HANDBOOK OF INCIDENCE GEOMETRY

Buildings and Foundations

edited by
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Université Libre de Bruxelles, Belgium

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Preface

Incidence Geometry has undergone deep and quick changes during the last 25 years. One of these was to find its identity. It had been spread over a series of subjects like geometry, finite geometry, combinatorics, graph theory and group theory. In the preceding century, the subject was integrated to projective geometry, via the unification process of classical geometries, due to F. Klein. The rise of buildings and still more general geometries with meaningful relations to group theory, has modified the landscape in its content, its meaning and its relationship to other fields, in particular, group theory.

This Handbook aims at giving a comprehensive introduction and a fairly concise picture of Incidence Geometry as it now stands, as well as an idea of its evolution for the next years. It is written for students and mathematicians who are not yet familiar with that field and who want to find, as we all do since Antiquity, a 'Royal Road' to its main ideas. It may help experts as well, since it is not easy to keep track of the explosion of developments.

The project was started in 1986. Choices of topics necessarily reflect a certain subjectivity and opportunism, since the task is inseparable of material limits and human factors, such as the need for authors willing to write at a given time and the limited size of the volume.

There is no doubt that some topics would have deserved longer developments, but the extensive lists of references should help the user of the Handbook to fill such gaps.

The theory of buildings has received particular attention in view of its interest in many other fields of mathematics. Several chapters are explicitly devoted to it, some chapters deal with particular classes of buildings, and a few more chapters are concerned with extensions of buildings or geometries that slightly generalize them.

The more classical Foundations of Geometry are represented in their latest developments and they are obviously more and more related to the building unification.

The first chapters may be considered as an effort to introduce to newcomers, especially students, the main ideas and to prepare them for somewhat harder reading in later chapters.

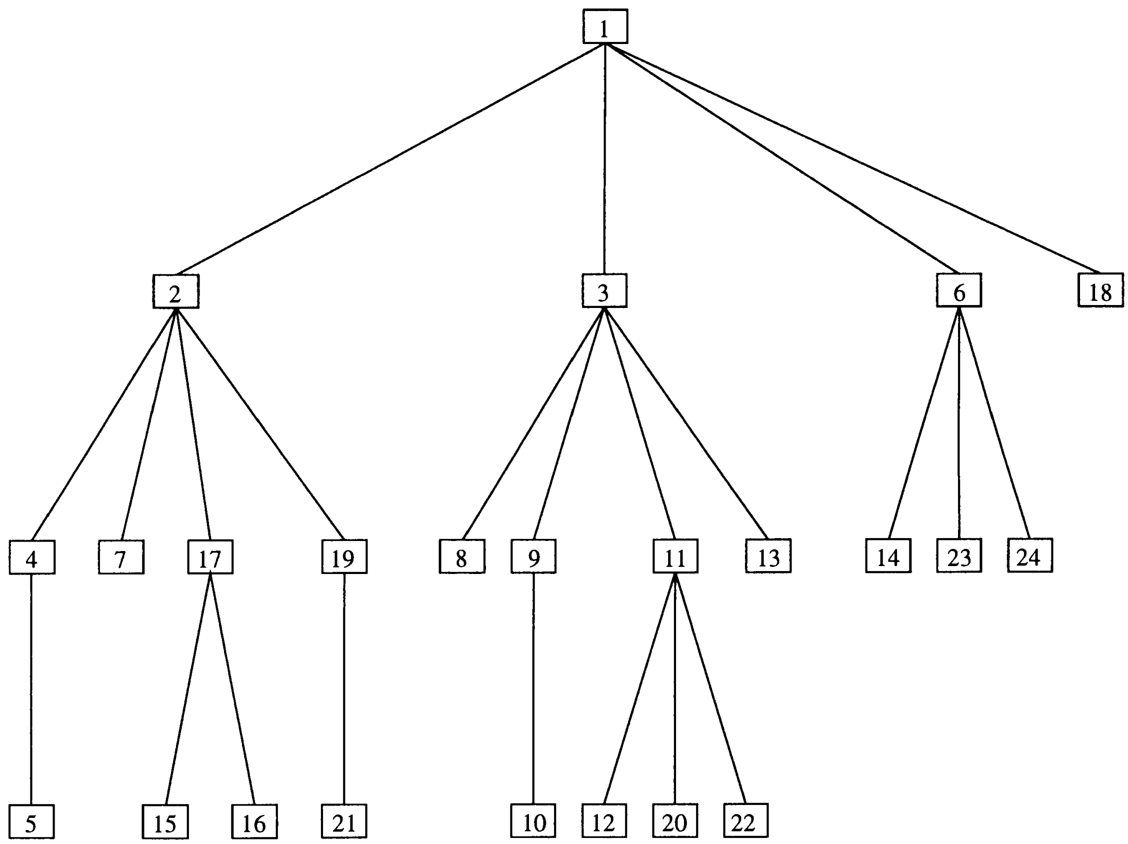
Most chapters are self-contained on the basis of fairly standard undergraduate mathematics. There are unavoidable exceptions to this general rule.

This volume is due to the combined efforts of a large team. I want to express my gratitude first of all to Alain Gottcheiner, who volunteered as an assistant in the most classical sense, to help efficiently with all tasks involved in the editorial process. I want to thank particularly W.M. Kantor, R. Scharlau, J.J. Seidel, J. Ueberberg and F.D. Veldkamp for their help in refereeing, advising and acting. My warmest thanks are due to all authors, including those who felt they had to give up, for their patience and care through the years. Finally, many persons, whose list is too long to be made explicit, have been helping various authors.

Francis Buekenhout

Guidelines Connecting the Chapters

These relationships should not be understood in too strict a way. On the one hand, there are definitely other connections. On the other hand, reading could start, in principle, in every chapter.



Contents

Preface v

Guidelines Connecting the Chapters vii

List of Contributors xi

1. An introduction to incidence geometry 1
   F. Buekenhout
2. Projective and affine geometry over division rings 27
   F. Buekenhout and P. Cameron
3. Foundations of incidence geometry 63
   F. Buekenhout
4. Projective planes 107
   A. Beutelspacher
5. Translation planes 137
   M. Kallaher
6. Dimensional linear spaces 193
   A. Delandtsheer
7. Projective geometry over a finite field 295
   J.A. Thas
8. Block designs 349
   A.E. Brouwer and H.A. Wilbrink
9. Generalized polygons 383
   J.A. Thas
10. Some classes of rank 2 geometries 433
    F. De Clerck and H. Van Maldeghem
11. Buildings 477
    R. Scharlau
12. Point-line spaces related to buildings 647
    A.M. Cohen
13. Free constructions 739
    M. Funk and K. Strambach

14. Chain geometries 781
A. Herzer
15. Discrete non-Euclidean geometry 843
J.J. Seidel
16. Distance preserving transformations 921
J.A. Lester
17. Metric geometry 945
E.M. Schröder
18. Pointless geometries 1015
G. Gerla
19. Geometry over rings 1033
F.D. Veldkamp
20. Applications of buildings 1085
J. Rohlfs and T.A. Springer
21. Projective geometry on modular lattices 1115
U. Brehm, M. Greferath and S.E. Schmidt
22. Finite diagram geometries extending buildings 1143
F. Buekenhout and A. Pasini
23. Linear topological geometries 1255
T. Grundhöfer and R. Löwen
24. Topological circle geometries 1325
G.F. Steinke

Author Index 1355

Symbol Index 1373

Subject Index 1381

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CHAPTER 1

An Introduction to Incidence Geometry

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Contents

1. Incidence geometry within mathematics and geometry ... 3
   1.1. The subject briefly explained ... 3
   1.2. On mathematics ... 3
   1.3. On geometry ... 5
   1.4. Incidence, related subjects and the foundations of geometry ... 6
   1.5. Is incidence geometry useful? ... 7
2. Incidence geometry as rooted in division rings and in dimension ... 8
   2.1. The reason for division rings (or skew fields) ... 8
   2.2. The role of dimension and vector spaces ... 8
   2.3. Projective spaces and affine spaces ... 9
   2.4. Topology, order, metric ... 12
   2.5. Linear spaces ... 12
   2.6. Linear spaces with dimension, matroids, geometric lattices ... 12
   2.7. Designs ... 13
   2.8. Finite Geometry ... 13
3. Excursion on the hill of the Handbook ... 14
   3.1. On this side of the hill: linear algebra and its zone of influence ... 14
   3.2. The other side of the hill: buildings and groups ... 15
4. Incidence geometry and groups ... 15
   4.1. The role of symmetry ... 15
   4.2. Klein's Erlangen Program (1872) ... 16
   4.3. Flag-transitivity and coset geometries ... 17
   4.4. Classical groups and Lie groups. The reappearance of $\mathbb{R}$ and $\mathbb{C}$ ... 18
   4.5. The meeting of projective geometry and Lie groups ... 20
   4.6. The rise of buildings ... 21
   4.7. Around buildings ... 22

References ... 22

HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

1. Incidence geometry within mathematics and geometry

1.1. *The subject briefly explained*

Incidence geometry arises from the points, lines and planes of elementary geometry, on the basis of properties stated in terms of inclusion and intersection. An example of such a property is the fact that two distinct points are elements of (are incident with) a unique line. The subject generalizes in various directions with various degrees of abstraction. Among the rather straightforward generalizations we mention affine spaces, projective spaces, linear spaces, linear spaces with dimension (about the same concept as geometric lattice or simple matroid) and among the generalizations at a greater distance from the first principles, buildings occupy a key position.

As in many other mathematical subjects, incidence may be somehow compatible with more structure. This leads in particular to ordered (incidence) geometry based on the properties related to segments of lines, half-lines, half-planes and half-spaces in elementary geometry, to topological (incidence) geometry when closed and open subsets of elementary geometry are taken into account and to metric (incidence) geometry based on the additional structure provided by perpendicularity, distance and motion.

1.2. *On mathematics*

This book is devoted to a body of knowledge belonging to mathematics. What is mathematics? Tits [1992] feels that mathematics is a natural science whose objects are discovered and studied as is usual in other sciences. There is no space here to deal further with that difficult philosophical and historical question. We shall rather go into a short pragmatic view on it, from within mathematics, inspired by the ideas of Bourbaki [1948, 1939] and the language of categories. Mathematics appears as a study of collections of sets endowed with structure (the objects of a category C) and of the relationships between these (the morphisms of C and the functors from C to other categories). Collections of sets are defined either by axioms (a definition in comprehension) or by construction from other collections of sets with structures (a definition in extension). A standard procedure while studying an object A from a known category C is to perform some construction Γ on it leading to an object $\Gamma(A)$ with a set P of properties. Afterwards P may be taken as a set of axioms to define a new category C' and Γ appears as a functor from C to C'. The choices made for the axioms and constructions and the path taken by the study are inspired in particular, by history, experience, personal taste, a feeling for promising relationships, and the pressure of mathematical objects themselves.

We shall not dwell on sets but we shall take the usual naive attitude on that matter and assume some familiarity with set theory. Given a set S it is endowed with a well-known internal structure (usually called trivial) based on inclusion (subsets) and operations such as the intersection and union. This structure is left invariant by the *symmetric group* $\mathrm{Sym}(S)$ consisting of all permutations (bijections of S with itself) and the law of composition. It is also left invariant or transported on S' by any bijection b of S onto a set S'. Thus b appears as an *isomorphism of sets*, or of S onto S', while the members of $\mathrm{Sym}(S)$ are the *automorphisms* of S. More generally, a mapping m from

S into a set S' conveys the idea of partial transportation of structure or *morphism* of sets since for any subset T of S, $m(T)$ is a subset of S' and for any subset T' of S', $m^{-1}(T')$ is a subset of S, while $T \subseteq U$ implies $m(T) \subseteq m(U)$, etc.

Nontrivial structure on S requires working outside of S. From S we can construct an infinite family $\boldsymbol{D}(S)$ of derived sets generated from S through the set-theoretical operations of *direct product* $\times$ and the function $\boldsymbol{P}$ assigning to S the set of all subsets of S. In $\boldsymbol{D}(S)$ we meet, in increasing order of complexity, the sets $\boldsymbol{P}(S)$, $S \times S = S^2$, $\boldsymbol{P}(\boldsymbol{P}(S))$, $\boldsymbol{P}(S) \times S$, $\boldsymbol{P}(S \times S)$, S^3, etc. Bourbaki's idea is that a *structure on* S can be seen as an element belonging to some set in $\boldsymbol{D}(S)$.

EXAMPLES. (1) In incidence geometry, we often deal with a block space, i.e. a set (of *points*) S together with a family $\boldsymbol{B}$ of subsets of S called blocks. This amounts to consider an element in $\boldsymbol{P}(\boldsymbol{P}(S))$.

(2) In elementary geometry, a particular problem like the bisector of two points is based on data provided by the Euclidean plane E_2 and a pair of distinct points A, B. As such, the problem conveys a structure represented by an element of $E_2 \times E_2$. The same problem involves distances. To avoid right now the difficulty due to the presence of an *auxiliary base set* $\mathbb{R}$, next to the *principal base set* E_2, distance can be seen at the very least as an equivalence relation on the set of pairs of points of E_2, i.e. as an element in $\boldsymbol{P}(E_2^2 \times E_2^2)$ and so the structure underlying the classical problem of the bisector of two points is represented by a member of $E_2 \times E_2 \times \boldsymbol{P}(E_2^2 \times E_2^2)$. Since lines will be involved in the solution, $\boldsymbol{P}(E_2)$ plays a role also.

(3) A *directed graph* on the set of vertices S is an element in $\boldsymbol{P}(S \times S)$.

(4) A *topological block space* $(S, \boldsymbol{B}, \tau)$, i.e. a block space $(S, \boldsymbol{B})$ with a topology on the set of points, appears as a member of $\boldsymbol{P}(\boldsymbol{P}(S)) \times \boldsymbol{P}(\boldsymbol{P}(S))$.

If f is a bijection of the set S onto the set S' then f induces a bijection of the class of sets $\boldsymbol{D}(S)$ onto the class of sets $\boldsymbol{D}(S')$ mapping $S \times S$ onto $S' \times S'$, $\boldsymbol{P}(S)$ onto $\boldsymbol{P}(S')$ and f induces a bijection of S^2 onto S'^2, of $\boldsymbol{P}(S)$ onto $\boldsymbol{P}(S')$, etc. The *isomorphism class* of a structured set (S, p) where $p \in T$ and $T \in \boldsymbol{D}(S)$ consists of all (S', p') where S' is a set and $p' = f(p)$ for some bijection f of S onto S'. In particular, $\mathrm{Sym}(S)$ acts on each member of $\boldsymbol{D}(S)$ as a permutation group isomorphic to $\mathrm{Sym}(S)$. For a structured set (S, p) the *automorphism group* $\mathrm{Aut}(S, p)$ is the stabilizer of p under the action of $\mathrm{Sym}(S)$ on p.

We simplified the exposition by taking a unique base set S to construct $\boldsymbol{D}(S)$. Actually we can start with a finite number r of *principal base sets* $S_1, \ldots, S_r$ and a finite number t of *auxiliary base sets* $A_1, \ldots, A_t$, generate $\boldsymbol{D}(S_1, \ldots, S_r, A_1, \ldots, A_t)$ by repeated use of $\boldsymbol{P}$ and $\times$ on the base sets, etc. The distinguishing feature of auxiliary sets is that their members appear as *constants*. This means that a transport of structure is allowed by arbitrary bijections f_i from S_i onto arbitrary S_i', $i = 1, \ldots, r$, while each A_j, $j = 1, \ldots, t$, must be fixed element-wise.

EXAMPLES. (1) The structure of a vector space over a field, involves a principal base set V, an auxiliary set F and a member of

$$\big(\boldsymbol{P}(F^3) \times \boldsymbol{P}(F^3)\big) \times \boldsymbol{P}(V^3) \times \boldsymbol{P}(V \times F \times V)$$

to represent the addition and multiplication on F (via $\boldsymbol{P}(F^3) \times \boldsymbol{P}(F^3)$), the addition on V (via $\boldsymbol{P}(V^3)$) and the multiplication of vectors by scalars (via $\boldsymbol{P}(V \times F \times V)$).

(2) The structure of a metric space requires a principal base set M, the auxiliary base set $\mathbb{R}$ and a member of $\mathbf{P}(M^2 \times \mathbb{R})$.

There is no room here, nor any need, to develop these views further. We shall in particular refrain from analyzing axioms, generality, functors and other relationships between structures in this context. Two remarks may be in order. First of all, the views developed here do not intend to cut mathematics from the rest of the world. Coming back to the simple situation provided by S and $\boldsymbol{D}(S)$, in actual mathematics we pay more attention to some $p \in T$ and some $T \in \boldsymbol{D}(S)$ than to other structures for reasons laying outside S and $\boldsymbol{D}(S)$. Next, given a class of structured sets, their study may take different paths according to the inspiration (or intuition) provided by models we have in mind. Considering for instance a block space structure $(S, \boldsymbol{B})$ we may think of the blocks as generalizations of lines, of hyperplanes, of open sets, of edges and on that basis the theory will be developed further in different directions.

REMARK. Since Pasch [1882], it has become standard to develop a mathematical theory on the basis of *primitive*, namely undefined, concepts and *axioms*, namely properties required from these concepts. It has been understood that primitive concepts are not totally undefined. For instance, a point of elementary geometry is most often assumed to be an element of some set and lines are subsets of this set. It may be interesting to point out, that an actual choice of primitive concepts consists most often of a set S (or several sets) together with some specified members of the family $\boldsymbol{D}(S)$ generated by S.

For a study of the philosophy of mathematics and of their history we refer to Kitcher [1984], Davis and Hersh [1982], Tymoczko [1985], Kline [1972], Dieudonné [1978]. Interesting and rather concise definitions of mathematics are given by Brieskorn [1983], Hirsch [1975] (see also Gochet [1986]).

1.3. *On geometry*

The nature of geometry as well as its existence, is a disputed matter (see, e.g., Logothetti [1981], Grünbaum [1981], Dieudonné [1981]). We often meet the idea that geometry is not so much a branch of mathematics but rather a language permeating all of mathematics and leading to transfers of intuition from elementary geometry to more general and abstract structure (see Atiyah [1982], Blumenthal [1961], Dieudonné [1981], Thom [1971]). We do not deny that geometry permeates all of mathematical branches. But the same holds true for algebra and yet we can recognize algebraic structures. There are indeed structured sets to which spatial (geometric) inspiration applies in a most efficient way. They deserve therefore to be called geometric objects. Would typical geometric work such as Grünbaum and Shephard [1986], to mention only one example, be ignored, or classified as algebra, combinatorics, topology?

Geometry has its roots in the physical, visual and muscular perception of spatial objects. Its main historical sources are Euclidean geometry and projective geometry.

The first arises from an idealized observation of solid bodies in motion while the second similarly derives from the optical one-eye observation of solid bodies. At our level of axiomatized or constructive mathematics, geometry appears as a study of structured sets with spatial inspiration or inspiration from former geometric subjects. Geometry studies *spaces*. A space can be seen as a structured set whose elements are called *points*. Typical examples are the usual (i.e. Euclidean) space, plane and line of elementary mathematics. The latter space structure, as a model of physical space, is not unique. Non-Euclidean geometry provides a classical illustration of this fact. Less classical, though well motivated, is the idea that physical space can be modelled without any appeal to points as primitive objects. Chapter 18 by G. Gerla aims at surveying this unusual approach. In a more classical setting, the buildings and geometries introduced by Tits suppress points too as absolute data, but they still do potentially exist and in more than one way. This point of view already goes back to Hilbert according to whom the words point, line, plane could be replaced in geometry by table, chair and beermug. Finally, we mention that large areas of mathematics belonging to geometry remain untouched in this Handbook even if they are not completely free from incidence. Typical examples are differential geometry and algebraic geometry for which we refer the reader to Spivak [1970], Brieskorn and Knörrer [1981], and Dieudonné [1974].

1.4. *Incidence, related subjects and the foundations of geometry*

The deep analysis of Euclidean geometry underwent throughout the 19th century got at an important turning point with Pasch [1882] who pointed at the explicit distinction between lines and segments and who gave to axiomatics its current status for the working mathematician. The final touch but not the least, was given to the efforts of many authors, by Hilbert [1899] in his famous 'Grundlagen der Geometrie'. Here the rather complicated structure of Euclidean space was made as simple as could be and fully understood. By the way, how many $\boldsymbol{P}$'s and $\times$'s would it require in Section 1.2? That structure was split up in three major parts:

(I) *the properties of incidence* based on points, lines, planes, the relation of inclusion (or rather its symmetrized version which is the incidence relation), and the relation of parallelism;

(II) *the properties of order* based on incidence enriched by line-segments (or incidence derived from the segments) as well as half-lines, half-planes, convex sets, angles, etc;

(III) *the properties of a metric nature* based on incidence enriched by congruence, perpendicularity, circles, motions, etc.

Hilbert saw that (I) is independent of $\mathbb{R}$ and that it corresponds to affine geometry over any division ring. Similarly, (I) and (II) roughly correspond to real affine geometry (or a variation of this which is affine geometry over an ordered division ring). Finally, (I), (II) and (III) give control over the traditional Euclidean geometry. Hilbert made it clear that the nature of points, lines and planes was irrelevant for their mathematical study. This claim gave a fantastic freedom of choice to the mathematicians and it was to lead to impressive consequences. After Hilbert, the further study of (I), (II) and (III) with various degrees of generality gave us the mathematical field called 'Foundations of Geometry'. A historical survey of this subject can be found in Karzel and Kroll [1988].

Incidence geometry appears as an effort to further investigate (I), at first in the tradition of affine and projective geometry. After 1935, it was increasingly influenced by applied statistics and combinatorics (see, e.g., Yates [1936], Bose [1939]). The combination of these influences with rather distinct origins, resulted in the large development of finite geometry masterly surveyed in Dembowski [1968]. From 1955 on, incidence geometry got a tremendous impulse from the theory of Lie groups through the work of Tits [1955] culminating with the theory of buildings and the rise of various more abstract and more general views on incidence (see Tits [1974]). This was not yet very much apparent in Dembowski [1968] but it would modify the situation soon afterwards, in a dramatic way. Roughly summarized, all of this stems from the points, lines and planes of elementary geometry and even from the points and lines. New useful views on this story may still be expected. We finish this section with some comments on the frontier of our subject and its interrelations with other fields. The geometry of order, metric and incidence got a brother namely topological geometry with Kolmogoroff [1932]. Topological incidence geometry is surveyed in Chapters 23 and 24. This includes also some material concerning incidence and order. These companions of incidence also give rise to large areas of study in Convexity and Differential Geometry, to mention only two examples. As a final example of interaction we briefly come back to algebraic geometry where, in the most elementary developments, incidence plays an obvious role in the study of curves and surfaces of projective and affine spaces.

1.5. *Is incidence geometry useful?*

We do not want to enter the old and deep philosophical question of the usefulness of mathematics but it may be in order to mention some matters that are specific to incidence geometry. This subject is not clearly recognized by the mathematical community. Even those who work in it, prefer a reference to combinatorics, finite geometry, designs, geometry, etc. In the AMS classification, Section 51 Geometry has a lot to do with incidence under various names but incidence geometry may as well occur in 05 Combinatorics and in 52 Convexity to mention but a few.

In many respects, incidence geometry grows out of classical projective geometry. Many mathematicians believe that projective geometry is dead and even of no particular use anymore. The key argument is that there is a functor (or just a dictionary) from projective geometry to linear algebra. This is only true at the surface of those subjects. If it was true in depth, the statement would apply to the various concepts of morphisms of projective geometry. Actually there are quite a number of open problems in this area in order to complete the dictionary if this is at all possible.

Linear algebra may actually benefit from projective geometry. A typical recent example is the concept of pseudoquadratic form (Tits [1974]).

A series of books like Segre [1961], Hirschfeld [1979, 1985], and Hirschfeld and Thas [1991] provide a neat contradiction to the common belief about the death of projective geometry. About usefulness, in a recognition problem, it is easier to check the three axioms of a projective space rather than the 20 usual axioms that cover a division ring and a vector space on it. This kind of observation is typical of recent genuine applications of results in incidence geometry for the characterization of classes of simple groups.

As pointed out to me by A. Beutelspacher, the language of incidence geometry provides an extremely flexible tool that allows for efficient modelization of situations with increasing complexity. Incidence geometry is one facet of the most elementary mathematics and therefore applies to many situations as other parts of mathematics do.

In another direction, the understanding of Space (or the Universe) has been a source of inspiration for geometry since 2600 years at least. Physics has no control over distances beyond some scale and within some other scale. Therefore, variations on the mathematical modelling of Space-Time are still likely to be needed.

A major reason for incidence geometry is group theory. Groups permeate all of mathematics in view of their close relationship to symmetry. A great deal of the present book is devoted to illustrate the usefulness of incidence geometry in the understanding of groups. Work involving no groups at all may be relevant to this purpose.

2. Incidence geometry as rooted in division rings and in dimension

2.1. *The reason for division rings (or skew fields)*

As mentioned earlier, Hilbert's analysis of Euclidean geometry isolated the properties of incidence of points, lines and planes. When coordinates were introduced in the traditional way of Fermat and in this context of new rigor, the algebraic concept of a division ring took over instead of the real or complex numbers. Projective geometry reinforced this fact. After the discovery of finite models and an analysis of the foundations of projective geometry, it became clear (see Veblen and Young [1910]) that few and simple axioms dealing with the incidence of projective planes and spaces were leading to division rings, and vice versa. As a matter of fact, still more generality is possible (see Chapters 19 and 21).

2.2. *The role of dimension and vector spaces*

During the first half of the twentieth century, linear algebra or the study of vector spaces (also called linear spaces) arrived at its maturity after a long history starting with proportion and linear equations. The appearance of infinite dimensional spaces such as Hilbert spaces, together with the classical examples related to $\mathbb{R}^2$ and $\mathbb{R}^3$ made it clear that the elementary theory of linear algebra did not require any restriction on the dimension. It also became clear that affine and projective geometry turn out to be mainly other facets of linear algebra (see Baer [1952]), and vice versa.

Now the powerful machinery provided by division rings whose theory is often based on 11 axioms and by vector spaces whose theory requires 9 further axioms turns out to be equivalent to the simple and natural structure of a projective space which uses only 3 axioms (see Section 2.3). This is a striking fact even if counting axioms is somewhat artificial. The close relationship between vector spaces, affine spaces and projective spaces is an early instance of categories and functors.

As to division rings, the algebraic analysis of $\mathbb{R}$ and other classical number systems, together with the evolution towards three equivalent categories of spaces as described above, imposed them on the mathematical scene, even if expository texts do often reduce to the commutative case and to the case where the characteristic is different from 2. While these restrictions can be motivated they are of little relevance for incidence geometry.

Moreover, a division ring very much appears as a vector space of dimension one provided with a basis of one vector, as an affine line provided with an affine basis consisting of an ordered pair of points and as a projective line provided with an ordered trio of points.

2.3. *Projective spaces and affine spaces*

The physical roots of projective geometry are visual (see, e.g., Kline [1963]) and summarized by a classical problem. Given an object A in space, how to draw A on a sheet of paper π, in a realistic way? This requires that the drawing of A gives the same visual impression to an eye o, as A itself. That problem leads to the mathematical concept of a *central projection* α in the Euclidean space E_3 determined by a *centre of projection* o (a point) and a plane π on which any point p of E_3 (and any object like A including $A = E_3$) has a *projection* $\alpha(p)$ defined as the intersection of π with the line op. Of course $\alpha(p)$ might not exist (if op is parallel to π or, worse, if $o = p$) but the resulting mess is at the origin of projective geometry. Namely, the image $\alpha(L)$ of a 'generic' line L is a line minus a point in π and parallel lines to L have images that are lines minus the same point q. This leads to the idea that q is the image of some (new or so far unknown) point common to L and all of its parallels. Thus, as in physics where elementary particles are 'seen' through their traces, central projections reveal *points at infinity* (also called improper or ideal but the least we can say is that they are no more ideal than other points and rather proper to the projective space). Systematizing this study one also gets the *line at infinity* common to a plane π and all planes parallel to π and the unique *plane at infinity* of E_3. We do thus obtain the projective space $P(E_3)$ whose points, lines and planes, respectively, consist of the points, lines, planes of E_3 completed by their elements at infinity and of the points, lines and planes at infinity. In the construction of $P(E_3)$, points bear so to speak two colours, but if we forget about colour both kinds of points cannot be distinguished from each other in the projective space.

The construction of $P(E_3)$ is exclusively based on the incidence structure of E_3: points, lines, planes, parallelism, inclusion and intersection. Hence the construction works in affine geometry as based on the incidence axioms of Hilbert. Here the dimension 3 of the geometry apparently plays a crucial role. But it does not, if the subject is conveniently generalized. In fact, each vector space V over some division ring, determines an affine space $A(V)$ whose (affine) subspaces are all cosets of subspaces of V while parallel affine subspaces are cosets of the same subspace of V. Then $A(V)$ can be extended by points and subspaces at infinity leading to a projective space $P(A(V))$. The lines are the smallest subspaces (in V, in $A(V)$, in $P(A(V))$ containing at least two points.

The class of spaces $P(A(V))$ has a most simple and natural axiomatization based on points and lines. Consider a pair $P = (P, L)$ consisting of a set P whose elements are called *points* and a set L of subsets of P whose elements are called *lines* such that the following axioms are fulfilled.

(1) Every line contains at least two points.

(2) Every pair of distinct points a, b is contained in one and only one line ab.

(3) Given distinct points a, b, c, d, e such that $ab = ac \neq ad = ae$, there is a point $x \in (bd \cap ce)$.[1]

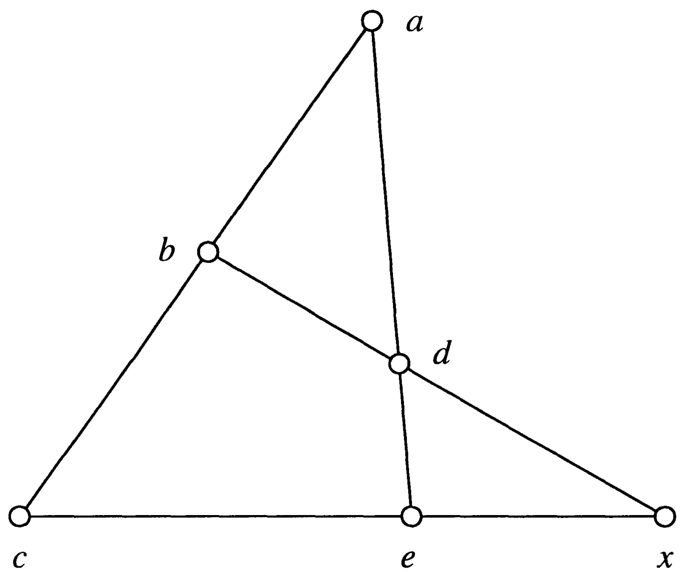


Figure 1.

Then P is called a projective space. The rather long theory of those spaces (see Chapter 2) provides a striking result.

THEOREM 1. *For each projective space P, with lines of cardinality three at least, containing a pair of disjoint lines, there is some division ring D and a vector space V on D, such that $P = P(A(V))$.*

The converse holds, as the reader will expect: for any vector space V (without any restriction), $P(A(V))$ is a projective space. What is the status of the theorem if any two lines of P have a common point? This apparently innocent question leads to fantastic developments. First we handle the freak cases: if there is at most one line then P has a trivial structure and the theorem does not necessarily remain true. So we introduce one further axiom.

(4) There are at least two lines in P and any two lines have a common point.

[1] This is often called Pasch's axiom. In Pasch [1882], the betweenness relation of Euclidean geometry is axiomatized and there appears a famous axiom for it whose projective counterpart (replacing segments by lines) is precisely (3). It is also called Veblen's axiom in view of Veblen and Bussey [1906] where it is explicitly introduced. It is called Pasch in Chapters 10 and 12, Pasch–Veblen in Chapter 3, Veblen in Chapter 2, etc.

Observe that (1), (2), (4) imply (3). Then P is called a *projective plane*. Theorem 1 is no longer true in that case. Hilbert [1899] realized this (even if vector spaces were not consciously known at that time). Analyzing the proof of Theorem 1, he found out that one further axiom is required, namely the famous *axiom of Desargues* named after the French geometer who discovered it as a theorem in the Euclidean plane.

(5) Given ten distinct points, $a, b, c, d, e, f, g, h, i, j$, such that the following trios are collinear on distinct lines: (a, b, c), (a, d, e), (a, f, g), (b, d, h), (c, e, h), (b, f, i), (c, g, i), (d, f, j), (e, g, j), it follows that h, i, j are collinear.

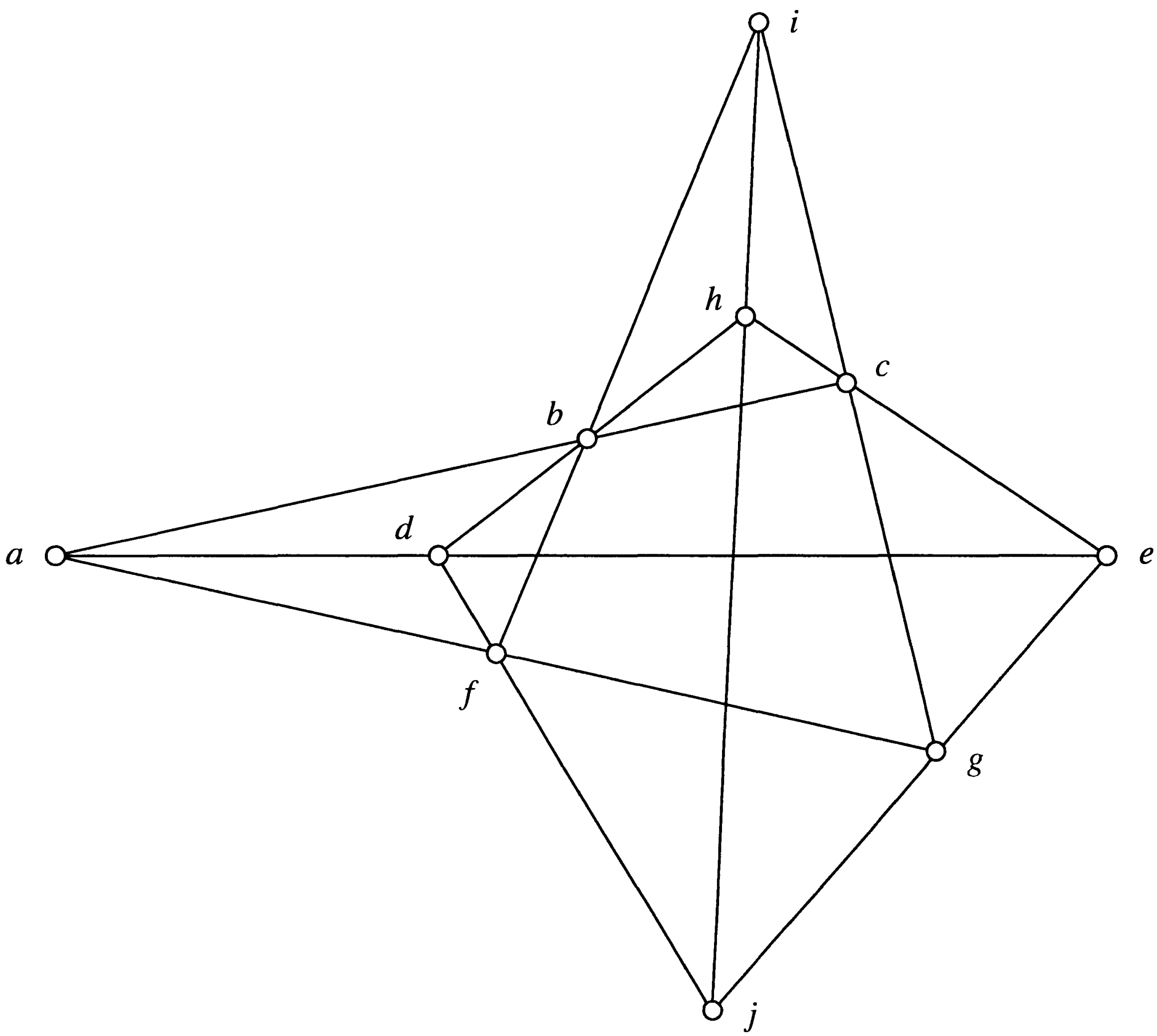


Figure 2.

THEOREM 2 (Hilbert [1899]). *For every projective plane P with lines of cardinality three at least in which Desargues' axiom holds, there is some division ring D and some vector space V over D such that $P = P(A(V))$.*

The converse holds: for any vector space V (of dimension 2) $P(A(V))$ is a projective plane satisfying Desargues' axiom. Are there non-Desarguesian projective planes? Hilbert answered that question as well, with an example which was soon somewhat simplified by Moulton [1902] (see also Priess-Crampe [1983]).

The existence of non-Desarguesian planes together with the simplicity of their definition (axioms (1), (2), (4)) drew a lot of attention and work in this area which appeared during more than half a century as a central theme of study in the Foundations of Geometry. Affine planes had a similar fate since such a plane appears as a projective plane from which a line (at infinity) and all of its points have been removed.

Meanwhile these classical projective planes and affine spaces have been generalized in various directions, as shown in later chapters such as 2, 19 and 21. Refer to Schmidt [1992] for a broad and successful analysis of 'general affine geometry'.

2.4. *Topology, order, metric*

In the context of vector spaces over a division ring D and of non-Desarguesian planes, topology, order and metric found suitable developments in particular through the case $D = \mathbb{R}$, the use of scalar products and generalizations of this. This is related to classical non-Euclidean geometry and the search for highly homogeneous substitutes for Euclidean geometry as in the Helmholz–Lie problem (see Tits [1955], Freudenthal [1956] and also Chapter 23, Section 2.13, for more details).

2.5. *Linear spaces*

In Section 2.3, axioms (1) and (2) are valid both for projective and affine spaces. So the trend towards generality and unity does naturally lead to the concept of a linear space (P, L) where P and L are as in Section 2.3 and satisfy axioms (1) and (2). The name was coined by Libois [1964] but the structure had been given attention earlier, in particular in a result of De Bruijn and Erdős [1948]. For the history of this subject, see Beutelspacher [1990] and Buekenhout [1993].

2.6. *Linear spaces with dimension, matroids, geometric lattices*

After 1935, it became clear that the elementary theory of vector spaces about linear independence and dimension, could be generalized in a rather strong way without using division rings nor any other scalars (see Whitney [1935]). This theory had various applications in algebra. Its geometric counterpart got known under various names whose content may slightly vary: simple matroids and geometric lattices. We prefer to use the expression 'linear space with dimension' (or dimensional linear space as in Chapter 6) which is closer to the content it represents. Linear spaces generalize affine and projective spaces equipped with their proper subspaces of all dimensions. If L is a linear space with (finite) dimension n, then L has subspaces of dimension 0 (the points), 1 (the lines), 2 (the planes), up to $n - 1$ (the hyperplanes). Any intersection of subspaces is a subspace. Given a subspace U of dimension i and a point p not in U, there is a unique subspace of dimension $i + 1$ containing U and p. For $n = 2$, L is just a linear space as in Section 2.5. The gain in generality attained with respect to a projective space is measured by an example. If L is a linear space with (finite) dimension n and X is any set of points of L (not contained in a hyperplane), then X together with the intersections $X \cap U$ where U is any subspace of L, is again a linear space with dimension n.

The theory of linear spaces with dimension has such a flavour of power, generality and necessity, and it is related to so many other mathematical fields that it could appear for a while as the fundamental object of incidence geometry. See Crapo and Rota [1970]. There were however still other and no less fascinating objects on the market, most remarkably buildings.

2.7. *Designs*

Again after 1935, statistical needs (see Section 1.4), led to the combinatorial and geometric development of so-called *block designs*, a subject generalizing mainly finite projective planes but also finite affine and projective spaces in more than one way. Here the basic structure is a *block space* (P, L) (see Section 1.2) where P is a finite set of points and L a family of subsets of P all of the same cardinality, such that

(i) all elements of L have at least 2 points;

(ii) the number λ of blocks $B \in L$ containing two points a, b is independent of the choice of a, b.

Many of the concepts and problems of finite projective geometry remain meaningful in the theory of designs. Here too, some concept of dimension can be introduced and for $t \in \mathbb{N}$, $t \geqslant 2$ a *t-design* is defined as a block space (P, L) such that for any point $p \in P$, the system $(P - p, Lp)$ where Lp consists of all $B - p$, with $p \in B \in L$, is a $(t-1)$-design. Here it is understood that a 2-design is just a design. The studies of linear spaces with dimension and of t-designs overlap in the study of so called Steiner systems, i.e. the case where $\lambda = 1$, the blocks being the hyperplanes of the linear space with dimension.

2.8. *Finite Geometry*

In preceding sections, finite structure appeared with an increasing weight. Once division rings replaced $\mathbb{R}$ and $\mathbb{C}$, finite fields and spaces were allowed. Finite non-Desarguesian planes arose in Veblen and Wedderburn [1907]. After 1935 they became the object of many efforts. Linear spaces with and without dimension find also a lot of inspiration in the finite case. Designs are necessarily reduced to the finite case in view of the methods, that can be used for their study.

Thus *Finite Geometry* has rapidly grown as a subject on its own. The famous and influential survey by Dembowski [1968] was soon followed by an explosion of the subject and another setting, related in particular to the finite buildings and finite simple groups.

The increasing role played by *discrete structures* in relationship with the rise of Computer Science had a strong influence on Finite Geometry. A typical example is provided by *coding theory* (see MacWilliams and Sloane [1977]), a theory born after 1945 for rather practical engineering purposes, which soon came in contact with finite geometry. Classical finite projective geometry has come back to the forefront as well. If there was only one reason to this, it would be the need for linear representations which has become such a powerful method whose typical examples are provided by the theories of finite groups and Lie groups. For a survey of finite projective geometry over fields, see Hirschfeld [1979], Hirschfeld [1985], Hirschfeld and Thas [1991].

3. Excursion on the hill of the Handbook

3.1. *On this side of the hill: linear algebra and its zone of influence*

At this stage, a large deal of the structure of the Handbook appears rather clearly. We call it this side of the hill because the other side, dominated by buildings, is not apparent yet. The main source for incidence geometry is to be found in linear algebra over a division ring.

Roughly speaking, this state of matters grew from Hilbert [1899] and Veblen and Young [1910]. Chapter 2 by Buekenhout and Cameron is devoted to this basic relationship between division rings and incidence geometry. The additional metric structure is taken into account in Chapter 17 by Schröder. The existence of non-Desarguesian projective planes and the rather extensive study of these constitutes a somewhat different world covered in Chapters 4 by Beutelspacher and 5 by Kallaher. The separate treatment of translation planes results partially from their closer relationship with linear algebra. We are still very close to this inspiration with Thas' Chapter 7 devoted to projective geometry over a finite field.

The generalization of projective geometry and affine geometry provided by linear spaces is dealt with in various chapters and the more refined structure of linear space with dimension (geometric lattice, matroid) is discussed by Delandtsheer in Chapter 6. Another generalization takes the direction of designs (Chapter 8 by Brouwer and Wilbrink). This explains more or less the first third of the book together with some later chapters. Chapter 3 has, in fact, a rather different flavour. It digs deeper into the present theory of incidence geometry and prepares, in particular, the second main zone of influence, namely geometry derived from groups and buildings. The projective spaces and planes are the first buildings to be met and they play a central role in the new theory. Lines play a crucial role in incidence geometry, especially the projective lines. Their own incidence structure is apparently trivial but if their ground ring or algebra itself acquires dimension over some field, then a rich geometric structure is inherited on the projective line. This structure has (locally) much in common with Euclidean geometry. This is the topic studied by Herzer in Chapter 14.

The interaction of incidence with topology is discussed in Chapter 23 by Grundhöfer and Löwen, and Chapter 24 by Steinke while its interaction with metric is the purpose of Chapters 15, 16 and 17 (Seidel, Lester, Schröder). As mentioned earlier (Section 1.3), the rather isolated Chapter 18 (Gerla) deals with an approach of elementary geometry which does not rest on the primitive concept of point but in which points are constructed from other objects.

Chapter 13 (Strambach and Funk) is devoted to free constructions of various classes of incidence geometries, inspired by the construction of groups freely generated by generators and relations.

Coming back to division rings, one of their most efficient algebraic generalization is provided by rings. It is no surprise that the transfer of geometry over division rings to geometry over rings leads to results related to a variety of mathematical subjects. This is discussed in Chapters 19 and 21 (Veldkamp and Brehm, Greferath and Schmidt).

Another important generalization is given by the concept of *near-field*, which unfortunately cannot find place here. Refer to the broad survey by Karzel and Thomsen [1992] and to Chapter 4.

3.2. *The other side of the hill: buildings and groups*

Getting close to the ridge of the hill on the side dominated by projective and Euclidean geometry, a glimpse on the other side reveals the more recent domination of buildings and the successful real-estate business ruled by Tits. The ramifications of buildings and their generalizations are numerous already. So far, the tourist does not see how this new world was erected but among the buildings he recognizes the projective spaces and planes, and he may observe that other important buildings involve them.

Buildings were born from constructions made on Lie–Chevalley groups. These were inspired by the derivation of projective geometries from the general linear groups. The ramifications and extensions of buildings are rather intricate.

We shall soon deal with the genesis of buildings. Before that, we shall end our description of the book's content.

The central Chapter 11 by Scharlau studies and surveys buildings in general. The non-trivial buildings of lowest rank (or dimension), namely 2, are the generalized polygons. They include the projective planes. They are studied in Chapter 9 by Thas. The simple characterization of projective spaces in terms of points and lines (Theorems 1 and 2 in Section 3.3) leads to similar views for more general buildings. This is the object of Chapter 12 by Cohen. The sporadic groups give rise to various geometries which are not buildings but that look more or less like them. This is surveyed in Chapter 22 by Buekenhout and Pasini. Rank 2 geometries close to buildings have been strongly developed. They are studied in Chapter 10 by De Clerck and Van Maldeghem.

Applications of the theory of buildings to algebraic groups are discussed in Chapter 20 by Springer and Rohlfs.

Buildings have modified the landscape of incidence geometry and of the foundations of geometry to a large extent, both in abstract and concrete ways. It becomes apparent that most chapters are or will be under the influence of buildings.

We shall now see how groups gave us buildings.

4. Incidence geometry and groups

4.1. *The role of symmetry*

The role of symmetry in science and of its mathematical counterparts: automorphisms, isomorphisms, morphisms, groups and categories, is well understood today. We know in particular, that the use of symmetry may simplify a problem and even be the key to its solution. While symmetry is always likely to be useful, a large amount of symmetry is even better. Whatever the meaning of 'large' in this statement, geometry is a typical place to exhibit it.

EXAMPLE. (1) Consider a cube C in the Euclidean space E_3 and the group G of the 48 isometries of E_3 leaving C invariant. G leaves other cubes invariant, as well as many semiregular polyhedra derived from C. If we know only G and have forgotten C, the latter cannot be uniquely derived from G but G determines nevertheless a clearcut family of polyhedra left invariant by it. Hence G brings unity among polyhedra and it appears as a kind of skeleton of C.

(2) The many thousands of varieties of crystals found in nature lead to only 219 different crystallographic groups (up to conjugacy in the affine group over $\mathbb{R}^3$).

4.2. *Klein's Erlangen Program (1872)*

At the elementary level of didactics it is often believed that Klein's role was to point to the interest of transformations (symmetries) for the study of geometry, i.e. the point developed in Section 4.1. Actually, Klein did much more. He was at the origin of a dialogue: from geometry to groups and from groups to geometry but we shall see that this dialogue did not come easily. A leading role in this evolution was played by Killing, S. Lie, E. Cartan and J. Tits, to mention but a few.

Analyzing the classical geometries of his time, with their large groups of automorphisms, Klein realized that a particular geometry Γ could simply be seen as the study of the invariants of the group $\mathrm{Aut}(\Gamma)$.

EXAMPLE. In elementary geometry we speak often about the sphere, the plane, the line, the cube, the angle of 60° as if they were unique and this is justified by the fact that the group of similarities of E_3 acts transitively on the spheres, on the planes, on the lines, etc. Klein did not stop here. He suggested that geometries could be forgotten at the beginning. In this view, we start with a permutation group G acting on a set E (we give ourselves the points and the symmetries). Then the geometry of the pair (G, E) is the study of the *invariants* of G on E.

EXAMPLE. Starting with a group G of 48 isometries in E_3, its geometry will reveal cubes, octahedra, cuboctahedra, etc.

Klein's program provides us with a fantastic tool to create geometry and it certainly brings a lot of unity among the classical geometries. But actually it did perhaps contribute to the relative decline of geometry during the first half of the century. Indeed, in the time of Klein, geometry appeared as an excellent excuse to do group theory. But so, groups became the important objects and geometry appeared mainly as a study of the invariants of a group action. The study of groups received a very strong impulse. The reduction of projective geometry to linear algebra went in the same direction. Hence there was a trend among mathematicians to forget projective geometry and related subjects. Another point is that the actual study of the invariants of some permutation group action, is not so easy in general. As far as E_3 was concerned, Poincaré [1898] showed how to reconstruct its geometry from the set of points and the action of the group of (direct) motions. Similar efforts were made in a more general setting using elements of order 2 as the main tool (see Bachmann [1959] for a survey). After 1955, Tits would find that Klein's data had to consist not just of a group G but also of some subgroups of G with a particular structure and relationship in order to get a useful setting to create geometry from groups and to have a dialogue on equal footing for the two subjects. Tits [1975] stated that in his lectures geometry would often take its revenge from the Erlangen Program and that group theory would now serve as an excuse for geometry.

Finally, the fight about which comes first, groups or geometry, leads to a compromise and a genuine mathematical subject, namely 'The groups of geometry and the geometry of groups'. This was the title of many Oberwolfach conferences lead by R. Baer and his followers during the period 1960–1980. It is obviously the point of view taken by most of the leading mathematicians who are active at the interface of groups and incidence geometry.

4.3. *Flag-transitivity and coset geometries*

Consider a geometry Γ of some sort, consisting of three types of objects, say points, lines and planes, and an incidence relation which is a binary symmetric relation defined on the set of those objects, with the property that two incident objects have distinct types. Let $\boldsymbol{A}$ be an unspecified set of axioms for Γ ($\boldsymbol{A}$ is chosen so as to make the following statements valid). Let G be some group of automorphisms of Γ (i.e. permutations of the points, lines and planes leaving incidence invariant) and let us assume that G acts transitively on flags (or chambers) of Γ, i.e. triples consisting of a point, a line and a plane which are pairwise incident. The next step we take has some analogy with the choice of a frame in order to introduce coordinates. Fix a flag (p_0, L_0, α_0) where p_0, L_0, α_0 are a point, a line and a plane, respectively. Consider the stabilizers G_{p_0}, G_{L_0}, G_{α_0} in G. Our goal is to describe the structure of Γ completely in terms of the group G and its subgroups G_{p_0}, G_{L_0}, G_{α_0} (Tits [1962]). How can we describe any object of Γ, say a line L? We consider the set $G(L_0, L)$ of elements of G mapping L_0 onto L. Is $G(L_0, L)$ nonempty? It is. By the hypothesized flag-transitivity and axioms in $\boldsymbol{A}$ there exists some flag (p, L, α) containing L and some $g \in G$ such that $g(p_0, L_0, \alpha_0) = (p, L, \alpha)$, hence $g(L_0) = L$. Now it is easy to check that $G(L_0, L) = gG_{L_0}$, i.e. $G(L_0, L)$ is a left coset of the stabilizer G_{L_0}. Conversely, each coset xG_{L_0} of G_{L_0} is the set $G(L_0, X)$ for some line X with $X = x(L_0)$. Hence we see that the choice of the origin (p_0, L_0, α_0) determines a bijection between the set of lines of Γ and the set of left cosets of G_L and so also a bijection between the set of objects of Γ and the set of left cosets of the subgroups G_{p_0}, G_{L_0}, G_{α_0}. What about incidence? Assume that (p', L') is a pair of incident objects. Then there is a flag (p', L', α') containing (p', L') by $\boldsymbol{A}$. Hence there is an element g of G mapping (p_0, L_0) on (p', L') and as a matter of fact $g \in (gG_{p_0}) \cap (gG_{L_0})$, so $(gG_{p_0}) \cap (gG_{L_0})$ is nonempty. Conversely, if cosets aG_{p_0} and bG_{L_0} have a nonempty intersection and if g is in it, then $gG_{p_0} = aG_{p_0}$, $gG_{L_0} = bG_{L_0}$ and g maps (p_0, L_0) on $(g(p_0), g(L_0))$ which is necessarily a pair of incident objects. Hence the objects represented by cosets whose intersection is nonempty are necessarily incident, and vice versa. Therefore our goal is attained. In the spirit of Klein, the geometry Γ has been converted to group theoretical data. Still in the spirit of Klein, we can reverse the procedure: start with data consisting of a group G and subgroups (G_0, G_1, G_2). Then define a geometry $\Gamma(G, G_0, G_1, G_2)$ with 3 types of objects which are the cosets gG_i, $i = 0, 1, 2$, and declare that gG_i and

hG_j are incident provided $gG_i \cap hG_j$ is nonempty. We get a geometry on which G acts as a group of automorphisms by left translation (g moves xG_i to gxG_i). However it is not necessarily true that G acts flag-transitively on Γ. So the process from geometry to group is only partially reversed from group to geometry. Neat necessary and sufficient conditions for flag-transitivity are available (see Chapter 3, Section 3.2.3).

The situation described here improves on Klein's search for invariants as far as incidence is concerned. We are nevertheless facing a wild situation, since the choice of a group and a collection of subgroups of it, is under very little control. We shall see how efficiently the theory of Lie groups deals with that question.

4.4. *Classical groups and Lie groups. The reappearance of* $\mathbb{R}$ *and* $\mathbb{C}$

We saw in Section 4.3 that the class of groups G of geometric relevance has to be restricted, as well as the choice of their 'geometric' subgroups, if we want a deep insight into the resulting geometry.

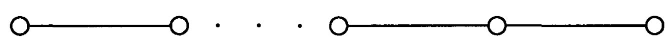
According to its founder Tits [1981], the theory of buildings is born from the meeting of projective geometry and of the theory of simple Lie groups. The story is rather involved and we shall deal first with the idea of a Lie group. Here the relevant ground field is $\mathbb{C}$ (or $\mathbb{R}$). The simple Lie groups arise as a generalization of the classical linear, orthogonal and symplectic groups over $\mathbb{C}$. They are groups of $n \times n$ matrices whose entries satisfy polynomial equalities and inequalities. Hence such a group can be identified with a subset G of the vector space $\mathbb{C}^{n\times n}$, and at the same time G bears a group structure and a (compatible) structure of analytic manifold (actually an algebraic manifold if only polynomial equations are used). These are the distinguished features of a Lie group. Why restrict ourselves to simple groups? The classical groups give an answer. If G is a classical group, consider its special subgroup SG consisting of those matrices in G whose determinant is equal to 1, and consider the centre Z of SG. Then SG/Z is the projective group PSG. It turns out that for all G and $n \geqslant 2$, PSG is a simple group . An interesting generalization is the study of all simple complex Lie groups; it turns out that they can be classified and that there are only 5 (so called exceptional groups E_6, E_7, E_8, F_4, G_2) which are not classical groups. We shall sketch the way to this classification. Given the Lie group G with identity element 1, the tangent space of G at 1 bears a structure of Lie algebra G whose product [,] is a local trace of the product in G. Killing [1890] (and Cartan [1894]) got a complete classification of all possible G, which was later completed by Cartan to a classification of all possible G. It suffices to state here that G is a minimal normal subgroup of Aut(G). How did Killing and Cartan classify the Lie algebras? In G, they observed the presence of a family of subalgebras now called Cartan subalgebras, which are characterized by the fact that they are selfnormalizing and nilpotent. They are conjugate under Aut(G) and the simplicity of G forces them to be maximal Abelian subalgebras. From the Lie algebra G and a Cartan subalgebra H, they produced a finite geometric structure R now called a *root system*. The set R appeared in H^*, the dual of H. For $f \in H^*$, let G_f, the set of elements of G of weight f, consist of all $x \in G$ such that $[h, x] = f(h)x$ for all $h \in H$. The roots of G with respect to H, i.e. the elements of R, are those $f \in H^*$ for which $G^f \neq 0$ and $f \neq 0$.

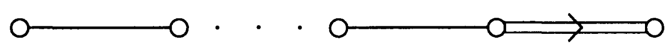
It turns out that G can be reconstructed from H and R; in particular G is the direct sum

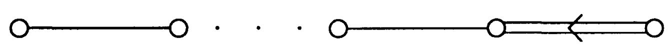
$$H \oplus \sum_{f \in R} G^f .$$

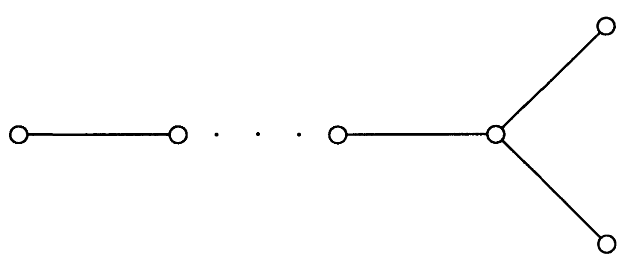
Today, finite root systems are studied and classified on an independent axiomatic basis. The root system R appears as a generating set of a real Euclidean vector space V and R contains so called 'simple root systems', i.e. bases S of V such that each member of R is a linear combination over $\mathbb{Z}$ of the members of S with entries having the same sign. The group Aut(R) is transitive on such bases and R can be reconstructed from S. The simple root system S has indeed a very simple structure which is reflected by the Dynkin diagram (1948). This is a directed graph with multiple edges whose vertices are the members of S, in which two members f, f' of S are joined by an edge of multiplicity $m - 2$ if and only if the angle of f and f', in V, is $2\pi/m$. There is an arrow on the edge, pointing from f to f', if and only if the length of f is greater than the length of f'.

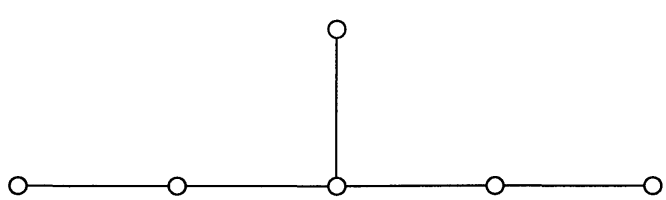
Finally, the rather involved structure of a complex simple Lie group is entirely contained somehow in their Dynkin diagram. The classification then consists of the list of all possible Dynkin diagrams and this goes as follows:

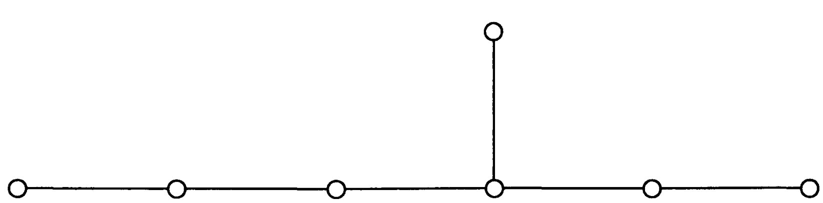
A_n  $n \geqslant 1$

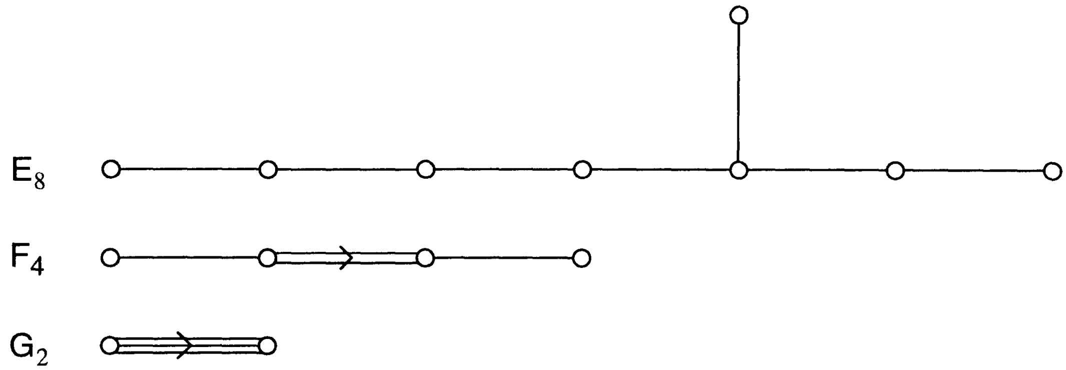
B_n  $n \geqslant 2$

C_n  $n \geqslant 2$

D_n  $n \geqslant 4$

E_6 

E_7 



The simple Lie groups of type A_n, B_n, C_n, D_n are respectively the projective linear group PSL$(n+1, \mathbb{C})$, the orthogonal group PSO$(2n+1, \mathbb{C})$, the symplectic group $\mathrm{PSp}_{2n}(\mathbb{C})$ and the orthogonal group PSO$(2n, \mathbb{C})$. The five remaining exceptional groups were first discovered by Killing [1890].

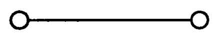
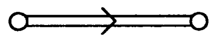
4.5. *The meeting of projective geometry and Lie groups*

In this meeting, the motivation came from the intriguing five exceptional groups. The theory of buildings was aimed primarily at a geometrical interpretation of those groups as groups of automorphisms of some geometries. Here we follow the historical account given by Tits [1981]. A heuristic argument went as follows (Tits [1955]). Since the structure of the groups was reflected by the Dynkin diagrams which are after all very simple pictures, the groups and geometries had to be rather simple as well. Looking at the best known and easiest case which is the group PSL$(n+1, \mathbb{C})$ of type A_n, Tits realized that the n-dimensional projective geometry $P_n(\mathbb{C}) = P(A(\mathbb{C}^n))$ could be reconstructed from the group as a coset geometry (see Section 4.3). Thanks to a basic theorem of Lie, the subgroups to choose in order to build the geometry, were the maximal connected nonsemi-simple subgroups containing a maximal connected solvable subgroup. It was tempting to push this further to the other Lie groups. Fortunately, the maximal connected nonsemi-simple subgroups, now called 'maximal parabolic subgroups', of any simple Lie group G, had been classified by Morozov [1943] and Karpelevic [1951]. These authors showed that the conjugacy classes of such subgroups are naturally related to the vertices of the Dynkin diagram of G. On this basis, Tits [1955] generalized the construction of $P_n(\mathbb{C})$ in terms of the group A_n. For every Lie group G over a Dynkin diagram M, he got a well defined coset geometry Γ whose objects were the left cosets of all maximal parabolic subgroups containing a given maximal connected solvable subgroup (now called a 'Borel subgroup'). That geometry was unique because all Borel subgroups of G are conjugate.

The duality principle of projective geometry (the hyperplanes play a role similar to the points) was reflected by the symmetry of the diagram A_n, and the triality principle first discovered by Study [1913] was reflected in the 3-fold symmetry of the diagram D_4. The construction derived its interest from other observations.

(i) First, if x is an element of Γ and $t(x)$ the corresponding vertex of M, then the residue geometry of x in Γ (see Chapter 3 for a definition of the residue) is the geometry associated with the diagram obtained from M by deleting $t(x)$ and all edges containing it. Observe that this forces us to consider disconnected diagrams and the corresponding groups.

(ii) This also means that the geometries associated with the rank 2 diagrams (i.e. those with 2 vertices) must give a good control over the geometries of larger rank. Forgetting the type G_2 which does not appear in higher ranks, this leaves well-known rank 2 'building stones', namely,

$\mathsf{A}_1 \times \mathsf{A}_1$		the direct sum of two projective lines $P_1(\mathbb{C})$
A_2		the projective plane $P_2(\mathbb{C})$
B_2		the quadric in $P_4(\mathbb{C})$

while the geometry for C_2 only is the dual of the last of these. Since these three geometries have obvious analogues over any field k it was natural to look for a geometry over k corresponding to every Dynkin diagram, with the use of (i) as an axiom. The automorphisms of that geometry would then produce an analogue of the Lie group of type M over k.

Meanwhile, Chevalley [1955] gave an algebraic uniform construction of those groups and then the corresponding geometries were easily derived as coset geometries. The theory of simple complex Lie groups described in Section 4.4 soon was taken over by a completely similar theory for the simple algebraic groups over an algebraically closed field. Now a given diagram M belongs to infinitely many simple groups and geometries, namely one over each field and so, the exceptional diagrams become less exceptional in this context.

4.6. *The rise of buildings*

The first attempts made by Tits in 1956, in order to axiomatize the geometries over $\mathbb{C}$, showed that the axioms became more symmetric if points were no longer distinguished from other subspaces. This was especially obvious for the geometry of type E_8 which has eight types of subspaces. This led to the definition of a geometry as a set of elements, partitioned into 'types', together with a binary symmetric relation of incidence defined on their union. This viewpoint gave a deeper geometric understanding of the new and the classical geometries, in contrast with the classical view of spaces defined on a set of points with inclusion as incidence. Coming back to the rank 2 geometries over a diagram with an edge of multiplicity $m-2$ (4-fold for the type G_2) Tits observed that they shared properties which gave him the concept of a *generalized m-gon* in 1959.

The definition of a building and the corresponding terminology did not yet exist. However, the main building geometries had been constructed and a fair amount of axiomatization was available. The missing link towards actual buildings was that of an apartment. The generalized 3-gons are the projective planes and the simplest of these is the usual triangle. Similarly, the simplest geometry of type B_2 (or C_2) is a quadrangle. In the heuristic developed by Tits, the triangle and quadrangle appeared as geometries over the 'field of order one', and he soon got a similar object for each Dynkin diagram.

These objects were later called 'Coxeter geometries'. He observed further that the geometry $\Gamma(M, k)$ of type M over a field k, has subgeometries isomorphic to the Coxeter geometry of type M and these subgeometries were later called *apartments*.

EXAMPLES. (1) In a projective space Γ of dimension n, an apartment is a set of $n+1$ linearly independent points together with all subspaces generated by some of these points.

(2) In a generalized m-gon, an apartment is a circuit of length $2m$, i.e. a subgeometry isomorphic to the usual m-gon. Some more observations were paving the road to an abstracted definition of buildings based on apartments.

(i) If Γ is the geometry $\Gamma(M, k)$ there are 'many' apartments in Γ; more precisely, any two distinct flags of Γ are always contained together in some apartment.

(ii) It is easy to pass from one apartment of Γ to another, more precisely, if A, A' are apartments in Γ and if F_1, F_2 are flags in $A \cap A'$, then there is a (type preserving) isomorphism of A onto A' fixing $F_1 \cup F_2$ elementwise.

(iii) For a given M, geometric properties common to all $\Gamma(M, k)$ can easily be read off from the particular case provided by the Coxeter geometry over M, i.e. the geometric behaviour of $\Gamma(M, k)$ is controlled by its apartments.

The monumental classification of all buildings of spherical type and rank $\leqslant 3$ (i.e. with finite apartments) and relationships with other subjects of mathematics were an impressive achievement (see Tits [1974]).

4.7. *Around buildings*

The classification of the finite simple groups (see Gorenstein [1982]) leads to the groups of Lie–Chevalley type whose corresponding natural geometries are buildings, the alternating groups whose geometry can be seen in relationship with the Coxeter geometries and 26 sporadic groups. Diagrams generalizing the Dynkin diagrams and geometries over these diagrams along the principles of Tits' early work, were introduced in Buekenhout [1979]. They led to several developments related to the idea of a building geometry which have already been briefly mentioned in the book description (see Section 3.2). All this requires renewed foundations of incidence geometry.

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CHAPTER 2

Projective and Affine Geometry over Division Rings

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Contents

Introduction 29
1. Projective and affine spaces 30
2. Axiomatizations 34
3. Collineations and correlations 41
4. Polar spaces 45
5. Buekenhout geometries 47
6. Grassmannians 49
7. Clifford algebras and spinors 51
8. Finite geometries 55
9. The projective line 59
References 61

HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

Introduction

This chapter describes the central topic of geometry, viz. projective and affine geometry. Many topics will be touched on quite briefly, and will be expanded in later chapters of the Handbook.

Projective and affine geometry is the geometry of subspaces (or cosets of subspaces) of vector spaces over division rings. It is basic in that we are concerned only with the incidence of subspaces; other properties which come later in the development of Euclidean geometry, such as order (or betweenness) and metric properties, play no part in it.

Let us outline the contents of the remaining sections of this chapter.

In Section 1, we construct the projective and affine spaces over division rings synthetically, and give the connection between them, with some historical comments.

Section 2 describes the axiomatic approach. We give the Veblen–Young axioms which characterize projective spaces of dimension at least 3. We define projective planes, and describe the connection between Desargues' Theorem and coordinatization, which is important in the general coordinatization theorem. We also describe Pappus' Theorem, which implies Desargues' Theorem and is equivalent to commutativity of the coordinatizing division ring. Finally, we explain how the axioms have to be modified to describe affine spaces.

In Section 3, we turn to the description of collineations and correlations (dualities) of projective and affine spaces, including the structure of the collineation groups and the classification of polarities.

The totally isotropic subspaces with respect to a polarity of a projective space form a geometry which is the prototypical *polar space*. There are, however, other kinds of polar spaces. All polar spaces of rank at least three have been determined by Tits, in the course of his classification of buildings of spherical type. We give, in Section 4, a summary of polar spaces and the work of Tits. The following section describes a geometric framework which includes all the types of incidence geometry so far described.

The remaining four sections discuss more specialized topics.

Section 6 describes some embeddings of geometries related to higher-dimensional subspaces: Grassmannians (embedded in exterior powers), and the Klein quadric. Section 7 is a brief account of Clifford algebras and spinors, which are used to construct embeddings of the geometry of maximal singular subspaces with respect to a quadratic form. This section concludes with a summary of isomorphism and duality relations among projective and polar spaces.

In Section 8, we discuss finite geometries, where the theorems of Wedderburn and Galois enable us to give a complete list of finite projective and affine geometries. We describe their numerical properties and give some characterizations in these terms, and tabulate the orders of their collineation groups.

So far our geometries have been defined in terms of incidence alone, so that projective and affine lines have no nontrivial structure. In the final section, we turn to some ways of giving structure to lines, notably cross-ratio on the projective line. We also discuss projectivities and the characterization of commutativity.

1. Projective and affine spaces

Projective geometry is a development of the theory of perspective during the Renaissance. To put matters very simply: An artist wishes to make a plane representation of a 3-dimensional scene. Any visible point of the scene is represented by the point at which the line joining it to the artist's eye (which we take as the origin) meets the picture plane. Thus, a point of the picture plane stands for a 1-dimensional subspace of the 3-dimensional vector space. Of course, 1-dimensional subspaces parallel to the picture plane will not be represented, unless we enlarge the picture plane by adding 'ideal' points to it to stand for these subspaces.

It is more convenient for our purposes to proceed in the reverse direction, defining the points of the projective plane to be the 1-dimensional subspaces of a 3-dimensional vector space, and then obtaining the affine plane by removing a distinguished line of ideal points.

Of course, there is no need to restrict the number of dimensions to 2 or the field to the real numbers. This leads to the following definition.

Let K be a division ring (a structure satisfying all the field axioms except possibly the commutativity of multiplication). Let V be a left vector space over K of dimension $n+1$. (This simply means that scalars act on the left, with the result that $c(c'v)$ is equal to $(cc')v$ rather than to $(c'c)v$ for $c, c' \in K$, $v \in V$.) Then V is conveniently represented as the space of row vectors of length $n+1$ over K. (Note that the dual space V' of V is a right vector space – we have $(cv)f = c(vf)$, not $(vf)c$ – and can be represented as the space of column vectors.)

Now the n-dimensional projective space $P(n, K)$ is the set of subspaces of V. The usual way to give structure to this set is by means of a binary incidence relation – two subspaces are incident provided one contains the other – as well as distinguishing types of subspaces according to their dimension. So a *projective k-space*, or *k-flat* is a subspace whose vector space dimension is $k+1$. We use familiar geometric terminology for subspaces of low dimension: points, lines, planes, solids have vector space dimension 1, 2, 3, 4 (and geometric dimension 0, 1, 2, 3), respectively, and *hyperplanes* are subspaces of codimension 1. We also use familiar words for incidence (a point lies on a line, a line passes through a point) and related properties (two lines are concurrent, or coplanar, etc.).

To try to avoid confusion between the two dimensions, we use the term 'rank' for vector space dimension, so that $\operatorname{rank}(U) = 1 + \dim(U)$.

We can immediately write down some simple properties: two points lie in a unique line; a line and a point not on it lie on a unique plane; and so on.

PROPOSITION 1.1. *If U is a k-flat and p a point not in U, then a unique $(k+1)$-flat contains U and p.*

In different language, this says that the projective geometry is a matroid, see Welsh [1976].

Further properties of projective spaces are not true in arbitrary matroids. For example, any two coplanar lines are concurrent. The generalization is most conveniently expressed in lattice terms.

We can regard a projective space as a lattice, where the lattice order is inclusion, and meet and join are given by

$$U \wedge W = U \cap W, \qquad U \vee W = \langle U, W \rangle.$$

This lattice has a number of properties.

PROPOSITION 1.2.
(i) $P(n, K)$ *is* atomic (*any element is a join of atoms*).
(ii) *If* $U \leqslant W$, *then any two maximal chains between* U *and* W *have the same length, i.e.* $\mathrm{rank}(W) = \mathrm{rank}(U)$.
(iii) *For any two flats* U *and* W,

$$\mathrm{rank}(U \wedge W) + \mathrm{rank}(U \vee W) = \mathrm{rank}(U) + \mathrm{rank}(W).$$

(iv) *If* $X \leqslant Z$, *then* $X \vee (Y \wedge Z) = (X \vee Y) \wedge Z$.

(A lattice satisfying (iv) is called *modular*.) All is clear except possibly (iv), whose equivalence to (iii) will appear in the next section.

Now we turn to some further geometric properties of projective spaces, expressible in terms of points and lines.

PROPOSITION 1.3.
(i) *If a line meets two sides of a triangle, not at their intersection, then it meets the third side too* (*Figure* 1).
(ii) *Desargues' Theorem holds* (*Figure* 2).
(iii) *Pappus' Theorem* (*Figure* 3) *holds if and only if the field* K *is commutative.*

Here, a configuration theorem described by a figure, such as Desargues' or Pappus' Theorem, is the assertion that, if all the points of the figure and all but one of the lines

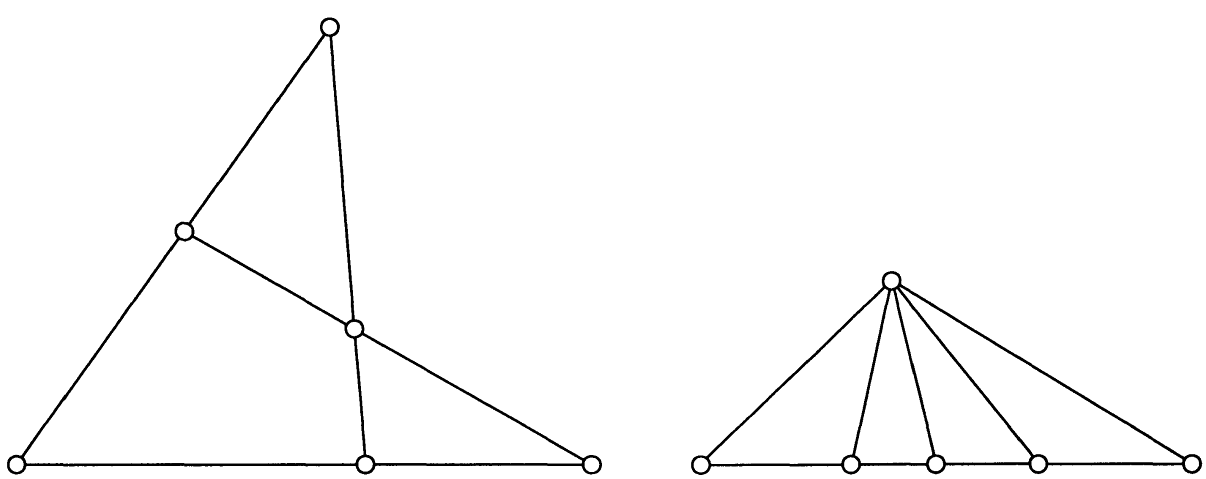


Figure 1. Veblen's Axiom.

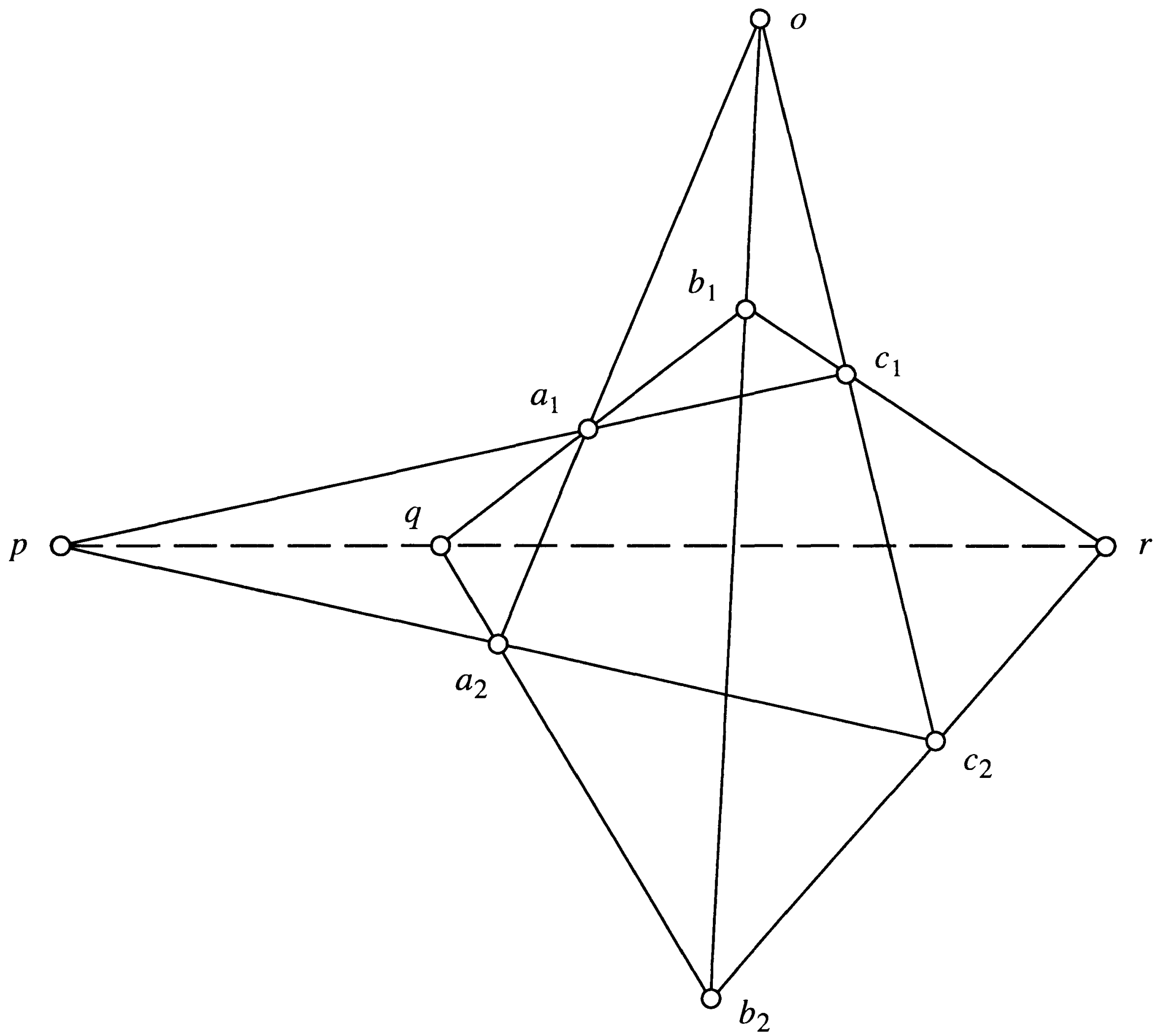


Figure 2. Desargues' Theorem.

exist satisfying the prescribed incidences, then the last line exists as well. In each case, the group of automorphisms (incidence-preserving maps) of the figure acts transitively on lines, so it does not matter which line is the 'last' one; and the figure is isomorphic to its dual[1], so the theorem is equivalent to its dual form. Note, however, that there may be 'degenerations', i.e. additional incidences not shown in the figures.

The proof of (i) is straightforward: the configuration lies in the plane spanned by the vertices of the triangle, and so any two of its lines meet. The proofs of (ii) and (iii) are exercises in coordinate geometry. But observe that, in the 'generic' case in which the figure is 3-dimensional, Desargues' Theorem follows by straightforward geometric reasoning. For the required line L must lie in the planes abc and $a'b'c'$; and, in a 3-space, two distinct planes really do meet in a line. It is also possible to prove the planar case by means of several applications of the 'generic' case; this fact will be relevant when we come to the axiomatization of projective spaces in the next section.

[1] This is not true for Figure 1 (Editor's notice).

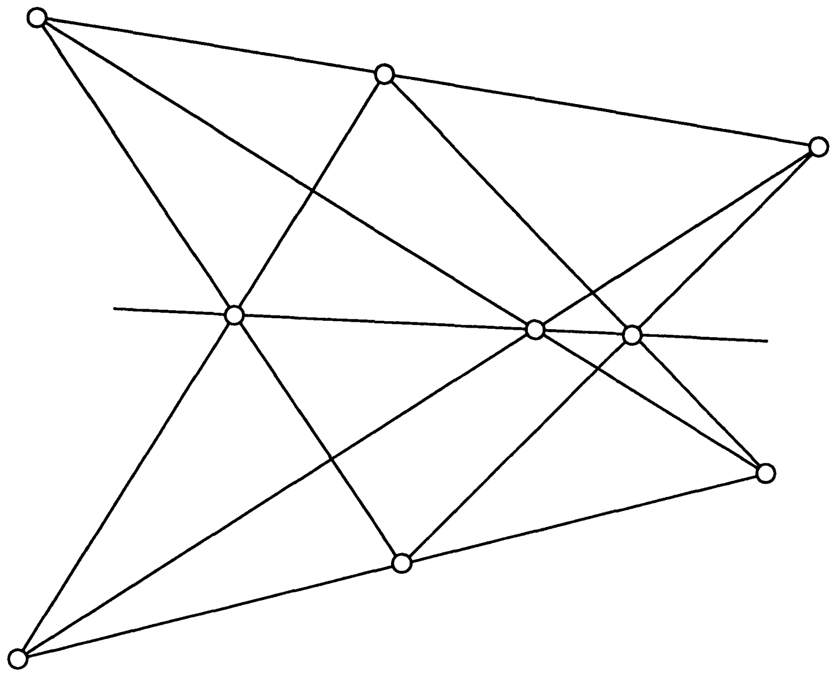


Figure 3. Pappus' Theorem.

Now we turn to affine geometry. The *affine space* $A(n, K)$ is most easily defined by removing a hyperplane (the *hyperplane at infinity*) and all of its subspaces from the projective space $P(n, K)$.

A synthetic definition runs as follows. Let V be a vector space of dimension n over K. Then the flats of the affine space flats of the affine space $A(n, K)$ consist of all the cosets of subspaces of V. In particular, the points are the cosets of the zero subspace, i.e. the singleton subsets of V, which can be identified with all the vectors in V. (Note that, in the affine case, vector space dimension is equal to geometric dimension.)

To see that the two definitions are the same, let W denote a K-vector space with dimension $n + 1$, and let $x_0, \ldots, x_n$ be coordinates in W. Let the hyperplane H at infinity have equation $x_0 = 0$. Then any 1-dimensional subspace of W which is not contained in H is spanned by a unique vector with $x_0 = 1$; and the map

$$\langle(1, x_1, \ldots, x_n)\rangle \mapsto (x_1, \ldots, x_n)$$

is a bijection between the point sets of the two descriptions, which is easily seen to be an isomorphism. Alternatively, if we adjoin to each coset $U + w \in W$ the subspace U (regarded as an element of $P(n - 1, K)$) we reconstruct $P(n, K)$.

Two flats of $A(n, K)$ are said to be *parallel* if, in the first description, they have the same intersection with the hyperplane at infinity, or in the second, they are cosets of the same vector subspace. Parallelism is an equivalence relation, and satisfies the Euclidean parallel postulate: given a flat U and point p, there is a unique flat U' which contains p and is parallel to U. In other words, each parallel class partitions the point set.

The affine space $A(n, K)$, like the projective space, is a lattice. It is not modular, but is locally modular; that is, the sublattice formed by all elements containing a fixed nonzero element of the lattice is a projective space (and hence modular).

Desargues' and (if K is commutative) Pappus' Theorems hold in $A(n, K)$ if slightly modified: where we required three lines to be concurrent in projective space, we must allow here the additional possibility that they are parallel.

2. Axiomatizations

Before discussing axioms for projective and affine spaces, we must briefly turn to projective and affine planes, a subject covered more fully elsewhere (e.g., Hughes and Piper [1973]).

A *projective plane* is a structure consisting of a set of points, with a collection of distinguished subsets called lines, satisfying the following conditions:

(a) any two points lie on a unique line;

(b) any two lines meet in a unique point;

(c) there exist four points with no three collinear.

The first two conditions are dual to each other; the last, a nondegeneracy condition, has many equivalent formulations, among them its own dual. Another equivalent form is that each line has at least three points. It is an easy exercise to check that, if a structure satisfies (a) and (b) but not (c), then either there is a line incident with all points and dually, or there is a line incident with all but one point, the remaining lines having two points each, and dually (Figure 4).

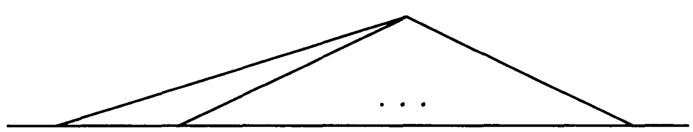


Figure 4. A degenerate projective plane.

An *affine plane* is a point-line structure satisfying:

(a) any two points lie on a unique line;

(b) if L is a line and p a point, then there is a unique line containing p and parallel to (i.e. either equal to or disjoint from) L;

(c) there exist three noncollinear points.

Note that it follows from (a) and (b) that parallelism is an equivalence relation on the set of lines, and that each equivalence class partitions the set of points.

From a projective plane, an affine plane is obtained by removing a line and all of its points. Conversely, given an affine plane, adjoin an 'ideal point' corresponding to each parallel class of lines, incident with every line in the class, and an 'ideal line' incident with all the ideal points; a projective plane is obtained. So an affine plane is equivalent to a projective plane with a distinguished line.

For any division ring K, $P(2, K)$ is a projective plane, and $A(2, K)$ an affine plane; the correspondence between projective and affine plane in this case is just the one described in the preceding section. However, not every projective or affine plane is of this form. Perhaps the easiest counterexample is the *Moulton plane*. The points are those of the familiar Euclidean plane (the affine plane over $\mathbb{R}$). However, lines with positive slope are modified by imagining that such a line is 'refracted' on crossing the x-axis so that its slope below the axis is twice as great as its slope above the axis. Lines with zero, negative or infinite slope are unaffected. We can see that Desargues' Theorem fails in the Moulton plane by choosing a Desargues configuration in the Euclidean plane having just one point below the x-axis, so that just one of the three lines through this point has positive slope. In the Moulton plane, this line is refracted, and fails to pass through the intersection of the other two.

In fact, we have the following important result.

THEOREM 2.1. *A projective plane is isomorphic to* $P(2, K)$ *for some division ring* K *if and only if it satisfies Desargues' Theorem.*

We have already seen the 'only if' part. A detailed proof of the converse will not be given here, but we give an indication of the argument.

First, it is clear that some collection of 'configuration theorems' will characterize the planes $P(2, K)$. For, given a projective plane Π, let L be a line and x, y, a three points of L, corresponding to three parallel classes in the derived affine plane, which we regard as horizontal, vertical, and with slope 1. Choose an origin $o = (0, 0)$, and a point on the horizontal line through o to receive the label $(1, 0)$. Now a bijection between the x and y axes, and a geometric definition of addition and multiplication, can be given (see Figures 5–7). Each division ring axiom will translate into a configuration theorem. What is unexpected is that a single configuration theorem suffices for all of them!

In fact, it is better to use central collineations here (see Section 3 for the definition of a collineation). In the notation of Figure 2 (Desargues' Theorem), suppose we seek a collineation fixing every line through p and every point on L. (Such a collineation is called *central*, and is an *elation* or a *homology* according as p is on L or not.) If a central collineation maps a_1 to a_2, it must map b_1 to b_2 and c_1 to c_2; Desargues' Theorem shows that the map defined in this way really is a collineation. Hence, if Desargues' Theorem holds, then the group of central collineations with centre p and axis L acts transitively on the points of any line M on p other than p and $L \cap M$. Now the additive and multiplicative group of K are the groups induced on M by the elations, resp., homologies. We have to prove just commutativity of addition and distributivity.

A companion result states:

THEOREM 2.2. *A projective plane is isomorphic to* $P(2, K)$ *for some commutative field* K *if and only if it satisfies Pappus' Theorem.*

This time, a geometric argument shows that Pappus' Theorem implies Desargues'; then (2.1) gives a coordinatizing division ring, and a simple argument with coordinates then shows that Pappus' Theorem requires commutativity of multiplication.

Now we turn to projective spaces of arbitrary dimension; the following result is the content of the Veblen–Young Theorem (1910). Some further terminology: in a geometry

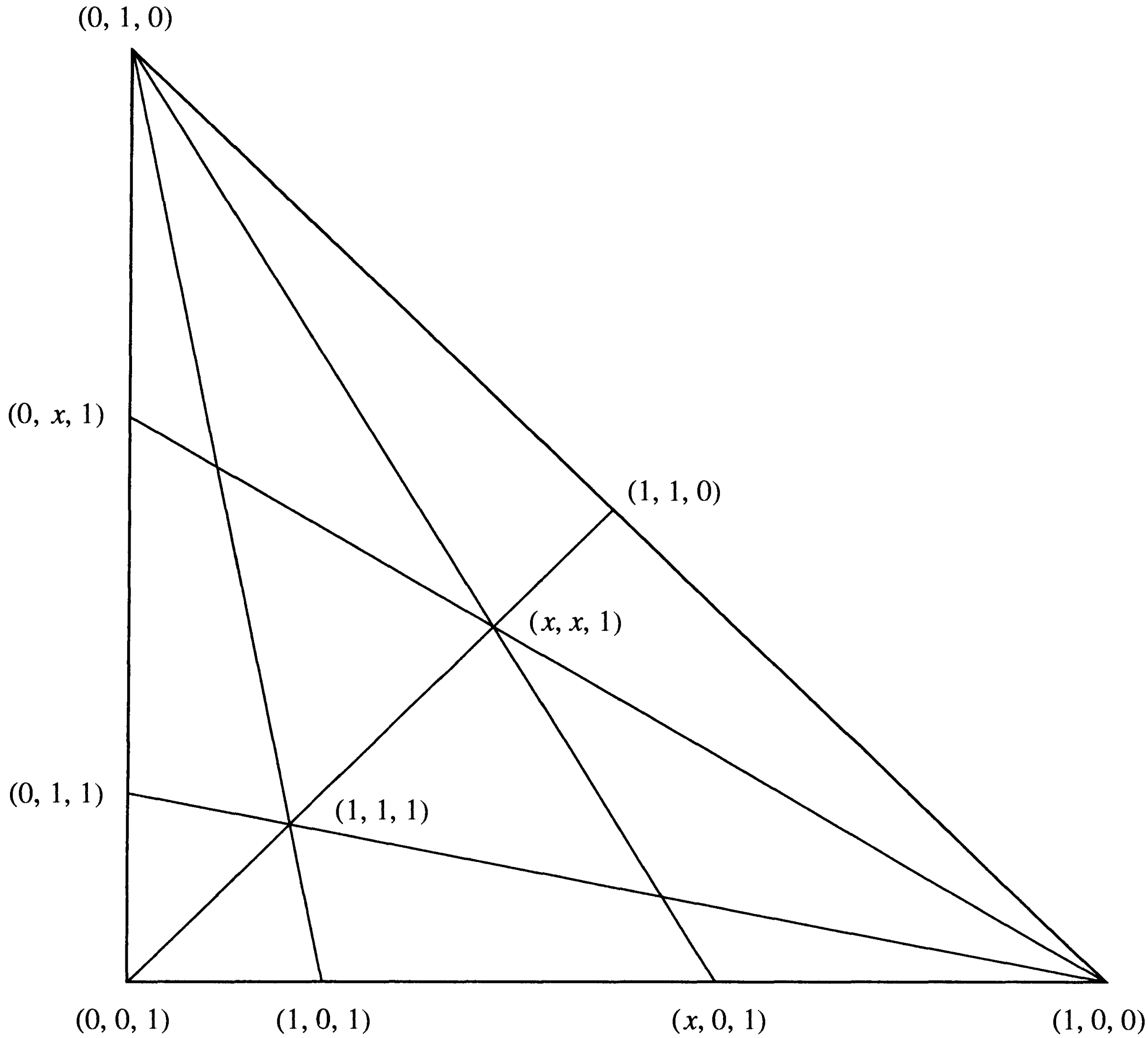


Figure 5. Coordinates.

of points and lines, a *subspace* is a set of points which contains any line through two of its points.

THEOREM 2.3. *Let X be a set of points, $\mathcal{L}$ a set of subsets of X called lines. Assume the following*:

(a) *two points lie on a unique line*;

(b) (*Veblen's Axiom*) *if a line meets two sides of a triangle, not at a vertex, then it meets the third side also*;

(c) *any line contains at least three points*;

(d) *there exist three noncollinear points*;

(e) *any chain of subspaces has finite length.*

Then the point set, equipped with all its subspaces, is either a projective plane, or $P(n, K)$ for some integer $n \geqslant 3$ and some division ring K.

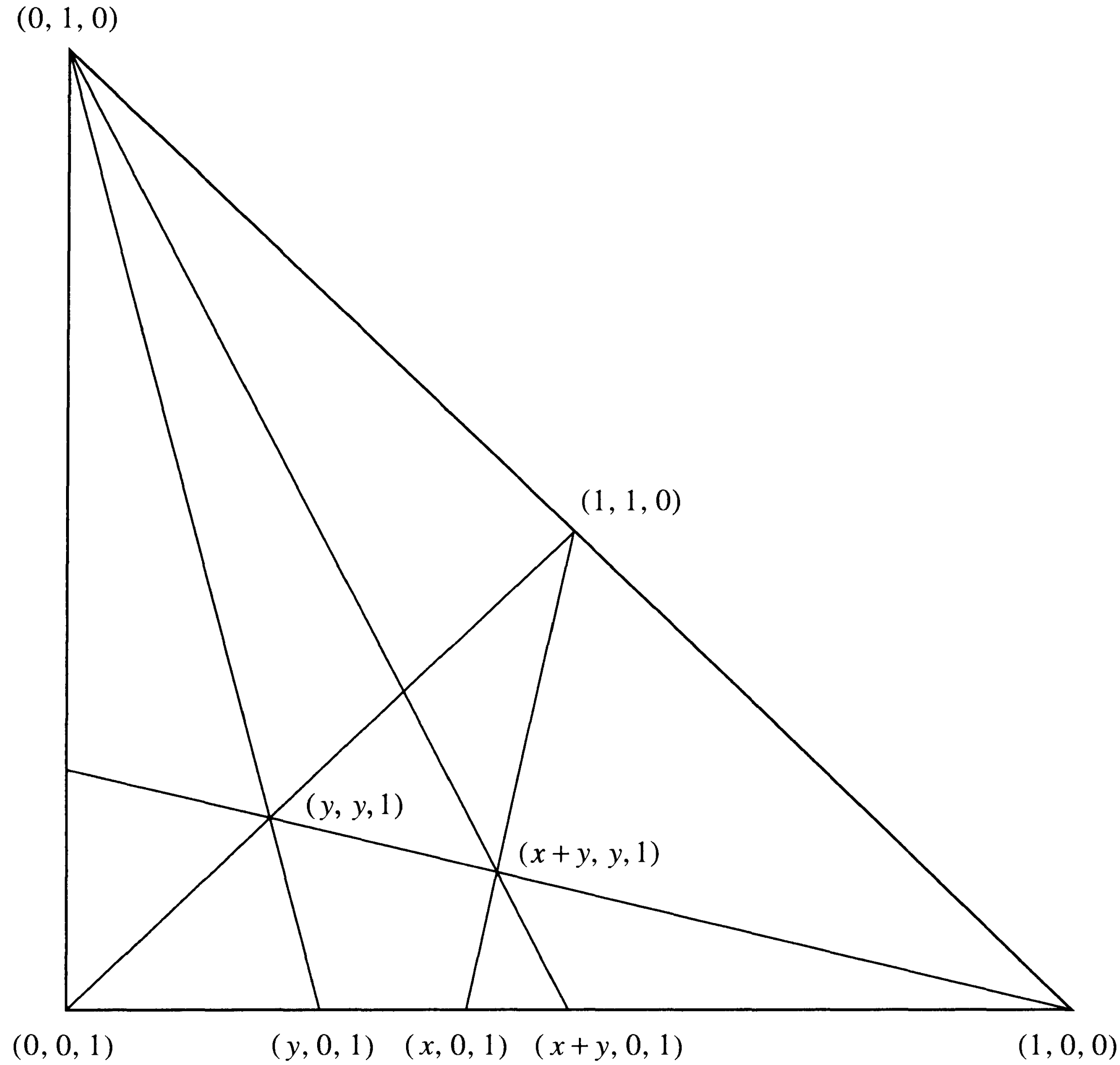


Figure 6. Addition.

In outline, the proof proceeds as follows. First, it is shown that, if U is a subspace and p a point not in U, then the union of all the lines joining p to points of U is a subspace $\langle U, p\rangle$, and every subspace arises in this way. In particular, for any nonincident point-line pair p, L, $\langle L, p\rangle$ is a projective plane. We may assume that there are points not in this plane.

(Note that, at this point, it is easy to see that the lattice of subspaces is modular.)

Next, Desargues' Theorem is verified. We have already observed that Desargues' Theorem for nonplanar configurations has a geometric proof, and this proof is valid in the present situation. The general case is shown by using the connection between Desargues' Theorem and central collineations, and showing that the required nongeneric collineations lie in the group generated by generic ones.

Now, by Theorem 2.1, each plane is coordinatized by a division ring. It is now possible to show that the coordinatizations of the planes can be done consistently, so as to give

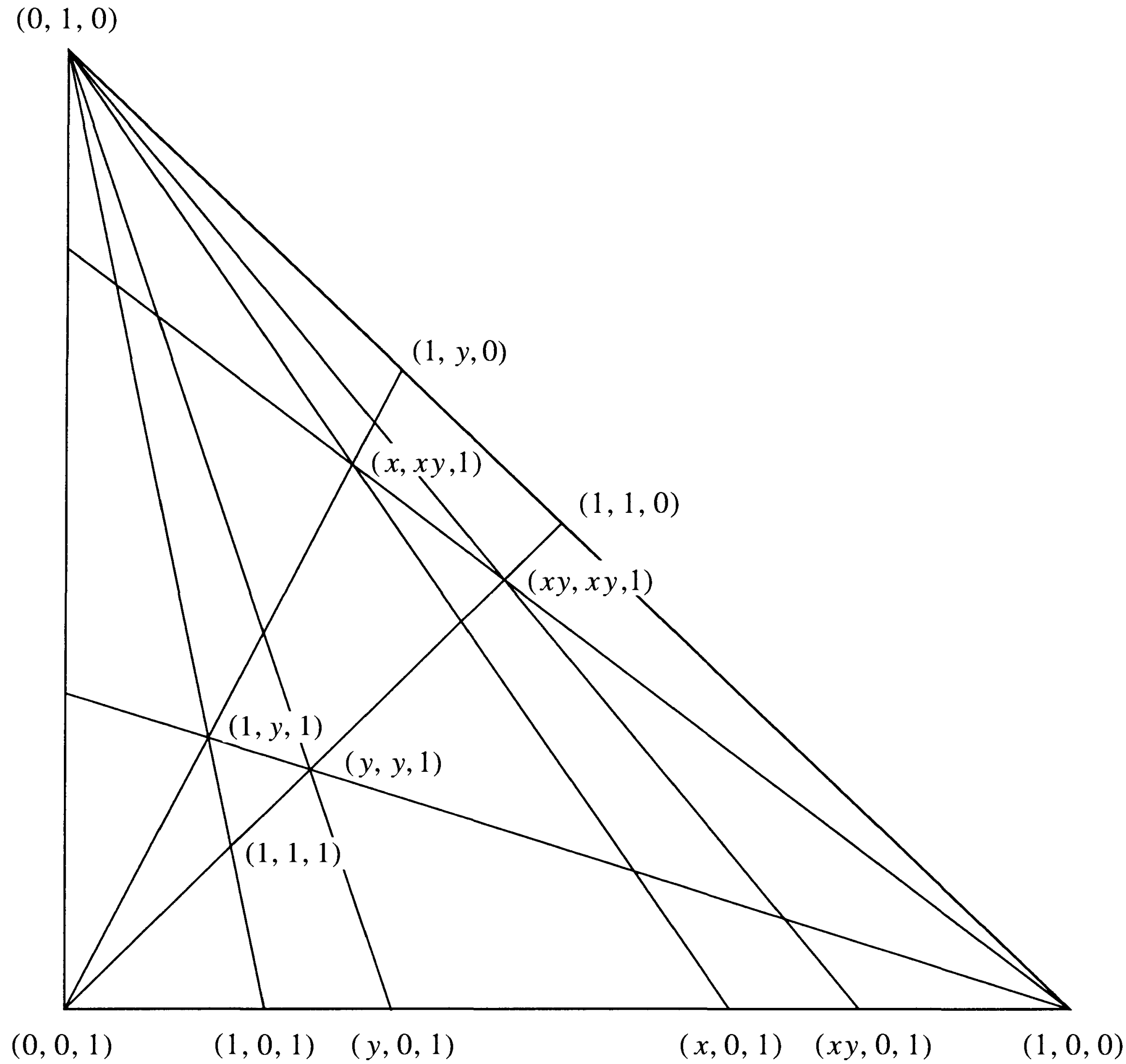


Figure 7. Multiplication.

the result.

A corollary of Theorem 2.3 is:

THEOREM 2.4. *$P(n, K)$ and $P(n', K')$ are isomorphic if and only if $n = n'$ and $K \cong K'$.*

For n is determined as one less than the longest chain of subspaces; and K is given by the coordinatization procedure, which as we saw can be defined geometrically.

Are the conditions of Theorem 2.3 really necessary?

Clearly, we do not want to abandon the crucial conditions (a) and (b). If we negate (c), the only additional possibilities are the empty set, a point, or a single line containing at least three points.

The *direct sum* of a family of point-line geometries is defined as follows: the point set is the disjoint union of the point sets of the summands; the line set is the disjoint union of their line sets, together with all pairs of points from different summands. Thus, a two-point line is the direct sum of its two points, while the degenerate projective plane of Figure 4 is the direct sum of a point and a line. A (possibly reducible) *projective space* is defined to be a direct sum of finitely many geometries of the following types: a point; a line with more than two points; a projective plane; and $P(n, K)$ for some n and some division ring K. The subspaces are as defined earlier.

THEOREM 2.5. *A geometry is a possibly reducible projective space if and only if it satisfies conditions* (a), (b) *and* (e) *of Theorem* 2.3, *and each line contains at least two points.*

If Veblen's Axiom holds, then the relation $P \sim Q$ if $P = Q$ or P and Q lie on a line of cardinality greater than 2, is an equivalence relation; its equivalence classes are subspaces satisfying (a), (b), (d) and (e), and the whole geometry is their direct sum.

The 'finite rank' condition (e) can also be weakened, but we shall not pursue this here.

