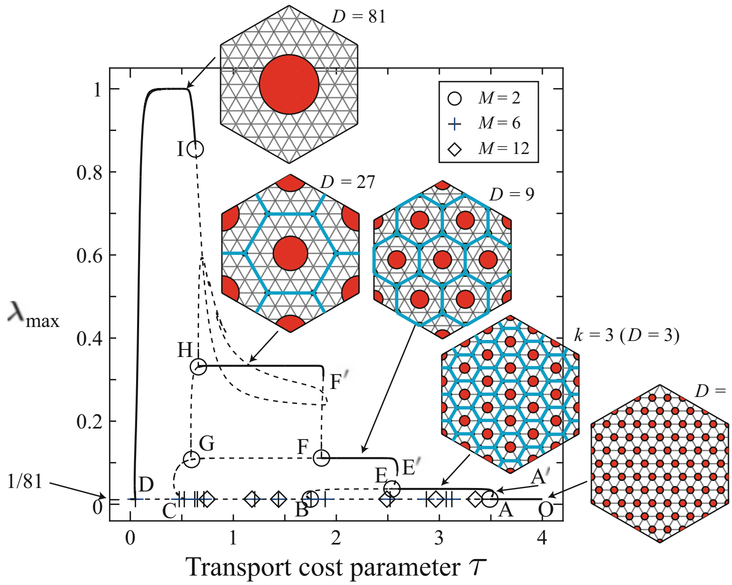

Fig. 1.9 Emergence of hexagonal population distributions on the 9 × 9 hexagonal lattice. *Solid curves* mean stable equilibria; *dashed curves*, unstable equilibria; $\lambda_{\max}$, the maximum population among the places; the area of each circle is proportional to the population size at that place; *M* means the multiplicity of a critical point to be defined in Sect. 2.4.1

The progress of stable equilibria observed in association with the decrease of τ captures important agglomeration properties for southern Germany (Sect. 1.3) that had three major states:

- Emergence of Christaller's $k = 3$ system.
- Transition via the hexagon with $D = 9$.
- Agglomeration and redispersion state.

This is explained below in detail.

Emergence of Christaller's $k = 3$ System

To begin with, let us look at the critical points A and B. A bifurcated equilibrium path branching from the point A returned to the flat earth equilibria at another bifurcation point B to form a loop of bifurcated path AEB. The agglomeration patterns found on this path AEB were hexagonal distributions of Christaller's $k = 3$ system, as portrayed in the hexagonal window for $k = 3$ in Fig. 1.9.

Recall that, in Sect. 1.3, such self-organization of the $k = 3$ system was also observed for the agglomeration analysis of southern Germany (without periodic boundaries) in Fig. 1.7b–d. In addition, the curve OAA′E has a step-like shape, which is quite similar to the shape of the curve AA′B′BC of the analysis for southern Germany in Fig. 1.6a. This indicates the generality of the emergence of the $k = 3$ system in spatial agglomeration, regardless of the difference in shape and the boundary conditions of the domain. This shows the role and the importance of the theoretical analysis on the hexagonal lattice with periodic boundaries presented in this book in the understanding of agglomeration behaviors.

Transition via the Hexagon with $D = 9$

Next, we look at the bifurcated path from the critical point C, on which Lösch's hexagon with $D = 9$ was observed. This path was connected, at the bifurcation point E, with another bifurcated path AEB associated with Christaller's $k = 3$ system. When τ is reduced monotonically, a possible progress of stable states of agglomeration is

$$\text{A}'\text{E } (k = 3 \text{ system}) \rightarrow \text{(dynamical shift)} \rightarrow \text{E}'\text{F (Lösch's hexagon with } D = 9),$$

following the curve OAA′E of a step-like shape. Recall that the budding of Lösch's hexagon with $D = 9$, following the stable state of Christaller's $k = 3$ system, was observed also in the agglomeration analysis of southern Germany (Fig. 1.7f).

Agglomeration and Redispersion State

Last, we observe the bifurcated path from the critical point D, on which Lösch's hexagon with $D = 81$ was observed. This path was stable at the beginning, lost stability at the point I, formed a complicated unstable loops, and arrived at the bifurcation point H on the bifurcated path FF′HG associated with Lösch's hexagon with $D = 27$. Along the stable curve ID, in association with the decrease of τ, a complete agglomeration at the peak of the curve, redispertion thereafter, and the flat earth state at D would be observed in this order.

1.5 Appendix: Krugman's Core–Periphery Model

Krugman's core–periphery model is introduced briefly. This model has a microeconomic structure comprising an ensemble of several economic principles, and only a fragment of this structure is presented here. Readers who are interested in this structure are suggested to consult Krugman, 1991 [21] and Fujita, Krugman, and Venables, 1999 (Chaps. 4 and 5) [16]. An emphasis is placed on formulating a general form of its governing equation so that the bifurcation mechanism of this equation can be described by the group-theoretic bifurcation theory presented in Chap. 2.

1.5.1 Basic Assumptions

We consider an economy comprising n locations (places) labeled $i = 1, \dots, n$, two industrial sectors (agriculture and manufacturing), and two factors of production (agricultural labor and manufacturing labor). The agricultural sector is perfectly competitive and produces a homogeneous good, whereas the manufacturing sector is imperfectly competitive with increasing returns, producing horizontally differentiated goods.

A consumer, who is either an agricultural or manufacturing worker, behaves to maximize the utility under the budget constraint of a given income Y. Every consumer shares the same Cobb–Douglas tastes that are expressed by the utility function as

$$U = M^{\mu} A^{1-\mu}, \tag{1.5}$$

where M means a composite index of the consumption of manufactured goods, the price of which is characterized by the so-called *price index* G, A is the consumption of agricultural goods, and μ $(0 < \mu < 1)$ is a constant representing the *expenditure share* of manufactured goods. By the maximization of the utility function U in (1.5), the consumptions M and A, as well as the utility U, are expressed as functions in the income Y and the price index G. The utility after maximization, which is called *indirect utility*, reads

$$U = \mu^{\mu}(1-\mu)^{1-\mu} Y G^{-\mu} (p^{\mathrm{A}})^{-(1-\mu)}.$$

Here p^{A} is the price for the agricultural good, which is assumed to be equal to unity, and $G^{-\mu}(p^{\mathrm{A}})^{-(1-\mu)}$ is the cost-of-living index that is used to scale the wage of workers.

Agricultural workers are immobile. Manufacturing workers are allowed to migrate between locations. Distribution of the *mobile population* of manufacturing workers is expressed by $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)^{\top}$, where λ_i is their population at the ith location normalized as

$$\sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \dots, n \tag{1.6}$$

based on an assumption of no population growth. The domain of $\boldsymbol{\lambda}$, accordingly, is the $(n-1)$-dimensional *simplex*.

For simplicity, we assume that agricultural goods are transported freely between locations. For manufactured goods, transport costs are defined by the *iceberg transport technology*, which means that if a manufacturing variety produced at location i is shipped to another location j $(\neq i)$, only a fraction $1/T_{ij}$ (< 1) arrives and the rest melts away en route, where T_{ij} (> 1) is a parameter that indirectly denotes the *transport cost* from location i to location j. If the variety produced at location i is sold at price[11] p_i^{M} at location i, then the price p_{ij}^{M} of that delivered variety at consumption location j is given as

$$p_{ij}^{\mathrm{M}} = p_i^{\mathrm{M}} T_{ij}, \quad i, j = 1, \ldots, n \tag{1.7}$$

with $T_{ii} = 1$ $(i = 1, \ldots, n)$. The transport cost is parameterized as

$$T_{ij} = T_{ij}(\tau) \tag{1.8}$$

using a single parameter τ, called the *transport cost parameter*. The transport cost is assumed to increase monotonically with an increase of τ. The concrete form of $T_{ij}(\tau)$ is specified depending on the geometry of the economy (Sects. 3.2 and 4.2).

The equilibrium equation of Krugman's model is formulated on the basis of several concepts, such as consumer behavior, transport costs, and producer behavior. There are two stages of equilibrium: market equilibrium and spatial equilibrium. The population distribution $\boldsymbol{\lambda}$ is kept constant in the market equilibrium, whereas $\boldsymbol{\lambda}$ is considered to be changeable in the spatial equilibrium.

1.5.2 Market Equilibrium

The market equilibrium gives a set of relations among the price index G_i, the wage rate w_i, and the total income Y_i of all the workers at the ith location:

- The *price index* G_i at the ith location is expressed in terms of λ_j and w_j $(j = 1, \ldots, n)$ as

$$G_i = [\sum_{j=1}^{n} \lambda_j (w_j T_{ji})^{1-\sigma}]^{\frac{1}{1-\sigma}}, \quad i = 1, \ldots, n. \tag{1.9}$$

- The *wage rate* w_i at the ith location is given in terms of Y_j and G_j $(j = 1, \ldots, n)$ by the *wage equation*:

$$w_i = [\sum_{j=1}^{n} Y_j {T_{ij}}^{1-\sigma} {G_j}^{\sigma-1}]^{1/\sigma}, \quad i = 1, \ldots, n. \tag{1.10}$$

- The total *income* Y_i of all the workers at the ith location is expressed in terms of λ_i and w_i by

$$Y_i = \mu \lambda_i w_i + \frac{1-\mu}{n}, \quad i = 1, \ldots, n. \tag{1.11}$$

[11] The price p_i^{M} at location i is assumed to be equal to the wage w_i of the manufacturing workers.

- The *real wage* is given in terms of w_i and G_i by

$$\omega_i = w_i G_i^{-\mu}, \quad i = 1, \ldots, n. \tag{1.12}$$

Here $\sigma(> 1)$ is an auxiliary parameter expressing the *elasticity of substitution* between differentiated goods and μ $(0 < \mu < 1)$ is another auxiliary parameter representing the *expenditure share* introduced in (1.5). These auxiliary parameters, which are specified and fixed in each problem, are suppressed in the following. *No-black-hole condition*,

$$\frac{\sigma - 1}{\sigma} > \mu, \tag{1.13}$$

is assumed to avoid economies with too strong increasing returns leading to excessive agglomeration.

Among many variables and parameters of these equations, we regard

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\top$$

as the independent variable vector and τ as the main parameter. Then from (1.9)–(1.11), the variables G_i, w_i, and Y_i are expressed implicitly as functions in $\boldsymbol{\lambda}$ and τ:

$$G_i = G_i(\boldsymbol{\lambda}, \tau), \quad w_i = w_i(\boldsymbol{\lambda}, \tau), \quad Y_i = Y_i(\boldsymbol{\lambda}, \tau), \quad i = 1, \ldots, n.$$

From (1.12), the real wages can be represented implicitly as functions in $\boldsymbol{\lambda}$ and τ:

$$\omega_i = \omega_i(\boldsymbol{\lambda}, \tau), \quad i = 1, \ldots, n, \tag{1.14}$$

which are assumed to be sufficiently smooth.

1.5.3 Spatial Equilibrium

We have formulated the market equilibrium for a fixed population distribution $\boldsymbol{\lambda}$ of manufacturing workers. Hereafter manufacturing workers are allowed to migrate among locations to obtain spatial equilibrium, by which the distribution of manufacturing workers is determined.

Definition

A population distribution $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\top$, normalized as (1.6), is called a *spatial equilibrium* if it satisfies the so-called *complementarity* condition:

$$(\omega_i(\boldsymbol{\lambda}, \tau) - \hat{\omega})\lambda_i = 0, \quad \lambda_i \geq 0, \quad \omega_i(\boldsymbol{\lambda}, \tau) - \hat{\omega} \leq 0, \quad i = 1, \ldots, n \tag{1.15}$$

for some $\hat{\omega}$, where $\omega_i(\boldsymbol{\lambda}, \tau)$ is given in (1.14). Note that $\hat{\omega}$ must be equal to the maximum value among $\omega_i = \omega_i(\boldsymbol{\lambda}, \tau)$ over $i = 1, \ldots, n$, since for some i we have $\lambda_i > 0$ and hence $\omega_i - \hat{\omega} = 0$ by the complementarity. Therefore, we can equivalently say that $\boldsymbol{\lambda}$ is a spatial equilibrium if (1.15) is true for

$$\hat{\omega} = \max(\omega_1, \ldots, \omega_n). \tag{1.16}$$

Thus, $\hat{\omega}$ means the *highest real wage*. When the system is in a spatial equilibrium, we have $\omega_i = \hat{\omega}$ for every i with $\lambda_i > 0$, which means that no individual worker can improve his/her real wage by changing his/her location.

Equilibria are classified into two types, interior equilibria and corner equilibria, according to whether they are located in the interior or on the boundary of the simplex (1.6). An equilibrium $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\top$ is called an *interior equilibrium* if $\lambda_i > 0$ for all $i = 1, \ldots, n$; otherwise, it is a *corner equilibrium*. At an interior equilibrium, all locations have positive populations whereas one or more locations have zero population at a corner equilibrium. An interior point $\boldsymbol{\lambda}$ of the simplex (1.6) is an equilibrium if and only if

$$\omega_i(\boldsymbol{\lambda}, \tau) - \hat{\omega} = 0, \quad i = 1, \ldots, n$$

for some $\hat{\omega}$, i.e., $\omega_i(\boldsymbol{\lambda}, \tau)$, is a constant across i.

Dynamics and Rest Point

A dynamics of the form

$$\frac{d\lambda_i}{dt} = F_i(\boldsymbol{\lambda}, \tau), \quad i = 1, \ldots, n \tag{1.17}$$

is often considered to express an adjustment process towards the spatial equilibrium. Consideration of a dynamics of migration is intended to capture the spatial equilibrium as a *long-term equilibrium*, in which workers have completed their migration and are not motivated to migrate any more. In vector notation, the system (1.17) is written as

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \tau), \tag{1.18}$$

where $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = (F_1(\boldsymbol{\lambda}, \tau), \ldots, F_n(\boldsymbol{\lambda}, \tau))^\top$. The function $\boldsymbol{F}(\boldsymbol{\lambda}, \tau)$ is chosen so that a rest point of this dynamics is a reasonable candidate for spatial equilibrium, where a *rest point* (or *stationary point*) of the dynamical system (1.18) means a vector $\boldsymbol{\lambda}$ such that

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0} \tag{1.19}$$

for a given τ.

Among other possibilities, Krugman, 1991 [21] adopted

$$F_i(\boldsymbol{\lambda}, \tau) = (\omega_i(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau))\lambda_i, \quad i = 1, \dots, n \tag{1.20}$$

for the adjustment process (1.18). Here

$$\overline{\omega} = \sum_{i=1}^{n} \lambda_i \omega_i \tag{1.21}$$

denotes the *average real wage* and $F_i(\boldsymbol{\lambda}, \tau)$ is defined for $\boldsymbol{\lambda}$ belonging to the $(n-1)$-dimensional simplex (1.6) and $\tau > 0$. A *rest point* $\boldsymbol{\lambda}$ of this dynamics is characterized as a point $\boldsymbol{\lambda}$ on this simplex that satisfies a system of nonlinear equilibrium equations

$$(\omega_i(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau))\lambda_i = 0, \quad i = 1, \dots, n. \tag{1.22}$$

The dynamics (1.17) with (1.20) is called *replicator dynamics* in the field of population dynamics or evolutionary game theory (Weibull, 1995 [31]; Hofbauer and Sigmund, 1998 [19]; and Sandholm, 2010 [27]).

Remark 1.1. The replicator dynamics (1.17) with (1.20) is defined on the simplex (1.6), and this constraint is indeed inherent in the dynamics. By adding (1.17) over $i = 1, \dots, n$, we obtain

$$\frac{d}{dt} \sum_{i=1}^{n} \lambda_i = \overline{\omega}(\boldsymbol{\lambda}, \tau)(1 - \sum_{i=1}^{n} \lambda_i), \tag{1.23}$$

which shows that the condition $\sum_{i=1}^{n} \lambda_i = 1$ in (1.6) is preserved in the dynamics. Also, for a rest point, we see from (1.23) that $\overline{\omega}(\boldsymbol{\lambda}, \tau)(1 - \sum_{i=1}^{n} \lambda_i) = 0$ holds, which implies $\sum_{i=1}^{n} \lambda_i = 1$ since $\overline{\omega} > 0$. It is noted, in passing, that nonnegativity of λ_i is not implied by (1.22). □

Remark 1.2. We have here focused on replicator dynamics as the adjustment dynamics. This is not the unique choice although it certainly has a historical significance and is convenient for mathematical treatment. Another adjustment dynamics, which guarantees the nonnegativity of λ_i, will be considered in Sect. 4.2.1. □

Relation Between Spatial Equilibrium and Rest Point

As expected, the spatial equilibrium is closely related to the rest point of the replicator dynamics. The former is a concept of economic importance, while the latter is more amenable to mathematical analysis.

Proposition 1.1. *A spatial equilibrium is a rest point of the replicator dynamics.*

Proof. Suppose that (1.15) holds. Then $\omega_i = \hat{\omega}$ for every i with $\lambda_i > 0$, which implies $\overline{\omega} = \hat{\omega}$. Hence, the complementarity (1.15) implies (1.22). □

The converse is not true in general. That is, a rest point of the replicator dynamics is not necessarily a spatial equilibrium, as the following example shows.

Example 1.1. The replicator dynamics for $n = 3$ reads

$$\frac{d\lambda_1}{dt} = (\omega_1 - \overline{\omega})\lambda_1, \quad \frac{d\lambda_2}{dt} = (\omega_2 - \overline{\omega})\lambda_2, \quad \frac{d\lambda_3}{dt} = (\omega_3 - \overline{\omega})\lambda_3.$$

Suppose that $(\omega_1, \omega_2, \omega_3) = (1, 1, 2)$ for $(\lambda_1, \lambda_2, \lambda_3) = (1/2, 1/2, 0)$. We then have $\overline{\omega} = 1$, and the equations

$$(\omega_1 - \overline{\omega})\lambda_1 = 0, \quad (\omega_2 - \overline{\omega})\lambda_2 = 0, \quad (\omega_3 - \overline{\omega})\lambda_3 = 0$$

in (1.22) are satisfied. This shows that $(\lambda_1, \lambda_2, \lambda_3) = (1/2, 1/2, 0)$ is a rest point of the replicator dynamics. This point, however, is not a spatial equilibrium since $\hat{\omega} = 2$, $(\omega_1 - \hat{\omega})\lambda_1 < 0$, and $(\omega_2 - \hat{\omega})\lambda_2 < 0$, and therefore the first condition in (1.15) is not met by this point. □

Proposition 1.2 below shows that a rest point with an additional property is a spatial equilibrium.

Proposition 1.2. *A rest point of the replicator dynamics with* (1.20) *is a spatial equilibrium, if and only if an additional condition*

$$\omega_i(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau) \leq 0, \quad i = 1, \ldots, n \tag{1.24}$$

is satisfied.

Proof. Let $\boldsymbol{\lambda}$ be a rest point of the replicator dynamics, satisfying (1.6) and (1.22). If (1.24) is true, then $\hat{\omega} = \max_i \omega_i \leq \overline{\omega}$, whereas $\hat{\omega} \geq \overline{\omega}$ by (1.16) and (1.21). Hence, it follows that $\hat{\omega} = \overline{\omega}$. Therefore, (1.24) implies the complementarity (1.15). Conversely, suppose that $\boldsymbol{\lambda}$ is a spatial equilibrium satisfying (1.15). Then $\omega_i = \hat{\omega}$ for every i with $\lambda_i > 0$, from which follows $\overline{\omega} = \hat{\omega}$. Therefore, (1.24) is implied by the complementarity (1.15). □

Remark 1.3. Proposition 1.2 provides us with a computational procedure to find a spatial equilibrium. We first solve the system (1.22) of nonlinear equations for a vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\top$. Then we check for nonnegativity of the components λ_i for all i, whereas the normalization condition $\sum_{i=1}^{n} \lambda_i = 1$ is always satisfied; see Remark 1.1. Next we check for the condition (1.24) and use Proposition 1.2 to decide whether the obtained nonnegative $\boldsymbol{\lambda}$ is indeed a spatial equilibrium or not. □

Sustainability

Proposition 1.2 above shows that a rest point with the additional property (1.24) is *sustainable*. Let us dwell on the sustainability implied by the additional condition (1.24). Failure of the condition (1.24) means that $\omega_i - \overline{\omega} > 0$ for some i. On the other hand, $\lambda_j > 0$ for some j, and for such j we must have $\omega_j - \overline{\omega} = 0$ by (1.22). Hence $\omega_i > \omega_j$. Since $\omega_i > \omega_j$, the workers in the jth location, where $\lambda_j > 0$, are motivated to migrate to the ith location. Obviously, such distribution cannot be sustained.

Example 1.2. The concept of sustain point is explained here with reference to the two-place economy of the Krugman model (Sect. 2.2.1). For the core–periphery pattern $(\lambda_1, \lambda_2) = (1, 0)$, the equations (1.9)–(1.12) with $T_{11} = T_{22} = 1$ and $T_{12} = T_{21} = T$ are satisfied by

$$Y_1 = \frac{1+\mu}{2}, \quad Y_2 = \frac{1-\mu}{2}, \quad G_1 = 1, \quad G_2 = T,$$

$$w_1 = 1, \quad w_2 = \left[\frac{1+\mu}{2}T^{1-\sigma} + \frac{1-\mu}{2}T^{\sigma-1}\right]^{\frac{1}{\sigma}},$$

$$\omega_1 = 1, \quad \omega_2 = T^{-\mu}\left[\frac{1+\mu}{2}T^{1-\sigma} + \frac{1-\mu}{2}T^{\sigma-1}\right]^{\frac{1}{\sigma}}.$$

Under the no-black-hole condition (1.13), a unique $T_{\text{sustain}} > 1$ exists such that $\omega_1 = \omega_2$ for $T = T_{\text{sustain}}$, as is shown in Fujita, Krugman, and Venables, 1999, Sect. 5.4 [16]. This value T_{sustain} is called the *sustain point*. □

1.5.4 Stability

Recall from (1.18) the dynamics of migration

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \tau),$$

of which the rest points are closely related to the spatial equilibria. By definition, a rest point is a vector $\boldsymbol{\lambda}^*$ that satisfies $\boldsymbol{F}(\boldsymbol{\lambda}^*, \tau) = \boldsymbol{0}$ for a given τ. Some issues related to stability of rest points are discussed here.

A rest point $\boldsymbol{\lambda}^*$ of the differential equation above is said to be *asymptotically stable* if every solution $\boldsymbol{\lambda}(t)$ with the initial state $\boldsymbol{\lambda}(0)$ sufficiently close to $\boldsymbol{\lambda}^*$ stays in a neighborhood of $\boldsymbol{\lambda}^*$ for all t and decays to $\boldsymbol{\lambda}^*$ as $t \to +\infty$. A rest point $\boldsymbol{\lambda}^*$ is termed *linearly stable* if every eigenvalue of the Jacobian matrix

$$J(\boldsymbol{\lambda}, \tau) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}, \tau) \tag{1.25}$$

evaluated at $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ has a negative real part, and *linearly unstable* if at least one eigenvalue has a positive real part. In this book "stability" means linear stability and a rest point that is not (linearly) stable is termed *unstable*.

Remark 1.4. Replicator dynamics is defined on the simplex (1.6). The constraint $\sum_{i=1}^{n} \lambda_i = 1$ is not explicit in (1.17) with (1.20), but only implicit in the dynamics (Remark 1.1). The hyperplane $\sum_{i=1}^{n} \lambda_i = 1$ is an invariant subspace, as shown by (1.23). Moreover, the dynamics is stable in the direction perpendicular to this hyperplane. In discussing stability, therefore, we may ignore the constraint $\sum_{i=1}^{n} \lambda_i = 1$ and investigate stability in the n-dimensional space. □

The following proposition shows that the spatial equilibrium can be captured, conceptually and computationally, as a stable rest point of the replicator dynamics.

Proposition 1.3. *A stable rest point of the replicator dynamics* (1.17) *with* (1.20) *satisfies the condition* (1.24)*, and hence is a spatial equilibrium in the sense of* (1.15).

Proof. For (1.20) the Jacobian matrix (1.25) is given by

$$
\begin{aligned}
J &= \begin{bmatrix} \omega_1 - \overline{\omega} & & O \\ & \ddots & \\ O & & \omega_n - \overline{\omega} \end{bmatrix} + \begin{bmatrix} \Omega_{11}\lambda_1 & \cdots & \Omega_{1n}\lambda_1 \\ \vdots & \ddots & \vdots \\ \Omega_{n1}\lambda_n & \cdots & \Omega_{nn}\lambda_n \end{bmatrix} \\
&= \operatorname{diag}(\omega_1 - \overline{\omega}, \ldots, \omega_n - \overline{\omega}) + \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\Omega,
\end{aligned}
$$

where diag$(\cdots)$ denotes a diagonal matrix with diagonal entries therein and

$$
\Omega_{ij} = \frac{\partial(\omega_i - \overline{\omega})}{\partial \lambda_j}, \quad i, j = 1, \ldots, n,
$$
$$
\Omega = (\Omega_{ij} \mid i, j = 1, \ldots, n).
$$

For a rest point $\boldsymbol{\lambda}$, which satisfies (1.22), we may assume, by renumbering of the locations, that

$$
\lambda_i > 0,\ \omega_i - \overline{\omega} = 0, \quad i = 1, \ldots, m, \tag{1.26}
$$
$$
\lambda_i = 0, \qquad\qquad i = m+1, \ldots, n \tag{1.27}
$$

for some m with $1 \leq m \leq n$. Then the above Jacobian matrix becomes

$$
J = \left(\begin{array}{c|ccc} \Phi_1 & & \Phi_2 & \\ \hline & \omega_{m+1} - \overline{\omega} & & 0 \\ O & & \ddots & \\ & 0 & & \omega_n - \overline{\omega} \end{array} \right), \tag{1.28}
$$

where $\Phi_i = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)\Omega_i$ $(i = 1, 2)$ and

$$\Omega_1 = \begin{bmatrix} \Omega_{11} & \cdots & \Omega_{1m} \\ \vdots & \ddots & \vdots \\ \Omega_{m1} & \cdots & \Omega_{mm} \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} \Omega_{1,m+1} & \cdots & \Omega_{1n} \\ \vdots & \ddots & \vdots \\ \Omega_{m,m+1} & \cdots & \Omega_{mn} \end{bmatrix}.$$

By (1.28), $\omega_i - \overline{\omega}$ $(i = m+1, \ldots, n)$ are eigenvalues of J, and hence negative due to the assumed stability. On the other hand, we have $\omega_i - \overline{\omega} = 0$ $(i = 1, \ldots, m)$ by (1.26). Therefore, $\omega_i - \overline{\omega} \leq 0$ for all $i = 1, \ldots, n$, i.e., (1.24) holds. The second statement about spatial equilibrium follows from this by Proposition 1.2. $\square$

Useful sufficient conditions for stability for a fairly general class of dynamics including replicator dynamics are derived in Tabuchi and Zeng, 2004 [29]. Also see Ginsburgh, Papageorgiou, and Thisse, 1985 [17].

1.6 Appendix: Preexisting Study of Self-organization of Cities

Self-organization of cities is one of the most important topics in *economic geography*, which studies the location, distribution, and spatial organization of economic activities across the world. Modern location economics began with a basic analytical model of the relationship among markets, production, and distance by von Thünen, 1826 [30]. When the land surrounding the market is entirely flat and its fertility uniform, the land use of several agricultural goods is expected to form famous von Thünen's rings.

Through a study of urban data in southern Germany, Christaller, 1933 [8] developed *central place theory* for describing self-organization of the assemblage of hexagonal market areas of different sizes and for describing a hierarchy of scales in urbanization (cities, towns, villages, etc.). Christaller posed a question: "Are there laws which determine the number, size, and distributions of towns?" Hexagonal market areas were generalized by Lösch, 1940 [23] from a geometrical standpoint. The Zipf law (rank-size rule) for the size distribution of cities was invented (Zipf, 1949 [34]). Empirical investigations of several places in America, such as Snohomish County (Washington), Southwestern Iowa, Southwestern Ontario, and the Niagara Peninsula, have been conducted (Lloyd and Dicken, 1972 [22]). Sanglier and Allen, 1989 [28] used a dynamic model based on central place theory and successfully calibrated the model with socioeconomic data for Belgium, 1970–1984. Early studies of self-organizing patterns in geography and regional science were conducted by Clarke and Wilson, 1985 [9]; Munz and Weidlich, 1990 [26]; and Weidlich, 2000 [32].

Central place theory laid the foundation of economic geography. Indeed Fujita, Krugman, and Mori, 1999 [15] stated "it [central place theory] is a powerful idea too good for being left as an obscure theory." This theory, however, is based only on a normative and geometrical approach and is not derived from market equilibrium conditions. An early attempt to provide central place theory with a microeconomic foundation was made by Eaton and Lipsey, 1975 [13], 1982 [14], and a hexagonal distribution of mobile production factors (e.g., firms and workers) in two dimensions was shown to exist as an economic equilibrium for spatial competition.

Based on a full-fledged general equilibrium approach, Krugman, 1991 [21] developed a core–periphery model, incorporating the monopolistic competition model of Dixit–Stiglitz, 1977 [12] into spatial economics. This model considered an economy with two sectors (agriculture and manufacturing) and assumed an upper-tier utility function of the Cobb–Douglas type with CES sub-preferences over manufacturing varieties. The model provided a new framework to explain interactions that occur among increasing returns at the level of firms, transport costs, and factor mobility. He demonstrated the predominant role of bifurcation in agglomeration: two identical cities are in a stable state with high transport costs, and, when the costs are reduced to a certain level (in association with progress of technology), tomahawk bifurcation triggers a spontaneous concentration to a single city. Since then, the study of such microeconomic mechanism has blossomed into *new economic geography*,[12] which is acknowledged as an important branch of international, regional, and urban economics.

Also in physics, self-organization of cities has drawn keen attention. The distributions of areas of satellite cities, towns, and villages around Berlin and London have been demonstrated to follow a universal law (Makse et al., 1995 [24], 1998 [25]). This behavior is related to fractal structures in urban patterns (Batty and Longley, 1994 [3]) and has been reproduced by correlated diffusion-limited aggregation models (Benguigui, 1995 [5]). Intermittent spatiotemporal structures for urban development have been generated to model large-scale city formation (Zanette and Manruiba, 1997 [33]). Spatial interaction models in the study of urban growth and transportation for social modeling have been presented to investigate interactions between activities on a lattice (Helbing, 1995 [18]). A Markov random field has been used to depict human actions and decisions (Anderson et al., 2002 [1]). The formation of a city has been studied using the percolation theory and GIS data of 29 cities (Binter, Hołyst, and Fiałkowski, 2009 [6]).

1.7 Summary

- Hexagonal distributions of Christaller and Lösch have been introduced.
- Self-organization of central places has been demonstrated for a domain with the shape of southern Germany.

[12] For overviews of the core–periphery models with variants, see Brakman, Garretsen, and van Marrewijk, 2001 [7]; Baldwin et al., 2003 [2]; and Combes, Mayer, and Thisse, 2008 [10].

- Emergence of a hexagonal distribution called Christaller's $k = 3$ system on the hexagonal lattice with periodic boundaries has been observed.
- Krugman's core–periphery model has been presented.
- History of the study of self-organization of cities has been reviewed.

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Chapter 2
Group-Theoretic Bifurcation Theory

Abstract Fundamentals of group-theoretic bifurcation theory are introduced as a mathematical methodology to deal with agglomeration in economic geography, which is captured in this book as bifurcation phenomena of systems with dihedral or hexagonal symmetry. The framework of group-theoretic bifurcation analysis of economic agglomeration is illustrated for the two-place economy in new economic geography. Symmetry of a system is described by means of a group, and group representation is introduced as the main tool used to formulate the symmetry of the equilibrium equation of symmetric systems. Group-theoretic bifurcation analysis procedure under group symmetry is presented with particular emphasis on Liapunov–Schmidt reduction under symmetry. Bifurcation equation, equivariant branching lemma, and block-diagonalization are introduced as mathematical tools used to tackle bifurcation of a symmetric system in Chaps. 3–9.

Keywords Bifurcation • Bifurcation equation • Block-diagonalization • Equivariant branching lemma • Group representation theory • Group-theoretic bifurcation analysis • Liapunov–Schmidt reduction • Symmetry

2.1 Introduction

In this book, the mechanism of self-organization of spatial distributions of cities is elucidated in light of symmetry-breaking bifurcations. This chapter is a succinct introduction to the group-theoretic bifurcation theory, which serves as our main mathematical tool for the analysis of agglomerations in new economic geography.

Bifurcation means the emergence of multiple solutions to a system of equations depending on a parameter. Usually it is presumed that such a system of equations admits a unique solution for a given parameter value, at least locally in an appropriate sense, and therefore, the solutions for varying parameter values form a single solution branch. Non-uniqueness of the solution can be induced by the

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singularity of the Jacobian matrix of the system of equations. Multiple solutions for the same parameter value can exist near a critical point, at which the Jacobian matrix is singular by definition.

Bifurcation often occurs in systems with symmetry. Bifurcated solutions from a fully symmetric state retain partial symmetry, that is, symmetry is partially broken at the onset of bifurcation. Symmetry is described by a group, and a hierarchy of subgroups $G_1 \to G_2 \to G_3 \to \cdots$ characterizes a recursive occurrence of bifurcations, where $\to$ denotes a bifurcation and G_i $(i = 1, 2, \ldots)$ stand for the nesting subgroups that describe the reduced symmetry of the bifurcated solutions. Such a mathematical mechanism affords a fairly universal explanation of pattern formations observed in many physical phenomena.[1]

As a systematic means to describe and analyze qualitative aspects of symmetry-breaking bifurcations, group-theoretic bifurcation theory has been developed in nonlinear mathematics.[2] An extremely important finding of this theory is that the mechanism of such bifurcation does not depend on individual systems but mostly on the symmetry of the system under consideration. The main ideas of group-theoretic bifurcation analysis are explained in this chapter as an informal introduction for non-mathematicians.[3]

In view of the symmetry of the system under consideration, possible critical points and bifurcated solutions can be classified with reference to group-theoretic concepts. Bifurcating solutions in a neighborhood of each critical point can be investigated through the Liapunov–Schmidt reduction, which is a systematic reduction procedure based on the implicit function theorem. The full system of equations is reduced to another system consisting of a few equations, and this reduction is done in a way compatible with the symmetry. The obtained reduced system is called the bifurcation equation.

The bifurcation equation is extremely useful in that it retains all the essential local characteristics of the full system of equations around a critical point. In particular, the symmetry of bifurcating solutions can be identified as the symmetry of solutions to the bifurcation equation. This fact enables us to find possible agglomeration patterns in new economic geographical models on the hexagonal lattice, as well as on the racetrack. Moreover, equivariant branching lemma is introduced as the most pertinent means to investigate the existence of a bifurcating solution with a specified symmetry.

[1] In fluid mechanics, for example, the *Couette–Taylor flow* in a hollow cylinder, which is a rotating annular fluid, displays wave patterns with various symmetries through pattern selection (e.g., Taylor, 1923 [19]). The convective motion of fluid in the *Bénard problem* displays regularly arrayed hexagons (e.g., Koschmieder, 1974 [11]). These phenomena can be explained successfully in terms of symmetry-breaking bifurcations.

[2] See, for example, Sattinger, 1979 [17], 1980; Chow and Hale, 1982 [2]; Golubitsky and Schaeffer, 1985 [6]; Golubitsky, Stewart, and Schaeffer, 1988 [7]; Mitropolsky and Lopatin, 1988 [14]; Allgower, Böhmer, and Golubitsky, 1992 [1]; Marsden and Ratiu, 1999 [12]; Olver, 1995 [15]; and Hoyle, 2006 [9].

[3] We restrict ourselves to finite-dimensional equations and finite groups, which are sufficient for our purpose. In so doing we can avoid mathematical sophistications necessary to address differential equations and continuous groups.

This chapter is organized as follows. The formation of a core–periphery pattern of Krugman's model for a two-place economy is demonstrated and the framework of group-theoretic bifurcation analysis for this economy is illustrated in Sect. 2.2. The theory of representations of finite groups is explained in Sect. 2.3 to formulate symmetry in mathematical terms. A group-theoretic bifurcation analysis procedure is given in Sect. 2.4.

2.2 Bifurcation of the Two-Place Economy

We explain the formation of a *core–periphery pattern* by way of bifurcation in the *two-place economy* in Fig. 2.1 consisting of two places, Place 1 and Place 2.

2.2.1 Economic Understanding

Understanding of agglomeration phenomena of the two-place economy in new economic geography is introduced. This two-place economy will be reexamined in Sect. 2.2.2 with an emphasis on mathematical mechanism of bifurcation.

Basic Assumptions and Governing Equation

The two-place economy in Krugman's modeling (Sect. 1.5) has two industrial sectors (agriculture and manufacturing) and two factors of production (agricultural labor and manufacturing labor). Agricultural workers are immobile. Manufacturing workers are allowed to migrate between places, and their distribution is expressed by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^\top$, where $(\cdot)^\top$ means the transpose and λ_i is their population at the ith place ($i = 1, 2$) satisfying

$$\lambda_1 + \lambda_2 = 1, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0, \tag{2.1}$$

which expresses the assumption of no population growth. The agglomeration pattern of migrating manufacturing workers is determined by Krugman's modeling and is parameterized by a single parameter τ with $0 < \tau < 1$, called the transport cost parameter. The cost of transportation between two places is reduced with a decrease of τ associated with the progress of transportation technology.

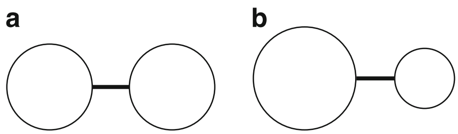

Fig. 2.1 Geometry of two-place economy. The area of a *circle* denotes the population size λ_i ($i = 1, 2$). (**a**) $\lambda_1 = \lambda_2$. (**b**) $\lambda_1 \neq \lambda_2$

As a special case ($n = 2$) of Krugman's modeling in Sect. 1.5, the two-place economy is described by

$$Y_1 = \mu\lambda_1 w_1 + \frac{1-\mu}{2}, \tag{2.2}$$

$$Y_2 = \mu\lambda_2 w_2 + \frac{1-\mu}{2}, \tag{2.3}$$

$$G_1 = [\lambda_1 {w_1}^{1-\sigma} + \lambda_2 (w_2 T)^{1-\sigma}]^{\frac{1}{1-\sigma}}, \tag{2.4}$$

$$G_2 = [\lambda_1 (w_1 T)^{1-\sigma} + \lambda_2 {w_2}^{1-\sigma}]^{\frac{1}{1-\sigma}}, \tag{2.5}$$

$$w_1 = [Y_1 {G_1}^{\sigma-1} + Y_2 {G_2}^{\sigma-1} T^{1-\sigma}]^{\frac{1}{\sigma}}, \tag{2.6}$$

$$w_2 = [Y_1 {G_1}^{\sigma-1} T^{1-\sigma} + Y_2 {G_2}^{\sigma-1}]^{\frac{1}{\sigma}}, \tag{2.7}$$

$$\omega_1 = w_1 {G_1}^{-\mu}, \tag{2.8}$$

$$\omega_2 = w_2 {G_2}^{-\mu}, \tag{2.9}$$

where

- Y_i is the total income of the workers in the ith place,
- G_i is the price index in the ith place,
- w_i is the nominal wage rate of workers in the ith place,
- ω_i is the real wage of workers in the ith place,
- $T = 1/(1-\tau)$ represents the transportation cost between Places 1 and 2 in terms of the transport cost parameter τ with $0 < \tau < 1$, and
- μ and σ are parameters with $0 < \mu < 1$ and $\sigma > 1$, which are assumed to satisfy the *no-black-hole condition* $(\sigma - 1)/\sigma > \mu$.

From the system of equations (2.2)–(2.9), the real wages can be represented, under some regularity conditions, uniquely as

$$\omega_1 = \omega_1(\lambda_1, \lambda_2, \tau), \quad \omega_2 = \omega_2(\lambda_1, \lambda_2, \tau), \tag{2.10}$$

where the dependence on μ and σ is suppressed in the notation.

The equilibrium equation of this economy reads as

$$F_i(\boldsymbol{\lambda}, \tau) = (\omega_i(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau))\lambda_i = 0, \quad i = 1, 2, \tag{2.11}$$

which is derived in Sect. 1.5 in a general setting. Here $\overline{\omega} = \overline{\omega}(\boldsymbol{\lambda}, \tau)$ is the average real wage defined as

$$\overline{\omega} = \lambda_1\omega_1 + \lambda_2\omega_2. \tag{2.12}$$

It is noted that $\lambda_1 + \lambda_2 = 1$ in (2.1) follows from (2.11).[4]

[4] By adding $\lambda_1\omega_1 = \lambda_1\overline{\omega}$ and $\lambda_2\omega_2 = \lambda_2\overline{\omega}$, we obtain $\overline{\omega} = \lambda_1\omega_1 + \lambda_2\omega_2 = (\lambda_1 + \lambda_2)\overline{\omega}$, which implies $\lambda_1 + \lambda_2 = 1$ since $\overline{\omega} > 0$.

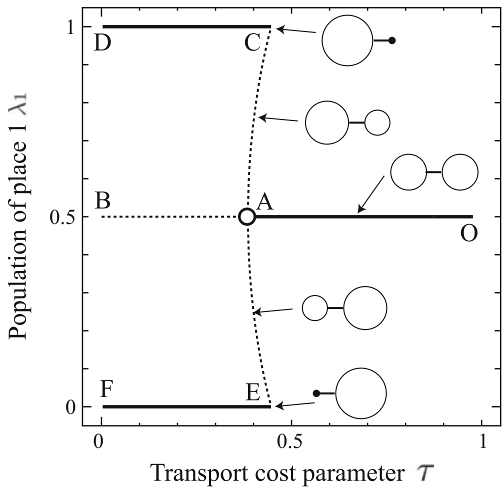

Fig. 2.2 Bifurcation behavior of the two-place economy with the Krugman model computed for specific parameter values of $\mu = 0.4$ and $\sigma = 5.0$. *Solid curves* mean stable equilibria; *dotted curves*, unstable equilibria; *white circle*, bifurcation point

Bifurcation Behavior

The population λ_1 versus transport cost parameter τ curves, obtained by solving (2.11) numerically, are shown in Fig. 2.2. Here the three solid curves denote stable equilibria and the dotted curves mean unstable equilibria.[5] There are three "obvious" equilibrium paths, on which the population distribution remains unchanged with respect to the change of τ:

- The horizontal line OAB representing the flat earth (uniformly-distributed) state of equilibrium with $\lambda_1 = \lambda_2 = 1/2$.
- The horizontal line CD representing a core–periphery pattern with complete agglomeration to Place 1 ($\lambda_1 = 1,\ \lambda_2 = 0$).
- The horizontal line EF representing another core–periphery pattern with complete agglomeration to Place 2 ($\lambda_1 = 0,\ \lambda_2 = 1$).

Those three obvious equilibria are connected by an unstable bifurcated curve CAE, which intersects with OAB at the subcritical pitchfork bifurcation point A, with the line CD at the transcritical bifurcation point C, and with the line EF at the transcritical bifurcation point E. Since stability of the obvious equilibria changes at these bifurcation points A, C, and E, the values τ_{A} and τ_{C} $(= \tau_{\mathrm{E}})$ of the transport cost parameter τ at these points play a predominant role in the progress of agglomeration.

The flat earth equilibrium with $\lambda_1 = \lambda_2 = 1/2$ is stable for an undeveloped state of transportation with large τ. When the transport cost parameter τ is reduced

[5]See Sect. 2.4.1, as well as Sect. 1.5.4, for introductory issues of bifurcation and the definition of stability.

in association with the progress of technology to the level of $\tau = \tau_{\mathrm{A}}$, the flat earth equilibrium ceases to be stable.[6]

The unstable transient state on the bifurcated path AC with $0 < \lambda_2 < 1/2 < \lambda_1 < 1$ is asymmetric, reaching the core–periphery pattern with complete agglomeration to Place 1 ($\lambda_1 = 1, \lambda_2 = 0$). This pattern is stable for a developed state of transportation with small τ. When τ is increased to the level of $\tau = \tau_{\mathrm{C}}$, the core–periphery pattern ceases to be stable. On the curve DC, $\omega_1 > \omega_2$ is satisfied, that is, Place 1 offers a higher real wage than Place 2 does. In this economic environment, the manufacturing workers completely agglomerated to Place 1 would not migrate to Place 2 and the core–periphery pattern is preserved. A pattern preserved in this manner is called *sustainable* in new economic geography (Sect. 1.5.3). At $\tau = \tau_{\mathrm{C}}$, we have $\omega_1 = \omega_2$, and this pattern of the core (Place 1) and the periphery (Place 2) becomes unsustainable since some workers at Place 1 starts to migrate to Place 2.[7]

Remark 2.1. Fujita, Krugman, and Venables, 1999 [5] restricted themselves to sustainable equilibria, and regarded each of the points C and E as a kink that connects two half branches. In this book, however, these points are regarded as transcritical bifurcation points, as explained in Sect. 3.5.3. □

2.2.2 *Mathematical Treatment*

Prior to a systematic presentation of group-theoretic bifurcation theory, a glimpse of the mathematical formulation of symmetry of an economic system is given here for the two-place economy. Introductory issues of bifurcation can be consulted with Sect. 2.4.1.

Equivariance and Governing Equation

The equilibrium equation in (2.11) is rewritten as

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \begin{bmatrix} F_1(\lambda_1, \lambda_2, \tau) \\ F_2(\lambda_1, \lambda_2, \tau) \end{bmatrix} = \mathbf{0} \tag{2.13}$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^\top$ and

$$F_1(\lambda_1, \lambda_2, \tau) = (\omega_1(\lambda_1, \lambda_2, \tau) - \overline{\omega}(\lambda_1, \lambda_2, \tau))\lambda_1, \tag{2.14}$$

---

[6]In new economic geography (Fujita, Krugman, and Venables, 1999 [5]), the value $\tau = \tau_{\mathrm{A}}$ is called the *break point* and the bifurcation at point A is called *tomahawk bifurcation*.

[7]The value $\tau = \tau_{\mathrm{C}}$ is an important index of the sustainability of the core–periphery pattern called the *sustain point*.

$$F_2(\lambda_1, \lambda_2, \tau) = (\omega_2(\lambda_1, \lambda_2, \tau) - \overline{\omega}(\lambda_1, \lambda_2, \tau))\lambda_2, \tag{2.15}$$

where $\overline{\omega} = \overline{\omega}(\lambda_1, \lambda_2, \tau)$ denotes the average real wage given by $\overline{\omega} = \lambda_1\omega_1 + \lambda_2\omega_2$ in (2.12).

The two-place economy is apparently invariant to the *permutation* of place numbers

$$(1, 2) \mapsto (2, 1). \tag{2.16}$$

Equations (2.2)–(2.9) are indeed invariant to (2.16), and therefore

$$\omega_2(\lambda_1, \lambda_2, \tau) = \omega_1(\lambda_2, \lambda_1, \tau). \tag{2.17}$$

As a consequence, the equations (2.14) and (2.15) are also invariant to (2.16), i.e.,

$$F_2(\lambda_1, \lambda_2, \tau) = F_1(\lambda_2, \lambda_1, \tau). \tag{2.18}$$

It follows from the symmetry above, (2.17) in particular, that $(\lambda_1, \lambda_2, \tau) = (1/2, 1/2, \tau)$ is a solution to the equilibrium equation, i.e.,

$$F_1(1/2, 1/2, \tau) = F_2(1/2, 1/2, \tau) = 0.$$

It is noted that the solution $(1/2, 1/2, \tau)$ has been derived solely from symmetry arguments, without reference to specific functional forms.

The bifurcation behavior of the two-place economy observed in Sect. 2.2.1 can be described using mathematical notations, to be presented in this chapter, as follows.

Denote the permutation (2.16) as s, and also consider the *identity transformation*

$$e: \ (1, 2) \mapsto (1, 2). \tag{2.19}$$

Then we have $s^2 = e$ and

$$G = \{e, s\} \tag{2.20}$$

forms a *group* (see Sect. 2.3.1 for the definition of a group). We represent the action of e and that of s on place numbers $(1, 2)$ in (2.19) and (2.16) by a pair of 2×2 matrices

$$T(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.21}$$

To express the symmetry condition (2.18) in a general form, first note that it is equivalent to

$$F_2(\lambda_1, \lambda_2, \tau) = F_1(\lambda_2, \lambda_1, \tau), \tag{2.22}$$

$$F_1(\lambda_1, \lambda_2, \tau) = F_2(\lambda_2, \lambda_1, \tau). \tag{2.23}$$

Using a column vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^\top$ in the argument of F_i, this may be further rewritten as

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_1(\boldsymbol{\lambda}, \tau) \\ F_2(\boldsymbol{\lambda}, \tau) \end{bmatrix} = \boldsymbol{F} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{\lambda}, \tau \right).$$

Hence, by (2.21), the symmetry of the two-place economy is expressed as

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \tau), \quad g \in G \tag{2.24}$$

for $G = \{e, s\}$. Note that the condition for $g = e$ in (2.24) is trivially satisfied. The equation of the form of (2.24), described with group G and matrices $T(g)$ representing the action of each element g of G, represents the symmetry condition for a system in general. This is called *equivariance* (Sect. 2.4).

Asymptotic Bifurcation Behavior

The equivariance (2.24) can be exploited in the local analysis as follows. On the path of flat earth (uniform) equilibria $(\lambda_1, \lambda_2, \tau) = (1/2, 1/2, \tau)$, a critical point $(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c}) = (1/2, 1/2, \tau_\mathrm{c})$ might be encountered at some value of $\tau = \tau_\mathrm{c}$. In view of the symmetry, it is natural to introduce a new variable $w = \lambda_1 - \lambda_2$; then, we have

$$\lambda_1 = \frac{1+w}{2}, \quad \lambda_2 = \frac{1-w}{2}.$$

From $\boldsymbol{F} = (F_1, F_2)^\top = \boldsymbol{0}$ with the symmetry condition (2.18), an equation in incremental variables w and $\tilde{f} = \tau - \tau_\mathrm{c}$ can be obtained as

$$\begin{aligned} \tilde{F}(w, \tilde{f}) &= F_1\left(\frac{1+w}{2}, \frac{1-w}{2}, \tilde{f}\right) - F_2\left(\frac{1+w}{2}, \frac{1-w}{2}, \tilde{f}\right) \\ &= w[A\tilde{f} + Bw^2 + \text{(higher order terms)}] = 0, \end{aligned} \tag{2.25}$$

where A and B are some constants. This equation is termed a *bifurcation equation*.

The bifurcation equation has two kinds of solutions (equilibria):

$$\begin{cases} w = 0, & \text{trivial equilibria } (\lambda_1 = \lambda_2), \\ \tilde{f} = -\frac{B}{A}w^2 + \text{(higher order terms)}, & \text{bifurcated equilibria } (\lambda_1 \neq \lambda_2). \end{cases}$$

The curve of the bifurcated equilibria is a parabola, and, therefore, this is a pitchfork bifurcation, subcritical or supercritical according to the signs of A and B (see Fig. 2.3).

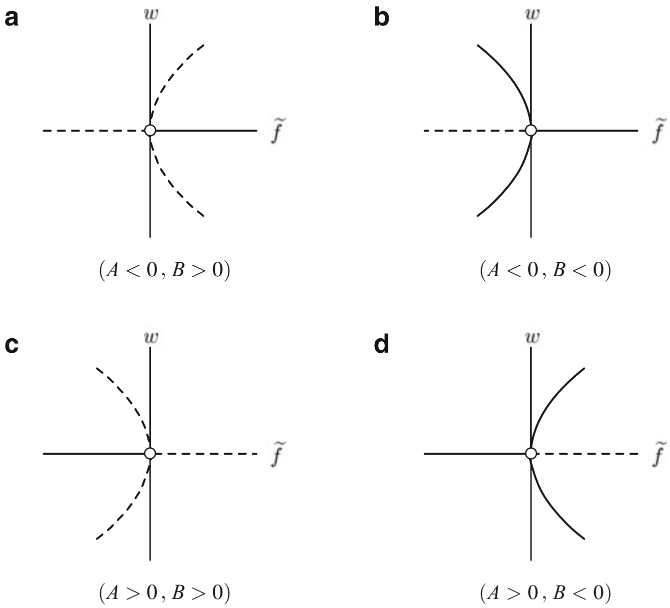

Fig. 2.3 Subcritical and supercritical pitchfork bifurcations. (**a**) Subcritical (new economic geography). (**b**) Supercritical (new economic geography). (**c**) Subcritical (physics). (**d**) Supercritical (physics). *Solid curves* mean stable equilibria; *dotted curves*, unstable equilibria; *white circle*, bifurcation point

As for the symmetry of these equilibria, the pre-bifurcation flat earth equilibria $w = 0$ ($\lambda_1 = \lambda_2 = 1/2$) have the permutation symmetry labeled $G = \{e, s\}$, and the bifurcated equilibria do not have any symmetry (labeled by a trivial group $\{e\}$).

Stability

To investigate the stability, the Jacobian matrix $\tilde{J}$ of the bifurcation equation (2.25) is evaluated at the flat earth equilibria as

$$\tilde{J}(0, \tilde{f}) = \frac{\partial \tilde{F}}{\partial w}(0, \tilde{f}) = A\tilde{f} + \text{(higher order terms)}.$$

Hence the flat earth equilibria are stable for $A\tilde{f} < 0$ and unstable for $A\tilde{f} > 0$ (see Sect. 1.5.4). For the bifurcated equilibria,

$$\tilde{J}\left(w, -\frac{B}{A}w^2\right) = \frac{\partial \tilde{F}}{\partial w}\left(w, -\frac{B}{A}w^2\right) = 2Bw^2 + \text{(higher order terms)}.$$

Hence the flat earth equilibria are stable for $B < 0$ and unstable for $B > 0$ (see Sect. 1.5.4). By virtue of these, the stability of these equilibria is classified in accordance with the signs of A and B. See Remark 2.12 in Sect. 2.4.3 for a mathematical background of this argument for stability.

In Krugman's core–periphery model in new economic geography, the governing equation is formulated in such a way that $A < 0$ and $B > 0$ (stable for $\tilde{f} > 0$). That is, the flat earth equilibria are stable when the transport cost is sufficiently large. This is based on the economic observation that the transport cost is reduced due to the progress of technology with the progress of time. As shown in Fig. 2.3a, the bifurcated path is unstable and the associated bifurcation is called tomahawk bifurcation or subcritical pitchfork bifurcation.[8]

On the other hand, in physics, it is customary to formulate the governing equation such that $A > 0$ (stable for $\tilde{f} < 0$). Then subcritical pitchfork bifurcation corresponds to $B > 0$ in Fig. 2.3c, and supercritical pitchfork bifurcation corresponds to $B < 0$ in Fig. 2.3d.

Discussion

To sum up, there are several aspects that can be determined by group-theoretic bifurcation analysis:

1. The pre-bifurcation equilibria have the form of $\lambda_1 = \lambda_2 = 1/2$.
2. A bifurcating solution necessarily has an asymmetric distribution.
3. The bifurcating curve is a parabola and the bifurcation is a subcritical or supercritical pitchfork.

In contrast, there are several aspects that vary with individual economic models with particular parameter values, and, therefore, cannot be captured by group-theoretic analysis:

1. stability of equilibria,
2. a specific value of τ_c, and
3. specific values and signs of A and B.

These aspects can be clarified by computational analysis.

[8]A simple, analytically-solvable model for the two-place economy in Pflüger, 2004 [16] has a supercritical pitchfork bifurcation with $A < 0$ and $B < 0$ in Fig. 2.3b.

2.3 Group Representation

Basics in the theory of group representation for finite groups are presented.[9]

2.3.1 Group

A set G is called a *group* if, for any pair of elements g and h of G, an element of G denoted as gh and called the *product* of g and h is specified, and if the following (i) through (iii) are satisfied.

(i) The *associative law* holds as

$$(g\,h)\,k = g\,(h\,k), \quad g, h, k \in G.$$

(ii) There exists an element $e \in G$ (called the *identity element*) such that

$$e\,g = g\,e = g, \quad g \in G.$$

(iii) For any $g \in G$ there exists $h \in G$ (called the *inverse* of g) such that

$$g\,h = h\,g = e.$$

It can be shown that the identity element e in (ii) is uniquely determined. The inverse of g is unique for each g and denoted as g^{-1}. In this book we assume that G is a *finite group* (i.e., a group consisting of a finite number of elements) and denote its *order* (= the number of elements) by $|G|$.

The structure of a group can be expressed explicitly by the *multiplication table*, which contains the product gh of elements g and h as the (g, h) entry. For example, a group

$$G = \{g_1, g_2, g_3\}$$

consisting of three elements g_1, g_2, and g_3 with relations

$$g_1 g_1 = g_2 g_3 = g_3 g_2 = g_1, \quad g_1 g_2 = g_2 g_1 = g_3 g_3 = g_2,$$
$$g_3 g_1 = g_2 g_2 = g_1 g_3 = g_3$$

[9]For details, the reader is referred to textbooks, such as Curtis and Reiner, 1962 [4]; Hamermesh, 1962 [8]; Miller, 1972 [13]; and Serre, 1977 [18].

Table 2.1 Multiplication tables for the groups in Example 2.1

(a) $G_1 = \{e, s\}(= \mathrm{D}_1)$

	e	s
e	e	s
s	s	e

(b) $G_2 = \{e, r, r^2\}(= \mathrm{C}_3)$

	e	r	r^2
e	e	r	r^2
r	r	r^2	e
r^2	r^2	e	r

(c) $G_3 = \{e, r, r^2, s, sr, sr^2\}(= \mathrm{D}_3)$

	e	r	r^2	s	sr	sr^2
e	e	r	r^2	s	sr	sr^2
r	r	r^2	e	sr^2	s	sr
r^2	r^2	e	r	sr	sr^2	s
s	s	sr	sr^2	e	r	r^2
sr	sr	sr^2	s	r^2	e	r
sr^2	sr^2	s	sr	r	r^2	e

can be represented by the following multiplication table:

	g_1	g_2	g_3
g_1	g_1	g_2	g_3
g_2	g_2	g_3	g_1
g_3	g_3	g_1	g_2

A group G is called *abelian* if $gh = hg$ for every $g, h \in G$. The multiplication table of an abelian group is symmetric with respect to its diagonal, as in the above example.

Two groups G_1 and G_2 are said to be *isomorphic*, denoted as $G_1 \simeq G_2$, if there exists a one-to-one correspondence $\phi : G_1 \to G_2$ such that

$$\phi(g)\phi(h) = \phi(gh), \quad g, h \in G_1.$$

Example 2.1. Let s and r be such that

$$s^2 = e, \quad r^3 = e, \quad (sr)^2 = e. \tag{2.26}$$

Then we can define groups

$$G_1 = \langle s \rangle = \{e, s\},$$
$$G_2 = \langle r \rangle = \{e, r, r^2\},$$
$$G_3 = \langle r, s \rangle = \{e, r, r^2, s, sr, sr^2\}.$$

Here, $\langle \cdot \rangle$ denotes a *group generated* by the listed element(s). For example, G_3 is generated by the two elements r and s under the *fundamental relations* given by (2.26). Note, for instance, that $sr^2 = s \cdot r \cdot r$, as the notation indicates. Table 2.1 shows the multiplication tables for these three groups G_1 to G_3. □

Example 2.2. The *cyclic group* of degree n, conventionally denoted as C_n, is a group of order $|\mathrm{C}_n| = n$ consisting of power products of a single element r with $r^n = e$. That is,

$$\mathrm{C}_n = \langle r \rangle = \{e, r, r^2, \dots, r^{n-1}\}$$

with $r^i r^j = r^{i+j}$. This group is an abelian group. The group G_2 in Example 2.1 above is C_3. The cyclic group C_n is also denoted by $\mathbb{Z}_n$, which is the case in Part II of this book. □

Example 2.3. The *dihedral group* of degree n, denoted as D_n, is a group of order $|D_n| = 2n$ defined as

$$D_n = \langle r, s \rangle = \{e, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\},$$

where r and s are assumed to satisfy the fundamental relations

$$r^n = s^2 = (sr)^2 = e$$

and $r^i r^j = r^{i+j}$. The groups G_1 and G_3 in Example 2.1 above are D_1 and D_3, respectively. □

Subgroup

A *subgroup* of a group G means a nonempty subset H of G that also forms a group with respect to the same product operation defined in G. A subgroup H of G is called a *proper subgroup* if it is distinct from G. A subgroup H of G is called a *normal subgroup* if $g^{-1}hg \in H$ for every $h \in H$ and $g \in G$. Two subgroups H and K of G are said to be *conjugate*, if

$$K = \{g^{-1}hg \mid h \in H\} \tag{2.27}$$

holds for some element $g \in G$.

Example 2.4. $H = \{e, r^3\}$ is a normal subgroup of $D_6 = \{e, r, \ldots, r^5, s, sr, \ldots, sr^5\}$ as

$$\begin{aligned}
g^{-1}e\,g &= e, \quad g \in D_6,\\
r^{-j}r^3r^j &= r^3, \quad j = 0, 1, \ldots, 5,\\
(sr^j)^{-1}r^3sr^j &= r^{-j}(srs)^3r^j = r^{-j}r^{-3}r^j = r^{-3} = r^3, \quad j = 0, 1, \ldots, 5.
\end{aligned}$$

The subgroups $\{e, s\}$ and $\{e, sr^2\}$ of D_6 are conjugate to each other as

$$r^{-1} \cdot \{e, s\} \cdot r = \{e, sr^2\}.$$

□

Direct Product and Semidirect Product

For any two groups G_1 and G_2, the set-theoretic direct product

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$$

forms a group with the product operation defined as

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2), \quad g_1, h_1 \in G_1;\ g_2, h_2 \ \in G_2.$$

This is called the *direct product* of G_1 and G_2. The identity element of $G_1 \times G_2$ is given by (e_1, e_2), where e_1 and e_2 mean the identity elements of G_1 and G_2, respectively. The groups G_1 and G_2 can be identified, respectively, with subgroups $H_1 = \{(g_1, e_2) \mid g_1 \in G_1\}$ and $H_2 = \{(e_1, g_2) \mid g_2 \in G_2\}$ of $G = G_1 \times G_2$. Then we have

- $h_1 h_2 = h_2 h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$, and
- each element $g \in G$ is represented uniquely as $g = h_1 h_2$ with $h_1 \in H_1$ and $h_2 \in H_2$.

If the above two conditions are satisfied by subgroups H_1 and H_2 of a group G, we also say that G is a *direct product* of H_1 and H_2. If this is the case, both H_1 and H_2 are normal subgroups of G. Note that we have two different definitions of direct product; in the latter definition here, G is a given object and is represented (or decomposed) by its subgroups H_1 and H_2, whereas in the former definition, the group G is constructed (or composed) from the given groups G_1 and G_2. These two definitions, however, are consistent with each other.

A group G is said to be a *semidirect product* of a subgroup H by another subgroup K, denoted $G = K \rtimes H$, if

- K is a *normal subgroup* of G, and
- each element $g \in G$ is represented uniquely as $g = kh$ with $k \in K$ and $h \in H$.

Each element $g = kh \in G$ can also be represented uniquely in an alternative form of $g = hk'$ with $h \in H$ and $k' \in K$, since $g = kh = h(h^{-1}kh)$ and $k' = h^{-1}kh$ belongs to K by the normality of K. When the alternative form $g = hk'$ is used, it is convenient to write $G = H \ltimes K$ instead. See Curtis and Reiner, 1962 [4] for more on the definition of the semidirect product.

Example 2.5. The dihedral group $\mathrm{D}_n = \{e, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}$ in Example 2.3 is a semidirect product of $H = \{e, s\}$ by $K = \{e, r, \dots, r^{n-1}\}$. The representation $g = sr^j$ is consistent with the notation $\mathrm{D}_n = H \ltimes K$, whereas $\mathrm{D}_n = K \rtimes H$ leads to an alternative representation $g = r^{-j}s$ for the same element $g = sr^j$. □

2.3.2 Basic Concepts of Group Representation

Let $F = \mathbb{R}$ or $\mathbb{C}$, where $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ is the field of complex numbers. The set of all nonsingular matrices over F of order N is denoted by $\mathrm{GL}(N, F)$. A *matrix representation* (or simply *representation*) of G means a mapping $T : G \to \mathrm{GL}(N, F)$ such that

$$T(gh) = T(g)T(h), \quad g, h \in G. \tag{2.28}$$

In other words, a matrix representation of order N is a family of $N \times N$ nonsingular matrices $\{T(g) \mid g \in G\}$ that satisfies (2.28). We refer to N as the *dimension*, or the *degree*, of the representation T, and to $V = F^N$ as the *representation space*.

A representation T is said to be a *unitary representation* if

$$T(g)^* T(g) = I, \quad g \in G, \tag{2.29}$$

where $(\cdot)^*$ denotes the conjugate transpose of a matrix and I is the identity matrix. In the case of $F = \mathbb{R}$, the term *orthogonal representation* is often used instead of unitary representation.

We say that two matrix representations T_1 and T_2 of G are *equivalent* if there exists a nonsingular matrix Q such that

$$T_1(g) = Q^{-1} T_2(g) Q, \quad g \in G. \tag{2.30}$$

The matrix Q above should be independent of $g \in G$. It is known that any matrix representation is equivalent to a unitary representation. Two representations are said to be *inequivalent* if they are not equivalent.

Example 2.6. A one-dimensional representation can be obtained by defining $T(g) = 1$ for all $g \in G$. This is called the *unit representation*. □

Example 2.7. For the group $\mathrm{D}_3 = \{e, r, r^2, s, sr, sr^2\}$ (Table 2.1c in Sect. 2.3.1), let us define

$$T(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad T(r) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \qquad T(r^2) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},$$
$$T(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T(sr) = \begin{bmatrix} \alpha & -\beta \\ -\beta & -\alpha \end{bmatrix}, \quad T(sr^2) = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$$

with $\alpha = \cos(2\pi/3)$ and $\beta = \sin(2\pi/3)$. Then T is a matrix representation of the group D_3, satisfying (2.28). Moreover, it is a unitary representation. □

Example 2.8. A representation can be obtained from permutations as follows. Let P be a finite set, and assume that G acts on P through permutations, which means that a permutation $\pi(g)$ of P is given for each $g \in G$ and that $\pi(gh) = \pi(g)\pi(h)$ holds for all $g, h \in G$. For each $g \in G$ let $T(g)$ be the permutation matrix that represents $\pi(g)$. That is, $T(g)$ is a matrix of order $|P|$ with rows and columns indexed by P, and

$$T_{ij}(g) = \begin{cases} 1 & \text{if } i = \pi(g)j, \\ 0 & \text{otherwise,} \end{cases}$$

where "$i = \pi(g)j$" means that j is moved to i by the permutation $\pi(g)$ associated with $g \in G$. Then the condition (2.28) is satisfied as a consequence of $\pi(gh) = \pi(g)\pi(h)$, and therefore T is a representation of G. Such representation is called a *permutation representation*, which is a unitary representation. □

Remark 2.2. In this book, a representation is defined as a family of matrices $\{T(g) \mid g \in G\}$ that satisfies (2.28). In mathematics, however, a representation is more often defined in terms of a mapping, as follows. Let V be a finite-dimensional vector space over $F = \mathbb{R}$ or $\mathbb{C}$, and denote the group of all nonsingular linear transformations of V onto itself by $\mathrm{GL}(V)$. A *representation* of G on V is defined as a mapping $\hat{T} : G \to \mathrm{GL}(V)$ such that

$$\hat{T}(gh) = \hat{T}(g)\hat{T}(h), \quad g, h \in G. \tag{2.31}$$

We call V the *representation space*. The condition (2.31) means that $\hat{T}$ is a *homomorphism* from G to $\mathrm{GL}(V)$. If a basis of V is fixed arbitrarily, a representation $\hat{T}$ on V is associated with a matrix representation T. □

Invariant Subspace

The *direct sum* of two matrix representations T_1 and T_2 is given as the family of their direct sums (block-diagonal matrices)

$$T_1(g) \oplus T_2(g) = \begin{bmatrix} T_1(g) & O \\ O & T_2(g) \end{bmatrix}$$

indexed by $g \in G$. The dimension of the direct sum representation is equal to the sum of the dimensions of T_1 and T_2.

A subspace W of V is said to be $(G\text{-})$*invariant* with respect to T if $T(g)\boldsymbol{w} \in W$ for all $\boldsymbol{w} \in W$ and $g \in G$. For an invariant subspace W, the restriction of T (more precisely, the restriction of the linear map associated with T) to W for each $g \in G$ defines a representation of G on W, called the *subrepresentation* of T on W.

It is known as the *Maschke theorem* that, for any invariant subspace W, there exists another invariant subspace W' such that

$$V = W \oplus W'. \tag{2.32}$$

Therefore, the representation matrix T can be brought into a block-diagonal form with a suitable change of basis; that is,

$$Q^{-1}T(g)Q = \begin{bmatrix} T_1(g) & O \\ O & T_2(g) \end{bmatrix}, \quad g \in G$$

for some nonsingular matrix Q, where T_1 and T_2, respectively, are subrepresentations of T on W and W'. It is emphasized that the matrix Q is independent of $g \in G$.

If T is unitary, the *orthogonal complement* $W^{\perp}$ of an invariant subspace W is also an invariant subspace. Consequently, the orthogonal decomposition

$$V = W \oplus W^{\perp} \tag{2.33}$$

serves as the decomposition (2.32) into two invariant subspaces.

2.3.3 Irreducible Representation

A representation T on V is said to be *irreducible* if there exists no nontrivial invariant subspace W, where W is nontrivial if W is neither $\{\mathbf{0}\}$ nor V.

There exist a finite number of mutually inequivalent irreducible representations of G (over a fixed field F). We denote by

$$\{T^{\mu} \mid \mu \in R(G)\} \tag{2.34}$$

a family of representatives of all irreducible unitary matrix representations of G, where $R(G)$ denotes the *index set for the irreducible representations* of G (over F). The dimension of representation μ is hereafter denoted by N^{μ}; for each $g \in G$, $T^{\mu}(g)$ is an $N^{\mu} \times N^{\mu}$ unitary matrix. In the case of $F = \mathbb{C}$, we have an identity

$$\sum_{\mu \in R(G)} (N^{\mu})^2 = |G|. \tag{2.35}$$

For $\mu \in R(G)$, we define a subgroup

$$G^{\mu} = \{g \in G \mid T^{\mu}(g) = I\}, \tag{2.36}$$

where I is the identity matrix. This is the subgroup of elements $g \in G$ that behave in μ as if it were the identity element of G.

Remark 2.3. Determining the explicit forms of irreducible representations is a nontrivial task, especially for the group for the hexagonal lattice (Sect. 6.3), although, for dihedral groups, we can use standard results (Sect. 3.3). □

Remark 2.4. A representation over $\mathbb{R}$ is said to be *absolutely irreducible* if it is irreducible as a representation over $\mathbb{C}$. For example, the two-dimensional representation of D_3 given in Example 2.7 is absolutely irreducible. In contrast, the two-dimensional representation of a cyclic group $C_3 = \{e, r, r^2\}$ (cf., Table 2.1b in Sect. 2.3.1) defined by

$$T(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T(r) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad T(r^2) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

with $\alpha = \cos(2\pi/3)$ and $\beta = \sin(2\pi/3)$ is irreducible over $\mathbb{R}$, but is not absolutely irreducible. Indeed, with a complex (unitary) matrix

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{bmatrix},$$

we can decompose T as follows:

$$Q^{-1}T(e)Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q^{-1}T(r)Q = \begin{bmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{bmatrix}, \quad Q^{-1}T(r^2)Q = \begin{bmatrix} \overline{\gamma} & 0 \\ 0 & \gamma \end{bmatrix}$$

with $\gamma = \alpha + \mathrm{i}\beta$ and $\overline{\gamma} = \alpha - \mathrm{i}\beta$.

Some groups have the property that every irreducible representation over $\mathbb{R}$ is absolutely irreducible. This is the case, for example, with the dihedral group D_n for any n, and also with the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ which we introduce in Sect. 5.5 to describe the symmetry of a hexagonal lattice. □

Decomposition into Irreducible Representations

It follows from repeated application of the Maschke theorem (2.32) that any representation T of G on V can be expressed as a direct sum of irreducible representations. To be more precise, the representation space V is decomposed as

$$V = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} V_i^\mu, \tag{2.37}$$

where V_i^μ is an invariant subspace, the subrepresentation of T on V_i^μ is irreducible and equivalent to T^μ in (2.34), and a^μ is a nonnegative integer, called the *multiplicity* of μ in T. We refer to (2.37) as the *irreducible decomposition* of the representation space V and to each V_i^μ as an *irreducible component*.

By aggregating the a^μ irreducible components V_i^μ corresponding to the same μ, we define a subspace

$$V^\mu = \bigoplus_{i=1}^{a^\mu} V_i^\mu \tag{2.38}$$

and rewrite (2.37) as

$$V = \bigoplus_{\mu \in R(G)} V^\mu. \tag{2.39}$$

Decomposition (2.39) is uniquely determined and termed *isotypic decomposition*, each V^μ being called an *isotypic component* (or *homogeneous component*). In contrast, decomposition (2.38) is not unique although the multiplicity a^μ is uniquely determined. Consequently, decomposition (2.37) into irreducible components is not unique either.

The direct sum decomposition (2.37) means that, with a suitable nonsingular matrix Q, the matrix representation T can be put into a *block-diagonal form* as follows:

$$\overline{T}(g) \equiv Q^{-1}T(g)Q = \bigoplus_{\mu\in R(G)} \bigoplus_{i=1}^{a^\mu} T_i^\mu(g), \quad g \in G \tag{2.40}$$

with T_i^μ being irreducible. It is possible and often advantageous to choose an identical matrix representation for equivalent representations (cf., (2.34)):

$$T_i^\mu = T^\mu, \quad i = 1, \ldots, a^\mu. \tag{2.41}$$

With the choice of (2.41) in (2.40) we obtain

$$\overline{T}(g) = Q^{-1}T(g)Q = \bigoplus_{\mu\in R(G)} \bigoplus_{i=1}^{a^\mu} T^\mu(g), \quad g \in G. \tag{2.42}$$

Here the size of the matrix $T^\mu(g)$ is N^μ, and we have

$$\sum_{\mu\in R(G)} a^\mu N^\mu = N. \tag{2.43}$$

The decomposition (2.40), as well as (2.42), is hereafter called the *irreducible decomposition* of the matrix representation T, and each T_i^μ in (2.40) is referred to as an *irreducible component* of T.

As an aggregation of the a^μ irreducible matrix representations in (2.42) corresponding to the same μ, we define

$$\overline{T}^\mu(g) = \bigoplus_{i=1}^{a^\mu} T^\mu(g), \quad g \in G, \tag{2.44}$$

and rewrite (2.42) as

$$\overline{T}(g) = Q^{-1}T(g)Q = \bigoplus_{\mu\in R(G)} \overline{T}^\mu(g), \quad g \in G. \tag{2.45}$$

We call this the *isotypic decomposition* of the representation T, and each $\overline{T}^\mu(g)$ the *isotypic component* of T. Note that the size of the matrix $\overline{T}^\mu(g)$ is $a^\mu N^\mu$.

Example 2.9. If $R(G) = \{\mu, \nu\}$, $a^\mu = 2$, and $a^\nu = 3$, the irreducible decomposition (2.42) takes the following form:

$$\overline{T}(g) = Q^{-1}T(g)Q = \left[\begin{array}{cc|ccc} T^\mu(g) & & & & \\ & T^\mu(g) & & & \\ \hline & & T^\nu(g) & & \\ & & & T^\nu(g) & \\ & & & & T^\nu(g) \end{array}\right].$$

The isotypic components are given by

$$\overline{T}^\mu(g) = \begin{bmatrix} T^\mu(g) & \\ & T^\mu(g) \end{bmatrix}, \quad \overline{T}^\nu(g) = \begin{bmatrix} T^\nu(g) & & \\ & T^\nu(g) & \\ & & T^\nu(g) \end{bmatrix}.$$

□

Example 2.10. The decomposition into irreducible representations is illustrated here for the dihedral group $\mathrm{D}_3 = \{e, r, r^2, s, sr, sr^2\}$ (cf., Table 2.1c) for $F = \mathbb{R}$. The group D_3 has three inequivalent irreducible representations, two one-dimensional irreducible representations, denoted as μ_1 and μ_2, and a two-dimensional irreducible representation, denoted as μ_3. They are defined, respectively, by

$$T^{\mu_1}(e) = T^{\mu_1}(r) = T^{\mu_1}(r^2) = 1, \quad T^{\mu_1}(s) = T^{\mu_1}(sr) = T^{\mu_1}(sr^2) = 1;$$
$$T^{\mu_2}(e) = T^{\mu_2}(r) = T^{\mu_2}(r^2) = 1, \quad T^{\mu_2}(s) = T^{\mu_2}(sr) = T^{\mu_2}(sr^2) = -1;$$

and

$$T^{\mu_3}(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T^{\mu_3}(r) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad T^{\mu_3}(r^2) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},$$
$$T^{\mu_3}(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T^{\mu_3}(sr) = \begin{bmatrix} \alpha & -\beta \\ -\beta & -\alpha \end{bmatrix}, \quad T^{\mu_3}(sr^2) = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$$

with $\alpha = \cos(2\pi/3)$ and $\beta = \sin(2\pi/3)$. We have $R(\mathrm{D}_3) = \{\mu_1, \mu_2, \mu_3\}$ in our notation (2.34).

Consider, for example, a permutation representation T of D_3 given by

$$T(e) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad T(r) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \quad T(r^2) = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix},$$

$$T(s) = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \quad T(sr) = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \quad T(sr^2) = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}.$$

With the use of an orthogonal transformation matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix},$$

the matrices $T(g)$ for $g \in \mathrm{D}_3$ can be transformed as

$$\begin{aligned}
\overline{T}(e) &= Q^{-1}T(e)Q &= [1] \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= T^{\mu_1}(e) \oplus T^{\mu_3}(e), \\
\overline{T}(r) &= Q^{-1}T(r)Q &= [1] \oplus \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} &= T^{\mu_1}(r) \oplus T^{\mu_3}(r), \\
\overline{T}(r^2) &= Q^{-1}T(r^2)Q &= [1] \oplus \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} &= T^{\mu_1}(r^2) \oplus T^{\mu_3}(r^2), \\
\overline{T}(s) &= Q^{-1}T(s)Q &= [1] \oplus \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= T^{\mu_1}(s) \oplus T^{\mu_3}(s), \\
\overline{T}(sr) &= Q^{-1}T(sr)Q &= [1] \oplus \begin{bmatrix} \alpha & -\beta \\ -\beta & -\alpha \end{bmatrix} &= T^{\mu_1}(sr) \oplus T^{\mu_3}(sr), \\
\overline{T}(sr^2) &= Q^{-1}T(sr^2)Q &= [1] \oplus \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} &= T^{\mu_1}(sr^2) \oplus T^{\mu_3}(sr^2).
\end{aligned}$$

Thus, the representation T above splits into two irreducible representations T^{μ_1} and T^{μ_3} with multiplicities $a^{\mu_1} = a^{\mu_3} = 1$, whereas T^{μ_2} is missing since $a^{\mu_2} = 0$. This gives the irreducible decomposition (2.42). □

Remark 2.5. Representations T appearing in applications are often unitary representations. This is indeed the case in our analysis of spatial agglomeration in a racetrack economy in Chap. 3 and of an economy on the hexagonal lattice in Chaps. 5–9, where we use permutation representations, which are unitary (cf., Example 2.8). For a unitary representation T over $\mathbb{R}$, the transformation matrix Q in (2.40) can be chosen to be unitary or orthogonal (cf., (2.33)). The further condition (2.41) can also be realized by an orthogonal transformation because any two equivalent irreducible unitary representations are connected as (2.30) with a unitary Q. Thus, we can choose an *orthogonal transformation* with an orthogonal matrix Q in the irreducible decomposition (2.42) and in the isotypic decomposition (2.45) to obtain the following forms of block-diagonalization:

$$\overline{T}(g) = Q^\top T(g) Q = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} T^\mu(g), \quad g \in G, \tag{2.46}$$

$$\overline{T}(g) = Q^\top T(g) Q = \bigoplus_{\mu \in R(G)} \overline{T}^\mu(g), \quad g \in G. \tag{2.47}$$

□

Transformation Matrix for Isotypic and Irreducible Decompositions

An additional explanation is given here to indicate how to construct the transformation matrix Q for isotypic and irreducible decompositions.

First, recall the isotypic decomposition (2.45):

$$\overline{T}(g) = Q^{-1}T(g)Q = \bigoplus_{\mu \in R(G)} \overline{T}^{\mu}(g), \quad g \in G,$$

where Q is a nonsingular matrix of size N. Compatibly with this decomposition, the matrix Q can be partitioned columnwise as

$$Q = (Q^{\mu} \mid \mu \in R(G)) = \left[\ldots, Q^{\mu}, \ldots\right], \tag{2.48}$$

where Q^{μ} is the submatrix, of size $N \times a^{\mu}N^{\mu}$, that corresponds to μ. For each $\mu \in R(G)$ we have

$$T(g)Q^{\mu} = Q^{\mu}\overline{T}^{\mu}(g), \quad g \in G. \tag{2.49}$$

The column vectors of Q^{μ} constitute a basis of the isotypic component V^{μ} in (2.39) corresponding to μ.

Compatibly with the decomposition (2.44):

$$\overline{T}^{\mu}(g) = \bigoplus_{i=1}^{a^{\mu}} T^{\mu}(g) \tag{2.50}$$

of an isotypic component into irreducible components, the submatrix Q^{μ} is partitioned as

$$Q^{\mu} = (Q_i^{\mu} \mid i = 1, \ldots, a^{\mu}), \quad \mu \in R(G). \tag{2.51}$$

Here Q_i^{μ} is an $N \times N^{\mu}$ matrix and the column vectors of Q_i^{μ} form a basis of the irreducible component V_i^{μ} in (2.38). By (2.42), hereafter we have

$$T(g)Q_i^{\mu} = Q_i^{\mu}T^{\mu}(g), \quad g \in G. \tag{2.52}$$

Substitution of (2.51) into (2.48) yields a finer partition of the transformation matrix Q as

$$Q = (Q^{\mu} \mid \mu \in R(G)) = ((Q_i^{\mu} \mid i = 1, \ldots, a^{\mu}) \mid \mu \in R(G)). \tag{2.53}$$

The condition (2.52), for all i and μ, is a necessary and sufficient condition for a nonsingular matrix Q to be eligible for irreducible decomposition (2.42) for a given T and the specified family $\{T^{\mu} \mid \mu \in R(G)\}$. This fact can be used to determine a concrete form of Q; see Sects. 3.4.1 and 7.5.

Example 2.11. In Example 2.9 we have $T = T^{\mu} \oplus T^{\mu} \oplus T^{\nu} \oplus T^{\nu} \oplus T^{\nu}$ with $a^{\mu} = 2$ and $a^{\nu} = 3$. Accordingly, we have $Q = (Q_1^{\mu}, Q_2^{\mu}; Q_1^{\nu}, Q_2^{\nu}, Q_3^{\nu})$, where Q_1^{μ} and Q_2^{μ} are $N \times N^{\mu}$ matrices, and Q_1^{ν}, Q_2^{ν}, and Q_3^{ν} are $N \times N^{\nu}$ matrices. □

Example 2.12. In Example 2.10 we have $T = T^{\mu_1} \oplus T^{\mu_3}$ with $a^{\mu_1} = a^{\mu_3} = 1$ and $a^{\mu_2} = 0$. The transformation matrix Q is partitioned as

$$Q = \left[\begin{array}{c|cc} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \end{array}\right] = \left[Q_1^{\mu_1}, Q_1^{\mu_3}\right],$$

where the submatrix corresponding to μ_2 is missing. □

Character

For a representation T of G over $\mathbb{C}$, the *character* of T is a function $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \mathrm{Tr}\, T(g), \quad g \in G, \tag{2.54}$$

where $\mathrm{Tr}\, T(g)$ means the trace of matrix $T(g)$. The importance of this concept lies in the fact that two representations are equivalent if and only if they have the same character. It is noted that the necessity of this condition is obvious since $T_1(g) = Q^{-1}T_2(g)Q$ in (2.30) implies

$$\mathrm{Tr}\, T_1(g) = \mathrm{Tr}\,(Q^{-1}T_2(g)Q) = \mathrm{Tr}\, T_2(g).$$

The character of an irreducible representation is called an *irreducible character*. For $\mu \in R(G)$ let χ^{μ} denote the character of T^{μ}, i.e.,

$$\chi^{\mu}(g) = \mathrm{Tr}\, T^{\mu}(g), \quad g \in G. \tag{2.55}$$

Irreducible characters of G are endowed with an *orthogonality* property. For irreducible characters χ^{μ} and χ^{ν}, we have

$$\sum_{g \in G} \chi^{\mu}(g)\overline{\chi^{\nu}(g)} = |G|\delta_{\mu\nu}, \tag{2.56}$$

where $\delta_{\mu\nu}$ is the Kronecker delta and $\overline{\cdot}$ means the complex conjugate.

This orthogonality yields a convenient formula for the multiplicity a^{μ} in the irreducible decomposition (2.42) of a representation T over $\mathbb{C}$. By taking the trace of both sides of (2.42), we obtain a decomposition of the character χ of T as

$$\chi(g) = \sum_{\mu \in R(G)} a^{\mu} \chi^{\mu}(g), \quad g \in G. \tag{2.57}$$

It then follows that

$$\sum_{g \in G} \chi(g)\overline{\chi^{\nu}(g)} = \sum_{\mu \in R(G)} a^{\mu} \sum_{g \in G} \chi^{\mu}(g)\overline{\chi^{\nu}(g)} = |G| \sum_{\mu \in R(G)} a^{\mu}\delta_{\mu\nu} = |G|a^{\nu},$$

where the orthogonality (2.56) is used. We thus obtain the formula

$$a^{\mu} = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi^{\mu}(g)} \tag{2.58}$$

for the multiplicity a^{μ}. This formula will be used in Chap. 7 in determining the multiplicities in the representation matrix of the hexagonal lattice.

2.3.4 Block-Diagonalization of Commuting Matrices

Matrices that are commutative with group actions can be transformed to block-diagonal forms through a suitable transformation. Block-diagonalization is used in our analysis of the racetrack economy in Sect. 3.4.1 and economy on the hexagonal lattice in Sect. 7.5. See also Remark 2.8 in Sect. 2.4.2. This useful mathematical fact is explained in this section for real matrices.

Let A be a real matrix such that

$$T(g)A = AT(g), \quad g \in G, \tag{2.59}$$

where T is an N-dimensional representation over $\mathbb{R}$ of a group G, and A is an $N \times N$ matrix. This condition represents the *commutativity* of A with the action of G, or the *invariance* to G. It is not assumed that T is a unitary representation.

To simplify the presentation, in this section we assume that[10]

$$\text{Every irreducible representation } \mu \text{ of } G \text{ is absolutely irreducible.} \tag{2.60}$$

Here, the definition of absolute irreducibility is given in Remark 2.4 in Sect. 2.3.2. This assumption is not restrictive for the purpose of this book, since it is satisfied both by the dihedral group D_n used for the racetrack economy in Chap. 3 and by the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ introduced in Sect. 5.5 to describe the symmetry of the hexagonal lattice.

[10] See Ikeda and Murota, 2010 [10] for the general treatment without the assumption (2.60).

The following fundamental fact is known as *Schur's lemma*.

Proposition 2.1 (Schur's lemma). *Let $F = \mathbb{R}$ or $\mathbb{C}$, and let B be a (possibly rectangular) matrix over F. Assume that T_1 and T_2 are irreducible matrix representations of group G over F, and*

$$T_1(g)B = BT_2(g), \quad g \in G.$$

Then the following hold.

(i) *$B = O$ or else B is square and nonsingular.*
(ii) *If T_1 and T_2 are inequivalent, then $B = O$.*
(iii) *If $T_1(g) = T_2(g)$ for all $g \in G$, and T_1 is absolutely irreducible, then $B = cI$ for some $c \in F$. Here I is the identity matrix.*

Construction of the block-diagonal form of A in (2.59) consists of two stages:

- Transformation to a block-diagonal form corresponding to isotypic decomposition (2.45).
- Transformation to a finer block-diagonal form with a nested block structure related to irreducible decomposition (2.42).

According to (2.53) in Sect. 2.3.3, let

$$Q = (Q^\mu \mid \mu \in R(G)) = ((Q_i^\mu \mid i = 1, \ldots, a^\mu) \mid \ \mu \in R(G)) \tag{2.61}$$

be the transformation matrix for the isotypic decomposition (2.45) and the irreducible decomposition (2.42). Here $R(G)$ is the index set for the irreducible representations of G over $\mathbb{R}$, and a^μ is the multiplicity of $\mu \in R(G)$ in (2.59).

For the block-diagonalization of A, we transform A to

$$\overline{A} = Q^{-1}AQ. \tag{2.62}$$

Since the columns of Q are partitioned into blocks Q^μ, the rows and the columns of $\overline{A}$ are partitioned accordingly as

$$\overline{A} = (\overline{A}^{\mu\nu} \mid \mu, \nu \in R(G)). \tag{2.63}$$

The commutativity condition $T(g)A = AT(g)$ in (2.59) can be rewritten as

$$\overline{T}(g)\, \overline{A} = \overline{A}\, \overline{T}(g), \quad g \in G, \tag{2.64}$$

in which

$$\overline{T}(g) = \bigoplus_{\mu \in R(G)} \overline{T}^\mu(g)$$

as (2.45) and $\overline{A}$ is partitioned as (2.63). Therefore, (2.64) is equivalent to

$$\overline{T}^{\mu}(g)\,\overline{A}^{\mu\nu} = \overline{A}^{\mu\nu}\,\overline{T}^{\nu}(g), \quad g \in G;\ \mu, \nu \in R(G). \tag{2.65}$$

This condition for $\mu \neq \nu$ implies, by Schur's lemma (Proposition 2.1(ii)), that

$$\overline{A}^{\mu\nu} = O \quad \text{if} \quad \mu \neq \nu. \tag{2.66}$$

Hence, $\overline{A}$ is a block-diagonal matrix:

$$\overline{A} = \bigoplus_{\mu \in R(G)} \overline{A}^{\mu}, \tag{2.67}$$

where $\overline{A}^{\mu} = \overline{A}^{\mu\mu}$ is of size $a^{\mu} N^{\mu}$. This is the block-diagonal form associated with the isotypic decomposition (2.45).

The diagonal block $\overline{A}^{\mu}$ in (2.67) has a further block-diagonal structure when the associated irreducible representation μ is absolutely irreducible, which is assumed in (2.60).

The finer partition of Q in (2.61) induces a partition of $\overline{A}^{\mu}$ into blocks as

$$\overline{A}^{\mu} = (\overline{A}^{\mu}_{ij} \mid i, j = 1, \ldots, a^{\mu}). \tag{2.68}$$

With the use of this and the decomposition

$$\overline{T}^{\mu}(g) = \bigoplus_{i=1}^{a^{\mu}} T^{\mu}(g)$$

in (2.44) into irreducible components, the condition (2.65) for $\mu = \nu$ can be rewritten as

$$T^{\mu}(g)\,\overline{A}^{\mu}_{ij} = \overline{A}^{\mu}_{ij}\,T^{\mu}(g), \quad g \in G,$$

where $i, j = 1, \ldots, a^{\mu}$. This implies, by Schur's lemma (Proposition 2.1(iii)), that $\overline{A}^{\mu}_{ij}$ is a scalar multiple of the identity matrix. That is,

$$\overline{A}^{\mu}_{ij} = \alpha^{\mu}_{ij} I_{N^{\mu}} \tag{2.69}$$

for some $\alpha^{\mu}_{ij} \in \mathbb{R}$, where N^{μ} denotes the dimension of T^{μ} and $I_{N^{\mu}}$ the identity matrix of size N^{μ}. Let us define

$$\tilde{A}^{\mu} = (\alpha^{\mu}_{ij} \mid i, j = 1, \ldots, a^{\mu}), \tag{2.70}$$

which is a matrix of size $a^{\mu} \times a^{\mu}$.

The expression (2.69) shows that the matrix $\overline{A}^{\mu}$ is equal, up to a simultaneous permutation of rows and columns, to a block-diagonal matrix consisting of N^{μ} copies of the matrix $\tilde{A}^{\mu}$, i.e.,

$$(\Pi^{\mu})^{-1}\overline{A}^{\mu}\Pi^{\mu} = \bigoplus_{k=1}^{N^{\mu}} \tilde{A}^{\mu} \tag{2.71}$$

with a suitable permutation matrix Π^{μ} for each $\mu \in R(G)$; see Example 2.13 below. The aggregation of these permutation matrices is denoted by Π, i.e.,

$$\Pi = \bigoplus_{\mu \in R(G)} \Pi^{\mu}. \tag{2.72}$$

The combination of (2.62), (2.67), (2.71), and (2.72) yields a finer block-diagonal form

$$(Q\Pi)^{-1}AQ\Pi = \Pi^{-1}\overline{A}\Pi = \bigoplus_{\mu \in R(G)} (\Pi^{\mu})^{-1}\overline{A}^{\mu}\Pi^{\mu} = \bigoplus_{\mu \in R(G)} \bigoplus_{k=1}^{N^{\mu}} \tilde{A}^{\mu} \tag{2.73}$$

with $\tilde{A}^{\mu}$ given in (2.70). This is the block-diagonal form associated with the irreducible decomposition (2.42).

Example 2.13. The decomposition (2.73) is illustrated. Suppose that $N = 7$, $R(G) = \{\mu, \nu\}$, $N^{\mu} = 2$, $N^{\nu} = 1$, $a^{\mu} = 2$, and $a^{\nu} = 3$ (Examples 2.9 and 2.11). By (2.68) with (2.69), the block-diagonal form in (2.67) is expressed as

$$\overline{A} = \left[\begin{array}{cc|ccc} \alpha^{\mu}_{11}I_2 & \alpha^{\mu}_{12}I_2 & & & \\ \alpha^{\mu}_{21}I_2 & \alpha^{\mu}_{22}I_2 & & & \\ \hline & & \alpha^{\nu}_{11}I_1 & \alpha^{\nu}_{12}I_1 & \alpha^{\nu}_{13}I_1 \\ & & \alpha^{\nu}_{21}I_1 & \alpha^{\nu}_{22}I_1 & \alpha^{\nu}_{23}I_1 \\ & & \alpha^{\nu}_{31}I_1 & \alpha^{\nu}_{32}I_1 & \alpha^{\nu}_{33}I_1 \end{array}\right] = \left[\begin{array}{cccc|ccc} \alpha^{\mu}_{11} & & \alpha^{\mu}_{12} & & & & \\ & \alpha^{\mu}_{11} & & \alpha^{\mu}_{12} & & & \\ \alpha^{\mu}_{21} & & \alpha^{\mu}_{22} & & & & \\ & \alpha^{\mu}_{21} & & \alpha^{\mu}_{22} & & & \\ \hline & & & & \alpha^{\nu}_{11} & \alpha^{\nu}_{12} & \alpha^{\nu}_{13} \\ & & & & \alpha^{\nu}_{21} & \alpha^{\nu}_{22} & \alpha^{\nu}_{23} \\ & & & & \alpha^{\nu}_{31} & \alpha^{\nu}_{32} & \alpha^{\nu}_{33} \end{array}\right].$$

With a suitable rearrangement of rows and columns, represented by a permutation matrix Π, the above matrix can be put into a block-diagonal form of (2.73) as

$$\Pi^{-1}\overline{A}\Pi = \left[\begin{array}{cc|cc|ccc} \alpha^{\mu}_{11} & \alpha^{\mu}_{12} & & & & & \\ \alpha^{\mu}_{21} & \alpha^{\mu}_{22} & & & & & \\ \hline & & \alpha^{\mu}_{11} & \alpha^{\mu}_{12} & & & \\ & & \alpha^{\mu}_{21} & \alpha^{\mu}_{22} & & & \\ \hline & & & & \alpha^{\nu}_{11} & \alpha^{\nu}_{12} & \alpha^{\nu}_{13} \\ & & & & \alpha^{\nu}_{21} & \alpha^{\nu}_{22} & \alpha^{\nu}_{23} \\ & & & & \alpha^{\nu}_{31} & \alpha^{\nu}_{32} & \alpha^{\nu}_{33} \end{array}\right] = (\tilde{A}^{\mu} \oplus \tilde{A}^{\mu}) \oplus \tilde{A}^{\nu}$$

with

$$\tilde{A}^{\mu} = \begin{bmatrix} \alpha_{11}^{\mu} & \alpha_{12}^{\mu} \\ \alpha_{21}^{\mu} & \alpha_{22}^{\mu} \end{bmatrix}, \quad \tilde{A}^{\nu} = \begin{bmatrix} \alpha_{11}^{\nu} & \alpha_{12}^{\nu} & \alpha_{13}^{\nu} \\ \alpha_{21}^{\nu} & \alpha_{22}^{\nu} & \alpha_{23}^{\nu} \\ \alpha_{31}^{\nu} & \alpha_{32}^{\nu} & \alpha_{33}^{\nu} \end{bmatrix}.$$

□

2.4 Group-Theoretic Bifurcation Analysis

The general mathematical framework of group-theoretic bifurcation analysis is presented. The symmetry of the equilibrium equation and that of the solutions are expressed in terms of a representation of the underlying group of symmetry.

2.4.1 Equilibrium Equation and Critical Point

We now consider a system of nonlinear *equilibrium (governing) equation*

$$\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{0}, \tag{2.74}$$

where $\boldsymbol{u} \in \mathbb{R}^N$ indicates an N-dimensional independent variable vector and $f \in \mathbb{R}$ denotes an auxiliary parameter. We assume that $\boldsymbol{F} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a sufficiently smooth nonlinear function in $\boldsymbol{u}$ and f. It is emphasized that there is a distinguished single parameter f, which plays the role of *bifurcation parameter*. In economic geography models that appear in this book, $\boldsymbol{u}$ is the vector of mobile population $\boldsymbol{\lambda}$ and f is the transport cost parameter τ.

Solutions $(\boldsymbol{u}, f)$ of the system of equations (2.74) make up solution curves called *equilibrium paths* or *loci of equilibria*. The solution points are divided into two types, *ordinary* or *critical* (*singular*) points, according to whether the *Jacobian matrix*

$$J = J(\boldsymbol{u}, f) = (J_{ij}) = \left(\frac{\partial F_i}{\partial u_j} \right), \tag{2.75}$$

which is an $N \times N$ matrix, is nonsingular or singular. That is,

$$\det(J) = \begin{cases} \text{nonzero} & \text{at an ordinary point,} \\ 0 & \text{at a critical (singular) point,} \end{cases} \tag{2.76}$$

where $\det(\cdot)$ denotes the determinant of the matrix therein.

In a sufficiently small neighborhood of an ordinary point, the implicit function theorem applies. For each f, there is a unique $\boldsymbol{u} = \boldsymbol{u}(f)$ such that $(\boldsymbol{u}(f), f)$ is a solution to (2.74).