Theorem 2.5 translates naturally into lattice-theoretic terms. A lattice is said to be *atomic* if every element is a join of atoms; *ranked* if all maximal chains from 0 to x have the same length (this length being called the *rank* of x); and *modular* if $x \leqslant z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all x, y, z.

THEOREM 2.6. *The following are equivalent for the lattice* $\mathcal{L}$:

(i) $\mathcal{L}$ *is a possibly reducible projective space*;

(ii) $\mathcal{L}$ *is atomic and ranked, and*

$$\operatorname{rank}(x \vee y) + \operatorname{rank}(x \wedge y) = \operatorname{rank}(x) + \operatorname{rank}(y)$$

for all $x, y \in \mathcal{L}$;

(iii) $\mathcal{L}$ *is atomic and modular.*

We noted in Section 1 that projective spaces satisfy all these conditions, and they are clearly preserved under direct sums.

Conditions (ii) and (iii) are easily seen to be equivalent. In any lattice, $x \leqslant z$ implies $x \vee (y \wedge z) \leqslant (x \vee y) \wedge z$; if (ii) holds, then the two sides have the same rank, and so are equal. Conversely, in a modular lattice, the map

$$c \mapsto b \vee (a \wedge c)$$

is an isomorphism between the intervals $[b, a \vee b]$ and $[a \wedge b, a]$ for any a, b; this implies both that the lattice is ranked and that the rank function satisfies (ii).

Now, in a lattice satisfying (ii), two points lie on a unique line, and two coplanar lines are concurrent, so that Veblen's axiom holds. By (2.5), the points and lines are those of a (possibly reducible) projective geometry, of which all lattice elements are subspaces; the rank condition then shows that every subspace is a lattice element, and the theorem is proved.

Theorem 2.3 can be stated in the form: a point-line geometry in which any two points lie on a unique line and any three noncollinear points lie in a projective plane (and such planes exist), is a projective plane or a projective space over a division ring. For affine spaces, the characterization is a little more elaborate; in general, the concept of parallelism must enter. Consider a geometry in which any triangle lies in an affine plane. Call two lines *parallel* if either they are equal, or there is an affine plane in which they are both contained and parallel.

THEOREM 2.7. *A point-line geometry is either an affine plane or* $A(n, K)$ *for some division ring* K, *if and only if the following conditions hold*:

(a) *two points lie in a unique line*;
(b) *three noncollinear points lie in a subspace which is an affine plane*;
(c) *parallelism is an equivalence relation on lines*;
(d) *there exist three noncollinear points*;
(e) *every chain of subspaces is finite.*

Condition (c) enables us to adjoin an ideal point to each equivalence class of lines; we can then define 'ideal lines' corresponding to 'parallel classes' of affine planes, and verify that the enlarged structure satisfies the hypotheses of Theorem 2.3.

The transitivity of parallelism cannot be dispensed with. Consider the case when lines have two points. Then any two points form a line, and any four points form an affine plane; even if we added as an assumption that a triangle lies in a unique distinguished affine plane, the conclusion would not follow, since any finite Steiner quadruple system (see Section 8) satisfies the conditions. However, there is a simple axiomatization of affine spaces over the field of order 2 in terms of points and planes, as follows:

(i) every plane has four points; three points lie in a unique plane; and the number of points is finite (i.e. the points and planes form a Steiner quadruple system);

(ii) if two planes have two common points, then their symmetric difference is a plane.

See Cameron [1976].

Hall [1967] gave the first example of a Steiner triple system (three points per line) in which any triangle lies in a 9-point affine plane, which is not an affine space; and much is known now about such systems (for example, the number of points must be a power of 3); see Bénéteau [1986], Young [1973]. In brief, if an origin is chosen in an affine space over the field of three elements, then the point set has the structure of an elementary Abelian group of exponent 3 (the translation group); under that weaker hypothesis, the group must be replaced by a commutative Moufang loop of exponent 3 (Bruck [1958]). For a thorough survey on these questions, refer to Deza and Sabidussi [1990].

On the other hand, we have:

THEOREM 2.8. *A point-line geometry having at least four points on some line is an affine plane or an* $A(n, K)$ *if it satisfies* (a), (b), (d), (e) *of Theorem* 2.7.

In other words, transitivity of parallelism can be proved if some (and hence every) line has at least four points. This was shown by Buekenhout [1969].

3. Collineations and correlations

A *collineation* of $P(n, K)$ is a permutation of the points which maps every flat to a flat of the same dimension.

Let V be the underlying vector space. Any nonsingular linear transformation of V induces a collineation. More generally, a function f on V is θ*-semilinear* (where θ is a field automorphism of K) if

$$(cv + c'v')f = c^{\theta}vf + c'^{\theta}v'f.$$

Any semilinear transformation induces a collineation of V. The next theorem, sometimes referred to as the *Fundamental Theorem of Projective Geometry*, is the converse of this assertion.

THEOREM 3.1. *If $n > 1$, then every collineation of $P(n, K)$ is induced by a semilinear transformation of the underlying vector space.*

A proof will be outlined after some discussion of the result.

First note that for every automorphism θ of K, there are θ-semilinear transformations: represent vectors of V by row vectors, and take the map

$$(v_0, \dots, v_k) \mapsto \left(v_0^{\theta}, \dots, v_k^{\theta}\right).$$

This particular map will also be denoted by θ. Now any θ-semilinear transformation is uniquely expressible as the product of a linear transformation and θ.

In group-theoretic language, let $\mathrm{GL}(n+1, K)$ be the group of nonsingular linear transformations, $\Gamma\mathrm{L}(n+1, K)$ the group of semilinear transformations, and $\mathrm{Aut}(K)$ the group of field automorphisms of K. Then $\Gamma\mathrm{L}(n+1, K)$ is the semidirect product of $\mathrm{GL}(n+1, K)$ by $\mathrm{Aut}(K)$. By (3.1), the full collineation group is induced by $\Gamma\mathrm{L}(n+1, K)$; we have to calculate the kernel of the action of this group on $P(n, K)$.

To this end, let p_i be spanned by the i-th basis vector e_i, for $i = 0, \dots, n$, and let $q = \langle e_0 + \cdots + e_n \rangle$. Then it is easily checked that the stabilizer of the points $p_0, \dots, p_n, q$ consists of transformations of the form

$$f\colon\ (v_0, \dots, v_n) \mapsto \left(v_0^{\theta}\lambda, \dots, v_n^{\theta}\lambda\right)$$

for some nonzero $\lambda \in K$ and some $\theta \in \mathrm{Aut}(K)$. Let r be the point spanned by $v = (1, c, 0, \dots, 0)$. If r is fixed, then $vf = \lambda v$, whence $c^{\theta} = \lambda c \lambda^{-1}$. If this holds for all $c \in K$, then θ is the inner automorphism induced by λ, and hence f is left multiplication by λ. Conversely, left multiplication by scalars obviously fixes every subspace of V.

Let Z denote the group of left multiplications by nonzero scalars, and $\mathrm{P}\Gamma\mathrm{L}(n+1, K)$ the quotient group $\Gamma\mathrm{L}(n+1, K)/Z$. Now we have

THEOREM 3.2. *For $n > 1$, the collineation group of $P(n, K)$ is* $\mathrm{P}\Gamma\mathrm{L}(n+1, K)$.

Note that $Z \cap \mathrm{GL}(n+1, K)$ consists of those scalars for which the inner automorphism of K is the identity, that is, the centre of K; so the group $\mathrm{PGL}(n+1, K)$ of collineations induced by linear collineations is $\mathrm{GL}(n+1, K)$ modulo central scalars.

Now we outline the proof of Theorem 3.1. A *simplex* is a set of $n+2$ points, any $n+1$ independent (such as $p_0, \ldots, p_n, q$ above). Since $\mathrm{P\Gamma L}(n+1, K)$ is transitive on simplices, it suffices to prove that any collineation fixing a simplex lies in $\mathrm{P\Gamma L}(n+1, K)$. This follows from careful inspection of the coordinatization theorem (3.3): once a simplex is fixed, the field operations can be defined geometrically, and so collineations must induce field automorphisms.

Next we describe the action and the structure of the collineation groups; we include the case $n = 1$ in our discussion.

PROPOSITION 3.3.

(i) *For* $n > 1$, $\mathrm{PGL}(n+1, K)$ *is* 2-*transitive but not* 3-*transitive on the points of* $P(n, K)$.

(ii) $\mathrm{PGL}(2, K)$ *is always* 3-*transitive*; *it is* 4-*transitive if and only if all elements of* $K \setminus \{0, 1\}$ *are conjugate in the multiplicative group of* K; *and it is never* 5-*transitive.*

PROOF. The 2-transitivity holds because any two independent vectors form part of a basis. For $n > 1$, some triples of points are collinear and others are not. For $n = 1$, on the other hand, any three points form a simplex, and their stabiliser is the group of inner automorphisms of K. □

Division rings with the property that all elements except 0 and 1 are conjugate do exist, see Cohn [1971]. They clearly must have characteristic 2.

The transitivity properties of $\mathrm{P\Gamma L}(n+1, K)$ are easily worked out.

Now we turn to the normal structure, and for convenience assume that K is commutative. There is an obvious homomorphism from $\mathrm{GL}(n+1, K)$ to K^* given by the determinant; its kernel (the set of linear transformations with determinant 1) is denoted by $\mathrm{SL}(n+1, K)$. Thus we have

$$\mathrm{GL}(n+1, K)/\mathrm{SL}(n+1, K) \cong K^\times,$$

and hence

$$\mathrm{PGL}(n+1, K)/\mathrm{PSL}(n+1, K) \cong K^\times / K^{\times(n+1)}.$$

The group $\mathrm{PSL}(n+1, K)$ is almost always simple. To see this, we consider *transvections*, linear maps of the form $I+E$, where E has rank 1 and $E^2 = 0$. The corresponding collineations of $P(n, K)$ are *elations*, fixing every subspace in a hyperplane H and every subspace containing a point p, where $p \in H$. (We met transvections in a projective plane in Section 2.) Note that transvections have determinant 1. They are all conjugate in $\mathrm{GL}(n+1, K)$, and even in $\mathrm{SL}(n+1, K)$ if $n > 1$.

THEOREM 3.3.
(i) $\mathrm{SL}(n+1, K)$ *is generated by transvections.*
(ii) $\mathrm{PSL}(n+1, K)$ *is simple unless* $n = 1$ *and* $|K| \leqslant 3$.

Elementary linear algebra shows that any element of $\mathrm{GL}(n+1, K)$ is a product of elementary matrices, these being either transvections or *reflections*, i.e. diagonalizable transformations with all eigenvalues but one equal to 1 (these induce *homologies* of the projective space). With a little more care, one can see that at most one reflection is needed in each such product; considering determinants, the first part follows. Now, for the second, if $n > 1$, all we need prove is that a normal subgroup of $\mathrm{SL}(n+1, K)$ which is not contained in the group Z of scalar matrices must contain a transvection. The case $n = 1$ is handled in a similar way.

When $|K| = 2$ or 3, $\mathrm{PSL}(2, K)$ is not simple, being isomorphic to the symmetric group of degree 3 or the alternating group of degree 4, respectively.

Now we turn to the affine geometry $A(n, K)$. Let V be the underlying vector space. It is clear that the group T of translations of V is contained in the collineation group.

THEOREM 3.4. *For* $n \geqslant 2$, *the collineation group of* $A(n, K)$ *is the semidirect product of the translation group* T *by the group* $\Gamma\mathrm{L}(n, K)$.

(This group is denoted by $\mathrm{A}\Gamma\mathrm{L}(n, K)$.)

This can be proved either by identifying the stabilizer of a hyperplane in $\mathrm{P}\Gamma\mathrm{L}(n+1, K)$ with $\mathrm{A}\Gamma\mathrm{L}(n, K)$, or by observing that any affine collineation is uniquely the product of a translation and a collineation fixing 0 and directly showing that the latter is semilinear.

We now turn to correlations.

A *correlation* or *duality* of $P(n, K)$ is an inclusion-reversing permutation of the subspaces. It is a *polarity* if it has order 2.

There is a familiar order-reversing correspondence between the subspaces of a finite-dimensional vector space V and those of its dual space V': to the subspace U of V corresponds its *annihilator*

$$\mathrm{Ann}(U) = \{f \in V' \colon (\forall v \in U)\, vf = 0\}.$$

Now V' is a right vector space over K, and so is naturally a left vector space over the *opposite* division ring K^{o}.

(This has the same elements and the same addition as K, but multiplication $*$ defined by $c * d = dc$.) So the dual of $P(n, K)$ is isomorphic to $P(n, K^{\mathrm{o}})$. By Theorem 2.4, we conclude:

THEOREM 3.5. *For* $n > 1$, $P(n, K)$ *admits a correlation if and only if* $K \cong K^{\mathrm{o}}$.

In particular, this holds if K is commutative; but it also holds, e.g., for the division ring of quaternions, where an anti-automorphism is given by

$$a + bi + cj + dk \mapsto a - bi - cj - dk.$$

Let σ be an anti-automorphism of K. A function b: $V \times V \to K$ is a *σ-sesquilinear form* if

$$b(cv + c'v', w) = cb(v, w) + c'b(v', w)$$

and

$$b(v, cw + c'w') = c^{\sigma}b(v, w) + c'^{\sigma}b(v, w').$$

It is *nonsingular* if

$$(\forall v \in V)\; b(v, w) = 0 \;\Rightarrow\; w = 0$$

and

$$(\forall w \in V)\; b(v, w) = 0 \;\Rightarrow\; v = 0.$$

Given a sesquilinear form b, there is a map f on subspaces of V defined by

$$f(U) = \{w \in V\colon\; (\forall v \in U)\; b(v, w) = 0\}.$$

If b is nonsingular, then f is one-to-one and onto; it is obviously inclusion-reversing, and so defines a correlation. Conversely,

THEOREM 3.6. *Every correlation of $P(n, K)$ is induced by a σ-sesquilinear form b, where σ is an anti-automorphism of K. The correlation is a polarity if and only if the form satisfies*

$$(\forall v, w \in V)\; b(v, w) = 0 \;\Rightarrow\; b(w, v) = 0.$$

This is a consequence of the Fundamental Theorem 3.1. First, there is a correlation f of this form (given that K is isomorphic to K°): take a fixed anti-automorphism α of K, and map U to the annihilator of U^{α}, where α is applied coordinate-wise. The appropriate form is given by

$$b(v, w) = \sum_{i=0}^{n} v_i w_i^{\alpha}.$$

If f_1 is another correlation, then ff_1^{-1} is a collineation, say h; and then f_1 is represented by the form $b'(v, w) = b(v, wh)$, which is sesquilinear since h is semilinear. The condition for a polarity is clear.

A bilinear form b is *alternating* if $b(v, v) = 0$ for all v; this implies that $b(v, w) = -b(w, v)$ for all v, w (and hence that the form is reflexive), and is implied by it if the characteristic is not 2. A σ-sesquilinear form b is *Hermitian* if $b(w, v) = b(v, w)^{\sigma}$ for all v, w (which again implies that the form is reflexive). In the commutative case, these are essentially all the reflexive forms:

THEOREM 3.7. *Suppose that K is commutative and $n > 1$. Then any polarity of $P(n, K)$ is induced by a Hermitian form or an alternating bilinear form.*

4. Polar spaces

Let f be a reflexive sesquilinear form on the vector space V, defining a polarity σ of the derived projective space. A subspace W of V is called *totally isotropic* (t.i.) if f vanishes identically on W, i.e. if $W \subseteq W^\sigma$. The totally isotropic subspaces of V form a subgeometry of the projective space, called a *polar space*. It has the following properties.

(P1) Each t.i. space, equipped with its t.i. subspaces, is isomorphic to a projective space of dimension at most $n-1$.
(P2) The intersection of any family of t.i. subspaces is a t.i. subspace.
(P3) If W is a t.i. subspace of dimension $n-1$, and p a point not in W, then the set

$$\{q \in W\colon \text{ the line } pq \text{ is t.i.}\}$$

is a hyperplane in W, and the union of the lines pq just described is a t.i. subspace of dimension $n-1$.
(P4) There exist two disjoint t.i. subspaces of dimension $n-1$.

Of these, (P1) and (P2) are clear. (P3) holds because, if $u \notin W$, then the function

$$w \mapsto f(w,u)$$

on W is linear, and is not identically zero (otherwise $\langle W,u\rangle$ would be a t.i. subspace of dimension n); so its kernel is a hyperplane in W. The final condition (P4) uses the nondegeneracy of f; we show by induction on i that we can find two t.i. spaces of dimension $n-1$ whose intersection has codimension i in each for all $i \leqslant n$.

Now a geometry consisting of a set of points with a collection of distinguished subsets called subspaces, satisfying conditions (P1)–(P4) (with the words 't.i.' deleted throughout), is called an *abstract polar space*. Usually the word 'abstract' is dropped here; we have a generalized, but not conflicting, use of the term 'polar space'.

Of course, it is a real generalization; not every structure satisfying (P1)–(P4) comes from a polarity. We now give five further types of polar spaces. The reference for what follows is Tits [1974].

1. A *quadratic form* on a vector space V over a field K is a function $q\colon V \to K$ which is of degree 2 in the coordinates. In other words, it satisfies

$$q(\lambda v) = \lambda^2 q(v) \quad \text{and} \quad q(v+w) = q(v) + q(w) + f(v,w)$$

for all $v, w \in V$, $\lambda \in K$, where f is bilinear. The form q is *nonsingular* if $q(v) \neq 0$ for all nonzero vectors v in the radical of f. (In particular, q is nonsingular if f is nondegenerate.) A subspace W of V is *totally singular* (t.s.) for q if q vanishes identically on it. Now the totally singular subspaces for a nonsingular quadratic form constitute a polar space.

If the characteristic of K is not 2, then q can be recovered from f by the rule $q(v) = f(v,v)/2$. In particular, q is nonsingular if and only if f is nondegenerate, and nothing new is obtained. However, in characteristic 2, the form f is alternating

($f(v,v) = 0$ for all $v \in V$), and we do get a new geometry, consisting of some of the subspaces of the polar space defined by f.

2. Let V be a vector space over a division ring K of characteristic 2, and σ an anti-automorphism of K satisfying $\sigma^2 = 1$. Let K_0 be the additive subgroup $\{x+x^\sigma\colon x \in K\}$ of K, and $K^* = K/K_0$. A function $f\colon V \to K^*$ is called a *pseudoquadratic form* relative to σ provided there is a σ-sesquilinear form g such that $f(v) = g(v,v) \bmod K_0$. The subspaces on which f vanish (mod K_0) define a polar space. If σ is the identity, then $K_0 = \{0\}$, and f is quadratic; if $g(v,v) \in K_0$ for all v (we say that g is *trace-valued* in this case), then the form f is degenerate. It can be shown that, if K is commutative, then one of these alternatives holds. So new examples are obtained only over noncommutative division rings.

3. Consider the Grassmannian of lines in the 3-dimensional projective space over K. The 'points' of this geometry are the lines of $P(3,K)$; 'lines' are plane pencils; and there are two types of 'planes', viz. all lines through a point (isomorphic to $P(2,K)$), and all lines in a plane (the dual of the plane, isomorphic to $P(2,K^{\mathrm{o}})$). This is easily shown to be a polar space of rank 3. If K is not isomorphic to K^{o}, this polar space contains nonisomorphic planes; this cannot happen in any of the earlier examples. However, if K is commutative, we will see that it is the geometry of totally singular subspaces on the Klein quadric in $P(5,K)$.

4. Another class of rank 3 polar spaces can be constructed from algebraic groups of type E_6. These are the only polar spaces in which the planes are non-Desarguesian; in fact they satisfy a weakening of Desargues' Theorem known as the *Moufang condition*, and can be coordinatized over alternative division rings (which generalize the Cayley numbers or octonions).

5. Finally, consider polar spaces of rank 2. Taking, as usual, 'projective space of dimension 1' to mean 'line with at least three points', we see that a polar space of rank 2 is a geometry of points and lines satisfying

(Q1) any line has at least three points;
(Q2) two points lie on at most one line;
(Q3) if a point p is not on a line L, then p is collinear with a unique point of L;
(Q4) no point is collinear with all others.

Such a geometry is called a *generalized quadrangle*, or GQ. Not all GQs arise from polarities or quadratic forms (the ones which do are sometimes called 'classical'). Indeed, GQs can be produced by a free construction (see Chapter 13), so we don't expect a classification to be possible.

As with projective spaces, we might expect a classification to be possible for sufficiently large rank. Tits [1974], building on work of Veldkamp, showed that this is indeed the case:

THEOREM 4.1 (Tits). *All polar spaces of rank at least* 3 *are known. In particular, all those of rank at least* 4 *arise either from polarities of projective spaces, or from quadratic or pseudoquadratic forms.*

An important simplification of the axioms (P1)–(P4) was obtained by Buekenhout and Shult [1974]. Like the Veblen–Young axioms (2.1) for projective spaces, the Buekenhout axioms only involve points and lines. In a point-line geometry, a *subspace* is a point set which contains any line through two of its points; it is *singular* provided any two of its points are collinear.

THEOREM 4.2 (Buekenhout–Shult). *Suppose that a point-line geometry has the following properties*:

(i) *if p is a point not on a line L, then p is collinear with one or all points of L*;

(ii) *any line contains at least three points*;

(iii) *no point is collinear with all others*;

(iv) *any chain of singular subspaces is finite.*

Then the singular subspaces constitute a polar space.

Note that axiom (i) implies that a clique in the collinearity graph (a maximal set of pairwise collinear points) is a singular subspace. The first step of the proof is to show that two points lie on at most one line – this makes essential use of the nondegeneracy condition (iii). Then the polar space axioms must be verified.

5. Buekenhout geometries

Polar spaces lend themselves to an inductive study. If p is a point in a polar space of rank n, then the subspaces containing p are easily seen to constitute a polar space of rank $n - 1$. It is possible to define a very general framework in which this inductive reasoning can be applied to classes of geometries. Moreover, this framework has been shown to include many more interesting geometries, for example, geometries on which many of the sporadic finite simple groups act (see Chapter 22).

A geometry has 'varieties' of various 'types' (e.g., subspaces of various dimensions), with an incidence relation holding between certain pairs of varieties. Accordingly, let Δ be a set of *types*. A *geometry over* Δ is a set X of *varieties*, with a *type map* tp: $X \to \Delta$ and a reflexive and symmetric *incidence relation* on Δ, such that two varieties of the same type which are incident are necessarily equal. It can be regarded simply as a $|\Delta|$-partite graph, with a loop at each vertex. Usually, we assume connectedness, and often even a stronger form of connectedness defined below. The *rank* of the geometry is $|\Delta|$.

A *flag* is a set of pairwise incident varieties (necessarily of different types). The geometry is *transversal* if every maximal flag contains a variety of each type, and is *firm* (resp. *thick*) if every nonmaximal flag is contained in at least two (resp. three) maximal flags. Most interesting geometries are transversal and thick, or at least firm.

The *residue* of a flag F is the set of varieties not in F which are incident with every variety in F. It is a geometry over the type set $\Delta \setminus \mathrm{tp}(F)$. A geometry is said to be *residually connected* if every residue of rank at least 2 is connected.

From now on, unless specified otherwise, all geometries are transversal and residually connected.

We give axioms for classes of geometries in a uniform way by assuming general hypotheses (transversality, residual connectedness) and specifying the possible rank 2 residues. This is done as follows. A *diagram* consists of a set Δ and, for each $i, j \in \Delta$ with $i \neq j$, a class $\mathcal{G}_{ij}$ of rank 2 geometries (whose varieties are called 'points' and 'lines'). We assume that the geometries in $\mathcal{G}_{ji}$ are the duals of those in $\mathcal{G}_{ij}$. Now a geometry over the type set Δ is said to *belong to the diagram* if every residue of type $\{i, j\}$ is in the class $\mathcal{G}_{ij}$, where varieties of type i and j in the residue are identified with points and lines respectively.

In order to use this concept, we must specify some classes of rank 2 geometries. We use three classes: digons; projective planes; and generalized quadrangles. (A *digon* consists of at least three points and at least three lines, any point being incident with any line.) We represent diagrams using these classes pictorially as follows. Take a vertex or node for each member of Δ; join nodes i, j by 0, 1 or 2 arcs according as $\mathcal{G}_{ij}$ is the class of digons, projective planes, or GQs. (Note that all three classes are self-dual.)

To illustrate:

THEOREM 5.1. *A geometry with diagram*

A_n = 

(*n nodes*)

is the same thing as an n*-dimensional projective geometry, if* $n \geqslant 3$.

First, take a projective space, and let F be a flag of cotype $\{i, j\}$, where the type of a subspace is its dimension. There are two cases:

1. $j = i + 1$. F has the form

$$U_0 \subseteq \cdots \subseteq U_{i-1} \subseteq U_{i+2} \subseteq \cdots,$$

and its residue consists of all i- and $(i+1)$-spaces between U_{i-1} and U_{i+2}; these clearly form a projective plane.

2. $j > i + 1$. Now F has the form

$$\cdots \subseteq U_{i-1} \subseteq U_{i+1} \subseteq \cdots \subseteq U_{j-1} \subseteq U_{j+1} \subseteq \cdots;$$

any i-space and j-space in the residue are incident, and the residue is a digon.

The converse is proved by induction. It is only necessary to show that any two points are collinear; then the fact that all planes are projective means that the *Veblen–Young axioms* apply, and it is an easy exercise to show that every subspace is a variety of the geometry and conversely. By connectedness, it suffices to prove that if two points p, r are collinear with a common point q, then p and r are collinear with one another. But the lines qp and qr are 'collinear' in the residue of q (by the inductive hypothesis), so that p, q and r are coplanar; the conclusion follows since the planes are projective.

We already observed that polar spaces of rank 2 are GQs. The inductive principle can be expressed as follows:

THEOREM 5.2. *A polar space belongs to the diagram*

C_n = 

(*n nodes*)

By contrast with what happens for projective spaces, the converse of this result is false. The failure has been carefully analyzed by Tits [1981]. We describe the situation briefly. There are two ways in which failure can occur. There is a notion of '2-covering', so that if one geometry is a 2-cover of another, they have the same rank 2 residues. Any geometry has a universal 2-cover; the best we could hope for is that the universal 2-cover[2] of a geometry with diagram C_n be a polar space. But even this is false; both finite and infinite counterexamples exist.

On the positive side, we have the following special case of a theorem of Tits.

THEOREM 5.3. *If a geometry of type C_n has the property that all of its C_3 residues are 2-covered by polar spaces, then the whole geometry is 2-covered by a polar space.*

In view of this, much attention has been paid to the 'simply-connected' C_3 geometries which are not polar spaces. We mention one finite example, Neumaier's *A_7 geometry*. Let Ω be a set of 7 points. The points of the geometry are the points of Ω, and the lines are the sets of three points. There are 30 ways in which Ω can be given the structure of a projective plane over the field of 2 elements; these fall into two orbits of length 15 under the alternating group A_7. The planes of the geometry are the elements of one of these orbits. Incidence between points and lines is membership; a line is incident with a plane if it is a line of that plane; and every point and plane are incident. (The last assertion says that the geometry is *flat*.)

It is clear that the residue of a plane is a projective plane, while the residue of a line is a digon. Let p be a point. The lines incident with p can be identified with the 15 2-subsets of $S = \Omega \setminus \{p\}$. Given three 3-subsets of Ω intersecting pairwise in p, there are just two ways to complete to a projective plane; these two lie in different A_7-orbits. So the planes incident with p can be identified with the 15 partitions of S into three 2-sets. This is a well-known representation of the GQ associated with a symplectic form on a 4-dimensional vector space over the field of 2 elements.

Some of the polar spaces have traditional names: those derived from quadratic forms are *orthogonal*; those derived from alternating bilinear forms are *symplectic*; and those derived from σ-Hermitian sesquilinear forms are *unitary* (this last term is often reserved for the case when $\sigma \neq 1$, that is, when the geometry is not orthogonal). These names are taken from the names of the 'classical groups' which preserve the forms in question.

6. Grassmannians

Let $\mathcal{P}$ be the set of k-flats in $P(n, K)$. In this section we assign geometric structure to the set $\mathcal{P}$.

There is a natural way to define 'lines' on $\mathcal{P}$ so as to form a partial linear space. Let U be a $(k-1)$-flat and W a $(k+1)$-flat with $U \subset W$. Then the *Grassmann line* $L(U, W)$ defined by U and W is the set

$$\{X \in \mathcal{P}\colon\ U \subset X \subset W\}.$$

[2] See Chapter 3.

Given two elements $X_1, X_2 \in \mathcal{P}$, if $X_1 \cap X_2$ is a $(k-1)$-flat then $\langle X_1, X_2 \rangle$ is a $(k+1)$-flat, and $L(X_1 \cap X_2, \langle X_1, X_2 \rangle)$ is the unique Grassmann line containing X_1 and X_2; otherwise, no Grassmann line contains these two elements.

If K is commutative, this Grassmannian space is itself embeddable in a projective space

$$P\left(\binom{n+1}{k+1} - 1, K\right),$$

as follows. Let V be the $n+1$-dimensional vector space underlying our projective space. Let $W = \bigwedge^{k+1} V$ be its $(k+1)$-st *exterior power*, spanned by the $(k+1)$-tuples of vectors of V subject to multilinearity and skew-symmetry. (This is the quotient of the $(k+1)$-st tensor power of V by the subspace spanned by the elements

$$\big((cv_0 + c'v_0') \otimes v_1 \otimes \cdots \otimes v_k\big) - c\big(v_0 \otimes \cdots \otimes v_k\big) - c'\big(v_0' \otimes \cdots \otimes v_k\big)$$

and

$$\big(v_0 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k\big) + \big(v_0 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k\big)$$

for $i = 0, \ldots, k-1$, where $v_0, v_0', \ldots, v_k \in V$ and $c, c' \in K$.) Then $\bigwedge^{k+1} V$ has dimension $\binom{n+1}{k+1}$, and has basis consisting of all vectors $e_{i_0} \wedge \cdots \wedge e_{i_k}$, where $\{e_0, \ldots, e_n\}$ is a basis for V and $i_0 < \cdots < i_k$. (Here $v_0 \wedge \cdots \wedge v_k$ is the image of $v_0 \otimes \cdots \otimes v_k$.)

In $\bigwedge^{k+1} V$, a vector $v_0 \wedge \cdots \wedge v_k$ is nonzero if and only if $v_0, \ldots, v_k$ are linearly independent. Moreover, replacing $v_0, \ldots, v_k$ by vectors $v_0', \ldots, v_k'$ spanning the same subspace merely multiplies their exterior product by a scalar (the determinant of the matrix expressing the primed vectors in terms of the unprimed ones). Hence we have a map from the k-flats of $P(n, K)$ to the points of $P(N, K)$, where

$$N = \binom{n+1}{k+1} - 1.$$

The significant properties of this map ϕ are:

(i) ϕ is one-to-one;

(ii) a line of $P(N, K)$ is contained in the image of ϕ if and only if it is the image under ϕ of a Grassmann line;

(iii) any other line of $P(N, K)$ contains at most two points of the image of ϕ.

There are two kinds of maximal singular subspaces of a Grassmannian. The first consists of all the k-flats contained in a given $(k+1)$-flat; the second, the k-flats containing a given $(k-1)$-flat. Note that each Grassmann line lies in a unique subspace of each type; and that each type of maximal singular subspace is embedded by ϕ as a subspace of the projective space $P(N, K)$.

On the other hand, the first type is a projective space over K°, whereas the second is a projective space over K; so, if K is not isomorphic to K°, the Grassmannian is not embeddable in any projective space.

In the case $k = 1$, there is a convenient representation of the exterior square of V. If v, w are (row) vectors in V, then $v^T w - w^T v$ is a skew-symmetric matrix; the map

$$v \wedge w \mapsto v^T w - w^T v$$

is an isomorphism from $V \wedge V$ to the space of skew-symmetric matrices.

The determinant of a skew-symmetric matrix A of even order m is a square. (There is a polynomial of degree $m/2$ in the entries of A, called the *Pfaffian* of A and denoted $\mathrm{Pf}(A)$, such that $\det(A) = \mathrm{Pf}(A)^2$.) Hence, if $m > 2$, the Grassmannian is embedded in a hypersurface of degree $m/2$ defined by the equation $\mathrm{Pf}(A) = 0$. If the order m is odd, the determinant is zero. (More generally, the rank of a skew-symmetric matrix is always even.)

Now let us specialize further to the case $k = 1$, $n = 3$. By the above remarks, a nonzero skew-symmetric matrix A of order 4 is of the form $v^T w - w^T v$ if and only if it has rank 2, i.e. if and only if it is singular, i.e. if and only if $\mathrm{Pf}(A) = 0$. So the image of the Grassmannian of lines in $P(3, k)$ is the quadric in $P(5, K)$ defined by the equation $\mathrm{Pf}(A) = 0$. This set is known as the *Klein quadric.*

Many features of the geometry of projective 3-space can be elucidated in terms of the Klein quadric $\mathcal{Q}$. For example, three pairwise nonperpendicular points of the quadric span a nonsingular plane, which meets $\mathcal{Q}$ in a conic; the perpendicular plane also meets $\mathcal{Q}$ in a conic. The set of lines corresponding to a conic on $\mathcal{Q}$ is called a *regulus*. So we see that three pairwise skew lines lie in a unique regulus, and that any regulus has an *opposite* regulus so that each line of the regulus meets each line of its opposite. Thus, a regulus and its opposite have the structure of a square grid. (In fact, they are the two families of rulings of a hyperbolic quadric.)

Another characterization of Grassmannians is as incidence structures with three kinds of varieties, the 'points' and the two kinds of maximal singular subspaces (corresponding to the $(k - 1)$-, k-, and $(k + 1)$-flats in $P(n, K)$). See Sprague [1981], Tallini [1981].

7. Clifford algebras and spinors

The space of spinors provides a projective embedding of the geometry of maximal t.s. subspaces for a quadratic form, rather similar to the embedding of Grassmann spaces in exterior powers.

Let q be a quadratic form on the vector space V. The *Clifford algebra* $C(q)$ of q is the largest algebra (with identity) generated by V in which the relation

$$v^2 = q(v).1$$

holds for all $v \in V$. It is obtained from the tensor algebra of V by factoring out the ideal generated by the elements $v \otimes v - q(v).1$ for $v \in V$.

Note that the relation

$$vw + wv = f(v, w).1$$

holds in $C(q)$, where f is the bilinear form obtained by polarizing q (that is, $q(v+w) = q(v)+q(w)+f(v,w)$). In particular, if q is identically 0, then $wv = -vw$ for all v, w. In this case, the Clifford algebra is just the *exterior algebra* of V, which (as vector space) is the direct sum of all the exterior powers of V.

We will now discuss a special class of quadratic forms, the *split* forms. Let V have even dimension $2n$. A split quadratic form has the shape

$$q(v) = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}$$

in coordinates relative to some basis. A nonsingular form is split if and only if its maximal t.s. subspaces have the largest possible dimension, i.e. $n = \dim(V)/2$.

PROPOSITION 7.1. *Let q be a split quadratic form on a $2n$-dimensional vector space. Then $C(q)$ is isomorphic to the algebra* End(S) *of linear transformations of a vector space S of dimension 2^n.*

This follows easily by induction from the facts:

1. The result is true for $n = 1$. For let v and w be basis vectors relative to which q has the form x_1x_2. Map v to $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ and w to $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$; the images satisfy $v^2 = w^2 = 0$, $vw + wv = 1$.

2. If V is the orthogonal direct sum of V_1 and V_2, carrying forms q_1 and q_2, respectively, then

$$C(q) = C(q_1) \otimes C(q_2)$$

and

$$\mathrm{End}(S_1 \otimes S_2) = \mathrm{End}(S_1) \otimes \mathrm{End}(S_2).$$

The elements of S are called *spinors*. As our discussion suggests, they are somewhat abstract, and cannot be unambiguously constructed from the original vector space V. However, certain 1-dimensional subspaces of S do correspond to recognizable objects in V:

PROPOSITION 7.2. *There is a* 1–1 *map from the set of maximal t.s. subspaces of V to the set of* 1*-dimensional subspaces of S.*

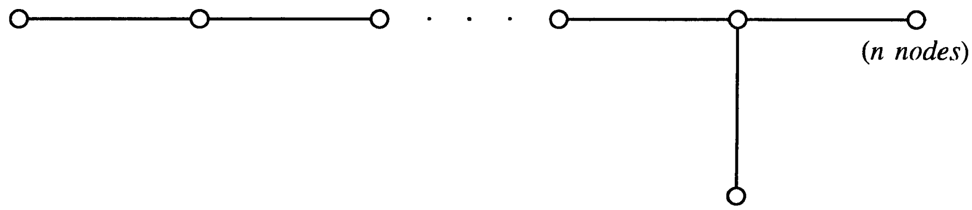
Briefly: Let U be maximal t.s. in V. Then there is a corresponding subalgebra of $C(q)$, isomorphic to the exterior algebra of U. This subalgebra contains a 'maximal' element $c(U)$ (the product of the vectors in a basis of U), uniquely defined up to scalar multiples. Now $c(U)$ generates a minimal left ideal of $C(q)$. Any minimal left ideal of End(S) is the set of endomorphisms whose range is contained in a fixed 1-dimensional subspace of S. This subspace is the required image of U.

The spinors contained in subspaces in the image of this map (or the subspaces themselves, if we are thinking projectively) are called *pure spinors*.

To explore further, we examine the polar space.

PROPOSITION 7.3. *Let q be a split quadratic form on V. Then the maximal t.s. subspaces for q fall into two families, two subspaces lying in the same family if and only if their intersection has even codimension in each. A t.s. $(n-1)$-space is contained in a unique member of each family. If we delete the $(n-1)$-spaces from the polar space, separate the n-spaces into two types corresponding to the two families, and let two n-spaces be incident if their intersection is a hyperplane in each, we obtain a geometry belonging to the diagram*

D_n =



PROPOSITION 7.4. *The spinor space S is the direct sum of two subspaces S^+, S^-, each of dimension 2^{n-1} and spanned by the pure spinors corresponding to one of the families of t.s. subspaces. Moreover, if a line in projective spinor space entirely consists of pure spinors, then these correspond to all the t.s. n-spaces in one family containing a fixed t.s. $(n-2)$-space (and conversely); and any other line contains at most two projective pure spinors.*

Here S^+ and S^- are known as *half-spinor spaces*. This proposition gives us two (isomorphic) new point-line geometries, the so-called *half-spinor geometries.*

We turn now to the low-dimensional cases.

For $n = 2$, the polar space consists of points and lines, each point lying on a unique line of each family – this is the ruled quadric in projective 3-space, familiar (in the real case) from geometric models. The pure spinors comprise two skew lines in 3-space.

For $n = 3$, the polar space is the Klein quadric. Each half-spinor space has dimension 4, and all of its elements are pure spinors. By Proposition 7.4, projective lines in half-spinor space are identified with points on the Klein quadric. This of course is a manifestation of the Klein correspondence. This can be interpreted in terms of the isomorphism between the diagrams A_3 and D_3.

The case $n = 4$ is even more interesting. Here, each half-spinor space has dimension 8, the same as that of the original space V; and the pure spinors of each type are the points of a quadric in this space, isomorphic to the quadric in V with which we began. Moreover, lines lying in the quadric of pure spinors in either half-spinor space correspond to lines on the original quadric. Thus, we have a *triality*, an incidence-preserving map of the D_4 geometry which carries the point set to one family of t.s. solids, this family to the other, and the second family back to points, and maps lines to lines. (This, too, can be seen in terms of the additional symmetry of the diagram D_4.)

Returning to the general set-up for a moment, let W be a subspace of V of codimension 1, on which the restriction q' of q is nonsingular (the subspace orthogonal to a nonsingular vector of V.) Then a maximal t.s. subspace for q' has dimension $n-1$, and lies in a unique maximal t.s. subspace of each family for q, and conversely any t.s. n-space for q intersects W in a t.s. $(n-1)$-space for q'. Thus we have a bijection between the maximal t.s. subspaces for q' and the projective pure spinors of each type. In this way, the dual of the polar space for q' is embedded in a half-spinor space; its lines (corresponding to the t.s. $(n-2)$-spaces for q') are some of the lines of the half-spinor geometry.

For $n = 3$, a nonsingular vector for the Klein quadric corresponds to a nonsingular skew-symmetric 4×4 matrix, defining a nondegenerate alternating bilinear form on the 4-dimensional space. In other words, we impose a symplectic structure on the half-spinor space. The points of half-spinor space now correspond to the t.s. lines for the form q', and the t.i. lines for the symplectic form to the points. In other words, we have established that the generalized quadrangles of symplectic type (in $P(3, K)$) and orthogonal type (in $P(4, K)$) are dual to each other (see Chapter 9).

Now consider $n = 4$. We have seen that the dual of the orthogonal polar space in $P(6, K)$ is embeddable in $P(7, K)$, using all the points and some of the t.s. lines. If we now intersect this configuration with a nonsingular hyperplane, we obtain a geometry consisting of all the points and some of the lines of a quadric in $P(6, K)$. This geometry turns out to be a *generalized hexagon* of type $\mathsf{G}_2(K)$.

A further modification is obtained by 'twisting' the constructions by means of a field automorphism. Without giving details, we summarize the results.

The twisted version of the Klein correspondence shows that the dual of the unitary GQ in $P(3, K)$ is an orthogonal GQ in $P(5, K)$ whose maximal t.s. subspaces are lines. (This is closely related to the 'twistor' construction of mathematical physics.) In a similar way, twisting the construction of the $\mathsf{G}_2(K)$ generalized hexagon gives further generalized hexagons (of type ${}^3\mathsf{D}_4(K)$).

Another phenomenon occurs over a perfect field K of characteristic 2. Let V be a vector space of odd dimension, and q a nonsingular quadratic form. Then q polarizes to an alternating bilinear form f which must have nonzero radical R, necessarily 1-dimensional (such that q is nonvanishing on R); f induces a nondegenerate alternating form $\bar{f}$ on V/R. The map

$$U \mapsto (U \oplus R)/R$$

defines a bijection between the t.s. subspaces of V (with respect to q) and the t.i. subspaces of V/R (with respect to $\bar{f}$). Thus the corresponding polar spaces (orthogonal and symplectic) are isomorphic.

Now we have established, by quite different but geometric means, both an isomorphism and a duality between the symplectic GQ in $P(3, K)$ and the orthogonal GQ in $P(4, K)$ (when K is perfect of characteristic 2). Hence the symplectic quadrangle is self-dual in this case.

It always has a polarity whose absolute points form an elliptic quadric. However, if the field K has an automorphism θ satisfying

$$x^{\theta^2} = x^2$$

for all $x \in K$, then there is a 'twisted' version of the polarity, whose absolute points form the *Suzuki–Tits ovoid*, on which the *Suzuki group* acts.

8. Finite geometries

Obviously a projective or affine geometry is finite if and only if the division ring is finite. The classification of finite fields is given by the following results:

THEOREM 8.1 (Wedderburn). *A finite division ring is commutative.*

THEOREM 8.2 (Galois). *The order of any finite field is a prime power; and, for any prime power q, there is a unique field with q elements.*

In view of this, finite fields are called *Galois fields*, and the unique field of order q is denoted by $\mathrm{GF}(q)$. Moreover, we abbreviate $P(n, \mathrm{GF}(q))$ and $A(n, \mathrm{GF}(q))$ to $P(n, q)$ and $A(n, q)$, respectively, with similar conventions for the collineation group and its relatives.

PROPOSITION 8.3.

(i) *The multiplicative group of* $\mathrm{GF}(q)$ *is cyclic of order* $q - 1$.

(ii) *If* $q = p^e$ *with* p *prime, the automorphism group of* $\mathrm{GF}(q)$ *is cyclic of order* e, *generated by the map* $x \mapsto x^p$.

The number of k-dimensional subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$ is called the *Gaussian* or *q-binomial coefficient*, denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

PROPOSITION 8.4.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{(q^n - q^{i-1})}{(q^k - q^{i-1})}.$$

To see this, note that the number of linearly independent k-tuples is

$$\prod_{i=1}^{k} (q^n - q^{i-1}),$$

since the i-th can be any of the q^n vectors of the n-dimensional vector space V except for the q^{i-1} lying in the subspace spanned by the first $(i - 1)$ vectors. Any such k-tuple spans a k-dimensional subspace. But each k-dimensional subspace has

$$\prod_{i=1}^{k} (q^k - q^{i-1})$$

spanning k-tuples, by the same argument.

PROPOSITION 8.5.

(i) *The number of k-flats of $P(n, q)$ is* $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.

(ii) *The number of k-flats of $A(n, q)$ is* $q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q$.

It is possible to do more recondite counting in finite projective and affine spaces. We content ourselves by stating one result for projective spaces. Moreover, we state it for vector space rather than geometric dimension.

PROPOSITION 8.6. *Let $U \subseteq W \subseteq V$ be vector spaces over* GF(q), *with dimensions l, k, n respectively. Then the number of subspaces X of V which have dimension m and satisfy $X \cap W = U$ is given by*

$$\prod_{i=1}^{m-l} \frac{q^n - q^{k+i-1}}{q^m - q^{l+i-1}}.$$

We now turn to design properties of projective and affine spaces. A *t-(v, k, λ) design* is a set of v points, with a collection of k-element subsets called blocks, such that any t points are contained in λ blocks. If $\lambda = 1$, the design is called a t-uple *Steiner system.* Steiner systems were investigated by Kirkman in the mid-19th century; he was aware that projective planes and affine spaces over fields of prime order give examples. More generally, we have:

PROPOSITION 8.7.

(i) *The points and k-flats in $P(n, q)$ form a* 2-$\left(\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q, \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q, \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\right)$ *design.*

(ii) *The points and k-flats in $A(n, q)$ form a* 2-$\left(q^n, q^k, \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\right)$ *design. If $q = 2$ and $k > 1$, it is a* 3-$\left(2^n, 2^k, \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_2\right)$ *design.*

The proof for projective spaces is a simple application of Proposition 8.6. For affine spaces, just use the fact that the subspaces of $A(n, q)$ containing a fixed d-space ($d \geqslant 0$) form a $P(n - d - 1, q)$. If $q = 2$, affine lines have just two points, and so any three points lie in an affine plane.

The points and hyperplanes of $P(n, q)$ form a *symmetric 2-design*: this means that there are equally many points and blocks, with the consequence that any two blocks intersect in λ points. This particular design is the subject of a characterization by Dembowski and Wagner [1960]. In any design, the *line* containing two points is the intersection of all blocks containing those points. Then any two points lie on a unique line, though in general the number of points on a line need not be constant. If $\lambda = 1$, the lines are just the blocks. In the case of the designs of Proposition 8.7, the lines are precisely the projective or affine lines of the geometry.

PROPOSITION 8.8. *For the symmetric 2-design $\mathcal{D}$ with $\lambda > 1$, the following conditions are equivalent:*

(i) *$\mathcal{D}$ is the point-hyperplane design of a projective space $P(n, q)$, $n \geqslant 3$;*

(ii) *every line meets every block;*

(iii) *the number of blocks containing three noncollinear points is constant;*

(iv) *the automorphism group is transitive on noncollinear triples of points.*

Clearly (i) implies (iv) and (iv) implies (iii). A counting argument shows the equivalence of (ii) and (iii). Finally, if (ii) and (iii) hold, then planes can be defined as intersections of the blocks containing noncollinear triples, and shown to be projective planes. By Section 2.3, the points and lines form a projective space. The blocks are subspaces and (by (ii)) hyperplanes, and counting shows that all hyperplanes must occur.

In fact, the group-theoretic assumption in (iv) can be considerably weakened. The classification of finite simple groups leads to a determination of all 2-transitive groups, and hence all symmetric 2-designs admitting 2-transitive groups.

Dembowski [1964] found a similar characterization of the point-hyperplane designs in affine spaces, in terms of Bose's concept of *affine* (or *affine resolvable*) 2-designs. However, this theorem admits exceptions in the case where lines contain just three points, the so-called *Hadamard 3-designs*, which are conjectured to exist for all orders divisible by 4.

Many of our themes in earlier chapters can be pushed further in the finite case. For example,

PROPOSITION 8.9.

$$|\,\mathrm{PSL}(n+1,q)| = \frac{1}{d}\prod_{i=0}^{n-1}\left(q^{n+1-i}-1\right), \quad \textit{where } d=(n+1,q-1).$$

The order of $\mathrm{GL}(n+1,q)$ is the number of bases of the vector space, viz.

$$\prod_{i=0}^{n}\left(q^{n+1-i}-1\right),$$

since any basis is carried to any other by a unique invertible linear map. As we noted in Section 3, we have to divide this by $q-1$, the order of the group of nonzero scalars, to get $|\mathrm{PGL}(n+1,q)|$, and then by $\mathrm{GF}(q)^{\times}/\,\mathrm{GF}(q)^{\times(n+1)} = (q-1,n+1)$, using Proposition 3.3.

We saw that $\mathrm{PSL}(n+1,q)$ is simple for $n \geqslant 1$ except for $n=1$, $q=2$ or 3; and $\mathrm{PSL}(2,2) \cong S_3$, $\mathrm{PSL}(2,3) \cong A_4$. Other identifications of small projective groups are

$$\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong A_5,$$

$$\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2),$$

$$\mathrm{PSL}(4,2) \cong A_8.$$

It is also possible to give a complete classification of the polarities of finite projective spaces, and detailed numerical information about the polar spaces of totally isotropic subspaces which arise: see Dieudonné [1955].

Another major line of development is *Galois geometry*, the study of configurations in finite projective and affine spaces in terms of their intersections with flats of various types. The classic example of such a theorem is *Segre's theorem* [1954]: if q is odd, then any *oval* in $P(2,q)$ (a set of $q+1$ points, no three collinear) must be a *conic*, the set of zeros of a nonsingular quadratic form. See Hirschfeld [1979] for more on this topic.

Yet another important topic is the *Hasse–Weil theorem*, which asserts that the number N of points on a nonsingular algebraic curve of genus g in a projective space over GF(q) satisfies

$$|N - (q+1)| \leqslant 2g\sqrt{q}.$$

This has been generalized by Deligne to algebraic varieties of higher dimension; see Katz [1976].

A further topic is the connection between configurations in projective spaces and error-correcting codes, which has been fruitful for both disciplines. See Cameron and Van Lint [1992].

We turn now to polar spaces.

It follows from Tits' Theorem 4.1 and the classification of polarities that there are six types of polar spaces of rank at least 3 over a finite field. In this list, q is the field order, r the polar space rank (the dimension of a maximal t.i. or t.s. subspace), and n the dimension of the vector space. The significance of ε will appear shortly.

Type	q	n	ε
Symplectic		$2r$	0
Orthogonal		$2r$	-1
Orthogonal		$2r+1$	0
Orthogonal		$2r+2$	$+1$
Unitary	square	$2r$	$-1/2$
Unitary	square	$2r+1$	$1/2$

Here, the number of maximal subspaces containing a subspace of dimension $r-1$ is equal to $q^{1+\varepsilon}+1$.

An outline of the classification in the orthogonal case follows. It is easy to see that, if there is a nonzero vector v which is singular, then v lies in a 2-dimensional subspace W (a *hyperbolic line*) spanned by vectors v_1, v_2 with $q(v_1) = q(v_2) = 0$, $f(v_1, v_2) = 1$; and V is the orthogonal direct sum of W and its orthogonal complement. By induction, V is the direct sum of r hyperbolic lines and a subspace X on which q does not vanish except at 0. So the classification is reduced to finding all possible such spaces X. By a theorem of Chevalley, a quadratic form in three or more variables over a finite field has a zero; so $\dim(X) \leqslant 2$; there is a unique possibility for each possible dimension (up to scalar multiples). Similar arguments apply in the other cases. Moreover, the values of ε are verified by counting the isotropic (or singular) points in the orthogonal direct sum of a hyperbolic line and a subspace of each possible type containing no zeros of the form. (For example, the quadratic forms x_1x_2, $x_1x_2 + x_3^2$, and $x_1x_2 + h(x_3, x_4)$, where h is an irreducible quadratic form over GF(q), have 2, $q+1$ and q^2+1 zeros, respectively, in the appropriate projective space; in the second and third cases, these zeros form a conic or an elliptic quadric, respectively.)

In particular, if $r = 2$ (so that the polar space is a GQ), each line has $q+1$ points, and each point lies on $q^{1+\varepsilon}+1$ lines. In terms of the more usual parameters s and t for GQs (see Chapter 9), $s = q$ and $t = q^{1+\varepsilon}$.

Many numerical properties of polar spaces can be expressed in terms of these parameters. For example, the number of points is

$$\frac{(q^r-1)(q^{r+\varepsilon}+1)}{(q-1)}$$

and the number of maximal subspaces is

$$\prod_{i=1}^{r}\left(q^{i+\varepsilon}+1\right).$$

(Use the fact that, for any point p, the subspaces containing p form a polar space with the same type and the same value of ε, but with rank $r-1$.)

We saw that the symplectic and orthogonal polar spaces with $\varepsilon = 0$ are isomorphic if q is even; and also that the symplectic and orthogonal GQs with $r = 2$, $\varepsilon = 0$ are dual for all q, as are the unitary GQ with $\varepsilon = -1/2$ and the orthogonal GQ with $\varepsilon = 1$ (both with $r = 2$, of course).

It is possible to write down the order and the normal subgroup structure of the automorphism group of each of these polar spaces. We do not give details here; they may be found in Dieudonné [1955].

9. The projective line

The structure of the projective line, as incidence geometry, is trivial: it has only points. In effect, its structure is given by the action of PGL(2, K) on it.

We can identify $P(1,K)$ with $K \cup \{\infty\}$: the point $\langle(1,x)\rangle$ is identified with x, and the point $\langle(0,1)\rangle$ with ∞. (These are just the slopes of the 1-dimensional subspaces of V.)

As we observed in Section 3, PGL(2, K) acts 3-transitively on $P(1,K)$, and the stabilizer of the three points $\infty, 0, 1$ is the group of inner automorphisms of the multiplicative group of K. In particular, the group is sharply 3-transitive if and only if K is commutative. For the rest of this section, we assume that K is commutative. We can regard PGL(2, K) as the group of *linear fractional transformations*

$$z \mapsto \frac{az+b}{cz+d}$$

of $K \cup \{\infty\}$. (We use the usual conventions about infinity: $\infty - x = \infty$, $\infty/\infty = 1$, etc.)

The *cross-ratio* is a function from ordered quadruples of distinct points of $P(1,K)$ to K, given by

$$\mathrm{cr}(x_1,x_2,x_3,x_4) = \frac{(x_1-x_2)(x_3-x_4)}{(x_2-x_3)(x_4-x_1)}.$$

THEOREM 9.1. *The group of permutations of projective space $P(1,K)$ preserving the cross-ratio is* PGL(2, K).

A calculation shows that linear fractional transformations do preserve cross-ratio; clearly any permutation preserving the cross-ratio is determined by its values on three points (since $\mathrm{cr}(x_1, x_2, x_3, x_4)$ is a one-to-one function of x_4).

The setwise stabilizer of any four points contains the Klein group of order 4; so the 24 values obtained by permuting the arguments include at most six distinct values, which can be expressed in terms of one of them as follows:

$$\{x, 1-x, 1/x, 1/(1-x), (x-1)/x, x/(x-1)\}.$$

In general these six values are all distinct. There are two possibilities for further coincidences:

(a) $\{-1, 1/2, 2\}$ (quadruples realizing these values are called *harmonic*, and are stabilized by the dihedral group of order 8);

(b) $\{-\omega, -\omega^2\}$, where ω is a primitive cube root of unity (quadruples satisfying this are called *equi-anharmonic*, and are stabilized by the alternating group of degree 4).

In characteristic 3, the two exceptional types happen to coincide; the corresponding quadruples admit the symmetric group of degree 4, and form sub-projective lines over the field of 3 elements, the cross-ratio taking the single value -1. In characteristic 2, harmonic quadruples do not occur.

A related topic concerns projectivities, by means of which structure is given to a line in a projective plane. Let L and M be two lines in a projective plane, p a point lying on neither. The map

$$x \mapsto \langle p, x \rangle \cap M$$

from L to M is called a *perspectivity*. Any composite of perspectivities is called a *projectivity*. The set of all projectivities from a line L to itself is a group of permutations of L. This *group of projectivities* is an invariant of the plane: distinct lines carry isomorphic permutation groups of projectivities.

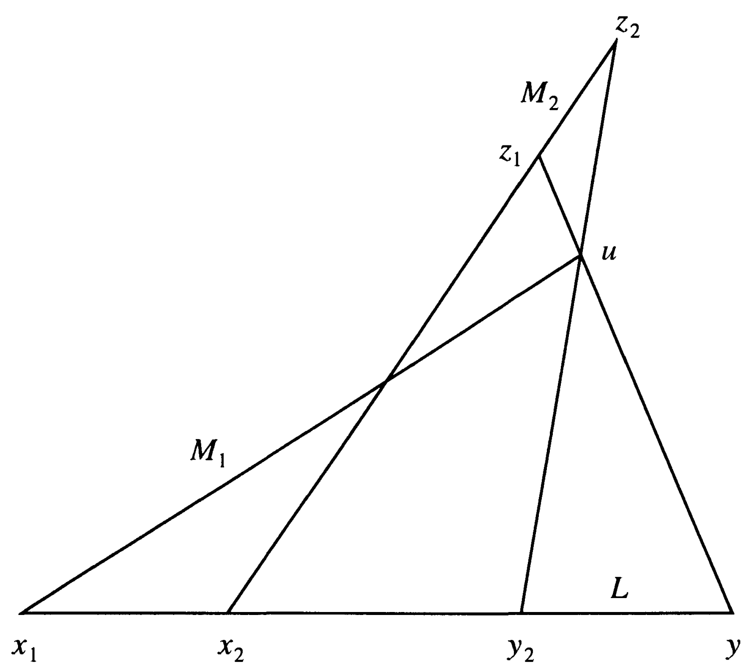


Figure 8. 3-transitivity.

THEOREM 9.2. *The group of projectivities in a projective plane is 3-transitive. It is sharply 3-transitive if and only if the plane satisfies Pappus' Theorem, in which case the group of projectivities is* PGL$(2, K)$, *where K is the coordinatizing field.*

The 3-transitivity follows from the fact that the stabilizer of any to points acts transitively on the remainder, which is shown in Figure 8, which demonstrates that there is a projectivity fixing any two points x_1, x_2 and having any prescribed effect on a third point (taking y_1 to y_2). It is clear that sharp 3-transitivity gives rise to a huge number of configuration theorems in the plane; Pappus' Theorem turns out to be one of these and, remarkably, to imply all of them.

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CHAPTER 3

Foundations of Incidence Geometry

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Contents

1. Different viewpoints 65
2. The block space approach 66
   2.1. Block spaces in general 66
   2.2. Line spaces 68
   2.3. Hyperplane spaces 71
   2.4. Closure spaces 73
   2.5. Other inspiration 74
3. The graph theoretic approach 75
   3.1. Geometries 75
   3.2. Special morphisms 78
   3.3. Diagrams 81
   3.4. Linear diagrams 85
   3.5. Apartments 88
   3.6. Shadows 93
   3.7. The intersection property 94
   3.8. Complexes 96
   3.9. Chamber systems 96
4. The functional approach 97
   4.1. Functional spaces 97
   4.2. Join geometries and convexity spaces 97
   4.3. André geometry 97
5. Incidence with other structure 98
   5.1. Parallelism on block spaces 98
   5.2. Hjelmslev–Klingenberg geometry 99
   5.3. Incidence groups 100
   5.4. Half ordered planes 100
References 101

HANDBOOK OF INCIDENCE GEOMETRY
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1. Different viewpoints

As mentioned in Section 2 of Chapter 1, our main source for incidence geometry is linear algebra over a division ring, namely the affine spaces, projective spaces and polar spaces as embedded in projective spaces. Various generalizations of division rings, like rings and near-fields, extend this algebraic source (see Chapter 19). As a matter of fact there are other sources provided by combinatorics, polytopes, etc. When it comes to generalize, axiomatize, characterize the classical geometries, several quite different approaches can be used. We survey the leading trends.

(a) In an *atomic approach* there is a distinguished set S of elements called *points* endowed with further structure.

The structure may be of a *block space* type that is, it consists of a distinguished family of subsets called blocks whose role is to generalize lines or hyperplanes or subspaces, etc. This structure consists of an element in $\boldsymbol{P}(\boldsymbol{P}(S))$ (see Chapter 1, Section 1.2).

The structure may also be of a *functional type* and belong to universal algebra. Let us give a typical example. In order to imitate the lines or closed segments or closed half-lines of the Euclidean plane we may require that for each ordered pair of distinct points a, b in S we are given a subset ab of S. This structure consists of an element in $\boldsymbol{P}(S^2 \times \boldsymbol{P}(S))$.

This list of approaches is not exhaustive. Another rather interesting one is based on posets (see Section 2.5.5).

(b) In a *graph-theoretic approach* we basically consider a *geometry* (see Section 3.1) or a variation of this. It appears as a graph whose vertices are supposed to generalize the (totally isotropic) subspaces of the classical spaces and whose edges generalize pairs of incident subspaces. Moreover we keep a record of the type of those subspaces but, since inclusion is replaced by an incidence relation, we do not need any more to distinguish points among the other elements. This point-free view has advantages explained in Chapter 1, Section 4.6.

(c) A geometry Γ leads to a generalization in terms of block spaces that we can see as a second degree generalization of the classical geometries. A *flag* of Γ is any set of pairwise incident elements. Any subset of a flag is itself a flag. Here is the generalization. A *complex* is a block space $(S, \boldsymbol{B})$ with the property that for any block B (flag) in $\boldsymbol{B}$, every subset of B is also a block. This structure is close to algebraic topology and so incidence geometry takes benefit of many ideas arising from that field (see Section 3.8).

(d) A geometry Γ also leads to a graphic generalization that we can see as another second degree generalization of the classical geometries. A *chamber* of Γ is any maximal set of pairwise incident elements, and two chambers are *adjacent* if they share all but one of their elements. In the classical cases, the graph whose vertices are the chambers and whose edges are the pairs of adjacent chambers, allows us to reconstruct Γ. This leads to a generalization called *chamber systems*, essentially based on a graph whose vertices are called chambers (see Section 3.1) and whose edges have types (as the elements of a geometry). To illustrate the respective powers of these various concepts we mention that there are different but equivalent definitions of buildings starting either with a complex, with a geometry or with a chamber system and all of these are useful (Chapter 11).

There are also useful approaches to some classes of buildings in terms of block spaces (Chapter 12).

(e) One of the preceding types of structure may go along with some additional compatible structure. We mention some typical examples.

Parallelism is best suited to block spaces. It arises from affine spaces where it can be entirely reduced to the block space structure (of all affine subspaces).

The tradition of *Hjelmslev–Klingenberg* structures based on neighbour relations and epimorphisms, whose algebraic origin lies in the classical geometries over rings, can be extended into the context of geometries. Finally, the tradition of incidence groups can be described in the setting of a group equipped with an extra block space structure.

2. The block space approach

2.1. *Block spaces in general*

2.1.1. Let $(\boldsymbol{P}, \boldsymbol{B})$ be a *block space*[1] (or *hypergraph*), i.e. a set of *points* $\boldsymbol{P}$ together with a family $\boldsymbol{B}$ of subsets of $\boldsymbol{P}$ called *blocks*. There are concepts about $(\boldsymbol{P}, \boldsymbol{B})$ which are useful whatever the inspiration source or models behind the blocks.

We constantly assume that distinct members of $\boldsymbol{B}$ are distinct subsets of $\boldsymbol{P}$, i.e. there are *no repeated blocks* although the more general case, in which repeated blocks are allowed, is considered in the literature, especially with block designs (Chapter 8) and geometries (Section 3.1).

EXAMPLE. Let $\boldsymbol{P}$ be a set of points, G a permutation group on $\boldsymbol{P}$, H a subgroup of G, B an orbit of H on $\boldsymbol{P}$. An interesting block space admits all gB, $g \in G$, as blocks. G acts *flag-transitively* as a group of automorphisms on this block space, that is G has a unique orbit on the pairs (p, C) where C is a block and p a point in C.

We do not consider separately the more general viewpoint in which several families of subsets are given. However, incidence geometry has a clear tendency to build several such families from a given one as we shall often see.

2.1.2. Given a block space $(\boldsymbol{P}, \boldsymbol{B})$ and a subset $\boldsymbol{M}$ of $\boldsymbol{P}$, there is a block space obtained by *restriction of* $(\boldsymbol{P}, \boldsymbol{B})$ *to* $\boldsymbol{M}$ whose points are those of $\boldsymbol{M}$ and whose blocks are the sets $\boldsymbol{M} \cap B$ with $B \in \boldsymbol{B}$. This procedure providing new block spaces from known ones is a rich source of examples. It is also a unifying tool among classical and other geometries. For instance, each affine space is a restriction of a projective space, the blocks of the latter being the subspaces. The hyperbolic spaces of non-Euclidean geometry (Chapters 15, 17) are likewise restrictions of projective spaces.

In the opposite direction, it is not true that the classical polar spaces, though embedded in projective spaces, are restrictions of the latter.

[1] In Chapter 8, this is called a *design*.

Let us observe that the definition of a restriction may slightly vary according to the inspiration behind the choice of blocks. For instance, if $\boldsymbol{B}$ is seen as a set of lines it may be required that the blocks of $\boldsymbol{M}$ be the sets $\boldsymbol{M} \cap B$, $B \in \boldsymbol{B}$, having at least two points. Another interpretation is to define the blocks of $\boldsymbol{M}$ as all $B \in \boldsymbol{B}$ such that $B \subseteq \boldsymbol{M}$. Then all polar spaces in projective spaces (except the symplectic polar spaces) become restrictions of the latter.

2.1.3. We come to quotients of a block space $(\boldsymbol{P}, \boldsymbol{B})$. Let R be an equivalence relation defined on a subset $\boldsymbol{M}$ of $\boldsymbol{P}$ and let i be the canonical mapping of $\boldsymbol{M}$ onto the set $\boldsymbol{M}/R$ of equivalence classes of R. The *quotient* of $(\boldsymbol{P}, \boldsymbol{B})$ by R is the block space whose set of points is $\boldsymbol{M}/R$ and whose blocks are all sets $i(B)$ with $B \in \boldsymbol{B}$ and $\boldsymbol{P} - \boldsymbol{M} \subseteq B$.

EXAMPLE 1. Let G be a group of automorphisms acting on $(\boldsymbol{P}, \boldsymbol{B})$ and let R be defined on $\boldsymbol{M} = \boldsymbol{P}$ by aRb if and only if a and b are in the same G-orbit. Then G acts as the identity on the quotient $(\boldsymbol{P}, \boldsymbol{B})/R$, i.e. two blocks B, B' in $\boldsymbol{B}$ which are in the same G-orbit have the same image in the quotient.

A typical illustration of this situation is provided by some classical polyhedra and polytopes like the cube, dodecahedron, hypercube, etc., and a group G of order 2 including the mapping which sends every vertex to the 'antipodal' vertex.

EXAMPLE 2. Let $(\boldsymbol{P}, \boldsymbol{B})$ be a block space, p a point in it. We are interested in a quotient by p. Therefore we define an equivalence relation R on $\boldsymbol{P} - \{p\}$ by putting aRb if and only if every block on p and a contains b and every block on p and b contains a. Here the blocks of the quotient are the sets $i(B - p)$ where B is any block on p.

As an illustration, in a projective or affine space whose blocks are either the lines or the hyperplanes or the subspaces, the quotient by a point always is a projective space. Another interesting illustration is provided by a t-design $(\boldsymbol{P}, \boldsymbol{B})$ with $t \geqslant 3$ (Chapter 8). Here the quotient by a point p is the usual 'residual' $(t-1)$-design.

EXAMPLE 3. Let A be an affine space considered as a block space with its points and lines. Let d be a direction of lines. Let R be defined on the set of points by aRb if and only if $a = b$, or $a \neq b$ and the line ab belongs to d. Then A/R is an affine space whose blocks are its lines and points. This suggests that slight variations on the definition of the blocks in a quotient might be suitable.

2.1.4. Whatever the inspiration behind the choice of blocks of a block space $(\boldsymbol{P}, \boldsymbol{B})$, popular axioms are:

(B1) all blocks have the same cardinality;
(B2) $\boldsymbol{P}$ is finite;
(B3) *flag-transitivity* of the group $A = \mathrm{Aut}(\boldsymbol{P}, \boldsymbol{B})$, i.e. A has a single orbit on the pairs (p, B) with $p \in B$ and $B \in \boldsymbol{B}$ (see the example in Section 2.1.1).
(B4) *antiflag-transitivity* of the group $A = \mathrm{Aut}(\boldsymbol{P}, \boldsymbol{B})$, i.e. A has a single orbit on the pairs (p, B) with $p \in \boldsymbol{P}$, $B \in \boldsymbol{B}$ and $p \notin B$.

If (B1) and (B2) hold then $\boldsymbol{B}$ is finite and every point is on a constant number of flags. If (B4) holds then (B3) holds on $(\boldsymbol{P}, \boldsymbol{B}^*)$ where $\boldsymbol{B}^*$ is the set of all $\boldsymbol{P} - B$, $B \in \boldsymbol{B}$ and conversely (B3) implies (B4) on $(\boldsymbol{P}, \boldsymbol{B}^*)$. Also, (B3) (resp., (B4)) implies (B1) unless there is a simultaneous presence of an empty block and a nonempty block (resp., a block equal to $\boldsymbol{P}$ and a block distinct from $\boldsymbol{P}$).

The classical affine, projective and polar spaces satisfy (B3) if the blocks are the singular subspaces of any given dimension.

The classical affine and projective spaces satisfy (B4) with the same blocks as above and this also holds for the polar spaces with the maximal singular subspaces as blocks.

2.1.5. Any block space $(\boldsymbol{P}, \boldsymbol{B})$ is giving rise to an *incidence graph* $\Gamma(\boldsymbol{P}, \boldsymbol{B})$ whose set of vertices is $\boldsymbol{P} \cup \boldsymbol{B}$ while its edges are the pairs $\{x, y\}$ where $x \in \boldsymbol{P}$, $y \in \boldsymbol{B}$ and $x \in y$, i.e. the *flags* of $(\boldsymbol{P}, \boldsymbol{B})$. There is an immediate transfer of standard notions about graphs to $(\boldsymbol{P}, \boldsymbol{B})$ via $\Gamma(\boldsymbol{P}, \boldsymbol{B})$. Examples of these are concepts such as connectedness, distance, valency, circuits. Another graph is used in the literature. Call it the *collinearity graph* $\overline{\Gamma}(\boldsymbol{P}, \boldsymbol{B})$. Its vertices are the elements of $\boldsymbol{P}$ and a pair of vertices $\{x, y\}$ is an edge if and only if there is some block containing x and y.

For still another variation of the idea of graph and connectedness, see Section 3.1.2.

2.1.6. Given a block space $(\boldsymbol{P}, \boldsymbol{B})$ a *blocking set* is a set of points S such that every block has at least one point on S. This concept is surveyed in Chapter 4. For a fairly general approach see Drake [1986].

2.2. *Line spaces*

2.2.1. A *line space*[2] is a block space $(\boldsymbol{P}, \boldsymbol{B})$ whose blocks, called *lines*, have at least two points. A set of points contained in some line is called *collinear*. In particular a point a is *collinear* with a point b if there is a line containing a and b.

The classical models lead to tempting definitions and axioms. Here are some possible axioms.

(L1) Any pair of points is contained in at least one line.

(L2) Any pair of points is contained in at most one line.

(L3) Any two distinct lines have one and only one common point.

(L4) Given a line L and a point p not on L, there is a unique line on p disjoint from L.

(L5) (Pasch–Veblen) If a, b, c, d, e are distinct points such that the sets $\{a, b, c\}$, $\{a, d, e\}$, $\{b, d\}$, $\{c, e\}$ are collinear on distinct lines, then there is a point f such that $\{b, d, f\}$, $\{c, e, f\}$ are collinear (see Figure 1).

[2] In Chapter 12, this is called a *space*.

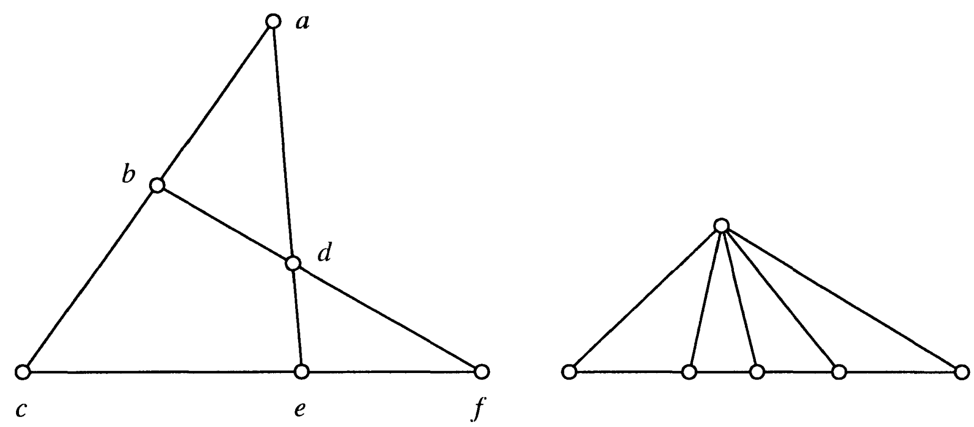


Figure 1.

(L6) (Desargues) If 12, 13, ..., 45 are ten distinct points organized in nine sets of three collinear points as in Figure 2, then $\{34, 35, 45\}$ are collinear.

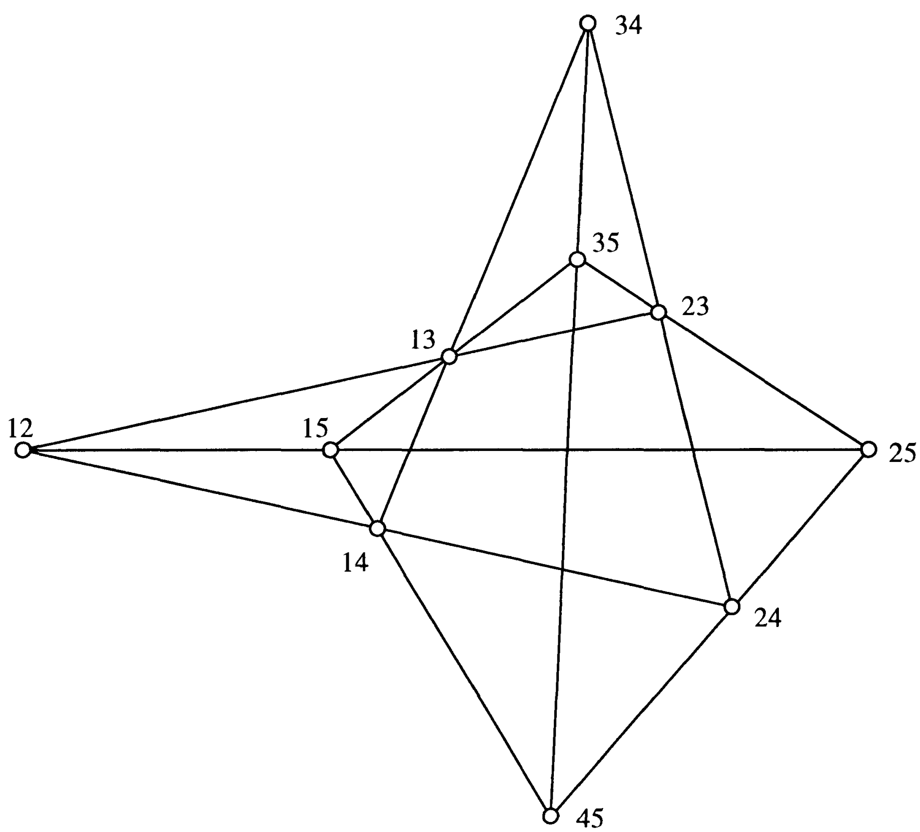


Figure 2.

(L7) ((0,1,all)-property) If L is a line and p is a point not on L then either no, one or all points of L are collinear to p.

Axiom (L1) is one of the main axioms in the definition of a building as given by Tits [1974]. In this context $\boldsymbol{P}$ is the set of chambers and $\boldsymbol{B}$ is the set of apartments of the building. In Teirlinck [1980] a line space verifying (L1) is called a 2-covering but we shall avoid this terminology since it has been used also in the important 'local approach' to buildings (Tits [1982]) with a completely different meaning. See Chapter 11. Axiom (L2) is included in lots of interesting geometric theories. The line spaces characterized by (L2) bear various names in the literature, for example, semilinear space, partial plane. We recommend '*partial linear space*'.

Axioms (L1) and (L2) characterize the *linear spaces* studied in Chapter 6.

Axiom (L3) is a dual for the union of (L1) and (L2). Together with (L1) and (L2) it characterizes the projective planes (see Chapters 4 and 5) up to some additional trivial cases in which there are lines of two points (see Figure 1).

Axioms (L1), (L2) and (L4) characterize the affine planes (see Chapter 4).

Axioms (L1), (L2) and (L5) characterize the projective spaces.

Axiom (L6) characterizes those projective planes P for which there is a vector space V with $P = P(V)$, the space of all subspaces of V.

Axiom (L7) together with (L2), (L5), (L6) holds in many line spaces derived from building geometries (Chapter 12).

2.2.2. Here are some tempting definitions.

(1) A *subspace* is a set of points S such that for all ab in S each line on $\{a, b\}$ is contained in S.

Observe that a line is not necessarily a subspace. Any intersection of subspaces is a subspace and so the developments of Section 2.5 apply to them.

The next concepts are particular cases of subspaces.

(2) A *linear subspace* (or *singular subspace* or *totally isotropic subspace*) is a set of points S such that for all $a \neq b$ in S there is one and only one line containing a and b and this line is contained in S.

(3) A *projective hyperplane* (or *geometric hyperplane*) (see Section 2.3) is a set of points $H \subsetneq \boldsymbol{P}$ such that for each line ℓ, $\ell \cap H$ is either a point or ℓ.

This concept appears in many contexts (e.g., Veldkamp [1959], Buekenhout and Hubaut [1977], Teirlinck [1980], Ronan [1987]). A rather general theory is developed in Teirlinck [1980] who deals, in an ingenious way, with *affine hyperplanes* when (L1) holds. His theory of projective hyperplanes is applied in Buekenhout [1990] and Percsy [1989] to improve on the Buekenhout–Shult theorems for polar spaces. There are further and recent spectacular developments on this theme (see Chapter 12).

2.2.3. The use of subspaces used to be restricted to the smooth case of linear spaces with dimension (matroids, geometric lattices; (see Chapter 6)). It is now growing in a broader context under the influence of geometries arising from groups.

The use of projective hyperplanes is illustrated in the next section. As to linear subspaces they bring us back to buildings.

The point-line geometries of buildings are line spaces in which (L2) and (L7) hold and in which the linear subspaces satisfy (L5), i.e. they are projective spaces. Such a line space has an important property: if p is a point then the quotient of $(\boldsymbol{P}, \boldsymbol{B})$ by p (see Example 2 in Section 2.1.3) satisfies the same properties, i.e. (L2), (L7) hold and (L5) holds in all linear subspaces. A general investigation of the line spaces satisfying these conditions and some nondegeneracy requirements (connectedness for instance) remains suitable (see Chapter 12).

For a detailed survey of classical linear spaces, refer to Karzel and Kroll [1988]. Kinematic geometry (see Karzel and Kist [1985]) leads to an interesting concept of weak projective space, characterized in Karzel and Marchi [1988].

2.3. *Hyperplane spaces*

2.3.1. A *hyperplane space* is a block space $(\boldsymbol{P}, \boldsymbol{B})$ whose blocks are called *hyperplanes* with the restriction that no hyperplane contains another one.

EXAMPLE 1. If $(\boldsymbol{P}, \boldsymbol{B})$ is any block space let $\boldsymbol{B}_H$ be the set of blocks $B \neq \boldsymbol{P}$ which are maximal for that property. Then $(\boldsymbol{P}, \boldsymbol{B}_H)$ is a hyperplane space.

EXAMPLE 2. If $(\boldsymbol{P}, \boldsymbol{B})$ is any line space let $\boldsymbol{B}_p$ be the set of projective hyperplanes of $(\boldsymbol{P}, \boldsymbol{B})$. Then it is not necessarily true that $(\boldsymbol{P}, \boldsymbol{B}_p)$ is a hyperplane space. To obtain a counter-example, consider a graph Γ and the complementary graph $\overline{\Gamma}$, whose vertices are those of Γ and whose edges are the pairs of vertices which are not on an edge in Γ. Now $\boldsymbol{P}, \boldsymbol{B}$ are the set of vertices and edges of Γ, respectively. A set of vertices H is a projective hyperplane of $(\boldsymbol{P}, \boldsymbol{B})$ if and only if $\boldsymbol{P} - H$ is a clique (complete subgraph) in $\overline{\Gamma}$ and so we obtain plenty of counter-examples. This unfortunate situation is avoided under the assumption of ample connectedness that we now define.

Call the line space $(\boldsymbol{P}, \boldsymbol{B})$ *amply connected* if for every subspace $U \neq \boldsymbol{P}$ and every pair of points p, q in $\boldsymbol{P} - U$ there is a sequence $p = p_1, p_2, \ldots, p_n = q$ of points in $\boldsymbol{P} - U$, such that any two consecutive members are collinear.

If $(\boldsymbol{P}, \boldsymbol{B})$ is amply connected then $(\boldsymbol{P}, \boldsymbol{B}_p)$ is a hyperplane space (see also Chapter 12).

2.3.2. Tempting axioms are:

(H1) there is an integer λ such that any pair of points is contained in exactly λ hyperplanes;
(H2) (resp., (H2)′) for any distinct hyperplanes H_1, H_2 and point p not in $H_1 \cup H_2$, there is at least one hyperplane H containing p such that $H \cap H_1 = H \cap H_2 = H_1 \cap H_2$ (resp., $H \supseteq H_1 \cap H_2$);
(H3) (resp., (H3)′) for any distinct hyperplanes H_1, H_2 and point p not in $H_1 \cup H_2$, there is at most one hyperplane H containing p such that $H \cap H_1 = H \cap H_2 = H_1 \cap H_2$ (resp., $H \supseteq H_1 \cap H_2$);
(H4) (resp., (H4)′) for every pair of points p, q there exists a hyperplane containing p but not q (resp., containing neither p nor q).

If $\boldsymbol{P}$ is finite and if all blocks have the same size, (H1) characterizes the 2-designs (see Chapter 8). The influence of the other axioms is discussed after the definitions given now.

Tempting definitions are:

(1) a *subspace* of $(\boldsymbol{P}, \boldsymbol{B})$ is the intersection of any family of hyperplanes;

(2) a *pencil* is a maximal set of hyperplanes π such that for any three distinct H_1, H_2, H_3 in π, $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3$;
(3) a *flock* (see Chapter 7) is a pencil π such that the union of all members of π is $\boldsymbol{P}$;
(4) the *dual* of $(\boldsymbol{P}, \boldsymbol{B})$ is the line space $(\boldsymbol{P}^*, \boldsymbol{B}^*)$ where $\boldsymbol{P}^* = \boldsymbol{B}$ and $\boldsymbol{B}^*$ is the set of pencils of $(\boldsymbol{P}, \boldsymbol{B})$;
(5) given $x \in \boldsymbol{P}$, x^{**} is the subset of $\boldsymbol{P}^*$ consisting of all hyperplanes $H \in B$ such that $x \in H$.

The dual space $(\boldsymbol{P}^*, \boldsymbol{B}^*)$ does always satisfy (L2) (see Section 2.2.1). (H3) holds in $(\boldsymbol{P}, \boldsymbol{B})$ if and only if $(\boldsymbol{P}^*, \boldsymbol{B}^*)$ is a linear space (i.e. (L1) and (L2) hold in it).

For each $x \in \boldsymbol{P}$, x^{**} is a subspace of the line space $(\boldsymbol{P}^*, \boldsymbol{B}^*)$. If (H4) holds in $(\boldsymbol{P}, \boldsymbol{B})$ and if p, q are distinct points then the subspaces p^{**}, q^{**} of the dual space, are distinct. If (H2) holds in $(\boldsymbol{P}, \boldsymbol{B})$ then, for every $x \in \boldsymbol{P}$, x^{**} is a projective hyperplane of $(\boldsymbol{P}^*, \boldsymbol{B}^*)$. If (H4)$'$ is applied to the projective hyperplanes of a linear space, then the lines of the latter have at least 3 points.

2.3.3. We shall now introduce more axioms and state some results.

(H5) let every hyperplane H equipped with the set $\boldsymbol{B}H$ of all sets $H \cap X$ where $X \in \boldsymbol{B}$, $X \neq H$, be a hyperplane space.

Assume that (H5) holds in $(\boldsymbol{P}, \boldsymbol{B})$, in every $(H, \boldsymbol{B}H)$ with $H \neq \boldsymbol{B}$ and in every hyperplane of the latter. Let (H3) hold in $(\boldsymbol{P}, \boldsymbol{B})$. Then $(\boldsymbol{P}^*, \boldsymbol{B}^*)$ is a projective space ((L1), (L2) and (L5) are satisfied). Let us prove (L5). We have

$$a \cap b \cap d = a \cap c \cap e \subset c \cap e.$$

By (H5) applied to $c \cap e$, $a \cap c \cap e$ is a hyperplane in $c \cap e$ and so every hyperplane $H \neq \boldsymbol{B}$, containing $a \cap c \cap e$ and one more point of $c \cap e$, must contain $c \cap e$. Let $p \in (c \cap e) - (a \cap b \cap d)$. Then

$$p \notin b \cap d \quad \text{since} \quad (b \cap d) \cap (c \cap e) = a \cap b \cap d.$$

By (H3) there is a hyperplane h containing $b \cap d$ and p. Hence h contains $c \cap e$ and (L5) holds. □

Let $(\boldsymbol{P}, \boldsymbol{B})$ be a line space and let $\boldsymbol{B}_H$ be its set of projective hyperplanes. Assume that $(\boldsymbol{P}, \boldsymbol{B})$ is amply connected and that the same property holds in every $(H, \boldsymbol{B}H)$ (see Section 2.3.3) and again in all $(X, \boldsymbol{B}(H, X))$ where $X \in \boldsymbol{B}H$ and $\boldsymbol{B}(H, X)$ is the set of all $H \cap X \cap Y$ where Y is a hyperplane of $(\boldsymbol{P}, \boldsymbol{B})$ not containing $H \cap X$. Let (H3) and (H4) hold in $(\boldsymbol{P}, \boldsymbol{B})$. Then $(\boldsymbol{P}^*, \boldsymbol{B}^*)$ is a projective space and if $\boldsymbol{P}^{**}$ denotes the set of projective hyperplanes of it, there is a canonical embedding (injective mapping on the points sending every line onto a line) of $(\boldsymbol{P}, \boldsymbol{B})$ into $(\boldsymbol{P}^{**}, \boldsymbol{B}^{**})$ where $\boldsymbol{B}^{**}$ is the set of pencils in $(\boldsymbol{P}^*, \boldsymbol{B}^*)$. Consequently (L2) holds in $(\boldsymbol{P}, \boldsymbol{B})$.

For an application of projective hyperplanes to designs, see Dehon [1977], Teirlinck [1979, 1980]. An early powerful application is the Dembowski–Wagner theorem, characterizing finite projective spaces as 2-designs (see Chapter 2, Section 9, and Chapter 8, Theorem 6.6).

2.4. *Closure spaces*

The inspiration provided by subspaces leads to the definition of a *closure space* as a block space $(\boldsymbol{P}, \boldsymbol{B})$ whose blocks are called *subspaces* with the requirement that any intersection of subspaces is a subspace. In particular, P is a subspace. As a consequence, every subset X of P *generates* a subspace $\langle X\rangle$ which is the intersection of all subspaces containing X. Moreover, every subspace equipped with the subspaces contained in it is again a closure space. If M is a subspace of $\boldsymbol{P}$ then the restriction (see Section 2.1.2) of $(\boldsymbol{P}, \boldsymbol{B})$ to M is a closure space. A geometric approach of closure spaces is developed in Buekenhout [1967] (with complements in Boffa [1968]).

Tempting definitions are as follows.

(1) A *free* (*or independent*) *set* is a set of points F such that F is a minimal generating set of $\langle F\rangle$.
(2) A *min basis* is a minimal generating set of $\boldsymbol{P}$.
(3) A *max basis* is a maximal free set.

Every min basis is also a max basis but the converse is not necessarily true.

As to the concept of dimension, we can approach it via min basis, max basis and maximal chains of subspaces other than $\boldsymbol{P}$. For each of these three concepts we get a set of cardinal numbers, their minimum (or lower dimension) and their supremum (or upper dimension). The meaning of the following notation should now be obvious: $\dim\min b^-$, $\dim\min b^+$, $\dim\max b^-$, $\dim ch^+$, etc. We have the inequalities:

$$\begin{aligned}\dim ch^- &\leqslant \dim\max b^- \leqslant \dim\min b^- \leqslant \dim\min b^+\\ &\leqslant \dim\max b^+ \leqslant \dim ch^+.\end{aligned}$$

A tempting axiom is:

(C1) All dimensions are equal (Jordan–Dedekind condition or graded closure space).

More on this is in Batten [1983, 1984] who characterizes (C1) in the finite-dimensional case by a weak exchange property.

Some caution with this concept is needed when $\dim ch^+$ is infinite. For projective spaces of infinite dimension (C1) holds and it also does for polar spaces of finite rank (the subspaces being the singular subspaces and $\boldsymbol{P}$) but it does not necessarily hold for a polar space of infinite rank. Spaces of finite dimension with (C1) can easily be related to geometries over a linear diagram (see Section 3.4). Developments in this direction are due to Danzer and Schulte [1982].

Various finiteness conditions can be imposed on $(\boldsymbol{P}, \boldsymbol{B})$ such as the finiteness of $\dim ch^+$, the Noetherian condition, etc. (Buekenhout [1967]). Here we need mention only one.

(C2) (finite character property) A set of points F is free if and only if every finite subset of F is free.

EXAMPLE. Let V be an infinite-dimensional vector space and let $P(V)$ be the corresponding projective space. Let B be a basis of V and $P(B)$ its image.

Consider $FP(V)$ namely the set of points of $P(V)$ equipped with all linear subspaces of finite dimension. Then $FP(V)$ is a graded closure space (a matroid, a linear space with dimension, a geometric lattice (see Chapter 6)). Now every finite subset of $P(B)$ is free in $FP(V)$ but $P(B)$ itself is not free. Hence (C2) does not hold. Moreover $FP(V)$ has no basis.

Let (C2) hold in $(\boldsymbol{P}, \boldsymbol{B})$. Then the following are equivalent.

$(C3)_1$ For every free set F and every point p not in $\langle F\rangle$, the union of F and p is a free set.
$(C3)_2$ For every subspace U and every point p not in U, the subspace $\langle U, p\rangle$ is minimal among the subspaces containing U properly.
$(C3)_3$ For every subset S of $\boldsymbol{P}$ containing a free set F, there exists a subset B of S containing F such that B is a min basis of $\langle S\rangle^*$.

In Buekenhout [1967] two more properties equivalent to the above are given (a correction and further developments are given in Ohn [1990]). Closure spaces satisfying (C2) and $(C3)_i$, $i = 1, 2, 3$, are the linear spaces with dimension (essentially also the matroids or geometric lattices). They are further studied in Chapter 6.

The topics of this section can be developed in the 'pointfree' context of lattices (Birkhoff [1967]). This observation also holds for partially ordered sets.

2.5. *Other inspiration*

2.5.1. *A unified approach of subspaces*

In Sections 2.2.2, 2.3.2, 2.4 we get three different views on subspaces. These can be unified to some extent along the ideas in Buekenhout [1977b] presented here in a slightly different way. These methods are useful in the construction of geometries from graphs, in particular graphs on which finite groups are acting.

Let $(\boldsymbol{P}, \boldsymbol{B})$ be a block space. We associate to it $(\boldsymbol{P}, \boldsymbol{B}_i)$ where $\boldsymbol{B}_i$ is the set of intersections of any family of blocks in $\boldsymbol{B}$. So $(\boldsymbol{P}, \boldsymbol{B}_i)$ already is a closure space. But potentially large subspaces may perhaps still be expected. Hence, we consider $(\boldsymbol{P}, \overline{\boldsymbol{B}}_i)$ where $X \in \overline{\boldsymbol{B}}_i$ if for any subset Y of X contained in some $B_i \in \boldsymbol{B}_i$ with $B_i \neq \boldsymbol{P}$, the closure $\langle Y\rangle$ in $(\boldsymbol{P}, \boldsymbol{B}_i)$ is contained in X. Now $\boldsymbol{B}_i \subseteq \overline{\boldsymbol{B}}_i$, $\overline{\overline{\boldsymbol{B}}}_i = \overline{\boldsymbol{B}}_i$, and if U is any member of $\overline{\boldsymbol{B}}_i - \boldsymbol{B}_i$ then U is not contained in any member of $\boldsymbol{B}_i$.

The passage of $\boldsymbol{B}_i$ to $\overline{\boldsymbol{B}}_i$ allows to define a canonical closure space structure on the family of all closure spaces whose set of points is P.

2.5.2. *Independent sets*

Here the blocks of a block space $(\boldsymbol{P}, \boldsymbol{B})$ are considered as *independent* or *free* sets. For developments on this see for instance Welsh [1976] and White [1986].

2.5.3. *Convex sets*

Here one starts with a closure space as in Section 1.3 and the blocks are considered as convex sets rather than subspaces. For a survey of this, see Edelman and Jamison [1985].

2.5.4. *Incidence*
Let $(\boldsymbol{P}, \boldsymbol{B})$ be a block space. How to define an incidence relation on the members of $\boldsymbol{B}$, in order to get closer to geometries? The first and most natural idea is to declare B_1 incident to B_2 if either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. But rather interesting examples like the geometries of type D_n, E_6, E_7, E_8 over a nonlinear diagram led Tits to another view.

Assume that $(\boldsymbol{P}, \boldsymbol{B})$ is equipped with some type function, i.e. a mapping t of $\boldsymbol{B}$ onto some set I. In a classical context $t(B)$ could be the dimension of B. More generally, blocks with the same type can be supposed to share some property. Now, if A, B are in $\boldsymbol{B}$, Tits (unpublished) declares A and B incident if $A \cap B$ is maximal among all sets $A' \cap B'$ with $t(A') = t(A)$ and $t(B') = t(B)$. This is the *principle of maximal intersection* (see Section 3.7). Incidence in the first sense is usually preserved in this broader sense (it suffices that blocks of the same type do not contain each other).

2.5.5. *Posets*
So far, closure spaces appear as a fairly general and appropriate setting for the concept of a 'space' provided with a set of 'subspaces'. This concept is crucial in order to develop incidence geometry. There are of course situations, e.g., projective planes in which it has little power.

One more step can be made in the direction of generality. We can replace a closure space by a poset (partially ordered set) and possibly get rid of points as in the famous von Neumann continuous geometries (see Chapter 21). This orientation is often related to lattice theory. A classical reference is Birkhoff [1967].

3. The graph theoretic approach

3.1. *Geometries*

Fairly comprehensive references are Buekenhout and Cohen [1994], Pasini [1994].

3.1.1. *Incidence*
In a geometry, the incidence structure is expressed by a graph on a set of elements which generalizes the set of subspaces of a classical geometry. Each element bears a type which is inspired by names such as 'point', 'line', 'plane' in elementary geometry or by the dimension of a subspace in classical geometries.

Let I be a set whose elements are called *types*. Consider a triple $\Gamma = (X, *, t)$ where X is a set, $*$ is a binary, symmetric, reflexive relation defined on X, and t is a mapping of X into I.

Members of X are called *elements of* Γ, $*$ is called the *incidence relation* of Γ and t is the *type function* of Γ. If x, y are in X with $x * y$ we call x and y *incident*. If $A \subseteq X$, $t(A)$ is the *type* of A and its *cotype* is $I \setminus t(A)$. The *rank* of A is $|t(A)|$ and its *corank* is $|I \setminus t(A)|$. The rank of Γ is the cardinality of I, that is the number of distinct types of elements.

A *flag* of Γ is a (possibly empty) set of pairwise incident elements of Γ. A *chamber* of Γ is a flag C such that $t(C) = I$. Two chambers are i-adjacent, $i \in I$, if all of their elements of type other than i coincide.

Let I and $\Gamma = (X, *, t)$ be as above. Then, we call Γ a *pregeometry over* I if

$$\big(x * y \text{ and } t(x) = t(y)\big) \Rightarrow (x = y).$$

The pregeometry Γ is a *geometry* (or I-*geometry*) provided t is surjective and every maximal flag of Γ is a chamber. It is called *firm* (resp., *thick*, *thin*) provided every flag of corank 1 is contained in at least 2 (resp., at least 3, exactly 2) chambers. Furthermore, Γ is called *thin* (resp., *thick*) at a type i provided every flag of cotype i is contained in exactly 2 (resp., at least 3) chambers.

3.1.2. *Some definitions*

While the elements of a pregeometry Γ over I remind us of the subspaces of a classical geometry, the latter are also most naturally represented by the set of elements incident with them. This leads to the concept of a residue which is central to any theory of geometries.

Let F be a flag of Γ. The *residue of* F *in* Γ is the pregeometry

$$\mathrm{Res}_\Gamma(F) = (X_F, *|X_F, t|X_F)$$

over $I \setminus t(F)$ where X_F is the set of all members of $X \setminus F$ incident with each element of F. Each residue of a geometry is also a geometry.

A property of a pregeometry Γ is called *hereditary* if it also holds in all residues of it. A property of Γ is called local if it puts requirements on rank 2 residues of Γ only.

Let $\Gamma = (X, *, t)$ be a geometry over I and let $\Gamma' = (X', *', t')$ be a geometry over I'. Then Γ' is a *subgeometry* (resp., *induced subgeometry*) of Γ if $I' \subseteq I$, $X' \subseteq X$, $*' \subseteq *$ (resp., $*' = *|X'$) and $t' = t|X'$.

A geometry Γ is *firm* (resp., *thick*, *thin*) if every rank one residue of a flag of Γ has at least 2 (resp., 3, exactly 2) elements.

A pregeometry $\Gamma = (X, *, t)$ is *connected* if the graph $(X, *)$ is connected and nonempty.

A pregeometry Γ is *residually connected* if for any subset $J \subseteq I$ of cardinality 2 at least, and every flag F of type $I \setminus J$, the graph $(X_F, *_F)$ is connected. This and earlier concepts arise in Tits [1956, 1962].

Let Γ be a pregeometry of finite rank. Then Γ is residually connected if and only if for any subset $J \subseteq I$ of cardinality 2 at least and every flag F of type $I \setminus J$, the graph Cham($\mathrm{Res}_\Gamma F$) whose vertices are the chambers of $\mathrm{Res}_\Gamma F$ and whose edges are the pairs of adjacent chambers, is connected (Valette [1982]).

If Γ is a residually connected pregeometry of finite rank, let i, j be distinct elements in I and p, q elements of Γ. Then there exists an $\{i, j\}$-path from p to q, i.e. a sequence

$$p * a_1 * a_2 * \cdots * a_n * q$$

in which each $a_1, a_2, \ldots, a_n$ is of type either i or j (Buekenhout [1979]). Moreover, Γ (resp., Γ_F for any flag F of Γ) is a geometry over I (resp., $I \setminus t(F)$).

3.1.3. *Morphisms*

Let $\Gamma = (X, *, t)$ be a pregeometry over I and $\Gamma' = (X, *', t')$ be a pregeometry over I', for some sets I, I'. A *morphism* from Γ to Γ' is a mapping α: $X \to X'$ such that

$$x * y \Rightarrow \alpha(x) *' \alpha(y) \quad \text{and} \quad t(x) = t(y) \Leftrightarrow t'\big(\alpha(x)\big) = t'\big(\alpha(y)\big).$$

The image of a flag by α is a flag.

If $I = I'$, we define a *homomorphism* (resp., *epimorphism*) as a morphism (resp., surjective morphism) α of Γ to Γ' such that $t(x) = t(\alpha(x))$ for each x. A bijective morphism α such that α^{-1} also is a morphism is called a *correlation*. If α is a homomorphism and a correlation then we call α an *isomorphism*. An isomorphism of Γ onto itself is an *automorphism*.

A *cover* of a pregeometry Γ onto a pregeometry Γ' is an epimorphism α such that for each element x of Γ, α restricted to $\mathrm{Res}_\Gamma x$ is an isomorphism on $\mathrm{Res}_{\Gamma'}\alpha(x)$. A geometry Γ is *simply connected* (resp., *residually simply connected*) if it is connected (resp., residually connected) and if every cover of a connected geometry onto Γ is an isomorphism (resp., all flag-residues of rank $\geqslant 3$ are simply connected). These concepts appear in Tits [1982] where we find also useful variations on it. Let Γ be a geometry.

A 2-*cover* (resp., m-*cover*, m an integer $\geqslant 3$) of Γ consists of a geometry $\overline{\Gamma}$ and an epimorphism $\bar{\alpha}$: $\overline{\Gamma} \to \Gamma$ such that α restricted to any rank 2 (resp., m) residue of $\overline{\Gamma}$, is an isomorphism. Then, Γ is called a 2-*quotient* of $\overline{\Gamma}$, and $\overline{\Gamma}$ is called a 2-*cover* of Γ. Also, if α is not an isomorphism, it is called a *proper cover*. A *universal* 2-*cover* (resp., m-*cover*) of Γ is a 2- (resp., m-) cover $(\widetilde{\Gamma}, \tilde{\alpha})$ of Γ such that for any 2- (resp., m-) cover $(\overline{\Gamma}, \bar{\alpha})$ of Γ we have a 2- (resp., m-) cover $(\widetilde{\Gamma}, \beta)$ of $\overline{\Gamma}$ such that $\tilde{\alpha} = \bar{\alpha}\beta$. It is not known in all cases whether a universal 2- (resp., m-) cover exists. If it exists, it is unique.

Every rank 3 geometry Γ has a universal 2-cover and more generally, a rank n geometry has a universal $(n-1)$-cover.

PROBLEM. Does any geometry have a universal 2-cover? For the development of this important theory, see Tits [1982], Ronan [1980, 1981], Pasini [1985, 1986, 1994].

It should be observed that such developments can be more properly dealt with in the context of chamber systems, rather than that of geometries (see Chapter 11).

3.1.4. *Quotients*

Let A be a group of automorphisms of the pregeometry $\Gamma = (X, *, t)$ over I. Then we define a *quotient pregeometry of* Γ *by* A, over I, say $\Gamma/A = (X/A, */A, t/A)$ where X/A is the set of orbits of A on X, two such orbits are incident (under $*/A$) if they are of the form Ax, Ay with $x * y$ in Γ and t/A maps every orbit of A on the type of its elements. The mapping $x \mapsto Ax$ is an epimorphism of Γ onto Γ/A called the *canonical projection*. The following properties play a role for the important theories developed in Tits [1982].

(Q1) For every flag F of Γ, the canonical projection π: $\Gamma \to \Gamma/A$ induces an isomorphism of $\mathrm{Res}_\Gamma F/AF$ onto $\mathrm{Res}_{\Gamma/A}(\pi(F))$ where A_F is the stabilizer of F in A.

(Q2′) The elements of an orbit of A in X which are incident to a given flag F form at most one orbit of the stabilizer of F in A.
(Q2″) If x, y are elements of Γ with $x * y$ and if F is a flag incident to some element of Ax and to some element of Ay, then there exists $a \in A$ such that F is incident to ax and to ay.
(Q3) In the graph $(X, *)$, the distance between two distinct vertices belonging to the same orbit of A is at least 4.

If I is finite (Q1) is equivalent to the union of (Q2′) and (Q2″). Condition (Q3) implies (Q2′) and (Q2″) hence (Q1). This is stated in Tits [1982]. For detailed proofs, see Gelbgras [1988].