In the vicinity of a critical point, say $(\boldsymbol{u}_{\rm c}, f_{\rm c})$, an interesting phenomenon called *bifurcation* can occur, which means the emergence of multiple solution paths. The Jacobian matrix

$$J_{\rm c} = J(\boldsymbol{u}_{\rm c}, f_{\rm c})$$

at $(\boldsymbol{u}_{\rm c}, f_{\rm c})$ is singular by the definition of a critical point; that is,

$$\det[J(\boldsymbol{u}_{\rm c}, f_{\rm c})] = 0. \tag{2.77}$$

The *multiplicity* M of a critical point $(\boldsymbol{u}_{\rm c}, f_{\rm c})$ is defined as the *rank deficiency* of the Jacobian matrix; that is,

$$M = \dim(\ker(J_{\rm c})) = N - \operatorname{rank}(J_{\rm c}), \tag{2.78}$$

where $\ker(\cdot)$ denotes the *kernel space* of the matrix in parentheses, $\dim(\cdot)$ means the dimension of a linear space, and $\operatorname{rank}(\cdot)$ denotes the *rank* of the matrix. The critical point $(\boldsymbol{u}_{\rm c}, f_{\rm c})$ is a *simple* critical point if $M = 1$ and a *multiple* critical point if $M \geq 2$.

Let $\{\boldsymbol{\xi}_i \mid i = 1, \ldots, M\}$ and $\{\boldsymbol{\eta}_i \mid i = 1, \ldots, M\}$ be two families of independent vectors of $\mathbb{R}^N$ such that

$$\boldsymbol{\xi}_i^\top J_{\rm c} = \boldsymbol{0}^\top, \quad J_{\rm c}\boldsymbol{\eta}_i = \boldsymbol{0}, \quad i = 1, \ldots, M. \tag{2.79}$$

Such vectors $\boldsymbol{\xi}_i$ $(i = 1, \ldots, M)$ are called the *left critical vectors*, and the vectors $\boldsymbol{\eta}_i$ $(i = 1, \ldots, M)$ are called the *right critical vectors*. Here $\{\boldsymbol{\xi}_i \mid i = 1, \ldots, M\}$ and $\{\boldsymbol{\eta}_i \mid i = 1, \ldots, M\}$ span the kernel of $J_{\rm c}^\top$ and $J_{\rm c}$, respectively. Note that orthogonality is not imposed in general although in some cases it is a natural and convenient requirement. Critical vectors play a crucial role in deriving a reduced system of equations, called the bifurcation equation, in Sect. 2.4.3.

Stability of a solution $\boldsymbol{u}$ to $\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{0}$ can be defined with reference to the associated dynamical system

$$\frac{d\hat{\boldsymbol{u}}}{dt} = \boldsymbol{F}(\hat{\boldsymbol{u}}, f). \tag{2.80}$$

A solution $\boldsymbol{u}$ to $\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{0}$ is said to be *asymptotically stable* if every solution $\hat{\boldsymbol{u}}(t)$ to (2.80) that is initially close to $\boldsymbol{u}$ stays in a neighborhood of $\boldsymbol{u}$ for all t and decays to $\boldsymbol{u}$ as $t \to +\infty$. The solution $\boldsymbol{u}$ is said to be *linearly stable* if every eigenvalue of $J(\boldsymbol{u}, f)$ has a negative real part, and *linearly unstable* if at least one eigenvalue has a positive real part. A linearly stable solution is asymptotically stable. In this book, "stability" means linear stability and a solution $\boldsymbol{u}$ that is not (linearly) stable is called *unstable*.

2.4.2 Group Equivariance of Equilibrium Equation

We are interested in the case where the system of equilibrium equations $\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{0}$ in (2.74) is endowed with an additional mathematical structure of group symmetry. Recall that f denotes a bifurcation parameter and $\boldsymbol{u} \in \mathbb{R}^N$ indicates an independent variable vector. The function $\boldsymbol{F} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is assumed to be sufficiently smooth.

Following the standard setting in group-theoretic bifurcation theory, we describe the symmetry of the system by the *equivariance*:

$$T(g)\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{F}(T(g)\boldsymbol{u}, f), \quad g \in G \tag{2.81}$$

of $\boldsymbol{F}(\boldsymbol{u}, f)$ to a group G in terms of a unitary matrix representation T of G on the N-dimensional space of the independent variable vector $\boldsymbol{u}$. For a critical point $(\boldsymbol{u}_{\mathrm{c}}, f_{\mathrm{c}})$ of the system, we always assume that $\boldsymbol{u}_{\mathrm{c}}$ is G*-symmetric* in the sense that

$$T(g)\boldsymbol{u}_{\mathrm{c}} = \boldsymbol{u}_{\mathrm{c}}, \quad g \in G. \tag{2.82}$$

Remark 2.6. It is emphasized that the equivariance (2.81) does not impose symmetry on the solution $\boldsymbol{u}$, but it denotes the symmetry of the system of equations $\boldsymbol{F}$ as a whole under the transformations with respect to G, which often represent geometric transformations. □

Remark 2.7. A seemingly more general formulation of equivariance (2.81) would be

$$T_1(g)\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{F}(T_2(g)\boldsymbol{u}, f), \quad g \in G \tag{2.83}$$

with two matrix representations T_1 and T_2. We may assume that T_1 and T_2 are equivalent in the case we are interested in. To see why, note that (2.83) implies

$$T_1(g)J(\boldsymbol{u}, f) = J(\boldsymbol{u}, f)T_2(g), \quad g \in G,$$

if $T_2(g)\boldsymbol{u} = \boldsymbol{u}$ for all $g \in G$, where $J(\boldsymbol{u}, f)$ is the Jacobian matrix (2.75) of $\boldsymbol{F}$. It is natural to assume that we do have a G-symmetric solution $\boldsymbol{u}$ at which $J(\boldsymbol{u}, f)$ is nonsingular. This implies that T_1 and T_2 are equivalent. Therefore, we can assume that $T_1(g) = T_2(g)$ for all $g \in G$ by a suitable basis change. We can further assume that T_1 $(= T_2)$ is unitary on the basis of the fact that any representation is equivalent to a unitary representation. □

Equivariance of Linear Part

The equivariance (2.81) is inherited by the Jacobian matrix as follows. Recall from (2.75) that the Jacobian matrix is an $N \times N$ matrix defined as

$$J = J(\boldsymbol{u}, f) = \left(\frac{\partial F_i}{\partial u_j} \,\middle|\, i, j = 1, \ldots, N \right). \tag{2.84}$$

Differentiation of (2.81) with respect to $\boldsymbol{u}$ yields

$$T(g)J(\boldsymbol{u}, f) = J(T(g)\boldsymbol{u}, f)T(g), \quad g \in G, \tag{2.85}$$

which is the *equivariance* of $J(\boldsymbol{u}, f)$ for general $(\boldsymbol{u}, f)$. If $\boldsymbol{u}$ is G-symmetric, that is, if

$$T(g)\boldsymbol{u} = \boldsymbol{u}, \quad g \in G, \tag{2.86}$$

it holds that

$$T(g)J(\boldsymbol{u}, f) = J(\boldsymbol{u}, f)T(g), \quad g \in G, \tag{2.87}$$

which shows the *commutativity* of J with $T(g)$ for all $g \in G$, or the invariance of J to G. In particular, at a critical point $(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})$, which is assumed to satisfy (2.82), we have

$$T(g)J(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c}) = J(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})T(g), \quad g \in G. \tag{2.88}$$

Remark 2.8. Commutativity (2.87) of the Jacobian matrix $J(\boldsymbol{u}, f)$ with the representation matrices implies, by the standard result explained in Sect. 2.3.4, that $J(\boldsymbol{u}, f)$ can be transformed to block-diagonal forms through a suitable transformation independent of $(\boldsymbol{u}, f)$. This fact is exploited in Sect. 3.4.1 for the racetrack economy and in Sect. 7.5 for economy on the hexagonal lattice. □

Group-Theoretic Critical Point

Let $(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})$ be a critical point that has G-symmetry in (2.82). Recall from Sect. 2.4.1 that the Jacobian matrix $J_\mathrm{c} = J(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})$ at this point is singular by definition, and the rank deficiency $M = \dim[\ker(J_\mathrm{c})] = N - \mathrm{rank}(J_\mathrm{c})$ is called the multiplicity of the critical point. The critical point $(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})$ is *simple* if $M = 1$ and *multiple* if $M \geq 2$.

The kernel space of J_c is a G-invariant subspace, as follows.

Proposition 2.2. *The kernel space* $\ker(J_\mathrm{c})$ *is G-invariant.*

Proof. Take any $\boldsymbol{\eta} \in \ker(J_\mathrm{c})$ and $g \in G$. It follows from (2.88) that $J_\mathrm{c}(T(g)\boldsymbol{\eta}) = T(g)(J_\mathrm{c}\boldsymbol{\eta}) = \boldsymbol{0}$, which means $T(g)\boldsymbol{\eta} \in \ker(J_\mathrm{c})$. Hence, $\ker(J_\mathrm{c})$ is G-invariant. □

By the above proposition, we can consider the subrepresentation of T on $\ker(J_\mathrm{c})$, which we denote by $\tilde{T}$. If the subrepresentation $\tilde{T}$ is an irreducible representation, the critical point $(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})$ is called *group-theoretic*; otherwise, it is called *parametric*.

This categorization primarily applies to multiple critical points although any simple critical point is (trivially) group-theoretic since a one-dimensional representation is necessarily irreducible.

In this book, our interest is focused on group-theoretic multiple critical points, which generically appear in a system with group symmetry. In customary terminology, "genericity" is defined in relation to a parameterized family of systems in question.[11] Here we implicitly suppose a family of economic systems with specified symmetry.

The concept of a group-theoretic critical point is fundamental in this book, so that we highlight its significance in the form of a proposition for later references.

Proposition 2.3. *A critical point of a generic G-equivariant system is a group-theoretic critical point, with which an irreducible representation of G is associated.*

2.4.3 Liapunov–Schmidt Reduction

We explain here a standard procedure, the *Liapunov–Schmidt reduction with symmetry*,[12] which reduces the full system of equations to a few bifurcation equations. The pivotal mathematical fact is the inheritance of symmetry in this reduction process.

Inheritance of Symmetry

To be specific, we consider a critical point $(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})$ of the system (2.74) of equations of multiplicity M that has symmetry (2.82). In a neighborhood of $(\boldsymbol{u}_\mathrm{c}, f_\mathrm{c})$, the full system of equations

$$\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{0} \tag{2.89}$$

in $\boldsymbol{u} \in \mathbb{R}^N$ is reduced to a system of M equations

$$\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = \boldsymbol{0} \tag{2.90}$$

in $\boldsymbol{w} \in \mathbb{R}^M$, where $\tilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function to be specified later in (2.103), and

$$\tilde{f} = f - f_\mathrm{c} \tag{2.91}$$

[11] Loosely speaking, the term "generically" might be replaced by "unless the parameters take special values."

[12] For details, see Sattinger, 1979 [17]; Chow and Hale, 1982 [2]; and Golubitsky, Stewart, and Schaeffer, 1988 [7].

denotes the increment of f. The reduced system (2.90) of equations is called the *bifurcation equation*. It must be emphasized that this reduction is valid only locally in a neighborhood of $(\boldsymbol{u}_{\rm c}, f_{\rm c})$.

The equivariance of the full system, formulated in (2.81) as

$$T(g)\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{F}(T(g)\boldsymbol{u}, f), \quad g \in G, \tag{2.92}$$

is inherited by the bifurcation equation (2.90) in the following form:

$$\tilde{T}(g)\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = \tilde{\boldsymbol{F}}(\tilde{T}(g)\boldsymbol{w}, \tilde{f}), \quad g \in G, \tag{2.93}$$

where $\tilde{T}$ is the subrepresentation of T on the kernel space of $J_{\rm c} = J(\boldsymbol{u}_{\rm c}, f_{\rm c})$. It is this *inheritance of symmetry* that plays a key role in determining the symmetry of bifurcating solutions.

The rest of this subsection is devoted to deriving the bifurcation equation (2.90) and the inheritance of symmetry (2.93); see (2.103) and Proposition 2.6. The reader may skip the derivations and jump to Sect. 2.4.4.

Reduction Procedure

The Liapunov–Schmidt reduction procedure proceeds as follows. Recall that $(\boldsymbol{u}_{\rm c}, f_{\rm c})$ is a critical point[13] of multiplicity M, and

$$\dim[\ker(J_{\rm c})] = M, \quad \dim[\mathrm{range}(J_{\rm c})] = N - M \tag{2.94}$$

for $J_{\rm c} = J(\boldsymbol{u}_{\rm c}, f_{\rm c})$, where range$(\cdot)$ denotes the range space of the matrix in the parentheses. It is assumed in (2.82) that $\boldsymbol{u}_{\rm c}$ is G-invariant; that is,

$$T(g)\boldsymbol{u}_{\rm c} = \boldsymbol{u}_{\rm c}, \quad g \in G. \tag{2.95}$$

Consider a direct sum decomposition

$$\mathbb{R}^N = \ker(J_{\rm c}) \oplus U \tag{2.96}$$

of the space to which $\boldsymbol{u}$ belongs, and another direct sum decomposition

$$\mathbb{R}^N = V \oplus \mathrm{range}(J_{\rm c}) \tag{2.97}$$

of the space in which $\boldsymbol{F}$ takes values. Note that $\dim(U) = N - M$ and $\dim(V) = M$. The complementary subspaces U and V here are not determined uniquely,

[13] The reduction procedure is valid for a critical point $(\boldsymbol{u}_{\rm c}, f_{\rm c})$ that is not necessarily group-theoretic although in later chapters we always treat group-theoretic critical points.

but any choice of them is valid for the derivation of the bifurcation equation. See Remark 2.9 below for a possible choice.

According to (2.96), we decompose $\boldsymbol{u} - \boldsymbol{u}_{\rm c}$ into two components as

$$\boldsymbol{u} = \boldsymbol{u}_{\rm c} + \boldsymbol{w} + \overline{\boldsymbol{w}}, \tag{2.98}$$

where $\boldsymbol{w} \in \ker(J_{\rm c})$ and $\overline{\boldsymbol{w}} \in U$. With reference to (2.97), let P be the *projection* onto V along the subspace range$(J_{\rm c})$, where $P^2 = P$. Then the full system (2.89) of equations can be decomposed into two parts[14]

$$P \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \overline{\boldsymbol{w}}, f_{\rm c} + \tilde{f}) = \boldsymbol{0}, \tag{2.99}$$

$$(I - P) \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \overline{\boldsymbol{w}}, f_{\rm c} + \tilde{f}) = \boldsymbol{0}. \tag{2.100}$$

The Jacobian matrix of (2.100) with respect to $\overline{\boldsymbol{w}}$, evaluated at $(\boldsymbol{w}, \overline{\boldsymbol{w}}, \tilde{f}) = (\boldsymbol{0}, \boldsymbol{0}, 0)$, is invertible as a mapping from U to range$(J_{\rm c})$. Consequently, by the implicit function theorem, (2.100) can be solved for $\overline{\boldsymbol{w}}$ as

$$\overline{\boldsymbol{w}} = \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}) \tag{2.101}$$

uniquely in some neighborhood of $(\boldsymbol{w}, \overline{\boldsymbol{w}}, \tilde{f}) = (\boldsymbol{0}, \boldsymbol{0}, 0)$. Substitution of this into (2.99) yields

$$P \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\rm c} + \tilde{f}) = \boldsymbol{0}. \tag{2.102}$$

This yields the reduced system (2.90) with

$$\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = P \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\rm c} + \tilde{f}) \tag{2.103}$$

(Remark 2.10 below).

The solutions $(\boldsymbol{w}, \tilde{f})$ to the bifurcation equation (2.90) are in one-to-one correspondence through (2.101) with the solutions $(\boldsymbol{u}, f)$ of the original system (2.89) in a neighborhood of $(\boldsymbol{u}_{\rm c}, f_{\rm c})$. For a solution $(\boldsymbol{w}, \tilde{f})$ to (2.90), the corresponding $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{w}, \tilde{f}) = \boldsymbol{u}_{\rm c} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}). \tag{2.104}$$

Conversely, for a solution $(\boldsymbol{u}, \tilde{f})$ to (2.89), the corresponding $\boldsymbol{w}$ is given by (2.98). Hence, the qualitative picture of the solution set of the original system (2.89) is isomorphic to that of the bifurcation equation (2.90).

[14]Since P is the projection onto an M-dimensional subspace, the equation (2.99) effectively represents M constraints and (2.100) represents $(N - M)$ constraints.

If we take a basis $\{\boldsymbol{\eta}_i \mid i = 1, \ldots, M\}$ of $\ker(J_{\rm c})$ and a basis $\{\boldsymbol{\eta}_j \mid j = M+1, \ldots, N\}$ of its complementary subspace U, then we obtain expansions

$$\boldsymbol{w} = \sum_{i=1}^{M} w_i \boldsymbol{\eta}_i, \quad \overline{\boldsymbol{w}} = \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}) = \sum_{j=M+1}^{N} \varphi_j(\boldsymbol{w}, \tilde{f})\boldsymbol{\eta}_j. \tag{2.105}$$

Accordingly, we can rewrite (2.104) as

$$\boldsymbol{u} = \boldsymbol{u}_{\rm c} + \sum_{i=1}^{M} w_i \boldsymbol{\eta}_i + \sum_{j=M+1}^{N} \varphi_j(\boldsymbol{w}, \tilde{f})\boldsymbol{\eta}_j. \tag{2.106}$$

Then, as a succinct representation of (2.99) and (2.100), we obtain

$$\boldsymbol{F}\left(\boldsymbol{u}_{\rm c} + \sum_{i=1}^{M} w_i \boldsymbol{\eta}_i + \sum_{j=M+1}^{N} \varphi_j(\boldsymbol{w}, \tilde{f})\boldsymbol{\eta}_j, f_{\rm c} + \tilde{f}\right) = \boldsymbol{0}. \tag{2.107}$$

Remark 2.9. A possible (and natural) choice of U and V in (2.96) and (2.97) is

$$U = (\ker(J_{\rm c}))^{\perp}, \quad V = (\mathrm{range}(J_{\rm c}))^{\perp}. \tag{2.108}$$

This choice is justified by the orthogonal decomposition (2.33) for a unitary representation. For an orthonormal basis $\{\boldsymbol{\xi}_i \mid i = 1, \ldots, N\}$ of $\mathbb{R}^N$ which satisfies $\boldsymbol{\xi}_i^{\top} J_{\rm c} = \boldsymbol{0}^{\top}$ for $i = 1, \ldots, M$, the projection P onto V is given by

$$P = \sum_{i=1}^{M} \boldsymbol{\xi}_i \boldsymbol{\xi}_i^{\top}, \tag{2.109}$$

which is an orthogonal projection. Then (2.99) is equivalent to $\boldsymbol{\xi}_i^{\top} \boldsymbol{F} = 0$ for $i = 1, \ldots, M$, and (2.100) is equivalent to $\boldsymbol{\xi}_j^{\top} \boldsymbol{F} = 0$ for $j = M+1, \ldots, N$. □

Remark 2.10. The right-hand side of (2.103) effectively stands for M constraints on $(\boldsymbol{w}, \tilde{f})$. A concrete vector representation of these constraints with respect to a certain basis of V is represented by an M-dimensional vector-valued function, which is denoted here as $\tilde{\boldsymbol{F}}$. It should be understood that the equality in (2.103) designates this correspondence, although the left-hand side is an M-dimensional vector and the right-hand side is an N-dimensional vector. □

Remark 2.11. The point $(\boldsymbol{w}, \tilde{f}) = (\boldsymbol{0}, 0)$ is a critical point of multiplicity M for the bifurcation equation $\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f})$. Accordingly, the Jacobian matrix $\tilde{J}(\boldsymbol{w}, \tilde{f})$ of the bifurcation equation, which is an $M \times M$ matrix defined as

$$\tilde{J}(\boldsymbol{w}, \tilde{f}) = \left(\frac{\partial \tilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \ldots, M \right), \tag{2.110}$$

vanishes at $(\boldsymbol{w}, \tilde{f}) = (\boldsymbol{0}, 0)$, i.e.,

$$\tilde{J}(\boldsymbol{0}, 0) = O. \tag{2.111}$$

The Jacobian matrix $\tilde{J}(\boldsymbol{w}, \tilde{f})$ of the bifurcation equation can be obtained from the full Jacobian matrix $J(\boldsymbol{u}, f)$ with $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{w}, \tilde{f})$ in (2.104) through the so-called *Schur complement*; see Lemma 7.5 of Ikeda and Murota, 2010 [10] for a precise statement. □

Group Equivariance

We proceed to consider the group equivariance (2.93) of the bifurcation equation. The following facts are fundamental here.

- Both $\ker(J_{\mathrm{c}})$ and $\mathrm{range}(J_{\mathrm{c}})$ are G-invariant subspaces, and therefore
- the complementary subspaces U and V in (2.96) and (2.97) can be chosen to satisfy G-invariance,[15] which is assumed throughout.

The G-invariance of $\ker(J_{\mathrm{c}})$ and $\mathrm{range}(J_{\mathrm{c}})$ is a consequence of the equivariance $T(g)J_{\mathrm{c}} = J_{\mathrm{c}}T(g)$ $(g \in G)$ in (2.88). Indeed, this is proved for $\ker(J_{\mathrm{c}})$ in Proposition 2.2 in Sect. 2.4.2, whereas $\mathrm{range}(J_{\mathrm{c}})$ is treated in the following proposition.

Proposition 2.4. *The range space* $\mathrm{range}(J_{\mathrm{c}})$ *is a* G*-invariant subspace. If the complementary subspace* V *is also* G*-invariant, the projection* P *onto* V *along* $\mathrm{range}(J_{\mathrm{c}})$ *satisfies*

$$T(g)P = P\,T(g), \quad g \in G. \tag{2.112}$$

Proof. Take any $\boldsymbol{y} \in \mathrm{range}(J_{\mathrm{c}})$ and $g \in G$. Then $\boldsymbol{y} = J_{\mathrm{c}}\boldsymbol{u}$ for some $\boldsymbol{u}$. It follows from (2.88) that

$$T(g)\boldsymbol{y} = T(g)(J_{\mathrm{c}}\boldsymbol{u}) = J_{\mathrm{c}}(T(g)\boldsymbol{u}) \in \mathrm{range}(J_{\mathrm{c}}).$$

Hence, $\mathrm{range}(J_{\mathrm{c}})$ is an invariant subspace. As is shown below, the invariance of $\mathrm{range}(J_{\mathrm{c}})$ is equivalent to

$$(I - P)\,T(g)(I - P) = T(g)(I - P), \quad g \in G, \tag{2.113}$$

and that of V is equivalent to

[15] $U = (\ker(J_{\mathrm{c}}))^{\perp}$ and $V = (\mathrm{range}(J_{\mathrm{c}}))^{\perp}$ in Remark 2.9 are valid choices, since T is assumed to be unitary (cf., (2.33)).

$$P\,T(g)P = T(g)P, \quad g \in G. \tag{2.114}$$

Equation (2.112) follows from (2.113) and (2.114).

We now show (2.114) for the invariance of V; the proof of (2.113) is similar. By the definition of projection P, a vector $\boldsymbol{x}$ belongs to V if and only if $P\boldsymbol{x} = \boldsymbol{x}$. Since V is an invariant subspace, we have $T(g)\boldsymbol{v} \in V$ for all $\boldsymbol{v} \in V$ and $g \in G$, that is, $PT(g)\boldsymbol{v} = T(g)\boldsymbol{v}$ for all $\boldsymbol{v} \in V$ and $g \in G$. Since a vector $\boldsymbol{v}$ belongs to V if and only if $\boldsymbol{v} = P\boldsymbol{u}$ for some $\boldsymbol{u}$, the latter condition above is equivalent to $PT(g)P\boldsymbol{u} = T(g)P\boldsymbol{u}$ for all $\boldsymbol{u}$ and $g \in G$. This is equivalent to (2.114). □

As the first step toward the proof of the equivariance of $\tilde{\boldsymbol{F}}$, the equivariance of $\boldsymbol{\varphi}$ in (2.101) is shown.

Proposition 2.5.

$$T(g)\boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}) = \boldsymbol{\varphi}(T(g)\boldsymbol{w}, \tilde{f}), \quad g \in G. \tag{2.115}$$

Proof. Recall (2.100) with (2.101) to see

$$(I - P) \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\rm c} + \tilde{f}) = \boldsymbol{0}, \tag{2.116}$$

which defines the function $\boldsymbol{\varphi}$ through the implicit function theorem. For the left-hand side of this equation, we have

$$\begin{aligned}
&T(g)(I - P) \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\rm c} + \tilde{f}) \\
&= (I - P) \cdot T(g)\boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\rm c} + \tilde{f}) \\
&= (I - P) \cdot \boldsymbol{F}(T(g)[\boldsymbol{u}_{\rm c} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f})], f_{\rm c} + \tilde{f}) \\
&= (I - P) \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + T(g)\boldsymbol{w} + T(g)\boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\rm c} + \tilde{f})
\end{aligned}$$

for all $g \in G$, where (2.112), (2.92), and (2.95) are used in this order. This means, on replacing[16] $T(g)\boldsymbol{w}$ by $\boldsymbol{w}$, that

$$(I - P) \cdot \boldsymbol{F}(\boldsymbol{u}_{\rm c} + \boldsymbol{w} + T(g)\boldsymbol{\varphi}(T(g^{-1})\boldsymbol{w}, \tilde{f}), f_{\rm c} + \tilde{f}) = \boldsymbol{0}. \tag{2.117}$$

A comparison of (2.116) and (2.117), together with the uniqueness[17] of the implicit function, shows that

$$T(g)\boldsymbol{\varphi}(T(g^{-1})\boldsymbol{w}, \tilde{f}) = \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), \quad g \in G.$$

Since we can replace $T(g^{-1})\boldsymbol{w}$ by $\boldsymbol{w}$, we obtain (2.115). □

[16] We can replace $T(g)\boldsymbol{w}$ by $\boldsymbol{w}$ since $\boldsymbol{w}$ is arbitrary in $\ker(J_{\rm c})$ and $\ker(J_{\rm c})$ is G-invariant.

[17] The uniqueness assertion applies since $T(g)\boldsymbol{\varphi}(T(g^{-1})\boldsymbol{w}, \tilde{f}) \in U$ by the G-invariance of U.

The equivariance of bifurcation equation $\tilde{\boldsymbol{F}}$ is obtained from Propositions 2.4 and 2.5 above, as follows.

Proposition 2.6.

$$\tilde{T}(g)\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = \tilde{\boldsymbol{F}}(\tilde{T}(g)\boldsymbol{w}, \tilde{f}), \quad g \in G. \tag{2.118}$$

Proof. With the use of (2.112), (2.81), (2.82), and (2.115) in this order, we have

$$\begin{aligned} T(g)\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) &= T(g)P \cdot \boldsymbol{F}(\boldsymbol{u}_{\mathrm{c}} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\mathrm{c}} + \tilde{f}) \\ &= P \cdot T(g)\, \boldsymbol{F}(\boldsymbol{u}_{\mathrm{c}} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}), f_{\mathrm{c}} + \tilde{f}) \\ &= P \cdot \boldsymbol{F}(T(g)\,[\boldsymbol{u}_{\mathrm{c}} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f})], f_{\mathrm{c}} + \tilde{f}) \\ &= P \cdot \boldsymbol{F}(\boldsymbol{u}_{\mathrm{c}} + T(g)\boldsymbol{w} + \boldsymbol{\varphi}(T(g)\boldsymbol{w}, \tilde{f}), f_{\mathrm{c}} + \tilde{f}) \\ &= \tilde{\boldsymbol{F}}(T(g)\boldsymbol{w}, \tilde{f}), \end{aligned}$$

in which the definition of $\tilde{\boldsymbol{F}}$ in (2.103) is used. Let $\tilde{T}$ denote the subrepresentation of T on the kernel space $\ker(J_{\mathrm{c}})$, and rewrite $T(g)\boldsymbol{w}$ above as $\tilde{T}(g)\boldsymbol{w}$. The subrepresentation of T on the subspace U is equivalent to that on $\mathrm{range}(J_{\mathrm{c}})$ through the one-to-one correspondence established by the Jacobian matrix of $\boldsymbol{\varphi}$ in (2.101) evaluated at $(\boldsymbol{w}, \tilde{f}) = (\boldsymbol{0}, 0)$. On the other hand, the representations on $\mathbb{R}^N = \ker(J_{\mathrm{c}}) \oplus U$ and $\mathbb{R}^N = V \oplus \mathrm{range}(J_{\mathrm{c}})$ are equivalent. It then follows that the subrepresentation on V is equivalent to that on $\ker(J_{\mathrm{c}})$. This allows us to replace the expression $T(g)\tilde{\boldsymbol{F}}$ above by $\tilde{T}(g)\tilde{\boldsymbol{F}}$. We thus obtain the equivariance (2.118) of the bifurcation equation $\tilde{\boldsymbol{F}}$. □

Remark 2.12. The Liapunov–Schmidt reduction can be done consistently with the stability property in the sense that a solution to the bifurcation equation is (linearly) stable or unstable according to whether the corresponding solution to the original system is (linearly) stable or unstable (see Chap. I, Sect. 4, of Golubitsky and Schaeffer, 1985 [6] and Sect. 2.4.4 of Ikeda and Murota, 2010 [10]). □

2.4.4 *Symmetry of Solutions*

Attention is now shifted from the symmetry of equations to the symmetry of solutions.

Isotropy Subgroup

The symmetry of a solution $\boldsymbol{u}$ is described by a subgroup of G, called the *isotropy subgroup* of $\boldsymbol{u}$, defined by

$$\Sigma(\boldsymbol{u}) = \Sigma(\boldsymbol{u}; G, T) = \{g \in G \mid T(g)\boldsymbol{u} = \boldsymbol{u}\}. \tag{2.119}$$

A subgroup Σ of G is said to be an *isotropy subgroup* if $\Sigma = \Sigma(\boldsymbol{u})$ for some $\boldsymbol{u} \neq \boldsymbol{0}$.

The notation (2.119) is extended to a set of vectors, say, W as

$$\Sigma(W) = \bigcap_{\boldsymbol{u} \in W} \Sigma(\boldsymbol{u}) = \{g \in G \mid T(g)\boldsymbol{u} = \boldsymbol{u} \text{ for all } \boldsymbol{u} \in W\}. \tag{2.120}$$

We use this extended notation primarily for $W = \ker(J_{\mathrm{c}})$.

Near Ordinary Point

For an ordinary point, at which the Jacobian matrix is nonsingular by definition, the most important fact is that the symmetry of a solution remains invariant in the neighborhood of this point. Let $(\boldsymbol{u}_*, f_*)$ be an ordinary point, and let $(\boldsymbol{u}, f)$ be another solution point sufficiently close to $(\boldsymbol{u}_*, f_*)$. Then it can be shown that $\boldsymbol{u}_*$ and $\boldsymbol{u}$ share the same symmetry in the sense that

$$\Sigma(\boldsymbol{u}_*) = \Sigma(\boldsymbol{u}). \tag{2.121}$$

The proof is given in Remark 2.13 below.

Remark 2.13. The proof of (2.121) is as follows. Since the Jacobian matrix $J(\boldsymbol{u}_*, f_*)$ is nonsingular, by the implicit function theorem, the equation (2.74) can be solved for $\boldsymbol{u}$ as

$$\boldsymbol{u} = \boldsymbol{u}_* + \boldsymbol{\varphi}(\tilde{f}) \tag{2.122}$$

in a neighborhood of $(\boldsymbol{u}, \tilde{f}) = (\boldsymbol{u}_*, 0)$, where $\tilde{f} = f - f_*$ and $\boldsymbol{\varphi}(0) = \boldsymbol{0}$. Then we have

$$\boldsymbol{F}(\boldsymbol{u}_* + \boldsymbol{\varphi}(\tilde{f}), f_* + \tilde{f}) = \boldsymbol{0}.$$

Fix an arbitrary $g \in \Sigma(\boldsymbol{u}_*)$. Using the equivariance (2.81) we obtain

$$\begin{aligned}
\boldsymbol{0} &= T(g)\boldsymbol{F}(\boldsymbol{u}_* + \boldsymbol{\varphi}(\tilde{f}), f_* + \tilde{f}) \\
&= \boldsymbol{F}(T(g)[\boldsymbol{u}_* + \boldsymbol{\varphi}(\tilde{f})], f_* + \tilde{f}) \\
&= \boldsymbol{F}(\boldsymbol{u}_* + T(g)\boldsymbol{\varphi}(\tilde{f}), f_* + \tilde{f}).
\end{aligned}$$

Then the uniqueness of the implicit function shows

$$T(g)\boldsymbol{\varphi}(\tilde{f}) = \boldsymbol{\varphi}(\tilde{f}),$$

and, therefore, by (2.122), we have

$$T(g)\boldsymbol{u} = T(g)[\boldsymbol{u}_* + \boldsymbol{\varphi}(\tilde{f})] = \boldsymbol{u}_* + \boldsymbol{\varphi}(\tilde{f}) = \boldsymbol{u},$$

which shows $g \in \Sigma(\boldsymbol{u})$. Hence $\Sigma(\boldsymbol{u}_*) \subseteq \Sigma(\boldsymbol{u})$. The roles of $\boldsymbol{u}_*$ and $\boldsymbol{u}$ are interchangeable; therefore, we also have $\Sigma(\boldsymbol{u}_*) \supseteq \Sigma(\boldsymbol{u})$. The invariance of the symmetry (2.121) is thus proved. □

Near Critical Point

In the neighborhood of a critical point, the relation (2.121) does not always hold. Typically, a solution $\boldsymbol{u}$ on the bifurcated path, if any, is less symmetric with $\Sigma(\boldsymbol{u})$ being a proper subgroup of G, although the possibility of $\Sigma(\boldsymbol{u}) = G$ is not excluded (Remark 2.15).

Let $(\boldsymbol{u}_{\rm c}, f_{\rm c})$ be a critical point such that $\Sigma(\boldsymbol{u}_{\rm c}) = G$. The Jacobian matrix $J_{\rm c} = J(\boldsymbol{u}_{\rm c}, f_{\rm c})$ has a nontrivial kernel space; its symmetry is represented by

$$\Sigma(\ker(J_{\rm c})) = \{g \in G \mid T(g)\boldsymbol{\eta} = \boldsymbol{\eta} \ \text{ for all } \ \boldsymbol{\eta} \in \ker(J_{\rm c})\}. \tag{2.123}$$

For a solution $(\boldsymbol{u}, f)$ that is sufficiently close to $(\boldsymbol{u}_{\rm c}, f_{\rm c})$, we have

$$\Sigma(\ker(J_{\rm c})) \subseteq \Sigma(\boldsymbol{w}) = \Sigma(\boldsymbol{u}), \tag{2.124}$$

where $\boldsymbol{w}$ denotes the solution to the bifurcation equation that corresponds to $\boldsymbol{u}$. This is proved in Remark 2.14 below.

If $(\boldsymbol{u}_{\rm c}, f_{\rm c})$ is a group-theoretic critical point associated with an irreducible representation, say $\mu \in R(G)$, the symmetry of the kernel space of $J_{\rm c}$ coincides with the subgroup G^{μ} introduced in (2.36). That is to say, we have

$$\Sigma(\ker(J_{\rm c})) = G^{\mu}. \tag{2.125}$$

The combination of (2.124) and (2.125) yields the following important relation about the symmetry of solutions that is vital in determining the symmetry of bifurcating solutions in Chaps. 3–9.

Proposition 2.7. *For a bifurcating solution* $\boldsymbol{u}$ *that branches from a group-theoretic critical point associated with an irreducible representation* μ*, we have*

$$G^{\mu} = \Sigma(\ker(J_{\rm c})) \subseteq \Sigma(\boldsymbol{w}) = \Sigma(\boldsymbol{u}) \subseteq G, \tag{2.126}$$

where $J_{\rm c}$ *denotes the Jacobian matrix at the critical point, and* $\boldsymbol{w}$ *is the solution to the bifurcation equation that corresponds to* $\boldsymbol{u}$.

The following points are noteworthy in (2.126).

- The identity $\Sigma(\boldsymbol{w}) = \Sigma(\boldsymbol{u})$ is a crucial technical relation, showing that the symmetry of the solution $\boldsymbol{u}$ to the full system of equations is determined by that of the solution $\boldsymbol{w}$ to the bifurcation equation. This fact allows us to concentrate on the solution $\boldsymbol{w}$ of the bifurcation equation in discussing the symmetry of a bifurcating solution $\boldsymbol{u}$.

- The discrepancy between $\Sigma(\ker(J_{\mathrm{c}}))$ and $\Sigma(\boldsymbol{w})$ is attributable to the nonlinearity of the equation and, as such, the symmetry $\Sigma(\boldsymbol{w})$ of the solution $\boldsymbol{w}$ can be determined only through an analysis involving nonlinear terms.
- Whereas the kernel space $\ker(J_{\mathrm{c}})$ of the Jacobian matrix J_{c} is dependent on individual systems, its symmetry $\Sigma(\ker(J_{\mathrm{c}}))$, being equal to the subgroup G^{μ}, is determined solely by the associated irreducible representation μ. The subgroup G^{μ} can be identified via algebraic considerations, independently of individual systems of equations.
- The symmetry of any solution in the neighborhood of the critical point is at least as high as the symmetry represented by the subgroup G^{μ}.

Remark 2.14. The proof of (2.124) is as follows. Take $g \in G$ and recall $T(g)\boldsymbol{u}_{\mathrm{c}} = \boldsymbol{u}_{\mathrm{c}}$. By applying $T(g)$ to the identity $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{c}} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f})$ in (2.104), we obtain

$$\begin{aligned} T(g)\boldsymbol{u} &= T(g)[\boldsymbol{u}_{\mathrm{c}} + \boldsymbol{w} + \boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f})] \\ &= \boldsymbol{u}_{\mathrm{c}} + T(g)\boldsymbol{w} + T(g)\boldsymbol{\varphi}(\boldsymbol{w}, \tilde{f}) \\ &= \boldsymbol{u}_{\mathrm{c}} + T(g)\boldsymbol{w} + \boldsymbol{\varphi}(T(g)\boldsymbol{w}, \tilde{f}), \end{aligned} \tag{2.127}$$

where the third equality follows from (2.115) in Proposition 2.5 in Sect. 2.4.3.

If $T(g)\boldsymbol{w} = \boldsymbol{w}$ in (2.127), we have $T(g)\boldsymbol{u} = \boldsymbol{u}$. Conversely, if $T(g)\boldsymbol{u} = \boldsymbol{u}$ in (2.127), we have

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{c}} + T(g)\boldsymbol{w} + \boldsymbol{\varphi}(T(g)\boldsymbol{w}, \tilde{f}), \tag{2.128}$$

where $T(g)\boldsymbol{w} \in \ker(J_{\mathrm{c}})$ and $\boldsymbol{\varphi}(T(g)\boldsymbol{w}, \tilde{f}) \in U$. A comparison of (2.128) with (2.104) shows $T(g)\boldsymbol{w} = \boldsymbol{w}$ since the decomposition of the form of (2.98) is unique. Therefore, we have $T(g)\boldsymbol{u} = \boldsymbol{u}$ if and only if $T(g)\boldsymbol{w} = \boldsymbol{w}$. This shows $\Sigma(\boldsymbol{w}) = \Sigma(\boldsymbol{u})$.

Finally, if $g \in \Sigma(\ker(J_{\mathrm{c}}))$, then we have $T(g)\boldsymbol{w} = \boldsymbol{w}$ by $\boldsymbol{w} \in \ker(J_{\mathrm{c}})$ and (2.123). Hence, $\Sigma(\ker(J_{\mathrm{c}})) \subseteq \Sigma(\boldsymbol{w})$. This completes the proof of (2.124). □

Remark 2.15. If the unit representation (see Example 2.6) is associated with a group-theoretic critical point $\boldsymbol{u}_{\mathrm{c}}$, the critical point is most likely to be a *limit point*, which is a point on an equilibrium path at which the value of the bifurcation parameter f takes a local maximum or minimum, and at which no bifurcation occurs. In some (nongeneric) cases, however, bifurcation may possibly occur. In this case we have $G^{\mu} = G$ and hence, by (2.126), the bifurcating solution $\boldsymbol{u}$ is as symmetric as the critical point $\boldsymbol{u}_{\mathrm{c}}$, i.e., $\Sigma(\boldsymbol{u}) = \Sigma(\boldsymbol{u}_{\mathrm{c}}) = G$. □

Orbit

The equivariance (2.81) implies that if $(\boldsymbol{u}, f)$ is a solution to the system of equations $\boldsymbol{F}(\boldsymbol{u}, f) = \boldsymbol{0}$ in (2.74), then $(T(g)\boldsymbol{u}, f)$ is also a solution for every $g \in G$. Motivated by this fact, we introduce the concept of *orbit* of a vector $\boldsymbol{u}$ as the set of vectors defined by

$$\mathrm{orb}(\boldsymbol{u}) = \{T(g)\boldsymbol{u} \mid g \in G\}. \tag{2.129}$$

For a solution $\boldsymbol{u}$, every member of $\mathrm{orb}(\boldsymbol{u})$ is also a solution. It is natural to identify solutions on the same orbit, whereas solutions not belonging to the same orbit (i.e., $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ not related as $\boldsymbol{u}_2 = T(g)\boldsymbol{u}_1$ for any $g \in G$) should be regarded as essentially different.

As for the symmetry of a solution $T(g)\boldsymbol{u}$ on the same orbit, we have[18]

$$\Sigma(T(g)\boldsymbol{u}) = g \cdot \Sigma(\boldsymbol{u}) \cdot g^{-1}, \quad g \in G. \tag{2.130}$$

This means that the symmetry subgroups of solutions on the same orbit are mutually conjugate (cf., (2.27)). Hence, we may focus on a representative subgroup among conjugate subgroups when we discuss possible symmetries appearing in bifurcated solutions in the group-theoretic bifurcation analysis in Chaps. 3–9.

Hierarchy of Symmetry

Typically, $\Sigma(\boldsymbol{u}')$ for a solution $\boldsymbol{u}'$ on a bifurcated path is a proper subgroup of $\Sigma(\boldsymbol{u})$ for a solution $\boldsymbol{u}$ on the path before bifurcation. In association with the repeated occurrence of bifurcation, one obtains a hierarchy of subgroups

$$G = G_1 \to G_2 \to G_3 \to \cdots \tag{2.131}$$

that characterizes a recursive change of symmetries. Here, G_{i+1} is a proper subgroup of G_i for $i = 1, 2, \ldots$.

2.4.5 Equivariant Branching Lemma

Equivariant branching lemma is a useful mathematical means to prove the existence of a bifurcating solution of a given symmetry (Vanderbauwhede, 1980 [20]; Cicogna, 1981 [3]; see also Golubitsky, Stewart, and Schaeffer, 1988 [7]).

[18]Proof of (2.130): $h \in \Sigma(T(g)\boldsymbol{u}) \iff T(h)T(g)\boldsymbol{u} = T(g)\boldsymbol{u} \iff T(g)^{-1}T(h)T(g)\boldsymbol{u} = \boldsymbol{u} \iff T(g^{-1}hg)\boldsymbol{u} = \boldsymbol{u} \iff g^{-1}hg \in \Sigma(\boldsymbol{u}) \iff h \in g \cdot \Sigma(\boldsymbol{u}) \cdot g^{-1}$.

In applications, this lemma is often applied to the bifurcation equation, and accordingly, the lemma is explained below using our notation for the bifurcation equation in (2.90). It should be emphasized, however, that the fact stated in the equivariant branching lemma is independent of the Liapunov–Schmidt reduction.

Consider a system of equations

$$\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = \boldsymbol{0} \tag{2.132}$$

in $\boldsymbol{w} \in \mathbb{R}^M$, where $\tilde{f}$ is a bifurcation parameter and $\tilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a sufficiently smooth nonlinear function in $\boldsymbol{w}$ and $\tilde{f}$. We assume the equivariance

$$\tilde{T}(g)\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = \tilde{\boldsymbol{F}}(\tilde{T}(g)\boldsymbol{w}, \tilde{f}), \quad g \in G \tag{2.133}$$

with respect to an M-dimensional representation $\tilde{T}$ of G. It is assumed that $(\boldsymbol{w}, \tilde{f}) = (\boldsymbol{0}, 0)$ is a critical point of multiplicity M.

Let us denote the Jacobian matrix of this system of equations by $\tilde{J}(\boldsymbol{w}, \tilde{f})$, which is an $M \times M$ matrix defined as

$$\tilde{J}(\boldsymbol{w}, \tilde{f}) = \left(\frac{\partial \tilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \ldots, M \right). \tag{2.134}$$

Since $(\boldsymbol{w}, \tilde{f}) = (\boldsymbol{0}, 0)$ is a critical point of multiplicity M, the Jacobian matrix vanishes at this point, i.e.,

$$\tilde{J}(\boldsymbol{0}, 0) = O. \tag{2.135}$$

The equivariance (2.133) is inherited by the Jacobian matrix as

$$\tilde{T}(g)\tilde{J}(\boldsymbol{w}, \tilde{f}) = \tilde{J}(\tilde{T}(g)\boldsymbol{w}, \tilde{f})\tilde{T}(g), \quad g \in G, \tag{2.136}$$

similarly to (2.85). On putting $\boldsymbol{w} = \boldsymbol{0}$ in the above, we obtain the commutativity of $\tilde{J}(\boldsymbol{0}, \tilde{f})$ with $\tilde{T}(g)$ expressed as

$$\tilde{T}(g)\tilde{J}(\boldsymbol{0}, \tilde{f}) = \tilde{J}(\boldsymbol{0}, \tilde{f})\tilde{T}(g), \quad g \in G, \tag{2.137}$$

which holds for all $\tilde{f}$.

Recall that the symmetry of a vector $\boldsymbol{w}$ is described by its isotropy subgroup

$$\Sigma(\boldsymbol{w}) = \Sigma(\boldsymbol{w}; G, \tilde{T}) = \{g \in G \mid \tilde{T}(g)\boldsymbol{w} = \boldsymbol{w}\}$$

introduced in (2.119). A subgroup Σ of G is said to be an isotropy subgroup (with respect to $\tilde{T}$) if $\Sigma = \Sigma(\boldsymbol{w})$ for some $\boldsymbol{w} \neq \boldsymbol{0}$. For a subgroup Σ of G, its *fixed-point subspace* is defined as

$$\mathrm{Fix}(\Sigma) = \{\boldsymbol{w} \mid \tilde{T}(g)\boldsymbol{w} = \boldsymbol{w} \text{ for all } g \in \Sigma\}. \tag{2.138}$$

For an isotropy subgroup Σ, we have $\dim \mathrm{Fix}(\Sigma) \geq 1$. An isotropy subgroup Σ with

$$\dim \mathrm{Fix}(\Sigma) = 1 \tag{2.139}$$

is called an *axial subgroup*.

The equivariant branching lemma guarantees the existence of a bifurcating solution branch under two technical assumptions:

1. The first assumption is that the representation $\tilde{T}$ in the equivariance condition (2.133) is an absolutely irreducible representation different from the unit representation.
2. The second assumption is that the system of equations (2.132) is *generic*, the precise meaning of which is as follows. The commutativity (2.137) of the Jacobian matrix $\tilde{J}(\mathbf{0}, \tilde{f})$ implies,[19] under the assumed absolute irreducibility of $\tilde{T}$, that

$$\tilde{J}(\mathbf{0}, \tilde{f}) = c(\tilde{f})I \tag{2.140}$$

for some scalar function $c(\tilde{f})$. The assumption is that the derivative of $c(\tilde{f})$ at $\tilde{f} = 0$ does not vanish, i.e.,

$$c'(0) \neq 0. \tag{2.141}$$

In the first assumption, it should be recalled that a representation over $\mathbb{R}$ is called absolutely irreducible if it is irreducible as a representation over $\mathbb{C}$ (cf., Remark 2.4 in Sect. 2.3.3). In the second assumption, it is noted that the function $c(\tilde{f})$ vanishes at $\tilde{f} = 0$, i.e., $c(0) = 0$ by (2.135).

Before entering into the theorem, we note a simple but crucial fact that $\boldsymbol{w} = \mathbf{0}$ is a (trivial) solution to (2.132) for all $\tilde{f}$, i.e.,

$$\tilde{\boldsymbol{F}}(\mathbf{0}, \tilde{f}) = \mathbf{0} \quad \text{for all } \tilde{f}. \tag{2.142}$$

This can be proved as follows. The equivariance (2.133) with $\boldsymbol{w} = \mathbf{0}$ gives

$$\tilde{T}(g)\tilde{\boldsymbol{F}}(\mathbf{0}, \tilde{f}) = \tilde{\boldsymbol{F}}(\mathbf{0}, \tilde{f}), \quad g \in G. \tag{2.143}$$

Suppose that $\tilde{\boldsymbol{F}}(\mathbf{0}, \tilde{f}) \neq \mathbf{0}$ for some $\tilde{f}$. Then the one-dimensional subspace spanned by the nonzero vector $\tilde{\boldsymbol{F}}(\mathbf{0}, \tilde{f})$ is an invariant subspace, contradicting the assumed irreducibility of $\tilde{T}$ when $M \geq 2$. When $M = 1$, (2.143) with $\tilde{\boldsymbol{F}}(\mathbf{0}, \tilde{f}) \neq \mathbf{0}$ implies $\tilde{T}(g) = 1$ for all $g \in G$, contradicting the assumption that $\tilde{T}$ is distinct from the unit representation.

[19] See Proposition 2.1 (Schur's lemma) in Sect. 2.3.4.

We are now ready to state and prove the equivariant branching lemma.

Theorem 2.1 (Equivariant Branching Lemma). *Assume that the representation $\tilde{T}$ in (2.133) is absolutely irreducible and the system of equations (2.132) is generic in the sense of (2.141). For an isotropy subgroup Σ with* dim Fix$(\Sigma) = 1$*, there exists a unique smooth solution branch to $\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = \boldsymbol{0}$ such that $\Sigma(\boldsymbol{w}) = \Sigma$ for each solution $\boldsymbol{w}$ on the branch.*

Proof. Since dim Fix$(\Sigma) = 1$ by the assumption, Fix(Σ) is spanned by a single vector, say, $\boldsymbol{\eta}$ and every vector $\boldsymbol{w}$ in Fix(Σ) can be expressed as $\boldsymbol{w} = x\boldsymbol{\eta}$ with $x \in \mathbb{R}$. For $\boldsymbol{w} = x\boldsymbol{\eta} \in \text{Fix}(\Sigma)$, the equivariance (2.133) implies

$$\tilde{T}(g)\tilde{\boldsymbol{F}}(x\boldsymbol{\eta}, \tilde{f}) = \tilde{\boldsymbol{F}}(x\boldsymbol{\eta}, \tilde{f}), \quad g \in \Sigma. \tag{2.144}$$

This shows that the vector $\tilde{\boldsymbol{F}}(x\boldsymbol{\eta}, \tilde{f})$ belongs to Fix(Σ), and hence we can express

$$\tilde{\boldsymbol{F}}(x\boldsymbol{\eta}, \tilde{f}) = h(x, \tilde{f})\boldsymbol{\eta}$$

with some scalar function $h(x, \tilde{f})$. We have $h(0, \tilde{f}) = 0$ by (2.142), and therefore we can represent $h(x, \tilde{f})$ as $h(x, \tilde{f}) = x\,k(x, \tilde{f})$ with some smooth function $k(x, \tilde{f})$. Then we have

$$\tilde{\boldsymbol{F}}(x\boldsymbol{\eta}, \tilde{f}) = x\,k(x, \tilde{f})\boldsymbol{\eta}.$$

Differentiation with respect to x yields

$$\tilde{J}(x\boldsymbol{\eta}, \tilde{f})\boldsymbol{\eta} = [k(x, \tilde{f}) + x\,\frac{\partial k}{\partial x}(x, \tilde{f})]\boldsymbol{\eta}.$$

The substitution of $x = 0$ yields

$$\tilde{J}(\boldsymbol{0}, \tilde{f})\boldsymbol{\eta} = k(0, \tilde{f})\boldsymbol{\eta},$$

whereas $\tilde{J}(\boldsymbol{0}, \tilde{f}) = c(\tilde{f})I$ by (2.140). Therefore,

$$c(\tilde{f}) = k(0, \tilde{f}).$$

We have $k(0, 0) = 0$ since $c(0) = 0$ by (2.135). Furthermore, $(\partial k/\partial \tilde{f})(0, 0) = c'(0) \neq 0$ by the assumption in (2.141). Then, by the implicit function theorem, the equation $k(x, \tilde{f}) = 0$ can be solved uniquely for $\tilde{f}$ as $\tilde{f} = \varphi(x)$ in a neighborhood of $(x, \tilde{f}) = (0, 0)$. This represents a bifurcating solution $(\boldsymbol{w}, \tilde{f}) = (x\boldsymbol{\eta}, \varphi(x))$ with $\Sigma(\boldsymbol{w}) = \Sigma$. □

As already mentioned at the beginning of this section, the equivariant branching lemma is often applied to the bifurcation equation $\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{f}) = \boldsymbol{0}$ in (2.90). In this case, the assumptions made in the above formulation are satisfied quite naturally. By construction, $(\boldsymbol{w}, \tilde{f}) = (\boldsymbol{0}, 0)$ is a critical point of multiplicity

M for the bifurcation equation. The bifurcation equation at a group-theoretic bifurcation point is associated with an irreducible representation $\tilde{T}$, and the assumed equivariance (2.133) is a consequence of the equivariance of the original system of equations, as shown in (2.93). In addition, the associated irreducible representation $\tilde{T}$ is usually different from the unit representation. Thus, the essential assumption is that $\tilde{T}$ should be absolutely irreducible.

If $\dim \mathrm{Fix}(\Sigma) = 1$ for an isotropy subgroup Σ of G, a bifurcating solution with symmetry Σ is guaranteed to exist generically by the equivariant branching lemma. If $\dim \mathrm{Fix}(\Sigma) = 0$, a bifurcated solution branch with symmetry Σ does not exist. If $\dim \mathrm{Fix}(\Sigma) \geq 2$, no definite conclusion can be reached by the equivariant branching lemma. The isotropy subgroup $\Sigma(\boldsymbol{w})$ of a solution $\boldsymbol{w}$ for the bifurcation equation is identical to the isotropy subgroup $\Sigma(\boldsymbol{u})$ of the corresponding solution $\boldsymbol{u}$ of the original system of equations, i.e., $\Sigma(\boldsymbol{u}) = \Sigma(\boldsymbol{w})$, as shown in (2.126). This fact allows us to assert the existence of a bifurcating branch with the specified symmetry Σ in the space of $\boldsymbol{u}$ by working with the bifurcation equation in $\boldsymbol{w}$.

In Chap. 3 the use of this method is illustrated for bifurcation in the racetrack economy. Extensive use of this method is made in Chap. 8 to investigate hexagons of Christaller and Lösch on the hexagonal lattice.

Remark 2.16. Here are some remarks concerning technical issues in our applications of the equivariant branching lemma in Chaps. 3 and 8. First of all, the assumption of absolute irreducibility is satisfied, fortunately, by all real irreducible representations of the symmetry groups for the racetrack economy and the economy on the hexagonal lattice, which fact allows us to apply the equivariant branching lemma to our problems. Second, in our application of economic agglomeration, we are interested not only in the symmetry of the bifurcating solution but also in the direction of the bifurcating solution (see Sect. 7.6.1 for details). In the proof of Theorem 2.1 we have an expression $\boldsymbol{w} = x\boldsymbol{\eta}$ for the bifurcating solution, where the symmetry of $\boldsymbol{w}$ is determined solely by $\boldsymbol{\eta}$, independently of x. The emergence of hexagons of Christaller and Lösch corresponds to the existence of a bifurcating solution with a specified sign of x, say, $x > 0$. In the above proof, the existence of a solution path $\tilde{f} = \varphi(x)$ is guaranteed in a neighborhood of $x = 0$. This implies, in particular, that there exists a solution branch with $x > 0$. This solution branch corresponds to an agglomeration pattern of economic significance. □

2.5 Summary

- Fundamentals of group representation theory have been presented.
- The block-diagonalization method for matrices commuting with group actions has been presented.
- Group equivariance of the equilibrium equation, a mathematical formulation of the symmetry inherent in systems, has been introduced.

- The Liapunov–Schmidt reduction procedure for deriving the bifurcation equation under symmetry has been presented.
- Equivariant branching lemma to ensure the existence of bifurcating solutions with specified symmetry has been introduced.

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Chapter 3
Agglomeration in Racetrack Economy

Abstract The spatial agglomeration is investigated for a racetrack economy, comprising a number of cities equally spreading on the circumference of a circle, with micromechanism by Krugman's core–periphery model. The group-theoretic bifurcation analysis procedure presented in Chap. 2 is applied to a problem with the dihedral group, expressing the symmetry of the racetrack economy. The theoretically possible agglomeration (bifurcation) patterns of this economy are predicted by using block-diagonalization, bifurcation equation, and equivariant branching lemma. Spatial period doubling bifurcation cascade is highlighted as the most characteristic progress of agglomeration. This chapter, as a whole, serves as an introduction to the methodology for a more general analysis in Chaps. 5–9 in Part II of an economy on a hexagonal lattice with a larger and more complicated symmetry group.

Keywords Agglomeration of population • Bifurcation • Core–periphery model • Dihedral group • Economic agglomeration • Group theory • Krugman model • Racetrack economy • Spatial period doubling

3.1 Introduction

Agglomeration of the two-place economy in core–periphery models (Sect. 2.2) was studied extensively in new economic geography (NEG) as an idealized spatial platform of migration between identical places. Although this economy is endowed with analytical tractability, there is criticism that its geometry is oversimplified (see Remark 3.1). In this chapter, a racetrack economy in Fig. 3.1 comprising identical places spread uniformly on the circumference of a circle is considered as a more realistic spatial platform. This chapter serves as a prelude to the bifurcation analysis of the hexagonal lattice with a more complicated but further realistic spatial platform (Part II).

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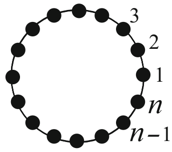


Fig. 3.1 Racetrack economy with n places ($n = 16$)

The main objective of this chapter is to demonstrate the framework and use of group-theoretic bifurcation theory (Chap. 2) through its application to the mathematical model of the racetrack economy of Krugman's core–periphery model. The regular-polygonal symmetry of this economy is described by the dihedral group, and possible agglomeration patterns in this economy, arising from bifurcations, are predicted[1] by group-theoretic bifurcation analysis using block-diagonalization, equivariant branching lemma, and bifurcation equation. As explained in Sect. 2.4, in the bifurcation analysis procedure under group symmetry, the following two different methods of analysis are available.

- The equivariant branching lemma (Sect. 2.4.5) is applied to the bifurcation equation associated with an irreducible representation of the group to see the existence of bifurcating solutions with a specified symmetry. This analysis is purely algebraic or group-theoretic, focusing on the symmetry of solutions. Since the bifurcation equation is not solved, the concrete form of this equation need not be derived.
- The bifurcation equation is obtained in the form of power series expansions and is solved asymptotically. This method is more complicated, treating nonlinear terms directly, but is more informative, giving asymptotic forms of the bifurcating solutions and their directions in addition to their existence.

Another objective of this chapter is to advance several characteristic agglomeration behaviors of the racetrack economy. The most characteristic agglomeration advanced is a spatial period doubling cascade that is encountered with the decrease of the transport cost (see Remark 3.1). An example of this cascade for the racetrack economy with eight places is presented in Fig. 3.2, in which the concentration progresses successively in association with the doubling of the spatial period. A system of $2^3 = 8$ identical places is concentrated into $2^2 = 4$ identical larger places en route to the concentration to a megalopolis. In addition, other bifurcations, such as period multiplying bifurcation and transcritical bifurcation, are studied.

This chapter is organized as follows. Racetrack economy is presented and its symmetry is investigated in Sect. 3.2. The irreducible representations of the dihedral

[1]Bifurcation behavior of systems with dihedral group symmetry has been studied fully in the literature of applied mathematics: e.g., Sattinger, 1979 [14], 1983 [15]; Healey, 1988 [7]; Golubitsky, Stewart, and Schaeffer, 1988 [6]; Dellnitz and Werner, 1989 [4]; and Ikeda and Murota, 2010 [8]. Sections 3.2–3.4 are adaptations of these results to the racetrack economy.

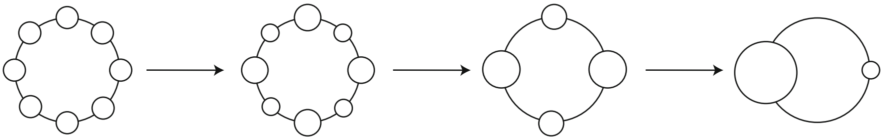


Fig. 3.2 Spatial period doubling cascade for the racetrack economy with eight places. See Sect. 3.5.2 for the precise meaning of this figure; the area of *white circle* denotes the population size of the associated place; the *arrow* denotes the occurrence of bifurcation

group are given in Sect. 3.3. Bifurcation from the flat earth equilibrium is studied in Sect. 3.4. Other bifurcations are studied in Sect. 3.5. Agglomerations in the racetrack economy are investigated numerically in Sect. 3.6.

Remark 3.1. Krugman's core–periphery model was used to describe microeconomic mechanism of the migration of population between places in the racetrack economy. Recall that this model has provided a new framework to explain interactions that occur among increasing returns at the level of firms, transport costs, and factor mobility, and the predominant role of bifurcation in agglomeration in the two-place economy was demonstrated using this model (see Sect. 1.5). Yet there is a criticism that economic agglomerations, in reality, would take place at more than two locations, which was empirically evidenced by Bosker et al., 2010 [3]. In an economy with more than two places, indirect spatial effects among places emerge, which deny a simple analysis. Behrens and Thisse, 2007 [2] stated "Dealing with these spatial interdependencies constitutes one of the main theoretical and empirical challenges NEG and regional economics will surely have to face in the future."

The racetrack economy is one of the most common spatial platforms in the literature (see, e.g., Krugman, 1993 [12]; Fujita, Krugman, and Venables, 1999 [5]; Picard and Tabuchi, 2010 [13]; Tabuchi and Thisse, 2011 [16]; Ikeda, Akamatsu, and Kono, 2012 [9]; and Akamatsu, Takayama, and Ikeda, 2012 [1]). Spatial period doubling cascade of the racetrack economy has been reported (Tabuchi and Thisse, 2011 [16]; Ikeda, Akamatsu, and Kono, 2012 [9]; and Akamatsu, Takayama, and Ikeda, 2012 [1]). □

3.2 Racetrack Economy and Symmetry

Let us consider the *racetrack economy* with n places (labeled $i = 1, \ldots, n$) that are spread equally on the circumference of a circle with the unit radius as shown in Fig. 3.1. This circle has a road along this circumference. The number n of places is set to be even for convenience of presentation. The dihedral group is advanced as the symmetry group of this economy, and its subgroups are presented to describe equilibria of various symmetries.

3.2.1 Equilibrium Equation

Let us employ Krugman's core–periphery model and recall from Sect. 1.5 the equilibrium equation in (1.19) with (1.20), which prescribes a rest point of the replicator dynamics as

$$F_i(\boldsymbol{\lambda}, \tau) = 0, \quad i = 1, \dots, n \tag{3.1}$$

with

$$F_i(\boldsymbol{\lambda}, \tau) = (\omega_i(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau))\lambda_i, \quad i = 1, \dots, n. \tag{3.2}$$

Here, $\boldsymbol{\lambda} = (\lambda, \dots, \lambda_n)^\top$ is the vector expressing distribution of a mobile population, τ is the transport cost parameter, $\omega_i(\boldsymbol{\lambda}, \tau)$ is the real wage at the ith place, and $\overline{\omega}(\boldsymbol{\lambda}, \tau)$ is the average real wage defined by

$$\overline{\omega} = \sum_{i=1}^{n} \lambda_i \omega_i. \tag{3.3}$$

The condition of no population growth

$$\sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \dots, n \tag{3.4}$$

is imposed. The stability of the equilibrium is classified by the eigenvalues of the Jacobian matrix $J(\boldsymbol{\lambda}, \tau)$ of the governing equation in (3.2), and a stable equilibrium is a spatial equilibrium (Proposition 1.3 in Sect. 1.5.4).

For the iceberg transport technology in the racetrack economy, we define the transport cost between place i and place j in (1.8) as

$$T_{ij} = T_{ij}(\tau) = (1 - \tau)^{-D_{ij}/\pi}, \quad i, j = 1, \dots, n, \tag{3.5}$$

where

$$D_{ij} = \frac{2\pi}{n} \min(|i - j|, n - |i - j|), \quad i, j = 1, \dots, n \tag{3.6}$$

represents the shortest distance between places i and j along the circle; $\min(\cdot, \cdot)$ denotes the smaller value of the variables in parentheses, and the transport cost parameter τ stays in a bounded range $0 < \tau < 1$. Note that the limit of $\tau \to 0$ corresponds to the state of no transport cost, and the limit of $\tau \to 1$ corresponds to the state of infinite transport cost.

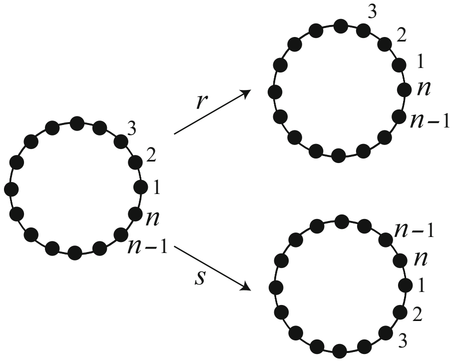


Fig. 3.3 Actions of the rotation r and the reflection s in the racetrack economy with n places ($n = 16$)

3.2.2 *Equivariance of Equilibrium Equation*

The regular n-gonal symmetry of the racetrack economy is described by the invariance with respect to the following two transformations (see Fig. 3.3):

r: counterclockwise *rotation* about the origin at an angle of $2\pi/n$, and
s: *reflection* $y \mapsto -y$.

We consider the group generated by these two transformations

$$\mathrm{D}_n = \langle r, s\rangle = \{r^i, sr^i \mid i = 0, 1, \ldots, n-1\},$$

where r^i denotes a counterclockwise rotation about the center of the circle at an angle of $2\pi i/n$ ($i = 0, 1, \ldots, n-1$), and sr^i is the combined action of the rotation r^i followed by the reflection s. We have the *fundamental relations*

$$r^n = s^2 = (sr)^2 = e, \tag{3.7}$$

as well as $r^i r^j = r^{i+j}$.

In our study of a system of n places on the racetrack, each element g of D_n acts as a permutation of place numbers $(1, \ldots, n)$. Consequently, the representation matrix $T(g)$ is an $n \times n$ permutation matrix for each $g \in \mathrm{D}_n$. With the use of the representation matrices

$$T(r) = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \quad T(s) = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & ⋰ & \\ & 1 & & \end{bmatrix} \tag{3.8}$$

for r and s, the representation matrices $T(g)$ for all $g \in \mathrm{D}_n$ can be generated as

$$T(r^i) = T(r)^i, \quad T(sr^i) = T(s)T(r)^i, \quad i = 0, 1, \ldots, n-1. \tag{3.9}$$

We have the following proposition for the equivariance.

Proposition 3.1. *The Krugman model* (3.2) *for the racetrack economy is equivariant to* D_n*, i.e.,*

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda},\tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda},\tau), \quad g \in \mathrm{D}_n \tag{3.10}$$

for the $n \times n$ *matrix representation* $T(g)$ *of* D_n *in* (3.8) *and* (3.9).

Proof. Each element g of D_n acts as a permutation of place numbers $(1,\dots,n)$. By expressing the action of $g \in \mathrm{D}_n$ as $g: i \mapsto i^*$, we have $\omega_i(T(g)\boldsymbol{\lambda},\tau) = \omega_{i^*}(\boldsymbol{\lambda},\tau)$ because of the form of the transport cost (3.5) with (3.6), as well as (1.9)–(1.12) for the market equilibrium in Sect. 1.5.2; we also have $\overline{\omega}(T(g)\boldsymbol{\lambda},\tau) = \overline{\omega}(\boldsymbol{\lambda},\tau)$ by (3.3). Therefore, we have

$$F_i(T(g)\boldsymbol{\lambda},\tau) = (\omega_{i^*}(\boldsymbol{\lambda},\tau) - \overline{\omega}(\boldsymbol{\lambda},\tau))\lambda_{i^*} = F_{i^*}(\boldsymbol{\lambda},\tau)$$

for the function F_i in (3.2). This proves the equivariance (3.10). □

Remark 3.2. The racetrack economy with $n = 2$ is reduced to the two-place economy. In this sense, the symmetry of the two-place economy is labeled $\mathrm{D}_2 = \{e, s, r, sr\}$. Yet it suffices to consider the group $\{e, sr\}$ since this economy always remains invariant to the horizontal reflection s since $T(s) = I$ (the identity matrix of order 2) in (3.8). This group $\{e, sr\}$ is isomorphic to $\mathrm{D}_1 = \{e, s\}$, and we may alternatively consider the equivariance with respect to $\mathrm{D}_1 = \{e, s\}$ as in (2.20). □

3.2.3 Distribution-Preserving Equilibria

The equilibrium equation

$$(\omega_i(\boldsymbol{\lambda},\tau) - \overline{\omega}(\boldsymbol{\lambda},\tau))\lambda_i = 0, \quad i = 1,\dots,n, \tag{3.11}$$

which is obtained from (3.1) with (3.2), has characteristic equilibria such that the population $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\tau)$ of the places remains unchanged in association with the change of the transport cost parameter τ. Such equilibria are here termed *distribution-preserving equilibria.*

It is readily apparent that the racetrack economy has the *flat earth equilibrium*

$$\boldsymbol{\lambda}_0 = (1/n,\dots,1/n)^\top \tag{3.12}$$

with D_n-symmetry and a spatial period $L_n = 2\pi/n$. All places have the same economic environments with the same real wage $\omega_i = \overline{\omega}$ $(i = 1,\dots,n)$. This is an interior equilibrium.[2]

[2]See Sect. 1.5.3 for the definition of interior and corner equilibria.

Examples of other distribution-preserving equilibria, which are corner equilibria, are as follows[3]:

- *Period doubling equilibria*

$$\boldsymbol{\lambda} = (2/n, 0, \ldots, 2/n, 0)^\top \text{ and } (0, 2/n, \ldots, 0, 2/n)^\top. \tag{3.13}$$

  These equilibria have an alternation of concentrated places and extinguished places along the circle.
- Equilibria with m (≥ 3) peaks:

$$\lambda_i = \begin{cases} 1/m & (i = k, k + n/m, \ldots, k + (m-1)n/m), \\ 0 & \text{otherwise}, \end{cases} \tag{3.14}$$

  which contain (3.13) as a special case, where m is a divisor of n and k is an integer with $1 \leq k \leq n/m$. The population is concentrated at m equally spaced places with the same population $1/m$ and with the same real wage

$$\omega_k = \omega_{k+n/m} = \cdots = \omega_{k+(m-1)n/m} = \overline{\omega}. \tag{3.15}$$

- *Twin peaks equilibria*

$$\boldsymbol{\lambda} = (0, \ldots, 0, \overset{i}{1/2}, 0, \ldots\ldots, 0, \overset{j}{1/2}, 0, \ldots, 0)^\top, \quad 1 \leq i < j \leq n. \tag{3.16}$$

  These equilibria express the concentration of the population to two different places, i and j, with the same population $1/2$ and with the same real wage $\omega_i = \omega_j = \overline{\omega}$.
- *Concentrated equilibria*

$$\boldsymbol{\lambda} = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)^\top, \quad i = 1, \ldots, n. \tag{3.17}$$

  These equilibria express the concentration of the population to a single place with $\omega_i = \overline{\omega}$ for some i.

Remark 3.3. The distribution in (3.14) is proved to be an equilibrium as follows. First note that, for any τ, (3.15) follows from (3.14) by the symmetry of the model in (1.9)–(1.12) with (3.5) and (3.6). Then (3.15) implies $(\omega_i - \overline{\omega})\lambda_i = 0$ for $i = k, k + n/m, \ldots, k + (m-1)n/m$. For other i we have $\lambda_i = 0$, which obviously implies $(\omega_i - \overline{\omega})\lambda_i = 0$. Therefore, (3.14) is a solution to the equilibrium equation (3.11).

[3] See Remark 3.3 for the proof.

For the twin peaks equilibria in (3.16), $\omega_i = \omega_j = \overline{\omega}$ follows from the symmetry of the model and $\lambda_k = 0$ for $k \neq i, j$. Then we have $(\omega_i - \overline{\omega})\lambda_i = 0$ and $(\omega_j - \overline{\omega})\lambda_j = 0$. For other places k $(\neq i, j)$, we obviously have $(\omega_k - \overline{\omega})\lambda_k = 0$ since $\lambda_k = 0$. Therefore, (3.16) is a solution to the equilibrium equation (3.11). □

Subgroups Expressing Reduced Symmetries

The symmetry of the period doubling distribution-preserving equilibria in (3.13) is expressed by the subgroup of D_n given by

$$\mathrm{D}_{n/2} = \{r^{2i}, sr^{2i} \mid i = 0, 1, \ldots, n/2 - 1\}.$$

The symmetries of the equilibria with m (≥ 3) peaks in (3.14) are expressed by the subgroups

$$\mathrm{D}_m^{k,n} = \{r^{in/m}, sr^{k-1+in/m} \mid i = 0, 1, \ldots, m-1\},$$

which are isomorphic to D_m, where the subscript m is an integer that divides n, $1 \leq m \leq n/2$, and superscript k $(= 1, \ldots, n/m)$ expresses the directions of the reflection axes.

The twin peaks equilibrium in (3.16) and the concentrated equilibrium in (3.17) both have the symmetry of $\mathrm{D}_1^{k,n}$ for some k. For example, the twin peaks equilibrium (3.16) with $i = 2$ and $j = n$ and the concentrated equilibrium in (3.17) with $i = 1$ both have the symmetry of $\mathrm{D}_1^{1,n} = \mathrm{D}_1$.

Example 3.1. The racetrack economy with four places $(n = 4)$ has several distribution-preserving equilibria, e.g.,

- the flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/4, 1/4, 1/4, 1/4)^\top$,
- a period doubling equilibrium $\boldsymbol{\lambda} = (1/2, 0, 1/2, 0)^\top$,
- a twin peaks equilibrium $\boldsymbol{\lambda} = (0, 1/2, 1/2, 0)^\top$, and
- a concentrated equilibrium $\boldsymbol{\lambda} = (1, 0, 0, 0)^\top$.

The symmetries of some of these equilibria, as well as those which are not distribution-preserving equilibria, are listed in Table 3.1. The patterns for subgroups D_1 and $\mathrm{D}_1^{3,4}$ have the same economic meaning. Such is also the case for $\mathrm{D}_1^{2,4}$ and $\mathrm{D}_1^{4,4}$. □

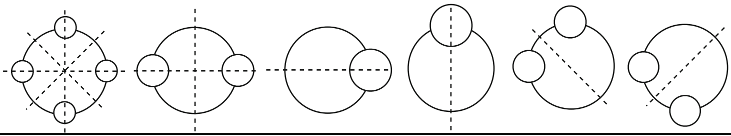
3.3 Irreducible Representations

The irreducible representations of the dihedral group D_n and the irreducible decomposition of the representation matrix for the racetrack economy are presented in preparation for the theoretical analysis in Sect. 3.4.

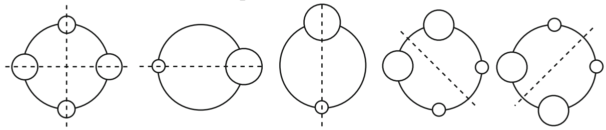
Table 3.1 Examples of distribution-preserving and other equilibria for $n = 4$ (*dashed line* means axis of reflection symmetry)

Symmetry groups					
D_4	D_2	D_1	$D_1^{3,4}$	$D_1^{2,4}$	$D_1^{4,4}$

Distribution-preserving equilibria



Other equilibria



3.3.1 Irreducible Representations of the Dihedral Group

The irreducible representations of the dihedral group D_n for even n are presented herein. There are four one-dimensional and $n/2 - 1$ two-dimensional irreducible representations over $\mathbb{R}$, which are absolutely irreducible. The one-dimensional irreducible representations are denoted as $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$ and the two-dimensional irreducible representations as (1), (2), ..., $(n/2-1)$ (see Remark 3.4 below for the notation). Then the index set $R(D_n)$ of irreducible representations (see (2.34) for the notation $R(G)$) is given by

$$R(D_n) = \{(+,+),(+,-),(-,+),(-,-)\} \cup \{(j) \mid j = 1,\ldots,n/2-1\}, \quad (3.18)$$

which, for $n = 2$, is reduced to

$$R(D_2) = \{(+,+),(+,-),(-,+),(-,-)\}.$$

The one-dimensional irreducible representations are defined by

$$\begin{aligned} T^{(+,+)}(r) &= 1, & T^{(+,+)}(s) &= 1, \\ T^{(+,-)}(r) &= 1, & T^{(+,-)}(s) &= -1, \\ T^{(-,+)}(r) &= -1, & T^{(-,+)}(s) &= 1, \\ T^{(-,-)}(r) &= -1, & T^{(-,-)}(s) &= -1. \end{aligned} \quad (3.19)$$

The two-dimensional representations, which are not unique, are chosen here to be unitary representations defined by

$$T^{(j)}(r) = \begin{bmatrix} \cos(2\pi j/n) & -\sin(2\pi j/n) \\ \sin(2\pi j/n) & \cos(2\pi j/n) \end{bmatrix}, \quad T^{(j)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$j = 1,\ldots,n/2-1. \quad (3.20)$$

We introduce an index

$$\hat{n} = \frac{n}{\gcd(n, j)} \tag{3.21}$$

with $\gcd(n, j)$ denoting the greatest common divisor of n and j. We have $3 \leq \hat{n} \leq n$, since $1 \leq j \leq n/2 - 1$. This index will play an important role in characterizing bifurcations.

Remark 3.4. The correspondence of the present notations for D_n to conventional notations in physics and chemistry (Kettle, 1995 [10] and Kim, 1999 [11]) is given for $n = 6$ as

Present notation	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$	(1)	(2)
Conventional notation	A_1	A_2	B_1	B_2	E_1	E_2

□

3.3.2 *Irreducible Representations for Racetrack Economy*

The matrix representation $T(g)$ of the racetrack economy specified by (3.8) has the irreducible decomposition (2.42) with the following multiplicities:

$$a^{(+,+)} = 1, \quad a^{(+,-)} = 0, \quad a^{(-,+)} = 1, \quad a^{(-,-)} = 0, \quad a^{(j)} = 1, \tag{3.22}$$

where $j = 1, \ldots, n/2 - 1$. It is noteworthy that the multiplicity is either 1 or 0 for each irreducible representation, that is, the representation $T(g)$ is *multiplicity-free*. Note that irreducible representations $(+,-)$ and $(-,-)$, with $a^{(+,-)} = a^{(-,-)} = 0$, are missing in $T(g)$, whereas the other irreducible representations $(+,+)$, $(-,+)$, and (j) for $j = 1, \ldots, n/2 - 1$ with $a^{\mu} = 1$ are contained in $T(g)$ and are potentially associated with critical points of the equilibrium equation (3.2) for the racetrack economy.

3.4 Bifurcation from Flat Earth Equilibria

Bifurcation from the flat earth equilibria is investigated according to the group-theoretic bifurcation analysis procedure given in Sect. 2.4. Recall that the number n of places is assumed to be even.

3.4.1 *Use of Block-Diagonalization*

Block-diagonalization expounded in Sect. 2.3.4 is convenient in the detection and classification of singularity. At the flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/n, \ldots, 1/n)^{\top}$ in (3.12), the Jacobian matrix $J(\boldsymbol{\lambda}_0, \tau)$ for (3.2) is endowed with equivariance

$$T(g)J(\boldsymbol{\lambda}_0, \tau) = J(\boldsymbol{\lambda}_0, \tau)T(g), \quad g \in \mathrm{D}_n \tag{3.23}$$

as a consequence of the D_n-equivariance in (3.10) (see (2.87)).

Remark 3.5. As a consequence of the commutativity (3.23), the Jacobian matrix $J(\boldsymbol{\lambda}_0, \tau) = (J_{ij})$ is a symmetric *circulant matrix* with entries J_{ij} depending only on $|i - j|$ as

$$J_{ij} = k_l, \quad l = \min\{|i-j|,\ n - |i-j|\}, \tag{3.24}$$

where each k_l is a function of τ for $l = 0, 1, \ldots, n/2$. □

By (3.23), it is possible to construct a matrix that transforms the Jacobian matrix $J = J(\boldsymbol{\lambda}_0, \tau)$ to a block-diagonal form that is common to all τ (Sect. 2.3.4). Indeed, the transformation matrix Q for block-diagonalization in (2.73) is given by[4]

$$Q = \begin{cases} \left[\boldsymbol{q}^{(+,+)}, \boldsymbol{q}^{(-,+)}\right], & \text{for } n = 2, \\ \left[\boldsymbol{q}^{(+,+)}, \boldsymbol{q}^{(-,+)}, \boldsymbol{q}^{(1),1}, \boldsymbol{q}^{(1),2}, \ldots, \boldsymbol{q}^{(n/2-1),1}, \boldsymbol{q}^{(n/2-1),2}\right], & \text{for } n \geq 4, \end{cases} \tag{3.25}$$

where $(+,+)$, $(-,+)$, and (j) for $j = 1, \ldots, n/2 - 1$ are the irreducible representations contained in $T(g)$ of (3.8) with multiplicities 1 in (3.22). Recall that the irreducible representations $(+,-)$ and $(-,-)$ are missing. The column vectors of this matrix Q are expressed as

$$\boldsymbol{q}^{(+,+)} = \frac{1}{\sqrt{n}}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}, \quad \boldsymbol{q}^{(-,+)} = \frac{1}{\sqrt{n}}\begin{bmatrix}\cos\pi\cdot 0\\ \cos\pi\cdot 1\\ \vdots\\ \cos(\pi(n-1))\end{bmatrix} = \frac{1}{\sqrt{n}}\begin{bmatrix}1\\ -1\\ \vdots\\ 1\\ -1\end{bmatrix}, \tag{3.26}$$

$$\boldsymbol{q}^{(j),1} = \sqrt{\frac{2}{n}}\begin{bmatrix}\cos(2\pi j\cdot 0/n)\\ \cos(2\pi j\cdot 1/n)\\ \vdots\\ \cos(2\pi j(n-1)/n)\end{bmatrix}, \quad \boldsymbol{q}^{(j),2} = \sqrt{\frac{2}{n}}\begin{bmatrix}\sin(2\pi j\cdot 0/n)\\ \sin(2\pi j\cdot 1/n)\\ \vdots\\ \sin(2\pi j(n-1)/n)\end{bmatrix},$$
$$j = 1, \ldots, n/2 - 1. \tag{3.27}$$

Since the multiplicities of irreducible representations appearing in (3.25) are all equal to one ($a^\mu = 1$) by (3.22), the block-diagonal form in (2.73) is reduced to a diagonal matrix as

[4] The transformation matrix Q used here is an orthogonal matrix.

$$\overline{J} = Q^{-1} J Q = \mathrm{diag}(e^{(+,+)},\ e^{(-,+)},\ e^{(1)},\ e^{(1)},\ \ldots,\ e^{(n/2-1)},\ e^{(n/2-1)}). \quad (3.28)$$

The diagonal entries, which are the eigenvalues of J, are

$$e^{(+,+)} = e(0), \quad e^{(-,+)} = e(n/2), \quad e^{(j)} = e(j),\ j = 1, \ldots, n/2-1 \quad (3.29)$$

with a function $e(i)$ being defined as

$$e(i) = k_0 + (-1)^i k_{n/2} + 2 \sum_{l=1}^{n/2-1} [\cos(2\pi i l/n)]\, k_l$$

for $i = 0, 1, \ldots, n/2$. It is noteworthy that $e^{(+,+)}$ and $e^{(-,+)}$ are simple eigenvalues and that $e^{(j)}$ ($j = 1, \ldots, n/2-1$) are double eigenvalues that are repeated twice. It should be clear that the column vectors of the matrix Q are the eigenvectors of J.

It will turn out (Sects. 3.4.2 and 3.4.3) that critical points on the flat earth equilibria of uniform population can be classified, according to which eigenvalue vanishes, as

$$\begin{cases} e^{(+,+)} = 0: & \text{limit point of } \tau, \\ e^{(-,+)} = 0: & \text{simple pitchfork bifurcation point}, \\ e^{(j)} = 0: & \text{double bifurcation point}. \end{cases} \quad (3.30)$$

These points have the following characteristics:

- A *limit point* is a critical point on an equilibrium path at which the value of the bifurcation parameter (τ) takes a local maximum or minimum.
- A simple pitchfork bifurcation point with $e^{(-,+)} = 0$ in (3.28) corresponds to the spatial period doubling bifurcation. Its critical eigenvector is given uniquely as the $\mathrm{D}_{n/2}$-symmetric vector $\boldsymbol{q}^{(-,+)}$ in (3.26) with components of alternating signs expressing the bifurcation mode of spatial period doubling.
- A double bifurcation point associated with $e^{(j)} = 0$ for some j in (3.28) corresponds to the bifurcation, in which the spatial period becomes $\hat{n}$ times, where $\hat{n} = n/\gcd(n, j)$. This bifurcation point exists in the racetrack economy with four or more places ($n \geq 4$).

Example 3.2. The use of the diagonalization (3.28) in stability analysis is illustrated for the two-place economy ($n = 2$). The Jacobian matrix for (3.2) evaluated at the flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/2, 1/2)^\top$ takes the form of

$$J(\boldsymbol{\lambda}_0, \tau) = \begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

where the entries a and b are functions in τ given as

$$a = \frac{1}{4}\left(\frac{\partial \omega_1}{\partial \lambda_1} - \frac{\partial \omega_2}{\partial \lambda_1} - 2\omega_1\right)\bigg|_{(\boldsymbol{\lambda},\tau)=(\boldsymbol{\lambda}_0,\tau)},$$

$$b = \frac{1}{4}\left(\frac{\partial \omega_2}{\partial \lambda_1} - \frac{\partial \omega_1}{\partial \lambda_1} - 2\omega_1\right)\bigg|_{(\boldsymbol{\lambda},\tau)=(\boldsymbol{\lambda}_0,\tau)}.$$

The transformation matrix Q for the diagonalization and the transformed Jacobian matrix $\overline{J}(\boldsymbol{\lambda}_0, \tau)$ are

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \overline{J}(\boldsymbol{\lambda}_0, \tau) = Q^{-1} J(\boldsymbol{\lambda}_0, \tau) Q = \begin{bmatrix} -\omega_1 & 0 \\ 0 & \frac{1}{2}\left(\frac{\partial \omega_1}{\partial \lambda_1} - \frac{\partial \omega_2}{\partial \lambda_1}\right) \end{bmatrix}.$$

The first diagonal entry $-\omega_1 = -\omega_1(\boldsymbol{\lambda}_0, \tau)$ of $\overline{J}(\boldsymbol{\lambda}_0, \tau)$ remains negative for all τ. The second diagonal entry $\frac{1}{2}(\frac{\partial \omega_1}{\partial \lambda_1} - \frac{\partial \omega_2}{\partial \lambda_1})$ can possibly become zero for some τ with the critical eigenvector $\boldsymbol{q}^{(-,+)} = (1/\sqrt{2}, -1/\sqrt{2})^\top$, in which case we have a pitchfork bifurcation. □

3.4.2 Use of Equivariant Branching Lemma

Equivariant branching lemma in Sect. 2.4.5 is used to investigate bifurcations from the D_n-symmetric flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/n, \ldots, 1/n)^\top$ in (3.12). The use of equivariant branching lemma is a convenient means to search for bifurcating solutions with certain symmetries, and serves as a preliminary step for the analysis of bifurcation equation (Sect. 3.4.3).

Simple Critical Point

Consider a simple critical point associated with the one-dimensional irreducible representation $(-, +)$, which is contained in the representation (3.8) for the racetrack economy; see (3.22). The irreducible representation $(-, +)$ is defined in (3.19) as

$$T^{(-,+)}(r) = -1, \quad T^{(-,+)}(s) = 1.$$

The general procedure in Sect. 2.4.5 is applied to $\tilde{T} = T^{(-,+)}$ and $\Sigma = \langle r^2, s \rangle = \mathrm{D}_{n/2}$. The fixed-point subspace $\mathrm{Fix}(\Sigma)$ with respect to $T^{(-,+)}$ is certainly a one-dimensional space since $T^{(-,+)}(r^2) = T^{(-,+)}(s) = 1$. Moreover, $\Sigma = \mathrm{D}_{n/2}$ is obviously an isotropy subgroup. Then, by the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5), a bifurcating path with the symmetry of $\mathrm{D}_{n/2}$ exists generically.

Double Critical Point

Consider a double critical point associated with the two-dimensional irreducible representation (j), which is contained in the representation (3.8) for the racetrack economy; see (3.22). The irreducible representation (j) is defined in (3.20) as

$$T^{(j)}(r) = \begin{bmatrix} \cos(2\pi j/n) & -\sin(2\pi j/n) \\ \sin(2\pi j/n) & \cos(2\pi j/n) \end{bmatrix}, \quad T^{(j)}(s) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix},$$

where $1 \leq j \leq n/2-1$. The general procedure in Sect. 2.4.5 is applied to $\tilde{T} = T^{(j)}$ and $\Sigma = \langle r^{\hat{n}}, s \rangle = \mathrm{D}_{n/\hat{n}}$. The fixed-point subspace $\mathrm{Fix}(\Sigma)$ with respect to $T^{(j)}$ is a one-dimensional space spanned by $(1,0)^\top$ since

$$T^{(j)}(r^{\hat{n}}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$T^{(j)}(s) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T^{(j)}(s) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \neq \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{if } w_2 \neq 0.$$

Moreover, $\Sigma = \mathrm{D}_{n/\hat{n}}$ is the isotropy subgroup of $(1,0)^\top$. Then, by the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5), a bifurcating path with the symmetry of $\mathrm{D}_{n/\hat{n}}$ exists generically.

Remark 3.6. All the irreducible representations of the dihedral group D_n are absolutely irreducible. Hence, the technical assumption of absolute irreducibility in the equivariant branching lemma is met. □

3.4.3 *Use of the Bifurcation Equation*

The bifurcation equation explained in Sect. 2.4.3 is used here to investigate the bifurcation from the D_n-symmetric flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/n, \ldots, 1/n)^\top$ in (3.12).

Simple Critical Point

The period doubling distribution-preserving equilibria in (3.13) with the spatial period of $L_{n/2} = 4\pi/n$ are engendered via the bifurcation at a simple critical point, as expounded Proposition 3.2 below.

Proposition 3.2. *At a simple critical point associated with* $e^{(-,+)} = 0$ *in* (3.28)*, we have a pitchfork bifurcation with a bifurcating path with the symmetry of* $\mathrm{D}_{n/2}$*. This bifurcation point is either supercritical or subcritical.*

Proof. The proof is outlined here in accordance with the flow of the bifurcation analysis procedure in Sect. 2.4, whereas a complete proof can be found in Chap. 8 of Ikeda and Murota, 2010 [8].

Using the column vectors $\boldsymbol{q}^{(-,+)}$, $\boldsymbol{q}^{(j),1}$, and $\boldsymbol{q}^{(j),2}$ in (3.26) and (3.27) of Q for block-diagonalization, we consider the expansion[5]

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + w^{(-,+)}\boldsymbol{q}^{(-,+)} + \sum_{j=1}^{n/2-1} (w^{(j),1}\boldsymbol{q}^{(j),1} + w^{(j),2}\boldsymbol{q}^{(j),2}) \tag{3.31}$$

around $\boldsymbol{\lambda}_0 = (1/n, \dots, 1/n)^\top$. Here $w^{(-,+)}$, $w^{(j),1}$, and $w^{(j),2}$ $(j = 1, \dots, n/2-1)$ are coefficients.

For a simple bifurcation point associated with $(-,+)$, the bifurcation equation (2.90) is reduced to a single equation

$$\tilde{F}(w, \tilde{\tau}) = 0 \tag{3.32}$$

in incremental variables $w = w^{(-,+)}$ and $\tilde{\tau} = \tau - \tau_c$, where τ_c is the value of τ at the bifurcation point and $(w, \tilde{\tau}) = (0, 0)$ corresponds to the bifurcation point. By (3.19) we have

$$T^{(-,+)}(r) = -1, \quad T^{(-,+)}(s) = 1, \tag{3.33}$$

and hence the equivariance (2.93) of the bifurcation equation is given by

$$\tilde{F}(-w, \tilde{\tau}) = -\tilde{F}(w, \tilde{\tau}).$$

As a consequence of this, the bifurcation equation takes the form of

$$\tilde{F}(w, \tilde{\tau}) = w[A\,\tilde{\tau} + Bw^2 + (\text{higher order terms})] = 0 \tag{3.34}$$

for some constants A and B in a neighborhood of $(w, \tilde{\tau}) = (0, 0)$. Generically, $A \neq 0$ and $B \neq 0$. The solutions of (3.34) are

$$\begin{cases} w = 0, & \text{flat earth equilibria,} \\ \tilde{\tau} = -\frac{B}{A}w^2 + (\text{higher order terms}), & \text{bifurcated equilibria.} \end{cases}$$

This bifurcation, accordingly, is either a supercritical or subcritical pitchfork depending on the signs of A and B.

From (3.33), we have the symmetry of w as

---

[5]This corresponds to (2.98) with $\boldsymbol{u} = \boldsymbol{\lambda}$, $\boldsymbol{u}_\mathrm{c} = \boldsymbol{\lambda}_0$, $\boldsymbol{w} = w^{(-,+)}\boldsymbol{q}^{(-,+)}$, and $\overline{\boldsymbol{w}} = \sum_{j=1}^{n/2-1}(w^{(j),1}\boldsymbol{q}^{(j),1} + w^{(j),2}\boldsymbol{q}^{(j),2})$. The term $w^{(+,+)}\boldsymbol{q}^{(+,+)}$ is lacking due to the condition of no population growth in (3.4).

$$\Sigma(w) = \langle r^2, s\rangle = \mathrm{D}_{n/2}$$

in the notation of (2.119). Then Proposition 2.7 in Sect. 2.4.4 gives the symmetry of $\boldsymbol{\lambda}$ as

$$\Sigma(\boldsymbol{\lambda}) = \Sigma(w) = \mathrm{D}_{n/2}. \qquad \square$$

Double Critical Point

At a double critical point, the spatial period becomes m times by bifurcation ($3 \leq m < n/2$), as stated in Proposition 3.3 below. Indeed, the spatial period tripling, with $m = 3$, is observed numerically (Sect. 3.6.3).

Proposition 3.3. *At the double critical point with $e^{(j)} = 0$ for some j $(1 \leq j \leq n/2-1)$, we have a bifurcation with $\hat{n}$ bifurcating paths with the symmetry of* $\mathrm{D}_{n/\hat{n}}$, *where $\hat{n} \geq 3$.*

Proof. The proof is outlined here in accordance with the flow of the bifurcation analysis procedure in Sect. 2.4, whereas a complete proof can be found in Chap. 8 of Ikeda and Murota, 2010 [8].

Since the double bifurcation point is associated with the two-dimensional representation (j), the bifurcation equation (2.90) is reduced to

$$\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{\tau}) = \boldsymbol{0}$$

in incremental variables $\boldsymbol{w} = (w_1, w_2)^\top = (w^{(j),1}, w^{(j),2})^\top$ in (3.31) and $\tilde{\tau} = \tau - \tau_{\mathrm{c}}$, where $(\boldsymbol{w}, \tilde{\tau}) = (\boldsymbol{0}, 0)$ corresponds to the bifurcation point. The equivariance of the bifurcation equation is given by

$$T^{(j)}(g)\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{\tau}) = \tilde{\boldsymbol{F}}(T^{(j)}(g)\boldsymbol{w}, \tilde{\tau}), \quad g \in \mathrm{D}_n, \tag{3.35}$$

which is the equivariance (2.93) with $\tilde{T}(g)$ replaced by $T^{(j)}(g)$, whereas $T^{(j)}(g)$ is specified by (3.20):

$$T^{(j)}(r) = \begin{bmatrix} \cos(2\pi j/n) & -\sin(2\pi j/n) \\ \sin(2\pi j/n) & \cos(2\pi j/n) \end{bmatrix}, \quad T^{(j)}(s) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{3.36}$$

The complex coordinates $(z, \overline{z})$ instead of (w_1, w_2) are adopted; that is,

$$z = w_1 + \mathrm{i}w_2, \quad \overline{z} = w_1 - \mathrm{i}w_2, \tag{3.37}$$

where i denotes the imaginary unit and $\overline{\cdot}$ denotes the complex conjugate. Using the components $\tilde{F}_1$ and $\tilde{F}_2$ of $\tilde{\boldsymbol{F}} = (\tilde{F}_1, \tilde{F}_2)^\top$, let us define

$$F(z, \overline{z}, \tilde{\tau}) = \tilde{F}_1(w_1, w_2, \tilde{\tau}) + \mathrm{i}\, \tilde{F}_2(w_1, w_2, \tilde{\tau}). \tag{3.38}$$

Then the bifurcation equation $\tilde{\boldsymbol{F}} = \boldsymbol{0}$ is equivalent to

$$F(z, \overline{z}, \tilde{\tau}) = \overline{F(z, \overline{z}, \tilde{\tau})} = 0. \tag{3.39}$$

The action given in (3.36) on the two-dimensional vector $\boldsymbol{w} = (w_1, w_2)^\top$ can be expressed for the complex variable $z = w_1 + \mathrm{i}w_2$ as

$$r : z \mapsto \omega z, \quad s : z \mapsto \overline{z} \tag{3.40}$$

with $\omega = \exp(\mathrm{i}2\pi j/n)$. The equivariance (3.35) is equivalent to

$$\omega F(z, \overline{z}, \tilde{\tau}) = F(\omega z, \overline{\omega}\, \overline{z}, \tilde{\tau}), \tag{3.41}$$

$$\overline{F(z, \overline{z}, \tilde{\tau})} = F(\overline{z}, z, \tilde{\tau}). \tag{3.42}$$

Let us expand F into a power series in $(z, \overline{z})$:

$$F(z, \overline{z}, \tilde{\tau}) \approx \sum_{a=0} \sum_{b=0} A_{ab}(\tilde{\tau})\, z^a \overline{z}^b \tag{3.43}$$

involving an appropriate number of terms. Since $(z, \overline{z}, \tilde{\tau}) = (0, 0, 0)$ corresponds to the double critical point, we have

$$A_{00}(0) = A_{10}(0) = A_{01}(0) = 0. \tag{3.44}$$

The equivariance (3.41)–(3.42) is expressed for the power series expansion (3.43) as follows. By (3.42) for (3.43), we see that all the coefficients $A_{ab}(\tilde{\tau})$ $(a, b = 0, 1, \ldots)$ are real. For the index (a, b) of the nonvanishing terms in (3.43), the condition (3.41) gives

$$a - b - 1 = p\hat{n}, \quad p \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes the set of integers. Using this relation in (3.43), we obtain the bifurcation equation (3.38) as

$$\begin{aligned} F(z, \overline{z}, \tilde{\tau}) \approx & \sum_{a=0} A_{a+1,a}(\tilde{\tau})\, z^{a+1} \overline{z}^a \\ & + \sum_{p=1} \sum_{a=0} [A_{a+1+p\hat{n},a}(\tilde{\tau})\, z^{a+1+p\hat{n}} \overline{z}^a + A_{a,a-1+p\hat{n}}(\tilde{\tau})\, z^a \overline{z}^{a-1+p\hat{n}}]. \end{aligned} \tag{3.45}$$

Herein, $\hat{n} = n/\mathrm{gcd}(n, j)$ as defined in (3.21), and $A_{a+1,a}(\tilde{\tau})$, $A_{a+1+p\hat{n},a}(\tilde{\tau})$, and $A_{a,a-1+p\hat{n}}(\tilde{\tau})$ are generically distinct from zero (as there is no reason for the vanishing of these terms).