3.2. *Special morphisms*

If $\Gamma = (X, *, t)$ is a geometry over I and α is a morphism of Γ into a geometry Γ' over I' then $\alpha(\Gamma)$ need not be a geometry over the set of types of elements in $\alpha(\Gamma)$. We avoid this situation as follows. The morphism α is a *lifting* provided for every flag F of Γ and every flag Y of $\alpha(X)$ containing $\alpha(F)$ there is a flag F' containing F in Γ with $\alpha(F') = Y$ (flags can be lifted).

Now $\alpha(\Gamma) = (\alpha(X), \alpha(*), \alpha(t))$ is a $\alpha(I)$-geometry where $i \in I$, $x \in X$, $t(x) = i$ gives $\alpha(i) = t'(\alpha(x))$, $x * y$ gives $\alpha(x)\alpha(*)\alpha(y)$, etc. We call $\alpha(\Gamma)$ the *image geometry*. The image of a chamber of Γ by α is a chamber of $\alpha(\Gamma)$. If F is a flag of Γ, then α restricted to $\mathrm{Res}_\Gamma(F)$ is a lifting onto $\mathrm{Res}_{\alpha(\Gamma)}(\alpha(F))$.

3.2.1. *Embeddings*

An embedding of an I-geometry Γ into an I'-geometry Γ' is a correlation of Γ onto a subgeometry of Γ'.

ILLUSTRATIONS.

(1) In the theory of polar spaces, a crucial step is to prove the embedding in a projective space under suitable conditions (see Chapter 12).

(2) A standard method intended to prove the embedding of some geometry in a projective geometry is to use projective hyperplanes of the geometry.

(3) A famous open problem about finite linear spaces is to prove that any one of them can be embedded in a finite projective plane (see Chapter 4, Section 5).

3.2.2. *Correlations*

Let $\Gamma = (X, *, t)$ be an I-geometry and α a correlation of Γ onto itself. Then α induces a permutation α_I on I. Then α_I is the identity if and only if α is an automorphism. If α_I is of order 2, α is called a *duality* or *reciprocity*. If α is a duality of order 2, α is called a *polarity*. If α and α_I are of order 3, α is called a *triality*. Let A be the group generated by α and consider the quotient geometry Γ/A defined in Section 3.1.4. In that geometry, an important theoretical role is played by the absolute of α, i.e. the induced subgeometry whose elements are those orbits of A which are flags of Γ.

EXAMPLES.

(1) The absolutes of polarities in finite-dimensional projective spaces give us all quadrics (in characteristic $\neq 2$), all symplectic and all unitary polar spaces. Each absolute element can be identified here with a totally singular subspace, and vice versa.

(2) The absolutes of polarities and trialities in finite building geometries are building geometries on which the centralizer of the polarity acts. All 'twisted' groups of Lie type arise in this way.

3.2.3. *Coset geometries*

Let Γ be an I-geometry and G a group of automorphisms of Γ acting transitively on the chambers. Sometimes it is said that G is flag-transitive because G acts transitively on all flags of a given type. For any flag F, the stabilizer G_F is called a *parabolic subgroup*. The stabilizer of a chamber is called a *Borel subgroup*.

Fix a chamber $C = \{x_i\colon i \in I,\ t(x) = i\}$. Let $P_i = G_{x_i}$. For each element y_i of type i in Γ, there is some $g \in G$ with $g(x_i) = y_i$, and y_i can be identified with the coset gP_i. If y_i is identified with gP_i and z_j with hP_j then $y_i * z_j$ if and only if $gP_i \cap hP_j$ is nonempty in G. Moreover, if $a \in G$ then a transforms $y_i = gP_i$ in agP_i. Hence the geometry Γ is entirely described by the data of G and of the subgroups P_i, $i \in I$.

Conversely, given a group G and a family of subgroups $(G_i)_{i\in I}$, we can define a pregeometry $\Gamma = \Gamma(G, (G_i)_{i\in I})$ over I as follows. The elements of type i, for $i \in I$, are all left cosets gG_i. Two elements gG_i and hG_j are incident if and only if $gG_i \cap hG_j$ is nonempty. The group G acts on Γ, by left translation, as a group of automorphisms. The action of G is transitive on all flags of cardinality 2 of a given type but it need not be chamber-transitive (nor flag-transitive).

(1) The action of G is flag-transitive if and only if every family of cosets $(x_iG_i)_{i\in J}$, $J \subseteq I$, which have pairwise a nonempty intersection, have some common element.

Then Γ is a geometry. If $I = \{1, 2, 3\}$, (1) is equivalent to each of

(2) $(G_1G_2) \cap (G_1G_3) = G_1(G_2 \cap G_3)$,

(3) $(G_1 \cap G_2)(G_1 \cap G_3) = G_1 \cap (G_2G_3)$ (Tits [1974]).

In the general case, let us order I (finite) by putting $I = \{1, 2, \ldots, n\}$. If J is a nonempty subset of I let $m(J)$ be the smallest element in J. Now (1) holds if and only if, for each $J \subseteq I$, with $|J| \geqslant 3$,

$$\bigcap_{j\in J\setminus m(J)} G_{m(J)}G_j = G_{m(J)} \bigcap_{j\in J\setminus m(J)} G_j$$

(see Buekenhout and Hermand [1994]).

If $I = \{1, 2, \ldots, n\}$, n finite, (1) is equivalent to the following system of conditions generalizing (2):

$$(G_1G_2) \cap (G_1G_3) \cap \cdots \cap (G_1G_n) = G_1(G_2 \cap \cdots \cap G_n),$$

$$(G_2G_3) \cap \cdots \cap (G_2G_n) = G_2(G_3 \cap \cdots \cap G_n) \cdots,$$

(4) $(G_{n-2}G_{n-1}) \cap (G_{n-2}G_n) = G_{n-2}(G_{n-1} \cap G_n)$.

The pregeometry Γ is connected if and only if the subgroups G_i generate G. Assuming that G acts flag-transitively on Γ, Γ is firm if and only if, for each $i \in I$,

$$\bigcap_{j \in I-i} G_j \nsubseteq G_i.$$

Other conditions are due to Aschbacher [1983] in the following particular case. For $J \in I$ put

$$G_J = \bigcap_{i \in J} G_i.$$

Call $i \neq j$ in I *joined* if for $J = \{i, j\}$, $\Gamma(G_{I-J}, (G_{I-i}, G_{I-j}))$ is not a generalized digon (see Section 3.3.1).

Assume:

(5) I is finite;

(6) for each $J \subseteq I$ with $|I_J| \geqslant 2$, G_J is generated by the $G_{J \cup k}$ where $k \in I - J$;

(7) the graph on I determined by the relation 'joined' is linear.

Then:

(8) G acts flag-transitively on Γ;

(9) Γ is residually connected and

(10) for $i \neq j$ in I and i, j joined the residues of Γ of type $\{i, j\}$ are isomorphic to $\Gamma(G_{I-J}, (G_{I-i}, G_{I-j}))$.

3.2.4. *Direct sums and products of geometries*

Let J be a set of indices and $(I_j)_{j \in J}$ a family of sets. For each $j \in J$, let $\Gamma_j = (X_j, *_j, t_j)$ be an I_j-geometry. Assume that the sets X_j, $j \in J$, are pairwise disjoint, as well as the sets I_j, $j \in J$. The *direct sum geometry*

$$\Gamma = \bigoplus_{j \in J} \Gamma_j$$

defined by: $\Gamma(X, *, t)$ is an I-geometry where

$$I = \bigcup_{j \in J} I_j, \qquad X = \bigcup_{j \in J} X_j, \qquad *|X_j = *_j$$

and $x * y$ whenever x and y belong to distinct Γ_j, Γ_k, respectively, and $t|X_j = t_j$. This arises in Tits [1956]. For a detailed study, see Valette [1982].

For different kinds of products of geometries, see Rees [1985] and Hillebrandt [1988].

3.2.5. *Incidence but no types*

It is possible to study incidence quite well without giving types. So an *incidence* is a graph without loops and with undirected edges. A *flag* is a complete subgraph, etc. For a study of this structure following the pattern of geometries, see Buekenhout and Buset [1988].

3.3. *Diagrams*[3]

The rank 2 geometries are building blocks for higher rank geometries. If Γ is an I-geometry and i, j any distinct elements of I we may be interested in the family Γ_{ij} of all residues of Γ of type $\{i, j\}$ and consider the data consisting of all Γ_{ij} as a structure on I. This idea goes back at least to Schläfli for the study of regular convex polytopes. It was systematically developed in the context of geometries by Tits [1956, 1962] and Buekenhout [1979a]. There are various concepts of diagrams. Here we present some of them.

3.3.1. *The basic diagram*

First we need the simple idea of a *generalized digon* or rank 2 geometry over $J = \{0, 1\}$ in which each element of type 0 is incident with each element of type 1.

Let Γ be any I-geometry, I any set. For $i \neq j$ in I we call i and j joined if there is at least one rank 2 residue of type $\{i, j\}$ in Γ which is not a generalized digon.

The *basic diagram* BD(Γ) is the graph whose vertices are the elements of I and whose edges are the joined pairs of vertices (Buekenhout [1981]). An I-geometry Γ is called *pure* (Pasini [1982]) if for any $i \neq j$ in I which are joined no residue of Γ of type $\{i, j\}$ is a generalized digon.

If this is the case, then every residue $\mathrm{Res}_\Gamma(F)$, F a flag of Γ, is pure, and the basic diagram BD($\mathrm{Res}_\Gamma(F)$) is the induced subgraph of BD(Γ) whose vertices are the types of elements in $\mathrm{Res}_\Gamma(F)$. A theorem of fundamental importance is related to the basic diagram (Tits [1956], Buekenhout [1979a]).

Let Γ be a residually connected I-geometry of finite rank. Let i, j be elements of I contained in distinct connected components of the basic diagram BD(Γ)*. Then every element of type i of Γ is incident with every element of type j of Γ. This theorem expresses that Γ is a direct sum of geometries over basic diagrams which are the connected components of* BD(Γ).

3.3.2. *Diagrams without a preassigned geometry*

Let I be a set. A diagram on I is a pair (I, f) where f is a function which assigns to every unordered pair of distinct elements i, j in I a class $f(i, j)$ of rank 2 geometries over $\{i, j\}$. Here $f(i, j) = f(j, i)$.

An I-geometry Γ *belongs to the diagram* (I, f) if for every i, j in I and every flag F of Γ such that $t(F) = I - \{i, j\}$, there follows $\mathrm{Res}_\Gamma(F) \in f(i, j)$.

These definitions allow the greatest freedom of choice for a diagram. In this context a diagram appears as a set of axioms (for those geometries belonging to it). Many axiom systems in the literature can be totally or partially reduced to data consisting of a diagram.

[3] The matter of this section is developed, with many examples, in Chapter 22 (Editor's note).

3.3.3. *More on rank 2 geometries*

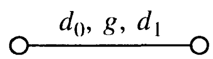
The definition of a diagram (I, f) in Section 3.3.2 suggests to put restrictions and some kind of regularity on each class $f(i, j)$. This requires some developments on rank 2 geometries.

Let $\Gamma = (X, *, t)$ be a firm, residually connected geometry over $I = \{0, 1\}$. The *dual geometry* is $\Gamma^* = (X, *, t^*)$ with $t^*(x) = 0$ if and only if $t(x) = 1$.

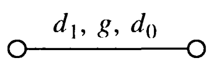
The *gonality* of Γ is equal to g, if $2g$ is the smallest cardinality of a thin connected subgeometry of Γ. If g is finite, $2g$ is indeed even, so g is an integer.

We shall say that Γ has an 0-*diameter* (resp., 1-*diameter*) of cardinality d_0 (resp., d_1) if for very element p of Γ of type 0 (resp., 1) d_0 is the least upper bound of the distances from p to any other element in the graph $(X, *)$. Clearly $|d_0 - d_1| \leqslant 1$ and $g \leqslant d_0$, $g \leqslant d_1$. Given integers (or ∞), g, d_0, d_1 with $2 \leqslant g \leqslant \min\{d_0, d_1\}$ we define a (g, d_0, d_1)-*gon* as a firm, residually connected $\{0, 1\}$-geometry of gonality g, 0-diameter d_0 and 1-diameter d_1 (Buekenhout [1983]).

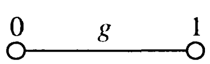
We use the picture



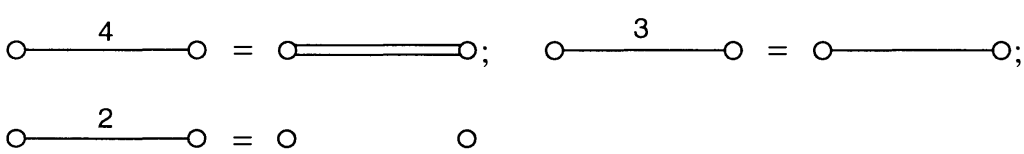
to represent the class of all (g, d_0, d_1)-gons. The class of their dual geometries is thus represented by



Important particular cases are those where $d_0 = g = d_1$. Then Γ is called a *generalized g-gon* (Tits [1959]). For that case we use the picture



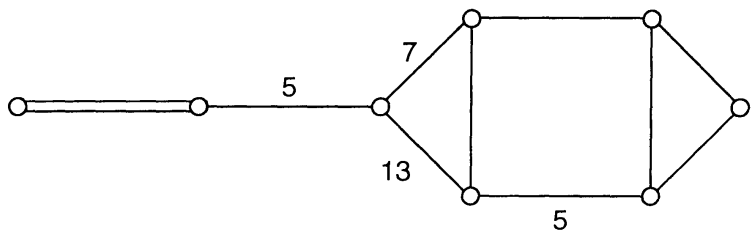
If g is equal, respectively, to 4, 3, 2 further simplifications in the pictures are given by

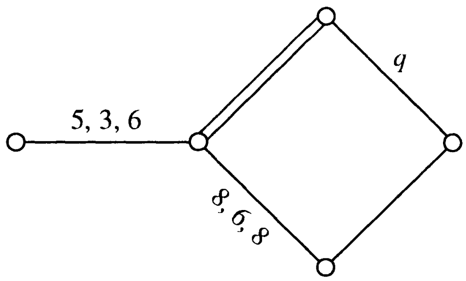


These are motivated in various ways on which we cannot dwell here (see Tits [1981], Buekenhout [1979a]).

3.3.4. *Special diagrams*

Let I be a set. A diagram (I, f) (see Section 3.3.2) is called *special* if for each $\{i, j\}$ in I, $i \neq j$, there are g_{ij}, d_{ij}, d_{ji} in $\mathbb{Z} \cup \{\infty\}$ with $2 \leqslant g_{ij} \leqslant \{d_{ij}, d_{ji}\}$ and if $f(i, j)$ is the class of all (g_{ij}, d_{ij}, d_{ji})-gons. Observe that $g_{ij} = g_{ji}$. A special diagram is conveniently described by a picture using the conventions of Section 3.3.3. Here are some examples.



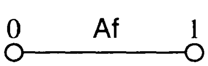


COXETER DIAGRAM. Let (I, f) be a special diagram. It is called a Coxeter diagram if for all $i \neq j$ in I, $d_{ij} = g_{ij} = g_{ji} = d_{ji}$.

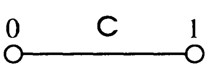
The upper example above is a Coxeter diagram.

3.3.5. *Some particular diagrams*
We use some more conventions.

denotes the class of *linear spaces*, or firm, connected rank 2 geometries of gonality 3 in which any two elements of type 0 are incident with some (by definition unique) element of type 1.

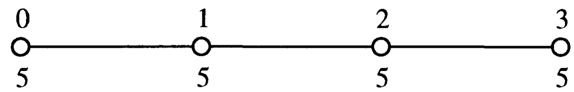


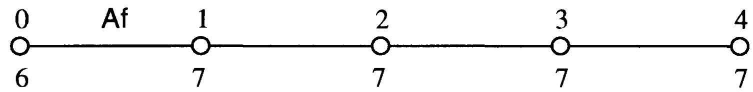
denotes the class of *affine planes*, or linear spaces such that for any element x of type 1 and any element y of type 0 at distance 3 of x there is a unique element z with $y * z$ and z at distance 4 from x.

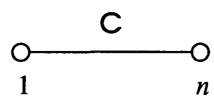


denotes the class of *circles*, or linear spaces all of whose elements of type 1 are incident with exactly two elements of type 0.

While describing a finite geometry Γ belonging to a diagram (I, f) it may be useful to indicate more information on a picture for (I, f). In particular, if for each $i \in I$, and each flag F of Γ of type $I - i$, $\mathrm{Res}_\Gamma F$ has a number $q_i + 1$ of elements which does not depend on F then we say that the i-order of Γ is q_i. On a picture for (I, f) we shall then write the letter i and the number q_i close to the node representing i. In the case of a linear diagram we write i above and q_i under that node. Here are some examples.

$P_4(5) =$ 

$A_5(7) =$ 

A circle of $(n+2)$ points $=$ 

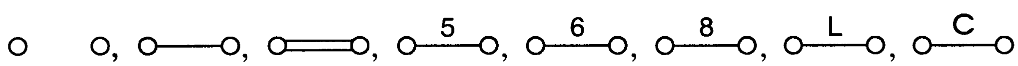
3.3.6. *Morphisms and diagrams*

Let Γ be a (g, d_0, d_1)-gon over $I = \{0, 1\}$ and α a lifting homomorphism of Γ onto an I-geometry Γ' which is a (g', d'_0, d'_1)-gon. Then $d'_0 \leqslant d_0$, $d'_1 \leqslant d_1$. If, e.g., $g = d_0$, then $g' \leqslant g$.

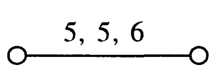
3.3.7. *Examples*

(1) If (Γ, G) is a pair consisting of a geometry Γ and a group G of automorphisms of Γ acting flag-transitively then Γ belongs to a special diagram and Γ has i-orders (see Section 3.3.5). Most interesting geometries (but not all) arise in this way. Many examples of such are given in Buekenhout [1979a, 1985], Ronan and Smith [1981], Ronan and Stroth [1984], Aschbacher [1986], Neumaier [1982]. See also Chapter 22.

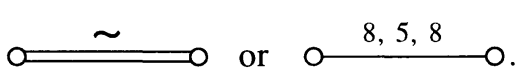
(2) In the examples of the references mentioned above most rank 2 residues are of one of the types



Some important exceptions are listed now. The Petersen graph (the vertices are the elements of type 0, the edges are the elements of type 1):

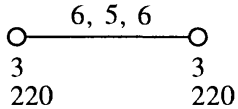


A triple one-cover of the generalized quadrangle $QSp_4(2)$:



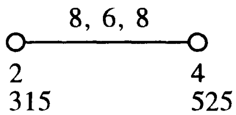
It has 45 points and 45 lines and automorphism group 3.Sym(6).

The diagram



with the Mathieu group M_{12} acting. There are 220 points and 220 lines.[4]

The diagram



with the Hall–Janko group $J_2.2$ acting. There are 315 points and 525 lines.

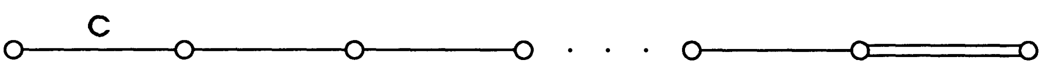
3.4. *Linear diagrams*

So far a majority of geometries that are met in the literature are linear, i.e. their basic diagram is of type

There are of course important exceptions such as the building geometries of types D_n, E_6, E_7, E_8, and many more examples are encountered in Chapter 22.

As mentioned earlier, various geometric theories found in the literature have an interpretation in terms of diagrams and geometries. We consider some of these as illustrations.

The locally polar spaces (see Buekenhout and Hubaut [1977] and Buekenhout [1977a]) turned out to be geometries belonging to a diagram



Actually this was one of the origins of the application of geometries and general diagrams to sporadic groups, in Buekenhout [1979a]. For further work on this topic see Kantor [1979], Del Fra, Ghinelli, Meixner and Pasini [1992] and Chapter 22.

3.4.1. *Incidence complexes*

See Danzer and Schulte [1982] and Schulte [1983]. This theory is very close to the developments on geometries but its inspiration is rather found in the theory of polytopes. It turns out that an *incidence complex of dimension* d as defined by axioms (I1) to (I4) by Danzer and Schulte is the same as a firm, residually connected I-geometry of rank d, having an i-order for each $i \in I$ and a linear basic diagram.

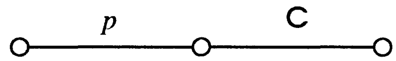
The regularity of such a geometry Γ is expressed by the existence of a flag-transitive group of automorphisms.

DANZER'S PROBLEM. For any regular incidence complex Γ of rank d, does there exist a regular complex of rank $d + 1$ in which the residue of some element is isomorphic to Γ?

Existing literature on extensions of geometries finds its place in this context. An example is Section 3.3.2. Danzer's problem may inspire much further work (see Chapter 22).

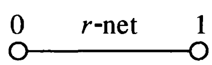
[4] Here, for the first time, the diagram is enriched with a mention of the number of elements of each type.

3.4.2. *Polygonal graphs*
These incidence structures introduced by Perkel turn out to be geometries belonging to the diagram

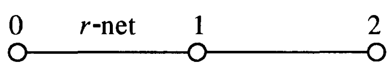


for some integer $p \geqslant 3$. A characterization of the Janko group J_1 as the group of a geometry belonging to the case $p = 5$ is obtained in Perkel [1980].

3.4.3. *Geometries involving rank 2 residues which are partial geometries*
For partial geometries, see Chapter 10. For higher rank geometries involving partial geometries, see Laskar and Dunbar [1978], Meyerowitz and Miskimins [1987], Meyerowitz [1985], Liebler and Meyerowitz [1988], and Chapter 22. A particularly interesting case was entirely worked out in Sprague [1979]. Let $r \geqslant 2$ be some integer. An *r-net* is a rank 2 geometry over $\{0, 1\}$ whose elements of type 1 fall in r equivalency classes (r directions) such that each class partitions the set of elements of type 0 (each element of type 0 is incident with one member of each class) and such that elements of type 1 in distinct classes are incident to a unique common point. Let



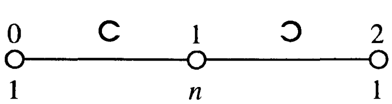
denote the class of r-nets. Let Γ be a firm, residually connected geometry belonging to



and let there be given a relation $\|$ on the set of 2-elements such that $\|$ is an equivalence, each parallel class partitions the set of points and two non parallel 2-elements intersect in a line. Then either $r = 2$ and Γ is a 'cubic lattice' (i.e. the obvious geometry derived from a set X and the Cartesian product X^3) or $r \geqslant 3$ and there is a projective space P of dimension 3, a plane π in P and a subplane π_0 in π such that the 0-elements of Γ are the points of $P - \pi$, the lines and planes of Γ are the lines (resp., planes) of $P - \pi$ whose intersection with π is a point (resp., line) of π_0.

Some authors do either explicitly start with a diagram or find inspiration in it to state axioms. We give some examples.

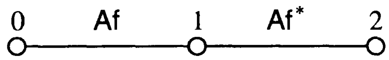
3.4.4. *Semibiplanes*
This is one of the most early examples. It was first studied by D. Hughes. Consider a geometry belonging to



with the intersection property (IP) (see Section 3.7). Then the set of *points* (elements of type 0) and the set of *blocks* (elements of type 2) constitute a block space $(\boldsymbol{P}, \boldsymbol{B})$

such that all blocks have the same cardinality $n+2$, any pair of points is on zero or two blocks, and any two blocks intersect in 0 or 2 points. These properties are used to define a *semibiplane* and they lead back to the diagram given at the beginning. For work on semibiplanes and somewhat more general diagrams, see Hughes [1978, 1981, 1983], Wild [1984], Hughes and Piper [1985].

3.4.5. Let Γ be a firm, residually connected geometry belonging to

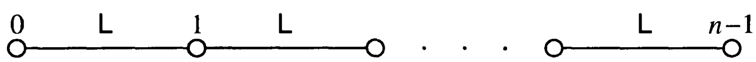


in which any two points are incident with at most one line. Then Γ is derived from some 3-dimensional affine space A and Γ satisfies one of the following:

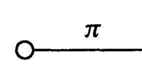
(1) there is a point o in A such that the elements of Γ are all elements of A which are not incident to o;

(2) there is a point at infinity ∞ of A such that the elements of Γ are all elements of A which are not incident to ∞ (Lefèvre-Percsy [1990]). On a similar problem, see Hale [1977].

3.4.6. *Linear spaces with dimension*
A firm, residually connected geometry belonging to the diagram



turns out to be a linear space with dimension n, see Chapter 6. For a quite similar treatment of geometries over

where


is the class of partial linear spaces (see Section 2.2.1), and other interesting related matters, see Deza and Laurent [1987].

3.4.7. *The work of Sprague*
A major result (see Sprague [1981]) has a variety of consequences. Unfortunately it assumes finiteness and it may call for a generalization.

Let Γ be a finite, residually connected, firm rank 2 *geometry in which*:

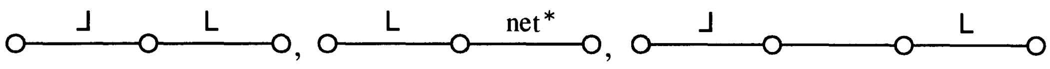
(1) *any two points are incident with at most one line*;

(2) Γ *and* Γ^* *satisfy Pasch's axiom* L5 (see Section 2.2.1);

(3) *for every point* p *and line* A *there is at least one line incident with* p *and intersecting* A.

Then there exists a (*generalized*) *projective space* π *and some integer* i *such that the points* (*resp., lines*) *of* Γ *are all subspaces of dimension* i (*resp.*, $i+1$ *or* $i-1$) *of* π.

The applications include very strong results on finite, firm, residually connected geometries over the diagrams



(see Sprague [1983, 1984a,b, 1985]).

3.5. *Apartments*

The unique and beautiful situation of apartments in buildings, especially those of spherical type (Tits [1974]), may be expected to give various interesting generalizations. There are two possible approaches. One of them is to point at the properties of a particular object as an apartment (such as being a thin subgeometry for instance). The other one is to consider systems of apartments. We start with the first approach and in this context, following a suggestion of G. Stroth, we speak of a *preapartment*.

Let $\Gamma = (X, *, t)$ be a firm, residually connected I-geometry and A some subgeometry of Γ that we call a preapartment of Γ. We always assume that A is an induced subgeometry, that it is firm, residually connected and that $t(A) = I$.

For a matter of convenience, we first restrict to the rank 2 case.

3.5.1. *Axioms*

Let Γ, A be as above with Γ of rank 2 and let d be the diameter of the graph $(X, *)$. Tempting axioms for A are:

(A1) A is thin;
(A2) A is a $2g$-gon in the incidence graph of Γ where g is the gonality of Γ;
(A3) A is a *geodesic circuit* in the incidence graph of Γ, i.e. for any two elements x, y in A, the distances of x and y in A and in Γ are equal;
(A4) A is an *optimal circuit*, i.e. A contains elements x, y whose distance in Γ and in A is equal to d and if a, b are elements of A with $t(a) = t(x)$, $t(b) = t(y)$, and $d_A(a, b) = d$ then $d_\Gamma(a, b) = d_A(a, b)$.

EXAMPLES. If we require (A2), all preapartments are isomorphic (to a g-gon). If Γ is for instance a generalized (4,10,10)-gon, and if (A3) is required then we can expect preapartments of $2n$ elements for n varying from 4 to 10 but some of these values need not be achieved. Axiom (A4) is rather subtle. It is inspired by two interesting examples:

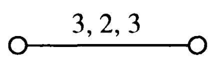
(i) if Γ is an affine plane, then $d = 4$ and the preapartments with (A4) are the parallelograms;
(ii) if Γ is the Petersen graph then the circuits of length 6 are optimal.

Coming back to the general situation we can consider the following conditions on A.

(A1)′ A is thin.
(A2)′ Every rank 2 residue of A satisfies (A2) (*minimality condition*).
(A3)′ Every rank 2 residue of A satisfies (A3) (*geodesic condition*).

(A4)$'$ Every rank 2 residue of A satisfies (A4) (*optimal condition*).
(A5)$'$ For any two chambers C, C' in A and for any minimal gallery of chambers stretched from C to C' in Γ, i.e. a sequence $C \sim C_1 \sim C_2 \cdots \sim C_n \sim C'$ where $\sim$ means adjacent, all of the C_i are in A (*convexity condition*).
(A6)$'$ Every minimal gallery of chambers in A is minimal in Γ, too.

Condition (A1)$'$ goes without saying for most experts. Here is however an argument against it. Let P be a 3-dimensional projective geometry and let Γ be its truncation to all points and all planes of P. Would it not be natural to require that an apartment A of P induce a (pre)apartment A' on Γ? If so A' belongs to the diagram



and A' is not thin. Here A' has one further advantage. Its gonality and i-diameters ($i = 0, 1$) are equal to those of Γ. So one could be tempted by a condition expressed in two steps.

(A7) If Γ is of rank 2, A is minimal subject to have the same gonality and i-diameters as Γ
(A7)$'$ If Γ is of any rank, every rank 2 residue of Γ satisfies (A7).

No exploration of this situation has been made so far. From now on we assume that (A1)$'$ holds always.

PROBLEM (raised by G. Stroth). Is there any Γ with no preapartments at all satisfying (A1)$'$? Let us further restrict it and ask whether there is any firm, residually connected, rank 3 geometry Γ with no thin, rank 3, residually connected subgeometry?

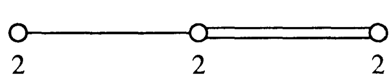
Requirements stronger than (A1) may be motivated by the following example.

EXAMPLE. Let P be a projective plane of order 25 with an oval of 26 points. Then we can construct thin preapartments with a number of points from 3 up to 26. This is somewhat annoying. But there may be other ways to restrict the apartment structure, via our second approach.

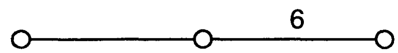
Conditions (A2)$'$, (A3)$'$, (A4)$'$, (A7)$'$ have an obvious local nature and are thus inherited by residues: if A is a preapartment in Γ satisfying (A3)$'$, say, and if F is a flag in A then $\mathrm{Res}_A\, F$ is a preapartment with (A3)$'$ in $\mathrm{Res}_\Gamma\, F$.

Many examples with (A2)$'$ are known (Buekenhout [1985]). See also Gerner [1990].

COUNTEREXAMPLE. There is a geometry of diagram



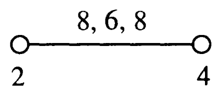
(not a polar space, see Neumaier [1984]) on which Alt(7) acts flag-transitively as a group of automorphisms. Its only preapartments are of type



and so (A2)′, (A3)′, (A4)′, (A7)′ are not satisfied in it (Buekenhout [1988]). Few examples with (A3)′, (A4)′, (A7)′ are known except in the rank 2 case where they are rather straightforward.

The conditions (A5)′, (A6)′ are not explored so far. Are they hereditary? If not there is an obvious hereditary version of each. In the rank 2 case, (A2)′ implies (A5)′ and each of (A2)′, (A3)′, (A4)′ implies (A6)′.

COUNTEREXAMPLES. (1) (A3)′ $\not\Rightarrow$ (A5)′. In the geometry over



for the group $J_2 \cdot 2$ (see Section 3.3.7) there are geodesic circuits of 8 points and 8 lines which do not satisfy (A5)′.

(2) (A4)′ $\not\Rightarrow$ (A5)′. This is obtained from an affine plane together with its parallelograms, as described earlier in this section.

(3) Does (A5)′ imply (A2)′? It does obviously in the rank 2 case and would do so in general if (A5)′ was hereditary.

3.5.2. *Systems of apartments and buildings*

Let Γ be as above. Here a system of apartments is a family $\boldsymbol{A}$ of preapartments satisfying (A1)′.

The most tempting axioms are those of Tits [1974] namely:

(B1) for any two chambers C, C' in Γ there is some $A \in \boldsymbol{A}$ with C and C' in A (this is sometimes called the *BNB condition*);

(B2) for any two members A, A' of $\boldsymbol{A}$ and flags F, F' in $A \cap A'$ there exists an isomorphism of A onto A' fixing $F \cup F'$ elementwise.

Both these conditions are hereditary. If we require these conditions simultaneously, Γ is a building and so all of the theory in Tits [1974] holds, in particular (A2)′, ..., (A7)′ for each $A \in \boldsymbol{A}$ and there is a unique maximal A on Γ.

Let $\Gamma, \boldsymbol{A}$ be as before. We want to assume (B1) alone now. A disturbing observation is that for any preapartment A not in $\boldsymbol{A}$, $\boldsymbol{A} \cup \{A\}$ satisfies (B1) too. We can think of a minimality condition on A but this is not obviously a hereditary property or is it?

But we observe an interesting weakening of (B2).

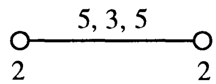
(B3) For any two members A, A' of $\boldsymbol{A}$ and any flag F in $A \cap A'$ there exists an isomorphism of A onto A' fixing F elementwise.

If (B3) (and (B1)) are satisfied we call $\boldsymbol{A}$ a *system of large apartments*.

PROBLEM. If $\boldsymbol{A}$ is a system of large apartments and (A2)′ holds, then Γ is a geometry over a Coxeter diagram. Is Γ necessarily a building?

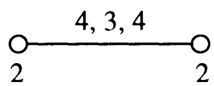
EXAMPLES. (1) In the Petersen graph-(5,5,6)-gon, the circuits of 6 points and 6 lines satisfy (B1) and (B3).

(2) In the Desargues configuration of 10 points and 10 lines (see Section 2.2.1) which has diagram



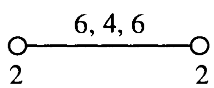
the pentagons satisfy (B1) and (B3).

(3) In the Pappus configuration of 9 points and 9 lines (see Chapter 2) which has diagram



the quadrangles satisfy (B1) and (B3).

(4) In the cube geometry of 8 points and 12 lines which has diagram



the hexagons satisfy (B1) and (B3).

(5) In the (6,8,8)-gon for J_2 (see Section 3.3.7), (B1) does not hold for octagons.

(6) In an affine space, the parallelohedra satisfy (B1) and (B3).

(7) In the geometry of type

(see Section 3.3.7) there is a class of 12-gons satisfying (B1) and (B3) (Heiss, private communication, 1988).

If we get rid of (B1) we can keep some of its power and put conditions that remain close to it. Consider the graph $\mathcal{G}(\Gamma)$ whose vertices are the chambers of Γ and in which two chambers are on an edge whenever they are together in some apartment. Now (B1) does just mean that $\mathcal{G}(\Gamma)$ is of diameter one. We can ask instead that $\mathcal{G}(\Gamma)$ be of diameter 2 (or 3, etc.).

(B4) If C, C' are chambers of Γ then there exist apartments A, A' with $C \in A$, $C' \in A'$ such that $A \cap A'$ contains some chamber.

Another idea is to require that chambers 'at small' distance of each other in the adjacency graph be necessarily together in some apartment.

Let the *apartment diameter* adiam(π, a) of (Γ, a) be the smallest upper bound of the $d(C, C')$ where C, C' are chambers of Γ contained together in some member of A.

(B5) For any pair of chambers C, C' of Γ such that $d(C, C') <$ adiam(Γ) there exists a member of A containing C and C'.

We call (Γ, A) a *semi-building* if (A5)$'$, (B2), (B4) and (B5) are satisfied.

PROBLEM. Is this hereditary?

A rank one semi-building is just a set provided with a graph structure of diameter $d \leqslant 2$. As to rank 2 semi-buildings they are characterized as follows. Let Γ be a generalized (g, d_0, d_1)-gon such that

(i) every path of g elements in the incidence graph of Γ is contained in at least one circuit of $2g$ elements;

(ii) $d_0 \leqslant 2g - 1$, $d_1 \leqslant 2g - 1$.

Let A be the set of $2g$-circuits of Γ. Then $(\Gamma, \boldsymbol{A})$ is a semi-building. Conversely, if $(\Gamma, \boldsymbol{A})$ is a rank 2 semi-building and if Γ is a generalized (g, d_0, d_1)-gon then $\boldsymbol{A}$ is the set of $2g$-circuits of Γ and (i), (ii) hold.

There are rather obvious examples of semi-buildings among the rank 2 geometries listed in this section so far. We have no other examples.

3.5.3. *The group case without* (B1) *and* (B2)

There is apparently little to do with systems of apartments in which neither (B1) nor (B2) hold. This may change if a group structure is introduced.

Let Γ be a firm residually connected I-geometry and G a group of I-automorphisms acting transitively on the chambers of Γ. A preapartment A of Γ is *regular* if the stabilizer G_A of A acts transitively on the chambers of A.

Now Γ must belong to a special diagram (see Section 3.3.4) because all of its residues of the same type $\{i, j\} \subseteq I$ are transitively permuted by G. In this context a *system of regular apartments* can be defined as the set of all regular preapartments satisfying (A2)$'$. In Buekenhout [1985] a series of 110 sporadic geometries is briefly described. For many of these there are regular apartments and they fall in the same orbit under G. There are some counterexamples as well and open cases of which some have meanwhile been settled (Gerner [1990]). So far, this is the main source of examples of apartments for geometries other than buildings. They are applied in Pasini [1985] in order to determine universal covers (see Chapter 22).

3.5.4. *The group case with* (B1)

Let Γ, G be as in Section 3.5.1. Let $\boldsymbol{A}$ be a system of regular preapartments satisfying (B1). Then (B3) is satisfied as well. Let $A \in \boldsymbol{A}$ and let C be a chamber in A. Let N (resp., B) be the stabilizer of A (resp., C) in G. Then $G = BNB$.

In Section 3.5.2 each of the examples (1) to (4) and (6) satisfies these properties (*How about* (7)?). The rank one case amounts to the doubly transitive groups.

In the general case, we can attach an interesting block space to $(\Gamma, G, \boldsymbol{A})$, namely, $(\boldsymbol{P}, \boldsymbol{B})$ where $\boldsymbol{P}$ is the set of chambers of Γ and a block is the set of chambers of some $A \in \boldsymbol{A}$. Then any two points are on some block and G acts flag-transitively on $(\boldsymbol{P}, \boldsymbol{B})$. There is an additional block space structure on $\boldsymbol{P}$, say $(\boldsymbol{P}, \boldsymbol{B}')$ where a block is a maximal set of i-adjacent chambers for some $i \in I$. Let the blocks of B' be called ribs. Since each rib goes along with some $i \in I$ this induces a parallelism on B', an equivalence relation on the ribs each of whose classes partitions the points. On two points there is at most one rib. Any $g \in G$ maps every rib R on a rib R' parallel to R.

PROBLEM. Classify all finite $(\Gamma, \boldsymbol{A}, G)$.

See Stroth and Weiss [1990] and Stroth [1992] for deep results in this direction.

3.5.5. *Cartan apartments*
Let Γ, G be as in Section 3.5.1 with G finite. Let C be a chamber and B its stabilizer in G. Consider the maximal nilpotent normal subgroup U of B. In many examples, in particular finite buildings, U has a complement H in B. It turns out quite often, that the set of fixed elements of H reveals a thin subgeometry but this is not always the case, in particular H may be the identity.

Here, assuming U to have a complement H in B, a *Cartan apartment* is any preapartment fixed elementwise by H (or by a conjugate of H in G). Most systems of regular apartments described in Buekenhout [1985] are obtained as Cartan apartments.

3.5.6. *More than one system of apartments*
The rank 2 examples given earlier suggest that there may be some interest in considering a system of large apartments together with a system of 'small' apartments.

3.6. *Shadows*

3.6.1. Let $\Gamma = (X, *, t)$ be a firm, residually connected I-geometry with I finite. We may like to replace Γ by some block space $(\boldsymbol{P}, \boldsymbol{B})$. We could take all elements of Γ of a given type $i \in I$ as points but actually a somewhat more general viewpoint may be usefully developed.

Let I_0 be a nonempty given subset of I. We define the I_0-space $\Gamma(I_0) = (\boldsymbol{P}, \boldsymbol{B})$ as follows. The *points* or members of $\boldsymbol{P}$ are the flags of Γ of type I_0. For any flag F of Γ, the I_0-*shadow* of F, $\sigma(F)$ or $\sigma_{I_0}(F)$, is the set of all points in $\mathrm{Res}_\Gamma(F)$. Now $\boldsymbol{B}$ is the set of all I_0-shadows of flags of Γ. Its members are called subspaces.

For building geometries the I_0-spaces have a rich collection of properties (Tits [1974]). This theory extends to geometries in which the *intersection property* (IP) holds (see Section 3.7) at least in the case where $I_0 = \{i\}$, $i \in I$ (Buekenhout [1979a,b, 1981], Pasini [1982]). The general case has not been completely studied yet.

Observe that nonisomorphic I-geometries Γ, Γ' may determine the same i-space for some $i \in I$. This is the case for a geometry of type D_n; it can be seen at the same time as a geometry of type C_n. The explanation of this phenomenon is that a block space does not lead uniquely to a definition of types on its blocks. Hence to go over from a block space to a geometry, a choice of types on the blocks must be made.

The main themes of the theory of I_0-spaces are:

(I) does every element of Γ uniquely determine a subspace and how do we recognize subspaces representing elements?

(II) are there lines in $\Gamma(I_0)$ and how do they behave with respect to other subspaces?

(III) relate the incidence relation of Γ to the inclusion of i-shadows and get control over each of these in terms of the other (example: the principle of maximal intersection stated in Sections 2.5 and 3.7.4);

(IV) distinct flags may have the same i-shadow. How does one get control over this from the structure of the diagram?

3.6.2. We start with problem (IV). Assume that F is a flag of type K, that $F' \subset F$ is of type $K' \subset K$ and that K' *separates* I_0 *from* K in the basic diagram (I, f), i.e. for the graph induced on $I - K'$, any 2 elements of the sets $I_0 - (I_0 \cap K')$ and $K - K'$ are in distinct connected components. Then $\sigma(F') = \sigma(F)$. This phenomenon does not depend on Γ but only on the graph (I, f) and I_0. In our example, we see that F' may be reduced to F but this reduction is explained in the graph. We are interested in subsets K of I that cannot be reduced anymore.

All of this leads to a theory of separation and reduction in a graph, independent of geometries. See Tits [1974], Buekenhout [1979b] for the case where $|I_0| = 1$, Scharlau [1990] for the general case.

3.7. *The intersection property*

Let (I, f) be a finite graph and I_0 a nonempty set of vertices of it. For subsets K, L of I we write $K \leqslant L$ if and only if I_0 and L are separated by K.

We have $K \leqslant L$ and $L \leqslant M$ implies $K \leqslant M$. Now

$$K \leqslant L, \qquad K' \leqslant L, \qquad K \cup K' \subseteq L$$

implies $K \cap K' \leqslant L$. So, for each $L \subseteq I$ there exists a unique smallest subset $L_{\text{red}} \subseteq L$ such that $L_{\text{red}} \leqslant L$. Vertex sets of the form L_{red} are called reduced (or I_0-reduced). Clearly $(L_{\text{red}})_{\text{red}} = L_{\text{red}}$. All subsets of L_{red} are reduced as well.

REMARK. All maximal chains of reduced sets do not necessarily have the same cardinality as stated incorrectly in Buekenhout [1979b, 1981] (Theorem 9.3).

The set of reduced sets together with the relation $\leqslant$ is a poset. More on this can be found in the references given in Section 3.6.2.

3.7.1. *Reduction of flags*

Let Γ, I be as earlier and let I_0 be a nonempty subset of I. A flag F is called I_0*-reduced* (or *reduced*) if no proper subset of F has the same shadow as F. The flag F is called *strongly* I_0*-reduced* if there is no flag E with $|E| < |F|$ and $\sigma(E) = \sigma(F)$. If F is strongly reduced it is reduced. Let F be of type L and let F_{red} be the subset of F whose type is L_{red}. Then as observed in Section 3.6.2, $\sigma(F_{\text{red}}) = \sigma(F)$. We will be interested in the converse:

(Red) if $F' \subseteq F_{\text{red}}$ and $\sigma(F') = \sigma(F_{\text{red}})$ then $F' = F_{\text{red}}$ (and so F has a 'unique reduction');
(S.Red) every reduced flag is strongly reduced.

3.7.2. *Lines*

We come to problem (II) of Section 3.6.1. Let Γ, I, I_0 be as earlier. If F is a flag such that $\sigma(F)$ contains at least two points and such that $\sigma(F)$ is minimal for this property then there is no point in F itself. If $i \in I_0$ and P is a flag of type $I - i$ containing F, then $\sigma(P) \subseteq \sigma(F)$ hence $\sigma(P) = \sigma(F)$.

In view of this observation a line (of kind $i \in I_0$) is any subspace $\sigma(P)$ where P is a flag of type $I - i$. In general, P will not be reduced and we are interested in a view on its reduction. Let I_0^1 be the set of vertices of $I - I_0$ at distance 1 from I_0. Then $(I_0 - i) \cup I_0^1 \leqslant I - i$. Hence, even if they are not necessarily reduced (but they often are) the subsets $(I_0 - i) \cup I_0^1$, $i \in I_0$, are quite interesting to analyze lines. We are interested in the following properties:

(LL) if two lines are both incident to two distinct points, then they coincide (this is (L2) of Section 2.2.1);
(LS) if $\sigma(F)$ is a subspace and $\sigma(P)$ is a line with two distinct points in $\sigma(F)$ then $\sigma(P) \subseteq \sigma(F)$.

3.7.3. Now we examine problem (I). For every $x \in X$, $\{x\}$ is of course strongly reduced. We wonder however whether $x \neq y$ in X may imply $\sigma(x) = \sigma(y)$. This is property:

(OO) distinct elements of Γ have distinct shadows.

Let E, F be reduced flags and assume that $\sigma(E) \subset \sigma(F)$. Then we are interested in

(U) $E \subset F$ if and only if F is the unique flag of its type whose shadow contains $\sigma(E)$.

3.7.4. Let us examine problem (III). We are interested in the following properties.

(IC) (inclusion control) If F, G are I_0-reduced flags then $\sigma(F) \subseteq \sigma(G)$ if and only if $F \cup G$ is a flag and $t(F) \leqslant t(G)$. Moreover $\sigma(F) = \sigma(G)$ if and only if $F = G$;
(PMI) (principle of maximal intersection) If x, y are elements of Γ then $x * y$ if and only if $\sigma(x) \cap \sigma(y)$ is maximal among all sets $\sigma(x') \cap \sigma(y')$ where $t(x') = t(x)$ and $t(y') = t(y)$.

Observe that (IC) implies (OO).

3.7.5. *The intersection property*
This powerful property was observed and stated in Tits [1956].

(IP) For each $i \in I$, $x \in X$ and flag F of Γ either $\sigma_i(x) \cap \sigma_i(F) = \phi$ or there is a flag F' incident with x and F such that $\sigma_i(x) \cap \sigma_i(F) = \sigma_i(F')$.

Usually (IP) also includes the requirement that the preceding condition hold as well in every residue of Γ. However, it has been shown that the above definition for (IP) implies its inheritance for each residue (Buekenhout and Hermand [1994]).

Let Γ be a firm, residually connected, pure I-geometry with I finite, in which (IP) holds. Then each of (Red), (S.Red), (LL), (LS), (OO), (IC), (PMI) hold and there are other such properties (Buekenhout [1979a, 1981]). It is not known whether (U) holds in general and the situation where i is replaced by a subset I_0 remains open as well.

Property (IP) holds in all building geometries and in many of the 110 sporadic geometries listed in Buekenhout [1985] but there are also important counterexamples such as the Alt(7) geometry of Section 3.5.1.

For a detailed analysis of (IP) and its relation to other properties see Biliotti and Pasini [1982], and Pasini [1994].

3.8. *Complexes*

Let $\Gamma = (X, *, t)$ be an I-geometry. Then Γ determines a *complex* $C(\Gamma)$ whose *faces* are the flags of Γ (Tits [1974]). We can *realize* $C(\Gamma)$ as a simplicial complex if we fill in each face with a real simplex, as is usual in algebraic topology.

The subject of geometries can now be generalized to complexes. We follow this path very briefly.

A *complex* is a block space $(\boldsymbol{P}, \boldsymbol{B})$ whose blocks are called *faces* and with the property that every subset of a face is a face while every point (called *vertex*) is a face.

We can imitate the residue of a flag in this context. If $B \in \boldsymbol{B}$, *the star of* B is the set of all faces containing B; it is a complex whose vertices are the faces containing B and one additional vertex in P.

A complex need not be of the form $C(\Gamma)$ where Γ is a geometry. Actually a complex $(\boldsymbol{P}, \boldsymbol{B})$ is isomorphic to some $C(\Gamma)$ if and only if every set of faces having pairwise an upper bound in $\boldsymbol{B}$ has a common upper bound in $\boldsymbol{B}$.

A complex $C(\Gamma)$ for some geometry Γ, is called a *flag complex*.

In Tits [1974] buildings are introduced as complexes with an additional property (chamber complexes) and a system of apartments (see Chapter 11).

3.9. *Chamber systems*

Let $\Gamma = (X, *, t)$ be an I-geometry. Then Γ determines a *chamber system* $C(\Gamma)$ which is the set of chambers of Γ provided with the various i-adjacency relations, denoted by $\tilde{i}$, for $i \in I$.

This leads to another generalization of geometries due to Tits [1982].

Let I be a set. A *chamber system* C over I consists of a set whose elements are called *chambers* and a collection of equivalence relations indexed by I, defined on the set of chambers. This concept can be developed very much in the same spirit as geometries (Tits [1982], Buekenhout and Cohen [1994]).

Besides its generality, it offers various advantages with respect to geometries. This appears for instance while working with groups. See also Chapter 11.

EXAMPLE. Let G be a group, B a subgroup of G and $(P_i)_{i \in I}$ a family of subgroups of G containing B. The chamber system $C(G, B, (P_i)_{i \in I})$ is defined as follows. The chambers are the left cosets of B in G and two chambers gB and hB are j-adjacent for $j \in I$ if $h^{-1}g \in P_j$.

A chamber system C allows to define a geometry $\Gamma(C)$. For the study of $\Gamma = \Gamma(C(\Gamma))$, see Tits [1982], and for $C = C(\Gamma(C))$, see Meixner and Timmesfeld [1983].

4. The functional approach

4.1. *Functional spaces*

A *functional space* is a triple $(S, \boldsymbol{S}, f)$ where S is a set, $\boldsymbol{S}$ is a family of subsets of S and f is a mapping from $\boldsymbol{S}$ to $\boldsymbol{P}(S)$ or $\boldsymbol{P}(\boldsymbol{P}(S))$ assigning to every member of $\boldsymbol{S}$ a subset of S (more generally, a collection of subsets of S). If $\boldsymbol{S}$ is the set of ordered pairs of distinct points of S, $(S, \boldsymbol{S}, f)$ is called a 2-*functional space*.

EXAMPLES of 2-functional spaces. S is the set of points of the Euclidean space E_3 and $f(a, b)$ is one of:
(1) the segment $[a, b]$;
(2) the line ab;
(3) the closed half-line with origin a containing b;
(4) the sphere with centre a and radius $[a, b]$;
(5) the sphere of diameter $[a, b]$;
(6) the plane bisecting $[a, b]$.

EXAMPLES of functional spaces. (1) In a 3-dimensional projective space over a field, three pairwise skew lines determine a 'regulus' (a quadric and a partition of its points by lines).
(2) In E_3, two skew lines determine a line (perpendicular to each of them).

4.2. *Join geometries and convexity spaces*

Here the inspiration is found in the segments of the Euclidean spaces. The basic structure (either join geometry or convexity space) turns out to be that of a linear space each of whose lines is endowed with two opposite total order relations and additional assumptions such as Pasch's axiom (for segments).

For join geometries, see Prenowitz and Jantosciak [1979]. For convexity spaces, see Bryant and Webster [1972] and Doignon [1976] who characterizes the convex sets of real affine spaces as convexity spaces. Strong improvements on the latter result are due to Kreuzer [1989b].

4.3. *André geometry*

This is basically 'noncommutative' affine geometry with inspiration from various sources. Consider a 2-functional space with point set S in which distinct points a, b determine a unique subset ab of S (called a line).

In addition to this let $\|$ be an equivalence relation on the sets ab (called *parallelism*). This is called a *skew affine* or *quasi-affine space* (André [1976, 1981]) if the following conditions hold:
(1) for $a \neq b$ in S, $a \in ab$ and $b \in ab$;
(2) $c \in ab - a$ implies $ab = ac$;
(3) $ab = ba = ac$ implies $ac = ca$;

(4) for every ab and every point c there exists exactly one set cd with $cd\|ab$ (cd is denoted by $c\|ab$;)

(5) $ab\|cd$ implies $ba\|dc$;

(6) for any distinct points a, b, c and distinct points a', b' with $ab\|a'b'$ there exists at least one point c' distinct from a', b' with $ac\|a'c'$ and $bc\|b'c'$.

EXAMPLE (actual source of motivation). Let E be a nonempty set and G a permutation group on E acting transitively but not 2-transitively. Assume that $p \neq q$ in E implies that the stabilizers G_p, G_q are distinct. Now pq is defined as the union of $\{p\}$ and the orbit of q under G_p. Here $pq\|p'q'$ if and only if there is some $g \in G$ such that $g(pq) = p'q'$.

Let us observe also that a real (or ordered) affine space in which ab would be defined as the closed halfline of origin a, containing b gives rise to a 2-functional space with an obvious parallelism which is a skew affine space. This structure has been axiomatized in Szmielew [1983] (see also Grochowska [1986]). For further developments on skew affine spaces we refer to André [1975, 1976, 1981], Wilbrink [1982], Tecklenburg [1982].

5. Incidence with other structure

5.1. *Parallelism on block spaces*

There are two classical ways to define a *parallelism*, or *resolution*, on the blocks of a block space $(\boldsymbol{P}, \boldsymbol{B})$ (or a functional space). In one of these, two blocks are called parallel if they are either disjoint or equal. The inspiration for it lays in the hyperplanes of affine spaces.

A slight variation is to consider $(\boldsymbol{P}, \boldsymbol{B})$ as a line space and to decide that lines B_1, B_2 are parallel if the subspace generated by B_1 and some point of B_2, contains B_2 and if B_1, B_2 are either disjoint or equal.

Whatever the definition, parallelism should usually be an equivalence relation in order to be useful.

The other approach is to give together with $(\boldsymbol{P}, \boldsymbol{B})$ an equivalence relation on $\boldsymbol{B}$ such that each equivalence class partitions $\boldsymbol{P}$.

5.1.1. *The case of linear spaces*

The latter viewpoint has been developed in a major way in the case where $(\boldsymbol{P}, \boldsymbol{B})$ is a linear space. This structure may be called ‘a linear space with parallelism’. It is known under various names like ‘Parallelstrukturen’ (used since 1961 by André and his followers) and ‘Sperner spaces’ (or ‘schwach affinen Räume’) that are used in 1962 by Sperner and his followers, with the additional assumptions that all lines have the same cardinality.

EXAMPLES. (1) Cartesian products of two or more sets provide obvious lines, planes, hyperplanes and a parallelism on them.

(2) If P is a 3-dimensional projective space and q some anisotropic quadratic form defined on it, then q determines two relations of parallelism on the lines of P (Clifford parallelism). (See Veblen and Young [1910].)

(3) 'Affine spaces' over ternary rings, near fields, etc. may have a natural parallelism (see, e.g., Nizette [1972]).

(4) If G is a permutation group on Ω, a *system of blocks of imprimitivity* is a partition of Ω, invariant by G. The set of all members of such systems provides a block space on Ω and the members of one system constitute a parallel class.

(5) The nets met earlier have a parallelism on their set of lines.

(6) Parallelism is part of the structure of skew affine spaces (see Section 4.3).

For other references on this theme, see Dembowski [1968] and Cameron [1976], Herzer [1977], Karzel and Kroll [1988], Beth, Jungnickel and Lenz [1985]. A more recent approach to parallelism in linear spaces is developed in Karzel, Kroll and Sörensen [1973] who define *double spaces* in a natural way (see also Karzel [1990, 1989a,b]).

5.1.2. *A relationship with groups*

In a linear space with parallelism it is possible to develop the concepts of *translation* and *dilatation* in various ways. This is an interesting theme in the direction of affine space characterizations and similar themes. The context of 'Translationsstrukturen' is often used. Refer to Karzel and Kroll [1988], Herzer [1979], Biliotti and Herzer [1984] and Herzer [1986].

5.2. *Hjelmslev–Klingenberg geometry*

The main source of inspiration (examples) is found in the incidence geometry over rings (see Chapters 19 and 21) and more recently in the theory of affine buildings. In Chapter 19, a neat distinction is made between geometry over a ring, Klingenberg geometry and Hjelmslev geometry. Here, we very briefly discuss the subject, from the point of view of functional incidence geometry. For detailed surveys, refer to Kreuzer [1990] and to Chapter 19.

The classical situation in this area is that of projective (resp., affine) Hjelmslev planes. For a survey of early work on these matters, see Dembowski [1968]. For a detailed bibliography, see Artmann et al. [1976]. A recent survey is due to Drake and Jungnickel [1985].

Interesting attempts towards more general viewpoints have been made by various authors (see Drake [1983], Hanssens and Van Maldeghem [1989, 1990], Keppens [1988a,b]) but so far there is an obvious lack of unity in this field. Hence we shall only point at some leading ideas and raise questions. We restrict ourselves to rank 2 geometries but once this case is clarified there are obvious extensions to higher rank geometries, in particular questions about 'Hjelmslev diagrams'.

Let Γ be a rank 2 I-geometry. The flavour of Hjelmslev–Klingenberg geometry is to consider a pair (Γ', φ) consisting of an I-geometry Γ' and a t-epimorphism φ of Γ' onto Γ. Variations occur as to the way that $\varphi(p) = \varphi(q)$ for $p \neq q$ in Γ' is influenced by the structure of Γ' and of Γ. Here are some examples of such requirements.

(1) Every flag of Γ is the image of some flag of Γ' (Drake [1983]). We can ask much more.

(2) For every p in Γ' and every path P in $(\Gamma, *)$ starting in $\varphi(p)$ there is a path P' in $(\Gamma', *)$ starting in p, with $\varphi(P') = P$. Here, $*$ denotes the incidence relation.

(3) If $p \neq q$ in Γ' and $\varphi(p) = \varphi(q)$ then there are at least two elements of Γ' incident with p and with q (this allows theoretical developments; see Drake [1983]).

(4) If $p \neq q$ in Γ' are elements of the same type which are incident to at least two elements of Γ' then $\varphi(p) = \varphi(q)$ (this is the choice in Keppens [1988a] for the case of generalized quadrangles and it works on very natural examples; it is not true in the choice of Hanssens and Van Maldeghem [1990] again for the case of generalized quadrangles and this is likewise justified by natural examples).

(5) For any pair of elements p, q in Γ' with $\varphi(p) \neq \varphi(q)$, $d(p,q) = d(\varphi(p), \varphi(q))$. Compare to (2). This property holds in the projective Hjelmslev planes.

(6) For any pair of elements p, q in Γ' with $\varphi(p) = \varphi(q)$, there exists x of the same type as p, with $\varphi(x) \neq \varphi(p)$ and $d(p, x) = d(p, q)$.

(7) For any pair of elements p, q with $\varphi(p) \neq \varphi(q)$ in Γ', there is a unique shortest path from p to q in Γ' if and only if there is a unique shortest path from $\varphi(p)$ to $\varphi(q)$ in Γ.

An important feature of Hjelmslev–Klingenberg geometry are sequences, finite or not, of the type:

$$\cdots \to^{\varphi_n} \Gamma^{(n)} \to^{\varphi_{n-1}} \Gamma^{(n-1)} \to^{\varphi_{n-2}} \cdots \to^{\varphi_1} \Gamma' \to^{\varphi} \Gamma$$

where each $(\Gamma^{(i)}, \varphi_{i-1})$ is a pair as above, over $\Gamma^{(i-1)}$, for each i. The sequence may have an *inverse limit* $(\overline{\Gamma}, \bar{\varphi})$ where $\bar{\varphi}$ is a t-epimorphism of $\overline{\Gamma}$ on Γ.

Hjelmslev–Klingenberg geometry goes along with a neighbour relation $\sim$ for which $p \sim q$ in Γ' if and only if $\varphi(p) = \varphi(q)$. Here $\sim$ is an equivalence relation. A more general viewpoint is to start with Γ', $\sim$ (and no more with Γ, Γ', φ) where $\sim$ is a binary symmetric relation such that $p \sim q$ implies $t(p) = t(q)$. For developments on this, see Chapter 19. We also mention the remarkable case of a geometry of type E_6 with its points and hyperlines as elements of Γ' and $\sim$ (on points) as collinearity (see Chapter 19).

For recent literature on this subject, see Hanssens and Van Maldeghem [1989, 1990, 1991], Keppens [1987, 1988a,b,c], Törner [1985], Van Maldeghem [1989a,b, 1990a,b].

5.3. *Incidence groups*

These consist primarily of a group G endowed with a structure of block space $(G, \boldsymbol{B})$ such that the left translations $x \mapsto ax$ leave $\boldsymbol{B}$ invariant. There may be more restrictions on $\boldsymbol{B}$ (like $(G, \boldsymbol{B})$ being a linear space) and on the group action. For a recent survey on this subject and related matters, see Karzel and Kist [1985], Karzel and Kroll [1988].

This material is related to Section 5.1.2.

5.4. *Half ordered planes*

This structure is a weakening of the classical betweenness relation. Refer to Kreuzer [1989a] and to the literature mentioned there.

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CHAPTER 4

Projective Planes

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Contents

Introduction 109
1. Constructions 109
2. Characterizations 113
   2.1. Collineation groups 113
   2.2. Projectivities 115
   2.3. Dualities 117
3. Combinatorics 119
   3.1. $\{k; n\}$-arcs 119
   3.2. Ovals and hyperovals 120
   3.3. Arcs 122
   3.4. Unitals 123
   3.5. Nuclei 126
   3.6. Blocking sets 126
4. Nonexistence theorems 128
5. Embeddings 129
References 131

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Introduction

A *projective plane* is an incidence structure (i.e. a rank 2 geometry) $\boldsymbol{P} = (P, L, \mathrm{I})$ consisting of a set P of *points*, a set L of *lines* and an *incidence relation* I such that the following three axioms are fulfilled.

– Any two distinct points are incident with precisely one common line.
– Any two distinct lines are incident with a common point.
– There exist at least two lines; any line is incident with at least three points.

Projective planes can be described as diagram geometries (cf. Chapter 3); they are denoted by the following diagram:

Projective planes are one of the most classical objects in geometry. We mention here two monographs on projective planes, the famous book of Pickert [1975] and the more modern text book by Hughes and Piper [1973]. Also Dembowski's book [1968] deals at a large percentage with projective planes. From this it follows trivially that it is impossible to present a panorama of projective planes on a couple of pages. Necessarily, one has to restrict oneself to a few topics. My selection is based on the following grounds.

– The material presented is not yet easily accessible in books so far. In particular, most of the results we shall see have been discovered after 1968 (the date of Dembowski's 'Finite Geometries') or are not dealt with in this book.
– Translation planes are not dealt with, as they are studied in Chapter 5.
– There will be nearly no proofs, but many references.
– It was my aim to nevertheless produce a readable, self-contained text.

1. Constructions

First of all, let us recall the definition of the most classical projective planes, namely the Desarguesian ones.

Consider the following 'configuration theorem'. Fix a point P and a line l of a projective plane $\boldsymbol{P}$. We say that $\boldsymbol{P}$ satisfies the *(P,l)-theorem of Desargues*, if for any six points $A_1, A_2, A_3, B_1, B_2, B_3$ with

a) $P \neq A_i \neq B_i \neq P$ $(i = 1,2,3)$, $A_i \neq A_j$, $B_i \neq B_j$ $(1 \leqslant i < j \leqslant 3)$, and
b) P, A_i, B_i are collinear $(i = 1,2,3)$

one has: If the points $A_1A_2 \cap B_1B_2$ and $A_2A_3 \cap B_2B_3$ lie on the line l, then also the point $A_1A_3 \cap B_1B_3$ lies on l (cf. Figure 1).

The projective plane $\boldsymbol{P}$ is called *Desarguesian* if the (P,l)-theorem of Desargues holds for any point-line pair (P,l).

The fact that a projective plane is Desarguesian is intimately related to the existence of 'all possible' central collineations. To be precise, we say that $\boldsymbol{P}$ is *(P,l)-transitive* for a point-line pair (P,l) if for some line $m \neq l$ through P and any two points X, Y on m with $X, Y \neq P, l \cap m$ there exists a central collineation with axis l and centre P mapping X onto Y. It can easily be seen that if $\boldsymbol{P}$ is (P,l)-transitive, then the group of central collineations with axis l and centre P acts transitively on the set of points not equal to $P, l \cap m'$ for any line $m' \neq l$ through P.

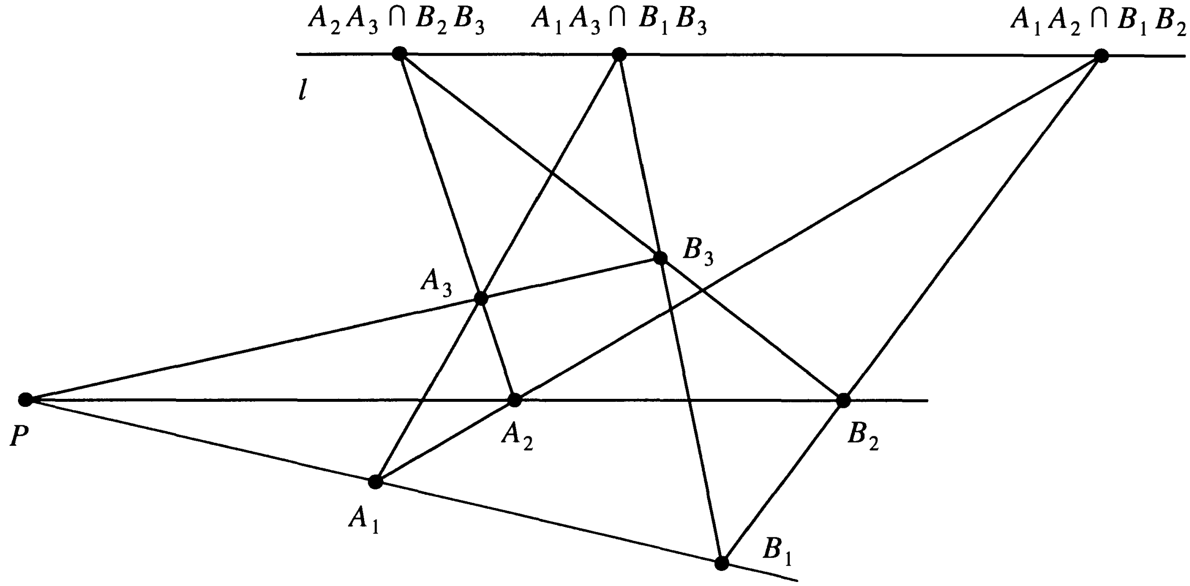

Figure 1. The Theorem of Desargues.

1.1. THEOREM (Baer [1942]). *A projective plane* $\boldsymbol{P}$ *satisfies the* (P, l)*-theorem of Desargues if and only if it is* (P, l)*-transitive. In particular,* $\boldsymbol{P}$ *is Desarguesian if and only if it is* (P, l)*-transitive for all point-line pairs* (P, l).

Given a vector space V of dimension 3 over a (not necessarily commutative) field K, we define $\boldsymbol{P}(V)$ to be the incidence structure whose points are the 1-dimensional subspaces of V and whose lines are the 2-dimensional subspaces of V, incidence being set-theoretical inclusion. It is easy to verify that $\boldsymbol{P}(V)$ is a projective plane, which we will denote by $\mathrm{PG}(2, K)$. If $K = \mathrm{GF}(q)$ is the finite field of order q, then $\boldsymbol{P}(V)$ has order q and we write also $\mathrm{PG}(2, q)$ instead of $\mathrm{PG}(2, K)$.

1.2. THEOREM. *The Desarguesian planes are precisely the planes* $\boldsymbol{P}(V)$, *where* V *is a* 3*-dimensional vector space over a (not necessarily commutative) field* K.

The history of this theorem does not seem to be quite clear; von Staudt has proved it in the real and complex case; later on, Hilbert proved it in his 'Grundlagen der Geometrie'.

In the last years there have been constructed many projective planes, mainly translation planes (cf. Chapter 5 of this Handbook). But there are very few 'proper' projective planes known to exist. A projective plane is called *proper*, if its group of collineations does not fix any line or any point. In other words, a projective plane is proper, if it is not sufficient to consider this plane as an affine plane, or a dual affine plane.

Of course, any Desarguesian plane is proper, since its collineation group acts transitively on the ordered quadrangles. The second class of proper finite projective planes are the *Hughes planes*, which have square order and are constructed using nearfields (see Hughes and Piper [1973]). Let us define these planes.

A *nearfield* is an algebraic structure F with addition and multiplication such that all axioms for a division ring are satisfied except possibly for the left distribution law

$$k(x+y) = kx + ky.$$

The set $K(F)$ of all $k \in F$ satisfying the above equation for all $x, y \in F$ is called the *kernel* of F, which is a division ring. If K is a subfield of the kernel, we may regard F as a vector space over K. Let us assume that K is a subfield of $K(F)$ such that F has vector space dimension 2 over K.

Consider the group $\mathrm{GL}(3, K)$ of all nonsingular 3×3 matrices with entries in K. For each $A = (a_{ij}) \in \mathrm{GL}(3, K)$ we define the map y_A from F^3 into F^3 by

$$y_A(x_1, x_2, x_3) := (a_{11}x_1 + a_{12}x_2 + a_{13}x_3,\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3,\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3).$$

Let Γ denote the group of all those y_A.

Now we are ready to define the incidence structure $H(F, K)$. The points of $H(F, K)$ are the sets

$$(x_1, x_2, x_3)F \ (= \{(x_1 f, x_2 f, x_3 f)\colon\ f \in F\})$$

for $(x_1, x_2, x_3) \in F^3$, $(x_1, x_2, x_3) \neq (0, 0, 0)$. In order to define the lines of $H(F, K)$ we first introduce the sets

$$L(f) := \{(x_1, x_2, x_3) \in F^3\colon\ (x_1, x_2, x_3) \neq (0,0,0) \text{ and}\\ x_1 + f x_2 + x_3 = 0\} \quad (f \in F).$$

The *lines* of $H(F, K)$ are then defined as the images of the sets $L(f)$ under the elements of the group Γ. One can easily show that any line consists of points. The *incidence* of $H(F, K)$ is set-theoretical inclusion.

A point of $H(F, K)$ is called *interior* if it can be written as $(x_1, x_2, x_3)F$ with $x_1, x_2, x_3 \in K$. Similarly, a line is *interior* if it is an image under an element of Γ of a set $L(k)$ with $k \in K$. The incidence structure induced by the interior points and lines is denoted by $H_0(F, K)$. Clearly, $H_0(F, K)$ is isomorphic to the Desarguesian plane $\boldsymbol{P}(K^3)$.

1.3. THEOREM (Dembowski [1971]). *If the nearfield F has dimension 2 over the subfield K of the kernel $K(F)$, then the incidence structure $H(F, K)$ is a projective plane. Moreover, $H_0(F, K)$ is a Desarguesian subplane of $H(F, K)$. Also, if F is a proper nearfield, then the full automorphism group of $H(F, K)$ leaves $H_0(F, K)$ invariant and is isomorphic to* $\mathrm{GL}(3, K)$.

The projective planes $H(F, K)$ are called *generalized Hughes planes*. For a discussion of the Hughes planes, see also Hughes and Piper [1973].

It was a sensation, when Figueroa [1982] constructed proper projective planes of cubic order q^3. His original construction worked only for prime powers q with $q \equiv 1 \pmod 3$. By a slight modification, Hering and Schaeffer [1982] were able to generalize the construction to any prime power q. Finally, Grundhöfer [1986] gave a very nice synthetic construction which also works in the infinite case. In this construction, the Figueroa planes are built from the points and lines of a Desarguesian plane, the incidence being (slightly) changed. Here, we will report on this construction.

Let $\boldsymbol{P}$ be a projective plane with a collineation α of order 3. Then the points of $\boldsymbol{P}$ fall into three classes:

$$P_1 = \{\mathrm{P}\colon\ \alpha(\mathrm{P}) = \mathrm{P}\},$$

$$P_2 = \{\mathrm{P}\colon\ \alpha(\mathrm{P}) \neq \mathrm{P} \text{ and } \mathrm{P}, \alpha(\mathrm{P}), \alpha^2(\mathrm{P}) \text{ collinear}\}, \text{ and}$$

$$P_3 = \{\mathrm{P}\colon\ \alpha(\mathrm{P}) \neq P \text{ and } \mathrm{P}, \alpha(\mathrm{P}), \alpha^2(\mathrm{P}) \text{ noncollinear}\}.$$

Dually, the lines fall into three disjoint classes L_1, L_2 and L_3:

$$L_1 = \{l\colon\ \alpha(l) = l\},$$

$$L_2 = \{l\colon\ \alpha(l) \neq l \text{ and } l, \alpha(l), \alpha^2(l) \text{ pass through a common point}\}, \text{ and}$$

$$L_3 = \{l\colon\ \alpha(l) \neq l \text{ and } l, \alpha(l), \alpha^2(l) \text{ do not pass through a common point}\}.$$

Let us define an involutory bijection $\mu\colon P_3 \to L_3$ by

$$\mu(\mathrm{P}) = \alpha(\mathrm{P})\alpha^2(\mathrm{P}) \quad (\mathrm{P} \in P_3), \text{ and}$$

$$\mu(l) = \alpha(l) \cap \alpha^2(l) \quad (l \in L_3).$$

Now we define the incidence structure $\boldsymbol{F} = \boldsymbol{F}(\alpha, \boldsymbol{P}) = (P, L, \mathrm{I}^*)$ as follows: $\boldsymbol{F}$ has the same points and the same lines as $\boldsymbol{P}$; the *incidence* of $\boldsymbol{F}$ is in nearly all cases the same as in $\boldsymbol{P}$, the exception being the incidence between points of P_3 and lines of L_3. In this case, the incidence of $\boldsymbol{P}$ is a little bit changed. More precisely, we have the following rule:

$$P\,\mathrm{I}^*\,l \Leftrightarrow \begin{cases} P\,\mathrm{I}\,l & \text{if } P \notin P_3 \text{ or } l \in L_3, \\ \mu(l)\,\mathrm{I}\,\mu(P) & \text{otherwise.} \end{cases}$$

1.4. THEOREM (Grundhöfer [1986]). *Let $\boldsymbol{P}$ be a projective plane with a collineation α of order* 3.

(a) *If $\boldsymbol{P}$ is Pappian, then $\boldsymbol{F} = \boldsymbol{F}(\alpha, \boldsymbol{P})$ is a projective plane.*

(b) *Let $\boldsymbol{P}$ be Pappian. If the order of $\boldsymbol{P}$ is greater than* 8 *and α is planar* (*i.e. the structure of its fixed points and lines is a subplane*), *then $\boldsymbol{F}$ is a non-Desarguesian projective plane, the so-called* Figueroa plane.

If α is not planar, then one gets nothing new, since in this case $\boldsymbol{F}$ is isomorphic to $\boldsymbol{P}$.

Let $\boldsymbol{P}_0$ be the subplane of a Figueroa plane $\boldsymbol{F}$ fixed by α. Then $\boldsymbol{P}_0$ is a Pappian plane. Note that the incidence in $\boldsymbol{F}$ is in particular the same as in $\boldsymbol{P}$ for

– all points and lines of $\boldsymbol{P}_0$,
– all points on lines of $\boldsymbol{P}_0$,
– all lines through points of $\boldsymbol{P}_0$.

If $\boldsymbol{P}_0$ is finite of order q, then $\boldsymbol{F}$ has order q_3.

Algebraic constructions of the Figueroa planes can be found in Brown [1983], Dempwolff [1984]. The full collineation groups of the Figueroa planes have been determined in Dempwolff [1985] and Hering and Schaeffer [1982]; they are semidirect products $A \times \langle\alpha\rangle$, where A is the collineation group of P_0.

Quite a few people attempted to generalize Grundhöfer's construction in order to 'extend' PG$(2, q)$ to non-Desarguesian planes of order q^h, $h \geqslant 3$. It was shown by Brown [1992] that the 'obvious way' of doing this works only in the case $h = 3$, that is in the case of the Figueroa planes. More precisely, Brown has proved the following theorem.

1.5. THEOREM. *Let F be a field with an automorphism α^* of odd order h. Let α be the planar automorphism induced by α on the Pappian plane coordinatized by F. Suppose that any point orbit of $\langle\alpha\rangle$ that consists of more than one point consists of collinear points or of h points no three of which are collinear* (*type* III). *Dually suppose that any line orbit with more than one element consists of lines through a common point or of h lines no three of which pass through a common point* (*type* III). *Define a bijection f between type* III *points and type* III *lines by*

$$\mu(P) = \alpha^{(h-1)/2}(P)\alpha^{(h+1)/2}(P) \quad \textit{and} \quad \mu(l) = \alpha^{(h-1)/2}(l)\alpha^{(h+1)/2(l)}.$$

Define an incidence structure as explained before Theorem 1.4. *Then this incidence structure is a projective plane if and only if $h = 3$.*

2. Characterizations

2.1. *Collineation groups*

Classical structures usually have the remarkable property of having many automorphisms. In the case of projective planes one knows that the full collineation group of a Desarguesian projective plane acts transitively on the set of ordered quadrangles. Conversely, one would like to characterize the finite Desarguesian projective planes by the property that they have a highly transitive group of collineations. (Note that in the infinite case transitivity on quadrangles does not imply any structural property, cf. Kegel and Schleiermacher [1973].) In this section we will report on the progress of this problem. We begin with the famous result of Ostrom and Wagner, which is one of the most beautiful and important results concerning finite projective planes.

2.1.1. THEOREM (Ostrom and Wagner [1959]). *Let $\boldsymbol{P}$ be a finite projective plane which a group of collineations acting transitively on pairs of distinct points. Then $\boldsymbol{P}$ is Desarguesian.*

Actually it has been conjectured that a finite projective plane having a group of collineations acting just transitively on the points must be Desarguesian. This conjecture is yet unproven. A result which strongly supports this conjecture is the following theorem of Wagner.

2.1.2. THEOREM (Wagner [1959]). *Let $\boldsymbol{P}$ be a finite projective plane and let Π be a group of collineations of $\boldsymbol{P}$. If Π acts transitively on the points of $\boldsymbol{P}$ and contains a nontrivial central collineation, then $\boldsymbol{P}$ is Desarguesian.*

Now we turn to special types of collineation groups, namely those acting transitively on the flags and Singer groups.

The above mentioned conjecture has been weakened to flag-transitive groups (Dembowski [1968], pp. 208–214). Using the classification of finite simple groups, Kantor [1987] could prove the following deep theorem.

2.1.3. THEOREM (Kantor [1987]). *Let $\boldsymbol{P}$ be a finite projective plane of order q, and let Π be a collineation group which acts transitively on the incident point-line pairs (the 'flags'). Then either*

- *$\boldsymbol{P}$ is Desarguesian, or*
- *Π is a Frobenius group of order $(q^2+q+1)(q+1)$, and q^2+q+1 is a prime.*

Note that no example of the second case is known. Actually, it turns out that the weaker condition that Π acts primitively on points (see, e.g., Ott [1977], Lemma 1) is sufficient.

2.1.4. THEOREM (Kantor [1987]). *Let $\boldsymbol{P}$ be a finite projective plane of order q, and let Π be a collineation group which acts primitively on the set of points. Then either*

- *$\boldsymbol{P}$ is Desarguesian, or*
- *Π is regular or a Frobenius group of order dividing $(q^2+q+1)(q+1)$ or q^2+q+1, and q^2+q+1 ~~is~~ a prime.*

A group of collineations of a projective plane $\boldsymbol{P}$ is called a *Singer group* if it acts regularly (namely, it is sharply transitive) on the points of $\boldsymbol{P}$. The name stems from the famous theorem of Singer [1938] which says that any finite Desarguesian projective plane (actually, every projective space) has a cyclic Singer group. It has been conjectured that any finite projective plane with a Singer group is Desarguesian. An important step in proving this conjecture is

2.1.5. THEOREM (Ott [1975]). *If a finite projective plane $\boldsymbol{P}$ has at least two cyclic Singer groups, then $\boldsymbol{P}$ is Desarguesian.*

The only known examples of Singer groups are cyclic or non-Abelian. It is known (cf. Ellers and Karzel [1963]) that any Abelian, noncyclic Singer group leads to a non-Desarguesian plane. There are few restrictions on the possible order of planes having Abelian Singer group.

2.1.6. THEOREM. *Let $\boldsymbol{P}$ be a finite projective plane of order q having an Abelian Singer group.*

(a) *If q is even, then $q = 2$, $q = 4$ or q is a multiple of* 8.
(b) *If q is divisible by* 3, *then $q = 3$ or q is divisible by* 9.

Part (a) is due to Jungnickel and Vedder [1984], part (b) has been proved by Wilbrink [1985].

Singer groups (mainly under the name 'planar difference sets') have been studied thoroughly during the last years. (For surveys on difference sets, cf. Hall [1975a] and Arasu [1988].)

Result 2.1.3 cannot be extended to infinite planes.

2.1.7. THEOREM (Karzel [1964]). *If an infinite projective plane $\boldsymbol{P}$ possesses a cyclic Singer group, then $\boldsymbol{P}$ is not Desarguesian.*

Finally, we shall deal with a common characterization of finite Desarguesian planes and the generalized Hughes planes.

2.1.8. THEOREM (Lüneburg [1976]). *Let $\boldsymbol{P}$ be a finite projective plane of order q. Then the following assertions are equivalent.*

(a) *$\boldsymbol{P}$ is a Desarguesian projective plane or a generalized Hughes plane.*
(b) *$\boldsymbol{P}$ contains a Baer subplane $\boldsymbol{B}$ (i.e. a subplane of order $\sqrt{q}$) with the property that for each line l of $\boldsymbol{B}$, there are exactly q elations of $\boldsymbol{B}$ (induced by elations of $\boldsymbol{P}$) with axis l.*
(c) *$\boldsymbol{P}$ has a proper subplane $\boldsymbol{B}$ such that the subgroup of the full collineation group of $\boldsymbol{P}$ which leaves $\boldsymbol{B}$ invariant operates flag-transitively on $\boldsymbol{P} - \boldsymbol{B}$.*
(d) *$\boldsymbol{P}$ contains a Baer subplane $\boldsymbol{B}$ with the property that the subgroup of the full collineation group of $\boldsymbol{P}$ which leaves $\boldsymbol{B}$ invariant is doubly transitive on the set of points of $\boldsymbol{B}$.*

2.2. *Projectivities*

Given two lines l_1, l_2 of a projective plane $\boldsymbol{P}$ and a point X outside l_1 and l_2, the perspectivity with centre X from l_1 onto l_2 is defined by

$$\pi\colon\ P \to PX \cap l_2 \quad (P \mathrm{I} l_1).$$

A *projectivity* is any product of perspectivities. Of particular interest is the set Π_l of all projectivities from the line l onto itself. We note some facts about Π_l.

– Π_l forms a group of permutations with respect to composition.

– Π_l is an invariant of the plane $\boldsymbol{P}$; i.e. for any two lines l and l' of $\boldsymbol{P}$, the groups Π_l and $\Pi_{l'}$ are isomorphic (even as permutation groups). Therefore, Π_l is called *the group of projectivities of* $\boldsymbol{P}$.

– Π_l acts triply transitively on the points of l.

The 'fundamental theorem of projective geometry' reads as follows.

2.2.1. THEOREM. *A projective plane* $\boldsymbol{P}$ *is Pappian if and only if its group of projectivities* Π_l *acts sharply triply transitively on the points of* l.

In other words, a plane is Pappian if its group of projectivities is as small as possible: only the identity fixes three points of l. This theorem goes back to von Staudt [1847], who proved it for the plane over the reals. For a history of this theorem, see Pickert [1981].

Barlotti [1964] introduced the conditions (Pn) in a projective plane $\boldsymbol{P}$:

(Pn) If a projectivity in Π_l fixes n points of l, then it is the identity.

So, the fundamental theorem can be reformulated as '(P3) implies the theorem of Pappus'. Barlotti [1964] also showed that condition (P6) is valid in any 'free plane'. This means, loosely speaking, that (P6) alone does not imply any interesting theorem. But, as a surprise, the following theorem holds.

2.2.2. THEOREM (Schleiermacher [1967]). *If a projective plane satisfies* (P5), *it is Pappian.*

For Desarguesian planes one has the following result.

2.2.3. THEOREM. *The group* Π_l *of projectivities of a Desarguesian plane over a skewfield* K *is isomorphic to* PGL$(2, K)$ *in its natural representation on the projective line over* K.

This theorem relies on the fact that any projectivity of a Desarguesian plane is induced by a central collineation; conversely any central collineation fixing a line l induces a projectivity of l. Thus Π_l is isomorphic to the stabilizer PGL$(3, K)_l$ of the line l, hence $\Pi_l \simeq$ PGL$(2, K)$. (Cf. also Grundhöfer [1988].)

By the classification of all finite simple groups one knows all finite 3-transitive groups. Using this, Grundhöfer proved the following spectacular theorem which says that *natura facit saltus*.

2.2.4. THEOREM (Grundhöfer [1988]). *If* $\boldsymbol{P}$ *is a non-Desarguesian finite projective plane of order* $q \neq 23$, *then* Π_l *is the alternating or the symmetric group on* $q + 1$ *symbols.*

So, the group of projectivities is a very bad invariant for finite projective planes. But in some cases, one could at least decide whether the group of projectivities is symmetric or alternating. For instance, in the finite non-Desarguesian André planes of even order,

Π_l is alternating (Herzer [1974]). There are also planes of even order with $\Pi_l = S_{q+1}$; Kilmer [1989] has constructed such planes of orders 16, 32 and 64.

2.3. *Dualities*

A famous theorem due to Baer [1946] says that a projective plane is Desarguesian if and only if it has 'all possible' central collineations. (Cf. Section 2.1.) Then the theorem of Baer reads as follows.

2.3.1. THEOREM (Baer [1946]). *In projective plane* $\boldsymbol{P}$ *the theorem of Desargues is universally valid if and only if* $\boldsymbol{P}$ *is* (P, l)*-transitive for any point-line pair* (P, l).

This is in the usual approach the first step in the proof of the first structure theorem (our Theorem 1.2): Any Desarguesian projective plane can be coordinatized over a (not necessarily commutative) field. Remember that the Pappian planes are exactly those planes which can be coordinatized over a commutative field.

In an interesting paper (Herzer [1972]), A. Herzer has obtained an analogue to the theorem of Baer for Pappian planes. In order to state his results, we have to introduce some notions.

Let us first recall the theorem of Pappus. Let l and l' be any two lines in a projective plane $\boldsymbol{P}$ and denote by P_1, P_2, P_3 three points on l and by P_1', P_2', P_3' three points on l'. We say that the *Theorem of Pappus* holds in $\boldsymbol{P}$ provided the points $P_1P_2' \cap P_1'P_2$, $P_2P_3' \cap P_2'P_3$ and $P_1P_3' \cap P_1'P_3$ are collinear (cf. Figure 2).

A *duality* of $\boldsymbol{P}$ is a bijective map δ from the set of points of $\boldsymbol{P}$ onto the set of lines of $\boldsymbol{P}$ which preserves incidence, that is which satisfies

$$\delta(PQ) = \delta(P) \cap \delta(Q) \quad \text{for all points } P, Q \text{ of } \boldsymbol{P}.$$

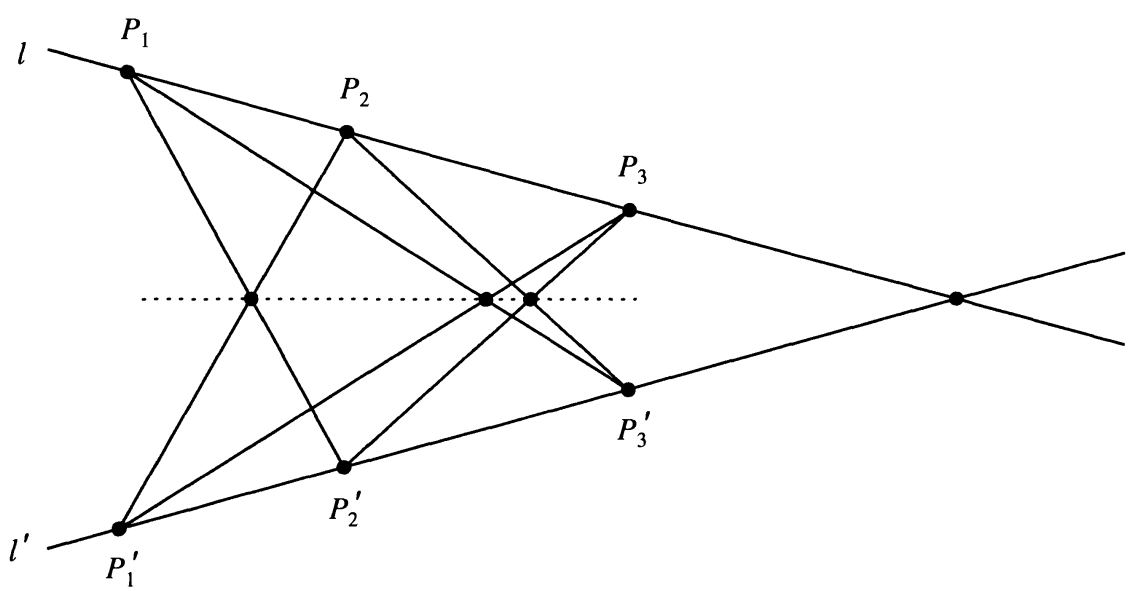

Figure 2. The Theorem of Pappus.

A point P (or a line l) is called *absolute* (with respect to the duality δ) if P lies on $\delta(P)$ (or l passes through $\delta(l)$, respectively).

Let P_1, P_2 be two points and l_1, l_2 be two lines of the projective plane $\boldsymbol{P}$. We call (P_1, P_2, l_1, l_2) an *H-configuration* (Herzer calls it 'Gestell'), if

– P_1, P_2 and $l_1 \cap l_2$ are distinct collinear points, and

– l_1, l_2 and P_1P_2 are distinct confluent lines (cf. Figure 3).

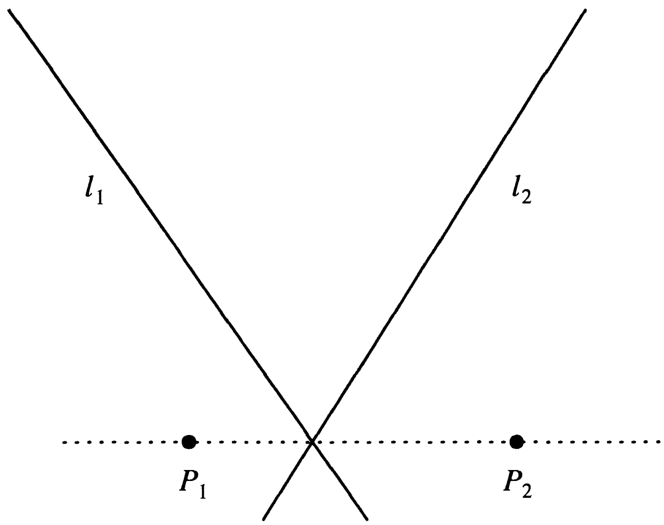

Figure 3. H-duality.

A duality δ is called an *H-duality* with respect to an H-configuration (P_1, P_2, l_1, l_2) if

– every line through P_i is an absolute line ($i = 1, 2$), and

– every point on l_i is an absolute point ($i = 1, 2$).

It is easy to show that there is at most one H-duality to any given H-configuration (P_1, P_2, l_1, l_2). In fact, one can prove more: δ maps P_i on l_i and interchanges the points $X \neq l_1 \cap l_2$ on l_i with XP_i. From this, the uniqueness follows easily.

The main theorem of Herzer reads as follows.

2.3.2. THEOREM (Herzer [1972]). *A projective plane $\boldsymbol{P}$ is Pappian if and only if there exists an H-duality with respect to any H-configuration of $\boldsymbol{P}$.*

One interesting feature of this theorem is that it allows an extremely neat proof of Hessenberg's theorem.

2.3.3. THEOREM (Hessenberg [1905]). *If in a projective plane $\boldsymbol{P}$ the theorem of Pappus holds, then also the theorem of Desargues holds.*

PROOF. For the proof one first uses the fact that all H-dualities exist in $\boldsymbol{P}$. If δ is an H-duality with respect to (P_1, P_2, l_1, l_2) and δ is an H-duality with respect to (P_1, P_2', l_1, l_2), then $\delta \cdot \delta'$ is a collineation of $\boldsymbol{P}$ which maps P_2 onto P_2', has P_1 as its centre and l_1 as its axis. Thus, $\boldsymbol{P}$ is (P, l)-transitive for any nonincident point-line pair (P, l). From this it follows (cf., e.g., Pickert [1975], p. 68) that $\boldsymbol{P}$ is (P, l)-transitive for any point-line pair (P, l), and then Baer's theorem shows that $\boldsymbol{P}$ is Desarguesian. □

There is the famous converse of Hessenberg's theorem: *any finite Desarguesian projective plane is Pappian.* This is traditionally proved by applying Wedderburn's theorem that any finite division ring is commutative. Recently, Tecklenburg [1987] gave a purely geometrical proof of this fact.

There is also a nice characterization of finite Desarguesian projective planes using polarities. A *polarity* is a duality π satisfying $\pi \cdot \pi = \mathrm{id}$. Any polarity of a projective plane of order q has at least $q+1$ absolute points (see Theorem 3.2.1 below). If it has exactly $q+1$ absolute points, it is called *orthogonal.* The absolute points of an orthogonal polarity form an oval (i.e. a set of $q+1$ points, no three of which are collinear) if q is odd and a line if q is even. If $\boldsymbol{P}$ is a Desarguesian projective plane of odd order q, then, by Segre's theorem, every oval is a conic. So for any 'pentagon' (five points, no three of which are collinear) there exists a unique orthogonal polarity having these points as absolute points. Also the converse is true.

2.3.4. THEOREM (Seib [1973]). *Let $\boldsymbol{P}$ be a projective plane of odd order $q \geqslant 5$. Then $\boldsymbol{P}$ is Desarguesian if and only if for every pentagon $\{P_1, \ldots, P_5\}$ there exists precisely one orthogonal polarity of $\boldsymbol{P}$ having $P_1, \ldots, P_5$ as absolute points.*

The nontrivial part of the proof consists in first showing that $\boldsymbol{P}$ has a group of collineations transitive on the points of $\boldsymbol{P}$ and then applying Theorem 2.1.2.

3. Combinatorics

3.1. $\{k;n\}$-*arcs*

Arcs in finite projective planes have mainly been investigated by the Italian school of Segre, Barlotti, Tallini and their students.

A $\{k;n\}$-arc in a projective plane $\boldsymbol{P}$ of order q is a nonempty set K of k points, with the property that n is the maximum number of points of K which are collinear. A $\{k;2\}$-arc is simply called a k-arc.

3.1.1. THEOREM (Barlotti [1956]). *Let K be a $\{k;n\}$-arc in a projective plane $\boldsymbol{P}$ of order q. Then*

(a) $k \leqslant (q+1)(n-1)+1$.

(b) *Suppose that $k = (q+1)(n-1)+1$. Then any line of $\boldsymbol{P}$ contains either* 0 *or n points of K. In particular, q is a multiple of n if n is less than $q+1$, i.e. if K is not the complete plane.*

(c) *Let $n = 2$. Then $k \leqslant q+2$ and $k \leqslant q+1$, if q is odd.*

The proof is straightforward. Any line through a point P of K contains at most $n-1$ points of $K - \{P\}$. Since these lines cover all points of K, (a) is proved.

(b) The above argument shows that in case of equality, every line through a point P of K has exactly n points of K. So, the first assertion of (b) is proved. Consider now a point Q outside K. Since any line through Q contains either 0 or exactly n points of K, k must be a multiple of n. Since $k = (q+1)n - q$, also q must be a multiple of n.

(c) is a special case of (b). □

A $\{k; n\}$-arc is called *maximal*, if $k = (q+1)(n-1)+1$. A $(q+1)$-arc is called an *oval* and a $(q+2)$-arc a *hyperoval*. The $\{k; n\}$-arc K is called *complete*, if it is not contained in another $\{k'; n\}$-arc.

The question, which planes admit maximal arcs has attracted much attention, but it is still one of the big unsolved problems in finite geometry. First we shall present a positive result.

3.1.2. THEOREM (Denniston [1971]). *Let $\boldsymbol{P} = PG(d, q)$ be a Desarguesian projective plane. If the order q of $\boldsymbol{P}$ is even, then for any divisor n of q, $\boldsymbol{P}$ contains a maximal $\{k; n\}$-arc.*

PROOF is surprisingly simple. Consider a corresponding affine plane on the point set $\{(x, y)\colon x, y \in \mathrm{GF}(q)\}$. Choose an nondegenerate quadratic form Φ over GF(q). Since n divides q, there is a subgroup H of order n of the additive group of GF(q). Then $K := \{(x, y)\colon \Phi(x, y) \in H\}$ is the desired maximal arc. □

Thas [1974] has constructed maximal arcs in translation planes; in particular he proved that the Lüneburg plane of order $q = 2^{2m}$ contains a maximal $\{k; 2m\}$-arc.

In Thas [1975], Desarguesian planes of order $q = 3^h$ with $h > 1$ are studied; it is shown that these planes do not contain maximal $\{k; 3\}$-arcs, and therefore also no maximal $\{k; q/3\}$-arcs. (Actually, it is shown that $k \leqslant 2q + 1$.) As a special case it follows a theorem of Cossu [1961] which says that the Desarguesian plane of order 9 does not contain a (maximal) $\{21; 3\}$-arc. In the above mentioned paper, Thas states the following

3.1.3. CONJECTURE (Thas [1975]). *In $\boldsymbol{P} = \mathrm{PG}(2, q)$, q odd the only maximal $\{k; n\}$-arcs are trivial, i.e. $n = 1$ (one point) or $n = q + 1$ (all points) or $n = q$ (all points outside one line).*

3.2. *Ovals and hyperovals*

In recent years there has been considerable progress to determine ovals and hyperovals in finite Desarguesian planes.

Let $\boldsymbol{P}$ be an arbitrary projective plane. An *oval* of $\boldsymbol{P}$ is a nonempty set O of points no three of which are collinear such that through any point of O there is precisely one tangent. A *hyperoval* of $\boldsymbol{P}$ is a nonempty set H of points such that any line intersects H in 0 or exactly 2 points. In a finite projective plane of order q an oval can also be described as a $(q+1)$-gon and a hyperoval is a $(q+2)$-gon.

The following facts are well known and easy to prove (see, e.g., Hirschfeld [1979]). Let O be an oval in a finite projective plane $\boldsymbol{P}$ of order q. Then through any point of O is on exactly one tangent. If q is odd, then any point of $\boldsymbol{P}$ outside O is either on 0 or on 2 tangents. If q is even, then all tangents pass through a common point, the *nucleus* (or *knot*) of O. In particular any oval in a projective plane of even order can uniquely be extended to a hyperoval (theorem of Qvist [1952]).

Examples of ovals in Desarguesian planes are provided by nondegenerate quadrics (*conics*). The famous theorem of B. Segre states that in finite projective planes of odd order there are no other ovals.

3.2.1. THEOREM (Segre [1955]). *Any oval in* PG$(2, q)$ *with q odd is a conic.*

It is worth noting that Buchanan proved a similar result for the complex plane.

3.2.2. THEOREM (Buchanan [1979]). *Every closed oval in the projective plane over the complex numbers is a conic.*

The conclusion of Segre's theorem does not remain true if the plane has even order. Therefore, research has very much concentrated on planes of even order. Since any oval in a plane of even order can uniquely be extended to a hyperoval, one is particularly interested in determining all hyperovals.

Let $\boldsymbol{P}$ be a finite Desarguesian plane of order $q = 2^h$. Any hyperoval Ω has at least four points, and we may assume (by the Fundamental Theorem of Projective Geometry, see Chapter 2, Section 3) that it passes through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$. Therefore, any hyperoval is completely defined by its affine points $(x, y, 1)$. We define the permutation polynomial f corresponding to Ω by the following rule

$$y = f(x) \quad \text{if and only if} \quad (x, y, 1) \quad \text{is a point of } \Omega.$$

It can easily be checked that f is indeed a permutation polynomial. Permutation polynomials that give rise to a hyperoval are sometimes called *o-polynomials*. Most results have been expressed in the language of o-polynomials. We give some examples.

– $f(x) = x^2$ is always an o-polynomial; the corresponding hyperoval is a conic together with its nucleus.

– $f(x) = x^{2^i}$ is an o-polynomial if and only if i and h are relatively prime (Segre [1957]). The corresponding hyperovals admit a group of translations transitive on the affine points of the hyperoval; therefore these hyperovals are called *translation hyperovals*.

– Segre [1962] showed that x^6 is an o-polynomial if h is odd.

– Glynn [1982] proved that $x^{\delta+\lambda}$ and $x^{3\delta+4}$ are o-polynomials for h odd, where $\lambda^4 \equiv \sigma^2 \equiv 2 \bmod (2^h - 1)$.

– The only other infinite family of hyperovals was discovered by Payne [1985]. He showed that for h odd the polynomial

$$f(x) = x^{1/6} + x^{3/6} + x^{5/6},$$

where the exponents are considered modulo $2^h - 1$, is an o-polynomial.

– A possible family of o-polynomials was found by Cherowitzo [1988]. For odd h the polynomial is given by

$$f(x) = x^{\sigma} + x^{\sigma+2} + x^{3\sigma+4},$$

where $\sigma^2 \equiv 2 \bmod (2^h - 1)$; for $h \in \{3, 5, 7, \ldots, 15\}$ it has been verified by computer that $f(x)$ is in fact an o-polynomial.

Finally we shall review what is known on hyperovals in small Desarguesian planes. The hyperovals in the Desarguesian planes of orders 2, 4, and 8 all consist of a conic plus its nucleus (see Hirschfeld [1979]).

In PG(2, 16) there are only two classes of hyperovals, the conics plus nucleus and the so-called *Lunelli–Sce hyperovals*. (See Lunelli and Sce [1958], who constructed the new class, Hall [1975b], who proved that there are only two classes and O'Keefe and Pentilla [1991], who proved this without a computer.)

In PG(2, 32) there are exactly six classes of hyperovals; see O'Keefe and Pentilla [1992] and Pentilla and Royle [1992].

In PG(2, 64) we know at present four families of hyperovals, the complete conics, two hyperovals found by Pentilla and Pinneri [1992] and one found by Pentilla and Royle [1992]. Thus a conjecture of Segre that PG(2, 64) contains hyperovals that do not arise from conics is settled in the affirmative.

For excellent surveys on ovals and hyperovals in finite projective planes, see Cherowitzo [1992] and Korchmáros [1991].

3.3. *Arcs*

The main question concerning arcs in a projective plane $\boldsymbol{P}$ is: For which values k does there exist a complete k-arc in $\boldsymbol{P}$? This question seems to be very ambitious, even in the case where $\boldsymbol{P}$ is Desarguesian. We shall concentrate on the following question: What can be said about the maximum number k of points in a complete k-arc with $k < q+1$ in a Desarguesian projective plane of order q? Here, B. Segre did quite a lot of pioneering work.

3.3.1. THEOREM (Segre [1967]). *Let $\boldsymbol{P}$ be a Desarguesian projective plane of order q, and let K be a complete k-arc with $k < q + 1$ in $\boldsymbol{P}$.*

(a) *If q is even, then $k \leqslant q - \sqrt{q} + 1$.*

(b) *If q is odd and $q > 49$, then $k \leqslant q - \sqrt{q}/4 + 7/4$.*

This theorem was proved by Segre using the Hasse–Weil theorem. Thas [1983] proved the even part without the deep machinery of algebraic geometry. Thas [1987] improved the constant in part (b) to 25/16 and was also able to prove the result for q odd without using the Hasse–Weil bound.

Part (b) can be reformulated using the celebrated theorem of Segre.

3.3.2. THEOREM (Segre [1954]). *Let K be a $(q + 1)$-arc in a Desarguesian projective plane of order q. If q is odd, then K is a conic.*

As a corollary we have

3.3.3. THEOREM. *Let K be a k-arc in a Desarguesian projective plane of odd order q. If $k > q - \sqrt{q}/4 + 7/4$ and $q > 49$, then K is contained in a conic.*

If $k = q$, then the conclusion of Theorem 3.3.3 is true without any further hypothesis.

3.3.4. THEOREM (Segre [1967]). *Any q-arc in a Desarguesian projective plane of odd order q is contained in a conic.*

These are important results, but the question remains whether the bounds of Theorem 3.3.1 are best possible. This question has been answered for q even in the affirmative.

3.3.5. THEOREM. *In any Desarguesian projective plane $\boldsymbol{P}$ of order q^2 (in fact in any plane having a cyclic Singer group, and of order q^2) there exist complete $(q^2 - q + 1)$-arcs.*

These arcs were first constructed by Kestenband [1981]; later on, Fisher, Hirschfeld and Thas [1986] and Boros and Szönyi [1986] rediscovered these arcs and noticed that they are complete. (They also showed that a Hermitian curve in PG$(2, q^2)$ is the union of $q + 1$ of these arcs.)

These arcs are amazingly simple to construct. Take a cyclic Singer group G of $\boldsymbol{P}$ which is a cyclic group of collineations of order $q^4 + q^2 + 1$ acting transitively on the points. Since $q^4 + q^2 + 1 = (q^2 - q + 1)(q^2 + q + 1)$, G has a subgroup A of order $q^2 - q + 1$. Any orbit of A is a complete $(q^2 - q + 1)$-arc.

Finally, we would like to mention two important results due to T. Szönyi, who constructed 'small' complete arcs guided by an idea of B. Segre and L. Lombardo-Radice: 'the points of the k-arc should be chosen, with some exceptions, among the points of a conic or a high-order algebraic curve'. In the following theorems Szönyi has for the first time constructed complete k-arcs for which asymptotically k is not of the order of q.

3.3.6. THEOREM (Szönyi [1985]). *Let n be an integer and $p \neq 3$ be a prime. If $q = p^{10n} \geqslant 7^{10}$, then there exists a complete k-arc in* PG$(2, q)$ *with $k \leqslant 2 \cdot q^{9/10}$.*

3.3.7. THEOREM (Szönyi [1987]). *Define the set A of rational numbers by*

$$A := \{k/q\colon \text{ there exists a complete } k\text{-arc in the projective plane } \mathrm{PG}(2, q)\}.$$

Then the set $A \cap [0, 1/2]$ is dense in $[0, 1/2]$.

3.4. *Unitals*

In the Desarguesian projective plane $\boldsymbol{P}$ over the field GF(q^2), any Hermitian curve (i.e. a set of points with homogeneous coordinates (x, y, z) satisfying an equation $xx^q + yy^q + zz^q = 0$) has exactly $q^3 + 1$ points and has the property that any line of $\boldsymbol{P}$ intersects it either in 1 or in $q+1$ points. In other words, the points of a Hermitian curve together with the nontangents of $\boldsymbol{P}$ form a 2-$(q^3+1, q+1, 1)$ design (or an $S(2, q+1, q^3+1)$ Steiner system). Conversely any design with these parameters is called a *unital* of *order* q. Another way of describing the above construction is the following: The set of absolute points and the set of nonabsolute lines of a unitary polarity in the plane $\boldsymbol{P}$ over GF(q^2) is a unital. These unitals are often called *classical*. In this connection the following theorem deserves some interest.

3.4.1. THEOREM (Baer [1946], Seib [1970]). *Let Π be a polarity in a finite projective plane of order q. Then the number $a(\Pi)$ of absolute points of Π satisfies*

$$q+1 \leqslant a(\Pi) \leqslant q\sqrt{q}+1.$$

Equality holds on the left-hand side if and only if Π is orthogonal (*which means that the absolute points of Π form a line if q is even or an oval if q is odd*); *equality holds on the right-hand side if and only if Π is* unitary, *i.e. if the absolute points of Π form a unital.*

The characterization of orthogonal polarities is due to Baer [1946], whereas the other part has been proved by Seib [1970]. Seib [1970] and Ganley [1972] have constructed unitary polarities in non-Desarguesian planes. Ganley has also constructed a class of planes which have three types of polarities: orthogonal, unitary, and a new type having $q^{5/4}+1$ absolute points.

Apart from the classical unitals and the Ree unitals (cf. Dembowski [1968]), the following unitals obtained by Buekenhout [1976] and Metz [1979] have proved to be very interesting. They are constructed as follows.

In order to construct these unitals, we represent PG$(2, q^2)$ in a 4-dimensional projective space. This representation is due to André [1954]; it was rediscovered by Bruck and Bose [1964] and Segre [1964].

Let Π = PG$(4, q)$ be the 4-dimensional projective space of order q, let H be a hyperplane of Π, and consider a spread S of H. (A *spread* is a set of skew lines which cover the points of H; simple counting yields that a spread of H consists of exactly q^2+1 lines. A spread S of H is called *regular* if for any line $m \notin S$ of H, the set of lines of S meeting m forms a regulus. (Cf. Dembowski [1968], p. 29.)

We define the incidence structure $\boldsymbol{P}(S)$ as follows. The *points* of $\boldsymbol{P}(S)$ are the points of $\Pi - H$ and the elements of the spread S; its *lines* are the planes of $\Pi - H$ that intersect H in an element of S along with the 'special line' H; *incidence* is set-theoretical inclusion.

3.4.2. THEOREM (André [1954]). *$\boldsymbol{P}(S)$ is a projective plane*; *it is Desarguesian if and only if S is regular.*

The planes $\boldsymbol{P}(S)$ always are translation planes. This construction (indeed in a more general variation) is of great importance in the theory of translation planes.

Let Π, H and S be as above. Suppose furthermore that $q \geqslant 3$. Fix a line $s \in S$ and consider a point $P \in s$. Then there exists an *ovoid* O in the quotient geometry Π/P (i.e. a set of q^2+1 lines through P, no three of which are coplanar) such that

– $s \in O$, and

– no line of $O - \{s\}$ is in H (i.e. H is a tangent plane in Π/P of O).

3.4.3. THEOREM (Buekenhout [1976], Metz [1979]). *Let S be an arbitrary spread of H, and denote by U' the set of points on the lines of O outside H. Then $U := U' \cup \{s\}$ is a unital. If S is regular (i.e. $\boldsymbol{P}(S)$ is Desarguesian) one can choose O in such a way that U is not a Hermitian unital.*

The unitals constructed in the above theorem are called *Buekenhout–Metz unitals*.

A question which has attracted much interest is the following: which unitals of order q (i.e. which 2-$(q^3+1, q+1, 1)$ designs) are embeddable in a projective plane $\boldsymbol{P}$ of order q^2? By *embeddable* we mean that the unital consist of some points and some lines of $\boldsymbol{P}$, the incidence being induced by $\boldsymbol{P}$. By definition, any classical unital of order q is embeddable in a (Desarguesian) projective plane of order q^2. For a general discussion of embeddings, see Section 5.

Piper [1979] has posed three questions.

– Are there unitals of order q not embeddable in a projective plane of order q^2?

– Are there unitals of order q which are embeddable in a projective plane of order q^2, but are not the set of absolute points of a polarity?

– Are there unitals of order q which can be embedded in two different projective planes of order q^2?

Brouwer [1981] showed that the answer to all these questions is 'yes'. More precisely, he constructed by computer a lot of 2-(28,4,1) designs (hence unitals of order 3) which provide counterexamples to a too naive conjecture.

The third question was again considered by Grüning. He proved the following theorem.

3.4.4. THEOREM (Grüning [1987]). *For any prime-power $q \geqslant 3$ there exists a unital of order q which is embeddable in the Hall plane of order q^2 and its dual (which are not isomorphic!). Brouwer's examples are the first instance of this series.*

The unitals of the above theorem are similar to the Buekenhout–Metz unitals; one could call them 'parabolic' Buekenhout–Metz unitals, as they intersect the line at infinity in $q+1$ points.

One also has studied embeddings of the complement of a unital in a projective plane. This problem is a lot easier, since one has more points at disposal. We mention

3.4.5. THEOREM (Beutelspacher [1984]). *Let $\boldsymbol{S}$ be an incidence structure consisting of points and lines with the following properties.*

– Any two distinct points are on a unique common line.

– Any line has m^2 or $m^2 - m$ points (where m is a positive integer).

– $\boldsymbol{S}$ has exactly $m^4 + m^2 + 1$ lines.

Then $\boldsymbol{S}$ is the complement of a unital or the complement of a Baer subplane in a projective plane of order m^2, or $\boldsymbol{S}$ is the complete graph K_7 on 7 vertices.

Given an 'abstract unital', i.e. a 2-$(q^3+1, q+1, 1)$ design, one may wonder what are the possible numbers q (the possible *orders of a unital*). For a long time, one knew only examples, where q is a prime power. Yet, there is another example, a 2-$(6^3+1, 6+1, 1)$ design constructed by Mathon [1987], cf. also Bagchi and Bagchi [1989], hence a unital of order 6.

3.5. *Nuclei*

Let us mention a beautiful theorem which has attracted much attention.

In a projective plane $\boldsymbol{P}$, let B be a set of points. A point $P \in B$ is called a *nucleus* (with respect to B) if any line through P contains exactly one point of B. Note that if $\boldsymbol{P}$ is finite of order q, then the existence of a nucleus implies that $|B| = q + 1$.

3.5.1. THEOREM (Blokhuis and Wilbrink [1987]). *Let $\boldsymbol{P}$ be the Desarguesian projective plane of order q. Consider a set B of points and a set A of nuclei with respect to B. If $|A| \geqslant q$, then B is the set of points on a line.*

This theorem has been proved in the special case where A is a conic by Bruen and Thas [1975] (q even) and by Segre and Korchmáros [1977].

The proof of 3.5.1 is beautiful, but it relies heavily on the fact that $\boldsymbol{P}$ can be coordinatized by a (commutative) field.

Assume that B is not a line. Then there exists a line l_∞ of $\boldsymbol{P}$ disjoint from B (and A) (see the next section). Consider the points of the corresponding affine plane as elements of $\mathrm{GF}(q^2)$. Define the polynomial $f \in \mathrm{GF}(q^2)[x]$ of degree $q - 1$ by

$$f(x) = \sum_{b \in B} (x - b)^{q-1}.$$

For each $a \in A$ one shows $f(a) = 0$. Hence $|A| \leqslant \deg f = q - 1$, a contradiction. □

3.6. *Blocking sets*

Blocking sets have found considerable interest in the last 20 years. Most of the literature can be found in Berardi and Eugeni [1988]. Here we will deal only with the fundamental properties.

Consider a set A of points in a projective plane $\boldsymbol{P}$ of order q such that any line contains at least one point of A. Then it is easy to show that A has at least $q + 1$ points, with equality if and only if the points of A form a line. (This has been proved in a much more general context by Tallini [1956] and Bose and Burton [1966].) So, the extreme examples are rather easy to describe. If one asks for sets which have also the 'complementary' property, the extremal examples are hard to describe. A *blocking set* of $\boldsymbol{P}$ is a set B of points such that any line contains a point of B and a point outside B.

The nicest examples of blocking sets occur in planes of square order. Any Baer subplane and any unital is a blocking set with $q + \sqrt{q} + 1$ and $q\sqrt{q} + 1$ points, respectively. But one can easily convince oneself that any projective plane of order $q \neq 2$ has a blocking set: consider two lines l_1, l_2 of $\boldsymbol{P}$ which intersect in a point P. Let $Q_i \neq P$ be a point on l_i and denote by Q a point on Q_1Q_2 different from Q_1 and Q_2. Then Q together with the points of l_1 and l_2 different from Q_1 and Q_2 form a blocking set.

The first important result on blocking sets, which still is the most important result, is due to Bruen [1970, 1971].

3.6.1. THEOREM. *Let B be a blocking set in a projective plane of order q. Then*

$$q + \sqrt{q} + 1 \leqslant |B| \leqslant q^2 - \sqrt{q}.$$

Equality holds on the left-hand side (right-hand side) if and only if B is the set of points of a Baer subplane (or, outside a Baer subplane, respectively).

A blocking set B is called *irreducible* (or *minimal*), if through any point P of B there is a line intersecting B just in P.

3.6.2. THEOREM (Bruen and Thas [1977]). *Let B be an irreducible blocking set in a projective plane of order q. Then*

$$|B| \leqslant q\sqrt{q} + 1$$

with equality if and only if B is the point set of a unital.

So, the extremal blocking sets in planes of square order are relatively easy to describe. In other planes, the situation is much more difficult and only partly solved. In 1993 there was a breakthrough by A. Blokhuis. He could settle the question of blocking sets of minimal cardinality in Desarguesian planes. In view of Bruen's result one has to consider only Desarguesian planes of order p^{2e+1}, where p is a prime. The results of Blokhuis read as follows.

3.6.3. THEOREM (Blokhuis [1994a,b]). *Let B be a blocking set in a Desarguesian projective plane of order p^{2e+1}, where p is a prime. Then*

(a) *If $e = 0$, then $|B| \geqslant p + (p + 3)/2$.*

(b) *If $e \geqslant 1$, then $|B| \geqslant p^{2e+1} + p^{e+1} + 1$.*

These bounds are sharp for $e = 0$ and $e = 1$.

P. Erdős has asked whether there exists an absolute constant c such that in any projective plane $\boldsymbol{P}$ there is a blocking set B such that any line of $\boldsymbol{P}$ intersects B in at most c points. This question is also interesting if we consider only certain classes of projective planes (e.g., Desarguesian planes). For some planes, this question has been answered.

3.6.4. THEOREM (Bruen and Fisher [1974]). *Any Desarguesian projective plane of order $q = 3^a$ has a blocking set which intersects each line in at most 4 points.*

The construction essentially works by joining two suitable cubic curves. It is also shown that any blocking set in a finite projective plane of order $q \geqslant 5$ necessarily has a line which shares at least four points with the blocking set.

Boros [1988] has generalized 3.6.4 by showing that any Desarguesian plane of order p^a (p an odd prime) has a blocking set B such that no line has more than $p + 1$ points in common with B.

4. Nonexistence theorems

The question, which positive integers q arise as the order of a projective plane still is one of the most fascinating topics in the field of projective planes. Recall the classical theorem of Bruck and Ryser.

4.1. THEOREM (Bruck and Ryser [1949]). *Let q be a positive integer with $q \equiv 1$ or 2 (mod 4). If there exists a projective plane of order q, then there is no prime number $p \equiv 3$ (mod 4) dividing the square free part of q, or, equivalently, q is the sum of two integral squares.*

As a corollary one has that there is no plane of an order $q \equiv 6 \pmod 8$. The first numbers not covered by the Bruck–Ryser theorem are $q = 10, 12, 15, \ldots$.

There was a big hope in the last years to handle at least some of these cases – in particular by the help of a computer. There have been many papers ruling out the possibility of collineations of a certain order in planes of order $10, 12, \ldots$ (for investigations of planes of order 10, see, e.g., Janko and Tran [1981] and Whitesides [1979]). In order to tackle the existence problem itself, the most promising way seems to be the coding theoretic approach, which was established in the early seventies by Assmus and Mattson [1970] and MacWilliams, Sloane and Thompson [1973]. The idea is simple. Consider an incidence matrix A of a hypothetical projective plane $\boldsymbol{P}$ of order 10 and consider the GF(2)-vector space $\boldsymbol{C}$ generated by the rows of A. This vector space $\boldsymbol{C}$ is also called a *code*; the weight of a vector in $\boldsymbol{C}$ (a *codeword*) is the number of its nonzero coordinates. Then one hopes that the weight distribution of this code $\boldsymbol{C}$ provides enough information for either a construction or a contradiction. It was shown in Assmus and Mattson [1970] that the weight distribution of the whole code is known if one knows the number of codewords of weight 12, 15 and 16. MacWilliams et al. [1973] showed that there are no codewords of weight 15; Bruen and Fisher [1973] observed that a result of Denniston [1969] implies that there are no words of weight 16. Also, the codewords of weight 12 correspond in a 1–1 manner to the hyperovals in $\boldsymbol{P}$.

The breakthrough was a result announced in Lam, Thiel, Swiercz and McKay [1985] that there are no hyperovals in any plane of order 10. Apart from a theoretical reduction, the proof involved 183 days (!) of computer time!

As a consequence, the weight distribution of $\boldsymbol{C}$ is known; in particular one can compute that there are 24,675 codewords of weight 19. The final contradiction was obtained by ruling out certain 19-point configurations in $\boldsymbol{P}$:

4.2. THEOREM (Lam, Thiel and Swiercz [1989]). *There does not exist a projective plane of order* 10.

Again, several thousand hours of computer time have been used. Of course, there might be software and hardware errors; for a thorough discussion of the probability of those errors, see the paper of Lam et al. [1989].

One might wonder, how this development will proceed. Two remarks are in order.

1. Certainly it is unfeasible to show the nonexistence of planes of big order using a computer.

2. Due to this result, most people now believe that perhaps every finite projective plane has prime power order.

5. Embeddings

Linear spaces have been formally introduced by Libois [1964]. A *linear space* (see also Chapter 6) is a rank 2 geometry consisting of points and lines such that the following axioms are satisfied:

– Any two distinct points are incident with precisely one common line.

– Any line is incident with at least two points.

If $\boldsymbol{P}$ denotes a projective plane, then we obtain a linear space by removing some of its points. A linear space that can be obtained in this way is called *embeddable* in the projective plane $\boldsymbol{P}$. Hall [1943] proved that any linear space $\boldsymbol{S}$ can be embedded in some projective plane. Let us recall this process. Let $\boldsymbol{S}$ be a linear space containing four points, no three of which are collinear. (Such a linear space is called *nondegenerate*.)

If $\boldsymbol{S}$ already is a projective plane, then there is nothing to show.

If $\boldsymbol{S} = \boldsymbol{S}_0$ is not a projective plane, then there are pairs of lines which have no point in common. To any such pair we adjoin a *new point* being incident with precisely those two lines.

So we get a structure $\boldsymbol{S}_0'$ which has the property that any two lines meet uniquely. But there are now pairs of points which have no line in common. (Otherwise $\boldsymbol{S}_0'$ would be a projective plane, but any new point is on just two lines, which is impossible in a projective plane).

Now we join any two points P, Q which are not on a common line by the *new line* $\{P, Q\}$. So we obtain a new linear space $\boldsymbol{S}_1$, which has pairs of nonintersecting lines.

Repeating this process we get a series

$$\boldsymbol{S} = \boldsymbol{S}_0 \subseteq \boldsymbol{S}_1 \subseteq \boldsymbol{S}_2 \subseteq \cdots$$

of linear spaces, none of which is a projective plane. Let now $\boldsymbol{P}$ be the union of all these (infinitely many) linear spaces. Then $\boldsymbol{P}$ is a projective plane. (For any two points P, Q of $\boldsymbol{P}$ are contained in some $\boldsymbol{S}_i$, so they are joined in $\boldsymbol{S}_{i+1}$; dually, any two lines are contained in some $\boldsymbol{S}_j$, so they intersect in any $\boldsymbol{S}_k$ with $k \geqslant j+1$. Hence in $\boldsymbol{P}$ any two points are joined and any two lines meet; thus $\boldsymbol{P}$ is a projective plane.)

To sum up, *any linear space* $\boldsymbol{S}$ *can be embedded in a projective plane*. However, if $\boldsymbol{S}$ is not itself a projective plane, then the above constructed projective plane $\boldsymbol{P}$ is always infinite, even if $\boldsymbol{S}$ is finite.

It is tempting to ask whether a finite linear space can be embedded in a *finite* projective plane. In fact, this is a well-known conjecture which probably goes back to Hall's paper (although it is not stated there).

CONJECTURE. Any finite linear space is embeddable in a finite projective plane. One would even like to ask a more precise question: Given a finite linear space $\boldsymbol{S}$, what is the least order of a projective plane $\boldsymbol{P}$ such that $\boldsymbol{S}$ is embeddable in $\boldsymbol{P}$?

The above conjecture is the leitmotif in the theory of finite linear spaces. If the conjecture was to be true, then today we are still far apart from answering it completely, although we shall see that there have been obtained many interesting and deep results.

From now on, we shall exclusively consider *finite* linear spaces. These are linear spaces with a finite number v of points and hence also a finite number b of lines. The Hanani–De Bruijn–Erdős theorem says that a nontrivial finite linear space has at least as many lines as points.

5.1. THEOREM (De Bruijn and Erdős [1948]). *Let $\boldsymbol{S}$ be a finite linear space with v points and $b > 1$ lines. Then $b \geqslant v$, with equality if and only if $\boldsymbol{S}$ is a projective plane or a near-pencil.*

A *near-pencil* is a linear space on v points which has a line with $v - 1$ points (all other lines having just two points).

A deep generalization of the Hanani–De Bruijn–Erdős theorem was obtained by Erdős, Mullin, Sós and Stinson [1983] and Metsch [1991]. In order to formulate the result we need a definition.

Let $\boldsymbol{S}$ be a finite linear space with v points. Denote by n the uniquely defined positive integer satisfying

$$n^2 - n + 1 = (n-1)^2 + (n-1) + 1 < v \leqslant n^2 + n + 1.$$

We define the number $B(v)$ as follows:

$$B(v) = \begin{cases} n^2 + n - 1 & \text{if } v = n^2 - n + 2 \neq 4, \\ n^2 + n & \text{if } n^2 - n + 3 \leqslant v \leqslant n^2 + 1 \text{ or } v = 4, \\ n^2 + n + 1 & \text{if } n^2 + 2 \leqslant v. \end{cases}$$

Now the above mentioned result reads as follows

5.2. THEOREM. *Let $\boldsymbol{S}$ be a finite linear space with v points and b lines and let $B(v)$ be the above defined number. Then $b \geqslant B(v)$.*

Moreover, equality implies that either $\boldsymbol{S}$ is embeddable in a projective plane of order n, or $\boldsymbol{S}$ is the following exceptional linear space which satisfies $n = 3$ and is embeddable in the projective plane of order 4 (see Figure 4).

A relatively old but very useful result is due to Vanstone [1973].

5.3. THEOREM. *Let $\boldsymbol{S}$ be a linear space of order n. If $v \geqslant n^2$, then $\boldsymbol{S}$ is embeddable in a projective plane of order n.*

The proof of Theorem 5.3 is relatively simple and consists of two steps. First one has to establish the existence of a line having exactly n points. Then, any such line lies in a unique parallel class.

This theorem has attracted much attention and was generalized several times. The latest version reads as follows.

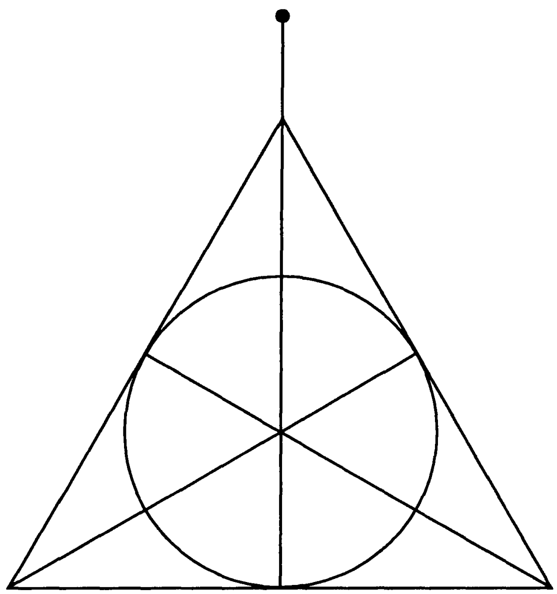

Figure 4. An exceptional linear space with $v = 8$. (Lines of size 2 are not drawn.)

5.4. THEOREM (Metsch [1991]). *Let* S *be a linear space of order* n. *If* $v \geqslant n^2 - n/2 + 1$ *and* $n > 222$, *then* S *is embeddable in a projective plane of order* n.

Finally we turn to the so-called *restricted* linear spaces; these are linear spaces satisfying $(b - v)^2 \leqslant v$. In the classification, inflated affine planes play an important role. A *completely projectively inflated affine plane* is an affine plane together with a (possibly degenerate) projective plane defined on all its points at infinity. The following theorem is a corollary of Totten's classification theorem.

5.5. THEOREM (Totten [1976]). *Let* S *be a restricted linear space. Then either* S *is embeddable (in a very natural way) in a finite projective plane or* S *is a completely projectively inflated affine plane.*

We mention that recently, Metsch [1991] has also determined all linear spaces satisfying $(b - v)^2 \leqslant b$.

The theory of linear spaces can be studied in Batten and Beutelspacher [1993]; Beutelspacher [1990] is a survey on embeddings of linear spaces in projective planes.

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CHAPTER 5

Translation Planes

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Contents

Introduction and historical survey 139
1. Basic theory 141
2. Classes of translation planes 149
 2.1. Generalized André systems and planes 149
 2.2. André systems and planes 151
 2.3. Nearfields and their planes 151
 2.4. Semifields and their planes 152
 2.5. Derived translation planes 156
 2.6. Generalized Hall planes, Hall planes 158
 2.7. Johnson planes 159
 2.8. Ostrom–Foulser planes 160
 2.9. Hering–Ott–Schaeffer planes 161
 2.10. Likeable planes and Walker planes 162
 2.11. Lüneburg planes 163
 2.12. Kantor planes 164
 2.13. Suetake planes. Hering plane of order 27 165
 2.14. Bartolone–Ostrom planes 165
 2.15. Flocks and translation planes. The Fisher planes 166
3. Dimension and collineations 169
4. Collineation groups 172
 4.1. Affine perspectivities 173
 4.2. Planar collineations 177
 4.3. Transitive collineation groups 180
5. Conclusion: two important problems 184
References 184

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Introduction and historical survey

Geometries* (including block designs) occupy a central position today in mathematics. In the theoretical realm geometry is the study of the possibilities arising from given postulates concerning the incidence of points and lines; in the practical realm geometries arise in the study of many diverse subjects such as coding theory and statistical designs. Historically, projective planes and translation planes were among the first geometries studied, and they occupy the central position in the subject. For the last thirty years the emphasis in this field of study has been on finite planes.

The first reference to finite geometries occurs in Von Staudt [1856]. The classical geometries PG(n, q), including the classical Desarguesian translation planes, appeared in Fano [1892] (with q a prime), Hessenberg [1902/03], and Veblen and Bussey [1906] who give a systematic exposition. The work of Hilbert [1899] was influential at these early stages.

During this period a major impetus for studying projective planes in general and translation planes in particular was the fact that all proofs of the classical Desargues configuration theorem for (projective) planes depended upon the plane being embeddable in a three-dimensional space. The work of Veblen and Wedderburn [1907] showed that embeddability was essential by constructing two non-Desarguesian projective planes of order 9 – one a translation plane.

The next major period of research was the 1930's. In the first half of the 1930's R. Moufang investigated planes coordinatized by alternative semifields (see p. 15). This work was continued by others, and their efforts resulted in the famous Bruck–Kleinfeld–Skornyakov theorem classifying all alternative semifields and their planes (see Kleinfeld [1963]). This in turn generated many results concerning semifields (and nonassociative rings) satisfying one or more 'alternative-like' identities.

In the second half of the 1930's M. Hall began his fundamental investigations into projective planes and their coordinatization, the results of which appeared in Hall [1943]; and in a series of articles beginning in 1939 R. Baer initiated the study of planes by means of their collineations. All subsequent work depends heavily on one or both of these viewpoints, and the works of J. André, R.H. Bruck, R.C. Bose, and T.G. Ostrom combine both for translation planes using linear algebra. The linear algebra representation of translation planes using spreads and group representations is now the standard way to treat translation planes.

The classification by H. Lenz [1954] and A. Barlotti [1957] of projective planes according to (P, l)-transitivity was followed by much work on the determination of the size of each Lenz–Barlotti class. The translation planes (and their duals) fall into four of these classes (IV–VII), of which one (class IV) contains all translation planes not coordinatized by a semifield. Many believed this class to be relatively small, but the work of Ostrom, Hering, and their students in the 1960's and 1970's showed conclusively that this class is rich in nontrivial planes.

Bose [1983] showed that projective planes have connections with Latin squares and experimental designs. Indeed, projective planes lie in the centre of the class of block

* Editor's note: the word 'geometry' is not given the same meaning here as in Chapter 3.

designs[1], and many block designs are closely tied to planes. These ties are exhibited, e.g., in Hughes and Piper [1985]. Furthermore, the class of translation planes lies in the centre of the class of projective planes.[2] A second modern stimulus for studying translation planes is the representation of certain finite groups as collineation groups of such planes; examples include SL$(2, q)$ and Sz(q). Thus, results on translation planes often lead to interesting results about other designs and/or algebraic structures. For example, recent investigations of collineation groups of translation planes have raised interesting questions about group representations over finite fields (Ostrom [1983a]).

On the other hand, ovoids in projective spaces can give rise to translation planes. (See Kantor [1982d], and Thas [1972] and chapter 7 of this book.) Also, certain translation planes correspond to flocks in finite projective spaces and hence to certain generalized quadrangles. (See Gevaert, Johnson and Thas [1988], Johnson [1987, 1989, 1990a], Thas [1975], and Walker [1976a].) Besides giving interesting results and holding promise for other investigations on translation planes, this correspondence has connections with the geometry of buildings.

Recently much interest has been expressed in the connection between translation planes and packings, or parallelisms of projective geometries. This connection has been studied by Lunardon [1984] and Jha and Johnson [1986c,d].

Dembowski [1968] gives a very accurate survey of the state of knowledge on translation planes up to the beginning of 1968, and its bibliography is complete up to that point. Therefore, the bibliography at the end of this chapter will be restricted to works published after 1967, except for those referenced directly.

Because of the encyclopedic nature of Dembowski's book the emphasis in this chapter will be on developments in the field since 1967. Section 1 gives the basic definitions in the theory; it will emphasize the linear algebra view of translation planes. Section 2 discusses examples and various construction techniques. Section 3 examines the relationship between the dimension (of the underlying vector space) and the collineation group, while Section 4 discusses collineation groups (and the underlying planes) having special transitivity properties or containing certain types of collineations. Finally, Section 5 treats two important problems in the field on which little progress has been made.

Limitation of space precludes a complete listing of all the developments in the field since 1967. Thus, only the major ones (in the opinion of the author) are discussed, and suggestions for further study and investigation are given. Outlines of some proofs are provided; for others references will be indicated. It is hoped that the reader will be encouraged to proceed beyond this chapter and consult other works.

Besides Dembowski [1968] someone studying translation planes should consult the following monographs: Hirschfeld [1979], Hirschfeld and Thas [1989], Hughes and Piper [1973], Johnson [1986b], Kallaher [1982], Lüneburg [1980], Ostrom [1970c], and Pickert [1975]. Also, Ostrom [1968] gives a very nice introduction to the subject. (See also Ostrom [1977b].) In the last ten years several articles reviewing results on specific topics have appeared, these include Johnson [1983b, 1986a], Goodaire and Kallaher [1990], Ostrom [1983a,b]. Additionally, several conferences in recent years have

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[1] In Dembowski [1968] his discussion of projective planes occupies the middle three chapters, thereby accurately reflecting where they lie within the class of block designs.

[2] See the well-written Chapter 3 of this monograph for developments in the general theory of projective planes since 1967.

devoted time to papers on translation planes; see, e.g., Baker and Batten [1985] or Johnson, Kallaher, and Long [1983].

Goodaire and Kallaher [1990] survey the present-day knowledge of quasifields, the coordinatizing algebras of translation planes. Overlap between this chapter and that article is minimized and the reader is encouraged to consult that article.

1. Basic theory

A *translation plane* is an affine plane Π whose translation group $T(\Pi)$ is (sharply) transitive on the affine points. (By a *translation* is meant a (P, ℓ_∞)-elation, where ℓ_∞ is the *line at infinity*, or *improper line*, of Π and $P \in \ell_\infty$.) Every coordinatizing ternary ring of a translation plane (with respect to the line ℓ_∞) is a *quasifield.*

A *quasifield*[3] is an algebra $(Q; +, \cdot)$, where Q is a nonempty set and $+$ and $\cdot$ are binary operations on Q, satisfying

$$\begin{array}{ll} \text{(q1)} & Q \text{ is a group under } + \text{ with identity } 0, \\ \text{(q2)} & Q - \{0\} \text{ is a loop under } \cdot \text{ with identity } 1, \\ \text{(q3)} & \text{For all } a \in Q, \quad a \cdot 0 = 0 \cdot a = 0, \\ \text{(q4)} & \text{For all } a, b, c \in Q, \ (a + b) \cdot c = (a \cdot c) + (b \cdot c), \\ \text{(q5)} & \text{Given } a, b, c \in Q \text{ with } a \neq b \text{ there exists a unique} \\ & x \in Q \text{ such that } -(x \cdot a) + (x \cdot b) = c.^4 \end{array} \tag{1.1}$$

When confusion will not result the quasifield will be simply denoted by Q, and the standard symbols $+$ and $\cdot$ (or juxtaposition) will be used for the operations. In this case, as usual, multiplication will take precedence over addition unless otherwise indicated. For example, the equation in (q4) above will read: $(a + b)c = ac + bc$.

Given a quasifield Q a translation plane $\Pi = \Pi(Q)$ is constructed as follows. *Points* are the ordered pairs (a, b) with $a, b \in Q$, *lines* are the point sets

$$[m, b] = \{(a, am + b)\colon\ a \in Q\}, \tag{1.2}$$

$$[c] = \{(c, a)\colon\ a \in Q\}. \tag{1.3}$$

Lines of the form (1.2) can be represented by the equations

$$y = xm + b, \tag{1.4}$$

[3] These algebras are frequently called 'Veblen–Wedderburn systems' after Veblen and Wedderburn [1907] who found the first finite ones that are not fields. The term 'quasifield' is due to Pickert [1975].

[4] This condition is known as the planar condition. If Q is finite-dimensional over its kernel (cf. paragraph following Statement 1.1 below) this condition follows from (q1)–(q4). See Zemmer [1964] for an example of a nonplanar quasifield.

while lines of the form (1.3) can be represented by the equations

$$x = c. \tag{1.5}$$

In this notation the points on ℓ_∞ are given coordinates as follows. The point of intersection of ℓ_∞ with the line (1.2) (or (1.4)), which is the same for every line in given family, is assigned the coordinate (m), and the point of intersection of ℓ_∞ with a line (1.3) ((1.5)) is assigned (∞), where ∞ is not an element of Q.

Furthermore, with this representation the translations are given by

$$\tau_{a,b}\colon\ (x, y) \mapsto (x + a, y + b), \tag{1.6}$$

one translation for each ordered pair a, b in $Q \times Q$. It follows that $T(\Pi) \cong Q \oplus Q$. Thus, the following results about quasifields also give information about translation groups.

(1) *The following statements hold for every quasifield. The additive group of Q is Abelian. If Q is finite then the additive group of Q is an elementary Abelian p-group for some prime p. If the order of Q is a prime then Q is a field.*

Pickert [1975], p. 91, gives an ingenious proof of (a). To prove (b) and (c), let $K = K(Q)$ be the set of all elements a in Q such that

$$a(xy) = (ax)y \quad \text{and} \quad a(x + y) = ax + ay \quad \text{for all } x, y \in Q. \tag{1.7}$$

The set K is called the *kernel of Q*, and it follows easily that K is a division ring and Q is a left vector space over K. Statements (b) and (c) are then direct consequences. Statement 1 gives the following information about translation planes.

(2) *Let Π be a translation plane.*
- (a) *If Π is finite then it has prime power order.*
- (b) *If Π has prime order then it is Desarguesian.*

Geometrically, the kernel K of Q gives the dilatations of $\Pi = \Pi(Q)$ fixing an affine point. Specifically, if $a \in K$ with $a \neq 0$ then

$$\sigma_a\colon\ (x, y) \mapsto (ax, ay) \tag{1.8}$$

is a collineation of Π fixing $(0, 0)$ and all points on ℓ_∞. Conversely, every such collineation of Π can be represented by σ_a for some nonzero a in K. Furthermore, the group $K(\Pi)$ of such collineations is isomorphic to the multiplicative group of K. For this reason $K(\Pi)$ is also called the *kernel*, or sometimes the *kern*, of Π.

(3) *Let Π be an affine plane coordinatized by the quasifield Q.*
- (a) *The group $T(\Pi)K(\Pi)$ consists of all dilatations of Π.*
- (b) *The group $T(\Pi)$ is isomorphic to the group $(Q, +) \oplus (Q, +)$ and the group $K(\Pi)$ is isomorphic to $K(Q)^*$, the multiplicative group of $K(Q)$.*

Since the quasifield Q is a left vector space over its kernel $K = K(Q)$, the translation plane $\Pi = \Pi(q)$ is also a left vector space over K (having twice the dimension of Q when that dimension is finite). Furthermore, the lines of Π through the origin $(0,0)$, called the *components* of Π and given by the equations

$$y = xm, \quad m \in Q, \tag{1.9}$$

$$x = 0, \tag{1.10}$$

are subspaces of Π having the same dimension as Q. In addition, the set $\mathcal{S}$ of these subspaces satisfies:

(i) Two distinct elements of $\mathcal{S}$ are isomorphic subspaces,
(ii) Every point except $\mathcal{O} = (0,0)$ of Π is on exactly one subspace in $\mathcal{S}$,
(iii) For any $U_1, U_2 \in \mathcal{S}$ with $U_1 \neq U_2$, the plane Π is the direct (vector space) sum of U_1 and U_2.

Given a (left) vector space V over a division ring K, a *spread* $\mathcal{S}$ in V is a collection of at least two subspaces, called the *components* of $\mathcal{S}$, of V satisfying properties (i), (ii), (iii) of the previous paragraph.[5] For a given spread $\mathcal{S}$ a translation plane $\Pi = \Pi(\mathcal{S})$ is constructed by taking the points of Π to be the vectors in V and the lines to be the components U of $\mathcal{S}$ together with their translates $U + v$, $v \in V$.

By the use of vector space isomorphisms it may be assumed that V has a subspace Q such that $V = Q \oplus Q$ and that the spread $\mathcal{S}$ contains the subspaces

$$V(0) = \{(x,0)\colon\ xQ\},\ V(\infty) = \{(0,x)\colon\ xQ\},\ \text{and}\ V(1) = \{(x,x)\colon\ x \in Q\}.$$

Taking the elements of Q to be the elements of a quasifield coordinatizing $\Pi(\mathcal{S})$ with the above components as x-axis ($y = 0$), y-axis ($x = 0$), and the line $y = x$, respectively, the remaining components are then represented by equations of the form (1.9) over Q. Furthermore, the division ring K is in the kernel of Q and Π. The spread $\mathcal{S}$ is *proper* if $K = K(\Pi)$; this is equivalent to the nonexistence of a division ring L with $K < L$ and V a vector space over L such that $\mathcal{S}$ is a spread of V over L. Hence:

(4) *There is a canonical correspondence*[6] *between translation planes and proper spreads of vector spaces over division rings.*

PROOF. André [1954] or Lüneburg [1980]; see also Bruck and Bose [1964] who discuss spreads from the projective geometry viewpoint. □

A second way of viewing spreads is as follows. Note that multiplication in the quasifield Q is a linear transformation on Q as a vector space over its kernel K: given $m \in Q$, define

$$\rho_m\colon\ x \mapsto xm. \tag{1.11}$$

[5] André [1954] uses the word *congruence* for this concept. The connection between spreads and translation planes is given by him.

[6] By 'canonical correspondence' is meant that the isomorphism classes of translation planes can be put into a (1-1) correspondence with the isomorphism classes of vector spaces having proper spreads.

If $m \neq 0$ then $\rho_m \in \mathrm{GL}(Q, K)$; furthermore, the component $V(m)$ of $\mathcal{S}$ represented by (1.9) consists of the points $(x, \rho_m(x))$ in Π. The properties of the spread $\mathcal{S}$, which are equivalent to the geometric properties of the translation plane Π, imply the set $\mathcal{M} \equiv \{\rho_m\colon\ m \in Q,\ m \neq 0\}$ forms an m-spread[7] in the following sense.

Given a vector space U over a division ring K an *m-spread* $\mathcal{M}$ of U is a collection of nonsingular linear transformations on U such that:

(i) $1 \in \mathcal{M}$,

(ii) Given $u, w \in U - \{0\}$ there exists $\rho \in \mathcal{M}$ such that $\rho(u) = w$,

(iii) If $\rho, \sigma \in \mathcal{M}$ with $\rho \neq \sigma$ then $\rho - \sigma$ is nonsingular.[8]

The connection between m-spreads and translation planes is given by the following result of Bruck and Bose [1964]. Here $V(0)$ and $V(\infty)$ have the same meaning as before, and $V(\rho) \equiv \{(x, \rho(x))\colon\ x \in U\}$ for $\rho \in \mathcal{M}$.

(5) *If $\mathcal{M}$ is an m-spread on the vector space U over the division ring K, then*

$$\mathcal{S}(\mathcal{M}) \equiv \{V(0), V(\infty)\} \cup \{V(\rho)\colon\ \rho \in \mathcal{M}\}$$

is a spread on the vector space $V = U \oplus U$ over K.

Furthermore, a quasifield Q coordinatizing the resulting translation plane $\Pi(\mathcal{M})$ is obtained by taking $Q = U$, the addition that of U, and multiplication as follows. Choose a nonzero element $e \in Q$; then for each $x \in Q$ with $x \neq 0$ there exists a unique $\rho_x \in \mathcal{M}$ with $x = \rho_x(e)$. Multiplication is then defined by

$$xy \equiv \begin{cases} 0 & \text{if } y = 0, \\ \rho_y(x) & \text{if } y \neq 0. \end{cases} \tag{1.12}$$

In addition, the element e will be the multiplicative identity of Q and the division ring K will be contained in Q by identifying $\alpha \in K$ with $\alpha e \in Q$.

In this context the subspaces $V(0)$, $V(\infty)$, and $V(1)$ will be the x-axis, the y-axis, and the line $y = x$, respectively. The m-spread $\mathcal{M}$ will be *proper* if the spread $\mathcal{S}(\mathcal{M})$ is a proper spread on V; i.e. if K is the kernel of Q. See Lüneburg [1980], Chapter 1, for more details.

Returning to the representation of a translation plane Π in terms of its coordinatization by a quasifield Q, assume Q has finite dimension d over its kernel $K = K(Q)$. Then Π has dimension $2d$ over K. Condition (i) for a spread $\mathcal{S}$ can then be replaced by:

(i′) Each element of $\mathcal{S}$ has dimension d over K.

Also, the elements of the m-spread can be viewed as nonsingular d by d matrices over K (with respect to a basis of Q).

If in addition K is a finite field $\mathrm{GF}(q)$ then condition (ii) for the spread $\mathcal{S}$ can be replaced with:

(ii′) $|\mathcal{S}| = q^d + 1$,

and condition (ii) for the m-spread $\mathcal{M}$ can be replaced with:

(ii′) $|\mathcal{M}| = q^d + 1$.

Thus, in the finite case the following result holds.

[7] The letter 'm' stands for the word 'matrix'. If Q is finite dimensional over K then the linear transformations can be represented by matrices relative to a chosen basis, and this is a standard procedure.

[8] Note that (iii) is equivalent to: (iii′) If $\rho, \sigma \in \mathcal{M}$ with $\rho \neq \sigma$ then $\sigma^{-1}\rho$ is fixed-point-free on $U - \{0\}$.

(6) *Let $q = p^k$ be a prime power, and let d be a positive integer. The following statements are equivalent.*

There exists a translation plane Π of order q^d with kernel $K(\Pi)$ of order $q - 1$.

There exists a quasifield Q of dimension d over its kernel $K(Q) = \mathrm{GF}(q)$.

There exists a proper spread $\mathcal{S}$ on the vector space $K^d \oplus K^d$, where K^d is the space of d-tuples over $K = \mathrm{GF}(q)$.

There exists a proper m-spread $\mathcal{M}$ on K^d, where $K = \mathrm{GF}(q)$.

The above linear algebra approach also has implications for the collineation group $G = G(\Pi)$ of the translation plane Π. Viewing Π as the vector space sum $Q \oplus Q$, where Q is a quasifield coordinatizing Π, with spread $\mathcal{S} = \{V(m)\colon\ m \in Q \cup \{\infty\}\}$, the translations in $T = T(\Pi)$ can be represented as the mappings

$$\tau(\boldsymbol{v})\colon\ \boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{v}, \quad \boldsymbol{v} \in \Pi. \tag{1.13}$$

(See Statement 1.3 and (1.6).) Furthermore, if $\nu \in G$ then $\tau(\boldsymbol{v})^{-1}\nu$, where $\boldsymbol{v} = \mathbf{0}\nu$, is in $C = C(\Pi)$, the subgroup of G consisting of all collineations fixing the origin $\mathbf{0} = (0, 0)$. The group C is called the *translation complement* of Π.

Consider the group C. The $(\mathbf{0}, \ell_\infty)$-homologies are in C, and by (1.8) they can be represented as the mappings

$$\sigma(a)\colon\ \boldsymbol{x} \mapsto a\boldsymbol{x}, \quad a \in K(Q) - \{0\}. \tag{1.14}$$

Let $\varphi \in C$. If $\boldsymbol{v} \in \Pi$ then $\varphi\tau(\boldsymbol{v})\varphi^{-1} = \tau(\varphi(\boldsymbol{v}))$ since T is normal in G. Also, for $a \in K(Q) - \{0\}$ it follows that $\varphi\sigma(a)\varphi^{-1} = \sigma(\overline{a})$ for some $\overline{a}K(Q) - \{0\}$. The mapping $\overline{\varphi}\colon\ K(Q) \mapsto K(Q)$ given by

$$\overline{\varphi}(0) = 0 \quad \text{and} \quad \overline{\varphi}(a) = \overline{a} \quad \text{for } a \neq 0 \tag{1.15}$$

then is an automorphism of $K(Q)$. A direct computation then shows that φ is a semilinear transformation on Π with associated automorphism $\overline{\varphi}$.

(7) *If Π is a translation plane defined on the vector space $V = Q \oplus Q$ by the spread $\mathcal{S}$, then the translation complement $C = C(Q)$ consists of those semilinear transformations on V which preserve the spread $\mathcal{S}$.*

The *linear translation complement of Π* is the subgroup $L = L(Q)$ of C consisting of linear transformations on Π; i.e. $L = C \cap \mathrm{GL}(\Pi, K(q))$. If $K(Q) = \mathrm{GF}(q)$ then C/L is a cyclic group. Thus, in this situation the structure of the full collineation group $G(\Pi)$ is essentially known if the subgroup L is determined.

Putting this together with Statement 1.1 gives the following complementary result to Statement 1.6.

(8) *Let Π be a finite translation plane over the quasifield Q of dimension d over its kernel* $K(Q) = \mathrm{GF}(q)$ *with* $q = p^k$, *where p is a prime and* $k \geqslant 1$. *The following statements hold*:

The collineation group $G(\Pi) = T(\Pi)C(\Pi)$, *where $T(\Pi)$ is the translation group of Π and $C(\Pi)$ is the translation complement of Π.*

The translation group $T(\Pi)$ is an elementary Abelian p-group of order q^{2d} isomorphic to $Q \oplus Q$.

The translation complement $C(\Pi)$ is the subgroup of $\Gamma\mathrm{L}(2d, q)$ *consisting of all semi-linear transformations fixing the spread* $\mathcal{S} = \{V(m)\colon\ mQ \cup \{\infty\}\}$.

If $L(\Pi)$ is the linear translation complement of Π then $C(\Pi)/L(\Pi)$ is a cyclic group of order dividing k.

Thus, in the context of Statement 1.8 the structure of the collineation group $G(\Pi)$ is known once the group $L(\Pi)$ is determined. This raises the fundamental question.

QUESTION. What groups can be, or can not be, subgroups of $L(\Pi)$ for some finite translation plane Π?

If the word 'finite' is removed in the above question, then the answer is that every group can be a subgroup of $L(\Pi)$ for some translation plane Π. (See Mendelsohn [1972].) In the finite case many groups – for example, the groups $\mathrm{SL}(2, p^t)$ and $\mathrm{Sz}(2^2)$ – have been shown to be subgroups of $L(\Pi)$ for one or more planes Π, but they fall into only a small number of classes. On the other hand, no finite group has yet been shown not to be a subgroup of $L(\Pi)$ for any finite translation plane Π. This question will again be discussed in Section 3.

In the context of Statements 1.6 and 1.8 the integer d is called the *dimension* of the plane Π; note that Π actually has dimension $2d$ over $K(Q)$. The field $K(Q)$ will also be called the *kernel* of Π as well as the kernel of Q. These concepts go back to André [1954], who showed the connection between translation planes and spreads. Ostrom [1979, 1980a, 1981a,b] has made extensive use of the concept of dimension, and he has shown that a small dimension restricts the possibilities for $L(\Pi)$. Before Ostrom's work a distinction was made in many investigations between square order and nonsquare order; Ostrom showed that the important distinction was between d even and d odd. Because of his work several people have called d the *Ostrom dimension*.

For a complete discussion of the connection between translation planes and spreads, see Chapter I of Lüneburg [1980]. Examples of particular types of spreads and their translation planes will be given and discussed in the next section.

For a quasifield Q three other subsets of Q, besides the kernel $K(Q)$, have geometric significance. The *middle nucleus* $N_m(Q)$ of Q, respectively the *right nucleus* $N_r(Q)$ of Q, is the set of all $n \in Q$ such that

$$(an)b = a(nb), \tag{1.16}$$

respectively,

$$(ab)n = a(bn), \tag{1.17}$$

for all $a, b \in Q$. The *distributor* $D(Q)$ of Q is the set of all $d \in Q$ such that

$$a(d + b) = ad + ab \tag{1.18}$$

for all $a, b \in Q$. The following result holds.

(9) *For a quasifield Q and its associated translation plane $\Pi = \Pi(Q)$ the following statements hold.*

The affine elations[9] *of Π with axis $(0,0)(\infty)$ are the mappings*

$$\delta_d\colon (x, y) \mapsto (x, xd + y), \quad d \in D(Q). \tag{1.19}$$

The affine homologies of Π with axis $(0,0)(\infty)$ and centre (0) are the mappings

$$\mu_n\colon (x, y) \mapsto (xn, y), \quad 0 \neq n \in N_m(Q). \tag{1.20}$$

The affine homologies of Π with axis $(0,0)(0)$ and centre (∞) are the mappings

$$\nu_n\colon (x, y) \mapsto (x, yn), \quad 0 \neq n \in N_r(Q). \tag{1.21}$$

PROOF. The proofs are straightforward. Note that $D(Q)$ is an additive subgroup of Q while $N_m(Q) - \{0\}$ and $N_r(Q) - \{0\}$ are multiplicative subgroups of Q. Furthermore, the group of homologies in statement (b), respectively statement (c), is anti-isomorphic to the group $N_m(Q) - \{0\}$, respectively, $N_r(Q) - \{0\}$.[10] □

This section closes with a brief discussion of the isomorphism problem for translation planes. An *isomorphism* of two affine planes Π_1 and Π_2 is a bijection φ of the points of Π_1 onto the points of Π_2 such that (i) the image $\varphi(\ell)$ of a line ℓ of Π_1 is a line of Π_2, and (ii) the point $\mathcal{P}$ is on the line ℓ in Π_1 if and only if $\varphi(\mathcal{P})$ is on $\varphi(\ell)$ in Π_2.

ISOMORPHISM PROBLEM. Given two translation planes, under what conditions are they isomorphic?

If Q_1 and Q_2 are isomorphic quasifields coordinatizing Π_1 and Π_2 respectively, then it follows easily that Π_1 and Π_2 are isomorphic. However, Π_1 and Π_2 can be isomorphic even when Q_1 and Q_2 are not isomorphic. Indeed, the non-Desarguesian translation plane of order nine is coordinatized by two nonisomorphic quasifields.

A more general sufficient condition involves the concept of isotopism. An *isotopism* of a quasifield Q_1 onto a quasifield Q_2 is a triple (α, β, γ) of bijections of Q_1 onto Q_2 which are isomorphisms of $(Q_1, +)$ onto $(Q_2, +)$ satisfying[11]

$$\gamma(ab) = \alpha(a)\beta(b) \quad \text{for all } a, b \in Q_1. \tag{1.22}$$

(10) *If Q_1 and Q_2 are isotopic quasifields with isotopism (α, β, γ) then the translation planes $\Pi_1 = \Pi(Q_1)$ and $\Pi_2 = \Pi(Q_2)$ are isomorphic.*

---

[9] An *affine perspectivity* is a perspectivity whose axis ℓ is an affine line (as opposed to ℓ_∞); for such a perspectivity the centre $\mathcal{P}$ is on ℓ_∞. If $\mathcal{P} = \ell \cap \ell_\infty$ it is called an *affine elation*; if $\mathcal{P} \neq \ell \cap \ell_\infty$ it is called an *affine homology*.

[10] The *left nucleus* $N_\ell(Q)$, which is defined in the obvious way, has no known geometric interpretation except when it equals $K(Q)$.

[11] Here the multiplication on the left is that of Q_1 while the multiplication on the right is that of Q_2.

PROOF. Define $\varphi\colon \Pi_1 \mapsto \Pi_2$ by

$$\varphi\colon\ (x,y) \mapsto \big(\alpha(x), \gamma(y)\big), \quad (m) \mapsto \big(\beta(m)\big), \quad (\infty) \mapsto (\infty). \tag{1.23}$$

It follows that φ is an isomorphism. Note that φ maps the points $(0,0)$, (0), and (∞) of Π_1 onto the corresponding points in Π_2. □

With regards to the converse of Statement 1.10 the two nonisomorphic quasifields coordinatizing the non-Desarguesian translation plane of order nine are also nonisotopic. There is a partial converse which is stated for use later.

(11) *If $\Pi_1 = \Pi(Q_1)$ and $\Pi_2 = \Pi(Q_2)$ are translation planes over the quasifields Q_1 and Q_2, and if $\varphi\colon \Pi_1 \to \Pi_2$ is an isomorphism with the points $(0,0),(0),(\infty)$ in Π_1 mapped onto the corresponding points in Π_2, then Q_1 and Q_2 are isotopic.*

PROOF. The isotopism (α, β, γ) of Q_1 onto Q_2 is defined by means of (1.23). □

In the case where $Q_1 = Q_2$ (and $\Pi_1 = \Pi_2$) an isotopism is called an *autotopism*. By the proofs of Statements 10 and 11 the collineation group of Π_1 fixing the points $(0,0)$, (0), and (∞) is isomorphic to the group of autotopisms of Q_1. For obvious reasons this collineation group is called the *autotopism group* of Π_1 with respect to the points $(0,0),(0),(\infty)$. The number of orbits of this collineation group on the set of points (x,y) with $xy \neq 0$ equals the number of distinct nonisomorphic isotopic images of Q_1.

Using Statement 1.9 a geometric argument gives that isotopism preserves the kernel, the distributor, the middle nucleus, and the right nucleus.[12] There is also an easy algebraic argument which proves in addition that the left nucleus, the *nucleus*

$$N(Q) = N_1(Q) \cap N_m(Q) \cap N_r(Q),$$

and the *centre*

$$C(Q) = \big\{c\colon\ cN(Q),\ ca = ac \text{ for all } aQ\big\}$$

are preserved by an isotopism. (See Bruck [1958], Chapter 3.) This means that the last three algebraic concepts have a geometric interpretation.[13]

The isomorphism problem has not been solved completely except in a few special cases where the planes or the quasifields have special properties. For example, if the planes are Desarguesian (i.e. the quasifields are division rings) then the converse of Statement 1.10 is also true. Other examples are given in the next section.

[12] By this is meant that isotopic quasifields have isomorphic kernels, distributors, and so forth.

[13] What the geometric interpretation is for these three subalgebras is not known for quasifields in general.

2. Classes of translation planes

This section discusses classes of finite translation planes with emphasis on those that have played an important role in the subject. Dembowski [1968], Sections 5.2 and 5.3, lists the translation planes known in 1967; while these will be mentioned they will not be discussed except for new information discovered since 1967. Many of these examples have infinite analogues, and this will be indicated where appropriate.

2.1. *Generalized André systems and planes*

Consider the field $F = \mathrm{GF}(q^f)$ where $f \geqslant 1$ and $q = p^s$ with p a prime and $s \geqslant 1$. Let λ: $F \to I_f = \{0, 1, \dots, f-1\}$ be a mapping satisfying: (i) $\lambda(0) = \lambda(1) = 0$, and (ii) given $a, b \in F - \{0\}$ there exists $x \neq 0$ with

$$x^{q^{\lambda(a)}} a = x^{q^{\lambda(b)}} b$$

if and only if $a = b$.[14] Then the triple $(F_\lambda; +, \circ)$, where F_λ consists of the elements of F, the addition $+$ is the field addition of F, and $\circ$ is given by the rule

$$x \circ y = x^{q^{\lambda(y)}} y \tag{2.1}$$

is a quasifield called a *generalized André system*, and the associated translation plane $\mathcal{F}_\lambda$ is called a *generalized André plane.* (See Foulser [1967a,b] or Chapter 2 of Lüneburg [1980].)

The kernel K_λ of F_λ is easily computed to be the set of all $a \in F_\lambda$ with $a^{q^{\lambda(b)}} = a$ for all $b \in F_\lambda$. Hence $\mathrm{GF}(q) \leqslant K_\lambda$ and the dimension d of $\mathcal{F}_\lambda$ is at most f. Note also that for each $a \in F_\lambda$ with $a \neq 0$ the mapping

$$x \mapsto (x \circ a)a^{-1}, \tag{2.2}$$

where a^{-1} is the multiplicative inverse of a in the field F, is a (field) automorphism of F. This characterizes the generalized André systems.

(1) *Let $(Q; +, \circ)$ be a quasifield of dimension d over its kernel $K = \mathrm{GF}(q)$. The quasifield $(Q; +, \circ)$ is a generalized André system if and only if Q admits a binary operation $\cdot$ such that $(Q; +, \cdot)$ is a field and the mapping* (2.2) *is an automorphism of $(Q; +, \cdot)$ for all $a \in Q - \{0\}$.*

For

$$u = \mathrm{lcm}\{q^e - 1:\ e \text{ divides } f \text{ and } e < f\},$$

[14] If ω is a generator of F then every nonzero element of F has the form ω^i, and λ can be considered a mapping from I_u, where $u = q^f - 1$, onto I_f. Condition (ii) is then equivalent to: Given integers i, j with $0 \leqslant i, j \leqslant q^f - 1$ the congruence $i \equiv j \pmod{(q^e - 1)}$, where $e = \gcd(f, \lambda(i) - \lambda(j))$ holds if and only if $i = j$. (Take $a = \omega^i$ and $b = \omega^j$.)

let U be the unique cyclic subgroup of $(F - \{0\}, \cdot)$ having order $u^{-1}(q^f - 1)$. It follows that for $a \in U$ the equation $x \circ a = xa$ holds for all $x \in F_\lambda$. Thus, the mappings

$$(x, y) \mapsto (x \circ a, y), \quad a \in U, \tag{2.3}$$

are affine homologies of $\mathcal{F}_\lambda$ with centre (0) and axis $V(\infty)$. Similarly, the mappings

$$(x, y) \mapsto (x, y \circ a), \quad a \in U, \tag{2.4}$$

are affine homologies with centre (∞) and axis $V(0)$.

More importantly, if A is the collineation group generated by the affine homologies (2.3) and (2.4), that is A consists of the collineations

$$(x, y) \mapsto (x \circ a, y \circ b), \tag{2.5}$$

then A is Abelian and for every component $V(m)$ of $\mathcal{F}_\lambda$ the stabilizer A_m of $V(m)$ in A acts irreducibly on $V(m)$ as a group of linear transformations. This property characterizes generalized André planes.

(2) *A translation plane Π of order q^d is a generalized André plane if and only if Π has an Abelian collineation group A fixing the points* $(0, 0)$, (0), (∞) *and for each component $V(m)$ of Π the stabilizer A_m of $V(m)$ in A induces an irreducible group of semilinear transformations on $V(m)$.*

PROOF. See Lüneburg [1976b]. Ostrom [1969] has a restricted version of this result. □

The necessary and sufficient condition given in Statement 2.1 can be used to define infinite generalized André systems and their planes, but Statement 2.2 does not generalize. See Lüneburg [1980], Chapter 2, or Rink [1977].

Another characterization of generalized André systems Q uses the subgroup of

$$\Gamma\mathrm{L}(d, K(Q))$$

generated by the m-spread $\mathcal{M} = \{\rho_m \colon m \in Q - \{0\}\}$. Each ρ is in $\Gamma\mathrm{L}(1, q^d)$, and hence $\mathcal{M}$ generates a solvable transitive group of linear transformations on Q as a vector space over $K(Q)$.

(3) *A quasifield Q of dimension d over its kernel $K = \mathrm{GF}(q)$ is a generalized André system if and only if the group $\langle \mathcal{M} \rangle$, where $\mathcal{M}$ is the m-spread defining Q, is solvable unless $d = 2$, $q = 5, 7, 11, 23$ or $d = 4$ and $q = 3$.*

PROOF. Kallaher [1987]. The proof involves Huppert's classification of solvable doubly transitive groups. The numbers listed are exceptions. □

2.2. *André systems and planes*

An *André system* is a generalized André system F_λ in which λ is given by

$$\lambda(a) = \mu\nu(a),$$

where μ: $\mathrm{GF}(q)^*(= \mathrm{GF}(q) - \{0\}) \to I_f$ is an arbitrary mapping with $\mu(1) = 0$ and

$$\nu(a) = \prod_{i=0}^{f-1} a^{q^i}, \qquad a \in \mathrm{GF}\left(q^f\right).$$

André systems and their planes were first discovered by André [1954] and then generalized by Foulser.

The characterization of André planes corresponding to Statement 2.2 is due to Ostrom [1969].

(4) *A finite translation plane Π is an André plane if and only if Π has an Abelian collineation group A fixing the points* $(0,0), (0), (\infty)$ *and for each component $V(m)$ of Π the stabilizer A_m of $V(m)$ in A is transitive on the affine points other than* $(0,0)$ *of* $V(m)$.

2.3. *Nearfields and their planes*

A *nearfield* is a quasifield N in which the multiplication is associative; that is, in which $(N - \{0\}, \cdot)$ is a group. The following necessary and sufficient condition is immediate.

(5) *A quasifield described by an m-spread $\mathcal{M}$ is a nearfield if and only if $\mathcal{M}$ is a group under composition.*

A class of nearfields is obtained as follows. Consider the field $\mathrm{GF}(q^d)$, where $q = p^k$ for some prime p and $k \geqslant 1$, and assume every prime divisor of d divides $q - 1$. Also, assume $d \not\equiv 0 \pmod 4$ if $q \equiv 3 \pmod 4$. Choosing a primitive element ω of $\mathrm{GF}(q^d)$, define λ: $\mathrm{GF}(q^d) - \{0\} \to I_d$ by

$$\left(q^{\lambda(a)} - 1\right)(q-1)^{-1} \equiv i \pmod d, \quad a = \omega^i \in \mathrm{GF}(q).$$

With $\lambda(0) = 0$ the mapping λ satisfies conditions (i) and (ii) for a generalized André system. This system, denoted by $N(q,d)$, is a nearfield with kernel $K = \mathrm{GF}(q)$ and is called a *regular nearfield* (or sometimes, *Dickson nearfield*).

Dickson [1905], besides discovering the regular nearfields, also found seven other finite nearfields called the *irregular nearfields*. Each has order p^2, where p is an odd prime, and are descri‿ɜd in Dembowski [1968], Section 5.2.[15] Zassenhaus [1935] discovered the following major classification result. A proof can also be found in Sections 18–20 of Passman [1967].

[15] There is an error in Dembowski's description; the (2,2) entry of the matrix A on the bottom of page 230 should be 1, not -1.

(6) *Every finite nearfield is either a regular nearfield $N(q,d)$ or an irregular nearfield $N(p)$.*

It follows from Statement 2.6 that a finite nearfield is regular if and only if its multiplication group is metacyclic.[16] A similar theorem holds for infinite nearfields.

(7) *An infinite nearfield is a regular nearfield (i.e. a generalized André system) if and only if its multiplicative group is solvable.*

PROOF. Grundhöfer [1987]. Here the definition of a generalized André system is that given by Statement 2.1. □

André [1954] investigated the translation planes over nearfields and determined their collineation groups. His results are given by Dembowski.

2.4. *Semifields and their planes*

A *semifield* is a quasifield S in which the left distributive law

$$a(b+c) = ab + ac \tag{2.6}$$

holds for all a, b, c in S. This is equivalent to the mappings

$$\delta_a\colon\ (x,y) \mapsto (x, xa+y), \quad a \in S, \tag{2.7}$$

being collineations of the associated translation plane $\Pi = \Pi(S)$, which is called a *semifield plane*. In fact, they will be affine elations with axis $x = 0$ and centre (∞). Similar to Statement 2.5 is the following easily proved result.

(8) *A quasifield described by an m-spread $\mathcal{M}$ is a semifield if and only if $\mathcal{M}$ is a group under matrix addition.*

For the various algebraic substructures of a semifield the following statement holds.

(9) *The left, middle, and right nucleus and the centre of a semifield are division rings, and the semifield is a vector space*[17] *over each of them.*

Let Π be a semifield plane with respect to the points $\mathcal{O}$, $\mathcal{U}$, $\mathcal{V}$; i.e. Π is a translation plane coordinatized by a semifield S with

$$\mathcal{O} = (0,0), \qquad \mathcal{U} = (0), \qquad \mathcal{V} = (\infty).$$

[16] A group is called metacyclic provided it has a normal subgroup N, which is cyclic, such that G/N is cyclic, too.

[17] Not necessarily a left vector space. The semifield will be a right vector space over the right nucleus and both a left and right vector space over the middle nucleus.

Let $C = C(\Pi)$ be the translation complement of Π, and define C_0 to be the autotopism group of Π. If every element of C fixes the point $\mathcal{V}$ – in which case the plane Π is called a *proper semifield plane* – then

$$C = EC_0,$$

where E is the group of $(\mathcal{V}, \mathcal{OV})$-elations.

If C has a collineation which moves the point $\mathcal{V}$, it can then be shown that Π is (W, ℓ)-transitive for all points $W \in \ell_\infty$ and all lines ℓ through W; i.e. the plane Π is a *Moufang plane*. The following statement can then be proven (Hughes and Piper [1973], Chapter 8).

(10) *Two translation planes coordinatized by semifields S and S' are isomorphic if and only if S and S' are isotopic.*

For all known finite proper semifield planes the autotopism group C_0 is solvable. This has given rise to the following conjecture.

CONJECTURE. For every finite semifield plane the autotopism group is solvable. In particular, if the plane is not Desarguesian then the complete collineation group is solvable.

The most general result concerning this conjecture is due to Burmeister and Hughes [1965].

(11) *If a semifield plane Π is coordinatized by a semifield S having odd dimension over one of its nuclei, then the autotopism group C_0 of Π is solvable.*

PROOF. See Hughes and Piper [1973], Section 8.6, for the details. □

Statement 2.11 shows that the dimension of a semifield over one of its nuclei can affect the nature of the autotopism group. The next result shows that, conversely, the autotopism group can affect the size of the dimension. This important result is due to Liebler [1981], who uses group representation theory together with the observation (Cronheim [1965]) that the group $E.T(\mathcal{O})$ determines the semifield plane Π.

(12) *If Π is a semifield plane of order p^r, where p is a prime and $r \geqslant 1$, and if G is a subgroup of the autotopism group C_0 of Π which fixes a subplane pointwise, then $|G|$ divides r.*

PROOF. Let $T(\Pi) = T(\mathcal{V})T(\mathcal{U})$, where $T(\mathcal{V})$ consists of the $(\mathcal{V}, \ell_\infty)$-translations and $T(\mathcal{U})$ consists of the $(\mathcal{U}, \ell_\infty)$-translations. The three groups E, $T(\mathcal{V})$, $T(\mathcal{U})$ are FG-modules, where $F = \mathrm{GF}(p)$. By results from linear algebra and group representation theory these modules are free, and the statement follows. □

Liebler [1981] uses Statement 2.12 to show that if the alternating group A_5 is a subgroup of C_0, then 60 divides r. Similarly, if p is odd and SL$(2, p^m)$ is a subgroup of C_0, then $p^m(p^{2m} - 1)$ divides r. These results give strong support to the conjecture on the solvability of C_0.

It is now time to give some classes of proper semifield planes. This will be accomplished by giving the semifields. Except for the Boerner and Kantor semifields these semifields were all known 20 years ago.

Twisted fields and generalized twisted fields. These are proper semifields of order q^n with $n > 2$ and $q > 2$ a prime power, and they are discussed in Section 5.3 of Dembowski [1968]. The importance of twisted fields is given by the following statement conjectured by Kaplansky [1975] and proven by Menichetti [1977]. (Sherk [1990] gives a different proof.)

(13) *Every finite semifield which is an algebra of dimension three over one of its nuclei is a twisted field.*

Sandler semifields. These are described and discussed in Dembowski [1968], Section 5.3. An interesting problem is:

PROBLEM. Investigate the geometric properties of the semifield planes coordinatized by the Sandler semifields.

Knuth semifields. These semifields of even order are given in Dembowski [1968], Section 5.3.

Kantor semifields. Let $q = 2^k$ with $k \geqslant 1$, let $K =$ GF(q), and let $L =$ GF(q^d) with d odd, $d > 1$, and $q^d > 8$. If τ: $L \to K$ is the trace map then for each $x \in L$ there exists a unique element $x^* \in L$ such that

$$x = (x^*)^2 + x^* + \tau(x^*).$$

Note that $(x + y)^* = x^* + y^*$. Define the binary operation $\odot$ on L by

$$x \odot y = \tau(xy^*) + \tau(x)y^* + \left(y^*\right)^2.$$

Then under the field addition $+$ and this multiplication the set L becomes a semifield with kernel K and right nucleus GF(2). Replacing the multiplication $\odot$ with the multiplication $x \circ y = y \odot x$ also gives a semifield with kernel GF(2) and right nucleus K. In both cases the translation complement has order $q^d dk(q - 1)$. (See Kantor [1983].)

Boerner semifields. Let $p > 3$ and choose $f \in K$ such that f and $1 + 4f$ are nonsquares. (Such an element always exists if $p > 3$.) Identify GF(q^2) as the set of elements $a_1 + a_2 t$ with $a_i \in K$ and t a root of $x^2 - f$. If $\mathcal{B}$ is the (left) vector space over GF(q^2) with basis 1, u define $+$ to be vector addition and define multiplication $\circ$ by

$$(A + Bu) \circ (C + Du) = AC + B\left(D^q t - d_1\right) + \left(AD + BC^q\right)u,$$

where $D = d_1 + d_2t$. The triple $(\mathcal{B}; +, \circ)$ is a semifield of dimension 2 over its left nucleus $\mathrm{GF}(q^2)$; furthermore, the field K is both the middle and right nucleus of $\mathcal{B}$. (See Boerner-Lantz [1986].)

The above examples do not include proper semifields of dimension 2 over a nucleus. The remaining examples will be of dimension 2 over a nucleus. Let $K = \mathrm{GF}(q)$ with $q = p^k$, where p is a prime and $k \geqslant 1$, and let S be the two-dimensional vector space over K with basis $1, t$.

Dickson–Knuth semifields. These semifields of odd order can be found in Section 5.3 of Dembowski [1968].

Hughes–Kleinfeld and Knuth semifields. Let σ be a nontrivial automorphism of K, and choose elements f, g in K such that $\sigma(x)x + gx - f \neq 0$ for all $x \in K$. The vector space S is a proper semifield of dimension 2 over K under the multiplication

$$(a+bt)(c+dt) = ac + \sigma^{-2}(b)\sigma^{-1}(d)f + \left[ad + b\sigma(c) + \sigma^{-1}(b)dg\right]t. \tag{2.8}$$

These semifields, first discovered by Hughes and Kleinfeld [1960], are precisely those semifields with $N_r(S) = N_m(S) = K$ and having dimension 2 over $N_r(S)$. Hughes [1960] showed that their autotopism groups are solvable.

Knuth [1965] showed that the multiplication rule (2.8) can be replaced by any one of three other rules with the resulting semifield S having dimension 2 over K as a subfield of S. These rules are

$$(a+bt)(c+dt) = ac + b\sigma(d)f + [ad + b\sigma(c) + b\sigma(d)g]t, \tag{2.9}$$

$$(a+bt)(c+dt) = ac + b\sigma^{-1}(d)f + [ad + b\sigma(c) + bdg]t, \tag{2.10}$$

$$(a+bt)(c+dt) = ac + b\sigma^{-2}(b)\sigma(d)f + \left[ad + b\sigma(c) + b\sigma^{-1}\sigma(d)g\right]t. \tag{2.11}$$

The semifields corresponding to rule (2.9) are precisely those of dimension 2 over $N_1(S) = N_m(S) = K$ (and their planes are dual to the Hughes–Kleinfeld planes), while those with rule (2.10) are precisely the semifields of dimension 2 over $N_l(S) = N_r(S) = K$.

Ganley semifields. Ganley [1981] discovered three classes of semifields that are closely allied with the Knuth semifields given by (2.9)–(2.11). Let $q = 3^k \geqslant 9$. Then the set S becomes a proper semifield of dimension 2 over K under any one of the following multiplications:

$$(a+bt)(c+dt) = ac + fb^9 + fg^2bd + \left[ad + bc + hb^3d\right]t, \tag{2.12}$$

$$(a+bt)(c+dt) = ac + fb^9d^9 + fg^2bd + \left[ad + bc + hb^3d^3\right]t, \tag{2.13}$$

$$(a+bt)(c+dt) = ac + fb^9d + fg^2bd^9 + \left[ad + bc + hb^3d^3\right]t. \tag{2.14}$$

Here f and g are nonsquares in K and $h^2 = fg$. The Ganley semifields in which the multiplication is given by (2.13) are commutative, and they are also discussed in Cohen and Ganley [1982].

The semifields constructed above prove half of the following statement; the other half is proven in Dembowski [1968], p. 244.

(14) *Let $n = p^r$ be a prime power. There exists a proper semifield of order n if and only if $r \geqslant 3$ and $n \geqslant 16$.*

For $n = 16$ Kleinfeld [1960] has shown that there are exactly 2 nonisotopic semifields; one is a Hughes–Kleinfeld semifield and the other is a semifield with multiplication rule (2.11). Walker [1963] showed that there are 5 nonisotopic semifields of order 32, one of which is a Knuth semifield of characteristic 2.

The planes coordinatized by proper semifields are precisely the planes contained in Lenz–Barlotti class V.1. (See Lenz [1954], Barlotti [1957].) Prior to 1965 this was known to contain more nonisomorphic planes than Lenz–Barlotti class IV.a, which contains all translation planes not coordinatized by semifields. This encouraged some to conjecture that, relatively speaking, semifields were much more plentiful than quasifields that were not semifields. However, since 1965, only the Boerner, Ganley and Kantor semifields and a recent class discovered recently by Jha and Johnson [1989a][18] have been found while the techniques of derivation and homology replacement discovered by Ostrom [1968, 1970c] have given a profusion of nonisomorphic quasifields.

2.5. *Derived translation planes*

The first general construction method for affine planes, and not just translation planes, was discovered by Ostrom [1964] and is called *derivation.*[19] Johnson [1972] extended it to infinite affine planes. Geometrically, derivability is a process which, in a plane of order n^2, replaces the lines of a net of order n with Baer subplanes. (See Kallaher [1982], Chapter 7, for details.) In the case of translation planes the process can frequently be described succinctly via the quasifields.[20] Here, derivation is described for quasifields satisfying a sufficient (but not necessary) condition.

Let Π be a translation plane of order q^2 for some prime power q, and assume Π is coordinatized by a quasifield Q containing a subfield F (relative to the operations of Q) over which Q is a right vector space of dimension 2. (Thus, $F = \mathrm{GF}(q)$.) A new quasifield Q^* is defined by replacing the multiplication $\cdot$ of Q with the multiplication $\circ$ defined as follows. Choose a basis $\{1, t\}$ for Q over F. If $a \in F$ and x is an element of Q then

$$x \circ a = xa.$$

[18] The semifields of Jha and Johnson have close connections to the Sandler semifields.

[19] This is a special case of a more general method called *net replacement*, also discovered by Ostrom. See Ostrom [1968, 1970].

[20] Grundhöfer [1981] and Lunardon [1979] have given necessary and sufficient conditions on a quasifield for the associated translation plane to be derivable. Lunardon assumes finiteness, while Grundhöfer does not.

For $y \in Q - F$ and $x \in Q$, let $x = ta_1 + a_2$ and $y = tb_1 + b_2$. Choose $tc_1 + c_2$ in Q such that

$$(tb_2 + 1)(tc_1 + c_2) = tb_1;$$

then choose d_1, d_2 in F such that

$$(td_2 + a_2)(tc_1 + c_2) = td_1 + a_1.$$

Define

$$x \circ y = td_1 + d_2.$$

The triple $Q^* = (Q; +, \circ)$, where $+$ is the addition of the quasifield Q, is a quasifield called the *derived quasifield* of Q, and the translation plane Π^* coordinatized by Q^* is the *derived plane* of Π. The original plane Π and the original quasifield are called *derivable* provided that the construction is possible.

Geometrically, the lines of Π with slope in F have been replaced by the Baer subplanes

$$\Pi(x, y, z) = \{(xa + y, xb + z)\colon\ a, b \in F\},$$

where x, y, z is a triple of elements in Q. The algebraic formulation given here is called the *Albert switch* and is due to Albert [1966]. The original lines of Π which have been replaced become Baer subplanes in the new plane Π^*. Thus, the derivation process is involutory. This proves the first part of the following.

(15) *If Π is a derivable translation plane of order q^2, and if Π^* is the derived plane, then Π^* is derivable and $(\Pi^*)^* = \Pi$. Furthermore, the group of collineations of Π which preserves the set of points (m) with $m \in F$ is also a collineation group of Π^*.*

PROOF. For the second statement see Kallaher [1982], pp. 81, 82. The collineation group of the second statement is called the *inherited* (*collineation*) *group* of the derived plane Π^*. □

In the context of Statement 2.15 the inherited group may or may not be the full collineation group of Π^*. The next class of planes gives examples of both situations. Johnson and Ostrom [1990] give a sufficient condition (on the Baer subplanes used in the derivation process) for the inherited group to be the full collineation group of Π^*.

It is clear that the derivation process as defined by the Albert switch does not require finiteness, but only that the quasifield have dimension 2 over a subfield. Note also that a translation plane may possibly be derivable in more than one way (relative to different coordinatizing quasifields). Similarly, a quasifield may have more than one subfield with respect to which it is derivable. In each case the resulting derived planes need not be isomorphic. (See the next class of examples.)

2.6. *Generalized Hall planes, Hall planes*

A *generalized Hall system* is a quasifield derived from a semifield, and a *generalized Hall plane* is a translation plane coordinatized by a generalized Hall system.[21] If the semifield is itself a field, i.e. the plane is Desarguesian, then the derived quasifield is called a *Hall system*, and the derived plane a *Hall plane*. Using Statement 2.15 above, the following facts do hold for generalized Hall planes.

(16) *Let Π^* be a generalized Hall plane of order q^2 derived from the semifield plane Π using the semifield S and subfield F. The following statements hold.*

(a) *The plane Π^* contains a Baer subplane Π_0 fixed pointwise by a collineation subgroup of $L(\Pi^*)$ having order q.*

(b) *If in addition the subfield F is contained in the middle or right nucleus of S, then the Baer subplane Π_0 of* (a) *is fixed*[22] *by a collineation group of order $q(q-1)$ in $L(\Pi^*)$ which is transitive on the set $\ell_\infty - \Pi_0$.*

(c) *If Π^* is a Hall plane and $q > 2$, then it has a collineation group $G = \mathrm{SL}(2, q)$ in $L(\Pi^*)$ which has two orbits on the line ℓ_∞: $\ell_\infty \cap \Pi_0$ and $\ell_\infty - \Pi_0$.*

PROOF. The Baer subplane is the y-axis $V(\infty)$ of Π. The group in (a) is the group of affine elations δ_a with $a \in F$. (See (2.7).) For the group in (b) add the group of affine homologies obtained from F. Under the hypothesis of (c) each line of Π replaced in the derivation process is the axis of q affine elations which are inherited by Π^*, and these collineations generate G. □

Hall planes were discovered early in the study of projective planes (Hall [1943]) and have been extensively studied. If $q = 2$ then the Hall plane is the Desarguesian plane of order 4, and for $q = 3$ the Hall plane is the nearfield plane of order 9. For $q > 3$ a Hall system H is not associative; also, for $q > 2$ the field F is the kernel of H, and hence the Hall plane has dimension 2 over its kernel. The collineation group of a Hall plane has the following properties.

(17) *Let Π be a Hall plane of order q^2 with $q > 3$, and let Π_0 be the Baer subplane of Statement* 2.16(a). *The translation complement of Π has the two orbits $\ell_\infty \cap \Pi_0$ and $\ell_\infty - \Pi_0$ on the line ℓ_∞, and on Π it has the orbits $\{(0,0)\}$, the set $\mathcal{A}$ of points $P \neq (0,0)$ on lines with slope in F, and the set $\mathcal{B}$ consisting of the remaining affine points.*

PROOF. See, e.g., pp. 88–90 of Kallaher [1982]. □

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[21] This definition is slightly more general than the original definition due to Kirkpatrick [1971] and the one given by Johnson [1975]. Hiramine [1985b] has generalized the definition of Hall planes in a slightly different direction.

[22] Pointwise if the field F is contained in the middle nucleus of S. See Statement 1.9(b).

An interesting fact concerning Hall planes is that, even though a Hall system is not generalized André, every Hall plane is a generalized André plane. Indeed, a generalized André plane of dimension 2 over its kernel is a Hall plane if and only if its collineation group is nonsolvable. (See Hughes [1959].)

Consider the Hughes–Kleinfeld semifield S of order 16 and the translation plane Π it coordinatizes. This plane is also coordinatized by the semifield S' dual to S. The multiplication in semifield S is given by (2.8) with σ the mapping $x \mapsto x^2$, and that of S' by (2.9). The plane Π can be derived in three different ways giving three distinct translation planes: Π_1, Π_2, and Π_3.

The plane Π_1 obtained by deriving Π with respect to the semifield S and its middle nucleus admits affine homologies. The plane Π_2, obtained by deriving S' with respect to its middle nucleus, admits affine elations and has $S_3 \times \mathrm{PSL}(2,7)$ as its full collineation group. (This plane was discovered independently by Lorimer [1974] and Rahilly [1973] and is called the *Lorimer–Rahilly* plane. Lorimer constructed the spread directly using the fact that A_8 is isomorphic to GL(4, 2).) The plane Π_3, obtained by deriving S' with respect to its right nucleus, also admits $S_3 \times \mathrm{PSL}(2,7)$ as a collineation group. (Walker [1976b] discovered this plane using a construction similar to that of Lorimer. It was also found by Johnson, and it is known as the *Johnson–Walker* plane.)

2.7. *Johnson planes*

Generalized André planes can be constructed with the dimension over the kernel as large as desired. These planes are well known and their structure, and the structure of their collineation groups, is understood. Other planes with large dimension can be constructed using a more general form of derivation. These examples show that there are many planes of large dimension about which little is known.

Let Π be a translation plane of dimension 2 over its kernel $K = \mathrm{GF}(q)$, where $q = p^k$ with p a prime and $k \geqslant 1$, and assume $\mathcal{M}$ is the m-spread defining Π over K. The elements of a quasifield coordinatizing Π can be considered as 2-tuples over K, and the points of Π are then 4-tuples over K. If $\mathcal{M}$ contains the matrices

$$N(\sigma, a) = \begin{bmatrix} a & 0 \\ 0 & \sigma(a) \end{bmatrix},$$

where σ is a (fixed) automorphism of K and a runs over the nonzero elements of K, a new plane can be constructed as follows.

Let $\mathcal{N}$ consist of the components $x = 0$, $y = 0$, $y = xN(\sigma, a)$, and define $\overline{\mathcal{N}}$ to consist of the $q+1$ subsets (in Π)

$$\overline{N}_\infty = \{(0, a, 0, b)\colon\ a, b \in K\},$$

$$\overline{N}_c = \{(a, \sigma(a)c, b, \sigma(b)c)\colon\ a, b, \in K\}, \quad c \in K.$$

Each element $\overline{N}$ of $\overline{\mathcal{N}}$ is a subspace over GF(p) and forms a Baer subplane of Π. Replacing the components of $\mathcal{N}$ (while keeping the other components of Π) with the elements of $\overline{\mathcal{N}}$ gives a new translation plane $\Pi(\sigma)$.

The kernel of $\Pi(\sigma)$ is not in general completely known, but it does contain the fixed field E of the automorphism σ. On the other hand, if the original plane Π has a group of q affine elations with axis $x = 0$ which leaves invariant the components of $\mathcal{N}$ (and hence those of $\overline{\mathcal{N}}$) then it can be shown that E is the kernel of the new plane $\Pi(\sigma)$. Thus, in this case $\Pi(\sigma)$ has dimension $2(k/e)$, where $E = \mathrm{GF}(p^e)$. See Johnson [1988a] for details.

For a specific example the dual Hughes–Kleinfeld semifield planes given by rule (2.9) have m-spreads which contain the matrices $N(\sigma, a)$ for the defining automorphism σ. Furthermore, these planes have the appropriate set of affine elations, and thus the new planes obtained by the above process have kernel E, the fixed field of σ. By choosing k to be large, and σ so that $e = 1$, translation planes of arbitrarily large (even) dimension can be constructed.

2.8. *Ostrom–Foulser planes*

Ostrom [1970a] constructed a class of translation planes of dimension 4 over the kernel $K = \mathrm{GF}(q)$, q odd, with the interesting property that they admit affine elations in the translation complement with different axes, and these affine elations generate $\mathrm{SL}(2, q)$. Assume $q \equiv 3 \pmod 4$ and let $F = \mathrm{GF}(q^4)$; furthermore, choose a primitive element w in F, let $v = w^e$ with $e = (q^2 - 1)/2$, and let S be a subset of $K \times (K - \{0\})$. In the spread for the Desarguesian plane Π over K replace the subspaces given by the equations

$$y = x\big(a + b^2 v^{2i+1}\big), \quad i = 0, 1, \dots, q,$$

with the subspaces (in Π)

$$L(a, b, i) = \big\{(x, y)\colon\ y = xa + v^{2i+1} b^2 x^2\big\}.$$

Here the pair (a, b) runs over the elements of S^2. The result is a translation plane $\Pi(K, S)$.

If $S = K \times (K - \{0\})$ then all mappings

$$(x, y) \mapsto (xc + yd, xg + yh),$$

with $c, d, g, h \in K$ and $ch - dg \neq 0$, are collineations of $\Pi(K, S)$. If S consists only of pairs (a, b) with b fixed, then $\Pi(K, S)$ admits the affine elations

$$(x, y) \mapsto (x, xg + y).$$

Foulser [1973a] shows that the above construction can be extended to replace other sets of lines; also, he extends the process to Desarguesian planes with q even. In another article (Foulser [1973b]) he proves that the planes constructed by him and Ostrom are derivable, and in the derivation process the affine elations become collineations of order p fixing Baer subplanes pointwise. Foulser also shows that these planes contain Baer subplanes which are Hall planes. These latter planes are called *Foulser planes*.

2.9. *Hering–Ott–Schaeffer planes*

Hering [1970] gives a class of translation planes having dimension 2 over their kernel $K = \mathrm{GF}(q)$, where $q \equiv 5 \pmod 6$. This construction was generalized independently by Ott [1975] and Schaeffer [1975] to q even.

Let $K = \mathrm{GF}(q)$ with $q > 2$ and $q \equiv 2 \pmod 3$; let V be the vector space of 4-tuples over K, and let $\mathcal{B}$ be the projective space over K having dimension 3. Assume first that q is odd. If $\langle x, y, z, w\rangle$ represents coordinates in $\mathcal{B}$, let $\mathcal{C}$ be the set of $q+1$ points $\langle x^3, x^2y, xy^2, y^3\rangle$, where $x, y \in K$ with $(x, y) \neq (0, 0)$, in $\mathcal{B}$. (The set $\mathcal{C}$ is called a *twisted cubic*.) The set $\mathcal{C}$ has $q + 1$ tangents in $\mathcal{B}$; each of these tangents is a 2-dimensional subspace of V.

The vector space V has a group S of nonsingular linear transformations isomorphic to $\mathrm{SL}(2, q)$ fixing the set $\mathcal{C}$ and its tangents. In S there are $q(q-1)/2$ subgroups of order three, and each of them fixes exactly 2 lines of $\mathcal{B}$ which are 2-dimensional subspaces of V. The $q + 1$ tangents of $\mathcal{C}$ together with these $q(q-1)$ lines form a spread in V which is preserved by the group S.

Assume now that $q = 2^k$; the restriction on q implies that k is odd. Let α be a generator of the automorphism group of the field K, and let $\mathcal{C}$ be the set of $q+1$ points $\langle x\alpha(x), y\alpha(x), x\alpha(y), y\alpha(y)\rangle$ in $\mathcal{B}$, where $x, y \in K$ with $(x, y) \neq (0, 0)$. As before, there is a group S of nonsingular linear transformations on V preserving $\mathcal{C}$, and S is isomorphic to $\mathrm{SL}(2, q)$. If T is a Sylow 2-subgroup of S then T fixes a point $\mathcal{P}$ of $\mathcal{C}$, and its nonidentity elements are elations of $\mathcal{B}$ with axes through $\mathcal{P}$. The remaining two lines through $\mathcal{P}$ are each in an orbit of length $q(q-1)$ under the group S.

As in the case of q odd the subgroups of order three in S give $q(q-1)$ lines in $\mathcal{B}$ which together with one of the orbits from the preceding paragraph form a spread in V which is preserved by S. Thus two translation planes are obtained. Since an automorphism α and its inverse α^{-1} give the same two planes a total of k planes are obtained.

The *Hering planes* are those obtained when q is odd, and the *Ott–Schaeffer planes* are those obtained when q is even. As already indicated these planes have $\mathrm{SL}(2, q)$ as a collineation group, a property shared with the Hall planes and the Desarguesian planes. This essentially characterizes these planes.

(18) *Let Π be a translation plane of order q^2, where $q = p^k$ with p prime and $k \geqslant 1$. If Π admits* $\mathrm{SL}(2, q)$ *as a collineation group then Π is either Desarguesian, Hall, Hering, Ott–Schaeffer, one of two Walker planes of order* 25, *or the Dempwolff plane of order* 16.

PROOF. See Foulser and Johnson [1982, 1984]; their work is the culmination of work done by themselves and others including Ostrom, Schaeffer, Walker. □

The Dempwolff plane of order 16 is the translation plane obtained by deriving the semifield plane of order 16 coordinatized by the semifield of order 16 with multiplication rule (2.9), where $f = g = 1$ and $\sigma\colon x \mapsto x^2$. (See Johnson [1983a] and also De Resmini [1990].) The two Walker planes are described in Lüneburg [1980], Section 46.

The planes are distinguished by the possible action of the group $G = \mathrm{SL}(2,q)$. If G is completely reducible but not irreducible, then Π is Desarguesian (and the p-elements are affine elations), Hall (and the p-elements are Baer elements), or the Dempwolff plane. If G is irreducible then Π is Hering or Ott–Schaeffer. If G is reducible but not completely reducible then Π has order 25 and is either the Hering plane or one of the Walker planes.

Johnson [1988b] has given an interesting generalization of the Ott–Schaeffer planes. It is based on the fact that the Ott–Schaeffer planes can be defined using the tensor product of $\mathrm{SL}(2,q)$ by a twisted version of the same group. However, it is not clear whether there are any nontrivial generalized Ott–Schaeffer planes. Also, Barriga and Mason [1991] give a characterization of some Ott–Schaefer planes.

2.10. *Likeable planes and Walker planes*

Let $K = \mathrm{GF}(q)$, where $q = p^k$ with p a prime and $k \geqslant 1$. Assume also that $q > 3$ and $p \neq 3$. A mapping $\lambda\colon\ K \to K$ is *likeable* if it satisfies the two properties:

(a) For all $x, y \in K$, $\lambda(x+y) = \lambda(x) + \lambda(y)$;

(b) If $y^2 = x^2 y - (1/3)x^4 + x\lambda(x)$ with $x, y \in K$ then $x = y = 0$.

Kantor [1982a] defines a quasifield as follows. Let L be the (left) vector space over K with basis $t, 1$. (Here 1 is the identity of K.) Then L is a quasifield under vector addition $+$ and a multiplication $\cdot$ defined by

$$(at+b)(ct+d) = \big[a(d-c^2)+bc\big]t + \big[-(1/3)ac^3 + a\lambda(c) + bd\big],$$

where λ is a likeable mapping on K. The quasifield L is called *likeable*; the corresponding translation plane is also called *likeable.*

If $q \equiv -1 \pmod 6$ and $\lambda\colon\ K \to K$ maps every $x \in K$ to 0, then the resulting quasifields and planes are the *Walker quasifields* and *planes* first found by Walker [1976a].[23] If $p = 5$ and $k > 1$, and if $\alpha \in K$ is a nonsquare, then the mapping $\lambda\colon\ K \to K$ given by

$$\lambda(x) = \alpha x^5 + \alpha^{-1} x$$

is a likeable mapping. A third example is obtained when $p = 2$, $k > 1$ and odd, by choosing an element $\gamma \in K$ and defining $\lambda\colon\ K \to K$ to be the mapping

$$\lambda(x) = \gamma^2 x + \gamma x^2.$$

Ganley [1983] proved that no other likeable quasifields with $p = 2$ occur.

The likeable planes all share a very interesting property given in the following result. This follows partly from the fact that in every likeable quasifield L the kernel is K, and hence L has dimension 2; but more importantly, the distributor (see Definition 1.18) of L is also K.

[23] These planes were also discovered independently and at approximately the same time by Betten [1973] while studying topological translation planes.

(19) *Let L be a likeable quasifield of dimension* 2 *over its kernel* $K = \mathrm{GF}(q)$, *where* $q = p^k > 3$ *and* $p \neq 3$, *and let* Π *be the corresponding likeable translation plane. The following statements hold.*

(a) *The linear translation complement* $L(\Pi)$ *contains an Abelian group* S *of order* q^2 *with a subgroup* E *of order* q *consisting of affine elations with axis* $(0,0)(\infty)$.

(b) *The group* S *fixes* (∞) *and is transitive on* $\ell_\infty - \{(\infty)\}$.

(c) *If* q *is odd then* S *is elementary Abelian and contains a subgroup* F *such that* S *is the direct sum of* E *and* F *and the elements of* F *fix every point of a one-dimensional subspace of* $(0,0)(\infty)$ *and no other point of* Π.

(d) *If* L *is the Walker quasifield of order* 5^2 *then* $L(\Pi)$ *contains* $\mathrm{SL}(2,5)$ *as a subgroup.*

PROOF. See Kantor [1982a] and Walker [1976a]. □

Result 2.19 is interesting for the following reason. The projective planes obtained from the likeable planes by dualizing with respect to the point (∞) and the line ℓ_∞ are derivable in the more general sense (Ostrom [1968]), and the resulting derived planes have the property that they are (P,ℓ)-transitive for exactly one point-line pair P, ℓ (and $P \in \ell$).

Likeable planes have been generalized by Fink, Johnson and Wilke [1983], and this generalization has been studied by Biliotti and Menichetti [1985]. In the first article the possible planes are determined when the kernel has prime order, while the authors in the second article determine the possible planes when the order is even.

2.11. *Lüneburg planes*

Lüneburg [1965] presented a class of quasifields and translation planes that also have a nontrivial distributor and nontrivial affine elations, respectively. These planes have as kernel a field K of characteristic 2 with the property that GF(4) is not a subfield. The existence of these planes, but not their properties, had been observed previously by Tits [1962]. Dembowski [1968], Section 5.2, describes the finite Lüneburg planes by means of their quasifields[24], and Lüneburg [1980], Chapter 4, describes the finite and infinite Lüneburg planes by means of their spreads in the vector space of 4-tuples over K, the kernel.

The finite Lüneburg planes have been characterized by Liebler [1972] using the groups $\mathrm{Sz}(q)$.

(20) *Let* $q = 2^k$ *with* $k = 2m+1 > 1$, *and let* Π *be a translation plane of order* q^2. *The plane* Π *is a Lüneburg plane if and only if its linear translation complement contains a subgroup* $S \cong \mathrm{Sz}(q)$, *the simple Suzuki group.*

In the context of 2.20 Lüneburg has determined the possible actions of $\mathrm{Sz}(q)$. See Section 28 of Lüneburg [1980].

[24] Dembowski's rule for multiplication in the finite Lüneburg quasifields is not correct; in his notation the correct rule is: $(x,y)(u,v) = (xu + y(u + u^\sigma + v^{\sigma+1}), xv + y(u+v))$.

2.12. *Kantor planes*

Kantor [1982b,c] discovered a host of translation planes by considering certain collections of subspaces in $\Omega^+(4m, q)$ vector spaces, where q is even. In Kantor [1983] many of these planes are described using spreads or quasifields. Some of these are given here. Let $K = \mathrm{GF}(q)$ and $L = \mathrm{GF}(q^m)$, where $q = 2^k$ with $k \geqslant 1$ and $m \geqslant 3$. Furthermore, let τ: $F \to K$ be the trace map for an intermediate subfield F.

Kantor flag-transitive planes. Let $m = 4n - 2$ with $n > 2$, let $F = \mathrm{GF}(q^{2n-1})$, let W be the kernel of τ, and fix an element $a \in \mathrm{GF}(q^2) - K$. The set

$$\{bW + baK\colon\ b \in F,\ |b| \mid (q^{2n-1} + 2)\}$$

forms a spread in the vector space L over K. The resulting translation plane has order $q^{2n-1} > 8$, and the kernel is the field K.

The number of distinct planes given by this construction is at least $q/2k$, and for each plane the collineation group is flag-transitive. The translation complement has order $(q-1)(q^{2n-1}+1)(2n-1)u$, where u divides k, and it contains a cyclic subgroup that is sharply transitive on ℓ_∞, the line at infinity.

Kantor type I planes. Let $q > 2$, and take $F = L$ with $m = 2n - 1$. Choose an element $a \in K - \mathrm{GF}(2)$. The mappings

$$\alpha\colon\ x \mapsto [x + a\tau(x)]/(a+1),$$

$$\beta\colon\ x \mapsto [x^2 + ax\tau(x)]/(a+1)$$

are bijections on F. On F define the binary operation $\circ$ by

$$x \circ y = \left[\left(\alpha^{-1}(x)\right)^2 \beta^{-1}(y) + a\alpha^{-1}(x)\tau\left(\alpha^{-1}(x)\beta^{-1}(y)\right)\right]/(a+1).$$

The algebra $(F; +, \circ)$, where $+$ is field addition, is a quasifield with kernel K and order q^{2n-1}.

The number of distinct planes so constructed is at least $(q-2)/2k$. The translation complement has order $(q^{2n-1}-1)(2n-1)u$, where u divides k, and it contains a cyclic subgroup (isomorphic to the multiplicative group of F) which fixes the points $\mathcal{O} = (0,0)$, $\mathcal{U} = (0)$, $\mathcal{V} = (\infty)$ and is transitive on the three sets $\mathcal{OU} - \{\mathcal{O}, \mathcal{U}\}$, $\mathcal{OV} - \{\mathcal{O}, \mathcal{V}\}$, $\mathcal{UV} - \{\mathcal{U}, \mathcal{V}\}$.

Kantor type II planes. For these planes let k be odd, choose $F = \mathrm{GF}(q^2)$, and define V to be the set of 3-tuples over F considered as a (six-dimensional) vector space over K. Choose $\omega \in F$ such that $\omega^3 = 1 \neq \omega$. Also, the elements of $\mathrm{SU}(2, q) \cong \mathrm{SL}(2, q)$ can be represented as 2 by 2 matrices over F having the form

$$M(\alpha, \beta) = \begin{bmatrix} \alpha & \beta^q \\ \beta & \alpha^q \end{bmatrix},$$

where α, β range over F.

A spread can be defined in V as follows. For each matrix $M(\alpha, \beta)$ consider the subspace (over K)

$$\{((\gamma\alpha^2 + \gamma^q\beta^2), a\omega\alpha + \gamma^q\beta, a\omega\beta + \gamma^q\alpha^2)\colon\ \gamma \in F,\ a \in K\}.$$

There are $q^3 - q$ such subspaces. For $\delta \in F$ with $\delta \neq 0$ consider the subspace

$$\{(a\delta^2, \gamma\delta, \gamma\delta^q)\colon\ a \in K,\ \gamma \in F\}.$$

There are $q + 1$ such subspaces. These $q^3 + 1$ subspaces form a spread $\mathcal{K}$ in V, and hence a translation plane of order q^3 with kernel K is obtained.

The plane constructed in the preceding paragraph has $\mathrm{SL}(2, q)$ in its translation complement. On the line ℓ_∞ this group has two orbits of lengths $q+1$ and $q^3 - q$. Furthermore, the translation complement has order $q(q^2 - 1)^2k$ and contains elements of order $q + 1$ fixing a Desarguesian subplane of order q on which the group $\Gamma\mathrm{L}(2, q)$ is induced.

Kantor discovered many other planes, including some with trivial translation complement. For a description of them, see one of the references listed above. Also, Liebler [1983] gives a second way for constructing the Kantor type II planes.

2.13. *Suetake planes. Hering plane of order 27*

Let q be an odd power with the property that 2 is not a square in $\mathrm{GF}(q)$. For each nonzero square $a \in \mathrm{GF}(q^3)$ define the matrices $M(a)$, $N(a)$ by

$$M(a) = \begin{bmatrix} a^q & 0 & a^{q^2} \\ a & a^{q^2} & 0 \\ 0 & a^q & a \end{bmatrix},$$

$$N(a) = \begin{bmatrix} a^q & -a^{q+q^2-1} & a^{q^2} \\ a & a^{q^2} & -a^{q^2+1-q} \\ -a^{1+q-q^2} & a^q & a \end{bmatrix}.$$

The set of all such matrices $M(a)$ and $N(a)$ form a m-spread $\mathcal{M}$ for the 3-dimensional space U over $\mathrm{GF}(q)$. This yields a translation plane Π of dimension 3 over $\mathrm{GF}(q)$. The case $q = 3$, in particular, gives the Hering plane[25] of order 27. (See Hering [1969].) Suetake [1985] shows that the linear translation complement $L(\Pi)$ has order $3(q-1)(q^3 - 1)$ with two orbits on ℓ_∞, one of length 2 and the other of length $q^3 - 1$.

2.14. *Bartolone–Ostrom planes*

Bartolone and Ostrom [1986] give a class of translation planes having dimension 3 and possessing the group $\mathrm{SL}(2, q)$ in the linear translation complement. Let $q = p^k$ with p

[25] This plane is incorrectly described on p. 236 of Dembowski [1968].

a prime and $k > 1$, let $K = \mathrm{GF}(q)$ and $F = \mathrm{GF}(q^3)$, and let V be the vector space of 2-tuples over F.