For a bifurcating solution ($z \neq 0$), the bifurcation equation can be rewritten as

$$\frac{1}{z} F(z, \bar{z}, \tilde{\tau}) = 0, \tag{3.46}$$

and the leading part of the equation is expressed as

$$\frac{1}{z} F(z, \bar{z}, \tilde{\tau}) \approx A'_{10}(0)\tilde{\tau} + A_{21}(0)\,|z|^2 + A_{0,\hat{n}-1}\,(0)\,\bar{z}^{\hat{n}}/|z|^2, \tag{3.47}$$

where $A'_{10}(0)$ means the derivative of $A_{10}(\tilde{\tau})$ with respect to $\tilde{\tau}$ evaluated at $\tilde{\tau} = 0$. Since the coefficients in (3.47) are all real, the solution z of (3.46) must satisfy $\bar{z}^{\hat{n}} \in \mathbb{R}$, i.e., $z = |z| \cdot \exp(-\mathrm{i}\pi(k-1)/\hat{n}\,)$ for some integer k with $1 \leq k \leq 2\hat{n}$. For such z, the asymptotic relation between $\tilde{\tau}$ and $|z|$ is determined by the implicit function theorem; see Chap. 8 of Ikeda and Murota, 2010 [8] for more details about this asymptotic form.

Therefore, we can conclude that the solutions of equation (3.39) with (3.45) are given (and exhausted) by

$$\begin{cases} z = 0, & \text{flat earth equilibrium (trivial solution)}, \\ z \text{ with } z^{\hat{n}} \in \mathbb{R}, & \text{bifurcating equilibria}. \end{cases}$$

Thus the use of bifurcation equation engenders more information than the use of equivariant branching lemma (Sect. 3.4.2).

To consider the symmetry of the bifurcating equilibria with $z^{\hat{n}} \in \mathbb{R}$, the solution with $z \in \mathbb{R}$ is chosen as a representative, which corresponds to $\boldsymbol{w} = (w_1, 0)^\top$. From (3.36), we have the symmetry of $\boldsymbol{w}$ as

$$\Sigma(\boldsymbol{w}) = \langle r^{\hat{n}}, s\rangle = \mathrm{D}_{n/\hat{n}}$$

in the notation of (2.119). Then Proposition 2.7 gives the symmetry of $\boldsymbol{\lambda}$ as

$$\Sigma(\boldsymbol{\lambda}) = \Sigma(\boldsymbol{w}) = \mathrm{D}_{n/\hat{n}}.$$

□

Remark 3.7. Propositions 3.2 and 3.3 indicate the dependence of the symmetry of bifurcating solutions on the number n of places in the racetrack economy. For example, $n = 12$ that can encompass bifurcation modes with as many as five different symmetries D_m ($m = 1, 2, 3, 4, 6$) might be a wise choice (Krugman, 1993 [12]). □

3.5 Other Bifurcations

In addition to the bifurcation from the flat earth equilibrium, there exist bifurcations from other equilibria. Important bifurcations are explained below.

3.5.1 Bifurcation from Distribution-Preserving Equilibria

A distribution-preserving equilibrium, other than the flat earth equilibrium, has symmetry expressed by a subgroup of D_n isomorphic to D_m for some m. The further bifurcations from such equilibria can be treated by the same method of group-theoretic bifurcation analysis given in Sect. 3.4 by replacing n with m.

3.5.2 Spatial Period Doubling Bifurcation Cascade

The racetrack economy with $n = 2^k$ places (k is some positive integer) can undergo a cascade of pitchfork bifurcations (Proposition 3.2) leading to the loss of symmetries as

$$\mathrm{D}_n \longrightarrow \mathrm{D}_{n/2} \longrightarrow \mathrm{D}_{n/4} \longrightarrow \cdots \longrightarrow \mathrm{D}_1, \tag{3.48}$$

which is associated with successive change of the spatial period:

$$L = \frac{2\pi}{n} \longrightarrow \frac{2\pi}{n/2} \longrightarrow \frac{2\pi}{n/4} \longrightarrow \cdots \longrightarrow 2\pi.$$

This is called *spatial period doubling bifurcation cascade*.

Recall Fig. 3.2 in Sect. 3.1, which depicts this bifurcation cascade for $n = 8 = 2^3$ places. The cascade in (3.48) is encountered in computational analysis (Sect. 3.6.2).

3.5.3 Transcitical Bifurcation

The racetrack economy has a bifurcation behavior, for which the bifurcating solution $\boldsymbol{\lambda}$ is as symmetric as the critical point (Remark 2.15 in Sect. 2.4.4).

By virtue of the factorized form $(\omega_i - \overline{\omega})\lambda_i$ on the left-hand side of the equilibrium equation (3.11), a *transcritical bifurcation* point (Remark 3.8 below) exists at an intersection of two equilibrium paths: a path with $\omega_i - \overline{\omega} = 0$ and another path with $\lambda_i = 0$ for some i, as shown in Fig. 3.4. The sign of λ_i changes at the transcritical point along the former path with $\omega_i - \overline{\omega} = 0$, whereas the sign of $\omega_i - \overline{\omega}$ changes at the transcritical point along the latter path with $\lambda_i = 0$.

The transcritical bifurcations exist at the points C and E in Fig. 3.5a (cf., Fig. 2.2 in Sect. 2.2.1) for the two-place economy. Fujita, Krugman, and Venables, 1999 [5] restricted their attention to sustainable equilibria with $\omega_i - \overline{\omega} \leq 0$ (cf., (1.24)) and regarded this type of transcritical bifurcation as a kink that connects two half branches. In contrast, as shown in Fig. 3.5b, this point is considered here as a transcritical bifurcation point in light of the group-theoretic viewpoint of this book.

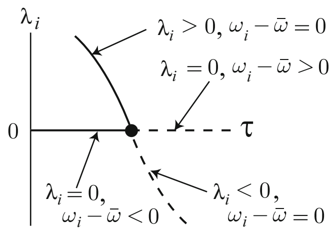


Fig. 3.4 Bifurcation behavior at a transcritical bifurcation point. *Solid curves* mean stable; *dashed curves*, unstable; *black circle*, transcritical bifurcation point

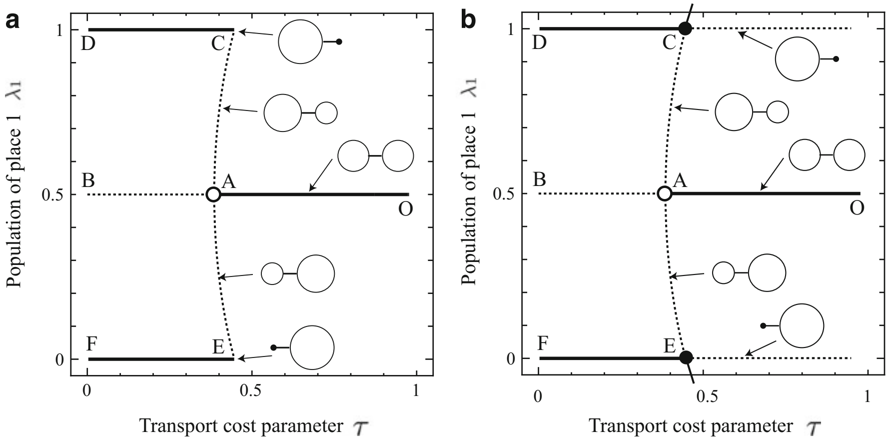


Fig. 3.5 Two kinds of interpretation of the transcritical bifurcations in the two-place economy with the Krugman model computed for specific parameter values of $\mu = 0.4$, $\sigma = 5.0$. (**a**) Kinks at C and E. (**b**) Transcritical bifurcations at C and E. See Sect. 1.5.2 for the definition of the parameters μ and σ; *solid curves* mean stable; *dotted curves*, unstable; *white circle*, pitchfork bifurcation point; *black circle*, transcritical bifurcation point

Along the curve ACD, the population at Place 2 vanishes at the point C via the migration to Place 1.

Remark 3.8. The transcritical bifurcation is investigated for the core–periphery pattern $(\lambda_1, \lambda_2, \tau) = (1, 0, \tau)$ of the two-place economy. For the governing equation $F_i(\boldsymbol{\lambda}, \tau) = (\omega_i(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau))\lambda_i$ for $i = 1, 2$ in (3.2), the Jacobian matrix at $(\lambda_1, \lambda_2, \tau) = (1, 0, \tau)$ is evaluated to

$$J(1, 0, \tau) = \begin{pmatrix} -\omega_1 & -\omega_2 \\ 0 & \omega_2 - \omega_1 \end{pmatrix}.$$

Let τ_c be the value of τ such that $\omega_1(1, 0, \tau_c) = \omega_2(1, 0, \tau_c)$ and put $\boldsymbol{\lambda}_c = (1, 0)^\top$. Then $(\boldsymbol{\lambda}, \tau) = (\boldsymbol{\lambda}_c, \tau_c)$ is a critical point at which

$$J_c = J(\boldsymbol{\lambda}_c, \tau_c) = \begin{pmatrix} -\omega_1(1, 0, \tau_c) & -\omega_1(1, 0, \tau_c) \\ 0 & 0 \end{pmatrix}$$

is singular. For $\boldsymbol{\xi}_1 = (0, 1)^\top$ and $\boldsymbol{\eta}_1 = (1, -1)^\top$ we have

$$\boldsymbol{\xi}_1^\top J_\mathrm{c} = \mathbf{0}^\top, \quad J_\mathrm{c}\boldsymbol{\eta}_1 = \mathbf{0}$$

((2.79) with $M = 1$). We also have

$$\frac{\partial \boldsymbol{F}}{\partial \tau} = \begin{bmatrix} [(\partial\omega_1/\partial\tau)(1-\lambda_1) - (\partial\omega_2/\partial\tau)\lambda_2]\lambda_1 \\ [-(\partial\omega_1/\partial\tau)\lambda_1 + (\partial\omega_2/\partial\tau)(1-\lambda_2)]\lambda_2 \end{bmatrix},$$

which is evaluated at $(\boldsymbol{\lambda}, \tau) = (\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c})$ to

$$\frac{\partial \boldsymbol{F}}{\partial \tau}(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c}) = \mathbf{0}.$$

Hence we have

$$\boldsymbol{\xi}_1^\top \frac{\partial \boldsymbol{F}}{\partial \tau}(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c}) = 0,$$

which means that this critical point is a bifurcation point.[6]

The variables for the bifurcation equations are $w = (\lambda_1 - 1) - \lambda_2$ and $\tilde{\tau} = \tau - \tau_\mathrm{c}$, and the bifurcation equation is given as[7]

$$\tilde{F}(w, \tilde{\tau}) = w[A\,\tilde{\tau} + Cw + (\text{higher order terms})] = 0 \tag{3.49}$$

with some constants A and C. The bifurcation equation (3.49) has two solutions: the trivial solution $w = 0$ and a bifurcating solution $A\,\tilde{\tau} + Cw \approx 0$. Such bifurcation is called a *transcritical bifurcation*. □

3.5.4 Hierarchical Bifurcations

Further bifurcations from bifurcated equilibria can be treated by the group-theoretic bifurcation analysis. For example, in the racetrack economy with four places ($n = 4$), a possible course of hierarchical bifurcations is presented in Fig. 3.6. From the flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/4, 1/4, 1/4, 1/4)^\top$ shown at the left, population distribution patterns of various kinds are engendered via hierarchical bifurcations comprising pitchfork bifurcation (period doubling), transcritical bifurcation, and bifurcation at a double critical point.

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[6]By (2.76) of Ikeda and Murota, 2010 [8], a simple critical point is classified as a bifurcation point if $\boldsymbol{\xi}_1^\top(\partial \boldsymbol{F}/\partial\tau)_\mathrm{c} = 0$, and a limit point of τ if $\boldsymbol{\xi}_1^\top(\partial \boldsymbol{F}/\partial\tau)_\mathrm{c} \neq 0$.

[7]In the bifurcation equation (3.34) of pitchfork bifurcation, the term Cw in (3.49) is replaced by Bw^2.

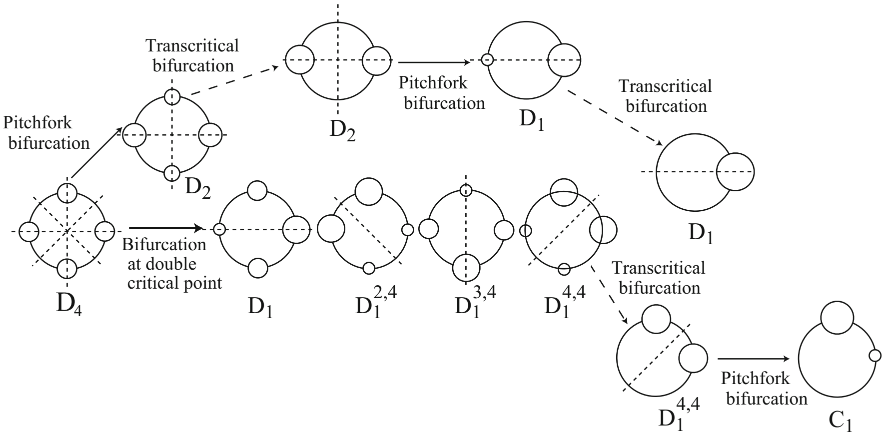

Fig. 3.6 Possible course of hierarchical bifurcations of the racetrack economy with four places ($n = 4$). *Thick solid arrow* means bifurcation at a double critical point; *thin solid arrow*, pitchfork bifurcation; *dashed arrow*, transcritical bifurcation; $\mathrm{C}_1 = \{e\}$

3.6 Numerical Analysis of Racetrack Economy

Agglomerations in the racetrack economy with the core–periphery model are investigated for $n = 4$, 6, and 16 places by numerical bifurcation analysis.

3.6.1 Computational Bifurcation Analysis Procedure

A systematic numerical procedure to obtain equilibrium paths of the core–periphery model by solving the set of nonlinear equations (1.9)–(1.12) is presented here. First, an elaborate search for all the equilibrium paths is conducted using the following procedure:

1. Obtain all distribution-preserving equilibria (Sect. 3.2.3).
2. Investigate the stability of these distribution-preserving equilibria and obtain the critical points on these equilibria. In the eigenanalysis of the Jacobian matrix, the formulas in (3.29) for the eigenvalues are used for the flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/n, \ldots, 1/n)^\top$, whereas the numerical eigenanalysis is conducted for other distribution-preserving equilibria.
3. Obtain bifurcating equilibria branching from all these distribution-preserving equilibria. Investigate the stability of these bifurcating equilibria and obtain the critical points on the equilibria. Equilibrium points with negative population $\lambda_i < 0$ for some i must be discarded as meaningless ones.
4. Repeat Step 3 to exhaust all equilibrium paths.

Next, among these equilibrium paths, we select stable ones that are to be encountered when the transport cost parameter τ is decreased from 1 to 0.

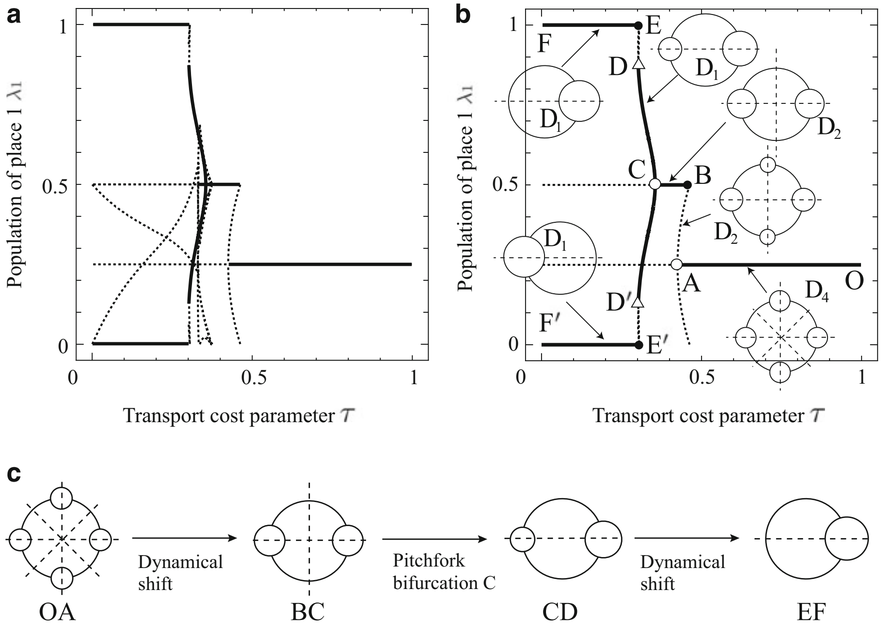

Fig. 3.7 Equilibrium paths for the racetrack economy with four places ($n = 4$) and predicted shifts of stable equilibria that are expected to be followed in association with the decrease of transport cost. (**a**) All equilibrium paths. (**b**) Stable distribution-preserving paths and bifurcated paths connecting them. (**c**) Predicted shifts of stable equilibria as τ decreases. *Solid curves* mean stable; *dotted curves*, unstable; *white circle*, pitchfork bifurcation point; *open triangle*, limit point; *black circle*, transcritical bifurcation point

In our computations in Sects. 3.6.2 and 3.6.3, the elasticity of substitution was set as $\sigma = 10.0$ and the expenditure share was set as $\mu = 0.4$. These parameter values satisfy the no-black-hole condition in (1.13) with $(\sigma - 1)/\sigma = 0.9 > \mu = 0.4$.

3.6.2 Spatial Period Doubling Bifurcation Cascade

The occurrence of a spatial period doubling cascade and other bifurcations is demonstrated for the racetrack economy with $n = 4$ and 16 places.

Figure 3.7a shows λ_1 versus τ curves of equilibrium for $n = 4$, which are apparently complicated. Stable equilibria are shown by the solid curves and unstable ones by the dotted curves. To support the economic interpretation, among such complicated paths, stable distribution-preserving paths and bifurcated paths connecting them are chosen as those most likely to occur (Fig. 3.7b).

Distribution-preserving equilibrium paths with D_m-symmetries ($m = 1, 2, 4$) exist at the horizontal lines at $\lambda_1 = 0$, 1/4, 1/2, and 1, and several bifurcated paths connect these distribution-preserving paths. Stable parts (shown as solid lines) of these distribution-preserving equilibria are

- OA: flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/4, 1/4, 1/4, 1/4)^\top$ (D_4-symmetry),
- BC: period doubling equilibrium $\boldsymbol{\lambda} = (1/2, 0, 1/2, 0)^\top$ (D_2-symmetry),
- EF: concentrated equilibrium $\boldsymbol{\lambda} = (1, 0, 0, 0)^\top$ (D_1-symmetry), and
- E′F′: another concentrated equilibrium $\boldsymbol{\lambda} = (0, 0, 1, 0)^\top$ (D_1-symmetry).

Note that CEF and CE′F′ are symmetric counterparts with the same economic meaning.

Bifurcation points on the paths of the stable distribution-preserving equilibria are pitchfork and transcritical points. Symmetries of the system are reduced at pitchfork bifurcation points A and C (denoted as *white circle*):

- At point A, we encounter a period doubling bifurcation associated with the loss of symmetry: $D_4 \longrightarrow D_2$.
- At point C, we encounter another period doubling bifurcation associated with the loss of symmetry: $D_2 \longrightarrow D_1$.

Symmetries are preserved at the transcritical bifurcation points B, E, and E′ denoted as *black circle* (Fig. 3.4 in Sect. 3.5.3).

Among the bifurcated paths, the path CD was found to be stable, as was its symmetric counterpart CD′. The stable path CD became unstable at the limit (minimal) point of τ at D denoted by *open triangle*. Let us specifically examine stable paths obtained herein. In association with the decrease of τ, we predict a possible course of the accumulation of population that traces four stable stages OA, BC, CD, and EF, as presented in Fig. 3.7c. Dynamical shifts[8] are assumed between OA and BC and between CD and EF. From the flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/4, 1/4, 1/4, 1/4)^\top$, via bifurcations and dynamical shifts, a complete concentration $\boldsymbol{\lambda} = (1, 0, 0, 0)^\top$ is realized. A spatial period doubling cascade associated with the loss of symmetry $D_4 \longrightarrow D_2 \longrightarrow D_1$ occurs, as explained in Sect. 3.5.2.

Similarly, the racetrack economy with 16 places displays, as shown in Fig. 3.8, the spatial period doubling cascade associated with the loss of symmetry $D_{16} \longrightarrow D_8 \longrightarrow D_4 \longrightarrow D_2 \longrightarrow D_1$.

[8]When a stable equilibrium path becomes unstable at a critical point where a stable bifurcated path does not exist, the stable path often shifts dynamically to another stable path. This is called *dynamical shift* in this book.

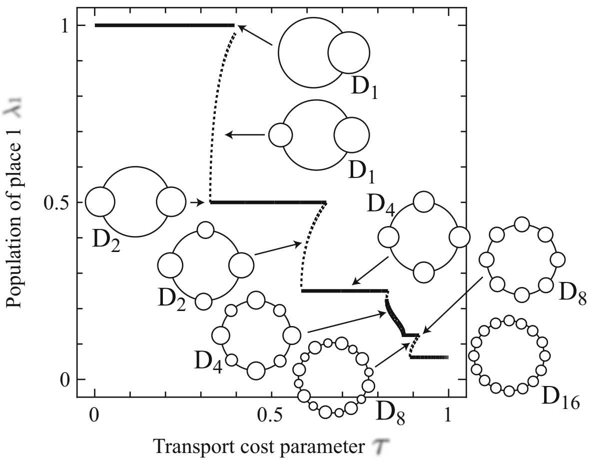

Fig. 3.8 Stable distribution-preserving paths and bifurcated paths connecting them in the racetrack economy with 16 places ($n = 16$) that are expected to be followed in association with the decrease of transport cost. *Solid curves* mean stable; *dotted curves*, unstable

3.6.3 Spatial Period Multiplying Bifurcations

Period tripling,[9] as well as doubling, was observed in the racetrack economy with six places ($n = 6$). Among the equilibrium paths shown in Fig. 3.9a, stable paths and some paths connecting them were chosen, as shown in Fig. 3.9b.

Distribution-preserving equilibria, for example, are

- $\boldsymbol{\lambda}_0 = (1/6, \dots, 1/6)^\top$: D_6-symmetric flat earth equilibrium,
- $\boldsymbol{\lambda} = (1/3, 0, 1/3, 0, 1/3, 0)^\top$: D_3-symmetric period doubling equilibrium,
- $\boldsymbol{\lambda} = (1/2, 0, 0, 1/2, 0, 0)^\top$: D_2-symmetric equilibrium,
- $\boldsymbol{\lambda} = (1/2, 0, 1/2, 0, 0, 0)^\top$: $D_1^{5,6}$-symmetric twin peaks equilibrium, and
- $\boldsymbol{\lambda} = (1, 0, 0, 0, 0, 0)^\top$: D_1-symmetric concentrated equilibrium.

We observed the following:

- period doubling bifurcations associated with the loss of symmetry: $D_6 \longrightarrow D_3$ and $D_2 \longrightarrow D_1$;
- period tripling bifurcations associated with the loss of symmetry: $D_6 \longrightarrow D_2$ and $D_3 \longrightarrow D_1$,

in accordance, respectively, with Propositions 3.2 and 3.3 in Sect. 3.4.3. This results from the fact that $n = 6$ has two divisors: 2 and 3.

Predicted shifts of stable equilibria occurring in association with the decrease of τ are presented in Fig. 3.9c:

- The D_6-symmetric state might dynamically shift into either the D_2- or D_3-symmetric state.

[9] *Period tripling* here means a special case with $m = 3$ of the bifurcation in (3.14) in Sect. 3.2.3, in which the spatial period becomes m times.

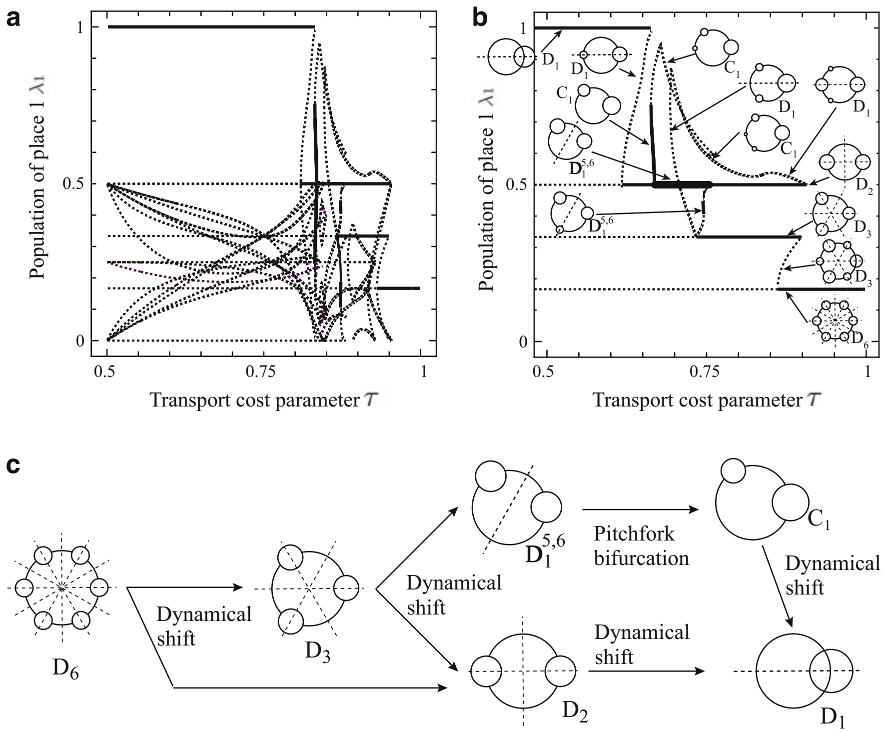

Fig. 3.9 Equilibrium paths for the racetrack economy with six places ($n = 6$). (**a**) All equilibrium paths. (**b**) Stable distribution-preserving paths and bifurcated paths connecting them. (**c**) Predicted shifts of stable equilibria as τ decreases. *Solid curves* mean stable; *dotted curves*, unstable; $C_1 = \{e\}$

- The D_3-symmetric state might dynamically shift into the D_2- or D_1-symmetric state.

The mixed occurrence of the period doubling and tripling led to the more complicated predicted shifts in the racetrack economy with six places ($n = 6$) than those with four places portrayed in Fig. 3.7b.

3.7 Summary

- The dihedral group has been presented as the symmetry group of the racetrack economy.
- Bifurcation mechanism of racetrack economy has been elucidated.
- Spatial period doubling bifurcation cascade and period tripling bifurcation have been advanced as important agglomeration behaviors.

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Part II
Theory of Economic Agglomeration on Hexagonal Lattice

Chapter 4
Introduction to Economic Agglomeration on Hexagonal Lattice

Abstract At the beginning of Part II, several fundamental issues are presented and major findings of Part II are previewed. The equilibrium equation of variants of core–periphery models and the framework for theoretical treatment of the economic model on the hexagonal lattice are given. As a summary of the theoretical results in Chaps. 5–9, a list of bifurcating hexagonal distributions and the associated sizes of the hexagonal lattice are provided. The existence of hexagonal distributions is demonstrated by numerical bifurcation analysis for a specific core–periphery model. These distributions are the ones envisaged by central place theory and also envisaged to emerge by Krugman, 1996 for a core–periphery model in two dimensions. The missing link between central place theory and new economic geography has thus been discovered.

Keywords Agglomeration of population • Bifurcation • Christaller's hexagonal distributions • Core–periphery model • Economic agglomeration • Group-theoretic bifurcation analysis • Hexagonal lattice • Lösch's hexagons

4.1 Introduction

Let us now shift our attention from introductory issues in Part I to the main objective of this book: the search for hexagonal distributions of Christaller and Lösch based on microeconomic structure of core–periphery models.[1] See Fig. 4.1 for Christaller's famous hexagonal distributions called the $k = 3$ system, the $k = 4$ system, and the $k = 7$ system.

In Chap. 3 in Part I, bifurcation in the racetrack economy with Krugman's core–periphery model was studied and several agglomeration patterns that emerge

[1] Hexagonal distributions of Christaller and Lösch (Christaller, 1933 [4] and Lösch, 1940 [10]) are introduced in Sect. 1.2 and Krugman's new economic geographical model is given in Sect. 1.5.

K. Ikeda and K. Murota, *Bifurcation Theory for Hexagonal Agglomeration in Economic Geography*, DOI 10.1007/978-4-431-54258-2_4,

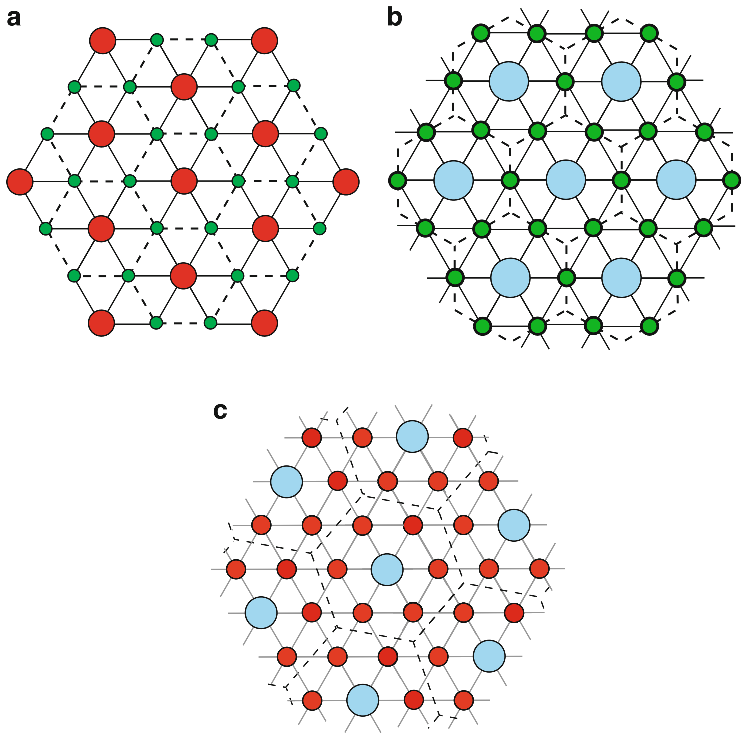

Fig. 4.1 Hexagonal distributions on the hexagonal lattice. The *larger circles* represent the first-level centers and the *smaller ones* represent the second-level centers; the *dashed lines* denote hexagonal market areas. (**a**) Christaller's $k = 3$ system. (**b**) Christaller's $k = 4$ system. (**c**) Christaller's $k = 7$ system

by bifurcations were presented. Krugman, 1996, p. 91 [9] regarded the racetrack economy as one-dimensional and inferred its extendibility to a two-dimensional economy to engender hexagonal distributions:

> I have demonstrated the emergence of a regular lattice only for a one-dimensional economy, but I have no doubt that a better mathematician could show that a system of hexagonal market areas will emerge in two dimensions.

The need to extend core–periphery models to two dimensions was thus acknowledged.[2]

In order to arrive at hexagonal distributions, an adequate two-dimensional space is established to express the flat earth, a completely homogeneous infinite

[2]Neary, 2001, p. 551 [11] stated "Perhaps it will prove possible to extend the Dixit–Stiglitz approach to a two-dimensional plain." Stelder, 2005 [14] conducted a simulation of agglomeration for cities in Europe using a grid of points. Barker, 2012 [2] extended the racetrack geometry to two dimensions and conducted a simulation and compared the results with real cities.

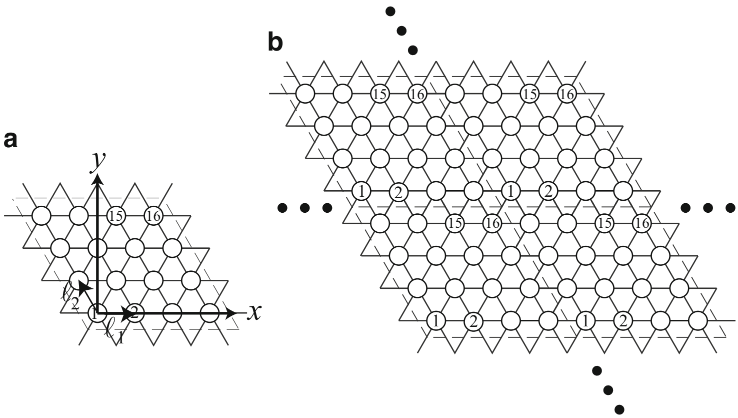

Fig. 4.2 A system of places on the 4×4 hexagonal lattice with periodic boundaries. (**a**) 4×4 hexagonal lattice. (**b**) Periodically repeated 4×4 hexagonal lattice

two-dimensional land surface in central place theory (Sect. 1.2). Since finite degrees-of-freedom are used in the analysis of core–periphery models, an $n \times n$ finite hexagonal lattice[3] with periodic boundaries is introduced as a discretized finite counterpart of the flat earth. Nodes on this lattice represent uniformly spread places of economic activities. These places are connected by roads of the same length d forming a regular-triangular mesh (see Fig. 4.2 for an example of $n = 4$), and goods are transported along these roads.

While a theoretical treatment of this hexagonal lattice is given in Chaps. 5 and 7, the aim of this chapter is to introduce necessary backgrounds of variants of core–periphery models and to preview the development of Part II. The equivariance of the equilibrium equation of these models on the hexagonal lattice is proved as a vital prerequisite for the group-theoretic bifurcation analysis in Chaps. 8 and 9. Theoretical results that are developed in Chaps. 5–9 are vital in the search for hexagonal distributions.

Bifurcating hexagonal distributions are demonstrated, for example, for the 3×3 hexagonal lattice. Figure 4.3a depicts equilibrium paths (the maximum population $\lambda_{\max}$ versus the transport cost parameter τ curves) of this lattice, where $\lambda_{\max} = \max(\lambda_1, \dots, \lambda_N)$ ($N = 3^2$). Two bifurcated solutions, i.e., ACD and AE, branch from the flat earth equilibrium (uniform population distribution), and a pattern of interest can be seen for ACD. The left side of Fig. 4.3b depicts the typical population distribution on ACD, where the area of a circle is proportional to the population. This distribution is repeated spatially in the right side of Fig. 4.3b, which is nothing but the hexagonal distribution of Christaller's $k = 3$ system in Fig 4.1a. Hexagonal distributions of Christaller and Lösch are observed in numerical

[3]The term of *hexagonal lattice* is commonly used in many fields of mathematical sciences, although it is also called *regular-triangular lattice*.

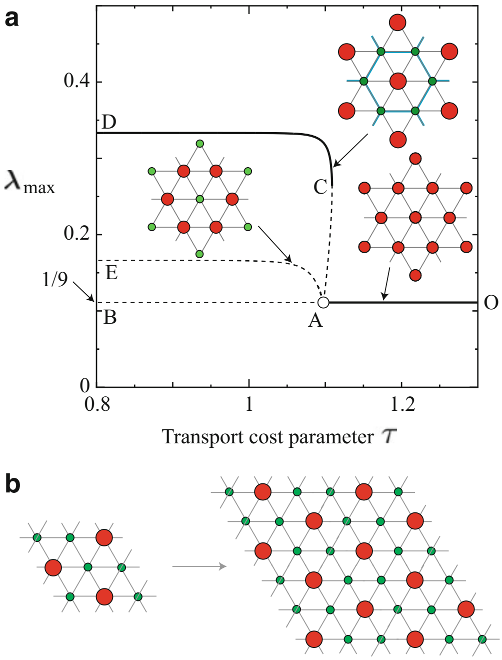

Fig. 4.3 Emergence of hexagonal population distributions on the 3×3 hexagonal lattice. (**a**) Equilibrium paths. (**b**) Bifurcating pattern on ACD. *Solid curves* mean stable equilibria; *dashed curves*, unstable equilibria; λ_{max}, the maximum population among the places; the area of each circle is proportional to the population size at that place

examples in Sect. 4.5 to demonstrate the relevance of theoretical predictions in Chaps. 5–9.

This chapter is organized as follows. The equilibrium equation of core–periphery models is introduced in Sect. 4.2. A framework for theoretical treatment of an economic model on the hexagonal lattice is given in Sect. 4.3. Theoretical predictions by group-theoretic analysis in the following chapters are previewed in Sect. 4.4. Numerical examples of economic agglomeration on the hexagonal lattice are presented in Sect. 4.5.

4.2 Core–Periphery Models

Variants of Krugman's original core–periphery model (Sect. 1.5) are introduced. It is emphasized that the methodology presented here is independent of specific models and is valid for all models insofar as they are consistent with the symmetry of the hexagonal lattice.

4.2.1 Equilibrium Equation

Core–periphery models (Sect. 1.5) for N places are described by the adjustment dynamics

$$\frac{d\lambda_i}{dt} = F_i(\boldsymbol{\lambda}, \tau), \quad i = 1, \ldots, N, \tag{4.1}$$

and the equilibrium equation

$$F_i(\boldsymbol{\lambda}, \tau) = 0, \quad i = 1, \ldots, N, \tag{4.2}$$

in which $\boldsymbol{\lambda} = (\lambda, \ldots, \lambda_N)^\top$ is the vector expressing distribution of mobile population, and τ is the transport cost parameter. We assume the normalization (1.6):

$$\sum_{i=1}^{N} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \ldots, N. \tag{4.3}$$

As a concrete form of $F_i(\boldsymbol{\lambda}, \tau)$ above, Krugman's original model uses (1.20):

$$F_i(\boldsymbol{\lambda}, \tau) = (\omega_i(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau))\lambda_i, \quad i = 1, \ldots, N, \tag{4.4}$$

where $\omega_i(\boldsymbol{\lambda}, \tau)$ is the real wage at the ith place determined implicitly as (1.14) from the market equilibrium, and $\overline{\omega}(\boldsymbol{\lambda}, \tau)$ is the average real wage defined by

$$\overline{\omega}(\boldsymbol{\lambda}, \tau) = \sum_{i=1}^{N} \lambda_i \omega_i(\boldsymbol{\lambda}, \tau). \tag{4.5}$$

There are variants of core–periphery models.[4] In numerical examples in Part II, we adopt a core–periphery model that is formulated using the *logit choice function* in terms of the *indirect utility function* $v_i(\boldsymbol{\lambda}, \tau)$ at the ith place ($i = 1, \ldots, N$). The equilibrium equation of this model is given by[5]

$$F_i(\boldsymbol{\lambda}, \tau) = \frac{\exp[\theta v_i(\boldsymbol{\lambda}, \tau)]}{\sum_{j=1}^{N} \exp[\theta v_j(\boldsymbol{\lambda}, \tau)]} - \lambda_i, \quad i = 1, \ldots, N, \tag{4.6}$$

where $\theta \in (0, \infty)$ is a parameter denoting the inverse of variance of the idiosyncratic tastes. The function (4.6) guarantees nonnegativity (positivity) of λ_i

[4]For overviews of core–periphery models, see Brakman, Garretsen, and van Marrewijk, 2001 [3]; Baldwin et al., 2003 [1]; and Combes, Mayer, and Thisse, 2008 [5].

[5]This model is developed on the basis of Forslid and Ottaviano, 2003 [6]. For details of this model, see Ikeda et al., 2010 [7] and Ikeda, Murota, and Akamatsu, 2012 [8]. See, e.g., Sandholm, 2010 [13] for the logit choice function.

at an equilibrium, which is particularly convenient in numerical computations. In contrast, such nonnegativity is not guaranteed by the function (4.4) of the Krugman model (Remark 1.1 in Sect. 1.5.3).

For general $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = (F_i(\boldsymbol{\lambda}, \tau) \mid i = 1, \dots, N)$, the Jacobian matrix is an $N \times N$ matrix defined as

$$J = J(\boldsymbol{\lambda}, \tau) = (J_{ij}) = \left(\frac{\partial F_i}{\partial \lambda_j} \right). \tag{4.7}$$

The modeling by (4.6) is used in the analysis of the 3×3 hexagonal lattice in Fig. 4.3 and in numerical examples in Part II.

Remark 4.1. The normalization (4.3) is also assumed for (4.6), and the dynamics (4.1) with (4.6) is defined on this simplex. This constraint is indeed inherent in the dynamics since the sum of (4.1) over $i = 1, \dots, N$ with (4.6) yields

$$\frac{d}{dt} \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} F_i(\boldsymbol{\lambda}, \tau) = 1 - \sum_{i=1}^{N} \lambda_i. \qquad \square$$

4.2.2 *Iceberg Transport Technology*

Let us recall the iceberg transport technology (Sect. 1.5.1), which means that if a manufacturing variety produced at place i is shipped to another place j $(\neq i)$, only a fraction $1/T_{ij}$ (< 1) arrives and the rest melts away en route, where T_{ij} (> 1) is a parameter that indirectly denotes the transport cost from place i to place j. We have $T_{ii} = 1$ $(i = 1, \dots, N)$.

Compatibly with the geometry of the hexagonal lattice, the transport cost is defined hereafter as

$$T_{ij} = T_{ij}(\tau) = \exp(\tau D_{ij}), \quad i, j = 1, \dots, N, \tag{4.8}$$

where D_{ij} represents the shortest distance between places i and j along the grid of the hexagonal lattice and the transport cost parameter τ stays in the range $\tau > 0$. Note that the limit of $\tau \to 0$ corresponds to the state of no transport cost, and the limit of $\tau \to \infty$ corresponds to the state of infinite transport cost.

4.3 Framework of Group-Theoretic Bifurcation Analysis

In Chaps. 5–9 of Part II, the group-theoretic bifurcation analysis of hexagonal distributions of Christaller and Lösch is conducted using the mathematical tools presented in Chap. 2. The framework for this theoretical treatment is introduced in Sects. 4.3.1 and 4.3.2, followed by a succinct description of the bifurcation analysis procedure in Sect. 4.3.3. Predictions by the theoretical analysis are previewed in Sect. 4.4.

4.3.1 Hexagonal Lattice and Symmetry

Let us consider an $n \times n$ finite hexagonal lattice with periodic boundaries (see Fig. 4.2 for an example of $n = 4$) spanned by oblique basis vectors $\boldsymbol{\ell}_1 = d(1, 0)^\top$ and $\boldsymbol{\ell}_2 = d(-1/2, \sqrt{3}/2)^\top$ with some positive constant d. This lattice represents uniformly distributed $n \times n$ discrete places that are connected by roads of the same length d forming a regular-triangular mesh.

The symmetry of the $n \times n$ hexagonal lattice is described by the invariance with respect to the following four transformations:

r: counterclockwise rotation about the origin at an angle of $\pi/3$,
s: reflection $y \mapsto -y$,
p_1: periodic translation along the $\boldsymbol{\ell}_1$-axis (i.e., the x-axis), and
p_2: periodic translation along the $\boldsymbol{\ell}_2$-axis.

We consider the group generated by these four transformations

$$G = \langle r, s, p_1, p_2 \rangle, \tag{4.9}$$

with fundamental relations[6]

$$\begin{aligned} &r^6 = s^2 = (rs)^2 = p_1{}^n = p_2{}^n = e, \qquad p_2 p_1 = p_1 p_2, \\ &rp_1 = p_1 p_2 r, \quad rp_2 = p_1^{-1} r, \quad sp_1 = p_1 s, \quad sp_2 = p_1^{-1} p_2^{-1} s, \end{aligned} \tag{4.10}$$

where e is the identity element. Each element of G can be represented uniquely as $s^l r^m p_1{}^i p_2{}^j$ with $l \in \{0, 1\}$, $m \in \{0, 1, \ldots, 5\}$, and $i, j \in \{0, 1, \ldots, n-1\}$.

4.3.2 Equivariance of Equilibrium Equation

We have the following proposition for the equivariance of core–periphery models on the $n \times n$ hexagonal lattice ($N = n^2$). It is again emphasized that the methodology

[6]These relations can be derived based on geometrical consideration; for an algebraic understanding of these relations, see (7.11)–(7.14) in Sect. 7.3. Note that $\langle r, s \rangle$ is isomorphic to the dihedral group D_6 and $\langle p_1, p_2 \rangle$ is isomorphic to the direct product $\mathbb{Z}_n \times \mathbb{Z}_n$ of two cyclic groups $\mathbb{Z}_n$ (denoted as C_n in Sect. 2.3). The group G is in fact the semidirect product of D_6 by $\mathbb{Z}_n \times \mathbb{Z}_n$, i.e., $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$; see Sect. 2.3.1 for this concept and notation. The structure of this group is discussed in Sect. 6.2, and its irreducible representations in Sect. 6.3.

presented in Part II is independent of specific models, so far as they are consistent with the symmetry of the hexagonal lattice.

Proposition 4.1. *The Krugman model with* (4.4) *and the model with* (4.6) *for the hexagonal lattice are equivariant to* $G = \langle r, s, p_1, p_2 \rangle$ *in* (4.9)*, i.e.,*

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \tau), \quad g \in G \tag{4.11}$$

for some N*-dimensional permutation representation*[7] $T(g)$ *of* G.

Proof. The proof of this proposition is essentially the same as the proof of Proposition 3.1 in Sect. 3.2.2 for the racetrack economy, but it is repeated here in view of its importance. Each element g of G acts as a permutation of place numbers $(1, \ldots, N)$, and the action of $g \in G$ is expressed as $g : i \mapsto i^*$. In the Krugman model, we have $\omega_i(T(g)\boldsymbol{\lambda}, \tau) = \omega_{i^*}(\boldsymbol{\lambda}, \tau)$ because of the form of the transport cost in (4.8); we also have $\overline{\omega}(T(g)\boldsymbol{\lambda}, \tau) = \overline{\omega}(\boldsymbol{\lambda}, \tau)$ by (4.5). Therefore, we have

$$F_i(T(g)\boldsymbol{\lambda}, \tau) = (\omega_{i^*}(\boldsymbol{\lambda}, \tau) - \overline{\omega}(\boldsymbol{\lambda}, \tau))\lambda_{i^*} = F_{i^*}(\boldsymbol{\lambda}, \tau)$$

for the function F_i in (4.4). This proves the equivariance (4.11) for the Krugman model. The proof for the other model with (4.6) can be conducted in a similar manner on the basis of the symmetry $v_i(T(g)\boldsymbol{\lambda}, \tau) = v_{i^*}(\boldsymbol{\lambda}, \tau)$ of the indirect utility functions. □

Let us note that, for the hexagonal lattice, the *flat earth equilibrium*

$$\boldsymbol{\lambda}_0 = (1/N, \ldots, 1/N)^\top \tag{4.12}$$

satisfies the equilibrium equation (4.2) for any τ both for (4.4) and (4.6). This equilibrium is invariant to $G = \langle r, s, p_1, p_2 \rangle$.

4.3.3 Bifurcation Analysis Procedure

The theoretical analysis of a mathematical model of an economy on the hexagonal lattice proceeds as follows.

First, the structures of the group $G = \langle r, s, p_1, p_2 \rangle$ in (4.9) and the N-dimensional permutation representation $T(g)$ in Proposition 4.1 are analyzed.

- A list of all irreducible representations μ is obtained (Sects. 6.3 and 6.4). The irreducible representations vary with the lattice size n.
- The concrete form of the N-dimensional permutation representation $T(g)$ for the $n \times n$ hexagonal lattice is determined (Sect. 7.3).

[7]The concrete form of $T(g)$ is given in Sects. 7.2 and 7.3.

- The multiplicity a^μ of irreducible representation μ in $T(g)$ is determined for each μ (Sect. 7.4). Only those μ with $a^\mu \geq 1$ are to be considered in the subsequent analysis.

Next, geometric properties of the hexagonal distributions of interest are analyzed.

- Subgroups G' of G expressing the symmetry of the hexagonal distributions of Christaller and Lösch are determined (Sect. 5.5.2).
- Pertinent sizes n of the underlying hexagonal lattice that can possibly engender the hexagonal distributions of Christaller and Lösch are determined (Sect. 5.4).

For each μ with $a^\mu \geq 1$, bifurcation from a critical point associated with μ is investigated by using the analysis procedure under group symmetry (Sect. 2.4). Two different methods of analysis are employed.

- The equivariant branching lemma (Sect. 2.4.5) is applied to the bifurcation equation associated with μ to determine the existence of bifurcating solutions with a specified symmetry G' (Chap. 8). The concrete form of the bifurcation equation need not be derived. This analysis is purely algebraic or group-theoretic, focusing on the symmetry of solutions without solving the bifurcation equation.
- The bifurcation equation is obtained in the form of power series expansions and is solved asymptotically (Chap. 9). This method demands more calculations, treating nonlinear terms directly, but is more informative, giving asymptotic forms of the bifurcating solutions, as well as their existence.

In principle, the latter method, explicitly treating the bifurcation equation, is capable of obtaining all bifurcating solutions. This is not the case with the former method based on the equivariant branching lemma, which gives a sufficient condition to ensure bifurcating solutions with a specified symmetry.

4.4 Theoretical Predictions by Group-Theoretic Analysis

Important theoretical predictions obtained in the following chapters are previewed. These predictions are put to use in the computational analysis in Sect. 4.5.

4.4.1 Christaller's Hexagonal Distributions

In Chap. 5, compatibility of the lattice size n with hexagonal distributions is studied. That is, pertinent sizes n that would engender a hexagonal distribution of interest are given. For example, Christaller's distributions (Fig. 4.1) are shown to emerge for the lattice size

$$n = \begin{cases} 3m & \text{for Christaller's } k = 3 \text{ system,} \\ 2m & \text{for Christaller's } k = 4 \text{ system,} \\ 7m & \text{for Christaller's } k = 7 \text{ system} \end{cases} \quad (4.13)$$

($m = 1, 2, \ldots$). In agreement with (4.13), the lattice size was chosen as $n = 3$ in the numerical bifurcation analysis in Fig. 4.3a in search of the $k = 3$ system.

The subgroups expressing the symmetry of hexagonal distributions of interest are obtained by simple geometrical observations. For example, the subgroups for Christaller's distributions are obtained as

$$\begin{cases} \langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle & \text{for Christaller's } k = 3 \text{ system,} \\ \langle r, s, p_1^2, p_2^2 \rangle & \text{for Christaller's } k = 4 \text{ system,} \\ \langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle & \text{for Christaller's } k = 7 \text{ system.} \end{cases} \quad (4.14)$$

In Chap. 6, the irreducible representations μ of the group $G = \langle r, s, p_1, p_2 \rangle$ in (4.9) are derived, and the matrix representation $T(g)$ that appeared in the equivariance (4.11) for the hexagonal lattice is obtained in Chap. 7.

In Chaps. 8 and 9, the existence of the hexagonal distributions is verified along with the relevant irreducible representations μ of $G = \langle r, s, p_1, p_2 \rangle$, where the dimension of the irreducible representation μ equals the multiplicity M of the associated bifurcation point. Christaller's hexagonal distributions (Fig. 4.1) emerge from a bifurcation point of multiplicity M given as

$$M = \begin{cases} 2 & \text{for Christaller's } k = 3 \text{ system,} \\ 3 & \text{for Christaller's } k = 4 \text{ system,} \\ 12 & \text{for Christaller's } k = 7 \text{ system.} \end{cases} \quad (4.15)$$

It is noteworthy that Christaller's $k = 7$ system arises from a bifurcation point of multiplicity as high as $M = 12$. Recall that the bifurcating solution for the $k = 3$ system in the numerical results in Fig. 4.3a emerged from a bifurcation point of multiplicity $M = 2$, in agreement with (4.15).

4.4.2 Lösch's Hexagonal Distributions

Theoretical predictions of Lösch's hexagons are presented similarly in Chaps. 5–9. The normalized spatial period L/d in (1.2) will be given a geometrical interpretation in Sect. 5.2.2. To sum up, we have

$$\frac{L}{d} = \sqrt{D} \quad (4.16)$$

Table 4.1 Predictions for Lösch's ten smallest hexagons

Spatial period L/d	D	(α, β)	Subgroup	Compatible size n (multiple of)	M
$\sqrt{3}$	3	$(2, 1)$	$\langle r, s, p_1^2 p_2, p_1^{-1} p_2\rangle$	3	2
$\sqrt{4}$	4	$(2, 0)$	$\langle r, s, p_1^2, p_2^2\rangle$	2	3
$\sqrt{7}$	7	$(3, 1)$	$\langle r, p_1^3 p_2, p_1^{-1} p_2^2\rangle$	7	12
$\sqrt{9}$	9	$(3, 0)$	$\langle r, s, p_1^3, p_2^3\rangle$	3	6
$\sqrt{12}$	12	$(4, 2)$	$\langle r, s, p_1^4 p_2^2, p_1^{-2} p_2^2\rangle$	6	6
$\sqrt{13}$	13	$(4, 1)$	$\langle r, p_1^4 p_2, p_1^{-1} p_2^3\rangle$	13	12
$\sqrt{16}$	16	$(4, 0)$	$\langle r, s, p_1^4, p_2^4\rangle$	4	6
$\sqrt{19}$	19	$(5, 2)$	$\langle r, p_1^5 p_2^2, p_1^{-2} p_2^3\rangle$	19	12
$\sqrt{21}$	21	$(5, 1)$	$\langle r, p_1^5 p_2, p_1^{-1} p_2^4\rangle$	21	12
$\sqrt{25}$	25	$(5, 0)$	$\langle r, s, p_1^5, p_2^5\rangle$	5	6

with

$$D = \alpha^2 - \alpha\beta + \beta^2$$

for some integers α and β. The predictions for Lösch's ten smallest hexagons are summarized in Table 4.1. Lösch's hexagons with $D = 3$, 4, and 7 correspond, respectively, to Christaller's $k = 3$, 4, and 7 systems. It is seen that Lösch's ten smallest hexagons emerge from bifurcation points of multiplicity $M = 2$, 3, 6, and 12 as follows:

$$D = \begin{cases} 3 & \text{for } M = 2, \\ 4 & \text{for } M = 3, \\ 9, 12, 16, 25 & \text{for } M = 6, \\ 7, 13, 19, 21 & \text{for } M = 12. \end{cases} \tag{4.17}$$

4.5 Numerical Examples of Hexagonal Economic Agglomeration

The emergence of hexagonal economic agglomerations among places on hexagonal lattices is demonstrated in this section by computational bifurcation analysis to ensure the validity and capability of the theoretical development in Part II.

With reference to (4.13) and Table 4.1, we chose the sizes of the $n \times n$ hexagonal lattice as $n = 9$, 16, and 7 to obtain Christaller's three systems and some of Lösch's hexagons. For microeconomic modeling, we use the core–periphery model with (4.6), utilizing the following parameter values:

- The length d of the road connecting neighboring places is $d = 1/n$.
- The expenditure share μ on industrial varieties is $\mu = 0.4$ (Sect. 1.5.2).

- The elasticity σ of substitution between any two varieties is $\sigma = 5.0$.
- The inverse θ of variance of the idiosyncratic tastes is $\theta = 1,000$.

The values of μ and σ satisfy the no-black-hole condition in (1.13) as $(\sigma - 1)/\sigma = 0.8 > \mu = 0.4$.

4.5.1 Christaller's $k = 3$ System and a Lösch's Hexagon

Let us demonstrate the emergence of Christaller's $k = 3$ system and Lösch's hexagon with $D = 9$ on the 9×9 hexagonal lattice. Figure 4.4a shows the equilibrium paths (the maximum population $\lambda_{\max}$ versus the transport cost parameter τ curves) for this lattice.

The flat earth equilibrium with $\boldsymbol{\lambda}_0 = (1/81, \ldots, 1/81)^\top$ corresponds to the horizontal line at $\lambda_{\max} = 1/81$ $(= 0.0123)$. Among bifurcation points on this line, we specifically examined[8] the bifurcation points A and B of multiplicity $M = 2$, from which a hexagonal distribution with $D = 3$ emanated, and the bifurcation point C of multiplicity $M = 6$, from which a hexagonal distribution with $D = 9$ emanated.

The path ADB, bifurcated from the bifurcation points A and B of multiplicity $M = 2$, corresponds to Christaller's $k = 3$ system (Lösch's smallest hexagon with $D = 3$) with $\langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$-symmetry and the spatial period $L/d = \sqrt{D} = \sqrt{3}$.

The path DEC, bifurcated from the bifurcation point C of multiplicity $M = 6$, corresponds to Lösch's fourth smallest hexagon with $D = 9$ with $\langle r, s, p_1^3, p_2^3 \rangle$-symmetry and the spatial period $L/d = \sqrt{D} = \sqrt{9}$.

A secondary path bifurcated at the bifurcation point D of multiplicity $M = 2$ on the primary bifurcated path ADB. By this cascade of bifurcations at points A and D, the spatial period became $\sqrt{3}$ times repeatedly as $L/d = \sqrt{D} : 1 \to \sqrt{3} \to 3$. Recall that such cascade was observed also in the agglomeration analysis of the domain of southern Germany with an irregular shape and without periodic boundaries (Sect. 1.3). Thus the present analysis on the hexagonal lattice captures much desired realism at the expense of idealization of periodic boundaries. In the hexagonal lattice, which is endowed with uniformity, the self-organization by way of bifurcation can be observed clearly, whereas the self-organization appeared less clearly in the domain of southern Germany.

Remark 4.2. A criticism may be raised about the stability of the bifurcating curve AA′D (see Sect. 1.5.4 for the definition of stability). This curve lost the stability at the onset of bifurcation at A. Such spontaneous loss of stability throws a doubt on the usefulness of the local bifurcation analysis at the critical point A. Nonetheless

[8] Although a large number of bifurcation points exist on this line, the theoretical development in Chap. 8 was useful in pinpointing a bifurcation point that produces a specified hexagonal pattern.

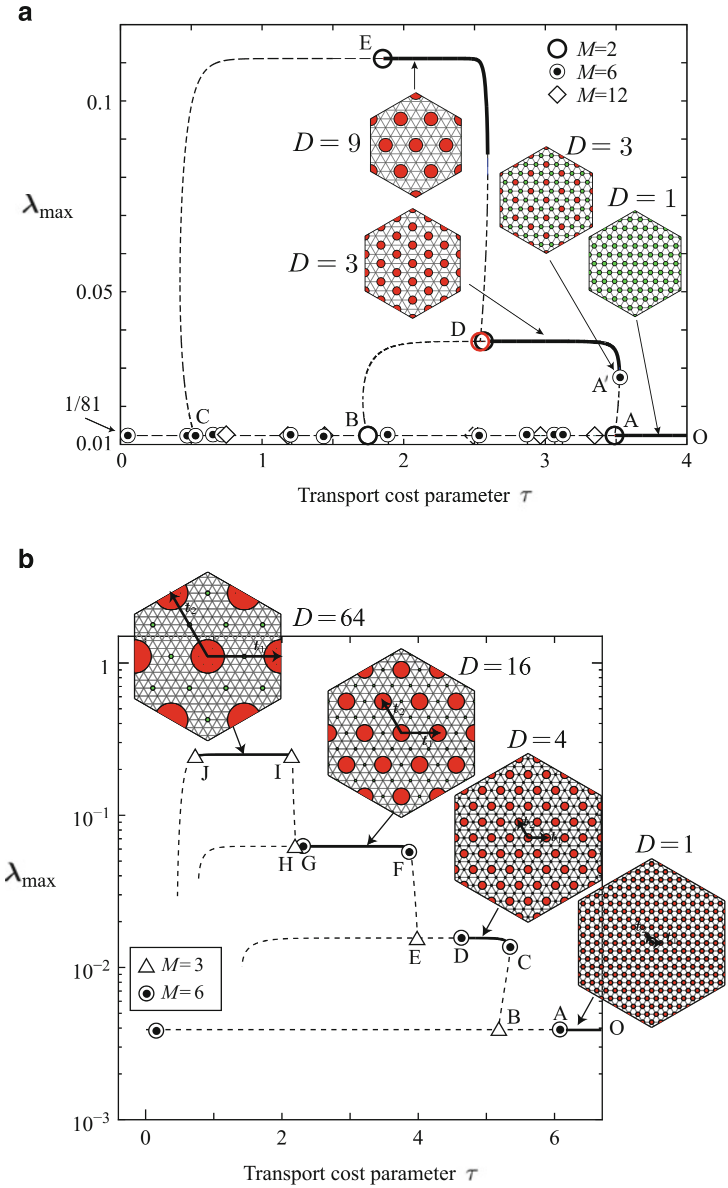

Fig. 4.4 Bifurcated paths for the 9 × 9 and 16 × 16 hexagonal lattices and associated population distributions in hexagonal windows. *Solid curves* mean stable equilibria; *dashed curves*, unstable equilibria; $\lambda_{\max}$, the maximum population among the places; M is the multiplicity of the bifurcation point; the area of each circle in the windows is proportional to the population size at that place. (**a**) 9 × 9 hexagonal lattice. (**b**) 16 × 16 hexagonal lattice

the stability of the bifurcated path was recovered at point A′ (located at the maximal point of τ). Although a continuous path of stable solutions leading to the hexagonal agglomeration is thus absent, an unstable discontinuous *dynamical shift* from the flat earth state OA to the hexagonally agglomerated state A′D is presumed to take place. Such loss and recovery of stability was also observed for the agglomeration analysis of southern Germany in Fig. 1.6, as well as for other hexagonal distributions of Christaller and Lösch on the hexagonal lattice in Fig. 4.4b. Moreover, the unstable state AA′ has the same spatial pattern as that of the unstable state A′D with regard to the geometrical symmetry. Since the symmetry of a bifurcating solution is preserved until undergoing another bifurcation, the local bifurcation analysis at a critical point that elucidates the symmetry of possible bifurcating solutions is vital in the successful search of stable bifurcating equilibria. Such analysis is a major target of this book (Chaps. 8 and 9). □

4.5.2 Christaller's $k = 4$ System and Lösch's Hexagons

Let us demonstrate the emergence of Christaller's $k = 4$ system and Lösch's hexagons with $D = 16$ and 64 on the 16×16 hexagonal lattice. The ratio of the number of first-level centers, that of the second-level centers, and so on is studied in light of central place theory (Sect. 1.2).

Period Doubling Bifurcation Cascade

For the 16×16 hexagonal lattice, Fig. 4.4b depicts the path bifurcated from the triple bifurcation point B ($M = 3$) that produces the hexagonal patterns related to the $k = 4$ system (see (4.15)) and further bifurcated paths. The bifurcation mechanism of the triple bifurcation point B will be studied theoretically in Sect. 8.5.

From the flat earth equilibria OA of this system, a hierarchy of bifurcating paths was found as follows:

$$\text{OAB} \rightarrow \text{BCDE} \rightarrow \text{EFGH} \rightarrow \text{HIJ},$$

branching at a series of triple bifurcation points B, E, and H. The path BCDE is associated with Christaller's $k = 4$ system with $D = 4$, EFGH with Lösch's seventh smallest hexagon with $D = 16$, and HIJ with $D = 64$. This is the spatial period doubling bifurcation cascade, in which the spatial period L is doubled repeatedly as $L/d = \sqrt{D}\ :\ 1 \rightarrow 2 \rightarrow 2^2 \rightarrow 2^3$. The stable parts OA, CD, FG, and IJ of these paths serve as an economically feasible process of agglomeration that is encountered as τ decreases.

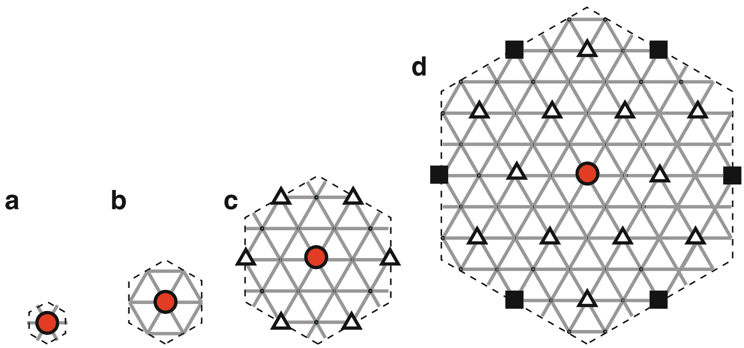

Fig. 4.5 Distribution of different level centers in market areas predicted by central place theory. Places with the same symbol have the same population, but the sizes of the symbols are not related to the population size; *dashed lines* express market area. (**a**) One-level hierarchy. (**b**) Two-level hierarchy. (**c**) Three-level hierarchy. (**d**) Four-level hierarchy

Number of Different Level Centers

In central place theory (Sect. 1.2), a hierarchy of places with different levels exists in the market area governed by the highest-level (first-level) center with the largest population, the second-level center with the second largest population, and so on. Let us compare the computed distributions of different level centers in market areas in Fig. 4.4b with those predicted by central place theory in Fig. 4.5. We can find some commonality between the computational results for the core–periphery model and the prediction by central place theory.

The ratio of the number N_1 of the first-level centers, the number N_2 of the second-level centers, and so on, are given by the following formulas (the recurrence formula (1.1) for $k = 4$):

$$\begin{cases} \text{one-level hierarchy}: & N_1 = 1, \\ \text{two-level hierarchy}: & N_1 : N_2 = 1 : 3, \\ \text{three-level hierarchy}: & N_1 : N_2 : N_3 = 1 : 3 : 12, \\ \text{four-level hierarchy}: & N_1 : N_2 : N_3 : N_4 = 1 : 3 : 12 : 48. \end{cases} \tag{4.18}$$

Figure 4.5 displays these centers in the market area for hierarchies of several levels.[9]

[9]For the two-level hierarchy, for example, each second-level center is shared by two neighboring market areas. In effect, $6/2 = 3$ second-level centers exist in the market area, thereby leading to $N_1 : N_2 = 1 : 3$.

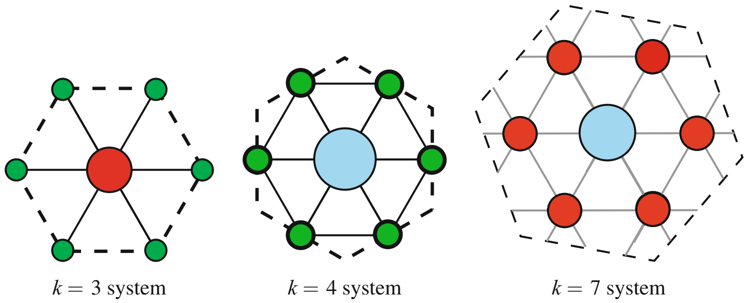

Fig. 4.6 Market areas for Christaller's $k = 3, 4, 7$ systems (indicated by the *dashed lines*)

In contrast, the ratio of the number of different level centers for the computed distributions is[10]

$$\begin{cases} \text{flat earth solution OA :} & N_1 = 1, \\ \text{direct bifurcated solution CD :} & N_1 : N_2 = 1 : 3, \\ \text{secondary bifurcated solution FG :} & N_1 : N_2 : N_3 = 1 : 3 : 12, \\ \text{tertiary bifurcated path IJ :} & N_1 : N_2 : N_3 = 1 : 3 : 60. \end{cases}$$

The ratio agrees with (4.18) for OA, CD, and FG, whereas such agreement is not observed for the tertiary bifurcated path IJ.

Remark 4.3. For the computed hexagonal distributions in core–periphery models, a set of identical hexagons can be obtained as a Voronoi tessellation (Okabe et al. 2000 [12]) of two-dimensional space for a system of first-level centers. In the two-level hierarchy, for example, the first-level center at the center of a hexagon is surrounded by six second-level centers within this hexagon or on its border (Fig. 4.6). This hexagon is termed the (predominant) *market area*, meaning that the first-level center has the maximum market share in relation to each second-level center. The market area is enlarged as Christaller's k value increases. □

4.5.3 Christaller's k = 7 System

Now let us demonstrate the emergence of Christaller's $k = 7$ system on the 7×7 hexagonal lattice. Figure 4.7 depicts equilibrium paths for this hexagonal lattice. Among the solution paths bifurcating from the flat earth equilibria OABC, we plot only the path BDEF from the bifurcation point B of multiplicity $M = 12$ that

[10]In the secondary bifurcated solution FG, six of the 12 third-level centers had slightly larger population than the other six, but they are considered to be identical herein.

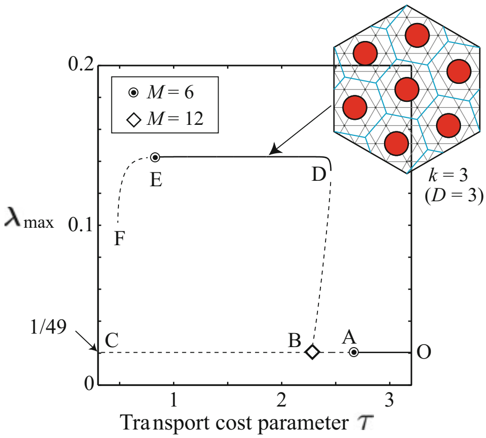

Fig. 4.7 Bifurcated paths for the 7 × 7 hexagonal lattice and associated population distribution in a hexagonal window. *Solid curves* mean stable equilibria; *dashed curves*, unstable equilibria; $\lambda_{\max}$, the maximum population among the places; M is the multiplicity of the bifurcation point; the area of each circle in the windows is proportional to the population size at that place

produces the hexagonal patterns related to the $k = 7$ system (see (4.15)). This bifurcated path has $\langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle$-symmetry[11] in (4.14) with the spatial period $L/d = \sqrt{7}$.

4.6 Summary

- An overview of Part II has been presented.
- Equivariance of core–periphery models on the hexagonal lattice has been proved.
- Theoretical predictions by group-theoretic analysis have been previewed.
- Emergence of hexagons of Christaller and Lösch on the hexagonal lattice has been demonstrated by numerical analysis.
- The role and the importance of the theoretical and computational analysis on the hexagonal lattice with periodic boundaries have been demonstrated.

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[11] In this case with $n = 7$, the expression $\langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle$ can be reduced to a simpler form $\langle r, p_1^3 p_2 \rangle$.

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Chapter 5
Hexagonal Distributions on Hexagonal Lattice

Abstract An infinite hexagonal lattice is introduced as a two-dimensional discretized uniform space for economic agglomeration. This chapter focuses on an analysis of geometrical characteristics of the lattice, as a vital prerequisite for the group-theoretic bifurcation analysis of this lattice that will be conducted in Chaps. 6–9. Hexagonal distributions on this lattice, corresponding to those envisaged by Christaller and Lösch in central place theory (Sect. 1.2), are explained, parameterized, and classified, and their two-dimensional periodicities are presented. A finite $n \times n$ hexagonal lattice with periodic boundaries is introduced as a spatial platform for economic activities of core–periphery models. The symmetry of the $n \times n$ hexagonal lattice is described by the group composed of the dihedral group D_6 expressing local regular-hexagonal symmetry and the group $\mathbb{Z}_n \times \mathbb{Z}_n$ (direct product of two cyclic groups of order n) expressing translational symmetry in two directions. Subgroups relevant to hexagonal distributions of this group are obtained by geometrical consideration and classified in accordance with the study of central place theory.

Keywords Christaller's hexagonal distributions • Group • Hexagonal lattice • Lösch's hexagons • Periodic boundaries • Symmetry

5.1 Introduction

As a spatial configuration of a system of places, we use the hexagonal lattice in view of its geometrical consistency with the hexagonal market areas predicted in the literature of economic geography (Lösch, 1940; 1954, pp. 133–134 [10]). As a first step towards the theoretical prediction of bifurcating hexagons, the geometrical platform of the analysis is constructed in this chapter. We first consider an infinite hexagonal lattice spreading over the entire plane. Basis vectors spanning these lattices are introduced and hexagonal distributions on this lattice are parameterized and classified into several types.

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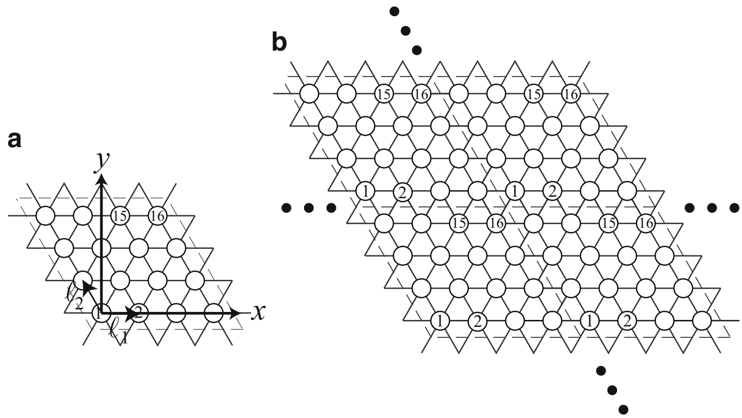

Fig. 5.1 A system of places on the 4 × 4 hexagonal lattice with periodic boundaries. (**a**) 4 × 4 hexagonal lattice. (**b**) Periodically repeated 4 × 4 hexagonal lattice

As a two-dimensional space for economic activities of core–periphery models, we further introduce an $n \times n$ finite hexagonal lattice with periodic boundaries comprising a system of uniformly distributed $n \times n$ places. See, for example, Fig. 5.1a for the hexagonal lattice with 4 × 4 places. Discretized degrees-of-freedom are allocated to each node of the lattice. These places are connected by roads of the same length forming a regular-triangular mesh, and goods are transported along these roads. Periodic boundaries are used to express infiniteness and uniformity and to avoid heterogeneity due to the boundaries by spatially repeating the finite lattice periodically to cover an infinite two-dimensional domain (see Fig. 5.1b).

The symmetry of this lattice with discretized degrees-of-freedom is expressed by the group consisting of D_6 and $\mathbb{Z}_n \times \mathbb{Z}_n$ introduced in Sect. 4.3.1. We are concerned with compatibility of n with the hexagonal patterns of interest, which is important in computational studies. Subgroups expressing the symmetries of these patterns are presented herein and classified in accordance with the study of Christaller and Lösch (Sect. 1.2). The study conducted in this chapter is purely geometric and involves no bifurcation mechanism, but forms an important foundation of the group-theoretic bifurcation analysis in Chaps. 8 and 9.

This chapter is organized as follows. The infinite hexagonal lattice is introduced in Sect. 5.2. Hexagonal distributions are parameterized and classified into several types in Sect. 5.3. The finite hexagonal lattice is given in Sect. 5.4. The group associated with the hexagonal lattice and the groups expressing the symmetry of hexagonal distributions are given in Sect. 5.5.

5.2 Infinite Hexagonal Lattice

Infinite hexagonal lattice is introduced and hexagonal distributions on this lattice are described.

5.2.1 Basis Vectors

Figure 5.2 portrays the infinite hexagonal lattice, which comprises regular triangles and covers the infinite two-dimensional domain. The hexagonal lattice is given as a set of integer combinations of oblique *basis vectors*

$$\boldsymbol{\ell}_1 = d\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\ell}_2 = d\begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \tag{5.1}$$

where $d > 0$ means the length of these vectors. That is, the infinite hexagonal lattice $\mathscr{H}$ is expressed as

$$\mathscr{H} = \{n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z}\}, \tag{5.2}$$

where $\mathbb{Z}$ denotes the set of integers.

To represent hexagonal distributions on the lattice $\mathscr{H}$, we consider a *sublattice* spanned by

$$\boldsymbol{t}_1 = \alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2, \quad \boldsymbol{t}_2 = -\beta\boldsymbol{\ell}_1 + (\alpha - \beta)\boldsymbol{\ell}_2, \tag{5.3}$$

where α and β are integer-valued parameters with $(\alpha, \beta) \neq (0, 0)$. We denote this sublattice by $\mathscr{H}(\alpha, \beta)$, that is,

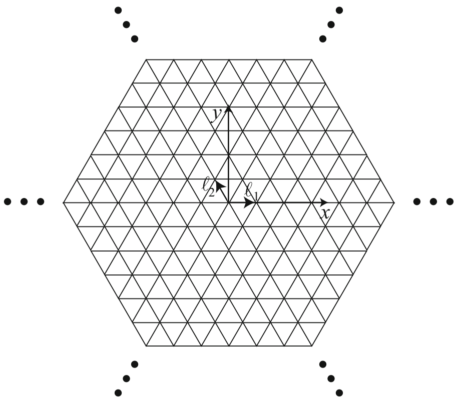

Fig. 5.2 Hexagonal lattice

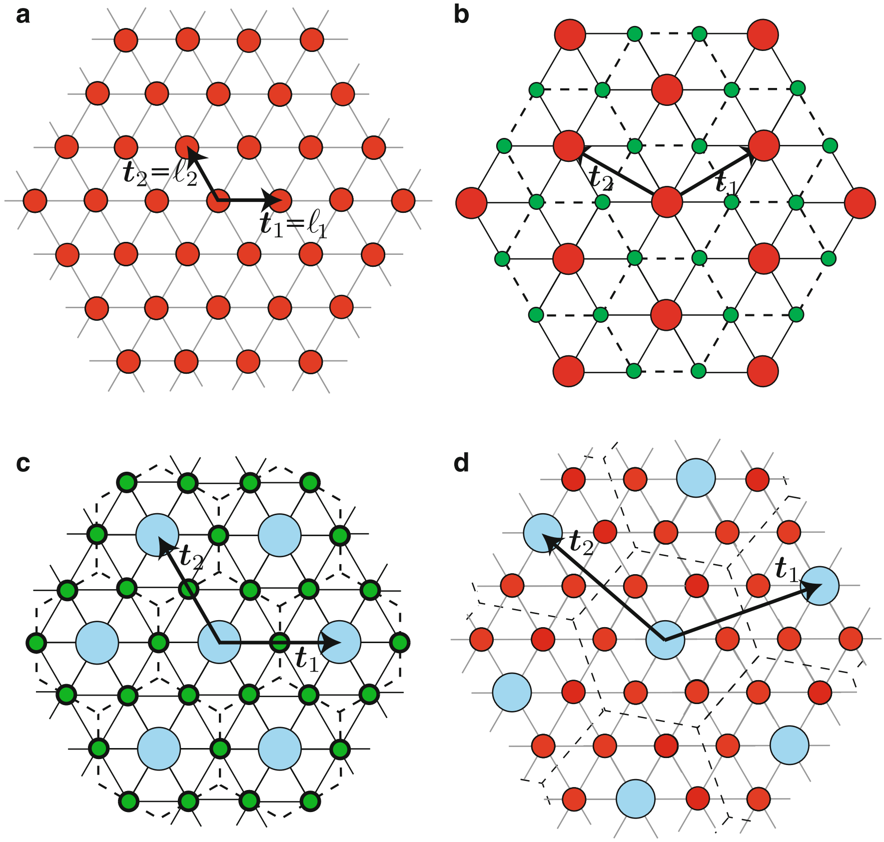

Fig. 5.3 Hexagonal distributions of Christaller on the hexagonal lattice. Types V, M, and T are defined in Sect. 5.3.2. (**a**) $(\alpha, \beta) = (1, 0)$, $L/d = 1$, type V. (**b**) $(\alpha, \beta) = (2, 1)$, $L/d = \sqrt{3}$, type M (Christaller's $k = 3$ system). (**c**) $(\alpha, \beta) = (2, 0)$, $L/d = \sqrt{4}$, type V (Christaller's $k = 4$ system). (**d**) $(\alpha, \beta) = (3, 1)$, $L/d = \sqrt{7}$, type T (Christaller's $k = 7$ system)

$$
\begin{aligned}
\mathscr{H}(\alpha, \beta) &= \{n_1 \boldsymbol{t}_1 + n_2 \boldsymbol{t}_2 \mid n_1, n_2 \in \mathbb{Z}\} \\
&= \{(n_1\alpha - n_2\beta)\boldsymbol{\ell}_1 + (n_1\beta + n_2(\alpha - \beta))\boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z}\} \\
&= \left\{ [\, \boldsymbol{\ell}_1 \ \boldsymbol{\ell}_2 \,] \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \,\middle|\, n_1, n_2 \in \mathbb{Z} \right\}. \qquad (5.4)
\end{aligned}
$$

Since $\|\boldsymbol{t}_1\| = \|\boldsymbol{t}_2\|$ and the angle between $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ is $2\pi/3$, the lattice $\mathscr{H}(\alpha, \beta)$ indeed represents a hexagonal distribution (see Fig. 5.3).

The *spatial period* L is defined to be the (common) length of the basis vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ and is given by

$$L = d\sqrt{\alpha^2 - \alpha\beta + \beta^2}. \tag{5.5}$$

We refer to

$$\frac{L}{d} = \sqrt{\alpha^2 - \alpha\beta + \beta^2} \tag{5.6}$$

as the *normalized spatial period*, which is an important index for characterizing the size of a hexagonal distribution. Although the definition here refers to the basis vectors, the spatial period L, as well as the normalized spatial period L/d, is in fact determined by the sublattice $\mathscr{H}(\alpha, \beta)$, as seen from (5.10) with (5.9) below.

5.2.2 Hexagons of Christaller and Lösch

Lösch, 1940 [10] noted that the normalized spatial period L/d in (5.6) takes specific values $\sqrt{1}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{7}$, ... as a consequence of the fact that α and β are integers. The hexagonal distribution with $L/d = 1$ is the uniform distribution (Fig. 5.3a), and the distributions with $L/d = \sqrt{3}$, $\sqrt{4}$, $\sqrt{7}$ correspond, respectively, to Christaller's $k = 3, 4, 7$ systems (Fig. 5.3b–d). The parameter values for these systems are given as follows:

$$(\alpha, \beta) = \begin{cases} (1, 0): & \text{uniform distribution } (L/d = 1), \\ (2, 1): & k = 3 \text{ system } (L/d = \sqrt{3}), \\ (2, 0): & k = 4 \text{ system } (L/d = \sqrt{4}), \\ (3, 1): & k = 7 \text{ system } (L/d = \sqrt{7}). \end{cases} \tag{5.7}$$

Lösch's formula (1.2) for the normalized spatial period is obtained from (5.6) as

$$\begin{aligned} \frac{L}{d} &= \sqrt{\alpha^2 - \alpha\beta + \beta^2} \\ &= \sqrt{1}, \sqrt{3}, \sqrt{4}, \sqrt{7}, \sqrt{9}, \sqrt{12}, \sqrt{13}, \sqrt{16}, \sqrt{19}, \sqrt{21}, \sqrt{25}, \ldots \\ &= \begin{cases} 1, 2, 3, 4, 5, \ldots, \\ \sqrt{3}, \sqrt{7}, \sqrt{12}, \sqrt{13}, \sqrt{19}, \sqrt{21}, \ldots. \end{cases} \end{aligned} \tag{5.8}$$

Lösch's first three hexagons correspond to Christaller's $k = 3, 4, 7$ systems, and their normalized spatial period L/d and the parameter (α, β) are given in (5.7). For the remainder of ten smallest hexagons, we have

$$(\alpha, \beta) = \begin{cases} (3,0): & L/d = 3, \\ (4,2): & L/d = \sqrt{12}, \\ (4,1): & L/d = \sqrt{13}, \\ (4,0): & L/d = 4, \\ (5,2): & L/d = \sqrt{19}, \\ (5,1): & L/d = \sqrt{21}, \\ (5,0): & L/d = 5. \end{cases}$$

5.3 Description of Hexagonal Distributions

Hexagonal distributions are parameterized and classified into several types.

5.3.1 Parameterization of Hexagonal Distributions

In the parameterization (α, β) of the lattice, let us note its non-uniqueness that different parameter values of (α, β) can sometimes result in the same lattice $\mathscr{H}(\alpha, \beta)$. Define

$$D = D(\alpha, \beta) = \alpha^2 - \alpha\beta + \beta^2, \tag{5.9}$$

which is a positive integer for $(\alpha, \beta) \neq (0,0)$. It will be shown later in this subsection that D is an invariant in this parameterization, that is, we have the following implication:

$$\mathscr{H}(\alpha, \beta) = \mathscr{H}(\alpha', \beta') \implies D(\alpha, \beta) = D(\alpha', \beta'). \tag{5.10}$$

The converse, however, is not true, as the following example shows.

Example 5.1. For $(\alpha, \beta) = (7,0)$ and $(\alpha', \beta') = (8,3)$, we have $D(\alpha, \beta) = D(\alpha', \beta') = 49$. But the lattices $\mathscr{H}(\alpha, \beta)$ and $\mathscr{H}(\alpha', \beta')$ are different (Fig. 5.4). □

The parameter space for the hexagonal sublattices is given as follows, and the proof is given later in this subsection.

Proposition 5.1. *Hexagonal sublattices $\mathscr{H}(\alpha, \beta)$ are parameterized, one-to-one, by*

$$\{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha > \beta \geq 0\}. \tag{5.11}$$

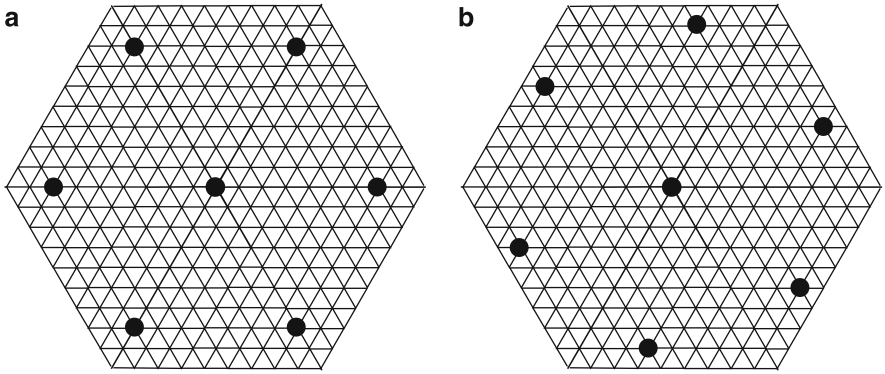

Fig. 5.4 Two different hexagonal distributions with $D = 49$. Types V and T are defined in Sect. 5.3.2. (**a**) $(\alpha, \beta) = (7, 0)$, type V. (**b**) $(\alpha, \beta) = (8, 3)$, type T

Two lattices $\mathscr{H}(\alpha, \beta)$ and $\mathscr{H}(\beta, \alpha)$ are not identical in general, but are mirror images with respect to the y-axis. As such they are naturally regarded as essentially the same. Let us call two hexagonal lattices *essentially different* if they are neither identical nor mirror images with respect to the y-axis. Essentially different hexagonal sublattices are parameterized as follows, the proof being given later in this subsection.

Proposition 5.2. *Essentially different hexagonal sublattices $\mathscr{H}(\alpha, \beta)$ are parameterized, one-to-one, by*

$$\{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha \geq 2\beta \geq 0,\ \alpha \neq 0\}. \tag{5.12}$$

Table 5.1 shows the values of $D = D(\alpha, \beta)$ for (α, β) with $11 \geq \alpha \geq 2\beta \geq 0$, $\alpha \neq 0$. It is worth noting that the values of D in this table are all distinct with the exceptions of $D(7, 0) = D(8, 3) = 49$ and $D(10, 1) = D(11, 5) = 91$. This means, in particular, that smaller hexagonal distributions (with $D < 49$) are uniquely determined by their spatial period L, which is related to D as

$$\frac{L}{d} = \sqrt{D} \tag{5.13}$$

by (5.6) and (5.9).

Proofs of (5.10) and Propositions 5.1 and 5.2

First, recall that $\mathscr{H}(\alpha, \beta)$ is generated by $(\boldsymbol{t}_1, \boldsymbol{t}_2) = (\boldsymbol{t}_1(\alpha, \beta), \boldsymbol{t}_2(\alpha, \beta))$ in (5.3), which can be expressed as

Table 5.1 The values of $D(\alpha, \beta)$ for (α, β) in (5.12)

$\alpha \setminus \beta$	0	1	2	3	4	5
1	1					
2	4	3				
3	9	7				
4	16	13	12			
5	25	21	19			
6	36	31	28	27		
7	$\boxed{49}$	43	39	37		
8	64	57	52	$\boxed{49}$	48	
9	81	73	67	63	61	
10	100	$\boxed{91}$	84	79	76	75
11	121	111	103	97	93	$\boxed{91}$

$$[\boldsymbol{t}_1 \; \boldsymbol{t}_2] = [\boldsymbol{\ell}_1 \; \boldsymbol{\ell}_2] \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix}. \tag{5.14}$$

The determinant of this coefficient matrix coincides with $D(\alpha, \beta)$ introduced in (5.9), i.e.,

$$D(\alpha, \beta) = \alpha^2 - \alpha\beta + \beta^2 = \det \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix}. \tag{5.15}$$

If $\mathcal{H}(\alpha', \beta') \subseteq \mathcal{H}(\alpha, \beta)$, then

$$\begin{bmatrix} \alpha' & -\beta' \\ \beta' & \alpha' - \beta' \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

for some integers $x_{11}, x_{12}, x_{21}, x_{22}$, and therefore, $D(\alpha', \beta')$ is a multiple of $D(\alpha, \beta)$. By exchanging the roles of (α, β) and (α', β'), we can show (5.10).

Next, the parameter spaces (5.11) and (5.12) for $\mathcal{H}(\alpha, \beta)$ are derived. We observe geometrically (see Fig. 5.5a) that $\mathcal{H}(\alpha', \beta') = \mathcal{H}(\alpha, \beta)$ if and only if $\boldsymbol{t}'_1 = \alpha'\boldsymbol{\ell}_1 + \beta'\boldsymbol{\ell}_2$ is obtained from $\boldsymbol{t}_1 = \alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2$ by a rotation at an angle that is a multiple of $2\pi/6$, i.e., $\boldsymbol{t}'_1 = {R_6}^k \boldsymbol{t}_1$ with

$$R_6 = \begin{bmatrix} \cos(2\pi/6) & -\sin(2\pi/6) \\ \sin(2\pi/6) & \cos(2\pi/6) \end{bmatrix}$$

for some $k \in \{0, 1, \ldots, 5\}$. Since

$$R_6 \boldsymbol{t}_1 = R_6(\alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2) = \alpha(\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2) + \beta(-\boldsymbol{\ell}_1) = [\boldsymbol{\ell}_1 \; \boldsymbol{\ell}_2] \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

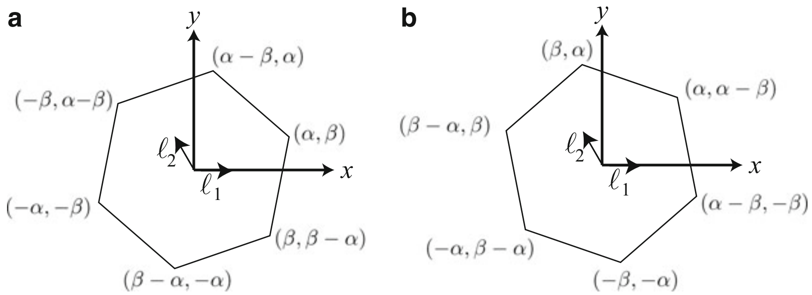

Fig. 5.5 Hexagons associated with (5.16) and (5.17). (**a**) Hexagon for (5.16). (**b**) Hexagon for (5.17)

we have $\mathscr{H}(\alpha', \beta') = \mathscr{H}(\alpha, \beta)$ if and only if

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for some $k \in \{0, 1, \ldots, 5\}$. Therefore, we obtain the same lattice for the following six parameter values:

$$(\alpha, \beta),\ (\alpha - \beta, \alpha),\ (-\beta, \alpha - \beta),\ (-\alpha, -\beta),\ (\beta - \alpha, -\alpha),\ (\beta, \beta - \alpha). \qquad (5.16)$$

This allows us to adopt (5.11) as the parameter space for $\mathscr{H}(\alpha, \beta)$, by which we mean that, for every $(\alpha', \beta') \neq (0, 0)$ in $\mathbb{Z}^2$, the sublattice $\mathscr{H}(\alpha', \beta')$ is the same as the sublattice $\mathscr{H}(\alpha, \beta)$ for some (uniquely determined) (α, β) in (5.11). It should be mentioned, in particular, that $\mathscr{H}(\alpha, \alpha) = \mathscr{H}(\alpha, 0)$ by (5.16), and hence we have strict inequality $\alpha > \beta$ in (5.11).

Geometrically, the lattices for (α, β) and (β, α) are mirror images with respect to the y-axis. In this sense, we regard $\mathscr{H}(\alpha, \beta)$ and $\mathscr{H}(\beta, \alpha)$ as essentially the same. Thus, we regard the following six parameter values as essentially equivalent to (α, β):

$$(\beta, \alpha),\ (\beta - \alpha, \beta),\ (-\alpha, \beta - \alpha),\ (-\beta, -\alpha),\ (\alpha - \beta, -\beta),\ (\alpha, \alpha - \beta). \qquad (5.17)$$

See Fig. 5.5b for the hexagon of (5.17). If $\beta = 0$ or $\alpha = 2\beta$, the set of six parameters in (5.17) is identical to the set in (5.16). This is because the lattices for $\beta = 0$ or $\alpha = 2\beta$ are symmetric with respect to the y-axis.

Essentially equivalent parameter values can thus be summarized as follows:

$$\begin{array}{l} (\alpha, \beta),\ (\alpha - \beta, \alpha),\ (-\beta, \alpha - \beta),\ (-\alpha, -\beta),\ (\beta - \alpha, -\alpha),\ (\beta, \beta - \alpha), \\ (\beta, \alpha),\ (\beta - \alpha, \beta),\ (-\alpha, \beta - \alpha),\ (-\beta, -\alpha),\ (\alpha - \beta, -\beta),\ (\alpha, \alpha - \beta), \end{array} \qquad (5.18)$$

which reduces in a special case of $\beta = 0$ to

$$(\alpha, 0),\ (\alpha, \alpha),\ (0, \alpha),\ (-\alpha, 0),\ (-\alpha, -\alpha),\ (0, -\alpha) \tag{5.19}$$

or in another special case of $\alpha = 2\beta$ to

$$(2\beta, \beta),\ (\beta, 2\beta),\ (\beta, -\beta),\ (-2\beta, -\beta),\ (-\beta, -2\beta),\ (-\beta, \beta). \tag{5.20}$$

On the basis of the observations above, (5.12) can be adopted as the parameter space for essentially different sublattices. This means that every $(\alpha, \beta) \neq (0, 0)$ in $\mathbb{Z}^2$ is essentially equivalent to some (uniquely determined) member in (5.12).

5.3.2 Types of Hexagonal Distributions

The *tilt angle* φ of the sublattice $\mathscr{H}(\alpha, \beta)$ is defined as the angle between $\boldsymbol{\ell}_1$ and $\boldsymbol{t}_1$, i.e., by

$$\cos\varphi = \frac{(\boldsymbol{\ell}_1)^\top \boldsymbol{t}_1}{\|\boldsymbol{\ell}_1\| \cdot \|\boldsymbol{t}_1\|}, \tag{5.21}$$

where (α, β) is chosen from the parameter space in (5.11) or (5.12). This is equivalent to defining φ by

$$\varphi = \arcsin\left(\frac{\sqrt{3}\beta}{2\sqrt{\alpha^2 - \alpha\beta + \beta^2}}\right). \tag{5.22}$$

We have $0 \leq \varphi < \pi/3$ in the case of (5.11) and $0 \leq \varphi \leq \pi/6$ in the case of (5.12).

With reference to the tilt angle φ, hexagonal distributions can be classified into three types:

$$\begin{cases} \text{type V} & \text{if } \varphi = 0, \\ \text{type M} & \text{if } \varphi = \pi/6, \\ \text{type T} & \text{otherwise.} \end{cases} \tag{5.23}$$

Figure 5.6 depicts hexagons of these three types that are centered at the origin, where “V” indicates that the x-axis contains a *v*ertex of the hexagon, “M” denotes that the x-axis contains the *m*idpoint of two neighboring vertices of that hexagon, and “T” means “*t*ilted.” In terms of the parameter (α, β), this classification is expressed as

$$\begin{cases} \text{type V} & \text{if } (\alpha, \beta) = (\alpha, 0) & (\alpha \geq 1), \\ \text{type M} & \text{if } (\alpha, \beta) = (2\beta, \beta) & (\beta \geq 1), \\ \text{type T} & \text{otherwise,} \end{cases} \tag{5.24}$$

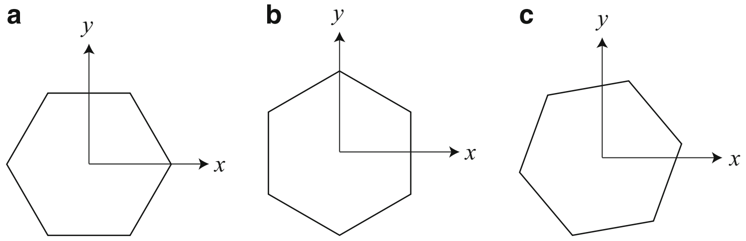

Fig. 5.6 Hexagons of three types that are centered at the origin. (**a**) Type V. (**b**) Type M. (**c**) Type T

where the parameter space for type T depends on the choice of (5.11) or (5.12) as

$$\text{For (5.11)}: \ \{(\alpha, \beta) \mid \alpha > \beta \geq 1, \alpha \neq 2\beta\}, \tag{5.25}$$

$$\text{For (5.12)}: \ \{(\alpha, \beta) \mid \beta \geq 1, \alpha > 2\beta\}. \tag{5.26}$$

Accordingly, the parameter spaces in (5.11) and (5.12) are divided, respectively, into three parts:

$$\{(\alpha, 0) \mid \alpha \geq 1\} \cup \{(2\beta, \beta) \mid \beta \geq 1\} \cup \{(\alpha, \beta) \mid \alpha > \beta \geq 1, \alpha \neq 2\beta\}, \tag{5.27}$$

$$\{(\alpha, 0) \mid \alpha \geq 1\} \cup \{(2\beta, \beta) \mid \beta \geq 1\} \cup \{(\alpha, \beta) \mid \beta \geq 1, \alpha > 2\beta\}. \tag{5.28}$$

The types V, M, and T are correlated with the normalized spatial period as

$$L/d = \begin{cases} \sqrt{4}, \sqrt{9}, \sqrt{16}, \sqrt{25}, \ldots & : \text{type V}, \\ \sqrt{3}, \sqrt{12}, \sqrt{27}, \sqrt{48}, \ldots & : \text{type M}, \\ \sqrt{7}, \sqrt{13}, \sqrt{19}, \sqrt{21}, \ldots & : \text{type T}. \end{cases}$$

It should be emphasized, however, that the type does not always determine, nor is determined by, the spatial period. This is demonstrated by the two lattices $\mathscr{H}(7,0)$ and $\mathscr{H}(8,3)$ mentioned in Example 5.1. These lattices share the same normalized spatial period $L/d = \sqrt{49}$, but of different types; the former is of type V and the latter of type T.

5.4 Finite Hexagonal Lattice

In the previous subsections we have considered the infinite hexagonal lattice spreading over the entire plane. We now introduce an $n \times n$ *finite hexagonal lattice* with periodic boundaries (Fig. 5.1).

Recall first that the hexagonal lattice $\mathscr{H}$ is defined in (5.2) as the set of all integer combinations of basis vectors

$$\boldsymbol{\ell}_1 = d \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\ell}_2 = d \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}. \tag{5.29}$$

We now consider a subset $\mathscr{H}_n$ of $\mathscr{H}$ that consists of integer combinations with coefficients between 0 and $n-1$:

$$\mathscr{H}_n = \{n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 \mid n_i \in \mathbb{Z},\ 0 \le n_i \le n-1\ (i = 1, 2)\}. \tag{5.30}$$

This is a finite set comprising n^2 elements, where n represents the size of $\mathscr{H}_n$.