Now F can be considered as a three-dimensional vector space over K. Choose $a, b, c \in F$ with $abc \notin K$, and let ρ be the linear transformation on F represented by the matrix

$$\begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}.$$

For $\alpha \in K$ with $\alpha \neq 0$ let ρ_α be the linear transformation on F represented by the diagonal matrix $\mathrm{diag}[\alpha, \alpha^q, \alpha^{q^2}]$. Furthermore, define G to be the group of mappings on V given by

$$(x, y) \mapsto \bigl(\rho_\alpha(x) + \rho_\beta(y), \rho_\gamma(x) + \rho_\delta(y)\bigr),$$

where $\alpha\delta - \beta\gamma = 1$. Using the notation of Section 1 define

$$\mathcal{S}_0 = \bigl\{V(0), V(\infty), V(1)\bigr\} \cup \bigl\{V(\rho_\alpha)\colon\ \alpha \in K,\ \alpha \neq 0\bigr\},$$

$$\mathcal{S}_1 = \bigl\{\beta(v(\rho))\colon\ \beta \in G\bigr\}, \quad \text{and} \quad \mathcal{S}_2 = \bigl\{\beta(V(\rho_\xi \rho))\colon\ \beta \in G\bigr\},$$

where ξ is a fixed nonsquare in K.

For q even the set $\mathcal{S}_0 \cup \mathcal{S}_1$ is a spread on V, while for q odd the set $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ is a spread on V; each of these spreads gives a new non-Desarguesian translation plane Π of order q^3 with the group G contained in $L(\Pi)$. Moreover, the group $G = \mathrm{SL}(2, q)$, and the p-elements are affine elations.

Jha and Johnson [1989b] investigate translation planes Π of order q^3 admitting $\mathrm{GL}(2, q)$ as a subgroup of $L(\Pi)$; furthermore, they assume the p-elements of $\mathrm{GL}(2, q)$ are affine elations. These planes include the Kantor type II planes and the Bartolone–Ostrom planes. Jha and Johnson determine all such planes; the interesting point is that they make no assumption about the kernel, and they show there are such planes of any finite dimension over the kernel.

2.15. *Flocks and translation planes. The Fisher planes*

Potentially, one of the most important discoveries in the last fifteen years is the connection between flocks[26] and translation planes. Let $\mathcal{B}$ be the projective space of dimension 3 over $\mathrm{GF}(q)$. A *flock* in $\mathcal{B}$ is a maximal collection $\mathcal{F}$ of disjoint conics (i.e. planar sections each having $q + 1$ points, no three collinear) on a quadratic surface $\mathcal{Q}$ of $\mathcal{B}$.

There are three types of quadratic surfaces.

(I) Elliptic quadratic surface (ovoid). Here $\mathcal{Q}$ has $q^2 + 1$ points, and the flock $\mathcal{F}$ has size $q - 1$; hence two points of $\mathcal{Q}$ are not covered by conics in $\mathcal{F}$.

(II) Hyperbolic quadratic surface. Here $\mathcal{Q}$ has $(q + 1)^2$ points, and $\mathcal{F}$ has size $q + 1$; thus every point of $\mathcal{Q}$ is covered by a conic in $\mathcal{F}$.

[26] Flocks are also discussed in Section 7.11 of this text.

(III) Quadratic cone. Here $\mathcal{Q}$ has $q^2 + q + 1$ points, one of which is a special point called the *vertex* of $\mathcal{Q}$, and $\mathcal{F}$ has size q; in this case the vertex is not covered by a conic in $\mathcal{F}$.

Walker [1976a] showed that for a given flock $\mathcal{F}$ on a quadratic surface $\mathcal{Q}$ in $\mathcal{B}$ a translation plane can be constructed as follows. Embed $\mathcal{Q}$ in the Klein quadric $\mathcal{K}$ of the 5-dimensional projective span over GF(q). Furthermore, let $\mathcal{P}_i$ be the plane in $\mathcal{B}$ containing the i-th conic $\mathcal{C}_i$ of $\mathcal{F}$. If ρ is the polarity associated with $\mathcal{K}$, then ρ interchanges points and 4-dimensional subspaces, linear and 3-dimensional subspaces and ρ maps planes to planes. In particular, let $\mathcal{P}_i^* = \rho(\mathcal{P}_i)$, let $\mathcal{C}_1^* = \mathcal{P}_i^* \cap \mathcal{K}$, and consider the points of $\bigcup \mathcal{C}_i^*$ plus the points of $\mathcal{Q}$ not covered by the conics (when $\mathcal{Q}$ is of type I or type III). There are always $q^2 + q + 1$ such points with no two of them together on a ruling line of $\mathcal{K}$. Using Plücker coordinates, these points yield $q^2 + q + 1$ lines in $\mathcal{B}$ which in turn gives a spread in 4-dimensional vertex space over GF(q) and hence a translation plane $\Pi(\mathcal{F})$ of order q^2 with kernel containing GF(q). Conversely, given such a translation plane, a flock $\mathcal{F}$ can be constructed on an appropriate quadratic surface. Moreover, Gevaert and Johnson [1988] showed that two flocks are isomorphic if and only if the corresponding translation planes are isomorphic.

A flock is *linear* if the planes of the conics contain a common line. This happens if and only if the corresponding translation plane is Desarguesian. If the underlying quadratic surface is elliptic or hyperbolic with q even, then the flock $\mathcal{F}$ is linear (Orr [1973], Thas [1973, 1975]), and thus $\Pi(\mathcal{F})$ is Desarguesian. Thus, non-Desarguesian translation planes can arise only when the underlying quadratic is either hyperbolic with q odd or a quadratic cone (Thas [1975], Fisher and Thas [1979]).

Many nonlinear flocks have been discovered, but most of them give rise by the above construction to translation planes already described, including the Walker planes, the Knuth semifield planes, and the likeable planes. J.C. Fisher discovered a class of nonlinear flocks described as follows.

Let q be odd with $q > 3$ and consider the quadratic cone $\mathcal{Q}$ arising from the cylinder $x^2 - \alpha y^2 = 1$ of 3-dimensional affine space $\mathcal{A}$ over GF(q) where α is a nonsquare of GF(q). The conics $\mathcal{Q} \cup \mathcal{P}_a$, where $\mathcal{P}_a$ is the horizontal plane $z = a$ of $\mathcal{A}$ give a linear flock of $\mathcal{Q}$. Consider the plane $\mathcal{P}_1$ and the regular $(q+1)$-gon $\{P_{1i}\colon\ i = 0, 1, \ldots, q\}$ in $\mathcal{P}_1$. Define $\mathcal{P}$ to be the plane through the origin $\mathcal{O}$ and the points P_{10}, P_{11}. If ϕ is the rotation about the z-axis mapping P_{1i} to $P_{1,i+2}$, then the planes $\phi^i(\mathcal{P})$, where $i = 1, \ldots, (q+1)/2$, together with the $(q-1)/2$ horizontal planes containing no point of $\mathcal{P} \cap \mathcal{Q}$ generate a nonlinear flock on $\mathcal{Q}$.

The translation planes associated with the Fisher flocks are, naturally, called the *Fisher* planes.[27] These planes are derivable (Gevaert and Johnson [1988]), and the derived Fisher planes have properties placing them close to the Hall planes which are derived from the Desarguesian planes).

[27] Payne [1988] shows that the Fisher planes are isomorphic to the translation planes discovered by Baker and Ebert [1988] using a net replacement method.

Bader, Lunardon and Thas [1990] discuss a construction process (which they call *derivation*[28]) that produces q new flocks from a given flock on a quadratic cone in the case where q is odd. The construction method applied to the flocks corresponding to the likeable planes with $q = 5^e$ and $e > 1$ yields new flocks. On the other hand, Johnson, Lunardon, and Wilke [1991] show that the flocks corresponding to the Walker planes and the Knuth semifield planes are precisely those flocks that admit a transitive (on the cones) automorphism group and whose derived flocks are all isomorphic to the original. Bader [1990] and Payne and Thas [1991] show that a flock derived from a Fisher flock is again a Fisher flock.[29]

Flocks have played an important role in investigation of certain types of translation planes; see, e.g., Jha and Johnson [1989c] and Johnson [1991]. Johnson [1990b] investigates the translation planes derived (in the sense of Ostrom) from the Fisher planes and shows their similarity to the Hall planes.

This completes our list of planes. Although not complete, it is representative of the variety among the known planes. Recent works by Charnes [1990], Lüneburg [1980], Sections 18 and 19, Mason [1985], Ostrom [1984], and Mason and Ostrom [1985] describe several planes of small order with interesting collineation groups acting in interesting ways. Mention also must be made of the class of planes discovered by Narayana Rao [1973] which have solvable collineation groups transitive on affine lines.

Many known translation planes not listed here can be obtained from the given ones by derivation or the more general process of net replacement. (See Ostrom [1968].) Examples include the classes of planes found by Bruen [1978] and Crismale [1985]. Some classes originally found by different methods have later been shown to be obtainable from known planes by derivation; examples include the class found by Narayana Rao and Satyanarayana [1983] and one of the two classes found by Cohen and Ganley [1984]. (Jha and Johnson [1986b] show both classes are derivable from the Walker planes.)

Of course, it is also possible to perform a sequence of constructions to obtain possibly new planes. One example of such a sequence is given by Johnson [1988c] which is based on the work of Hiramine, Matsumoto, and Oyama [1987].

Some attempts have been made to reasonably estimate the number of nonisomorphic translation planes of a given order. Dempwolff and Reifart [1983] determined all translation planes of order 16 (there are 8 of them), and Davis [1979] and Oakden [1973] have investigated the number of translation planes of order 25. (Also, see Czerwinski [1991].) Ebert [1978] obtained asymptotic estimates of the number of translation planes of order q^2 with kernel GF(q).

Contrary to the situation of the 1960's large number of examples of translation planes are known, and the existence of new planes is by itself of little interest. New examples with previously unseen properties, or new properties of known planes, are needed. For example, the list of known nonsolvable groups acting on finite translation planes has not increased since 1975. Note also that few classes of planes of odd dimension are known.

[28] It is unfortunate that this term is used, since the construction does not correspond to the derivation method of Ostrom.

[29] See Chapter 7 by Thas who discusses the technique of derivation move fully and describes some new flocks obtained by using it.

3. Dimension and collineations

In this section the interplay between the dimension of a translation plane and collineations of the plane will be examined. Work of Ostrom and others has shown that odd dimension substantially restricts the possibilities for collineations and collineation groups contained in the linear translation complement. This is a sequel to Baer's famous theorem that an involutory collineation is either a perspectivity or a Baer involution, and that the latter can only occur if the plane has square order.

Throughout this section let Π be a finite translation plane of dimension d over its kernel $K = \mathrm{GF}(q)$, where $q = p^k$ with p a prime and $k \geqslant 1$, let $C(\Pi)$ be the translation complement, and let $L(\Pi)$ be the linear translation complement of Π. A simple indication of the impact of the integer d is given in the following generalization of Statement 2.15. The first part is true for finite dimensional planes over infinite kernels.

(1) *If d is odd then $L(\Pi)$ contains no Baer involutions. Furthermore, the subgroup of $C(\Pi)$ fixing a component is solvable.*

The second part of Statement 1 goes back to Hering [1972a]. The first part has been generalized by Ostrom [1981a] and Kallaher and Ostrom [1982] in the case when Π has odd order.

(2) *Assume the plane Π has odd characteristic p, and let S be an elementary Abelian 2-group contained in $L(\Pi)$ with $|S| = 2^e$.*
- (a) *If the nonidentity elements of S are Baer involutions, then 2^e divides d.*
- (b) *If $e \geqslant 2$ then S has a subgroup of order 2^{e-2} all of whose involutions are Baer, and 2^{e-2} divides d.*

PROOF. Statement (a) follows from the fact[30] that the associated Baer subplanes intersect in a subplane, fixed by S, of dimension $d \mid |S|$. Statement (b) is a consequence of the fact that S can contain at most 2 affine homologies whose composition has axis ℓ_∞. The condition in (a) can be replaced by the condition that the involutions in S are conjugate in $L(\Pi)$ (provided $e \geqslant 2$). □

Ostrom [1981b] considers a 2-subgroup S which induces on the line ℓ_∞ an elementary Abelian 2-group S^*, and he obtains a conclusion[31] similar to (a). Jha and Ostrom [1981] consider subgroups U of $L(\Pi)$ having order u^e, where u is a prime.

(3) *Let U be an elementary Abelian u-group contained in $L(\Pi)$ with $|U| = u^e$.*
- (a) *If U fixes a subplane pointwise, then $d \geqslant 2^e$; in any case, $d \geqslant 2^{e-2}$.*
- (b) *If the origin is the only point fixed by U, then $2d$ is divisible by the multiplicative order of q* (mod u).

[30] This fact follows from Foulser's work on collineation groups fixing a Baer subplane pointwise; see Foulser [1972].

[31] Ostrom's result is correct but the proof is erroneous.

Fink and Kallaher [1987] and Ostrom [1986] have used statement 2 to investigate noncyclic simple groups acting on the plane Π, when q is odd.[32] Fink and Kallaher showed that, for q odd, if a simple group G acts on Π then $L(\Pi)$ contains a subgroup isomorphic to G, and all involutions in G are Baer involutions.

(4) *Let G be a noncyclic simple group in $L(\Pi)$, and assume q is odd.*

(a) *If 2^e is the order of a maximal elementary Abelian 2-subgroup of G, then d is divisible by 2^e. In any case, 4 divides d.*

(b) *If 8 does not divide d, then either $G = \mathrm{PSU}(3,4)$ or $G = \mathrm{PSL}(2,u)$ with $u \equiv \pm 3 \pmod 8$.*

No known translation plane of odd order has $\mathrm{PSL}(2,u)$ as a collineation group. Indeed, the Lorimer–Rahilly and Johnson–Walker planes of order 16 are the only planes known to have such a group with u prime to q; both have $\mathrm{PSL}(2,7) \cong \mathrm{SL}(3,2)$ as a collineation group. Of course, the Desarguesian planes of order 2^k and the Hall planes of order 2^{2k} have $\mathrm{PSL}(2,2^k) = \mathrm{SL}(2,2^k)$ in the linear translation complement.

The above raises the following question which is a special case of the question given after Statement 1.8.

QUESTION. What noncyclic simple groups act as subgroups of $L(\Pi)$ for one or more finite translation planes Π?

Besides the above examples the Suzuki group $\mathrm{Sz}(2^s)$ is in $L(\Pi)$ for the Lüneburg plane of order 2^{2s} and for Lüneburg planes of higher order. Note that all of these examples have even order. No known translation plane of odd order has a noncyclic simple group as a collineation group!

The above question has been investigated by several people in recent years. Mason [1983] and Foulser, Mason and Walker [1984] have considered the possibility of $\mathrm{PSL}(n,w)$ acting on a translation plane Π, while Fink and Kallaher [1987] have considered the sporadic simple groups acting on Π. In all three articles a critical assumption is that the simple groups, as subgroups of $L(\Pi)$, act irreducibly on Π.

(5) *Assume the group $\mathrm{SL}(n,q)$ induces a group G of collineations contained in $L(\Pi)$, and assume G acts irreducibly on Π. If (a) $p = 2$ or (b) k is even, and 3 does not divide $p-1$, then $n = 2$ and $G = \mathrm{SL}(2,q)$.*

PROOF. Mason [1983] and Foulser et al. [1984]. Assuming that $n \neq 2$ it can be shown that only the case $n = 3$ need be considered. This case is excluded by using the Steinberg tensor product theorem to describe the irreducible KG-module, and then the conditions on p show that a certain element of $\mathrm{SL}(3,q)$ cannot induce a collineation on Π. (The power q can be replaced by any power p^e such that $\mathrm{GF}(p^e)$ is contained in K.) □

[32] However, the main results of Ostrom [1986] are false at least in the case where q is a power of 3; see Ostrom [1988].

Fink and Kallaher [1987] also assumed irreducibility, but used statement 2 and the fact that involutions must have character value 0 together with character tables for the sporadic simple groups to obtain the next result.

(6) *Let p be an odd prime, and let G be a sporadic simple group contained in $L(\Pi)$. Assume G acts irreducibly on Π.*

(a) *If p does not divide $|G|$, then $G \neq \mathsf{J}_1, \mathsf{J}_2, \mathsf{M}_{12}, \mathsf{M}_{24}$, HS, Sz, Ru, .3, .1, or F_{23}.*

(b) *If G is a Mathieu group then either $G = \mathsf{M}_{11}$ or $G = \mathsf{M}_{23}$.*

Ostrom [1977a, 1980b] and Johnson and Ostrom [1983] discuss the case $d = 2$ and delineate the possible subgroups of $L(\Pi)$. Among the possibilities are the groups $\mathrm{SL}(2, p^s)$, $\mathrm{SL}(2, 5)$, $\mathrm{SL}(2, 9)$, and (extraspecial) groups G with a normal subgroup $Q \cong Q_8 \cdot D_8$.

Examples of such planes with such collineation groups occur. They include the Hall planes, the Hering–Ott–Schaeffer planes, the Ostrom planes, and the Foulser planes (for $\mathrm{SL}(2, p^s)$), and certain irregular nearfield planes and the two Walker planes of order 25 (for $\mathrm{SL}(2, 5)$). Mason [1985] constructed two planes of order 49 with $\mathrm{SL}(2, 9)$ in the linear translation complement, while Mason and Ostrom [1985] constructed planes of orders 49 and 121 with an extraspecial group as collineation group.[33]

Several people have studied the possibility of $\mathrm{SL}(n, w)$, where $n \geqslant 2$ and w is a prime power (not necessarily of p), being contained in $L(\Pi)$. Under additional hypotheses it is shown that the plane is known. For example, Kallaher and Ostrom [1982] derived – without the assumption of irreducibility – some general conclusions, of which the following is an example.

(7) *Let $\mathrm{SL}(n, w)$, where $n \geqslant 3$ and w is a prime power, be contained in $L(\Pi)$. If q is odd then the following statements hold.*

(a) *If both n and w are odd then 2^{n-1} divides d.*

(b) *If n is even and w is odd then 2^{n-2} divides d.*

(c) *If $w = 2^s$ then $2^{s(n-1)}$ divides d.*

Furthermore, if $d = 4$ then q odd implies $n = 3$ and $w = 2$, while q even implies either $n = w = 3$ or $n \geqslant 8$ and $w - 2^s$ for $s \geqslant 1$.

The work of Jha [1982], and Jha and Kallaher [1981, 1982] on this question is discussed in the next section. Note that by the above, if $d = 2$ then $\mathrm{SL}(n, w)$ in $L(\Pi)$ implies $n = 2$. Johnson [1986a] gives an overview of the known information about this case.

Ostrom in a series of articles [1978, 1979, 1980a, 1981c, 1985] continues his investigations of the effect of the dimension d on the group $L(\Pi)$. In these articles, as well as those discussed above, a frequent hypothesis is that the group under consideration is (group) irreducible on the plane Π. Geometrically, this assumption appears to be unnatural. Ostrom [1981c] discusses a natural condition he calls geometrical irreducibility.

A subgroup G of $L(\Pi)$ is *geometrically irreducible* if no invariant vector subspace of G is a proper subplane or a component of Π. Note that if G is irreducible in the group theory sense, then it is geometrically irreducible.

[33] Other examples of such planes are contained in Mason and Shult [1986].

QUESTION. Under what additional 'geometrical' conditions, if any, does geometrical irreducibility imply (group) irreducibility?

A geometrically irreducible subgroup G of $L(\Pi)$ is *geometrically primitive* if Π (as a vector space over K) is not a direct sum of proper subspaces which are subplanes or components of Π and which are permuted by G. Ostrom [1981c] shows that a solvable geometrically primitive subgroup of $L(\Pi)$ must be fixed-point-free on Π or metacylic or contain a normal subgroup of order w^{2a+b}, where w is a prime, with w^a dividing d.

In closing this section consider briefly the case $d = 3$. The result of Menichetti – Statement 2.13 – indicates that the following is within reach.

QUESTION. What are the possible planes if $d = 3$?

The Kantor type II planes and the Bartolone–Ostrom planes show that there are translation planes with $d = 3$ and having $L(\Pi)$ nonsolvable. These planes all have SL$(2, q)$, where GF(q) is the kernel, contained in $L(\Pi)$. The translation plane of order 27 discovered by Hering [1969] has SL$(2, 13)$ contained in $L(\Pi)$. This is the only known translation plane having SL$(2, w)$ as a collineation group where w is relatively prime to the order.

QUESTION. If Π is a translation plane of dimension d over its kernel GF(q) having SL$(2, w)$, where w and q are relatively prime, in $L(\Pi)$, is Π the Hering plane of order 27? What if $d = 3$?

Finally, all known nonsemifield planes of odd order and odd dimension with solvable collineation groups are generalized André planes. This gives rise to the question.

QUESTION. If Π is a nonsemifield translation plane of odd order and odd dimension with $L(\Pi)$ solvable, is Π a generalized André plane?

4. Collineation groups

Let Π be a finite translation plane, and let $G = G(\Pi)$ be its collineation group. By statement 1.8 the structure of G is determined when the structure of the linear translation complement $L(\Pi)$ is found. Also, by 1.8 the group $L(\Pi)$ is a subgroup of GL$(2d, q)$, where $K =$ GF(q) is the kernel and d is the dimension of π.

Investigations of the possible structures for the group $L(\Pi)$ have imposed additional hypotheses either on it or on a subgroup. Beginning with the work of Baer in the early 1940's possible actions of a specific type of collineation (elation, homology, involution) on projective planes were considered. The work of Baer, André, Dembowski, Hughes, Piper, and others is important in this respect. (See Dembowski [1968], especially Sections 4.1 and 4.3.) Although for general projective planes this is not quite complete, it is for translation planes.

In the 1950's and 1960's, as already indicated in Sections 2 and 3, several investigators considered the question of whether specific groups – e.g., $\mathrm{SL}(2, q^d)$ or $\mathrm{Sz}(q)$ – could be collineation groups of projective planes. Also considered were projective planes having the property that every geometric configuration of a certain type (e.g., point, line, flag, antiflag) is fixed by a specific type of collineation (e.g., elation or homology); the resulting theorems, although very important for the general theory, do not give much insight into translation planes. One exception is the following result which is a consequence of Lüneburg [1965] and 2.20.

(1) *If Π is a projective plane of order n such that for every flag (P, ℓ) there is a nontrivial elation fixing (P, ℓ), then Π is either Desarguesian, a Lüneburg plane, or dual to a Lüneburg plane.*

PROOF. See 4.3.25 in Dembowski [1968]; this is Lüneburg's result that Π must be either Desarguesian, a translation plane with $\mathrm{Sz}(\sqrt{n})$ contained in $L(\Pi)$, or the dual of such a plane. An outline of the proof of Lüneburg's result is given by Dembowski. Note also that the possibilities for the group generated by the nontrivial elations are listed.

A third type of investigation in the 1960's involved collineation groups being transitive or doubly transitive on certain subsets of points (or lines) of projective planes. Results include the Higman–McLaughlin theorem on flag-transitive collineation groups, the Ostrom–Wagner theorem on doubly transitive collineation groups, and the Wagner theorem on affine line-transitive collineation groups. (See Statements 4.4.14, 4.4.20, and 4.4.22 in Dembowski [1968].) Also included is work of Cofman [1967a,b,c, 1968]. As the hypothesis in most cases implied there was no line fixed by the collineations (and the conclusion usually was that the projective plane is Desarguesian), little insight into the structure of translation planes was obtained. Wagner's theorem is an exception – as is Cofman's work – and this will be discussed below.[34]

During the 1970's investigators of collineation groups of finite translation planes usually imposed hypotheses which took one of two forms: the existence of certain types of collineations, e.g., affine elations, Baer elements or certain transitivity properties, e.g., transitivity or double transitivity on certain subsets of points. The goal in all cases was both a description of the collineation group and a list of the possible translation planes.

4.1. *Affine perspectivities*

An *affine perspectivity* in a translation plane is a perspectivity whose axis is an affine line; an *affine elation* (*homology*) is an affine perspectivity which is an elation (homology). Note that the centre P of a nontrivial affine perspectivity ρ must lie on the line ℓ_∞; thus, if ℓ is the axis of ρ than $P = \ell \cap \ell_\infty$ is called the *cocentre of* ρ.[35]

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[34] Cofman's work will not be discussed, but it is an interesting line of inquiry which is still incomplete. Recent work on doubly transitive groups should be helpful.

[35] An affine elation is also called a *shear*, and an affine homology is sometimes called a *strain*. This nomenclature arose in the early study of Desarguesian projective geometries and is used mostly by European geometers.

Statement 1.9 implies that every affine perspectivity of a translation plane Π whose axis is a component is contained in the linear translation component. Thus, in discussing groups generated by affine perspectivities attention may be restricted to $L(\Pi)$. Also, if a set of affine perspectivities of Π forms a group then they must all have either the same axis or the same centre. The following facts, except (d), are easy to derive.

(2) *Let Π be a translation plane.*

(a) *Two affine elations of Π commute if and only if they have the same axis or the same centre.*

(b) *If P is a point on ℓ_∞ and ℓ is an affine line with $P \notin \ell$, then there is at most one involutory homology with centre P and axis ℓ.*

(c) *A finite group of affine homologies of Π with the same centre and axis is a Frobenius complement.*[36]

(d) *If Π is finite and it has a nonsolvable group of affine homologies, then Π has even dimension over its kernel.*

PROOF. Lüneburg [1980], Section 3. □

A major result in the early 1970's was the determination of the subgroup of the linear translation complement generated by affine elations. It is due to Ostrom [1970b, 1974] and Hering [1972b, 1976].

(3) *Let Π be a translation plane of dimension d over its kernel $K = \mathrm{GF}(q)$, where $q = p^k$ with a p a prime and $k \geqslant 1$, let $\mathcal{E}$ be the set of all affine elations contained in $L(\Pi)$, and let $\mathcal{L}$ be the set of all components of Π which are axes of nontrivial elations in $\mathcal{E}$. If $G = \langle \mathcal{E} \rangle$ then one of the following holds:*

(a) *The group G is trivial and $|\mathcal{L}| = 0$.*

(b) *The group G is an elementary p-group of affine elations and $|\mathcal{L}| = 1$.*

(c) *The group G is $\mathrm{SL}(2, q^s)$ and $|\mathcal{L}| = q^s + 1$ for some divisor $s \geqslant 1$ of d, and Π contains a Desarguesian subplane Π_0 of order q^s on which G acts faithfully.*

(d) *The prime $p = 2$, $G = \mathrm{Sz}(q^r)$ and $|\mathcal{L}| = q^{2r} + 1$ for some odd $r \geqslant 3$ with $2r$ dividing d, and Π contains a Lüneburg subplane Π_0 of order q^{2r} on which G acts faithfully.*

(e) *The prime $p = 3$, $G = \mathrm{SL}(2, 5)$ and $|\mathcal{L}| = 10$, and Π contains a Desarguesian subplane Π_0 of order 9 on which G acts faithfully.*

(f) *The prime $p = 2$, $|G|$ is not divisible by 4, $|\mathcal{L}|$ is odd, and G is transitive on $\mathcal{L}$.*

The components of the subplane Π_0 in cases (c), (d), and (e) are just the sets $\ell \cap \Pi_0$ for $\ell \in \mathcal{L}$.

PROOF. Using the discussion before Statement 1.4, the plane Π has a subspace Q such that $\Pi = Q \oplus Q$ with x-axis $V(0)$, y-axis $V(\infty)$, and $V(1)$ is the line $y = x$. (The

[36] The Frobenius kernel is the translation group of Π.

subspace Q can be thought of as a quasifield coordinatizing the plane Π.) Let $r = kd$; then Q is an r-dimensional vector space over $F = \mathrm{GF}(p)$ and the elements of G are linear transformations on Π as a $2r$-dimensional vector space over F. If $|\mathcal{L}| = 1$ then G is an elementary Abelian p-group by Statement 1.9(a). Thus, assume $|\mathcal{L}| > 1$.

Assume first that $p > 2$; let ρ_1 and ρ_2 be two nontrivial affine elations of G with distinct axes. Without loss of generality it may be assumed that ρ_1 has axis $V(0)$ and maps $V(\infty)$ into $V(1)$ while ρ_2 has axis $V(\infty)$ and maps $V(0)$ into a component $V(a)$ for some $a \in Q$. Simple algebra shows that ρ_1 on Π can be represented with respect to the coordinatization by

$$\rho_1\colon\ (x, y) \mapsto (x + y, y),$$

and, similarly, ρ_2 is represented by

$$\rho_2\colon\ (x, y) \mapsto (x, xa + y).$$

It follows that as linear transformations on Π as a vector space over F the collineations ρ_1 and ρ_2 are represented by the $2r \times 2r$ matrices

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & A \\ 0 & I \end{bmatrix},$$

respectively; here I is the $r \times r$ identity matrix, 0 is the $r \times r$ zero matrix, and A is the $r \times r$ matrix representing the multiplicative mapping $x \mapsto xa$.

Let λ be an eigenvalue of A, and let $M = F(\lambda)$. Furthermore, let $\mathcal{L}^*$ be the set of components of Π that are axes of affine elations in the group $H = \langle \rho_1, \rho_2 \rangle$. If $\Pi_M = \Pi \otimes M$, then H induces on Π_M, a group $\langle \rho_1 \otimes 1,\ \rho_2 \otimes 1 \rangle$ isomorphic to H, and each component ℓ in $\mathcal{L}^*$ gives rise to an r-dimensional subspace $\ell \otimes M$ in Π_M. It can be shown that Π_M has a subspace U of dimension 2 over M such that H fixes U and on U the mappings ρ_1 and ρ_2 are represented by the 2 by 2 matrices (over L)

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.$$

If $\overline{H}$ is the induced group on U then a theorem of L.E. Dickson (Theorem 2.8 of Gorenstein [1968]) says that either $\overline{H} = \mathrm{SL}(2, M)$ or $\overline{H} = \mathrm{SL}(2, 5)$ and $L = \mathrm{GF}(9)$. By various arguments the set of elements $m \in Q$ such that $y = xm$ is a component in $\mathcal{L}^*$ is a field (under the operations of Q) isomorphic to M, and from this it follows that $\overline{H} \cong H$.[37]

Group theoretic considerations show that if $\langle \sigma, \tau \rangle = \mathrm{SL}(2, p^n)$ for any two noncommuting affine elations σ, τ in $\mathcal{E}$, where n depends on σ and τ, then $G = \langle \mathcal{E} \rangle = \mathrm{SL}(2, p^s)$. Thus, if $p \geqslant 3$ and $|\mathcal{L}| > 1$ then either statement (c) or (e) holds.

Assume now that $p = 2$. The following result of Hering [1972a] can be applied to the subgroup of all affine elations with a common axis.

[37] The proof to this point is Ostrom's contribution; the remaining part is due to Hering.

(4) *Let H be a subgroup of a finite group G such that:*
(a) $N_G(H) \cap (g^{-1}Hg) = 1$ *for all* $g \in G - N_g(H)$,
(b) $N_g(H) \neq G$, *and*
(c) $2 \mid |H|$.

If S is the normal closure of H in G, then $S = H \cdot O(S)$ and H is a Frobenius complement, unless $S = \mathrm{SL}(2, u)$, $\mathrm{Sz}(u)$, $\mathrm{SU}(3, u)$ or $\mathrm{PSU}(3, u)$ where u is a power of 2 at least 4.

If $S = SU(3, u)$ or $S = \mathrm{PSU}(3, u)$ it can be shown that $\langle\sigma, \tau\rangle$ is $\mathrm{SL}(2, p^s)$ for two noncommuting elations σ, τ in $\mathcal{E}$; this leads to a contradiction. If $S = \mathrm{SL}(2, u)$ or $S = \mathrm{Sz}(u)$ these obviously give statements (c) and (d), respectively. If $S = NT$, where N is the largest normal subgroup of odd order in L and T is a Sylow 2-subgroup of S, then T has order 2 since it is elementary Abelian. Each component in $\mathcal{L}$ is then the axis of exactly one nontrivial affine elation in $\mathcal{E}$. It follows that case (f) holds. □

Full details of this proof can be found in Section 35 of Lüneburg [1980]. This is the only complete and fully correct proof in the literature; the articles of Hering and Ostrom contain minor errors which Lüneburg corrects. The statements on the subplane Π_0 follow from Hering [1976].

Examples for all the possibilities listed in Statement 4.3 exist. The likeable planes and the non-Desarguesian semifield planes satisfy (b); the Desarguesian planes, the Kantor type II planes, the Bartolone–Ostrom planes, and the Ostrom planes all satisfy (c). Clearly, the Lüneburg planes satisfy (d), while nearfield planes of even order and Hall planes of even order have groups generated by affine elations satisfying (f). Finally, Prohaska [1977] exhibited a plane of order 81 having a group generated by affine elations isomorphic to $\mathrm{SL}(2, 5)$.

Groups generated by affine homologies have not yet been completely determined. Investigations have been carried out by Ostrom [1973], Kallaher [1975], and Johnson and Kallaher [1976]. Ostrom considered the case of nontrivial affine homologies with distinct axes.

(5) *Let Π be a translation plane of odd dimension d over its kernel* $\mathrm{GF}(q)$, *where* $q = p^k$ *with p a prime and $k \geqslant 1$. If $L(\Pi)$ has an affine homology σ of prime order u with centre P and cocentre Q on ℓ_∞, then one of the following statements holds.*
(a) *The points P and Q are either fixed or interchanged by all collineations of Π.*
(b) *The group $L(\Pi)$ contains affine elations. If p is odd then $L(\Pi)$ has affine elations with centre P or Q.*
(c) *The prime u is at most* 5.

The hypothesis on the dimension d is essential.[38] For example, a Hall plane Π of odd order has dimension 2 over its kernel $\mathrm{GF}(q)$ and $q^2 - q$ components which are axes of

[38] Ostrom assumes the order is nonsquare in order to avoid Baer involutions. But odd dimension insures that Baer involutions do not occur in $L(\Pi)$; see Statement 3.2.

homologies of order $q+1$; these components form an orbit under $L(\Pi)$. Ostrom [1970d] considers affine homologies of order a, where a is large relative to the order of the plane; sufficient conditions on a are given that force the plane to be a generalized André plane.

There is no theorem for affine homologies similar to the Hering–Ostrom Theorem 4.3. The situation is different if both affine elations and affine homologies exist, and this has been investigated by Kallaher [1975] and Johnson and Kallaher [1976] who determine the possible structures for the group generated by the affine perspectivities and other results, of which the following is an example.

(6) *If Π is a translation plane of odd dimension over its kernel in which every component is the axis of a nontrivial affine homology and at least one has prime order $u \geqslant 7$, then Π is either a nearfield plane or a semifield plane.*

PROOF. Johnson and Kallaher assume every hypothesized homology has prime order $u \geqslant 7$, and they conclude that Π is a semifield plane. Using a result of Kallaher and Ostrom [1979] on Bol quasifields, see Goodaire and Kallaher [1990], the present generalization holds. □

The restriction on the dimension is necessary; e.g., Hall planes of odd order satisfy the hypothesis concerning affine homologies. Johnson [1991] considers planes of dimension at most 2 over the kernel GF(q) with a group of affine homologies having order $q-1$.

4.2. *Planar collineations*

A *planar collineation* of a translation plane Π is a collineation which fixes a subplane pointwise. A *planar collineation group* of Π is a collineation group which fixes a subplane pointwise. If the subplane is a Baer subplane the following result of Foulser [1972] applies.

(7) *Let Π be a finite translation plane. If G is a subgroup of $L(\Pi)$ fixing a Baer subplane pointwise then G is a subgroup of a one-dimensional affine group, and hence G is solvable.*

Foulser [1976] investigated the more general case of a p-group G, where Π has order p^r with p a prime, fixing a subplane Π_0 pointwise. He proves the dimension d_0 of Π_0 divides the dimension d of Π over the kernel K and G is a subgroup of $\mathrm{GL}(d/d_0, K)$.

The planar collineations which have been extensively studied are the Baer involutions and *Baer p-elements*; the latter are collineations of order p, where Π has order p^r with p a prime, fixing a Baer subplane pointwise. Foulser [1974] made the observation that, as linear transformations, a Baer p-element and an affine elation in $L(\Pi)$ have the same minimal polynomial. The proof of Statement 4.3 can then be used, with only slight modifications, to give the following information about Baer p-elements in the linear translation complement.

(8) *Let Π be a finite translation plane of dimension d over its kernel $K = \mathrm{GF}(q)$ with $q = p^k$ and p a prime, and let G be the subgroup of $L(\Pi)$ generated by all Baer p-elements in $L(\Pi)$. If $p > 3$ and G is nontrivial then G is either an elementary Abelian p-group, in which case the elements of G fix the same Baer subplane pointwise, or $G = \mathrm{SL}(2, p^e)$ for some $e \geqslant 1$. Furthermore, the Baer subplanes of the Baer p-elements in G contain a common set of $q^{d/2} + 1$ components of Π.*

Foulser also shows that in the context of Statement 4.8 the fixed Baer subplanes of two Baer p-elements in $L(\Pi)$ must either coincide or intersect only in the origin. Additionally, he proves the following, which is false if $p = 2$.

(9) *In the context of Statement* 4.8 *above, if $p > 2$ then Π does not have both nontrivial affine elations and Baer p-elements.*

Note that the Hall planes as well as the Foulser planes given in Section 4.2 are examples for Statement 4.8; Hall planes of even order as well as certain Hughes–Kleinfeld planes of even order show that Statement 4.9 is false for $p = 2$.

Recently, Dempwolff [1982], Jha [1984], Jha and Johnson [1987a] have studied collineation groups whose elements each fix a Baer subplane – possibly different subplanes for different collineations – pointwise. Such groups are called *B-groups*.[39] There are many examples of planes with B-groups; examples include generalized Hall planes, the planes derived from the Walker planes, and the Dempwolff plane of order 16. Using Statements 4.7–4.9 above as well as Statement 4.2 the following holds.

(10) *Let Π be a translation plane of dimension d over its kernel $K = \mathrm{GF}(q)$, where $q = p^k$ with p prime and $k \geqslant 1$, and let G be a B-group in $C(\Pi)$. If $p \nmid |G|$ then G is a planar group and one of the following holds.*

(a) *The group G is cyclic and fixes pointwise a Baer subplane of Π.*
(b) *The group G is an elementary Abelian 2-group and fixes pointwise a subplane of order $q^{d/e}$ with $e = |G|$.*
(c) *The group G fixes pointwise a subplane of order $q^{d/4}$, and either G is dihedral or $G \leqslant S_5$.*

If $p = 2$ then only case (a) *can occur.*

PROOF. Jha [1984]. Examples exist for (a) and (b); they include the Desarguesian and generalized Hall planes. □

Assume now that $p \mid |G|$; no complete result similar to 4.10 above exists, but some results have been derived under additional assumptions. The generalized Hall planes have B-groups G in $L(\Pi)$ with $p \mid |G|$; for these planes the groups G fix a Baer subplane pointwise and are transitive on the set of components intersecting the Baer subplane in only the origin.

[39] Actually, Dempwolff, Jha, and Jha and Johnson each impose an additional, but not the same, restriction on B-groups.

A *strong B-group* – called a Baer group by Dempwolff – is a B-group H in which all nonidentity elements are p-elements fixing the same Baer subplane pointwise, such that $|H| > q^{d/4}$. Dempwolff [1982] considered collineation groups G in $C(\Pi)$ generated by strong B-groups; such groups are called *Dempwolff groups*. Combining results of Dempwolff [1982] and Jha and Johnson [1986a, 1987a] gives the following information.

(11) *Let Π be a translation plane of dimension d over its kernel $K = \mathrm{GF}(q)$, where $q = p^k$ with p a prime and $k \geqslant 1$, and let G be a Dempwolff group in $C(\Pi)$. One of the following statements holds.*

(a) *The order $q^d = 16$, the group $G = \mathrm{SL}(3,2)$, and Π is the Lorimer–Rahilly plane or the Johnson–Walker plane.*

(b) *The plane Π is a Hall plane and $G = \mathrm{SL}(2,q)$.*

(c) *The prime p is odd, and G is an elementary Abelian B-group.*

PROOF. Dempwolff determines the possible structures for G, and Jha and Johnson determine the possible planes. Note that the Foulser planes of order q^4 have B-groups of order q which generate $\mathrm{SL}(2,q)$. □

The main tool of Jha and Johnson in the above proof is a statement concerning the possible sizes of strong B-groups and affine elation groups in the case $p = 2$ when both exist. (Recall that for $p > 2$ Statement 4.9 shows that planes cannot have both strong B-groups and groups of affine elations.)

(12) *Let Π be a translation plane of dimension d over its kernel $K = \mathrm{GF}(2^k)$, let B be a strong B-group and E a group of affine elations in $C(\Pi)$, and assume E normalizes B.*

(a) *If $|B| > 2^{(k/2)+1}$ then $|E| \leqslant 2$.*

(b) *If $|E| \geqslant 2^k$ then $|B| \leqslant 2$.*

PROOF. Jha and Johnson [1985, 1987b]. This generalizes Ganley [1973] who proved that in a semifield plane of even order the largest subgroup fixing a Baer subplane pointwise has order 2. □

Johnson [1990b] has investigated case (i) of Statement 4.11 when the plane has dimension 2 over its kernel.

(13) *Let Π be a translation plane of dimension 2 over its kernel $\mathrm{GF}(q)$, when q is odd and $q > 29$. If Π has a B-group of order q and $L(\Pi)$ contains at least $q+1$ involutory perspectivity, then Π is either a Hall plane or a translation plane derived from a Fisher plane.*

Johnson shows that every derived Fisher plane for which $q \equiv 3 \pmod 4$ satisfies the hypothesis of Statement 4.13, and he conjectures that no derived Fisher plane for which $q \equiv 1 \pmod 4$ has $q+1$ involutory homologies. Also, the restriction $q > 29$ is only needed to exclude the situation where the involutory homologies induce on the line ℓ_∞

one of the groups A_4, S_4, or A_5. It would be interesting to see whether exceptions occur for $q \leqslant 29$.

Statement 4.13 naturally generates the following questions.

QUESTION. What happens in Statement 4.13 if it is only assumed that Π has a B-group of order q? What happens in higher dimensions?

This subsection closes with a pertinent result due to Biliotti, Jha, Johnson and Menichetti [1989]. Huang and Johnson [1990] give nontrivial examples of order 64.

(14) *Let Π be a translation plane of dimension* 2 *over its kernel* GF(q), *where $q = p^k$ with p a prime and $k \geqslant 1$. If $L(\Pi)$ has a p-subgroup G of order at least pq^2, then $p = 2$, $|G| = 2q^2$, and Π is a semifield plane.*

4.3. *Transitive collineation groups*

In the 1970's several studies dealt with collineation groups transitive on certain subsets of points or lines. One of the most important was the study of collineation groups transitive on affine lines. Such groups must contain the group of translations and also be transitive on the points of ℓ_∞.[40] Many known translation planes have such groups: The Desarguesian planes, the Lüneburg planes, a class of translation planes discovered by Narayana Rao [1973], a class found by Durgaprasad, Kuppuswamy Rao, and Narayana Rao [1989], and several sporadic planes. Hering [1973] gives the best result on collineation groups transitive on affine lines and their planes.

(15) *Let Π be a translation plane of dimension d over its kernel $K =$ GF(q), where $q = p^k$ with p a prime and $k \geqslant 1$, and assume G is a collineation group of Π which is transitive on affine lines. If one of the following conditions holds*:

(a) *d is odd,*

(b) *G contains a nontrivial affine elation,*

(c) *$p = 2$ and G contains a nontrivial affine perspectivity,*

(d) *G has a composition factor isomorphic to* PSL(2, w), *where w is a prime power at least* 4, *or to A_m, where $m \geqslant 6$,*

then one of the following statements holds:

(e) *Π is Desarguesian,*

(f) *Π is a Lüneburg plane,*

(g) *G is a subgroup of* ΓL$(1, q^{2d})$ *and contains no nontrivial affine perspectivity,*

(h) *$q^d = 9$ and Π is the nearfield plane,*

(i) *$q^d = 27$ and Π is the Hering plane.*

[40] By Wagner's theorem this is a necessary and sufficient condition (Dembowski [1968], p. 214).

PROOF. The proof can be found in Hering [1973] where other hypotheses are considered with the same conclusions. The fact that in conclusion (i) the plane must be the Hering plane of order 27 follows from Barriga and Pomareda [1985]. Also, the statement includes the results of Foulser [1964a,b] which are used in the proof. □

A second study of collineation groups involved those which are doubly transitive on the points of some line. Consider first the case where the line in question is the line ℓ_∞. The following elegant result, which is essentially due to Czerwinski [1972] and Schultz [1971], holds.

(16) *If Π is a finite translation plane with a collineation group G doubly transitive on the points of the line ℓ_∞, then Π is either Desarguesian or a Lüneburg plane.*

PROOF. Without loss of generality it may be assumed that G fixes the origin. If G has a nontrivial affine perspectivity or if Π has even order then the conclusion follows from Statement 4.15. Thus, assume Π has odd order.

Assume Π has even dimension d. The classification of the finite simple groups gives a classification of the finite doubly transitive groups. Such a group has a minimal normal subgroup, called the *socle*, which either is elementary Abelian or is one of the groups listed in the table in Cameron [1981], p. 8. Each of these possibilities leads to a contradiction when d is even.

For example, if the socle[41] is elementary Abelian then, assuming q^d is the order of Π, it follows that $q^d + 1$, the number of points of ℓ_∞, must be 2^e for some e. This in turn implies that $d = 1$ and q is prime by Lüneburg [1980], Statement 6.2, a contradiction. Similarly, the possibilities M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, HS, and Co_3 cannot occur since each of them would imply that the order is prime, contradicting the assumption that d is even.

The other possibilities given by Cameron are handled in a similar, but more involved, manner. For example, if the socle is $\mathrm{PSU}(3, w^2)$ for some prime power w then it is shown that the action (on ℓ_∞) is the same as the action of $\mathrm{PSU}(3, 2^2)$ on the associated unital, and from that a contradiction arises.

Thus, it may be assumed that the dimension d is odd. If G is solvable then it can be shown that Π is Desarguesian of order 3. If G is nonsolvable it can be shown that G contains $\mathrm{SL}(2, q^d)$, where q^d is the order of Π. By a result of Lüneburg [1964] and Yaqub [1966] the plane is Desarguesian. See Section 39 of Lüneburg [1980] for full details. □

Consider now collineation groups which are doubly transitive on an affine line.[42] Most of the known planes have a collineation group which is doubly transitive on at least one affine line, so additional conditions must be imposed. Kallaher and Ostrom [1980] proved the following.

[41] This is referring to the socle of the doubly transitive permutation group induced on ℓ_∞ by G.

[42] By this is meant that the stabilizer of some affine line ℓ is doubly transitive on ℓ.

(17) *If Π is a finite translation plane of odd dimension* d *over its kernel* K *and* Π *has a collineation group* G *which is doubly transitive on each affine line, then* Π *is a generalized André plane, a semifield plane, or the Hering plane of order* 27.

Kallaher and Ostrom only showed that the third possibility was a plane of order 27 with SL(2, 13) acting on it. The fact that this is the Hering plane follows from Barriga and Pomareda [1985]. A corollary to 4.17 discusses rank 3 collineation groups.

(18) *If Π is a translation plane of odd dimension* d *over its kernel* K *and* Π *has a collineation group* G *which is a rank* 3 *group on the affine points of* Π, *then* Π *is either a generalized André plane or a semifield plane.*

The proof uses Foulser and Kallaher [1978] as well as 4.17. The Hering plane of order 27 does not occur since in that plane SL(2, 13) fixes the origin and is transitive on the remaining affine points. Although examples are known which satisfy 4.18, including the twisted field planes, no one has characterized the generalized André planes and semifield planes satisfying the hypothesis of 4.18. In the case where the rank 3 group G of 4.13 has an orbit of length 2 on ℓ_∞ (without any restriction on the dimension d) the plane must be a nearfield plane or have order 7^2; see Section 20 of Lüneburg [1980].

A second study considered groups which are transitive on proper subsets of the line ℓ_∞. One model is the collineation group of a semifield plane which is transitive on $\ell_\infty - \{(\infty)\}$. Other planes with such a collineation group include the likeable planes of even order, the Lüneburg planes, and the Walker planes. Bartolone [1983], Johnson and Wilke [1984], and Jha, Johnson and Wilke [1984] have studied translation planes with such collineation groups. Among their results is the following characterization.

(19) *Let Π be a translation plane of dimension* 2 *over its kernel* $K = \mathrm{GF}(q)$. *If* $L(\Pi)$ *has a subgroup* G *that induces on* ℓ_∞ *a group of order* $q^2(q-1)$, *then* Π *is a semifield plane, a Lüneburg plane, a Walker plane, or a likeable plane of even order.*

In this connection Johnson and Wilke [1984] and Jha and Johnson [1988] have shown that if Π is as in Statement 4.19, then a collineation group of order q^2 in $C(\Pi)$ is Abelian if and only if it is in $L(\Pi)$.

The most complete result on collineation groups transitive on $\ell_\infty - \{(\infty)\}$ is that due to Ganley and Jha [1986].

(20) *If Π is a finite translation plane admitting a collineation group fixing a point* $\mathcal{V}$ *of* ℓ_∞ *and doubly transitive on* $\mathcal{L}_\infty - \{\mathcal{V}\}$, *then* Π *is a semifield plane.*

PROOF. The proof is very similar to that of Statement 4.16 and uses the classification of doubly transitive groups. □

A reasonable question to ask in light of Statements 4.16 and 4.20 is the following.

QUESTION. What are the possibilities for the plane Π if the hypothesis 'doubly transitive' in 4.16 and 4.20 is replaced by 'primitive?'

A third study has as its model the Hall planes. These planes have a collineation group $H = \mathrm{SL}(2, q)$, where q^2 is the order of the plane, fixing pointwise a subset Δ of length $q + 1$ on ℓ_∞ and transitive on $\Phi = \ell_\infty - \Delta$. Furthermore, the full collineation group G has the sets Δ and Φ as orbits. Note also that Δ is the intersection with ℓ_∞ of a Baer subplane Π_0 fixed pointwise by H.

Let Π be a translation plane with a subset Δ of ℓ_∞ such that $|\Delta| \geqslant 3$. A collineation group G of Π is *transitive relative to* Δ if G leaves Δ invariant and is transitive on $\ell_\infty - \Delta$. Note that in studying such groups it may be assumed that G is contained in $C(\Pi)$.

Special cases of the above concept include the following due to Jha [1975, 1985]. If Π_0 is an (affine) subplane of Π the collineation group G is *semitransitive relative to* Π_0 if G fixes Π_0, fixes $\Delta = \Pi_0 \cap \ell_\infty$ pointwise, and is transitive relative to Δ. If G in addition fixes Π_0 pointwise then G is said to be *tangentially transitive*[43] *relative to* Π_0.

(21) *Let Π be a finite translation plane with a collineation group G that is semitransitive relative to a subplane Π_0. Assume Π has order $q^d \neq 16$.*

(a) *If Π has a nontrivial kernel then Π_0 is a Desarguesian Baer subplane of Π and Π is derivable.*

(b) *If G is tangentially transitive relative to Π_0 then Π is a generalized Hall plane.*

PROOF. Jha [1975, 1985]. If $q^d = 16$ then Π is either the Hall plane or the Lorimer–Rahilly plane (Johnson and Ostrom [1977], Walker [1976]). Also, if Π can be coordinatized so that Π_0 is coordinatized by the kernel, then Π is a Hall plane. Jha [1985] gives examples of planes with semitransitive groups but not possessing tangentially transitive groups. □

Several articles consider collineation groups transitive relative to a set Δ on ℓ_∞ where $|\Delta| = q^* + 1$ with q^* dividing the order of the kernel. These include the articles Cohen, Ganley and Jha [1980], Hiramine [1985a,c,d], and Jha and Kallaher [1981, 1982]. A typical result is the following.

(22) *Let Π be a translation plane of dimension d over its kernel $K = \mathrm{GF}(q)$, where $q = p^k$ with p a prime and $k \geqslant 1$, and let G be a collineation group transitive relative to the set Δ with $|\Delta| = q^* + 1$, where $q^* = p^e$ with $e \leqslant k$.*

(a) *If $G \cong \mathrm{PSL}(n, w)$ with $n \geqslant 3$, then $q^d = 16$, the group $G \cong \mathrm{PSL}(3, 2)$, and Π is either the Lorimer–Rahilly plane or the Johnson–Walker plane.*

(b) *If $q^d \equiv 3 \pmod 4$ and $O_p(G)$ is nontrivial, then $dk/e = 3$.*

PROOF. See Jha and Kallaher [1981] for (a) and Hiramine [1985d] for (b). □

[43] This means that for every affine point P of Π_0 the stabilizer G_P is transitive on the lines through P which meet no other point of Π_0.

5. Conclusion: two important problems

The material discussed in the previous sections does not cover all the results obtained in the field since 1967, but it gives the important trends and results of the last twenty years. Furthermore, it shows that in many ways there is, as yet, no general theory for the field. In particular, no meaningful classification for finite translation planes has appeared. Indeed, this statement is even true if restriction is made to planes of dimension 2. These facts are brought out by the many articles whose purpose is to demonstrate that the planes found by one person are really those found by another or are easily obtained by derivation from those found by another.

Ostrom [1973b] called for a reasonable classification of translation planes and suggested the rudiments of one based upon types of collineations existing in the plane. The term 'reasonable' includes the attributes: (a) disjoint classes, (b) small number of classes, (c) classes correspond to real differences among planes. However, the number and variety of planes has exploded since 1973. Thus, the first problem is:

PROBLEM 1. Find a reasonable classification of finite translation planes.

Because most known planes, and – except for the Hering plane of order 27 – the most interesting ones occur at dimension 2, the following special case of Problem 1 is of interest.

PROBLEM 2. Find a reasonable classification of finite translation planes of dimension 2.

Progress toward the solution of these problems will help reduce the chaos that now exists in the field. It will also help in shaping solutions to other problems like those mentioned previously.

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CHAPTER 6

Dimensional Linear Spaces

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Contents

1. Introduction and definitions 195
2. Some examples and characterizations 207
3. Enumeration 221
4. Planar spaces 229
5. Benz planes 237
6. Embeddings, coordinatizations, subgeometries and extensions 255
7. Automorphisms 267
8. Permutation geometries and squashed geometries 276

References 278

HANDBOOK OF INCIDENCE GEOMETRY
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1. Introduction and definitions

1.1. *Introduction*

Dimensional linear spaces (shorthand: DLS's) were already introduced to the reader as a rather straightforward generalization of the basic structure of elementary geometry in Chapter 1, Section 2.6, as a class of closure spaces in Chapter 3, Section 2.4, and as a class of geometries on linear diagrams in Chapter 3, Section 3.3. This prefigures the fact that in the literature DLS's are hidden under such diverse names as closure space with the exchange property, atomic semimodular lattice or geometric lattice, simple (or geometric) matroid, independence space, combinatorial geometry,

diagram geometry, or sometimes even merely ... geometry! All these notions are essentially equivalent, the main difference coming from the introduction of some kind of finiteness condition (see Section 1.8). Linear spaces, perfect matroid designs or regular geometric designs, t-$(v,k,1)$-designs, $(t,1)$-quasidesigns or t-partitions, projective and affine spaces, inversive, Laguerre and Minkowski planes or spaces, Wille incidence geometries of grade n, all of them are DLS's. The prototypes of DLS's are projective spaces provided with all their subspaces up to a certain dimension, as well as the structures induced by these spaces on a subset of their point-set. Key notions are those of closure, span, independence, semimodular lattice (of subspaces), dimension, base, etc. Many equivalent (or cryptomorphic) axiom systems can be developed and this is often put forward as a testimony of the richness and universality of this concept. However, this also often hampers entering the core of the subject. Therefore we refer the reader to the matroid literature for a systematic study of these cryptomorphisms (see, e.g., White [1986]) and advise him not to dwell on this in the beginning. Although the notion of DLS already appeared (implicitly) in 1871 in Jordan's work on permutation groups and (quite explicitly) in 1910 in Steinitz's investigation of field extensions, it is usually credited to Whitney [1935], Birkhoff [1935] and Mac Lane [1938] because they proved the first deep results on DLS's. Their respective motivations came from graph theory (and the famous four-colour theorem), geometrically blended lattice theory, and the study of transcendental field extensions. Close connections between DLS's and coding theory, design theory, combinatorial optimization, convexity, rigidity of spatial structures, model theory[1] (among other fields) emerged and are still popping up from time to time: e.g., Brickell and Davenport [1990] recently showed how closely DLS's are related to ideal secret sharing schemes.

A general treatment of DLS's (from different viewpoints) can be found in Birkhoff [1967], Crapo and Rota [1970], Tutte [1971], Welsh [1976], Aigner [1979] and in the recent collective books White [1986, 1987, 1992].

However the viewpoint adopted here is that of incidence geometry as presented in the first chapters of this book. So the main themes of this chapter are characterization and embeddability problems for DLS's, i.e.:

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[1] The relationship with model theory is treated in Chapter 13 (Editor's note).

(1) characterize classical DLS's in terms of local or global, combinatorial, geometrical or group-theoretical conditions;

(2) find sufficient conditions for DLS's to be embeddable (or coordinatizable, or representable) into projective spaces (or into algebraic spaces).

Most results concern finite-dimensional linear spaces (FDLS). Therefore, we first propose a descriptive definition of FDLS's in Section 1.2, and then a diagram definition of FDLS's in the language of Chapter 3, Section 1.5. However, the reader is free to go immediately to the general definition of DLS's in terms of closure spaces (Section 1.6).

1.2. *Descriptive definition of an n-DLS*

Let n be a positive integer. An *n-dimensional linear space* (or n-DLS)

$$\boldsymbol{S} = (S^i;\ i = 0, \ldots, n-1)$$

is a set S^0 of *points* (also called *0-varieties*), together with $n-1$ families $S^i (1 \leqslant i < n)$ of proper subsets of S^0 called *i-varieties*, such that

(i) for any i-variety $V (\forall i < n-1)$ and any point $x \in V$, there is a unique $(i+1)$-variety containing x and V, denoted by $\langle x, V \rangle$;

(ii) if W is a j-variety containing V and x as in (i), then W contains $\langle x, V \rangle$;

(iii) any i-variety $(1 \leqslant i < n)$ contains at least one $(i-1)$-variety as a proper subset.

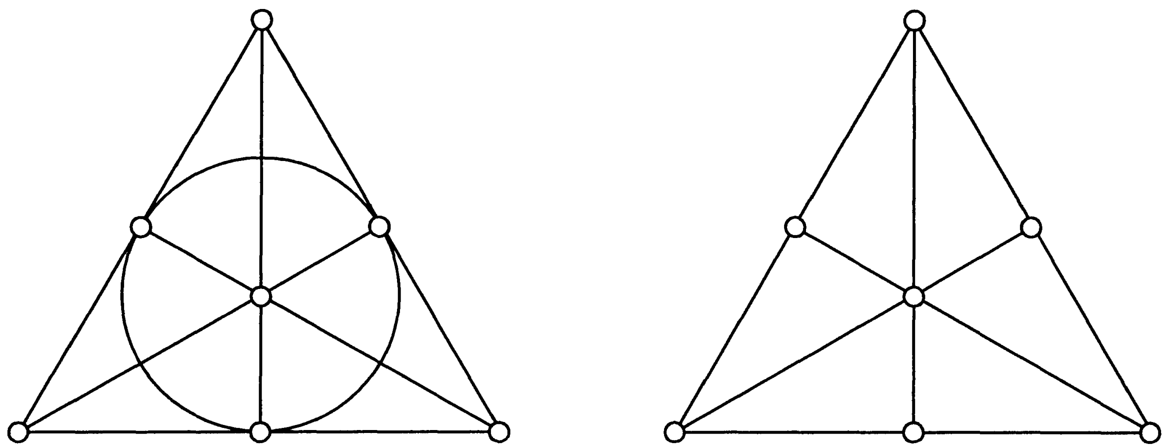
If we do not want to specify the dimension n, we say that $\boldsymbol{S}$ is a *finite dimensional linear space* (FDLS). Since the structure of a 1-DLS is trivial, we implicitly assume $n \geqslant 2$ unless otherwise mentioned. The elements of

$$\bigcup_{0 \leqslant i < n} S^i$$

are called *proper varieties* and a variety V is said to have *dimension i* ($\dim V = i$) whenever V is an i-variety. The empty set and S^0 are called *improper varieties*, of dimension -1 and n, respectively. It is easily seen that $2\text{-}\boldsymbol{S} := (S^0, S^1)$ is a nontrivial linear space (called the *2-truncation* of $\boldsymbol{S}$) and that every variety is a linear subspace of (S^0, S^1). Therefore the 1-varieties of $\boldsymbol{S}$ are called *lines*. The 2-DLS's are merely the linear spaces. Note that if $V \subset W$ are two distinct varieties, then $\dim V < \dim W$, and the intersection of any family of varieties is a (possibly improper) variety, so that $(S^0, \bigcup_i S^i)$ is a closure space (see Section 1.6).

1.3. *DLS versus LS with some notion of dimension*

Note also that the notion of n-DLS is quite different from that of linear space whose dimension (defined in terms of chains of subspaces) is n. For example, given any $d \geqslant n$, the d-dimensional projective space PG(d, F) provided with its subspaces of dimension $< n$ is an n-DLS (called the *n-truncation* n-PG(d, F) of PG(d, F)). On the other hand, certain linear spaces contain nested linear subspaces of the same dimension. This occurs, e.g., in the linear space derived from PG(d, F) ($d \geqslant 3$) as follows: its points are those



The Fano plane PG(2, 2) The non-Fano plane

Figure 1.1.

of PG(d, F) and its lines are those of PG(d, F), except for one line which is replaced by lines of size 2. If $d = 2$, this construction derives the so-called non-Fano plane from PG(2, 2) (which is called the Fano plane), as shown in Figure 1.1. These two 2-DLS's play an important role in coordinatization problems, in relation with the characteristic of the coordinatizing field (see Section 6).

Moreover the linear space 2-PG(d, F) ($d > n \geqslant 3$) can be erected (in the sense of Section 1.10) in many ways into an n-DLS, as suggested by the numerous variations over the following easy construction: take as i-varieties, with $i < n - 1$, all i-dimensional subspaces of PG(d, F), and as $(n-1)$-varieties a given hyperplane H of PG(d, F), together with all $(n-1)$-dimensional subspaces which are not contained in H. We will sometimes refer to such FDLS's, whose 2-truncation (or linear structure) is that of some projective space, but at least one of whose i-varieties is a projective subspace of dimension $> i$, as *missed projective spaces*. Some 'missed affine spaces' can even have a 2-transitive automorphism group (see Section 7.7).

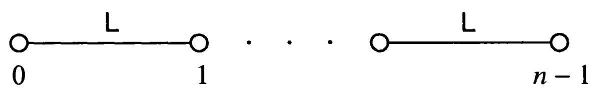
1.4. *Truncations and residues of an n-DLS*

Let $\boldsymbol{S} = (S^i;\ 0 \leqslant i < n)$ be an n-DLS. Given $2 \leqslant m \leqslant n$, the *$m$-truncation* of $\boldsymbol{S}$, denoted by m-$\boldsymbol{S}$, is the m-DLS $(S^i;\ 0 \leqslant i < m)$. Given two varieties $V \subset W$ such that $\dim W - \dim V = l + 1 \geqslant 3$, let T^j be the set of all $(\dim V + j + 1)$-varieties containing V and contained in W, where $j \leqslant l - 1$. Then the (*top*) *residue of V in W*, denoted by $\boldsymbol{S}_V^W$ (or simply $\boldsymbol{W}_V$), is the l-DLS $(T^j;\ 0 \leqslant j \leqslant l - 1)$. In particular, all such 2-dimensional residues are nontrivial linear spaces. This property characterizes FDLS's as will be seen below.

1.5. *n-DLS as a rank n diagram geometry*

An incidence structure is canonically associated to any n-DLS as follows: call two distinct varieties of that n-DLS *incident* if and only if one of them contains the other one. Conversely, given an incidence structure as in Chapter 3, define S^i to be the set of all 0-shadows of the elements of type i.

THEOREM (Buekenhout [1979]). *The* n-DLS*'s* ($n \geqslant 2$) *are precisely the firm, residually connected, simple incidence structures belonging to the diagram*



where

denotes the class of all linear spaces (see Chapter 3 for the terminology).

The crux of Buekenhout's proof is, starting from an incidence structure as above, to force (i) of Definition 1.2. This is done by induction on the rank and uses the strong connectedness (which follows from the finiteness of the rank).

1.6. *Closure space, simplification, exchange, restriction*

We will now define dimensional linear spaces whose dimension is not necessarily finite. A *closure space* (on $\boldsymbol{P}$) is a pair $(\boldsymbol{P}, \boldsymbol{C})$, where $\boldsymbol{P}$ is a set whose elements are called *points*, and $\boldsymbol{C}$ is a family of subsets of $\boldsymbol{P}$, called *closed sets*, such that any intersection of closed sets is a closed set. In particular $\boldsymbol{P}$ is necessarily a closed set, since it is the intersection of an empty family of closed sets. The *closure* $\langle X \rangle$ of any subset X of $\boldsymbol{P}$ is the intersection of all closed sets containing X, and the closure operator $\langle \cdot \rangle$ satisfies the following properties, where X and Y are any subsets of $\boldsymbol{P}$:

(C1) $X \subseteq \langle X \rangle$,
(C2) $X \subseteq Y \Rightarrow \langle X \rangle \subseteq \langle Y \rangle$,
(C3) $\langle \langle X \rangle \rangle = \langle X \rangle$.

A closure space is *simple* if the empty set and all singletons are closed. A simple closure space is canonically associated to any closure space by neglecting the points which are in the closure of the empty set and by identifying all points which are in the closure of each other. For example, a vector space provided with all its subspaces is a closure space on its set of vectors, but it is not simple because the closure of the empty set is $\{0\}$ and the closure of a nonzero vector is a 1-dimensional subspace. The associated simple closure space (also called the *simplified* closure space) is the derived projective space.

A *dimensional linear space* (DLS) on $\boldsymbol{P}$ is a simple closure space $(\boldsymbol{P}, \boldsymbol{C})$ (whose closed sets are preferably called varieties) satisfying the *strong exchange axiom*

(SX) if $X \subseteq Y \subseteq \boldsymbol{P}$ and $z \in \langle Y \rangle \backslash \langle Y \backslash X \rangle$, then there is $x \in X$ such that $x \in \langle z \cup (Y \backslash x) \rangle$.

Possibly nonsimple closure spaces satisfying (SX) will be called *pre*-DLS'*s*. In the important finitary case (Section 1.8) this axiom is equivalent to the classical (Mac Lane–Steinitz) *exchange axiom*:

(X) if $X \subseteq \boldsymbol{P}$ and $y, z \in \boldsymbol{P} \setminus \langle X \rangle$, then $y \in \langle X \cup z \rangle$ forces $z \in \langle X \cup y \rangle$.

This crucial axiom is also called the *partition axiom* because it corresponds to the following fact: for any closed set C, the closed sets of the form $\langle C \cup x \rangle$ with $x \notin C$ induce a partition on $\boldsymbol{P} \setminus C$. Note that in topological simple closure spaces (X) always holds while (SX) holds iff all closed subsets are discrete (Klee [1971]). Hence the usual topology on $\mathbb{R}^n$ provides an example of a simple closure space satisfying (X) but not (SX).

Both axioms (X) and (SX) are hereditary under the operation of restriction, which we define now. Given $X \subseteq \boldsymbol{P}$, the *restriction* $(\boldsymbol{P}, \boldsymbol{C})|_X$ of $(\boldsymbol{P}, \boldsymbol{C})$ to X is the closure space *induced* on X by $(\boldsymbol{P}, \boldsymbol{C})$, that is the closure space with point-set X and whose closed sets are the intersections with X of the elements of $\boldsymbol{C}$.

Two DLS's $(\boldsymbol{P}, \boldsymbol{C})$ and $(\boldsymbol{P}', \boldsymbol{C}')$ are *isomorphic* provided there is a bijection from $\boldsymbol{P}$ onto $\boldsymbol{P}'$ that induces a bijection from $\boldsymbol{C}$ to $\boldsymbol{C}'$.

1.7. *Lattice of varieties, dimension, independence, basis, hyperplane*

Given any closure space $(\boldsymbol{P}, \boldsymbol{C})$, we define Lat$(\boldsymbol{P}, \boldsymbol{C})$ to be the lattice on $\boldsymbol{C}$ obtained by ordering it by inclusion. This lattice is complete, atomic, (upper) semimodular, and would be called geometric if it were finite dimensional (Birkhoff [1967]). Note that the lattice of proper closed sets or proper varieties of a DLS is illustrated by its incidence graph when seen as a diagram geometry (see Figure 1.2; lattice-theoretists would probably prefer to have the highest dimensional varieties on top rather than to the right).

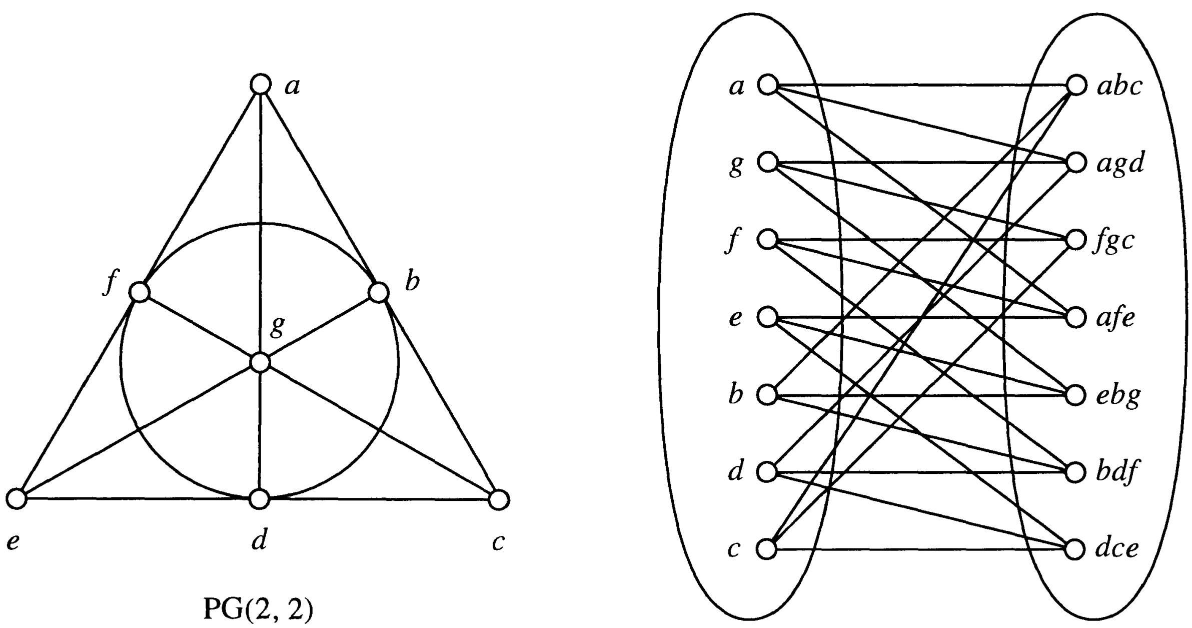


Figure 1.2.

A closed set is said to *cover* another closed set D provided there is no closed set between C and D. The *height* $h(X)$ of $X \subseteq \boldsymbol{P}$ is that of $\langle X \rangle$ in Lat$(\boldsymbol{P}, \boldsymbol{C})$, i.e. the least upper bound of the lengths of all chains from $\langle\phi\rangle$ to $\langle X\rangle$. The set X is said to *span* (or *generate*) $\langle X\rangle$, and X is *spanning* if $\langle X\rangle = \boldsymbol{P}$.

The *dimension* $\dim C$ of $C \in \boldsymbol{C}$ is one less than the smallest cardinal of the subsets spanning C. Clearly if $h(C)$ is finite, then $\dim C \leqslant h(C) - 1$, and equality holds in DLS's. The closed sets of finite dimension i are then called *i-varieties*. The n-DLS's are merely the DLS's of finite dimension n (or equivalently of finite height $n+1$). Unless otherwise mentioned, we implicitly assume $n \geqslant 2$ since only these DLS's can have a nontrivial structure. Note that the DLS's on $\boldsymbol{P}$ whose dimension is -1 or 0 satisfy $|\boldsymbol{P}| = 0$ or 1, respectively.

$X \subseteq \boldsymbol{P}$ is *independent* if $x \notin \langle X \setminus x\rangle$ for any $x \in X$, and X is *dependent* otherwise. A *basis* is a minimal spanning set, or equivalently an independent and spanning set.

A *hyperplane* is a maximal proper closed set. The infinite-dimensional projective spaces provided only with their finite-dimensional subspaces have neither bases nor hyperplanes. Sufficient conditions for the existence of bases and hyperplanes are given in Section 1.8.

For a study of independence and dimension in a more general context, see Rado [1949] and Robertson and Weston [1958].

1.8. *Finitary DLS, variations on the exchange axiom, $(\boldsymbol{P}, \boldsymbol{H})$-truncation*

$(\boldsymbol{P}, \boldsymbol{C})$ is *finitary* if and only if for any $X \subseteq \boldsymbol{P}$ and $x \in \langle X\rangle$, there is a finite subset Y of X such that $x \in \langle Y\rangle$. In finitary closure spaces the independence has *finite character*, i.e. the independent sets are precisely the sets all of whose finite subsets are independent. These two properties coincide for DLS's (Buekenhout [1967]), are satisfied in all FDLS's and allow the following variations on the exchange axiom.

PROPOSITION (Buekenhout [1967], Klee [1971], Ohn [1990]). *If $(\boldsymbol{P}, \boldsymbol{C})$ is a closure space whose independence has finite character, then the following properties are equivalent and force all bases to have the same cardinality.*

(1) *The exchange property* (X) (see Section 1.6).

(2) *The strong exchange property* (SX) (see Section 1.6).

(3) *If $C \in \boldsymbol{C}$ and $x \in \boldsymbol{P} \setminus C$, then $\langle C \cup x\rangle$ covers C in* $\mathrm{Lat}(\boldsymbol{P}, \boldsymbol{C})$.

(4) *If I is an independent set and $x \in \boldsymbol{P} \setminus \langle I\rangle$, then $I \cup x$ is an independent set.*

(5) *If $C, D \in \boldsymbol{C}$ have finite height, then*

$$h(C \cup D) + h(C \cap D) \leqslant h(C) + h(D)$$

and if $x \in \boldsymbol{P}$, then $h(x) \leqslant 1$.

(6) *For any $X \subseteq \boldsymbol{P}$, every independent subset of X is contained in a basis of $(\boldsymbol{P}, \boldsymbol{C})|_X$.*

(7) *For any $X \subseteq \boldsymbol{P}$, all maximal independent subsets of X are bases of $(\boldsymbol{P}, \boldsymbol{C})|_X$.*

(8) *Any finite set has finite height and for any closed set C of finite height, all maximal independent sets in C have cardinality $h(C)$.*

(9) *Every closed set is an intersection of hyperplanes and for any two distinct hyperplanes H_1, H_2 and any point $x \notin H_1 \cup H_2$, there is a hyperplane containing x and $H_1 \cap H_2$.*

Note that the name *combinatorial geometry* is often used for finitary DLS, the name *independence space* is used for finitary pre-DLS, *geometric lattice* for FDLS (actually for the lattice of an FDLS, but these notions are equivalent since every geometric lattice is the lattice of a unique (up to isomorphism) FDLS), and *simple matroid* for finite pre-DLS. However, this terminology is not fixed either. The FDLS's are finitary, but infinite-dimensional linear spaces can be finitary as well: this is the case for projective spaces (provided with all their subspaces), since vector spaces are finitary closure spaces. In a finitary DLS $\boldsymbol{S}$, the hyperplanes are the closures of the independent sets of type $B \setminus x$, where B is a basis and $x \in B$; moreover the *(point, hyperplane)-truncation* of $\boldsymbol{S} = (\boldsymbol{P}, \boldsymbol{C})$, i.e. the incidence structure $(\boldsymbol{P}, \boldsymbol{H})$, where $\boldsymbol{H}$ is the set of all hyperplanes of $\boldsymbol{S}$ completely determines $\boldsymbol{S}$ (see Sections 2.3 to 2.5 for illustrations of this fact). Actually a set $\boldsymbol{P}$ of points together with a family $\boldsymbol{H}$ of subsets of $\boldsymbol{P}$ is the $(\boldsymbol{P}, \boldsymbol{H})$-truncation of a finitary DLS precisely when the following three conditions hold:

(i) no element of $\boldsymbol{H}$ is contained in another element of $\boldsymbol{H}$,

(ii) for any two distinct elements H and H' of $\boldsymbol{H}$ and any point $x \notin H \cup H'$ there is an element H'' of $\boldsymbol{H}$ such that $\{x\} \cup (H \cap H') \subseteq H''$,

(iii) the intersections of hyperplanes are the closed sets of a closure space whose independence has finite character.

Further reading on finitary, cofinitary and nonfinitary DLS's is provided by Oxley [1978], Duchamp [1989], Teirlinck [1982a], Welsh [1976] and references therein.

1.9. *Jordan–Dedekind's condition, independence sets and bases*

If $\boldsymbol{S}$ is an FDLS, then the lattice $\operatorname{Lat} \boldsymbol{S}$ satisfies the *Jordan–Dedekind chain condition*, i.e. all maximal chains with the same endpoints have the same length (Birkhoff [1967], II.8). However, this important property does not force the exchange condition: the 'n-dimensional nearfield affine spaces' introduced by Sperner [1960] and further investigated and generalized by Nizette [1970] are linear spaces whose lattice of all linear subspaces is complemented, graded (i.e. satisfies the Jordan–Dedekind condition) and has dimension n. However, if $n > 2$ and if the associated nearfield is not a field, then this linear space with such a well-behaved dimension is not a DLS, because the subspace generated by a point and a line may have a dimension higher than 2 (Nizette [1970]).

Unlike finitary DLS's, nonfinitary ones are not uniquely determined by their set of independent sets and may have bases of different cardinalities, as shown by the examples below.

Let $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ be two disjoint infinite sets and let $\boldsymbol{P}_3 = \boldsymbol{P}_1 \cup \boldsymbol{P}_2$. Define $\boldsymbol{S}_i$ $(i = 1, 2, 3)$ to be the DLS on $\boldsymbol{P}_i$ whose proper varieties are the nonempty finite subsets of $\boldsymbol{P}_i$. The independent sets of the two nonisomorphic DLS's $\boldsymbol{S}_3$ and $\boldsymbol{S}_1 \oplus \boldsymbol{S}_2$ (see Section 1.12) are precisely the finite subsets of their common point-set $\boldsymbol{P}_3$.

Now suppose that the two disjoint sets $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ have distinct infinite cardinalities. It is easily checked that there is a DLS $\boldsymbol{S}$ whose bases are all sets B for which there is an integer m such that either

$$|\boldsymbol{P}_1 \cap B| = |\boldsymbol{P}_2 \setminus B| = m \quad \text{or} \quad |\boldsymbol{P}_1 \setminus B| = |\boldsymbol{P}_2 \cap B| = m,$$

so that the bases of S have cardinalities $|P_1|$ or $|P_2|$ (Dlab [1965]). This also illustrates the fact that nonfinitary DLS's can have bases.

For a study of mappings leaving independent sets invariant (weak homomorphisms) see Wenzel [1990].

1.10. *Truncations and residues*

An (*upper*) *truncation* of $\boldsymbol{S} = (\boldsymbol{P}, \boldsymbol{C})$ is a closure space $(\boldsymbol{P}, \boldsymbol{C}')$ such that $\boldsymbol{C}' \subseteq \boldsymbol{C}$ and any subset of an element $\neq \boldsymbol{P}$ of $\boldsymbol{C}'$ which is in $\boldsymbol{C}$ also belongs to $\boldsymbol{C}'$. Conversely $(\boldsymbol{P}, \boldsymbol{C})$ is said to be an *erection* of $(\boldsymbol{P}, \boldsymbol{C}')$. In particular if $n \leqslant \dim \boldsymbol{S}$, then the proper closed sets of the n-*truncation* of $\boldsymbol{S}$ are the elements of $\boldsymbol{C}'$ whose dimension is less than n. (Upper) truncations of DLS's are still DLS's but the finitary property need not be preserved, as shown by the hyperplane-free truncations of infinite-dimensional projective spaces. However, lower truncations of DLS's, obtained by deleting all elements of dimension $\leqslant i$, for some fixed i, are not DLS's in general (see Section 2.10).

The *bottom residue* of a variety V of $\boldsymbol{S} = (\boldsymbol{P}, \boldsymbol{C})$ is just the restriction $\boldsymbol{V} = \boldsymbol{S}|_V$ of $\boldsymbol{S}$ to V (cf. Section 1.6) and will often be identified with V. However, the (*top*) *residue* $\boldsymbol{S}_V$ of V is the simple closure space associated to $(\boldsymbol{P}, \boldsymbol{C}_V)$, where $\boldsymbol{C}_V$ consists of the elements of $\boldsymbol{C}$ containing V. So if $V \subseteq W$ are two varieties of $\boldsymbol{S}$, the (*top*) *residue of V in W*, denoted by $\boldsymbol{W}_V$, consists of the varieties containing V and contained in W. Such residues of a DLS $\boldsymbol{S}$ are still DLS's, they are finitary if $\boldsymbol{S}$ is finitary and have dimension $\dim W - \dim V - 1$ if V and W have finite dimension. When we speak of the residues of a DLS we always mean residues of this type (connected residues in the diagram geometry and intervals in the lattice of varieties).

1.11. *Duality*

In projective spaces, duality reverses the inclusion on subspaces; more generally, it reverses the order on the elements in lattice theory and reverses the order on the types in incidence geometry with linear diagram, interchanging among other things points and hyperplanes (copoints). Hints of these related notions can be found in our terminology when we define varieties of *codimension* j to be those whose (upper) residue has dimension j and call *copoints, colines* and *coplanes* varieties of codimension 0, 1 and 2, respectively.

Although the dual of a modular lattice (resp., projective space) is still modular (resp., a projective space), upper semimodularity is not preserved by duality: the (lattice-theoretical) dual L^* (or inverted lattice) of the lattice of an n-DLS $\boldsymbol{S}$ need not even have a rank function. However, if L^* can be extended into the lattice of an n-dimensional geometric lattice L^Δ without adding new points, then the associated n-DLS $\boldsymbol{S}^\Delta$ is called an *adjoint* of $\boldsymbol{S}$. This notion is closely related to the study of the extension lattice of $\boldsymbol{S}$ (Bachem and Kern [1986b]) and can be used in some coordinatization problems (Kern [1988], Bachem and Kern [1986b]). Nevertheless most DLS's have no adjoint (Cheung [1974], Bachem and Wanka [1989]).

Usually quite a different notion of duality is preferred, which is also called *orthogonality* or *orthogonal duality*, and has proved remarkably fruitful in matroid theory.

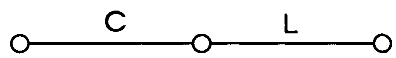
Let $(\boldsymbol{P}, \boldsymbol{C})$ be a pre-DLS. For any subset X of $\boldsymbol{P}$, set

$$\langle X\rangle^* = X \cup \{x\colon\ x \notin \langle \boldsymbol{P} \setminus (X \cup x)\rangle\}.$$

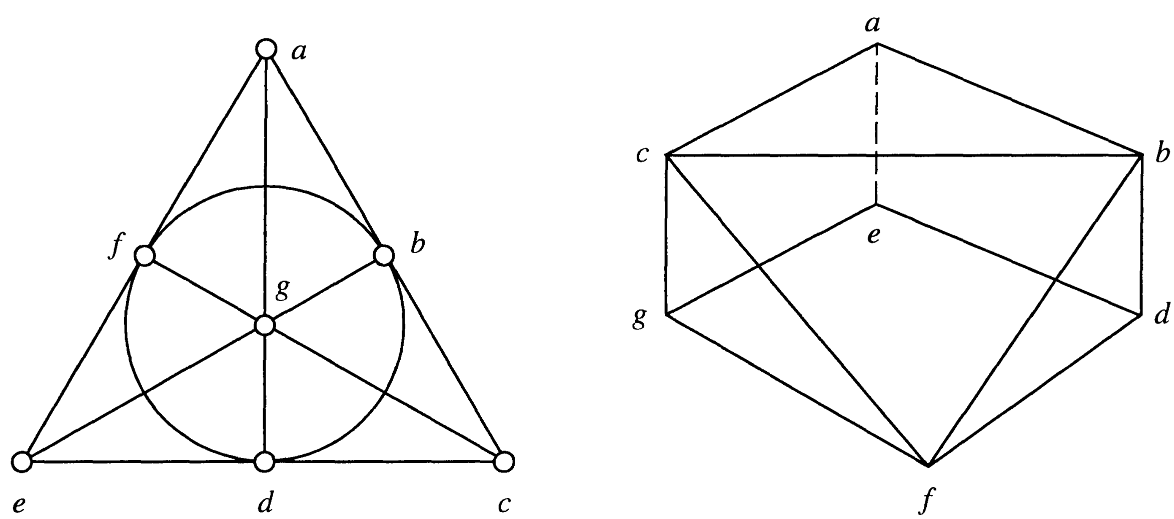
Then $\langle\cdot\rangle^*$ is a closure operator on $\boldsymbol{P}$, the corresponding closure space $(\boldsymbol{P}, \boldsymbol{C}^*)$ satisfies the strong exchange property (SX) (Klee [1971], see also Welsh [1976], p. 395), so that it is a pre-DLS, called the *dual* of $(\boldsymbol{P}, \boldsymbol{C})$. Moreover $(\boldsymbol{P}, \boldsymbol{C}^{**}) = (\boldsymbol{P}, \boldsymbol{C})$.

The dual of a DLS needs not be a DLS because it may be nonsimple. The dual $\boldsymbol{S}^*$ of a finitary DLS $\boldsymbol{S}$ on $\boldsymbol{P}$ is said to be *cofinitary*; the bases of $\boldsymbol{S}^*$ are precisely the complements in $\boldsymbol{P}$ of the bases of $\boldsymbol{S}$, and the spanning (resp., independent) sets are the complements in $\boldsymbol{P}$ of the independent (resp., spanning) sets of $\boldsymbol{S}$. Cofinitary DLS's are those determined by their point-hyperplane truncation and in which the complements of hyperplanes are finite.

For example, PG(2, 2)*, suggested in Figure 1.3, belongs to



has 7 planes of size 4 (the complements of the lines of PG(2, 2)) and 7 planes of size 3 (the lines of PG(2, 2)). It can be seen as obtained from AG(3, 2) by deleting a point, more precisely as the restriction of AG(3, 2) to any 7-subset of its point-set.



PG(2, 2) PG (2, 2)* (incompletely described)

Figure 1.3.

The notion dual to restriction is that of contraction: if $X \subseteq \boldsymbol{P}$, the *contraction* $\boldsymbol{S}/X$ of $\boldsymbol{S}$ *by* X (or *to* $\boldsymbol{P} \setminus X$) is the restriction to $\boldsymbol{P} \setminus X$ of the closure space whose closed sets are the closed sets of $\boldsymbol{S}$ containing X. Then $(\boldsymbol{S}|_X)^* = \boldsymbol{S}^*/(\boldsymbol{P} \setminus X)$ and $(\boldsymbol{S}/X)^* = \boldsymbol{S}^*|_{\boldsymbol{P}\setminus X}$. Moreover the operations of restriction and contraction commute in the following sense: if $X \subseteq Y \subseteq \boldsymbol{P}$, then $(\boldsymbol{S}|_Y)/X = (\boldsymbol{S}/X)|_{Y\setminus X}$.

The pre-DLS's obtained from $\boldsymbol{S}$ by successive restrictions and contractions are called the *minors* of S. Note that if $V \subseteq W$ are varieties of $\boldsymbol{S}$, then the residue $\boldsymbol{W}_V$ is the 'simplification' of $(S|_W)/V$. Hence a simplified minor of S is merely a *subgeometry* of S, i.e. some restriction of some residue of S (cf. Section 6.4). See also Sections 2.12 and 2.8 for illustrations of (orthogonal) duality.

1.12. *Direct sum*

If $\{S_i;\ i \in I\}$ is a family of DLS's on pairwise disjoint point-sets P_i, the *direct sum* $\bigoplus_{i\in I} S_i$ is the DLS on $\bigcup_{i\in I} P_i$ whose varieties are all unions $\bigcup_{i\in I} V_i$, where V_i is a variety of S_i. For example the finite-dimensional generalized projective spaces are direct sums of nondegenerate (but possibly 0- or 1-dimensional) projective spaces. If $|I| = t$ and if each S_i has finite dimension d_i, then

$$\dim\left(\bigoplus_i S_i\right) = t - 1 + \sum_i d_i.$$

The direct sum of finitary DLS's is finitary.

1.13. *Supersums*

Needed in Section 7.9, they generalize direct sums, introducing a superstructure of pre-DLS on the t-set of all S_i's, in order to select and keep only some of the varieties of $\bigoplus_i S_i$.

Let T be a d_0-pre-DLS on $\{1,\ldots,t\}$, let $P_1,\ldots,P_t$ be pairwise disjoint sets, let S_i be a d_i-DLS ($0 \leqslant d_i < +\infty$) on P_i and let S_i' be a $(d_i - 1)$-pre-DLS on P_i such that every basis of S_i' is contained in some basis of S_i, every basis of S_i contains a basis of S_i', and the hyperplanes of S_i' are certain $(d_i - 2)$- or $(d_i - 1)$-varieties of S_i. For shortness such a pair (S_i, S_i') will be called a $(d, d-1)$-DLS's *association*. The simplest possibility is that S_i' is the $(d_i - 1)$-truncation of S_i, but there are other possibilities, e.g., S_i' can be any $(d_i - 1)$-DLS erected on $(d_i - 2)$-S_i, the hyperplanes of S_i' being certain hyperplanes and colines of S_i (compare with the 'missed projective spaces' of Section 1.3).

In order to get a DLS, assume further that if $d_i = d_j = 0$ for distinct i, j, then the pair $\{i, j\}$ is independent in T. Then the *supersum* of the (S_i, S_i')'s *over* T is the $(\sum_{i=0}^{t} d_i)$-DLS $\bigoplus_T(S_i, S_i')$ with point-set

$$\bigcup_{i=1}^{t} P_i$$

and whose bases are the sets B for which there is a basis J of T, such that $B \cap P_j$ is a basis of S_j if $j \in J$ and $B \cap P_j$ is a basis of S_j' if $j \notin J$. Note that in the notation $\bigoplus_T(S_i, S_i')$, the pre-DLS T is *provided with a numbering* of its points. Figure 1.4

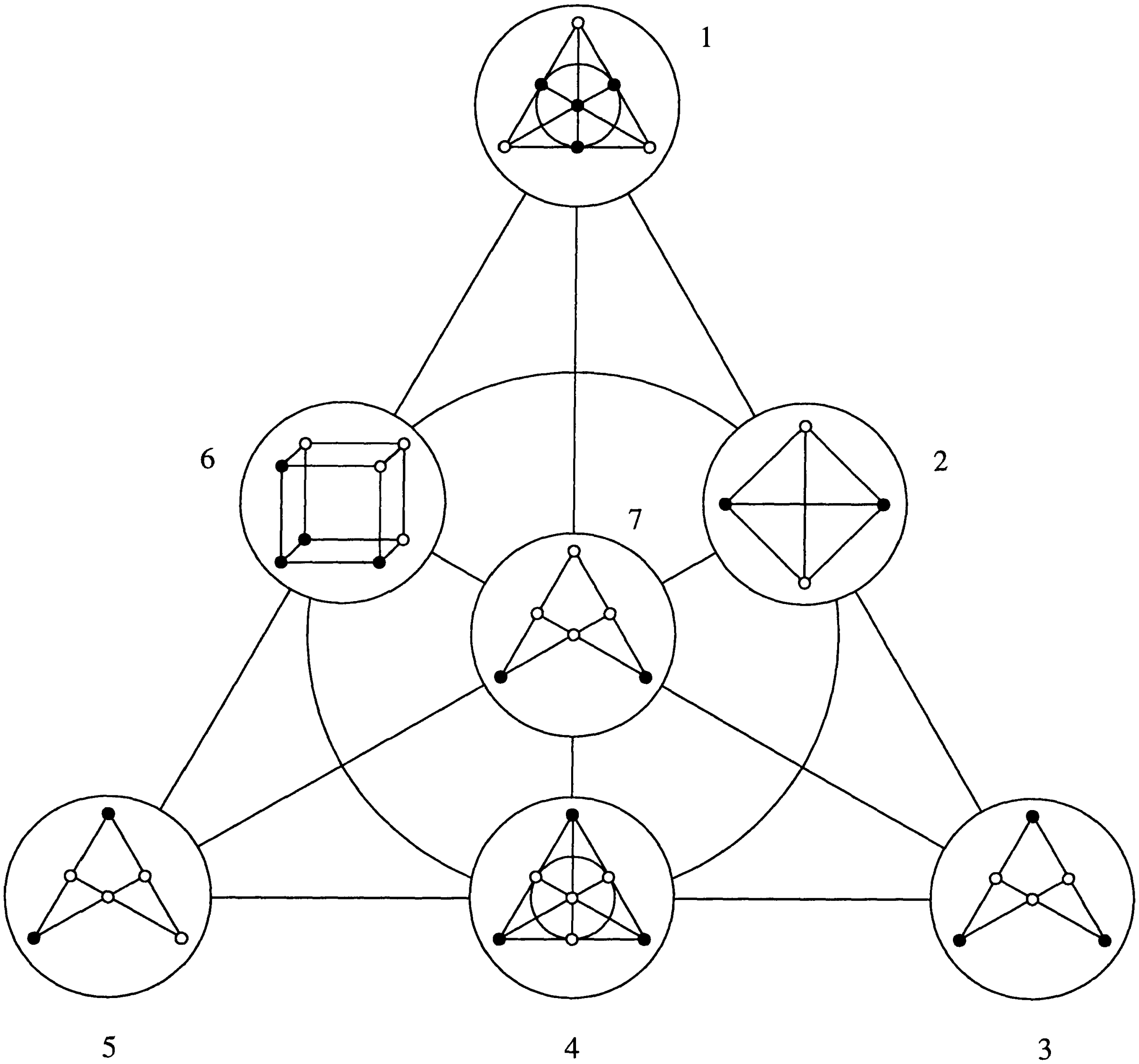


Figure 1.4.

suggests such a supersum for

$$\boldsymbol{T} = \mathrm{PG}(2,2) \quad \text{with points numbered as indicated,}$$

$$\boldsymbol{S}_1 \simeq \boldsymbol{S}_4 \simeq \mathrm{PG}(2,2)^*, \quad \boldsymbol{S}_1' \simeq \boldsymbol{S}_4' \simeq \mathrm{PG}(2,2);$$

$$\boldsymbol{S}_2 \simeq \mathrm{PG}(3,1), \quad \boldsymbol{S}_2' = 2\text{-}\mathrm{PG}(3,1);$$

$$\boldsymbol{S}_3 \simeq \boldsymbol{S}_5 \simeq \boldsymbol{S}_7 \simeq \boldsymbol{\Pi}_4; \quad \boldsymbol{S}_3' \simeq \boldsymbol{S}_5' \simeq \boldsymbol{S}_7' \simeq \mathrm{PG}(1,5);$$

$$\boldsymbol{S}_6 \simeq \mathrm{AG}(3,2), \quad \boldsymbol{S}_6' \simeq 2\text{-}\mathrm{AG}(3,2);$$

where the partition DLS $\boldsymbol{\Pi}_4$, corresponding to the lattice of partitions of a 4-set, is the punctured projective plane of order 2 (see Section 2.6). In this figure, all points and all lines of size $\geqslant 3$ are drawn, other varieties are usually omitted. The black points form a basis corresponding to the choice $J = \{1, 3, 6\}$.

If $\boldsymbol{T}$ is the completely trivial DLS (whose only basis is the full point-set), then $\boldsymbol{S}_i'$ is useless and we get back the direct sum. Our notation $\bigoplus_{\boldsymbol{T}}(\boldsymbol{S}_i, \boldsymbol{S}_i')$ thins down into $\bigoplus_{\boldsymbol{T}} \boldsymbol{S}$ in the special case where all $\boldsymbol{S}_i'$s are isomorphic to some d-DLS $\boldsymbol{S}$ and where $\boldsymbol{S}_i'$ is the $(d-1)$-truncation of $\boldsymbol{S}_i$. This DLS coincides with the direct product $\boldsymbol{T} \otimes \boldsymbol{S}$ of the two matroids $\boldsymbol{T}$ and $\boldsymbol{S}$ defined by Lim [1977], who proved that if $\boldsymbol{S}$ and $\boldsymbol{T}$ are two nonsingleton matroids, then

$$\operatorname{Aut}\left(\bigoplus_{\boldsymbol{T}} \boldsymbol{S}\right) = \operatorname{Aut} \boldsymbol{S} \operatorname{\mathtt{wr}} \operatorname{Aut} \boldsymbol{T} \ ^{2}$$

if and only if the following two implications hold

(i) if $\boldsymbol{S}$ is a 0-DLS, then $\boldsymbol{T}$ is a DLS, and

(ii) if $\boldsymbol{S}$ is a direct sum, then for any point-pair (x, y) in $\boldsymbol{T}$, there is a dependent set containing x but not y.

Note that the supersum of FDLS's over a finitary DLS $\boldsymbol{T}$ is also finitary.

1.14. *Connectivity and generalizations*

A DLS which is not a direct sum is called *connected* (resp., *irreducible*) in matroid (resp., lattice) terminology. A generalization of the decomposition of a nonconnected DLS into a direct sum of irreducible DLS's is provided by the following notion which will be used in Theorem 6.6. A *k-separation* of $\boldsymbol{S}$ is a partition of its point-set S^0 into two subsets A, B of size $\geqslant k$ such that

$$\dim\langle A\rangle + \dim\langle B\rangle \leqslant \dim \boldsymbol{S} + k - 2;$$

the *parts* of this k-separation are the restrictions of $\boldsymbol{S}$ to $A \setminus \langle B\rangle$ and $B \setminus \langle A\rangle$, respectively. Hence given an n-DLS $\boldsymbol{S}$ and an n'-DLS $\boldsymbol{S}'$ with $n, n' \geqslant 1$, the direct sum $\boldsymbol{S} \oplus \boldsymbol{S}'$ and its $(n+n')$-truncation admit 2-separations with parts $\boldsymbol{S}$ and $\boldsymbol{S}'$. Also the $(n+n')$-DLS obtained by gluing together $\boldsymbol{S}$ and $\boldsymbol{S}'$ at one point x and defining the missing varieties to be the smallest possible ones, admits a 2-separation with parts $\boldsymbol{S}|_{S\setminus\{x\}}$ and $\boldsymbol{S}'|_{S'\setminus\{x\}}$. The supersums defined in Section 1.13 also admit obvious 2-separations. A DLS $\boldsymbol{S}$ is called *k-connected* if it has no k-separation. Hence a DLS is *connected* (or 1-connected) if and only if it is not a direct sum.

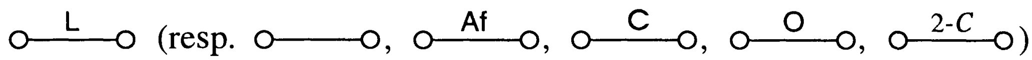
The reader can find further information about direct sums and their generalizations in White [1986], Section 7.6.

[2] wr stands for wreath product (Editor's note).

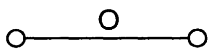
2. Some examples and characterizations

2.1. *Truncations of generalized projective or affine spaces*

We extend the notation introduced in Chapter 3, Section 3.2, as follows. The DLS's belonging to the diagram

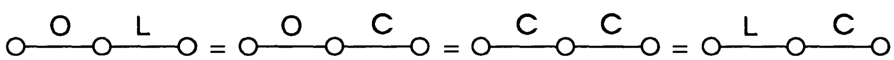


are (up to isomorphism) the linear spaces (resp., the generalized projective planes, the affine planes, the linear spaces all of whose lines have two points, the triangle, the 2-truncations of DLS's belonging to the class C. We introduce the notation



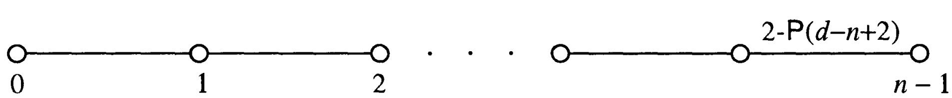
for the 'self-dual' subclass of

in order to stress the fact that it corresponds to a generalized projective plane, but its use is often superfluous because

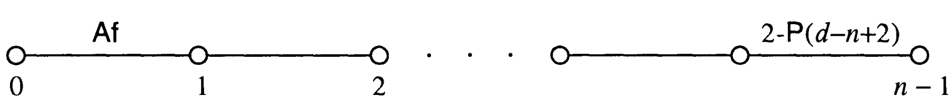


(see Section 3.1).

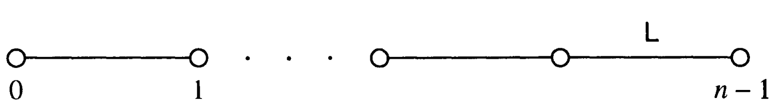
Let $n \geqslant 3$ be an integer and let d be a cardinal number $\geqslant n$. The n-truncations of generalized projective or affine spaces of dimension $d \geqslant n$ belong to the following diagrams:



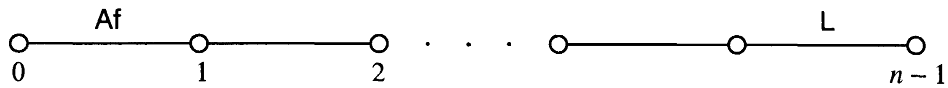
and



where $P(d)$ denotes the class of d-dimensional generalized projective spaces. These diagrams, and even the more general diagrams



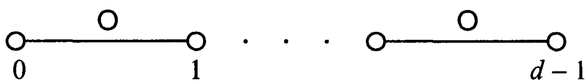
and



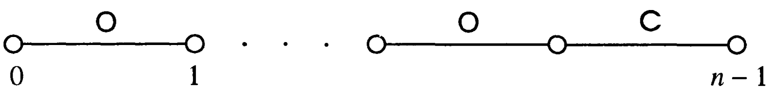
actually characterize these n-DLS's except in the cases where $n = 3$ and all planes are affine planes of order $\leqslant 3$ (compare Sections 4.2, 4.4 and Chapter 2, Section 6.1).

2.2. *Trivial DLS's*

In particular, if d is finite, we call *completely trivial* or *Boolean* the DLS's belonging to



They correspond to PG(d, 1), to *Boolean lattices* (i.e. distributive complemented lattices) and to *free matroids*. For $n \leqslant d$, their n-truncations are called *trivial n-DLS's*: they correspond to the diagram

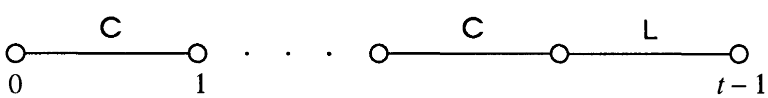


and are also called *uniform matroids*. All their proper varieties are *thin* (i.e. their size (or cardinality) is one more than their dimension).

2.3. *Hypercircular DLS's*

A DLS is called *circular* if it belongs to a diagram admitting at least one c-stroke, or equivalently if all its lines have just two points. A *circular space* is a circular 3-DLS. A *noncircular* DLS has at least one *thick* line (i.e. a line of size $\geqslant 3$), while a *thick* DLS has thick lines only.