The infinite lattice $\mathscr{H}$ is regarded as a periodic extension of $\mathscr{H}_n$ with the two-dimensional period of $(n\boldsymbol{\ell}_1, n\boldsymbol{\ell}_2)$. In other words, $\mathscr{H}$ is regarded as being covered by translations of $\mathscr{H}_n$ by vectors of the form $m_1(n\boldsymbol{\ell}_1) + m_2(n\boldsymbol{\ell}_2)$ with integers m_1 and m_2. A point $n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2$ in $\mathscr{H}$ corresponds to $n_1'\boldsymbol{\ell}_1 + n_2'\boldsymbol{\ell}_2$ in $\mathscr{H}_n$ for (n_1', n_2') given by

$$n_1' \equiv n_1 \mod n, \quad n_2' \equiv n_2 \mod n. \tag{5.31}$$

For the sublattice $\mathscr{H}(\alpha, \beta)$ of $\mathscr{H}$, defined in (5.4), we may consider its portion $\mathscr{H}(\alpha, \beta) \cap \mathscr{H}_n$ contained in $\mathscr{H}_n$, expecting that the periodic extension of this portion coincides with $\mathscr{H}(\alpha, \beta)$ itself. If this is the case, we say that (α, β) is *compatible* with n, or n is compatible with (α, β). Using the Minkowski sum[1] of $\mathscr{H}(\alpha, \beta) \cap \mathscr{H}_n$ and $\mathscr{H}(n, 0)$, the *condition for compatibility* can be expressed as

$$(\mathscr{H}(\alpha, \beta) \cap \mathscr{H}_n) + \mathscr{H}(n, 0) = \mathscr{H}(\alpha, \beta), \tag{5.32}$$

which is equivalent to

$$\mathscr{H}(n, 0) \subseteq \mathscr{H}(\alpha, \beta). \tag{5.33}$$

The compatibility condition is given as follows.

Proposition 5.3. *The size n of $\mathscr{H}_n$ is compatible with (α, β) if and only if n is a multiple of $D(\alpha, \beta)/\gcd(\alpha, \beta)$, that is,*

$$n = m\,\frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad m = 1, 2, \ldots. \tag{5.34}$$

[1]For two sets $X, Y \subseteq \mathbb{Z}^2$, their *Minkowski sum* $X + Y$ is defined as $X + Y = \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in X, \boldsymbol{y} \in Y\}$.

Proof. By (5.33), the size n is compatible with (α, β) if and only if

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix}$$

for some integers $x_{11}, x_{12}, x_{21}, x_{22}$, where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are defined in (5.3). Substituting

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix}$$

in (5.14) into the above equation and multiplying the inverse of $\begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix}$ from the left, we obtain

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

from which

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix}^{-1} = \frac{n}{D(\alpha, \beta)} \begin{bmatrix} \alpha - \beta & \beta \\ -\beta & \alpha \end{bmatrix} = \frac{n \ \gcd(\alpha, \beta)}{D(\alpha, \beta)} \begin{bmatrix} \hat{\alpha} - \hat{\beta} & \hat{\beta} \\ -\hat{\beta} & \hat{\alpha} \end{bmatrix},$$

where $\hat{\alpha} = \alpha / \gcd(\alpha, \beta)$ and $\hat{\beta} = \beta / \gcd(\alpha, \beta)$. This shows[2] that $x_{11}, x_{12}, x_{21}, x_{22}$ are integers if and only if n is a multiple of $D(\alpha, \beta) / \gcd(\alpha, \beta)$. □

For example, the following can be seen from Proposition 5.3.

- For $(\alpha, \beta) = (2, 1)$ we have $D(\alpha, \beta) = 3$ and $D(\alpha, \beta) / \gcd(\alpha, \beta) = 3/1 = 3$. Therefore, a compatible n is a multiple of 3. Recall that $(\alpha, \beta) = (2, 1)$ corresponds to Christaller's $k = 3$ system (Fig. 5.3b).
- For $(\alpha, \beta) = (2, 0)$ we have $D(\alpha, \beta) = 4$ and $D(\alpha, \beta) / \gcd(\alpha, \beta) = 4/2 = 2$. Therefore, a compatible n is a multiple of 2. Recall that $(\alpha, \beta) = (2, 0)$ corresponds to Christaller's $k = 4$ system (Fig. 5.3c).
- For $(\alpha, \beta) = (3, 1)$ we have $D(\alpha, \beta) = 7$ and $D(\alpha, \beta) / \gcd(\alpha, \beta) = 7/1 = 7$. Therefore, a compatible n is a multiple of 7. Recall that $(\alpha, \beta) = (3, 1)$ corresponds to Christaller's $k = 7$ system (Fig. 5.3d).

For Lösch's ten smallest hexagons, compatible n's are a multiple of 3, 2, 7, 3, 6, 13, 4, 19, 21, 5, respectively, as listed in Table 5.2, which also shows the values of $G(\alpha, \beta)$ to be defined later in Sect. 5.5.2.

When combined with the three types in (5.24), the compatibility condition (5.34) in Proposition 5.3 shows the following.

[2]For rigorous reasoning we can use Theorem 8.2 in Sect. 8.7.9. It can be shown that the determinantal divisors of $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix}$ and $\begin{bmatrix} \alpha & -\beta & n & 0 \\ \beta & \alpha - \beta & 0 & n \end{bmatrix}$ are the same if and only if n is a multiple of $D(\alpha, \beta) / \gcd(\alpha, \beta)$.

Table 5.2 Lösch's ten smallest hexagons

Spatial period L/d	D	(α, β)	Tilt angle φ	Type of hexagon	Symmetry group $G(\alpha, \beta)$	Compatible size n (multiple of)
$\sqrt{3}$	3	$(2, 1)$	$\pi/6$	M	$\Sigma(2, 1)$	3
$\sqrt{4}$	4	$(2, 0)$	0	V	$\Sigma(2, 0)$	2
$\sqrt{7}$	7	$(3, 1)$	0.106π	T	$\Sigma_0(3, 1)$	7
$\sqrt{9}$	9	$(3, 0)$	0	V	$\Sigma(3, 0)$	3
$\sqrt{12}$	12	$(4, 2)$	$\pi/6$	M	$\Sigma(4, 2)$	6
$\sqrt{13}$	13	$(4, 1)$	0.077π	T	$\Sigma_0(4, 1)$	13
$\sqrt{16}$	16	$(4, 0)$	0	V	$\Sigma(4, 0)$	4
$\sqrt{19}$	19	$(5, 2)$	0.130π	T	$\Sigma_0(5, 2)$	19
$\sqrt{21}$	21	$(5, 1)$	0.061π	T	$\Sigma_0(5, 1)$	21
$\sqrt{25}$	25	$(5, 0)$	0	V	$\Sigma(5, 0)$	5

(α, β) is chosen from (5.12)

- For a distribution $\mathscr{H}(\alpha, \beta)$ of type V, parameterized by $(\alpha, \beta) = (\alpha, 0)$ with $\alpha \geq 1$, a compatible n is a multiple of α.
- For a distribution $\mathscr{H}(\alpha, \beta)$ of type M, parameterized by $(\alpha, \beta) = (2\beta, \beta)$ with $\beta \geq 1$, a compatible n is a multiple of 3β.
- For a distribution $\mathscr{H}(\alpha, \beta)$ of type T, with (α, β) in (5.25) or (5.26), a compatible n is a multiple of $D(\alpha, \beta)/\gcd(\alpha, \beta)$.

To sum up, we have

$$n = \begin{cases} m\alpha & (\alpha \geq 1) \quad \text{for type V}, \\ 3m\beta & (\beta \geq 1) \quad \text{for type M}, \\ mD(\alpha, \beta)/\gcd(\alpha, \beta) & \qquad\qquad \text{for type T}, \end{cases} \tag{5.35}$$

where $m = 1, 2, \ldots$.

5.5 Groups Expressing the Symmetry

The first step of the bifurcation analysis of the hexagonal agglomeration pattern of population distribution on the $n \times n$ hexagonal lattice is to identify the subgroup expressing the symmetry of this pattern.

5.5.1 Symmetry of the Finite Hexagonal Lattice

The symmetry of the $n \times n$ hexagonal lattice $\mathscr{H}_n$ in (5.30) is characterized by invariance with respect to

- r: counterclockwise rotation about the origin at an angle of $\pi/3$,
- s: reflection $y \mapsto -y$,
- p_1: periodic translation along the $\boldsymbol{\ell}_1$-axis (i.e., the x-axis), and
- p_2: periodic translation along the $\boldsymbol{\ell}_2$-axis.

Consequently, the symmetry of the hexagonal lattice $\mathscr{H}_n$ is described by the group

$$G = \langle r, s, p_1, p_2 \rangle, \tag{5.36}$$

which is generated by r, s, p_1, and p_2 with the fundamental relations :

$$\begin{aligned} r^6 = s^2 = (rs)^2 = p_1{}^n = p_2{}^n = e, \quad & p_2 p_1 = p_1 p_2, \\ rp_1 = p_1 p_2 r, \quad rp_2 = p_1^{-1} r, \quad sp_1 = p_1 s, \quad & sp_2 = p_1^{-1} p_2^{-1} s, \end{aligned} \tag{5.37}$$

where e is the identity element. Each element of G can be represented uniquely in the form of

$$s^l r^m p_1{}^i p_2{}^j, \quad l \in \{0, 1\},\ m \in \{0, 1, \dots, 5\},\ i, j \in \{0, 1, \dots, n-1\}. \tag{5.38}$$

The group G contains the *dihedral group*

$$\langle r, s \rangle \simeq \mathrm{D}_6$$

and *cyclic groups*

$$\langle p_1 \rangle \simeq \mathbb{Z}_n, \quad \langle p_2 \rangle \simeq \mathbb{Z}_n$$

as its subgroups, where $\mathbb{Z}_n$ means the cyclic group of order n, which is denoted as C_n in Sect. 2.3 in Part I. The group G has the structure of the semidirect product of D_6 by $\mathbb{Z}_n \times \mathbb{Z}_n$, that is, $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$; see Sect. 2.3.1 for the definition of the semidirect product.

Remark 5.1. The symmetry of the hexagonal lattice $\mathscr{H}_n$ with discretized degrees-of-freedom is expressed by group G in (5.36) consisting of D_6 and $\mathbb{Z}_n \times \mathbb{Z}_n$. This group is naturally regarded as a discretized version of the group $\mathrm{D}_6 \ltimes \mathrm{T}^2$, where T^2 denotes the two-dimensional torus. The group $\mathrm{D}_6 \ltimes \mathrm{T}^2$ was treated in connection with Bénard convection (e.g., Bénard, 1900 [1]; Chandrasekhar, 1961 [3]; and Koschmieder, 1966 [7], 1993 [8]) and the Faraday experiment (Kudrolli, Pier, and Gollub, 1998 [9]). The bifurcation for this group can be consulted with, for example, Sattinger, 1978 [11], 1979 [12], 1980 [13]; Buzano and Golubitsky, 1983 [2]; Golubitsky, Stewart, and Schaeffer, 1988 [6]; Dionne, Silber, and Skeldon, 1997 [4]; and Golubitsky and Stewart, 2002 [5]. Nonlinear competition between hexagonal and triangular patterns was studied (Silber and Proctor, 1998 [14]; Skeldon and Silber, 1998 [15]). □

5.5.2 Subgroups for Hexagonal Distributions

As discussed in Sects. 5.3 and 5.4, a hexagonal distribution is described by a sublattice $\mathscr{H}(\alpha, \beta)$ in (5.4) with the parameter (α, β) belonging to (5.11) or (5.12). We assume that n is compatible with (α, β) in the sense of (5.32) that $\mathscr{H}(\alpha, \beta)$ is obtained as the periodic extension of $\mathscr{H}(\alpha, \beta) \cap \mathscr{H}_n$.

The symmetry of $\mathscr{H}(\alpha, \beta) \cap \mathscr{H}_n$ is described by a subgroup of $G = \langle r, s, p_1, p_2 \rangle$, which we denote by $G(\alpha, \beta)$. With notations[3]

$$\Sigma(\alpha, \beta) = \langle r, s, p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha-\beta} \rangle, \tag{5.39}$$

$$\Sigma_0(\alpha, \beta) = \langle r, \quad p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha-\beta} \rangle, \tag{5.40}$$

the subgroup $G(\alpha, \beta)$ is given as follows:

$$G(\alpha, \beta) = \begin{cases} \langle r, s, p_1^{\alpha}, p_2^{\alpha} \rangle = \Sigma(\alpha, 0) & (\alpha \geq 1, \beta = 0) \quad : \text{type V}, \\ \langle r, s, p_1^{2\beta} p_2^{\beta}, p_1^{-\beta} p_2^{\beta} \rangle = \Sigma(2\beta, \beta) & (\beta \geq 1, \alpha = 2\beta) : \text{type M}, \\ \langle r, \quad p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha-\beta} \rangle = \Sigma_0(\alpha, \beta) & (\text{otherwise}) \qquad : \text{type T}, \end{cases} \tag{5.41}$$

where the parameter (α, β) for type T runs over $\{(\alpha, \beta) \mid \alpha > \beta \geq 1, \alpha \neq 2\beta\}$ in (5.25) or $\{(\alpha, \beta) \mid \beta \geq 1, \alpha > 2\beta\}$ in (5.26), depending on the adopted parameter space (5.11) or (5.12).

The parameter (α, β) must be compatible with the lattice size n via (5.35), which restricts (α, β) to stay in a bounded range. Among the hexagonal distributions of type V on the $n \times n$ lattice, we exclude those with $\Sigma(1, 0)$ from our consideration of subgroups since $\Sigma(1, 0) = \langle r, s, p_1, p_2 \rangle$ represents the symmetry of the underlying $n \times n$ hexagonal lattice. That is, we consider $\Sigma(\alpha, 0)$ for $2 \leq \alpha \leq n$ since n is divisible by α by (5.35). A hexagon with the symmetry of $\Sigma(n, 0) = \mathrm{D}_6$, which lacks translational symmetry, is included here as a type V hexagon for theoretical consistency. As for type M, we must have $1 \leq \beta \leq n/3$ in $\Sigma(2\beta, \beta)$ since n is divisible by 3β ($\beta \geq 1$) by (5.35). The parameter for type T, which is dependent on the choice of (5.11) or (5.12), must stay in the range

$$\text{For (5.11)}: \ \{(\alpha, \beta) \mid 1 \leq \beta < \alpha \leq n - 1, \ \alpha \neq 2\beta\}, \tag{5.42}$$

$$\text{For (5.12)}: \ \{(\alpha, \beta) \mid 1 \leq \beta, \ 2\beta < \alpha \leq n - 1\}. \tag{5.43}$$

To sum up, the relevant subgroups of our interest are given by

$$\begin{cases} \Sigma(\alpha, 0) = \langle r, s, p_1^{\alpha}, p_2^{\alpha} \rangle & (2 \leq \alpha \leq n) \qquad : \text{type V}, \\ \Sigma(2\beta, \beta) = \langle r, s, p_1^{2\beta} p_2^{\beta}, p_1^{-\beta} p_2^{\beta} \rangle & (1 \leq \beta \leq n/3) \quad : \text{type M}, \\ \Sigma_0(\alpha, \beta) = \langle r, p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha-\beta} \rangle & ((5.42) \text{ or } (5.43)) : \text{type T}. \end{cases} \tag{5.44}$$

Recall that (α, β) must also satisfy the compatibility condition (5.35).

[3]Subscript "0" to $\Sigma_0(\alpha, \beta)$ indicates the lack of s.

The symmetry of Christaller's hexagonal distributions in Fig. 5.3b–d is expressed as

$$\begin{cases} \Sigma(2,1) & \text{for Christaller's } k = 3 \text{ system,} \\ \Sigma(2,0) & \text{for Christaller's } k = 4 \text{ system,} \\ \Sigma_0(3,1) & \text{for Christaller's } k = 7 \text{ system} \end{cases} \tag{5.45}$$

and the symmetry of Lösch's ten smallest hexagons is shown in Table 5.2.

Lösch's hexagons on the hexagonal lattices that satisfy the compatibility condition (5.35) are listed in Table 5.3 for several lattice sizes n. Recall that the study conducted in this chapter is purely geometric and involves no bifurcation mechanism, whereas the group-theoretic bifurcation analysis for this lattice will be conducted in Chaps. 8 and 9.

5.5.3 *Conjugate Patterns*

For a proper understanding of bifurcation phenomena, it is vital to identify patterns that belong to the same orbit. Let us recall from (2.130) that

$$\Sigma(T(g)\boldsymbol{\lambda}) = g \cdot \Sigma(\boldsymbol{\lambda}) \cdot g^{-1}, \quad g \in G,$$

which means that the symmetry subgroups of solutions on the same orbit are mutually conjugate. Hence, we may focus on a representative subgroup among conjugate subgroups when we discuss possible symmetries appearing in bifurcated solutions in the group-theoretic bifurcation analysis in Chaps. 8 and 9. Patterns with the symmetry of (5.44) serve as the most natural and pertinent choice, as explained below for Christaller's $k = 3$, 4, and 7 systems (Lösch's hexagons with $D = 3$, 4, and 7).

Christaller's $k = 3$ System

The symmetry of Christaller's $k = 3$ system is labeled by the subgroup $\Sigma(2,1) = \langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$ in (5.45). Subgroups that are conjugate to $\Sigma(2,1)$ are

$$p_1 \cdot \Sigma(2,1) \cdot p_1^{-1} = \langle p_2^{-1} r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle,$$
$$p_1^2 \cdot \Sigma(2,1) \cdot p_1^{-2} = \langle p_2^{-2} r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle.$$

The patterns expressed by these conjugate subgroups are illustrated in Fig. 5.7a for the lattice size $n = 6$. It is most natural and pertinent to choose $\Sigma(2,1)$ as a representative among the conjugate subgroups.

Table 5.3 Possible hexagons for several lattice sizes n

n	(α, β)	D	Type	$G(\alpha, \beta)$
2	$(2, 0)$	4	V	$\Sigma(2, 0)$
3	$(3, 0)$	9	V	$\Sigma(3, 0)$
	$(2, 1)$	3	M	$\Sigma(2, 1)$
4	$(2, 0)$	4	V	$\Sigma(2, 0)$
	$(4, 0)$	16	V	$\Sigma(4, 0)$
5	$(5, 0)$	25	V	$\Sigma(5, 0)$
6	$(2, 0)$	4	V	$\Sigma(2, 0)$
	$(3, 0)$	9	V	$\Sigma(3, 0)$
	$(6, 0)$	36	V	$\Sigma(6, 0)$
	$(2, 1)$	3	M	$\Sigma(2, 1)$
	$(4, 2)$	12	M	$\Sigma(4, 2)$
7	$(7, 0)$	49	V	$\Sigma(7, 0)$
	$(3, 1)$	7	T	$\Sigma_0(3, 1)$
	$(3, 2)$	7	T	$\Sigma_0(3, 2)$
8	$(2, 0)$	4	V	$\Sigma(2, 0)$
	$(4, 0)$	16	V	$\Sigma(4, 0)$
	$(8, 0)$	64	V	$\Sigma(8, 0)$
9	$(3, 0)$	9	V	$\Sigma(3, 0)$
	$(9, 0)$	81	V	$\Sigma(9, 0)$
	$(2, 1)$	3	M	$\Sigma(2, 1)$
	$(6, 3)$	27	M	$\Sigma(6, 3)$
10	$(2, 0)$	4	V	$\Sigma(2, 0)$
	$(5, 0)$	25	V	$\Sigma(5, 0)$
	$(10, 0)$	100	V	$\Sigma(10, 0)$
11	$(11, 0)$	121	V	$\Sigma(11, 0)$
12	$(2, 0)$	4	V	$\Sigma(2, 0)$
	$(3, 0)$	9	V	$\Sigma(3, 0)$
	$(4, 0)$	16	V	$\Sigma(4, 0)$
	$(6, 0)$	36	V	$\Sigma(6, 0)$
	$(12, 0)$	144	V	$\Sigma(12, 0)$
	$(2, 1)$	3	M	$\Sigma(2, 1)$
	$(4, 2)$	12	M	$\Sigma(4, 2)$
	$(8, 4)$	48	M	$\Sigma(8, 4)$
13	$(13, 0)$	169	V	$\Sigma(13, 0)$
	$(4, 1)$	13	T	$\Sigma_0(4, 1)$
	$(4, 3)$	13	T	$\Sigma_0(4, 3)$
21	$(3, 0)$	9	V	$\Sigma(3, 0)$
	$(7, 0)$	49	V	$\Sigma(7, 0)$
	$(21, 0)$	21^2	V	$\Sigma(21, 0)$
	$(2, 1)$	3	M	$\Sigma(2, 1)$
	$(14, 7)$	147	M	$\Sigma(14, 7)$
	$(3, 1)$	7	T	$\Sigma_0(3, 1)$
	$(3, 2)$	7	T	$\Sigma_0(3, 2)$
	$(5, 1)$	21	T	$\Sigma_0(5, 1)$
	$(5, 4)$	21	T	$\Sigma_0(5, 4)$
	$(9, 3)$	63	T	$\Sigma_0(9, 3)$
	$(9, 6)$	63	T	$\Sigma_0(9, 6)$
39	$(3, 0)$	9	V	$\Sigma(3, 0)$
	$(13, 0)$	169	V	$\Sigma(13, 0)$
	$(39, 0)$	39^2	V	$\Sigma(39, 0)$
	$(2, 1)$	3	M	$\Sigma(2, 1)$
	$(26, 13)$	507	M	$\Sigma(26, 13)$
	$(4, 1)$	13	T	$\Sigma_0(4, 1)$
	$(4, 3)$	13	T	$\Sigma_0(4, 3)$
	$(7, 2)$	39	T	$\Sigma_0(7, 2)$
	$(7, 5)$	39	T	$\Sigma_0(7, 5)$
	$(12, 3)$	117	T	$\Sigma_0(12, 3)$
	$(12, 9)$	117	T	$\Sigma_0(12, 9)$
42	$(2, 0)$	4	V	$\Sigma(2, 0)$
	$(3, 0)$	9	V	$\Sigma(3, 0)$
	$(6, 0)$	36	V	$\Sigma(6, 0)$
	$(7, 0)$	49	V	$\Sigma(7, 0)$
	$(14, 0)$	196	V	$\Sigma(14, 0)$
	$(21, 0)$	441	V	$\Sigma(21, 0)$
	$(42, 0)$	42^2	V	$\Sigma(42, 0)$
	$(2, 1)$	3	M	$\Sigma(2, 1)$
	$(4, 2)$	12	M	$\Sigma(4, 2)$
	$(14, 7)$	147	M	$\Sigma(14, 7)$
	$(28, 14)$	588	M	$\Sigma(28, 14)$
	$(3, 1)$	7	T	$\Sigma_0(3, 1)$
	$(3, 2)$	7	T	$\Sigma_0(3, 2)$
	$(5, 1)$	21	T	$\Sigma_0(5, 1)$
	$(5, 4)$	21	T	$\Sigma_0(5, 4)$
	$(6, 2)$	28	T	$\Sigma_0(6, 2)$
	$(6, 4)$	28	T	$\Sigma_0(6, 4)$
	$(9, 3)$	63	T	$\Sigma_0(9, 3)$
	$(9, 6)$	63	T	$\Sigma_0(9, 6)$
	$(10, 2)$	84	T	$\Sigma_0(10, 2)$
	$(10, 8)$	84	T	$\Sigma_0(10, 8)$
	$(18, 6)$	252	T	$\Sigma_0(18, 6)$
	$(18, 12)$	252	T	$\Sigma_0(18, 12)$
49	$(7, 0)$	49	V	$\Sigma(7, 0)$
	$(49, 0)$	49^2	V	$\Sigma(49, 0)$
	$(3, 1)$	7	T	$\Sigma_0(3, 1)$
	$(3, 2)$	7	T	$\Sigma_0(3, 2)$
	$(8, 3)$	49	T	$\Sigma_0(8, 3)$
	$(8, 5)$	49	T	$\Sigma_0(8, 5)$
	$(21, 7)$	343	T	$\Sigma_0(21, 7)$
	$(21, 14)$	343	T	$\Sigma_0(21, 14)$

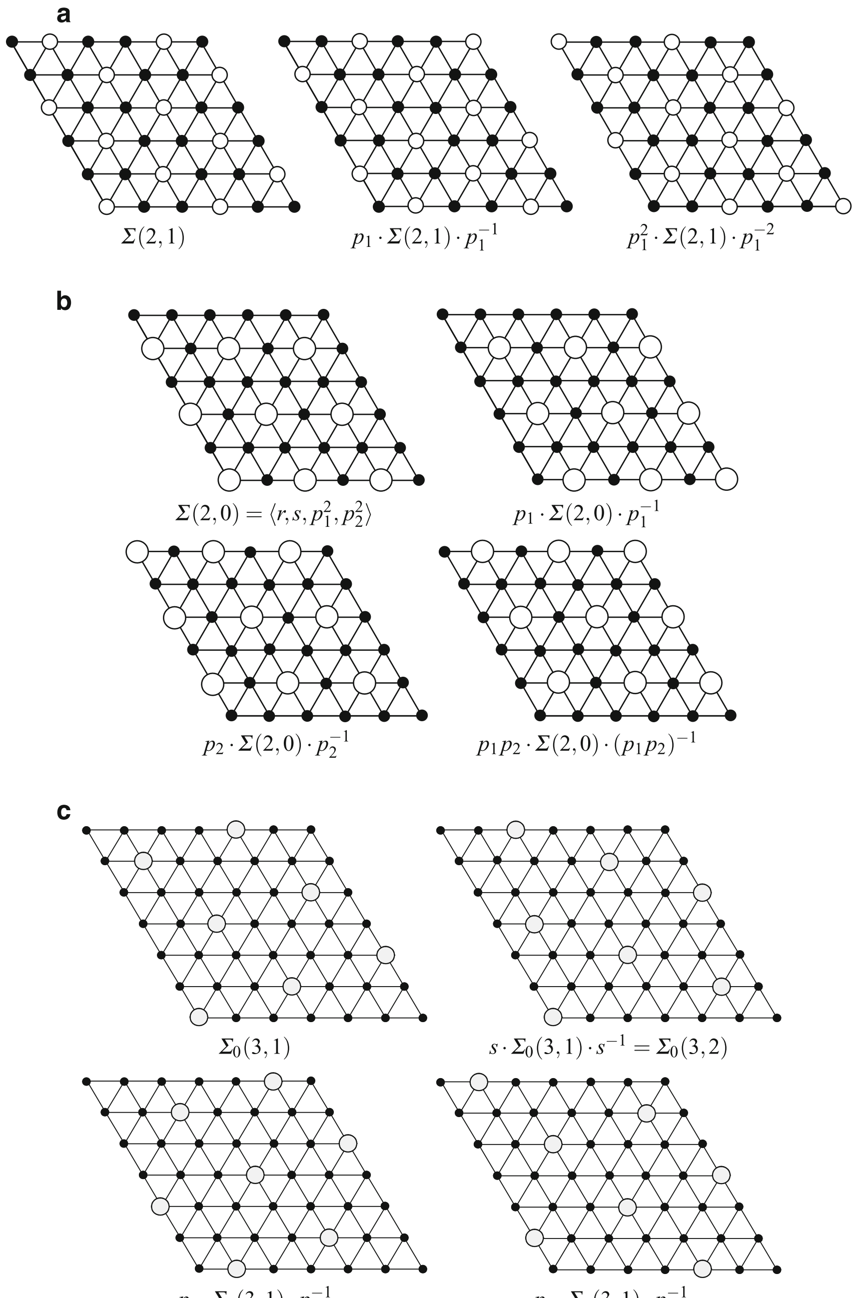

Fig. 5.7 Conjugate patterns for Christaller's $k = 3$, 4, and 7 systems on hexagonal lattices. A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) Christaller's $k = 3$ system on the 6×6 hexagonal lattice. (**b**) Christaller's $k = 4$ system on the 6×6 hexagonal lattice. (**c**) Christaller's $k = 7$ system on the 7×7 hexagonal lattice

Christaller's $k = 4$ System

The symmetry of Christaller's $k = 4$ system is labeled by the subgroup $\Sigma(2,0) = \langle r, s, p_1^2, p_2^2 \rangle$ in (5.45). Subgroups that are conjugate to $\Sigma(2,0)$ are

$$\begin{aligned} p_1 \cdot \Sigma(2,0) \cdot p_1^{-1} &= \langle p_2^{-1} r, s, p_1^2, p_2^2 \rangle, \\ p_2 \cdot \Sigma(2,0) \cdot p_2^{-1} &= \langle p_1 p_2 r, s, p_1^2, p_2^2 \rangle, \\ p_1 p_2 \cdot \Sigma(2,0) \cdot (p_1 p_2)^{-1} &= \langle p_1 r, s, p_1^2, p_2^2 \rangle. \end{aligned}$$

The patterns expressed by these conjugate subgroups are illustrated in Fig. 5.7b for the lattice size $n = 6$. It is most natural and pertinent to choose $\Sigma(2,0)$ as a representative among the conjugate subgroups.

Christaller's $k = 7$ System

The symmetry of Christaller's $k = 7$ system is labeled by the subgroup $\Sigma_0(3,1) = \langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle$ in (5.45). Subgroups that are conjugate to $\Sigma_0(3,1)$, for example, are

$$\begin{aligned} s \cdot \Sigma_0(3,1) \cdot s^{-1} &= \langle r, p_1^3 p_2^2, p_1^{-2} p_2 \rangle = \Sigma_0(3,2), \\ p_1 \cdot \Sigma_0(3,1) \cdot p_1^{-1} &= \langle p_2^{-1} r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle, \\ p_2 \cdot \Sigma_0(3,1) \cdot p_2^{-1} &= \langle p_1 p_2 r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle. \end{aligned}$$

The patterns expressed by these conjugate subgroups are illustrated in Fig. 5.7c for the lattice size $n = 7$. It is most natural and pertinent to choose $\Sigma_0(3,1)$ as a representative among the conjugate subgroups.

5.6 Summary

- Hexagonal lattices have been presented.
- Hexagonal distributions on these lattices are parameterized and classified into several types.
- Symmetry group of the finite hexagonal lattice with periodic boundaries has been presented.
- Subgroups representing hexagonal distributions have been advanced.

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Chapter 6
Irreducible Representations of the Group for Hexagonal Lattice

Abstract An $n \times n$ hexagonal lattice was introduced as a two-dimensional discretized uniform space for economic agglomeration, and the symmetry of this lattice was described in Chap. 5 by the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, which is the semidirect product of D_6 by $\mathbb{Z}_n \times \mathbb{Z}_n$. In this chapter, the irreducible representations of this group are found according to a standard procedure in group representation theory known as the method of little groups, which exploits the semidirect product structure of the group. The group has a structure that admits irreducible representations of various kinds; the dimensions of the irreducible representations over $\mathbb{R}$ are 1, 2, 3, 4, 6, or 12. The concrete forms of these irreducible representations are presented. These forms will be used in the group-theoretic bifurcation analysis in Chaps. 8 and 9 to prove the existence of the hexagonal distributions of Christaller and Lösch.

Keywords Character • Group • Hexagonal lattice • Induced representation • Irreducible representation • Method of little groups • Representation matrix

6.1 Introduction

In the previous chapter, an $n \times n$ hexagonal lattice was introduced as a two-dimensional discretized uniform space for economic agglomeration, and the symmetry group of this lattice was identified as the group (5.36):

$$G = \langle r, s, p_1, p_2 \rangle, \tag{6.1}$$

composed of the dihedral group $\langle r, s \rangle \simeq \mathrm{D}_6$ expressing local regular-hexagonal symmetry and the group $\langle p_1, p_2 \rangle \simeq \mathbb{Z}_n \times \mathbb{Z}_n$ (direct product of two cyclic groups of order n) expressing translational symmetry in two directions. In group-theoretic bifurcation analysis in Chaps. 8 and 9, bifurcating solutions will be found for each irreducible representation of this group, as each irreducible representation is associated with possible bifurcating solutions with certain symmetries. It is,

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therefore, the first step of the analysis to obtain all irreducible representations of the group. This is not difficult for groups with simple structures, such as the dihedral and cyclic groups.

Yet for the group in (6.1) with a far more complicated structure, it might be difficult to list all irreducible representations in an ad hoc way. Fortunately, by the method of little groups in group representation theory, all irreducible representations can be obtained in a systematic manner. In this chapter, this method is described and a complete list of irreducible representations of this group is constructed. It turns out that the irreducible representations over $\mathbb{R}$ are one-, two-, three-, four-, six-, or twelve-dimensional, and all of them are absolutely irreducible. Irreducible representations derived in this manner will be used in the group-theoretic bifurcation analysis in Chaps. 8 and 9 to prove the existence of the hexagonal distributions of Christaller and Lösch.

This chapter is organized as follows. The structure of the group for the hexagonal lattice is studied in Sect. 6.2, the matrix forms of the irreducible representations are listed in Sect. 6.3, and the irreducible representations are derived in Sect. 6.4.

6.2 Structure of the Group for the Hexagonal Lattice

It was observed in Sect. 5.5 that the $n \times n$ hexagonal lattice $\mathscr{H}_n$ in (5.30) enjoys invariance with respect to

- r: counterclockwise rotation about the origin at an angle of $\pi/3$,
- s: reflection $y \mapsto -y$,
- p_1: periodic translation along the $\boldsymbol{\ell}_1$-axis (i.e., the x-axis), and
- p_2: periodic translation along the $\boldsymbol{\ell}_2$-axis.

Accordingly, the symmetry of the hexagonal lattice $\mathscr{H}_n$ is described by the group

$$G = \langle r, s, p_1, p_2 \rangle, \tag{6.2}$$

which is generated by r, s, p_1, and p_2 with the following fundamental relations:

$$\begin{aligned} &r^6 = s^2 = (rs)^2 = {p_1}^n = {p_2}^n = e, \quad p_2 p_1 = p_1 p_2, \\ &rp_1 = p_1 p_2 r, \quad rp_2 = p_1^{-1} r, \quad sp_1 = p_1 s, \quad sp_2 = p_1^{-1} p_2^{-1} s, \end{aligned} \tag{6.3}$$

where e is the identity element. Each element of G can be represented uniquely in the form of

$$s^l r^m {p_1}^i {p_2}^j, \quad l \in \{0, 1\},\ m \in \{0, 1, \ldots, 5\},\ i, j \in \{0, 1, \ldots, n-1\}. \tag{6.4}$$

The group G contains, as its subgroups, the *dihedral group*

$$\langle r, s \rangle \simeq \mathrm{D}_6$$

and *cyclic groups*

$$\langle p_1 \rangle \simeq \mathbb{Z}_n, \quad \langle p_2 \rangle \simeq \mathbb{Z}_n,$$

where $\mathbb{Z}_n$ means the cyclic group of order n. Since $p_1 p_2 = p_2 p_1$ by (6.3), these two cyclic groups form a direct product $\mathbb{Z}_n \times \mathbb{Z}_n$, which is an abelian group.

Moreover, the group G has the structure of the *semidirect product* of D_6 by $\mathbb{Z}_n \times \mathbb{Z}_n$, which means, by definition (Remark 6.1), that $\mathbb{Z}_n \times \mathbb{Z}_n$ is a normal subgroup of G, and each element $g \in G$ is represented uniquely as $g = ha$ with $h \in \mathrm{D}_6$ and $a \in \mathbb{Z}_n \times \mathbb{Z}_n$. We denote this by

$$G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s \rangle \ltimes \langle p_1, p_2 \rangle \tag{6.5}$$

or, in another notation,[1] by

$$G = \mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s \rangle \dot{+} \langle p_1, p_2 \rangle. \tag{6.6}$$

Remark 6.1. Recall from Sect. 2.3.1 that a group G is said to be a *semidirect product* of a subgroup H by another subgroup A, denoted $G = A \rtimes H$ if

- A is a *normal subgroup* of G, and
- each element $g \in G$ is represented uniquely as $g = ah$ with $a \in A$ and $h \in H$.

It should be clear, in the former condition, that a subgroup A of G is called a *normal subgroup* if $g^{-1}ag \in A$ for every $a \in A$ and $g \in G$. The unique representability in the latter condition implies that $A \cap H = \{e\}$, where e is the identity element.

Each element $g \in G$ can also be represented uniquely in an alternative form of $g = ha$ with $h \in H$ and $a \in A$, since $g = ah = h(h^{-1}ah)$ and $h^{-1}ah \in A$ by normality of A. Then it is convenient to write $G = H \ltimes A$ instead. Our group $G = \langle r, s, p_1, p_2 \rangle$ is a semidirect product of $H = \mathrm{D}_6$ by $A = \mathbb{Z}_n \times \mathbb{Z}_n$, and we have adopted the notation $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in (6.5) in accordance with the representation $g = s^l r^m {p_1}^i {p_2}^j$ in (6.4) with $s^l r^m \in \mathrm{D}_6$ and ${p_1}^i {p_2}^j \in \mathbb{Z}_n \times \mathbb{Z}_n$. See Curtis and Reiner, 1962 [1] for more on the definition of the semidirect product. $\square$

6.3 List of Irreducible Representations

The irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ over $\mathbb{R}$ are listed in this section, whereas their derivation is given in Sect. 6.4.

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[1] Notation $\dot{+}$ in (6.6) for semidirect product is used, e.g., in Golubitsky, Stewart, and Schaeffer, 1988 [2] and Ikeda and Murota, 2010 [3].

Table 6.1 Number N_d of d-dimensional irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

$n \setminus d$	1	2	3	4	6	12		$n \setminus d$	1	2	3	4	6	12	
	N_1	N_2	N_3	N_4	N_6	N_{12}	$\sum N_d$		N_1	N_2	N_3	N_4	N_6	N_{12}	$\sum N_d$
1	4	2					6	13	4	2			24	8	38
2	4	2	4				10	14	4	2	4		24	10	44
3	4	4		1	2		11	15	4	4		1	26	12	47
4	4	2	4		4		14	16	4	2	4		28	14	52
5	4	2			8		14	17	4	2			32	16	54
6	4	4	4	1	6	1	20	18	4	4	4	1	30	19	62
7	4	2			12	1	19	19	4	2			36	21	63
8	4	2	4		12	2	24	20	4	2	4		36	24	70
9	4	4		1	14	3	26	21	4	4		1	38	27	74
10	4	2	4		16	4	30	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
11	4	2			20	5	31	42	4	4	4	1	78	127	218
12	4	4	4	1	18	7	38								

6.3.1 Number of Irreducible Representations

The irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ over $\mathbb{R}$ are one-, two-, three-, four-, six-, or twelve-dimensional. The number N_d of d-dimensional irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ depends on n, as shown below:

	1	2	3	4	6	12
$n \setminus d$	N_1	N_2	N_3	N_4	N_6	N_{12}
$6m$	4	4	4	1	$2n-6$	$(n^2-6n+12)/12$
$6m \pm 1$	4	2	0	0	$2n-2$	$(n^2-6n+5)/12$
$6m \pm 2$	4	2	4	0	$2n-4$	$(n^2-6n+8)/12$
$6m \pm 3$	4	4	0	1	$2n-4$	$(n^2-6n+9)/12$

(6.7)

where m denotes a positive integer. For some values of n, the concrete numbers N_d of the d-dimensional irreducible representations are listed in Table 6.1. This table for $n = 1$ shows that $\mathrm{D}_6 \ltimes (\mathbb{Z}_1 \times \mathbb{Z}_1)$, being isomorphic to D_6, has four one-dimensional irreducible representations and two two-dimensional ones, which is consistent with (3.18) for D_6. Six-dimensional irreducible representations exist for $n \geq 3$ and 12-dimensional ones appear for $n \geq 6$.

We have the relation

$$\sum_d d^2 N_d = 1^2 N_1 + 2^2 N_2 + 3^2 N_3 + 4^2 N_4 + 6^2 N_6 + 12^2 N_{12} = 12n^2, \tag{6.8}$$

Table 6.2 Irreducible representations of $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

$n \setminus d$	1	2	3	4	6	12
$6m$	$(1;+,+)$, $(1;+,-)$	$(2;1), (2;2)$	$(3;+,+)$, $(3;+,-)$	$(4;1)$	$(6;k,0;+)$, $(6;k,0;-)$	$(12;k,\ell)$
	$(1;-,+)$, $(1;-,-)$	$(2;+), (2;-)$	$(3;-,+)$, $(3;-,-)$		$(6;k,k;+)$, $(6;k,k;-)$	
$6m \pm 1$	$(1;+,+)$, $(1;+,-)$	$(2;1), (2;2)$			$(6;k,0;+)$, $(6;k,0;-)$	$(12;k,\ell)$
	$(1;-,+)$, $(1;-,-)$				$(6;k,k;+)$, $(6;k,k;-)$	
$6m \pm 2$	$(1;+,+)$, $(1;+,-)$	$(2;1), (2;2)$	$(3;+,+)$, $(3;+,-)$		$(6;k,0;+)$, $(6;k,0;-)$	$(12;k,\ell)$
	$(1;-,+)$, $(1;-,-)$		$(3;-,+)$, $(3;-,-)$		$(6;k,k;+)$, $(6;k,k;-)$	
$6m \pm 3$	$(1;+,+)$, $(1;+,-)$	$(2;1), (2;2)$		$(4;1)$	$(6;k,0;+)$, $(6;k,0;-)$	$(12;k,\ell)$
	$(1;-,+)$, $(1;-,-)$	$(2;+), (2;-)$			$(6;k,k;+)$, $(6;k,k;-)$	

$(6;k,0;+), (6;k,0;-)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor$ in (6.26)
$(6;k,k;+), (6;k,k;-)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor,\ k \ne n/3$ in (6.27)
$(12;k,\ell)$ with $1 \le \ell \le k-1,\ 2k+\ell \le n-1$ in (6.35)

which is a special case of the well-known general identity (2.35) for the number of irreducible representations over $\mathbb{C}$. This formula applies since all the irreducible representations over $\mathbb{R}$ of $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ are absolutely irreducible.[2]

In Sects. 6.3.2–6.3.7, the matrix forms of the irreducible representations of respective dimensions are shown together with their characters. Table 6.2 is a preview summary, referring to names of irreducible representations, such as $(1;+,+)$ and $(12;k,\ell)$, to be introduced in the following subsections.

6.3.2 One-Dimensional Irreducible Representations

The group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (6.5) has four one-dimensional irreducible representations. They are labeled $(1;+,+)$, $(1;+,-)$, $(1;-,+)$, $(1;-,-)$ and are given by

$$\begin{aligned}
&T^{(1;+,+)}(r) = 1, \quad T^{(1;+,+)}(s) = 1, \quad T^{(1;+,+)}(p_1) = 1,\ T^{(1;+,+)}(p_2) = 1,\\
&T^{(1;+,-)}(r) = 1, \quad T^{(1;+,-)}(s) = -1,\ T^{(1;+,-)}(p_1) = 1,\ T^{(1;+,-)}(p_2) = 1,\\
&T^{(1;-,+)}(r) = -1,\ T^{(1;-,+)}(s) = 1, \quad T^{(1;-,+)}(p_1) = 1,\ T^{(1;-,+)}(p_2) = 1,\\
&T^{(1;-,-)}(r) = -1,\ T^{(1;-,-)}(s) = -1,\ T^{(1;-,-)}(p_1) = 1,\ T^{(1;-,-)}(p_2) = 1.
\end{aligned} \tag{6.9}$$

[2] See Remark 2.4 in Sect. 2.3.3 for the definition of absolute irreducibility.

The characters $\chi^{\mu}(g) = \mathrm{Tr}\, T^{\mu}(g)$, which are equal to $T^{\mu}(g)$ for one-dimensional representations, are given as follows for $\mu = (1;+,+), (1;+,-), (1;-,+), (1;-,-)$:

$$
\begin{array}{l|cccc}
g & \chi^{(1;+,+)}(g) & \chi^{(1;+,-)}(g) & \chi^{(1;-,+)}(g) & \chi^{(1;-,-)}(g) \\
\hline
{p_1}^i\,{p_2}^j & 1 & 1 & 1 & 1 \\
r\,{p_1}^i\,{p_2}^j & 1 & 1 & -1 & -1 \\
r^2\,{p_1}^i\,{p_2}^j & 1 & 1 & 1 & 1 \\
r^3\,{p_1}^i\,{p_2}^j & 1 & 1 & -1 & -1 \\
r^4\,{p_1}^i\,{p_2}^j & 1 & 1 & 1 & 1 \\
r^5\,{p_1}^i\,{p_2}^j & 1 & 1 & -1 & -1 \\
\hline
s\,r^m\,{p_1}^i\,{p_2}^j\ \ (m: \text{even}) & 1 & -1 & 1 & -1 \\
\qquad\qquad\quad (m: \text{odd}) & 1 & -1 & -1 & 1 \\
\hline
\end{array}
\tag{6.10}
$$

where $i, j = 0, 1, \dots, n-1$ and $m = 0, 1, \dots, 5$.

6.3.3 Two-Dimensional Irreducible Representations

The group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (6.5) has four or two two-dimensional irreducible representations depending on whether n is a multiple of 3 or not. Two two-dimensional irreducible representations, denoted as $(2; j)$ $(j = 1, 2)$, exist for all n and are defined by

$$
T^{(2;j)}(r) = \begin{bmatrix} \cos(2\pi j/6) & -\sin(2\pi j/6) \\ \sin(2\pi j/6) & \cos(2\pi j/6) \end{bmatrix}, \quad T^{(2;j)}(s) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \tag{6.11}
$$

$$
T^{(2;j)}(p_1) = T^{(2;j)}(p_2) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{6.12}
$$

The other two two-dimensional irreducible representations, denoted as $(2; \sigma)$ $(\sigma \in \{+, -\})$, exist when n is a multiple of 3, and are defined by

$$
T^{(2;\sigma)}(r) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad T^{(2;\sigma)}(s) = \sigma \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \tag{6.13}
$$

$$
T^{(2;\sigma)}(p_1) = T^{(2;\sigma)}(p_2) = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}. \tag{6.14}
$$

The characters $\chi^{\mu}(g) = \mathrm{Tr}\, T^{\mu}(g)$ are given as follows for $\mu = (2; 1), (2; 2), (2; +), (2; -)$. For the representations $\mu = (2; 1), (2; 2)$ in (6.11) and (6.12), we have

g	$\chi^{(2;1)}(g)$	$\chi^{(2;2)}(g)$
$p_1{}^i p_2{}^j$	2	2
$r p_1{}^i p_2{}^j$	1	−1
$r^2 p_1{}^i p_2{}^j$	−1	−1
$r^3 p_1{}^i p_2{}^j$	−2	2
$r^4 p_1{}^i p_2{}^j$	−1	−1
$r^5 p_1{}^i p_2{}^j$	1	−1
$s r^m p_1{}^i p_2{}^j$	0	0

(6.15)

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, \ldots, 5$. For the representations $\mu = (2; +), (2; -)$ in (6.13) and (6.14), we have

g		$\chi^{(2;+)}(g)$	$\chi^{(2;-)}(g)$
$r^m p_1{}^i p_2{}^j$	(m : even)	$2\cos(2(i+j)\pi/3)$	$2\cos(2(i+j)\pi/3)$
	(m : odd)	0	0
$s r^m p_1{}^i p_2{}^j$	(m : even)	$2\cos(2(i+j)\pi/3)$	$-2\cos(2(i+j)\pi/3)$
	(m : odd)	0	0

(6.16)

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, \ldots, 5$.

6.3.4 Three-Dimensional Irreducible Representations

The group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (6.5) has four three-dimensional irreducible representations when n is even. They are labeled $(3; +, +)$, $(3; +, -)$, $(3; -, +)$, $(3; -, -)$ and are given by

$$T^{(3;+,+)}(r) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \quad T^{(3;+,+)}(s) = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}; \tag{6.17}$$

$$T^{(3;+,-)}(r) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \quad T^{(3;+,-)}(s) = \begin{bmatrix} & & -1 \\ & -1 & \\ -1 & & \end{bmatrix}; \tag{6.18}$$

$$T^{(3;-,+)}(r) = \begin{bmatrix} & & -1 \\ 1 & & \\ & 1 & \end{bmatrix}, \quad T^{(3;-,+)}(s) = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}; \tag{6.19}$$

$$T^{(3;-,-)}(r) = \begin{bmatrix} & & -1 \\ 1 & & \\ & 1 & \end{bmatrix}, \quad T^{(3;-,-)}(s) = \begin{bmatrix} & & -1 \\ & -1 & \\ -1 & & \end{bmatrix}; \tag{6.20}$$

and

$$T^{(3;\sigma_r,\sigma_s)}(p_1) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \quad T^{(3;\sigma_r,\sigma_s)}(p_2) = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \tag{6.21}$$

for $\sigma_r, \sigma_s \in \{+, -\}$. The representation matrices for p_1 and p_2 are independent of (σ_r, σ_s).

The characters $\chi^{(3;\sigma_r,\sigma_s)}(g) = \mathrm{Tr}\, T^{(3;\sigma_r,\sigma_s)}(g)$ for $\sigma_r, \sigma_s \in \{+, -\}$ are given as follows:

(6.22)

g	$\chi^{(3;\sigma_r,\sigma_s)}(g)$
${p_1}^i {p_2}^j$	$(-1)^i + (-1)^j + (-1)^{i+j}$
$r^3 {p_1}^i {p_2}^j$	$\sigma_r[(-1)^i + (-1)^j + (-1)^{i+j}]$
$r^m {p_1}^i {p_2}^j$ $(m = 1, 2, 4, 5)$	0
$s {p_1}^i {p_2}^j$	$\sigma_s(-1)^j$
$s r {p_1}^i {p_2}^j$	$\sigma_r\sigma_s(-1)^{i+j}$
$s r^2 {p_1}^i {p_2}^j$	$\sigma_s(-1)^i$
$s r^3 {p_1}^i {p_2}^j$	$\sigma_r\sigma_s(-1)^j$
$s r^4 {p_1}^i {p_2}^j$	$\sigma_s(-1)^{i+j}$
$s r^5 {p_1}^i {p_2}^j$	$\sigma_r\sigma_s(-1)^i$

where $i, j = 0, 1, \ldots, n-1$.

Remark 6.2. Three-dimensional irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ for $n = 2$ are treated in Ikeda and Murota, 2010, Chap. 16 [3], where the following matrix representations are adopted:

$$T^{(3,1)}(r) = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \quad T^{(3,1)}(s) = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix};$$

$$T^{(3,2)}(r) = \begin{bmatrix} & -1 & \\ & & -1 \\ -1 & & \end{bmatrix}, \quad T^{(3,2)}(s) = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix};$$

$$T^{(3,3)}(r) = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \quad T^{(3,3)}(s) = \begin{bmatrix} -1 & & \\ & & -1 \\ & -1 & \end{bmatrix};$$

$$T^{(3,4)}(r) = \begin{bmatrix} & -1 & \\ & & -1 \\ -1 & & \end{bmatrix}, \quad T^{(3,4)}(s) = \begin{bmatrix} -1 & & \\ & & -1 \\ & -1 & \end{bmatrix};$$

and

$$T^{(3,j)}(p_1) = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad T^{(3,j)}(p_2) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \qquad j = 1, \ldots, 4.$$

It can be verified that the representations $(3; \sigma_r, \sigma_s)$ are equivalent to $(3, j)$ with the following correspondences:

$$(3; +, +) \simeq (3, 1), \quad (3; +, -) \simeq (3, 3), \quad (3; -, +) \simeq (3, 2), \quad (3; -, -) \simeq (3, 4).$$

For example, we have

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}^{-1} T^{(3;+,+)}(g) \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} = T^{(3,1)}(g), \qquad g = r, s, p_1, p_2.$$

□

6.3.5 Four-Dimensional Irreducible Representations

The group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (6.5) has one four-dimensional irreducible representation when n is a multiple of 3. This irreducible representation, denoted as $(4; 1)$, is defined by

$$T^{(4;1)}(r) = \begin{bmatrix} & R_6 \\ R_6 & \end{bmatrix}, \quad T^{(4;1)}(s) = \begin{bmatrix} & S \\ S & \end{bmatrix}, \tag{6.23}$$

$$T^{(4;1)}(p_1) = \begin{bmatrix} R_6^2 & \\ & R_6^{-2} \end{bmatrix}, \quad T^{(4;1)}(p_2) = \begin{bmatrix} R_6^2 & \\ & R_6^{-2} \end{bmatrix}, \tag{6.24}$$

where

$$R_6 = \begin{bmatrix} \cos(2\pi/6) & -\sin(2\pi/6) \\ \sin(2\pi/6) & \cos(2\pi/6) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

The character $\chi^{(4;1)}(g) = \mathrm{Tr}\, T^{(4;1)}(g)$ is given as follows:

(6.25)

g		$\chi^{(4;1)}(g)$
$r^m {p_1}^i {p_2}^j$	$(m:\text{even})$	$4\cos(m\pi/3)\cos(2(i+j)\pi/3)$
	$(m:\text{odd})$	0
$s r^m {p_1}^i {p_2}^j$		0

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, \ldots, 5$.

6.3.6 Six-Dimensional Irreducible Representations

The group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (6.5), with $n \geq 3$, has six-dimensional irreducible representations. We can designate them by

$$(6; k, 0, \sigma) \ \text{ with } \ 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \ \sigma \in \{+, -\}; \ \text{and} \tag{6.26}$$

$$(6; k, k, \sigma) \ \text{ with } \ 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \ k \neq \frac{n}{3}; \ \sigma \in \{+, -\}. \tag{6.27}$$

Here for a real number x, $\lfloor x \rfloor$ denotes the largest integer not larger than x. The number of six-dimensional irreducible representations is given by

$$N_6 = \begin{cases} 2n-6 \ (n = 6m), \\ 2n-2 \ (n = 6m \pm 1), \\ 2n-4 \ (n = 6m \pm 2), \\ 2n-4 \ (n = 6m \pm 3). \end{cases} \tag{6.28}$$

The irreducible representation $(6; k, 0, \sigma)$ is given by

$$T^{(6;k,0,\sigma)}(r) = \begin{bmatrix} & & S \\ S & & \\ & S & \end{bmatrix}, \quad T^{(6;k,0,\sigma)}(s) = \sigma \begin{bmatrix} & & S \\ & S & \\ S & & \end{bmatrix}, \tag{6.29}$$

$$T^{(6;k,0,\sigma)}(p_1) = \begin{bmatrix} R^k & & \\ & I & \\ & & R^{-k} \end{bmatrix}, \quad T^{(6;k,0,\sigma)}(p_2) = \begin{bmatrix} I & & \\ & R^{-k} & \\ & & R^k \end{bmatrix}, \tag{6.30}$$

where

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{6.31}$$

The other type of irreducible representation $(6; k, k, \sigma)$ is given by

$$T^{(6;k,k,\sigma)}(r) = \begin{bmatrix} & & S \\ S & & \\ & S & \end{bmatrix}, \quad T^{(6;k,k,\sigma)}(s) = \sigma \begin{bmatrix} & I & \\ I & & \\ & & I \end{bmatrix}, \tag{6.32}$$

$$T^{(6;k,k,\sigma)}(p_1) = \begin{bmatrix} R^k & & \\ & R^k & \\ & & R^{-2k} \end{bmatrix}, \quad T^{(6;k,k,\sigma)}(p_2) = \begin{bmatrix} R^k & & \\ & R^{-2k} & \\ & & R^k \end{bmatrix}. \tag{6.33}$$

The characters $\chi^{(6;k,0,\sigma)}(g) = \mathrm{Tr}\, T^{(6;k,0,\sigma)}(g)$ and $\chi^{(6;k,k,\sigma)}(g) = \mathrm{Tr}\, T^{(6;k,k,\sigma)}(g)$ for $\sigma \in \{+,-\}$ are given as follows:

g	$\chi^{(6;k,0,\sigma)}(g)$	$\chi^{(6;k,k,\sigma)}(g)$
${p_1}^i {p_2}^j$	$2[\cos(ki\theta) + \cos(-kj\theta) + \cos(-k(i-j)\theta)]$	$2[\cos(k(i+j)\theta) + \cos(k(i-2j)\theta) + \cos(-k(2i-j)\theta)]$
$r^m {p_1}^i {p_2}^j$ $(m = 1, \dots, 5)$	0	0
$s {p_1}^i {p_2}^j$	0	$2\sigma\cos(-k(2i-j)\theta)$
$sr {p_1}^i {p_2}^j$	$2\sigma\cos(-k(i-j)\theta)$	0
$sr^2 {p_1}^i {p_2}^j$	0	$2\sigma\cos(k(i-2j)\theta)$
$sr^3 {p_1}^i {p_2}^j$	$2\sigma\cos(-kj\theta)$	0
$sr^4 {p_1}^i {p_2}^j$	0	$2\sigma\cos(k(i+j)\theta)$
$sr^5 {p_1}^i {p_2}^j$	$2\sigma\cos(ki\theta)$	0

(6.34)

where $\theta = 2\pi/n$ and $i, j = 0, 1, \dots, n-1$.

Remark 6.3. We note the following:

- $T^{(6;k,0,+)}(r) = T^{(6;k,0,-)}(r) = T^{(6;k,k,+)}(r) = T^{(6;k,k,-)}(r)$ for each k.
- $T^{(6;k,0,\sigma)}(s)$ and $T^{(6;k,k,\sigma)}(s)$ are independent of k.
- $T^{(6;k,0,\sigma)}(p_1)$ and $T^{(6;k,k,\sigma)}(p_1)$ are independent of σ.
- $T^{(6;k,0,\sigma)}(p_2)$ and $T^{(6;k,k,\sigma)}(p_2)$ are independent of σ.
- By defining

$$T^{(6;k,\ell)}(p_1) = \begin{bmatrix} R^k & & \\ & R^\ell & \\ & & R^{-k-\ell} \end{bmatrix}, \quad T^{(6;k,\ell)}(p_2) = \begin{bmatrix} R^\ell & & \\ & R^{-k-\ell} & \\ & & R^k \end{bmatrix},$$

  we can represent

$$T^{(6;k,0,\sigma)}(p_1) = T^{(6;k,0)}(p_1), \quad T^{(6;k,0,\sigma)}(p_2) = T^{(6;k,0)}(p_2),$$
$$T^{(6;k,k,\sigma)}(p_1) = T^{(6;k,k)}(p_1), \quad T^{(6;k,k,\sigma)}(p_2) = T^{(6;k,k)}(p_2).$$

- By defining

$$\chi^{(6;k,\ell)}({p_1}^i {p_2}^j) = 2[\cos((ki + \ell j)\theta) + \cos((\ell i - (k+\ell)j)\theta) + \cos((-(k+\ell)i + kj)\theta)]$$

  with $\theta = 2\pi/n$, we have

$$\chi^{(6;k,0,\sigma)}({p_1}^i {p_2}^j) = \chi^{(6;k,0)}({p_1}^i {p_2}^j), \quad \chi^{(6;k,k,\sigma)}({p_1}^i {p_2}^j) = \chi^{(6;k,k)}({p_1}^i {p_2}^j).$$

□

6.3.7 Twelve-Dimensional Irreducible Representations

The group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (6.5), with $n \geq 6$, has 12-dimensional irreducible representations. We can designate them by $(12; k, \ell)$ with

$$1 \leq \ell \leq k-1, \quad 2k+\ell \leq n-1. \tag{6.35}$$

The number of 12-dimensional irreducible representations is given by

$$N_{12} = \begin{cases} (n^2-6n+12)/12 & (n=6m), \\ (n^2-6n+5)/12 & (n=6m\pm 1), \\ (n^2-6n+8)/12 & (n=6m\pm 2), \\ (n^2-6n+9)/12 & (n=6m\pm 3). \end{cases} \tag{6.36}$$

The irreducible representation $(12; k, \ell)$ is defined as

$$T^{(12;k,\ell)}(r) = \left[\begin{array}{ccc|ccc} & & S & & & \\ S & & & & & \\ & S & & & & \\ \hline & & & S & & \\ & & & & S & \\ & & & S & & \end{array}\right], \quad T^{(12;k,\ell)}(s) = \left[\begin{array}{ccc|ccc} & & & I & & \\ & & & & I & \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \\ & & I & & & \end{array}\right], \tag{6.37}$$

$$T^{(12;k,\ell)}(p_1) = \left[\begin{array}{ccc|ccc} R^k & & & & & \\ & R^\ell & & & & \\ & & R^{-k-\ell} & & & \\ \hline & & & R^k & & \\ & & & & R^\ell & \\ & & & & & R^{-k-\ell} \end{array}\right],$$

$$T^{(12;k,\ell)}(p_2) = \left[\begin{array}{ccc|ccc} R^\ell & & & & & \\ & R^{-k-\ell} & & & & \\ & & R^k & & & \\ \hline & & & R^{-k-\ell} & & \\ & & & & R^k & \\ & & & & & R^\ell \end{array}\right] \tag{6.38}$$

with

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{6.39}$$

The characters $\chi^{(12;k,\ell)}(g) = \mathrm{Tr}\, T^{(12;k,\ell)}(g)$ are given as follows. For $g = {p_1}^i {p_2}^j$, being free from r and s, we have

$$\begin{aligned}\chi^{(12;k,\ell)}({p_1}^i {p_2}^j) = 2[&\cos((ki + \ell j)\theta) + \cos((\ell i - (k+\ell)j)\theta)\\ &+ \cos((-(k+\ell)i + kj)\theta) + \cos((ki - (k+\ell)j)\theta)\\ &+ \cos((\ell i + kj)\theta) + \cos((-(k+\ell)i + \ell j)\theta)],\end{aligned} \quad (6.40)$$

where $\theta = 2\pi/n$ and $i, j = 0, 1, \ldots, n-1$. For other g, we have $\chi^{(12;k,\ell)}(g) = 0$.

6.4 Derivation of Irreducible Representations

The group $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in (6.5) for the $n \times n$ hexagonal lattice is a semidirect product of $H = \langle r, s\rangle = \mathrm{D}_6$ by an abelian group $A = \langle p_1, p_2\rangle = \mathbb{Z}_n \times \mathbb{Z}_n$. The irreducible representations over $\mathbb{C}$ of such a group can be constructed in a systematic manner by the method of little groups of Wigner and Mackey. It turns out that the irreducible representations over $\mathbb{C}$ are one-, two-, three-, four-, six-, or twelve-dimensional, and they are representable over $\mathbb{R}$.

6.4.1 Method of Little Groups

A systematic method, called the *method of little groups*, for constructing irreducible representations of a general group with the structure of semidirect product by an abelian group is described in this subsection. See Sect. 8.2 of Serre, 1977 [4] for the theoretical background of this procedure.

Let G be a group that is a semidirect product[3] of a group H and an abelian group A. This means that A is a normal subgroup of G, and each element $g \in G$ is represented uniquely as $g = ah$ with $a \in A$ and $h \in H$.

Since A is abelian, every irreducible representation of A over $\mathbb{C}$ is one-dimensional and is identified with its character χ. Accordingly, the set of all irreducible representations of A over $\mathbb{C}$ can be denoted as

$$X = \{\chi^i \mid i \in R(A)\} \quad (6.41)$$

with a suitable index set $R(A)$ as in (2.34) in Sect. 2.3.3. For $\chi \in X$ and $g \in G$, we define a function ${}^g\chi$ on A by

$${}^g\chi(a) = \chi(g^{-1}ag), \quad a \in A, \quad (6.42)$$

which is also a character of A, belonging to X. This defines an action of G on X.

[3] See Remark 6.1 in Sect. 6.2 for the definition of semidirect product.

With reference to the action of G on X, we classify the elements of X into orbits. It should be noted that, for $g = bh$ with $b \in A$ and $h \in H$, we have

$$^{g}\chi(a) = \chi((bh)^{-1}a(bh)) = \chi(h^{-1}ah) = {}^{h}\chi(a), \quad a \in A,$$

in which $b^{-1}ab = a$ since A is abelian. Hence, the orbits can in fact be obtained by the action of the subgroup H on X, instead of that of G. Denote by

$$\{\chi^i \mid i \in R(A)/H\} \tag{6.43}$$

a system of representatives from the orbits, where $R(A)/H$ is an index set, or the set of "names" of the orbit. This means that

- $\chi^i \in X$ for each $i \in R(A)/H$,
- for distinct i and j in $R(A)/H$, $\chi^i \neq {}^{h}(\chi^j)$ for any $h \in H$, and
- for each $\chi \in X$, there exist some $i \in R(A)/H$ and $h \in H$ such that $\chi = {}^{h}(\chi^i)$.

For each $i \in R(A)/H$, we define

$$H^i = \{h \in H \mid {}^{h}(\chi^i) = \chi^i\}, \tag{6.44}$$

which is a subgroup of H associated with the orbit i, and

$$G^i = \{ah \mid a \in A,\ h \in H^i\}, \tag{6.45}$$

which is a subgroup of G, called the *little group*. Noting that each element of G^i can be represented as ah with $a \in A$ and $h \in H^i$, we define a function $\tilde{\chi}^i$ on G^i by

$$\tilde{\chi}^i(ah) = \chi^i(a), \quad a \in A,\ h \in H^i, \tag{6.46}$$

which is a one-dimensional representation (a character of degree one) of G^i.

Let T^μ be an irreducible representation of H^i over $\mathbb{C}$ indexed by $\mu \in R(H^i)$. Then the matrix-valued function $T^{(i,\mu)}$ defined on G^i of (6.45) by

$$T^{(i,\mu)}(ah) = \chi^i(a)T^\mu(h), \quad a \in A,\ h \in H^i \tag{6.47}$$

is an irreducible representation of G^i. Denote by $\tilde{T}^{(i,\mu)}$ the induced representation of G obtained from $T^{(i,\mu)}$ (see Remark 6.4 below). Then $\tilde{T}^{(i,\mu)}$ is an irreducible representation of G. Moreover, all the irreducible representations of G can be obtained in this manner, and $\tilde{T}^{(i,\mu)}$'s are mutually inequivalent for different (i, μ). Thus, the irreducible representations of G are indexed by (i, μ), i.e.,

$$R(G) = \{(i, \mu) \mid i \in R(A)/H,\ \mu \in R(H^i)\} \tag{6.48}$$

in the notation of Sect. 2.3.3 and

$$\{\tilde{T}^{(i,\mu)} \mid i \in R(A)/H,\ \mu \in R(H^i)\} \tag{6.49}$$

gives a complete list of irreducible representations of G over $\mathbb{C}$.

Remark 6.4. The induced representation is explained here. Let G be a group, G' be a subgroup of G, and T' be a representation of G' of dimension N'. Consider the *coset decomposition*

$$G = g_1 G' + g_2 G' + \cdots + g_m G', \tag{6.50}$$

where $g_j \in G$ for $j = 1, \ldots, m$ and $m = |G|/|G'|$. Each $g \in G$ causes a permutation of $(g_1, g_2, \ldots, g_m)$ to $(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(m)})$ according to the equation

$$g g_j = g_{\pi(j)} f_j, \quad f_j \in G' \tag{6.51}$$

for $j = 1, \ldots, m$. Note that the choice of $(g_1, g_2, \ldots, g_m)$ is not unique, but once this is fixed, f_j is uniquely determined for each g.

Define $\tilde{T}(g)$ to be an $mN' \times mN'$ matrix with rows and columns partitioned into m blocks of size N' such that the $(\pi(j), j)$-block of $\tilde{T}(g)$ equals $T'(f_j)$, whereas the (i, j)-block of $\tilde{T}(g)$ equals O if $i \neq \pi(j)$. Note that this is well defined, since f_j and $\pi(j)$ are uniquely determined from g, and $T'(f_j)$ for $j = 1, \ldots, m$ are assumed to be given. The family of matrices $\{\tilde{T}(g) \mid g \in G\}$ is a representation of G of dimension mN', called the *induced representation*. For example, if $m = 3$, $(\pi(1), \pi(2), \pi(3)) = (2, 3, 1)$, we have

$$\tilde{T}(g) = \begin{bmatrix} & & T'(f_3) \\ T'(f_1) & & \\ & T'(f_2) & \end{bmatrix}.$$

We shall apply this construction to $T' = T^{(i,\mu)}$ on $G' = G^i$ to obtain $\tilde{T} = \tilde{T}^{(i,\mu)}$, where the dimension N' of $T^{(i,\mu)}$ is equal to that of T^{μ} by (6.47). □

6.4.2 Orbit Decomposition and Little Groups

We apply the method of little groups in Sect. 6.4.1 to

$$A = \mathbb{Z}_n \times \mathbb{Z}_n = \langle p_1 \rangle \times \langle p_2 \rangle, \quad H = \mathrm{D}_6 = \langle r, s \rangle$$

to obtain the irreducible representations of $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in (6.5).

As the first step we determine the system of representatives (6.43) in the orbit decomposition of X. Since $A = \mathbb{Z}_n \times \mathbb{Z}_n$ is an abelian group, all the irreducible representations are one-dimensional. The set X of irreducible representations of $A = \mathbb{Z}_n \times \mathbb{Z}_n$ is indexed by

$$R(A) = \{(k, \ell) \mid 0 \leq k \leq n-1,\ 0 \leq \ell \leq n-1\}, \tag{6.52}$$

where (k, ℓ) denotes a one-dimensional representation (or character) $\chi^{(k,\ell)}$ defined by

$$\chi^{(k,\ell)}(p_1) = \omega^k, \qquad \chi^{(k,\ell)}(p_2) = \omega^\ell \tag{6.53}$$

with

$$\omega = \exp(2\pi \mathrm{i}/n). \tag{6.54}$$

We extend the notation (k, ℓ) for any integers, to designate the element (k', ℓ') of $R(A)$ with $k' \equiv k \mod n$ and $\ell' \equiv \ell \mod n$.

For the orbit decomposition of X by H, we compute $h^{-1} p_1 h$ and $h^{-1} p_2 h$ for $h \in H$, to obtain

h	e	r	r^2	r^3	r^4	r^5	s	sr	sr^2	sr^3	sr^4	sr^5
$h^{-1} p_1 h$	p_1	p_2^{-1}	$p_1^{-1} p_2^{-1}$	p_1^{-1}	p_2	$p_1 p_2$	p_1	p_2^{-1}	$p_1^{-1} p_2^{-1}$	p_1^{-1}	p_2	$p_1 p_2$
$h^{-1} p_2 h$	p_2	$p_1 p_2$	p_1	p_2^{-1}	$p_1^{-1} p_2^{-1}$	p_1^{-1}	$p_1^{-1} p_2^{-1}$	p_1^{-1}	p_2	$p_1 p_2$	p_1	p_2^{-1}

(6.55)

For example, for $h = s$, we have $(h^{-1} p_1 h, h^{-1} p_2 h) = (p_1, p_1^{-1} p_2^{-1})$, and we see, by (6.53), that the action of s in (6.42) is given as ${}^{s}\chi^{(k,\ell)} = \chi^{(k,-k-\ell)}$, which is expressed symbolically as $(k, \ell) \Rightarrow (k, -k - \ell)$. In this manner, we can obtain the following orbit containing (k, ℓ):

$$\boxed{\begin{array}{lll} (k+\ell, -k) & \leftarrow (\ell, -k-\ell) & \leftarrow (-k, -\ell) \\ \downarrow & & \uparrow \\ (k, \ell) & \rightarrow (-\ell, k+\ell) & \rightarrow (-k-\ell, k) \\ \Downarrow & & \\ (k, -k-\ell) & \rightarrow (k+\ell, -\ell) & \rightarrow (\ell, k) \\ \uparrow & & \downarrow \\ (-\ell, -k) & \leftarrow (-k-\ell, \ell) & \leftarrow (-k, k+\ell) \end{array}} \tag{6.56}$$

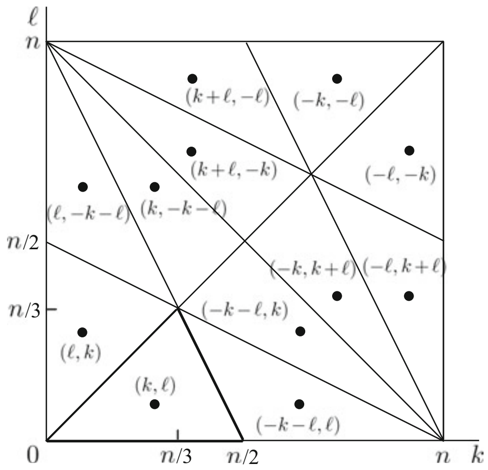
where "$\Downarrow$" means the action of s, and "$\rightarrow$" (or "$\leftarrow$", "$\uparrow$", "$\downarrow$") means the action of r. It should be clear that $(k + \ell, -k)$, for example, is understood as $(k + \ell \bmod n, -k \bmod n)$. The orbit (6.56) is illustrated in Fig. 6.1.

The system of representatives in (6.43) in the orbit decomposition of X with respect to the action of G is given as follows. In view of Fig. 6.1, it is natural to take

$$R(A)/H = \{(k, \ell) \mid 0 \le \ell \le k,\ 2k + \ell \le n\}, \tag{6.57}$$

which corresponds to the set of integer lattice points (k, ℓ) contained in the triangle with vertices at $(k, \ell) = (0, 0), (n/2, 0), (n/3, n/3)$, where the points on the edges

Fig. 6.1 Orbit of (k, ℓ) in (6.56)



of the triangle are included. But it turns out to be more convenient to replace the points $(k, \ell) = (k, n - 2k)$ on the line $2k + \ell = n$ with $n/3 < k < n/2$ by the points $(k, \ell) = (k, k)$ on the line $k = \ell$ with $n/3 < k < n/2$; note that $(k, n - 2k)$ and (k, k) belong to the same orbit by (6.56). This modification results in our choice of

$$
\begin{aligned}
R(A)/H = \ & \{(k, 0) \mid 0 \leq k \leq n/2\} \ \cup \ \{(k, k) \mid 0 < k < n/2\} \\
& \cup \{(k, \ell) \mid 0 < \ell < k, \ 2k + \ell < n\}.
\end{aligned} \tag{6.58}
$$

See Fig. 6.2.

The subgroup $H^i = H^{(k,\ell)}$ in (6.44) for $i = (k, \ell)$, which is expressed as

$$
\begin{aligned}
H^{(k,\ell)} &= \{h \in \mathrm{D}_6 \mid {}^h(\chi^{(k,\ell)}) = \chi^{(k,\ell)}\} \\
&= \{h \in \mathrm{D}_6 \mid \chi^{(k,\ell)}(h^{-1}ah) = \chi^{(k,\ell)}(a) \text{ for all } a \in \mathbb{Z}_n \times \mathbb{Z}_n\},
\end{aligned}
$$

is obtained with reference to (6.53) and (6.55). For $h \in \mathrm{D}_6$, we have $h \in H^{(k,\ell)}$ if and only if

$$
\chi^{(k,\ell)}(hp_1h^{-1}) = \chi^{(k,\ell)}(p_1), \quad \chi^{(k,\ell)}(hp_2h^{-1}) = \chi^{(k,\ell)}(p_2).
$$

For $(k, \ell) = (0, 0)$, for example, this condition is satisfied by all $h \in \mathrm{D}_6$, and hence $H^{(0,0)} = \langle r, s \rangle$. For $(k, \ell) = (n/3, n/3)$, where n is a multiple of 3, the above condition does not hold for all $h \in \mathrm{D}_6$; this condition fails, for example, for $h = r$, since $r^{-1}p_1r = p_2^{-1}$ and $\chi^{(n/3,n/3)}(rp_1r^{-1}) = \omega^{-n/3}$, whereas $\chi^{(n/3,n/3)}(p_1) = \omega^{n/3}$. Therefore, $r \notin H^{(n/3,n/3)}$. In this manner, we obtain

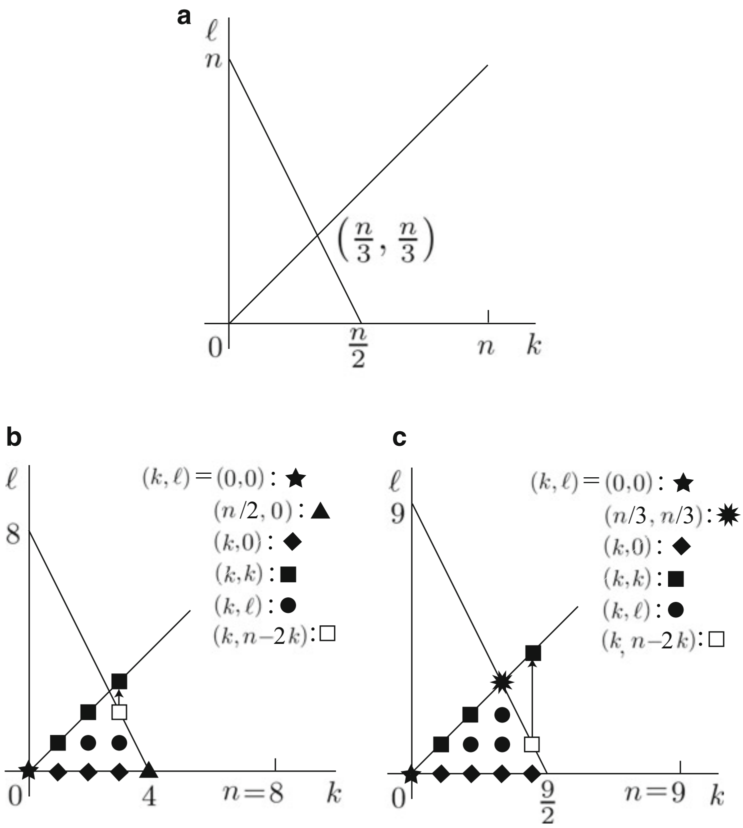


Fig. 6.2 System of representatives $R(A)/H$. (**a**) General n. (**b**) $n = 8$. (**c**) $n = 9$

$$H^{(k,\ell)} = \begin{cases} \langle r, s\rangle & \text{for } (k,\ell) = (0,0), \\ \langle r^2, s\rangle & \text{for } (k,\ell) = (n/3, n/3) \quad (n\text{: multiple of } 3), \\ \{e, r^3, sr^2, sr^5\} & \text{for } (k,\ell) = (n/2, 0) \quad (n\text{: even}), \\ \{e, sr^5\} & \text{for } (k,\ell) = (k, 0) \quad (1 \le k \le \lfloor \frac{n-1}{2} \rfloor), \\ \{e, sr^4\} & \text{for } (k,\ell) = (k, k) \quad (1 \le k \le \lfloor \frac{n-1}{2} \rfloor,\ k \ne \frac{n}{3}), \\ \{e\} & \text{for } (k,\ell) \quad (1 \le \ell \le k-1,\ 2k+\ell \le n-1). \end{cases} \tag{6.59}$$

The little group $G^i = G^{(k,\ell)}$ in (6.45) for $i = (k,\ell)$ is obtained as the semidirect product of $H^{(k,\ell)}$ by $A = \langle p_1, p_2\rangle$.

Example 6.1. For $n = 3, 4, 7, 8, 9$, the system of representatives $R(A)/H$ in (6.58) and the associated subgroups $H^{(k,\ell)}$ in (6.59) are given as follows.

$n=3$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(1,1)$	$\langle r^2,s\rangle$
$(1,0)$	$\{e,sr^5\}$

$n=4$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(2,0)$	$\{e,r^3,sr^2,sr^5\}$
$(1,0)$	$\{e,sr^5\}$
$(1,1)$	$\{e,sr^4\}$

$n=7$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(1,0),(2,0),(3,0)$	$\{e,sr^5\}$
$(1,1),(2,2),(3,3)$	$\{e,sr^4\}$
$(2,1)$	$\{e\}$

$n=8$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(4,0)$	$\{e,r^3,sr^2,sr^5\}$
$(1,0),(2,0),(3,0)$	$\{e,sr^5\}$
$(1,1),(2,2),(3,3)$	$\{e,sr^4\}$
$(2,1),(3,1)$	$\{e\}$

$n=9$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(3,3)$	$\langle r^2,s\rangle$
$(1,0),(2,0),(3,0),(4,0)$	$\{e,sr^5\}$
$(1,1),(2,2),(4,4)$	$\{e,sr^4\}$
$(2,1),(3,1),(3,2)$	$\{e\}$

$\square$

6.4.3 Induced Irreducible Representations

The procedure for constructing irreducible representations of $G = \langle r,s,p_1,p_2\rangle$ using the orbit decomposition and little groups in Sect. 6.4.2 is as follows.

For each $(k,\ell) \in R(A)/H$ in (6.58) we have the associated subgroup $H^{(k,\ell)}$ in (6.59). Let T^{μ} be an irreducible representation of $H^{(k,\ell)}$ indexed by $\mu \in R(H^{(k,\ell)})$, and define $T^{(k,\ell,\mu)}$ by

$$T^{(k,\ell,\mu)}({p_1}^i {p_2}^j h) = \chi^{(k,\ell)}({p_1}^i {p_2}^j)T^{\mu}(h) = \omega^{ki+\ell j}T^{\mu}(h),$$
$$0 \le i,j \le n-1,\ h \in H^{(k,\ell)}, \quad (6.60)$$

which is an irreducible representation of the little group $G^{(k,\ell)}$.

The coset decomposition (6.50) takes the form of

$$G = g_1 G^{(k,\ell)} + g_2 G^{(k,\ell)} + \cdots + g_m G^{(k,\ell)} \quad (6.61)$$

with $m = |G|/|G^{(k,\ell)}| = |\mathrm{D}_6|/|H^{(k,\ell)}| = 12/|H^{(k,\ell)}|$. Since $G^{(k,\ell)} \supseteq \langle p_1,p_2\rangle$, we may assume that $g_j \in \langle r,s\rangle$ for $j = 1,\ldots,m$ and $g_1 = e$.

The induced representation $\tilde{T}^{(k,\ell,\mu)}(g)$ is determined by its values at $g = p_1, p_2, r, s$ that generate the group G. Hence, it suffices to consider $g = p_1, p_2, r, s$ in (6.51):

$$g g_j = g_{\pi(j)} f_j, \quad (6.62)$$

Table 6.3 Induced irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

(k,ℓ)	$H^{(k,\ell)}$	m	Induced irreducible representations
$(0,0)$	$\langle r, s\rangle$	1	$(1;+,+), (1;+,-), (1;-,+), (1;-,-), (2;1), (2;2)$
$(n/3, n/3)$	$\langle r^2, s\rangle$	2	$(2;+), (2;-), (4;1)$
$(n/2, 0)$	$\{e, r^3, sr^2, sr^5\}$	3	$(3;+,+), (3;+,-), (3;-,+), (3;-,-)$
$(k,0)$	$\{e, sr^5\}$	6	$(6;k,0,+), (6;k,0,-)$
(k,k)	$\{e, sr^4\}$	6	$(6;k,k,+), (6;k,k,-)$
(k,ℓ)	$\{e\}$	12	$(12;k,\ell)$

$(k,\ell) = (n/3, n/3)$ exists for n a multiple of 3; $(n/2, 0)$ exists for n even
$(k,0)$ for $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$ in (6.26); (k,k) for $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$, $k \neq \frac{n}{3}$ in (6.27)
(k,ℓ) for $1 \le \ell \le k-1$, $2k+\ell \le n-1$ in (6.35)

where $\pi(j)$ and $f_j \in G^{(k,\ell)}$ are to be found for $j = 1,\ldots,m$. The induced representation $\tilde{T}^{(k,\ell,\mu)}$ is an irreducible representation of dimension $mN^\mu = 12N^\mu/|H^{(k,\ell)}|$ over $\mathbb{C}$, where N^μ denotes the dimension of T^μ.

According to the general theory, $\tilde{T}^{(k,\ell,\mu)}$ obtained in this manner is not a representation over $\mathbb{R}$ but over $\mathbb{C}$, as is evident from the fact that ω appearing in (6.60) is a complex number defined by (6.54). Fortunately, however, all irreducible representations thus obtained are representable over $\mathbb{R}$. We can thus determine a complete list of irreducible representations over $\mathbb{R}$ of the group $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Table 6.3 is a summary of the derivations below.

Case of $(k,\ell) = (0,0)$

For $(k,\ell) = (0,0)$, $\chi^{(k,\ell)}$ is the unit representation by (6.53), and therefore

$$H^{(k,\ell)} = \langle r, s\rangle = \mathrm{D}_6,$$

as is shown in (6.59). We recall from Sect. 3.3 that D_6 has four one-dimensional irreducible representations $\mu = (+,+), (+,-), (-,+)$, and $(-,-)$, and two two-dimensional irreducible representations $\mu = (1)$ and (2).

Since $G^{(k,\ell)} = G$, the coset decomposition (6.61) is trivial with $m = 1$ and $g_1 = e$, and (6.62) reads $g \cdot g_1 = g_1 \cdot g$ for every $g \in G$. For each μ, the induced representation $\tilde{T}^{(0,0,\mu)}(g)$ for $g = {p_1}^i {p_2}^j h$ with $h \in \mathrm{D}_6$ is given by (6.60) as

$$\tilde{T}^{(0,0,\mu)}(g) = \tilde{T}^{(0,0,\mu)}({p_1}^i {p_2}^j h) = \chi^{(0,0)}({p_1}^i {p_2}^j) T^\mu(h) = T^\mu(h).$$

Hence the induced representations $\tilde{T}^{(0,0,\mu)}$ for $\mu = (+,+), (+,-), (-,+), (-,-)$ are the one-dimensional irreducible representations $(1;+,+), (1;+,-), (1;-,+), (1;-,-)$ in Sect. 6.3.2, and those for $\mu = (1), (2)$ are the two-dimensional irreducible representations $(2;1), (2;2)$ in Sect. 6.3.3.

Case of $(k, \ell) = (n/3, n/3)$

The case of $(k, \ell) = (n/3, n/3)$ occurs when n is a multiple of 3. In this case, $\chi = \chi^{(k,\ell)} = \chi^{(n/3,n/3)}$ is given by (6.53) as $\chi(p_1) = \chi(p_2) = \omega^{n/3} = \zeta$ with $\zeta = \exp(2\pi \mathrm{i}/3)$, and therefore

$$H^{(k,\ell)} = \langle r^2, s\rangle \simeq \mathrm{D}_3,$$

as is shown in (6.59). This group has two one-dimensional irreducible representations, namely, $\mu_1 = (+)$ and $\mu_2 = (-)$ given by

$$\begin{aligned} T^{\mu_1}(r^2) &= 1, \quad T^{\mu_1}(s) = 1, \\ T^{\mu_2}(r^2) &= 1, \quad T^{\mu_2}(s) = -1, \end{aligned}$$

and one two-dimensional irreducible representation given by

$$T^{\mu_3}(r^2) = \begin{bmatrix} \zeta & \\ & \overline{\zeta} \end{bmatrix}, \quad T^{\mu_3}(s) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

Since $G^{(k,\ell)} = \langle r^2, s, p_1, p_2\rangle$, the coset decomposition in (6.61) is given by

$$G = g_1 G^{(k,\ell)} + g_2 G^{(k,\ell)} = e \cdot \langle r^2, s, p_1, p_2\rangle + r \cdot \langle r^2, s, p_1, p_2\rangle$$

with $m = 2$, $g_1 = e$ and $g_2 = r$. Equation (6.62) for $g = p_1, p_2, r, s$ reads as follows:

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$	$s \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$	$s \cdot e = e \cdot s$
$p_1 \cdot r = r \cdot p_2^{-1}$	$p_2 \cdot r = r \cdot p_1 p_2$	$r \cdot r = e \cdot r^2$	$s \cdot r = r \cdot s r^2$

For the one-dimensional representation $\mu = (\sigma)$ with $\sigma \in \{+, -\}$, the induced representation $\tilde{T} = \tilde{T}^{(n/3,n/3,\mu)}$ is given by

$$\begin{aligned} \tilde{T}(p_1) &= \begin{bmatrix} \chi(p_1)T^{\mu}(e) & \\ & \chi(p_2^{-1})T^{\mu}(e) \end{bmatrix} = \begin{bmatrix} \zeta & \\ & \overline{\zeta} \end{bmatrix}, \\ \tilde{T}(p_2) &= \begin{bmatrix} \chi(p_2)T^{\mu}(e) & \\ & \chi(p_1 p_2)T^{\mu}(e) \end{bmatrix} = \begin{bmatrix} \zeta & \\ & \overline{\zeta} \end{bmatrix}, \\ \tilde{T}(r) &= \begin{bmatrix} & \chi(e)T^{\mu}(r^2) \\ \chi(e)T^{\mu}(e) & \end{bmatrix} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \\ \tilde{T}(s) &= \begin{bmatrix} \chi(e)T^{\mu}(s) & \\ & \chi(e)T^{\mu}(s r^2) \end{bmatrix} = \sigma \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \end{aligned}$$

where (6.60) is used and the nonzero blocks here are determined with reference to $\pi(j)$ and f_j computed above (Remark 6.4 in Sect. 6.4.1). These representations are equivalent to the real two-dimensional irreducible representations $(2;\sigma) = (2;+)$ and $(2;-)$ in (6.13) and (6.14) in Sect. 6.3.3. Indeed, with the use of

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix},$$

we obtain

$$\begin{aligned} Q^{-1}\tilde{T}(p_1)Q &= Q^{-1}\tilde{T}(p_2)Q = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}, \\ Q^{-1}\tilde{T}(r)Q &= \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad Q^{-1}\tilde{T}(s)Q = \sigma \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \end{aligned}$$

For the two-dimensional representation $\mu = \mu_3$, the induced representation $\tilde{T} = \tilde{T}^{(n/3,n/3,\mu_3)}$ is a four-dimensional representation given by

$$\tilde{T}(p_1) = \begin{bmatrix} \chi(p_1)T^{\mu}(e) & \\ & \chi(p_2^{-1})T^{\mu}(e) \end{bmatrix} = \begin{bmatrix} \zeta I_2 & \\ & \overline{\zeta} I_2 \end{bmatrix},$$

$$\tilde{T}(p_2) = \begin{bmatrix} \chi(p_2)T^{\mu}(e) & \\ & \chi(p_1 p_2)T^{\mu}(e) \end{bmatrix} = \begin{bmatrix} \zeta I_2 & \\ & \overline{\zeta} I_2 \end{bmatrix},$$

$$\tilde{T}(r) = \begin{bmatrix} & \chi(e)T^{\mu}(r^2) \\ \chi(e)T^{\mu}(e) & \end{bmatrix} = \begin{bmatrix} & & \zeta & \\ & & & \overline{\zeta} \\ 1 & & & \\ & 1 & & \end{bmatrix},$$

$$\tilde{T}(s) = \begin{bmatrix} \chi(e)T^{\mu}(s) & \\ & \chi(e)T^{\mu}(sr^2) \end{bmatrix} = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & \overline{\zeta} \\ & & \zeta & \end{bmatrix}.$$

This representation over $\mathbb{C}$ can be transformed to a real representation. By permuting the rows and columns as $(1, 4, 3, 2)$, we obtain

$$\hat{T}(p) = \left[\begin{array}{cc|cc} \zeta & & & \\ & \overline{\zeta} & & \\ \hline & & \overline{\zeta} & \\ & & & \zeta \end{array}\right], \quad \hat{T}(r) = \left[\begin{array}{cc|cc} & & \zeta & \\ & & & 1 \\ \hline 1 & & & \\ & \overline{\zeta} & & \end{array}\right], \quad \hat{T}(s) = \left[\begin{array}{cc|cc} & & & 1 \\ & & \zeta & \\ \hline & \overline{\zeta} & & \\ 1 & & & \end{array}\right],$$

where $p = p_1, p_2$. A further transformation $\check{T}(g) = U^{-1}\hat{T}(g)U$ using $U = \mathrm{diag}(1, \xi, \overline{\xi}, 1)$ with $\xi = \exp(2\pi \mathrm{i}/6)$ yields

$$\check{T}(p) = \left[\begin{array}{cc|cc} \zeta & & & \\ & \overline{\zeta} & & \\ \hline & & \overline{\zeta} & \\ & & & \zeta \end{array}\right], \quad \check{T}(r) = \left[\begin{array}{cc|cc} & & \xi & \\ & & & \overline{\xi} \\ \hline \xi & & & \\ & \overline{\xi} & & \end{array}\right], \quad \check{T}(s) = \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right].$$

It is apparent that $\check{T}$ is equivalent to the four-dimensional real irreducible representation (4; 1) in (6.23) and (6.24) in Sect. 6.3.5. Indeed, for

$$Q = \frac{1}{\sqrt{2}} \left[\begin{array}{cc|cc} 1 & \mathrm{i} & & \\ 1 & -\mathrm{i} & & \\ \hline & & 1 & \mathrm{i} \\ & & 1 & -\mathrm{i} \end{array}\right],$$

we have $Q^{-1}\check{T}(g)Q = T^{(4;1)}(g)$ for $g = p_1, p_2, r, s$.

Case of $(k, \ell) = (n/2, 0)$

The case of $(k, \ell) = (n/2, 0)$ occurs when n is even. In this case, $\chi = \chi^{(k,\ell)} = \chi^{(n/2,0)}$ is given by (6.53) as $\chi(p_1) = -1$ and $\chi(p_2) = 1$, and therefore

$$H^{(k,\ell)} = \{e, r^3, sr^2, sr^5\} = \langle r^3, sr^2 \rangle \simeq \mathrm{D}_2,$$

as is shown in (6.59). This group has four one-dimensional irreducible representations, say, $\mu = (\sigma_r, \sigma_s) = (+,+), (+,-), (-,+), (-,-)$ defined by

$$T^{\mu}(r^3) = \sigma_r = \pm 1, \quad T^{\mu}(sr^2) = \sigma_s = \pm 1.$$

Since $G^{(k,\ell)} = \langle r^3, sr^2, p_1, p_2 \rangle$, the coset decomposition in (6.61) is given by

$$\begin{aligned} G &= g_1 G^{(k,\ell)} + g_2 G^{(k,\ell)} + g_3 G^{(k,\ell)} \\ &= e \cdot \langle r^3, sr^2, p_1, p_2 \rangle + r \cdot \langle r^3, sr^2, p_1, p_2 \rangle + r^2 \cdot \langle r^3, sr^2, p_1, p_2 \rangle \end{aligned}$$

with $m = 3$, $g_1 = e$, $g_2 = r$, and $g_3 = r^2$. Equation (6.62) for $g = p_1, p_2, r, s$ reads as follows:

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$	$s \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$	$s \cdot e = r^2 \cdot sr^2$
$p_1 \cdot r = r \cdot p_2^{-1}$	$p_2 \cdot r = r \cdot p_1 p_2$	$r \cdot r = r^2 \cdot e$	$s \cdot r = r \cdot sr^2$
$p_1 \cdot r^2 = r^2 \cdot p_1^{-1} p_2^{-1}$	$p_2 \cdot r^2 = r^2 \cdot p_1$	$r \cdot r^2 = e \cdot r^3$	$s \cdot r^2 = e \cdot sr^2$

The induced representation $\tilde{T} = \tilde{T}^{(n/2,0,\mu)}$ is given as follows:

$$\tilde{T}(p_1) = \begin{bmatrix} \chi(p_1) & & \\ & \chi(p_2^{-1}) & \\ & & \chi(p_1^{-1}p_2^{-1}) \end{bmatrix} = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix},$$

$$\tilde{T}(p_2) = \begin{bmatrix} \chi(p_2) & & \\ & \chi(p_1 p_2) & \\ & & \chi(p_1) \end{bmatrix} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix},$$

$$\tilde{T}(r) = \begin{bmatrix} & & T^\mu(r^3) \\ T^\mu(e) & & \\ & T^\mu(e) & \end{bmatrix} = \begin{bmatrix} & & \sigma_r \\ 1 & & \\ & 1 & \end{bmatrix} = \begin{bmatrix} & & \pm 1 \\ 1 & & \\ & 1 & \end{bmatrix},$$

$$\tilde{T}(s) = \begin{bmatrix} & & T^\mu(sr^2) \\ & T^\mu(sr^2) & \\ T^\mu(sr^2) & & \end{bmatrix} = \begin{bmatrix} & & \sigma_s \\ & \sigma_s & \\ \sigma_s & & \end{bmatrix} = \pm \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

These are the four three-dimensional irreducible representations $(3; \sigma_r, \sigma_s) = (3; +, +), (3; +, -), (3; -, +), (3; -, -)$ in Sect. 6.3.4.