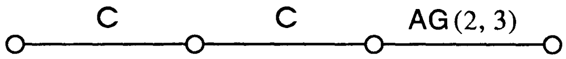
We call *hypercircular* t-DLS (or *paving matroid*) any DLS belonging to



All its i-varieties with $i \leqslant t - 2$ are thin and its (point, hyperplane)-truncation $(\boldsymbol{P}, \boldsymbol{H})$ is a $(t, 1)$-*quasi-design* (or a t-*partition*), which means that any t-subset of $\boldsymbol{P}$ is in exactly one hyperplane (see Hartmanis [1959, 1961]). If moreover all hyperplanes have the same size k, then $(\boldsymbol{P}, \boldsymbol{H})$ is a t-$(v, k, 1)$ *design* (or *Steiner system* $S(t, k, v)$) with $t \leqslant k < v \leqslant \infty$ (see Chapter 8). Since any t-$(v, k, 1)$ design is the (point, hyperplane)-truncation of a unique DLS (see Section 1.8), we will identify each such design with its associated DLS. Hence Steiner systems coincide with the regular hypercircular finite DLS's (Section 3.1). Note that any $(t - i)$-dimensional top residue of a t-$(v, k, 1)$ design is a $(t - i)$-$(v - i, k - i, 1)$ design.

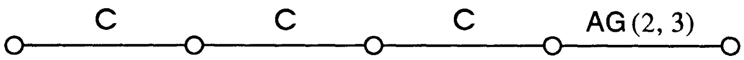
The most famous examples of regular hypercircular DLS's are the five *Mathieu–Witt designs* $\boldsymbol{M}_i$ related to the Mathieu groups M_i (see also Chapters 9 and 22).

The 4-(11,5,1) design $\boldsymbol{M}_{11}$ belongs to



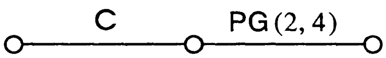
and Aut $\boldsymbol{M}_{11} = \mathsf{M}_{11}$.

The 5-(12,6,1) design $\boldsymbol{M}_{12}$ belongs to



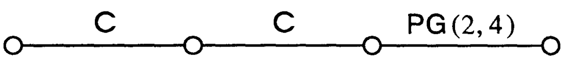
and Aut $\boldsymbol{M}_{12} = \mathsf{M}_{12}$.

The 3-(22,6,1) design $\boldsymbol{M}_{22}$ belongs to



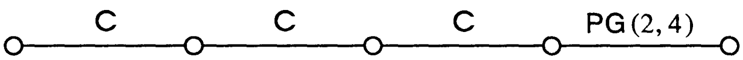
and Aut $\boldsymbol{M}_{22} = 2 \cdot \mathsf{M}_{22}$.

The 4-(23,7,1) design $\boldsymbol{M}_{23}$ belongs to



and Aut $\boldsymbol{M}_{23} = \mathsf{M}_{23}$.

The 5-(24,8,1) design $\boldsymbol{M}_{24}$ belongs to



and Aut $\boldsymbol{M}_{24} = \mathsf{M}_{24}$.

Though free constructions (see Chapter 13) provide for any n lots of infinite nontrivial *regular* hypercircular n-DLS's, no finite example is known for $n \geqslant 6$. By contrast again finite *nonregular* hypercircular DLS's exist in profusion (cf. 3.14).

If all $(n-i)$-dimensional top residues of an n-DLS $\boldsymbol{S}$ are isomorphic to some fixed $(n-i)$-DLS $\boldsymbol{R}$, then $\boldsymbol{S}$ is an *i-fold L-extension* or an L^i-extension of $\boldsymbol{R}$ (an *i-fold* c*-extension* or c^i-extension if $(i+1)$-$\boldsymbol{S}$ is trivial). Although truncations of Desarguesian projective spaces admit L^i-extensions for all i's, the extendability of a given DLS $\boldsymbol{R}$ seems to be rather exceptional. For example, it will follow from 4.8 and 5.28 that, for $d \geqslant 3$, AG(d,q) has no finite regular L-extension, while its truncation 2-AG(d,q) can be c-extended (see 5.28).

2.4. *Design DLS's*

The (point, hyperplane)-truncation of any regular hypercircular n-DLS is an n-$(v,k,1)$ design, but is a 2-(v,k,λ) design as well. A *design*-DLS is a DLS whose (point, hyperplane)-truncation is a 2-(v,k,λ) design. If $d \geqslant n \geqslant 3$, n-PG(d,q) and n-AG(d,q) provide examples of design-DLS's with $\lambda > 1$ which are not hypercircular except if $\boldsymbol{S}$ = 3-AG$(d,2)$ (see Chapter 2, Proposition 8.7). More generally the (point, hyperplane)-truncation of any regular DLS is a 2-(v,k,λ) design with $k = s_{n-1}$ and $\lambda = t(1, n-1, n)$ (cf. Sections 3.1 and 3.2).

We do not know any example of a nonregular design-DLS.

The constancy of s_{n-1} and $t(1, n-1, n)$ forces the constancy of s_1, but the constancy of s_{n-1} and s_1 does not imply that of $t(1, n-1, n)$, as shown, e.g., by $\boldsymbol{M}_{22} \oplus \boldsymbol{M}_{22}$.

Note also that a 2-(v, k, λ) design is not necessarily the (point, hyperplane)-truncation of some DLS. Indeed remember the following:

PROPOSITION (see 1.8(9) and, e.g., White [1986]). *A finite (point, block) incidence structure* $(\boldsymbol{P}, \boldsymbol{B})$ *without repeated blocks is the (point, hyperplane)-truncation of a* DLS $\boldsymbol{S}$ *iff*

(i) *for any two distinct blocks and any point, there is a block containing both the point and the intersection of the blocks, and*

(ii) *for any two distinct points* x *and* y, *there is a block on* x *but not on* y.

Most 2-(v, k, λ) designs do not satisfy (i). Even if (i) happens to hold, it will usually fail to hold in the complementary design. It is easy to check that no nontrivial 2-$(v, k, 2)$ design is the (point, hyperplane)-truncation of a DLS since $s_1 \leqslant t(1, n-1, n) = 2$, forcing $s_1 = s_{n-2} = 2$ and $k = s_{n-1} = 3$, $v = 4$.

2.5. *Generously intersecting hyperplanes*

Here are some crucial characterizations of projective and affine spaces by means of hyperplane intersections. Such results were initiated by the Dembowski–Wagner theorem for 2-designs (see Chapter 2, Theorem 8.8), whose statement requires the following definitions. A *line* of a (point, block) incidence structure $(\boldsymbol{P}, \boldsymbol{B})$ is the intersection of all blocks containing any two given points; points lying on a common line are called *collinear*. If $(\boldsymbol{P}, \boldsymbol{B})$ is the (point, hyperplane)-truncation of some DLS $\boldsymbol{S}$, then the lines of $(\boldsymbol{P}, \boldsymbol{B})$ coincide with those of $\boldsymbol{S}$.

DEMBOWSKI–WAGNER THEOREM (Dembowski and Wagner [1960]). *A finite (point, block) incidence structure* $(\boldsymbol{P}, \boldsymbol{B})$ *without repeated blocks is the (point, hyperplane)-truncation of a finite projective space iff* $(\boldsymbol{P}, \boldsymbol{B})$ *is a* 2-*design in which every line meets every block, iff* $(\boldsymbol{P}, \boldsymbol{B})$ *is a symmetric (i.e.* $|\boldsymbol{P}| = |\boldsymbol{B}|$*)* 2-*design in which any three noncollinear points are on a constant number of blocks.*

The assumption that $(\boldsymbol{P}, \boldsymbol{B})$ be a 2-design is slightly weakened in Kantor [1976] (Theorem 5).

In the following two variations of the Dembowski–Wagner theorem, the equicardinality of the hyperplanes is no longer assumed.

TEIRLINCK'S THEOREM (Teirlinck [1980]). *A finite* DLS *is a generalized projective space if and only if every line meets every hyperplane.*

HERON'S THEOREM. *A finite* DLS *is a projective space iff any two hyperplanes intersect in a constant number of points.*

As noticed in Heron [1973], this result follows from Isbell [1959] (or Majumdar [1953]) and the results of 3.6.

Finally Kantor dropped the assumption that any two hyperplanes intersect.

KANTOR'S THEOREM (Kantor [1968]). *A finite* DLS $\boldsymbol{S}$ *on* v *points is a projective or affine space of dimension* $n \geqslant 4$ *iff there are integers* μ *and* k *with* $1 < \mu < k < v - 1$ *and* $(\mu - 1)(v - k) \neq (k - \mu)^2$ *such that all hyperplanes have* k *points and any two intersecting hyperplanes have* μ *common points.*

The assumptions $\mu > 1$ and $k < v - 1$ exclude the linear spaces with constant line-size k and the Boolean spaces, respectively, while the assumption $(\mu - 1)(v - k) \neq (k - \mu)^2$ excludes the 3-DLS's whose point-residues are projective planes (we will see in 4.8 that these are not necessarily projective or affine spaces).

As noticed by Kantor, if the top coplane-residues of a DLS are thick projective planes and if all hyperplanes have the same size, then all colines have the same size, so that Kantor's theorem, 4.8 and 4.6 prove the following.

COROLLARY. *Let* $\boldsymbol{S}$ *be a finite* n-DLS *with* $n \geqslant 3$. *If any two nondisjoint hyperplanes have the same size and intersect in a coline, then* $\boldsymbol{S}$ *is an* n*-dimensional generalized projective or affine space, or* $\boldsymbol{S}$ *is a* 3-DLS *whose point-residues are projective planes of order as prescribed by* 4.8.

It is clear that dropping all the equicardinality hypotheses in these results would allow many other spaces, namely restrictions of affine or projective spaces (see Section 6, this was indeed the origin of Kantor's notion of strong embedding). Even worse: dropping the finiteness assumption brings up a lot of nonembeddable n-DLS's for each $n \geqslant 3$ (Kantor [1974a], Example 5). However, Kantor [1974a] generalized the above intersection hypothesis and proved the following.

DIMENSION LEMMA. *Given an* n-DLS $\boldsymbol{S}$ $(3 \leqslant n < \infty)$ *and two positive integers* i, j *less than* n *with* $i + j \geqslant n$, *if no two varieties generating* $\boldsymbol{S}$ *and of respective dimensions* i, j *intersect in an* $(i + j - n - 1)$*-variety, then the top residue of any* $(i + j - n - 1)$*-variety is a generalized projective space* (*of dimension* $2n - i - j$).

Hence $\boldsymbol{S}$ is 'locally projective'; according to our terminology in 4.4 and 5.28, we could say more precisely that $\boldsymbol{S}$ is *sharply locally projective* (shLP) with respect to $(i + j - n - 1)$-varieties, or that the truncation $(i + j - n + 2)$-$\boldsymbol{S}$ is *strongly locally projective* (SLP). As we shall see in 6.9, except in the case $i = j = n - 1$, this forces $\boldsymbol{S}$ to be embeddable in an n-dimensional generalized projective space $\boldsymbol{T}$ in such a way that the top $(2n - i - j)$-dimensional residues in $\boldsymbol{S}$ and $\boldsymbol{T}$ agree. A similar result is proved in the case $i = j = n - 1$ under additional assumptions requiring that $\boldsymbol{S}$ be a 'large chunk' of the projective space $\boldsymbol{T}$ ('large' relatively to the hyperplane-order, see 4.10 and 6.12–15).

In 1978 another intersection property has been introduced in the literature: for some $i \leqslant n - 2$, any two i-varieties in a common $(i + 1)$-variety intersect in an $(i - 1)$-variety.

This forces the 3-truncation of the top residue of any $(i-2)$-variety to be the 3-truncation of some generalized projective space (of dimension $\geqslant 3$), and the best one can do is to embed the $(i+2)$-truncation of $\boldsymbol{S}$ in the $(i+2)$-truncation of some generalized projective space of dimension $\geqslant n$, as shown by the missed projective spaces (see 1.3).

Cameron [1977] introduces still another intersection property: 'if H_1, H_2 and H_3 are hyperplanes such that $H_1 \cap H_2$ and $H_1 \cap H_3$ are disjoint colines, then $H_2 \cap H_3$ is empty or a coline'. He notes that this condition is trivially satisfied if $n = 2$ and that, if $n = 3$, it is equivalent to requiring the planar space to be sharply locally projective, i.e. to be the 3-truncation of a DLS whose point-residues are generalized projective spaces (see Proposition 4.5 for the equivalence and Sections 4 and 6 for results in that direction). Cameron characterized the hypercircular n-DLS's with $n \geqslant 3$ satisfying this intersection condition as being the 3-truncations of affine spaces over GF(2), the Mathieu–Witt designs 3-(22,6,1) or 5-(24,8,1) and the n-$(v, k, 1)$ designs such that $k < 2n - 2$. Still on this theme, see Cameron [1974b, 1976b].

Note that all the above assumptions on the dimension of intersections of varieties hold under the stronger hypothesis that every variety V of the FDLS $\boldsymbol{S}$ is *modular*, which means that for any other variety W,

$$\dim V + \dim W = \dim\langle V \cup W\rangle + \dim(V \cap W).$$

The FDLS's satisfying this strong hypothesis are precisely the finite dimensional generalized projective spaces (see, e.g., Birkhoff [1967]).

In particular, a hyperplane of an FDLS is *modular* if and only if it intersects every line. We use this latter characterization to define *modular hyperplanes* in any finitary DLS (alternative terminologies are 'projective' or 'geometric' hyperplanes, often used in more general structures than DLS's).

ALL MODULAR HYPERPLANES THEOREM. *The finitary* DLS'*s all of whose hyperplanes are modular are precisely the generalized projective spaces.*

See Teirlinck [1980] where this is presented in a more general setting.

ENOUGH MODULAR HYPERPLANES THEOREM (Kahn and Kung [1986]). *Let $\boldsymbol{S}$ be a connected n-*DLS *with $2 < n < \infty$. If the intersection of all modular hyperplanes of $\boldsymbol{S}$ is empty, then either $\boldsymbol{S}$ is a Dowling lattice $\boldsymbol{Q}_{n+1}(G)$* (see 2.7) *or $\boldsymbol{S}$ can be embedded in some projective space* PG(n, F) *in such a way that the point-set of $\boldsymbol{S}$ contains the union of $n+1$ hyperplanes of* PG(n, F) *having an empty intersection.*

The case $n = 2$ was excluded from this theorem because then Dowling's construction for $\boldsymbol{Q}_3(G)$ still holds even if G is no longer a group but only a quasigroup, i.e. combinatorially a Latin square, and gives linear spaces having three nonconcurrent modular lines, so that the range of examples is dramatically enlarged.

For further work in the spirit of this section, see Gorn [1940].

2.6. *Partition DLS's*

Let n be an integer $\geqslant 2$ and let Ω_n be an n-set. The lattice of all partitions of Ω_n, ordered by the relation 'is a refinement of', is called the *partition lattice* $\boldsymbol{\Pi}_n$ and is the lattice of an $(n-2)$-DLS which we also denote by $\boldsymbol{\Pi}_n$. The points of $\boldsymbol{\Pi}_n$ can be identified with the pairs of elements of Ω_n, the points of a variety π being the pairs of elements of Ω_n lying in a common class of the partition π. It is easily seen that lines have size 2 or 3 and that the 3-truncation of $\boldsymbol{\Pi}_n$ is a Fischer space without affine planes (see 4.12). All residues of $\boldsymbol{\Pi}_n$ are direct sums of $\boldsymbol{\Pi}_i$'s, more precisely if π is a refinement of the partition σ in $\boldsymbol{\Pi}_n$ and if $c(\pi)$ denotes the number of classes of π, then

$$\dim \pi = n - 1 - c(\pi) \quad \text{and} \quad \left(\boldsymbol{\Pi}_n\right)_\pi \cong \boldsymbol{\Pi}_{c(\pi)},$$

$$\sigma = \bigoplus_{i=1}^{c(\sigma)} \boldsymbol{\Pi}_{s_i} \quad \text{and} \quad \sigma_\pi \cong \bigoplus_{i=1}^{c(\sigma)} \boldsymbol{\Pi}_{m_i},$$

where the s_i's are the sizes of the classes of σ and the m_i's are such that

$$\sum_{i=1}^{c(\sigma)} m_i = c(\pi).$$

Aigner [1974] characterized $\boldsymbol{\Pi}_n$ as the only DLS all of whose point-residues are isomorphic to $\boldsymbol{\Pi}_{n-1}$ and which contain a modular hyperplane (i.e. a hyperplane meeting any i-variety in an i- or $(i-1)$-variety). The automorphism group of $\boldsymbol{\Pi}_n$ is Sym(n) if $n \geqslant 3$. Note that $\boldsymbol{\Pi}_1$ is empty, $\boldsymbol{\Pi}_2$ is the 0-DLS and $\boldsymbol{\Pi}_3$ is the 1-DLS on three points. The 2-DLS $\boldsymbol{\Pi}_4$ is also called the *Pasch* (*or Veblen*) *configuration* while the 3-DLS $\boldsymbol{\Pi}_5$ is (the 3-dimensional) *Desargues configuration* (see Figure 2.1 and Chapters 2 and 4). Moreover any partition lattice is binary (cf. 6.5).

Grieser [1991] counts the number of modular complements σ of a given variety π in $\boldsymbol{\Pi}_n$ (i.e. $\pi \cap \sigma = \phi$, $\langle \pi, \sigma \rangle = \boldsymbol{\Pi}_n$ and $\dim \pi + \dim \sigma = n - 3$).

Finally, let us mention the remarkable fact that every finite lattice can be embedded as a sublattice of some finite (but often enormous) partition lattice (Pudlák and Tůma [1980]); however, even when restricted to DLS's, such embeddings are not isometric (see 6.2).

2.7. *Dowling lattices*

To any finite group G and any positive integer n Dowling [1973] associated an $(n-1)$-DLS $\boldsymbol{Q}_n(G)$. These spaces generalize in a nice way the partition spaces: $\boldsymbol{Q}_n(1) = \boldsymbol{\Pi}_{n+1}$ and every residue of $\boldsymbol{Q}_n(G)$ is a direct sum of various $\boldsymbol{\Pi}_i$'s and at most one $\boldsymbol{Q}_j(G)$. Kahn and Kung's theorems quoted in Sections 2.5 and 6.7 both highlight Dowling spaces as well as projective spaces. Here is Dowling's construction:

Let X be an n-set and define a *partial G-partition of X* as a set

$$\alpha = \{a_j\colon \operatorname{dom} a_j \to G\colon\ j = 1, \ldots, r\}$$

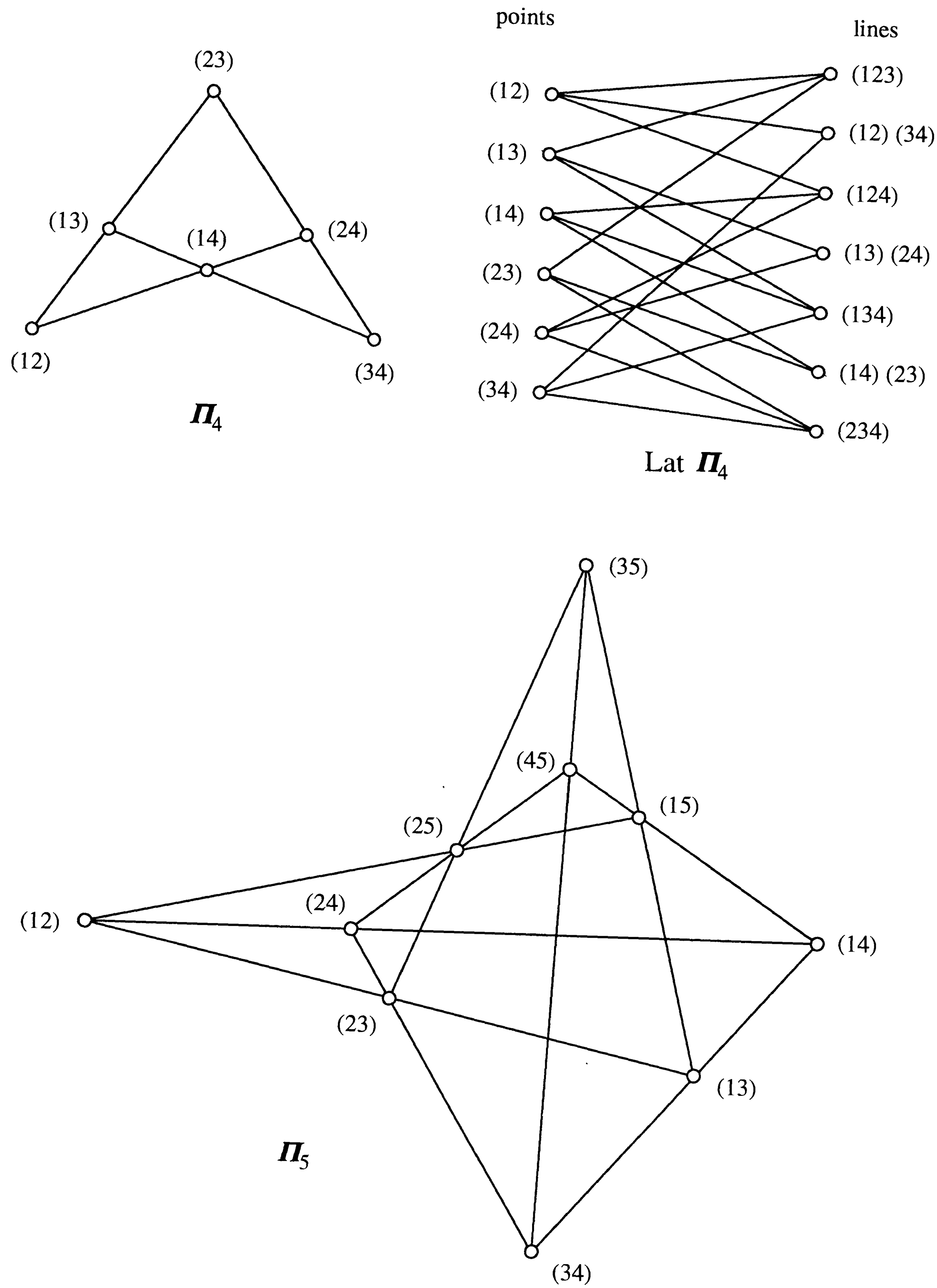

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Figure 2.1.

of functions into G whose domains $\mathrm{dom}\, a_j$ are disjoint nonempty subsets of X. Given a nonempty subset α' of α, a *linear combination of* α' is a function

$$b = \sum_{\alpha'} \lambda_i a_i,$$

where a_i ranges over α' and λ_i belongs to G, defined as follows: its domain is the union of the domains of the a_i's and its restriction to $\mathrm{dom}\, a_i$ is $\lambda_i a_i$, i.e. $b(x) = \lambda_i a_i(x)$ for any $x \in \mathrm{dom}\, a_i$.

Now define an order relation on the set $Q'_n(G)$ of partial G-partitions of X as follows: $\alpha \leqslant \beta$ if and only if for each $b \in \beta$, there is a subset α' of α and there are elements $\lambda_i \in G$ such that

$$b = \sum_{\alpha'} \lambda_i a_i.$$

Finally the quotient set obtained after identifying any two elements α and β for which $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$, provided with the partial order induced by $\leqslant$, is the lattice of varieties of the $(n-1)$-DLS $\boldsymbol{Q}_n(G)$.

In his original paper, Dowling established many remarkable properties of these spaces. Here are some of them.

For $n \geqslant 3$, $\boldsymbol{Q}_n(G)$ is coordinatizable (see 6.2) iff G is cyclic. More precisely, if G is cyclic of order m, then $\boldsymbol{Q}_n(G)$ is

(i) unimodular (see 6.5) iff $m = 1$,
(ii) GF(q)-coordinatizable iff $m \mid q - 1$,
(iii) $\mathbb{Q}$- or $\mathbb{R}$-coordinatizable iff $m = 1$ or 2,
(iv) $\mathbb{C}$-coordinatizable for every m.

$\boldsymbol{Q}_n(G)$ is supersolvable (see 2.9). Dowling also noticed that if $n = 3$ then his construction also works for any quasigroup G and that in any case the linear space $\boldsymbol{Q}_3(G)$ is the union of three nonconcurrent modular lines L_1, L_2, L_3 of size $|G| + 2$ and that all the other lines missing the three joints $L_1 \cap L_2$, $L_2 \cap L_3$, $L_3 \cap L_1$ have 3 points, and so allow us to reconstruct the multiplication of G.

Bonin and Bogart [1991] provide a geometric characterization of Dowling lattices using the Kahn and Kung determination of DLS's with enough modular hyperplanes (see 2.5).

2.8. *Graphic matroids*

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be a finite undirected graph with vertex-set $\boldsymbol{V}$ and edge-set $\boldsymbol{E}$. Then $\boldsymbol{E}$, provided with the closure operator defined by

$$\langle \boldsymbol{A} \rangle = \big\{ e = \{u, v\} \colon\ u, v \in \boldsymbol{V} \text{ are in the same connected component of the subgraph } (\boldsymbol{V}, \boldsymbol{A}) \big\},$$

is a pre-DLS, called the *polygon* (or *cycle* or *circuit*) *matroid* G of $\boldsymbol{G}$, which is simple iff $\boldsymbol{G}$ is *simple* (i.e. free of loops and multiple edges) and has dimension $|\boldsymbol{V}| - 1 - c(\boldsymbol{G})$ where $c(\boldsymbol{G})$ is the number of connected components of $\boldsymbol{G}$. This result, due to Whitney, was one of the starting points of matroid theory (as witnessed by much of the matroidal terminology), which rose as an important algebraization tool for the study of graphs. So these matroids, called *graphic*, and their duals called *cographic*, occupy a central position in the matroid literature, and we refer the reader to the numerous specialized texts on this subject (see, e.g., White [1986], Chapter 6; Tutte [1971], Biggs [1974]). Graphic and cographic matroids are unimodular (i.e. coordinatizable over every field, see 6.5) and conversely they are, together with some 10-points DLS $\boldsymbol{R}_{10}$, the building blocks of any unimodular DLS (see 6.6).

Let us mention some easy examples. The polygon matroid $M(\boldsymbol{C}_n)$ associated with the connected graph $\boldsymbol{C}_n$ of degree 2 with n vertices and n edges is the trivial $(n-2)$-DLS on n points, and the one associated with the complete graph $\boldsymbol{K}_n$ on n vertices is the partition $(n-2)$-DLS $\boldsymbol{\Pi}_n$ (see 2.6).

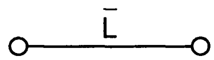
The cycle matroid $M(\boldsymbol{K}_{3,3})$ associated with the complete bipartite graph $\boldsymbol{K}_{3,3}$, as well as its dual $M(\boldsymbol{K}_{3,3})^*$, play an important role in characterizations of cographic and graphic DLS's (see 6.5).

Generalizing one of his own results in graph theory, Tutte [1966] proved that the fundamental building blocks for the class of 3-connected DLS's (see 1.14) are cycle matroids of graphs called *wheels*, as well as *whirls*, which are self-dual nongraphic DLS's that can be easily defined from wheels (see, e.g., Welsh [1976]). The wheel $\boldsymbol{W}_n$ of order n ($n \geqslant 3$) is obtained from the circuit $\boldsymbol{C}_n$ (called the *rim*) by joining each vertex of the rim to a new vertex c by a simple edge, called a *spoke*. The whirl $\boldsymbol{W}^n$ of order n is the DLS whose points are the edges of $\boldsymbol{W}_n$ and whose minimal dependent sets are all cycles of $\boldsymbol{W}_n$ but the rim, and all sets of edges formed by adding one spoke to the set of edges of the rim. Since Seymour [1980] showed that every non-3-connected DLS is a direct sum or a 2-sum of two smaller DLS's, quite a few papers investigate classes of DLS's of which no subgeometry is $M(\boldsymbol{W}_n)$ or $\boldsymbol{W}^n$ for some small n (see 6.6. and Oxley [1990]).

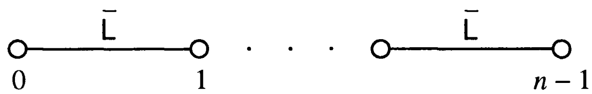
2.9. *m-closed DLS's*

Given a DLS $\boldsymbol{S}$ whose closure operator is denoted by $\langle\cdot\rangle$, a subset A of t he point-set S^0 is called *m-closed* ($m \leqslant \dim \boldsymbol{S}$) if and only if for every $X \subseteq A$ either $\dim\langle X\rangle \geqslant m$ or $\langle X\rangle \subseteq A$. In particular, the 2-closed sets of $\boldsymbol{S}$ are the linear subspaces of the linear space 2-$\boldsymbol{S}$ and the 3-closed sets of $\boldsymbol{S}$ are the planar subspaces of the planar space 3-$\boldsymbol{S}$ (see 4.5). An m-closed subset A is said to be *proper* in $\boldsymbol{S}$ if it is not a variety of $\boldsymbol{S}$. Trivially the existence of n-closed (or hyperplane-closed) proper subsets is required for $\boldsymbol{S}$ to be erectable into an $(n+1)$-DLS.

S^0 provided with all its m-closed sets is a closure space, but not necessarily a DLS, called the *m-closure space over m-$\boldsymbol{S}$* (with closure $\langle\cdot\rangle_m$). A DLS $\boldsymbol{S}$ is said to be *m-closed* if and only if it coincides with the m-closure space over m-$\boldsymbol{S}$. In particular, n-closed (or hyperplane-closed) n-DLS's cannot be erected into higher dimensional DLS's. The projective and affine spaces (of order $\neq 2$ if affine), the Dowling DLS's (and hence the partition DLS's) and more generally all *supersolvable* FDLS's (i.e. having a maximal flag all of whose members are modular varieties (see 2.5)) are 2-closed (or line-closed) (Halsey [1987]). If we denote by



the class of all 2-closed linear spaces (i.e. linear spaces having no proper linear subspaces), then the 2-closed n-DLS's are precisely those belonging to the diagram



(Halsey [1987]; see also Halsey [1990] for further developments). Note that the DLS's all of whose planes are projective planes are not known, although the 2-closed ones are the projective spaces.

2.10. *Dilworth truncation*

As mentioned in 1.10, lower truncations of DLS's of dimension > 2 are not DLS's. In particular if we delete all points from the lattice of an n-DLS $\boldsymbol{S}$ ($n \geqslant 3$), we get a nongeometric lattice because $\boldsymbol{S}$ contains noncoplanar lines. However this lattice Lat(S^i; $1 \leqslant i \leqslant n-1$) can be completed (by adding as few elements as possible) into a geometric lattice, called the *first Dilworth truncation* of Lat($\boldsymbol{S}$) or the *first Dilworth completion* of Lat(S^i; $1 \leqslant i \leqslant n-1$). If $n = 3$, it simply consists of adding lines of two points to the semilinear space (S^1, S^2), so that the Dilworth truncation of the trivial 3-DLS on v points is 2-$\boldsymbol{\Pi}_v$, while that of the Boolean DLS on v points is the partition DLS $\boldsymbol{\Pi}_v$ (see 2.6). Slightly more generally the $(n-2)$-Dilworth truncation consists of adding lines of two points to the semilinear space (S^{n-2}, S^{n-1}). For example, the second Dilworth truncation of the completely trivial DLS on 5 points is the following 'Pentagram linear space' on 10 points.

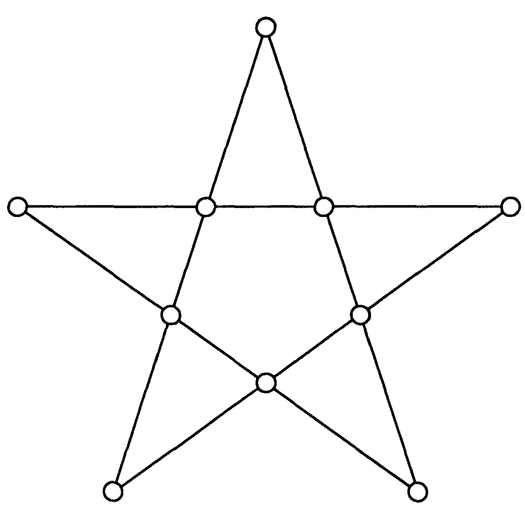
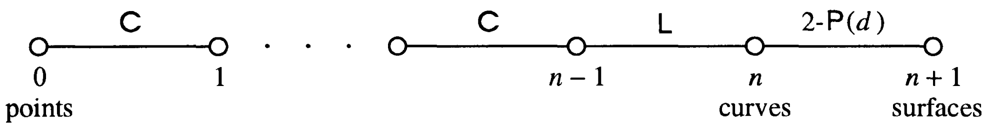


Figure 2.2.

For more on Dilworth truncations, see, e.g., Crawley and Dilworth [1973], White [1986] (7.7 and 10.2).

2.11. *Wille incidence geometries of grade n, Möbius geometries, etc.*

Wille [1967, 1971] defines incidence geometries of grade n ($n \geqslant 0$) as a certain type of (point, curve, surface) incidence structures and proves that these can be seen as the DLS's of dimension $\geqslant n+2$ whose $(n-1)$-varieties are thin and have top residues isomorphic to generalized projective spaces. More precisely an incidence geometry of grade n is the (point, coline, hyperplane)-truncation of an $(n+2)$-DLS $\boldsymbol{S}$ belonging to



which can be erected into a $(d+n)$-DLS (d may be infinite). Remember that

denotes the class of 2-truncations of the d-dimensional projective (resp., affine) spaces. If $d \geqslant 3$, then $\boldsymbol{S}$ is always embeddable into a projective space (see 4.4, 4.7–10 and 6.12). Isomorphism between such $(n+2)$-DLS's having at least one pair of isomorphic point-residues is studied in Metz [1973].

Replace now the 2-$P(d)$ stroke in the above diagram by a 2-$A(d)$ stroke. If we choose moreover $n = 1$ (i.e. we remove all c-strokes), then we get the *d-dimensional Möbius geometries* of Mäurer [1968]. If to the contrary we keep all c-strokes but remove the L-stroke, then we get the *inversive spaces* of Buekenhout [1971b] (Section 5.28(3)). In both cases the hypothesis $d > 2$ ensures the embeddability into a projective space (5.28(iii)).

Related structures are the inversive (or Möbius) planes, Laguerre planes and Minkowski planes (collectively named Benz planes), studied in Section 5.

2.12. *Function spaces, chain groups and embeddability*

Given a set $\boldsymbol{P}$ of points, a field F and a vector space $\boldsymbol{V}$ over F of functions from $\boldsymbol{P}$ to F, the *function space* $\boldsymbol{FS}(\boldsymbol{P},\boldsymbol{V})$ associated with $(\boldsymbol{P},\boldsymbol{V})$ is the pre-DLS on $\boldsymbol{P}$ whose varieties are all intersections of kernels of functions in $\boldsymbol{V}$, while the *chain group* $\boldsymbol{CG}(\boldsymbol{P},\boldsymbol{V})$ is the dual of $\boldsymbol{FS}(\boldsymbol{P},\boldsymbol{V})$. A first step towards simplification of the pre-DLS $\boldsymbol{FS}$ on $\boldsymbol{P}$ is to gather in a common class any two points p, p' of $\boldsymbol{P}$ such that $f(p) = f(p')$ for every $f \in \boldsymbol{V}$ and call $\boldsymbol{X}$ any subset of $\boldsymbol{P}$ containing exactly one representative of each class. Evaluation at a point provides an isomorphism from $\boldsymbol{FS}|_{\boldsymbol{X}}$ to the restriction $\boldsymbol{V}^*|_{\boldsymbol{X}}$ of the dual vector space $\boldsymbol{V}^*$. Conversely, any restriction $\boldsymbol{V}|_{\boldsymbol{X}}$ of a vector space $\boldsymbol{V}$ to a spanning subset $\boldsymbol{X}$ is isomorphic to the function space $\boldsymbol{FS}(\boldsymbol{X},\boldsymbol{V}^*)$. Hence function spaces and chain groups provide nothing new but another point of view and terminology which prove helpful in some embedding and coordinatization problems (see 6.5). However the notion of function space has been more or less generalized by working over a subtractive algebra or a commutative integral domain rather than over a field (Crapo and Rota [1970]).

An example of function space is $\boldsymbol{FS}(\mathbb{R}^3, \mathbb{R}_2[x, y, z])$, where the index 2 means that we only consider polynomials of degree at most 2. This is the 9-DLS on $\mathbb{R}^3$ whose hyperplanes are those quadrics that are not contained in any other.

Finally let us mention that if $\boldsymbol{FS}(\boldsymbol{P}, \boldsymbol{V})$ is a function space on a finite set $\boldsymbol{P}$, then its dual pre-DLS is isomorphic to the function space $\boldsymbol{FS}(\boldsymbol{P}, \boldsymbol{V}^{\perp})$, where $\boldsymbol{V}^{\perp}$ is the orthogonal complement of $\boldsymbol{V}$ in $F^{\boldsymbol{P}}$ relative to the inner product

$$(f, g) = \sum_{p \in \boldsymbol{P}} f(p)g(p)$$

(see White [1986], pp. 11, 85; Aigner [1979]; Welsh [1976], p. 145).

2.13. *Algebraic dependence and algebraic DLS's*

We have highlighted projective spaces as the prototypical examples of DLS's. Considering algebraic dependence in field extensions instead of linear dependence in vector spaces provides another important (though often neglected) class of finitary DLS's. Let K be a field extending another field F with (possibly infinite) transcendence degree $d + 1$. A subset X of K is said to be *algebraically independent over* F if and only if for any positive integer n, each nonzero polynomial function from X^n to K with coefficients in F has no root in X^n.

MAC LANE'S THEOREM [1938]. *The algebraically independent sets over F are the independent sets of a finitary pre-d-*DLS *on K, whose closed sets are the relatively algebraically closed subfields L of K containing F (i.e. every element of K which is algebraic over L is in L).*

See, e.g., Lang [1965], Chapter 10.

The associated DLS will be called the *algebraic space* $\mathrm{Alg}(K/F)$. If K is algebraically closed we will say that $\mathrm{Alg}(K/F)$ is a *full algebraic space.* If moreover d is finite we write $\mathrm{FAlg}(d, F)$ instead of $\mathrm{FAlg}(K/F)$ (cf. Bastida [1984], Corollary 4.2.8); its i-varieties are algebraically closed fields $F(x_1, \ldots, x_{i+1})$ of transcendence degree $i + 1$. $\mathrm{FAlg}(n, F)$ is best described as the simplification of the algebraic closure of the field of rational fractions over F in the indeterminates $X_1, \ldots, X_{n+1}$, i.e. as the simplification of the pre-DLS $F(X_1, \ldots, X_{n+1})$.

Via an embedding, $\mathrm{FAlg}(n, F)$ not only contains $\mathrm{PG}(n, F)$ but also $\mathrm{PG}(n, \mathbb{Q})$ (even if F is finite), as well as an n-dimensional non-Pappian projective space if F is a nonprime field with prime characteristic. In order to embed $\mathrm{PG}(n, F)$ into $\mathrm{FAlg}(n, F)$, it suffices to take $(a_1, \ldots, a_{n+1}) \in F^{n+1}$ to $F(a_1X_1 + \cdots + a_{n+1}X_{n+1})$.

Indeed, in order to embed $\mathrm{PG}(n, \mathbb{Q})$ into $\mathrm{FAlg}(n, F)$, first note that since the coordinates of a projective point are defined up to a scalar, we only need to consider vectors in $\boldsymbol{Z}^{n+1}$. Map $(a_1, \ldots, a_{n+1})$ to $F(X_1^{a_1}, \ldots, X_{n+1}^{a_{n+1}})$ and check that linear independence of the vectors of $\mathbb{Q}^{n+1}$ corresponds to algebraic independence of the associated 'monomials' for every field F. Finally if char $F = p$, define a *p-polynomial* to be a linear combination of terms $X_i^{p^h}$ $(1 \leqslant i \leqslant n+1,\ h \geqslant 0)$. Then the restriction of FAlg(n, F) to the points corresponding to p-polynomials is a projective space over a skew-field $Q(R)$, namely the quotient skew-field of the Ore domain R consisting of the p-polynomials in $F[X]$ with the usual sum and the composition $P(Q(X))$ as product. This skew-field $Q(R)$ is commutative iff $F = \mathrm{GF}(p)$ (Lindström [1988b]).

Note also that FAlg(n, F) is highly transitive since it is Jordan-transitive, which implies in particular that its automorphism group acts transitively on ordered bases (see 7.5).

2.14. *Geometric properties of algebraic spaces*

Note that coplanar lines may be disjoint in an FAlg(n, F): e.g., in a plane $\langle x, y, z\rangle$, the lines $\langle x, y\rangle$ and $\langle z, xz + y\rangle$ are always disjoint (Lindström [1985b]). However weakened forms of some crucial properties of projective spaces hold.

THEOREM. *The following hold in* FAlg(n, F):

(1) *Bundle with three* (Ingleton and Main [1975]). *If every* 2 *out of* 3 *lines are coplanar, then the* 3 *lines are either coplanar or concurrent.*

(2) *Pseudo-modularity* (Björner and Lovász [1987]). *If* U, V, W *are three varieties such that the first three of the following numbers are equal, then all four are equal*:

$$\dim\langle U, W\rangle - \dim U,$$

$$\dim\langle V, W\rangle - \dim V,$$

$$\dim\langle U, V, W\rangle - \dim\langle U, V\rangle,$$

$$\dim(\langle U, W\rangle \cap \langle V, W\rangle) - \dim(U \cap V).$$

(3) *Dual Desargues theorem in planes* (Lindström [1985b]). *If* $x_1, x_2, x_3, y_1, y_2, y_3$ *are six coplanar points such that* $\langle x_i, x_j\rangle \cap \langle y_i, y_j\rangle$ *is a point* z_k *for all distinct* i, j, k *ranging from* 1 *to* 3 *and if the points* z_1, z_2, z_3 *are collinear, then the three lines* $\langle x_i, y_i\rangle$ *are concurrent* (Figure 2.3)

(4) *Diagonal points of planar quadrangles* (Ash and Rosenthal [1986], see Lindström [1988a] where this property is conjectured for any characteristic). *No planar quadrangle has exactly* 2 *diagonal points if* char $F = 0$ (*in other words*: *if no* 3 *of the* 4 *coplanar points* x, y, z, t *are collinear and if two of the following three intersections are nonempty, then the third one also is*: $\langle x, y\rangle \cap \langle z, t\rangle$, $\langle x, z\rangle \cap \langle y, t\rangle$, $\langle x, t\rangle \cap \langle y, z\rangle$).

(1) forces the bundle condition (see 4.9) but deals with three lines instead of four. As a consequence, the bundle condition will hold in the restriction of Alg(d, F) to any subset.

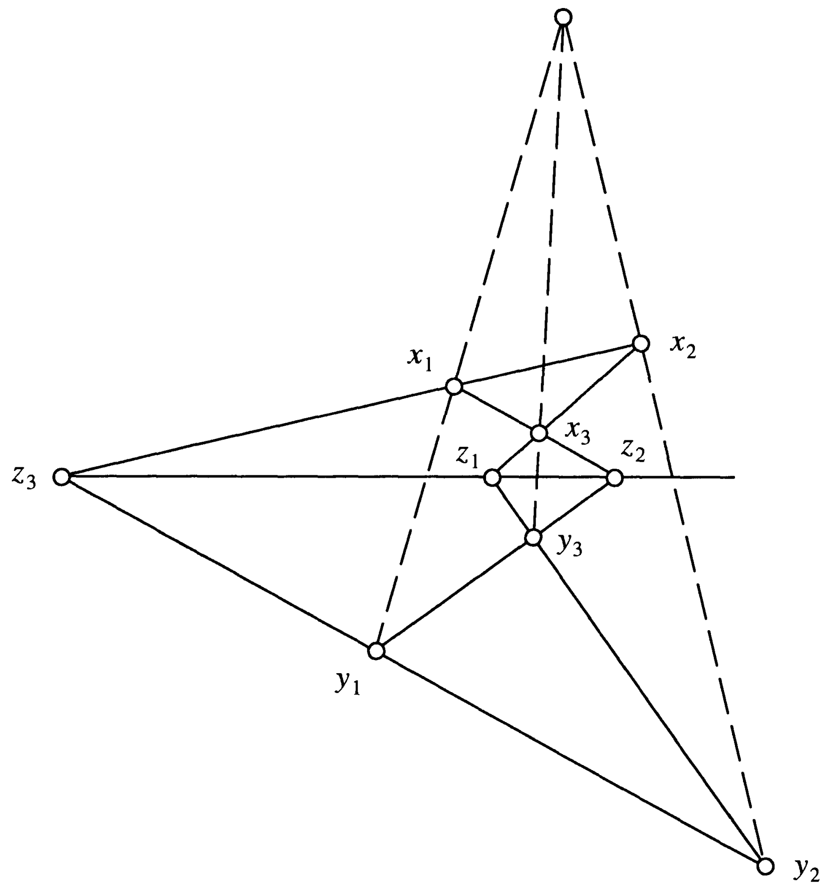


Figure 2.3.

(1) and (2) together force the dual Desargues condition (i.e. if all lines and all but one point of a Desargues configuration can be embedded in Alg(d, F), then there is a point of Alg(d, F) completing this configuration into a Desargues one, cf. Section 2.2.3). With the notation of Section 1.6, this forces the lines $\langle a', c' \rangle$, $\langle a, c \rangle$ and $\langle d, c \rangle$ to be concurrent, showing that Desargues theorem holds in the restriction of FAlg(d, F) to any subset.

However, Pappus theorem does not hold in FAlg(n, F) when $\operatorname{char} F = p \neq 0$ (for $F \neq$ GF(p), this follows from the p-polynomial embedding of Section 2.13). More can be found about algebraic DLS's in Sections 6.9–10.

3. Enumeration

3.1. *Sizes, orders, regularity*

Let S be an n-DLS and let $0 \leqslant i \leqslant n-1$. The *size* of an i-variety is the number of points incident to it, an *i-size* is the size of some i-variety. The *order* of a flag of cotype i is one less than the number of maximal flags containing this flag, and an *i-order* is the order of some flag of cotype i. One can speak of infinite i-sizes or i-orders, but in this section we assume S to be finite. S is *regular* if, for every i, it has a unique i-size

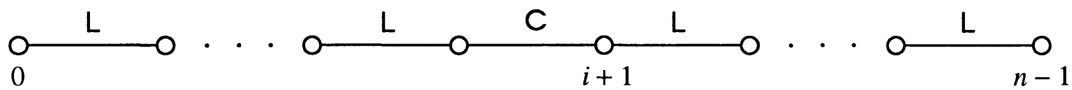
s_i, or equivalently, for every i, it has a unique i-order q_i. Note that the finite projective spaces of order q are precisely the finite DLS's all of whose i-orders are equal to q (see Theorem 5.1 of Chapter 2). The regular linear spaces are those in which all lines have the same size $k = s_1 = q_0 + 1$, so that the number of lines through a given point is the constant $r = (v-1)/(k-1) = q_1 + 1 \geqslant q_0 + 1 = k$. Since the residues of cotype $\{i, i+1\}$ of a regular DLS are regular linear spaces with $k = q_i + 1$ and $r = q_{i+1} + 1$, it follows that

(i) the sequence (q_i) is monotonic nondecreasing, while trivially
(ii) the sequence (s_i) is strictly increasing.

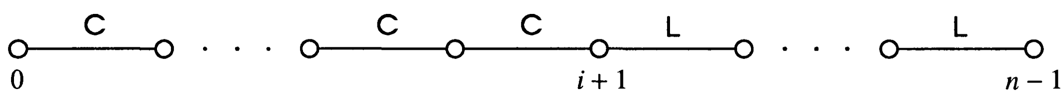
For a thorough study of n-DLS's all of whose hyperplanes have equal size, see Young [1973a].

3.2. *Parameters $t(j, i, k)$ and their existence*

By 3.1(i) if $\boldsymbol{S}$ has i-order $q_i = 1$, then for all $j \leqslant i$, $\boldsymbol{S}$ has j-order 1. Pictorially, if $\boldsymbol{S}$ belongs to the diagram



then it actually belongs to



and is called an $(i+1)$-*fold circular* n-DLS or $c^{i+1}L^{n-i-2}$-geometry. When $i + 1 = n - 1$, $\boldsymbol{S}$ is trivial and when $i+1 = n-2$, we speak of *hypercircular* DLS's. A diagram geometry is said to be *thick* if all residues of flags of corank 1 contain at least 3 elements (see Chapter 3, Section 3.1.2); for a DLS this merely means that all lines contain at least 3 points, (i.e. all lines are thick, see 2.3).

Given two incident varieties J, K with $\dim J = j < i < k = \dim K$, let $t(J, i, K)$ be the number of i-varieties incident with both J and K (see Figure 3.1).

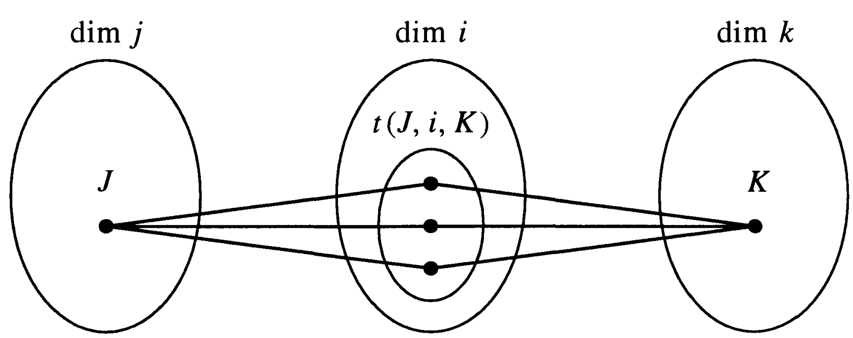


Figure 3.1.

If this number depends only on j, i, k, we write $t(j, i, k)$. This holds in any regular DLS; in particular $q_i + 1 = t(i-1, i, i+1)$, $s_i = t(-1, 0, i)$, and $t(-1, i, n) = b_i$ is the number of i-varieties in $\boldsymbol{S}$. In finite regular DLS's easy counting arguments yield relations between the parameters $t(j,i,k)$ (see, e.g., Welsh [1976], Murty, Edmonds and Young [1970], Cameron and Deza [1979]):

(i) $q_i = (s_{i+1} - s_{i-1})/(s_i - s_{i-1})$,

(ii) $t(j,i,k) = \prod_{l=j}^{i-1}(s_k - s_l)/(s_i - s_l)$.

The necessary conditions for the existence of a regular DLS with size sequence (s_i) implied by these two conditions together with 3.1 (i) and (ii) are far from sufficient. For example, the occurrence $q_0 = q_1$ together with $n \geqslant 3$ forces

$$3\text{-}\boldsymbol{S} = 3\text{-PG}(d, q_0) \quad \text{for some } d \geqslant n$$

(see 4.2), so that q_0 must be a prime power q and $s_n = (q^{d+1} - 1)/(q-1)$.

The *design* DLS's introduced in 2.4 are precisely the n-DLS's in which both parameters s_{n-1} and $t(1, n-1, n)$ are constant. We do not know of any nonregular design DLS.

Note that the constancy of some of the parameters q_i or s_i does not in general guarantee the constancy of other ones. For example, the constancy of the i-sizes for every $i < n-1$ does not imply that of the $(n-1)$-sizes, as shown by the 'missed projective spaces' (see 1.3). The constancy of s_i does not force that of s_{i-1} either (think of certain generalized projective spaces).

However it is known that in a linear space, the constancy of the line-size q_0+1 implies the constancy of the point-degree q_1+1. The corresponding statement in an n-DLS is that the constancy of the $(n-2)$-order q_{n-2} forces that q_{n-1}. More generally the constancy of the i-order $q_i = t(i-1,i,i+1) - 1$ forces the constancy of both $t(i-1,i,n)$ and $t(i,i+1,n)$. Indeed consider a pair (I^-, I) of incident varieties of respective dimensions $i-1$ and i. Since any i-variety I' distinct from I and incident with I^- generates with I an $(i+1)$-variety I^+, counting in two ways the number of pairs (I', I^+) yields

$$t(I^-, i, n) - 1 = t(I, i+1, n)\big(t(I^-, i, i+1) - 1\big) = t(I, i+1, n)q_i.$$

The constancy of $t(i-1,i,n)$ and $t(i,i+1,n)$ now follow from the connectivity of the $(i-1,i)$-truncation of the incidence graph of $\boldsymbol{S}$ (Theorem 1.5).

3.3. *DLS's with large hyperplanes*

The *regular* DLS's whose hyperplanes are 'very large' are classified below. Note however that direct sums and supersums (see 1.12 and 1.13) provide lots of *nonregular* DLS's with very large hyperplanes.

PROPOSITION (Kantor [1979] and, for a weaker statement, Young and Edmonds [1972]). *If $\boldsymbol{S}$ is a regular n-DLS on v points, then $v \geqslant 3s_{n-1}$ unless*

(i) *$\boldsymbol{S}$ is trivial (i.e. $s_{n-1} = n$), or*

(ii) *$\boldsymbol{S}$ is hypercircular (i.e. its (point, hyperplane)-truncation is an n-$(v, s_{n-1}, 1)$ design, so that $v \geqslant 2s_{n-1}$ if $\boldsymbol{S}$ is nontrivial), or*

(iii) *$\boldsymbol{S} = \text{PG}(n,2)$ or $\text{AG}(n,2)$, so that $v = 2s_{n-1}+1$ or $2s_{n-1}$.*

Kantor's (unpublished) proof implicitly uses the Scum theorem (6.4).

3.4. *Whitney numbers b_i and conjectures on (b_i)*

Let $\boldsymbol{S}$ be a finite n-DLS and let $b_i = t(-1, i, n)$ (b_i is also denoted by W_{i+1} and called *Whitney number of the second kind*, see White [1986]).

By 3.2(ii),

$$b_i = v(v-1)(v-s_1)\cdots\big(v-s_{i-1}\big)/s_i(s_i-1)\cdots\big(s_i-s_{i-1}\big)$$

if $\boldsymbol{S}$ is a regular DLS on v points. In particular, if $\boldsymbol{S}$ is trivial then b_i is the binomial coefficient $\binom{v}{i+1}$. If $\boldsymbol{S} = \mathrm{PG}(n, q)$, then b_i is the *Gaussian or q-binomial coefficient* $\left[{n+1 \atop i+1}\right]_q$. If $\boldsymbol{S} = \mathrm{AG}(n, q)$, then $b_i = q^{n-i}\left[{n \atop i}\right]_q$ (see Chapter 2, Propositions 8.4 and 8.5). If $\boldsymbol{S}$ is the partition DLS $\boldsymbol{\Pi}_{n+2}$, then b_i is the *Stirling number of the second kind* $S_{n+2,n+1-i}$. In all these cases (and more generally in all regular DLS's, see 3.5) the sequence (b_i) satisfies the following properties.

(A) *Logarithmic concavity*: $b_i^2 \geqslant b_{i-1}b_{i+1}$.
(B) *Unimodality*: $b_j \geqslant \min\{b_i, b_k\}$ whenever $i \leqslant j \leqslant k$.
(C) *Top heaviness*: $b_i \leqslant b_{n-1-i}$.
(D) *Half way increase*: $v = b_0 \leqslant b_1 \leqslant \cdots \leqslant b_{[(n-1)/2]}$.
(E) For any $i < (n-1)/2$, the incidence matrix of i varieties versus $(i+1)$-varieties has rank b_i (hence $b_i \leqslant b_{i+1}$).

Note that property (A) implies (B) and that property (C) implies (D) by considering truncations of $\boldsymbol{S}$. These four properties were checked in DLS's on at most 8 points (Blackburn, Crapo and Higgs [1973]). Property (A) also holds in partition lattices (Lieb [1968]) and in Hartmanis partition lattices (Aigner [1979], p. 258).

Property (D) is also a consequence of (E), which was conjectured to hold in any n-DLS by Kung [1994]. The reason for the restriction $i < (n-1)/2$ clearly appears when considering $\mathrm{PG}(n, q)$. (E) holds for Boolean DLS's, affine and projective spaces (Kantor [1972]) and for partition spaces (Kung [1994]). Further contribution, including unifying proofs of several of the above-mentioned results is proved by Damiani, D'Antona and Regonati [1994].

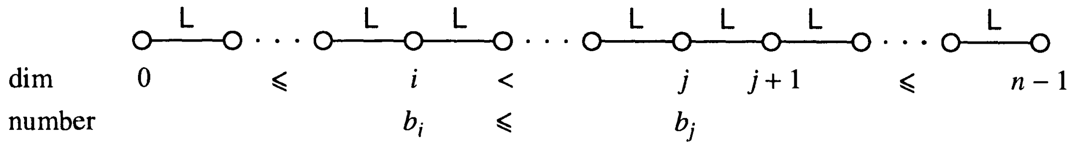
Finally, we mention another inequality (Seymour [1982]). In any n-DLS in which no line has five or more points, $b_1^2 \geqslant (3b_0 - 1)b_0b_2(2b_0 - 2)^{-1}$ with equality if and only if every plane has three points.

3.5. *Theorems on (b_i) for regular DLS's*

THEOREM (Murty, Edmonds and Young [1970], Kung [1994]). *Properties* (A) *to* (E) *hold in any regular n-DLS.*

More precisely Kung [1994] proves that given integers i, j, n such that $0 \leqslant i \leqslant j \leqslant n - i - 1$, if $\boldsymbol{S}$ is an n-DLS having a constant parameter $t(i, j, l)$ (see 3.1) for every l

such that $j + 1 \leqslant l \leqslant j + i$, then the rank of the incidence matrix of i-varieties versus j-varieties is b_i, and so $b_i \leqslant b_j$.



Note that this 'local regularity' assumption is satisfied in any n-DLS all of whose top-residues of i-varieties in $(j + i)$-varieties are isomorphic to a fixed regular $(j - 1)$-DLS (as in any locally projective or locally affine n-DLS).

The next proposition is a strengthening of (A).

PROPOSITION. *In any regular* DLS *and for any* i $(0 \leqslant i \leqslant n - 1)$

$$b_i^2 > q_i^2 b_{i-1} b_{i+1}.$$

PROOF (inspired by Kantor [1979]). Counting in two ways the number of $(i, i+1)$-flags yields

$$b_i t(i, i + 1, n) = b_{i+1} t(-1, i, i + 1).$$

Similarly

$$b_{i-1} t(i - 1, i, n) = b_i t(-1, i - 1, i).$$

And so

$$b_i^2 / b_{i-1} b_{i+1} = \big(t(-1, i, i + 1)/t(-1, i - 1, i)\big)\big(t(i - 1, i, n)/t(i, i + 1, n)\big).$$

We shall prove that both factors on the right-hand side are greater than q_i. Let (X, Y, Z) be an $(i - 1, i, i + 1)$-flag. The number of i-varieties in Z intersecting Y in a given $(i - 1)$-variety is q_i and any two i-varieties intersecting Y in distinct $(i - 1)$-varieties are distinct, so that $t(-1, i, i + 1) \geqslant q_i t(-1, i - 1, i) + 1$, where the 1 stands for X. On the other hand, the i- and $(i + 1)$-varieties through X form a linear space (the 2-truncation of the top residue of X) and so

$$t(i - 1, i, n) = t(i, i + 1, n) q_i + 1 \qquad \square$$

We now turn to properties weaker than (A) and (D), which are proved true for all DLS's.

3.6. *Bounds for* b_i

PROPOSITION. *In any* n-DLS $\boldsymbol{S}$,

$$v \leqslant b_i \leqslant \binom{v}{i+1} \quad \textit{whenever } 1 \leqslant i \leqslant n - 1.$$

Equality holds on the left iff $\boldsymbol{S}$ *is an* $(i + 1)$*-dimensional generalized projective space. Equality holds on the right iff the* $(i + 1)$*-truncation of* $\boldsymbol{S}$ *is trivial.*

The right-hand inequality of this proposition is trivial in all respects. Many different proofs of the left-hand inequality have been published (e.g., Motzkin [1951], Greene [1970, 1975], Heron [1973], Woodall [1976], Kung [1979], Kołodziejczyk and Romanowicz [1988]). An elegant and quite elementary one is based on an idea of J.H. Conway (see Basterfield and Kelly [1968]) from which it is easy to derive a complete proof of this proposition (consider first the $(i+1)$-truncation of $\boldsymbol{S}$).

This proposition yields a lower bound for b_i, but it can be reached only if $i = n-1$. Dowling and Wilson found a sharper bound which, in some spaces, is attained for every i between 0 and n.

3.7. PROPOSITION (Dowling and Wilson [1974]). *In an n-DLS $\boldsymbol{S}$ on v points,*

$$\binom{n-1}{i}(v-n-1)+\binom{n+1}{i+1} \leqslant b_i \quad \textit{for any } i \leqslant n-1.$$

Equality holds for $i = n-1$ iff $\boldsymbol{S}$ is an n-dimensional generalized projective space. Equality holds for some i with $1 \leqslant i \leqslant n-2$ iff equality holds for every i iff $\boldsymbol{S}$ is the direct sum of a generalized projective plane and a completely trivial $(n-2)$-dimensional space (see 2.2).

This is not surprising since, given n, the smallest possible b_i is that of the trivial DLS on $n+1$ points. Here, in order to keep the b_i's as low as possible, the $(v-n-1)$ points in excess are added to a unique plane in such a way as to get a generalized projective plane, the only kind of DLS in which the number of i-varieties never exceeds the number of points. Note that given any n and $v \geqslant n+1$, this construction is always possible.

The case $i = 1$ has been further investigated by Metsch [1994] who proves that if $n \geqslant 3$ and $v > 4$, if q is the unique positive real number such that $v = q^3+q^2+q+1$ and if no plane is so big as to have more than q^3+q^2+1 points, then $(q^2+q+1)(q^2+1) \leqslant b_1$, with equality iff $\boldsymbol{S} = \mathrm{PG}(3,q)$.

Proposition 3.7 has been applied to arrangements of hyperplanes in Zaslavski [1981, 1983]. Another related work is Björner [1980].

3.8. Dowling and Wilson also strengthened Proposition 3.6 towards property (C) as follows:

PROPOSITION (Dowling and Wilson [1975]). *In any n-DLS $\boldsymbol{S}$, for any l with $0 \leqslant l \leqslant (n/2)-1$,*

$$b_0+b_1+\cdots+b_l \leqslant b_{n-l-1}+\cdots+b_{n-2}+b_{n-1}.$$

Moreover equality holds for some l if and only if $b_i = b_{n-i-1}$ for every i if and only if $\boldsymbol{S}$ is an n-dimensional generalized projective space.

That $b_i = b_{n-i-1}$ holds in any n-dimensional generalized projective space was proved in Dilworth [1954].

Write

$$\mathrm{Bot}_l(\boldsymbol{S}) = \bigcup_{i=0}^{l} S^i \quad \text{and} \quad \mathrm{Top}_l(\boldsymbol{S}) = \bigcup_{i=n-1-l}^{n-1} S^i.$$

Let V be any j-variety with $j \leqslant n-l-1$ and remember that $\boldsymbol{S}_V$ denotes the top residue of V. Dowling and Wilson [1975] sharpened the above inequality into

$$|\mathrm{Bot}_l(\boldsymbol{S})| \leqslant |\mathrm{Top}_l(\boldsymbol{S})| - |\mathrm{Top}_l(\boldsymbol{S}_V)| + |\mathrm{Bot}_l(\boldsymbol{S}_V)| \leqslant |\mathrm{Top}_l(\boldsymbol{S})|.$$

3.9. In the same paper Dowling and Wilson prove also the following proposition (compare with (E), in Section 3.4).

PROPOSITION. *Over a field of characteristic zero, the incidence matrix of elements of* $\mathrm{Bot}_l(\boldsymbol{S})$ *versus elements of* $\mathrm{Top}_l(\boldsymbol{S})$ *has rank* $|\mathrm{Bot}_l(\boldsymbol{S})|$.

This result has several consequences, including the inequality in Proposition 3.8, a similar inequality for orbit-numbers in 7.4, and also Proposition 3.10(a).

3.10. The inequality $b_0 \leqslant b_{n-1}$ raises the question whether there is an injection from the point-set into the hyperplane-set of a DLS mapping every point to a hyperplane containing it.

PROPOSITION (Dowling and Wilson [1975], Aigner [1987], Kung [1986b]). *Let* $\boldsymbol{S}$ *be an* n-DLS *and* $0 \leqslant l \leqslant n-1$. *Then there exist injections* f, g, h *as follows.*

(a) f: $\mathrm{Bot}_l(\boldsymbol{S}) \to \mathrm{Top}_l(\boldsymbol{S})$ *such that* $V \subseteq f(V)$ *for every* $V \in \mathrm{Bot}_l(\boldsymbol{S})$.

(b) g: $\mathrm{Bot}_l(\boldsymbol{S}) \to \mathrm{Top}_l(\boldsymbol{S})$ *such that* $V \nsubseteq g(V)$ *for every* $V \in \mathrm{Bot}_l(\boldsymbol{S})$.

(c) h: $\mathrm{Bot}_l(\boldsymbol{S}) \to \mathrm{Top}_l(\boldsymbol{S})$ *such that* V *and* $h(V)$ *generate* $\boldsymbol{S}$ *for every* $V \in \mathrm{Bot}_l(\boldsymbol{S})$.

Note also that any n-DLS on v points contains a family of v pairwise disjoint maximal flags (Mason [1973]).

3.11. *On the number* m_i *of modular* i*-varieties*

PROPOSITION (Kung [1987b]). *In any* n-DLS $\boldsymbol{S}$,

$$b_0 + b_1 + \cdots + b_l \geqslant m_{n-l-1} + \cdots + m_{n-2} + m_{n-1} \quad \textit{for } 0 \leqslant l \leqslant \frac{n}{2} - 1.$$

3.12. *On the number* i_l *of independent* l*-sets*

In a regular n-DLS, $i_l = s_n(s_n - 1)(s_n - s_1)(s_n - s_2) \cdots (s_n - s_{l-2})/l!$

Questions similar to those asked about the sequence (b_l) have been raised for the sequence (i_l), see Mason [1972], Dowling [1980], Mahoney [1985], Stanley [1981], Welsh [1976], p. 296, and Zhao [1985].

3.13. *Comments on the methods of proof*

In the proofs of Propositions 3.7 to 3.10, the basic tool is the Möbius function μ defined on the set of ordered pairs of (possibly improper) varieties of $\boldsymbol{S}$ by

$$\mu(V,W)=\begin{cases} -\sum\limits_{V\subseteq U\subset W}\mu(V,U) & \text{if } V\subset W,\\ 1 & \text{if } V=W,\\ 0 & \text{if } V\not\subseteq W,\end{cases}$$

whose success is due to the following *Möbius inversion principle.*

Let f, g be functions from $\boldsymbol{S}$ into a field of characteristic 0. *Then*

$$g(V)=\sum_{W:\ W\subseteq V} f(W) \Leftrightarrow f(V)=\sum_{W:\ W\subseteq V}\mu(W,V)g(W),$$

and dually (*i.e. reversing the inclusions and hence writing $\mu(V,W)$ in place of $\mu(W,V)$*).

Another crucial property of the Möbius function in DLS's is the following: if $V\subseteq W$ then $\mu(V,W)\neq 0$ and has sign $(-1)^{\dim W-\dim V}$ (Rota [1964]).

For developments on the Möbius function, see Rota [1964], Aigner [1979], White [1987] (Chapters 7 and 8).

The particular case $l=0$ in Proposition 3.9 has also been proved elegantly by means of a finite Radon transform (Kung [1979, 1985]). Later on, Kung extended his Radon transform approach thanks to a systematic use of the Möbius function. He introduced the technical notion of *concordant sets* (Kung [1987b]), which are pairs (J,M) of subsets of a finite lattice for which he can prove that the incidence matrix $I(J|M)$ has rank $|J|$. In that way he found new proofs of Propositions 3.7 to 3.10 together with several similar results, often extended to more general lattices than the geometric ones. He also used this method to prove his result on locally regular DLS's (see 3.5). A good review of these techniques and their applications can be found in Kung [1986a], see also Grinberg [1990]. Cameron [1984] suggests to generalize this Radon transform approach in order to compare the numbers (if finite) of orbits on i-varieties of an automorphism group of an infinite DLS (see 7.4).

3.14. *Enumeration of DLS's*

For early work in the direction of this section, see Hanani [1954]. The number $N(v)$ of nonisomorphic DLS's on v points cannot exceed 2^{2^v}, since a DLS on v points is defined by specifying a set of subsets of a v-set, namely the hyperplane-set. $N(v)$ grows extremely fast indeed, to wit:

$$v-(3/2)\log_2 v+O(\log\log v)\leqslant \log_2\log_2 N(v)\leqslant v-\log_2 v+O(\log\log v).$$

The upper bound is due to Piff [1973] while the lower bound is a consequence of the following result.

THEOREM (Knuth [1974]). *The number of nonisomorphic hypercircular n-DLS's on v points with $n = [v/2] - 1$ and hyperplane-sizes n and $n + 1$ is at least*

$$2^{\binom{v}{n+1}/2v}/v!.$$

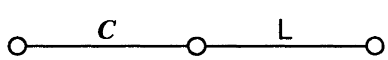
The proof consists in showing the existence of at least $\binom{v}{n+1}/2v$ binary words of length v and weight n, which form a single error correcting code. Note by contrast that no finite nontrivial hypercircular n-DLS with constant hyperplane size is known for $n \geqslant 6$.

4. Planar spaces

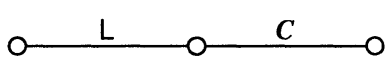
4.1. *Introduction*

Planar spaces (PS's) are the 3-DLS's. Their role is crucial because the (bottom) 3-truncation of every n-DLS with $n \geqslant 3$ is a planar space and also because their structure, which is richer and more restricted than that of linear spaces, still allows enough freedom to include interesting and diverse spaces, such as Fischer spaces and Benz planes. Curious things can happen in dimension 3 that are known or conjectured to be imposible in higher dimensions, e.g., DLS's all of whose point-residues are projective (resp., affine) spaces but cannot be embedded in a projective space, and DLS's all of whose point-residues are truncations of finite dimensional projective (resp., affine) spaces but are not truncations of a DLS whose point-residues are projective (resp., affine) spaces.

In all what follows $\boldsymbol{S}$ denotes a planar space and we identify the lines and planes both with their point-sets and with their bottom residues, so that each plane appears as a linear space. The main theme of this section is the following: given a class $\boldsymbol{C}$ of linear spaces, determine (or embed in known spaces) the planar spaces belonging to



or to



The most famous result of this type is undoubtedly that of Veblen and Young, generalized later to the following theorem.

4.2. *PS's whose planes are projective or affine planes*

THEOREM (Veblen and Young [1910], Buekenhout [1969b], Teirlinck [1975]).

(i) *If each plane of $\boldsymbol{S}$ is a projective or affine plane, then either $\boldsymbol{S}$ is the* 3-*truncation of a projective or affine space of dimension* $\geqslant 3$, *or all planes of $\boldsymbol{S}$ are affine planes of the same order $q = 3$ or 2 and 2-$\boldsymbol{S}$ is a Hall triple system* (see 4.12) *if $q = 3$, while $\boldsymbol{S}$ is merely a Steiner quadruple system if $q = 2$.*

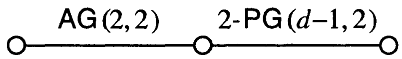
(ii) *If every plane of $\boldsymbol{S}$ is a generalized projective plane, then $\boldsymbol{S}$ is the* 3-*truncation of a generalized projective space.*

Compare Chapter 2, Theorems 2.3, 2.5 and 2.8 where a finiteness assumption is made on the dimension.

See also Hall [1974] and Sörensen [1988b]. For a thorough elementary treatment of these and other matters in the present section and Section 5, see Beutelspacher [1982/83].

Theorem 4.2 could as well be presented as a result on linear spaces, because the hypothesis forces all planes to be minimal linear subspaces of $\boldsymbol{S}$, up to the case where all lines have size 2. By contrast to this result, the planar spaces whose point-residues are projective or affine planes are not known (see 4.4.).

Note that the diagram



does not characterize 3-PG(d, q) as shown by Teirlinck's construction in 4.4. The investigation of Hall Triple Systems should provide the answer to the similar problem over GF(3). For an early characterization of affine spaces in terms of planar spaces, see Sasaki [1952].

4.3. *PS's whose planes are 2-truncations of affine and projective spaces*

It follows from 4.2 that any planar space whose planes are linear truncations of projective or affine spaces (of order > 3 in the affine case) is an erection of the linear truncation of some projective or affine space $\boldsymbol{T}$. For any family $\boldsymbol{F}$ of proper subspaces of $\boldsymbol{T}$ of dimension at least 2 such that every plane of $\boldsymbol{T}$ is contained in precisely one member of $\boldsymbol{F}$, $\boldsymbol{S} = (T^0, T^1, \boldsymbol{F})$, with the incidence induced by that of $\boldsymbol{T}$ is such a planar space. Examples exist in profusion and can even have an automorphism group 2-transitive on the points (see 7.7). However the existence of such a family $\boldsymbol{F}$ all of whose members have the same dimension d in $\boldsymbol{T}$ is still unsettled if $\boldsymbol{T}$ is finite (see Kantor [1974b], p. 67, and the final section of Thomas [1987]), whilst it is known to exist if $\boldsymbol{T}$ is at least $(2d-1)$-dimensional over a countable field (Ceccherini and Tallini [1986]).

4.4. *Sharply, strongly, weakly locally projective PS's*

We now turn to the intensively investigated but still mysterious *locally projective* (LP) planar spaces, i.e. those all of whose point-residues are linear truncations of projective spaces (*locally affine* (LA) planar spaces are similarly defined and will also be considered in 5.28). For convenience, we distinguish three types of LP planar spaces $\boldsymbol{S}$.

$\boldsymbol{S}$ is *sharply locally projective* (shLP) provided all its point-residues are projective planes,

$\boldsymbol{S}$ is *strongly locally projective* (SLP) provided either $\boldsymbol{S}$ is shLP or $\boldsymbol{S}$ can be erected into a 4-DLS all of whose point-residues are planar truncations of projective spaces,

$\boldsymbol{S}$ is *weakly locally projective* (WLP) provided all its point-residues are linear truncations of projective spaces.

Each of these conditions is stronger than the next one. Teirlinck [1986] gives the following example of a WLP but not SLP planar space of order 2 using a well-known doubling construction for Steiner quadruple systems. Let $\boldsymbol{T} = (T^0, T^1, T^2)$ be the planar truncation of AG(d, 2) with $d \geqslant 3$. Define $S^0 = T^0 \times \{0,1\}$, S^1 to be the set of all pairs in S^0, and S^2 to be the set of all 4-subsets of S^0 of the form

$$\{(a,i),(b,i),(c,i),(d,j)\} \quad \text{where } \{a,b,c,d\} \in T^2 \text{ and } \{i,j\} = \{0,1\}$$

or of the form

$$\{(a,0),(a,1),(b,0),(b,1)\} \quad \text{where } a,b \in T^0,\ a \neq b.$$

Then the planar space $\boldsymbol{S} = (S^0, S^1, S^2)$ (with the natural incidence) has all its point-residues isomorphic to 2-PG(d, 2) but is not SLP (the parallelism of lines is not transitive, see 4.5) and does not satisfy the bundle condition (see 4.9). Examples of WLA but not SLA planar spaces of any prime-power order q and preserved by PSL$(2, q^d)$ are given in 5.28.

OPEN QUESTIONS. Are there other examples of WLP but not SLP planar spaces, or WLA but not SLA? Is there any DLS that is WLP but not SLP and whose lines have at least 3 points?

4.5. *Characterizations of strongly local projectiveness*

A *planar subspace* of $\boldsymbol{S}$ is a 3-closed subset X of S^0, which means that the line or plane generated by any nonsingleton coplanar subset of X is entirely contained in X.

PROPOSITION (Wyler [1953], Teirlinck [1986]). *The following statements are equivalent:*

(a) $\boldsymbol{S}$ *is* SLP;

(b) $\boldsymbol{S}$ *is* WLP *and for any point x of $\boldsymbol{S}$, the linear subspaces of $\boldsymbol{S}_x$ are precisely those induced in $\boldsymbol{S}_x$ by the planar subspaces of $\boldsymbol{S}$ containing x;*

(c) *no plane is the union of two intersecting lines, and for any two disjoint coplanar lines L, L' and any point x outside the plane $\langle L, L'\rangle$ the two planes $\langle x, L\rangle$ and $\langle x, L'\rangle$ intersect in a line.*

In particular, if $\boldsymbol{S}$ is SLP and if, for every point x, the projective space on $\boldsymbol{S}_x$ has dimension $\leqslant \aleph_0$, then $\boldsymbol{S}$ can be erected to a DLS all of whose point-residues are projective spaces (see also Buekenhout [1971b] for a result similar to the first equivalence, about locally affine circular spaces). Note also that if all planes are affine planes, then the last condition is equivalent to the 'transitivity of parallelism'.

4.6. *Strongly locally generalized projective PS's*

The definitions in 4.4 can be extended by allowing degenerate projective spaces; we denote it by adding the word 'generalized' or the letter 'G'. Proposition 4.5 still holds in this context if we allow planes which are the union of two lines. The 'incidence geometries' of Wyler [1953] are merely the SLGP planar spaces. He proved that any SLGP but not SLP planar space is the planar truncation of a degenerate projective space (see also Batten [1987b]), so that we may focus on SLP planar spaces.

4.7. *Embeddability of some strongly locally projective PS's*

This is the subject of Sections 4.7 to 4.11.

SLP planar spaces are not necessarily 3-truncations of projective spaces. Indeed given $\mathrm{PG}(d, F)$ with $d \geqslant 3$ and given any subset E of its point-set such that any line intersects E at least twice, the restriction to E of the planar truncation of $\mathrm{PG}(d, F)$ is an SLP planar space. Alternatively, E can be chosen to be the complement of the union of a family of hyperplanes having a common coline. Assuming finiteness and regularity as in the next section excludes such situations, but in general the best we could hope is embeddability theorems. As in Section 6, we shall say that a planar (resp., linear) space is *embeddable* in $\mathrm{PG}(d, F)$ if and only if it is isomorphic to the planar (resp., linear) truncation of a restriction of $\mathrm{PG}(d, F)$. Since planar spaces are not determined by their linear truncation, the embeddability of their linear truncation does not force their embeddability. A planar space is called *coordinatizable* provided it is embeddable in some $\mathrm{PG}(3, F)$ where, unless otherwise mentioned, F is a field (see also Section 6.2).

THEOREM (Wyler [1953]; see also Wille [1971] and Kantor [1974a]). *Any* SLP *but not* shLP *planar space* $\boldsymbol{S}$ *is embeddable in a projective space* (*whose point-residues have linear truncations isomorphic to the point-residues of* $\boldsymbol{S}_x$).

Note that in Kantor [1974a] the projective space erected on $\boldsymbol{S}_x$ must have finite dimension. Actually this is also a theorem on n-DLS's with $n > 3$. By contrast, shLP planar spaces are not necessarily embeddable, as shown by the famous Mathieu–Witt space $S(3, 6, 22)$, already mentionned in Wyler [1953], and which remains the only known finite noncoordinatizable shLP planar space (see, e.g., Kantor [1974a] for a proof of its nonembeddability). Kantor [1974a] also provides a (free) construction of infinite circular shLP planar spaces which are nonembeddable since their point-residues are not Desarguesian. Similarly Ewald [1960] provided a construction of an infinite circular shLA (inversive plane) which is not embeddable since its point-residues are not isomorphic to each other.

4.8. *Regular locally projective or affine planar spaces*

Before tackling the embeddability problem for shLP and shLA planar spaces, let us point out that a constant size assumption severely restricts the range of parameters of such a space.

THEOREM (Doyen and Hubaut [1971], Cameron [1974a], Kantor [1974a]). *Let* $\boldsymbol{S}$ *be finite, with constant line-size* k.

(i) *If* $\boldsymbol{S}$ *is* SLP, *then either* $\boldsymbol{S} = 3\text{-PG}(n, k-1)$, *or* $3\text{-AG}(n, k)$ *with* $n \geqslant 3$, *or else* $\boldsymbol{S}$ *is* shLP *and the common order of the point-residues is* k^2 *or* $k^3 + k$.

(ii) *If* $\boldsymbol{S}$ *is* SLA, *then it is* shLA *and* $k = 2$, *so that* $\boldsymbol{S}$ *is an inversive plane* (see 5.2).

For variations on this result see Beutelspacher [1985].

It is equivalent to assume a constant plane-size h, since all point-residues are easily seen to have the same order in a locally projective (or affine) planar space, so that the number $(v - |L|)/(h - |L|)$ of planes containing a line L must be independent of L, forcing the constancy of the line-size.

As noticed by Percsy, Kantor's [1974a] proof of the noncoordinatizability of the Mathieu–Witt design 3-(22,6,1) extends to any other hypothetical finite nonclassical regular shLP planar space.

4.9. *Bundle condition and bundle theorem for shLP planar spaces*

The following *bundle condition* (**B**) holds in any restriction of a projective space and so is necessary for coordinatizability:

(**B**) if L_1, L_2, L_3, L_4 are lines, no three in a common plane, and if five of the six pairs $\{L_i, L_j\}$ $(1 \leqslant i \neq j \leqslant 4)$ are coplanar, then the sixth pair is also coplanar.

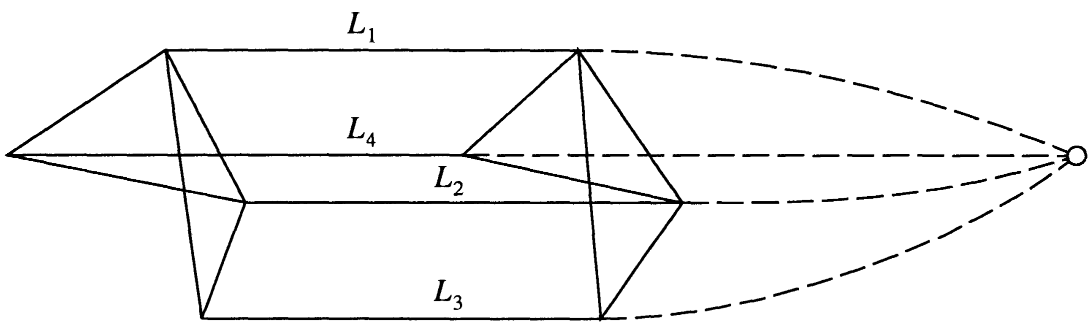


Figure 4.1.

Indeed, suppose that $\boldsymbol{S}$ is some restriction of a projective space $\boldsymbol{S}'$ and that the lines L_i of $\boldsymbol{S}$ are as assumed in (**B**), the pair $\{L_3, L_4\}$ being possibly noncoplanar. Denote by L_i' the line of $\boldsymbol{S}'$ containing L_i. By hypothesis the three lines L_1', L_2', L_3' are pairwise coplanar, hence intersecting, but not all three coplanar, so that they are concurrent. Similarly L_1', L_2' and L_4' are concurrent, so that L_3' and L_4' intersect, forcing L_3 and L_4 to be coplanar (see Figure 4.1).

(**B**) does not force coordinatizability of planar spaces in general (Bachem and Wanka [1989]) but the following holds.

BUNDLE THEOREM (Kahn [1980b]). *A sharply locally projective planar space* $\boldsymbol{S}$ *is coordinatizable over some skewfield iff* $\boldsymbol{S}$ *satisfies the bundle condition* (**B**).

Actually Kahn's result is more general since he does not assume $\boldsymbol{S}$ to be a planar space but only a semimodular 3-dimensional lattice in which not too many lines are on only one point. More precisely, he assumes that one can associate with any point x a set $\boldsymbol{T}_x$ of planes with $|\boldsymbol{T}_x| < f(x)$ such that any line whose only point is x lies on some plane of $\boldsymbol{T}_x$, where $f(x)$ is defined from the order n of the projective plane $\boldsymbol{S}_x$ as follows: $f(x) = \infty$ if n is infinite, $f(x) = 1 + \max\{1, [\sqrt{n-1}/2]\}$ if n is finite. See in Section 5.9 the reason for the terminology 'bundle condition' and for a specialization of this major result to Benz planes.

4.10. *Corollaries of the bundle theorem*

We now state a few sufficient conditions for embeddability, whose proofs partly rely on Kahn's result. The assumptions will always force $\boldsymbol{S}$ to be a rather large chunk of some PG$(3, F)$, e.g., requiring the line-orders to be relatively small with respect to the line-sizes.

(1) If $\boldsymbol{S}$ is a finite shLP planar space with line-order n and with at least n^3 points (resp., with at least $n^3 - 3n^2 + 9n + 12$ points and without disjoint planes), then S is coordinatizable over GF(n) (Metsch [1988], Kern [1988], resp., Metsch [1989]).

(2) Given a nonincident point-line pair (x, L) in a linear space, denote by $\delta(x, L)$ the (possibly infinite) number of lines through x disjoint from L. Let $\boldsymbol{C}_2$ be the class of all linear spaces in which for any line L,

$$\max\left\{\delta(x, L), (\delta(x, L) - 1)^2;\ x \notin L\right\} < |L|.$$

If $\boldsymbol{S}$ is a shLP planar space all of whose planes belong to $\boldsymbol{C}_2$, then $\boldsymbol{S}$ is embeddable in some 3-dimensional generalized projective space (Frank [1985]).

(3) Let $\boldsymbol{C}_3$ be the class of all finite linear spaces $\boldsymbol{R}$ with a constant line-order $r(\boldsymbol{R}) - 1$, all point-orders being at least $2r(\boldsymbol{R})/3$. If all planes of $\boldsymbol{S}$ belong to $\boldsymbol{C}_3$, then $\boldsymbol{S}$ is SLP and embeddable in some projective space (Teirlinck [1986]).

(4) Let $\boldsymbol{C}_4$ be the class of all linear spaces $\boldsymbol{R}$ for which there is a projective plane $\boldsymbol{T}$ and a subset E of its point-set such that $\boldsymbol{R} = \boldsymbol{T}|_E$ and, for any line L of $\boldsymbol{T}$ intersecting E,

$$3|L \setminus E| + 2 < |L|.$$

If all planes of $\boldsymbol{S}$ belong to $\boldsymbol{C}_4$, then $\boldsymbol{S}$ is SLP and embeddable in some projective space (Frank [1988]).

Note that all finite linear spaces belonging to $\boldsymbol{C}_4$ also belong to $\boldsymbol{C}_3$. Moreover the characterization 4.2(i) of affine spaces of order $\geqslant 4$ can be deduced from this, provided the order is not 4. In the finite case, both (3) and (4) generalize 4.2(i) by allowing planes having a constant line-order, which is rather small with respect to all line-sizes. The next result also generalizes 4.2(i), but allows planes which, while being nearly projective, can nevertheless have a thin line however large their line-orders may be.

A deep analysis of this theme, in the context of *bundle spaces*, has recently been developed by Kreuzer [1989, 1992].

4.11. *PS's whose planes are affino-projective*

An *affino-projective space* is the restriction of some projective space $\boldsymbol{T}$ to any subset whose complement lies in a hyperplane of $\boldsymbol{T}$. Its *dimension* and *order* are those of the projective space $\boldsymbol{T}$. An *affino-projective plane* is a 2-dimensional affino-projective space.

THEOREM (Teirlinck [1976]). *Let $\boldsymbol{S}$ be a planar space all of whose planes are affino-projective planes.*

(i) *If the maximal line-order is greater than* 3, *then the linear truncation of $\boldsymbol{S}$ is universally embeddable in a Desarguesian projective space.*

(ii) *If moreover the linear (or planar) dimension of* $\boldsymbol{S}$ *is finite, then the linear truncation of* $\boldsymbol{S}$ *is that of some affino-projective space.*

(iii) *If* $\boldsymbol{S}$ *is finite and if all its planes have the same order greater than* 3, *then* $\boldsymbol{S}$ *is the planar truncation of some affino-projective space.*

In (i) 2-$\boldsymbol{S}$ *is universally embeddable in* $\boldsymbol{T}$ means that there is an embedding ε: 2-$\boldsymbol{S} \to \boldsymbol{T}$ such that for any other embedding ε' of 2-$\boldsymbol{S}$ in a Desarguesian projective space $\boldsymbol{T}'$, there is a unique embedding η of $\boldsymbol{T}$ in $\boldsymbol{T}'$ such that ε' and $\varepsilon\eta$ coincide on $\boldsymbol{S}$.

Results (i) and (ii) can be presented in terms of linear spaces and this is their right framework, since the embeddability conclusion holds only for the linear (and not the planar) structure of $\boldsymbol{S}$, as shown by the following example due to Teirlinck [1986].

Let $\boldsymbol{T} = \mathrm{PG}(d, q^n)$ with $d, q, n \geqslant 3$, and let π be a plane of $\boldsymbol{T}$. Thas [1970] showed that the restriction of $\pi = \mathrm{PG}(2, q^n)$ to the complement of some subset D is isomorphic to the 2-truncation of $\mathrm{AG}(n, q)$. Hence the restriction of $\boldsymbol{T}$ to the complement of D is a d-DLS $\boldsymbol{T}'$ having more linear subspaces than varieties. However if we provide the 2-truncation of $\boldsymbol{T}'$ with planes defined to be all linear closures of triangles, we get a planar space $\boldsymbol{S}$ all of whose planes are affino-projective planes (having different orders). Such a planar space does not satisfy the bundle condition, and so is not embeddable in any projective space.

Kreuzer [1994] uses Teirlinck's theorem in order to investigate more closely the planar spaces all of whose planes are *semi-affine planes* (i.e. for each (point, line) pair (x, L) of π, there is at most one line of π through x and disjoint from L).

4.12. *Hall triple systems and Fischer spaces*

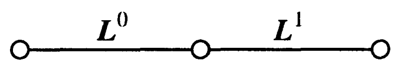
The planar spaces all of whose planes are affine planes of order 3 are uniquely determined by their linear truncations, named *affine triple systems* (see Young [1973b,c]). They are called *Hall triple systems* if they are finite and are not truncations of affine spaces of order 3 (see 4.2). They also appear as special *Fischer spaces*, which are the planar spaces all of whose planes are *Fischer planes*, namely degenerate projective planes on 3 or 4 points, affine planes of order 3, or Pasch configurations $\boldsymbol{\Pi}_4$ (see 2.6). For a thorough survey on Hall triple systems, see Deza and Sabidussi [1990]. See also Sörensen [1988a].

4.13. *PS's with isomorphic planes*

In view of 4.2, it is natural to investigate the planar spaces (called $\boldsymbol{L}^0$*-planar spaces*) all of whose planes are isomorphic to a given linear space $\boldsymbol{L}^0$. Examples are the 3-truncations of affine and projective spaces, Hall triple systems, circular spaces with constant plane size. If infinite planes are allowed, many other examples arise, such as Klein's model of hyperbolic spaces (see Chapter 15, Section 4.1.3) and the planar spaces mentioned in 4.3, all of whose planes are isomorphic to 2-PG(d, F) or 2-AG(d, F), where d is finite and F is a countable field.

If $\boldsymbol{L}^0$ is finite and has thin lines only, or if $\boldsymbol{L}^0$ is a finite Desarguesian projective or affine plane, then there are infinitely many finite $\boldsymbol{L}^0$-planar spaces. However in general, given a finite linear space $\boldsymbol{L}^0$, the existence of a finite $\boldsymbol{L}^0$-planar space is quite exceptional. For example, there is no planar space all of whose planes are isomorphic to a non-Desarguesian affine or projective plane (see Section 4.2 and Chapter 2, Sections 1.3, 2.3). Finite $\boldsymbol{L}^0$-planar spaces were investigated in Brouwer [1978], Delandtsheer [1982], Leonard [1982]. In such a space, the number of k-lines through a point is independent of the point. If $\boldsymbol{L}^0$ has nonconstant line-size, then the number of points of a possible finite $\boldsymbol{L}^0$-planar space is uniquely determined by some parameters of $\boldsymbol{L}^0$, in such a way that the existence of a finite $\boldsymbol{L}^0$-planar space with more than two line-sizes seems to be very unlikely. The only known finite $\boldsymbol{L}^0$-planar spaces with two line-sizes are the 3-dimensional degenerate projective spaces which are the direct sum of two k-lines ($k \geqslant 3$; their planes are degenerate projective planes with one k-line). If $\boldsymbol{L}^0$ is not a degenerate projective plane, if $\boldsymbol{L}^0$ has at most 60 points and precisely two line-sizes, then there is no finite $\boldsymbol{L}^0$-planar space except possibly if the two line-sizes are 2 and 3, and the parameters of $\boldsymbol{L}^0$ lie in a list of only 17 entries. The smallest linear space $\boldsymbol{L}^0$ for which the existence of a finite $\boldsymbol{L}^0$-planar space is unsettled has 7 points, 4 lines of size 3 (three of which are concurrent) and 9 lines of size 2, while the $\boldsymbol{L}^0$-planar space would have 47 points and would be rigid (i.e. have no nontrivial automorphism, Delandtsheer [1983b]).

Any finite planar space whose automorphism group is transitive on planes (and so on points, by 7.4) is an $\boldsymbol{L}^0$-planar space all of whose point-residues are isomorphic to some linear space $\boldsymbol{L}^1$. Not much is known about those spaces nor about the class



where $\boldsymbol{L}^0$ and $\boldsymbol{L}^1$ are two given linear spaces. However, it is not difficult to prove that the only finite planar spaces all of whose 2-dimensional residues are isomorphic are the 3-dimensional generalized projective spaces with isomorphic planes (Delandtsheer [1983b]), i.e. the Desarguesian projective space $\mathrm{PG}(3, q)$ and the full Matchstick spaces $\boldsymbol{M}_4(q)$ (cf. 6.7).

4.14. *Some other regularity conditions*

See also Section 2.5.

The 3-dimensional generalized projective spaces are exactly the planar spaces in which for every pair (π, π') of planes intersecting in a line, any line intersecting π intersects π'. In Delandtsheer [1984c, 1985], the finite planar spaces in which for every pair (π, π') of planes intersecting in precisely one point (resp., of disjoint planes) any line intersecting π intersects π', are proved (by further assuming the existence of at least two such planes) to be essentially the planar spaces obtained by deleting certain points from a 3-dimensional projective space, the two most interesting exceptions being closely related to the Fischer spaces on 18 and 36 points constructed from Hermitian quadrics in $\mathrm{PG}(3, 4)$ (see Section 4.12).

Dehon [1979] proves that if $\boldsymbol{S}$ is a finite planar space with constant line-size 3, for which there is a constant number α such that, for any plane π and any line L disjoint from π, there are exactly α lines in π coplanar with L, then either 2-$\boldsymbol{S}$ = 2-$\mathrm{PG}(d, 2)$

($d > 2$) or 2-AG(3, 3), or else its number of points is $2(6m+7)(3m^2+3m+1)$ for $m \geqslant 1$, or 171 or 183 or 2055.

For other topics in this area, we refer to Teirlinck [1979], Buekenhout and Metz [1975], Herzer [1988].

5. Benz planes[3]

5.1. *Introduction*

Benz planes arise from a common axiomatization of the structure induced by a 3-dimensional projective space on one of its quadrics having an empty or singleton radical, or more generally on one of its quadratic sets (see Chapter 17). They naturally split into 3 main families, which actually were introduced separately: inversive (or Möbius) planes (see Van der Waerden and Smid [1935], Benz [1958, 1960]), Laguerre planes and Minkowski planes (Benz [1968]), and their prototypes are provided by elliptic quadrics, quadratic cones and hyperbolic quadrics, respectively (see 5.8). The classical real inversive plane (provided with the appropriate metric) is well known as 'the geometry' of the $\mathbb{R}$-algebra $\mathbb{C}$, while the classical real Minkowski (resp., Laguerre) plane (provided with the appropriate metric) is the 2-dimensional analogue of the 4-dimensional Minkowski spacetime of special relativity (resp., Einstein's cylindrical universe).

As exemplified above, Benz planes are closely related to 2-dimensional algebras (see Chapter 14 and Benz [1973]) and to quadratic sets in 3-dimensional projective spaces although not all of them fit into these frameworks. Benz planes, also called circle planes, were named after Benz' famous book 'Vorlesungen über der Geometrie der Algebren' where he actually studied more general geometries than the present Benz planes (called 'in the narrow sense' in the terminology of Benz [1973]). Indeed one could more generally study Benz planes 'in the broad sense', corresponding to semiquadratic rather than quadratic sets in PG($3, K$), and higher dimensional 'Benz spaces'. However most of the results concern Benz planes 'in the narrow sense' and those concerning higher dimensional 'Benz spaces' state that they are embeddable as quadratic sets of suitable projective spaces (see 5.29).

Finally we recall that the algebraic approach to such spaces yields 'chain geometries' which are developed in Chapter 14.

The reader unacquainted with Benz planes is advised to read Sections 5.8(a) and 5.6 before proceeding to the general presentation below.

5.2. *Complete Benz planes*

The following presentation of Benz planes is inspired by Buekenhout [1981]. In a planar space $\boldsymbol{S}$, call *circular plane* any plane all of whose lines are thin, and call *radical* of $\boldsymbol{S}$ (Rad $\boldsymbol{S}$) the set of all points which lie on thick lines only. A *complete Benz plane* is a (nontrivial) planar space $\boldsymbol{S}$ satisfying the following two conditions:

[3] This subject is also covered in Chapter 24, with a different approach (Editor's note).

(1) every noncircular plane is the union of two intersecting thick lines,

(2) for every point $x \notin \operatorname{Rad} \boldsymbol{S}$, the residue $\boldsymbol{S}_x$ is an affino-projective plane whose points at infinity are the thick lines through x.

Note that $\boldsymbol{S}_x$ can only be an affine plane extended by at most two ideal points. Hence the ideal points of $\boldsymbol{S}_x$ are uniquely defined except in the case of the punctured projective plane of order 2 in which any thin line may be chosen to be the line at infinity. Only three situations can occur (Buekenhout [1981]):

(i) $\boldsymbol{S}$ has no thick line, so that all point-residues are affine planes; $\boldsymbol{S}$ is then called a *inversive* (or *Möbius*) *plane*;

(ii) all thick lines are concurrent in a point p, so that all point-residues other than $\boldsymbol{S}_p$ are affine planes extended by one point at infinity; $\boldsymbol{S}$ is then called a *complete Laguerre plane*;

(iii) the set of thick lines splits into two subsets $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$, each of which partitions the point-set $\boldsymbol{P}$, and every line of $\boldsymbol{L}_1$ intersects every line of $\boldsymbol{L}_2$; in this case all point-residues are affino-projective planes with two points at infinity. $\boldsymbol{S}$ is then called a *Minkowski plane*.

To prove the above assertion, it suffices to notice that if neither (i) nor (ii) holds, then $\boldsymbol{S}$ has at least one thick line L and $\operatorname{Rad} \boldsymbol{S}$ is empty, so that the thick lines intersecting L in exactly one point partition $\boldsymbol{P}$ and every point is on exactly two thick lines. Note also that in a Benz plane, every circular plane C meets every thick line L. Indeed, if x is a point on C but not on L, then in $\boldsymbol{S}_x$ the lines $\langle x, L\rangle$ and C intersect in a point which is not a infinity, and so C and L intersect in $\boldsymbol{S}$. The *order* of a finite (complete) Benz plane is the common order of all its point-residual affino-projective planes.

For another approach in this spirit, see Heise and Seybold [1976].

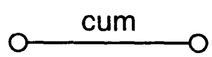
5.3. *Benz planes*

A *Benz plane* is a complete Benz plane minus its radical, so that the Laguerre planes are the only Benz planes that are not complete: all their noncircular planes are the union of two disjoint thick lines and every point is on exactly one thick line, except in the Laguerre plane LP(2) of order 2, because of which we first introduced complete Benz planes. This planar space LP(2) has a Euclidean representation as a triangular prism on 6 points (vertices): all its lines are thin and all its planes are circular. However, it also has two types of lines and two types of planes: call *circles* the thin planes and *singular lines* those lines which are not contained in any circle. Then the singular lines and the circles play the role that thick lines and circular planes play in other Laguerre planes.

In all the other Benz planes, *singular lines* are the same as thick lines and *circles* are the same as circular planes. Two points are said to be (*singularly*) *collinear* if they lie on a common singular line, otherwise they are called *concircular*. More generally any set of points which is contained in some circle is said to be concircular.

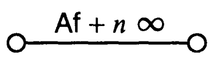
5.4. *Diagrams for Benz planes*

With the notation

(resp. 

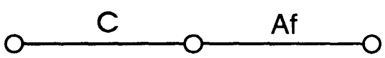
)

for the class of all linear spaces which are either trivial or the union of two disjoint (resp., meeting) thick lines, and

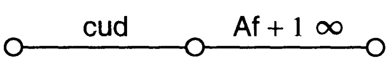


for the class of all affine planes completed by n points at infinity, Benz planes can be defined directly as follows.

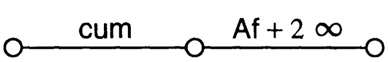
(I) An *inversive plane* (IP) is a planar space belonging to



(L) A *Laguerre plane* (LP) is a planar space belonging to

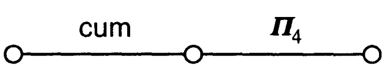


in which either all lines and some planes are thin, or the thick lines correspond precisely to the points at infinity of the point-residues (the latter condition being equivalent to requiring that any thick line intersects any circular plane),
(M) A *Minkowski plane* (MP) is a planar space belonging to



and in which every point is on two thick lines.

Note that in particular the Minkowski plane MP(2) of order 2 belongs to



(see 2.6 for the definition of $\boldsymbol{\Pi}_4$).

5.5. *Residual plane model at a point*

Actually a Benz plane S induces more than a point-line structure on each of its point-residues S_x. Note first that S_x has 0, 1 or 2 distinguished points, called *singular ideal points*, corresponding to the singular lines through x, and these points are uniquely determined by the linear space S_x except if $S = \mathrm{MP}(2)$. Also the traces of the singular lines missing x are the restrictions to the nonideal part of S_x of the lines through any of the singular ideal points, while the traces of the noncircular planes missing x are unions of the traces of two singular lines missing x.

Now the circles missing x induce in $\boldsymbol{S}_x$ ovals of its projective extension $\overline{\boldsymbol{S}}_x$, passing through all its singular ideal points. All this follows immediately from the axioms. For embeddable Benz planes, this is pictured by stereographic projection (see 5.8 (a) and (b)). $\boldsymbol{S}_x$ provided with this family of ovals is called the *residual plane model* $\pi(\boldsymbol{S}_x)$ *of* $\boldsymbol{S}$ *at* x. The reader should be aware that these structures differ from the usual plane models of Benz planes, which provide a complete description of them. For inversive planes, it suffices to add a new point, called ∞, to $\pi(\boldsymbol{S}_x)$, and to call circles all lines of $\pi(\boldsymbol{S}_x)$ completed by ∞, together with all distinguished ovals of $\pi(\boldsymbol{S}_x)$. For the usual plane models of Laguerre planes, see 5.8(g).

5.6. *Classical axioms for Benz planes*

A Benz plane is completely determined by its points and circles (seen as sets of points), the singular lines being the maximal sets of pairwise nonconcircular points.

The finite inversive planes of order n can be identified with the 3-$(n^2+1, n+1, 1)$ designs (see Chapter 8). Axioms for inversive planes are discussed in Biscarini [1973a,b]. The classical axioms for Laguerre and Minkowski planes in terms of points, singular lines and circles are as follows.

A *Laguerre plane* is an incidence structure $(\boldsymbol{P}, \boldsymbol{L}_1, \boldsymbol{C})$ whose members are called *points, singular lines* and *circles*, respectively, such that the following hold:

(1) $\boldsymbol{L}_1$ is a partition of $\boldsymbol{P}$,

(2) any three points, no two of which are on a singular line, are on a unique circle,

(3) each circle intersects each singular line in exactly one point,

(4) there are at least two circles and each circle contains at least 3 points,

(5) if C is a circle, if x and y are points not on a common singular line and such that $x \in C$ and $y \notin C$, then there is a unique circle D through y intersecting C in x only (*tangency axiom*).

A *Minkowski plane* is an incidence structure $(\boldsymbol{P}, \boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{C})$ whose members are called *points, singular lines of the first and second type* and *circles*, respectively, such that

(1) to (5) hold, together with

(1′) $\boldsymbol{L}_2$ is a partition of $\boldsymbol{P}$ and any two singular lines of different types intersect in exactly one point.

5.7. *Sharply* 3*-transitive sets*

Benz planes can be presented as sharply 3-transitive sets of functions from a set A to a set A'. Since in the case of inversive planes we should allow multivalued functions, let us focus on Minkowski and Laguerre planes.

Incidence structures $(\boldsymbol{P}, \boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{C})$ satisfying (1), (1′), (2), (3), (4) but not necessarily the tangency axiom (5), are also called B^*-geometries (Benz [1971]) or 'Hyperbelstruktur' (Heise and Karzel [1973a,b], see Chapter 14). By (1), (1′) and (3), all singular lines and circles are equipotent to a common set, say A. Then $\boldsymbol{P}$ can be identified with $A \times A$, the singular lines being lines parallel to the coordinate axes, the circles (whose set is also denoted by $\boldsymbol{C}$) being graphs of certain permutations of A. We may assume, without loss of generality, that the identity belongs to C. Moreover, by (2), C is a sharply 3-transitive set. Conversely any sharply 3-transitive set of permutations containing the identity provides a B^*-geometry. If $\boldsymbol{P}$ is finite, then all singular lines have the same size l, so that for every point x the residual linear space (whose points are the $l^2 - 2l + 1$ points that are not (singularly) collinear with x and whose lines are the circles through x and the singular lines off x) has line-size $l - 1$, and so is an affine plane. Therefore (5) follows from the other axioms (Heise and Karzel [1973a]). However there are examples of B^*-geometries of any infinite cardinality which are not Minkowski planes because any two circles intersect in two distinct points (Valette [1979]). Note that these geometries have been further generalized to (B)-geometries, whose automorphisms have been studied, e.g., by Bonisoli [1991a,b] and Quattrocchi and Rinaldi [1991].

Similarly the finite Laguerre planes of order n can be described as the sharply 3-transitive sets M of mappings from a set A of size $n+1$ to a set B of size n. Remember that a set M of mappings from A to B is *sharply* 3-*transitive* if and only if for any 3-set $\{(a_i, b_i)\colon\ i = 1,2,3\}$ of elements of $A \times B$ such that the a_i's are pairwise distinct, there is precisely one element of M whose graph contains this 3-set. The points of the Laguerre plane are then the elements of $A \times B$, the circles are the graphs of the elements of M, while the singular lines are the verticals $\{(a, y)\colon\ y \in B\}$.

This introduces Laguerre and Minkowski planes as geometries of mappings, namely as certain transversal geometries and permutation geometries, respectively. Such geometries are no longer DLS's but are *d-injection geometries* with $d = 1$ and 2, respectively, while $d = 0$ leads to DLS's. All this is encapsulated by the notion of *squashed geometries*, (see Section 8).

5.8. *Examples of Benz planes*

(a) *The Miquelian* (*or classical*) *Benz planes*
Let F be a commutative field and let $\boldsymbol{Q}$ be an elliptic quadric, a quadratic cone without its vertex or a hyperbolic quadric in PG(3, F). The planar space $\boldsymbol{S}(\boldsymbol{Q})$ induced by PG(3, F) on $\boldsymbol{Q}$ is an inversive plane, a Laguerre plane or a Minkowski plane, respectively, and will be denoted by IP(F), LP(F), MP(F), respectively, or simply by IP(q), LP(q) or MP(q) if $F = \mathrm{GF}(q)$.

The residual plane model of such a Benz plane is obtained from $\boldsymbol{S}(\boldsymbol{Q})$ by 'stereographic' projection, i.e. given any point x on $\boldsymbol{Q}$ and any plane π not on x, by the projection of $\boldsymbol{Q} \setminus \{x\}$ from x to π. Let $L = \pi \cap \pi_x$ where π_x denotes the tangent plane to $\boldsymbol{Q}$ at the point x. Then $\boldsymbol{Q} \setminus \{x\}$ is mapped onto the point-set $\boldsymbol{P} = \pi \setminus L$ in the inversive case, on $\boldsymbol{P} = (\pi \setminus L) \cup \{t\}$ with $t \in L$ in the Laguerre case, and on $\boldsymbol{P} = (\pi \setminus L) \cup \{r, s\}$ with $r, s \in L$ $(r \neq s)$ in the Minkowski case. The planes of $\boldsymbol{S}(\boldsymbol{Q})$ through x are mapped onto the lines of the affino-projective plane induced by π on $\boldsymbol{P}$, in particular the circles through x are mapped onto the lines having no point at infinity (i.e. onto L). The circles not through x are mapped onto the irreducible conics of π passing through r and s in the Minkowski case and tangent to L at t in the Laguerre case. To characterize these conics in the inversive case, we need to work over a quadratic extension $\overline{F}$ of F: they are then the irreducible conics 'passing' through the two points $\bar{r}$ and $\bar{s}$ of intersection of the extensions of $\boldsymbol{Q}$ and L (these points are conjugate with respect to $\overline{F}$). The residual *plane model* of $\boldsymbol{S}(\boldsymbol{Q})$ is the (Desarguesian) affino-projective plane induced on $\boldsymbol{P}$, provided with this family of irreducible conics. For the Minkowski case, see for instance Dicuonzo [1975].

We cannot resist mentioning the elegant algebraic presentation of IP(F) (see Chapter 14). As above let $\overline{F}$ be a quadratic extension of F: just call *points* the elements of $\overline{F} \cup \{\infty\}$ and *circles* the images of $F \cup \{\infty\}$ under PGL$(2, \overline{F})$. Here the 3-transitivity (on points) of Aut IP(F) = PΓL$(2, \overline{F})$ is immediate.

(b) *Embeddable* (*i.e. coordinatizable*) *Benz planes*

The classical construction in (a) can be generalized by introducing quadratic sets in place of quadrics. Let $\boldsymbol{Q}$ be a subset of PG$(3, F)$, F a (possibly skew) field. $\boldsymbol{Q}$ is a *semi-ovoid* if and only if for every point x in $\boldsymbol{Q}$, the union of all lines intersecting $\boldsymbol{Q}$ in and only in x is a plane. $\boldsymbol{Q}$ is an *ovoid* if moreover no three of its points are collinear (see Segre [1959] for a pioneering work on ovoids). A *semi-oval* $\boldsymbol{O}$ is a set of coplanar points such that, through each of its points x, there is precisely one line coplanar with $\boldsymbol{O}$ and intersecting $\boldsymbol{O}$ in and only in x. $\boldsymbol{O}$ is an *oval* if moreover no three of its points are collinear. The *(semi-)oval cylinder* over $\boldsymbol{O}$ with vertex t is the set

$$\boldsymbol{Q} = \left(\bigcup_{x \in \boldsymbol{O}} \langle x, t \rangle \right) \setminus \{t\},$$

where t is not coplanar with $\boldsymbol{O}$. On ovals, see, e.g., Chapter 4 and Hartmann [1981b].

If $\boldsymbol{Q}$ is a *quadratic set* in PG$(3, F)$, i.e. an ovoid, an oval cylinder or a hyperbolic quadric, then the planar space $\boldsymbol{S}(\boldsymbol{Q})$ induced on $\boldsymbol{Q}$ is an inversive, Laguerre or Minkowski plane, respectively.

A Benz plane $\boldsymbol{S}$ is called *embeddable* (or *egglike* (Dembowski and Hughes [1965]) or *coordinatizable*) if and only if it can be planarly embedded in some PG$(3, F)$. It is not difficult to check that the embedding necessarily maps the point-set of $\boldsymbol{S}$ onto a quadratic set of PG$(3, F)$.

As in the Miquelian case, stereographic projection pictures the point-residue $\boldsymbol{S}_x$ provided with the circles missing x.

For explicit examples, see Ewald [1967]. Ovoids over skew fields appear in a remarkable paper by Yaqub [1990]. An approach to inversive planes via ovoids in linear spaces, is studied in Sherman [1990]. For characterizations of ovoids with specified automorphisms see Mäurer [1991]. A survey can be found in Faina [1992].

Still more general objects related to polar spaces (see Chapter 2, Section 4) and to quadratic sets are the Hermitian varieties. For a study and characterizations of these, see Mäurer [1970, 1972], Dienst and Mäurer [1974], Dienst [1977c,d].

(c) *Nonembeddable Benz planes*

All known finite inversive and Laguerre planes are embeddable. However, infinite nonembeddable inversive and Laguerre planes were constructed by Ewald [1960], Schleiermacher and Strambach [1967, 1970], Groh [1973, 1974], Hartmann [1976, 1979], Krier [1979] and Mäurer [1987]. The latter example even has a 2-transitive automorphism group! The reader is also referred to Chapter 13 for free constructions of Benz planes.

For the study of these, see also Hotje [1986], Iden and Moe [1978], Heise and Sörensen [1973].

The situation is quite different for Minkowski planes, of which the only embeddable ones are classical (see Theorem 5.11(i)). Hence any Minkowski plane nonisomorphic to a classical one is nonembeddable (see the paragraph below and also Artzy [1979, 1980], Hartmann [1981a]).

(d) *Minkowski planes and quasifields*

A permutation set consisting of a set Γ of permutations of the elements of some set Ω will be denoted by (Γ, Ω). As mentioned in 5.7 any B^*-geometry (a kind of pre-Minkowski plane) is equivalent to a sharply 3-transitive permutation set (Γ, Ω) containing the identity, and conversely. Moreover all finite B^*-geometries are Minkowski planes, so that the finiteness of Ω guarantees that we actually get a Minkowski plane, denoted by $\mathrm{MP}(\Gamma, \Omega)$. (Γ, Ω) is a permutation group iff the following *rectangle property* holds in $\mathrm{MP}(\Gamma, \Omega)$: for any points x_i, y_i with $i = 1, 2, 3, 4$ such that the x_i (resp., y_i) are four distinct concircular points, the points (x_i, y_i) are concircular iff the points (x_i, y_i) are concircular. Here (x_i, y_i) denotes $X_i^1 \cap Y_i^2$ where X_i^1 is the line of $\boldsymbol{L}_1$ through x_i and Y_i^2 is the line of $\boldsymbol{L}_2$ through y_i (Benz [1973], III, § 4; Heise and Karzel [1973a]).

If $\Omega = \mathrm{GF}(q) \cup \{\infty\}$, $\sigma \in \mathrm{Aut}\,\mathrm{GF}(q)$ and $\sigma(\infty) = \infty$, then

$$\Gamma = \mathrm{PSL}(2, q) \cup (\mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q))^\sigma$$

is a sharply 3-transitive permutation set on Ω, which is merely $\mathrm{PSL}(2, q)$ if q is even and which, for q odd, is a group iff $\sigma^2 = 1$. The corresponding Minkowski plane $\mathrm{MP}(\Gamma, \Omega)$ is also denoted by $\mathrm{MP}(q, \sigma)$ and is not Miquelian unless $\Gamma = \mathrm{PSL}(2, q)$ (Pedrini [1966], Wilbrink [1978]). In the other cases, the residue of the point (∞, ∞) in $\mathrm{MP}(q, \sigma)$ is a translation plane which can be coordinatized by the quasifield Q_q^σ obtained from $\mathrm{GF}(q)$ by replacing its multiplication by $\circ$ as follows:

$$x \circ y = \begin{cases} xy & \text{if } y \text{ is a square in } \mathrm{GF}(q), \\ x^\sigma y & \text{otherwise.} \end{cases}$$

If σ has order 2, then Q_q^σ is merely the regular nearfield of rank 2 over its kernel (remember that q is odd) (see Percsy [1981] for this coordinatization and for a description of the circles in terms of Q_q^σ; see also Bonisoli and Korchmáros [1990] for an improvement of that paper). For a characterization of the known finite Minkowski planes, see Wilbrink [1982a], and for the general case see also Artzy [1977].

(e) *The Suzuki–Tits inversive planes*

Let F be a commutative field of characteristic 2 admitting an endomorphism σ whose square is the Frobenius endomorphism, i.e. $(x^\sigma)^\sigma = x^2$ (if $F = \mathrm{GF}(q)$, then $q = 2^{2m+1}$ and $x^\sigma = x^{2^{m+1}}$). The subset of $\mathrm{AG}(3, F)$ with the equation

$$z = xy + x^{\sigma+2} + y^\sigma$$

together with the point at infinity of the z-axis is an ovoid of PG(3, F), called a *perfect σ-quadric* or *Suzuki–Tits ovoid.* The corresponding inversive planes are the only embeddable inversive planes of Hering type VI.1 (Tits [1966], see 5.23 for a characterization of these).

If $F = \mathrm{GF}(q)$, then the automorphism group of this ovoid is the semidirect product

$$\mathrm{Sz}(q).\,\mathrm{Aut}\,\mathrm{GF}(q)$$

(where Sz(q) denotes Suzuki's simple group of order $q^2(q^2+1)(q-1)$) which acts 2- (but not 3-) transitively on points, and transitively on circles but not on point-circle flags. It contains no inversion and all its dilatations are translations (see 5.17 and 21).

(f) *Tits translation inversive planes*
These are infinite and are the only embeddable inversive planes of Hering type VII.1 (Tits [1966]), the corresponding ovoids being named *translation ovoids* by Tits.

(g) *Plane models for Laguerre planes*
Let F be a field and f a function such that the point-set with the equation $y = f(x)$ in AG(2, F), together with the ideal point of the y-axis, is an oval $\boldsymbol{O}$ containing (1,1) and tangent to the x-axis in (0,0). Then there is a Laguerre plane $\boldsymbol{S}$ having $(F \cup \{\infty\}) \times F$ (where $\infty \notin F$) as point-set and

$$\{(x, y) \in F^2\colon\ y = af(x) + bx + c\} \cup \{(\infty, a)\},$$

where a, b, c range over F, as circles. Moreover, this Laguerre plane can be embedded in PG(3, F) so that its point-set is mapped onto a cylinder over on oval $\boldsymbol{O}'$ which is projectively equivalent to $\boldsymbol{O}$. Conversely, any embeddable Laguerre plane over F can be presented by the above *plane-model* (Mäurer [1977]). $\boldsymbol{S}$ is Miquelian iff F is commutative and $f(x) = x^2$. $\boldsymbol{S}$ is embeddable as a cylinder over a Moufang oval iff $\mathrm{char}\, F = 2$, f is injective and satisfies the functional equations

(1) $f(x + x') = f(x) + f(x')$, and

(2) $f\big(xf(x)^{-1}\big) = f(x)^{-1}$ (Hartmann [1981c]).

If $F = \mathrm{GF}(2^n)$ and $\boldsymbol{S}$ is embeddable as a cylinder over a translation oval, then f can be chosen such that $f(x) = x^{2k}$, where $k \in \{1, \ldots, n-1\}$ and k, and n are relatively prime (Payne [1971]).

Finally note that a similar plane model was used by Hartmann [1979] to introduce a nonembeddable Laguerre plane having a real affino-projective plane as a point-residue.

5.9. *The bundle embedding theorem and local projectiveness*

Before stating this fundamental result, we note that Definitions 5.4 allow us to extend the lattice of varieties of a Benz plane $\boldsymbol{S}$ to a (nonatomic) semimodular lattice $\overline{\boldsymbol{L}}$ with the same point-set, in such a way that, for every point x, the interval $[x, 1]$ is the projective plane extending S_x. This makes us introduce lines covering only one point and possibly planes whose point-sets are reduced to one point or one singular line, so that $\overline{L}$ is no longer geometric. The bundle condition (**B**), given in 4.9, is so crucial that we restate it here in terms of bundles in S, without any reference to $\overline{L}$.

A *bundle* (resp., *pencil*) in S is a maximal set of planes such that the intersection of any two of them is a given line L (resp., a given point x). The points belonging to that common intersection are called *carriers* of the bundle or pencil. Hence if B is a bundle carried by the points of L and if x is on L, then B corresponds to the pencil of all lines of S_x through the point L_x. Similarly, if P is a pencil carried by x, then P corresponds to some parallel class of lines in S_x (together with the line at infinity if S is Minkowskian). In any case, a bundle or a pencil having the point x as a carrier corresponds to a point in the projective plane completing S_x. A *fan* is either a bundle or a pencil. Two fans are *coplanar* provided they have a plane in common. Now the bundle condition for Benz planes reads:

(**BB**) Given any four fans, no three of which are coplanar and no two of which have a common carrier, if five of the six unordered pairs of those fans are coplanar, then the sixth pair also is coplanar.

The embeddable Benz planes satisfy quite obviously (**BB**) (see also 4.9). The converse remained a famous open problem for nearly half a century, until 1980, when Kahn proved its validity for a class of geometries wider than the Benz planes class (see 4.9).

THEOREM (Kahn [1980a]). *A Benz plane is embeddable iff it satisfies the bundle condition.*

See also Kahn [1982b].

Other necessary and sufficient conditions for the coordinatizability of Benz planes (and of more general semimodular lattices) are available in Herzer [1976, 1981] and Batten [1987a].

The role of the embeddable (resp., Miquelian) Benz planes is to some extent similar to that of the Desarguesian (resp., Pappian) projective planes, the Bundle (resp., Miquel's) theorem corresponding to Desargues (resp., Pappus) theorem. In particular, in analogy with Desargues theorem, the Bundle condition holds in the higher dimensional generalizations of Benz planes, which are indeed all embeddable in a projective space. For a recent study of the Bundle theorem, both from an algebraic and a geometric viewpoint, see Benz [1988]. For a related result, see Willems [1986].

5.10. *Miquel's theorem*

See also Chapter 17, 3.4.8–10 and 4.4.8.

THEOREM (Van der Waerden and Smid [1935] for IP and LP, Kaerlein [1970] for MP). *A Benz plane is Miquelian iff it satisfies Miquel's condition.*

Here Miquelian means classical, i.e. embeddable as a quadric in a projective space over a commutative field. *Miquel's condition* reads:

(M) Let C_1, C_2, C_3, C_4 be circles no three of which have a common point and such that $C_i \cap C_{i+1} = \{x_i, y_i\}$ for every i (subscripts are modulo 4), where the points x_i and y_i need not be distinct. Then the four points x_i are concircular iff the four points y_i are.

Note that if the x_i and y_i are assumed to be pairwise distinct, Miquel's condition takes the following appealing 'cubic' form.

(M8) If eight points can be arranged as the vertices of a cube in such a way that each of five faces is concircular, then the sixth one is also.

Several variations on **(M)** have been studied by Chen [1970], Schaeffer [1974a,b,c], Lozanov [1987] for inversive planes and by Artzy [1974], Samaga [1991] for Laguerre planes. In particular **(M8)** is equivalent to **(M)** for inversive planes. For more on Miquelian inversive planes, see Orr [1973], Hoffman [1951a], Bruck [1973c].

5.11. *When embeddable means Miquelian*

Just as Desargues and Pappus conditions are equivalent for finite projective planes, so are **(B)** and **(M)** for all Minkowski planes and for finite inversive or Laguerre planes of odd order, as shown by the following theorems.

THEOREM.

(i) *If F is commutative, any quadratic set of index* 2 *in* PG(3, F) *is a quadric* (Buekenhout [1969a]).

(ii) *If q is odd, any ovoid in* PG(3, q) *is a quadric* (Barlotti [1955], Panella [1955]).

(iii) *If q is odd, any oval in* PG(2, q) *is a conic* (Segre [1955]).

The existence of nonelliptic ovoids and ovals in projective spaces of even or infinite order is well known (see 5.8 (e)–(g)).

5.12. *Even order forces embeddability*

Many nonembeddable Minkowski planes of odd order are known (see 5.8(d)), while the existence of nonembeddable inversive or Laguerre planes of odd order is still in doubt. However, by a fundamental result of Dembowski, the study of even order inversive planes amounts to that of ovoids. The analogous result for Minkowski planes was presented by Heise [1974] as a trivial consequence of results of Artzy [1973] and Heise and Karzel [1973a,b] (see also Dicuonzo [1975], Percsy [1974]).

THEOREM (Dembowski [1964] for IP, Heise [1974] for MP). *Any finite inversive or Minkowski plane of even order is embeddable.*

Table 1

Benz plane of order n	inversive	Laguerre	Minkowski
n even			
necessarily Miquelian	no true for $n = 2, 4, 16$ false for $n = 2^{2m+1} \geqslant 8$	no true for $n = 2, 4$	yes
necessarily embeddable	yes	?	yes
n odd			
Miquelian iff embeddable iff some point-residue is Desarguesian	yes	yes	yes
necessarily Miquelian	? true for $n = 3, 5, 7$	? true for $n = 3, 5, 7$	no true for $n = 3, 5, 7$
n infinite			
necessarily embeddable	no	no	no

Table 1 summarizes the situation. (The uniqueness of the inversive plane of order 16 is proved with the aid of a computer in O'Keefe and Penttila [1990]). For inversive planes of order 5 (resp., 7), the uniqueness proof is due to Chen [1972] and Denniston [1973a] (resp., Denniston [1973b]).

5.13. *Desarguesian point-residues and odd order implies Miquelian*

If a Benz plane is embeddable, then all its point-residues are Desarguesian. The converse has been proved to be true for finite odd order Benz planes (up to a few undecided cases). Actually, the assumption can be weakened to only one point-residue.

THEOREM (Chen and Karlein [1973] for LP and MP, Thas [1994], Fisher, Penttila, Praeger and Royle [1989] for IP). *If S is a Benz plane of odd order n with at least one Desarguesian point-residue S_x then S is Miquelian.*

This theorem was proved by showing that the traces in S_x of the circles of S not through x form a set of irreducible conics as in the plane models of Miquelian planes (see 5.8(a)) and the odd order assumption allows the use of Segre's theorem (see 5.11(iii)).

Surprisingly Thas' proof for inversive planes relies on results concerning flocks (see 5.14) of Miquelian Laguerre and Minkowski planes. Flocks are indeed related to many interesting geometric structures, among others translation planes, generalized quadrangles, maximal exterior sets (and hence nonbuilding C_3-geometries). See Thas [1991] and Chapters 7 and 9.

5.14. *Flocks*

Let S be a Benz plane with point-set P. If S is inversive and $x, y \in P$, a *flock* with *carriers* x and y is a set of circles of S partitioning $P \setminus \{x, y\}$. If S is Laguerrian or Minkowskian, then a *flock* is a set of circles of S partitioning P. The existence of flocks is not clear if S is nonembeddable. However if S is embedded in PG$(3, F)$ and if L is a line of PG$(3, F)$ disjoint from $P \cup (\text{Rad}\, P)$, then the pencil of planes through L defines a flock of S. Such a flock is called *linear*linear (compare with bundles and pencils). A crucial challenge is to classify flocks of embeddable Benz planes or at least to decide when nonlinear flocks can occur (but this is a problem about quadratic sets, see Chapters 2 and 7). We refer the reader to the excellent survey of Thas [1991], see also Thas [1990], and references therein or directly to Chapter 7. For an early result on the inversive plane case, see Thas [1973].

5.15. *Automorphisms of Benz planes*

First of all, set your mind at ease: the automorphisms of embeddable Benz planes are exactly what you expect.

THEOREM (Mäurer [1967] for IP and LP). *Let $S = S(Q)$ be an embedded Benz plane, where Q is a quadratic set in* PG$(3, F)$. *Then every automorphism of S is induced by a collineation of* PG$(3, F)$ *which preserves Q.*

5.16. *Central automorphisms*

The main theorems involving automorphisms provide characterizations and classifications of Benz planes. Any automorphism φ of a Benz plane S which fixes a point x induces a collineation φ_x in the projective plane $\overline{S}_x$ completing S_x and, in turn, φ is uniquely determined by φ_x. Those φ for which some φ_x is a central collineation of $\overline{S}_x$ play a central role and are also called *central* (with respect to x). Since φ_x must stabilize the ideal line of S_x, φ_x must be either a shear or a strain, or else a dilatation (translation or homothety). Those cases have to be further subdivided in the Laguerre and Minkowski cases because the set of singular ideal points (corresponding to the singular lines through x) must also be preserved.

5.17. *Inversions*

Note that Fix φ (the fixed-point-set of φ) is a circle iff for some (or equivalently every) point $x \in \text{Fix}\,\varphi$, φ_x is a shear or a strain whose axis misses the singular ideal points. More precisely φ_x must be a strain centred at the singular ideal point in the Laguerre case, and a shear interchanging the two ideal points in the Minkowski case. However in the inversive case, φ_x may either be a strain or a shear but, as φ (and so φ_x) is involutory (Dembowski [1964]) either all φ_x are strains or all φ_x are shears as soon as the inversive plane is finite. Note that φ also has to be involutory in the Minkowski case but not in the Laguerre case. An involutory central automorphism whose fixed-point-set is a circle C will be called an *inversion* with *axis* C (these coincide with the usual inversions in classical inversive planes but differ from the B-inversions defined in 5.19). For inversions of finite inversive planes, see Cofman [1970].

THEOREM. *Let S be a Benz plane each of whose circles is the axis of an inversion. Then*

(a) *S is finite inversive iff S is finite Miquelian inversive* (Dembowski [1965]);

(b) *S is a finite Minkowskian iff S is a finite Miquelian or a nearfield Minkowski plane* (see 5.8, Percsy [1981], Hartmann [1981c], and Bonisoli and Quattrocchi [1987] for a generalization);

(c) *if S is inversive embeddable with* char $\neq 2$, *then S is Miquelian* (Mäurer [1971]);

(d) *if S is a Minkowski plane, then every point-residue is a translation plane* (Dienst [1977a], in which an additional property allows a characterization of Miquelian planes).

5.18. *Configurational symmetry in Minkowski planes*

In a Minkowski plane, any inversion is a *reflection with respect to a circle C*, i.e. the involution ρ_C fixing C pointwise and mapping any point x on the only other point x' such that the singular lines through x and those through x' intersect C in the same pair of points. Hence C is the axis of an inversion iff the reflection ρ_C is an automorphism.

Two circles C, D are *opposite* (resp., *symmetric*) if $\rho_D(x) \in C$ for some (resp., every) point x on C but not on D. The following nice geometrical (as opposed to group-theoretical) characterization of Miquelian Minkowski planes in due to Artzy [1973].

THEOREM. *A Minkowski plane is Miquelian iff any two opposite circles are symmetric.*

5.19. *B-inversions*

Here and in 5.20, permutations are introduced which generalize the automorphisms (of embedded Benz planes) induced by homologies. A *B-inversion* (Buekenhout [1971a]) is any automorphism σ of an inversive or Minkowski plane S such that, for any two distinct concircular points x and y, the points x, y, x^σ, y^σ are concircular. Except in MP(2), B-inversions are involutory, but need not be inversions (since they can be fixed-point-free) and conversely inversions need not be B-inversions! However in Miquelian planes, inversions are B-inversions.

THEOREM (Buekenhout [1971a] for IP, Percsy [1983] for MP).

(a) *An inversive or Minkowski plane is Miquelian iff for every 4-tuple of points x, x', y, y' pairwise concircular and pairwise distinct (except possibly y and y'), there is a B-inversion mapping x onto x' and y onto y'.*

(b) *A Minkowski plane is Miquelian iff every circle is the fixed-point-set of a B-inversion.*

Actually, (b) is an easy consequence of Theorem 5.18. Buekenhout [1971a] also contains a classification of inversive planes based on their set of 'Miquelian pairs of points' (compare 5.23).

5.20. *Quasi-inversions*

The definition of a *quasi-inversion* in an inversive plane can be derived from that of a B-inversion by replacing the requirement that σ be an automorphism by the following one: every fixed point carries a pencil all of whose circles are invariant under σ.

THEOREM (Freudenthal and Strambach [1975]). *Let* $\boldsymbol{S}$ *be an inversive plane.*

(a) $\boldsymbol{S}$ *is embeddable iff for all points* x, x', y, y' *which are pairwise distinct (except possibly* y *and* y'*), there is a quasi-inversion mapping* x *onto* x' *and* y *onto* y'.

(b) *If* $\boldsymbol{S}$ *is embeddable, then it is Miquelian iff every quasi-inversion is an automorphism.*

5.21. *Dilatations*

A *dilatation* of a Benz plane $\boldsymbol{S}$ is any central automorphism δ inducing a dilatation in some point-residue $\boldsymbol{S}_x$. Hence there is precisely one fan carried by x which is fixed element- and carrier-wise by δ (unless δ is the identity). Fix δ can either consist of two concircular points x and y (*homothety* with *centre* x) or of singularly collinear points (*translation*). In inversive planes, the nontrivial dilatations are precisely the automorphisms fixing precisely one fan element- and carrier-wise (homothety if the fan is a bundle, translation if it is a pencil).

5.22. *Characterizations by dilatations*

Here are a few characterizations of Benz planes by means of the existence of certain dilatations.

THEOREM.

(a) *A finite inversive plane is Miquelian iff for every pencil* P, *there is a nontrivial translation fixing* P *element-wise* (Cofman [1968]).

(b) *An embeddable inversive plane is Miquelian of characteristic* $\neq 2$ *iff there is a point* x *such that for every* $y \neq x$, *there is an involutory homothety with centres* x *and* y (Mäurer [1973]).

(c) *A Laguerre plane is Miquelian of characteristic* $\neq 2$ *iff for every pair of concircular points* x *and* y, *there is an involutory homothety with centres* x *and* y (Mäurer [1978], see also Kleinewillinghöfer [1979], p. 141, for a slight improvement).

(d) *A Minkowski plane is Miquelian iff for any four distinct concircular points x, y, z, z', there is a homothety with centres x and y mapping z onto z'* (Hartmann [1982b]).

5.23. *F-transitivity, Hering's classification of inversive planes*

We now turn to Hering's classification of inversive planes (see Chapter 24), which is similar to the Lenz–Barlotti classification of projective planes (see Chapter 5, Introduction and historical survey).

Let $\boldsymbol{I}$ be an inversive plane and F a fan of $\boldsymbol{I}$. Denote by $\mathrm{Dil}(F)$ the set of all dilatations fixing F element- and carrier-wise. $\boldsymbol{I}$ is *F-transitive* if $\mathrm{Dil}(F)$ is transitive on the noncarrier points of some (or equivalently every) circle of F. If x is a carrier point of F and if p is the point of $\overline{\boldsymbol{S}}_x$ corresponding to F, then F-transitivity is the same as (p, L_∞)-transitivity in $\overline{\boldsymbol{S}}_x$ (where L_∞ denotes the ideal line of $\boldsymbol{S}_x$). Hering [1965] classified all possibilities for the set of all fans F such that $\boldsymbol{I}$ is F-transitive. Actually he did it more generally for subgroups of $\mathrm{Aut}\,\boldsymbol{I}$ rather than for $\mathrm{Aut}\,\boldsymbol{I}$ itself and found 18 possibilities. Much work has been done to find out which types can occur in the three main cases: even order, odd order and infinite order. For even order inversive planes the situation was remarkably cleaned up by Glynn [1984] using Dembowski's embeddability Theorem (5.12).

THEOREM (Glynn [1984]). *Any inversive plane $\boldsymbol{I}$ of even order is either Miquelian* (*type* VII.2) *or Suzuki–Tits* (*type* VI.1) *or* $\mathrm{Aut}\,\boldsymbol{I}$ *is never F-transitive for any fan F* (*type* I.1).

The reader can find relevant results for the odd and infinite cases in Dembowski [1968], pp. 261, 262, Krier [1973, 1976], Krier and Liebler [1975], Yaqub [1975, 1977, 1978], Ewald [1967], Tits [1962a,b, 1966], Maurer [1987], Kroll [1988].

5.24. *Kleinewillinghöfer's classification of Laguerre planes*

A similar job can be done for other Benz planes. For Laguerre planes however the situation is much more complicated since there are many more types of dilatations. Kleinewillinghöfer [1980] presents two classifications: one based on homotheties fixing element- and carrier-wise one bundle carried by two concircular points, and the other based on translations whose fixed-point-set is a singular line. Refinements involving other central automorphisms are in Kleinewillinghöfer [1979].

Let us mention two results characterizing Laguerre planes by properties similar to F-transitivity in inversive planes (many more can be found in Hartmann [1982b], some of them concern Laguerre planes over Moufang ovals or translation ovals).

THEOREM.

(1) *A Laguerre plane $\boldsymbol{S}$ is embeddable iff there is a point x such that for every circle C on x, the pointwise stabilizer of C in* $\mathrm{Aut}\,\boldsymbol{S}$ *is transitive on the points* (*not on* C) *of some singular line* (Krebs [1976], Mäurer [1977]).

(2) *A finite Laguerre plane S is Miquelian iff for every triple (F, x, L), where F is a pencil, x is its carrier-point, and L is the singular line on x, the group of all dilatations fixing F elementwise and L point-wise is transitive on the points (distinct from x) of some circle of F* (Hartmann [1982b]).

5.25. *Klein's classification of Minkowski planes*

A Hering type classification for Minkowski planes has been initiated by Meschiari and Quattrocchi [1975] and pushed further by Klein and Kroll [1989] and Klein [1992]. Here is an F-transitivity-like characterization of Miquelian Minkowski planes.

THEOREM. *A Minkowski plane is Miquelian iff for every pair of concircular points x and y, the group of all homotheties fixing element- and carrier-wise the bundle F carried by x and y acts transitively on the points $\neq x, y$ of some circle of F* (Hartmann [1982a]).

5.26. *Transitivity on points or on pairs of concircular points*

Let S be an inversive or Minkowski plane of finite order n and let G be its automorphism group.

If S is inversive and n is even, then G is point-transitive iff S is Miquelian or Suzuki–Tits (Bagchi and Narasimha Sastry [1987]).

If S is inversive and n is odd, then G is 2-transitive iff S is Miquelian (Hering [1967]).

If S is Minkowskian, then G is transitive on ordered pairs of concircular points iff $n = q$ and S is a quasifield Minkowski plane MP(q, σ) (see 5.8(d)) (Wilbrink [1982b]).

For finite inversive planes, Block's [1967] lemma yields the following corollaries:

(a) the circle-transitive finite inversive planes are the Miquelian and Suzuki–Tits planes (see also Lüneburg [1966, 1967]);

(b) the (point, circle)-flag-transitive finite inversive planes are the Miquelian ones (use Block's lemma in S_x).

Note that the proofs of these results do not invoke the classification of all finite simple groups. However the finiteness assumption is essential, as illustrated by the 2-transitive nonembeddable inversive planes constructed by Mäurer [1987].

Note that the two Miquelian inversive planes IP(3) and IP(4) (related to the Mathieu groups M_{12} and M_{24}) are the only finite 2-transitive inversive planes such that any point-subset is stabilized by at least one nontrivial automorphism (Key and Rostom [1989]).

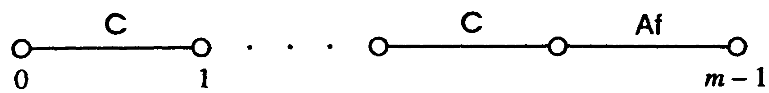
5.27. Finally we cannot close this section on characterizations of Benz planes by transitivity properties without mentioning that a Benz plane (of order $\neq 9$ if inversive) is Miquelian iff its group of projectivities stabilizing a circle C is sharply 3-transitive on C (Freudenthal and Strambach [1975], Kroll [1977], see Chapter 23 for more details, including the definition of projectivity).

For other work involving automorphisms, we refer to Mäurer [1976a,b, 1979, 1981] and to Bagchi and Narasimha Sastri [1993].

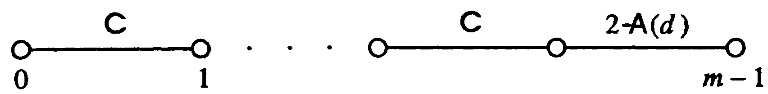
5.28. *Inversive spaces and locally affine hypercircular spaces*

For an early study of inversive spaces over an ordered field, see Hoffman [1951b]. We use here the terminology introduced in 4.4. Since the inversive planes are the sharply locally affine (shLA) circular spaces, it is natural to investigate also the weakly locally affine (WLA) circular spaces, the strongly locally affine (SLA) circular spaces, or even the SLA planar spaces. However 4.8 suggests that we focus on circular spaces. These structures can be further generalized to the locally affine m-dimensional hypercircular spaces defined below (see also Heise and Timm [1971]).

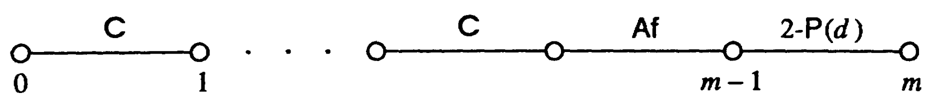
(1) *The* shLA *hypercircular spaces* are the m-DLS's belonging to



(2) *The* WLA *hypercircular spaces* are the m-DLS's belonging to



(3) *The* SLA *hypercircular spaces* also belong to the above diagram, but are best described as the m-truncations of $(m+d-2)$-DLS's whose $(m-2)$-varieties are thin and whose top residues of $(m-3)$-varieties are d-dimensional affine spaces, or equivalently (Buekenhout [1971b]) as the m-truncations of the $(m+1)$-DLS's belonging to



The terminology *inversive spaces* or *Möbius spaces* often refers to SLA circular spaces (Buekenhout [1971b, 1976], Kroll [1987]), but 'Möbius spaces' refers to WLA circular spaces in Chapter 14. The *d-dimensional Möbius geometries* of Mäurer [1968] are SLA planar spaces whose point-residues are 2-truncations of d-dimensional affine spaces.

The existence of finite SLA and shLA hypercircular spaces is severely restricted by the following three results.

(i) Every finite SLA DLS is shLA (cf. Kantor [1974a], Remark 2; Buekenhout [1976], Theorem 24, together with Thas [1974]),

(ii) Every finite shLA hypercircular space is either a trivial m-$(m+2, m, 1)$ design or an inversive plane or one of the two Mathieu–Witt designs 4-(11,5,1) and 5-(12,6,1), or a hypothetical 4-(171,15,1) design all of whose point-residues are non-Miquelian inversive planes of order 13 (Kantor [1974a], Remark 2, and Penttila [to appear]).

However, finite WLA non-SLA circular spaces are obtained by taking as point-set $GF(q^d) \cup \{\infty\}$ and as plane-set the orbit of $GF(q) \cup \{\infty\}$ under $PGL(2, q^d)$ for $d \geqslant 3$. Bruck [1973a,b] defines such a WLA circular space for every pair (K, F) of finite fields where K is a d-dimensional extension of F, and investigates the case where $|F| \geqslant d$ and d is an odd prime number. Infinite examples of shLA hypercircular spaces of any dimension are given in Barlotti [1969], and infinite examples of WLA hypercircular spaces for any (m, d) with $m \leqslant d$ are given in Heise [1971].

(iii) All SLA but not shLA planar spaces are embeddable (Mäurer [1968]) and more generally all SLA but not shLA DLS are embeddable (see, e.g., Section 6.12).

However WLA but not SLA DLS's are not embeddable, so that Hölz [1980] investigates another kind of representation of these spaces in projective spaces.

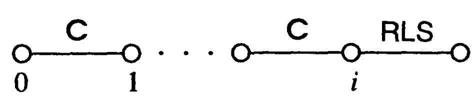
Kroll [1987] generalized Hering's classification for inversive planes to SLA hypercircular spaces. On inversive spaces, see also Heise [1970].

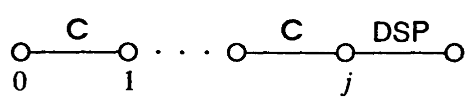
5.29. *Benz spaces*

The *cycle geometries* of Dienst [1977b] generalize SLA planar spaces just as Benz planes generalize inversive planes. Here the 'point-residues' are not affine spaces, but certain structures derived from d-dimensional affine spaces together with a proper semiquadratic set at infinity. We refer the reader to the very complete paper of Dienst, where the cycle geometries for which $d \geqslant 3$ are embedded as semiquadratic sets in projective spaces. Compare also with Section 6 for more general embedding problems. Generalizations of shLA hypercircular spaces are considered below (see also Heise and Karzel [1972b]). For another approach to 'Benz spaces' see Schultze [1976]. For developments on Laguerre geometry, see also Mäurer [1965, 1966, 1969], Hartmann [1975, 1982b].

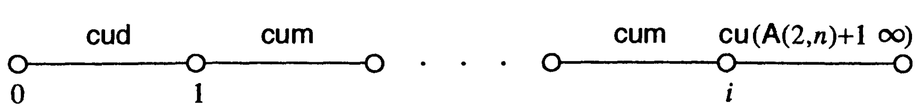
5.30. *Other variations on Benz planes and related topics*

As finite affine planes are restricted linear spaces (RLS, see Chapter 4, 6.4) and are also *Dembowski semiplanes* (DSP, i.e. finite semiplanes which are either hyperbolic, parabolic, a projective plane or the complement of a Baer subset of a projective plane, see Dembowski [1968], Section 7.4), similarly the shLA hypercircular spaces (5.28) belong to

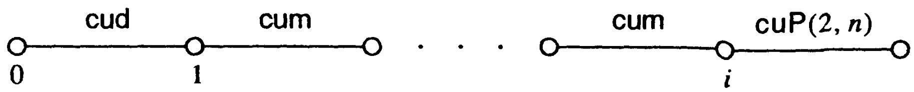


and


Thas and Willems [1982], Willems and Mielants [1982, 1984], Willems [1984] have investigated these two classes and shown that their members, called *restricted* $D(i+1)$-*spaces* and *Dembowski semi*-$(i+1)$-*spaces*, either derive (in a known way) from a finite shLA or shLP hypercircular space (5.28, 4.7) or are $(i+2)$-DLS on $i+3$ or $i+4$ points all of whose (except possibly one) hyperplanes have $i+2$ points. Analogous results of Laguerre and Minkowski type were obtained by Willems and Thas [1982, 1983], Willems and Depunt [1983], Willems [1983a,b, 1984, 1985]. See also Olanda [1988], Rinaldi [1989] and Sprague [1984a,b]. Let us only mention that, for $i \geqslant 1$, *Laguerre* i-*structures* (resp., *special Laguerre* i-*structures*) of order n belong to



resp.,



(cf. 5.4) and correspond to optimal $(n+i, i+2)$- (resp., $(n+i+1, i+2)$-) codes of order n. 'Classical' special Laguerre i-structures are constructed like embeddable Laguerre planes, starting from a complete oval instead of an oval. In fact each Laguerre i-structure of even order is obtained from a special Laguerre i-structure by deleting a line and its points.

The material in this section is also related to Heise and Karzel [1972a].

6. Embeddings, coordinatizations, subgeometries and extensions

6.1. *Introduction*

DLS's (and quite independently Benz planes) were introduced as abstract generalizations of substructures of projective spaces. This raises the problem of determining which DLS's are embeddable in projective spaces. Moreover, since an embeddable DLS may be given some additional structure from the host projective space (coordinates, erection to a higher dimension, ...), it is useful to know whether the embedding is essentially unique. This information is also helpful in the investigation of automorphism groups. But what is an embedding? If we allow as embeddings all order-preserving monomorphisms of the associated geometric lattices, then translation planes of order q^d with kernel GF(q) would be embeddable in AG($2d, q$) (points and lines being mapped onto points and d-dimensional subspaces, respectively, see Chapter 5), the Mathieu–Witt 5-(24, 8, 1) design would be embeddable in PG(11, 2) (any i-variety with $i < 4$ (resp., $i = 4$) being mapped onto an i-subspace (resp., 6-subspace), see Todd [1959]) and any erection of the linear truncation of PG(d, F) would be embeddable in PG(d, F).

If we consider lattice embeddings (join- and meet-preserving monomorphisms), then Dilworth proved in the early forties that any finite lattice is embeddable (as a sublattice) in some finite geometric lattice (see, e.g., Crawley and Dilworth [1973]). Whitman [1946] proved the embeddability of any finite lattice into some infinite partition lattice (see 2.6) and conjectured that 'infinite' could be replaced by 'finite'. This was eventually proved true by Pudlák and Tûma [1980].

Standard arguments provide an order- and equidimensionality-preserving monomorphism from any finite geometric lattice G into the lattice of subspaces of any $n(G)$-dimensional projective space, for some integer $n(G)$ (see, e.g., Percsy [1980]).

Since we do not want to encounter non-Desarguesian projective planes, nor failed bundle configurations (4.9) embedded in Desarguesian projective spaces, we require all embeddings to be isometric, i.e. to preserve the dimension of proper varieties.

6.2. *Embedding, projective embedding, F-coordinatization, algebraic representation*

Let $S = (S^i)$ be an n-DLS and $T = (T^j)$ be an m-DLS ($n \leqslant m$ need not be finite). From a nonatomic viewpoint, an (*isometric*) *embedding*isometric of S in T is an injective map

$$\varepsilon\colon \bigcup_{i=-1}^{n} S^i \to \bigcup_{j=-1}^{m} T^j$$

such that

(i) for any proper variety $V \in S^i$, $\varepsilon(V) \in T^i$;

(ii) V are V' incident in $\boldsymbol{S} \Leftrightarrow \varepsilon(V)$ are $\varepsilon(V')$ incident in $\boldsymbol{T}$ for any two varieties V, V' of $\boldsymbol{S}$;

(iii) if $n = 1$, then $\varepsilon(S^0)$ is collinear in $\boldsymbol{T}$.

$\boldsymbol{T}$ is called the *host space*. From now on embedding will always mean isometric embedding. This definition of embedding in terms of DLS's is easily translated in terms of the associated lattices to order-, join- and nonmaximal dimension-preserving monomorphism mapping the minimal and maximal elements $0_{\boldsymbol{S}}$ and $1_{\boldsymbol{S}}$ of Lat $\boldsymbol{S}$ onto $0_{\boldsymbol{T}}$ and $1_{\boldsymbol{T}}$, respectively.

The surjective embeddings are precisely the isomorphisms of DLS's. $\boldsymbol{S}$ is said to be *embedded* in $\boldsymbol{T}$ when it is identified with $\varepsilon(\boldsymbol{S})$. Turning to the atomic viewpoint, a variety of $\varepsilon(\boldsymbol{S})$ can be identified either with its point-set in $\boldsymbol{T}$ (since it is a variety of $\boldsymbol{T}$) or with its point-set in $\varepsilon(\boldsymbol{S})$ (since points from the outside are never atoms of a DLS). We choose the second interpretation, so that $\varepsilon(\boldsymbol{S})$ is the n-truncation of the restriction $\boldsymbol{T}|_{\varepsilon(S^0)}$ of $\boldsymbol{T}$ to the point-set $\varepsilon(S^0)$ (however the first choice is used, e.g., in Kantor's strong embedding definition, see Kantor [1974a, 1976].)

A *(generalized) projective embedding* is an embedding in a (generalized) projective space, while an F-embedding is an embedding in a projective space over F. A *coordinatization*, or *linear, or projective representation* (resp., *F-coordinatization*) of a finitary DLS $\boldsymbol{S}$ is an embedding of $\boldsymbol{S}$ in a Desarguesian projective space $\boldsymbol{T}$ (over F) mapping bases of $\boldsymbol{S}$ onto bases of $\boldsymbol{T}$. Most authors require F to be a field, while others allow noncommutative skewfields. In this latter case left- and right-coordinatizations should be distinguished: Lindström [1988b] proves that if $\boldsymbol{S}$ has a left-F-coordinatization, then the simplification of its dual has a right F-coordinatization. Unless otherwise mentioned we will assume F to be a field. F-coordinatizable DLS's correspond to function spaces over F (see 2.12), but more general representations can be defined by considering function spaces over subtractive algebras (Crapo and Rota [1970]). On the other hand, Brickell and Davenport [1990] define representability over a nearfield (and derive ideal secret sharing schemes from near-representable DLS's).

The epithet *linear* refers to the basic notion of linear dependence for projective spaces as opposed to the algebraic dependence used in algebraic spaces. An *algebraic representation* of $\boldsymbol{S}$ is an embedding of $\boldsymbol{S}$ in a full algebraic space $\boldsymbol{T}$ (see 2.13), mapping bases of $\boldsymbol{S}$ onto bases of $\boldsymbol{T}$.

For example, AG$(3, q)$ and 3-AG(d, q) with $d > 3$ are embeddable but not coordinatizable in PG$(4, q)$ and PG(d, q), resp., (however 3-AG(d, q) is coordinatizable over sufficiently large fields, cf. 6.3). The planar spaces in which the bundle condition (4.9) fails (the smallest ones have 8 points, e.g., the 'twisted cube' or the Vámos DLS) are

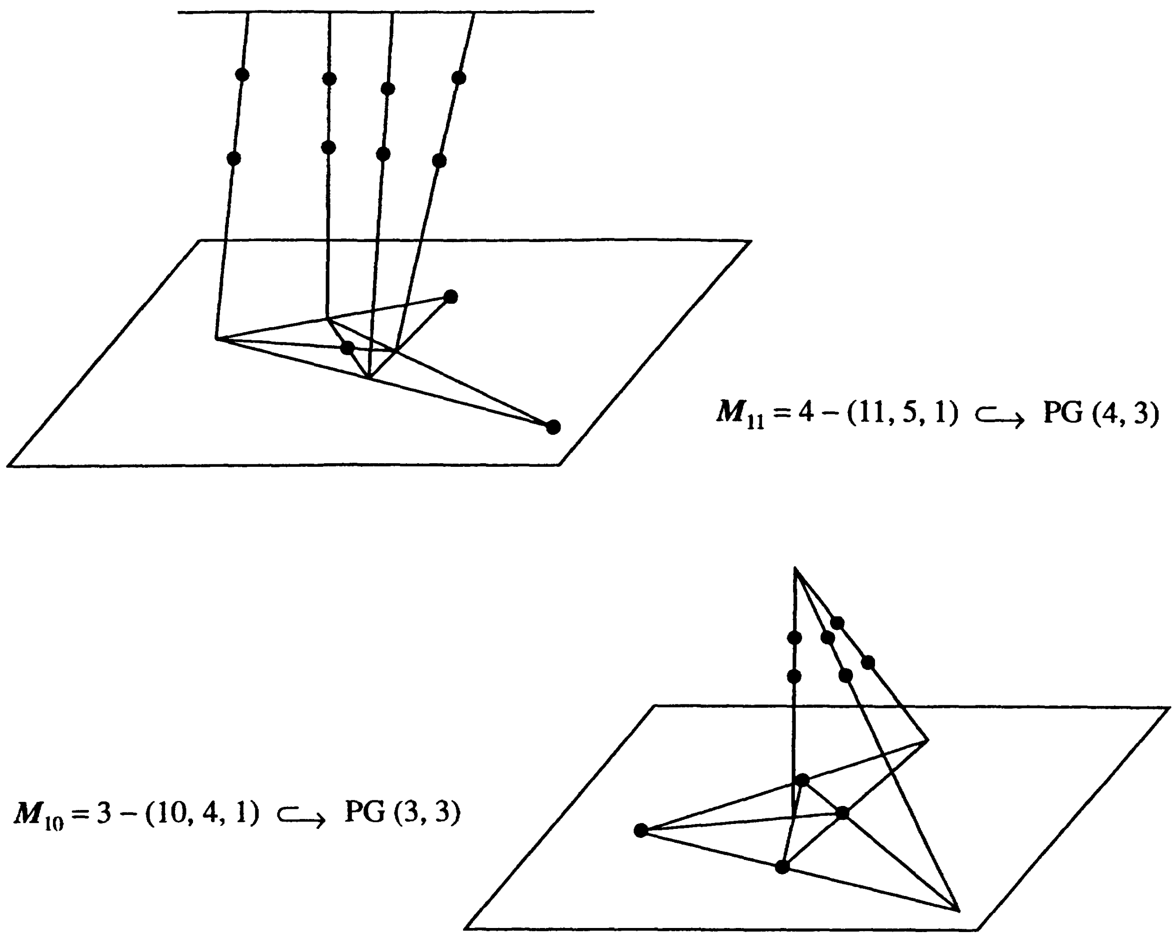


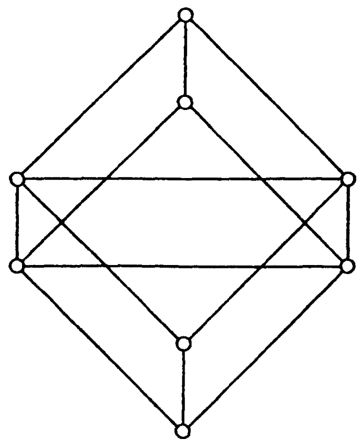
Figure 6.1.

not projectively embeddable, however all linear spaces are projectively embeddable (in a sufficiently large and possibly non-Desarguesian projective plane) (cf. Hall [1943]).

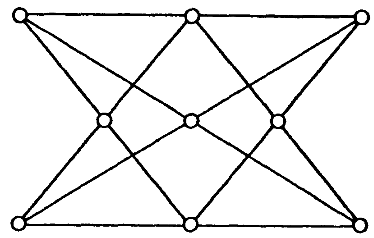
Figure 6.1 suggests a coordinatization of the Mathieu–Witt space M_{11} and its point-residue $M_{10} = \mathrm{IP}(3)$ in PG$(n, 3)$ $(n = 4, 3)$. Figure 6.2 represents three noncoordinatizable n-DLS's $(n = 3, 2)$.

Hall [1943] made the long standing conjecture that any finite linear space can be embedded into a finite projective plane. Wanner and Ziegler [1991] enlarged the class C of host spaces, usually generalized projective (i.e. modular) DLS's in order to guarantee embeddability of any finite n-DLS into a finite n-DLS of C. They proved that for $n = 2$, it suffices to take for C the class of n-DLS that are *modularity complemented* (for $n = 2$ this means that for any point x there is a modular line off x). They also proved that for $n > 2$, C must still be enlarged, e.g., to the wider class of supersolvable n-DLS's (i.e. containing a flag of n modular varieties).

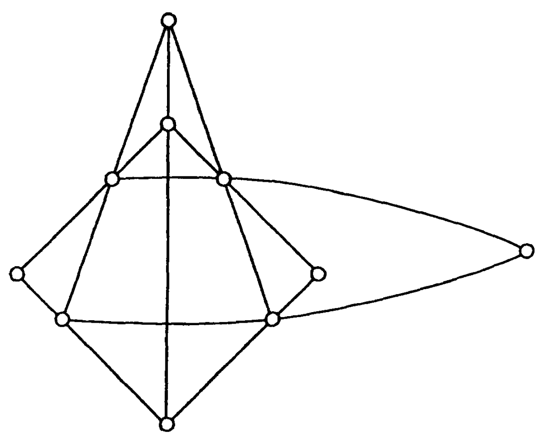
Actually, the problem of coordinatizability was already considered by Whitney [1935] who noticed that PG(2, 2) is coordinatizable over a field F only if $\operatorname{char} F = 2$ and by Mac Lane [1936] who constructed infinite families of noncoordinatizable DLS's based on the failed Desargues or Pappus configurations. The Dowling spaces associated to noncyclic groups provide another class of noncoordinatizable DLS's (see 2.7).



The Vámos 3-DLS
(twisted cube, or failed bundle DLS)



The failed Pappus linear space



The failed Desargues linear space

Figure 6.2.

6.3. *Fundamental properties and references*

Here are some general properties of embeddings:

(i) the product of two embeddings is still an embedding;

(ii) for any variety V of $\boldsymbol{S}$, any embedding (coordinatization) ε of $\boldsymbol{S}$ in $\boldsymbol{T}$ induces embeddings (coordinatizations) of the residues $\boldsymbol{S}^V = \boldsymbol{V}$ and $\boldsymbol{S}_V$ in $\varepsilon(\boldsymbol{V})$ and $\boldsymbol{T}_{\varepsilon(\boldsymbol{V})}$, respectively;

(iii) if $\boldsymbol{S}$ is finite and coordinatizable over F then its simplified dual $\boldsymbol{S}^*$ is also;

(iv) if $\boldsymbol{S}$ is embeddable as a sufficiently small part of PG(d, F), then $\boldsymbol{S}$ is coordinatizable over F (i.e. embeddable in PG(n, F)) (White [1987], Proposition 1.3.3(3)), however the truncations of a large chunk of PG(d, F) are not F-coordinatizable.

Theoretical and technically useful necessary and sufficient conditions for F-coordinatizability, including chain groups, bracket rings and Vámos rings can be found in Welsh [1976], White [1987] and references therein. We now turn to more geometrical considerations.

6.4. *Scum theorem and subgeometries*

Note that the bottom residue of any i-variety V of an m-DLS $\boldsymbol{T}$ can be embedded in the top i-dimensional residue $\boldsymbol{T}_W$ of $\boldsymbol{T}$, where W is any $(m-i-1)$-variety such that V and W generate $\boldsymbol{T}$: it suffices to map each variety X included in V onto the variety $\langle W, X\rangle$ of $\boldsymbol{T}_W$. This simple observation generalizes to the

SCUM THEOREM (Crapo and Rota [1970]). *Let $\boldsymbol{T}$ be an m-*DLS. *If an n-*DLS *$\boldsymbol{S}$ is embeddable in some n-dimensional residue of $\boldsymbol{T}$, then $\boldsymbol{S}$ is also embeddable in some n-dimensional top residue of $\boldsymbol{T}$.*

For $n = 1$ and $\boldsymbol{T}$ finite, this simply means that if $f(i)$ denotes the largest size of the residues of flags of cotype i in $\boldsymbol{T}$, then $f(i)$ is an increasing function of i. The Scum theorem is useful in forbidden configuration theorems (e.g., to be sure that $\boldsymbol{T}$ contains no failed Desargues configuration, it suffices to check the 2- and 3-dimensional top residues, see also 6.5). In view of this theorem it seems natural to investigate DLS's with prescribed top-residues, another major theme of this section (see 6.12, 14).

We name *subgeometry* of $\boldsymbol{T}$ any n-DLS $\boldsymbol{S}$ as in the Scum theorem, or equivalently any DLS which is isomorphic to some restriction of some residue $\boldsymbol{T}_V^W$ (shorthand $\boldsymbol{W}_V$). For example, every F-coordinatizable n-DLS is a subgeometry of PG(n, F) (hence of PG(m, F) for any $m \geqslant n$), but truncations of subgeometries are not necessarily subgeometries, so that embeddability in $\boldsymbol{T}$ is not sufficient for being a subgeometry. The matroidal terminology is *minor* for subgeometry and *strong map* for the map from $\boldsymbol{S}$ to $\boldsymbol{T}$ corresponding to the embedding of $\boldsymbol{S}$ in $\boldsymbol{T}_V^W$. (Note however that the notion of subgeometry differs from that of submatroid or sublattice; for surveys on strong maps, and more generally, on weak maps, see White [1986]. In Duchamp [1989], these notions are extended to finitary and cofinitary matroids.)

6.5. *Forbidden subgeometry theorems (Tutte, Vámos, etc.)*

As noted in 6.2, some DLS's can never show up as subgeometries of coordinatizable DLS's, e.g., the Vámos 3-DLS, the failed Desargues linear space and, if we consider commutative fields only, the failed Pappus linear space (Figure 6.2). These prevent coordinatization over any commutative field. Some other ones prevent coordinatization over specific fields. For example, if in an F-coordinatizable DLS four points have three diagonal points, then these three points are collinear iff the characteristic of F is 2. By contrast, the Fano plane PG(2, 2) and the non-Fano plane (see 1.3) can coexist in algebraic DLS's.

In 1958, Tutte gave a forbidden subgeometries characterization of the finite DLS's which are coordinatizable over every field, and of the GF(2)-coordinatizable ones. His results led to the following

THEOREM. *Let $\boldsymbol{S}$ be a finitary* DLS.

(i) *$\boldsymbol{S}$ is coordinatizable over a finite field F iff every finite subgeometry of $\boldsymbol{S}$ is F-coordinatizable* (Piff [1971]).

(ii) *S is coordinatizable over all fields (i.e. S is unimodular) iff none of its subgeometries is the 4-point line* PG(2, 2), *or its dual* PG(2, 2)* (Tutte [1958]).

(iii) *S is* GF(2)-*coordinatizable (i.e. S is binary) iff none of its subgeometries is the 4-point line, iff none of its colines is in more than 3 hyperplanes* (Tutte [1958]).

(iv) *S is* GF(3)-*coordinatizable (i.e. S is ternary) iff none of its subgeometries is the 5-point line or its dual, the* 2-(5, 2, 1) *design,* PG(2, 2) *or its dual* PG(2, 2)* (Bixby [1977], Seymour [1979]).

(v) *S is graphic (resp., cographic) iff S is unimodular and none of its subgeometries is Π_5^* or $M(K_{3,3})^*$ (resp., Π_5 or $M(K_{3,3})$)* (cf. 2.8) (Tutte [1958]).

Tutte's and Bixby's proofs are based on Tutte's homotopy theorem (see, e.g., White [1987]) which can now be avoided (a remarkably nice proof of (ii) is offered by Gerards [1989]).

Note however that Tutte's homotopy theory (Tutte [1965]) has been used and reinterpreted in algebraic form by Dress and Wenzel [1989], who associated with any finite DLS S a certain Abelian group, called *Tutte's group* of S, which controls the coordinatizability, as well as the binarity, ternarity, unimodularity and orientability of S. In this framework Wenzel [1989] provides a new and rather conceptual proof of (iv). See also Dress and Wenzel [1988, 1990, 1991], Wenzel [1991], Dress [1986], and references given therein.

The language of *chain groups* (see, e.g., Welsh [1976] and 2.12) is frequently used when dealing with coordinatizability problems. The DLS's in (ii) are often called *unimodular* because they are precisely those whose v points can be represented in $\mathbb{Q}^{n+1}$ by the columns of a $(n+1) \times v$ totally unimodular matrix (i.e. whose every square submatrix has determinant 0 or ± 1). We prefer the epithet 'unimodular' rather than *regular* to avoid confusion with the DLS's having constant parameters (3.1). Note that S is unimodular iff it is GF(2)- and GF(3)-coordinatizable; more generally it can be proved that S is unimodular iff there is a field F with $\operatorname{char} F \neq 2$ such that S is both GF(2)- and F-coordinatizable. Binary, ternary and unimodular DLS's were studied extensively: see, e.g., White [1987] and the recent papers by Lee [1989], Duchamp [1989], Oxley and Reid [1990], and also Sections 6.6 and 6.7.

Given a finite field GF(q), does there exist a finite set of *obstructions* to GF(q)-coordinatizability, i.e. a finite family $\boldsymbol{F}_q$ of DLS's such that a finite DLS is GF(q)-coordinatizable iff none of its subgeometries belongs to $\boldsymbol{F}_q$? This fundamental problem is open for $q \geqslant 4$. It is worth mentioning that an important tool in the proofs of Theorem 6.5 is the uniqueness of GF(q)-coordinatization, which does not hold in general when $q \geqslant 4$, see Section 6.11.

Anyway there is no global forbidden subgeometry characterization of coordinatizability.

PROPOSITION (Vámos [1978]). *There is no sentence in the first order language of* DLS'*s theory and no finite list of excluded finite subgeometries characterizing those finitary* DLS'*s which are coordinatizable over a field (resp., over a skew field, over a field of characteristic zero).*

The proof is based on results on the (skew) characteristic set of a DLS and on the finite character of coordinatizability (see 6.8).

6.6. *Seymour's decomposition theorem*

Tutte's characterization of unimodular DLS's (6.5) now appears as a corollary of Seymour's decomposition theorem stating that the finite unimodular matroids are those which can be decomposed via operations closely related to 1-, 2- and 3-separations (cf. 1.14) into graphic matroids, cographic matroids and copies of $\boldsymbol{R}_{10}$. The 4-DLS $\boldsymbol{R}_{10}$ belongs to the diagram

,

it is the restriction of AG(4, 2) to the complement of any set of 6 vectors with null sum. Its point-residues are Minkowski planes of order 2. We refer the reader to Seymour [1980], to Kahn [1985], to the survey in White [1987] and to Truemper [1985–1988] for further decomposition theorems and their applications, e.g., a polynomial algorithm for recognizing whether a matrix is totally unimodular, linear programming and a characterization of space-filling zonotopes for the n-dimensional Euclidean space E^n.

6.7. *Subgeometry-closed and hereditary classes of DLS's*

A class C of DLS's is called *subgeometry-closed* if every subgeometry of every element of C belongs to C. Obvious examples are provided by the class of embeddable (resp., coordinatizable, F-coordinatizable, unimodular, ...) DLS's as well as by any class for which there is an excluded subgeometry characterization. Quite a few papers investigate some of these classes. Let us mention a few results in this vein.

First of all, let S be an n-DLS on v points, so that trivially $v \leqslant (q^{n+1}-1)/(q-1)$ if S is GF(q)-coordinatizable.

(1) (Heller [1957]) If S is both binary and F-coordinatizable for some field F with characteristic $\neq 2$, then $v \leqslant \binom{n+2}{2}$.

(2) (a) (Kung [1990a]) If S is both ternary and F-coordinatizable for some field F with characteristic $\neq 3$, then $v \leqslant (n+1)^2$.

(b) (Kung and Oxley [1988]) If moreover char $F \neq 2$, then $v = (n+1)^2$ iff S is the Dowling geometry $\boldsymbol{Q}_{n+1}(G)$, where G denotes the cyclic group of order 2.

(3) (Kung [1990b]) If S is both GF(q)- and F-coordinatizable, where $q > 2$ and char $F \mid q$, then

$$v \leqslant \left(q^e - q^{e-1}\right)\binom{n+2}{2} - (n+1), \quad \text{where } e = 2^{q-1} - 1.$$

(4) (Kung [1987a]) If S is binary and does not admit the Desargues configuration $\boldsymbol{\Pi}_5$ (see 2.6) as a subgeometry, then $v \leqslant 8n$.

Further results in this vein can be found in the above mentioned papers and in Kung [1986b, 1988]. Further excluded subgeometry characterizations can be found in Oxley [1989, 1990].

A class C of DLS's is called *hereditary* if C contains at least one 1-DLS, is subgeometry-closed and closed under direct summation. Kung [1986c] proved a few properties of hereditary classes. For example, define $h(x)$ to be the maximum number of points of an n-DLS in C and define C to have *rapid growth* if the growth function $g(n) = h(n) - h(n-1)$ is unbounded. Kung [1986d] proved that if an hereditary class C enjoys rapid growth and has a finite maximal line-size, then C has, for any natural number n, an *n-dimensional universal model*, i.e. an n-DLS $U_n \in C$ such that every n-DLS in C is a subgeometry of U_n.

To the contrary, note that the hereditary class of all DLS's and that of unimodular DLS's have no sequence (U_n) of universal models (this situation differs from the one in universal algebra). Kahn and Kung [1982] determined the hereditary classes of finite DLS's having an n-dimensional universal model for any n: these are the GF(q)-coordinatizable DLS's for some fixed q, the so-called *voltage-graphic* DLS's with voltages in a fixed finite group G, and a few degenerate cases. They are best defined by their sequences (U_n) of universal models, to wit:

(i) $U_n = \mathrm{PG}(n, q)$ (q a fixed prime power);

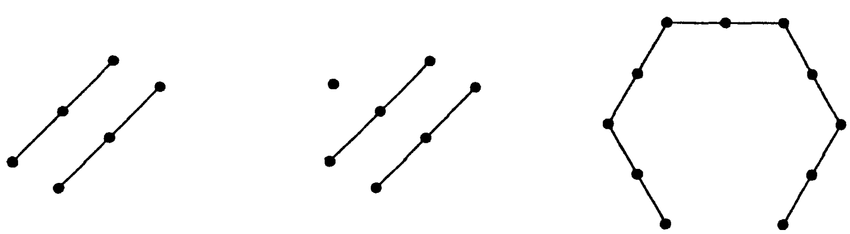
(ii) $U_n = Q_{n+1}(G)$ (*Dowling space*, G a fixed group, see 2.7);

(iii) $U_n = \mathrm{PG}(n, 1)$ (*Boolean space*);

(iv) $U_n = M_{n+1}(q)$ (*full Matchstick space*, q any fixed integer);

(v) $U_n = O_{n+1}(q)$ (*full Origami space*, q any fixed integer).

The full Matchstick space $M_{2n}(q)$ is the direct sum of n lines of size $q+1$, while $M_{2n+1}(q)$ is the direct sum of a point and n lines of size $q+1$. Hence $M_n(q)$ is a 2-closed $(n-1)$-DLS. Since the full Origami space $O_n(q)$ is also a 2-closed $(n-1)$-DLS, we only need to describe its 2-truncation 2-$O_n(q)$. The linear space 2-$O_n(q)$ consists of n distinct lines $L_1, \ldots, L_{n-1}$ of size $q+1$ placed in a chainlike fashion, i.e. L_i and L_j intersect iff $j = \pm i$. See Figure 6.3.



2-$M_4(3)$ 2-$M_5(3)$ 2-$O_6(3)$

Figure 6.3.

Kahn and Kung note that their finiteness assumption is crucial: on the one hand the uniqueness of universal models might fail since if the field F is isomorphic to some subfield of F, then PG(n, F) is embeddable in AG(n, F), and on the other hand new hereditary classes would show up, including algebraic DLS's, whose natural universal models are not even supersolvable (cf. 2.9).

6.8. *Characteristic set*

The *(skew) characteristic set* $\chi(s)$ of a DLS S is the set of characteristics of all (skew) fields coordinatizing S. Let Π be the set whose elements are 0 and the prime numbers. The problem of determining which subsets of Π are characteristic sets of finite DLS's was solved by the work of Rado, Vámos, Reid, Mason and Kahn, but the similar problem for skew characteristic sets is not completely settled.

THEOREM (Kahn [1982a]). *A subset χ of Π is the characteristic set of some finite* DLS *iff it is the characteristic set of some finite linear space iff either χ or $\Pi \setminus \chi$ is finite and excludes* 0.

PROPOSITION (Vámos [1971], see also Piff [1971] and Leader [1989]). *Let S be a finitary* DLS.

(a) *If S is coordinatizable over fields of arbitrarily large characteristic, then S is coordinatizable over a field of characteristic zero.*
(b) *If, given a field F, every finite subset of S is coordinatizable over an extension field of F, then S is coordinatizable over an extension field of F.*

Note that the F-coordinatizability of every finite subset of S does not force the F-coordinatizability of S, as shown by any trivial 2-DLS S on an uncountable set: all finite subsets of S are $\mathbb{Q}$-coordinatizable but S is not.

The early paper Mac Lane [1936] assigns a DLS to any field algebraic over its prime field and shows that the fields which are transcendental over their prime field are not necessary – although they are useful – for coordinatization of finite DLS's. More precisely: K being a prime field,

(i) if L is a finite algebraic extension field of K, then there is an L-coordinatizable linear space which cannot be F-coordinatized unless F is an extension of L,
(ii) if a finite DLS S is coordinatizable over some extension field of K, then there is an algebraic extension L of K over which S can be coordinatized.

Let us now turn to another notion of coordinatization, introduced by Mac Lane [1938].

6.9. *Algebraicity versus coordinatizability*

Remember that an n-DLS S is *algebraic* (resp., *F-algebraic*) iff it is embeddable into some n-dimensional full algebraic space (resp., FAlg(n, F)). Nonalgebraic DLS's can be easily constructed from Theorem 2.13: the failed bundle (or Vámos) DLS and the failed Desargues DLS are nonalgebraic (as well as noncoordinatizable). Algebraicity does not force coordinatizability: Ingleton [1971] noticed that FAlg(2, GF(2)) contains both the Fano and non-Fano configurations which are, respectively, coordinatizable only over fields of characteristic 2 and only over fields of characteristic distinct from 2. Moreover the failed Pappus space (Figure 6.2) is algebraic over every finite field though noncoordinatizable over any field (Lindström [1986b], see also the p-polynomial embeddings for nonprime fields in 2.13 and Lindström [1983] for an explicit representation over GF(p^2)).

Algebraicity and coordinatizability are nevertheless closely related as the next theorem shows.

THEOREM. *Let S be a finite n-DLS.*

(1) *If S is F-coordinatizable, then S is F-algebraic.*

(2) *If* $\operatorname{char} F = 0$ *and if S is F-algebraic, then S is coordinatizable over some finite transcendental extension of F.*

(3) *If S is algebraic over some transcendental or algebraic extension of F, then S is F-algebraic.*

(4) *If S is F-algebraic with* $\operatorname{char} F = p$ *(resp.,* 0), *then S is* GF(p)- *(resp.,* $\mathbb{Q}$-) *algebraic.*

The proof of (1) is easy (see, e.g., Welsh [1971] or 2.13): mapping the vectors $\sum a_i e_i$ of F^n onto the elements $\sum a_i x_i$ of the transcendental field extension $F(x_1, \ldots, x_n)$ of degree n defines an embedding of PG($n-1, F$) into FAlg($n-1, F$). The proof of (2) is more sophisticated: it uses the notion of derivation of fields in order to convert algebraic dependence into linear dependence (see, e.g., Lang [1965], Chapter 10, Proposition 10). For instance, if $\operatorname{char} F = 0$ and if K is the field of rational functions over F in n variables, then m functions belonging to K are algebraically independent iff their gradients are linearly independent over K. (3) and its immediate corollary (4) are announced in Lindström [1988a] and proved in Lindström [1989].

6.10. *Algebraic characteristic set*

The *algebraic characteristic set* $\chi_A(S)$ of a DLS S is the set of characteristics of all fields over which S is algebraically representable. We have seen that $\chi(S) \subseteq \chi_A(S)$ and that $0 \in \chi(S)$ iff $0 \in \chi_A(S)$. Hence if $\chi_A(S)$ contains 0 then it also contains almost all primes, however $\chi_A(S)$ may be infinite and exclude 0 (failed Pappus space), or be finite (e.g., $\chi_A(\mathrm{PG}(2,2)) = \{2\}$, see White [1987] or Lindström [1985a, 1986a, 1988a] for other examples).

6.11. *F-unique embeddability*

An n-DLS S is *uniquely* GF(q)*-coordinatizable* if it has a GF(q)-coordinatization γ which is unique up to collineations of PG(n, q) (i.e. for every other GF(q)-coordinatization γ', there is a collineation α of PG(n, q) such that $\gamma\alpha = \gamma'$). Such a property is of independent interest, e.g., it forces the automorphism group of S to extend to a collineation group of PG(n, q). The first two propositions below follow now from well-known transitivity properties of PΓL($n+1, q$) and were proved by Brylawski and Lucas [1970]; (iii) is due to Kahn [1988].

THEOREM.

(i) *Every finite* GF(2)- *and* GF(q)*-coordinatizable* DLS *is uniquely* GF(q)*-coordinatizable.*

(ii) *Every finite* GF(3)*-coordinatizable* DLS *is uniquely* GF(3)*-coordinatizable.*

(iii) *A finite* GF(4)*-coordinatizable* DLS S *is uniquely* GF(4)*-coordinatizable iff* S *does not have a* 2*-separation with both parts non-*GF(2)*-coordinatizable.*

Note that the proof of (iii) uses the fact that in a 3-connected binary matroid, any 2 points are in a 4-point line subgeometry (Seymour [1981], see also Oxley [1985] for a nice generalization and Kahn [1984] where (ii) is proved using 6.13).

6.12. *Strongly locally projective (or affine) n-DLS's* ($n \geqslant 4$)

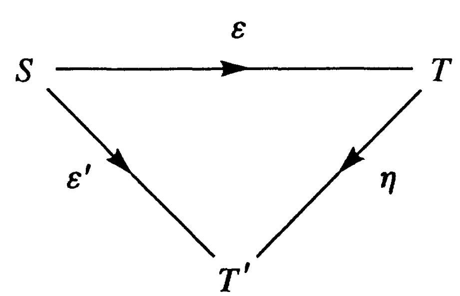
The disadvantage of coordinatizations and algebraic representations in general is that we are often led to embed a small DLS into some very huge one. In Sections 6.12–15 however we deal with tighter embeddings.

By 4.6 and 4.8, the finite regular DLS's all of whose point-residues are generalized projective or affine spaces of dimension $d > 2$ are generalized projective or affine spaces. Consequently the same conclusion holds if we replace 'point-residues' by 'top d-dimensional residues'. Let us now drop the finiteness and regularity assumptions.

Any n-DLS with $n > 3$ all of whose point-residues are generalized projective spaces (resp., projective spaces) is embeddable in a generalized projective space of the same dimension n (resp., coordinatizable over a skewfield), as follows immediately from 4.7. Moreover the embeddings provided are 'minimal' in the following sense: the point-residues of the embedded spaces coincide with those of the host space. The two major papers Kantor [1974a, 1975] (see also the survey Kantor [1976]) generalize this result, showing in particular that if the point-residues of S can be embedded as sufficiently large chunks of projective spaces, then S itself is a rather large chunk of a projective space. In the special case where all point-residues of S are affine spaces, S was already proved to be coordinatizable by Mäurer [1968], but this enabled Thas [1974] to deduce the nonexistence of S in the finite case. Let us emphasize the fact that the dimension of S is greater than 3 in most results here, since planar spaces behave differently and are treated separately in Sections 4 and 5.

6.13. *F-unique, universal or rigid embedding*

One way to specify the meaning of 'large chunk' is to introduce the notions of universality and rigidity of embeddings. If C is a class of embeddings, call *C-embedding* any member of C. A C-embedding ε: $S \to T$ is *C-unique* (resp., *C-universal*) if for any C-embedding ε': $S \to T'$, there is a (resp., one and only one) C-embedding η: $T \to T'$ making the following diagram commutative:



The coordinatizability of AG(2, 3) in both PG(2, 4) and PG(2, 3), which itself cannot be embedded in PG(2, 4), illustrates why uniqueness and universality are usually restricted to particular classes of embeddings. C is often chosen to be the class of all embeddings (resp., coordinatizations) into finite-dimensional projective spaces over a given field F, and we then speak of *F-unique* (or *F-universal*) embeddings (resp., coordinatizations). However, an F-unique coordinatization is not necessarily an F-unique embedding (embed $S(2,2,4)$ as four (coplanar) points in PG(2, 2) and as four noncoplanar points in PG(3, 2)). An embedding ε of S in PG(d, F) is *F-rigid* if and only if the identity is the only automorphism of PG(d, F) fixing $\varepsilon(S^0)$ pointwise. A DLS S which is embeddable as a generating subset of a projective F-space T is called *F-rigid* if for all such embeddings ε, the identity is the only collineation of T inducing the identity on $\varepsilon(S^0)$.

PROPOSITION (Kantor [1975]).

(i) *If* $\dim S \geqslant 2$ *and* ε: $S \to T$ *is an F-universal embedding, then* $\varepsilon(S)$ *spans* T.

(ii) *If, moreover, F is isomorphic to no proper subfield of itself, then S is F-rigid.*

Hence, under the above assumptions on F and $\dim S$, ε is F-universal iff ε is F-unique and S is F-rigid. However, if F has a proper subfield isomorphic to itself, then $S \cong$ PG(n, F) can be properly embedded in some $S' \cong$ PG(n, F), so that the identity on S is an F-universal embedding (of S into itself) while S' can have nontrivial collineations inducing the identity on $S \subset S'$.

F-uniqueness appears here for convenience but differs from the 'large chunk' notion, since any F-embedding of PG(n, 1) and of PG(n, 2) is F-unique (if n is finite and F has no subfield isomorphic to F), as seen from the transitivity properties of PGL($n+1, F$). As in 6.11, S is said to be *uniquely F-coordinatizable* if all its F-coordinatizations are F-unique (or equivalently if F has no proper subfield isomorphic to F, if S has an F-unique coordinatization).

6.14. *Kantor's embedding theorems*

The main result of Kantor [1975] is the following.

THEOREM. *Let F be a field isomorphic to no proper subfield of itself, let j and n be integers such that $1 \leqslant j \leqslant n-2$ and let S be an n-DLS. If every top-residue of dimension $j+1$ or $j+2$ has an F-universal embedding and if every top-residue of dimension j is F-rigid (and F-coordinatizable if $j=1$), then S has an F-universal embedding.*

COROLLARY 1. *If S is a finite n-DLS (with $4 \leqslant n$) all of whose top 1-, 2- and 3-dimensional residues are rigidly* GF(q)*-coordinatizable, then S is rigidly* GF(q)*-coordinatizable.*

The situation becomes remarkably simple over GF(p), p prime, since every GF(p)-coordinatization of a *top-thick* DLS (i.e. which is not the union of two hyperplanes) is necessarily rigid (Kantor [1979]).

COROLLARY 2. *Any finite top-thick n-DLS (with $n \geqslant 4$) all of whose top 3-dimensional residues are* GF(p)*-coordinatizable is* GF(p)*-coordinatizable.*

Let us mention that the first result of this type appeared in Kantor [1974a] (Theorem 2) and is contained in the above theorem, except that it also allows top 2-dimensional residues of S to be degenerate projective planes (but then Theorem 4.6 can be used). There the basic notion is Kantor's *strong embedding* which is indeed a very restrictive class of embeddings of DLS's whose top 2-dimensional residues are affine or generalized projective planes that coincide with the corresponding residues in the host space (up to a line in the case of affine planes). Finally note that, unlike universal or strong embedding theorems, Wyler's result 4.7 also deals with 4-DLS's whose point-residues are 3-truncations of projective spaces of possibly infinite dimension.

6.15. *Teirlinck's sufficient condition for F-universality*

Teirlinck [1982b] gives the following sufficient condition for F-universality, which does not involve all the other F-embeddings of S (again, we are back to the idea of a 'large chunk' of a projective space). Given $n, q \geqslant 2$ and a class C of Desarguesian projective spaces containing PG(n, q), denote by $\mu(C, n, q)$ the smallest integer m such that for every n-DLS S having at least m points, all C-coordinatizations are C-universal embeddings. Then

$$2q + 3 \leqslant \mu(C, 2, q) \leqslant \left(q^2 + 3q\right)/2,$$

and

$$\sum_{i=3}^{n-1} q^i + q^2 + 2q + 2 \leqslant \mu(C, n, q) \leqslant \sum_{i=3}^{n-1} q^i + 3\left(q^2 + q\right)/2 \quad \text{if } n \geqslant 3.$$

Let us close this section by noting that such embeddability theorems do not settle the existence problem of such DLS's but reduce the problem to the investigation of substructures of projective spaces (see, e.g., the nonexistence of finite DLS's whose point-residues are affine spaces of dimension $\geqslant 3$, cf. 6.12). Let us also mention the work of Bachem and Kern [1986a, 1988].

7. Automorphisms

7.1. *Introduction*

The most transitive automorphism groups of combinatorial or geometrical structures (designs, DLS's and many kinds of graphs) play an important role in group theory (see, e.g., Kantor [1969a,b, 1974b] and Tits [1962a, 1964] for founding papers, and Babai [1981], Cameron [1979, 1994] for more recent surveys). However our aim here is not group theory and most of this section will be reversing the process: DLS's will no longer be a tool for investigating groups but to the contrary the classification of finite simple groups will be used to determine the most highly transitive DLS's. Infinite permutation groups behave quite differently (see, e.g., Cameron [1990], and 7.4 for one illustration) and very little is known about highly symmetric infinite DLS's, the major exception being Theorem 7.5 (where some 'local' finiteness is nevertheless assumed).

Recall that the results on automorphism groups and high transitivity of Benz planes were presented separately in Sections 5.15 to 5.27.

An automorphism of a DLS $S = (\boldsymbol{P}, \boldsymbol{C})$ is a permutation on its point-set $\boldsymbol{P}$ inducing a permutation on its set $\boldsymbol{C}$ of varieties. In particular an automorphism g of an n-DLS $S = (S^i;\ i = 0, \ldots, n-1)$ defines n permutations, one on each S^i. Each of these n actions of g uniquely defines the permutation g of $\boldsymbol{P} = S^0$ since for any i, every point is an intersection of i-varieties. We denote by $\operatorname{Aut} S$ the full automorphism group of S. We have seen that the action $G(S^i)$ of G on the i-varieties is faithful. If X is a subset of the point-set $\boldsymbol{P}$, then G_X denotes the set-stabilizer of X in G and $G_X(X)$ denotes the action of G_X on the point-set X. Let S be a DLS and let $G \leqslant \operatorname{Aut} S$. If G acts transitively on the set of all configurations of type x, we shall say that G and the pair (S, G) are *type x-transitive*. For short we shall call *type x-transitive* any DLS whose full automorphism group is type x-transitive.

7.2. *Abstract groups and permutation groups as automorphism groups of DLS's*

Babai [1978] proved the following result, as well as other ones in the same vein found in Babai [1981].

THEOREM. *For any abstract group G and any integer $n \geqslant 2$, there is a cardinal number $f(|G|, n)$ which is finite if $|G|$ is finite and which coincides with $|G|$ if $|G|$ is infinite, such that there is an n-DLS S whose full automorphism group is isomorphic to G and which is coordinatizable over any field F of order $\geqslant f(|G|, n)$.*

However, given a permutation group $(G, \boldsymbol{P})$ on a set $\boldsymbol{P}$ and an integer $n \geqslant 2$, if often occurs that $(G, \boldsymbol{P})$ is n-DLS *primitive* (paraphrasing Cameron [1979]), i.e. there is no nontrivial n-DLS on $\boldsymbol{P}$ admitting $(G, \boldsymbol{P})$ as automorphism group. This is illustrated by Kantor's classification of finite n-transitive n-DLS imprimitive groups (see 7.3). Note that *n-homogeneous* permutation groups (i.e. those transitive on n-sets) where first used to construct n-DLS's: if $(G, \boldsymbol{P})$ is n-homogeneous on the v points of $\boldsymbol{P}$ and if the stabilizer of any n-tuple of points fixes exactly k points with $n < k < v$, then the G-orbit of this k-set is the block-set of a nontrivial n-$(v, k, 1)$ design. Witt [1938a,b] generalized this construction in order to derive the Witt designs from the Mathieu groups and the WLA planar spaces of Theorem 7.3 from $\mathrm{PSL}(2, q)$. However such constructions fail on infinite sets since it can occur that an n-point stabilizer in an n-transitive group properly contains another (Cameron [1979], see also Cameron [1990]).

7.3. *n-homogeneous n-DLS's*

Let G be an n-homogeneous permutation group on a finite set $\boldsymbol{P}$ of v points. $(G, \boldsymbol{P})$ is n-DLS *imprimitive* provided there is a nontrivial n-DLS S on $\boldsymbol{P}$ preserved by G. In this case S is a hypercircular n-DLS with constant hyperplane-size k (i.e. an n-$(v, k, 1)$ design).

Since S is nontrivial, the stabilizer of any n-set N should have an orbit included in the nonempty set $\langle N\rangle \setminus N$, whose length must therefore be less than $k-n$, which in turn is strictly less than $\sqrt{v-n}$. This inequality is obtained by noting that the 2-dimensional top-residues of S are nontrivial linear spaces 2-$(v-n+2, k-n+2, 1)$, so that $v-n+2 \geqslant (k-n+1)^2+(k-n+1)+1$, where the right-hand side is the number of points of a projective plane of order $k-n+1$. This is a crucial argument in Kantor [1985].

THEOREM (Kantor [1985], slightly improved by using Kantor [1972]). *Let S be a finite nontrivial n-DLS with $n \geqslant 3$. If G is n-homogeneous on the point-set of S, then one of the following holds*:

(i) $S=$ 3-AG$(d,2)$ *with* $d \geqslant 3$ *and* $G=\mathrm{AGL}(d,2)$, *or* $d=3$ *and* $G=\mathrm{AGL}(1,8)$ *or* $\mathrm{A\Gamma L}(1,8)$ *or* $d=5$ *and* $G=\mathrm{A\Gamma L}(1,32)$;

(ii) *S is the* WLA *planar space* 3-$(q^d+1, q+1, 1)$ *on the projective line* $\mathrm{PG}(1,q^d)$ *with a projective subline* $\mathrm{PG}(1,q)$ *as base-block, and* $\mathrm{PSL}(2,q^d) \leqslant G \leqslant \mathrm{P\Gamma L}(2,q^d)$ *and* $q \geqslant 3$;

(iii) *S is the locally Netto circular space* 3-$(q+1,4,1)$ *on the projective line* $\mathrm{PG}(1,q)$ *with* $\{\infty\} \cup K$ *as base-block, where K is the set of all third roots of unity in* $\mathrm{GF}(q)$, $\mathrm{PSL}(2,q) \leqslant G \leqslant \mathrm{P\Gamma L}(2,q)$, $G_\infty \leqslant \mathrm{A\Gamma^2L}(1,q)$ *and* $q \equiv 7 \pmod{12}$;

(iv) *S is a Mathieu–Witt design* 3-$(22,6,1)$, 4-$(11,5,1)$, 4-$(23,7,1)$, 5-$(12,6,1)$ *or* 5-$(24,8,1)$, *and* $\mathsf{M}_v \leqslant G \leqslant \mathrm{Aut}\,\mathsf{M}_v$ (*where v denotes the number of points*).

Before turning to other high transitivity conditions, let us mention some relations between orbit numbers, which also provide implications between certain transitivity conditions.

7.4. *Orbit numbers*

Applying Block's lemma to Proposition 3.9 and to Kung's result in 3.5, we get the following.

PROPOSITION. *Let S be a finite n-DLS, let $G \leqslant \mathrm{Aut}\,S$ and denote by β_i the number of G-orbits on the set S^i of i-varieties, then, for any j with $0 \leqslant j \leqslant (n-2)/2$,*

(i) $\beta_0+\beta_1+\cdots+\beta_j \leqslant \beta_{n-j-1}+\cdots+\beta_{n-2}+\beta_{n-1}$,

(ii) *if S is regular, then $\beta_i \leqslant \beta_j$ for any $i<j$.*

Note that the regularity assumption in (ii) can be weakened to a local regularity assumption as in 3.5. In particular we recall that in a finite DLS, hyperplane transitivity forces point transitivity. We emphasize the fact that this is no longer true in infinite DLS's, as shown by the n-DLS induced by $\mathbb{R}^n$ on a closed ball, which has two point-orbits, namely the interior and the boundary. Therefore in all what follows we will always assume S to be *finite*. We refer the reader to Cameron [1984] for some suggestions about the infinite cases and to Cameron [1990] for a first approach to the strange world of permutation groups on a countable set Ω having only finitely many orbits on Ω^n, for any positive integer n.

7.5. *Jordan groups*

One of the most severe transitivity conditions on $G \leqslant \operatorname{Aut} S$ is *Jordan-transitivity*, requiring that the pointwise stabilizer of any finite-dimensional variety V of S be transitive on the complement of V. If $\dim S$ is finite, this forces G to act transitively on the ordered bases of S. Examples of Jordan-transitive DLS's are the three large Mathieu–Witt designs 3-(22, 6, 1), 4-(23, 7, 1) and 5-(24, 8, 1), the Desarguesian projective or affine spaces, and the (see 2.13).

This transitivity condition comes from the following notions. A *Jordan set* of a permutation group G on $\boldsymbol{P}$ is a set J of (at least two) points such that the pointwise stabilizer of its complement $\boldsymbol{P} \setminus J$ is transitive on J. If G is n-transitive, then the complement of any $(n-1)$-subset of $\boldsymbol{P}$ is a Jordan set, called an *improper* Jordan set. A *Jordan group* is a permutation group on $\boldsymbol{P}$ having a proper Jordan set. The automorphism group of a Jordan-transitive nontrivial finitary DLS is a Jordan group, the Jordan sets being the complements of the varieties. Conversely, calling *variety* any complement of a Jordan set of a Jordan group G on $\boldsymbol{P}$ defines a finitary pre-DLS S if $\boldsymbol{P}$ is finite (Hall [1960], Kantor [1969b]), and more generally if every finite subset of $\boldsymbol{P}$ is contained in the complement of a cofinite Jordan set (Neumann [1985]). Moreover if G is primitive, then S is a DLS.

No classification of Jordan groups (nor of Jordan-transitive DLS's) is available; in spite of many partial results, starting with Jordan's work. Here is the best result for DLS's.

THEOREM (Zil'ber [1981], Kantor [1985]). *Any nontrivial Jordan-transitive finitary* DLS *all of whose finite-dimensional varieties are finite is a truncation of a Desarguesian projective or affine space over a finite field or one of the Mathieu–Witt designs* 3-(22, 6, 1), 4-(23, 7, 1) *or* 5-(24, 8, 1).

The finite part of this theorem follows from Kantor [1985] and its proof relies on the classification of finite 2-transitive groups. However the infinite part was proved by Zil'ber [1981] and later by Evans [1986] and Zil'ber [1988] without assuming the classification of the finite simple groups. The latter proofs are geometric and combinatorial, much of them dealing only with the finite part of the theorem, providing an elementary proof for finite DLS's of dimension $\geqslant 6$. Evans' proof relies on one of the deepest results on highly transitive finite DLS's that do not require the heavy machinery of the classification of finite simple groups, namely the theorem of Cameron and Kantor [1979] on the 2-transitive and antiflag-transitive collineation groups of finite projective spaces. Although the finite primitive Jordan groups are known (Kantor [1985], Neumann [1985]), this is not the case for the infinite ones (Neumann [1985], Remark 3).

Remarkably enough Jordan groups closely connect geometry and model theory, since Zil'ber's result was proved independently in this latter context by Cherlin, Harrington and Lachlan [1985]. More on this can be found in Cameron [1990], while a generalization of the above theorem, leading to objects having more structure than DLS's (essentially the vector spaces are allowed to carry sesquilinear forms of classical type) is given in Kantor, Liebeck and McPherson [1989].

7.6. *Geometric groups*

A *geometric group* is a Jordan group G whose action on the ordered bases of the associated pre-DLS $S(G)$ is sharply transitive and such that the fixed-point-set of any of its elements is a variety of $S(G)$. The only finite primitive geometric groups are $\mathrm{PSL}(d+1,2)$, $\mathrm{AGL}(d,q)$, A_7, $2^4 \cdot A_7$ and any sharply $(d+1)$-transitive group of degree v, the associated DLS's being $\mathrm{PG}(d,2)$, $\mathrm{AG}(d,q)$, 2-PG(3,2), 3-AG(4,2) and the trivial d-DLS on v points (Cameron and Deza [1979] for the explicit statement, Kantor [1975] for the proof). This notion of geometric group leads in a natural way to that of *permutation geometry* (Cameron and Deza [1979]). This all fits into the framework of *squashed geometries*, a nice generalization of pre-DLS's (see Section 8).

7.7. *Flag-transitive n-DLS's*

Given three distinct non-negative integers $i, j, l \leqslant n-1$, call (i,j)*-flag* (resp., (i,j,l)*-flag*) any flag consisting of varieties of dimensions i and j (resp., i, j and l).

Working with truncations of residues of varieties, one easily derives from 7.4 that (i,j)-flag-transitivity (i.e. transitivity on the flags of type (i,j), for $0 \leqslant i < j \leqslant n-1$) implies $(i,i+1)$-, $(0,j)$- and $(0,1)$-flag-transitivity, and that in turn (i,j,l)-flag-transitivity forces (i',j',l')-flag-transitivity for 6 possibly distinct triples (i',j',l'); in particular it implies $(0,1,l)$-flag-transitivity, and so the transitivity of G on the $(1,l)$-flags. This fact motivates the search for (line, hyperplane)-flag-transitive DLS's (see the theorem below). A recent classification of finite flag-transitive linear spaces (Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [1990]) (see Chapter 22, Section 1.10) easily yields the following lemma.

LEMMA. *If S is a finite nontrivial n-DLS with $n \geqslant 3$ on v points and if $G \leqslant \mathrm{Aut}\, S$ acts transitively on the (i,j)-flags for some i, j $(0 \leqslant i < j \leqslant n-1)$, then S has a constant line-size k and one of the following holds*

(i) $k = q+1 \geqslant 3$, $S = n\text{-PG}(d,q)$, $d \geqslant n$ *and* $G \geqslant \mathrm{PSL}(d+1,q)$ *or* $G = A_7$,

(ii) $k = q \geqslant 3$, $2\text{-}S = 2\text{-AG}(d,q)$, $d \geqslant n$ *and G lies in a known list,*

(iii) $k \geqslant 3$ *and* $G \leqslant \mathrm{A\Gamma L}(1,v)$,

(iv) $k = 2$ *and G is 2-transitive.*

This calls for improvements. For example, $d \leqslant 4$ or $i \neq 0$ forces $S = n\text{-AG}(d,q)$ in case (ii). By contrast, the case $(i,j) = (0,1)$ allows *missed affine spaces* erected on $2\text{-AG}(d,q)$, described in the example below.

EXAMPLE. Suppose d has a proper divisor $t > 2$ and let $G = \mathrm{AGL}(1,q^d)$. The point-set of S is $\mathrm{GF}(q^d)$ and the line-set of S is the orbit of the subfield $\mathrm{GF}(q)$ under G. Now let O_t

be the orbit of $\mathrm{GF}(q^t)$ under G and define the planes of S to be the elements of O_t together with the planes of $\mathrm{AG}(d,q)$ which are in no element of O_t. Then $3\text{-}S \neq 3\text{-AG}(d,q)$ although $2\text{-}S = 2\text{-AG}(d,q)$ and S admits a 2-transitive automorphism group.

This suggests that we assume stronger transitivities, i.e. transitivities involving hyperplanes since any DLS is determined by its (point, hyperplane)-truncation (see 1.8). Here are a complete classification of finite $(1, n-1)$-flag-transitive n-DLS's ($n \geqslant 3$) and derived classifications of (i,j,l)-flag-transitive n-DLS's and of (i,j)-flag-transitive thick n-DLS's. This settles some problems on hypercircular n-DLS's tackled in Hughes [1965], Cofman [1967] and Buekenhout [1968].

THEOREM (Delandtsheer [1991, 1992]). *Let S be a finite nontrivial n-DLS with $n \geqslant 3$ and let $G \leqslant \mathrm{Aut}\, S$. Then the following are equivalent*

(a) *G acts transitively on the (line, hyperplane)-flags of S,*

(b) *for some $i < j < n-1$, G acts $(i,j,n-1)$-flag-transitively,*

(c) *G acts transitively on the chambers (i.e. the maximal flags) of S,*

(d) *one of the following holds:*

(i) $S = n\text{-PG}(d,q)$ *with* $d \geqslant n, q \geqslant 2$ *and* $\mathrm{PSL}(d+1,q) \leqslant G \leqslant \mathrm{P\Gamma L}(d+1,q)$, *or* $G = A_7$ *and* $d = 3,\ q = 2$,

(ii) $S = n\text{-AG}(d,q)$ *with* $d \geqslant n, q \geqslant 2$ *and* $\mathrm{ASL}(d,q) \leqslant G \leqslant \mathrm{A\Gamma L}(d,q)$,

(iii) $S = \mathrm{AG}(3,2)$ *or* $\mathrm{AG}(3,8)$ *and* $G = \mathrm{A\Gamma L}(1,8)$ *or* $\mathrm{A\Gamma^7 L}(1,512) \leqslant G \leqslant \mathrm{A\Gamma L}(1,512)$, (*where* $\mathrm{A\Gamma^7 L}(1,512)$ *consists of all permutations* $x \mapsto a^7x^\sigma + b$, *where* $a, b \in \mathrm{GF}(512)$, $a \neq 0$, $\sigma \in \mathrm{Aut}\,\mathrm{GF}(512)$),

(iv) S *is the* WLA *circular space* $3\text{-}(q^d+1, q+1, 1)$ *on* $\mathrm{PG}(1,q^d)$ *with* $\mathrm{PG}(1,q)$ *as base-block and* $\mathrm{PSL}(2,q^d) \leqslant G \leqslant \mathrm{P\Gamma L}(2,q^d)$,

(v) S *is the locally Netto circular space* $3\text{-}(q+1,4,1)$ *on* $\mathrm{PG}(1,q)$ (*for any* $q \equiv 7 \pmod{12}$) *with* $\{\infty\} \cup K$ *as a plane* (*where* K *is the set of third roots of unity in* $\mathrm{GF}(q)$) *and* $\mathrm{PSL}(2,q) \leqslant G \leqslant \mathrm{P\Sigma L}(2,q)$,

(vi) S *is a Mathieu–Witt design* $4\text{-}(11,5,1)$, $5\text{-}(12,6,1)$, $3\text{-}(22,6,1)$, $4\text{-}(23,7,1)$ *or* $5\text{-}(24,8,1)$ *and G is a Mathieu group* M_{11}, M_{12}, M_{22}, $\mathrm{Aut}\,\mathrm{M}_{22}$, M_{23} *or* M_{24}.

COROLLARY (Delandtsheer [1991, 1992, 1994]). *Let $1 < j < n \geqslant 3$, let S be a finite n-DLS and let $G \leqslant \mathrm{Aut}\, S$.*

(a) *If j-S is nontrivial and if G is $(1,j)$-flag-transitive, then* (i), (ii) *or* (iii) *of the above theorem holds.*

(b) *Given i such that $1 \leqslant i \leqslant n-3$, if S has a constant parameter $t(1,i,i-1)$* (*see* 3.2) *and if G is $(i,n-1)$-flag-transitive, then G is chamber-transitive.*

(c) *Given i such that $1 \leqslant i \leqslant j-2$, if j-S is nontrivial and has a constant parameter $t(1,i,i+1)$ and if G is (i,j)-flag-transitive, then G is chamber-transitive.*

Note that in a finite n-DLS and for $i \geqslant 1$, $(i, n-1)$-flag-transitivity forces $(1, n-1)$-flag-transitivity if i-S is trivial and $i \geqslant 3$ (use Ray-Chaudhuri and Wilson [1975] together with Block's lemma) and of course also if $(i+1)$-S is trivial, as well as if $i \leqslant n-4$ and S admits a constant parameter $t(1, i, i+1)$ (use 3.5).

Note also that a classification of the (point, hyperplane)-flag-transitive finite n-DLS's would include the longstanding and still open problem of classifying all (point, block)-flag-transitive n-$(v, k, 1)$ designs. The situation for (0, 1)-flag-transitive n-DLS's is still worse since it encapsulates not only the tantalizing case $n = 2$ with a 1-dimensional affine group, but also the hopeless case $n \geqslant 3$ with a 2-transitive group, illustrated by Example 7.7.

7.8. *Some other transitivities on pairs of varieties*

A finite n-DLS whose automorphism group acts transitively on the unordered pairs of intesecting hyperplanes is one of the following:

(i) if $n \geqslant 3$: a completely trivial DLS, a projective or affine space or the Mathieu–Witt 3-design M_{22}, or

(ii) if $n = 2$: the 2-truncation of a Desarguesian projective or affine space, a Lüneburg–Tits affine plane, or a 3-homogeneous trivial linear space (Delandtsheer [1984a,b, 1986a]).

If a finite planar space S has an automorphism group G acting transitively on the pairs (L, π), where L is a line intersecting the plane π in one point, then a point-stabilizer G_x acts transitively on the antiflags of the point-residue S_x, and so G_x acts 2-transitively on S_x. Therefore (S, G) is known by Theorem 7.7, namely S is either a projective or affine space of dimension $\geqslant 3$, the Mathieu–Witt design M_{22} a Miquelian inversive plane, or the WLA circular space 3-$(8^3+1,\ 8+1,\ 1)$ space (the last two cases are contained in (iv) of Theorem 7.7).

PROBLEM. Classify the finite (point, hyperplane)-antiflag-transitive n-DLS's. For $n = 2$, this hypothesis forces 2-transitivity on points, and so the antiflag-transitive finite linear spaces are known thanks to Kantor [1985] (see also Delandtsheer [1984d]). However, in general, (point, hyperplane)-antiflag-transitivity does not even force point primitivity, as shown by the point imprimitive though (point, hyperplane)-antiflag-transitive action of $\mathrm{P\Gamma L}((n+1)/2, 4)$ on $\mathrm{PG}(n, 2)$ with n odd (Cameron and Kantor [1979]). If S is not hyperplane-closed, then S can be erected into an $(n+1)$-DLS T which is $(n-1, n)$-flag-transitive and whose hyperplanes are $(0, n-1)$-antiflag-transitive.

7.9. *Basis-transitivity: reduction to the point-primitive case*

For early work on this theme, see Cameron [1976a]. The ordered basis-transitive nontrivial DLS's are the Hermitian unital $U_H(4)$ of order 4, the three Mathieu–Witt designs related to the groups M_{22}, M_{23} and M_{24}, and the truncations of Desarguesian affine or projective spaces (Kantor [1985], using the classification of the finite 2-transitive groups).