Case of $(k, \ell) = (k, 0)$ or (k, k)

For $(k, \ell) = (k, 0)$ in (6.26), we have $\chi^{(k,\ell)}(p_1) = \omega^k$ and $\chi^{(k,\ell)}(p_2) = 1$ by (6.53), and therefore

$$H^{(k,\ell)} = \{e, sr^5\},$$

as is shown in (6.59). For $(k, \ell) = (k, k)$ in (6.27), we have $\chi^{(k,\ell)}(p_1) = \chi^{(k,\ell)}(p_2) = \omega^k$, and therefore

$$H^{(k,\ell)} = \{e, sr^4\}.$$

Let $h_0 = sr^5$ for $(k, \ell) = (k, 0)$ and $h_0 = sr^4$ for $(k, \ell) = (k, k)$. In either case $H^{(k,\ell)} = \{e, h_0\}$ is isomorphic to D_1 and has two one-dimensional irreducible representations, say, $\mu = \mu_1, \mu_2$ defined by

$$T^{\mu_1}(h_0) = 1, \quad T^{\mu_2}(h_0) = -1.$$

That is, $T^\mu(h_0) = \sigma^\mu$ with $\sigma^{\mu_1} = 1$ and $\sigma^{\mu_2} = -1$. The notation is summarized as follows:

(k, ℓ)	$H^{(k,\ell)}$	h_0	$T^{\mu_1}(h_0)$	$T^{\mu_2}(h_0)$
$(k, 0)$	$\{e, sr^5\}$	sr^5	1	-1
(k, k)	$\{e, sr^4\}$	sr^4	1	-1

The coset decomposition in (6.61) is given by $G^{(k,\ell)} = \langle h_0, p_1, p_2\rangle$, $m = 6$, and $g_j = r^{j-1}$ for $j = 1, \ldots, 6$. The equation (6.62) for $g = p_1, p_2, r, s$ reads as follows (see (6.55) for p_1 and p_2):

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$
$p_1 \cdot r = r \cdot p_2^{-1}$	$p_2 \cdot r = r \cdot p_1 p_2$	$r \cdot r = r^2 \cdot e$
$p_1 \cdot r^2 = r^2 \cdot p_1^{-1} p_2^{-1}$	$p_2 \cdot r^2 = r^2 \cdot p_1$	$r \cdot r^2 = r^3 \cdot e$
$p_1 \cdot r^3 = r^3 \cdot p_1^{-1}$	$p_2 \cdot r^3 = r^3 \cdot p_2^{-1}$	$r \cdot r^3 = r^4 \cdot e$
$p_1 \cdot r^4 = r^4 \cdot p_2$	$p_2 \cdot r^4 = r^4 \cdot p_1^{-1} p_2^{-1}$	$r \cdot r^4 = r^5 \cdot e$
$p_1 \cdot r^5 = r^5 \cdot p_1 p_2$	$p_2 \cdot r^5 = r^5 \cdot p_1^{-1}$	$r \cdot r^5 = e \cdot e$

$s \cdot g_j = g_{\pi(j)} \cdot f_j$	
$(k, 0)$	(k, k)
$s \cdot e = r^5 \cdot sr^5$	$s \cdot e = r^4 \cdot sr^4$
$s \cdot r = r^4 \cdot sr^5$	$s \cdot r = r^3 \cdot sr^4$
$s \cdot r^2 = r^3 \cdot sr^5$	$s \cdot r^2 = r^2 \cdot sr^4$
$s \cdot r^3 = r^2 \cdot sr^5$	$s \cdot r^3 = r \cdot sr^4$
$s \cdot r^4 = r \cdot sr^5$	$s \cdot r^4 = e \cdot sr^4$
$s \cdot r^5 = e \cdot sr^5$	$s \cdot r^5 = r^5 \cdot sr^4$

For $(k, \ell) = (k, 0), (k, k)$ and $\mu = \mu_1, \mu_2$, the induced representation $\tilde{T}^{(k,\ell,\mu)}$ is given, with $\omega = \exp(2\pi \mathrm{i}/n)$, by

$$
\begin{aligned}
\tilde{T}^{(k,\ell,\mu)}(p_1) &= \mathrm{diag}(\chi(p_1),\ \chi(p_2^{-1}),\ \chi(p_1^{-1} p_2^{-1}),\ \chi(p_1^{-1}),\ \chi(p_2),\ \chi(p_1 p_2)) \\
&= \mathrm{diag}(\omega^k,\ \omega^{-\ell},\ \omega^{-k-\ell},\ \omega^{-k},\ \omega^{\ell},\ \omega^{k+\ell}), \\
\tilde{T}^{(k,\ell,\mu)}(p_2) &= \mathrm{diag}(\chi(p_2),\ \chi(p_1 p_2),\ \chi(p_1),\ \chi(p_2^{-1}),\ \chi(p_1^{-1} p_2^{-1}),\ \chi(p_1^{-1})) \\
&= \mathrm{diag}(\omega^{\ell},\ \omega^{k+\ell},\ \omega^k,\ \omega^{-\ell},\ \omega^{-k-\ell},\ \omega^{-k}),
\end{aligned}
$$

$$
\tilde{T}^{(k,\ell,\mu)}(r) = T^{\mu}(e) \begin{bmatrix} & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix} = \begin{bmatrix} & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix},
$$

and

$$\tilde{T}^{(k,0,\mu)}(s) = T^{\mu}(sr^5) \begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{bmatrix} = \sigma^{\mu} \begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{bmatrix},$$

$$\tilde{T}^{(k,k,\mu)}(s) = T^{\mu}(sr^4) \begin{bmatrix} & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \end{bmatrix} = \sigma^{\mu} \begin{bmatrix} & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \end{bmatrix}.$$

The above representation over $\mathbb{C}$ can be transformed to a real representation. By permuting the rows and columns as $(1, 4, 5, 2, 3, 6)$, we obtain

$$\hat{T}^{(k,\ell,\mu)}(p_1) = \left[\begin{array}{cc|cc|cc} \omega^{k} & & & & & \\ & \omega^{-k} & & & & \\ \hline & & \omega^{\ell} & & & \\ & & & \omega^{-\ell} & & \\ \hline & & & & \omega^{-k-\ell} & \\ & & & & & \omega^{k+\ell} \end{array}\right], \quad \hat{T}^{(k,\ell,\mu)}(p_2) = \left[\begin{array}{cc|cc|cc} \omega^{\ell} & & & & & \\ & \omega^{-\ell} & & & & \\ \hline & & \omega^{-k-\ell} & & & \\ & & & \omega^{k+\ell} & & \\ \hline & & & & \omega^{k} & \\ & & & & & \omega^{-k} \end{array}\right],$$

$$\hat{T}^{(k,\ell,\mu)}(r) = \left[\begin{array}{cc|cc|cc} & & & & & 1 \\ & & & & 1 & \\ \hline & 1 & & & & \\ 1 & & & & & \\ \hline & & & 1 & & \\ & & 1 & & & \end{array}\right], \quad \hat{T}^{(k,0,\mu)}(s) = \sigma^{\mu} \left[\begin{array}{cc|cc|cc} & & & & & 1 \\ & & & & 1 & \\ \hline & & & 1 & & \\ & & 1 & & & \\ \hline & 1 & & & & \\ 1 & & & & & \end{array}\right], \quad \hat{T}^{(k,k,\mu)}(s) = \sigma^{\mu} \left[\begin{array}{cc|cc|cc} & & 1 & & & \\ & & & 1 & & \\ \hline 1 & & & & & \\ & 1 & & & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array}\right].$$

It is apparent that these representations are equivalent, respectively, to the six-dimensional real irreducible representations $(6; k, 0, \sigma)$ and $(6; k, k, \sigma)$ in Sect. 6.3.6 with $\sigma = \sigma^{\mu}$.

Case of General (k, ℓ)

For (k, ℓ) in (6.35), $\chi = \chi^{(k,\ell)}$ is given by (6.53), and $H^{(k,\ell)} = \{e\}$. The unit representation μ is the only irreducible representation of $H^{(k,\ell)}$.

The coset decomposition in (6.61) is given by $G^{(k,\ell)} = \langle p_1, p_2 \rangle$, $m = 12$, and

$$g_1 = e,\ g_2 = r,\ g_3 = r^2,\ \ldots,\ g_6 = r^5,\ g_7 = s,\ g_8 = sr,\ \ldots,\ g_{12} = sr^5.$$

The equation (6.62) for $g = p_1, p_2, r, s$ reads as follows:

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$	$s \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$	$s \cdot e = s \cdot e$
$p_1 \cdot r = r \cdot p_2^{-1}$	$p_2 \cdot r = r \cdot p_1 p_2$	$r \cdot r = r^2 \cdot e$	$s \cdot r = sr \cdot e$
$p_1 \cdot r^2 = r^2 \cdot p_1^{-1} p_2^{-1}$	$p_2 \cdot r^2 = r^2 \cdot p_1$	$r \cdot r^2 = r^3 \cdot e$	$s \cdot r^2 = sr^2 \cdot e$
$p_1 \cdot r^3 = r^3 \cdot p_1^{-1}$	$p_2 \cdot r^3 = r^3 \cdot p_2^{-1}$	$r \cdot r^3 = r^4 \cdot e$	$s \cdot r^3 = sr^3 \cdot e$
$p_1 \cdot r^4 = r^4 \cdot p_2$	$p_2 \cdot r^4 = r^4 \cdot p_1^{-1} p_2^{-1}$	$r \cdot r^4 = r^5 \cdot e$	$s \cdot r^4 = sr^4 \cdot e$
$p_1 \cdot r^5 = r^5 \cdot p_1 p_2$	$p_2 \cdot r^5 = r^5 \cdot p_1^{-1}$	$r \cdot r^5 = e \cdot e$	$s \cdot r^5 = sr^5 \cdot e$
$p_1 \cdot s = s \cdot p_1$	$p_2 \cdot s = s \cdot p_1^{-1} p_2^{-1}$	$r \cdot s = sr^5 \cdot e$	$s \cdot s = e \cdot e$
$p_1 \cdot sr = sr \cdot p_2^{-1}$	$p_2 \cdot sr = sr \cdot p_1^{-1}$	$r \cdot sr = s \cdot e$	$s \cdot sr = r \cdot e$
$p_1 \cdot sr^2 = sr^2 \cdot p_1^{-1} p_2^{-1}$	$p_2 \cdot sr^2 = sr^2 \cdot p_2$	$r \cdot sr^2 = sr \cdot e$	$s \cdot sr^2 = r^2 \cdot e$
$p_1 \cdot sr^3 = sr^3 \cdot p_1^{-1}$	$p_2 \cdot sr^3 = sr^3 \cdot p_1 p_2$	$r \cdot sr^3 = sr^2 \cdot e$	$s \cdot sr^3 = r^3 \cdot e$
$p_1 \cdot sr^4 = sr^4 \cdot p_2$	$p_2 \cdot sr^4 = sr^4 \cdot p_1$	$r \cdot sr^4 = sr^3 \cdot e$	$s \cdot sr^4 = r^4 \cdot e$
$p_1 \cdot sr^5 = sr^5 \cdot p_1 p_2$	$p_2 \cdot sr^5 = sr^5 \cdot p_2^{-1}$	$r \cdot sr^5 = sr^4 \cdot e$	$s \cdot sr^5 = r^5 \cdot e$

The induced representation $\tilde{T} = \tilde{T}^{(k,\ell,\mu)}$, of dimension 12, is given in terms of $\omega = \exp(2\pi \mathrm{i}/n)$ as follows:

$$\tilde{T}(p_1) = \mathrm{diag}(\omega^k, \omega^{-\ell}, \omega^{-k-\ell}, \omega^{-k}, \omega^{\ell}, \omega^{k+\ell}, \omega^k, \omega^{-\ell}, \omega^{-k-\ell}, \omega^{-k}, \omega^{\ell}, \omega^{k+\ell}),$$
$$\tilde{T}(p_2) = \mathrm{diag}(\omega^{\ell}, \omega^{k+\ell}, \omega^k, \omega^{-\ell}, \omega^{-k-\ell}, \omega^{-k}, \omega^{-k-\ell}, \omega^{-k}, \omega^{\ell}, \omega^{k+\ell}, \omega^k, \omega^{-\ell}),$$
$$\tilde{T}(r) = \begin{bmatrix} C & O \\ O & C^\top \end{bmatrix}, \qquad \tilde{T}(s) = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$$

with

$$C = \begin{bmatrix} & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}, \quad I = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

The above representation over $\mathbb{C}$ can be transformed to a real representation. By permuting the rows and columns as $(1, 4, 5, 2, 3, 6; 7, 10, 11, 8, 9, 12)$, we obtain

$$\hat{T}(p_1)=\begin{bmatrix} \Omega_1 & \\ & \Omega_1 \end{bmatrix}, \quad \hat{T}(p_2)=\begin{bmatrix} \Omega_2 & \\ & \Omega_3 \end{bmatrix}, \quad \hat{T}(r)=\begin{bmatrix} D & \\ & D^\top \end{bmatrix}, \quad \hat{T}(s)=\begin{bmatrix} & I \\ I & \end{bmatrix},$$

where

$$\Omega_1 = \left[\begin{array}{cc|cc|cc} \omega^k & & & & & \\ & \omega^{-k} & & & & \\ \hline & & \omega^\ell & & & \\ & & & \omega^{-\ell} & & \\ \hline & & & & \omega^{-k-\ell} & \\ & & & & & \omega^{k+\ell} \end{array}\right], \quad \Omega_2 = \left[\begin{array}{cc|cc|cc} \omega^\ell & & & & & \\ & \omega^{-\ell} & & & & \\ \hline & & \omega^{-k-\ell} & & & \\ & & & \omega^{k+\ell} & & \\ \hline & & & & \omega^k & \\ & & & & & \omega^{-k} \end{array}\right],$$

$$\Omega_3 = \left[\begin{array}{cc|cc|cc} \omega^{-k-\ell} & & & & & \\ & \omega^{k+\ell} & & & & \\ \hline & & \omega^k & & & \\ & & & \omega^{-k} & & \\ \hline & & & & \omega^\ell & \\ & & & & & \omega^{-\ell} \end{array}\right], \quad D = \left[\begin{array}{cc|cc|cc} & & & & & 1 \\ & & & & 1 & \\ \hline & 1 & & & & \\ 1 & & & & & \\ \hline & & & 1 & & \\ & & 1 & & & \end{array}\right].$$

This representation is easily seen to be equivalent to the 12-dimensional real irreducible representation $(12; k, \ell)$ in Sect. 6.3.7.

6.5 Summary

- The irreducible representations of the group for the $n \times n$ hexagonal lattice have been derived and listed.
- A systematic method for constructing irreducible representations of a general group with the structure of semidirect product by an abelian group has been described.

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Chapter 7
Matrix Representation for Economy on Hexagonal Lattice

Abstract Group-theoretic properties of the symmetry group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ of the hexagonal lattice, such as subgroups and irreducible representations, were derived in Chaps. 5 and 6. In this chapter, the matrix representation of this group for the economy on the hexagonal lattice is investigated in preparation for the group-theoretic bifurcation analysis in search of bifurcating hexagonal patterns in Chaps. 8 and 9. Irreducible decomposition of the matrix representation is obtained with the aid of characters to identify irreducible representations that are relevant to our analysis of the mathematical model of an economy on the hexagonal lattice. Formulas for the transformation matrix for block-diagonalization of the Jacobian matrix of the equilibrium equation of the economy on the hexagonal lattice are derived and put to use in numerical bifurcation analysis of hexagonal patterns.

Keywords Block-diagonalization • Character • Christaller's hexagonal distributions • Hexagonal lattice • Irreducible decomposition • Lösch's hexagons • Multiplicity-free • Permutation representation • Representation matrix

7.1 Introduction

In preparation for the group-theoretic analysis in Chaps. 8 and 9, irreducible representations of the group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ were found in Chap. 6. Yet it is to be emphasized that not all irreducible representations are involved in the mathematical model of the economy on the hexagonal lattice. Consideration of relevant irreducible representations is vital in adequate group-theoretic analysis that yields accurate information about bifurcating solutions.

The main objective of this chapter is to identify the irreducible representations μ that are relevant to our analysis of the economy on the hexagonal lattice. For this purpose, we derive the explicit form of the permutation representation $T(g)$ of the group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in the equivariance (4.11) in Proposition 4.1 in Sect. 4.3.2. Then the irreducible decomposition (2.42) of this permutation representation $T(g)$

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is investigated. Irreducible representations not contained in $T(g)$ can be excluded from consideration in search of hexagonal bifurcating patterns in Chaps. 8 and 9. Although the irreducible representations μ over $\mathbb{R}$ are one-, two-, three-, four-, six-, or twelve-dimensional, it will be shown that the four-dimensional one is irrelevant to our analysis, only some of the one-, two-, three-, and six-dimensional ones are relevant and all of the twelve-dimensional ones are relevant.

Another objective of this chapter is to present the transformation matrix Q for irreducible decomposition. Since the irreducible representations are multiplicity-free ($a^\mu = 1$ or 0), the orthogonal transformation (2.62) of the Jacobian matrix J of the equilibrium equation takes a diagonal form

$$Q^{-1}JQ = \mathrm{diag}(e_1, \ldots, e_N).$$

This diagonal form is useful in the eigenanalysis of the computational bifurcation analysis of the economy on the hexagonal lattice.

This chapter is organized as follows. The representation matrix of the 3×3 hexagonal lattice is given in Sect. 7.2. The permutation representation for the hexagonal lattice is investigated in Sect. 7.3. The irreducible decomposition of the permutation representation is presented in Sect. 7.4. Transformation matrices for block-diagonalization are derived in Sect. 7.5. Geometrical implication and computational use of the transformation matrix are given in Sect. 7.6.

7.2 Simple Example of Representation Matrix

With the underlying group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ identified in Sect. 6.2, the next step of group-theoretic bifurcation analysis of economic agglomeration consists of finding a concrete form of the representation matrix $T(g)$ used in Proposition 4.1 in Sect. 4.3.2 and obtaining its irreducible decomposition. This amounts to identifying possible irreducible representations associated with the bifurcation points of this particular problem. Before starting a systematic investigation, we will have a quick look at this process using a simple example in this section.

In a system of $N = n^2$ places on the $n \times n$ hexagonal lattice, each element g of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ acts as a permutation of place numbers $(1, \ldots, N)$. Consequently, the representation matrix $T(g)$ is a permutation matrix for each g. Figure 7.1 illustrates the permutations caused by r, s, p_1, and p_2 in the case of $n = 3$. Accordingly, the associated representation matrices for r, s, p_1, and p_2 are given as

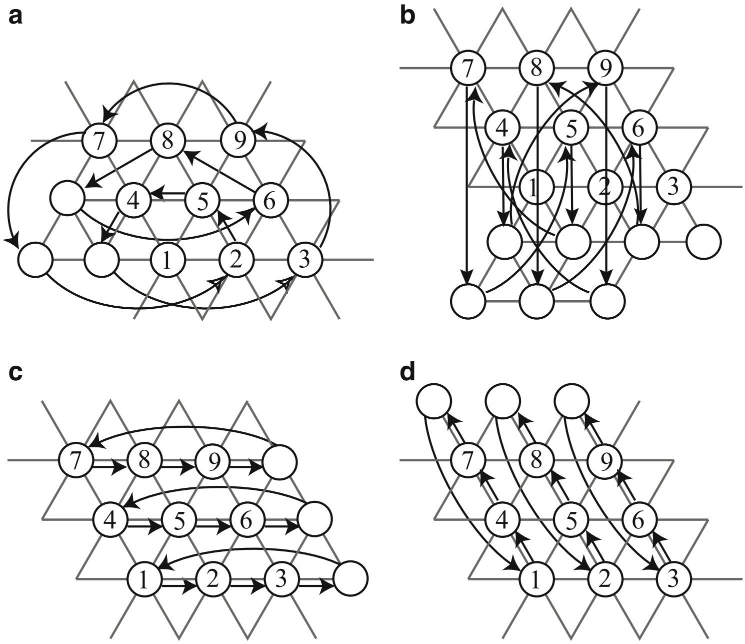

Fig. 7.1 Actions of (**a**) r, (**b**) s, (**c**) p_1, and (**d**) p_2 on the 3×3 hexagonal lattice

$$T(r) = \left[\begin{array}{ccc|ccc|ccc} 1 & & & & & & & & \\ & & & & & & 1 & & \\ & & & 1 & & & & & \\ \hline & & & & 1 & & & & \\ & 1 & & & & & & & \\ & & & & & & & 1 & \\ \hline & & & & & & & & 1 \\ & & & & & 1 & & & \\ & & 1 & & & & & & \end{array}\right], \quad T(s) = \left[\begin{array}{ccc|ccc|ccc} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ \hline & & & & & & & & 1 \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ \hline & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & 1 & & & & & \end{array}\right], \tag{7.1}$$

$$T(p_1) = \left[\begin{array}{ccc|ccc|ccc} & & 1 & & & & & & \\ 1 & & & & & & & & \\ & 1 & & & & & & & \\ \hline & & & & & 1 & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ \hline & & & & & & & & 1 \\ & & & & & & 1 & & \\ & & & & & & & 1 & \end{array}\right], \quad T(p_2) = \left[\begin{array}{ccc|ccc|ccc} & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \\ \hline 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ \hline & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \end{array}\right], \tag{7.2}$$

whereas the matrices $T(g)$ for other elements g can be generated from these four matrices. It should be clear, for example, that the second column of $T(r)$ has "1" at the fifth row because place 2 is moved to place 5 by the action of r, which is consistent with the definition of a permutation representation given in Example 2.8 in Sect. 2.3.2.

The irreducible decomposition of the representation $T(g)$ is a block-diagonal form

$$Q^{-1}T(g)Q = \bigoplus_{\mu \in R(\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n))} \bigoplus_{i=1}^{a^\mu} T^\mu(g), \qquad g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) \tag{7.3}$$

in (2.42). Here, Q is an orthogonal transformation matrix in (2.48), a^μ is the multiplicity (cf., (2.58)) of irreducible representation μ in $T(g)$, and $T^\mu(g)$ is the matrix representation of μ given in Sect. 6.3.

In the case of $n = 3$, we choose[1]

$$Q = [Q^{\mu_1}, Q^{\mu_2}, Q^{\mu_3}] = \frac{1}{3\sqrt{2}} \left[\begin{array}{c|cc|cccccc} \sqrt{2} & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ \sqrt{2} & -1 & \sqrt{3} & -1 & \sqrt{3} & 2 & 0 & -1 & -\sqrt{3} \\ \sqrt{2} & -1 & -\sqrt{3} & -1 & -\sqrt{3} & 2 & 0 & -1 & \sqrt{3} \\ \hline \sqrt{2} & -1 & \sqrt{3} & 2 & 0 & -1 & -\sqrt{3} & -1 & \sqrt{3} \\ \sqrt{2} & -1 & -\sqrt{3} & -1 & \sqrt{3} & -1 & -\sqrt{3} & 2 & 0 \\ \sqrt{2} & 2 & 0 & -1 & -\sqrt{3} & -1 & -\sqrt{3} & -1 & -\sqrt{3} \\ \hline \sqrt{2} & -1 & -\sqrt{3} & 2 & 0 & -1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{2} & 2 & 0 & -1 & \sqrt{3} & -1 & \sqrt{3} & -1 & \sqrt{3} \\ \sqrt{2} & -1 & \sqrt{3} & -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \end{array} \right] \tag{7.4}$$

to obtain a block-diagonal form

$$Q^{-1}T(g)Q = T^{\mu_1}(g) \oplus T^{\mu_2}(g) \oplus T^{\mu_3}(g) = \begin{bmatrix} T^{\mu_1}(g) & & \\ & T^{\mu_2}(g) & \\ & & T^{\mu_3}(g) \end{bmatrix} \tag{7.5}$$

for $g = r, s, p_1, p_2$, and hence for all $g \in \langle r, s, p_1, p_2 \rangle = \mathrm{D}_6 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$. Here $T^{\mu_1}(g)$ is the one-dimensional irreducible representation (unit representation) given by

$$T^{\mu_1}(r) = 1, \quad T^{\mu_1}(s) = 1, \quad T^{\mu_1}(p_1) = 1, \quad T^{\mu_1}(p_2) = 1, \tag{7.6}$$

$T^{\mu_2}(g)$ is the two-dimensional irreducible representation given by

$$T^{\mu_2}(r) = S, \quad T^{\mu_2}(s) = I, \quad T^{\mu_2}(p_1) = T^{\mu_2}(p_2) = R \tag{7.7}$$

[1] The choice of this matrix Q will be explained in Sect. 7.5, and μ_1, μ_2, and μ_3 are irreducible representations denoted in Sect. 6.3 as $(1; +, +)$, $(2; +)$, and $(6; 1, 0, +)$, respectively.

with

$$R = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \tag{7.8}$$

and $T^{\mu_3}(g)$ is the six-dimensional irreducible representation given by

$$T^{\mu_3}(r) = \begin{bmatrix} & & S \\ S & & \\ & S & \end{bmatrix}, \quad T^{\mu_3}(s) = \begin{bmatrix} & & S \\ & S & \\ S & & \end{bmatrix}, \tag{7.9}$$

$$T^{\mu_3}(p_1) = \begin{bmatrix} R & & \\ & I & \\ & & R^{-1} \end{bmatrix}, \quad T^{\mu_3}(p_2) = \begin{bmatrix} I & & \\ & R^{-1} & \\ & & R \end{bmatrix}. \tag{7.10}$$

The decomposition (7.5) reveals that there are two irreducible representations μ_2 and μ_3 that should be considered for possible bifurcation, whereas the unit representation μ_1 does not correspond to bifurcation. Analyses in Chaps. 8 and 9 will show that bifurcation for hexagonal patterns does occur at a critical point associated with μ_2 or μ_3.

The block-diagonal form (2.62) for the Jacobian matrix $J(\boldsymbol{\lambda}, \tau)$ in (4.7) reads

$$Q^{-1} J(\boldsymbol{\lambda}, \tau) Q = \mathrm{diag}(e^{\mu_1};\ e^{\mu_2}, e^{\mu_2};\ e^{\mu_3}, e^{\mu_3}, e^{\mu_3}, e^{\mu_3}, e^{\mu_3}, e^{\mu_3}),$$

in which the eigenvalue e^{μ_1} is repeated once, e^{μ_2} is repeated twice, and e^{μ_3} is repeated six times.

Hexagonal patterns are already apparent in the transformation matrix Q in (7.4). We denote by $\boldsymbol{q}_j^{\mu_i}$ the jth column vector of Q^{μ_i} as

$$Q^{\mu_1} = \boldsymbol{q}^{\mu_1}, \quad Q^{\mu_2} = [\boldsymbol{q}_1^{\mu_2}, \boldsymbol{q}_2^{\mu_2}], \quad Q^{\mu_3} = [\boldsymbol{q}_1^{\mu_3}, \boldsymbol{q}_2^{\mu_3}, \boldsymbol{q}_3^{\mu_3}, \boldsymbol{q}_4^{\mu_3}, \boldsymbol{q}_5^{\mu_3}, \boldsymbol{q}_6^{\mu_3}].$$

Then we find three hexagonal distributions:

- The vector $\boldsymbol{q}^{\mu_1}$ denotes the flat earth equilibrium (uniform state) in Fig. 5.3a with $\Sigma(1,0)$-symmetry ($\Sigma(1,0) = \langle r, s, p_1, p_2 \rangle$ as defined in (5.39)).
- The vector

$$\boldsymbol{q}_1^{\mu_2} = \frac{1}{3\sqrt{2}}(2, -1, -1, -1, -1, 2, -1, 2, -1)^\top,$$

depicted in Fig. 7.2a, denotes the Christaller's $k = 3$ system in Fig. 5.3b with $\Sigma(2,1)$-symmetry.
- The vector

$$\frac{1}{\sqrt{3}}(\boldsymbol{q}_1^{\mu_3} + \boldsymbol{q}_3^{\mu_3} + \boldsymbol{q}_5^{\mu_3}) = \frac{1}{\sqrt{6}}(2, 0, 0, 0, 0, -1, 0, -1, 0)^\top,$$

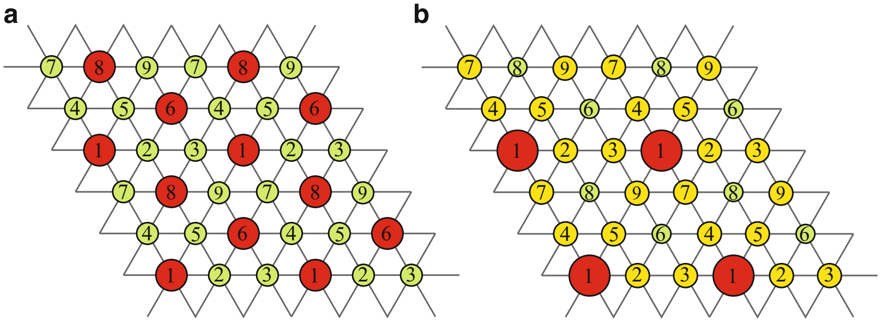

Fig. 7.2 Hexagonal patterns on the 3 × 3 hexagonal lattice. (**a**) $\boldsymbol{q}_1^{\mu_2}$. (**b**) $\frac{1}{\sqrt{3}}(\boldsymbol{q}_1^{\mu_3} + \boldsymbol{q}_3^{\mu_3} + \boldsymbol{q}_5^{\mu_3})$

shown in Fig. 7.2b, denotes Lösch's hexagon with $L/d = \sqrt{9}$ with $\Sigma(3,0)$-symmetry.

A more issue on hexagonal patterns is discussed in Sect. 7.6.

7.3 Representation Matrix

In our study of a system of $N = n^2$ places on the $n \times n$ hexagonal lattice, each element g of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ acts as a permutation of place numbers $(1, \ldots, N)$. Consequently, the representation matrix $T(g)$ is a permutation matrix for each g. By definition (Example 2.8 in Sect. 2.3.2), $T(g)$ has "1" at the (i, j) entry if place j is moved to place i by the action of g.

In Sect. 7.2 we have seen the representation matrices for $n = 3$. To be specific, the permutations caused by r, s, p_1, and p_2 are illustrated in Fig. 7.1 and the representation matrices for r, s, p_1, and p_2 are given in (7.1) and (7.2).

The representation matrix $T(g)$ for general n can be determined as follows. The coordinate of a place on the $n \times n$ hexagonal lattice is given by

$$\boldsymbol{x} = n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2, \quad n_1, n_2 = 0, 1, \ldots, n-1$$

with $\boldsymbol{\ell}_1 = d(1,0)^\top$, $\boldsymbol{\ell}_2 = d(-1/2, \sqrt{3}/2)^\top$ in (5.1), where d means the length of these vectors. Thus, the n^2 places are indexed by (n_1, n_2), and so are the rows and columns of the representation matrix $T(g)$. The action of r is expressed as

$$r \cdot \boldsymbol{\ell}_1 = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2, \quad r \cdot \boldsymbol{\ell}_2 = -\boldsymbol{\ell}_1.$$

Hence, we have

$$r \cdot \boldsymbol{x} = n_1(r \cdot \boldsymbol{\ell}_1) + n_2(r \cdot \boldsymbol{\ell}_2) = n_1(\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2) + n_2(-\boldsymbol{\ell}_1) = (n_1 - n_2)\boldsymbol{\ell}_1 + n_1 \boldsymbol{\ell}_2,$$

which means that the action of r on (n_1, n_2) is given by

$$r \cdot (n_1, n_2) \equiv (n_1 - n_2,\ n_1) \quad \bmod n. \tag{7.11}$$

This shows that the column of $T(r)$ indexed by (n_1, n_2) has "1" in the row indexed by $(n_1 - n_2 \bmod n, n_1)$. Similarly, the actions of s, p_1, and p_2 are expressed as

$$s \cdot (n_1, n_2) \equiv (n_1 - n_2,\ -n_2) \quad \bmod n, \tag{7.12}$$

$$p_1 \cdot (n_1, n_2) \equiv (n_1 + 1,\ n_2) \quad \bmod n, \tag{7.13}$$

$$p_2 \cdot (n_1, n_2) \equiv (n_1,\ n_2 + 1) \quad \bmod n. \tag{7.14}$$

The permutation representation $T(g)$ is specified by (7.11)–(7.14) above.

Example 7.1. The permutation representation for the 4×4 hexagonal lattice is given by (7.11)–(7.14) as follows:

$$T(r) = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&1&&& \\
&&&&&&&&1&&&&&&& \\
&&&&1&&&&&&&&&&& \\ \hline
&&&&&1&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&1&&&&&& \\ \hline
&&&&&&&&&&1&&&&& \\
&&&&&&1&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&1& \\ \hline
&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&1&&&& \\
&&&&&&&1&&&&&&&& \\
&&&1&&&&&&&&&&&&
\end{array}\right], \quad
T(s) = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\
&&&1&&&&&&&&&&&& \\ \hline
&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&&&1& \\ \hline
&&&&&&&&&&1&&&&& \\
&&&&&&&&&&&1&&&& \\
&&&&&&&&1&&&&&&& \\
&&&&&&&&&1&&&&&& \\ \hline
&&&&&1&&&&&&&&&& \\
&&&&&&1&&&&&&&&& \\
&&&&&&&1&&&&&&&& \\
&&&&1&&&&&&&&&&&
\end{array}\right],$$

$$T(p_1) = \left[\begin{array}{cccc|cccc|cccc|cccc}
&&&1&&&&&&&&&&&& \\
1&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\ \hline
&&&&&&&1&&&&&&&& \\
&&&&1&&&&&&&&&&& \\
&&&&&1&&&&&&&&&& \\
&&&&&&1&&&&&&&&& \\ \hline
&&&&&&&&&&&1&&&& \\
&&&&&&&&1&&&&&&& \\
&&&&&&&&&1&&&&&& \\
&&&&&&&&&&1&&&&& \\ \hline
&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&&&1&
\end{array}\right], \quad
T(p_2) = \left[\begin{array}{cccc|cccc|cccc|cccc}
&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&&&1& \\
&&&&&&&&&&&&&&&1 \\ \hline
1&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\
&&&1&&&&&&&&&&&& \\ \hline
&&&&1&&&&&&&&&&& \\
&&&&&1&&&&&&&&&& \\
&&&&&&1&&&&&&&&& \\
&&&&&&&1&&&&&&&& \\ \hline
&&&&&&&&1&&&&&&& \\
&&&&&&&&&1&&&&&& \\
&&&&&&&&&&1&&&&& \\
&&&&&&&&&&&1&&&&
\end{array}\right].$$

$\square$

7.4 Irreducible Decomposition

The irreducible decomposition (7.3) of the permutation representation $T(g)$ for the $n \times n$ hexagonal lattice is now investigated. The multiplicities of irreducible representations in this decomposition are determined. It is to be emphasized that irreducible representations lacking in the decomposition of $T(g)$ can be excluded from consideration in the search for hexagonal bifurcating patterns in Chaps. 8–9.

7.4.1 Simple Examples

Prior to the analyses for general n we present the results for $n = 3$ and $n = 4$.

We begin with the case of $n = 3$. First recall the permutation representation $T(g)$ for $n = 3$ from (7.1) and (7.2). The group $\mathrm{D}_6 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$ has 11 irreducible representations (Sect. 6.3):

$$\begin{aligned} R(\mathrm{D}_6 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)) = \{&(1;+,+), (1;+,-), (1;-,+), (1;-,-),\\ &(2;1), (2;2), (2;+), (2;-), (4;1), (6;1,0,+), (6;1,0,-)\}. \end{aligned}$$

Among these 11 irreducible representations, only three of them, $(1;+,+)$, $(2;+)$, and $(6;1,0,+)$, are contained in $T(g)$ with multiplicity 1, whereas the others are missing in $T(g)$. Indeed we have seen in (7.5) (and will see in Sect. 7.5 in a general setting) that

$$Q^{-1}T(g)Q = T^{(1;+,+)}(g) \oplus T^{(2;+)}(g) \oplus T^{(6;1,0,+)}(g), \quad g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) \tag{7.15}$$

for the orthogonal matrix Q in (7.4). Accordingly, the multiplicities a^{μ} for $\mu \in R(\mathrm{D}_6 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3))$ are given as follows:

$$\begin{aligned} &a^{(1;+,+)} = 1, \quad a^{(1;+,-)} = 0, \quad a^{(1;-,+)} = 0, \quad a^{(1;-,-)} = 0;\\ &a^{(2;1)} = 0, \quad a^{(2;2)} = 0, \quad a^{(2;+)} = 1, \quad a^{(2;-)} = 0;\\ &a^{(4;1)} = 0; \quad a^{(6;1,0,+)} = 1, \quad a^{(6;1,0,-)} = 0. \end{aligned}$$

We next show the case of $n = 4$. Recall the permutation representation $T(g)$ for $n = 4$ from Example 7.1. The group $\mathrm{D}_6 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ has 14 irreducible representations (Sect. 6.3):

$$\begin{aligned} R(\mathrm{D}_6 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)) = \{&(1;+,+), (1;+,-), (1;-,+), (1;-,-),\\ &(2;1), (2;2), (3;+,+), (3;+,-), (3;-,+), (3;-,-),\\ &(6;1,0,+), (6;1,0,-), (6;1,1,+), (6;1,1,-)\}. \end{aligned}$$

Among these 14 irreducible representations, only four of them, $(1;+,+)$, $(3;+,+)$, $(6;1,0,+)$, and $(6;1,1,+)$, are contained in $T(g)$ with multiplicity 1, whereas the others are missing in $T(g)$, as we will see in Sect. 7.5 in a general setting. This means that

$$Q^{-1}T(g)Q = T^{(1;+,+)}(g) \oplus T^{(3;+,+)}(g) \oplus T^{(6;1,0,+)}(g) \oplus T^{(6;1,1,+)}(g),$$
$$g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4) \qquad (7.16)$$

for some orthogonal matrix Q, the concrete form of which is given in Example 7.2 in Sect. 7.5.1. Accordingly, the multiplicities a^μ for $\mu \in R(\mathrm{D}_6 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4))$ are given as follows:

$$\begin{aligned}
a^{(1;+,+)} &= 1, \quad a^{(1;+,-)} = 0, \quad a^{(1;-,+)} = 0, \quad a^{(1;-,-)} = 0,\\
a^{(2;1)} &= 0, \quad a^{(2;2)} = 0,\\
a^{(3;+,+)} &= 1, \quad a^{(3;+,-)} = 0, \quad a^{(3;-,+)} = 0, \quad a^{(3;-,-)} = 0,\\
a^{(6;1,0,+)} &= 1, \quad a^{(6;1,0,-)} = 0, \quad a^{(6;1,1,+)} = 1, \quad a^{(6;1,1,-)} = 0.
\end{aligned}$$

7.4.2 Analysis for the Finite Hexagonal Lattice

For general n, the permutation representation $T(g)$ is specified by (7.11)–(7.14). We determine the irreducible decomposition of $T(g)$ with the aid of characters. Let $\chi(g)$ be the character of $T(g)$, which is defined (see (2.54)) by

$$\chi(g) = \mathrm{Tr}\, T(g), \quad g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n). \qquad (7.17)$$

Table 7.1 shows the values of $\chi(g)$ for all $g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, which are dependent on whether n is a multiple of 2 and/or 3.

In terms of characters, the irreducible decomposition of $T(g)$ can be expressed as

$$\chi(g) = \sum_{\mu} a^\mu \chi^\mu(g), \quad g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \qquad (7.18)$$

where χ^μ is the character of $\mu \in R(\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n))$, and the multiplicity a^μ of μ can be determined by the formula

$$a^\mu = \frac{1}{12n^2} \sum_{g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)} \chi(g)\chi^\mu(g); \qquad (7.19)$$

see (2.57) and (2.58). It is noted that the formula (2.58), valid for representations over $\mathbb{C}$, can be used here since all the irreducible representations are absolutely irreducible.

Table 7.1 The values of character χ of the permutation representation T

g		$\chi(g)$	
e		n^2	
${p_1}^i {p_2}^j$	$((i,j) \neq (0,0))$	0	
$r {p_1}^i {p_2}^j$		1	
$r^2 {p_1}^i {p_2}^j$	$(i+j=3k)$	3	1
	$(i+j \neq 3k)$	0	1
		$(n=3m)$	$(n \neq 3m)$
$r^3 {p_1}^i {p_2}^j$	$(i, j\text{: even})$	4	1
	(other (i,j))	0	1
		$(n=2m)$	$(n \neq 2m)$
$r^4 {p_1}^i {p_2}^j$	$(i+j=3k)$	3	1
	$(i+j \neq 3k)$	0	1
		$(n=3m)$	$(n \neq 3m)$
$r^5 {p_1}^i {p_2}^j$		1	
$s {p_1}^i {p_2}^j$	$(2i-j=nk)$	n	
	$(2i-j \neq nk)$	0	
$sr {p_1}^i {p_2}^j$	$(i-j=nk)$	n	
	$(i-j \neq nk)$	0	
$sr^2 {p_1}^i {p_2}^j$	$(i-2j=nk)$	n	
	$(i-2j \neq nk)$	0	
$sr^3 {p_1}^i {p_2}^j$	$(j=nk)$	n	
	$(j \neq nk)$	0	
$sr^4 {p_1}^i {p_2}^j$	$(i+j=nk)$	n	
	$(i+j \neq nk)$	0	
$sr^5 {p_1}^i {p_2}^j$	$(i=nk)$	n	
	$(i \neq nk)$	0	

$0 \leq i, j \leq n-1$; k, m: integers

In the case of $n = 6m$ (a multiple of 6), for example, we obtain

$$\chi(g) = \chi^{(1;+,+)}(g) + \chi^{(2;+)}(g) \;\; + \chi^{(3;+,+)}(g) + \sum_{k:(6.26)} \chi^{(6;k,0,+)}(g) + \sum_{k:(6.27)} \chi^{(6;k,k,+)}(g) + \sum_{(k,\ell):(6.35)} \chi^{(12;k,\ell)}(g)$$

as the decomposition (7.18). The term $\chi^{(2;+)}(g)$ appears only when n is a multiple of 3, and $\chi^{(3;+,+)}(g)$ appears only when n is even. Hence we may represent this succinctly as

$$\chi(g) = \chi^{(1;+,+)}(g) \;\; \left[+\chi^{(2;+)}(g)\right]_{\text{if } n=3m} \;\; \left[+\chi^{(3;+,+)}(g)\right]_{\text{if } n=2m} + \sum_{k:(6.26)} \chi^{(6;k,0,+)}(g) + \sum_{k:(6.27)} \chi^{(6;k,k,+)}(g) + \sum_{(k,\ell):(6.35)} \chi^{(12;k,\ell)}(g),$$

$$g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \qquad (7.20)$$

Table 7.2 The values of irreducible characters χ^{μ} appearing in (7.20)

g	$\chi^{(1;+,+)}$	$\chi^{(6;k,0,+)}$	$\chi^{(6;k,k,+)}$	$\chi^{(12;k,\ell)}$	$\chi^{(2;+)}$ $(n=3m)$	$\chi^{(3;+,+)}$ $(n=2m)$
${p_1}^i {p_2}^j$	1	$2[\cos(ki\theta) + \cos(-kj\theta) + \cos(-k(i-j)\theta)]$	$2[\cos(k(i+j)\theta) + \cos(k(i-2j)\theta) + \cos(-k(2i-j)\theta)]$	(6.40)	$2\cos(\frac{2}{3}(i+j)\pi)$	$(-1)^i+(-1)^j+(-1)^{i+j}$
$r{p_1}^i {p_2}^j$	1	0	0	0	0	0
$r^2{p_1}^i {p_2}^j$	1	0	0	0	$2\cos(\frac{2}{3}(i+j)\pi)$	0
$r^3{p_1}^i {p_2}^j$	1	0	0	0	0	$(-1)^i+(-1)^j+(-1)^{i+j}$
$r^4{p_1}^i {p_2}^j$	1	0	0	0	$2\cos(\frac{2}{3}(i+j)\pi)$	0
$r^5{p_1}^i {p_2}^j$	1	0	0	0	0	0
$s{p_1}^i {p_2}^j$	1	0	$2\cos(-k(2i-j)\theta)$	0	$2\cos(\frac{2}{3}(i+j)\pi)$	$(-1)^j$
$sr{p_1}^i {p_2}^j$	1	$2\cos(-k(i-j)\theta)$	0	0	0	$(-1)^{i+j}$
$sr^2{p_1}^i {p_2}^j$	1	0	$2\cos(k(i-2j)\theta)$	0	$2\cos(\frac{2}{3}(i+j)\pi)$	$(-1)^i$
$sr^3{p_1}^i {p_2}^j$	1	$2\cos(-kj\theta)$	0	0	0	$(-1)^j$
$sr^4{p_1}^i {p_2}^j$	1	0	$2\cos(k(i+j)\theta)$	0	$2\cos(\frac{2}{3}(i+j)\pi)$	$(-1)^{i+j}$
$sr^5{p_1}^i {p_2}^j$	1	$2\cos(ki\theta)$	0	0	0	$(-1)^i$

$\theta = 2\pi/n$ and (6.40) reads:

$$\chi^{(12;k,\ell)}({p_1}^i {p_2}^j) = 2[\cos((ki+\ell j)\theta) + \cos((\ell i-(k+\ell)j)\theta) + \cos((-(k+\ell)i+kj)\theta) + \cos((ki-(k+\ell)j)\theta) + \cos((\ell i+kj)\theta) + \cos((-(k+\ell)i+\ell j)\theta)]$$

where $[\ \cdot\ ]_{\text{if } n=3m}$ means that the term is included when n is a multiple of 3, and similarly for $[\ \cdot\ ]_{\text{if } n=2m}$. Table 7.2 shows the values of the irreducible characters $\chi^{\mu}(g)$ appearing on the right-hand side of (7.20) (see Sect. 6.3 for details about $\chi^{\mu}(g)$). The equality in (7.20) can be verified with the aid of Tables 7.1 and 7.2.

The decomposition (7.20) of the character $\chi(g)$ of $T(g)$ means that some orthogonal matrix Q exists such that

$$\begin{aligned} Q^{-1}T(g)Q = {} & T^{(1;+,+)}(g)\ \ [\oplus T^{(2;+)}(g)]_{\text{if } n=3m}\ \ [\oplus T^{(3;+,+)}(g)]_{\text{if } n=2m} \\ & \oplus \bigoplus_{k:(6.26)} T^{(6;k,0,+)}(g) \oplus \bigoplus_{k:(6.27)} T^{(6;k,k,+)}(g) \\ & \oplus \bigoplus_{(k,\ell):(6.35)} T^{(12;k,\ell)}(g), \\ & \qquad g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n). \qquad (7.21) \end{aligned}$$

This gives the irreducible decomposition (2.42) of $T(g)$. Accordingly, the multiplicities a^{μ} in the irreducible decomposition of $T(g)$ are given as follows:

$$a^{(1;+,+)} = 1, \quad a^{(1;+,-)} = 0, \quad a^{(1;-,+)} = 0, \quad a^{(1;-,-)} = 0,$$

$$a^{(2;1)} = 0, \quad a^{(2;2)} = 0, \quad a^{(2;+)} = \begin{cases} 1, & \text{if } n \text{ is a multiple of 3}, \\ 0, & \text{otherwise}, \end{cases} \quad a^{(2;-)} = 0,$$

$$a^{(3;+,+)} = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}, \end{cases} \quad a^{(3;+,-)} = 0, \quad a^{(3;-,+)} = 0, \quad a^{(3;-,-)} = 0,$$

$$a^{(4;1)} = 0,$$

$$a^{(6;k,0,+)} = 1, \quad a^{(6;k,0,-)} = 0, \quad 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$$

$$a^{(6;k,k,+)} = 1, \quad a^{(6;k,k,-)} = 0, \quad 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \ k \neq \frac{n}{3},$$

$$a^{(12;k,\ell)} = 1, \quad 1 \leq \ell \leq k-1, \ 2k + \ell \leq n-1.$$

The unit representation $(1;+,+)$ is the only one-dimensional irreducible representation with nonzero multiplicity, $(2;+)$ is the only two-dimensional one with nonzero multiplicity, and $(3;+,+)$ is the only three-dimensional one with nonzero multiplicity. Among the six-dimensional ones, $(6;k,0,\sigma)$ and $(6;k,k,\sigma)$ with k in (6.26) and (6.27), respectively, only those with σ = " + " have nonzero multiplicity, and those with σ = " − " have zero multiplicity. All 12-dimensional ones $(12;k,\ell)$ with (k,ℓ) in (6.35) have nonzero multiplicity. It is noteworthy that the multiplicity is either 0 or 1 for each irreducible representation, that is, the permutation representation $T(g)$ in (7.11)–(7.14) is *multiplicity-free* (Remark 7.1 below). Table 7.3 shows a summary.

By $\tilde{N}_d$, we denote the number of d-dimensional irreducible representations of $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ that exists in the permutation representation $T(g)$. We have the following expressions for $\tilde{N}_d$:

$n \setminus d$	1 $\tilde{N}_1$	2 $\tilde{N}_2$	3 $\tilde{N}_3$	4 $\tilde{N}_4$	6 $\tilde{N}_6$	12 $\tilde{N}_{12}$
$6m$	1	1	1	0	$n-3$	$(n^2-6n+12)/12$
$6m \pm 1$	1	0	0	0	$n-1$	$(n^2-6n+5)/12$
$6m \pm 2$	1	0	1	0	$n-2$	$(n^2-6n+8)/12$
$6m \pm 3$	1	1	0	0	$n-2$	$(n^2-6n+9)/12$

(7.22)

whereas Table 7.4 shows the values of $\tilde{N}_d$ for several n. Also note the relation

$$\sum_d d\,\tilde{N}_d = n^2. \tag{7.23}$$

Remark 7.1. The permutation representation $T(g)$ defined by (7.11)–(7.14) has been found to be multiplicity-free through an explicit computation of the multi-

Table 7.3 Irreducible representations contained in the permutation representation T

$n \setminus d$	1	2	3	4	6	12
$6m$	$(1;+,+)$	$(2;+)$	$(3;+,+)$		$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 1$	$(1;+,+)$				$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 2$	$(1;+,+)$		$(3;+,+)$		$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 3$	$(1;+,+)$	$(2;+)$			$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$

$(6;k,0;+)$ for k in (6.26); $(6;k,k;+)$ for k in (6.27); $(12;k,\ell)$ for (k,ℓ) in (6.35)

Table 7.4 Number $\tilde{N}_d$ of d-dimensional irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ contained in the permutation representation T for the hexagonal lattice

$n \setminus d$	1	2	3	4	6	12		$n \setminus d$	1	2	3	4	6	12	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_3$	$\tilde{N}_4$	$\tilde{N}_6$	$\tilde{N}_{12}$	$\sum \tilde{N}_d$		$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_3$	$\tilde{N}_4$	$\tilde{N}_6$	$\tilde{N}_{12}$	$\sum \tilde{N}_d$
1	1						1	13	1				12	8	21
2	1		1				2	14	1		1		12	10	24
3	1	1			1		3	15	1	1			13	12	27
4	1		1		2		4	16	1		1		14	14	30
5	1				4		5	17	1				16	16	33
6	1	1	1		3	1	7	18	1	1	1		15	19	37
7	1				6	1	8	19	1				18	21	40
8	1		1		6	2	10	20	1		1		18	24	44
9	1	1			7	3	12	21	1	1			19	27	48
								$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	1		1		8	4	14								
11	1				10	5	16	42	1	1	1		39	127	169
12	1	1	1		9	7	19								

plicities by means of characters. This can also be seen from a general fact[2] that a permutation representation $T(g)$ representing the action of a group G on a finite set P is multiplicity-free if, for any pair of p and q in P, there exists some $g \in G$ such that $g \cdot p = q$ and $g \cdot q = p$. This condition can be verified for the permutation representation $T(g)$ in (7.11)–(7.14) as follows. By (7.11), (7.13), and (7.14), we have

$$r^3 {p_1}^i {p_2}^j \cdot (n_1, n_2) \equiv (-n_1 - i, -n_2 - j) \mod n.$$

Hence, any pair of (n_1, n_2) and (n_1', n_2') can be interchanged as

$$g \cdot (n_1, n_2) \equiv (n_1', n_2'), \quad g \cdot (n_1', n_2') \equiv (n_1, n_2) \mod n$$

by $g = r^3 {p_1}^i {p_2}^j$ with $i = -n_1 - n_1'$ and $j = -n_2 - n_2'$. $\square$

---

[2]This is a basic fact, sometimes called *Gelfand's lemma*, in the context of *Gelfand pairs* (Diaconis, 1988 [2]; Macdonald, 1995 [3]; Ceccherini-Silberstein, Scarabotti, and Tolli, 2010 [1]). See Proposition 1.4.8 of [1], in particular.

7.5 Transformation Matrix for Irreducible Decomposition

Transformation matrix Q for the irreducible decomposition is derived for the hexagonal lattice, and examples of this matrix Q are presented.

7.5.1 Formulas for Transformation Matrix

For the economy on the $n \times n$ hexagonal lattice with the symmetry of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, we derive the transformation matrix

$$Q = (Q^{\mu} \mid \mu \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)) \tag{7.24}$$

for the irreducible decomposition (7.3). Note that the column set of Q is partitioned into blocks, each associated with an irreducible representation μ present in $T(g)$ (Table 7.3). Since such μ has $a^{\mu} = 1$ (multiplicity-free), the relation (2.49) with (2.50) is simplified to

$$T(g)Q^{\mu} = Q^{\mu}T^{\mu}(g), \quad g \in \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \tag{7.25}$$

where $T(g)$ is the permutation representation given in Sect. 7.3.

The vector $\boldsymbol{\lambda}$ expressing population distribution is defined as

$$\begin{aligned}
\boldsymbol{\lambda} &= (\lambda_1, \dots, \lambda_N)^{\top} \\
&= (\lambda_{00}, \dots, \lambda_{n-1,0};\ \lambda_{01}, \dots, \lambda_{n-1,1};\ \dots\ ;\ \lambda_{0,n-1}, \dots, \lambda_{n-1,n-1})^{\top} \\
&= (\lambda_{n_1 n_2} \mid n_1, n_2 = 0, \dots, n-1),
\end{aligned}$$

where $N = n^2$ and $(\lambda_{n_1 n_2} \mid n_1, n_2 = 0, \dots, n-1)$ is an N-dimensional column vector. For a vector on this lattice with the (n_1, n_2)-component $g(n_1, n_2)$, we express its normalization as[3]

$$\langle g(n_1, n_2) \rangle = \Big(g(n_1, n_2) / \big(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} g(i, j)^2\big)^{1/2} \mid n_1, n_2 = 0, \dots, n-1\Big). \tag{7.26}$$

Recall that the permutation representation $T(g)$ is specified by (7.11)–(7.14) above. The action of r on (n_1, n_2), for example, is expressed by

$$r \cdot (n_1, n_2) \equiv (n_1 - n_2,\ n_1) \mod n$$

[3] The notation $\langle \cdot \rangle$ here should not be confused with that for the generators of a group.

in (7.11), which shows that the column of $T(r)$ indexed by (n_1, n_2) has "1" in the row indexed by $(n_1 - n_2, n_1) \mod n$. For the present purpose, however, it is convenient to consider $T(g)$ row-wise. It is seen that the row of $T(r)$ indexed by (n_1, n_2) has "1" at the column indexed by $(n_2, n_2 - n_1) \mod n$, since

$$(n'_1, n'_2) \equiv (n_1 - n_2,\ n_1) \mod n$$

can be solved for (n_1, n_2) as

$$(n_1, n_2) \equiv (n'_2,\ n'_2 - n'_1) \mod n.$$

We denote this as

$$r * (n_1, n_2) \equiv (n_2, n_2 - n_1) \mod n. \tag{7.27}$$

For s, p_1, and p_2, a similar argument based on (7.12)–(7.14) yields

$$s * (n_1, n_2) \equiv (n_1 - n_2,\ -n_2) \mod n, \tag{7.28}$$

$$p_1 * (n_1, n_2) \equiv (n_1 - 1,\ n_2) \mod n, \tag{7.29}$$

$$p_2 * (n_1, n_2) \equiv (n_1,\ n_2 - 1) \mod n. \tag{7.30}$$

The submatrices Q^{μ} for μ are given by the following proposition, where the notation $\langle\cdot\rangle$ for normalization in (7.26) is used and the set of integers that are multiples of 3 is denoted by $3\mathbb{Z}$.

Proposition 7.1. *The submatrices Q^{μ} of the transformation matrix Q for the economy on the $n \times n$ hexagonal lattice are given by*

$$Q^{(1;+,+)} = \frac{1}{n}(1, \ldots, 1)^{\top} = \langle 1 \rangle, \tag{7.31}$$

$$Q^{(2;+)} = \begin{cases} [\ \langle \cos(2\pi(n_1 - 2n_2)/3) \rangle,\ \langle \sin(2\pi(n_1 - 2n_2)/3) \rangle\ ] & \text{for } n \in 3\mathbb{Z}, \\ \text{missing} & \text{for } n \notin 3\mathbb{Z}, \end{cases} \tag{7.32}$$

$$Q^{(3;+,+)} = \begin{cases} [\ \langle \cos(\pi n_1) \rangle,\ \langle \cos(\pi n_2) \rangle,\ \langle \cos(\pi(n_1 - n_2)) \rangle\ ] & \text{for } n \text{ even}, \\ \text{missing} & \text{for } n \text{ odd}, \end{cases} \tag{7.33}$$

$$\begin{aligned} Q^{(6;k,0,+)} = [\ & \langle \cos(2\pi k\ n_1/n) \rangle,\ \langle \sin(2\pi k\ n_1/n) \rangle, \\ & \langle \cos(2\pi k(-n_2)/n) \rangle,\ \langle \sin(2\pi k(-n_2)/n) \rangle, \\ & \langle \cos(2\pi k(-n_1 + n_2)/n) \rangle,\ \langle \sin(2\pi k(-n_1 + n_2)/n) \rangle\ ] \\ & \text{for } 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \end{aligned} \tag{7.34}$$

$$
\begin{aligned}
Q^{(6;k,k,+)} = [\, & \langle \cos(2\pi k(n_1+n_2)/n)\rangle,\ \langle \sin(2\pi k(n_1+n_2)/n)\rangle,\\
& \langle \cos(2\pi k(n_1-2n_2)/n)\rangle,\ \langle \sin(2\pi k(n_1-2n_2)/n)\rangle,\\
& \langle \cos(2\pi k(-2n_1+n_2)/n)\rangle,\ \langle \sin(2\pi k(-2n_1+n_2)/n)\rangle\,]\\
& \text{for } 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor,\ k \ne \frac{n}{3},
\end{aligned}
\tag{7.35}
$$

$$
\begin{aligned}
Q^{(12;k,\ell)} = [\, & \langle \cos(2\pi(kn_1+\ell n_2)/n)\rangle,\ \langle \sin(2\pi(kn_1+\ell n_2)/n)\rangle,\\
& \langle \cos(2\pi(\ell n_1-(k+\ell)n_2)/n)\rangle,\ \langle \sin(2\pi(\ell n_1-(k+\ell)n_2)/n)\rangle,\\
& \langle \cos(2\pi(-(k+\ell)n_1+kn_2)/n)\rangle,\ \langle \sin(2\pi(-(k+\ell)n_1+kn_2)/n)\rangle,\\
& \langle \cos(2\pi(kn_1-(k+\ell)n_2)/n)\rangle,\ \langle \sin(2\pi(kn_1-(k+\ell)n_2)/n)\rangle,\\
& \langle \cos(2\pi(\ell n_1+kn_2)/n)\rangle,\ \langle \sin(2\pi(\ell n_1+kn_2)/n)\rangle,\\
& \langle \cos(2\pi(-(k+\ell)n_1+\ell n_2)/n)\rangle,\ \langle \sin(2\pi(-(k+\ell)n_1+\ell n_2)/n)\rangle\,]\\
& \text{for } 1 \le \ell \le k-1,\ 2k+\ell \le n-1.
\end{aligned}
\tag{7.36}
$$

Proof. Proof is given in Sect. 7.5.2. □

Since the multiplicities of irreducible representations μ contained in T are all equal to unity ($a^\mu = 1$), the orthogonal transformation of the Jacobian matrix $J = J(\boldsymbol{\lambda}, \tau)$ of the equilibrium equation in (4.2) takes a diagonal form

$$
Q^{-1} J Q = \mathrm{diag}(e_1, \dots, e_N). \tag{7.37}
$$

This diagonal form is useful in the eigenanalysis in the computational bifurcation analysis of the economy on the hexagonal lattice (Sect. 7.6.2).

An example of the transformation matrix Q for $n = 4$ is presented below by assembling submatrices Q^μ in Proposition 7.1.

Example 7.2. The transformation matrix Q for the 4×4 hexagonal lattice reads

$$
\begin{aligned}
Q &= [Q^{(1;+,+)}, Q^{(3;+,+)}, Q^{(6;1,0,+)}, Q^{(6;1,1,+)}]\\
&= [\, \langle 1\rangle \mid \langle \cos(\pi n_1)\rangle,\ \langle \cos(\pi n_2)\rangle,\ \langle \cos(\pi(n_1-n_2))\rangle \mid\\
&\quad \langle \cos(2\pi\ n_1/4)\rangle,\ \langle \sin(2\pi\ n_1/4)\rangle,\ \langle \cos(2\pi\ (-n_2)/4)\rangle,\ \langle \sin(2\pi\ (-n_2)/4)\rangle,\\
&\quad \langle \cos(2\pi\ (-n_1+n_2)/4)\rangle,\ \langle \sin(2\pi\ (-n_1+n_2)/4)\rangle \mid\\
&\quad \langle \cos(2\pi(n_1+n_2)/4)\rangle,\ \langle \sin(2\pi(n_1+n_2)/4)\rangle,\ \langle \cos(2\pi(n_1-2n_2)/4)\rangle,\\
&\quad \langle \sin(2\pi(n_1-2n_2)/4)\rangle,\ \langle \cos(2\pi(-2n_1+n_2)/4)\rangle,\ \langle \sin(2\pi(-2n_1+n_2)/4)\rangle\,]
\end{aligned}
$$

$$
= \frac{1}{4}\left[\begin{array}{c|ccc|cccccc|cccccc}
1 & 1 & 1 & 1 & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & \\
1 & -1 & 1 & -1 & & \sqrt{2} & \sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & & \sqrt{2} & -\sqrt{2} & \\
1 & 1 & 1 & 1 & -\sqrt{2} & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & \\
1 & -1 & 1 & -1 & & -\sqrt{2} & \sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & -\sqrt{2} & \\
\hline
1 & 1 & -1 & -1 & \sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & & \sqrt{2} & -\sqrt{2} & & & \sqrt{2} \\
1 & -1 & -1 & 1 & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & & -\sqrt{2} & & & -\sqrt{2} & & -\sqrt{2} \\
1 & 1 & -1 & -1 & -\sqrt{2} & & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & \sqrt{2} & & & \sqrt{2} \\
1 & -1 & -1 & 1 & & -\sqrt{2} & & -\sqrt{2} & -\sqrt{2} & & \sqrt{2} & & & \sqrt{2} & & -\sqrt{2} \\
\hline
1 & 1 & 1 & 1 & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & & -\sqrt{2} & \\
1 & -1 & 1 & -1 & & \sqrt{2} & -\sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & \\
1 & 1 & 1 & 1 & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & \\
1 & -1 & 1 & -1 & & -\sqrt{2} & -\sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & \\
\hline
1 & 1 & -1 & -1 & \sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & -\sqrt{2} & & & -\sqrt{2} \\
1 & -1 & -1 & 1 & & \sqrt{2} & & \sqrt{2} & -\sqrt{2} & & \sqrt{2} & & & -\sqrt{2} & & \sqrt{2} \\
1 & 1 & -1 & -1 & -\sqrt{2} & & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & \sqrt{2} & & & -\sqrt{2} \\
1 & -1 & -1 & 1 & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & & -\sqrt{2} & & & \sqrt{2} & & \sqrt{2}
\end{array}\right].
$$

The diagonal form in (7.37) reads

$$
\begin{aligned}
Q^{-1}J(\boldsymbol{\lambda},\tau)Q = \mathrm{diag}(&e^{(1;+,+)}; e^{(3;+,+)}, e^{(3;+,+)}, e^{(3;+,+)}; \\
&e^{(6;1,0,+)}, e^{(6;1,0,+)}, e^{(6;1,0,+)}, e^{(6;1,0,+)}, e^{(6;1,0,+)}, e^{(6;1,0,+)}; \\
&e^{(6;1,1,+)}, e^{(6;1,1,+)}, e^{(6;1,1,+)}, e^{(6;1,1,+)}, e^{(6;1,1,+)}, e^{(6;1,1,+)}).
\end{aligned}
$$

□

7.5.2 *Proof of Proposition 7.1*

We will now show that the relation $T(g)Q^{\mu} = Q^{\mu}T^{\mu}(g)$ in (7.25) is satisfied by Q^{μ} in Proposition 7.1 for r, s, p_1, and p_2 that generate the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Recall the actions of r, s, p_1, and p_2 given in (7.27)–(7.30). We demonstrate the proof for $\mu = (2;+)$ and $(12;k,\ell)$, and the other cases can be treated similarly.

Two-Dimensional Irreducible Representation

We shall prove that

$$
Q^{(2;+)} = [\ \langle\cos(2\pi(n_1-2n_2)/3)\rangle,\ \langle\sin(2\pi(n_1-2n_2)/3)\rangle\ ] \qquad (7.38)
$$

satisfies (7.25) for $\mu = (2;+)$. Recall that $(2;+)$ exists when n is a multiple of 3 and $T^{(2;+)}(g)$ is defined by (6.13) and (6.14).

The action of r on the wave number $n_1 - 2n_2$ in (7.38) is given, by a formal calculation using (7.27), as

$$
r * (n_1 - 2n_2) = (r * n_1) - 2(r * n_2) = n_2 - 2(n_2 - n_1) \equiv 2n_1 - n_2 \mod n.
$$

In the matrix form, this gives

$$\begin{aligned} T(r)Q^{(2;+)} &= [\ \langle\cos(2\pi(2n_1-n_2)/3)\rangle,\ \langle\sin(2\pi(2n_1-n_2)/3)\rangle\ ] \\ &= [\ \langle\cos(2\pi(-(n_1-2n_2))/3)\rangle,\ \langle\sin(2\pi(-(n_1-2n_2))/3)\rangle\ ] \\ &= [\ \langle\cos(2\pi(n_1-2n_2)/3)\rangle,\ \langle\sin(2\pi(n_1-2n_2)/3)\rangle\ ] \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\ &= Q^{(2;+)}T^{(2;+)}(r). \end{aligned}$$

The action of p_1 on the wave number n_1-2n_2 is given by (7.29) as

$$p_1 * (n_1-2n_2) \equiv n_1-2n_2-1 \mod n,$$

which, in the matrix form, yields

$$\begin{aligned} T(p_1)Q^{(2;+)} &= [\ \langle\cos(2\pi(n_1-2n_2-1)/3)\rangle,\ \langle\sin(2\pi(n_1-2n_2-1)/3)\rangle\ ] \\ &= [\ \langle\cos(2\pi(n_1-2n_2)/3)\rangle,\ \langle\sin(2\pi(n_1-2n_2)/3)\rangle\ ] \\ &\quad\times \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} \\ &= Q^{(2;+)}T^{(2;+)}(p_1). \end{aligned}$$

The cases of s and p_2 can be treated similarly. Thus, we have

$$T(g)Q^{(2;+)} = Q^{(2;+)}T^{(2;+)}(g), \quad g = r, s, p_1, p_2.$$

This completes the proof for $\mu = (2;+)$.

Twelve-Dimensional Irreducible Representations

We shall prove that

$$\begin{aligned} Q^{(12;k,\ell)} = [\ &\langle\cos(2\pi(kn_1+\ell n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1+\ell n_2)/n)\rangle, \\ &\langle\cos(2\pi(\ell n_1-(k+\ell)n_2)/n)\rangle,\ \langle\sin(2\pi(\ell n_1-(k+\ell)n_2)/n)\rangle, \\ &\langle\cos(2\pi(-(k+\ell)n_1+kn_2)/n)\rangle,\ \langle\sin(2\pi(-(k+\ell)n_1+kn_2)/n)\rangle, \\ &\langle\cos(2\pi(kn_1-(k+\ell)n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1-(k+\ell)n_2)/n)\rangle, \\ &\langle\cos(2\pi(\ell n_1+kn_2)/n)\rangle,\ \langle\sin(2\pi(\ell n_1+kn_2)/n)\rangle, \\ &\langle\cos(2\pi(-(k+\ell)n_1+\ell n_2)/n)\rangle,\ \langle\sin(2\pi(-(k+\ell)n_1+\ell n_2)/n)\rangle\ ] \end{aligned} \tag{7.39}$$

satisfies (7.25) for $(12; k, \ell)$ where $1 \leq \ell \leq k-1$, $2k+\ell \leq n-1$, and $n \geq 6$. Recall the definition of $T^{(12;k,\ell)}(g)$ for $g = r, s, p_1, p_2$ in (6.37) and (6.38), as well as the notations

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

The action of r on the six wave numbers in (7.39) is given by (7.27) as

$$r * \begin{bmatrix} kn_1 + \ell n_2 \\ \ell n_1 - (k+\ell)n_2 \\ -(k+\ell)n_1 + kn_2 \\ kn_1 - (k+\ell)n_2 \\ \ell n_1 + kn_2 \\ -(k+\ell)n_1 + \ell n_2 \end{bmatrix} \equiv - \begin{bmatrix} \ell n_1 - (k+\ell)n_2 \\ -(k+\ell)n_1 + kn_2 \\ kn_1 + \ell n_2 \\ -(k+\ell)n_1 + \ell n_2 \\ kn_1 - (k+\ell)n_2 \\ \ell n_1 + kn_2 \end{bmatrix} \mod n,$$

which permutes and changes the sign of the column vectors of $Q^{(12;k,\ell)}$ in (7.39) as

$$T(r)Q^{(12;k,\ell)} = Q^{(12;k,\ell)} \left[\begin{array}{ccc|ccc} & & S & & & \\ S & & & & & \\ & S & & & & \\ \hline & & & & S & \\ & & & & & S \\ & & & S & & \end{array} \right] = Q^{(12;k,\ell)} T^{(12;k,\ell)}(r).$$

The action of s on the six wave numbers in (7.39) is given by (7.28) as

$$s * \begin{bmatrix} kn_1 + \ell n_2 \\ \ell n_1 - (k+\ell)n_2 \\ -(k+\ell)n_1 + kn_2 \\ kn_1 - (k+\ell)n_2 \\ \ell n_1 + kn_2 \\ -(k+\ell)n_1 + \ell n_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 - (k+\ell)n_2 \\ \ell n_1 + kn_2 \\ -(k+\ell)n_1 + \ell n_2 \\ kn_1 + \ell n_2 \\ \ell n_1 - (k+\ell)n_2 \\ -(k+\ell)n_1 + kn_2 \end{bmatrix} \mod n,$$

which gives

$$T(s)Q^{(12;k,\ell)} = Q^{(12;k,\ell)} \left[\begin{array}{ccc|ccc} & & & I & & \\ & & & & I & \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \\ & & I & & & \end{array} \right] = Q^{(12;k,\ell)} T^{(12;k,\ell)}(s).$$

The action of p_1 on the six wave numbers in (7.39) is given by (7.29) as

$$p_1 * \begin{bmatrix} kn_1 + \ell n_2 \\ \ell n_1 - (k+\ell)n_2 \\ -(k+\ell)n_1 + kn_2 \\ kn_1 - (k+\ell)n_2 \\ \ell n_1 + kn_2 \\ -(k+\ell)n_1 + \ell n_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 + \ell n_2 - k \\ \ell n_1 - (k+\ell)n_2 - \ell \\ -(k+\ell)n_1 + kn_2 + k + \ell \\ kn_1 - (k+\ell)n_2 - k \\ \ell n_1 + kn_2 - \ell \\ -(k+\ell)n_1 + \ell n_2 + k + \ell \end{bmatrix} \quad \text{mod } n,$$

which gives

$$T(p_1)Q^{(12;k,\ell)} = Q^{(12;k,\ell)} \left[\begin{array}{ccc|ccc} R^k & & & & & \\ & R^\ell & & & & \\ & & R^{-k-\ell} & & & \\ \hline & & & R^k & & \\ & & & & R^\ell & \\ & & & & & R^{-k-\ell} \end{array}\right] = Q^{(12;k,\ell)} T^{(12;k,\ell)}(p_1).$$

The action of p_2 on the six wave numbers in (7.39) is given by (7.30) as

$$p_2 * \begin{bmatrix} kn_1 + \ell n_2 \\ \ell n_1 - (k+\ell)n_2 \\ -(k+\ell)n_1 + kn_2 \\ kn_1 - (k+\ell)n_2 \\ \ell n_1 + kn_2 \\ -(k+\ell)n_1 + \ell n_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 + \ell n_2 - \ell \\ \ell n_1 - (k+\ell)n_2 + k + \ell \\ -(k+\ell)n_1 + kn_2 - k \\ kn_1 - (k+\ell)n_2 + k + \ell \\ \ell n_1 + kn_2 - k \\ -(k+\ell)n_1 + \ell n_2 - \ell \end{bmatrix} \quad \text{mod } n,$$

which gives

$$T(p_2)Q^{(12;k,\ell)} = Q^{(12;k,\ell)} \left[\begin{array}{ccc|ccc} R^\ell & & & & & \\ & R^{-k-\ell} & & & & \\ & & R^k & & & \\ \hline & & & R^{-k-\ell} & & \\ & & & & R^k & \\ & & & & & R^\ell \end{array}\right] = Q^{(12;k,\ell)} T^{(12;k,\ell)}(p_2).$$

Thus, we have the following relation to complete the proof for $\mu = (12; k, \ell)$:

$$T(g)Q^{(12;k,\ell)} = Q^{(12;k,\ell)} T^{(12;k,\ell)}(g), \quad g = r, s, p_1, p_2.$$

7.6 Geometrical Implication and Computational Use of Transformation Matrix

The explicit form of the transformation matrix Q for the hexagonal lattice has been derived in Sect. 7.5. Geometrical implication and computational use of this matrix are demonstrated here as a prelude to the group-theoretic bifurcation analysis of hexagonal bifurcating patterns in Chaps. 8 and 9.

7.6.1 Hexagonal Patterns Associated with Transformation Matrix

Let us investigate hexagonal patterns that are expressed by the column vectors of the submatrix Q^{μ} for each irreducible representation μ contained in the permutation representation T of the economy on the hexagonal lattice in Table 7.3. We deal with Christaller's $k = 3$, 4, and 7 systems and some of Lösch's hexagons. Discussion here is focused exclusively on geometrical patterns of the column vectors of Q, whereas the emergence of such patterns as bifurcating solutions will be studied in Chaps. 8 and 9.

Christaller's $k = 3$ System

In search of Christaller's $k = 3$ system (Lösch's hexagon with $D = 3$), let us consider[4] the submatrix

$$\begin{aligned} Q^{(2;+)} &= [\boldsymbol{q}_1^{(2;+)}, \boldsymbol{q}_2^{(2;+)}] \\ &= [\ \langle \cos(2\pi(n_1 - 2n_2)/3) \rangle,\ \langle \sin(2\pi(n_1 - 2n_2)/3) \rangle\ ] \end{aligned} \tag{7.40}$$

of the transformation matrix Q that is given by formula (7.32). It is easy to see that $\boldsymbol{q}_1^{(2;+)}$ is invariant to the subgroup $\Sigma(2, 1) = \langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$ in (5.45) for Christaller's $k = 3$ system. In contrast, $\boldsymbol{q}_2^{(2;+)}$ is invariant to another subgroup $\langle r^2, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$, and, hence, does not represent a hexagonal pattern of our interest. As an example, the spatial patterns for these two column vectors on the 6×6 hexagonal lattice are illustrated in Fig. 7.3a, b.

Let us mention an important fact about the irreversibility of the sign of the hexagonal pattern for $\boldsymbol{q}_1^{(2;+)}$. The pattern for $\boldsymbol{q}_1^{(2;+)}$ shown in Fig. 7.3a has a third of the places (denoted by *white circle*) growing into first-level centers and two-thirds (denoted by *black circle*) shrinking into second-level centers, thereby representing

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[4]The theoretical background of this consideration will be given in Sect. 8.4.

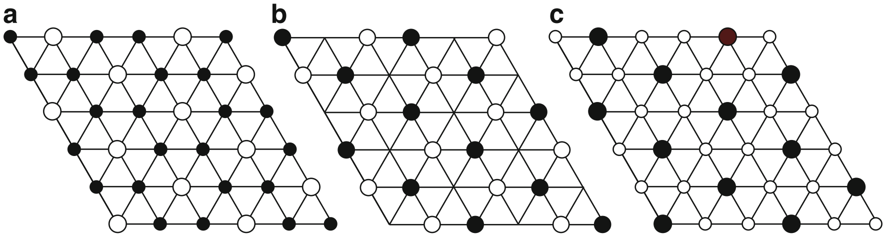

Fig. 7.3 Patterns on the 6 × 6 hexagonal lattice expressed by the column vectors of $Q^{(2;+)}$. A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_1^{(2;+)}$ $(\Sigma(2,1))$; (**b**) $\boldsymbol{q}_2^{(2;+)}$ $(\langle r^2, s, p_1^2 p_2, p_1^{-1} p_2\rangle)$; (**c**) $-\boldsymbol{q}_1^{(2;+)}$ $(\Sigma(2,1))$

the hexagonal pattern for Christaller's $k = 3$ system (Fig. 1.2a in Sect. 1.2). The vector $-\boldsymbol{q}_1^{(2;+)}$ with the sign reversed has the same symmetry group $\Sigma(2,1)$, but does not represent Christaller's $k = 3$ system because too many places, two-thirds of places (denoted by *white circle*), grow in population (Fig. 7.3c).

Christaller's $k = 4$ System

In search of Christaller's $k = 4$ system (Lösch's hexagonal with $D = 4$), let us consider[5] the spatial patterns for the three column vectors of

$$\begin{aligned} Q^{(3;+,+)} &= [\boldsymbol{q}_1^{(3;+,+)}, \boldsymbol{q}_2^{(3;+,+)}, \boldsymbol{q}_3^{(3;+,+)}] \\ &= [\ \langle\cos(\pi n_1)\rangle,\ \langle\cos(\pi n_2)\rangle,\ \langle\cos(\pi(n_1 - n_2))\rangle\ ] \end{aligned}$$

in (7.33). As shown in Fig. 7.4a–c for $n = 6$, the three vectors represent stripe patterns and do not represent hexagonal patterns of our interest.

Nonetheless, it turns out that the sum

$$\boldsymbol{q}_{\mathrm{sum}}^{(3;+,+)} = \boldsymbol{q}_1^{(3;+,+)} + \boldsymbol{q}_2^{(3;+,+)} + \boldsymbol{q}_3^{(3;+,+)}$$

of these three vectors (Fig. 7.4d) gives the hexagonal pattern of Christaller's $k = 4$ system (Fig. 1.2b), for which a fourth of the places (denoted by *white circle*) grow in population. Thus, some elaboration is required to obtain hexagonal patterns from the submatrix $Q^{(3;+,+)}$. The mechanism of this elaboration will be untangled in Chaps. 8 and 9 based on group-theoretic bifurcation analysis.

[5]The theoretical background of this consideration will be given in Sect. 8.5.

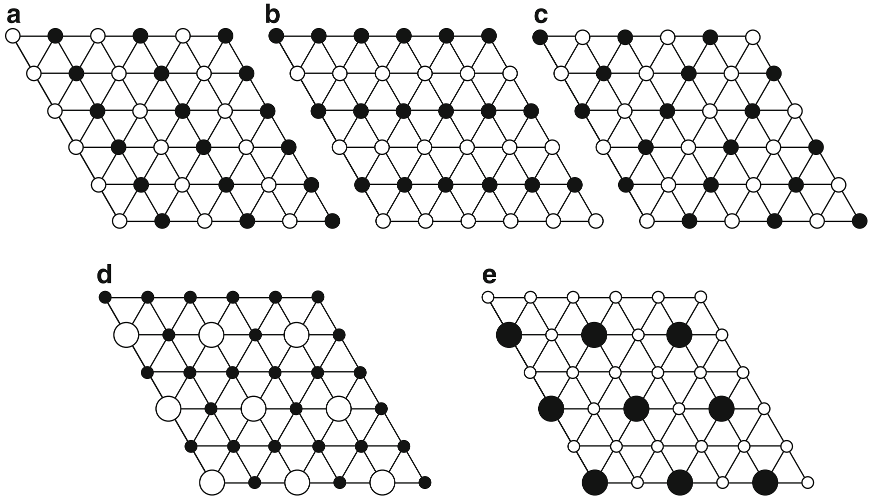

Fig. 7.4 Patterns on the 6 × 6 hexagonal lattice expressed by the column vectors of $Q^{(3;+,+)}$. A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_1^{(3;+,+)}$ ($\langle r^3, sr^2, p_1^2, p_2\rangle$); (**b**) $\boldsymbol{q}_2^{(3;+,+)}$ ($\langle r^3, s, p_1, p_2^2\rangle$); (**c**) $\boldsymbol{q}_3^{(3;+,+)}$ ($\langle r^3, sr, p_1p_2, p_2^2\rangle$); (**d**) $\boldsymbol{q}_{\mathrm{sum}}^{(3;+,+)}$ ($\Sigma(2,0) = \langle r, s, p_1^2, p_2^2\rangle$); (**e**) $-\boldsymbol{q}_{\mathrm{sum}}^{(3;+,+)}$ ($\Sigma(2,0) = \langle r, s, p_1^2, p_2^2\rangle$)

The hexagonal pattern is irreversible in sign. Although both $\boldsymbol{q}_{\mathrm{sum}}^{(3;+,+)}$ and $-\boldsymbol{q}_{\mathrm{sum}}^{(3;+,+)}$ have the same symmetry labeled $\Sigma(2,0) = \langle r, s, p_1^2, p_2^2\rangle$ (see Fig. 7.4d, e), only $\boldsymbol{q}_{\mathrm{sum}}^{(3;+,+)}$ represents Christaller's $k = 4$ system.

Christaller's $k = 7$ System

In search of Christaller's $k = 7$ system (Lösch's hexagon with $D = 7$), let us consider[6] the spatial patterns for the column vectors of $Q^{(12;2,1)} = [\boldsymbol{q}_1^{(12;2,1)}, \ldots, \boldsymbol{q}_{12}^{(12;2,1)}]$ in (7.36) for $(k, \ell) = (2, 1)$ for $n = 7$.

As shown in Fig. 7.5g, the sum of three vectors (among the 12 column vectors of $Q^{(12;2,1)}$) formed as

$$\boldsymbol{q}_{\mathrm{sum}}^{(12;2,1)} = \boldsymbol{q}_1^{(12;2,1)} + \boldsymbol{q}_3^{(12;2,1)} + \boldsymbol{q}_5^{(12;2,1)}$$

gives a hexagonal pattern of Christaller's $k = 7$ system in Fig. 1.2c with the symmetry labeled $\Sigma_0(3,1) = \langle r, p_1^3 p_2, p_1^{-1} p_2^2\rangle$. The hexagonal pattern of $\boldsymbol{q}_{\mathrm{sum}}^{(12;2,1)}$ is irreversible in sign in that $-\boldsymbol{q}_{\mathrm{sum}}^{(12;2,1)}$ does not represent Christaller's $k = 7$ system.

[6]The theoretical background of this consideration will be given in Sect. 8.7.

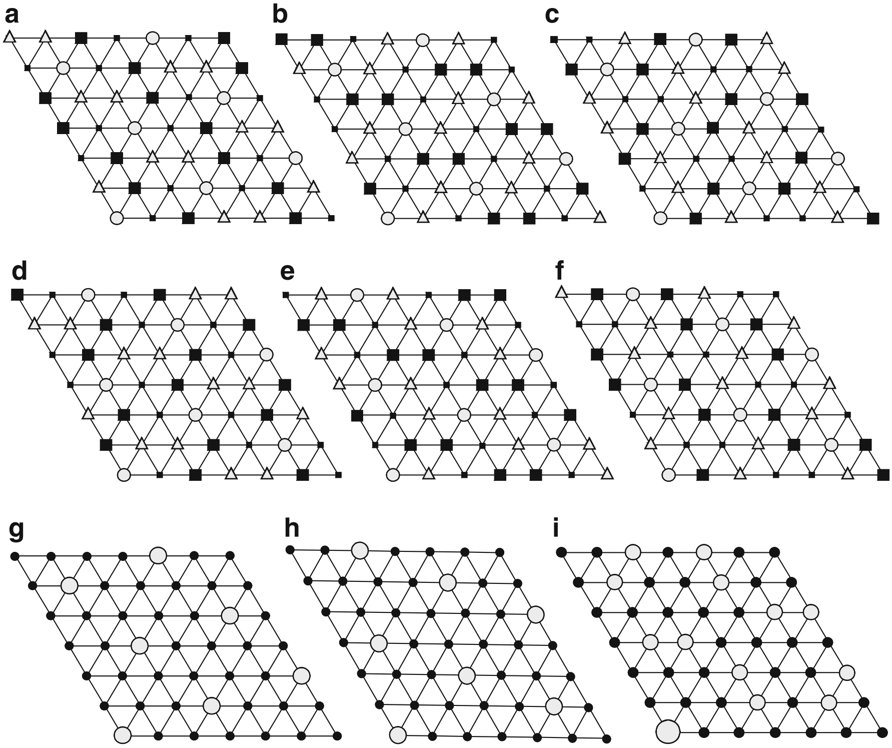

Fig. 7.5 Patterns on the 7 × 7 hexagonal lattice expressed by the column vectors of $Q^{(12;2,1)}$. A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_1^{(12;2,1)}$ $(\langle p_1^3 p_2\rangle)$; (**b**) $\boldsymbol{q}_3^{(12;2,1)}$ $(\langle p_1^3 p_2\rangle)$; (**c**) $\boldsymbol{q}_5^{(12;2,1)}$ $(\langle p_1^3 p_2\rangle)$; (**d**) $\boldsymbol{q}_7^{(12;2,1)}$ $(\langle p_1^3 p_2^2\rangle)$; (**e**) $\boldsymbol{q}_9^{(12;2,1)}$ $(\langle p_1^3 p_2^2\rangle)$; (**f**) $\boldsymbol{q}_{11}^{(12;2,1)}$ $(\langle p_1^3 p_2^2\rangle)$; (**g**) $\boldsymbol{q}_{\text{sum}}^{(12;2,1)}$ $(\Sigma_0(3,1) = \langle r, p_1^3 p_2, p_1^{-1} p_2^2\rangle)$; (**h**) $\boldsymbol{q}_{\text{sum}'}^{(12;2,1)}$ $(\Sigma_0(3,2) = \langle r, p_1^3 p_2^2, p_1^{-2} p_2\rangle)$; (**i**) $\boldsymbol{q}_{\text{sum}''}^{(12;2,1)}$ $(\mathrm{D}_6 = \langle r, s\rangle)$

As shown in Fig. 7.5h, another sum of three vectors formed as

$$\boldsymbol{q}_{\text{sum}'}^{(12;2,1)} = \boldsymbol{q}_7^{(12;2,1)} + \boldsymbol{q}_9^{(12;2,1)} + \boldsymbol{q}_{11}^{(12;2,1)}$$

gives another hexagonal pattern of Christaller's $k = 7$ system with the symmetry labeled $\Sigma_0(3,2) = \langle r, p_1^3 p_2^2, p_1^{-2} p_2\rangle$. The hexagonal pattern of $\boldsymbol{q}_{\text{sum}'}^{(12;2,1)}$ is irreversible in sign in that $-\boldsymbol{q}_{\text{sum}'}^{(12;2,1)}$ does not represent Christaller's $k = 7$ system.

In addition, the sum of six vectors

$$\boldsymbol{q}_{\text{sum}''}^{(12;2,1)} = \boldsymbol{q}_1^{(12;2,1)} + \boldsymbol{q}_3^{(12;2,1)} + \boldsymbol{q}_5^{(12;2,1)} + \boldsymbol{q}_7^{(12;2,1)} + \boldsymbol{q}_9^{(12;2,1)} + \boldsymbol{q}_{11}^{(12;2,1)}$$

gives a Lösch's hexagon with $D = 49$ with the symmetry labeled D_6 (Fig. 7.5i). In contrast, these six vectors by themselves do not represent hexagonal patterns of our interest, as shown in Fig. 7.5a–f.

Larger Hexagons

We have so far investigated Christaller's $k = 3$, 4, and 7 systems. Larger hexagons are introduced using several examples for the lattice size $n = 6$. This size is also used in the computational example in Sect. 7.6.2.

For $n = 6$, we have three six-dimensional irreducible representations, i.e., $\mu = (6; 1, 0, +)$, $(6; 2, 0, +)$, and $(6; 1, 1, +)$ with submatrices $Q^\mu = [\boldsymbol{q}_1^\mu, \ldots, \boldsymbol{q}_6^\mu]$ containing six vectors. It turns out that, for each $\mu = (6; 1, 0, +)$, $(6; 2, 0, +)$, $(6; 1, 1, +)$, the sum of three vectors (among these six vectors)

$$\boldsymbol{q}_{\mathrm{sum}}^\mu = \boldsymbol{q}_1^\mu + \boldsymbol{q}_3^\mu + \boldsymbol{q}_5^\mu$$

gives a Lösch's hexagonal pattern (see Fig. 7.6j–l). In contrast, these three vectors by themselves do not represent hexagonal patterns of our interest, as shown in Fig. 7.6a–i.

7.6.2 *Use in Computational Bifurcation Analysis*

Use of the transformation matrix Q in computational bifurcation analysis is demonstrated for the 6×6 hexagonal lattice. With the use of Table 7.3 listing the relevant irreducible representations, the transformation matrix Q in (7.24) reads

$$Q = [Q^{(1;+,+)}, Q^{(2;+)}, Q^{(3;+,+)}, Q^{(6;1,0;+)}, Q^{(6;2,0;+)}, Q^{(6;1,1;+)}, Q^{(12;2,1)}]. \tag{7.41}$$

The associated diagonal form in (7.37) of the Jacobian matrix becomes

$$\begin{aligned} Q^{-1} J(\boldsymbol{\lambda}, \tau) Q = \mathrm{diag}(&e^{(1;+,+)}; e^{(2;+)}, e^{(2;+)}; e^{(3;+,+)}, e^{(3;+,+)}, e^{(3;+,+)}; \\ &e^{(6;1,0,+)}, \ldots, e^{(6;1,0,+)};\ e^{(6;2,0,+)}, \ldots, e^{(6;2,0,+)}; \\ &e^{(6;1,1,+)}, \ldots, e^{(6;1,1,+)};\ e^{(12;2,1,+)}, \ldots\ldots, e^{(12;2,1,+)}) \end{aligned}$$

with diagonal entries being the eigenvalues of the Jacobian matrix $J(\boldsymbol{\lambda}, \tau)$. There are seven sets of distinct eigenvalues corresponding to the seven blocks in (7.41).

These eigenvalues were computed for the flat earth equilibrium $\boldsymbol{\lambda}_0 = (1/36, \ldots, 1/36)^\top$ for the economy on the 6×6 hexagonal lattice using the same core–periphery model and parameter setting as those used in Sect. 4.5. As shown in Fig. 7.7a, all eigenvalues were negative in the range of $\tau > 2.5$. The eigenvalue

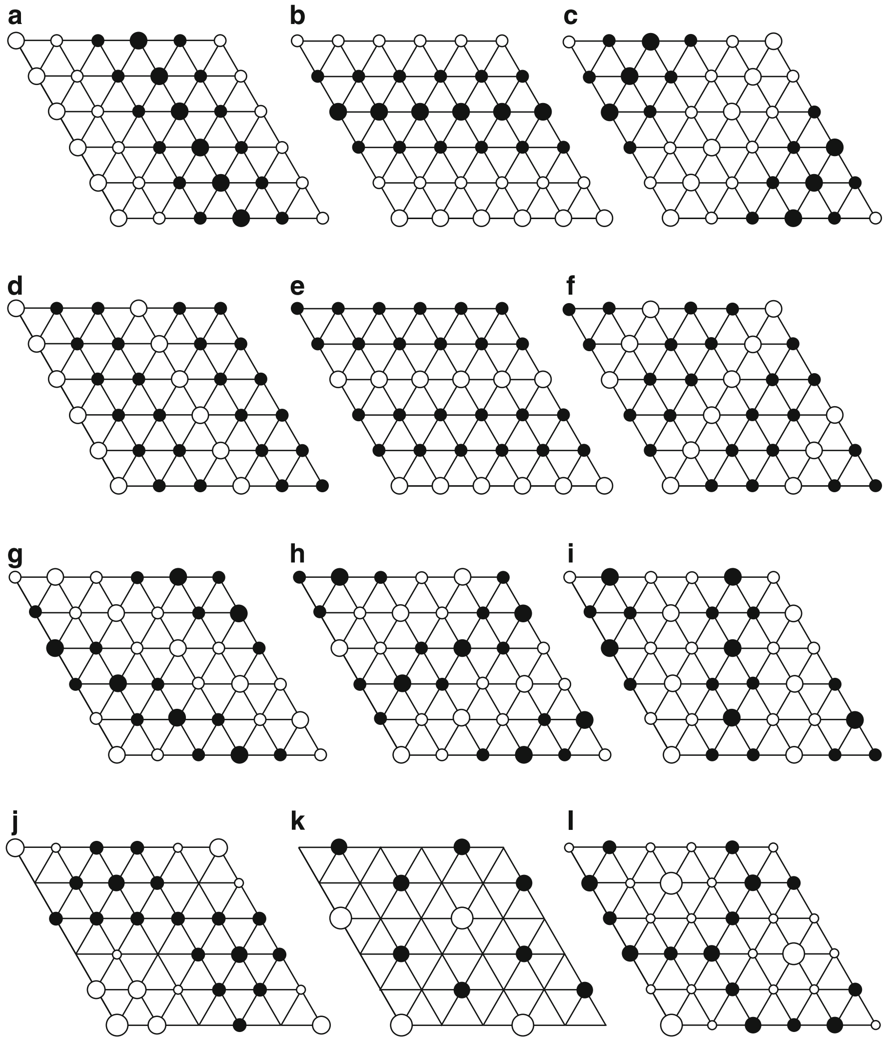

Fig. 7.6 Patterns on the 6×6 hexagonal lattice expressed by the column vectors of Q^{μ} ($\mu = (6;1,0,+),(6;2,0,+),(6;1,1,+)$). A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_1^{(6;1,0,+)}$ $(\langle r^3, sr^2, p_2\rangle)$; (**b**) $\boldsymbol{q}_3^{(6;1,0,+)}$ $(\langle r^3, s, p_1\rangle)$; (**c**) $\boldsymbol{q}_5^{(6;1,0,+)}$ $(\langle r^3, sr, p_1p_2\rangle)$; (**d**) $\boldsymbol{q}_1^{(6;2,0,+)}$ $(\langle r^3, sr^2, p_2, p_1^3\rangle)$; (**e**) $\boldsymbol{q}_3^{(6;2,0,+)}$ $(\langle r^3, s, p_1, p_2^3\rangle)$; (**f**) $\boldsymbol{q}_5^{(6;2,0,+)}$ $(\langle r^3, sr, p_1p_2, p_1^3\rangle)$; (**g**) $\boldsymbol{q}_1^{(6;1,1,+)}$ $(\langle r^3, sr, p_1^{-1}p_2, p_1^3p_2^3\rangle)$; (**h**) $\boldsymbol{q}_3^{(6;1,1,+)}$ $(\langle r^3, sr^2, p_1^2p_2, p_2^3\rangle)$; (**i**) $\boldsymbol{q}_5^{(6;1,1,+)}$ $(\langle r^3, s, p_1p_2^2, p_1^3\rangle)$; (**j**) $\boldsymbol{q}_{\mathrm{sum}}^{(6;1,0;+)}$ $(\mathrm{D}_6 = \langle r, s\rangle)$; (**k**) $\boldsymbol{q}_{\mathrm{sum}}^{(6;2,0;+)}$ $(\Sigma(3,0) = \langle r, s, p_1^3, p_2^3\rangle)$; (**l**) $\boldsymbol{q}_{\mathrm{sum}}^{(6;1,1;+)}$ $(\Sigma(4,2) = \langle r, s, p_1^4p_2^2, p_1^{-2}p_2^2\rangle)$

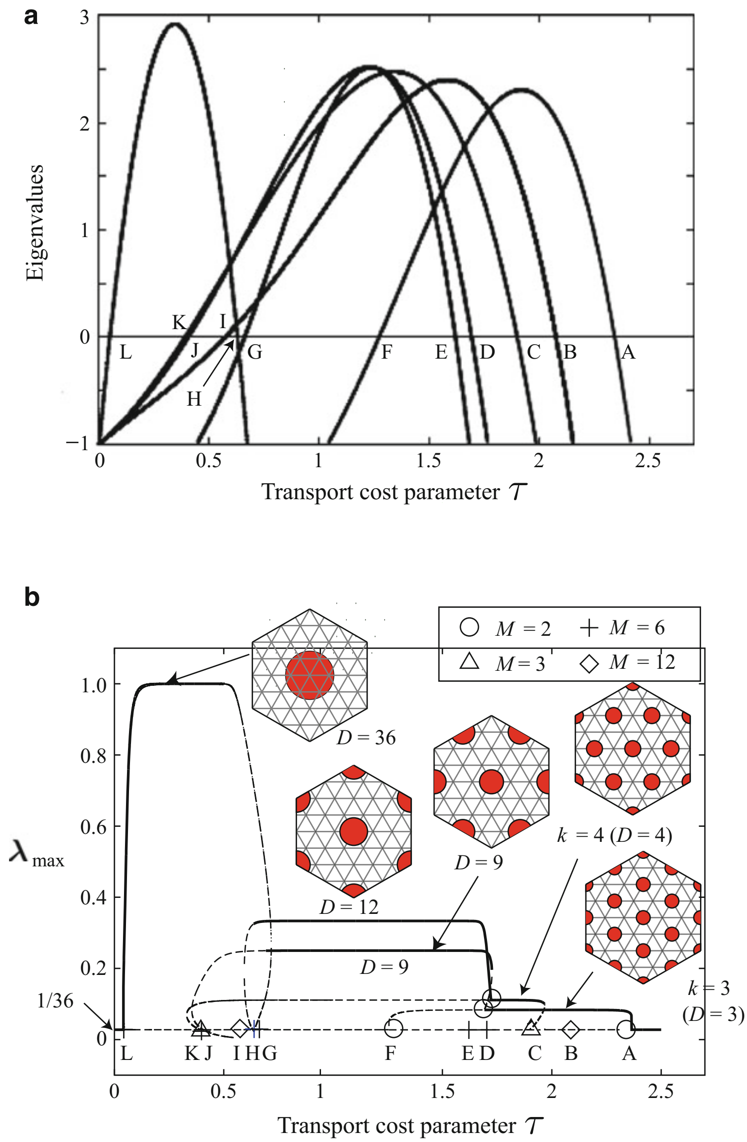

Fig. 7.7 (**a**) Eigenvalues and (**b**) bifurcated paths for the 6 × 6 hexagonal lattice and associated population distributions in hexagonal windows. *Solid curves* mean stable equilibria; *dashed curves*, unstable equilibria; $\lambda_{\max}$, the maximum population among the places; M is the multiplicity of the bifurcation point; the area of each circle in the windows is proportional to the population size at that place

$e^{(1;+,+)}$ remained negative throughout and never became zero (this eigenvalue is not plotted in Fig. 7.7a). In the range of $0 < \tau < 2.5$, each of the other six distinct eigenvalues $e^{(2;+)}$, $e^{(3;+,+)}$, $e^{(6;1,0;+)}$, $e^{(6;2,0;+)}$, $e^{(6;1,1;+)}$, and $e^{(12;2,1)}$ became zero twice. Accordingly, there are 12 critical points A–L on the flat earth equilibria. In view of the eigenvectors of the Jacobian matrix, the irreducible representations associated with these critical points can be classified as

Bifurcation point	Irreducible representation	Multiplicity	D	Type
A, F	$(2;+)$	2	3	M
C, K	$(3;+,+)$	3	4	V
D, J	$(6;2,0,+)$	6	9	V
E, G	$(6;1,1,+)$	6	12	M
H, L	$(6;1,0,+)$	6	36	V
B, I	$(12;2,1)$	12	36	V

From these critical points, bifurcating paths for Lösch's hexagons of various kinds were obtained as depicted in Fig. 7.7b. For example, from critical points A and F with the multiplicity $M = 2$, bifurcating branches for Christaller's $k = 3$ system (Lösch's hexagonal with $D = 3$) were found in the direction of $\boldsymbol{q}_1^{(2;+)}$ in (7.40). Other branches with $D = 4$, 9, 12, and 36 in Fig. 7.7b were found in a similar manner.

7.7 Summary

- Permutation representation for the hexagonal lattice has been investigated.
- Irreducible decomposition of the permutation representation has been presented.
- Transformation matrices for the irreducible decomposition have been derived.

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Chapter 8
Hexagons of Christaller and Lösch: Using Equivariant Branching Lemma

Abstract Following the framework of group-theoretic bifurcation analysis for the hexagonal lattice with $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$-symmetry in Chaps. 5–7, we highlight the equivariant branching lemma as a pertinent and sufficient means to test the existence of hexagonal bifurcating patterns on the hexagonal lattice. By the application of this lemma to the irreducible representations of the group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, all hexagonal distributions of Christaller and Lösch (Chaps. 1 and 4) are shown to appear at critical points of multiplicity 2, 3, 6, or 12 for appropriate lattice sizes n. A complete classification of these hexagonal distributions is presented. As a main technical contribution of this book, a complete analysis of bifurcating solutions for hexagonal distributions from critical points of multiplicity 12 is conducted. In particular, hexagons of different types are shown to emerge simultaneously at bifurcation points of multiplicity 12 of certain types.

Keywords Bifurcation • Christaller's hexagonal distributions • Equivariant branching lemma • Group • Group-theoretic bifurcation analysis • Hexagonal lattice • Irreducible representation • Isotropy subgroup • Lösch's hexagons

8.1 Introduction

Fundamental mathematical facts about the hexagonal lattice were presented in Chaps. 5–7. The $n \times n$ hexagonal lattice with periodic boundaries was introduced as a spatial platform for agglomeration (Chap. 5). The symmetry of this lattice was labeled by the group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, and the irreducible representations of this group were given in Chap. 6. The representation matrix for the hexagonal lattice was decomposed into irreducible components to determine the multiplicity a^{μ} of each irreducible representation μ (Chap. 7).

We are ready to investigate the existence of hexagonal distributions as bifurcating solutions on the hexagonal lattice. For each irreducible representation μ with $a^{\mu} \geq 1$, bifurcation from a critical point associated with μ is investigated by

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using the group-theoretic bifurcation analysis procedure under group symmetry (Sect. 2.4). As explained in Sect. 4.3.3, the following two different methods of the group-theoretic analysis are available.

- The equivariant branching lemma is applied to the bifurcation equation associated with μ to see the existence of bifurcating solutions with a specified symmetry. This analysis is purely algebraic or group-theoretic, focusing on the symmetry of solutions. Since the bifurcation equation is not solved, the concrete form of this equation need not be derived. Isotropy subgroups play the central role in the analysis.
- The bifurcation equation is obtained in the form of power series expansions and is solved asymptotically. This method is more complicated, treating nonlinear terms directly, but is more informative, giving asymptotic forms of the bifurcating solutions and their directions in addition to their existence.

In principle, the latter method, dealing explicitly with the bifurcation equation, is capable of capturing all bifurcating solutions. This is not the case with the former method based on the equivariant branching lemma, which gives a sufficient condition to ensure bifurcating solutions with a specified symmetry. Nonetheless, the former method, which demands less analytical effort than the latter method, fits the objective of this book: the pinpointing of the targeted hexagonal distributions among many other bifurcating solutions.

In this chapter, the first method using equivariant branching lemma is applied to the economy on the $n \times n$ hexagonal lattice with the symmetry of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. We obtain possible bifurcating hexagonal distributions and associated lattice sizes for all irreducible representations, which are related to group-theoretic critical points with multiplicity $M = 2$, 3, 6, and 12. All hexagons of Lösch and Christallers's three systems (Chap. 1) are verified to exist as bifurcating patterns (see Fig. 8.1 for the ten smallest hexagons of Lösch and Fig. 8.2 for Christaller's three systems). In particular, different hexagons are shown to emerge simultaneously at bifurcation points of multiplicity 12 of certain types. Recall that the emergence of some of these distributions was confirmed in Sect. 4.5 by the computational bifurcation analysis for various lattice sizes n, and the theoretical predictions to be obtained here played a pivotal role in that analysis. See also Remark 5.1 in Sect. 5.5.1.

As a main technical contribution of this book, a complete analysis of bifurcating solutions for hexagonal distributions from critical points of multiplicity 12 is conducted. Mathematically, the analysis of hexagonal distributions is carried out in a streamlined manner by means of fundamental theoretical tools for integer matrices, such as the Smith normal form and determinantal divisors. In particular, a solvability criterion for a system of linear equations in integer unknown variables that refers to determinantal divisors plays a significant role. Duality nature between the results obtained by the two methods, one by the equivariant branching lemma and the other by bifurcation equations, is revealed through a theorem for integrality of solutions, the so-called integer analogue of the Farkas lemma.

This chapter is organized as follows. Theoretically predicted bifurcating hexagonal distributions are previewed in Sect. 8.2. Fundamentals of bifurcation analysis are recapitulated in Sect. 8.3. Bifurcation points of multiplicity $M = 2$, 3, 6, and 12 are, respectively, studied in Sects. 8.4–8.7.

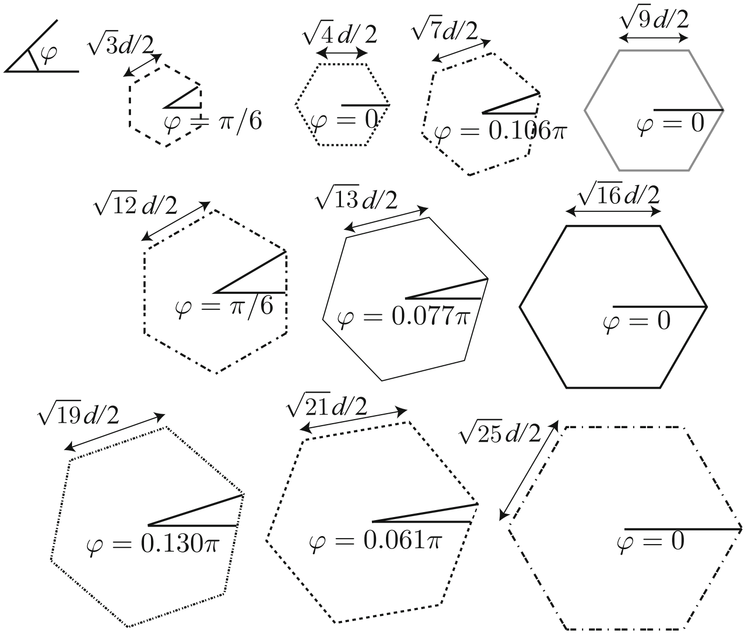

Fig. 8.1 Lösch's ten smallest hexagons

Remark 8.1. Equivariant branching lemma has been used in the study of pattern formation in nonlinear physics as a pertinent means to guarantee the existence of a bifurcating solution of a given symmetry (Vanderbauwhede, 1980 [10]; Golubitsky, Stewart, and Schaeffer, 1988 [5]). Group-theoretic bifurcation analysis of six- and 12-dimensional irreducible representations of the group $D_6 \ltimes T^2$ was conducted to search for possible bifurcating patterns:

- For six-dimensional ones, hexagons, as well as rolls and triangles, are shown to exist (Buzano and Golubitsky, 1983 [1]; Dionne and Golubitsky, 1992 [2]; Golubitsky and Stewart, 2002 [4]).
- For 12-dimensional ones, simple hexagons and super hexagons are shown to exist (Kirchgässner, 1979 [7]; Dionne, Silber, and Skeldon, 1997 [3]; Judd and Silber, 2000 [6]). □

8.2 Theoretically Predicted Bifurcating Hexagonal Distributions

A possible bifurcation mechanism that can produce Lösch's hexagons, such as those in Fig. 8.1, is presented as a preview of the group-theoretic bifurcation analyses in Sects. 8.4–8.7. All critical points are assumed to be group-theoretic (Sect. 2.4.2).

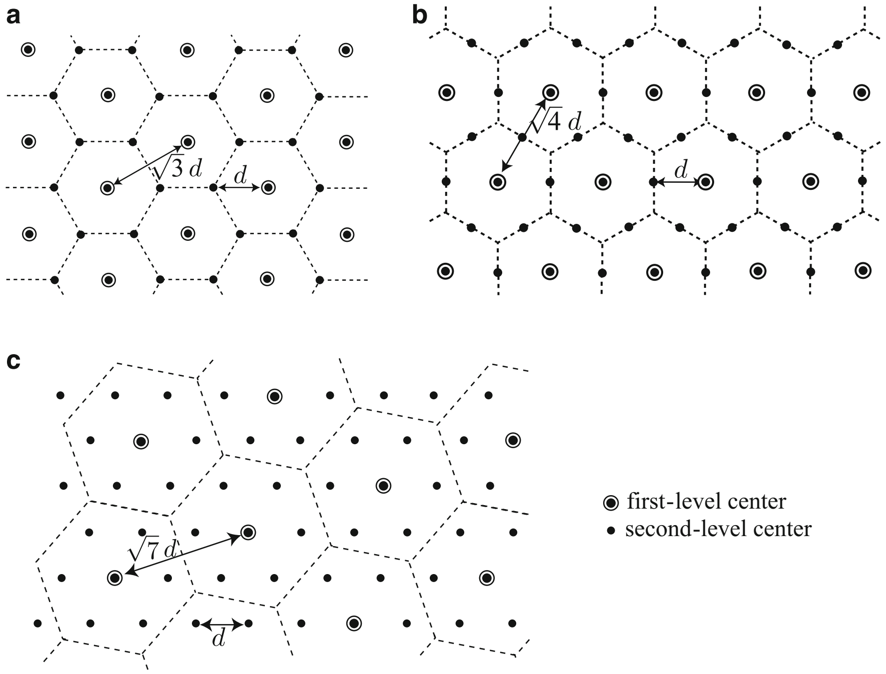

Fig. 8.2 Three systems predicted by Christaller, corresponding to Lösch's three smallest hexagons with $D = 3$, 4, and 7. (**a**) Christaller's $k = 3$ system. (**b**) Christaller's $k = 4$ system. (**c**) Christaller's $k = 7$ system

8.2.1 Symmetry of Bifurcating Hexagonal Distributions

Recall first that the symmetry of the $n \times n$ hexagonal lattice is labeled by the group

$$G = \langle r, s, p_1, p_2 \rangle = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) \tag{8.1}$$

in (5.36) with the fundamental relations (5.37):

$$\begin{aligned} r^6 = s^2 = (rs)^2 = p_1{}^n = p_2{}^n = e, \quad p_2 p_1 = p_1 p_2, \\ rp_1 = p_1 p_2 r, \quad rp_2 = p_1^{-1} r, \quad sp_1 = p_1 s, \quad sp_2 = p_1^{-1} p_2^{-1} s, \end{aligned} \tag{8.2}$$

where e is the identity element.

Recall also that the core–periphery model on the $n \times n$ hexagonal lattice is described by an equilibrium equation of the form

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \mathbf{0} \tag{8.3}$$

in (4.2), where $\boldsymbol{\lambda} = (\lambda, \ldots, \lambda_N)^\top$ with $N = n^2$ is the vector expressing a distribution of mobile population and τ is the transport cost parameter. Among many possible solutions $\boldsymbol{\lambda}$ to the equilibrium equation (8.3), we are particularly interested in those bifurcated solutions that represent the hexagonal distributions.

To describe the hexagonal distributions, we introduced in (5.4) a sublattice

$$\begin{aligned}\mathscr{H}(\alpha,\beta) &= \{n_1(\alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2) + n_2(-\beta\boldsymbol{\ell}_1 + (\alpha-\beta)\boldsymbol{\ell}_2) \mid n_1, n_2 \in \mathbb{Z}\}\\ &= \{\begin{bmatrix}\boldsymbol{\ell}_1 & \boldsymbol{\ell}_2\end{bmatrix}\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha-\beta\end{bmatrix}\begin{bmatrix}n_1\\ n_2\end{bmatrix} \mid n_1, n_2 \in \mathbb{Z}\},\end{aligned} \tag{8.4}$$

where

$$\boldsymbol{\ell}_1 = d\begin{bmatrix}1\\0\end{bmatrix}, \quad \boldsymbol{\ell}_2 = d\begin{bmatrix}-1/2\\ \sqrt{3}/2\end{bmatrix} \tag{8.5}$$

are basis vectors of length d of the underlying infinite hexagonal lattice

$$\mathscr{H} = \{n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z}\} \tag{8.6}$$

in (5.2). In this chapter we adopt the parameter space

$$\{(\alpha,\beta) \in \mathbb{Z}^2 \mid \alpha > \beta \geq 0\} \tag{8.7}$$

in (5.11) of Proposition 5.1, instead of $\{(\alpha,\beta) \in \mathbb{Z}^2 \mid \alpha \geq 2\beta \geq 0,\ \alpha \neq 0\}$ in (5.12), unless otherwise stated. The size of the hexagons in $\mathscr{H}(\alpha,\beta)$ is characterized by

$$D = D(\alpha,\beta) = \alpha^2 - \alpha\beta + \beta^2 \tag{8.8}$$

in (5.9).

We recall the $n \times n$ lattice

$$\mathscr{H}_n = \{n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 \mid n_i \in \mathbb{Z},\ 0 \leq n_i \leq n-1\ (i = 1,2)\} \tag{8.9}$$

in (5.30) and describe the symmetry of a hexagonal distribution $\mathscr{H}(\alpha,\beta) \cap \mathscr{H}_n$ on this lattice by the subgroup $G(\alpha,\beta)$. This subgroup is classified in (5.41) into three types:

$$\begin{aligned}
&\Sigma(\alpha,0) = \langle r,s\rangle \ltimes \langle p_1^{\alpha}, p_2^{\alpha}\rangle && (2 \leq \alpha \leq n) && :\text{type V},\\
&\Sigma(2\beta,\beta) = \langle r,s\rangle \ltimes \langle p_1^{2\beta} p_2^{\beta}, p_1^{-\beta} p_2^{\beta}\rangle && (1 \leq \beta \leq n/3) && :\text{type M},\\
&\Sigma_0(\alpha,\beta) = \langle r\rangle \ltimes \langle p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha-\beta}\rangle && (1 \leq \beta < \alpha < n,\ \alpha \neq 2\beta) && :\text{type T}.
\end{aligned} \tag{8.10}$$

Here it is convenient to introduce a convention

$$\Sigma_0(0,0) = \langle r \rangle, \quad \Sigma(0,0) = \langle r, s \rangle, \quad \Sigma(1,0) = \langle r, s, p_1, p_2 \rangle. \tag{8.11}$$

Recall the compatibility condition (5.35) between (α, β) and n given as

$$n = \begin{cases} m\alpha & \text{for type V}, \\ 3m\beta & \text{for type M}, \\ mD(\alpha,\beta)/\gcd(\alpha,\beta) & \text{for type T}, \end{cases} \tag{8.12}$$

where $m = 1, 2, \ldots$.

The objective of this chapter is to look for a solution $\boldsymbol{\lambda}$ to (8.3) such that the isotropy subgroup $\Sigma(\boldsymbol{\lambda})$ for the symmetry of $\boldsymbol{\lambda}$ coincides with one of the subgroups in (8.10).

The subgroups $G(\alpha, \beta)$ for Lösch's ten smallest hexagons with different values of D are given by

$$\begin{cases} \Sigma(2,1) & \text{for } D = 3, \\ \Sigma(2,0) & \text{for } D = 4, \\ \Sigma_0(3,1), \Sigma_0(3,2) & \text{for } D = 7, \\ \Sigma(3,0) & \text{for } D = 9, \\ \Sigma(4,2) & \text{for } D = 12, \end{cases} \quad \begin{cases} \Sigma_0(4,1), \Sigma_0(4,3) & \text{for } D = 13, \\ \Sigma(4,0) & \text{for } D = 16, \\ \Sigma_0(5,2), \Sigma_0(5,3) & \text{for } D = 19, \\ \Sigma_0(5,1), \Sigma_0(5,4) & \text{for } D = 21, \\ \Sigma(5,0) & \text{for } D = 25. \end{cases} \tag{8.13}$$

Table 8.1 summarizes the characteristics of Lösch's ten smallest hexagons classified into three types. The rightmost column shows the multiplicity M of relevant bifurcation points, which is explained in Sect. 8.2.2. It is noted that $\Sigma_0(\alpha, \beta)$ and $\Sigma_0(\alpha, \alpha - \beta)$, such as $\Sigma_0(3,1)$ and $\Sigma_0(3,2)$, represent hexagonal distributions that are conjugate to each other with respect to the reflection s. It will be appropriate to distinguish them in accordance with our choice of the parameter space in (8.7) in the theoretical derivation in Sect. 8.7.

8.2.2 Hexagons Engendered by Direct Bifurcations

The main message of this chapter is that bifurcating solutions for Lösch's hexagons do arise from the mathematical model of an economy on the hexagonal lattice with pertinent lattice sizes, and therefore these hexagons can be understood within the framework of group-theoretic bifurcation theory. The major results to be derived in Sects. 8.4–8.7 are summarized below.

Proposition 8.1. *A bifurcating solution with the hexagonal symmetry expressed by the subgroup in* (8.10) *exists for pertinent lattice sizes n. More specifically, we have the following, where m denotes a positive integer.*

Table 8.1 Lösch's ten smallest hexagons classified into three types

D	(α, β)	Tilt angle φ	Type of hexagons	Subgroup $G(\alpha, \beta)$	Lattice size n $(m = 1, 2, \ldots)$	Multiplicity M
4	$(2, 0)$	0	V	$\Sigma(2, 0)$	$2m$	3
9	$(3, 0)$	0	V	$\Sigma(3, 0)$	$3m$	6
16	$(4, 0)$	0	V	$\Sigma(4, 0)$	$4m$	6
25	$(5, 0)$	0	V	$\Sigma(5, 0)$	$5m$	6
3	$(2, 1)$	$\pi/6$	M	$\Sigma(2, 1)$	$3m$	2
12	$(4, 2)$	$\pi/6$	M	$\Sigma(4, 2)$	$6m$	6
7	$(3, 1)$	0.106π	T	$\Sigma_0(3, 1)$	$7m$	12
7	$(3, 2)$	0.227π	T	$\Sigma_0(3, 2)$	$7m$	12
13	$(4, 1)$	0.077π	T	$\Sigma_0(4, 1)$	$13m$	12
13	$(4, 3)$	0.137π	T	$\Sigma_0(4, 3)$	$13m$	12
19	$(5, 2)$	0.130π	T	$\Sigma_0(5, 2)$	$19m$	12
19	$(5, 3)$	0.203π	T	$\Sigma_0(5, 3)$	$19m$	12
21	$(5, 1)$	0.061π	T	$\Sigma_0(5, 1)$	$21m$	12
21	$(5, 4)$	0.273π	T	$\Sigma_0(5, 4)$	$21m$	12

- *For* $(\alpha, \beta; n) = (\alpha, 0; \alpha m)$ $(2 \leq \alpha \leq n)$*, a hexagonal distribution of type V with symmetry* $\Sigma(\alpha, 0)$ *branches at a bifurcation point with multiplicity* $M = 3$ $(\alpha = 2)$*,* $M = 6$ $(\alpha \geq 3)$*, or* $M = 12$ $(\alpha \geq 6)$.
- *For* $(\alpha, \beta; n) = (2\beta, \beta; 3\beta m)$ $(1 \leq \beta \leq n/3)$*, a hexagonal distribution of type M with symmetry* $\Sigma(2\beta, \beta)$ *branches at a bifurcation point with multiplicity* $M = 2$ $(\beta = 1)$*,* $M = 6$ $(\beta \geq 2)$*, or* $M = 12$ $(\beta \geq 4)$.
- *For* $(\alpha, \beta; n) = (\alpha, \beta; mD(\alpha, \beta)/\gcd(\alpha, \beta))$*, where* $1 \leq \beta < \alpha < n$ *and* $\alpha \neq 2\beta$*, a hexagonal distribution of type T with symmetry* $\Sigma_0(\alpha, \beta)$ *branches at a bifurcation point with multiplicity* $M = 12$.

Proof. This is proved in Sects. 8.4–8.7. □

Possible hexagonal distributions for each value of $(\alpha, \beta; n)$, given in Proposition 8.1, are summarized as follows:

$(\alpha, \beta; n)$		Multiplicity M	Type
	$\alpha = 2$	3	
$(\alpha, 0; \alpha m)$	$\alpha \geq 3$	6	V
	$\alpha \geq 6$	12	
	$\beta = 1$	2	
$(2\beta, \beta; 3\beta m)$	$\beta \geq 2$	6	M
	$\beta \geq 4$	12	
$(\alpha, \beta; \dfrac{mD(\alpha, \beta)}{\gcd(\alpha, \beta)})$	$1 \leq \beta < \alpha,\ \alpha \neq 2\beta$	12	T

$(m = 1, 2, \ldots)$

We shall see that the hexagons emerge from critical points of multiplicity $M = 2$, 3, 6, and 12, but not from those of $M = 1$ and 4. The dependence of the size D of the hexagons on the multiplicity M is expressed as

$$D = \begin{cases} 3 & \text{for } M = 2, \\ 4 & \text{for } M = 3, \\ 9, 12, 16, 25 & \text{for } M = 6, \\ 7, 13, 19, 21 & \text{for } M = 12. \end{cases}$$

The following proposition plays a pivotal role in the search for Lösch's hexagons.

Proposition 8.2. *The existence of hexagons depends on the divisors of the lattice size n as follows.*

(i) *If n has a divisor α $(2 \le \alpha \le n)$, a hexagonal distribution of type V with symmetry $\Sigma(\alpha, 0)$ exists.*
(ii) *If n has a divisor 3β $(1 \le \beta \le n/3)$, a hexagonal distribution of type M with symmetry $\Sigma(2\beta, \beta)$ exists.*
(iii) *If n has a divisor $D(\alpha, \beta)/\gcd(\alpha, \beta)$, where $1 \le \beta < \alpha < n$ and $\alpha \ne 2\beta$, a hexagonal distribution of type T with symmetry $\Sigma_0(\alpha, \beta)$ exists.*

Proof. This follows from Proposition 8.1. □

Possible hexagons emerging via direct bifurcations for several values of n, obtained from Proposition 8.2, are listed in Table 8.2.

8.3 Procedure of Theoretical Analysis: Recapitulation

Bifurcation analysis procedure resorting to the equivariant branching lemma is summarized in this section as an application of the general theoretical framework of Chap. 2 to the core–periphery model on the $n \times n$ hexagonal lattice described in Sect. 4.2. On the basis of the framework of group-theoretic bifurcation analysis given in Sect. 4.3 and the enumeration of irreducible representations for the symmetry group of the $n \times n$ hexagonal lattice in Chap. 7, the group-theoretic bifurcation analysis is carried out in Sects. 8.4–8.7.

8.3.1 Bifurcation and Symmetry of Solutions

Let us consider the system of equilibrium equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0} \tag{8.14}$$

of the economy in (8.3) endowed with the symmetry of, or *equivariance* to, $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ formulated as

Table 8.2 Possible hexagons for several lattice sizes n

n	(α, β)	D	Type	$G(\alpha, \beta)$	M
2	$(2, 0)$	4	V	$\Sigma(2, 0)$	3
3	$(3, 0)$	9	V	$\Sigma(3, 0)$	6
	$(2, 1)$	3	M	$\Sigma(2, 1)$	2
4	$(2, 0)$	4	V	$\Sigma(2, 0)$	3
	$(4, 0)$	16	V	$\Sigma(4, 0)$	6
5	$(5, 0)$	25	V	$\Sigma(5, 0)$	6
6	$(2, 0)$	4	V	$\Sigma(2, 0)$	3
	$(3, 0)$	9	V	$\Sigma(3, 0)$	6
	$(6, 0)$	36	V	$\Sigma(6, 0)$	6, 12
	$(2, 1)$	3	M	$\Sigma(2, 1)$	2
	$(4, 2)$	12	M	$\Sigma(4, 2)$	6
7	$(7, 0)$	49	V	$\Sigma(7, 0)$	6, 12
	$(3, 1)$	7	T	$\Sigma_0(3, 1)$	12
	$(3, 2)$	7	T	$\Sigma_0(3, 2)$	12
8	$(2, 0)$	4	V	$\Sigma(2, 0)$	3
	$(4, 0)$	16	V	$\Sigma(4, 0)$	6
	$(8, 0)$	64	V	$\Sigma(8, 0)$	6, 12
9	$(3, 0)$	9	V	$\Sigma(3, 0)$	6
	$(9, 0)$	81	V	$\Sigma(9, 0)$	6, 12
	$(2, 1)$	3	M	$\Sigma(2, 1)$	2
	$(6, 3)$	27	M	$\Sigma(6, 3)$	6
10	$(2, 0)$	4	V	$\Sigma(2, 0)$	3
	$(5, 0)$	25	V	$\Sigma(5, 0)$	6
	$(10, 0)$	100	V	$\Sigma(10, 0)$	6, 12
11	$(11, 0)$	121	V	$\Sigma(11, 0)$	6, 12
12	$(2, 0)$	4	V	$\Sigma(2, 0)$	3
	$(3, 0)$	9	V	$\Sigma(3, 0)$	6
	$(4, 0)$	16	V	$\Sigma(4, 0)$	6
	$(6, 0)$	36	V	$\Sigma(6, 0)$	6, 12
	$(12, 0)$	144	V	$\Sigma(12, 0)$	6, 12
	$(2, 1)$	3	M	$\Sigma(2, 1)$	2
	$(4, 2)$	12	M	$\Sigma(4, 2)$	6
	$(8, 4)$	48	M	$\Sigma(8, 4)$	6, 12
13	$(13, 0)$	169	V	$\Sigma(13, 0)$	6, 12
	$(4, 1)$	13	T	$\Sigma_0(4, 1)$	12
	$(4, 3)$	13	T	$\Sigma_0(4, 3)$	12
21	$(3, 0)$	9	V	$\Sigma(3, 0)$	6
	$(7, 0)$	49	V	$\Sigma(7, 0)$	6, 12
	$(21, 0)$	21^2	V	$\Sigma(21, 0)$	6, 12
	$(2, 1)$	3	M	$\Sigma(2, 1)$	2
	$(14, 7)$	147	M	$\Sigma(14, 7)$	6, 12
21	$(3, 1)$	7	T	$\Sigma_0(3, 1)$	12
	$(3, 2)$	7	T	$\Sigma_0(3, 2)$	12
	$(5, 1)$	21	T	$\Sigma_0(5, 1)$	12
	$(5, 4)$	21	T	$\Sigma_0(5, 4)$	12
	$(9, 3)$	63	T	$\Sigma_0(9, 3)$	12
	$(9, 6)$	63	T	$\Sigma_0(9, 6)$	12
39	$(3, 0)$	9	V	$\Sigma(3, 0)$	6
	$(13, 0)$	169	V	$\Sigma(13, 0)$	6, 12
	$(39, 0)$	39^2	V	$\Sigma(39, 0)$	6, 12
	$(2, 1)$	3	M	$\Sigma(2, 1)$	2
	$(26, 13)$	507	M	$\Sigma(26, 13)$	6, 12
	$(4, 1)$	13	T	$\Sigma_0(4, 1)$	12
	$(4, 3)$	13	T	$\Sigma_0(4, 3)$	12
	$(7, 2)$	39	T	$\Sigma_0(7, 2)$	12
	$(7, 5)$	39	T	$\Sigma_0(7, 5)$	12
	$(12, 3)$	117	T	$\Sigma_0(12, 3)$	12
	$(12, 9)$	117	T	$\Sigma_0(12, 9)$	12
42	$(2, 0)$	4	V	$\Sigma(2, 0)$	3
	$(3, 0)$	9	V	$\Sigma(3, 0)$	6
	$(6, 0)$	36	V	$\Sigma(6, 0)$	6, 12
	$(7, 0)$	49	V	$\Sigma(7, 0)$	6, 12
	$(14, 0)$	196	V	$\Sigma(14, 0)$	6, 12
	$(21, 0)$	441	V	$\Sigma(21, 0)$	6, 12
	$(42, 0)$	42^2	V	$\Sigma(42, 0)$	6, 12
	$(2, 1)$	3	M	$\Sigma(2, 1)$	2
	$(4, 2)$	12	M	$\Sigma(4, 2)$	6
	$(14, 7)$	147	M	$\Sigma(14, 7)$	6, 12
	$(28, 14)$	588	M	$\Sigma(28, 14)$	6, 12
	$(3, 1)$	7	T	$\Sigma_0(3, 1)$	12
	$(3, 2)$	7	T	$\Sigma_0(3, 2)$	12
	$(5, 1)$	21	T	$\Sigma_0(5, 1)$	12
	$(5, 4)$	21	T	$\Sigma_0(5, 4)$	12
	$(6, 2)$	28	T	$\Sigma_0(6, 2)$	12
	$(6, 4)$	28	T	$\Sigma_0(6, 4)$	12
	$(9, 3)$	63	T	$\Sigma_0(9, 3)$	12
	$(9, 6)$	63	T	$\Sigma_0(9, 6)$	12
	$(10, 2)$	84	T	$\Sigma_0(10, 2)$	12
	$(10, 8)$	84	T	$\Sigma_0(10, 8)$	12
	$(18, 6)$	252	T	$\Sigma_0(18, 6)$	12
	$(18, 12)$	252	T	$\Sigma_0(18, 12)$	12

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \tau), \quad g \in G \tag{8.15}$$

in (4.11) in Proposition 4.1. Recall that τ, being the transport cost parameter, serves as a bifurcation parameter, $\boldsymbol{\lambda} \in \mathbb{R}^N$ is an independent variable vector of dimension $N = n^2$ expressing a distribution of mobile population, $\boldsymbol{F} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is the nonlinear function in (4.4) or (4.6) for core–periphery models, and T is the N-dimensional permutation representation in Sect. 7.3 of the group $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Let $(\boldsymbol{\lambda}_{\mathrm{c}}, \tau_{\mathrm{c}})$ be a critical point of multiplicity M (≥ 1), at which the Jacobian matrix of $\boldsymbol{F}$ has a rank deficiency M. The critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \tau_{\mathrm{c}})$ is assumed to be G-symmetric in the sense of (2.82):

$$T(g)\boldsymbol{\lambda}_{\mathrm{c}} = \boldsymbol{\lambda}_{\mathrm{c}}, \quad g \in G. \tag{8.16}$$

Moreover, it is assumed to be *group-theoretic*, which means, by definition, that the M-dimensional kernel space of the Jacobian matrix at $(\boldsymbol{\lambda}_{\mathrm{c}}, \tau_{\mathrm{c}})$ is irreducible with respect to the representation T. Then the critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \tau_{\mathrm{c}})$ is associated with an irreducible representation μ of G, and the multiplicity M corresponds to the dimension of the irreducible representation μ. We denote the representation matrix for μ by $T^{\mu}(g)$.

By the Liapunov–Schmidt reduction with symmetry in Sect. 2.4.3, the full system of equilibrium equation (8.14) is reduced, in a neighborhood of the critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \tau_{\mathrm{c}})$, to a system of bifurcation equations (2.90):

$$\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{\tau}) = \boldsymbol{0} \tag{8.17}$$

in $\boldsymbol{w} \in \mathbb{R}^M$, where $\tilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function and $\tilde{\tau} = \tau - \tau_{\mathrm{c}}$ denotes the increment of τ. In this reduction process the equivariance (8.15) of the full system is inherited by the reduced system (8.17). With the use of the representation matrix $T^{\mu}(g)$ for the associated irreducible representation μ, the equivariance of the bifurcation equation can be expressed as

$$T^{\mu}(g)\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{\tau}) = \tilde{\boldsymbol{F}}(T^{\mu}(g)\boldsymbol{w}, \tilde{\tau}), \quad g \in G, \tag{8.18}$$

which is given in (2.93) for general cases. This inherited symmetry plays a key role in determining the symmetry of bifurcating solutions.

The reduced equation (8.17) can possibly admit multiple solutions $\boldsymbol{w} = \boldsymbol{w}(\tilde{\tau})$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, since $(\boldsymbol{w}, \tilde{\tau}) = (\boldsymbol{0}, 0)$ is a singular point of (8.17). This gives rise to bifurcation. Each $\boldsymbol{w}$ uniquely determines a solution $\boldsymbol{\lambda}$ to the full system (8.14), and moreover the symmetry of $\boldsymbol{w}$ is identical with that of $\boldsymbol{\lambda}$. Indeed, we have the following relation (Proposition 2.7 in Sect. 2.4.4):

$$G^{\mu} \subseteq \Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{\lambda}), \tag{8.19}$$

where G^{μ} is a subgroup of G defined in (2.36) as

$$G^{\mu} = \{g \in G \mid T^{\mu}(g) = I\}, \tag{8.20}$$

and $\Sigma(\boldsymbol{\lambda})$ and $\Sigma^{\mu}(\boldsymbol{w})$ are isotropy subgroups defined (cf., (2.119)), respectively, as

$$\Sigma(\boldsymbol{\lambda}) = \Sigma(\boldsymbol{\lambda}; G, T) = \{g \in G \mid T(g)\boldsymbol{\lambda} = \boldsymbol{\lambda}\}, \tag{8.21}$$

$$\Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{w}; G, T^{\mu}) = \{g \in G \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w}\}. \tag{8.22}$$

The significance of the relation (8.19) is twofold. First, unless a subgroup Σ is large enough to contain G^{μ}, no bifurcating solution $\boldsymbol{\lambda}$ exists such that $\Sigma = \Sigma(\boldsymbol{\lambda})$. Second, the symmetry of a bifurcating solution $\boldsymbol{\lambda}$ is known as $\Sigma(\boldsymbol{\lambda}) = \Sigma^{\mu}(\boldsymbol{w})$ through the analysis of the bifurcation equation in $\boldsymbol{w}$.

Remark 8.2. Although there exist various ways to define $\boldsymbol{w} = (w_1, \dots, w_M)^{\top}$ in the bifurcation equation (8.17) satisfying the equivariance condition (8.18), it turns out to be pertinent to define it consistently with the matrix Q in Sect. 7.5. That is, the components of $\boldsymbol{w} = (w_1, \dots, w_M)^{\top}$ are henceforth assumed to correspond to the column vectors of $Q^{\mu} = [\boldsymbol{q}_1^{\mu}, \dots, \boldsymbol{q}_M^{\mu}]$. □

8.3.2 *Use of Equivariant Branching Lemma*

Equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) is a useful mathematical means to prove the existence of a bifurcating solution with a specified symmetry without actually solving the bifurcation equation in (8.17). By the equivariant branching lemma, we shall demonstrate the emergence of Lösch's hexagons with symmetries specified by (8.10), that is,[1]

$$\Sigma = \begin{cases} \Sigma(\alpha, 0) = \langle r, s\rangle \ltimes \langle p_1^{\alpha}, p_2^{\alpha}\rangle & (2 \le \alpha \le n), \\ \Sigma(2\beta, \beta) = \langle r, s\rangle \ltimes \langle p_1^{2\beta} p_2^{\beta}, p_1^{-\beta} p_2^{\beta}\rangle & (1 \le \beta \le n/3), \\ \Sigma_0(\alpha, \beta) = \langle r\rangle \ltimes \langle p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha-\beta}\rangle & (1 \le \beta < \alpha < n,\ \alpha \ne 2\beta) \end{cases} \tag{8.23}$$

with (α, β) satisfying the compatibility condition (8.12) with the lattice size n.

Bifurcation Equation and the Associated Irreducible Representation

To investigate the existence of a bifurcating solution $\boldsymbol{\lambda}$ with a specified symmetry Σ to the equilibrium equation $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0}$ in (8.14), it suffices to apply the equivariant branching lemma to the bifurcation equation $\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{\tau})$ in (8.17). This is justified by the fact that the isotropy subgroup $\Sigma(\boldsymbol{\lambda})$ expressing the symmetry of a bifurcating solution $\boldsymbol{\lambda}$ is identical to the isotropy subgroup $\Sigma^{\mu}(\boldsymbol{w})$ of the corresponding solution $\boldsymbol{w}$ for the bifurcation equation, i.e., $\Sigma(\boldsymbol{\lambda}) = \Sigma^{\mu}(\boldsymbol{w})$ as shown in (8.19).

[1] Among conjugate subgroups, it is pertinent to choose those in (8.23) as representatives (see Sect. 5.5.3).

Table 8.3 Irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ to be considered in bifurcation analysis

$n \setminus M$	2	3	6	12
$6m$	$(2;+)$	$(3;+,+)$	$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 1$			$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 2$		$(3;+,+)$	$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 3$	$(2;+)$		$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$

$(6;k,0;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$
$(6;k,k;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor, k \neq n/3$
$(12;k,\ell)$ for (k,ℓ) with $1 \leq \ell \leq k-1, 2k+\ell \leq n-1$

The bifurcation equation is associated with an irreducible representation μ of $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ as in (8.18). The associated irreducible representation μ is restricted to

$$\mu = (2;+),\ (3;+,+),\ (6;k,0;+),\ (6;k,k;+),\ (12;k,\ell)$$

with k for $(6;k,0;+)$ in (6.26), k for $(6;k,k;+)$ in (6.27), and (k,ℓ) for $(12;k,\ell)$ in (6.35), as a consequence of the irreducible decomposition (7.21) of the permutation representation T for the economy on the $n \times n$ hexagonal lattice. The unit representation $(1;+,+)$ has been excluded since it does not correspond to a symmetry-breaking bifurcation point. Thus we only have to deal with critical points of multiplicity $M = 2, 3, 6, 12$, and not of $M = 1$ or 4. As a modified form of Table 7.3, therefore, we obtain Table 8.3, where the multiplicity M of a critical point is equal to the dimension d of the associated irreducible representation.

Isotropy Subgroup and Fixed-Point Subspace

In the analysis by the equivariant branching lemma, the isotropy subgroup of $\boldsymbol{w}$ with respect to T^{μ}:

$$\Sigma^{\mu}(\boldsymbol{w}) = \{g \in G \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w}\}$$

introduced in (8.22), and the fixed-point subspace (cf., (2.138)) of Σ for T^{μ}:

$$\mathrm{Fix}^{\mu}(\Sigma) = \{\boldsymbol{w} \in \mathbb{R}^M \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w} \text{ for all } g \in \Sigma\} \tag{8.24}$$

play the major roles. The following facts, though immediate from the definitions, are important and useful.

- By definition, Σ is an isotropy subgroup if and only if $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$ for some $\boldsymbol{w} \neq \boldsymbol{0}$.
- If $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$, then $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ and $\dim \mathrm{Fix}^{\mu}(\Sigma) \geq 1$.
- Not every Σ with the property of $\dim \mathrm{Fix}^{\mu}(\Sigma) \geq 1$ is an isotropy subgroup.
- $\Sigma \subseteq \Sigma^{\mu}(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$.

- Σ is an isotropy subgroup if and only if $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$ for some $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ with $\boldsymbol{w} \neq \boldsymbol{0}$.
- Unless Σ is an isotropy subgroup, there exists no bifurcating solution $\boldsymbol{w}$ with symmetry Σ.

Analysis Procedure Using Equivariant Branching Lemma

The analysis for the $n \times n$ hexagonal lattice based on the equivariant branching lemma follows the steps below.[2]

1. Specify an irreducible representation μ of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Table 8.3.
2. Specify a subgroup Σ in (8.23) as a candidate of an isotropy subgroup of a possible bifurcating solution.
3. Obtain the fixed-point subspace $\mathrm{Fix}^{\mu}(\Sigma)$ in (8.24) for the subgroup Σ with respect to the irreducible representation μ.
4. Search for some $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ such that $\Sigma^{\mu}(\boldsymbol{w}) = \Sigma$. If no such $\boldsymbol{w}$ exists, then[3] Σ is not an isotropy subgroup, and hence there exists no solution with the specified symmetry Σ for the bifurcation equation associated with μ. If such $\boldsymbol{w}$ exists, then we can ensure that Σ is an isotropy subgroup and can proceed to the next step.
5. Calculate the dimension $\dim \mathrm{Fix}^{\mu}(\Sigma)$ of the fixed-point subspace.
6. If $\dim \mathrm{Fix}^{\mu}(\Sigma) = 1$, a bifurcating solution with symmetry Σ is guaranteed to exist generically by the equivariant branching lemma. If $\dim \mathrm{Fix}^{\mu}(\Sigma) \geq 2$, no definite conclusion can be reached by means of the equivariant branching lemma.

As shown by the above procedure, the equivariant branching lemma is effective only if $\dim \mathrm{Fix}(\Sigma) \leq 1$. Fortunately, in Sects. 8.4–8.7, we have $\dim \mathrm{Fix}(\Sigma) \leq 1$ for every symmetry Σ of Lösch's hexagons, and therefore the equivariant branching lemma is capable of determining the existence or nonexistence of bifurcating solutions for Lösch's hexagons.

Remark 8.3. The equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) assumes two technical conditions concerning absolute irreducibility and genericity. The former condition of absolute irreducibility is concerned with an algebraic property of the underlying group, which is satisfied by the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ since all the irreducible representations over $\mathbb{R}$ of this group are absolutely irreducible (Sect. 6.3). The latter condition of genericity is a matter of modeling. It may be worth emphasizing that we assume genericity for two different reasons: (1) to allow us to assume (2.141) for the equivariant branching lemma and (2) to allow us to exclusively consider group-theoretic critical points. □

[2]The analysis procedure is illustrated for two- and three-dimensional irreducible representations in the proofs of Proposition 8.3 in Sect. 8.4 and Proposition 8.4 in Sect. 8.5.

[3]This is because, if $\Sigma^{\mu}(\boldsymbol{w}) = \Sigma$, then $\boldsymbol{w}$ belongs to $\mathrm{Fix}^{\mu}(\Sigma)$.

Remark 8.4. In our analysis for economic agglomeration, we are concerned not only with the symmetry of the bifurcating solution but also with the direction of the bifurcating solution (Sect. 7.6.1; see Figs. 7.3 and 7.4, in particular). In this connection, it is recalled from Remark 2.16 in Sect. 2.4.5 that the equivariant branching lemma guarantees the existence of a bifurcating path in a neighborhood of the critical point, and hence the existence of a bifurcating branch that corresponds to an agglomeration pattern of economic significance. We shall dwell on this issue in Remark 8.5 in Sect. 8.4 for Christaller's $k = 3$ system and Remark 8.6 in Sect. 8.5 for Christaller's $k = 4$ system. □

8.4 Bifurcation Point of Multiplicity 2

In this section Lösch's hexagon with $D = 3$ in Fig. 8.1 (Christaller's $k = 3$ system in Fig. 8.2a) is predicted to branch from a double critical point[4] with $M = 2$ when n is a multiple of 3.

As shown by Table 8.3 in Sect. 8.3.2, a critical point of multiplicity 2 is associated with the two-dimensional irreducible representation $(2; +)$, which exists only when n is a multiple of 3. Recall from (6.13) and (6.14) that this irreducible representation is given by

$$T^{(2;+)}(r) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad T^{(2;+)}(s) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \tag{8.25}$$

$$T^{(2;+)}(p_1) = T^{(2;+)}(p_2) = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}. \tag{8.26}$$

In view of Remark 8.2 in Sect. 8.3.1, let us assume that the variable $\boldsymbol{w} = (w_1, w_2)^\top$ for the bifurcation equation (8.17) corresponds to the column vectors of

$$Q^{(2;+)} = [\boldsymbol{q}_1, \boldsymbol{q}_2] = [\ \langle\cos(2\pi(n_1 - 2n_2)/3)\rangle,\ \langle\sin(2\pi(n_1 - 2n_2)/3)\rangle\ ] \tag{8.27}$$

in (7.32). The spatial patterns for these vectors are depicted in Fig. 8.3 for $n = 6$. The vector $\boldsymbol{q}_1 = \langle\cos(2\pi(n_1 - 2n_2)/3)\rangle$ represents Lösch's hexagon with $D = 3$ (Fig. 8.3a), but the other vector $\boldsymbol{q}_2 = \langle\sin(2\pi(n_1 - 2n_2)/3)\rangle$ does not (Fig. 8.3b). The vector $\boldsymbol{q}_1$ corresponds to the vector $\boldsymbol{w}_0 = (1, 0)^\top$ in the proof of Proposition 8.3 below.

[4]This critical point is different from the double critical point studied in Sect. 3.4.2.

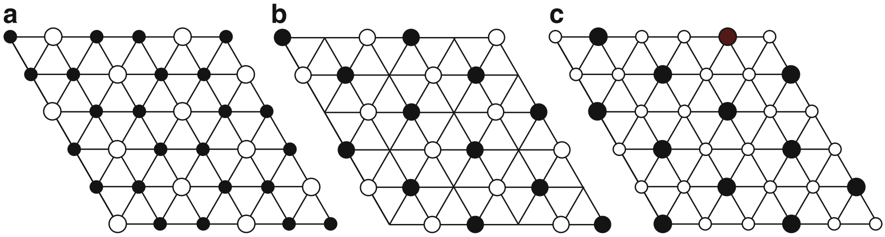

Fig. 8.3 Patterns on the 6×6 hexagonal lattice expressed by the column vectors of $Q^{(2;+)}$. A *white circle* denotes a positive component and a *black circle* denotes a negative component; $\Sigma(2,1) = \langle r, s, p_1^2 p_2, p_1^{-1} p_2\rangle$. (**a**) $\boldsymbol{q}_1$ $(\Sigma(2,1))$. (**b**) $\boldsymbol{q}_2$ $(\langle r^2, s, p_1^2 p_2, p_1^{-1} p_2\rangle)$. (**c**) $-\boldsymbol{q}_1$ $(\Sigma(2,1))$

Proposition 8.3.

(i) *When n is a multiple of 3, Lösch's hexagon of type M with $D = 3$ arises as a bifurcating solution with the symmetry of $\Sigma(2,1)$ from a critical point of multiplicity 2 associated with the irreducible representation* $(2;+)$.

(ii) *No hexagonal distribution with a symmetry in* (8.10) *other than $\Sigma(2,1)$ arises as a bifurcating solution from a critical point of multiplicity 2 for any n.*

Proof. (i) The general procedure in Sect. 8.3.2 is applied to $\mu = (2;+)$ and $\Sigma = \Sigma(2,1) = \langle r, s\rangle \ltimes \langle p_1^2 p_2, p_1^{-1} p_2\rangle$ under the compatibility condition (8.12) for type M with $\beta = 1$. First note

$$\mathrm{Fix}^{(2;+)}(\Sigma(2,1)) = \mathrm{Fix}^{(2;+)}(\langle r\rangle) \;\cap\; \mathrm{Fix}^{(2;+)}(\langle s, p_1^2 p_2, p_1^{-1} p_2\rangle).$$

Here we have

$$\mathrm{Fix}^{(2;+)}(\langle r\rangle) = \{c(1,0)^\top \mid c \in \mathbb{R}\}$$

since $T^{(2;+)}(r)(w_1, w_2)^\top = (w_1, -w_2)^\top$ by (8.25), whereas

$$\mathrm{Fix}^{(2;+)}(\langle s, p_1^2 p_2, p_1^{-1} p_2\rangle) = \mathbb{R}^2$$

since $T^{(2;+)}(s) = T^{(2;+)}(p_1^2 p_2) = T^{(2;+)}(p_1^{-1} p_2) = I$ by (8.25) and (8.26). Therefore,

$$\mathrm{Fix}^{(2;+)}(\Sigma(2,1)) = \{c(1,0)^\top \mid c \in \mathbb{R}\},$$

that is, $\Sigma(2,1) = \Sigma^{(2;+)}(\boldsymbol{w}_0)$ for $\boldsymbol{w}_0 = (1,0)^\top$. Thus the targeted symmetry $\Sigma(2,1)$ is an isotropy subgroup with

$$\dim \mathrm{Fix}^{(2;+)}(\Sigma(2,1)) = 1.$$

The equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) then guarantees the existence of a bifurcating path with symmetry $\Sigma(2,1)$.

(ii) It follows from (8.25) and (8.26) that the subgroup G^{μ} in (8.20) for $\mu = (2;+)$ is given by

$$G^{(2;+)} = \langle r^2, s, p_1^2 p_2, p_1^{-1} p_2 \rangle.$$

For each subgroup Σ in (8.10) other than $\Sigma(2,1)$, we do not have the inclusion $G^{(2;+)} \subseteq \Sigma$. Hence such Σ cannot be an isotropy subgroup by (8.19). $\square$

Remark 8.5. Here is a supplement to the proof of Proposition 8.3. The proof shows that a bifurcating path exists in the one-dimensional subspace spanned by $\boldsymbol{w}_0 = (1,0)^\top$, which corresponds to the first column vector $\boldsymbol{q}_1$ of $Q^{(2;+)}$ in (8.27). The vector $\boldsymbol{q}_1$ in Fig. 8.3a represents Christaller's $k = 3$ system (Fig. 8.2a), but the vector $-\boldsymbol{q}_1$ in the opposite direction in Fig. 8.3c does not represent this system. Because of our assumption in Remark 8.2 in Sect. 8.3.1, the system corresponds to a bifurcating solution with $w_1 > 0$. The existence of such bifurcating branch is guaranteed by the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5); see Remark 8.4 in Sect. 8.3.2. $\square$

8.5 Bifurcation Point of Multiplicity 3

In this section Lösch's hexagon with $D = 4$ in Fig. 8.1 (Christaller's $k = 4$ system in Fig. 8.2b) is predicted to branch from a triple critical point with $M = 3$ when n is a multiple of 2.

As shown by Table 8.3 in Sect. 8.3.2, a critical point of multiplicity 3 is associated with the three-dimensional irreducible representation $(3;+,+)$, which exists only when n is a multiple of 2. Recall from (6.17) and (6.21) that this irreducible representation is given by

$$T^{(3;+,+)}(r) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \quad T^{(3;+,+)}(s) = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}; \tag{8.28}$$

$$T^{(3;+,+)}(p_1) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \quad T^{(3;+,+)}(p_2) = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}. \tag{8.29}$$

In view of Remark 8.2 in Sect. 8.3.1, let us assume that the variable $\boldsymbol{w} = (w_1, w_2, w_3)^\top$ for the bifurcation equation (8.17) corresponds to the column vectors of

$$Q^{(3;+,+)} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3] = [\ \langle \cos(\pi n_1) \rangle,\ \langle \cos(\pi n_2) \rangle,\ \langle \cos(\pi(n_1 - n_2)) \rangle\ ] \tag{8.30}$$

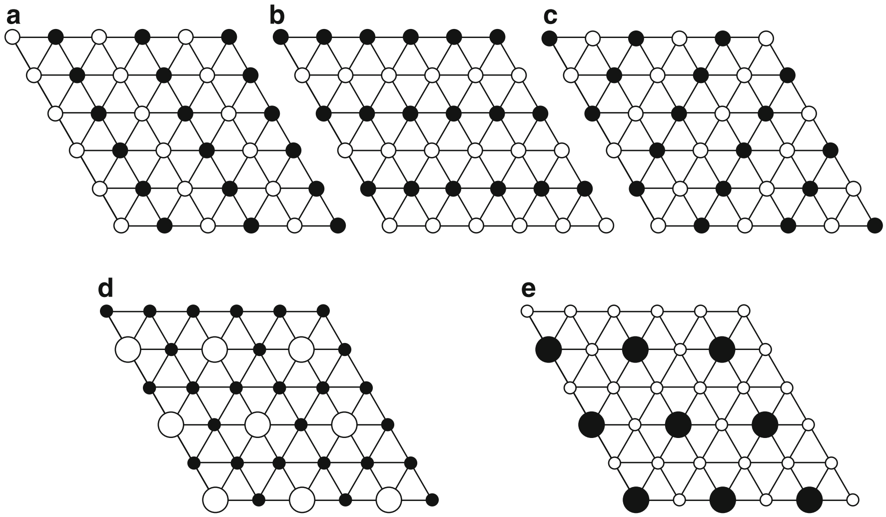

Fig. 8.4 Patterns on the 6×6 hexagonal lattice expressed by the column vectors of $Q^{(3;+,+)}$. A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_1$ $(\langle r^3, sr^2, p_1^2, p_2\rangle)$. (**b**) $\boldsymbol{q}_2$ $(\langle r^3, s, p_1, p_2^2\rangle)$. (**c**) $\boldsymbol{q}_3$ $(\langle r^3, sr, p_1p_2, p_2^2\rangle)$. (**d**) $\boldsymbol{q}_{\text{sum}} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3$ $(\Sigma(2,0) = \langle r, s, p_1^2, p_2^2\rangle)$. (**e**) $-\boldsymbol{q}_{\text{sum}}$ $(\Sigma(2,0) = \langle r, s, p_1^2, p_2^2\rangle)$

in (7.33). The spatial patterns for these vectors are depicted in Fig. 8.4 for $n = 6$. The three vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$ represent stripe patterns in different directions (Fig. 8.4a–c). The sum $\boldsymbol{q}_{\text{sum}} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3$ of these three vectors (Fig. 8.4d) represents Lösch's hexagon with $D = 4$. This vector $\boldsymbol{q}_{\text{sum}}$ corresponds to the vector $\boldsymbol{w}_0 = (1,1,1)^\top$ in the proof of Proposition 8.4 below.

Proposition 8.4.

(i) *When n is a multiple of 2, Lösch's hexagon of type V with $D = 4$ arises as a bifurcating solution with the symmetry of $\Sigma(2,0)$ from a critical point of multiplicity 3 associated with the irreducible representation* $(3;+,+)$.

(ii) *No hexagonal distribution with a symmetry in* (8.10) *other than $\Sigma(2,0)$ arises as a bifurcating solution from a critical point of multiplicity 3 for any n.*

Proof. (i) The general procedure in Sect. 8.3.2 is applied to $\mu = (3;+,+)$ and $\Sigma = \Sigma(2,0) = \langle r, s\rangle \ltimes \langle p_1^2, p_2^2\rangle$ under the compatibility condition (8.12) for type V with $\alpha = 2$. First note

$$\mathrm{Fix}^{(3;+,+)}(\Sigma(2,0)) = \mathrm{Fix}^{(3;+,+)}(\langle r, s\rangle) \cap \mathrm{Fix}^{(3;+,+)}(\langle p_1^2, p_2^2\rangle).$$

Here we have

$$\mathrm{Fix}^{(3;+,+)}(\langle r, s\rangle) = \{c(1,1,1)^\top \mid c \in \mathbb{R}\}$$

since, for $\boldsymbol{w} = (w_1, w_2, w_3)^\top$, we have

$$T^{(3;+,+)}(r)\boldsymbol{w} = (w_3, w_1, w_2)^\top, \quad T^{(3;+,+)}(s)\boldsymbol{w} = (w_3, w_2, w_1)^\top$$

by (8.28). We also have

$$\mathrm{Fix}^{(3;+,+)}(\langle p_1^2, p_2^2 \rangle) = \mathbb{R}^3$$

since $T^{(3;+,+)}(p_1^2) = T^{(3;+,+)}(p_2^2) = I$ by (8.28) and (8.29) . Therefore,

$$\mathrm{Fix}^{(3;+,+)}(\Sigma(2,0)) = \{c(1,1,1)^\top \mid c \in \mathbb{R}\},$$

that is, $\Sigma(2,0) = \Sigma^{(3;+,+)}(\boldsymbol{w}_0)$ for $\boldsymbol{w}_0 = (1,1,1)^\top$. Thus the targeted symmetry $\Sigma(2,0)$ is an isotropy subgroup with

$$\dim \mathrm{Fix}^{(3;+,+)}(\Sigma(2,0)) = 1.$$

The equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) then guarantees the existence of a bifurcating path with symmetry $\Sigma(2,0)$.

(ii) It follows from (8.28) and (8.29) that G^μ in (8.20) for $\mu = (3;+,+)$ is given by

$$G^{(3;+,+)} = \langle r^3, p_1^2, p_2^2 \rangle.$$

For each subgroup Σ in (8.10) other than $\Sigma(2,0)$, we do not have the inclusion $G^{(3;+,+)} \subseteq \Sigma$. Hence such Σ cannot be an isotropy subgroup by (8.19). □

Remark 8.6. Here is a supplement to the proof of Proposition 8.4. The proof shows that a bifurcating path exists in the one-dimensional subspace spanned by $\boldsymbol{w}_0 = c(1,1,1)^\top$, which corresponds to the sum $\boldsymbol{q}_{\mathrm{sum}} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3$ of the column vectors of $Q^{(3;+,+)}$ in (8.30). The vector $\boldsymbol{q}_{\mathrm{sum}}$ in Fig. 8.4d represents Christaller's $k = 4$ system (Fig. 8.2b), but the vector $-\boldsymbol{q}_{\mathrm{sum}}$ in the opposite direction in Fig. 8.4e does not represent this system. Because of our assumption in Remark 8.2 in Sect. 8.3.1, the system corresponds to a bifurcating solution with $c > 0$. The existence of such bifurcating branch is guaranteed by the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5); see Remark 8.4 in Sect. 8.3.2. □

8.6 Bifurcation Point of Multiplicity 6

Hexagons branching from critical points of multiplicity 6 are investigated. Among Lösch's ten smallest hexagons in Fig. 8.1, hexagons with $D = 9, 12, 16, 25$ are predicted to branch from critical points of multiplicity 6, whereas the other six hexagons never appear (Propositions 8.7, 8.9, and 8.11).

8.6.1 Representation in Complex Variables

As shown by Table 8.3 in Sect. 8.3.2, a critical point of multiplicity 6 is associated with one of the six-dimensional irreducible representations

$$(6;k,0,+) \ \text{with} \ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor; \tag{8.31}$$

$$(6;k,k,+) \ \text{with} \ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \ k \ne \frac{n}{3}, \tag{8.32}$$

where $n \ge 3$.

Recall from (6.29) and (6.30) that the irreducible representation $(6;k,0,+)$ is given by

$$T^{(6;k,0,+)}(r) = \begin{bmatrix} & & S \\ S & & \\ & S & \end{bmatrix}, \quad T^{(6;k,0,+)}(s) = \begin{bmatrix} & & S \\ & S & \\ S & & \end{bmatrix}, \tag{8.33}$$

$$T^{(6;k,0,+)}(p_1) = \begin{bmatrix} R^k & & \\ & I & \\ & & R^{-k} \end{bmatrix}, \quad T^{(6;k,0,+)}(p_2) = \begin{bmatrix} I & & \\ & R^{-k} & \\ & & R^k \end{bmatrix} \tag{8.34}$$

with

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

The other irreducible representation $(6;k,k,+)$ is given by (6.32) and (6.33):

$$T^{(6;k,k,+)}(r) = \begin{bmatrix} & & S \\ S & & \\ & S & \end{bmatrix}, \quad T^{(6;k,k,+)}(s) = \begin{bmatrix} & I & \\ I & & \\ & & I \end{bmatrix}, \tag{8.35}$$

$$T^{(6;k,k,+)}(p_1) = \begin{bmatrix} R^k & & \\ & R^k & \\ & & R^{-2k} \end{bmatrix}, \quad T^{(6;k,k,+)}(p_2) = \begin{bmatrix} R^k & & \\ & R^{-2k} & \\ & & R^k \end{bmatrix}. \tag{8.36}$$

It is noted in Remark 6.3 that (8.34) and (8.36) can be expressed in a unified manner as

$$T^{(6;k,0,+)}(p_i) = T^{(6;k,0)}(p_i), \quad T^{(6;k,k,+)}(p_i) = T^{(6;k,k)}(p_i), \quad i = 1,2$$

with the use of notations

$$T^{(6;k,\ell)}(p_1) = \begin{bmatrix} R^k & & \\ & R^\ell & \\ & & R^{-k-\ell} \end{bmatrix}, \quad T^{(6;k,\ell)}(p_2) = \begin{bmatrix} R^\ell & & \\ & R^{-k-\ell} & \\ & & R^k \end{bmatrix}. \tag{8.37}$$

Using complex variables

$$(z_1, z_2, z_3) = (w_1 + \mathrm{i}w_2, w_3 + \mathrm{i}w_4, w_5 + \mathrm{i}w_6),$$

we can express the actions in $(6; k, 0, +)$, given in (8.33) and (8.34) for the 6-dimensional vectors $(w_1, \dots, w_6)$, as

$$\begin{aligned} r &: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}, \qquad s : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_2 \\ \bar{z}_1 \end{bmatrix}, \\ p_1 &: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ z_2 \\ \omega^{-k} z_3 \end{bmatrix}, \ p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \omega^{-k} z_2 \\ \omega^k z_3 \end{bmatrix}, \end{aligned} \tag{8.38}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$. On the other hand, the actions in $(6; k, k, +)$, given in (8.35) and (8.36), are

$$\begin{aligned} r &: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}, \qquad s : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} z_2 \\ z_1 \\ z_3 \end{bmatrix}, \\ p_1 &: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^k z_2 \\ \omega^{-2k} z_3 \end{bmatrix}, \ p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-2k} z_2 \\ \omega^k z_3 \end{bmatrix}. \end{aligned} \tag{8.39}$$

The actions of p_1 and p_2 in $(6; k, \ell, +)$ for $\ell \in \{0, k\}$ are expressed in a unified form as

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^\ell z_2 \\ \omega^{-k-\ell} z_3 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ \omega^{-k-\ell} z_2 \\ \omega^k z_3 \end{bmatrix}. \tag{8.40}$$

8.6.2 Isotropy Subgroups

To apply the method of analysis in Sect. 8.3.2, we identify isotropy subgroups for $(6; k, 0, +)$ and $(6; k, k, +)$ that are relevant to Lösch's hexagons. We denote the isotropy subgroup of $z = (z_1, z_2, z_3)$ and the fixed-point subspace of Σ with respect to $T^{(6;k,\ell,+)}$ with $\ell \in \{0, k\}$ as

$$\Sigma^{(6;k,\ell,+)}(z) = \{g \in G \mid T^{(6;k,\ell,+)}(g) \cdot z = z\}, \tag{8.41}$$

$$\mathrm{Fix}^{(6;k,\ell,+)}(\Sigma) = \{z \mid T^{(6;k,\ell,+)}(g) \cdot z = z \text{ for all } g \in \Sigma\}, \tag{8.42}$$

where $T^{(6;k,\ell,+)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (8.38) and (8.39). We also define

$$\hat{k} = \frac{k}{\gcd(k,n)}, \quad \hat{n} = \frac{n}{\gcd(k,n)}, \tag{8.43}$$

where $\gcd(k, n)$ means the greatest common divisor of k and n.

The symmetries of $\langle r \rangle$ and $\langle r, s \rangle$ and the translational symmetry of $p_1^a p_2^b$ are dealt with in Propositions 8.5 and 8.6 below. In this connection, the isotropy subgroups of $z = (z_1, z_2, z_3) = (1, 1, 1)$ (i.e., $\boldsymbol{w} = (1, 0, 1, 0, 1, 0)^\top$) play a crucial role. Remark 8.7 given later should be consulted with regard to the geometrical interpretation of the propositions below.

Proposition 8.5. *For* $(6; k, 0, +)$ *in* (8.31)*, we have the following statements.*

(i) $\mathrm{Fix}^{(6;k,0,+)}(\langle r \rangle) = \mathrm{Fix}^{(6;k,0,+)}(\langle r, s \rangle) = \{c(1, 1, 1) \mid c \in \mathbb{R}\}$ *for each* k.

(ii) $p_1^a p_2^b \in \Sigma^{(6;k,0,+)}((1, 1, 1))$ *if and only if*

$$\hat{k}a \equiv 0, \quad \hat{k}b \equiv 0 \mod \hat{n}. \tag{8.44}$$

(iii) $\Sigma^{(6;k,0,+)}((1, 1, 1)) = \Sigma(\hat{n}, 0)$ *and* $\mathrm{Fix}^{(6;k,0,+)}(\Sigma(\hat{n}, 0)) = \{c(1, 1, 1) \mid c \in \mathbb{R}\}$*. That is,* $\Sigma(\hat{n}, 0)$ *is the isotropy subgroup of* $z = (1, 1, 1)$ *with* $\dim \mathrm{Fix}^{(6;k,0,+)}(\Sigma(\hat{n}, 0)) = 1$.

(iv) *If* $\Sigma(\alpha, \beta)$ *is an isotropy subgroup (for some* z*), then* $(\alpha, \beta) = (\hat{n}, 0)$ *and it is the isotropy subgroup of* $z = (1, 1, 1)$.

(v) $\Sigma_0(\alpha, \beta)$ *is not an isotropy subgroup (for any* z*) for any value of* (α, β).

Proof. (i) By (8.38), $z = (z_1, z_2, z_3)$ is invariant to r if and only if $(\bar{z}_3, \bar{z}_1, \bar{z}_2) = (z_1, z_2, z_3)$, which is equivalent to $z_1 = z_2 = z_3 \in \mathbb{R}$. Such z is also invariant to s.

(ii) By (8.40) for $(6; k, \ell, +)$, the invariance of $z = (1, 1, 1)$ to $p_1^a p_2^b$ is expressed as

$$ka + \ell b \equiv 0, \quad \ell a - (k + \ell)b \equiv 0, \quad -(k + \ell)a + kb \equiv 0 \mod n,$$

which is equivalent to

$$ka + \ell b \equiv 0, \quad \ell a - (k + \ell)b \equiv 0 \mod n. \tag{8.45}$$

For $\ell = 0$, this condition reduces to

$$ka \equiv 0, \quad kb \equiv 0 \mod n,$$

which is equivalent to (8.44).

(iii) (a, b) satisfies (8.44) if and only if both a and b are multiples of $\hat{n}$. The subgroup of G generated by $p_1^a p_2^b$ for such (a, b), together with r and s, coincides with $\Sigma(\hat{n}, 0)$.

(iv) This follows from (i) and (iii).

(v) This follows from (i). □

Proposition 8.6. *For* $(6; k, k, +)$ *in* (8.32)*, we have the following statements.*

(i) $\mathrm{Fix}^{(6;k,k,+)}(\langle r\rangle) = \mathrm{Fix}^{(6;k,k,+)}(\langle r, s\rangle) = \{c(1, 1, 1) \mid c \in \mathbb{R}\}$ *for each* k.

(ii) $p_1^a p_2^b \in \Sigma^{(6;k,k,+)}((1, 1, 1))$ *if and only if*

$$\hat{k}(a + b) \equiv 0, \quad \hat{k}(a - 2b) \equiv 0 \mod \hat{n}. \tag{8.46}$$

(iii) *If* $\hat{n}$ *is a multiple of 3, then we have*

$$\Sigma^{(6;k,k,+)}((1, 1, 1)) = \Sigma(2\hat{n}/3, \hat{n}/3),$$
$$\mathrm{Fix}^{(6;k,k,+)}(\Sigma(2\hat{n}/3, \hat{n}/3)) = \{c(1, 1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(2\hat{n}/3, \hat{n}/3)$ *is the isotropy subgroup of* $z = (1, 1, 1)$ *with* $\dim \mathrm{Fix}^{(6;k,k,+)}(\Sigma(2\hat{n}/3, \hat{n}/3)) = 1$. *If* $\hat{n}$ *is not a multiple of 3, then we have*

$$\Sigma^{(6;k,k,+)}((1, 1, 1)) = \Sigma(\hat{n}, 0),$$
$$\mathrm{Fix}^{(6;k,k,+)}(\Sigma(\hat{n}, 0)) = \{c(1, 1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\hat{n}, 0)$ *is the isotropy subgroup of* $z = (1, 1, 1)$ *with* $\dim \mathrm{Fix}^{(6;k,k,+)}(\Sigma(\hat{n}, 0)) = 1$.

(iv) *If $\Sigma(\alpha,\beta)$ is an isotropy subgroup (for some z), then*

$$(\alpha,\beta) = \begin{cases} (2\hat{n}/3, \hat{n}/3) & \text{if } \hat{n} \text{ is a multiple of 3,} \\ (\hat{n}, 0) & \text{otherwise.} \end{cases}$$

(v) *$\Sigma_0(\alpha,\beta)$ is not an isotropy subgroup (for any z) for any value of (α,β).*

Proof. (i) By (8.39), $z = (z_1, z_2, z_3)$ is invariant to r if and only if $(\bar{z}_3, \bar{z}_1, \bar{z}_2) = (z_1, z_2, z_3)$, which is equivalent to $z_1 = z_2 = z_3 \in \mathbb{R}$. Such z is also invariant to s.

(ii) The condition (8.45) for $\ell = k$ reduces to

$$k(a+b) \equiv 0, \quad k(a-2b) \equiv 0 \mod n,$$

which is equivalent to (8.46).

(iii) The condition (8.46) is equivalent to the existence of integers p and q such that

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} p \\ q \end{bmatrix}.$$

Hence a and b satisfy (8.46) if and only if they are integers expressed as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{\hat{n}}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

for some integers p and q. When $\hat{n}$ is not a multiple of 3, this is equivalent to $(a,b) = \hat{n}(p', q')$ for integers p' and q'. Therefore, the subgroup of G generated by $p_1^a p_2^b$ with such (a,b), together with r and s, is given by $\Sigma(2\hat{n}/3, \hat{n}/3)$ with $(p,q) = (1,0)$ or $\Sigma(\hat{n}, 0)$ with $(p', q') = (1,0)$ according to whether $\hat{n}$ is a multiple of 3 or not.

(iv) This follows from (i) and (iii).

(v) This follows from (i). □

The above propositions show that, in either case of $(6;k,0,+)$ and $(6;k,k,+)$, any isotropy subgroup Σ containing $\langle r \rangle$, which is of our interest, can be represented as $\Sigma = \Sigma^{(6;k,\ell,+)}(z)$ for $z = (1,1,1)$ and that $\dim \mathrm{Fix}^{(6;k,\ell,+)}(\Sigma) = 1$. On the basis of this fact, we will investigate possible occurrences of Lösch's hexagons for each of the three types V, M, and T in Sects. 8.6.3–8.6.5.

Remark 8.7. The six-dimensional space of $\boldsymbol{w} = (w_1, w_2, w_3, w_4, w_5, w_6)^\top$ for the bifurcation equation (8.17) is spanned by the column vectors of

$$Q^{(6;k,\ell,+)} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4, \boldsymbol{q}_5, \boldsymbol{q}_6], \tag{8.47}$$

the concrete form of which is given in (7.34) and (7.35). For example, the spatial patterns for these vectors are depicted in Fig. 8.5 for $(6;1,1,+)$ with $n = 6$.

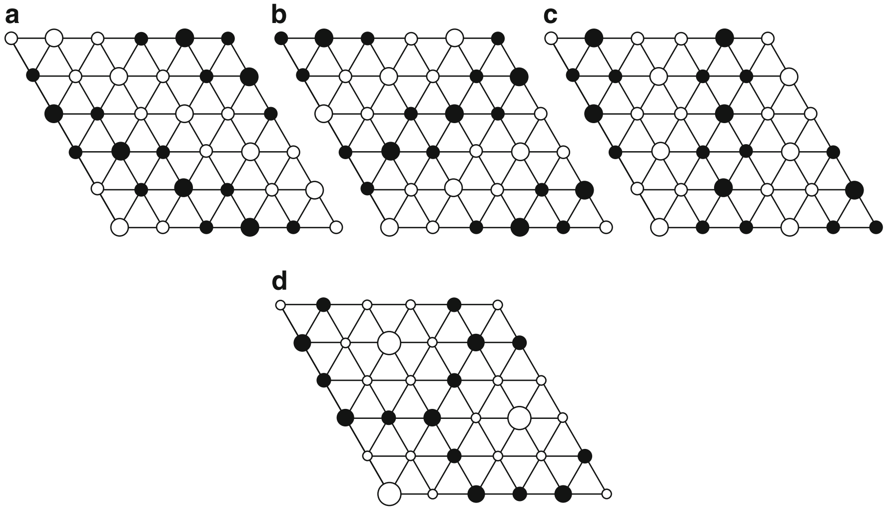

Fig. 8.5 Patterns on the 6 × 6 hexagonal lattice expressed by the column vectors of $Q^{(6;1,1,+)}$. A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_1$ $(\langle r^3, sr, p_1^{-1}p_2, p_1^3p_2^3\rangle)$. (**b**) $\boldsymbol{q}_3$ $(\langle r^3, sr^2, p_1^2p_2, p_2^3\rangle)$. (**c**) $\boldsymbol{q}_5$ $(\langle r^3, s, p_1p_2^2, p_1^3\rangle)$. (**d**) $\boldsymbol{q}_{\text{sum}} = \boldsymbol{q}_1 + \boldsymbol{q}_3 + \boldsymbol{q}_5$ $(\mathrm{D}_6 = \langle r, s\rangle)$

The three vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_3$, and $\boldsymbol{q}_5$ represent stripe patterns in different directions (Fig. 8.5a–c). The sum

$$\boldsymbol{q}_{\text{sum}} = \boldsymbol{q}_1 + \boldsymbol{q}_3 + \boldsymbol{q}_5 \tag{8.48}$$

of these three vectors, which is associated with $z = (1, 1, 1)$, represents Lösch's hexagon of type M with $D = 12$ when $n = 6$ (Fig. 8.5d). □

8.6.3 Hexagons of Type V

Lösch's hexagons of type V are here shown to branch from critical points of multiplicity 6. Recall that a hexagonal distribution of type V is characterized by the symmetry of $\Sigma(\alpha, 0)$ with $2 \le \alpha \le n$ compatible with n (see (8.10) and (8.12)) and that $D(\alpha, 0) = \alpha^2$.

The following proposition is concerned with the four hexagons of type V with $D = 4$, 9, 16, and 25, among Lösch's ten smallest hexagons.

Proposition 8.7.

(i) *Lösch's hexagons of type V with $D = 9$, 16, and 25 arise as bifurcating solutions from critical points of multiplicity 6 for specific values of n and associated irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(6;k,\ell,+)$	
$(3,0)$	9	$3m$	$(m,0)$	
$(4,0)$	16	$4m$	$(m,0)$;	(m,m)
$(5,0)$	25	$5m$	$(m,0),(2m,0)$;	$(m,m),(2m,2m)$

(8.49)

where $m \geq 1$ is an integer.

(ii) *Lösch's hexagon of type V with $D = 4$ does not arise as a bifurcating solution from a critical point of multiplicity 6 for any n.*

Proof. (i) These statements are special cases of Proposition 8.8.
(ii) See the end of the proof of Proposition 8.8. □

For larger hexagons, the above statement is extended as follows.

Proposition 8.8. *Lösch's hexagons of type V with the symmetry of $\Sigma(\alpha,0)$ $(\alpha \geq 3)$ arise as bifurcating solutions from critical points of multiplicity 6 for specific values of n and associated irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(6;k,\ell,+)$	
$(\alpha,0)$	α^2	αm	$(pm,0)$	
$(\alpha,0)$	α^2	αm	(pm,pm)	(α is not a multiple of 3)

(8.50)

where $m \geq 1$ and

$$\gcd(p,\alpha) = 1, \quad 1 \leq p < \alpha/2, \tag{8.51}$$

and the latter case of $(k,\ell) = (pm,pm)$ is excluded if α is a multiple of 3.

Proof. By Propositions 8.5 and 8.6, we have two possibilities: $(6;k,0,+)$ and $(6;k,k,+)$, where the latter is excluded if α is a multiple of 3. For $(6;k,0,+)$, we fix α and look for (k,n) that satisfies (8.31) and $\hat{n} = \alpha$. For such (k,n), $\Sigma(\alpha,0) = \Sigma(\hat{n},0)$ is an isotropy subgroup with dim $\mathrm{Fix}^{(6;k,0,+)}(\Sigma(\alpha,0)) = 1$ by Proposition 8.5. Then the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) guarantees the existence of a bifurcating solution with symmetry $\Sigma(\alpha,0)$; see Remark 8.4 in Sect. 8.3.2.

For $(6;k,k,+)$, we fix α that is not divisible by 3 and look for (k,n) that satisfies (8.32) and $\hat{n} = \alpha$, and proceed in a similar manner using Proposition 8.6.

Suppose that (k,n) is given by (8.50) with (8.51). Then $m = \gcd(k,n)$ by $\gcd(p,\alpha) = 1$ and $\hat{n} = n/\gcd(k,n) = n/m = \alpha$. In the case of $(6;pm,0,+)$, (8.31) follows from $k/n = p/\alpha < 1/2$. In the case of $(6;pm,pm,+)$, we have $k \neq n/3$ from $3k/n = 3p/\alpha \neq 1$ since α is not a multiple of 3, in addition to $k/n = p/\alpha < 1/2$, thereby showing (8.32).

Conversely, suppose that (k,n) satisfies $\hat{n} = \alpha$, and (8.31) or (8.32). Then we have $\alpha = \hat{n} = n/\gcd(k,n)$, which shows $\gcd(k,n) = n/\alpha$ is an integer, say m. We also have $k = \hat{k}\gcd(k,n) = mp$ for $p = \hat{k}$. Then $\gcd(p,\alpha) = \gcd(\hat{k},\hat{n}) = 1$ and $p/\alpha = k/n < 1/2$ by (8.31) or (8.32), thereby showing (8.51).

Proof of Proposition 8.7(ii): The above argument is in fact valid for $\alpha \geq 2$. For $\alpha = 2$, however, the condition $1 \leq p < \alpha/2$ is already a contradiction, which proves the nonexistence of the Lösch's hexagon with $D = 4$. $\square$

Example 8.1. The parameter values of (8.50) in Proposition 8.8 give the following:

(α,β)	D	n	(k,ℓ) in $(6;k,\ell,+)$	
$(6,0)$	36	$6m$	$(m,0)$	
$(7,0)$	49	$7m$	$(m,0),(2m,0),(3m,0)$;	$(m,m),(2m,2m),(3m,3m)$
$(8,0)$	64	$8m$	$(m,0),(3m,0)$;	$(m,m),(3m,3m)$

where $m \geq 1$. For each $\alpha \geq 3$, there exists at least one eligible (k,n) for $(6;k,0,+)$ in (8.50); for instance, $(k,n) = (m,\alpha m)$, which corresponds to $p = 1$. $\square$

Remark 8.8. In all cases in (8.50), the compatibility condition (8.12) for $\Sigma(\alpha,0)$ is satisfied as $n = \alpha m$ with $m = \gcd(k,n)$. $\square$

8.6.4 *Hexagons of Type M*

Lösch's hexagons of type M are shown here to branch from critical points of multiplicity 6. Recall that a hexagonal distribution of type M is characterized by the symmetry of $\Sigma(2\beta,\beta)$ with $1 \leq \beta \leq n/3$ compatible with n (see (8.10) and (8.12)) and $D(2\beta,\beta) = 3\beta^2$.

The following proposition is concerned with the two hexagons of type M with $D = 3$ and 12, among Lösch's ten smallest hexagons.

Proposition 8.9.

(i) *Lösch's hexagon of type M with $D = 12$ arises as a bifurcating solution from critical points of multiplicity 6 for specific values of n and associated irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(6;k,\ell,+)$
$(4,2)$	12	$6m$	(m,m)

(8.52)

where $m \geq 1$ is an integer.

(ii) *Lösch's hexagon of type M with $D = 3$ does not arise as a bifurcating solution from a critical point of multiplicity 6 for any n.*

Proof. (i) This statement is a special case of Proposition 8.10.
(ii) See the end of the proof of Proposition 8.10. $\square$

For larger hexagons, the above statement is extended as follows.

Proposition 8.10. *Lösch's hexagons of type M with the symmetry of $\Sigma(2\beta,\beta)$ ($\beta \geq 2$) arise as bifurcating solutions from critical points of multiplicity 6 for specific values of n and associated irreducible representations given by*

(α, β)	D	n	(k, ℓ) in $(6; k, \ell, +)$
$(2\beta, \beta)$	$3\beta^2$	$3\beta m$	(pm, pm)

(8.53)

where $m \geq 1$ *and*

$$\gcd(p, 3\beta) = 1, \quad 1 \leq p < 3\beta/2. \tag{8.54}$$

Proof. By Propositions 8.5 and 8.6, we can focus on $(6; k, k, +)$ and look for (k, n) that satisfies (8.32) and the condition that

$$\hat{n} \text{ is a multiple of } 3, \text{ and } \beta = \hat{n}/3. \tag{8.55}$$

For such parameter value (k, n), $\Sigma(2\beta, \beta) = \Sigma(2\hat{n}/3, \hat{n}/3)$ is an isotropy subgroup with $\dim \mathrm{Fix}^{(6;k,0,+)}(\Sigma(2\beta, \beta)) = 1$ by Proposition 8.6. Then the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) guarantees the existence of a bifurcating solution with symmetry $\Sigma(2\beta, \beta)$; see Remark 8.4 in Sect. 8.3.2.

Suppose that (k, n) is given by (8.53). Then $m = \gcd(k, n)$ by $\gcd(p, 3\beta) = 1$, and $\hat{n} = n/\gcd(k, n) = 3\beta$, which shows (8.55). As for the condition (8.32), we first observe that $k/n = p/(3\beta) < 1/2$, which shows $k < n/2$. We also note that $k \neq n/3$ if and only if $p \neq \beta$, which is implied by $\gcd(p, 3\beta) = 1$ when $\beta \geq 2$.

Conversely, suppose that (k, n) satisfies (8.32) and (8.55). Put $m' = \gcd(k, n)$ to obtain $n = m'\hat{n} = 3m'\beta$ and $k = m'\hat{k} = m'p$ for $p = \hat{k}$. Hence we have $(k, n) = (pm', 3\beta m')$, where $\gcd(p, 3\beta) = \gcd(\hat{k}, \hat{n}) = 1$ and $p/(3\beta) = k/n < 1/2$, thereby showing (8.54).

Proof of Proposition 8.9(ii): The above argument is almost valid also for $\beta = 1$, except that we need to impose $p \neq \beta$ to guarantee $k \neq n/3$. For $\beta = 1$, however, no p satisfies $p \neq \beta$ and $1 \leq p < 3\beta/2$. This proves the nonexistence of the Lösch's hexagon with $D = 3$. □

Example 8.2. The parameter values of (8.53) in Proposition 8.10 give the following:

(α, β)	D	n	(k, ℓ) in $(6; k, \ell, +)$
$(6, 3)$	27	$9m$	$(m, m), (2m, 2m), (4m, 4m)$,
$(8, 4)$	48	$12m$	$(m, m), (5m, 5m)$
$(10, 5)$	75	$15m$	$(m, m), (2m, 2m), (4m, 4m), (7m, 7m)$
$(12, 6)$	108	$18m$	$(m, m), (5m, 5m), (7m, 7m)$

where $m \geq 1$. For each $\beta \geq 2$, there exists at least one eligible (k, n) in (8.53); for instance, $(k, n) = (m, 3\beta m)$, which corresponds to $p = 1$. □

Remark 8.9. In all cases in (8.53), the compatibility condition (8.12) for $\Sigma(2\beta, \beta)$ is satisfied as $n = 3\beta m$ with $m = \gcd(k, n)$. □

8.6.5 Hexagons of Type T

It is shown that Lösch's hexagons of type T do not appear from critical points of multiplicity 6. Recall that a hexagonal distribution of type T is characterized by the symmetry of $\Sigma_0(\alpha, \beta)$ with $1 \leq \beta < \alpha \leq n-1$ and $\alpha \neq 2\beta$ (see (8.10)).

The following proposition shows the nonexistence of the four hexagons of type T with $D = 7$, 13, 19, and 21, among Lösch's ten smallest hexagons. Furthermore, this proposition denies the existence of larger hexagons of type T.

Proposition 8.11. *Lösch's hexagons of type T with the symmetry of* $\Sigma_0(\alpha, \beta)$ $(1 \leq \beta < \alpha \leq n-1,\ \alpha \neq 2\beta)$ *do not arise as bifurcating solutions from critical points of multiplicity 6 for any* n.

Proof. By Propositions 8.5 and 8.6, $\Sigma_0(\alpha, \beta)$ is not an isotropy subgroup with respect to $(6; k, 0, +)$ nor to $(6; k, k, +)$. □

8.6.6 Possible Hexagons for Several Lattice Sizes

In Sects. 8.6.3–8.6.5 we have investigated possible occurrences of Lösch's hexagons for each of the three types V, M, and T, and enumerated all possible parameter values of n for the lattice size and k for the associated irreducible representations $(6; k, 0, +)$ and/or $(6; k, k, +)$. By compiling the obtained facts, we can capture, for each n, all hexagons that can potentially arise from critical points of multiplicity 6. The result is given in Table 8.4 for several lattice sizes n and is also incorporated in Table 8.2.

Example 8.3. For $n = 6$, we have three six-dimensional irreducible representations, i.e., $\mu = (6; 1, 0, +)$, $(6; 2, 0, +)$, and $(6; 1, 1, +)$. A bifurcating hexagon exists in the direction associated with $z = (1, 1, 1)$ for all these irreducible representations; see Proposition 8.5 for $\mu = (6; 1, 0, +)$, $(6; 2, 0, +)$, and Proposition 8.6 for $\mu = (6; 1, 1, +)$. We further note, by Remark 8.7 in Sect. 8.6.2, that $z = (1, 1, 1)$ is associated with $\boldsymbol{q}^{\mu}_{\mathrm{sum}} = \boldsymbol{q}^{\mu}_1 + \boldsymbol{q}^{\mu}_3 + \boldsymbol{q}^{\mu}_5$ in (8.48) for each μ. Three Lösch's hexagonal patterns expressed by $\boldsymbol{q}^{\mu}_{\mathrm{sum}}$ are depicted in Fig. 8.6a–c. These three patterns have different symmetries, $\mathrm{D}_6 = \Sigma(6, 0)$, $\Sigma(3, 0)$, and $\Sigma(4, 2)$, in agreement with Table 8.4 for $n = 6$. □

8.7 Bifurcation Point of Multiplicity 12

Hexagons branching from critical points of multiplicity 12 are investigated. In particular, among Lösch's ten smallest hexagons in Fig. 8.1, hexagons with $D = 7$, 13, 19, and 21 are predicted to branch from critical points of multiplicity 12, whereas the other six never appear (Propositions 8.22, 8.24, and 8.26). The hexagons with

Table 8.4 Lösch's hexagons arising from critical points of multiplicity 6 for several lattice sizes n

n	(k,ℓ) in $(6;k,\ell,+)$	$\hat{n}$	(α,β)	D	Type
3	(1, 0)	3	(3, 0)	9	V
4	(1, 0)	4	(4, 0)	16	V
	(1, 1)				
5	(1, 0), (2, 0)	5	(5, 0)	25	V
	(1, 1), (2, 2)				
6	(2, 0)	3	(3, 0)	9	V
	(1, 0)	6	(6, 0)	36	V
	(1, 1)	6	(4, 2)	12	M
7	(1, 0), (2, 0), (3, 0)	7	(7, 0)	49	V
	(1, 1), (2, 2), (3, 3)				
8	(2, 0)	4	(4, 0)	16	V
	(1, 0), (3, 0)	8	(8, 0)	64	V
	(2, 2)	4	(4, 0)	16	V
	(1, 1), (3, 3)	8	(8, 0)	64	V
9	(3, 0)	3	(3, 0)	9	V
	(1, 0), (2, 0), (4, 0)	9	(9, 0)	81	V
	(1, 1), (2, 2), (4, 4)	9	(6, 3)	27	M
12	(4, 0)	3	(3, 0)	9	V
	(3, 0)	4	(4, 0)	16	V
	(2, 0)	6	(6, 0)	36	V
	(1, 0), (5, 0)	12	(12, 0)	144	V
	(3, 3)	4	(4, 0)	16	V
	(2, 2)	6	(4, 2)	12	M
	(1, 1), (5, 5)	12	(8, 4)	48	M
13	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)	13	(13, 0)	169	V
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)				
21	(7, 0)	3	(3, 0)	9	V
	(3, 0), (6, 0), (9, 0)	7	(7, 0)	49	V
	(1, 0), (2, 0), (4, 0), (5, 0), (8, 0), (10, 0)	21	(21, 0)	21^2	V
	(3, 3), (6, 6), (9, 9)	7	(7, 0)	49	V
	(1, 1), (2, 2), (4, 4), (5, 5), (8, 8), (10, 10)	21	(14, 7)	147	M
39	(13, 0)	3	(3, 0)	9	V
	(3, 0), (6, 0), (9, 0), (12, 0), (15, 0), (18, 0)	13	(13, 0)	169	V
	(1, 0), (2, 0), (4, 0), (5, 0), (7, 0), (8, 0), (10, 0)	39	(39,0)	39^2	V
	(11, 0), (14, 0), (16, 0), (17, 0), (19, 0)				
	(3, 3), (6, 6), (9, 9), (12, 12), (15, 15), (18, 18)	13	(13, 0)	169	V
	(1, 1), (2, 2), (4, 4), (5, 5), (7, 7), (8, 8), (10, 10)	39	(26,13)	507	M
	(11, 11), (14, 14), (16, 16), (17, 17), (19, 19)				

Table 8.4 (continued)

n	(k, ℓ) in $(6; k, \ell, +)$	$\hat{n}$	(α, β)	D	Type
42	$(14, 0)$	3	$(3, 0)$	9	V
	$(7, 0)$	6	$(6, 0)$	36	V
	$(6, 0), (12, 0), (18, 0)$	7	$(7, 0)$	49	V
	$(3, 0), (9, 0), (15, 0)$	14	$(14, 0)$	196	V
	$(2, 0), (4, 0), (8, 0), (10, 0), (16, 0), (20, 0)$	21	$(21,0)$	21^2	V
	$(1, 0), (5, 0), (11, 0), (13, 0), (17, 0), (19, 0)$	42	$(42,0)$	42^2	V
	$(6, 6), (12, 12), (18, 18)$	7	$(7, 0)$	49	V
	$(3, 3), (9, 9), (15, 15)$	14	$(14, 0)$	196	V
	$(7, 7)$	6	$(4, 2)$	12	M
	$(2, 2), (4, 4), (8, 8), (10, 10), (16, 16), (20, 20)$	21	$(14, 7)$	147	M
	$(1, 1), (5, 5), (11, 11), (13, 13), (17, 17), (19, 19)$	42	$(28, 14)$	588	M

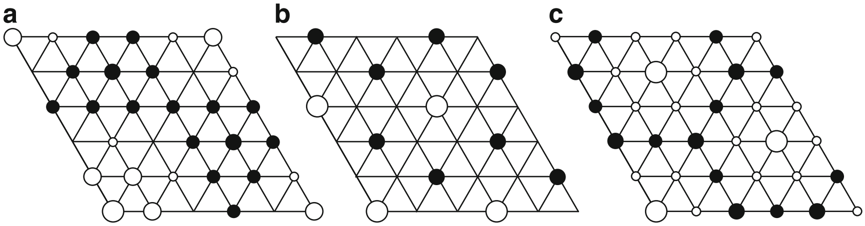

Fig. 8.6 Patterns on the 6×6 hexagonal lattice expressed by the column vectors of Q^{μ} ($\mu = (6; 1, 0, +), (6; 2, 0, +), (6; 1, 1, +)$). A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_{\text{sum}}^{(6;1,0;+)}$ ($\mathrm{D}_6 = \Sigma(6, 0) = \langle r, s \rangle$). (**b**) $\boldsymbol{q}_{\text{sum}}^{(6;2,0;+)}$ ($\Sigma(3, 0) = \langle r, s, p_1^3, p_2^3 \rangle$). (**c**) $\boldsymbol{q}_{\text{sum}}^{(6;1,1;+)}$ ($\Sigma(4, 2) = \langle r, s, p_1^4 p_2^2, p_1^{-2} p_2^2 \rangle$)

$D = 7$, 13, 19, and 21 are of type T and are tilted ($\varphi \neq 0, \pi/6$), whereas the other hexagons obtained in Sects. 8.4–8.6 are not. The emergence of such tilted hexagons is the most phenomenal finding of this book. In addition, larger hexagons of type V and M also branch. This theoretical prediction is confirmed in the computational result in Sect. 4.5.3 for a hexagon of type T with $D = 7$.

8.7.1 Representation in Complex Variables

As shown by Table 8.3 in Sect. 8.3.2, a critical point of multiplicity 12 is associated with one of the 12-dimensional irreducible representations $(12; k, \ell)$ with

$$1 \leq \ell \leq k - 1, \quad 2k + \ell \leq n - 1, \tag{8.56}$$

where $n \geq 6$.

Recall from (6.37)–(6.38) that the irreducible representation $(12; k, \ell)$ is given by

$$T^{(12;k,\ell)}(r) = \left[\begin{array}{ccc|ccc} & & S & & & \\ S & & & & & \\ & S & & & & \\ \hline & & & S & & \\ & & & & S & \\ & & & S & & \end{array}\right], \qquad T^{(12;k,\ell)}(s) = \left[\begin{array}{ccc|ccc} & & & I & & \\ & & & & I & \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \\ & & I & & & \end{array}\right], \quad (8.57)$$

$$T^{(12;k,\ell)}(p_1) = \left[\begin{array}{ccc|ccc} R^k & & & & & \\ & R^\ell & & & & \\ & & R^{-k-\ell} & & & \\ \hline & & & R^k & & \\ & & & & R^\ell & \\ & & & & & R^{-k-\ell} \end{array}\right],$$

$$T^{(12;k,\ell)}(p_2) = \left[\begin{array}{ccc|ccc} R^\ell & & & & & \\ & R^{-k-\ell} & & & & \\ & & R^k & & & \\ \hline & & & R^{-k-\ell} & & \\ & & & & R^k & \\ & & & & & R^\ell \end{array}\right] \qquad (8.58)$$

with

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

The action given above on 12-dimensional vectors $(w_1, \ldots, w_{12})$ can be expressed for complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \ldots, 6)$ as

$$r: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_5 \\ \bar{z}_6 \\ \bar{z}_4 \end{bmatrix}, \qquad s: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}, \qquad (8.59)$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^\ell z_2 \\ \omega^{-k-\ell} z_3 \\ \omega^k z_4 \\ \omega^\ell z_5 \\ \omega^{-k-\ell} z_6 \end{bmatrix}, \qquad p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ \omega^{-k-\ell} z_2 \\ \omega^k z_3 \\ \omega^{-k-\ell} z_4 \\ \omega^k z_5 \\ \omega^\ell z_6 \end{bmatrix}, \tag{8.60}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$.

8.7.2 Outline of Analysis

The major ingredients of our analysis for critical points of multiplicity 12 associated with $(12; k, \ell)$ are previewed.

We denote the isotropy subgroup of $z = (z_1, \ldots, z_6)$ with respect to $(12; k, \ell)$ as

$$\Sigma^{(12;k,\ell)}(z) = \{g \in G \mid T^{(12;k,\ell)}(g) \cdot z = z\}, \tag{8.61}$$

where $T^{(12;k,\ell)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (8.59) and (8.60). It turns out (Sect. 8.7.3) that the isotropy subgroup of $z = (1, 1, 1, 0, 0, 0)$ plays a crucial role in our analysis and that

$$\Sigma^{(12;k,\ell)}((1, 1, 1, 0, 0, 0)) = \Sigma_0(\alpha, \beta) \tag{8.62}$$

for a uniquely determined (α, β) with $0 \leq \beta < \alpha \leq n$ (see Proposition 8.18 in Sect. 8.7.3). We denote this correspondence $(k, \ell) \mapsto (\alpha, \beta) = (\alpha(k, \ell, n), \beta(k, \ell, n))$ by

$$\Phi(k, \ell, n) = (\alpha, \beta). \tag{8.63}$$

In a sense, (k, ℓ) and (α, β) are dual to each other; (k, ℓ) prescribes the action of the translations p_1 and p_2, and (α, β) describes the symmetry preserved under this action.[5]

Whereas the concrete form of the correspondence Φ is discussed in detail in Sect. 8.7.9, the following proposition shows the most fundamental formulas connecting (k, ℓ) and (α, β). We use the notations[6]:

$$\hat{k} = \frac{k}{\gcd(k, \ell, n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k, \ell, n)}, \quad \hat{n} = \frac{n}{\gcd(k, \ell, n)}, \tag{8.64}$$

[5] In an analogy with physics we may compare (k, ℓ) to frequency and (α, β) to wave length.

[6] The notations here should not be confused with (8.43) used in Sect. 8.6.

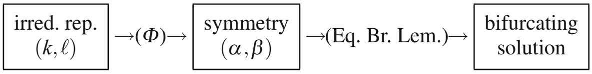

Fig. 8.7 Two stages of bifurcation analysis at a critical point of multiplicity 12

where $\gcd(k, \ell, n)$ means the greatest common divisor of k, ℓ, and n.

Proposition 8.12. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$.

(i)

$$\hat{n} = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}. \tag{8.65}$$

(ii)

$$\frac{\hat{n}}{\gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n})} = \gcd(\alpha, \beta). \tag{8.66}$$

Proof. The proof is given in Sect. 8.7.10; see Propositions 8.35(ii) and 8.36. It is mentioned here that the proof relies on the Smith normal form for integer matrices. □

Our analysis of bifurcation consists of two stages (see Fig. 8.7):

1. Connect the irreducible representation (k, ℓ) to the associated symmetry represented by (α, β) by obtaining the function $\Phi : (k, \ell) \mapsto (\alpha, \beta)$.
2. Connect the symmetry represented by (α, β) to the existence of bifurcating solutions on the basis of the equivariant branching lemma.

Proposition 8.13 below is a preview of a major result (Proposition 8.21 in Sect. 8.7.4) in a simplified form. For classification of bifurcation into several cases, we consider the condition

$$\textbf{GCD-div}: \quad (\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) \text{ is divisible by } \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}), \tag{8.67}$$

and the negation of this condition is referred to as $\overline{\textbf{GCD-div}}$. The set of integers that are multiples of 3 is denoted by $3\mathbb{Z}$ below.

Proposition 8.13. *For a critical point of multiplicity 12, let* $(12; k, \ell)$ *be the associated irreducible representation and* $(\alpha, \beta) = \Phi(k, \ell, n)$*. The bifurcation at this point is classified as follows.*

Case 1: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 3\mathbb{Z}$: *A bifurcating solution with symmetry* $\Sigma(\hat{n}, 0)$ *exists. This solution is of type V.*

Case 2: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z}$: *A bifurcating solution with symmetry* $\Sigma(2\hat{n}/3, \hat{n}/3)$ *exists. This solution is of type M.*

Case 3: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 3\mathbb{Z}$: *Bifurcating solutions with symmetries* $\Sigma(\hat{n},0)$, $\Sigma_0(\alpha,\beta)$, *and* $\Sigma_0(\beta,\alpha)$ *exist.*[7] *The first solution is of type V, and the other two solutions are of type T.*

Case 4: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k}-\hat{\ell},\hat{n}) \in 3\mathbb{Z}$: *Bifurcating solutions with symmetries* $\Sigma(2\hat{n}/3,\hat{n}/3)$, $\Sigma_0(\alpha,\beta)$, *and* $\Sigma_0(\beta,\alpha)$ *exist. The first solution is of type M, and the other two solutions are of type T.*

The classification criteria for the above four cases become more transparent when expressed in terms of (α,β) $(= \Phi(k,\ell,n))$ rather than (k,ℓ). The expressions in terms of (α,β) can be obtained from Proposition 8.14 below, where

$$\hat{\alpha} = \frac{\alpha}{\gcd(\alpha,\beta)}, \quad \hat{\beta} = \frac{\beta}{\gcd(\alpha,\beta)}, \tag{8.68}$$

$$\hat{D} = \hat{\alpha}^2 - \hat{\alpha}\hat{\beta} + \hat{\beta}^2 = \frac{D(\alpha,\beta)}{(\gcd(\alpha,\beta))^2}. \tag{8.69}$$

It is noted in passing that an alternative expression

$$\hat{D} = \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}) \tag{8.70}$$

results from (8.66) and (8.69).

Proposition 8.14. *Let* $(\alpha,\beta) = \Phi(k,\ell,n)$.

(i) $\gcd(\hat{k}-\hat{\ell},\hat{n}) \in 3\mathbb{Z} \iff \hat{D} \in 3\mathbb{Z}$.
(ii) **GCD-div** *in* (8.67) $\iff \beta = 0$ *or* $\alpha = 2\beta$.

Proof. The proof is given in Sect. 8.7.10; see Propositions 8.35(i) and 8.40. It is mentioned here that the proof of the equivalence in (ii) relies on the Smith normal form for integer matrices and the integer analogue of the Farkas lemma. □

Propositions 8.13 and 8.14 together yield Table 8.5 that summarizes the classification of bifurcation phenomena into the four cases in terms of both (k,ℓ) and (α,β).

An important observation here is that the classification into the four cases in Proposition 8.13, as well as in Table 8.5, can also be described in terms of the subgroup $\Sigma_0(\alpha,\beta)$. The following proposition shows how the conditions "$\beta = 0$ or $\alpha = 2\beta$" and "$\hat{D} \in 3\mathbb{Z}$" can be replaced by conditions for $\Sigma_0(\alpha,\beta)$.

Proposition 8.15.

(i) $\Sigma_0(\alpha,\beta) = \Sigma_0(\beta,\alpha) \iff \beta = 0$ *or* $\alpha = 2\beta$.

[7] To be precise, $\Sigma_0(\beta,\alpha)$ should be denoted as $\Sigma_0(\alpha',\beta')$ with (α',β') in (8.78), which lies in the parameter space of (8.7).

Table 8.5 Classification of bifurcation at a critical point associated with $(12; k, \ell)$ with $(\alpha, \beta) = \Phi(k, \ell, n)$

	$\gcd(\hat{k}-\hat{\ell}, \hat{n}) \notin 3\mathbb{Z}$ $\hat{D} \notin 3\mathbb{Z}$	$\gcd(\hat{k}-\hat{\ell}, \hat{n}) \in 3\mathbb{Z}$ $\hat{D} \in 3\mathbb{Z}$
GCD-div $\beta = 0$ or $\alpha = 2\beta$	Case 1: type V	Case 2: type M
$\overline{\textbf{GCD-div}}$ $\beta \neq 0$ and $\alpha \neq 2\beta$	Case 3: type V and type T	Case 4: type M and type T

Table 8.6 Bifurcation at a critical point associated with $(12; k, \ell)$ classified in terms of the subgroup $\Sigma_0(\alpha, \beta)$ for $(\alpha, \beta) = \Phi(k, \ell, n)$

	$\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha)$ $= \Sigma_0(\alpha'', 0)$	$\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha)$ $= \Sigma_0(2\beta'', \beta'')$
$\Sigma_0(\alpha,\beta) = \Sigma_0(\beta,\alpha)$	Case 1: type V	Case 2: type M
$\Sigma_0(\alpha,\beta) \neq \Sigma_0(\beta,\alpha)$	Case 3: type V and type T	Case 4: type M and type T

(ii)

$$\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha) = \begin{cases} \Sigma_0(\alpha'', 0) & \text{if } \hat{D} \notin 3\mathbb{Z}, \\ \Sigma_0(2\beta'', \beta'') & \text{if } \hat{D} \in 3\mathbb{Z} \end{cases} \tag{8.71}$$

with

$$\alpha'' = \frac{D(\alpha,\beta)}{\gcd(\alpha,\beta)}, \qquad \beta'' = \frac{D(\alpha,\beta)}{3\gcd(\alpha,\beta)}. \tag{8.72}$$

Proof. (i) This is obvious from the definition of $\Sigma_0(\alpha,\beta)$ in (8.10).
(ii) The proof is given in Proposition 8.32 in Sect. 8.7.10. □

By Proposition 8.15 above, we can rewrite Table 8.5 as Table 8.6. In particular, solutions of type T exist if and only if $\Sigma_0(\alpha,\beta)$ is asymmetric in the sense of $\Sigma_0(\alpha,\beta) \neq \Sigma_0(\beta,\alpha)$. Not only is this statement intuitively appealing, but it plays a crucial role in our technical arguments in Sect. 8.7.10.

Remark 8.10. Some comments are in order about (8.71) in each case corresponding to type V, type M, or type T.

- If $\beta = 0$, we have $\hat{D} = 1$ and $\alpha'' = D(\alpha, 0)/\gcd(\alpha, 0) = \alpha^2/\alpha = \alpha$.
- If $\alpha = 2\beta$, we have $\hat{D} = 3$ and $\beta'' = D(2\beta,\beta)/(3\gcd(2\beta,\beta)) = (3\beta^2)/(3\beta) = \beta$.
- For (α,β) with $1 \leq \beta < \alpha$ and $\alpha \neq 2\beta$, we have $\hat{D} = 7, 13, 19, 21$, and so on, some of which satisfy $\hat{D} \in 3\mathbb{Z}$, while others do not.

It should also be mentioned that the identity (8.71) is purely geometric in that it is valid for all (α, β) that may or may not be related to irreducible representation $(12; k, \ell)$. If (α, β) is associated with $(12; k, \ell)$, we have $\alpha'' = \hat{n}$ and $\beta'' = \hat{n}/3$ by (8.65) and (8.72), respectively. □

8.7.3 Isotropy Subgroups

To apply the method of analysis described in Sect. 8.3.2, we identify isotropy subgroups for $(12; k, \ell)$ related to Lösch's hexagons.

We denote the fixed-point subspace of Σ in terms of $z = (z_1, \dots, z_6)$ as

$$\mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{z \mid T^{(12;k,\ell)}(g) \cdot z = z \text{ for all } g \in \Sigma\}, \tag{8.73}$$

where $T^{(12;k,\ell)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (8.59) and (8.60). Also recall from (8.61) the notation $\Sigma^{(12;k,\ell)}(z)$ for the isotropy subgroup of z.

The symmetries of $\langle r \rangle$ and $\langle r, s \rangle$ are dealt with in Proposition 8.16 below, and the translational symmetry $p_1^a p_2^b$ is considered thereafter. Remark 8.14 below should be consulted with regard to the geometrical interpretation of the following discussion.

Proposition 8.16.

(i) $\mathrm{Fix}^{(12;k,\ell)}(\langle r \rangle) = \{c(1, 1, 1, 0, 0, 0) + c'(0, 0, 0, 1, 1, 1) \mid c, c' \in \mathbb{R}\}$.
(ii) $\mathrm{Fix}^{(12;k,\ell)}(\langle r, s \rangle) = \{c(1, 1, 1, 1, 1, 1) \mid c \in \mathbb{R}\}$.

Proof. (i) By (8.59), z is invariant to r if and only if $(\bar{z}_3, \bar{z}_1, \bar{z}_2, \bar{z}_5, \bar{z}_6, \bar{z}_4) = (z_1, z_2, z_3, z_4, z_5, z_6)$, which is equivalent to $z_1 = z_2 = z_3 \in \mathbb{R}$ and $z_4 = z_5 = z_6 \in \mathbb{R}$.
(ii) By (8.59), z is invariant to s if and only if $(z_4, z_5, z_6, z_1, z_2, z_3) = (z_1, z_2, z_3, z_4, z_5, z_6)$, which is equivalent to $z_1 = z_4$, $z_2 = z_5$, and $z_3 = z_6$. Hence z is invariant to both r and s if and only if $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 \in \mathbb{R}$. □

The above proposition implies that any isotropy subgroup Σ containing $\langle r \rangle$, which is of our interest, can be represented as $\Sigma = \Sigma^{(12;k,\ell)}(z)$ for some vector z of the form

$$z = c(1, 1, 1, 0, 0, 0) + c'(0, 0, 0, 1, 1, 1), \quad c, c' \in \mathbb{R}, \tag{8.74}$$

and that $\dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) \le 2$.

We now turn to the invariance to the translational symmetry $p_1^a p_2^b$.

Proposition 8.17.

(i) $p_1^a p_2^b \in \Sigma^{(12;k,\ell)}((1, 1, 1, 0, 0, 0))$ *if and only if*

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - (\hat{k} + \hat{\ell})b \equiv 0 \mod \hat{n}. \tag{8.75}$$

(ii) $p_1^a p_2^b \in \Sigma^{(12;k,\ell)}((0,0,0,1,1,1))$ *if and only if*

$$\hat{k}b + \hat{\ell}a \equiv 0, \quad \hat{\ell}b - (\hat{k} + \hat{\ell})a \equiv 0 \mod \hat{n}. \tag{8.76}$$

Proof. (i) By (8.60), the invariance of $z = (1,1,1,0,0,0)$ to $p_1^a p_2^b$ is expressed as

$$ka + \ell b \equiv 0, \quad \ell a - (k+\ell)b \equiv 0, \quad -(k+\ell)a + kb \equiv 0 \mod n,$$

which is equivalent to the first two conditions:

$$ka + \ell b \equiv 0, \quad \ell a - (k+\ell)b \equiv 0 \mod n.$$

This is further equivalent to (8.75) with the notations in (8.64).

(ii) By (8.60) the invariance of $z = (0,0,0,1,1,1)$ to $p_1^a p_2^b$ is expressed as

$$ka - (k+\ell)b \equiv 0, \quad \ell a + kb \equiv 0, \quad -(k+\ell)a + \ell b \equiv 0 \mod n,$$

which is equivalent to (8.76). □

The isotropy subgroup of $z = c(1,1,1,0,0,0) + c'(0,0,0,1,1,1)$ of the form of (8.74) is identified in the following two propositions: the case with $cc' = 0$ in Proposition 8.18 and the case with $cc' \neq 0$ in Proposition 8.19.

Proposition 8.18.

(i) *For each* (k,ℓ)*, we have*

$$\Sigma^{(12;k,\ell)}((1,1,1,0,0,0)) = \Sigma_0(\alpha,\beta) \tag{8.77}$$

for a uniquely determined (α,β) *with* $0 \le \beta < \alpha \le n$.

(ii) *For the* (α,β) *associated with* (k,ℓ) *as in* (i) *above, define*

$$(\alpha',\beta') = \begin{cases} (\alpha, \alpha-\beta) & \text{if } \alpha > \beta > 0, \\ (\alpha, 0) & \text{if } \alpha > \beta = 0. \end{cases} \tag{8.78}$$

Then we have

$$\Sigma^{(12;k,\ell)}((0,0,0,1,1,1)) = \Sigma_0(\alpha',\beta'). \tag{8.79}$$

Proof. (i) By (8.59), $\Sigma^{(12;k,\ell)}((1,1,1,0,0,0))$ contains r and not s. To investigate the translation symmetry, denote by $\mathscr{A}(k,\ell,n)$ the set of all (a,b) satisfying (8.75). That is,

$$\mathscr{A}(k,\ell,n) = \{(a,b) \in \mathbb{Z}^2 \mid \hat{k}a + \hat{\ell}b \equiv 0, \ \hat{\ell}a - (\hat{k}+\hat{\ell})b \equiv 0 \mod \hat{n}\}. \tag{8.80}$$

Then $\mathscr{A}(k,\ell,n)$ is closed under integer combination, i.e., if $(a_1,b_1), (a_2,b_2) \in \mathscr{A}(k,\ell,n)$, then $n_1(a_1,b_1)+n_2(a_2,b_2) \in \mathscr{A}(k,\ell,n)$ for any $n_1,n_2 \in \mathbb{Z}$. Next, if $(a,b) \in \mathscr{A}(k,\ell,n)$, then $(a',b') = (-b,a-b)$ also belongs to $\mathscr{A}(k,\ell,n)$ since

$$\begin{aligned}
\hat{k}a' + \hat{\ell}b' &= \hat{k}(-b) + \hat{\ell}(a-b) = \hat{\ell}a - (\hat{k}+\hat{\ell})b \equiv 0 \mod \hat{n},\\
\hat{\ell}a' - (\hat{k}+\hat{\ell})b' &= \hat{\ell}(-b) - (\hat{k}+\hat{\ell})(a-b)\\
&= -(\hat{k}a+\hat{\ell}b) - (\hat{\ell}a - (\hat{k}+\hat{\ell})b) \equiv 0 \mod \hat{n}.
\end{aligned}$$

The above argument shows that $\mathscr{A}(k,\ell,n)$ coincides with a set of the form

$$\mathscr{L}(\alpha,\beta)=\{(a,b) \in \mathbb{Z}^2 \mid (a,b)=n_1(\alpha,\beta)+n_2(-\beta,\alpha-\beta),\ n_1,n_2 \in \mathbb{Z}\} \tag{8.81}$$

for some appropriately chosen integers α and β. For such (α,β) we have

$$\Sigma^{(12;k,\ell)}((1,1,1,0,0,0)) = \langle r\rangle \ltimes \langle p_1^{\alpha}p_2^{\beta}, p_1^{-\beta}p_2^{\alpha-\beta}\rangle = \Sigma_0(\alpha,\beta).$$

To see the uniqueness of (α,β) we note the obvious correspondence between $\mathscr{L}(\alpha,\beta)$ and the hexagonal sublattice $\mathscr{H}(\alpha,\beta)$ in (5.4). By Proposition 5.1, $\mathscr{H}(\alpha,\beta)$ is uniquely parameterized by (α,β) with $0 \le \beta < \alpha$. Furthermore, we have $\alpha \le n$ as a consequence of the fact that $\mathscr{L}(\alpha,\beta)$ contains no point (a,b) of the form of $(a,b) = x(\alpha,\beta) + y(-\beta,\alpha-\beta)$ with $0 < x < 1$ and $0 < y < 1$, which lies in the interior of the parallelogram formed by its basis vectors (α,β) and $(-\beta,\alpha-\beta)$. To prove this by contradiction, suppose that $\alpha > n$ and consider the point $(a,b) = (\alpha-n,\beta)$. This point belongs to $\mathscr{L}(\alpha,\beta)$, satisfying the defining conditions in of $\mathscr{A}(k,\ell,n)$ in (8.80), whereas the corresponding (x,y) satisfies $0 < x < 1$ and $0 < y < 1$, which is a contradiction.

(ii) Since

$$(0,0,0,1,1,1) = T^{(12;k,\ell)}(s)\cdot(1,1,1,0,0,0),$$

it follows using (2.130), (8.77), (5.37), and (8.78) in this order that

$$\begin{aligned}
\Sigma^{(12;k,\ell)}((0,0,0,1,1,1)) &= s\cdot\Sigma^{(12;k,\ell)}((1,1,1,0,0,0))\cdot s^{-1}\\
&= s\cdot\Sigma_0(\alpha,\beta)\cdot s^{-1} = \Sigma_0(\beta,\alpha) = \Sigma_0(\alpha',\beta').
\end{aligned}$$

□

We denote the correspondence $(k,\ell) \mapsto (\alpha,\beta) = (\alpha(k,\ell,n),\beta(k,\ell,n))$ defined by (8.77) in Proposition 8.18 as

$$\Phi(k,\ell,n) = (\alpha,\beta). \tag{8.82}$$

Remark 8.11. A preliminary explanation is presented here about how the value of $(\alpha, \beta) = \Phi(k, \ell, n)$ can be determined, whereas a systematic method is given in Sect. 8.7.9.

The condition for $(a, b) \in \mathscr{A}(k, \ell, n)$ in (8.80) is equivalent to the existence of integers p and q satisfying

$$\begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} - \hat{\ell} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} p \\ q \end{bmatrix}. \tag{8.83}$$

Hence a pair of integers (a, b) belongs to $\mathscr{A}(k, \ell, n)$ if and only if

$$\begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} - \hat{\ell} \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{\hat{n}}{\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2} \begin{bmatrix} \hat{k} + \hat{\ell} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{8.84}$$

for some integers p and q. There are two cases to consider.

- If $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2)$ is an integer, a simpler method works. In this case, the right-hand side of (8.84) gives a pair of integers for any integers p and q. Therefore, we set $(p, q) = (1, 0)$ to obtain an integer vector

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\hat{n}}{\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2} \begin{bmatrix} \hat{k} + \hat{\ell} \\ \hat{\ell} \end{bmatrix} \tag{8.85}$$

  and note that the vectors $(a, b)^\top$ of integers satisfying (8.83) form a lattice spanned by $(\alpha, \beta)^\top$ and $(\beta, \beta - \alpha)^\top$. For $(k, \ell, n) = (3, 1, 26)$, for example, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (3, 1, 26)$ and $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2) = 26/13 = 2$, and hence (8.84) reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = 2 \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

  This shows $\Phi(3, 1, 26) = (\alpha, \beta) = (8, 2)$, corresponding to $(p, q) = (1, 0)$.
- If $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2)$ is not an integer, number-theoretic considerations are needed to determine $(\alpha, \beta) = \Phi(k, \ell, n)$. For $(k, \ell, n) = (18, 2, 42)$, for instance, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (9, 1, 21)$ and $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 91$, and (8.84) reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{21}{91} \begin{bmatrix} 10 & 1 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{3}{13} \begin{bmatrix} 10 & 1 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

  With some inspection we could arrive at $\Phi(18, 2, 42) = (\alpha, \beta) = (9, 3)$, which corresponds to $(p, q) = (4, -1)$. A systematic procedure based on the Smith normal form is given in Sect. 8.7.9 (Example 8.9). □

Remark 8.12. In the following arguments we shall make use of Propositions 8.12, 8.14, and 8.15 presented in Sect. 8.7.2. The readers may take these propositions for granted in the first reading, but those who are interested in mathematical issues are advised to have a look at their proofs given in Sect. 8.7.10. □

Proposition 8.19. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$*, and define*

$$\alpha'' = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad \beta'' = \frac{D(\alpha, \beta)}{3\gcd(\alpha, \beta)}. \tag{8.86}$$

For distinct nonzero real numbers c *and* c' $(c \neq c')$*, we have the following:*

(i)

$$\Sigma^{(12;k,\ell)}((c, c, c, c, c, c)) = \begin{cases} \Sigma(\alpha'', 0) & \text{if } \hat{D} \notin 3\mathbb{Z}, \\ \Sigma(2\beta'', \beta'') & \text{if } \hat{D} \in 3\mathbb{Z}, \end{cases}$$

where $\hat{D}$ *is defined in* (8.69) *and* $\hat{D} \in 3\mathbb{Z}$ *means that* $\hat{D}$ *is divisible by 3.*

(ii)

$$\Sigma^{(12;k,\ell)}((c, c, c, c', c', c')) = \begin{cases} \Sigma_0(\alpha'', 0) & \text{if } \hat{D} \notin 3\mathbb{Z}, \\ \Sigma_0(2\beta'', \beta'') & \text{if } \hat{D} \in 3\mathbb{Z}. \end{cases}$$

Proof. We first prove (ii). By (8.59), $\Sigma^{(12;k,\ell)}((c, c, c, c', c', c'))$ contains r and not s. We have

$$\begin{aligned} \Sigma^{(12;k,\ell)}((c, c, c, c', c', c')) &= \Sigma^{(12;k,\ell)}((1, 1, 1, 0, 0, 0)) \cap \Sigma^{(12;k,\ell)}((0, 0, 0, 1, 1, 1)) \\ &= \Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta'), \end{aligned}$$

where the second equality is due to Proposition 8.18. Then the claim follows from Proposition 8.15(ii).

Next we prove (i). By (8.59), $\Sigma^{(12;k,\ell)}((c, c, c, c, c, c))$ contains both r and s. We can proceed in a similar manner as above while including the element s. Therefore

$$\Sigma^{(12;k,\ell)}((c, c, c, c, c, c)) = \Sigma(\alpha, \beta) \cap \Sigma(\alpha', \beta'),$$

which implies the claim. □

In Proposition 8.20, we can present the isotropy subgroups containing $\langle r \rangle$, with a classification of the irreducible representations $(12; k, \ell)$ in terms of $(\alpha, \beta) = \Phi(k, \ell, n)$. See Fig. 8.8 for the classification.

Proposition 8.20. *For an irreducible representation* $(12; k, \ell)$*, let* $(\alpha, \beta) = \Phi(k, \ell, n)$*, and define* (α', β')*,* α'' *and* β'' *by* (8.78) *and* (8.86)*, respectively. Then the isotropy subgroups containing* $\langle r \rangle$ *are given by* Σ *listed below.*

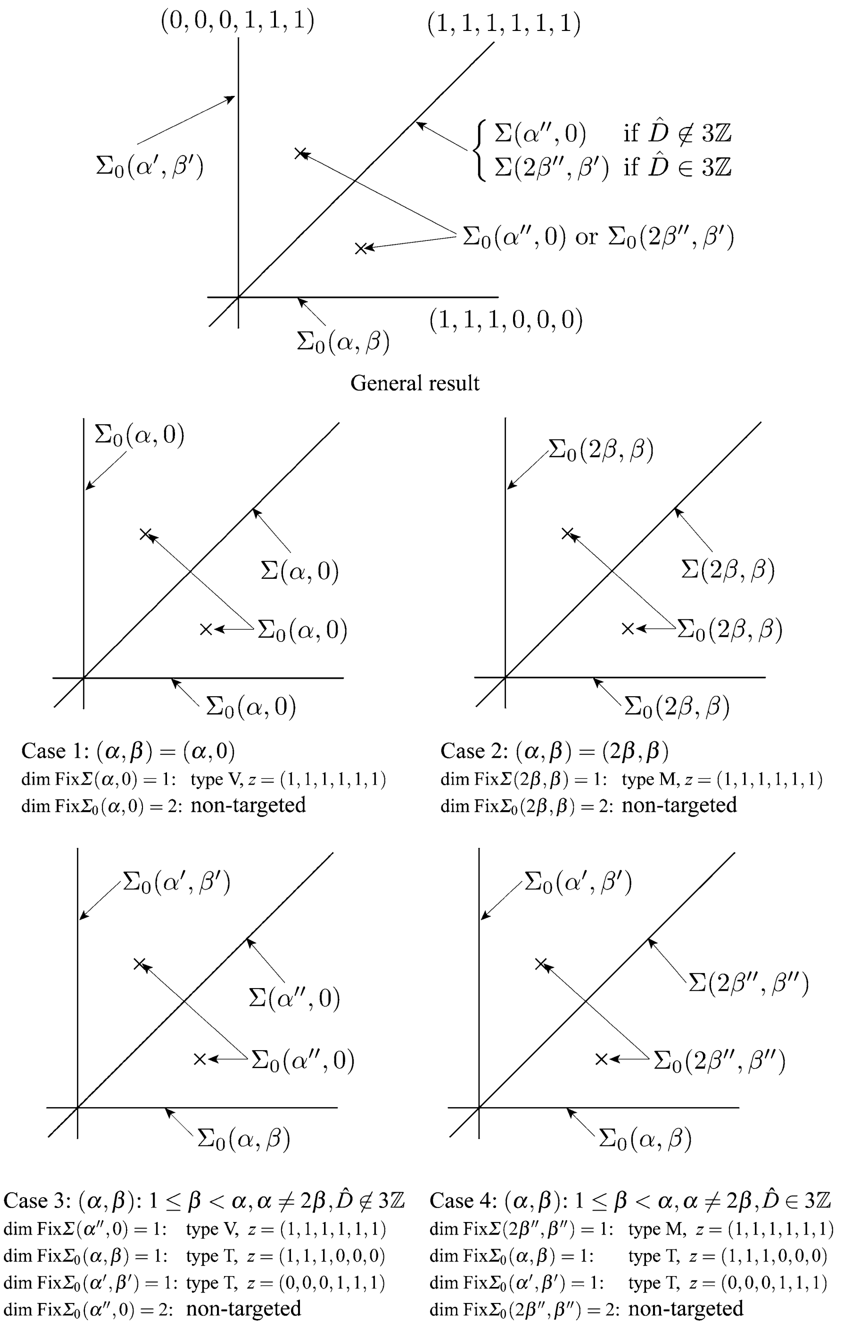

Fig. 8.8 Isotropy subgroups for $(12;k,\ell)$ with $(\alpha,\beta) = \Phi(k,\ell,n)$, (α',β') in (8.78), and (α'',β'') in (8.86)

Case 1: $(\alpha, \beta) = (\alpha, 0)$ *with* $1 \le \alpha \le n$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\alpha, 0) = \Sigma^{(12;k,\ell)}((1,1,1,1,1,1)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha, 0) = \Sigma^{(12;k,\ell)}((c,c,c,c',c',c')) \ (c \ne c', c \ne 0, c' \ne 0), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c',c',c') \mid c, c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 2: $(\alpha, \beta) = (2\beta, \beta)$ *with* $1 \le \beta \le n/3$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(2\beta, \beta) = \Sigma^{(12;k,\ell)}((1,1,1,1,1,1)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(2\beta, \beta) = \Sigma^{(12;k,\ell)}((c,c,c,c',c',c')) \ (c \ne c', c \ne 0, c' \ne 0), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c',c',c') \mid c, c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 3: (α, β) *with* $1 \le \beta < \alpha < n$, $\alpha \ne 2\beta$, *and* $\hat{D} \notin 3\mathbb{Z}$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\alpha'', 0) = \Sigma^{(12;k,\ell)}((1,1,1,1,1,1)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha, \beta) = \Sigma^{(12;k,\ell)}((1,1,1,0,0,0)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,0,0,0) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(c) } \Sigma = \Sigma_0(\alpha', \beta') = \Sigma^{(12;k,\ell)}((0,0,0,1,1,1)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(0,0,0,c',c',c') \mid c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(d) } \Sigma = \Sigma_0(\alpha'', 0) = \Sigma^{(12;k,\ell)}((c,c,c,c',c',c')) \ (c \ne c', c \ne 0, c' \ne 0), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c',c',c') \mid c, c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 4: (α, β) *with* $1 \le \beta < \alpha < n$, $\alpha \ne 2\beta$, *and* $\hat{D} \in 3\mathbb{Z}$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(2\beta'', \beta'') = \Sigma^{(12;k,\ell)}((1,1,1,1,1,1)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha, \beta) = \Sigma^{(12;k,\ell)}((1,1,1,0,0,0)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,0,0,0) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(c) } \Sigma = \Sigma_0(\alpha', \beta') = \Sigma^{(12;k,\ell)}((0,0,0,1,1,1)), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(0,0,0,c',c',c') \mid c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1. \\ \text{(d) } \Sigma = \Sigma_0(2\beta'', \beta'') = \Sigma^{(12;k,\ell)}((c,c,c,c',c',c')) \ (c \ne c', c \ne 0, c' \ne 0), \\ \quad \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = \{(c,c,c,c',c',c') \mid c, c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 2. \end{cases}$$

Proof. With an observation that $\Sigma_0(\alpha, \beta) \ne \Sigma_0(\alpha', \beta')$ in Cases 3 and 4, the above classification follows immediately from Propositions 8.18 and 8.19. □

Remark 8.13. In Case 1 of Proposition 8.20, we may have $\alpha = n$, in which case $\Sigma(\alpha, 0) = \Sigma(0, 0) = \langle r, s \rangle$ and $\Sigma_0(\alpha, 0) = \Sigma_0(0, 0) = \langle r \rangle$, and the translational symmetry is absent. □

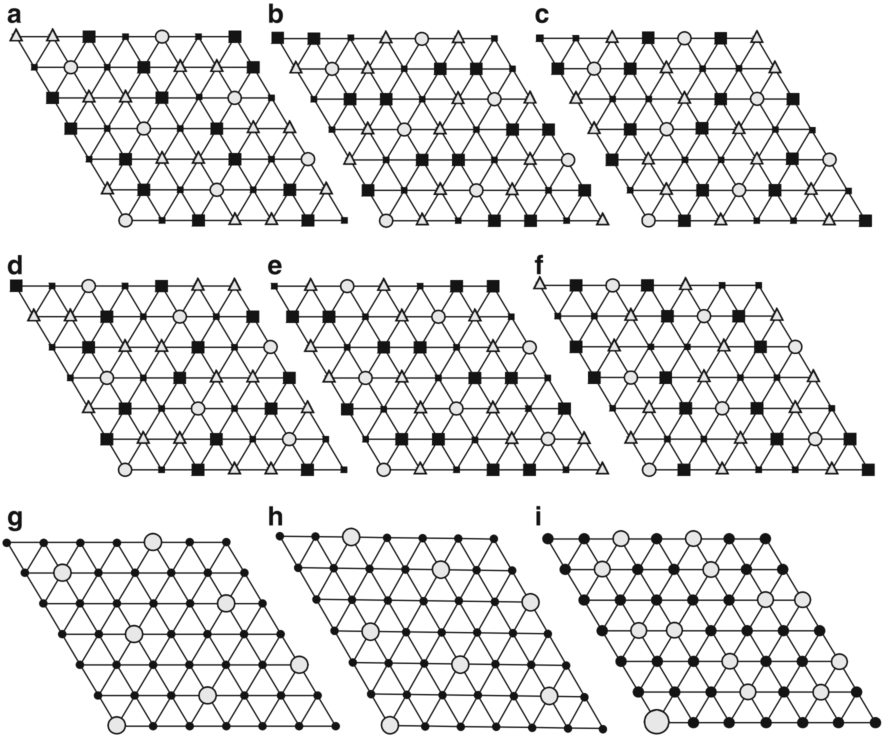

Fig. 8.9 Patterns on the 7×7 hexagonal lattice expressed by the column vectors of $Q^{(12;2,1)}$. A *white circle* denotes a positive component and a *black circle* denotes a negative component. (**a**) $\boldsymbol{q}_1$ $(\langle p_1^3 p_2 \rangle)$. (**b**) $\boldsymbol{q}_3$ $(\langle p_1^3 p_2 \rangle)$. (**c**) $\boldsymbol{q}_5$ $(\langle p_1^3 p_2 \rangle)$. (**d**) $\boldsymbol{q}_7$ $(\langle p_1^3 p_2^2 \rangle)$. (**e**) $\boldsymbol{q}_9$ $(\langle p_1^3 p_2^2 \rangle)$. (**f**) $\boldsymbol{q}_{11}$ $(\langle p_1^3 p_2^2 \rangle)$. (**g**) $\boldsymbol{q}_1+\boldsymbol{q}_3+\boldsymbol{q}_5$ $(\Sigma_0(3,1) = \langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle)$. (**h**) $\boldsymbol{q}_7+\boldsymbol{q}_9+\boldsymbol{q}_{11}$ $(\Sigma_0(3,2) = \langle r, p_1^3 p_2^2, p_1^{-2} p_2 \rangle)$. (**i**) $\sum_{i=1,3,5,7,9,11} \boldsymbol{q}_i$ $(\mathrm{D}_6 = \langle r, s \rangle)$

Remark 8.14. The isotropy subgroups in Proposition 8.20 can be understood quite naturally with reference to the column vectors of the matrix

$$Q^{(12;k,\ell)} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{12}]$$

given in (7.36). The spatial patterns for these vectors are depicted in Fig. 8.9, for example, for (12; 2, 1) with $n = 7$. Although the six vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_3$, $\boldsymbol{q}_5$, $\boldsymbol{q}_7$, $\boldsymbol{q}_9$, and $\boldsymbol{q}_{11}$ do not represent hexagonal patterns (Fig. 8.9a–f), the sum of these six vectors, which is associated with $z = (1,1,1,1,1,1)$ $(\boldsymbol{w} = (1,0,1,0,1,0,1,0,1,0,1,0)^{\top})$, represents Lösch's hexagon of type V with $D = 49$ (Fig. 8.9i). Moreover, the sum $\boldsymbol{q}_1 + \boldsymbol{q}_3 + \boldsymbol{q}_5$, which is associated with $z = (1,1,1,0,0,0)$, represents Lösch's hexagon of type T with $D = 7$ (Fig. 8.9g). On the other hand, the pattern in Fig. 8.9h, which is associated with $z = (0,0,0,1,1,1)$, represents another Lösch's hexagon of type T with $D = 7$. □

8.7.4 Existence of Bifurcating Solutions

A combination of Proposition 8.20 with the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) shows the existence of solutions with the targeted symmetry bifurcating from a critical point associated with $(12; k, \ell)$.

Bifurcating solutions can be classified in accordance with number-theoretic properties of (k, ℓ). To be specific, it depends on the following two properties:

$$(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) \text{ is divisible by } \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}), \tag{8.87}$$

$$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z}. \tag{8.88}$$

We refer to the condition (8.87) as **GCD-div** and its negation as $\overline{\textbf{GCD-div}}$. It should be mentioned that a simplified version of the following proposition has already been presented as Proposition 8.13 in Sect. 8.7.2. See also Table 8.5.

Proposition 8.21. *From a critical point associated with the irreducible representation* $(12; k, \ell)$*, solutions with the following symmetries emerge as bifurcating solutions, where* $(\alpha, \beta) = \Phi(k, \ell, n)$ *and* (α', β') *is defined in* (8.78)*. We have four cases.*

Case 1: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 3\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta) = (\hat{n}, 0)$*. A bifurcating solution with symmetry* $\Sigma(\hat{n}, 0)$*, which corresponds to* $z^{(1)} = c(1, 1, 1, 1, 1, 1)$*, exists. This solution is of type V.*

Case 2: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta) = (2\hat{n}/3, \hat{n}/3)$*. A bifurcating solution with symmetry* $\Sigma(2\hat{n}/3, \hat{n}/3)$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1, 1, 1)$*, exists. This solution is of type M.*

Case 3: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 3\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta)$ *with* $1 \le \beta < \alpha < n$*,* $\alpha \ne 2\beta$*, and* $\hat{D} \notin 3\mathbb{Z}$*. Bifurcating solutions with symmetries* $\Sigma(\hat{n}, 0)$*,* $\Sigma_0(\alpha, \beta)$*, and* $\Sigma_0(\alpha', \beta')$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1, 1, 1)$*,* $z^{(2)} = c(1, 1, 1, 0, 0, 0)$*, and* $z^{(3)} = c(0, 0, 0, 1, 1, 1)$*, respectively, exist. The first solution is of type V, and the other two solutions are of type T.*

Case 4: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta)$ *with* $1 \le \beta < \alpha < n$*,* $\alpha \ne 2\beta$*, and* $\hat{D} \in 3\mathbb{Z}$*. Bifurcating solutions with symmetries* $\Sigma(2\hat{n}/3, \hat{n}/3)$*,* $\Sigma_0(\alpha, \beta)$*, and* $\Sigma_0(\alpha', \beta')$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1, 1, 1)$*,* $z^{(2)} = c(1, 1, 1, 0, 0, 0)$*, and* $z^{(3)} = c(0, 0, 0, 1, 1, 1)$*, respectively, exist. The first solution is of type M, and the other two solutions are of type T.*

Proof. By Proposition 8.14, as well as Remark 8.10 in Sect. 8.7.2, the above four cases correspond to those in Proposition 8.20. In all cases, the relevant subgroup Σ is an isotropy subgroup with $\dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 1$ by Proposition 8.20. Then the equivariant branching lemma (Theorem 2.1 in Sect. 2.4.5) guarantees the existence of a bifurcating solution with symmetry Σ; see Remark 8.4 in Sect. 8.3.2. □

Remark 8.15. The subgroup $\Sigma = \Sigma_0(\alpha, 0)$, $\Sigma_0(2\beta, \beta)$, $\Sigma_0(\hat{n}, 0)$ or $\Sigma_0(2\hat{n}/3, \hat{n}/3)$ appearing in Proposition 8.20 is an isotropy subgroup with $\dim \mathrm{Fix}^{(12;k,\ell)}(\Sigma) = 2$, for which the equivariant branching lemma is not effective. It is emphasized that Proposition 8.21 does not assert the nonexistence of solutions of these symmetries. Nonetheless, we do not have to deal with these subgroups since none of these symmetries corresponds to Lösch's hexagons (see (8.10)). □

8.7.5 Hexagons of Type V

Lösch's larger hexagons of type V (with $D \geq 36$) are predicted to branch from critical points of multiplicity 12, whereas smaller hexagons of type V with $D = 4$, 9, 16, and 25 do not exist. Recall that a hexagonal distribution of type V is characterized by the symmetry of $\Sigma(\alpha, 0)$ with $2 \leq \alpha \leq n$ (see (8.10)) and that $D(\alpha, 0) = \alpha^2$.

The following propositions show such nonexistence and existence of hexagons of type V.

Proposition 8.22. *Lösch's hexagons of type V with $D = 4$, 9, 16, or 25 do not arise as bifurcating solutions from critical points of multiplicity 12 for any n.*

Proof. The proof is given at the end of the proof of Proposition 8.23. □

Proposition 8.23. *Lösch's hexagons of type V with the symmetry of $\Sigma(\alpha, 0)$ ($6 \leq \alpha \leq n$) arise as bifurcating solutions from critical points of multiplicity 12 for specific values of n and irreducible representations given by*

$$\begin{array}{cccc} (\alpha,\beta) & D & n & (k,\ell) \text{ in } (12;k,\ell) \\ \hline (\alpha,0) & \alpha^2 & \alpha m & ((p+q)m, qm) \\ \hline \end{array} \tag{8.89}$$

with $m \geq 1$ and

$$p \geq 1, \quad q \geq 1, \quad 2p+3q \leq \alpha - 1, \quad \gcd(p,q,\alpha) = 1, \quad \gcd(p,\alpha) \notin 3\mathbb{Z}. \tag{8.90}$$

Proof. Type V occurs in Case 1 and Case 3 in Proposition 8.21, characterized by the condition of $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 3\mathbb{Z}$. Put $\hat{k} = p + q$ and $\hat{\ell} = q$ for some $p, q \in \mathbb{Z}$ and note $\hat{n} = \alpha$. Since $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(p, \alpha)$, the condition $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 3\mathbb{Z}$ holds if and only if $\gcd(p, \alpha) \notin 3\mathbb{Z}$. We have $(k, \ell, n) = ((p + q)m, qm, \alpha m)$ for $m = \gcd(k, \ell, n)$. Here we must have

$$1 = \gcd(\hat{k}, \hat{\ell}, \hat{n}) = \gcd(p + q, q, \alpha) = \gcd(p, q, \alpha).$$

The inequality constraint in (8.56) is translated as

$$1 \leq \ell \leq k - 1, \quad 2k + \ell \leq n - 1 \iff p \geq 1, \quad q \geq 1, \quad 2p + 3q \leq \alpha - 1.$$

Table 8.7 Correspondence of irreducible representation $(12;k,\ell)$ to (α,β) for hexagons of type V

(α,β)	D	n	(k,ℓ) in $(12;k,\ell)$
(6,0)	36	$6m$	$(2m,m)$
(7,0)	49	$7m$	$(2m,m)^*$
(8,0)	64	$8m$	$(2m,m)$, $(3m,m)$
(12,0)	144	$12m$	$(2m,m)$, $(3m,m)$, $(3m,2m)$, $(4m,3m)$, $(5m,m)$
(13,0)	169	$13m$	$(2m,m)$, $(3m,m)^*$, $(3m,2m)$, $(4m,m)$, $(4m,2m)$, $(4m,3m)$, $(5m,m)$, $(5m,2m)^*$
(14,0)	196	$14m$	$(2m,m)^*$, $(3m,m)$, $(3m,2m)$, $(4m,m)^*$, $(4m,3m)$, $(5m,m)$, $(5m,2m)$, $(5m,3m)^*$, $(6m,m)$
(15,0)	225	$15m$	$(2m,m)$, $(3m,m)$, $(3m,2m)$, $(4m,2m)$, $(4m,3m)$, $(5m,m)$, $(5m,3m)$, $(5m,4m)$, $(6m,m)$, $(6m,2m)$
(18,0)	324	$18m$	$(2m,m)$, $(3m,m)$, $(3m,2m)$, $(4m,3m)$, $(5m,m)$, $(5m,3m)$, $(5m,4m)$, $(6m,m)$, $(6m,5m)$, $(7m,2m)$, $(7m,3m)$, $(8m,m)$
(21,0)	441	$21m$	$(2m,m)^*$, $(3m,m)$, $(3m,2m)$, $(4m,2m)^*$, $(4m,3m)$, $(5m,m)$, $(5m,3m)^*$, $(5m,4m)$, $(6m,m)$, $(6m,2m)$, $(6m,4m)$, $(6m,5m)^*$, $(7m,2m)$, $(7m,3m)$, $(7m,5m)$, $(7m,6m)$, $(8m,m)$, $(8m,3m)$, $(8m,4m)^*$, $(9m,m)^*$, $(9m,2m)$

$m = 1,2,\ldots$; $(\cdot)^*$ indicates coexistence of type T (Case 3)

Proposition 8.23 is thus obtained.

To prove Proposition 8.22, we note that, for $\alpha = 2,3,4,5$, no (p,q) satisfies (8.90), which proves the nonexistence of the smaller Lösch's hexagons claimed in Proposition 8.22. □

Example 8.4. The parameter values of (8.89) in Proposition 8.23 give Table 8.7. Here, the asterisk $(\cdot)^*$ indicates coexistence of type T (see (8.93)), i.e., Case 3 of Proposition 8.21, whereas unmarked cases correspond to Case 1 of Proposition 8.21, where no solution of type T coexists. There are several cases that are excluded by $\gcd(p,\alpha) \notin 3\mathbb{Z}$ in (8.90), such as $(k,\ell,n) = (4m,m,12m)$, $(4m,m,15m)$, and $(5m,2m,15m)$, corresponding, respectively, to $(p,q,\alpha) = (3,1,12)$, $(3,1,15)$, and $(3,2,15)$ with $\gcd(p,\alpha) = 3$. □

Remark 8.16. In all cases in (8.89), the compatibility condition (8.12) is satisfied for $\Sigma(\alpha,0)$ as $n = m\alpha$ with $m = \gcd(k,\ell,n)$, since we have

$$\gcd(k,\ell,n) = ((p+q)m, qm, \alpha m) = m\gcd(p+q,q,\alpha) = m\gcd(p,q,\alpha) = m$$

by (8.89) and (8.90). □

8.7.6 Hexagons of Type M

Lösch's larger hexagons of type M (with $D \geq 48$) are predicted to branch from critical points of multiplicity 12, whereas smaller hexagons of type M with $D = 3$, 12, and 27 do not exist. Recall that a hexagonal distribution of type M is characterized by the symmetry of $\Sigma(2\beta, \beta)$ with $1 \leq \beta \leq n/3$ (see (8.10)) and that $D(2\beta, \beta) = 3\beta^2$.

The following propositions show such nonexistence and existence of hexagons of type M.

Proposition 8.24. *Lösch's hexagons of type M with $D = 3$, 12, or 27 do not arise as bifurcating solutions from critical points of multiplicity 12 for any n.*

Proof. The proof is given at the end of the proof of Proposition 8.25. □

Proposition 8.25. *Lösch's hexagons of type M with the symmetry of $\Sigma(2\beta, \beta)$ $(4 \leq \beta \leq n/3)$ arise as bifurcating solutions from critical points of multiplicity 12 for specific values of n and irreducible representations given by*

$$\begin{array}{cccc} (\alpha,\beta) & D & n & (k,\ell) \text{ in } (12;k,\ell) \\ \hline (2\beta,\beta) & 3\beta^2 & 3\beta m & ((3p+q)m, qm) \\ \hline \end{array} \tag{8.91}$$

where $m \geq 1$ and

$$p \geq 1, \quad q \geq 1, \quad 2p+q \leq \beta - 1, \quad \gcd(p,q,\beta) = 1, \quad q \notin 3\mathbb{Z}. \tag{8.92}$$

Proof. Type M occurs in Case 2 and Case 4 in Proposition 8.21, characterized by the condition of $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z}$. For $\hat{k} - \hat{\ell} \in 3\mathbb{Z}$ to be true, we can put $\hat{k} = 3p + q$ and $\hat{\ell} = q$ for some $p, q \in \mathbb{Z}$. Then $(k, \ell, n) = ((3p+q)m, qm, 3\beta m)$ for $m = \gcd(k, \ell, n)$. Since

$$1 = \gcd(\hat{k}, \hat{\ell}, \hat{n}) = \gcd(3p+q, q, 3\beta) = \gcd(3p, q, 3\beta),$$

we must have $q \notin 3\mathbb{Z}$ and $\gcd(p, q, \beta) = 1$. The inequality constraint in (8.56) is translated as

$$1 \leq \ell \leq k-1, \quad 2k + \ell \leq n-1 \iff p \geq 1, \quad q \geq 1, \quad 2p+q \leq \beta - 1.$$

Proposition 8.25 is thus proved.

Finally, for $\beta = 1, 2, 3$, no (p, q) satisfies (8.92), which proves the nonexistence of the smaller Lösch's hexagons claimed in Proposition 8.24. □

Example 8.5. The parameter values of (8.91) in Proposition 8.25 give the following:

(α,β)	D	n	(k,ℓ) in $(12;k,\ell)$
$(8,4)$	48	$12m$	$(4m,m)$
$(10,5)$	75	$15m$	$(4m,m),(5m,2m)$
$(12,6)$	108	$18m$	$(4m,m),(5m,2m),(7m,m)$
$(14,7)$	147	$21m$	$(4m,m)^*,(5m,2m),(7m,m),(7m,4m),(8m,2m)^*$

where $m \geq 1$. The asterisk $(\cdot)^*$ indicates the coexistence of type T (see (8.93)), i.e., Case 4 of Proposition 8.21. The other (unmarked) cases correspond to Case 2 of Proposition 8.21, where no solution of type T coexists. The coexistence of type T is a relatively rare event; it does not occur for $n = 24m, 27m, 30m, 33m$, and $36m$, but it recurs for $n = 39m$. $\square$

Remark 8.17. In all cases in (8.91), the compatibility condition (8.12) for $\Sigma(2\beta,\beta)$ is satisfied as $n = 3m\beta$ with $m = \gcd(k,\ell,n)$, since

$$\begin{aligned}\gcd(k,\ell,n) &= \gcd((3p+q)m, qm, 3\beta m)\\ &= m\gcd(3p+q,q,3\beta) = m\gcd(3p,q,3\beta) = m\end{aligned}$$

by (8.91) and (8.92). $\square$

8.7.7 Hexagons of Type T

Lösch's hexagons of type T are shown here to branch from critical points of multiplicity 12. Recall that a hexagonal distribution of type T is characterized by the symmetry of $\Sigma_0(\alpha,\beta)$ with $1 \leq \beta < \alpha < n$ and $\alpha \neq 2\beta$ (see (8.10)).

The following proposition is concerned with the four hexagons of type T with $D = 7$, 13, 19, and 21, among Lösch's ten smallest hexagons.

Proposition 8.26. *Lösch's hexagons of type T with $D = 7$, 13, 19, and 21 arise as bifurcating solutions from critical points of multiplicity 12 for specific values of n and irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(12;k,\ell)$	
			$z^{(2)} = c(1,1,1,0,0,0)$	$z^{(3)} = c(0,0,0,1,1,1)$
$(3,1)$	7	$7m$	$(2m,m)$	None
$(3,2)$			None	$(2m,m)$
$(4,1)$	13	$13m$	$(3m,m)$	$(5m,2m)$
$(4,3)$			$(5m,2m)$	$(3m,m)$
$(5,2)$	19	$19m$	$(3m,2m),(6m,4m)$	$(7m,m)$
$(5,3)$			$(7m,m)$	$(3m,2m),(6m,4m)$
$(5,1)$	21	$21m$	$(4m,m),(8m,2m)$	None
$(5,4)$			None	$(4m,m),(8m,2m)$

(8.93)

where $m \geq 1$ is an integer.

Proof. By Proposition 8.21 (Case 3 and Case 4), a bifurcating solution with symmetry $\Sigma_0(\alpha, \beta)$ exists for (k, ℓ) such that $\Phi(k, \ell, n) = (\alpha, \beta)$, where the bifurcating solution corresponds to $z = c(1, 1, 1, 0, 0, 0)$. For such (k, ℓ), another bifurcating solution exists, which corresponds to $z = c(0, 0, 0, 1, 1, 1)$ and is endowed with the symmetry $\Sigma_0(\alpha', \beta')$ for (α', β') given by (8.78). The list of parameters in (8.93) is obtained by searching for such (k, ℓ) in the range of (8.56) using the method given in Sect. 8.7.9, which was previewed in Remark 8.11 in Sect. 8.7.3. □

For hexagons of type T, in general, the above statement extends as follows.

Proposition 8.27. *Assume* $1 \leq \beta < \alpha < n$ *and* $\alpha \neq 2\beta$ *for* (α, β).

(i) *Lösch's hexagons of type T with the symmetry of* $\Sigma_0(\alpha, \beta)$ *arise as bifurcating solutions from critical points of multiplicity 12 associated with the irreducible representation* $(12; k, \ell)$ *such that* $\Phi(k, \ell, n) = (\alpha, \beta)$ *or* (α', β')*, where* (α', β') *is defined by* (8.78).
(ii) *Some* (k, ℓ, n) *exist such that* $\Phi(k, \ell, n) = (\alpha, \beta)$ *or* (α', β').

Proof. (i) The proof is the same as the proof of Proposition 8.26.
(ii) We can assume $\alpha > 2\beta$ by replacing (α, β) by (α', β') if necessary. Take $(k, \ell, n) = m(\hat{\alpha} - \hat{\beta}, \hat{\beta}, D(\alpha, \beta)/\gcd(\alpha, \beta))$, for instance. Then $m = \gcd(k, \ell, n)$ and $(\hat{k}, \hat{\ell}, \hat{n}) = (\hat{\alpha} - \hat{\beta}, \hat{\beta}, D(\alpha, \beta)/\gcd(\alpha, \beta))$, and therefore

$$\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = \hat{\alpha}^2 - \hat{\alpha}\hat{\beta} + \hat{\beta}^2 = \hat{n}/\gcd(\alpha, \beta).$$

This shows that the simpler method of computing $\Phi(k, \ell, n)$, described in Remark 8.11 in Sect. 8.7.3, is applicable. The right-hand side of (8.85) is calculated as

$$\frac{\hat{n}}{\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2} \begin{bmatrix} \hat{k} + \hat{\ell} \\ \hat{\ell} \end{bmatrix} = \gcd(\alpha, \beta) \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

which shows $\Phi(k, \ell, n) = (\alpha, \beta)$.

We also note that the chosen parameter (k, ℓ) lies in the range of (8.56). The inequality $1 \leq \ell \leq k - 1$ is immediate from $\beta \geq 1$ and $\alpha > 2\beta$, whereas $2k + \ell \leq n - 1$ is shown by

$$\begin{aligned} \frac{n - (2k + \ell)}{m} &= \gcd(\alpha, \beta)(\hat{\alpha}^2 - \hat{\alpha}\hat{\beta} + \hat{\beta}^2) - (2\hat{\alpha} - \hat{\beta}) \\ &\geq (\hat{\alpha}^2 - \hat{\alpha}\hat{\beta} + \hat{\beta}^2) - 2\hat{\alpha} + \hat{\beta} \\ &= \left(\hat{\beta} - \frac{\hat{\alpha}}{2}\right)^2 + \frac{1}{4}\hat{\alpha}\,(3\hat{\alpha} - 8) + \hat{\beta} > 0, \end{aligned}$$

where $\hat{\alpha} \geq 3$ is used in the last inequality. □

Hexagons of type T appear in Cases 3 and 4 in Proposition 8.21, and these two cases are characterized by a single condition

$$\overline{\textbf{GCD-div}}: \quad (\hat{k}-\hat{\ell})\,\mathrm{gcd}(\hat{k},\hat{\ell}) \text{ is not divisible by } \mathrm{gcd}(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2,\hat{n}). \tag{8.94}$$

This observation yields the following statement.[8]

Proposition 8.28. *A bifurcating solution of type T exists if and only if* $\overline{\textbf{GCD-div}}$ *holds.*

Remark 8.18. The compatibility condition (8.12) for $\Sigma_0(\alpha,\beta)$ is satisfied as

$$n = m\,\frac{D(\alpha,\beta)}{\mathrm{gcd}(\alpha,\beta)}$$

with $m = \mathrm{gcd}(k,\ell,n)$ by (8.65) with (8.64). □

8.7.8 Possible Hexagons for Several Lattice Sizes

In Sects. 8.7.5–8.7.7 we have investigated possible occurrences of Lösch's hexagons for each of the three types V, M, and T and have enumerated all possible combinations of lattice size n and irreducible representation $(12;k,\ell)$ that can potentially engender Lösch's hexagons. By compiling these results, we can capture, for each n, all hexagons that can potentially arise from critical points of multiplicity 12. The results are given in Tables 8.8 and 8.9 for several lattice sizes. The results are also incorporated in Table 8.2. Recall from Proposition 8.21 in Sect. 8.7.4 that bifurcating hexagons are associated with

$$z = \begin{cases} z^{(1)} = c(1,1,1,1,1,1) & \text{corresponding to type V or type M,}\\ z^{(2)} = c(1,1,1,0,0,0) & \text{corresponding to type T,}\\ z^{(3)} = c(0,0,0,1,1,1) & \text{corresponding to type T.} \end{cases}$$

No hexagon of type M or T exists for the lattice sizes of $n = 6$. For $n = 7$, hexagons of type T exist for the irreducible representation $(12;k,\ell) = (12;2,1)$ with $\Sigma_0(\alpha,\beta) = \Sigma_0(3,1)$ and $\Sigma_0(3,2)$. For a composite number $n = 42$ with several divisors, hexagons of various kinds exist. Subgroups of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ expressing hexagonal distributions satisfy the inclusion relations given below.

[8] See Remark 8.20 at the end of Sect. 8.7.10 for conditions equivalent to $\overline{\textbf{GCD-div}}$.

Table 8.8 Hexagons of types V, M, and T arising from critical points of multiplicity 12 for the $n \times n$ hexagonal lattices with $n = 6, 7, 13, 21$, and 39 ($\hat{D}$ is defined in (8.69))

n	(k, ℓ) in $(12; k, \ell)$	$\hat{n}$	z	(α, β)	D	$\hat{D}$	Type
6	$(2, 1)$	6	$z^{(1)}$	$(6, 0)$	36	1	V
7	$(2, 1)$	7	$z^{(1)}$	$(7, 0)$	49	1	V
			$z^{(2)}$	$(3, 1)$	7	7	T
			$z^{(3)}$	$(3, 2)$	7	7	T
13	$(3, 1)$	13	$z^{(1)}$	(13,0)	169	1	V
			$z^{(2)}$	(4,1)	13	13	T
			$z^{(3)}$	(4,3)	13	13	T
	$(5, 2)$	13	$z^{(1)}$	(13,0)	169	1	V
			$z^{(2)}$	(4,3)	13	13	T
			$z^{(3)}$	(4,1)	13	13	T
	$(2, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1)$	13	$z^{(1)}$	$(13, 0)$	169	1	V
21	$(6, 3)$	7	$z^{(1)}$	$(7, 0)$	49	1	V
			$z^{(2)}$	$(3, 1)$	7	7	T
			$z^{(3)}$	$(3, 2)$	7	7	T
	$(4, 1), (8, 2)$	21	$z^{(1)}$	$(14, 7)$	147	3	M
			$z^{(2)}$	$(5, 1)$	21	21	T
			$z^{(3)}$	$(5, 4)$	21	21	T
	$(2, 1), (4, 2), (8, 4), (9, 1)$	21	$z^{(1)}$	$(21, 0)$	441	1	V
			$z^{(2)}$	$(9, 3)$	63	7	T
			$z^{(3)}$	$(9, 6)$	63	7	T
	$(5, 3), (6, 5)$	21	$z^{(1)}$	$(21, 0)$	441	1	V
			$z^{(2)}$	$(9, 6)$	63	7	T
			$z^{(3)}$	$(9, 3)$	63	7	T
	$(5, 2), (7, 1), (7, 4)$	21	$z^{(1)}$	$(14, 7)$	147	3	M
	Other (k, ℓ)'s	21	$z^{(1)}$	$(21, 0)$	441	1	V
39	$(9, 3)$	13	$z^{(1)}$	$(13, 0)$	169	1	V
			$z^{(2)}$	$(4, 1)$	13	13	T
			$z^{(3)}$	$(4, 3)$	13	13	T
	$(15, 6)$	13	$z^{(1)}$	$(13, 0)$	169	1	V
			$z^{(2)}$	$(4, 3)$	13	13	T
			$z^{(3)}$	$(4, 1)$	13	13	T
	$(5, 2), (10, 4)$	39	$z^{(1)}$	$(26, 13)$	507	3	M
			$z^{(2)}$	$(7, 2)$	39	39	T
			$z^{(3)}$	$(7, 5)$	39	39	T
	$(11, 8), (16, 1)$	39	$z^{(1)}$	$(26, 13)$	507	3	M
			$z^{(2)}$	$(7, 5)$	39	39	T
			$z^{(3)}$	$(7, 2)$	39	39	T
	$(3, 1), (6, 2), (8, 7), (12, 4), (14, 9), (15, 5)$	39	$z^{(1)}$	$(39, 0)$	39^2	1	V
			$z^{(2)}$	$(12, 3)$	117	13	T
			$z^{(3)}$	$(12, 9)$	117	13	T

Table 8.8 (continued)

n	(k,ℓ) in $(12;k,\ell)$	$\hat{n}$	z	(α,β)	D	$\hat{D}$	Type
	(6, 5), (9, 1), (11, 7), (12, 10), (14, 3), (18, 2)	39	$z^{(1)}$	(39, 0)	39^2	1	V
			$z^{(2)}$	(12, 9)	117	13	T
			$z^{(3)}$	(12, 3)	117	13	T
	(6, 3), (9, 6), (12, 3), (12, 6), (12, 9), (15, 3)	13	$z^{(1)}$	(13, 0)	169	1	V
	(4, 1), (7, 1), (7, 4), (8, 2), (8, 5), (10, 1), (10, 7) (11, 2), (11, 5), (13, 1), (13, 4), (13, 7), (13, 10) (14, 2), (14, 5), (14, 8), (16, 4), (17, 2)	39	$z^{(1)}$	(26, 13)	507	3	M
	Other (k,ℓ)'s	39	$z^{(1)}$	(39, 0)	39^2	1	V

Example 8.6. For $n = 21$, possible hexagons are of type V, M, and T. Subgroups for hexagons of type T have inclusion relations

$$\left.\begin{array}{l} \Sigma_0(3,1) \supset \Sigma_0(5,4) \supset \Sigma_0(9,3) \\ \Sigma_0(3,2) \supset \Sigma_0(5,1) \supset \Sigma_0(9,6) \end{array}\right\} \supset \Sigma_0(21,0) = \langle r \rangle,$$

and satisfy

$$\begin{aligned} \Sigma_0(3,1) \cap \Sigma_0(3,2) &= \Sigma_0(7,0), \\ \Sigma_0(5,1) \cap \Sigma_0(5,4) &= \Sigma_0(14,7), \\ \Sigma_0(9,3) \cap \Sigma_0(9,6) &= \Sigma_0(21,0) = \langle r \rangle. \end{aligned}$$

In addition, subgroups for hexagons of type V and type M satisfy

$$\Sigma(1,0) \supset \left\{\begin{array}{l} \Sigma(2,1) \supset \Sigma(3,0) \\ \Sigma(7,0) \supset \Sigma(14,7) \end{array}\right\} \supset \Sigma(21,0) = \langle r,s \rangle.$$

□

Example 8.7. For $n = 39$, possible hexagons are of type V, M, and T. Subgroups for hexagons of type T have inclusion relations

$$\left.\begin{array}{l} \Sigma_0(4,1) \supset \Sigma_0(7,5) \supset \Sigma_0(12,3) \\ \Sigma_0(4,3) \supset \Sigma_0(7,2) \supset \Sigma_0(12,9) \end{array}\right\} \supset \Sigma_0(39,0) = \langle r \rangle.$$

and satisfy

$$\begin{aligned} \Sigma_0(4,1) \cap \Sigma_0(4,3) &= \Sigma_0(13,0), \\ \Sigma_0(7,5) \cap \Sigma_0(7,2) &= \Sigma_0(26,13), \\ \Sigma_0(12,3) \cap \Sigma_0(12,9) &= \Sigma_0(39,0) = \langle r \rangle. \end{aligned}$$

In addition, subgroups for hexagons of type V and type M satisfy

Table 8.9 Hexagons of types V, M, and T arising from critical points of multiplicity 12 for the $n \times n$ hexagonal lattice with $n = 42$ ($\hat{D}$ is defined in (8.69))

n	(k, ℓ) in $(12; k, \ell)$	$\hat{n}$	z	(α, β)	D	$\hat{D}$	Type
42	(12, 6)	7	$z^{(1)}$	(7, 0)	49	1	V
			$z^{(2)}$	(3, 1)	7	7	T
			$z^{(3)}$	(3, 2)	7	7	T
	(8, 2), (16, 4)	21	$z^{(1)}$	(14, 7)	147	3	M
			$z^{(2)}$	(5, 1)	21	21	T
			$z^{(3)}$	(5, 4)	21	21	T
	(6, 3)	14	$z^{(1)}$	(14, 0)	196	1	V
			$z^{(2)}$	(6, 2)	28	7	T
			$z^{(3)}$	(6, 4)	28	7	T
	(12, 3), (15, 9)	14	$z^{(1)}$	(14, 0)	196	1	V
			$z^{(2)}$	(6, 4)	28	7	T
			$z^{(3)}$	(6, 2)	28	7	T
	(4, 2), (8, 4), (16, 8), (18, 2)	21	$z^{(1)}$	(21, 0)	441	1	V
			$z^{(2)}$	(9, 3)	63	7	T
			$z^{(3)}$	(9, 6)	63	7	T
	(10, 6), (12, 10)	21	$z^{(1)}$	(21, 0)	441	1	V
			$z^{(2)}$	(9, 6)	63	7	T
			$z^{(3)}$	(9, 3)	63	7	T
	(4, 1), (11, 8)	42	$z^{(1)}$	(28, 14)	588	3	M
			$z^{(2)}$	(10, 2)	84	21	T
			$z^{(3)}$	(10, 8)	84	21	T
	(11, 2), (13, 10), (16, 1), (17, 5)	42	$z^{(1)}$	(28, 14)	588	3	M
			$z^{(2)}$	(10, 8)	84	21	T
			$z^{(3)}$	(10, 2)	84	21	T
	(2, 1), (9, 1), (9, 8), (10, 5), (11, 9), (13, 3)	42	$z^{(1)}$	(42, 0)	42^2	1	V
	(15, 4), (15, 11)		$z^{(2)}$	(18, 6)	252	7	T
			$z^{(3)}$	(18, 12)	252	7	T
	(5, 3), (6, 5), (9, 4), (11, 1), (13, 5), (13, 12)	42	$z^{(1)}$	(42, 0)	42^2	1	V
	(15, 2), (17, 6), (18, 1), (19, 3)		$z^{(2)}$	(18, 12)	252	7	T
			$z^{(3)}$	(18, 6)	252	7	T
	(14, 7)	6	$z^{(1)}$	(6, 0)	36	1	V
	(9, 3), (9, 6), (12, 9), (15, 3), (15, 6), (18, 3)	14	$z^{(1)}$	(14, 0)	196	1	V
	(6, 2), (6, 4), (8, 6), (10, 2), (10, 8), (12, 2)	21	$z^{(1)}$	(21, 0)	441	1	V
	(12, 4), (12, 8), (14, 4), (14, 6), (14, 10)						
	(14, 12), (16, 2), (16, 6), (18, 4)						
	(10, 4), (14, 2), (14, 8)	21	$z^{(1)}$	(14, 7)	147	3	M
	(5, 2), (7, 1), (7, 4), (8, 5), (10, 1), (10, 7)	42	$z^{(1)}$	(28, 14)	588	3	M
	(11, 5), (13, 1), (13, 4), (13, 7), (14, 5)						
	(14, 11), (16, 7), (17, 2), (19, 1)						
	Other (k, ℓ)'s	42	$z^{(1)}$	(42, 0)	42^2	1	V

$$\Sigma(1,0) \supset \left\{ \begin{array}{l} \Sigma(2,1) \;\; \supset \Sigma(3,0) \\ \Sigma(13,0) \supset \Sigma(26,13) \end{array} \right\} \supset \Sigma(39,0) = \langle r, s \rangle. \qquad \square$$

Example 8.8. For $n = 42$, possible hexagons are of type V, M, and T. Subgroups for hexagons of type T have inclusion relations

$$\left. \begin{array}{l} \Sigma_0(3,1) \supset \Sigma_0(5,4) \supset \left\{ \begin{array}{l} \Sigma_0(9,3) \\ \Sigma_0(10,8) \end{array} \right\} \supset \Sigma_0(18,6) \\ \Sigma_0(3,2) \supset \Sigma_0(5,1) \supset \left\{ \begin{array}{l} \Sigma_0(9,6) \\ \Sigma_0(10,2) \end{array} \right\} \supset \Sigma_0(18,12) \end{array} \right\} \supset \Sigma_0(42,0) = \langle r \rangle,$$

$$\left. \begin{array}{l} \Sigma_0(3,1) \supset \Sigma_0(6,2) \supset \Sigma_0(10,8) \supset \Sigma_0(18,6) \\ \Sigma_0(3,2) \supset \Sigma_0(6,4) \supset \Sigma_0(10,2) \supset \Sigma_0(18,12) \end{array} \right\} \supset \Sigma_0(42,0) = \langle r \rangle,$$

and satisfy

$$\begin{array}{ll} \Sigma_0(3,1) \cap \Sigma_0(3,2) = \Sigma_0(7,0), & \Sigma_0(9,3) \cap \Sigma_0(9,6) = \Sigma_0(21,0), \\ \Sigma_0(5,1) \cap \Sigma_0(5,4) = \Sigma_0(14,7), & \Sigma_0(10,2) \cap \Sigma_0(10,8) = \Sigma_0(28,14), \\ \Sigma_0(6,2) \cap \Sigma_0(6,4) = \Sigma_0(14,0), & \Sigma_0(18,6) \cap \Sigma_0(18,12) = \Sigma_0(42,0) = \langle r \rangle. \end{array}$$

The inclusion relations among subgroups for hexagons of type V and type M are complicated and are omitted here. $\square$

8.7.9 Appendix: Construction of the Function Φ

A systematic construction procedure of the function Φ in (8.82) is given here.

Basic Facts About Integer Matrices

We present here some basic facts about integer matrices[9] that are used in the construction of the correspondence Φ and in the proofs in Sect. 8.7.10.

A square integer matrix U is called *unimodular* if its determinant is equal to ± 1; U is unimodular if and only if its inverse U^{-1} exists and is an integer matrix. For an integer matrix A, the kth *determinantal divisor*, denoted $d_k(A)$, is the greatest common divisor of all $k \times k$ minors (subdeterminants) of A. By convention we put $d_0(A) = 1$.

The first theorem states that every integer matrix can be brought to the *Smith normal form* by a bilateral unimodular transformation.

[9] See Newman, 1972 [8] and Schrijver, 1986 [9] for details about integer matrices.

Theorem 8.1. *Let A be an $m \times n$ integer matrix. There exist unimodular matrices U and V such that*

$$UAV = \left[\begin{array}{ccc|c} \alpha_1 & & 0 & \\ & \ddots & & 0_{r,n-r} \\ 0 & & \alpha_r & \\ \hline & 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right], \tag{8.95}$$

where $r = \operatorname{rank} A$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$ are positive integers with the divisibility property[10]*:*

$$\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r.$$

Such integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ are uniquely determined by A, and are expressed as

$$\alpha_k = \frac{d_k(A)}{d_{k-1}(A)}, \quad k = 1, \ldots, r,$$

in terms of the determinantal divisors $d_1(A), d_2(A), \ldots, d_r(A)$ of A.

The second theorem gives a solvability criterion for a system of linear equations in unknown integer vectors.

Theorem 8.2. *Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. The following two conditions* (a) *and* (b) *are equivalent.*

(a) *The system of equations $A\boldsymbol{x} = \boldsymbol{b}$ admits an integer solution $\boldsymbol{x}$.*
(b) *Two matrices A and $[A \mid \boldsymbol{b}]$ share the same determinantal divisors, i.e.,* $\operatorname{rank} A = \operatorname{rank} [A \mid \boldsymbol{b}]$ *and $d_k(A) = d_k([A \mid \boldsymbol{b}])$ for all k.*

As a corollary of Theorem 8.2 we can obtain the following facts.

Proposition 8.29. *Let $a_1, \ldots, a_n$ be integers.*

(i) $\gcd(a_1, \ldots, a_n) = 1$ *if and only if there exist some integers $x_1, \ldots, x_n$ such that $a_1x_1 + \cdots + a_nx_n = 1$.*
(ii) *An integer b is divisible by* $\gcd(a_1, \ldots, a_n)$ *if and only if there exist some integers $x_1, \ldots, x_n$ such that $a_1x_1 + \cdots + a_nx_n = b$.*

The third theorem is a kind of duality theorem, which is sometimes referred to as the *integer analogue of the Farkas lemma.*

Theorem 8.3. *Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. The following two conditions* (a) *and* (b) *are equivalent.*

[10] Notation "$a \mid b$" means that a divides b, that is, b is a multiple of a.

(a) *The system of equations* $A\boldsymbol{x} = \boldsymbol{b}$ *admits an integer solution* $\boldsymbol{x}$.
(b) *We have* "$\boldsymbol{y}^\top A \in \mathbb{Z}^n \implies \boldsymbol{y}^\top \boldsymbol{b} \in \mathbb{Z}$" *for any* m*-dimensional rational vector* $\boldsymbol{y}$.

Construction of Φ via the Smith Normal Form

The correspondence $\Phi : (k, \ell) \mapsto (\alpha, \beta)$ can be constructed with the aid of the Smith normal form. Recall notations

$$\hat{k} = \frac{k}{\gcd(k, \ell, n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k, \ell, n)}, \quad \hat{n} = \frac{n}{\gcd(k, \ell, n)}$$

in (8.64), for which

$$\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1. \tag{8.96}$$

By the definition of the correspondence Φ of (8.82) in Proposition 8.18, we have

$$\mathscr{A}(k, \ell, n) = \mathscr{L}(\alpha, \beta) \quad \text{for} \quad (\alpha, \beta) = \Phi(k, \ell, n), \tag{8.97}$$

where

$$\mathscr{A}(k, \ell, n) = \{(a, b) \in \mathbb{Z}^2 \mid \hat{k}a + \hat{\ell}b \equiv 0,\ \hat{\ell}a - (\hat{k} + \hat{\ell})b \equiv 0 \mod \hat{n}\}, \tag{8.98}$$

$$\mathscr{L}(\alpha, \beta) = \{(a, b) \in \mathbb{Z}^2 \mid (a, b) = n_1(\alpha, \beta) + n_2(-\beta, \alpha - \beta),\ n_1, n_2 \in \mathbb{Z}\}. \tag{8.99}$$

The condition in the definition of $\mathscr{A}(k, \ell, n)$ can be rewritten in a matrix form as

$$\begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} - \hat{\ell} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mod \hat{n}. \tag{8.100}$$

We define matrices K and A as

$$K = \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} - \hat{\ell} \end{bmatrix}, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix}, \tag{8.101}$$

which play the key role in our analysis. Note that

$$\mathscr{L}(\alpha, \beta) = \left\{(a, b) \mid \begin{bmatrix} a \\ b \end{bmatrix} = A \begin{bmatrix} n_1 \\ n_2 \end{bmatrix};\ n_1, n_2 \in \mathbb{Z}\right\} \tag{8.102}$$

by (8.99).

The condition for $\mathscr{A}(k, \ell, n)$ in (8.100) is equivalent to the existence of integers p and q such that

$$\left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{array}\right] \begin{bmatrix} a \\ b \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8.103}$$

Since the determinantal divisors d_1 and d_2 of this 2×4 coefficient matrix are

$$\begin{aligned} d_1 &= \gcd(\hat{k}, \hat{\ell}, \hat{k}+\hat{\ell}, \hat{n}) = \gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1, \\ d_2 &= \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{k}\hat{n}, \hat{\ell}\hat{n}, (\hat{k}+\hat{\ell})\hat{n}, \hat{n}^2) = \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}\gcd(\hat{k}, \hat{\ell}, \hat{n})) \\ &= \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}), \end{aligned}$$

the Smith normal form of that matrix is given (see Theorem 8.1) as

$$U \left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{array}\right] V = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array}\right], \tag{8.104}$$

where U and V are unimodular matrices and

$$\kappa = \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}). \tag{8.105}$$

The 4×4 matrix V for the Smith normal form in (8.104) affords an explicit representation of the correspondence Φ that is defined rather implicitly by the relationship in (8.97). As stated in the following proposition, the correspondence $(\alpha, \beta) = \Phi(k, \ell, n)$ is encoded in the upper-right block of a suitably chosen matrix V. Partition the matrix V into 2×2 submatrices as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

and recall the matrix A in (8.101) that is parameterized by (α, β).

Proposition 8.30. *We can take* V *such that* $V_{12} = A$ *for some* (α, β) *with* $\alpha > \beta \geq 0$. *Then* $\Phi(k, \ell, n) = (\alpha, \beta)$.

Proof. Putting

$$\boldsymbol{a} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = V^{-1} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix}$$

and using (8.104), we can rewrite (8.103) as

$$U[K \mid -\hat{n}I]V \cdot V^{-1} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{0}.$$

This shows that $\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{y}$ is free. Therefore, the solutions of (8.103) are given as

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix} = V \begin{bmatrix} \mathbf{0} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y} \in \mathbb{Z}^2.$$

This means, by (8.97), that

$$\mathscr{L}(\alpha, \beta) = \{\boldsymbol{a} = (a, b)^\top \mid \boldsymbol{a} = V_{12}\boldsymbol{y},\ \boldsymbol{y} \in \mathbb{Z}^2\}.$$

By comparing this with (8.102), we see that the column vectors of V_{12} and those of A are both basis vectors of the same lattice. As is well known, this implies that the matrices V_{12} and A are related as $V_{12} W = A$ for some unimodular matrix W. Therefore,

$$\tilde{V} = V \begin{bmatrix} I & O \\ O & W \end{bmatrix} = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix}$$

is also a valid choice for the Smith normal form (8.104), with the property that $\tilde{V}_{12} = A$. □

In what follows we assume $V_{12} = A$, i.e.,

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & A \\ V_{21} & V_{22} \end{bmatrix}. \tag{8.106}$$

Example 8.9. The construction of Φ above is illustrated for $(k, \ell, n) = (18, 2, 42)$, the second case treated in Remark 8.11 in Sect. 8.7.3. We have $(\hat{k}, \hat{\ell}, \hat{n}) = (9, 1, 21)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 91$, and $\kappa = \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}) = \gcd(91, 21) = 7$. The transformation to the Smith normal form in (8.104) is given as

$$\begin{bmatrix} 0 & 1 \\ 1 & -9 \end{bmatrix} \left[\begin{array}{cc|cc} 9 & 1 & -21 & 0 \\ 1 & -10 & 0 & -21 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 10 & 9 & -3 \\ 0 & 1 & 3 & 6 \\ \hline 0 & 4 & 4 & -1 \\ 0 & 0 & -1 & -3 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{array}\right].$$

This shows

$$\begin{bmatrix} 9 & -3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha - \beta \end{bmatrix}.$$

Therefore, $(\alpha, \beta) = (9, 3)$, i.e., $\Phi(18, 2, 42) = (9, 3)$. □

Remark 8.19. In Remark 8.11 in Sect. 8.7.3, we indicated a simpler construction of Φ that works when $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2)$ is an integer. This simpler construction can also be understood in the framework of the general method here. Let U and V_{11}

be some unimodular matrices that transform the matrix K in (8.101) to its Smith normal form: $UKV_{11} = \mathrm{diag}\,(1, \kappa)$. By choosing

$$V_{12} = \frac{\hat{n}}{\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2} \begin{bmatrix} \hat{k} + \hat{\ell} & -\hat{\ell} \\ \hat{\ell} & \hat{k} \end{bmatrix}, \quad V_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in (8.106), we obtain a unimodular matrix V since $|\det V| = |\det V_{11}| \cdot |\det V_{22}| = 1$. Then we have (8.104), and therefore $(\alpha, \beta) = \Phi(k, \ell, n)$ is obtained from the first column of V_{12}, i.e., $(\alpha, \beta) = m(\hat{k} + \hat{\ell}, \hat{\ell})$ with $m = \hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2)$. □

The use of the Smith normal form is demonstrated below when $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2)$ is not an integer, whereas when $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2)$ is an integer, the simpler method of construction in Remark 8.11 in Sect. 8.7.3 is used.

Examples

The following examples show the values of $\Phi(k, \ell, n)$ for several (k, ℓ, n), and demonstrate Proposition 8.15 and the four cases in Proposition 8.20 in Sect. 8.7.3.

The first example is a case with a solution of type V and without one of type T.

Example 8.10 (Case 1 of Proposition 8.20). For $(k, \ell, n) = (2m, m, 6m)$ with $m \geq 1$, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (2, 1, 6)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 7$, and $\kappa = \gcd(7, 6) = 1$. The transformation to the Smith normal form in (8.104) is given as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{cc|cc} 2 & 1 & -6 & 0 \\ 1 & -3 & 0 & -6 \end{array}\right] \left[\begin{array}{cc|cc} 3 & 1 & 6 & 0 \\ 1 & -2 & 0 & 6 \\ \hline 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -3 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

This shows $\mathscr{A}(2m, m, 6m) = \mathscr{L}(6, 0)$, i.e., $\Phi(2m, m, 6m) = (6, 0) = (\alpha, \beta)$. We have $\alpha = \hat{n} = 6$ and $(\alpha', \beta') = (6, 0)$ by (8.78). This is a case of $(\alpha, \beta) = (\alpha', \beta')$, and we have

$$\Sigma_0(\alpha, \beta) = \Sigma_0(\alpha', \beta') = \Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(6, 0).$$

When $m = 1$, $\Sigma_0(6, 0)$ reduces to $\{e\}$. We have $(\hat{\alpha}, \hat{\beta}) = (1, 0)$, $\hat{D} = 1 \notin 3\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(1, 6) = 1 \notin 3\mathbb{Z}$, and **GCD-div** since $(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) = (2 - 1) \gcd(2, 1) = 1$ is divisible by $\kappa = 1$. □

The second example is a case with a solution of type M and without one of type T.

Example 8.11 (Case 2 of Proposition 8.20). For $(k, \ell, n) = (4m, m, 12m)$ with $m \geq 1$, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (4, 1, 12)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 21$, and $\kappa = \gcd(21, 12) = 3$. The transformation to the Smith normal form in (8.104) is given as

$$\begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix} \left[\begin{array}{cc|cc} 4 & 1 & -12 & 0 \\ 1 & -5 & 0 & -12 \end{array}\right] \left[\begin{array}{cc|cc} 1 & -5 & 8 & -4 \\ 0 & -1 & 4 & 4 \\ \hline 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & -2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right].$$

This shows $\mathscr{A}(4m, m, 12m) = \mathscr{L}(8, 4)$, i.e., $\Phi(4m, m, 12m) = (8, 4) = (\alpha, \beta)$. We have $(\alpha', \beta') = (8, 4)$ by (8.78) (a case of $(\alpha, \beta) = (\alpha', \beta')$) and

$$\Sigma_0(\alpha, \beta) = \Sigma_0(\alpha', \beta') = \Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(8, 4).$$

Note that we have $4 = \beta \le n/3 = 4m$, $(\hat{\alpha}, \hat{\beta}) = (2, 1)$, $\hat{D} = 3 \in 3\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(3, 12) = 3 \in 3\mathbb{Z}$, and **GCD-div** since $(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) = (4 - 1) \gcd(4, 1) = 3$ is divisible by $\kappa = 3$. □

The following three examples are cases where solutions of type T exist together with another solution of type V without translation symmetry.

Example 8.12 (Case 3 of Proposition 8.20 with $\hat{n} = n$). For $(k, \ell, n) = (2, 1, 7)$, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (2, 1, 7)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 7$, and $\kappa = \gcd(7, 7) = 7$. Since $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2) = 1$ is an integer, the simpler method is applicable and (8.84) reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{7}{7} \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

which yields $\mathscr{A}(2, 1, 7) = \mathscr{L}(3, 1)$, i.e., $\Phi(2, 1, 7) = (3, 1) = (\alpha, \beta)$, corresponding to $(p, q) = (1, 0)$. We have $(\alpha', \beta') = (3, 2)$ by (8.78), and

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(3, 1) \cap \Sigma_0(3, 2) = \Sigma_0(7, 0) = \Sigma_0(0, 0) = \{e\}$$

by Proposition 8.15. We have $(\hat{\alpha}, \hat{\beta}) = (3, 1)$, $\hat{D} = 7 \notin 3\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(1, 7) = 1 \notin 3\mathbb{Z}$, and $\overline{\textbf{GCD-div}}$ since $(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) = (2 - 1) \gcd(2, 1) = 1$ is not divisible by $\kappa = 7$. □

Example 8.13 (Case 3 of Proposition 8.20 with $\hat{n} = n$). For $(k, \ell, n) = (3, 1, 26)$, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (3, 1, 26)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 13$, and $\kappa = \gcd(13, 26) = 13$. Since $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2) = 2$ is an integer, the simpler method is applicable and (8.84) reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{26}{13} \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = 2 \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

which yields $\mathscr{A}(3, 1, 26) = \mathscr{L}(8, 2)$, i.e., $\Phi(3, 1, 26) = (8, 2) = (\alpha, \beta)$, corresponding to $(p, q) = (1, 0)$. We have $(\alpha', \beta') = (8, 6)$ by (8.78), and

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(8, 2) \cap \Sigma_0(8, 6) = \Sigma_0(26, 0) = \Sigma_0(0, 0) = \{e\}$$

by Proposition 8.15. We have $(\hat{\alpha}, \hat{\beta}) = (4, 1)$, $\hat{D} = 13 \notin 3\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(2, 26) = 2 \notin 3\mathbb{Z}$, and $\overline{\textbf{GCD-div}}$ since $(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) = (3 - 1) \gcd(3, 1) = 2$ is not divisible by $\kappa = 13$. □

Example 8.14 (Case 3 of Proposition 8.20 with $\hat{n} = n$). For $(k, \ell, n) = (5, 2, 26)$, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (5, 2, 26)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 39$, and $\kappa = \gcd(39, 26) = 13$. The transformation to the Smith normal form in (8.104) is given as

$$\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \left[\begin{array}{cc|cc} 5 & 2 & -26 & 0 \\ 2 & -7 & 0 & -26 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 10 & 8 & -6 \\ 0 & 1 & 6 & 2 \\ \hline 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \end{array}\right].$$

This shows $\mathscr{A}(5, 2, 26) = \mathscr{L}(8, 6)$, i.e., $\Phi(5, 2, 26) = (8, 6)$. We have $(\alpha', \beta') = (8, 2)$ by (8.78), and

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(8, 6) \cap \Sigma_0(8, 2) = \Sigma_0(26, 0) = \Sigma_0(0, 0) = \{e\}$$

by Proposition 8.15. We have $(\hat{\alpha}, \hat{\beta}) = (4, 3)$, $\hat{D} = 13 \notin 3\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(3, 26) = 1 \notin 3\mathbb{Z}$, and $\overline{\textbf{GCD-div}}$ since $(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) = (5 - 2) \gcd(5, 2) = 3$ is not divisible by $\kappa = 13$. □

The next example is a case with solutions of type T and type V with translation symmetry.

Example 8.15 (Case 3 of Proposition 8.20 with $\hat{n} \neq n$). For $(k, \ell, n) = (18, 2, 42)$, we have seen that $(\hat{k}, \hat{\ell}, \hat{n}) = (9, 1, 21)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 91$, $\kappa = \gcd(91, 21) = 7$, and $\Phi(18, 2, 42) = (9, 3) = (\alpha, \beta)$ in Example 8.9. Hence $\mathscr{A}(18, 2, 42) = \mathscr{L}(9, 3)$. We have $(\alpha', \beta') = (9, 6)$ by (8.78), and

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(9, 3) \cap \Sigma_0(9, 6) = \Sigma_0(21, 0)$$

by Proposition 8.15. We have $(\hat{\alpha}, \hat{\beta}) = (3, 1)$, $\hat{D} = 7 \notin 3\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(8, 21) = 1 \notin 3\mathbb{Z}$, and $\overline{\textbf{GCD-div}}$ since $(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) = (9 - 1) \gcd(9, 1) = 8$ is not divisible by $\kappa = 7$. □

The last example is a case with solutions of type T and type M.

Example 8.16 (Case 4 of Proposition 8.20). For $(k, \ell, n) = (4m, m, 21m)$ with $m \geq 1$, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (4, 1, 21)$, $\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2 = 21$, and $\kappa = \gcd(21, 21) = 21$. Since $\hat{n}/(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2) = 1$ is an integer, the simpler method is applicable and (8.84) reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{21}{21} \begin{bmatrix} 5 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

which yields $\mathscr{A}(4m, m, 21m) = \mathscr{L}(5, 1)$, i.e., $\Phi(4m, m, 21m) = (5, 1)$, where $(\alpha, \beta) = (5, 1)$ corresponds to $(p, q) = (1, 0)$. We have $(\alpha', \beta') = (5, 4)$ by (8.78), and

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(5, 1) \cap \Sigma_0(5, 4) = \Sigma_0(14, 7)$$

by Proposition 8.15. We have $(\hat{\alpha}, \hat{\beta}) = (5, 1)$, $\hat{D} = 21 \in 3\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(3, 21) = 3 \in 3\mathbb{Z}$, and $\overline{\textbf{GCD-div}}$ since $(\hat{k} - \hat{\ell}) \gcd(\hat{k}, \hat{\ell}) = (4-1) \gcd(4, 1) = 3$ is not divisible by $\kappa = 21$. □

8.7.10 Appendix: Proofs of Propositions 8.12, 8.14, and 8.15

In this section we establish a series of propositions, which together serve as the proofs of Propositions 8.12, 8.14, and 8.15 presented in Sect. 8.7.2.

We first focus on Proposition 8.15.

Proposition 8.31.

(i) $\gcd(\hat{\alpha} - 2\hat{\beta}, \hat{\beta} - 2\hat{\alpha}) \in \{1, 3\}$.
(ii) $\gcd(\hat{\alpha} - 2\hat{\beta}, \hat{\beta} - 2\hat{\alpha}) = 3 \iff \hat{D} \in 3\mathbb{Z}$.
(iii) $\gcd(\hat{\alpha} - 2\hat{\beta}, \hat{\beta} - 2\hat{\alpha}) = 1 \iff \hat{D} \notin 3\mathbb{Z}$.

Proof. (i) Since $\gcd(\hat{\alpha}, \hat{\beta}) = 1$, Proposition 8.29(i) implies the existence of integers x and y such that $x\hat{\alpha} + y\hat{\beta} = 1$. For $p = -(x + 2y)$, $q = -(y + 2x)$, we have

$$p(\hat{\alpha} - 2\hat{\beta}) + q(\hat{\beta} - 2\hat{\alpha}) = 3(x\hat{\alpha} + y\hat{\beta}) = 3.$$

Then Proposition 8.29(ii) shows that 3 is divisible by $\gcd(\hat{\alpha} - 2\hat{\beta}, \hat{\beta} - 2\hat{\alpha})$, which is equivalent to the statement of (i) of this proposition.

(ii) We have $\{1, 3\} \ni \gcd(\hat{\alpha} - 2\hat{\beta}, \hat{\beta} - 2\hat{\alpha}) = \gcd(\hat{\alpha} + \hat{\beta}, \hat{\beta} - 2\hat{\alpha}) = \gcd(\hat{\alpha} + \hat{\beta}, 3\hat{\alpha})$. Therefore, $\gcd(\hat{\alpha} - 2\hat{\beta}, \hat{\beta} - 2\hat{\alpha}) = 3$ if and only if $\hat{\alpha} + \hat{\beta} \in 3\mathbb{Z}$. Finally we note a simple identity $\hat{D} = (\hat{\alpha} + \hat{\beta})^2 - 3\hat{\alpha}\hat{\beta}$ to see that $\hat{\alpha} + \hat{\beta} \in 3\mathbb{Z}$ if and only if $\hat{D} \in 3\mathbb{Z}$.

(iii) This is obvious from (i) and (ii) above. □

Proposition 8.32.

$$\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha) = \begin{cases} \Sigma_0(\alpha'',0) & \text{if } \hat{D} \notin 3\mathbb{Z}, \\ \Sigma_0(2\beta'',\beta'') & \text{if } \hat{D} \in 3\mathbb{Z} \end{cases} \tag{8.107}$$

with

$$\alpha'' = \frac{D(\alpha,\beta)}{\gcd(\alpha,\beta)}, \quad \beta'' = \frac{D(\alpha,\beta)}{3\gcd(\alpha,\beta)}. \tag{8.108}$$

Proof. First note that $\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha)$ is the subgroup generated by r and $p_1^a p_2^b$ for $(a,b) \in \mathscr{L}(\alpha,\beta) \cap \mathscr{L}(\beta,\alpha)$. In considering $\mathscr{L}(\alpha,\beta)$ of (8.99), it is convenient to have $\mathscr{H}(\alpha,\beta)$ of (8.4) in mind, as it has a natural correspondence with $\mathscr{L}(\alpha,\beta)$. The set $\mathscr{H}(\alpha,\beta) \cap \mathscr{H}(\beta,\alpha)$ is a hexagonal sublattice with the reflection symmetry with respect to the x-axis, and hence it can be represented as $\mathscr{H}(\alpha'',0)$ or $\mathscr{H}(2\beta'',\beta'')$ for some α'' or β''. Such α'' is determined as the minimum α'' satisfying $\mathscr{L}(\alpha'',0) \subseteq \mathscr{L}(\alpha,\beta)$, and β'' as the minimum β'' satisfying $\mathscr{L}(2\beta'',\beta'') \subseteq \mathscr{L}(\alpha,\beta)$. Then $\mathscr{L}(\alpha,\beta) \cap \mathscr{L}(\beta,\alpha)$ coincides with the larger of $\mathscr{L}(\alpha'',0)$ and $\mathscr{L}(2\beta'',\beta'')$.

The parameter α'' is determined as follows. The inclusion $\mathscr{L}(\alpha'',0) \subseteq \mathscr{L}(\alpha,\beta)$ holds if and only if integers n_1 and n_2 exist such that

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha-\beta \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \alpha'' \\ 0 \end{bmatrix}.$$

By the solvability criterion using determinantal divisors given in Theorem 8.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cc|c} \alpha & -\beta & \alpha'' \\ \beta & \alpha-\beta & 0 \end{array}\right]\right) \quad \text{equals} \quad d_1\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha-\beta \end{bmatrix}\right) = \gcd(\alpha,\beta),$$

$$d_2\left(\left[\begin{array}{cc|c} \alpha & -\beta & \alpha'' \\ \beta & \alpha-\beta & 0 \end{array}\right]\right) \quad \text{equals} \quad d_2\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha-\beta \end{bmatrix}\right) = D(\alpha,\beta).$$

The former condition is equivalent to α'' being a multiple of $\gcd(\alpha,\beta)$, and the latter to α'' being a multiple of $D(\alpha,\beta)/\gcd(\alpha,\beta)$. Hence we have $\alpha'' = D(\alpha,\beta)/\gcd(\alpha,\beta)$, which is a multiple of $\gcd(\alpha,\beta)$ since $D(\alpha,\beta)/\gcd(\alpha,\beta) = \hat{D}\gcd(\alpha,\beta)$.

The parameter β'' is determined as follows. The inclusion $\mathscr{L}(2\beta'',\beta'') \subseteq \mathscr{L}(\alpha,\beta)$ holds if and only if integers n_1 and n_2 exist such that

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha-\beta \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 2\beta'' \\ \beta'' \end{bmatrix}.$$

Again by Theorem 8.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cc|c} \alpha & -\beta & 2\beta'' \\ \beta & \alpha-\beta & \beta'' \end{array}\right]\right) \quad \text{equals} \quad d_1\left(\left[\begin{array}{cc} \alpha & -\beta \\ \beta & \alpha-\beta \end{array}\right]\right) = \gcd(\alpha,\beta),$$

$$d_2\left(\left[\begin{array}{cc|c} \alpha & -\beta & 2\beta'' \\ \beta & \alpha-\beta & \beta'' \end{array}\right]\right) \quad \text{equals} \quad d_2\left(\left[\begin{array}{cc} \alpha & -\beta \\ \beta & \alpha-\beta \end{array}\right]\right) = D(\alpha,\beta).$$

The former condition is equivalent to β'' being a multiple of $\gcd(\alpha,\beta)$, and the latter to β'' being a multiple of

$$\frac{D(\alpha,\beta)}{\gcd(\alpha-2\beta,\beta-2\alpha)} = \frac{D(\alpha,\beta)}{\gcd(\alpha,\beta)\gcd(\hat{\alpha}-2\hat{\beta},\hat{\beta}-2\hat{\alpha})}.$$

Then by Proposition 8.31, we obtain

$$\beta'' = \begin{cases} D(\alpha,\beta)/\gcd(\alpha,\beta) & \text{if } \hat{D} \notin 3\mathbb{Z}, \\ D(\alpha,\beta)/(3\gcd(\alpha,\beta)) & \text{if } \hat{D} \in 3\mathbb{Z}. \end{cases}$$

We have $\mathscr{L}(\alpha'',0) \supset \mathscr{L}(2\beta'',\beta'')$ (with $\beta''=\alpha''$) if $\hat{D} \notin 3\mathbb{Z}$, and $\mathscr{L}(2\beta'',\beta'') \supset \mathscr{L}(\alpha'',0)$ (with $\beta''=\alpha''/3$) if $\hat{D} \in 3\mathbb{Z}$. This completes the proof. □

Next we focus on Proposition 8.12(i). With this aim in mind, we rephrase (8.107) in Proposition 8.32 in terms of (k,ℓ) instead of (α,β).

Proposition 8.33.

(i) $\gcd(2\hat{k}+\hat{\ell},\hat{k}+2\hat{\ell},\hat{n}) \in \{1,3\}$.
(ii) $\gcd(2\hat{k}+\hat{\ell},\hat{k}+2\hat{\ell},\hat{n}) = 3 \iff \gcd(\hat{k}-\hat{\ell},\hat{n}) \in 3\mathbb{Z}$.
(iii) $\gcd(2\hat{k}+\hat{\ell},\hat{k}+2\hat{\ell},\hat{n}) = 1 \iff \gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 3\mathbb{Z}$.

Proof. (i) Since $\gcd(\hat{k},\hat{\ell},\hat{n}) = 1$, Proposition 8.29(i) implies the existence of integers a, b, and c such that $a\hat{k}+b\hat{\ell}+c\hat{n}=1$. For $p=2a-b$, $q=2b-a$, $r=3c$, we have

$$p(2\hat{k}+\hat{\ell})+q(\hat{k}+2\hat{\ell})+r\hat{n} = 3(a\hat{k}+b\hat{\ell}+c\hat{n}) = 3.$$

Then Proposition 8.29(ii) shows that 3 is divisible by $\gcd(2\hat{k}+\hat{\ell},\hat{k}+2\hat{\ell},\hat{n})$, which is equivalent to the claim in (i).

(ii) We have $\{1,3\} \ni \gcd(2\hat{k}+\hat{\ell},\hat{k}+2\hat{\ell},\hat{n}) = \gcd(\hat{k}-\hat{\ell},\hat{k}+2\hat{\ell},\hat{n}) = \gcd(\hat{k}-\hat{\ell},3\hat{\ell},\hat{n})$. Hence follows the claim.

(iii) This is obvious from (i) and (ii) above. □

Proposition 8.34.

$$\Sigma_0(\alpha,\beta)\cap\Sigma_0(\beta,\alpha) = \begin{cases} \Sigma_0(\hat{n},0) & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 3\mathbb{Z}, \\ \Sigma_0(2\hat{n}/3,\hat{n}/3) & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \in 3\mathbb{Z}. \end{cases} \tag{8.109}$$

Proof. Recall the notation $\mathscr{A}(k, \ell, n)$ in (8.98). By the same argument as in the proof of Proposition 8.32, we compute the minimum α'' satisfying $(\alpha'', 0) \in \mathscr{A}(k, \ell, n)$ and the minimum β'' satisfying $(2\beta'', \beta'') \in \mathscr{A}(k, \ell, n)$. Then $\mathscr{L}(\alpha, \beta) \cap \mathscr{L}(\beta, \alpha)$ coincides with the larger of $\mathscr{L}(\alpha'', 0)$ and $\mathscr{L}(2\beta'', \beta'')$.

By the definition of $\mathscr{A}(k, \ell, n)$ in (8.98) we have $(\alpha'', 0) \in \mathscr{A}(k, \ell, n)$ if and only if

$$\hat{k}\alpha'' \equiv 0, \quad \hat{\ell}\alpha'' \equiv 0 \mod \hat{n}.$$

Since $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$, the smallest α'' satisfying this condition is given by $\alpha'' = \hat{n}$. As for β'', we have $(2\beta'', \beta'') \in \mathscr{A}(k, \ell, n)$ if and only if

$$(2\hat{k} + \hat{\ell})\beta'' \equiv 0, \quad (\hat{k} + 2\hat{\ell})\beta'' \equiv 0 \mod \hat{n}.$$

The smallest β'' satisfying this condition is given by

$$\beta'' = \frac{\hat{n}}{\gcd(2\hat{k} + \hat{\ell}, \hat{k} + 2\hat{\ell}, \hat{n})} = \begin{cases} \hat{n} & \text{if } \gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 3\mathbb{Z}, \\ \hat{n}/3 & \text{if } \gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z}, \end{cases}$$

where Proposition 8.33 is used. We finally note $\mathscr{L}(2\hat{n}, \hat{n}) \subset \mathscr{L}(\hat{n}, 0)$ and $\mathscr{L}(\hat{n}, 0) \subset \mathscr{L}(2\hat{n}/3, \hat{n}/3)$ if $\hat{n} \in 3\mathbb{Z}$. This completes the proof. □

Proposition 8.35. (i) $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z} \iff \hat{D} \in 3\mathbb{Z}$.
(ii)

$$\hat{n} = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}. \tag{8.110}$$

Proof. This follows from a comparison of Proposition 8.32 with Proposition 8.34. □

We now focus on the second statement of Proposition 8.12.

Proposition 8.36.

$$\frac{\hat{n}}{\gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n})} = \gcd(\alpha, \beta). \tag{8.111}$$

Proof. We rely on the representation of Φ given in Proposition 8.30 in terms of the transformation matrix V in the Smith normal form of $[K \mid -\hat{n}I]$ in (8.104) with (8.101). Let

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

be the inverse of the matrix V in (8.104). We have $|\det V| = 1$ since V is unimodular. By a well-known formula in linear algebra and $V_{12} = A$ in (8.106), we have

$$|\det W_{12}| = |\det V_{12}|/|\det V| = |\det A| = D(\alpha, \beta). \tag{8.112}$$

On the other hand, it follows from (8.104) with $V = W^{-1}$ that

$$U \left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array}\right] \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

This implies

$$-\hat{n} U = \begin{bmatrix} 1 & 0 \\ 0 & \kappa \end{bmatrix} W_{12},$$

which shows

$$\hat{n}^2 = \kappa \, |\det W_{12}| \tag{8.113}$$

since $|\det U| = 1$.

Combining (8.112) and (8.113) with the expression (8.105) of κ, we obtain

$$\hat{n}^2 = \kappa \; D(\alpha, \beta) = \gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n}) \cdot D(\alpha, \beta).$$

By eliminating $D(\alpha, \beta)$ using (8.110), we obtain (8.111). □

Propositions 8.37–8.40 below are concerned with the symmetry of $\mathscr{A}(k, \ell, n)$ of (8.98), or that of $\Sigma_0(\alpha, \beta)$. Interestingly, such symmetry consideration leads to the proof of Proposition 8.14 of duality nature.

Proposition 8.37. *The four conditions* (a), (b), (c), *and* (d) *below are equivalent.*

(a) $(u_1, u_2) \in \mathbb{Z}^2$ *exists such that*

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k}-\hat{\ell} \end{bmatrix} \equiv \begin{bmatrix} \hat{\ell} & \hat{k} \end{bmatrix} \mod \hat{n}. \tag{8.114}$$

(b) *An integer matrix U exists such that*

$$U \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k}-\hat{\ell} \end{bmatrix} \equiv \begin{bmatrix} \hat{\ell} & \hat{k} \\ \hat{k} & -\hat{k}-\hat{\ell} \end{bmatrix} \mod \hat{n}. \tag{8.115}$$

(c) $\gcd(\hat{k}^2 + 2\hat{k}\hat{\ell},\ \hat{\ell}^2 + 2\hat{k}\hat{\ell})$ is divisible by $\gcd(\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \hat{n})$.

(d) **GCD-div** *in* (8.67)*:*

$$(\hat{k}-\hat{\ell})\,\mathrm{gcd}(\hat{k},\hat{\ell}) \text{ is divisible by } \mathrm{gcd}(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2,\hat{n}).$$

Proof. First, we show (a) $\Leftrightarrow$ (b). For $(u_1,u_2) \in \mathbb{Z}^2$ satisfying (8.114), the matrix $U = \begin{bmatrix} u_1 & u_2 \\ -u_2 & u_1-u_2 \end{bmatrix}$ is an integer matrix that satisfies (8.115). This shows (a) $\Rightarrow$ (b), whereas (b) $\Rightarrow$ (a) is obvious.

Next, we show (a) $\Leftrightarrow$ (c). The condition (a) is equivalent to the existence of integers u_1, u_2, p, and q that satisfy

$$\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \\ q \end{bmatrix} = \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix}.$$

By the solvability criterion using determinantal divisors given in Theorem 8.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cccc|c} \hat{k} & \hat{\ell} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} & \hat{k} \end{array}\right]\right) \quad \text{equals} \quad d_1\left(\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{bmatrix}\right) = 1,$$

$$d_2\left(\left[\begin{array}{cccc|c} \hat{k} & \hat{\ell} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} & \hat{k} \end{array}\right]\right) \quad \text{equals} \quad d_2\left(\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{bmatrix}\right).$$

The former condition imposes nothing and the latter reduces to (c). We have thus shown (a) $\Leftrightarrow$ (c).

Finally, we show (c) $\Leftrightarrow$ (d). Since $\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2$ is a multiple of $\kappa = \mathrm{gcd}(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2,\hat{n})$, $\hat{k}^2+2\hat{k}\hat{\ell}$ is divisible by κ if and only if $(\hat{k}^2+2\hat{k}\hat{\ell})-(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2) = \hat{\ell}(\hat{k}-\hat{\ell})$ is divisible by κ. Similarly, $\hat{\ell}^2+2\hat{k}\hat{\ell}$ is divisible by κ if and only if $\hat{k}(\hat{k}-\hat{\ell})$ is divisible by κ. Therefore, $\mathrm{gcd}(\hat{k}^2+2\hat{k}\hat{\ell},\ \hat{\ell}^2+2\hat{k}\hat{\ell})$ is divisible by κ if and only if $\mathrm{gcd}(\hat{\ell}(\hat{k}-\hat{\ell}),\ \hat{k}(\hat{k}-\hat{\ell})) = (\hat{k}-\hat{\ell})\,\mathrm{gcd}(\hat{k},\hat{\ell})$ is divisible by κ. ☐

Proposition 8.38. *The following two conditions are equivalent.*

(a) $\mathscr{A}(k,\ell,n) = \mathscr{A}(\ell,k,n)$.
(b) $(a,b) \in \mathscr{A}(k,\ell,n) \implies (b,a) \in \mathscr{A}(k,\ell,n)$.

Proof. The defining equations in (8.98) for $\mathscr{A}(k,\ell,n)$ are invariant under the change of variables $(a,b,k,\ell) \mapsto (b,a,\ell,k)$, and therefore, $\mathscr{A}(\ell,k,n) = \{(b,a) \mid (a,b) \in \mathscr{A}(k,\ell,n)\}$. This shows the equivalence of (a) and (b). ☐

Proposition 8.39. *The following two conditions are equivalent.*

(a) $\mathscr{A}(k,\ell,n) = \mathscr{A}(\ell,k,n)$.
(b) *An integer matrix* U *exists such that* (8.115) *holds.*

Proof. Although the claim is intuitively obvious from symmetry, we provide here a rigorous proof on the basis of Theorem 8.3 (the integer analogue of the Farkas lemma).

As in the proof of Proposition 8.37, the condition (b) is equivalent to the existence of integer tuples (u_1, u_2, p, q) and (u_1', u_2', p', q') such that

$$\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \\ q \end{bmatrix} = \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix}, \quad \begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ p' \\ q' \end{bmatrix} = \begin{bmatrix} \hat{k} \\ -\hat{k}-\hat{\ell} \end{bmatrix}.$$

By Theorem 8.3, the existence of such (u_1, u_2, p, q) is equivalent to the following condition:

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k}-\hat{\ell} & 0 & -\hat{n} \end{bmatrix} \in \mathbb{Z}^4 \quad \Longrightarrow \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix} \in \mathbb{Z},$$

which can be rewritten as

$$[\hat{k}y_1 + \hat{\ell}y_2,\ \hat{\ell}y_1 - (\hat{k}+\hat{\ell})y_2,\ -\hat{n}y_1,\ -\hat{n}y_2\,] \in \mathbb{Z}^4 \quad \Longrightarrow \quad \hat{\ell}y_1 + \hat{k}y_2 \in \mathbb{Z}.$$

Integrality condition for the third and fourth components allows us to put $y_1 = a/\hat{n}$ and $y_2 = b/\hat{n}$ with integers a and b. Then we can rewrite the above as

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - (\hat{k}+\hat{\ell})b \equiv 0 \mod \hat{n} \quad \Longrightarrow \quad \hat{\ell}a + \hat{k}b \equiv 0 \mod \hat{n}.$$

Similarly, the existence of (u_1', u_2', p', q') above is equivalent to the following condition:

$$\hat{k}a + \hat{\ell}b \equiv 0,\ \hat{\ell}a - (\hat{k}+\hat{\ell})b \equiv 0 \mod \hat{n} \Longrightarrow \hat{k}a - (\hat{k}+\hat{\ell})b \equiv 0 \mod \hat{n}.$$

Since

$$[\hat{\ell}a + \hat{k}b] + [ka - (\hat{k}+\hat{\ell})b] = -[\hat{\ell}b - (\hat{k}+\hat{\ell})a],$$

the above two conditions together are nothing but the statement that $(a, b) \in \mathscr{A}(k, \ell, n)$ implies $(b, a) \in \mathscr{A}(k, \ell, n)$, which is equivalent to (a) by Proposition 8.38. $\square$

Proposition 8.40. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$.

(i) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff \beta = 0$ *or* $\alpha = 2\beta$.
(ii) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff$ **GCD-div** *in* (8.67).

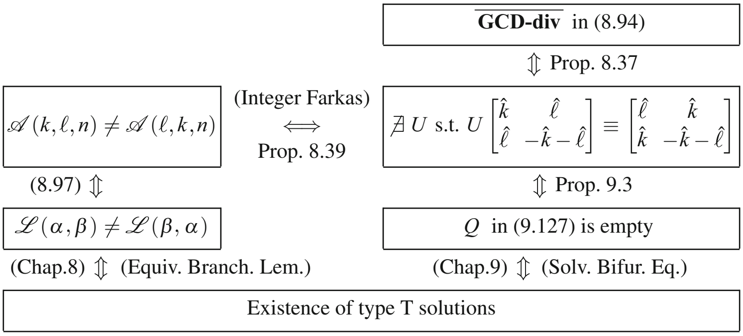

Fig. 8.10 Technical conditions for type T

Proof. (i) is obvious, and (ii) follows from Propositions 8.37 and 8.39. Note that $\Sigma_0(\alpha, \beta)$ is the subgroup generated by r and $p_1^a p_2^b$ for $(a, b) \in \mathscr{A}(k, \ell, n)$. $\square$

Remark 8.20. Figure 8.10 summarizes conditions characterizing the existence of type T. See Sect. 9.6.3 ((9.127) in particular) for the definition of the set Q of nonnegative integers and for the significance of Q being empty. $\square$

8.8 Summary

- For hexagonal distributions on the hexagonal lattice, the associated irreducible representations and the size of the hexagonal lattice have been advanced.
- Hexagonal distributions of Christaller and Lösch in central place theory have been guaranteed to exist for the economy on the hexagonal lattice.
- Use of the equivariant branching lemma has been demonstrated to be effective in the test of the existence of hexagonal bifurcating patterns.

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Chapter 9
Hexagons of Christaller and Lösch: Solving Bifurcation Equations

Abstract Based on the fundamental issues of the mathematical model of an economy on the hexagonal lattice with $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$-symmetry in Chaps. 5–7, group-theoretic bifurcation analysis by the equivariant branching lemma was conducted in Chap. 8. As a result, a complete classification of hexagonal distributions of Christaller and Lösch on this economy was presented. This analysis, however, showed only the existence of such distributions without reference to concrete forms of bifurcation equations. In this chapter, group-theoretic bifurcation analysis is conducted by using bifurcation equations. Expanded forms of these equations are derived by exploiting their symmetry and are solved to obtain information about bifurcating hexagonal patterns, such as asymptotic forms of bifurcating equilibrium paths and the directions of these paths, in addition to their existence. All critical points of the economy on the $n \times n$ hexagonal lattice, with multiplicity 2, 3, 6, and 12, are considered.

Keywords Bifurcation • Bifurcation equation • Christaller's hexagonal distributions • Group • Group-theoretic bifurcation analysis • Hexagonal lattice • Irreducible representations • Lösch's hexagons

9.1 Introduction

The $n \times n$ hexagonal lattice was introduced as a two-dimensional discretized uniform space, and the group $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ labeling the symmetry was presented in Chap. 5. Irreducible decomposition of the matrix representation was obtained to identify those irreducible representations which are relevant to the economy on the hexagonal lattice in Chap. 7. Equivariant branching lemma was presented in Chap. 8 as a pertinent and sufficient means to test the existence of a bifurcating pattern, and was used to show the existence of Lösch's hexagons, as well as Christallers's three systems, for particular irreducible representations and the size n of the lattice. See Fig. 9.1 for the ten smallest hexagons of Lösch and Fig. 9.2 for Christaller's three systems. See also Remark 5.1 in Sect. 5.5.1 for related studies.

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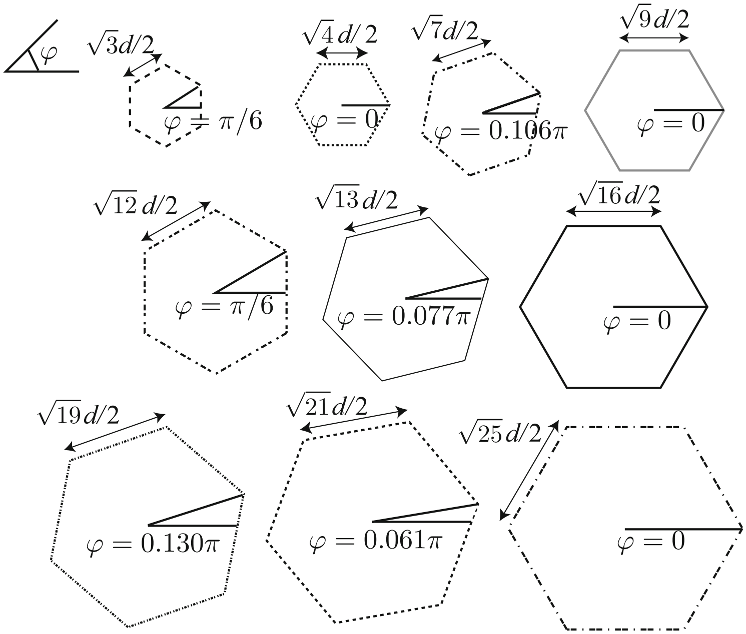

Fig. 9.1 Lösch's ten smallest hexagons

In this chapter, bifurcation analysis by solving bifurcation equations is advanced as a more complicated but more informative means to investigate the asymptotic properties of the bifurcating pattern, in addition to its existence. This analysis is conducted for the particular irreducible representations and the lattice sizes n that give Lösch's hexagons. The multiplicity M of a bifurcation point for Lösch's ten hexagons is given (Proposition 8.1) by[1]

$$M = \begin{cases} 2 & \text{for } D = 3, \\ 3 & \text{for } D = 4, \\ 6 & \text{for } D = 9, 12, 16, 25, \\ 12 & \text{for } D = 7, 13, 19, 21. \end{cases}$$

Expanded forms of the bifurcation equations are derived by exploiting their symmetry. These equations are solved to obtain information about bifurcating hexagonal distributions, such as asymptotic forms of bifurcating paths, in addition to their existence. In particular, for critical points of multiplicity $M = 12$, duality nature between the results obtained by the two methods, one by the equivariant

[1]The emergence of some of these hexagons was already confirmed numerically in Sect. 4.5 by computational bifurcation analysis for the hexagonal lattice with various sizes n.

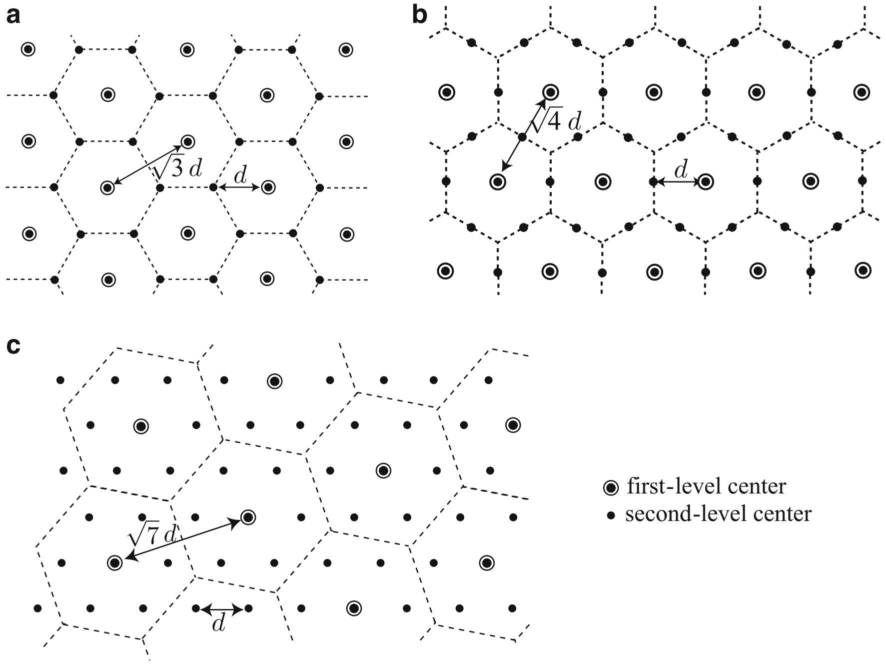

Fig. 9.2 Three systems predicted by Christaller, corresponding to Lösch's three smallest hexagons with $D = 3$, 4, and 7. (**a**) Christaller's $k = 3$ system. (**b**) Christaller's $k = 4$ system. (**c**) Christaller's $k = 7$ system

branching lemma and the other by bifurcation equations, is established through a theorem for integrality of solutions, the so-called integer analogue of the Farkas lemma.

This chapter is organized as follows. Solution procedure for the determination of bifurcating solutions is presented in Sect. 9.2. This procedure is applied to critical points of multiplicity $M = 2$, 3, 6, and 12 in Sects. 9.3–9.6, respectively, to obtain possible hexagonal distributions.

9.2 Bifurcation and Symmetry of Solutions

Let us consider the equilibrium equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0} \tag{9.1}$$

in (8.14) endowed with the symmetry of, or *equivariance* to, the group

$$G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle \tag{9.2}$$

Table 9.1 Irreducible representations of $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ to be considered in bifurcation analysis (Tables 7.3 and 8.3)

$n \setminus M$	2	3	6	12
$6m$	$(2;+)$	$(3;+,+)$	$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 1$			$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 2$		$(3;+,+)$	$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$
$6m \pm 3$	$(2;+)$		$(6;k,0;+), (6;k,k;+)$	$(12;k,\ell)$

$(6;k,0;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$
$(6;k,k;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, $k \neq n/3$
$(12;k,\ell)$ for (k,ℓ) with $1 \leq \ell \leq k-1$, $2k+\ell \leq n-1$

in (8.1) formulated as

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda},\tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda},\tau), \quad g \in G \tag{9.3}$$

in (8.15). Recall that τ, being the transport cost parameter, serves as a bifurcation parameter, $\boldsymbol{\lambda} \in \mathbb{R}^N$ is an independent variable vector of dimension $N = n^2$ expressing a distribution of mobile population, $\boldsymbol{F} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is the nonlinear function in (4.4) or (4.6) for core–periphery models, and T is the N-dimensional permutation representation (Sect. 7.3) of the group $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Let $(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c})$ be a critical point of multiplicity M (≥ 1), at which the Jacobian matrix of $\boldsymbol{F}$ has the rank deficiency M. The critical point $(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c})$ is assumed to be G-symmetric in the sense of (8.16):

$$T(g)\boldsymbol{\lambda}_\mathrm{c} = \boldsymbol{\lambda}_\mathrm{c}, \quad g \in G. \tag{9.4}$$

Moreover, it is assumed to be *group-theoretic*, which means, by definition, that the M-dimensional kernel space of the Jacobian matrix at $(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c})$ is irreducible with respect to the representation T. Then the critical point $(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c})$ is associated with one of the irreducible representations μ of G that appear in the economy on the hexagonal lattice (Table 9.1), and the multiplicity M corresponds to the dimension of the irreducible representation μ. We denote by $T^\mu(g)$ the representation matrix for μ.

By the Liapunov–Schmidt reduction with symmetry in Sect. 2.4.3, the full system of equilibrium equation (9.1) is reduced, in a neighborhood of the critical point $(\boldsymbol{\lambda}_\mathrm{c}, \tau_\mathrm{c})$, to a system of bifurcation equations (8.17):

$$\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{\tau}) = \boldsymbol{0} \tag{9.5}$$

in $\boldsymbol{w} \in \mathbb{R}^M$, where $\tilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function and $\tilde{\tau} = \tau - \tau_\mathrm{c}$ denotes the increment of τ. In this reduction process the equivariance (9.3) of the full system is inherited by the reduced system (9.5). Using the representation matrix $T^\mu(g)$ for the associated irreducible representation μ, we can express the equivariance of the bifurcation equation as

$$T^\mu(g)\tilde{\boldsymbol{F}}(\boldsymbol{w}, \tilde{\tau}) = \tilde{\boldsymbol{F}}(T^\mu(g)\boldsymbol{w}, \tilde{\tau}), \quad g \in G, \tag{9.6}$$

which is given in (8.18). This inherited symmetry plays a key role in determining the form of bifurcation equation and the symmetry of bifurcating solutions.

The reduced equation (9.5) can possibly admit multiple solutions $\boldsymbol{w} = \boldsymbol{w}(\tilde{\tau})$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, since $(\boldsymbol{w}, \tilde{\tau}) = (\boldsymbol{0}, 0)$ is a singular point of (9.5). This gives rise to bifurcation. Each $\boldsymbol{w}$ uniquely determines a solution $\boldsymbol{\lambda}$ to the full system (9.1), and moreover the symmetry of $\boldsymbol{w}$ is identical to that of $\boldsymbol{\lambda}$. As in (8.19)–(8.22), we have indeed the following relation

$$G^{\mu} \subseteq \Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{\lambda}) \tag{9.7}$$

for the subgroup

$$G^{\mu} = \{g \in G \mid T^{\mu}(g) = I\} \tag{9.8}$$

and the isotropy subgroups

$$\Sigma(\boldsymbol{\lambda}) = \Sigma(\boldsymbol{\lambda}; G, T) = \{g \in G \mid T(g)\boldsymbol{\lambda} = \boldsymbol{\lambda}\}, \tag{9.9}$$

$$\Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{w}; G, T^{\mu}) = \{g \in G \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w}\}. \tag{9.10}$$

The significance of the relation (9.7) is that the symmetry of a bifurcating solution $\boldsymbol{\lambda}$ is known through the analysis of bifurcation equations in $\boldsymbol{w}$.

Remark 9.1. Although there exist various ways to define $\boldsymbol{w} = (w_1, \ldots, w_M)^{\top}$ in the bifurcation equation (9.5) satisfying the equivariance condition (9.6), it turns out to be pertinent to relate the components of $\boldsymbol{w}$ to the column vectors of $Q^{\mu} = [\boldsymbol{q}_1^{\mu}, \ldots, \boldsymbol{q}_M^{\mu}]$. □

9.3 Bifurcation Point of Multiplicity 2

We derive bifurcating solutions at a group-theoretic critical point[2] of multiplicity $M = 2$ for the irreducible representation $\mu = (2; +)$ of the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (9.2) relevant to the analysis of the hexagonal lattice (Table 9.1). We assume that n is divisible by 3.

9.3.1 Irreducible Representation

The two-dimensional irreducible representation $\mu = (2; +)$ is defined by (6.13) and (6.14) as

[2]This critical point is different from the double critical point studied in Sect. 3.4.3.

$$T^{(2;+)}(r) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad T^{(2;+)}(s) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},$$

$$T^{(2;+)}(p_1) = T^{(2;+)}(p_2) = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}.$$

We adopt the complex coordinates $(z, \overline{z})$ instead of (w_1, w_2); that is,

$$z = w_1 + \mathrm{i}w_2, \quad \overline{z} = w_1 - \mathrm{i}w_2, \tag{9.11}$$

where i denotes the imaginary unit and $\overline{\cdot}$ denotes the complex conjugate. The action on the two-dimensional vector (w_1, w_2) can be expressed for the complex variable $(z, \overline{z})$ as

$$r: \quad z \mapsto \overline{z}, \tag{9.12}$$

$$s: \quad z \mapsto z, \tag{9.13}$$

$$p_1, p_2: z \mapsto \omega_3 z, \tag{9.14}$$

where $\omega_3 = \exp(\mathrm{i}2\pi/3)$.

9.3.2 Equivariance of Bifurcation Equation

The bifurcation equation for the group-theoretic critical point of multiplicity 2 is a two-dimensional equation over $\mathbb{R}$. This equation can be expressed as a complex-valued equation in complex variables as

$$F(z, \overline{z}, \tilde{\tau}) = 0, \tag{9.15}$$

where $(z, \overline{z}, \tilde{\tau}) = (0, 0, 0)$ is assumed to correspond to the critical point. We often omit $\tilde{\tau}$ in the subsequent derivation.

The equivariance of the bifurcation equation to the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, p_2 generating this group. Therefore, the equivariance condition (9.6) of the bifurcation equation can be written for (9.15) as

$$r: \quad \overline{F(z, \overline{z})} = F(\overline{z}, z), \tag{9.16}$$

$$s: \quad F(z, \overline{z}) = F(z, \overline{z}), \tag{9.17}$$

$$p_1, p_2: \omega_3 F(z, \overline{z}) = F(\omega_3 z, \overline{\omega}_3 \overline{z}). \tag{9.18}$$

We expand F into a power series as

$$F(z, \bar{z}, \tilde{\tau}) = \sum_{a=0} \sum_{b=0} A_{ab}(\tilde{\tau}) z^a \bar{z}^b. \tag{9.19}$$

The equivariance (9.16) to r gives $A_{ab}(\tilde{\tau}) = \overline{A_{ab}(\tilde{\tau})}$, i.e., all $A_{ab}(\tilde{\tau})$ are real, whereas the equivariance (9.17) to s imposes no condition. For the index (a, b) of nonvanishing terms in (9.19), the condition (9.18) gives

$$a - b - 1 = 3p, \quad p \in \mathbb{Z}.$$

Using this relation in (9.19), we obtain the bifurcation equation (9.15) as

$$\begin{aligned} F(z, \bar{z}, \tilde{\tau}) &= \sum_{a=0} A_{a+1,a}(\tilde{\tau})\, z^{a+1} \bar{z}^a \\ &\quad + \sum_{p=1} \sum_{a=0} [A_{a+1+3p,a}(\tilde{\tau})\, z^{a+1+3p} \bar{z}^a + A_{a,a-1+3p}\, (\tilde{\tau})\, z^a \bar{z}^{a-1+3p}] = 0. \end{aligned} \tag{9.20}$$

Therein, $A_{a+1,a}(\tilde{\tau})$, $A_{a+1+3p,a}(\tilde{\tau})$, and $A_{a,a-1+3p}\,(\tilde{\tau})$ are real and generically distinct from zero (because no reason exists for the vanishing of these terms). Because $(z, \bar{z}, \tilde{\tau}) = (0, 0, 0)$ corresponds to the double critical point, we have

$$A_{00}(0) = 0, \quad A_{10}(0) = 0, \quad A_{01}(0) = 0. \tag{9.21}$$

Therefore, we have

$$A_{10}(\tilde{\tau}) \approx A\tilde{\tau} \tag{9.22}$$

for $A = A'_{10}(0)$, which is generically nonzero.[3]

Equation (9.20) has the trivial solution $z = 0$ since each term in (9.20) vanishes if $z = 0$. Nontrivial solutions of (9.20) are analyzed hereafter.

9.3.3 *Bifurcating Solutions*

Nontrivial solutions of (9.20) are determined from $F/z = 0$. If we put

$$\hat{F}(\rho, \theta, \tilde{\tau}) = \frac{F(\rho \exp(\mathrm{i}\theta), \rho \exp(-\mathrm{i}\theta), \tilde{\tau})}{\rho \exp(\mathrm{i}\theta)} \quad \left(= \frac{F}{z} \right)$$

using the polar coordinates $z = w_1 + \mathrm{i}w_2 = \rho \exp(\mathrm{i}\theta)$ $(\rho > 0)$, then we have

---

[3] Notation $A'_{10}(0)$ means the derivative of $A_{10}(\tilde{\tau})$ with respect to $\tilde{\tau}$, evaluated at $\tilde{\tau} = 0$. Generically we have $A'_{10}(0) \neq 0$ since the group symmetry imposes no condition.

$$\begin{aligned}\mathrm{Re}(\hat{F}) &= \sum_{a=0} A_{a+1,a}(\tilde{\tau})\rho^{2a} \\ &\quad + \sum_{p=1}\sum_{a=0} [A_{a+1+3p,a}(\tilde{\tau})\rho^{2a+3p} + A_{a,a-1+3p}\,(\tilde{\tau})\,\rho^{2(a-1)+3p}]\cos(3p\theta), \\ \mathrm{Im}(\hat{F}) &= \sum_{p=1}\sum_{a=0} [A_{a+1+3p,a}(\tilde{\tau})\rho^{2a+3p} - A_{a,a-1+3p}\,(\tilde{\tau})\,\rho^{2(a-1)+3p}]\sin(3p\theta).\end{aligned}$$

The nontrivial solution of (9.20) is determined from $\mathrm{Re}(\hat{F}) = \mathrm{Im}(\hat{F}) = 0$.

Equation $\mathrm{Im}(\hat{F}) = 0$ is satisfied by

$$\theta = \pi\frac{j-1}{3}, \quad j = 1, \ldots, 6,$$

for which $\sin(3p\theta) = \sin(p(j-1)\pi) = 0$ and

$$\cos(3p\theta) = \cos(p(j-1)\pi) = (-1)^{p(j-1)}.$$

With this expression, the other equation $\mathrm{Re}(\hat{F}) = 0$ can be represented as

$$\begin{aligned}&\sum_{a=0} A_{a+1,a}(\tilde{\tau})\rho^{2a} \\ &+ \sum_{p=1}\sum_{a=0}(-1)^{p(j-1)}\,[A_{a+1+3p,a}(\tilde{\tau})\rho^{2a+3p}+A_{a,a-1+3p}\,(\tilde{\tau})\,\rho^{2(a-1)+3p}]=0. \quad (9.23)\end{aligned}$$

By $A_{10}(\tilde{\tau}) \approx A\tilde{\tau}$ in (9.22), the leading part of this equation is

$$\begin{cases} A\tilde{\tau} + A_{02}\rho = 0, & j = 1, 3, 5, \\ A\tilde{\tau} - A_{02}\rho = 0, & j = 2, 4, 6, \end{cases}$$

where $A_{02} = A_{02}(0) \neq 0$ (generically). Therefore, for each j, the equation (9.23) can be solved, by the implicit function theorem, for $\rho(> 0)$ as

$$\rho = \begin{cases} \psi_+(\tilde{\tau}) \;\; (\mathrm{sign}(\tilde{\tau}) = -\mathrm{sign}(AA_{02})) & j = 1, 3, 5, \\ \psi_-(\tilde{\tau}) \;\; (\mathrm{sign}(\tilde{\tau}) = \mathrm{sign}(AA_{02})) & j = 2, 4, 6, \end{cases}$$

where $\mathrm{sign}(\cdot)$ is the sign of the variable therein, and

$$\psi_+(\tilde{\tau}) \approx -\frac{A_{02}}{A}\tilde{\tau}, \quad \psi_-(\tilde{\tau}) \approx \frac{A_{02}}{A}\tilde{\tau}, \quad \tilde{\tau} \to 0. \quad (9.24)$$

Hence we obtain six half-branches of bifurcating solutions:

$$z = \begin{cases} \psi_{+}(\tilde{\tau})\ \exp\left(\frac{\mathrm{i}\pi(j-1)}{3}\right), \ j = 1, 3, 5, \\ \psi_{-}(\tilde{\tau})\ \exp\left(\frac{\mathrm{i}\pi(j-1)}{3}\right), \ j = 2, 4, 6. \end{cases} \tag{9.25}$$

Accordingly, the two half-branches in the opposite directions associated with $j = k,\ k+3$ $(k = 1, 2, 3)$ have different ρ versus $\tilde{\tau}$ curves. In this sense, the bifurcation point is asymmetric (Remark 9.2).

From the actions (9.12)–(9.14), we can determine the symmetry of the real solutions $z \in \mathbb{R}$ (corresponding to $j = 1, 4$) as[4]

$$\Sigma^{(2;+)}(z) = \langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle \quad \text{for } j = 1, 4,$$

which is $\Sigma(2, 1)$ in the notation of (8.10) and is in agreement with Proposition 8.3. Since the other solutions are obtainable from z as conjugate solutions $T^{(2;+)}(p_1) \cdot z$ or $T^{(2;+)}(p_1^2) \cdot z$, the symmetries of these solutions are given by the following subgroups

$$\begin{cases} p_1 \cdot \Sigma^{(2;+)}(z) \cdot p_1^{-1} = \langle p_1^2 r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle \text{ for } j = 3, 6, \\ p_1^2 \cdot \Sigma^{(2;+)}(z) \cdot p_1^{-2} = \langle p_1 r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle \text{ for } j = 2, 5 \end{cases}$$

by (2.130). See Remark 9.2.

We have thus shown, by solving bifurcation equations, that Lösch's hexagon with $D = 3$ in Fig. 9.1 (Christaller's $k = 3$ system in Fig. 9.2a) is predicted to branch from a double critical point.

Remark 9.2. As mentioned in Remark 9.1 in Sect. 9.2, it is pertinent to assume that the variable $\boldsymbol{w} = (w_1, w_2)^\top$ corresponds to the column vectors of

$$Q^{(2;+)} = [\boldsymbol{q}_1, \boldsymbol{q}_2] = [\ \langle \cos(2\pi(n_1 - 2n_2)/3) \rangle,\ \langle \sin(2\pi(n_1 - 2n_2)/3) \rangle\ ]$$

in (7.32). The spatial patterns for these vectors are depicted in Fig. 9.3a and b for $n = 6$. The vector $\boldsymbol{q}_1$ represents the Lösch's hexagon with $D = 3$ with $\Sigma(2, 1)$-symmetry (Christaller's $k = 3$ system in Fig. 9.2a) but the other vector $\boldsymbol{q}_2$ does not.

Bifurcating patterns shown in Fig. 9.4a–c are associated with Lösch's hexagon with $D = 3$ with a third of the places growing into first-level centers. In contrast, bifurcating patterns shown in Fig. 9.4d–f are not associated with Lösch's because two-thirds of the places grow in population and a third of the places shrink. □

[4] $\Sigma^{(2;+)}(z)$ means $\Sigma^{(2;+)}(\boldsymbol{w})$ for $z = w_1 + \mathrm{i}w_2$ and $\boldsymbol{w} = (w_1, w_2)^\top$.

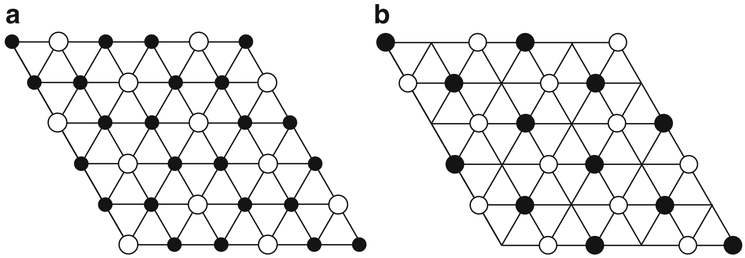

Fig. 9.3 Patterns on the 6×6 hexagonal lattice expressed by the column vectors of $Q^{(2;+)}$. (**a**) $\boldsymbol{q}_1$ ($\Sigma(2,1)$). (**b**) $\boldsymbol{q}_2$ ($\langle r^2, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$). A *white circle* denotes a positive component and a *black circle* denotes a negative component

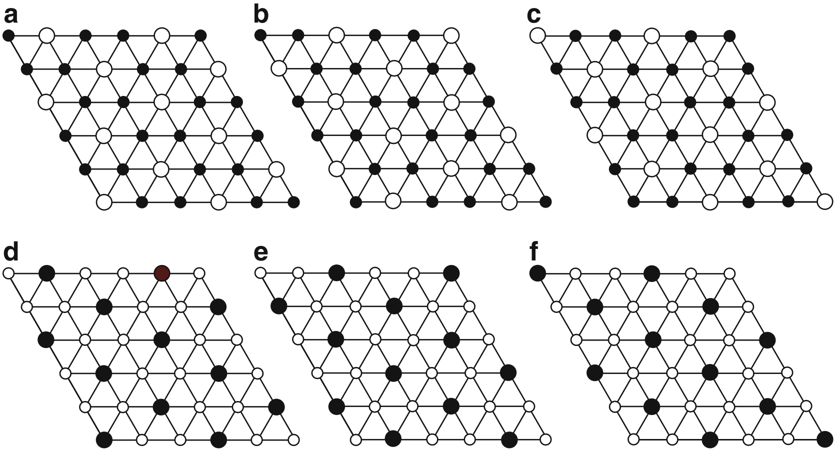

Fig. 9.4 Bifurcating patterns on the $6{\times}6$ hexagonal lattice associated with $\mu = (2;+)$. (**a**) $j = 1$ ($\Sigma(2,1)$); $(w_1, w_2) = (-\rho, 0)$. (**b**) $j = 3$ ($p_1 \cdot \Sigma(2,1) \cdot p_1^{-1}$); $(w_1, w_2) = (-\frac{1}{2}\rho, \frac{\sqrt{3}}{2}\rho)$. (**c**) $j = 5$ ($p_1^2 \cdot \Sigma(2,1) \cdot p_1^{-2}$); $(w_1, w_2) = (-\frac{1}{2}\rho, -\frac{\sqrt{3}}{2}\rho)$. (**d**) $j = 4$ ($\Sigma(2,1)$); $(w_1, w_2) = (-\rho, 0)$. (**e**) $j = 6$ ($p_1 \cdot \Sigma(2,1) \cdot p_1^{-1}$); $(w_1, w_2) = (\frac{1}{2}\rho, -\frac{\sqrt{3}}{2}\rho)$. (**f**) $j = 2$ ($p_1^2 \cdot \Sigma(2,1) \cdot p_1^{-2}$); $(w_1, w_2) = (\frac{1}{2}\rho, \frac{\sqrt{3}}{2}\rho)$. A *white circle* denotes a positive component and a *black circle* denotes a negative component

9.4 Bifurcation Point of Multiplicity 3

We derive bifurcating solutions[5] at a group-theoretic critical point of multiplicity $M = 3$ for the irreducible representation $\mu = (3;+,+)$ of the group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (9.2) relevant to the analysis of the hexagonal lattice (Table 9.1). We assume that n is even.

---

[5]Preexisting results for a triple critical point can be found, for example, in Sattinger, 1979 [3]; Golubitsky, Stewart, and Schaeffer, 1988 [1]; and Ikeda and Murota, 2010, Chap. 8 [2].

9.4.1 Irreducible Representation

The three-dimensional irreducible representation $(3;+,+)$ is given by (6.17) and (6.21) as

$$T^{(3;+,+)}(r) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \qquad T^{(3;+,+)}(s) = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}; \tag{9.26}$$

$$T^{(3;+,+)}(p_1) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \qquad T^{(3;+,+)}(p_2) = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}. \tag{9.27}$$

For this irreducible representation, the action of $\langle r, s, p_1, p_2 \rangle$ on $\boldsymbol{w} = (w_1, w_2, w_3)^\top$ is given as

$$\begin{aligned} r &: (w_1, w_2, w_3) \mapsto (w_3, w_1, w_2), \\ s &: (w_1, w_2, w_3) \mapsto (w_3, w_2, w_1), \\ p_1 &: (w_1, w_2, w_3) \mapsto (-w_1, w_2, -w_3), \\ p_2 &: (w_1, w_2, w_3) \mapsto (w_1, -w_2, -w_3). \end{aligned}$$

9.4.2 Equivariance of Bifurcation Equation

The bifurcation equation for the group-theoretic critical point of multiplicity 3 is a three-dimensional equation. This equation can be expressed as

$$F_1(w_1, w_2, w_3, \tilde{\tau}) = F_2(w_1, w_2, w_3, \tilde{\tau}) = F_3(w_1, w_2, w_3, \tilde{\tau}) = 0. \tag{9.28}$$

It is assumed that $(w_1, w_2, w_3, \tilde{\tau}) = (0, 0, 0, 0)$ corresponds to the triple critical point. We often omit $\tilde{\tau}$ in the subsequent derivation.

The equivariance of the bifurcation equation to the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, p_2 generating this group. Therefore, the equivariance condition (9.6) of the bifurcation equation (9.28) can be expressed as

$$\begin{aligned} r: \quad & F_3(w_1, w_2, w_3) = F_1(w_3, w_1, w_2), \\ & F_1(w_1, w_2, w_3) = F_2(w_3, w_1, w_2), \\ & F_2(w_1, w_2, w_3) = F_3(w_3, w_1, w_2), \\ s: \quad & F_3(w_1, w_2, w_3) = F_1(w_3, w_2, w_1), \\ & F_2(w_1, w_2, w_3) = F_2(w_3, w_2, w_1), \end{aligned}$$

$$
\begin{aligned}
F_1(w_1, w_2, w_3) &= F_3(w_3, w_2, w_1), \\
p_1 : -F_1(w_1, w_2, w_3) &= F_1(-w_1, w_2, -w_3), \\
F_2(w_1, w_2, w_3) &= F_2(-w_1, w_2, -w_3), \\
-F_3(w_1, w_2, w_3) &= F_3(-w_1, w_2, -w_3), \\
p_2 : \quad F_1(w_1, w_2, w_3) &= F_1(w_1, -w_2, -w_3), \\
-F_2(w_1, w_2, w_3) &= F_2(w_1, -w_2, -w_3), \\
-F_3(w_1, w_2, w_3) &= F_3(w_1, -w_2, -w_3).
\end{aligned}
$$

The above conditions reduce to three conditions

$$F_2(w_1, w_2, w_3) = F_2(-w_1, w_2, -w_3), \tag{9.29}$$

$$-F_2(w_1, w_2, w_3) = F_2(w_1, -w_2, -w_3), \tag{9.30}$$

$$F_2(w_1, w_2, w_3) = F_2(w_3, w_2, w_1) \tag{9.31}$$

imposed on F_2 and two others

$$F_1(w_1, w_2, w_3) = F_2(w_3, w_1, w_2), \tag{9.32}$$

$$F_3(w_1, w_2, w_3) = F_2(w_2, w_3, w_1) \tag{9.33}$$

connecting F_2 to F_1 and F_3.

We expand F_2 as[6]

$$F_2(w_1, w_2, w_3, \tilde{\tau}) = \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{abc}(\tilde{\tau}) {w_1}^a {w_2}^b {w_3}^c. \tag{9.34}$$

Because $(w_1, w_2, w_3, \tilde{\tau}) = (0, 0, 0, 0)$ corresponds to the triple critical point, we have

$$A_{000}(0) = 0, \quad A_{100}(0) = 0, \quad A_{010}(0) = 0, \quad A_{001}(0) = 0. \tag{9.35}$$

Therefore, we have

$$A_{010}(\tilde{\tau}) \approx A\tilde{\tau} \tag{9.36}$$

for $A = A'_{010}(0)$, which is generically nonzero.

For nonzero terms in (9.34), conditions (9.29) and (9.30) yield

$$(-1)^{a+c} = (-1)^{b+c-1} = 1,$$

[6]It is slightly more convenient to work with F_2, rather than F_1 or F_3, because the action of s in (9.26) leaves the second component unchanged.

which means that $(a, b, c) =$ (even, odd, even) or (odd, even, odd). Therefore, F_2 reduces to

$$
\begin{aligned}
F_2(w_1, w_2, w_3, \tilde{\tau}) &= \sum_{a:\text{even}\geq 0} \sum_{b:\text{odd}\geq 1} \sum_{c:\text{even}\geq 0} A_{abc}(\tilde{\tau})\, {w_1}^a {w_2}^b {w_3}^c \\
&\quad + \sum_{a:\text{odd}\geq 1} \sum_{b:\text{even}\geq 0} \sum_{c:\text{odd}\geq 1} A_{abc}(\tilde{\tau})\, {w_1}^a {w_2}^b {w_3}^c \\
&= w_2 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c}(\tilde{\tau})\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c} \\
&\quad + w_1 w_3 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c+1}(\tilde{\tau})\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c}. \qquad (9.37)
\end{aligned}
$$

The condition (9.31) yields[7]

$$
A_{abc}(\tilde{\tau}) = A_{cba}(\tilde{\tau}). \qquad (9.38)
$$

The expressions in (9.32) and (9.33) with the expansion of F_2 in (9.37) yield

$$
\begin{aligned}
F_1(w_1, w_2, w_3, \tilde{\tau}) &= w_1 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c}(\tilde{\tau})\, {w_3}^{2a} {w_1}^{2b} {w_2}^{2c} \\
&\quad + w_3 w_2 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c+1}(\tilde{\tau})\, {w_3}^{2a} {w_1}^{2b} {w_2}^{2c}, \qquad (9.39)
\end{aligned}
$$

$$
\begin{aligned}
F_3(w_1, w_2, w_3, \tilde{\tau}) &= w_3 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c}(\tilde{\tau})\, {w_2}^{2a} {w_3}^{2b} {w_1}^{2c} \\
&\quad + w_2 w_1 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c+1}(\tilde{\tau})\, {w_2}^{2a} {w_3}^{2b} {w_1}^{2c}. \qquad (9.40)
\end{aligned}
$$

The system of bifurcation equations is expressed as (9.28) with F_1, F_2, and F_3 given in (9.37), (9.39), and (9.40), respectively. Equation (9.28) has the trivial solution $w_1 = w_2 = w_3 = 0$. Nontrivial solutions of (9.28) are analyzed hereafter.

9.4.3 Bifurcating Solutions

We search for nontrivial solutions with $w_1 = w_2 = w_3 \neq 0$, which represent Lösch's hexagon with $D = 4$ in Fig. 9.1 (Christaller's $k = 4$ system in Fig. 9.2b). This is consistent with the analysis conducted in Sect. 8.5, where the existence of

[7] The condition (9.38) plays no critical role in solving the bifurcation equations although it puts some constraints on some terms.

such a bifurcating solution was shown by the equivariant branching lemma. See Remark 9.4 for other solutions.

With $w_1 = w_2 = w_3 \neq 0$, the three equations in (9.28) become identical and they reduce, when divided by w_1, to

$$\sum_{a=0}\sum_{b=0}\sum_{c=0} A_{2a,2b+1,2c}(\tilde{\tau})\, {w_1}^{2(a+b+c)} + w_1 \sum_{a=0}\sum_{b=0}\sum_{c=0} A_{2a+1,2b,2c+1}(\tilde{\tau})\, {w_1}^{2(a+b+c)} = 0. \tag{9.41}$$

The leading part of (9.41) is given as

$$A\tilde{\tau} + A_{101}(0)w_1 = 0$$

with $A = A'_{010}(0)$ in (9.36), where $A_{101}(0) \neq 0$ (generically). By the implicit function theorem, the equation (9.41) can be solved for w_1 as

$$w_1 = \psi(\tilde{\tau}),$$

where

$$\psi(\tilde{\tau}) \approx C\,\tilde{\tau}, \quad \tilde{\tau} \to 0,$$

with $C = -A/A_{101}(0) \neq 0$ (generically). Hence we obtain a bifurcating solution

$$w_1 = w_2 = w_3 = \psi(\tilde{\tau}).$$

By (9.26) and (9.27), the symmetry of the solution $\boldsymbol{w} = (\psi(\tilde{\tau}), \psi(\tilde{\tau}), \psi(\tilde{\tau}))^\top$ is given by

$$\Sigma^{(3;+,+)}(\boldsymbol{w}) = \langle r, s, p_1^2, p_2^2 \rangle.$$

This symmetry is $\Sigma(2,0)$ in the notation of (8.10) and is in agreement with Proposition 8.4 in Sect. 8.5.

Other conjugate solutions are obtainable from $\boldsymbol{w}$ as $T^{(3;+,+)}(p_1)\boldsymbol{w}$, $T^{(3;+,+)}(p_2)\boldsymbol{w}$, or $T^{(3;+,+)}(p_1p_2)\boldsymbol{w}$, and the symmetry of these solutions is given by (2.130) as

$$p_1 \cdot \Sigma^{(3;+,+)}(\boldsymbol{w}) \cdot p_1^{-1} = \langle p_2 r, s, p_1^2, p_2^2 \rangle, \tag{9.42}$$

$$p_2 \cdot \Sigma^{(3;+,+)}(\boldsymbol{w}) \cdot p_2^{-1} = \langle p_1 p_2 r, s, p_1^2, p_2^2 \rangle, \tag{9.43}$$

$$p_1 p_2 \cdot \Sigma^{(3;+,+)}(\boldsymbol{w}) \cdot (p_1 p_2)^{-1} = \langle p_1 r, s, p_1^2, p_2^2 \rangle. \tag{9.44}$$

Remark 9.3. As mentioned in Remark 9.1 in Sect. 9.2, it is pertinent to assume that the variable $\boldsymbol{w} = (w_1, w_2, w_3)^\top$ corresponds to the column vectors of

$$Q^{(3;+,+)} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3] = [\ \langle\cos(\pi n_1)\rangle,\ \langle\cos(\pi n_2)\rangle,\ \langle\cos(\pi(n_1 - n_2))\rangle\ ]$$

in (7.33). The spatial patterns for these vectors $\boldsymbol{q}_i$ $(i = 1, 2, 3)$ represent stripe patterns in different directions as shown in Fig. 9.5a–c for $n = 6$. Nonetheless, the sum $\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3$ of these three vectors represents a hexagonal pattern for Lösch's hexagon with $D = 4$ (Fig. 9.5d), for which a fourth of the places (denoted by ○) growing into first-level centers and three-fourths (denoted by *black circle*) shrinking into second-level centers. On the other hand, the vector $-(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3)$ with the sign reversed does not represent Lösch's hexagon with $D = 4$ because as many as three-fourths of places (denoted by *white circle* in Fig. 9.5e) grow in population. Spatial patterns for the conjugate solutions in (9.42)–(9.44) are shown in Fig. 9.5f–h. □

Remark 9.4. Let us search for a solution of the form[8] $w_2 \neq 0$ and $w_1 = w_3 = 0$. Such solution satisfies $F_1 = F_3 = 0$ in (9.39) and (9.40), whereas the remaining one $F_2 = 0$ in (9.37) becomes

$$\sum_{b=0} A_{0,2b+1,0}(\tilde{\tau})\, {w_2}^{2b} = 0. \tag{9.45}$$

The leading part of this equation is given as

$$A\tilde{\tau} + A_{030}(0) w_2^2 = 0$$

with $A = A'_{010}(0)$ in (9.36), where $A_{030}(0) \neq 0$ (generically). By the implicit function theorem, the equation (9.45) can be solved for w_2^2 as

$$w_2^2 = \psi(\tilde{\tau}),$$

that is,

$$w_2 = \pm\sqrt{\psi(\tilde{\tau})}\ \approx\ \pm\sqrt{C\,\tilde{\tau}}, \quad \tilde{\tau} \to 0,$$

with $C = -A/A_{030}(0)$. This is a pitchfork bifurcation (Fig. 2.3); we have $\tilde{\tau} \geq 0$ if $C > 0$, and $\tilde{\tau} \leq 0$ if $C < 0$. We thus obtain $\boldsymbol{w} = (0, \sqrt{\psi(\tilde{\tau})}, 0)^\top$ and $(0, -\sqrt{\psi(\tilde{\tau})}, 0)^\top$.

The symmetry of such solutions $\boldsymbol{w}$ is given as

[8] Such a solution for $n = 2$ is treated in Ikeda and Murota, 2010, Sect. 16.5.2 [2], where $(3; +, +)$ is denoted by $(3, 1)$ as mentioned in Remark 6.2 in Sect. 6.3.4.

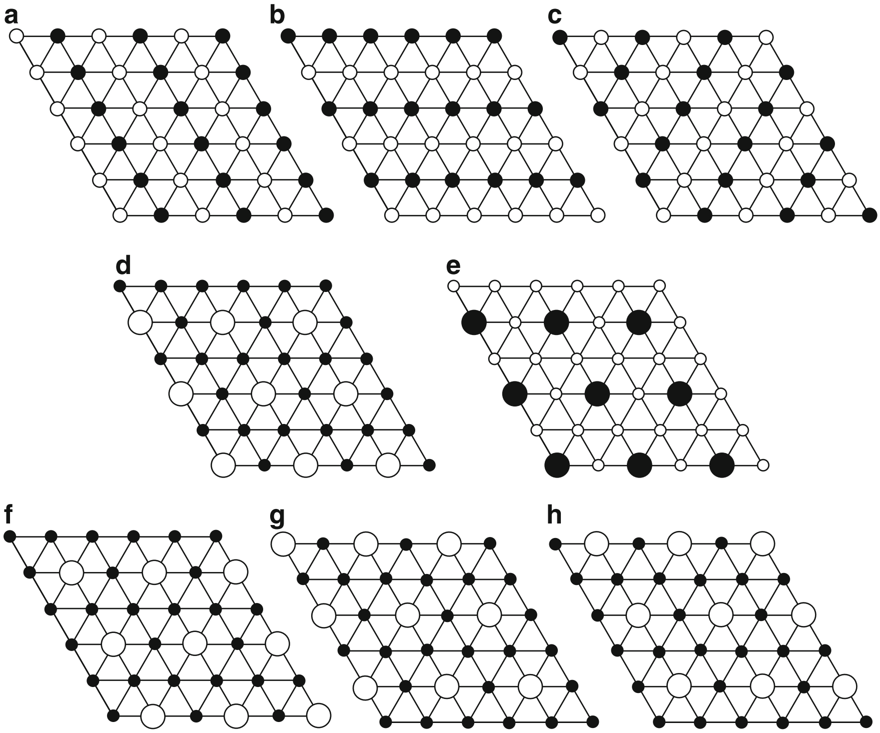

Fig. 9.5 Patterns on the 6×6 hexagonal lattice associated with $\mu = (3; +, +)$. $\rho = |\psi(\tilde{\tau})| > 0$. (**a**) $\boldsymbol{q}_1$ ($\langle r^3, sr^2, p_1^2, p_2\rangle$). (**b**) $\boldsymbol{q}_2$ ($\langle r^3, s, p_1, p_2^2\rangle$). (**c**) $\boldsymbol{q}_3$ ($\langle r^3, sr, p_1 p_2, p_2^2\rangle$). (**d**) $\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3$ ($\Sigma(2,0)$). (**e**) $-(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3)$ ($\Sigma(2,0)$). (**f**) $\boldsymbol{w} = (-\rho, \rho, -\rho)^\top$ ($p_1 \cdot \Sigma(2,0) \cdot p_1^{-1}$). (**g**) $\boldsymbol{w} = (\rho, -\rho, -\rho)^\top$ ($p_2 \cdot \Sigma(2,0) \cdot p_2^{-1}$). (**h**) $\boldsymbol{w} = (-\rho, -\rho, \rho)^\top$ ($p_1 p_2 \cdot \Sigma(2,0) \cdot (p_1 p_2)^{-1}$). A *white circle* denotes a positive component and a *black circle* denotes a negative component

$$\Sigma^{(3;+,+)}(\boldsymbol{w}) = \langle r^3, s, p_1, p_2^2\rangle.$$

This symmetry is different from those of Lösch's hexagons. $\square$

9.5 Bifurcation Point of Multiplicity 6

We derive bifurcating solutions at a group-theoretic critical point of multiplicity $M = 6$ for the irreducible representations $\mu = (6; k, 0, +)$ and $(6; k, k, +)$ of the group $D_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2\rangle$ in (9.2) relevant to the analysis of the hexagonal lattice (Table 9.1). We assume $n \geq 3$.

9.5.1 Representation in Complex Variables

We consider six-dimensional irreducible representations in (8.31) and (8.32):

$$(6;k,0,+) \text{ with } 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor;$$

$$(6;k,k,+) \text{ with } 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor,\ k \ne \frac{n}{3}.$$

The actions in $(6;k,0,+)$ on 6-dimensional vectors $(w_1,\dots,w_6)$ can be expressed for complex variables $(z_1,z_2,z_3) = (w_1+\mathrm{i}w_2, w_3+\mathrm{i}w_4, w_5+\mathrm{i}w_6)$ as

$$\begin{aligned} &r: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}, \qquad s: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_2 \\ \bar{z}_1 \end{bmatrix}, \\ &p_1: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ z_2 \\ \omega^{-k} z_3 \end{bmatrix}, \ p_2: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \omega^{-k} z_2 \\ \omega^k z_3 \end{bmatrix} \end{aligned} \tag{9.46}$$

by (8.38), where $\omega = \exp(\mathrm{i}2\pi/n)$. On the other hand, the actions in $(6;k,k,+)$ are given by (8.39) as

$$\begin{aligned} &r: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}, \qquad s: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} z_2 \\ z_1 \\ z_3 \end{bmatrix}, \\ &p_1: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^k z_2 \\ \omega^{-2k} z_3 \end{bmatrix}, \ p_2: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-2k} z_2 \\ \omega^k z_3 \end{bmatrix}. \end{aligned} \tag{9.47}$$

The actions of p_1 and p_2 in $(6;k,0,+)$ and $(6;k,k,+)$ are expressed for $\ell \in \{0,k\}$ in a unified manner as

$$p_1: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^\ell z_2 \\ \omega^{-k-\ell} z_3 \end{bmatrix}, \quad p_2: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ \omega^{-k-\ell} z_2 \\ \omega^k z_3 \end{bmatrix}. \tag{9.48}$$

9.5.2 Equivariance of Bifurcation Equation

The bifurcation equation for the group-theoretic critical point of multiplicity 6 is a 6-dimensional equation over $\mathbb{R}$. This equation can be expressed as a three-dimensional complex-valued equation in complex variables z_1, z_2, z_3 as

$$F_i(z_1,z_2,z_3,\bar{z}_1,\bar{z}_2,\bar{z}_3,\tilde{\tau}) = 0, \quad i = 1,2,3, \tag{9.49}$$

where

$$(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3, \tilde{\tau}) = (0, \dots, 0, 0)$$

is assumed to correspond to the critical point. We often omit $\tilde{\tau}$ in the subsequent derivation.

The equivariance of the bifurcation equation to the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, p_2 generating this group. Therefore, the equivariance condition of the bifurcation equation can be written as below.

The action of s is given for $(6; k, 0; +)$ as

$$\begin{aligned} s: \overline{F_3(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)} &= F_1(\bar{z}_3, \bar{z}_2, \bar{z}_1, z_3, z_2, z_1), \\ \overline{F_2(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)} &= F_2(\bar{z}_3, \bar{z}_2, \bar{z}_1, z_3, z_2, z_1), \\ \overline{F_1(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)} &= F_3(\bar{z}_3, \bar{z}_2, \bar{z}_1, z_3, z_2, z_1), \end{aligned} \tag{9.50}$$

and for $(6; k, k; +)$ as

$$\begin{aligned} s: F_2(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_1(z_2, z_1, z_3, \bar{z}_2, \bar{z}_1, \bar{z}_3), \\ F_1(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_2(z_2, z_1, z_3, \bar{z}_2, \bar{z}_1, \bar{z}_3), \\ F_3(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_3(z_2, z_1, z_3, \bar{z}_2, \bar{z}_1, \bar{z}_3). \end{aligned} \tag{9.51}$$

The actions of r, p_1, and p_2 are given, for $(6; k, \ell; +)$ ($\ell = 0$ or k), as

$$\begin{aligned} r: \overline{F_3(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)} &= F_1(\bar{z}_3, \bar{z}_1, \bar{z}_2, z_3, z_1, z_2), \\ \overline{F_1(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)} &= F_2(\bar{z}_3, \bar{z}_1, \bar{z}_2, z_3, z_1, z_2), \\ \overline{F_2(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)} &= F_3(\bar{z}_3, \bar{z}_1, \bar{z}_2, z_3, z_1, z_2); \end{aligned} \tag{9.52}$$

$$\begin{aligned} p_1: \quad \omega^k F_1(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_1(\omega^k z_1, \omega^\ell z_2, \omega^{-k-\ell} z_3, \omega^{-k} \bar{z}_1, \omega^{-\ell} \bar{z}_2, \omega^{k+\ell} \bar{z}_3), \\ \omega^\ell F_2(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_2(\omega^k z_1, \omega^\ell z_2, \omega^{-k-\ell} z_3, \omega^{-k} \bar{z}_1, \omega^{-\ell} \bar{z}_2, \omega^{k+\ell} \bar{z}_3), \\ \omega^{-k-\ell} F_3(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_3(\omega^k z_1, \omega^\ell z_2, \omega^{-k-\ell} z_3, \omega^{-k} \bar{z}_1, \omega^{-\ell} \bar{z}_2, \omega^{k+\ell} \bar{z}_3); \end{aligned} \tag{9.53}$$

$$\begin{aligned} p_2: \quad \omega^\ell F_1(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_1(\omega^\ell z_1, \omega^{-k-\ell} z_2, \omega^k z_3, \omega^{-\ell} \bar{z}_1, \omega^{k+\ell} \bar{z}_2, \omega^{-k} \bar{z}_3), \\ \omega^{-k-\ell} F_2(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_2(\omega^\ell z_1, \omega^{-k-\ell} z_2, \omega^k z_3, \omega^{-\ell} \bar{z}_1, \omega^{k+\ell} \bar{z}_2, \omega^{-k} \bar{z}_3), \\ \omega^k F_3(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) &= F_3(\omega^\ell z_1, \omega^{-k-\ell} z_2, \omega^k z_3, \omega^{-\ell} \bar{z}_1, \omega^{k+\ell} \bar{z}_2, \omega^{-k} \bar{z}_3). \end{aligned} \tag{9.54}$$

We expand F_1 as

$$F_1(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3) = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}\sum_{e=0}\sum_{g=0} A_{abcdeg}(\tilde{\tau}) z_1^a z_2^b z_3^c \overline{z}_1^d \overline{z}_2^e \overline{z}_3^g. \quad (9.55)$$

Since $(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3, \tilde{\tau}) = (0, \ldots, 0, 0)$ corresponds to the critical point, we have

$$A_{000000}(0) = 0, \quad A_{100000}(0) = A_{010000}(0) = \cdots = A_{000001}(0) = 0. \quad (9.56)$$

Therefore, we have

$$A_{100000}(\tilde{\tau}) \approx A\tilde{\tau} \quad (9.57)$$

for $A = A'_{100000}(0)$, which is generically nonzero.

The equivariance condition with respect to r is equivalent to

$$F_1(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3) = \overline{F_1(\overline{z}_1, \overline{z}_2, \overline{z}_3, z_1, z_2, z_3)}, \quad (9.58)$$

$$F_2(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3) = F_1(z_2, z_3, z_1, \overline{z}_2, \overline{z}_3, \overline{z}_1), \quad (9.59)$$

$$F_3(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3) = F_1(z_3, z_1, z_2, \overline{z}_3, \overline{z}_1, \overline{z}_2). \quad (9.60)$$

We see that all $A_{abcdeg}(\tilde{\tau})$ are real from (9.58).

The equivariance condition with respect to s, combined with (9.59) and (9.60), gives

$$F_1(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3) = F_1(z_1, z_3, z_2, \overline{z}_1, \overline{z}_3, \overline{z}_2)$$

for both $(6; k, 0, +)$ and $(6; k, k, +)$. This condition is equivalent to[9]

$$A_{abcdeg}(\tilde{\tau}) = A_{acbdge}(\tilde{\tau}) \quad (9.61)$$

for all (a, b, c, d, e, g).

Next we determine the set of indices (a, b, c, d, e, g) of nonvanishing coefficients $A_{abcdeg}(\tilde{\tau})$ in (9.55). The equivariance conditions (9.53) with respect to p_1 and (9.54) to p_2 are expressed, respectively, as

$$k(a-d) + \ell(b-e) - (k+\ell)(c-g) \equiv k \mod n, \quad (9.62)$$

$$\ell(a-d) - (k+\ell)(b-e) + k(c-g) \equiv \ell \mod n. \quad (9.63)$$

In the case of $(6; k, \ell, +) = (6; k, 0, +)$ the above conditions are equivalent to

$$a - d - c + g - 1 \equiv 0 \mod \hat{n}, \quad -b + e + c - g \equiv 0 \mod \hat{n}, \quad (9.64)$$

[9]The condition (9.61) plays no critical role in solving the bifurcation equations although it puts some constraints on some terms.

where

$$\hat{n} = \frac{n}{\gcd(k,n)}.$$

In the case of $(6; k, \ell, +) = (6; k, k, +)$, they are equivalent to

$$a - d + b - e - 2c + 2g \equiv 1 \mod \hat{n}, \quad a - d - 2b + 2e + c - g \equiv 1 \mod \hat{n},$$

which can be rewritten as

$$a - d - c + g - 1 \equiv 0 \mod \tilde{n}, \quad -b + e + c - g \equiv 0 \mod \tilde{n}, \tag{9.65}$$

where

$$\tilde{n} = \begin{cases} \hat{n}/3 & \text{if } \hat{n} \in 3\mathbb{Z}, \\ \hat{n} & \text{otherwise.} \end{cases}$$

We denote by P the set of nonnegative indices (a, b, c, d, e, g) that satisfy the above conditions, i.e.,

$$P = \begin{cases} \{(a, b, c, d, e, g) \in \mathbb{Z}_+^6 \mid (9.64)\} & \text{for } (6; k, 0, +), \\ \{(a, b, c, d, e, g) \in \mathbb{Z}_+^6 \mid (9.65)\} & \text{for } (6; k, k, +), \end{cases} \tag{9.66}$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Then (a, b, c, d, e, g) must belong to P if $A_{abcdeg}(\tilde{\tau}) \neq 0$, and hence (9.55) can be replaced by

$$F_1(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3) = \sum_P A_{abcdeg}(\tilde{\tau}) z_1^a z_2^b z_3^c \overline{z}_1^d \overline{z}_2^e \overline{z}_3^g. \tag{9.67}$$

We have $A_{abcdeg}(\tilde{\tau}) \neq 0$ (generically) for $(a, b, c, d, e, g) \in P$.

By the condition, (9.64) or (9.65), for P in (9.66), we have

$$(0, 0, 0, 0, 0, 0) \notin P, \tag{9.68}$$

$$(1, 0, 0, 0, 0, 0), \ (0, 0, 0, 0, 1, 1) \in P, \tag{9.69}$$

as well as

$$(0, 1, 0, 0, 0, 0), \ (0, 0, 1, 0, 0, 0), \ (0, 0, 0, 1, 0, 0),$$
$$(0, 0, 0, 0, 1, 0), \ (0, 0, 0, 0, 0, 1) \ \notin P.$$

The bifurcation equation F_1, accordingly, is restricted to a form of

$$F_1 = A_{100000} z_1 + A_{000011} \overline{z}_2 \overline{z}_3 + (\text{other terms}). \tag{9.70}$$

The form of "(other terms)" depends on the irreducible representation $(6; k, \ell, +)$ with $\ell \in \{0, k\}$ and the size n of the hexagonal lattice.

9.5.3 Bifurcating Solutions

We search for solutions with $z_1 = z_2 = z_3 = x$ for $x \in \mathbb{R}$ and $x \neq 0$, which represent Lösch's hexagons of type V or type M. This is consistent with the analysis conducted in Sect. 8.6, where the existence of such bifurcating solution was shown by the equivariant branching lemma. See Remark 9.6 for other solutions.

With $z_1 = z_2 = z_3 = x \in \mathbb{R}$, the three equations in (9.49) become identical. Then using (9.67), we have

$$F_i(x, x, x, x, x, x) = \sum_P A_{abcdeg}(\tilde{\tau}) x^{a+b+c+d+e+g}$$

for $i = 1, 2, 3$. Since $(0, 0, 0, 0, 0, 0) \notin P$ by (9.68), it is possible to divide this by x to arrive at

$$\frac{1}{x} F_i(x, x, x, x, x, x) = \sum_P A_{abcedg}(\tilde{\tau}) x^{a+b+c+d+e+g-1}$$

and the bifurcating solution is determined from

$$\sum_P A_{abcedg}(\tilde{\tau}) x^{a+b+c+d+e+g-1} = 0. \tag{9.71}$$

The leading part of (9.71) is given as

$$A\tilde{\tau} + Bx = 0 \tag{9.72}$$

with generically nonzero coefficients A and B (see Remark 9.5 below). By the implicit function theorem, the equation (9.71) can be solved for x as

$$x = \psi(\tilde{\tau}),$$

where

$$\psi(\tilde{\tau}) \approx C\,\tilde{\tau}, \quad \tilde{\tau} \to 0, \tag{9.73}$$

with $C = -A/B \neq 0$. Hence we obtain a bifurcating solution

$$z_1 = z_2 = z_3 = \psi(\tilde{\tau}). \tag{9.74}$$

It should be noted that the asymptotic form of the bifurcating solution, which cannot be known from the equivariant branching lemma (Chap. 8), has thus been obtained successfully.

The symmetry of the bifurcating solution $z = (z_1, z_2, z_3) = (\psi(\tilde{\tau}), \psi(\tilde{\tau}), \psi(\tilde{\tau}))$ in (9.74) is determined as follows. In the case of $(6; k, 0, +)$, the symmetry of z is given by Proposition 8.5(iii) in Sect. 8.6.2 as[10]

$$\Sigma^{(6;k,0,+)}(z) = \Sigma(\hat{n}, 0),$$

where $\hat{n} = n/\gcd(k, n) \geq 3$. In the case of $(6; k, k, +)$, the symmetry of z is given by Proposition 8.6(iii) as

$$\Sigma^{(6;k,k,+)}(z) = \begin{cases} \Sigma(2\hat{n}/3, \hat{n}/3) & \text{if } \hat{n} \text{ is a multiple of 3,} \\ \Sigma(\hat{n}, 0) & \text{otherwise.} \end{cases}$$

This is consistent with Propositions 8.8 and 8.10 showing, respectively, the existence of solutions of type V and type M by the equivariant branching lemma.

Remark 9.5. The concrete forms of the coefficients A and B in (9.72) are investigated here. First note that $a + b + c + d + e + g \geq 1$ for all $(a, b, c, d, e, g) \in P$ with the equality holding only for $(a, b, c, d, e, g) = (1, 0, 0, 0, 0, 0)$. This shows

$$A = A'_{100000}(0), \tag{9.75}$$

which is generically nonzero. The other coefficient B is given as the sum of $A_{abcdeg}(0)$ over all $(a, b, c, d, e, g) \in P$ with $a + b + c + d + e + g = 2$. As a summand in B, we surely have $A_{000011}(0)$ by (9.69). In addition, we have $A_{000200}(0)$ for $\hat{n} = 3$ and $(6; k, \ell, +) = (6; k, 0, +)$. This implies that B is generically distinct from zero since other summands, which may or may not exist, cannot cancel the term $A_{000011}(0)$ in the generic situation. □

Remark 9.6. Let us search for a solution of the form $z_1 = x$ with $x \in \mathbb{R}$ and $x \neq 0$ and $z_2 = z_3 = 0$. Such solution satisfies $F_2 = F_3 = 0$, whereas the remaining equation $F_1 = 0$ with F_1 in (9.67) reduces, when divided by $x \neq 0$, to

$$\frac{1}{x} F_1(x, 0, 0, x, 0, 0) = \sum_{(a,0,0,d,0,0) \in P} A_{a00d00}(\tilde{\tau}) x^{a+d-1} = 0. \tag{9.76}$$

The leading part of this equation is given as

$$\begin{cases} A\tilde{\tau} + A_{000200}(0)x = 0 & \text{for } \hat{n} = 3 \text{ and } (6; k, 0, +), \\ A\tilde{\tau} + (A_{200100}(0) + A_{000300}(0))x^2 = 0 & \text{for } \hat{n} = 4 \text{ and } (6; k, 0, +), (6; k, k, +), \\ A\tilde{\tau} + A_{200100}(0)x^2 = 0 & \text{for } \hat{n} \geq 5 \text{ and } (6; k, 0, +), (6; k, k, +), \end{cases}$$

with $A = A'_{100000}(0)$ in (9.57), where $A_{000200}(0)$, $A_{200100}(0)$, and $A_{000300}(0)$ are nonzero (generically).

[10] $\Sigma^{(6;k,\ell,+)}(z)$ is defined in (8.41).

By (9.46) and (9.47), the bifurcating solution $z = (x, 0, 0)$ has symmetries of

$$\Sigma^{(6;k,0,+)}(z) = \langle rs, r^3, p_1^{\hat{n}}, p_2 \rangle, \quad \Sigma^{(6;k,k,+)}(z) = \langle r^2 s, r^3, p_1^{\hat{n}}, p_1 p_2^{-1} \rangle.$$

These symmetries are different from those of Lösch's hexagons. □

9.6 Bifurcation Point of Multiplicity 12

We derive bifurcating solutions at a group-theoretic critical point of multiplicity $M = 12$ for the irreducible representation $\mu = (12; k, \ell)$ of the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ in (9.2) relevant to the analysis of the hexagonal lattice (Table 9.1). We assume $n \geq 6$.

9.6.1 Representation in Complex Variables

Recall from (6.35) that we can designate the twelve-dimensional irreducible representations by $(12; k, \ell)$ with

$$1 \leq \ell \leq k - 1, \quad 2k + \ell \leq n - 1. \tag{9.77}$$

The action in $(12; k, \ell)$ on 12-dimensional vectors $(w_1, \ldots, w_{12})$ can be expressed for complex variables $z_j = w_{2j-1} + \mathrm{i} w_{2j}$ $(j = 1, \ldots, 6)$ as

$$r : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_3 \\ \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_5 \\ \bar{z}_6 \\ \bar{z}_4 \end{bmatrix}, \quad s : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}, \tag{9.78}$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^\ell z_2 \\ \omega^{-k-\ell} z_3 \\ \omega^k z_4 \\ \omega^\ell z_5 \\ \omega^{-k-\ell} z_6 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ \omega^{-k-\ell} z_2 \\ \omega^k z_3 \\ \omega^{-k-\ell} z_4 \\ \omega^k z_5 \\ \omega^\ell z_6 \end{bmatrix} \tag{9.79}$$

by (8.59) and (8.60), where $\omega = \exp(\mathrm{i}2\pi/n)$.

9.6.2 Equivariance of Bifurcation Equation

The bifurcation equation for the group-theoretic critical point of multiplicity 12 is a 12-dimensional equation over $\mathbb{R}$. This equation can be expressed as a 6-dimensional complex-valued equation in complex variables $z_1, \dots, z_6$ as

$$F_i(z_1, \dots, z_6, \overline{z}_1, \dots, \overline{z}_6, \tilde{\tau}) = 0, \quad i = 1, \dots, 6, \tag{9.80}$$

where

$$(z_1, \dots, z_6, \overline{z}_1, \dots, \overline{z}_6, \tilde{\tau}) = (0, \dots, 0, 0, \dots, 0, 0)$$

is assumed to correspond to the critical point. For notational simplicity we write (9.80) as

$$F_i(z_1, \dots, z_6) = 0, \quad i = 1, \dots, 6 \tag{9.81}$$

by omitting $\overline{z}_1, \dots, \overline{z}_6$ and $\tilde{\tau}$ in the subsequent derivation.

The equivariance of the bifurcation equation to the group $\mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, p_2 generating this group. Therefore, the equivariance condition for (9.81) can be written as

$$r: \ \overline{F_3(z_1, z_2, z_3, z_4, z_5, z_6)} = F_1(\overline{z}_3, \overline{z}_1, \overline{z}_2, \overline{z}_5, \overline{z}_6, \overline{z}_4), \tag{9.82}$$
$$\overline{F_1(z_1, z_2, z_3, z_4, z_5, z_6)} = F_2(\overline{z}_3, \overline{z}_1, \overline{z}_2, \overline{z}_5, \overline{z}_6, \overline{z}_4), \tag{9.83}$$
$$\overline{F_2(z_1, z_2, z_3, z_4, z_5, z_6)} = F_3(\overline{z}_3, \overline{z}_1, \overline{z}_2, \overline{z}_5, \overline{z}_6, \overline{z}_4), \tag{9.84}$$
$$\overline{F_5(z_1, z_2, z_3, z_4, z_5, z_6)} = F_4(\overline{z}_3, \overline{z}_1, \overline{z}_2, \overline{z}_5, \overline{z}_6, \overline{z}_4), \tag{9.85}$$
$$\overline{F_6(z_1, z_2, z_3, z_4, z_5, z_6)} = F_5(\overline{z}_3, \overline{z}_1, \overline{z}_2, \overline{z}_5, \overline{z}_6, \overline{z}_4), \tag{9.86}$$
$$\overline{F_4(z_1, z_2, z_3, z_4, z_5, z_6)} = F_6(\overline{z}_3, \overline{z}_1, \overline{z}_2, \overline{z}_5, \overline{z}_6, \overline{z}_4); \tag{9.87}$$
$$s: \ F_{i+3}(z_1, z_2, z_3, z_4, z_5, z_6) = F_i(z_4, z_5, z_6, z_1, z_2, z_3), \quad i = 1, 2, 3, \tag{9.88}$$
$$F_i(z_1, z_2, z_3, z_4, z_5, z_6) = F_{i+3}(z_4, z_5, z_6, z_1, z_2, z_3), \quad i = 1, 2, 3; \tag{9.89}$$
$$p_1: \omega_{1i} F_i(z_1, \dots, z_6) = F_i(\omega_{11} z_1, \dots, \omega_{16} z_6), \quad i = 1, \dots, 6; \tag{9.90}$$
$$p_2: \omega_{2i} F_i(z_1, \dots, z_6) = F_i(\omega_{21} z_1, \dots, \omega_{26} z_6), \quad i = 1, \dots, 6, \tag{9.91}$$

where

$$(\omega_{11}, \dots, \omega_{16}) = (\omega^k, \omega^\ell, \omega^{-k-\ell}, \omega^k, \omega^\ell, \omega^{-k-\ell}),$$
$$(\omega_{21}, \dots, \omega_{26}) = (\omega^\ell, \omega^{-k-\ell}, \omega^k, \omega^{-k-\ell}, \omega^k, \omega^\ell).$$

We expand F_1 as

$$F_1(z_1, z_2, z_3, z_4, z_5, z_6)$$
$$= \sum_{a=0}\sum_{b=0}\cdots\sum_{u=0} A_{abcdeghijstu}(\tilde{\tau}) z_1^a z_2^b z_3^c z_4^d z_5^e z_6^g \bar{z}_1^h \bar{z}_2^i \bar{z}_3^j \bar{z}_4^s \bar{z}_5^t \bar{z}_6^u. \tag{9.92}$$

Since $(z_1, z_2, z_3, z_4, z_5, z_6, \tilde{\tau}) = (0, 0, 0, 0, 0, 0, 0)$ corresponds to the critical point of multiplicity 12, we have

$$A_{000000000000}(0) = 0, \tag{9.93}$$
$$A_{100000000000}(0) = A_{010000000000}(0) = \cdots = A_{000000000001}(0) = 0. \tag{9.94}$$

The equivariance conditions (9.82)–(9.84) with respect to r give

$$\begin{aligned} F_1(z_1, z_2, z_3, z_4, z_5, z_6) &= \overline{F_2(\bar{z}_3, \bar{z}_1, \bar{z}_2, \bar{z}_5, \bar{z}_6, \bar{z}_4)} \\ &= F_3(z_2, z_3, z_1, z_6, z_4, z_5) \\ &= \overline{F_1(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5, \bar{z}_6)}, \end{aligned}$$

from which we see that all $A_{ab\cdots tu}(\tilde{\tau})$ are real. Then, by the equivariance conditions (9.82)–(9.87) and (9.88)–(9.89) with respect to r and s, $F_2, \ldots, F_6$ are obtained from F_1 as

$$F_2(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_2, z_3, z_1, z_6, z_4, z_5), \tag{9.95}$$
$$F_3(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_3, z_1, z_2, z_5, z_6, z_4), \tag{9.96}$$
$$F_4(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_4, z_5, z_6, z_1, z_2, z_3), \tag{9.97}$$
$$F_5(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_5, z_6, z_4, z_3, z_1, z_2), \tag{9.98}$$
$$F_6(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_6, z_4, z_5, z_2, z_3, z_1). \tag{9.99}$$

Next we determine the set of indices $(a, b, \ldots, t, u)$ of nonvanishing coefficients $A_{ab\cdots tu}(\tilde{\tau})$ in (9.92). The equivariance conditions (9.90) by p_1 and (9.91) by p_2 yield

$$k(a-h) + \ell(b-i) - (k+\ell)(c-j) + k(d-s) + \ell(e-t) - (k+\ell)(g-u)$$
$$\equiv k \mod n, \tag{9.100}$$
$$\ell(a-h) - (k+\ell)(b-i) + k(c-j) - (k+\ell)(d-s) + k(e-t) + \ell(g-u)$$
$$\equiv \ell \mod n, \tag{9.101}$$

which are equivalent to

$$\hat{k}(a-h) + \hat{\ell}(b-i) - (\hat{k}+\hat{\ell})(c-j) + \hat{k}(d-s) + \hat{\ell}(e-t) - (\hat{k}+\hat{\ell})(g-u)$$
$$\equiv \hat{k} \mod \hat{n}, \tag{9.102}$$

$$\hat{\ell}(a-h)-(\hat{k}+\hat{\ell})(b-i)+\hat{k}(c-j)-(\hat{k}+\hat{\ell})(d-s)+\hat{k}(e-t)+\hat{\ell}(g-u) \\ \equiv \hat{\ell} \mod \hat{n}, \tag{9.103}$$

where

$$\hat{k}=\frac{k}{\gcd(k,\ell,n)}, \quad \hat{\ell}=\frac{\ell}{\gcd(k,\ell,n)}, \quad \hat{n}=\frac{n}{\gcd(k,\ell,n)}.$$

We denote by S the set of nonnegative indices $(a,b,\dots,t,u)$ that satisfy the above conditions, i.e.,

$$S=\{(a,b,\dots,t,u)\in\mathbb{Z}_+^{12} \mid (9.102) \text{ and } (9.103)\}, \tag{9.104}$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Then $(a,b,\dots,t,u)$ must belong to S if $A_{ab\cdots tu}(\tilde{\tau})\neq 0$, and hence (9.92) can be replaced by

$$F_1(z_1,z_2,z_3,z_4,z_5,z_6)=\sum_S A_{abcdeghijstu}(\tilde{\tau}) z_1^a z_2^b z_3^c z_4^d z_5^e z_6^g \bar{z}_1^h \bar{z}_2^i \bar{z}_3^j \bar{z}_4^s \bar{z}_5^t \bar{z}_6^u. \tag{9.105}$$

We have $A_{ab\cdots tu}(\tilde{\tau})\neq 0$ (generically) for $(a,b,\dots,t,u)\in S$.

By the conditions (9.102) and (9.103) for S in (9.104), we have

$$(0,0,0,0,0,0,0,0,0,0,0,0)\notin S, \tag{9.106}$$

$$(1,0,0,0,0,0,0,0,0,0,0,0),\ (0,0,0,0,0,0,0,1,1,0,0,0)\in S. \tag{9.107}$$

To prove (9.106) by contradiction, suppose $(0,0,0,0,0,0,0,0,0,0,0,0)\in S$. Then (9.102) and (9.103) imply that both $\hat{k}$ and $\hat{\ell}$ are multiples of $\hat{n}$. This is a contradiction since $\gcd(\hat{k},\hat{\ell},\hat{n})=1$ and $\hat{n}\geq 3$. In a similar manner, we can show that the following tuples do not belong to S:

$$\begin{aligned}
&(0,1,0,0,0,0,0,0,0,0,0,0),\ (0,0,1,0,0,0,0,0,0,0,0,0),\\
&(0,0,0,1,0,0,0,0,0,0,0,0),\ (0,0,0,0,1,0,0,0,0,0,0,0),\\
&(0,0,0,0,0,1,0,0,0,0,0,0),\ (0,0,0,0,0,0,1,0,0,0,0,0),\\
&(0,0,0,0,0,0,0,1,0,0,0,0),\ (0,0,0,0,0,0,0,0,1,0,0,0),\\
&(0,0,0,0,0,0,0,0,0,1,0,0),\ (0,0,0,0,0,0,0,0,0,0,1,0),\\
&(0,0,0,0,0,0,0,0,0,0,0,1)\ \notin S.
\end{aligned}$$

The bifurcation equation F_1, accordingly, is restricted to a form of

$$F_1=A_{100000000000}z_1+A_{000000011000}\bar{z}_2\bar{z}_3+(\text{other terms}). \tag{9.108}$$

The form of "(other terms)" depends on the irreducible representation $(12; k, \ell)$ and the size n of the hexagonal lattice. Those terms govern the existence or nonexistence of bifurcating solutions of respective types. The following examples give expanded forms of two different kinds that are studied in the remainder of this section.

Example 9.1. For $(k, \ell, n) = (2, 1, 6)$, the expanded form (9.108) (or (9.105)) becomes

$$\begin{aligned} F_1 = {} & A_1 z_1 + A_2 \bar{z}_2 \bar{z}_3 + (A_3 z_1^2 \bar{z}_1 + A_4 z_1 z_2 \bar{z}_2 + A_5 z_1 z_3 \bar{z}_3 + A_6 z_1 z_4 \bar{z}_4 + A_7 z_1 z_5 \bar{z}_5 \\ & + A_8 z_1 z_6 \bar{z}_6 + A_9 z_2 \bar{z}_4 z_6 + A_{10} z_3 \bar{z}_4 z_5 + A_{11} \bar{z}_1 z_2 \bar{z}_6 + A_{12} z_3^2 z_4 + A_{13} \bar{z}_1 \bar{z}_5^2) \\ & + [A_{14} z_4 \bar{z}_6^2 + A_{15} \bar{z}_5 z_6^3 + A_{16} \bar{z}_5 \bar{z}_6^3 + \cdots] + \cdots \end{aligned} \tag{9.109}$$

for some constants A_i $(i = 1, 2, \ldots)$. It is shown in Sect. 9.6.3, however, that this equation does not have a solution for a hexagon of type T due to the terms in $[\cdots]$. □

Example 9.2. For $(k, \ell, n) = (2, 1, 7)$, the expanded form (9.108) (or (9.105)) becomes

$$\begin{aligned} F_1 = {} & A_1 z_1 + (A_2 \bar{z}_2 \bar{z}_3 + A_3 \bar{z}_1 z_3 + A_4 z_2^2) \\ & + (A_5 z_1^2 \bar{z}_1 + A_6 z_1 z_2 \bar{z}_2 + A_7 z_1 z_3 \bar{z}_3 + A_8 z_1 z_4 \bar{z}_4 + A_9 z_1 z_5 \bar{z}_5 + A_{10} z_1 z_6 \bar{z}_6 \\ & + A_{11} \bar{z}_1 z_2 \bar{z}_3 + A_{12} z_2 z_3^2 + A_{13} \bar{z}_2^2 z_3 + A_{14} \bar{z}_1^2 \bar{z}_2 + A_{15} \bar{z}_3^3) + \cdots \end{aligned} \tag{9.110}$$

for some constants A_i $(i = 1, 2, \ldots)$. It is shown in Sect. 9.6.3 that this equation has a solution for a hexagon of type T. □

9.6.3 Bifurcating Solutions

The set of bifurcation equations $F_i(z_1, \ldots, z_6) = 0$ $(i = 1, \ldots, 6)$ in (9.81) for a critical point of the multiplicity 12 associated with a 12-dimensional irreducible representation $(12; k, \ell)$ can engender bifurcating solutions of several kinds. Among these solutions, Lösch's hexagons of all three types are considered here. As has already been made clear by Proposition 8.21 in Sect. 8.7.4, the existence and nonexistence of such hexagons depend on the value of (k, ℓ), as classified in Table 9.2, which is duplicated from Table 8.5 in Sect. 8.7.2. There are four cases to consider

- Case 1: Hexagon of type V exists.
- Case 2: Hexagon of type M exists.
- Case 3: Hexagons of type V and type T exist.
- Case 4: Hexagons of type M and type T exist.

Table 9.2 Classification of bifurcation at a critical point associated with $(12; k, \ell)$ with $(\alpha, \beta) = \Phi(k, \ell, n)$

	$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 3\mathbb{Z}$	$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 3\mathbb{Z}$
	$\hat{D} \notin 3\mathbb{Z}$	$\hat{D} \in 3\mathbb{Z}$
GCD-div	Case 1:	Case 2:
$\beta = 0$ or $\alpha = 2\beta$	type V	type M
$\overline{\textbf{GCD-div}}$	Case 3:	Case 4:
$\beta \neq 0$ and $\alpha \neq 2\beta$	type V and type T	type M and type T

The forms of the solution $z = (z_1, \ldots, z_6)$ for those hexagons have been presented in Proposition 8.20 in Sect. 8.7.3 as

1. Bifurcating solution I: $(z_1, z_2, z_3, z_4, z_5, z_6) = x(1, 1, 1, 1, 1, 1)$ with $x \in \mathbb{R}$.
2. Bifurcating solution II: $(z_1, z_2, z_3, z_4, z_5, z_6) = x(1, 1, 1, 0, 0, 0)$ with $x \in \mathbb{R}$.
3. Bifurcating solution III: $(z_1, z_2, z_3, z_4, z_5, z_6) = x(0, 0, 0, 1, 1, 1)$ with $x \in \mathbb{R}$.

Bifurcating solution I is a possible candidate for hexagons of types V or M in Cases 1–4 and bifurcating solution II is a possible candidate for a hexagon of type T in Cases 3 and 4. Bifurcating solutions II and III are mutually conjugate via the transformation by the reflection s in (9.78). Although we do not exclude the possibility of other bifurcating solutions, those bifurcating solutions I–III are sufficient for our purpose.

Bifurcating Solution I: Hexagons of Types V and M

Bifurcating solutions I of the form

$$z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = x \tag{9.111}$$

with $x \in \mathbb{R}$ and $x \neq 0$ are derived.

Using (9.95)–(9.99) and (9.105), we obtain

$$F_i(x, x, x, x, x, x) = \sum_S A_{abcdeghijstu}(\tilde{\tau}) x^{a+b+c+d+e+g+h+i+j+s+t+u}$$

for $i = 1, \ldots, 6$. Since $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \notin S$ by (9.106), it is possible to divide this by x to arrive at

$$\frac{1}{x} F_i(x, x, x, x, x, x) = \sum_S A_{ab\cdots tu}(\tilde{\tau}) x^{a+b+\cdots+t+u-1}$$

and the bifurcating solution is determined from

$$\sum_{S} A_{ab\cdots tu}(\tilde{\tau})x^{a+b+\cdots+t+u-1} = 0. \tag{9.112}$$

The leading terms of (9.112) are given as

$$A\tilde{\tau} + Bx = 0 \tag{9.113}$$

with generically nonzero coefficients A and B (see Remark 9.7 below). By the implicit function theorem, the equation (9.112) can be solved for x as

$$x = \psi(\tilde{\tau}),$$

where

$$\psi(\tilde{\tau}) \approx C\,\tilde{\tau}, \quad \tilde{\tau} \to 0 \tag{9.114}$$

with $C = -A/B \neq 0$. Hence we obtain a bifurcating solution

$$z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = \psi(\tilde{\tau}). \tag{9.115}$$

It should be noted that the asymptotic form (9.114) of the bifurcating solution, which cannot be known from the equivariant branching lemma (Chap. 8), has thus been obtained successfully.

The symmetry of the bifurcating solution in (9.115) is determined by Proposition 8.19(i) in Sect. 8.7.3 with Propositions 8.12(i) and 8.14(i) in Sect. 8.7.2 as[11]

$$\Sigma^{(12;k,\ell)}(z) = \begin{cases} \Sigma(\hat{n},0) & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 3\mathbb{Z}, \\ \Sigma(2\hat{n}/3,\hat{n}/3) & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \in 3\mathbb{Z}. \end{cases}$$

This is consistent with Propositions 8.23 and 8.25 showing, respectively, the existence of solutions of type V and type M by the equivariant branching lemma.

Remark 9.7. The concrete forms of the coefficients A and B in (9.113) are investigated here. First note that $a+b+\cdots+t+u \geq 1$ for all $(a,b,\ldots,t,u) \in S$ with the equality holding only for $(a,b,\ldots,t,u) = (1,0,0,0,0,0,0,0,0,0,0,0)$. This shows

$$A = A'_{100000000000}(0), \tag{9.116}$$

which is generically nonzero. The other coefficient B is given as the sum of $A_{ab\ldots tu}(0)$ over all $(a,b,\ldots,t,u) \in S$ with $a+b+\cdots+t+u = 2$. As a

[11] $\Sigma^{(12;k,\ell)}(z)$ is defined in (8.61).

summand in B, we surely have $A_{000000011000}(0)$ by (9.107). This implies that B is generically distinct from zero since other summands, which may or may not exist, cannot cancel this term in the generic situation. For most cases, such as $(k, \ell, n) = (2, 1, 6), (3, 1, 13), (3, 2, 19)$, and $(4, 1, 21)$, we have

$$B = A_{000000011000}(0), \tag{9.117}$$

but for $(k, \ell, n) = (2, 1, 7)$, we have

$$B = A_{000000011000}(0) + A_{001000100000}(0) + A_{020000000000}(0). \tag{9.118}$$

□

Bifurcating Solution II: Hexagon of Type T

Bifurcating solutions II of the form

$$z_1 = z_2 = z_3 = x, \quad z_4 = z_5 = z_6 = 0 \tag{9.119}$$

with $x \in \mathbb{R}$ and $x \neq 0$ are derived.

By (9.95)–(9.99), we have

$$F_i(x, x, x, 0, 0, 0) = F_1(x, x, x, 0, 0, 0), \quad i = 1, 2, 3, \tag{9.120}$$

$$F_i(x, x, x, 0, 0, 0) = F_1(0, 0, 0, x, x, x), \quad i = 4, 5, 6, \tag{9.121}$$

and therefore, the system of equations $F_i(z_1, \dots, z_6) = 0$ $(i = 1, \dots, 6)$ in (9.81) reduces to a pair of equations

$$F_1(x, x, x, 0, 0, 0) = 0, \tag{9.122}$$

$$F_1(0, 0, 0, x, x, x) = 0. \tag{9.123}$$

With the expansion of $F_1(z_1, \dots, z_6)$ in (9.105) we have

$$F_1(x, x, x, 0, 0, 0) = \sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tilde{\tau}) x^{a+b+c+h+i+j}, \tag{9.124}$$

$$F_1(0, 0, 0, x, x, x) = \sum_{(d,e,g,s,t,u)\in Q} A_{000deg000stu}(\tilde{\tau}) x^{d+e+g+s+t+u}, \tag{9.125}$$

where P and Q are defined using S in (9.104) as

$$P = \{(a, b, c, h, i, j) \mid (a, b, c, 0, 0, 0, h, i, j, 0, 0, 0) \in S\}, \tag{9.126}$$

$$Q = \{(d, e, g, s, t, u) \mid (0, 0, 0, d, e, g, 0, 0, 0, s, t, u) \in S\}. \tag{9.127}$$

It follows from (9.106) and (9.107) that

$$(0,0,0,0,0,0) \notin P, \tag{9.128}$$

$$(1,0,0,0,0,0),\ (0,0,0,0,1,1) \in P. \tag{9.129}$$

There are two equations, (9.122) and (9.123), to be satisfied by a single real variable x. Such x does not exist in general, but there are some values of (k,ℓ,n) for which the set Q defined in (9.127) is an empty set. Then the second equation (9.123) is actually void, and a bifurcating solution exists generically, determined from the first equation (9.122).

The following proposition shows that this is the case if and only if (k,ℓ,n) satisfies the condition

$$\overline{\textbf{GCD-div}}: \quad (\hat{k}-\hat{\ell})\gcd(\hat{k},\hat{\ell}) \ \text{ is not divisible by } \ \gcd(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2,\hat{n}), \tag{9.130}$$

which we already encountered in Sect. 8.7.4.

Proposition 9.1. *The set Q in* (9.127) *is empty if and only if* $\overline{\textbf{GCD-div}}$ *holds.*

Proof. The proof is given at the end of this section. □

The condition $\overline{\textbf{GCD-div}}$ does not hold for Example 9.3 but does hold for Example 9.4 below.

Example 9.3. For $(k,\ell,n) = (2,1,6)$, $(\hat{k}-\hat{\ell})\gcd(\hat{k},\hat{\ell}) = 1$ is divisible by $\gcd(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2,\hat{n}) = \gcd(7,6) = 1$, and hence $\overline{\textbf{GCD-div}}$ does not hold. By (9.109) in Example 9.1, on the other hand, we have

$$F_1(x,x,x,0,0,0) = A_1x + A_2x^2 + (A_3+A_4+A_5)x^3 + \cdots,$$
$$F_1(0,0,0,x,x,x) = A_{14}x^3 + (A_{15}+A_{16})x^4 + \cdots$$

for some constants A_i $(i=1,2,\ldots)$. Thus $Q \neq \emptyset$. □

Example 9.4. For $(k,\ell,n) = (2,1,7)$, $(\hat{k}-\hat{\ell})\gcd(\hat{k},\hat{\ell}) = 1$ is not divisible by $\gcd(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2,\hat{n}) = \gcd(7,7) = 7$, and hence $\overline{\textbf{GCD-div}}$ holds. By (9.110) in Example 9.2, on the other hand, we have

$$\begin{aligned} F_1(x,x,x,0,0,0) &= A_1x+(A_2+A_3+A_4)x^2 \\ &\quad +(A_5+A_6+A_7+A_{11}+A_{12}+A_{13}+A_{14}+A_{15})x^3+\cdots, \\ F_1(0,0,0,x,x,x) &= 0 \end{aligned}$$

for some constants A_i $(i=1,2,\ldots)$. Thus $Q = \emptyset$ □

We now focus on the first equation (9.122), assuming $\overline{\textbf{GCD-div}}$, i.e., $Q = \emptyset$, which means that the second equation (9.123) is void. We see from (9.124) that

$$\frac{1}{x}\,F_1(x,x,x,0,0,0) = \sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tilde{\tau})x^{a+b+c+h+i+j-1},$$

and the bifurcating solution is determined from

$$\sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tilde{\tau})x^{a+b+c+h+i+j-1} = 0. \tag{9.131}$$

The leading terms of (9.131) are given as

$$A\tilde{\tau} + Bx = 0 \tag{9.132}$$

with generically nonzero coefficients A and B (see Remark 9.8 below). By the implicit function theorem, the equation (9.131) can be solved for x as

$$x = \psi(\tilde{\tau}),$$

where

$$\psi(\tilde{\tau}) \approx C\,\tilde{\tau}, \quad \tilde{\tau} \to 0,$$

with $C = -A/B \neq 0$. Thus we obtain a bifurcating solution

$$z_1 = z_2 = z_3 = \psi(\tilde{\tau}), \quad z_4 = z_5 = z_6 = 0.$$

It should be noted that the asymptotic form of the bifurcating solution, which cannot be known from the equivariant branching lemma (Chap. 8), has thus been obtained successfully.

The symmetry of the bifurcating solution in (9.119) can be determined by Proposition 8.18(i) as $\Sigma^{(12;k,\ell)}(z) = \Sigma_0(\alpha,\beta)$ for $(\alpha,\beta) = \Phi(k,\ell,n)$.

We have shown that, generically, a bifurcating solution of type T exists if $Q = \emptyset$. The converse is also true. The reason is that the numerical values of the nonzero coefficients of the terms in P and Q are not subject to any constraints under the equivariance, and consequently, the solution to the first equation (9.122) cannot satisfy the second equation (9.123) in the generic situation.

Thus we obtain the following statement.

Proposition 9.2. *A bifurcating solution of type T exists if and only if the set Q in* (9.127) *is empty.*

Proposition 9.2 is consistent with Proposition 8.28 in Sect. 8.7.7 characterizing the existence of type T solution in terms of $\overline{\textbf{GCD-div}}$, which is equivalent to $Q = \emptyset$ by virtue of Proposition 9.1. It should be emphasized that Proposition 8.28 is derived from the equivariant branching lemma, whereas Proposition 9.2 is obtained by investigating nonvanishing terms in the bifurcation equations.

Remark 9.8. The coefficients A and B in (9.132) are investigated here. First note that $a+b+c+h+i+j \geq 1$ for all $(a,b,c,h,i,j) \in P$ with the equality holding only for $(a,b,c,h,i,j) = (1,0,0,0,0,0)$. This shows that A is given by (9.116). The other coefficient B is given as the sum of $A_{abc000hij000}(0)$ over all $(a,b,c,h,i,j) \in P$ with $a+b+c+h+i+j = 2$. As a summand in B, we surely have $A_{000000011000}(0)$ by (9.129). This implies that B is generically distinct from zero since other summands, which may or may not exist, cannot cancel this term in the generic situation. For most cases, such as $(k,\ell,n) = (3,1,13), (3,2,19)$, and $(4,1,21)$, we have $B = A_{000000011000}(0)$ as in (9.117), but for $(k,\ell,n) = (2,1,7)$, for example, we have B in (9.118) containing more terms. □

Proof of Proposition 9.1

Proposition 9.1 is proved by establishing Proposition 9.3 below. Since the condition (b) below is equivalent to **GCD-div** by Proposition 8.37, we obtain the equivalence: $Q \neq \emptyset \iff$ **GCD-div**, and hence $Q = \emptyset \iff \overline{\textbf{GCD-div}}$.

Proposition 9.3. *The following conditions* (a) *and* (b) *are equivalent.*

(a) *Q in* (9.127) *is nonempty.*

(b) *There exists $(u_1, u_2) \in \mathbb{Z}^2$ such that*

$$[u_1 \ u_2] \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k}-\hat{\ell} \end{bmatrix} \equiv [\hat{\ell} \ \hat{k}] \mod \hat{n}.$$

Proof. By the definition of Q in (9.127), we have $(d,e,g,s,t,u) \in Q$ if and only if

$$\hat{k}(d-s) + \hat{\ell}(e-t) - (\hat{k}+\hat{\ell})(g-u) \equiv \hat{k} \mod \hat{n},$$
$$-(\hat{k}+\hat{\ell})(d-s) + \hat{k}(e-t) + \hat{\ell}(g-u) \equiv \hat{\ell} \mod \hat{n}.$$

These conditions can be represented in a matrix form as

$$[e-t-d+s \ \ g-u-d+s] \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k}-\hat{\ell} \end{bmatrix} \equiv [\hat{\ell} \ \hat{k}] \mod \hat{n}.$$

This shows (a) $\Rightarrow$ (b) with $u_1 = e-t-d+s$ and $u_2 = g-u-d+s$. The converse is also true since for any integers u_1 and u_2, there exist tuples of nonnegative integers (e,t,d,s) and (g,u,d,s) such that $u_1 = e-t-d+s$ and $u_2 = g-u-d+s$. □

Remark 9.9. Figure 9.6 is a schematic diagram, already shown as Fig. 8.10 in Sect. 8.7.10, summarizing the relations among conditions equivalent to $\overline{\textbf{GCD-div}}$. □

$\boxed{\overline{\textbf{GCD-div}} \text{ in (8.94)}}$

$\Updownarrow$ Prop. 8.37

$\boxed{\mathscr{A}(k,\ell,n) \neq \mathscr{A}(\ell,k,n)}$ $\overset{\text{(Integer Farkas)}}{\underset{\text{Prop. 8.39}}{\Longleftrightarrow}}$ $\boxed{\nexists\, U \text{ s.t. } U \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k}-\hat{\ell} \end{bmatrix} \equiv \begin{bmatrix} \hat{\ell} & \hat{k} \\ \hat{k} & -\hat{k}-\hat{\ell} \end{bmatrix}}$

(8.97) $\Updownarrow$ $\Updownarrow$ Prop. 9.3

$\boxed{\mathscr{L}(\alpha,\beta) \neq \mathscr{L}(\beta,\alpha)}$ $\boxed{Q \text{ in (9.127) is empty}}$

(Chap.8) $\Updownarrow$ (Equiv. Branch. Lem.) (Chap.9) $\Updownarrow$ (Solv. Bifur. Eq.)

$\boxed{\text{Existence of type T solutions}}$

Fig. 9.6 Technical conditions for type T

9.7 Summary

- Bifurcating solutions for hexagonal distributions of Christaller and Lösch have been obtained by solving bifurcation equations.
- Asymptotic forms of bifurcating equilibrium paths and the directions of these paths have been obtained.
- Use of the bifurcation equation has been demonstrated to be effective in the investigation of bifurcating solutions.

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