The situation for (unordered) basis-transitivity appears to be much more chaotic since the group does not even need to be point-transitive. Before stating how the problem amounts to the determination of point-primitive and basis-transitive finite DLS's, let us give some tricks for manufacturing new examples from old ones.

GENERAL CONSTRUCTIONS.

(i) The simplification of the dual of a basis-transitive finite n-DLS on v points is basis-transitive $(v-n-2)$-DLS on $\leqslant v$ points. If G is point-primitive and basis-transitive on $\boldsymbol{S}$, then G also is point-primitive and basis-primitive on the DLS $\boldsymbol{S}^*$.

(ii) If G_i acts basis-transitively on $\boldsymbol{S}_i$ for each i, then the direct product $\prod_i G_i$ acts basis-transitively on the direct sum $\bigoplus_i \boldsymbol{S}_i$.

(iii) If $\boldsymbol{T}$ is a point-transitive and basis-transitive d_0-pre-DLS on t points, if $(\boldsymbol{S}, \boldsymbol{S}')$ is a $(d, d-1)$-DLS's association sharing a common point- and basis-transitive automorphism group G (see 1.13), then the supersum

$$\bigoplus_{\boldsymbol{T}}(\boldsymbol{S}, \boldsymbol{S}') := \bigoplus_{\boldsymbol{T}}(\boldsymbol{S}_i, \boldsymbol{S}_i'), \quad \text{where } \boldsymbol{S}_i \cong \boldsymbol{S} \text{ and } \boldsymbol{S}_i' \cong \boldsymbol{S}'$$

is a (d_0+td)-DLS admitting $G \operatorname{wr} \operatorname{Aut} \boldsymbol{T}$ as a basis-transitive automorphism group.

Note that starting from well-known DLS's, this process can be iterated again and again, producing quite unusual creatures.

Conversely, the following reduction theorem holds.

THEOREM (Li [1989] for (a), Delandtsheer [1986b] for $\lambda = 1$ in (c), Delandtsheer and Li [1994] for (b) and Delandtsheer [1991] for (c)). *Let $\boldsymbol{S}$ be a finite* DLS *and let $G \leqslant \operatorname{Aut} \boldsymbol{S}$ be basis transitive. Then*

(a) *$\boldsymbol{S}$ is a direct sum of point- and basis-transitive* DLS*'s.*

(b) *If $\boldsymbol{S}$ is point-transitive, then $\boldsymbol{S}$ is a supersum $\bigoplus_{\boldsymbol{T}}(\boldsymbol{R}, \boldsymbol{R}')$, where $\boldsymbol{T}$ is a point- and basis-transitive pre-d_0-*DLS *with $d_0 \geqslant 0$ and $(\boldsymbol{R}, \boldsymbol{R}')$ is a $(d, d-1)$-*DLS'*s association sharing a common point-primitive and basis-transitive automorphism group.*

(c) *Let λ be the smallest dimension of a thick variety (i.e. with size $\geqslant \lambda + 2$). If at least one λ-variety has size $\geqslant 2\lambda + 1$ and if $\boldsymbol{S}$ is point-transitive, then $\boldsymbol{S}$ is a supersum $\bigoplus_{\boldsymbol{T}}(\boldsymbol{R}, \boldsymbol{R}')$, where $\boldsymbol{T}$ is as in* (b) *and $(\boldsymbol{R}, \boldsymbol{R}')$ is a $(d, d-1)$-*DLS'*s association with $d-1 \geqslant \lambda$, where $\boldsymbol{R}$ and $\boldsymbol{R}'$ share a common 2-homogeneous and basis-transitive automorphism group* (see 7.1.13). *If moreover $\lambda = 1$, then $\boldsymbol{R}' = (d-1)$-$\boldsymbol{R}$.*

Note that whenever $\boldsymbol{R}' = (d-1)$-$\boldsymbol{R}$, the supersum $\boldsymbol{S} = \bigoplus_{\boldsymbol{T}}(\boldsymbol{R}, \boldsymbol{R}')$ is merely a direct product in the sense of Lim [1977]. For small values of d and d_0, we recall the following. The $(d-1)$-truncation of $\boldsymbol{R}$ is the pre-DLS on the same point-set as $\boldsymbol{R}$ and whose independent sets are the independent sets of size $\leqslant d$ in $\boldsymbol{R}$. If $d_0 = 0$, the bases of $\boldsymbol{T}$ are the singletons, and if $d_0 = 1$, then $\boldsymbol{T}$ can be described as a complete regular multipartite graph Γ (possibly with maximal clique[4]-size 1), the points of $\boldsymbol{T}$ being the vertices of Γ and the bases of $\boldsymbol{T}$ being the edges of Γ.

Note also that the classification of point- and basis-transitive pre-DLS's amounts to that of point- and basis-transitive DLS's in the following way. Let $\boldsymbol{S}$ be a DLS on the point-set $\boldsymbol{P}$ with $G \leqslant \mathrm{Aut}\,\boldsymbol{S}$ point- and basis-transitive on $\boldsymbol{S}$ and let H be a permutation group acting transitively on a set Ω. Then $\boldsymbol{S}$ defines on the point-set $\Omega \times \boldsymbol{P}$ a pre-DLS $\boldsymbol{T}$ whose classes of parallel points are the sets $\Omega \times \{x\}$ with $x \in \boldsymbol{P}$ and whose simplification is $\boldsymbol{S}$, and $K = H \,\mathrm{wr}\, G \leqslant \mathrm{Aut}\,\boldsymbol{T}$ acts point- and basis-transitively on $\boldsymbol{T}$. Conversely, any pair $(\boldsymbol{T}, K)$, where K acts point- and basis-transitively on $\boldsymbol{T}$, is obtained in this way from the simplification $\boldsymbol{S}$ of $\boldsymbol{T}$ and from the actions of K on $\boldsymbol{S}$ and on the classes of parallel points of $\boldsymbol{T}$.

7.10. *Point-primitive and basis-transitive DLS's*

THEOREM. *Let $\boldsymbol{S}$ be a finite n-DLS and let $G \leqslant \mathrm{Aut}\,\boldsymbol{S}$ act transitively on the bases and primitively on the points of $\boldsymbol{S}$.*

(a) *If $\boldsymbol{S}$ is noncircular, then one of the following occurs:*
  (i) $\boldsymbol{S} = n\text{-PG}(d,q)$ *and* $G \geqslant \mathrm{PSL}(d+1,q)$ $(d, q \geqslant 2)$ *or* $G = A_7$ *and* $(d,q) = (3,2)$,
  (ii) $\boldsymbol{S} = n\text{-AG}(d,q)$ *and* $G \geqslant \mathrm{ASL}(d,q)$ $(d, q \geqslant 2)$,
  (iii) $\boldsymbol{S} = U_H(4)$ *and* $G \geqslant \mathrm{PSU}(3,4)$ (7.9 and Delandtsheer [1986b]).

(b) *If $\boldsymbol{S}$ is circular and $n = 3$, then one of the following occurs:*
  (i) $\boldsymbol{S}$ *is the* 3-(10, 4, 1) *or* 3-(17, 5, 1) *inversive plane of order* 3 *or* 4,
  (ii) $\boldsymbol{S}$ *is the Mathieu–Witt design* 3-(22, 6, 1),
  (iii) $\boldsymbol{S} = \mathrm{PG}(2,2)^*$.

(c) *If $\boldsymbol{S}$ has* 12 *points and if G is the Mathieu group M_{11}, then $\boldsymbol{S}$ is the hypercircular* 4-DLS *whose thick hyperplanes are precisely the blocks of the Hadamard* 3-(12, 6, 2) *design; its top* 2*-dimensional residues are isomorphic to the Pentagram linear space (Figure* 2.2) *on* 10 *points, admitting* Sym(5) *as its full automorphism group* (Delandtsheer [1994]).

Other examples of point-primitive and basis-transitive circular DLS's are the four Mathieu–Witt designs 4-(11, 5, 1), 5-(12, 6, 1), 4-(23, 7, 1) and 5-(24, 8, 1) related to M_{11}, M_{12}, M_{23} and M_{24}, the double circular extension of the Pentagram linear space (described in Theorem 7.10(c)) and the duals of all aforementioned examples.

Li's classification of all basis-transitive 2- or 3-DLS's can be recovered from (b) by noting that if the group is imprimitive, then $\boldsymbol{S} = \bigoplus_{\boldsymbol{T}}(\boldsymbol{R}, \boldsymbol{R}')$, where $\boldsymbol{T}$ is a pre-d_0-DLS on t points and $(d_0, t) = (0,2)$, $(0,3)$ or $(1,2)$.

7.11. *Transitivity on independent sets*

Let m, n be two positive integers with $2 \leqslant m \leqslant n+1$. If some n-DLS $\boldsymbol{S}$ has an automorphism group G acting transitively on the independent m-sets, then G acts transitively

[4] A *coclique* is a clique of the complementary graph.

on the $(m-1)$-varieties of $\boldsymbol{S}$ and on the bases of $(m-1)$-$\boldsymbol{S}$, so that the classification of such spaces amounts to that of the point- and basis-transitive pairs $(\boldsymbol{S}, G)$ and of the G-invariant erections of $\boldsymbol{S}$.

The problem seems to be hopeless if $m = 1$ since any DLS induces a point-transitive DLS on each point-orbit of its automorphism group. Example 7.7 suggests that the case $m = 2$ would also require additional assumptions in order to be solved.

Finally the case $m = n + 1$ was dealt with in the preceding two sections. In all the remaining cases (i.e. $3 \leqslant m \leqslant n$) the problem has been solved for noncircular DLS's.

THEOREM (Delandtsheer [1986c]). *Let* $\boldsymbol{S}$ *be a finite noncircular* n-DLS *and let* $3 \leqslant m \leqslant n$. *If* $G \leqslant \operatorname{Aut} \boldsymbol{S}$ *is transitive on the independent* m*-sets of* $\boldsymbol{S}$, *then one of the following two possibilities occurs*:

(i) $\boldsymbol{S} = n\text{-AG}(d, q)$ *or* $n\text{-PG}(d, q)$ *with* $d \geqslant n$, *or*

(ii) $m = n = (d+1)t - 1$ *with* $d, t \geqslant 1$ *and* $\boldsymbol{S}$ *is a direct sum*

$$\bigoplus_{i=1}^{t} \boldsymbol{R}_i,$$

where $\boldsymbol{R}_i \cong \boldsymbol{R}$ *is a* 2*-homogeneous and basis-transitive* d-DLS. *Note that* $\boldsymbol{R}$ *is known by* 7.10: *if* $d \geqslant 2$, *then* $\boldsymbol{R} = U_H(4)$ *or* $\boldsymbol{R} = d\text{-AG}(\delta, q)$ *or* $d\text{-PG}(\delta, q)$ *with* $\delta \geqslant d \geqslant 2$.

8. Permutation geometries and squashed geometries

We will just touch on this topic since it goes far beyond the framework of DLS's. Hence we refer to the literature, especially to the founding papers by Cameron and Deza [1979], Deza and Frankl [1984], Deza and Laurent [1987], and to Deza, Laurent and Pasini [1994], Pasini [1992], Cameron [1992], Cameron, Deza and Frankl [1987], Cameron, Deza and Singhi [1988].

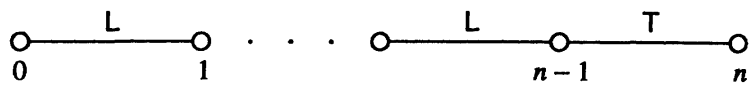
A permutation geometry is an analogue, for permutations, of a pre-DLS. If the set of permutations involved happens to be a group, then it is a geometric group.

We have seen in Section 7.6 that a geometric group is a permutation group (G, Ω) satisfying certain conditions such that the point-set Ω, provided with the fixed-point-sets of the elements of G, is a pre-DLS $\boldsymbol{S} = \boldsymbol{S}(G, \Omega)$ on which G acts as a Jordan transitive automorphism group (moreover G acts *sharply* transitively on the ordered bases of $\boldsymbol{S}(G, \Omega)$).

The *permutation geometry* $\boldsymbol{G} = \boldsymbol{G}(G, \Omega)$ associated with a permutation group (G, Ω) consists essentially of the set $\Omega \times \Omega$ provided with a family R of subsets called *roofs*, defined to be the graphs of the permutations of Ω belonging to G. The *positive automorphism group* of $\boldsymbol{G}(G, \Omega)$ consists of all elements (s, s') of $\operatorname{Sym}(\Omega) \times \operatorname{Sym}(\Omega)$ preserving roofs, i.e. such that for any g in G, $s^{-1}gs'$ belongs to G. Hence, positive automorphisms of $\boldsymbol{G}$ are not allowed to interchange the horizontal and vertical fibres of $\Omega \times \Omega$.

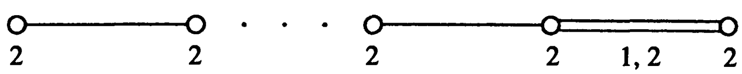
Formally $G(G, \Omega)$ consists of the ground-set $\Omega \times \Omega$ and distinguished subsets, called *flats*, defined to be all intersections of roofs. The *rank* of a flat f is the maximal length of chains of flats contained in f, while the rank of the geometry G is that of its roofs. The flats contained in the roof 1_Ω (the identity permutation on Ω) are precisely the subsets of the diagonal 1_Ω of $\Omega \times \Omega$ whose (vertical and horizontal) projections onto Ω are varieties of the pre-DLS $S(G, \Omega)$. More generally the flats of rank $i + 1$ (with $1 < n = \dim S$) are the sets $\{(x, g(x))\colon\ x \in V\}$, where V is an i-variety of S and $g \in G$.

The incidence geometry whose varieties of type i are the flats of rank $i + 1$ and whose incidence is inclusion belongs to the diagram

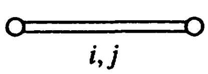


the residues of roofs being isomorphic to the simplification of $S(G, \Omega)$ and the rank 2 top-residues being isomorphic to a semilinear space T. Denote by q the hyperplane order q_{n-1} in S and let $s = s_{n-1} - s_{n-2}$, where s_i denotes the size of the i-varieties in the simplification of S. Then the point-set of T is partitioned into $(q + 1)^2$ classes of s points, forming a $(q + 1) \times (q + 1)$ grid such that two points of T are noncollinear if and only if their classes are on a common row or column in this grid.

For example, if $(G, \Omega) = (\mathrm{PSL}(n + 1, 2),\ 1\text{-}\mathrm{PG}(n, 2))$, then $G(G, \Omega)$ belongs to



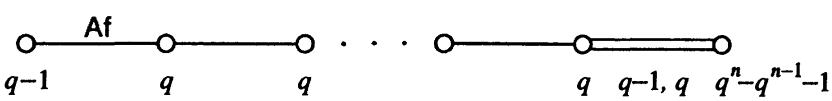
where



$(n + 1$ nodes)

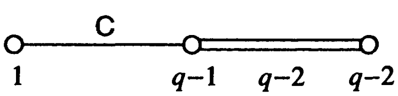
denotes the class of semilinear spaces in which, for every antiflag (x, L), the number of points of L that are collinear with x is either i or j.

If $(G, \Omega) = (\mathrm{AGL}(n, q),\ 1\text{-}\mathrm{AG}(n, q))$, then $G(G, \Omega)$ belongs to



$(n + 1$ nodes)

In the easier case where $(G, \Omega) = (\mathrm{PSL}(2, q),\ \mathrm{PG}(1, q))$, then $G(G, \Omega)$ belongs to

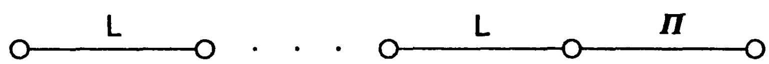


Hence the point-residues are affine planes from which two directions of lines have been deleted. In other words, $G(G, \Omega)$ is a Minkowski plane (here a Miquelian one) provided with its points, its pairs of concircular points (2-flats) and its circles (3-flats of roofs). From any finite sharply 3-transitive set of permutations we obtain in the same way a (not necessarily Miquelian) Minkowski plane (see also 5.7). Indeed the definition of $G(S, \Omega)$ may apply to any set S of permutations on Ω. Such a set S is called *geometric*, and $G(S, \Omega)$ is called a *permutation geometry of rank* $s+1$ if and only if the roofs have rank $s+1 < \infty$ and the condition (GEO) below holds (Cameron [1988]). Let f be a flat and let (ω, ω') belong to $(\Omega \times \Omega) \setminus f$; then the 'antiflag' $(f, (\omega, \omega'))$ is called admissible if and only if no vertical or horizontal fiber of $\Omega \times \Omega$ intersects $\{(\omega, \omega')\} \cup f$ more than once (remember that our aim is to eventually get a roof, i.e. the graph of a permutation). The condition (GEO) then reads:

(GEO): for any flat f and any element (ω, ω') of $\Omega \times \Omega$, if $(f, (\omega, \omega'))$ is an admissible antiflag, then there is exactly one flat containing $f \cup \{(\omega, \omega')\}$ and whose rank is $1 + \operatorname{rank} f$.

Hence the axioms for permutation geometries are very close to those for pre-DLS's, except that we only consider admissible antiflags.

This notion has been further generalized to ***F**-squashed geometries*, where admissibility means that the union of the singleton and the flat lies in some element of a certain family $\boldsymbol{F}$ of subsets of the ground-set. For a precise definition, refer to Deza and Laurent [1987], where it is proved that the important subclass of *simple well-cut squashed geometries of rank* n coincides with the class of firm strongly connected geometries satisfying (IP) (see Chapter 3, Section 3.7) and belonging to the diagram



(n nodes)

Compare 1.5. Hence this encapsulates not only DLS's and polar spaces, but also extended partial geometries, L^{n-2}-extensions of partial geometries, and other geometries investigated in many recent papers (see, e.g., Del Fra, Ghinelli and Pasini [1990, 1991] and references therein).

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CHAPTER 7

Projective Geometry over a Finite Field

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Contents

Introduction ... 297
1. k-arcs and normal rational curves ... 298
   1.1. Definitions ... 298
   1.2. The three problems of B. Segre ... 298
   1.3. k-arcs in PG$(2, q)$ ... 298
   1.4. k-arcs in PG$(3, q)$ ... 302
   1.5. k-arcs in PG(n, q) ... 303
   1.6. The nonclassical 10-arc of PG(4, 9) ... 304
   1.7. The nucleus or kernel of a normal rational curve, and $(q + 2)$-arcs in PG$(q - 2, q)$ ... 304
   1.8. The duality principle for k-arcs ... 305
   1.9. Open problems ... 306
2. k-arcs and MDS codes ... 307
   2.1. MDS codes ... 307
   2.2. The general case ... 308
   2.3. Linear MDS codes ... 308
3. k-caps and ovoids ... 309
   3.1. Definitions ... 309
   3.2. k-caps in PG$(3, q)$ ... 309
   3.3. Ovoids and inversive planes ... 310
   3.4. Caps in ovoids ... 311
   3.5. Maximum size of a cap in PG(n, q), $n \geqslant 4$ ... 311
   3.6. Open problems ... 313
4. Maximal arcs and Hermitian arcs ... 313
   4.1. Maximal arcs ... 313
   4.2. Maximal arcs in PG$(2, q)$, with q odd ... 314
   4.3. Maximal arcs in PG$(2, q)$, with q even ... 314
   4.4. Hermitian arcs and Hermitian curves ... 314
   4.5. Characterizations of nonsingular Hermitian curves ... 315

HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

4.6. Hermitian arcs other than nonsingular Hermitian curves 315
4.7. A characterization theorem of Tallini Scafati 316
4.8. Open problems 316
5. Semi-ovals, semi-ovoids and two combinatorial characterizations of ovoids 316
5.1. A combinatorial characterization of ovoids 316
5.2. Semi-ovals, semi-ovoids, and a second combinatorial characterization of ovoids 317
5.3. Regular semi-ovals 317
6. Sets of type $(1, m, q+1)$ 318
6.1. Introduction 318
6.2. The classification for $m = 1$, $m = 2$ and $m = q$ 318
6.3. Projections of quadrics 319
6.4. The classification for $3 \leqslant m \leqslant q-1$, $n \geqslant 3$, and $q \neq 4$ 320
6.5. The classification for $m = 3$, $n \geqslant 3$ and $q = 4$ 320
7. Blocking sets 321
7.1. Introduction 321
7.2. Baer cones 322
7.3. Main results 322
7.4. Hyperplane coverings and minimal blocking sets 323
7.5. Open problems 324
8. Spreads and partial spreads 324
8.1. t-spreads and partial t-spreads of PG(n, q) 324
8.2. Geometric t-spreads 325
8.3. t-spreads of PG$(2t+1, q)$ 325
8.4. Spreads and partial spreads of PG$(3, q)$ 327
8.5. Partition of PG(n, q^k) into subgeometries PG(n, q) 329
8.6. Open problems 329
9. Ovoids and spreads of classical polar spaces, hemisystems 329
9.1. Finite classical polar spaces 329
9.2. Ovoids and spreads of polar spaces 330
9.3. Existence and nonexistence of spreads 331
9.4. Existence and nonexistence of ovoids 332
9.5. Hemisystems 332
10. Flocks, partial flocks and maximal exterior sets 333
10.1. Flocks 333
10.2. Flocks, translation planes and generalized quadrangles 334
10.3. Flocks of ovoids 334
10.4. Flocks of hyperbolic quadrics 334
10.5. Flocks of cones 336
10.6. Maximal exterior sets 338
10.7. Partial flocks 339
10.8. Open problems 339
References 340

Introduction

In recent years there has been an increasing interest in finite spaces, and important applications to practical topics such as coding theory and design of experiments has made the field even more attractive. There have been many papers but few books on the subject. Basic works are 'Projective Geometries over Finite Fields', 'Finite Projective Spaces of Three Dimensions' and 'General Galois Geometries', the first two volumes being written by Hirschfeld [1979, 1985] and the third volume by Hirschfeld and Thas [1991]; in fact, it consists of a single work in three volumes. In the first volume Hirschfeld makes the following historical remarks:

'The first reference I can find on finite geometry in the literature is von Staudt [1856], pp. 86–91. He already has the idea of 'real' and 'complex' points of a finite space as defined in §2.6, but he restricts his attention to dimensions two and three. Then Fano [1892] defined PG(n, p) synthetically for an arbitrary prime p, while Hessenberg [1902/3] did it analytically. The first systematic account of PG(n, q) for arbitrary n and an arbitrary prime power q is due to Veblen and Bussey [1906], although the group of projectivities PGL$(n + 1, q)$ goes back to Jordan [1870].'

At the beginning of the 20th century the works on group theory by Dickson did also much for the knowledge of PG(n, q). Deep results in finite algebraic geometry and number theory on estimates for the number of points on an algebraic curve were obtained by Hasse [1934] and by Weil [1948]. In 1935, Yates, in a lecture to the Royal Statistical Society, introduced the notion of *design*, thus showing how finite geometries can be applied to the planning of experiments. A few years later the statistician Bose started with his investigations on graph theory, design theory and finite projective spaces, using mainly pure combinatorial arguments. Another great pioneer in finite projective geometry was the Italian geometer Beniamino Segre. His celebrated result of 1955 stating that in the projective plane PG$(2, q)$ over the Galois field GF(q) with q odd, every set of $q + 1$ points, no three of which are collinear, is a conic, stimulated the enthusiasm of many young geometers. In particular, it were the fundamental papers of Segre which persuaded me to work on Galois geometries, i.e. on the geometry of PG(n, q). Finally, the applications of finite projective geometry to the theory of error-correcting codes completed the deserved success of this beautiful part of the modern development of classical geometry.

This chapter contains many results but no proofs. It is a survey of the main results about Galois geometries in the spirit of Segre's work and his followers. Further, I have restricted myself to pure finite projective geometry, without devoting a chapter to finite algebraic geometry. Finally, there is a small deviation in Section 2 where I explain the equivalence of the theory of k-arcs and the theory of linear maximum distance separable codes.

1. k-arcs and normal rational curves

1.1. *Definitions*

A *k-arc* in PG(n,q) is a set K of k points with $k \geqslant n+1$ such that no $n+1$ points of K lie in a hyperplane. An arc K is *complete* if it is not properly contained in a larger arc. Otherwise, if $K \cup \{x\}$ is an arc for some point x of PG(n,q) we say that x *extends* K.

A *normal rational curve* of PG(n,q) is any set of points in PG(n,q) which is projectively equivalent to

$$\{(t^n, t^{n-1}, \ldots, t, 1)\colon\ t \in \mathrm{GF}(q)\} \cup \{(1,0,\ldots,0,0)\}.$$

Clearly any normal rational curve contains $q+1$ points. A normal rational curve of PG$(2,q)$ is an *irreducible conic*; a normal rational curve of PG$(3,q)$ is a *twisted cubic*. It is well known that any $(n+3)$-arc of PG(n,q) is contained in a unique normal rational curve of this space (see Hirschfeld [1985]). For $q > n+1$, the *osculating hyperplane* of the normal rational curve C at the point $x \in C$ is the unique hyperplane through x intersecting C at x with multiplicity n.

A section on k-arcs in PG$(2,q)$ is also contained in Chapter 4.

1.2. *The three problems of B. Segre*

In 1955 Segre posed the following three fundamental problems.

(a) For given n and q what is the maximum value of k for which there exist k-arcs in PG(n,q)?

(b) For what values of n and q, with $q > n+1$, is every $(q+1)$-arc of PG(n,q) a normal rational curve?

(c) For given n and q, with $q > n+1$, what are the values of k for which every k-arc of PG(n,q) is contained in a normal rational curve of this space?

1.3. *k-arcs in* PG$(2,q)$

Let K be a k-arc of PG$(2,q)$. Then clearly $k \leqslant q+2$. By Bose [1947], for q odd, $k \leqslant q+1$. Further, any irreducible conic of PG$(2,q)$ is a $(q+1)$-arc. It can be shown that each $(q+1)$-arc K of PG$(2,q)$, q even, extends to a $(q+2)$-arc $K \cup \{x\}$ (see, e.g., Hirschfeld [1979], p. 165); the point x, which is uniquely defined by K, is called the *kernel* or *nucleus* of K. The $(q+1)$-arcs of PG$(2,q)$ are called *ovals*; the $(q+2)$-arcs of PG$(2,q)$, q even, are called *complete ovals* or *hyperovals*.

The following celebrated theorem is due to Segre [1955a].

THEOREM 1. *In* PG$(2,q)$, *q odd, every oval is an irreducible conic.*

Let $K \cup \{x\}$ be a hyperoval in PG$(2,q)$, q even, with K an irreducible conic. If $y \in K$, then

$$(K \setminus \{y\}) \cup \{x\} = K'$$

is an oval of PG$(2,q)$. Clearly $|K \cap K'| = q$. So for $q > 4$ the oval K' cannot be an irreducible conic. Hence for $q \geqslant 8$, q even, the plane PG$(2,q)$ always contains ovals which are not irreducible conics. It is easy to show that for $q \in \{2,4\}$ any oval of PG$(2,q)$ is an irreducible conic. By Segre [1957, 1962], each hyperoval of PG$(2,8)$ is the union of a conic and its nucleus, and in PG$(2,2^h)$ with $h = 5$ and $h \geqslant 7$ there exist hyperovals not containing a conic as subset. Lunelli and Sce [1958] have shown that in PG$(2,16)$ there is a hyperoval which is not the union of a conic and its nucleus; a similar result for PG$(2,64)$ was shown by Penttila and Pinneri [to appear].

The known hyperovals of PG$(2,q)$, $q = 2^h$
Let $D(k)$, with $k \in \mathbb{N} \setminus \{0\}$, be the pointset

$$\{(0,1,0),(0,0,1)\} \cup \{(1,t,t^k)\colon\ t \in \mathrm{GF}(q)\}.$$

Now we list all known hyperovals of PG$(2,q)$, q even.

(a) $D(2^m)$, with $(m,h) = 1$; these are due to Segre [1957]. Note that $D(2)$ gives a conic union its nucleus.

(b) $D(6)$, with h odd; these are also due to Segre [1962].

(c) Let h be odd, $h \geqslant 3$. Define two automorphisms

$$x \mapsto x^\sigma \quad \text{and} \quad x \mapsto x^\gamma$$

of GF(q) as follows:

$$\sigma = 2^{(h+1)/2}, \qquad \gamma = \begin{cases} 2^m & \text{if } h = 4m-1, \\ 2^{3m+1} & \text{if } h = 4m+1. \end{cases}$$

Then it was shown by Glynn [1983a] that $D(\sigma+\gamma)$ and $D(3\sigma+4)$ are hyperovals.

(d) Now follows a description by Glynn [1983a] of the hyperoval O of Lunelli and Sce [1958]. Consider in PG$(2,16)$ the cubics C and C' with equations

$$X_0^3 + X_1^3 + X_2^3 + dX_0X_1X_2 = 0$$

and

$$X_0^3 + X_1^3 + X_2^3 + d^4X_0X_1X_2 = 0,$$

where $d \in \mathrm{GF}(16)$, $d^5 = 1$ and $d \neq 1$. Then $O = (C \cup C') \setminus (C \cap C')$.

(e) Let h be odd. Define δ: GF$(q) \to$ GF(q) by

$$\delta\colon\ x \mapsto x^{1/6} + x^{1/2} + x^{5/6}.$$

Then Payne [1985] has shown that

$$D(\delta) = \{(0,1,0),(0,0,1)\} \cup \{(1,t,t^\delta)\colon\ t \in \mathrm{GF}(q)\}$$

is a hyperoval of PG$(2,q)$.

(f) Next we describe the hyperovals of Cherowitzo [1986]. Let $h = 2s + 1$,

$$\sigma\colon \mathrm{GF}(q) \to \mathrm{GF}(q), \quad x \mapsto x^{2^{s+1}},$$

$$\zeta\colon \mathrm{GF}(q) \to \mathrm{GF}(q), \quad x \mapsto x^{\sigma} + x^{\sigma+2} + x^{3\sigma+4}.$$

Then

$$D(\zeta) = \{(0,1,0),(0,0,1)\} \cup \{(1,t,t^{\zeta})\colon t \in \mathrm{GF}(q)\}$$

is a hyperoval for $h \leqslant 9$. It does not belong to the previous classes for $h \in \{5,7,9\}$.

(g) Finally O'Keefe and Penttila [1992b] discovered a new hyperoval in PG(2, 32), Penttila and Pinneri [to appear] constructed two new hyperovals in PG(2, 64), and Penttila and Royle [1994] discovered another hyperoval in PG(2, 64).

Note that the classes (a)–(e) sometimes overlap for small values of q, but they are distinct for large values.

Let K be any k-arc of PG$(2,q)$. A *tangent* of K is a line of PG$(2,q)$ meeting K in a unique point, and a *secant* of K is a line meeting K in two points. At each point K has $t = q + 2 - k$ tangents, and the total number of tangents is equal to tk. By an ingenious trick (the lemma of tangents, see 8.2.2 in Hirschfeld [1979]) and applying a nice theorem of projective geometry, Segre [1967] proved the following theorem.

THEOREM 2.

(a) *Let K be a k-arc in* PG$(2,q)$, *with q even. Then the tk tangents of K belong to an algebraic envelope Γ_t of class t with the properties*:
 (i) *Γ_t is unique if $k > (q+2)/2$;*
 (ii) *Γ_t contains no secant of K and so no pencil*[1] *with vertex x in K;*
 (iii) *if Δ_x is the pencil with vertex x in K and if L is a tangent of K at x, then the intersection multiplicity of Γ_t and Δ_x at L is one.*

(b) *Let K be a k-arc in* PG$(2,q)$, *with q odd. Then the tk tangents of K belong to an algebraic envelope Γ_{2t} of class $2t$ with the properties*:
 (i) *Γ_{2t} is unique if $k > (2q+4)/3$;*
 (ii) *Γ_{2t} contains no secant of K and so no pencil with vertex x in K;*
 (iii) *if Δ_x is the pencil with vertex x in K and if L is a tangent of K at x, then the intersection multiplicity of Γ_{2t} and Δ_x at L is two;*
 (iv) *Γ_{2t} may contain components of multiplicity at most two, but does not consist entirely of double components.*

Now the following corollary is quite easy to prove.

[1] The *pencil* with vertex x is the set of lines containing x.

COROLLARY.

(a) *If q is even and $k > (q+2)/2$ then K is contained in a unique complete arc of* PG$(2, q)$;

(b) *if q is odd and $k > (2q+4)/3$ then K is contained in a unique complete arc of* PG$(2, q)$.

Using some fundamental theorems from algebraic geometry the following key result transpires, see Segre [1967] and Thas [1987a]. The odd case is a slight improvement by Thas of the original theorem due to Segre.

THEOREM 3. *Assume that $k > \lambda(q)$ where*

$$\lambda(q) = \begin{cases} q - \sqrt{q} + 1 & \text{for } q \text{ even,} \\ q - \sqrt{q}/4 + 25/16 & \text{for } q \text{ odd.} \end{cases}$$

Then

(a) *for q even, any k-arc K is embedded in a hyperoval,*

(b) *for q odd, any k-arc K is embedded in a unique conic.*

Fisher, Hirschfeld and Thas [1986], and independently Boros and Szönyi [1986], construct complete $(q - \sqrt{q} + 1)$-arcs for q an even square and $q > 4$; in fact these arcs were already constructed by Kestenband [1981], but not recognized to be complete. So for q an even square and $q \neq 4$ the bound of Segre is best possible. These $(q - \sqrt{q} + 1)$-arcs can be described as follows. Let G be a cyclic subgroup of PGL$(3, q)$ acting regularly on PG$(2, q)$, q square. Let G_1 be the subgroup of order $q - \sqrt{q} + 1$ of G. Then the orbits of G_1 are complete $(q - \sqrt{q} + 1)$-arcs when $q \geqslant 9$. Further, in the odd case the bound in Theorem 3 certainly is not best possible; also, examples show that the bound $q - \sqrt{q} + 1$ does not work for q an odd square (in PG$(2, 9)$ there exists a complete 8-arc, see Hirschfeld [1979]).

Further, for q an odd power of a prime, Voloch [1990, 1991] was able to improve the bound in Theorem 3.

THEOREM 4.

(a) *Every k-arc K of* PG$(2, p)$, *p an odd prime, with $k > (44p + 40)/45$, is embedded in a unique conic.*

(b) *Every k-arc K of* PG$(2, q)$, *$q = p^{2m+1}$, $m \geqslant 1$, p odd, with $k > q - \sqrt{pq}/4 + 29p/16 + 1$, is embedded in a unique conic.*

(c) *Every k-arc K of* PG$(2, q)$, *$q = 2^{2m+1}$, $m \geqslant 1$, with $k > q - \sqrt{2q} + 2$, is contained in a unique hyperoval.*

From Theorem 3 it follows that, for q even, any q-arc of PG$(2, q)$ is contained in a hyperoval, and that, for q odd, with $q \geqslant 41$, any q-arc of PG$(2, q)$ is contained in a conic. The following theorem is due to Segre [1955b] (see also Hirschfeld [1979], §8.6), but a short proof can be found in Thas [1987a].

THEOREM 5. *Any q-arc of* PG(2, q), *q odd, is contained in a* $(q+1)$*-arc.*

1.4. *k-arcs in* PG(3, q)

For $q > 2$ any twisted cubic of PG(3, q) is a $(q+1)$-arc.

THEOREM 6.

(a) (Segre [1955b].) *For any k-arc of* PG(3, q), *q odd and* $q > 3$, $k \leqslant q+1$; *any k-arc of* PG(3, 3) *has at most 5 points.*

(b) (Casse [1969].) *For any k-arc of* PG(3, q), *q even and* $q > 2$, $k \leqslant q+1$; *any k-arc of* PG(3, 2) *has at most 5 points.*

The following theorem gives the classification of all $(q+1)$-arcs of PG(3, q).

THEOREM 7.

(a) (Segre [1955b].) *Any* $(q+1)$*-arc of* PG(3, q), *q odd, is a twisted cubic.*

(b) (Casse and Glynn [1982].) *Every* $(q+1)$*-arc of* PG(3, q), $q = 2^h$, *is projectively equivalent to*

$$C = \{(1, t, t^e, t^{e+1}) \colon t \in \mathrm{GF}(q)\} \cup \{(0, 0, 0, 1)\}$$

where $e = 2^m$ *and* $(m, h) = 1$.

In Bruen, Thas and Blokhuis [1988] Theorem 2 is generalized to PG(3, q).

THEOREM 8. *Let* $K = \{p_1, p_2, \ldots, p_k\}$ *be a k-arc of* PG(3, q). *For distinct* i_1, i_2 *in* $\{1, 2, \ldots, k\}$, *let* $Z_{\{i_1,i_2\}}$ *be the set of* $t = q + 3 - k$ *planes through the line* $p_{i_1}p_{i_2}$ *that contain no other point of K.*

(a) *For q even there exists a dual algebraic surface* Φ_t *of degree* $t = q + 3 - k$ *in* PG(3, q) *which contains the planes of each set* $Z_{\{i_1,i_2\}}$ *as elements. This dual surface is unique if* $k > (q+4)/2$.

(b) *For q odd there exists a dual algebraic surface* Φ_{2t} *of degree* $2t = 2(q+3-k)$ *in* PG(3, q) *which contains the planes of each set* $Z_{\{i_1,i_2\}}$ *as elements. Also, the intersection multiplicity of* Φ_{2t} *and the pencil of planes* $p_{i_1}p_{i_2}$ *at each plane of* $Z_{\{i_1,i_2\}}$ *is two. This dual surface is unique if* $k > (2q+7)/3$.

COROLLARY (Bruen et al. [1988]).

(a) *Any k-arc K of* PG(3, q), *q even and* $k > (q+4)/2$, *is contained in a unique complete arc of* PG(3, q).

(b) *Any k-arc K of* PG(3, q), *q odd and* $k > (2q+7)/3$, *is contained in a unique complete arc of* PG(3, q).

Using the dual surface Φ_t for q even, Storme and Thas [1993] obtained the following partial answer to Problem (c) of Segre; part (b) was proved by Thas [1968a, 1987a] using totally different techniques.

THEOREM 9.

(a) *Let K be a k-arc of* PG(3, q), *q even and $q \neq 2$. If $k > q - \sqrt{q}/2 + 9/4$, then K can be completed to a $(q+1)$-arc which is uniquely determined by K.*

(b) *Let K be a k-arc of* PG(3, q), *q odd. If $k > q - \sqrt{q}/4 + 41/16$, then K is contained in a unique twisted cubic.*

1.5. *k-arcs in* PG(n, q)

For $q \geqslant n$ any normal rational curve of PG(n, q) is a $(q+1)$-arc.

THEOREM 10 (Kaneta and Maruta [1989]). *If every $(q+1)$-arc of* PG(n, q), *$n \geqslant 3$ and $q \geqslant n+3$, is a normal rational curve, then $q+1$ is the maximum value of k for which k-arcs exist in* PG($n+1, q$).

THEOREM 11.

(a) (Casse [1969].) *For any k-arc of* PG(4, q), *q even and $q > 4$, $k \leqslant q+1$ holds; any k-arc of either* PG(4, 2) *or* PG(4, 4) *has at most* 6 *points.*

(b) (Segre [1955b].) *For any k-arc of* PG(4, q), *q odd and $q \geqslant 5$, we have $k \leqslant q+1$; any k-arc of* PG(4, 3) *has at most* 6 *points.*

(c) (Casse and Glynn [1984].) *Any $(q+1)$-arc of* PG(4, q), *q even, is a normal rational curve.*

(d) (Kaneta and Maruta [1989].) *For any k-arc of* PG(5, q), *q even and $q \geqslant 8$, $k \leqslant q+1$ holds.*

The following theorem by Thas [1968a, 1987a] gives an answer to the problems of Segre, for q odd.

THEOREM 12.

(a) *For any k-arc of* PG(n, q), *q odd and $q > (4n - 39/4)^2$, $k \leqslant q+1$.*

(b) *In* PG(n, q), *q odd and $q > (4n - 23/4)^2$, every $(q+1)$-arc is a normal rational curve.*

(c) *In* PG(n, q), *q odd, every k-arc with $k > q - \sqrt{q}/4 + n - 7/16$ is contained in one and only one normal rational curve of this space.*

In Blokhuis, Bruen and Thas [1990] Theorem 2 is generalized to PG(n, q).

THEOREM 13. *Let $K = \{p_1, p_2, \ldots, p_k\}$ be a k-arc of* PG(n, q). *For distinct numbers $i_1, i_2, \ldots, i_{n-1}$ in $\{1, 2, \ldots, k\}$, let $Z_{\{i_1, i_2, \ldots, i_{n-1}\}}$ be the set of $t = q + n - k$ hyperplanes through the $(n-2)$-dimensional space $p_{i_1} p_{i_2} \cdots p_{i_{n-1}}$ that contain no other point of K.*

(a) *For q even there exists a dual algebraic hypersurface Φ_t of degree $t = q + n - k$ in* PG(n, q) *which contains the hyperplanes of each set $Z_{\{i_1, i_2, \ldots, i_{n-1}\}}$ as elements. This dual hypersurface is unique if $k > (q + 2n - 2)/2$.*

(b) *For q odd there exists a dual algebraic hypersurface Φ_{2t} of degree $2t = 2(q + n - k)$ in* PG(n,q) *which contains the hyperplanes of each set $Z_{\{i_1,i_2,\ldots,i_{n-1}\}}$ as elements. Also, the intersection multiplicity of Φ_{2t} and the pencil of hyperplanes $p_{i_1}p_{i_2}\ldots p_{i_{n-1}}$ at each hyperplane of $Z_{\{i_1,i_2,\ldots,i_{n-1}\}}$ is two. This dual hypersurface is unique if $k > (2q + 3n - 2)/3$.*

COROLLARY (Blokhuis et al. [1990]).

(a) *Any k-arc K of* PG(n,q), *q even and $k > (q+2n-2)/2$, is contained in a unique complete arc of* PG(n,q).

(b) *Any k-arc K of* PG(n,q), *q odd and $k > (2q+3n-2)/3$, is contained in a unique complete arc of* PG(n,q).

Using the dual algebraic hypersurface Φ_t for q even, the following answers to the problems of Segre were obtained.

THEOREM 14 (Storme and Thas [1993]).

(a) *In* PG(n,q), $n \geqslant 4$, *q even and* $q > (2n - 11/2)^2$, *the inequality* $k \leqslant q + 1$ *holds for every k-arc K.*

(b) *In* PG(n,q), $n \geqslant 4$, *q even and* $q > (2n - 7/2)^2$, *every* $(q + 1)$*-arc is a normal rational curve.*

(c) *Let K be a k-arc in* PG(n,q), $n \geqslant 4$, *q even,* $q > 4$ *and* $k > q - \sqrt{q}/2 + n - 3/4$. *Then K lies in a normal rational curve C of* PG(n,q). *Also, C is completely determined by K.*

1.6. *The nonclassical* 10*-arc of* PG(4,9)

By Theorem 12(b), in PG$(4,q)$, with q odd and $q \geqslant 107$, every $(q + 1)$-arc is a normal rational curve. In Glynn [1986] a 10-arc of PG(4,9) is constructed which is not a normal rational curve. This 10-arc K consists of the following points: $(0,0,0,0,1)$ and $(1,t,t^2 + mt^6, t^3, t^4)$, with $t \in$ GF(9) and m a nonsquare. Also, Glynn [1986] shows that, up to a projectivity, this nonclassical arc together with the normal rational curve are the only 10-arcs of PG(4,9). Finally it is noted that the projection of Glynn's arc K from the line p_1p_2, with $p_1, p_2 \in K$, onto a plane PG(2,9) skew to p_1p_2, is the unique complete 8-arc of PG(2,9) (see also §14.7 in Hirschfeld [1979]).

1.7. *The nucleus or kernel of a normal rational curve, and* $(q + 2)$*-arcs in* PG$(q - 2, q)$

Theorems 15 and 16 of this section are taken from Thas [1969b].

THEOREM 15. *Let C be a normal rational curve of* PG$(2^s - 2, q)$, *with* $q = 2^h$ *and* $h \geqslant s \geqslant 3$. *Then the intersection of the* $q + 1$ *osculating hyperplanes of C is a* PG$(2^{s-1} - 2, q)$. *Also, each of the* $q + 1$ *tangents of the algebraic curve C has a point in common with* PG$(2^{s-1} - 2, q)$. *Finally, these* $q + 1$ *points of* PG$(2^{s-1} - 2, q)$ *form a normal rational curve* C_1 *of this space.*

DEFINITIONS. The curve C_1 of Theorem 15 will be called the *tangent curve* of C. The tangent curve $(C_1)_1$ of C_1 will also be denoted by C_2, etc. The curve C_{s-2} is an irreducible conic of $\mathrm{PG}(2,q)$. The nucleus of C_{s-2} will be called the *nucleus* or *kernel* of the normal rational curve C.

THEOREM 16. *Let C be a normal rational curve of* $\mathrm{PG}(q-2,q)$, $q=2^h$, *with nucleus* x. *Then $C \cup \{x\}$ is a $(q+2)$-arc of* $\mathrm{PG}(q-2,q)$.

This section concludes with an interesting theorem of Seroussi and Roth [1986] on normal rational curves.

THEOREM 17. *For $n \geqslant 2$, and $n \neq 2$ if q is even, a k-arc in* $\mathrm{PG}(n,q)$ *not contained in a normal rational curve has at most* $(q+2n-1)/2$ *points in common with any normal rational curve.*

COROLLARY. *For q even, $n \geqslant 3$, $q > 2n-4$ and q odd, $n \geqslant 2$, $q > 2n-3$, any normal rational curve of* $\mathrm{PG}(n,q)$ *is complete.*

REMARK. Using the dual hypersurface Φ_t a short proof of Theorem 17 and its corollary, for q even, is given in Blokhuis et al. [1990]. Considerable improvements of the corollary are contained in Storme and Thas [1991] and in Storme [1992]. Storme [1992] proves that any normal rational curve of $\mathrm{PG}(n,q)$ is complete whenever q is prime with $q \geqslant p_0$ or $q = p^{2h+1}$, $h \geqslant 1$, with p an odd prime and $p \geqslant p_0(h)$.

1.8. *The duality principle for k-arcs*

This section is taken from Thas [1969a].

Let K be a k-arc of $\mathrm{PG}(n,q)$, $n \geqslant 2$ and $k \geqslant n+4$, and let K consist of the points

$$p_i\big(y_0^{(i)}, y_1^{(i)}, \ldots, y_n^{(i)}\big), \quad i = 0,1,\ldots,k-1.$$

Then each submatrix of order $n+1$ of the $k \times (n+1)$ matrix $[y_j^{(i)}]$ is nonsingular. Now consider the $n+1$ hyperplanes of $\mathrm{PG}(k-1,q)$ with equations

$$y_j^{(0)}X_0 + y_j^{(1)}X_1 + \cdots + y_j^{(k-1)}X_{k-1} = 0, \quad j = 0,1,\ldots,n.$$

These hyperplanes are linearly independent, and so they intersect in a $\mathrm{PG}(k-n-2,q)$. Now take $k-n-1$ linearly independent points of $\mathrm{PG}(k-1,q)$ in this $\mathrm{PG}(k-n-2,q)$:

$$q_i\big(z_0^{(i)}, z_1^{(i)}, \ldots, z_{k-1}^{(i)}\big), \quad i = 0,1,\ldots,k-n-2.$$

Now consider the following k points of $\mathrm{PG}(k-n-2,q)$:

$$p'_j\big(z_j^{(0)}, z_j^{(1)}, \ldots, z_j^{(k-n-2)}\big), \quad j = 0,1,\ldots,k-1.$$

Then it can be shown that each submatrix of order $k-n-1$ of the $k \times (k-n-1)$ matrix $[z_j^{(i)}]$ is nonsingular. Hence $\{p'_0, p'_1, \ldots, p'_k\}$ is a k-arc of PG$(k-n-2, q)$.

In particular, if

$$[y_j^{(i)}] = \begin{bmatrix} I_{n+1} \\ Y \end{bmatrix},$$

with I_{n+1} the identity matrix of order $n+1$ and Y a $(k-n-1) \times (n+1)$ matrix, then one can put

$$[z_j^{(i)}]^T = \begin{bmatrix} -Y & I_{k-n-1} \end{bmatrix}.$$

DUALITY PRINCIPLE FOR k-ARCS. *A k-arc of* PG(n,q), $n \geqslant 2$ *and* $k \geqslant n+4$, *exists if and only if a k-arc of* PG$(k-n-2,q)$ *exists.*

Further, the following formula can be proved: for $q \geqslant \max(n+2, k-n)$, $n \geqslant 2$ and $k \geqslant n+4$,

$$\frac{\text{number of } k\text{-arcs of PG}(n,q)}{\text{number of } k\text{-arcs of PG}(k-n-2,q)} = \frac{\text{number of normal rational curves of PG}(n,q)}{\text{number of normal rational curves of PG}(k-n-2,q)}.$$

This *duality* can be applied onto the results of the preceding sections. Two examples will be given:

(i) Since $(q+2)$-arcs exist in PG$(2,q)$, q even, $(q+2)$-arcs also do exist in PG$(q-2,q)$.

(ii) Duality applied to Theorem 14 gives:

(a) In PG(n,q), $q-4 \geqslant n > q-\sqrt{q}/2-11/4$ and $q=2^h$, every k-arc K satisfies $k \leqslant q+1$.

(b) In PG(n,q), $q-5 \geqslant n > q-\sqrt{q}/2-11/4$ and $q=2^h$, every $(q+1)$-arc is a normal rational curve.

(c) Let K be a k-arc in PG(n,q), $n > q-\sqrt{q}/2-11/4$, $q=2^h$, $h>2$ and $k \geqslant n+6$. Then K lies in a normal rational curve C of PG(n,q). Also, C is completely determined by K.

1.9. *Open problems*

(a) Classify all hyperovals of PG$(2,q)$, q even; see also the private communication in 10.8(b).

(b) Is the set $D(\zeta)$ of Cherowitzo (see 1.3) a hyperoval for $h > 9$?

(c) What is the best possible $\lambda(q)$, q odd or q an even nonsquare, in Theorem 3?

(d) Is every 6-arc of $\mathrm{PG}(3,q)$, $q=2^h$ and $h>2$, contained in exactly one $(q+1)$-arc, projectively equivalent to

$$C=\{(1,t,t^e,t^{e+1}):\ t\in \mathrm{GF}(q)\}\cup\{(0,0,0,1)\},$$

where $e=2^m$ and $(m,h)=1$?

(e) For $q\geqslant n+1$, are $(q+2)$-arcs in $\mathrm{PG}(n,q)$ only possible for q even with $n=2$ or $n=q-2$?

(f) For $q\geqslant n$ with q odd, are there $(q+1)$-arcs in $\mathrm{PG}(n,q)$, other than the 10-arc of Glynn, which are not normal rational curves?

(g) Is a normal rational curve of $\mathrm{PG}(n,q)$, with $q\geqslant n+1$ and $2<n<q-2$, always complete?

2. k-arcs and MDS codes

2.1. MDS *codes*

Let C be a *code* of *length* k over an *alphabet* A of size q, $q\geqslant 2$. In other words C is simply a set of (code) words where each *word* is a k-tuple over A. Having chosen m with $2\leqslant m\leqslant k$ we impose the following condition on C: No two words in C agree in as many as m positions. It then follows that $|C|\leqslant q^m$. If $|C|=q^m$, then C is called a *Maximum Distance Separable code* (=MDS code). There is a voluminous literature on the subject. We refer to MacWilliams and Sloane [1977] for references as well as to the work of Maneri and Silverman [1971] and to the book of Hill [1986]. MacWilliams and Sloane introduce their chapter on MDS codes as 'one of the most fascinating in all of coding theory'.

The *Hamming distance* between two code words

$$x=(x_1,x_2,\ldots,x_k)\quad\text{and}\quad y=(y_1,y_2,\ldots,y_k)$$

is the number of indices i for which $x_i\neq y_i$; it is denoted by $d(x,y)$.

The *minimum Hamming distance* of C is defined by

$$\min(d(x,y):\ x,y\in C \text{ and } x\neq y)$$

and denoted by $d(C)$. If C is an MDS code then the following interesting equality holds (see, e.g., Hill [1986]).

THEOREM 1. *For any* MDS *code* $d(C)=k-m+1$.

One of the main problems concerning such codes is to maximize $d(C)$, and so k, for given m and q. Also, what is the structure of C in the optimal case?

2.2. *The general case*

First, let $m = 2$. Then C gives a set of q^2 code words of length k, no two of which agree in as many as 2 positions. It is easily seen that this is equivalent to the existence of a net of order q and degree k (see Dembowski [1968], Ryser [1963], and Chapter 10). It follows that $k \leqslant q + 1$, the case of equality corresponding to an affine plane of order q. From this, by an inductive argument, the following result is obtained.

THEOREM 2. *For any* MDS *code* $k \leqslant q + m - 1$.

The case $m = 3$ and $k = q + 2$ is equivalent to the existence of an affine plane π of order q, q even, containing an elaborate system of hyperovals. For all known examples the plane π is Desarguesian and $q = 2^h$ (see Willems and Thas [1983]). For $m = 4$ and $k = q + 3$ one can only show that either $q = 2$ or 36 divides q (see Bruen and Silverman [1983]), even though (presumably) no examples with $q > 2$ exist. Accordingly, it seems that one cannot do much with the problem in its present generality.

2.3. *Linear* MDS *codes*

Now the problem will be formulated for the case when C is *linear*, i.e. for the case that C is a m-dimensional subspace of the k-dimensional vector space $V(k, q)$ over GF(q). It goes like this. Choose any basis for C and represent it as a $m \times k$ matrix X over GF(q) of rank m. Then C is MDS if and only if every set of m columns of X is linearly independent. One can multiply the columns of X by nonzero scalars and still preserve the desired property. Therefore, regard the columns of X as points $p_1, p_2, \ldots, p_k$ of PG($m - 1, q$). From what precedes it follows that C is MDS if and only if $\{p_1, p_2, \ldots, p_k\}$ is a k-arc of PG($m - 1, q$). This gives the following fundamental result.

THEOREM 3. *Linear* MDS *codes and arcs are equivalent objects.*

Hence all results on arcs can be translated in terms of linear MDS codes. We give some examples.

THEOREM 4.

(a) *Let q be odd. Then $k \leqslant q + 1$ for* (i) $m = 3$, (ii) $m = 4$ *with* $q > 3$ *and* (iii) $m \geqslant 5$ *with* $q > (4m - 55/4)^2$.

(b) *Let q be even. Then $k \leqslant q + 2$ for $m = 3$, and $k \leqslant q + 1$ for* (i) $m = 4$ *with* $q \neq 2$, (ii) $m = 5$ *with* $q \neq 2, 4$, *and* (iii) $m \geqslant 6$ *with* $q > (2m - 15/2)^2$.

Let K be a k-arc of PG($m - 1, q$), with $3 \leqslant m \leqslant k - 3$, and let K' be a k-arc of PG($k - m - 1, q$) obtained from K by duality. If C is the linear MDS code corresponding to K and C' is the linear MDS code corresponding to K', then each vector of C is orthogonal to each vector of C'. Since $\dim C' = k - m = k - \dim C$, it follows that $C' = C^{\perp}$, i.e. C' is the *dual code* of C. This also gives a proof, for $3 \leqslant m \leqslant k - 3$, of the following theorem.

THEOREM 5. *For $2 \leqslant m \leqslant k-2$ the dual of a linear* MDS *code is again a linear* MDS *code.*

A linear code C over GF(q) is called a *generalized Reed–Solomon* (GRS) *code* if it is represented by a matrix of the form

$$X = [g_{ij}] \quad \text{with } g_{ij} = v_j t_j^{i-1}, \ 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant k.$$

Here $t_1, t_2, \ldots, t_k$ are distinct elements of GF(q); $v_1, v_2, \ldots, v_k$ are nonzero (not necessarily distinct) elements of GF(q). We define $0^0 = 1$. If one adds an extra column of the form $(0, 0, \ldots, v)^T$, with $v \neq 0$, then the resulting linear code is called a *Generalized Doubly Extended Reed–Solomon* (GDRS) *code*. It is well known that GRS codes and GDRS codes are MDS codes. From the form of the matrices immediately follows that each corresponding arc is a subset of a normal rational curve. For more details, we refer to Seroussi and Roth [1986].

3. k-caps and ovoids

3.1. *Definitions*

In PG(n, q), $n \geqslant 3$, a set K of k points no three of which are collinear is a *k-cap*. A k-cap is *complete* if it is not contained in a $(k+1)$-cap. A line of PG(n, q) is a *secant*, a *tangent*, or an *external line* of a k-cap as it meets K in 2, 1, or 0 points.

3.2. *k-caps in* PG(3, q)

It is easy to show that any elliptic quadric of PG($3, q$) is a (q^2+1)-cap. The following theorem is due to Bose [1947] for q odd and to Qvist [1952] for q even.

THEOREM 1. *For any k-cap K of* PG($3, q$), *with $q \neq 2$, the cardinality k satisfies $k \leqslant q^2+1$. For any k-cap K of* PG(3, 2), *$k \leqslant 8$ holds; the 8-caps of* PG(3, 2) *are the complements of the planes.*

DEFINITION. A (q^2+1)-cap of PG($3, q$), $q \neq 2$, is called an *ovoid* of PG($3, q$); the *ovoids* of PG(3, 2) are the elliptic quadrics of this space.

Let O be an ovoid of PG($3, q$). At each point x of O, there is a unique *tangent plane* π such that $\pi \cap O = \{x\}$. Apart from the q^2+1 tangent planes, every plane meets O in a $(q+1)$-arc. If q is even, then the $(q^2+1)(q+1)$ tangents of O are the totally isotropic lines of some symplectic polarity ζ of PG($3, q$). The proofs of these properties can be found in Hirschfeld [1985].

The following fundamental theorem is due to Barlotti [1955] and Panella [1955].

THEOREM 2. *In* PG($3, q$), *q odd or $q = 4$, any ovoid K is an elliptic quadric.*

Apart from the elliptic quadrics only one other class of ovoids is known; it is due to Tits [1962]. Let $W(q)$ be the 1-design formed by the points of PG(3, q), q even, together with the totally isotropic lines of a given symplectic polarity ζ of PG(3, q). This design has parameters $v = b = (q^2+1)(q+1)$ and $k = r = q+1$. Tits [1962] shows that $W(q)$ admits a polarity if and only if $q = 2^{2e+1}$. Further, the absolute points of such a polarity form an ovoid of PG(3, q). An ovoid of this type is an elliptic quadric if and only if $q = 2$. For $q = 2^{2e+1}$ and $e \geqslant 1$, these ovoids are called *Tits ovoids*. An ovoid other than an elliptic quadric was also found for $q = 8$ by Segre [1959]. However, Fellagara [1962] showed that Segre's example is a Tits ovoid. Fellagara [1962] further showed, by computer, that every ovoid in PG(3, 8) is either the elliptic quadric or the Tits ovoid.

The canonical form of a Tits ovoid is

$$O = \{(0,1,0,0)\} \cup \{(1,z,y,x)\colon\ z = xy + x^{\sigma+2} + y^{\sigma}\},$$

where σ is the automorphism $t \mapsto t^{e+1}$ of GF(2^{2e+1}).

For $q = 2^{2e+1}$, $e \geqslant 1$, the group of all projectivities of PG(3, q) fixing the Tits ovoid O is the simple Suzuki group Sz(q); it acts doubly transitively on O.

In O'Keefe and Penttila [1990, 1992a] it is proved, with the aid of a computer, that in PG(3, 16) the only ovoids are the elliptic quadrics; in O'Keefe, Penttila and Royle [1994] it is proved, also with the aid of a computer, that in PG(3, 32) the only ovoids are the elliptic quadrics and the Tits ovoids.

3.3. *Ovoids and inversive planes*

Let O be an ovoid of PG(3, q). Then the points of O together with the intersections $\pi \cap O$, π a nontangent plane of O, form a 3-$(q^2+1, q+1, 1)$ design $I(O)$. A 3-$(n^2+1, n+1, 1)$ design is called an *inversive plane of order* n, and the inversive planes arising from the ovoids are called the *egglike* inversive planes. The following famous theorem is due to Dembowski [1964].

THEOREM 3. *Each inversive plane of even order is egglike.*

If the ovoid O is an elliptic quadric, then the inversive plane $I(O)$ is called *classical* or *Miquelian*. By Theorem 2 any egglike inversive plane of odd order is Miquelian; for odd order no other inversive planes are known.

Let I be an inversive plane of order n. For any point x of I, the points of I different from x, together with the circles containing x (minus x), form a 2-$(n^2, n, 1)$ design, i.e. an affine plane of order n. That affine plane is denoted by I_x, and is called the *internal plane* of I at p. For an egglike inversive plane $I(O)$, each internal plane is the Desarguesian affine plane AG(2, q). The following theorem, due to Thas [1990a], gives an answer to an old fundamental problem on inversive planes.

THEOREM 4. *Let I be an inversive plane of odd order n. If for at least one point x of I the internal plane I_x is Desarguesian, then I is Miquelian.*

Up to isomorphism, there is a unique inversive plane of order $n \in \{2,3,4,5,7\}$, see Chen [1972], Denniston [1973a], Denniston [1973b] and Witt [1938]. As a direct corollary of Theorem 4 and the uniqueness of the projective plane of order n, $n \in \{3,5,7\}$, a computer-free proof of the uniqueness of the inversive plane of order n, $n \in \{3,5,7\}$, is obtained.

For more information on inversive planes, we refer to Section 5 of Chapter 6, Section 8 of Chapter 8, Section 7 of Chapter 14, and Section 4 of Chapter 17.

3.4. *Caps in ovoids*

An interesting problem is the determination of the maximum size of a cap contained in no ovoid.

THEOREM 5.

(a) (Hirschfeld [1983], Segre [1967].) *In* PG(3, q) *with q odd, $q \geqslant 67$, a k-cap with*

$$k > q^2 - \tfrac{1}{4}q^{3/2} + \frac{31q + 14\sqrt{q} - 53}{16}$$

is contained in a unique elliptic quadric.

(b) (Barlotti [1956], Thas [1968b].) *Let K be a k-cap in* PG(3, q), *q odd. If $k \geqslant q^2 - q + 7$ for $q \geqslant 7$ and $k \geqslant q^2$ for $q = 3$ or 5, then K is contained in a unique elliptic quadric.*

THEOREM 6 (Hirschfeld and Thas [1987]). *In* PG(3, q) *with q even, $q > 2$, a k-cap with*

$$k > q^2 - \tfrac{1}{2}q - \tfrac{1}{2}q^{1/2} + 2$$

is contained in a unique ovoid.

THEOREM 7 (Hirschfeld and Thas [1987]). *In* PG(3, 4), 14 *is the maximum size of a complete cap other than an ovoid. This complete* 14*-cap is unique up to a projectivity.*

The complete 14-cap of PG(3, 4) can be constructed as follows. Embed PG(3, 2) in PG(3, 4), let PG(2, 2) be a plane of PG(3, 2) and let x be a point of PG(3, 2) not in PG(2, 2). In PG(3, 4) the plane PG(2, 2) is projected from x, giving a set K of 29 points. Finally, the complete 14-cap consists of the 14 points of K not in PG(3, 2).

3.5. *Maximum size of a cap in* PG(n, q), $n \geqslant 4$

The following theorems concern the maximum size of a cap in PG(n, q), $n \geqslant 4$.

THEOREM 8.

(a) (Bose [1947].) *The maximum size of a cap in* PG(n, 2) *is* 2^n.

(b) (Hirschfeld and Thas [1987].) *Let K be a k-cap of* PG(n,q), *q even, $q>2$ and $n \geqslant 4$. Then*

$$k \leqslant q^{n-1} - \tfrac{1}{2}q^{n-2} - \tfrac{1}{2}q^{n-5/2} + \tfrac{5}{2}q^{n-3} + \tfrac{1}{2}q^{n-7/2} + (c-1)q^{n-4} \underbrace{-2\left(q^{n-5}+q^{n-6}+\cdots+q+1\right)}_{(*)} +1,$$

where $c=0$ for $q \geqslant 8$, $c=-3$ for $q=4$, and where the term $(*)$ is discarded for $n=4$.

REMARK. Part (b) may be expressed more weakly but more compactly in the following form. For q even, $q>2$ and $n \geqslant 4$, any k-cap satisfies

$$k \leqslant q^{n-1} - \tfrac{1}{2}q^{n-2} + dq^{n-7/2},$$

where $d=0$ for $q \geqslant 32$, and $d=4$ for $q \leqslant 16$.

THEOREM 9.

(a) (Hirschfeld [1983], Hirschfeld and Thas [1991], Segre [1967].) *Let K be a k-cap of* PG(n,q), *$n \geqslant 4$, q odd and $q \geqslant 197$. Then*

$$k < q^{n-1} - \tfrac{1}{4}q^{n-3/2} + 2q^{n-2}.$$

In fact, for $q \geqslant 67$ and odd,

$$k < q^{n-1} - \tfrac{1}{4}q^{n-3/2} + \tfrac{1}{16}\left(31q^{n-2} + 22q^{n-5/2} - 112q^{n-3} - 14q^{n-7/2} + 69q^{n-4}\right) - 2\left(q^{n-5}+q^{n-6}+\cdots+q+1\right) + 1,$$

where there is no term $-2(q^{n-5}+\cdots+1)$ for $n=4$.

(b) (Barlotti [1965], Hill [1978].) *If K is a k-cap of* PG(n,q), *$n \geqslant 4$, q odd and $q>7$, then*

$$k \leqslant q^{n-1} - q^{n-2} + 8q^{n-3} - 15q^{n-4} \underbrace{-2\left(q^{n-5}+q^{n-6}+\cdots+q+1\right)}_{(*)} +1,$$

where the term $(*)$ is discarded for $n=4$.

(c) (Hill [1978].) *Let K be a k-cap of* PG(n,q), *with $n \geqslant 4$. Then*

$$k \leqslant 7^{n-1} - \tfrac{19}{2}7^{n-4}\left(\tfrac{3}{4}\sqrt{7} - \tfrac{1}{3}\right) + \tfrac{4}{3} \quad \textit{for } q=7,$$

$$k \leqslant 5^{n-1} - \tfrac{19}{2}5^{n-4} + \tfrac{3}{2} \quad \textit{for } q=5, \textit{ and}$$

$$k \leqslant 2.3^{n-2} + 2 \quad \textit{for } q=3.$$

Let $q = 3$. By Theorem 9(c) $k \leqslant 20$ holds for $n = 4$ and $k \leqslant 56$ for $n = 5$. The following theorem shows that these bounds are best possible (see Hill [1973, 1978] and Pellegrino [1970]).

THEOREM 10.

(a) *If K is a k-cap of* PG(4, 3), *then* $k \leqslant 20$; *moreover in* PG(4, 3) *there are nine projectively distinct* 20*-caps.*

(b) *If K is a k-cap of* PG(5, 3), *then* $k \leqslant 56$; *moreover in* PG(5, 3) *there is a projectively unique* 56*-cap.*

Let K be a 56-cap in PG(5, 3). Then K is a subset of an elliptic quadric Q^- of PG(5, 3), where each line of Q^- has exactly two points in common with K. Also $Q^- \setminus K$ is a 56-cap. Let $x \in K$ and let π_x be the tangent hyperplane of Q^- at x. Then it is easy to show that $\pi_x \cap (Q^- \setminus K)$ is a 20-cap of π_x. Further, it is shown in Thas [1981a] that $\pi_x \cap K$ is a 11-cap of π_x, with the property that any 4 points of $\pi_x \cap K$ are linearly independent and generate a 3-dimensional space which has exactly 5 points in common with the 11-cap. Hence this 11-cap, together with these sets of 5 points, form the 4-(11, 5, 1) design of Witt [1938]. If x varies, then K provided with the sets $\pi_x \cap K$ is the 2-(56, 11, 2) design first mentioned as a design by Hall, Lane and Wales [1970]. Finally, let two points of K be adjacent if they are on a common line of Q^-; then K provided with this adjacency relation is the well-known strongly regular graph of Gewirtz [1969].

3.6. *Open problems*

(a) Classify all ovoids of PG(3, q), q even.
(b) Is every inversive plane of odd order Miquelian?
(c) In PG(3, q) what is the maximum size of a complete cap other than an ovoid?
(d) What is the maximum size of a cap in PG(n, q), $n \geqslant 4$?

4. Maximal arcs and Hermitian arcs

4.1. *Maximal arcs*

In PG(2, q) any nonempty set of k points may be described as a $\{k; m\}$-arc, where m ($m \neq 0$) is the greatest number of collinear points in the set. For given q and m ($m \neq 0$), k can never exceed $mq - q + m$, and a $\{mq - q + m; m\}$-arc will be called a *maximal arc*. Equivalently, a maximal arc may be defined as a nonempty set of points meeting every line in just m points or in none at all. Trivial maximal arcs are the plane PG(2, q) ($m = q + 1$), the affine plane AG(2, q) obtained by deleting a line L from PG(2, q) ($m = q$), and a single point ($m = 1$).

If K is a $\{mq - q + m; m\}$-arc (i.e. a maximal arc) of PG(2, q), where $m \leqslant q$, then it is easy to show that the set

$$K' = \{\text{lines } L \text{ of PG}(2, q)\colon\ L \cap K = \varnothing\}$$

is a $\{q(q-m+1)/m; q/m\}$-arc (i.e. a maximal arc) of the dual plane. Hence, if the plane PG$(2,q)$ contains a $\{mq-q+m; m\}$-arc, $m \leqslant q$, then it also contains a $\{q(q-m+1)/m; q/m\}$-arc. It follows that a necessary condition for the existence of a maximal arc, with $m \leqslant q$, is that m should be a factor of q.

4.2. *Maximal arcs in* PG$(2,q)$, *with q odd*

In 1961 Cossu proved that in PG(2,9) there is no $\{21;3\}$-arc. The following generalization is due to Thas [1975a].

THEOREM 1. *In* PG$(2,q)$, $q = 3^h$ *and* $h > 1$, *there are no* $\{2q+3;3\}$-*arcs and no* $\{q(q-2)/3; q/3\}$-*arcs.*

No other nonexistence theorem is known.

CONJECTURE. In PG$(2,q)$, q odd, the only maximal arcs are the trivial ones.

4.3. *Maximal arcs in* PG$(2,q)$, *with q even*

Every hyperoval of PG$(2,q)$, q even, is a maximal arc with $m=2$. So every hyperoval of PG$(2,q)$ also defines a maximal arc with $m=q/2$.

Denniston [1969] proves that the condition $m \mid q$ does suffice in the case of any plane PG$(2,2^h)$. Consider an irreducible homogeneous quadratic polynomial $F(X,Y)$ over GF(2^h) and also a subgroup H, of order $m=2^e$ with $0 \leqslant e \leqslant h$, of the additive group A of GF(2^h). If we choose a system of nonhomogeneous coordinates (x,y) in PG$(2,2^h)$, then

$$K = \{(x,y)\colon\ F(x,y) \in H\}$$

is a $\{2^{h+e}-2^h+2^e; 2^e\}$-arc of PG$(2,2^h)$.

Another construction of maximal arcs in PG$(2,q)$, with q even, is due to Thas [1974a]. Let O be an ovoid of PG$(3,q)$, $q=2^h$. In Section 3.2 it was mentioned that the $(q^2+1)(q+1)$ tangents of O are the totally isotropic lines of some symplectic polarity ζ of PG$(3,q)$. Call S a regular spread (see Section 8.1) consisting of q^2+1 tangents of O. Further, let PG$(3,q)$ be embedded as a hyperplane in PG$(4,q)$, and let x be a point of PG$(4,q) \setminus$ PG$(3,q)$. Call C the set of all points of PG$(4,q) \setminus$ PG$(3,q)$ which are on the lines joining x to the points of O. Then C is a maximal $\{2^{3h}-2^{2h}+2^h; 2^h\}$-arc of the projective plane PG$(2,2^{2h})$ defined by the regular spread S (see Section 8.3 for the construction of PG$(2,2^{2h})$). If O is an elliptic quadric, then it can be shown that the maximal arc is of Denniston type; if O is a Tits ovoid (so h is odd), then the maximal arc is not of Denniston type.

No other maximal arcs (as proper subsets) of PG$(2,q)$ are known.

4.4. *Hermitian arcs and Hermitian curves*

A *Hermitian arc* of PG$(2,q)$, with q a square, is a $\{q\sqrt{q}+1; \sqrt{q}+1\}$-arc H of PG$(2,q)$ such that any line of PG$(2,q)$ intersects H in either 1 or $\sqrt{q}+1$ points. The lines intersecting H in one point are called the *tangent lines* of H. At each of its points H has a unique tangent line; each point not on H is on exactly $\sqrt{q}+1$ tangent lines. Clearly a Hermitian arc together with the nontangent lines form a 2-$(q\sqrt{q}+1,\ \sqrt{q}+1, 1)$ design, i.e. a unital.

Let ζ be a unitary polarity of PG$(2,q)$, q a square. Then the absolute points of ζ form a Hermitian arc. Such a Hermitian arc is called either a *nonsingular Hermitian curve*, or a *classical Hermitian arc*. Any nonsingular Hermitian curve is projectively equivalent to the algebraic curve represented by the equation

$$X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1} = 0.$$

A section on unitals is also contained in Chapter 4.

4.5. *Characterizations of nonsingular Hermitian curves*

The following characterization is due to Lefèvre-Percsy [1982], and independently to Faina and Korchmáros [1983]. Let H be a nonsingular Hermitian curve of PG$(2,q)$. If L is any nontangent line of H, then $L\cap H$ is a Baer subline of L, i.e. $L\cap H$ is a subline PG$(1,\sqrt{q})$ of L. Conversely, any Hermitian arc having that property is a nonsingular Hermitian curve.

Hirschfeld, Storme, Thas and Voloch [1991] obtained a characterization in terms of algebraic curves: In PG$(2,q)$, q a square and $q\neq 4$, any algebraic curve of degree $\sqrt{q}+1$, without linear components, and with at least $q\sqrt{q}+1$ points in PG$(2,q)$, must be a Hermitian curve.

In Thas [1992a] a well-known conjecture is proved, by showing that in PG$(2,q)$, q a square, a Hermitian arc H is a Hermitian curve if and only if tangents of H at collinear points of H are concurrent.

4.6. *Hermitian arcs other than nonsingular Hermitian curves*

The following interesting construction is due to Buekenhout [1976a]. Let S be a regular spread of PG$(3,q)$, let PG$(3,q)$ be embedded as a hyperplane in PG$(4,q)$, and let O be an ovoid of some hyperplane $\overline{\mathrm{PG}(3,q)}$ of PG$(4,q)$, where $O\cap \mathrm{PG}(3,q)=\{x\}$ and where x does not belong to the line of S in $\mathrm{PG}(3,q)\cap\overline{\mathrm{PG}(3,q)}$. Further, let L be the line of S through x and let $y\in L\setminus\{x\}$. The cone projecting O from y will be denoted by C. Then

$$\big(C\setminus \mathrm{PG}(3,q)\big)\cup\{L\} = (C\setminus L)\cup\{L\}$$

is a Hermitian arc H of the projective plane PG$(2,q^2)$ defined by the regular spread S (the line L is one of the points at infinity of the affine plane AG$(2,q^2)$ defined by S).

If $q=2^{2e+1}$, $e\geqslant 1$, and O is the Tits ovoid, then this construction yields a Hermitian arc which is not a nonsingular Hermitian curve. Further, Metz [1979] showed that for any prime power q, $q\neq 2$, there is at least one elliptic quadric O of $\overline{\mathrm{PG}(3,q)}$ which yields a nonclassical Hermitian arc.

Hermitian arcs of the type described in this section are called *Buekenhout–Metz Hermitian arcs*.

4.7. *A characterization theorem of Tallini Scafati*

A subset K of $\mathrm{PG}(n,q)$, $n \geqslant 1$, is of *type* $(1, m, q+1)$ if every line meets it in 1, m, or $q+1$ points. More particularly, K is a $k_{m,n,q}$ if m is a fixed integer satisfying $1 \leqslant m \leqslant q$ such that

(i) $|K| = k$,

(ii) $|L \cap K| = 1$, m, or $q+1$ for each line L,

(iii) $|L \cap K| = m$ for some line L.

The following important characterization theorem is due to Tallini Scafati [1967].

THEOREM 2. *If K is a $k_{m,2,q}$, then K is one of the following.*

(I) *A Hermitian arc, i.e. a $\{q\sqrt{q}+1; \sqrt{q}+1\}$-arc; here $k = q\sqrt{q}+1$ and $m = \sqrt{q}+1$.*

(II) *A Baer subplane* $\mathrm{PG}(2, \sqrt{q})$, *i.e. a $\{q+\sqrt{q}+1; \sqrt{q}+1\}$-arc; here $k = q+\sqrt{q}+1$ and $m = \sqrt{q}+1$.*

(III) *A maximal $\{(m-2)q+m-1; m-1\}$-arc plus an external line; here $k = (m-1)q+m$.*

(IV) *The complement in* $\mathrm{PG}(2,q)$ *of a $\{(q-m)q+q-m+1; q-m+1\}$-arc, $m \geqslant 2$; here $k = m(q+1)$.*

(V) *m concurrent lines, $m \geqslant 2$; here $k = mq+1$.*

(VI) *A single line; here $k = q+1$ and $m = 1$.*

In Section 6 all $k_{m,n,q}$'s, with $n \geqslant 3$, will be classified.

4.8. *Open problems*

(a) Classify all maximal arcs in $\mathrm{PG}(2,q)$, q even.

(b) Does $\mathrm{PG}(2,q)$, q odd, contain nontrivial maximal arcs?

(c) Are the Hermitian curves and the Buekenhout–Metz Hermitian arcs the only Hermitian arcs of $\mathrm{PG}(2,q^2)$?

5. Semi-ovals, semi-ovoids and two combinatorial characterizations of ovoids

5.1. *A combinatorial characterization of ovoids*

The hyperplane $\mathrm{PG}(n-1,q)$ of $\mathrm{PG}(n,q)$, $n \geqslant 2$, is called a *tangent hyperplane* of the pointset K in $\mathrm{PG}(n,q)$ if and only if K and $\mathrm{PG}(n-1,q)$ have exactly one point in common.

Let K be a pointset of $\mathrm{PG}(2,q)$ which possesses at least one tangent line and which has exactly m, $m > 1$, points in common with every nontangent line. From Theorem 2 in 4.7 follows that K is a Hermitian arc, a Baer subplane $\mathrm{PG}(2,\sqrt{q})$ or a single line. The following extension to n dimensions, $n \geqslant 3$, is contained in Thas [1973a].

THEOREM 1. *The only pointsets of* PG(n, q), $n > 2$, *which possess at least one tangent hyperplane and have exactly* m ($m > 1$) *points in common with every nontangent hyperplane, are the lines of* PG(n, q) *and the ovoids of* PG$(3, q)$.

TERMINOLOGY. Throughout the chapter '*tangent hyperplane*' and '*tangent* (*line*)' of K have different meanings according to whether K is just considered as a pointset or as an algebraic variety, i.e. an algebraic hypersurface, an algebraic curve, ...

5.2. *Semi-ovals, semi-ovoids, and a second combinatorial characterization of ovoids*

If K is a pointset of PG(n, q), $n \geqslant 2$, then a *tangent* (*line*) to K at $x \in K$ is a line L such that $L \cap K = \{x\}$. A *semi-oval* of PG$(2, q)$ is a nonempty set of points K in PG$(2, q)$ such that for all $x \in K$ there is exactly one tangent to K at x; a *semi-ovoid* of PG(n, q), $n \geqslant 3$, is a nonempty set of points K in PG(n, q) such that for all $x \in K$ the union of all tangents to K at x is a hyperplane. The semi-oval O (resp., semi-ovoid O) is an oval (resp., an ovoid) of PG$(2, q)$ (resp., PG$(3, q)$) if and only if each nontangent line of O intersects O in just two points or in none at all.

The following examples of semi-ovals are due to Buekenhout [1976b].

(1) Ovals and Hermitian arcs in PG$(2, q)$ are semi-ovals.

(2) Let O be a semi-oval of PG$(2, q)$ and let $x \in O$ be such that all nontangent lines of O through x intersect O in more than two points. Then $O \setminus \{x\}$ still is a semi-oval of PG$(2, q)$.

(3) In PG$(2, 3)$ any quadrangle together with any two of its diagonal points is a semi-oval with six points.

In Buekenhout [1976b] it was also conjectured that any semi-ovoid of PG$(3, q)$ is an ovoid and that in PG(n, q), $n > 3$, there are no semi-ovoids. In Thas [1974b] this conjecture was proved to be true.

THEOREM 2.

(i) *If O is a semi-oval of* PG$(2, q)$, *then* $q + 1 \leqslant |O| \leqslant q\sqrt{q} + 1$.

(ii) *The only semi-ovoids of* PG$(3, q)$ *are the ovoids.*

(iii) *In* PG(n, q), $n > 3$, *there are no semi-ovoids.*

5.3. *Regular semi-ovals*

A semi-oval K of PG$(2, q)$ is *regular* provided all nontangents intersect K in either 0 or a constant number m of points. Ovals and Hermitian arcs in PG$(2, q)$ are regular. The integer m is called the *character* of the semi-oval. In Blokhuis and Szönyi [1992] the following theorem is proved.

THEOREM 3. *Let K be a regular semi-oval in* PG$(2, q)$. *Then*

(i) *K is an oval, or for the character m of K we have* $m \mid (q - 1)$;

(ii) *K is a Hermitian arc, or for the character m of K we have* $(m - 1, q) = 1$;

(iii) *K is a Hermitian arc, or the tangents at collinear points of K are concurrent.*

CONJECTURE. The only regular semi-ovals of $PG(2,q)$ are the ovals and the Hermitian arcs.

6. Sets of type $(1, m, q+1)$

6.1. *Introduction*

Sets of type $(1, m, q+1)$ and sets $k_{m,n,q}$ in $PG(n,q)$, with $1 \leqslant m \leqslant q$, were introduced in 4.7. In Theorem 2 of Section 4 all sets $k_{m,2,q}$ of $PG(2,q)$ were classified. The restriction $1 \leqslant m \leqslant q$ implies that $k_{m,n,q}$ cannot be the whole space.

The line L is called an *i-secant* of $k_{m,n,q}$ if $|L \cap k_{m,n,q}| = i$. Also a 1-secant is sometimes called a *unisecant* or a *tangent* (*line*), and $(q+1)$-secants are *lines* of $k_{m,n,q}$.

There is one important definition to distinguish points of $K = k_{m,n,q}$. A point of K is *singular* if every line through it is a unisecant or a $(q+1)$-secant. Then K is called *singular* or *nonsingular* as it has singular points or not. If K is a $k_{m,n,q}$, then the singular points of K form a subspace of $PG(n,q)$; this subspace is the *singular space* of K.

In $PG(n,q)$, a *cone* $\pi_r V$, with $\pi_r = PG(r,q)$, is the set of points on the joins of π_r with points of V, where V is contained in a subspace $\pi_s = PG(s,q)$ skew to π_r. The set π_r is the *vertex* and the set V is the *base* of the cone.

THEOREM 1. *If* $\pi_r = PG(r,q)$ *is a subspace of* $PG(n,q)$, $0 \leqslant r \leqslant n-2$, *and if* K' *is a nonsingular* $k'_{m,n-r-1,q}$ *in a subspace* $\pi_{n-r-1} = PG(n-r-1,q)$ *of* $PG(n,q)$ *skew to* π_r, *then the cone* $\pi_r K'$ *is a singular* $k_{m,n,q}$ *of* $PG(n,q)$; *the singular space of* $k_{m,n,q}$ *is the vertex* π_r *of the cone. Conversely, if* K *is a singular* $k_{m,n,q}$, *then* K *is a hyperplane or a cone* $\pi_r K'$ *where* $0 \leqslant r \leqslant n-2$ *and* K' *is a nonsingular* $k'_{m,n-r-1,q}$.

6.2. *The classification for* $m = 1$, $m = 2$ *and* $m = q$

The following two theorems are easy to prove.

THEOREM 2. *If* K *is a* $k_{m,n,q}$, *then the following are equivalent*:

(i) $m = 1$;

(ii) K *is a hyperplane*;

(iii) *all points of* K *are singular.*

THEOREM 3. *If* K *is a* $k_{2,n,2}$, $n > 2$, *then no three points of* $PG(n,2) \setminus K = K'$ *are collinear; hence for* $|K'| \geqslant n+1$ *the complement of* K *is a cap. Conversely if* K' *is a nonempty set of points of* $PG(n,2)$, $n > 2$, *no three of which are collinear, then the complement of* K' *is a* $k_{2,n,2}$, *except in the case where* K' *is the complement of a hyperplane.*

For the proofs of Theorems 4 and 5, we refer to Tallini Scafati [1967] and Hirschfeld and Thas [1980a].

THEOREM 4. *If K is a $k_{2,n,q}$, $n > 2$ and $q > 2$, it consists of a hyperplane π_{n-1} and a subspace π_r of dimension r of* PG(n, q), $0 \leqslant r \leqslant n-1$, *where* $\pi_r \not\subset \pi_{n-1}$.

THEOREM 5. *If K is a $k_{q,n,q}$, for $n > 2$ and $q > 2$, it consists of the points not in a* PG(r, q), $0 \leqslant r \leqslant n-1$, *and of the points in a* PG$(r-1, q)$ *contained in* PG(r, q).

6.3. *Projections of quadrics*

Let Q be a nonsingular quadric in PG(n, q), with $n \geqslant 3$ and $q = 2^h$, and let x be a point off Q, other than the nucleus (or kernel) in the case that n is even. Then Hirschfeld and Thas [1980a] prove the following result.

THEOREM 6. *The projection of Q from x to a hyperplane π_{n-1} is a nonsingular $k_{m,n-1,q}$ with $m = q/2 + 1$ and*

$$k = \frac{1}{2}q^{n-1} + q^{n-2} + \cdots + q + 1 + \frac{1}{2}(w-1)q^{(n-1)/2},$$

where $w = 2, 1$, or 0 as n is odd and Q is hyperbolic, n is even, or n is odd and Q is elliptic.

For such a $k_{m,n-1,q}$ the following notation will be used: R_{n-1} if n is even, R_{n-1}^{-} if n is odd and Q is elliptic, R_{n-1}^{+} if n is odd and Q is hyperbolic.

To get a better view of R_{n-1}, R_{n-1}^{-} and R_{n-1}^{+}, a description will be given without using the projection.

Let

$$F(X_0, X_1, \ldots, X_{n-2}) = 0$$

be the equation of a nonsingular quadric Q of PG$(n-2, q)$, $n \geqslant 3$ and $q = 2^h$, and let H be an additive subgroup of GF(q) of index 2. Let Q_λ be the quadric of PG$(n-1, q)$ with equation

$$F(X_0, X_1, \ldots, X_{n-2}) + \lambda X_{n-1}^2 = 0;$$

Q_∞ is the quadric of PG$(n-1, q)$ with equation $X_{n-1}^2 = 0$.

THEOREM 7 (Hirschfeld and Thas [1980a]).

$$\bigcup_{\lambda \in H \cup \{\infty\}} Q_\lambda = \begin{cases} R_{n-1} & \text{for } n \text{ even}, \\ R_{n-1}^{-} & \text{for } n \text{ odd and } Q \text{ elliptic}, \\ R_{n-1}^{+} & \text{for } n \text{ odd and } Q \text{ hyperbolic}. \end{cases}$$

REMARK. When $n = 3$ and Q is hyperbolic, we obtain that R_2^{+} is a $k_{m,2,q}$ of type IV in Theorem 2 of 4.7; when $n = 3$ and Q is elliptic, R_2^{-} is a $k_{m,2,q}$ of type III in Theorem 2 of 4.7.

6.4. *The classification for* $3 \leqslant m \leqslant q-1$, $n \geqslant 3$, *and* $q \neq 4$

The complete classification of all nonsingular $k_{m,n,q}$ with $3 \leqslant m \leqslant q-1$, $n \geqslant 3$, and $q > 4$, is given by the following important theorem.

THEOREM 8. *If K is a nonsingular $k_{m,n,q}$, with $3 \leqslant m \leqslant q-1$, $n \geqslant 3$, and $q > 4$, then K is either a nonsingular Hermitian variety (i.e. the set of all absolute points of a unitary polarity) or one of the sets R_n, R_n^+, R_n^-.*

For $m \neq q/2+1$ the result is due to Tallini Scafati [1967] and for $m = q/2+1$, $n > 3$ and part of $n = 3$, to Hirschfeld and Thas [1980a,b]. The missing part in the case $m = q/2+1$ and $n = 3$ was done by Glynn [1983b].

6.5. *The classification for* $m = 3$, $n \geqslant 3$ *and* $q = 4$

First, note that a $k_{3,n,4}$, $n \geqslant 2$, is a set of type $(1,3,5)$, i.e. every line meets it in 1, 3, or 5 points. Accordingly a set K in PG$(n,4)$, $n \geqslant 2$, is a set of *odd type* if every line meets K in an odd number of points. This immediately implies a closure operation on such sets.

Let O_n be the set of all sets of odd type in PG$(n,4)$. For K, K' in O_n, define

$$K \bigtriangledown K' = \mathrm{PG}(n,4) \setminus (K \bigtriangleup K');$$

i.e. $K \bigtriangledown K'$ comprises the points in $K \cap K'$ and the points in PG$(n,4) \setminus (K \cup K')$. Let H_n be the set of Hermitian varieties in PG$(n,4)$, where PG$(n,4)$ is included in the set. The proof of the following theorem can be found in Hirschfeld [1985].

THEOREM 9. *For K, K' in O_n,*

(i) *$K \bigtriangledown K'$ is a set of odd type;*

(ii) $|K \bigtriangledown K'| = |\mathrm{PG}(n,4)| - |K| - |K'| + 2|K \cap K'|$;

(iii) *O_n is a binary vector space under $\bigtriangledown$ whose zero is* PG$(n,4)$;

(iv) *H_n is a subspace of dimension $(n+1)^2$ of O_n.*

For $n = 2$ the vector space O_2 has dimension 11, and in PG$(2,4)$ there are seven projectively distinct sets K of odd type. In Hirschfeld and Hubaut [1980], the case $n = 3$ is considered in detail. There it is proved that the vector space O_3 has dimension 24, and that in PG$(3,4)$ there are fourteen projectively distinct sets K of odd type. Seven of these sets are nonsingular $k_{3,3,4}$'s. For any nonsingular $k_{3,3,4}$, we have $k \in \{33, 37, 41, 45, 49, 53\}$; for $k = 45$ there are two projectively distinct nonsingular $k_{3,3,4}$'s.

The following notation will be used:

$$E = \sum c_{ijk} X_i X_j X_k,$$

with $c_{ijk} \in \mathrm{GF}(4)$ and where the summation is from 0 to n with $i < j < k$;

$$H = \sum_{i=0}^{r} L_i^3,$$

with L_i a homogeneous linear polynomial in $X_0, X_1, \ldots, X_n$ (so H is a Hermitian form). The next theorem is taken from Sherman [1983].

THEOREM 10. *The sets of odd type in* $\mathrm{PG}(n, 4)$ *are the sets which can be represented by an equation of the form* $E^2 + E + H = 0$; *the polynomials* E *and* H *are uniquely defined by the set of odd type.*

To illustrate Theorem 10, we give a form $E^2 + E + H$ for each type of element in O_2:

(I) $X_0^3 + X_1^3 + X_2^3$,

(II) $X_0^2X_1^2X_2^2 + X_0X_1X_2 + X_0^3 + X_1^3 + X_2^3 + (X_0 + X_1 + X_2)^3$,

(III) $X_0^2X_1^2X_2^2 + X_0X_1X_2 + (X_0 + X_1 + X_2)^3$,

(IV) $X_0^2X_1^2X_2^2 + X_0X_1X_2$,

(V) $X_0^3 + X_1^3$,

(VI) X_0^3,

(VII) 0.

Finally two interesting corollaries of Theorem 10 are mentioned, the proofs of which can be found in Hirschfeld [1985].

COROLLARY 1. *The dimension of* O_n *as a binary vector space is* $(n^3 + 3n^2 + 5n + 3)/3$.

COROLLARY 2. *The rank over* $\mathrm{GF}(2)$ *of the incidence matrix of points and lines in* $\mathrm{PG}(n, 4)$ *is* $(4^{n+1} - n^3 - 3n^2 - 5n - 4)/3$.

7. Blocking sets

7.1. *Introduction*

A *(t, s)-blocking set* of $\mathrm{PG}(n, q)$, where $n \geqslant 2$, $n \geqslant s \geqslant 1$, and $n - 1 \geqslant t \geqslant 0$, is a set B of points of $\mathrm{PG}(n, q)$ satisfying the following properties:

(i) any subspace of dimension $n - t$ of $\mathrm{PG}(n, q)$ intersects B in at least one point;

(ii) any s-dimensional subspace of $\mathrm{PG}(n, q)$ contains at least one point not in B.

If $t \leqslant s - 1$, with $0 \leqslant t$ and $n \geqslant s$, then any t-dimensional subspace of $\mathrm{PG}(n, q)$, $n \geqslant 2$, is a (t, s)-blocking set.

A *blocking set* of $\mathrm{PG}(n, q)$, $n \geqslant 2$, is a set B of points of $\mathrm{PG}(n, q)$ satisfying:

(i) any hyperplane of $\mathrm{PG}(n, q)$ intersects B in at least one point;

(ii) any line of $\mathrm{PG}(n, q)$ contains at least one point not in B.

So a blocking set is the same as a (1,1)-blocking set.

If B is a (t, s)-blocking set of $\mathrm{PG}(n, q)$, then clearly $\mathrm{PG}(n, q) \setminus B$ is a $(n - s, n - t)$-blocking set of $\mathrm{PG}(n, q)$.

Blocking sets in projective planes are also considered in Section 3.6 of Chapter 4.

7.2. *Baer cones*

Let PG(r,q) and PG(d,q) be skew subspaces of PG(n,q), with $n \geqslant 2$, $r \geqslant -1$, and $d \geqslant 1$. If q is a square, there is a *Baer subspace* PG$(d,\sqrt{q})$ in PG(d,q). The set

$$C = \mathrm{PG}(r,q)\mathrm{PG}(d,\sqrt{q}) = \bigcup_{x\in \mathrm{PG}(d,\sqrt{q})} \mathrm{PG}(r,q)x,$$

i.e. the cone with vertex PG(r,q) and base PG$(d,\sqrt{q})$, is called a *Baer cone*. We also say that C is a Baer cone of *type* (r,d).

The following result is not difficult to show: *any Baer cone of type* $(s-2, 2(t-s+1))$ *of* PG(n,q), *with* $n \geqslant 2$, $n \geqslant s \geqslant 1$, $n-1 \geqslant t \geqslant 0$, *and* $t \geqslant s$, *is a* (t,s)*-blocking set of* PG(n,q).

7.3. *Main results*

The following well-known result is due to Bose and Burton [1966].

THEOREM 1. *Let B be a (t,s)-blocking set in* PG(n,q) *with* $t \leqslant s-1$. *Then*

$$|B| \geqslant q^t + q^{t-1} + \cdots + q + 1$$

with equality if and only if B is the pointset of a t-dimensional subspace of PG(n,q).

COROLLARY. *Let B be a (t,s)-blocking set in* PG(n,q) *with* $t \leqslant s-1$. *Then*

$$|B| \leqslant q^n + q^{n-1} + \cdots + q^{n-s+1}$$

with equality if and only if B is the complement of a $(n-s)$-dimensional subspace of PG(n,q).

In Huber [1987] the next fundamental theorem is proved.

THEOREM 2. *Let B be a (t,s)-blocking set of* PG(n,q), *with* $t \geqslant s$ *and* $q \geqslant 5$. *Then*

$$|B| \geqslant q^t + q^{t-1} + \cdots + 1 + \sqrt{q}\big(q^{t-1} + q^{t-2} + \cdots + q^{s-1}\big),$$

with equality if and only if B is a Baer cone of type $(s-2, 2(t-s+1))$.

COROLLARY. *Let B be a (t,s)-blocking set of* PG(n,q), *with* $t \geqslant s$ *and* $q \geqslant 5$. *Then*

$$|B| \leqslant q^n + q^{n-1} + \cdots + q^{n-s+1} - \sqrt{q}\big(q^{n-s-1} + q^{n-s-2} + \cdots + q^{n-t-1}\big),$$

with equality if and only if the complement of B is a Baer cone of type $(n-t-2, 2(t-s+1))$.

The case $s = 1$ of Theorem 2 (without the restriction on q) is also due to Beutelspacher [1983]; the case $t = 1$ of Theorem 2 (without the restriction on q) is also due to Bruen [1980]. In fact, the first basic result on blocking sets is the following well-known theorem of Bruen [1971a].

THEOREM 3. *Let B be a $(1,1)$-blocking set of* PG$(2,q)$. *Then*

$$q + \sqrt{q} + 1 \leqslant |B| \leqslant q^2 - \sqrt{q},$$

with $|B| = q + \sqrt{q} + 1$ if and only if B is a Baer subplane of PG$(2,q)$ *and $|B| = q^2 - \sqrt{q}$ if and only if B is the complement of a Baer subplane of* PG$(2,q)$.

REMARK (Huber [1985]). For $q < 5$, Theorem 2 and its corollary remain valid if we add the following condition: there exists a point x of B such that through x there are at most

$$q^{s-2} + q^{s-3} + \cdots + q + 1$$

lines which are totally contained in B.

The following existence theorem was proved by Mazzocca and Tallini [1985].

THEOREM 4. *For any prime power q and any two natural numbers s and t, with $s \geqslant 1$, there exists an integer $f(s,t,q)$ such that* PG(n,q), *with $s \leqslant n$ and $t \leqslant n-1$, contains a (t,s)-blocking set if and only if $n \leqslant f(s,t,q)$.*

7.4. *Hyperplane coverings and minimal blocking sets*

A (t,s)-blocking set B is *minimal* (or *reduced*) if no proper subset of B also is a (t,s)-blocking set. This is equivalent to saying that for each point x of B there is at least one $(n-t)$-dimensional subspace π_{n-t} of PG(n,q) through x for which $\pi_{n-t} \cap B = \{x\}$. The following theorem on minimal blocking sets, i.e. minimal (1,1)-blocking sets, is due to Bruen and Thas [1982].

THEOREM 5. *Let B be a minimal blocking set in* PG(n,q). *Then we have the following.*

(i) *For $n = 2$, $|B| \leqslant q\sqrt{q} + 1$. If $|B| = q\sqrt{q} + 1$, then B is a Hermitian arc of* PG$(2,q)$.

(ii) *For $n = 3$, $|B| \leqslant q^2 + 1$. If $|B| = q^2 + 1$, then B is an ovoid of* PG$(3,q)$.

(iii) *For $n \geqslant 4$, $|B| < \sqrt{q^{n+1}} + 1$.*

Related to this is the theory of minimal sets of representatives of vector spaces and the theory of minimal hyperplane coverings of vector spaces. Let $V = V(n,q)$, $n \geqslant 2$, be the n-dimensional vector space over GF(q). A *hyperplane* of V is a translate of a subspace of co-dimension one in V. Following Jamison [1977] a subset S of V is a *set of representatives* provided that S has a nonempty intersection with each hyperplane in V. The set of representatives S is *minimal* (or *reduced*) if it has no proper subset of representatives. Let us denote by $N(q,n)$ the maximum size of a minimal set of representatives in V. A *covering* of the set $V \setminus \{\bar{0}\}$ of all nonzero vectors in V is a set T of hyperplanes not containing $\bar{0}$, such that each nonzero vector is in at least one element of T. The covering is called *minimal* (or *reduced*) if no proper subset of it is also a covering. Following Jamison [1977] we denote by $M(q,n)$ the maximal cardinality of a minimal covering of $V \setminus \{\bar{0}\}$ by hyperplanes not containing the zero vector.

THEOREM 6 (Bruen and Thas [1982]). *Either* $M(q,n) = N(q,n)$ *or* $M(q,n) = N(q,n) - 1$. *Moreover* $N(q,n) < \sqrt{q^{n+1}} + 1$, *and*

(1) $N(q,2) = q\sqrt{q}$ *if* q *is a square, the bound being attained by the affine part of a unital in* PG$(2,q)$, *the projective plane defined by* $V(2,q)$, *that has the line at infinity as a tangent line,*

(2) $N(q,3) = q^2$, *the bound being attained by the affine part of an ovoid in* PG$(3,q)$, *the projective space defined by* $V(3,q)$, *which has the plane at infinity as a tangent plane.*

7.5. *Open problems*

(a) Does Theorem 2 hold for $q < 5$?
(b) What is $f(s,t,q)$ in Theorem 4?
(c) What is the maximum size of a minimal blocking set in PG(n,q), $n > 3$?
(d) For which values of q and n is $M(q,n) = N(q,n)$?
(e) What is the maximum size of a minimal set of representatives in $V(n,q)$, $n > 3$?

8. Spreads and partial spreads

8.1. *t-spreads and partial t-spreads of* PG(n,q)

A *t-spread* of PG(n,q), with $1 \leqslant t < n$, is a set of t-dimensional subspaces of PG(n,q) which partitions PG(n,q). A *partial t-spread* of PG(n,q) is a set S of mutually skew t-dimensional subspaces; the partial t-spread S is said to be *maximal* if there is no partial t-spread of PG(n,q) containing S as a proper subset.

In PG(n,q), with $n+1 = r(t+1)$ and $1 \leqslant t < n$, a t-spread can be constructed as follows. Let $P_0, P_1, \ldots, P_t$ be $t+1$ $(r-1)$-dimensional subspaces of PG(n, q^{t+1}) which generate the space and are conjugate with respect to the $(t+1)$-th extension GF(q^{t+1}) of GF(q). Further, let x_0 be any point of P_0 and let $x_1, x_2, \ldots, x_t$ be the points conjugate to x_0. Then $x_i \in P_i$, with $i = 0, 1, \ldots, t$, and the points $x_0, x_1, \ldots, x_t$ generate a t-dimensional subspace PG(t, q^{t+1}) of PG(n, q^{t+1}). The intersection of PG(t, q^{t+1}) and the space PG(n,q) is a t-dimensional subspace PG(t,q) of PG(n,q). The set of these $(q^{n+1} - 1)/(q^{t+1} - 1)$ spaces PG(t,q) is a t-spread of PG(n,q) which is denoted by $\mathcal{S}(P_0, P_1, \ldots, P_t)$. Such a t-spread $\mathcal{S}(P_0, P_1, \ldots, P_t)$ will be called *regular*. Hence a t-spread of PG(n,q) exists if $t+1$ divides $n+1$, with $1 \leqslant t < n$. Conversely if a t-spread of PG(n,q) exists, then the number of points of PG(t,q) divides the number of points of $\mathrm{PG}(n,q)$; so $q^{t+1}-1$ divides $q^{n+1}-1$, and consequently $t+1$ divides $n+1$. This gives

THEOREM 1. *A t-spread of* $\mathrm{PG}(n,q)$, $1 \leqslant t < n$, *exists if and only if* $t+1$ *divides* $n+1$.

8.2. *Geometric t-spreads*

Let S be a t-spread of $\mathrm{PG}(n,q)$ and let π be a subspace of $\mathrm{PG}(n,q)$. We say that S induces a spread in π, if for any element $\mathrm{PG}(t,q)$ of S either $\mathrm{PG}(t,q) \cap \pi = \varnothing$ or $\mathrm{PG}(t,q) \subset \pi$. The spread S is called *geometric* if S induces a spread in the space generated by any two elements of S. Clearly any regular t-spread is geometric. For a geometric t-spread S in $\mathrm{PG}(r(t+1)-1,q)$, we denote by $\mathcal{I}(S)$ the following incidence structure: the points of $\mathcal{I}(S)$ are the elements of S; the blocks of $\mathcal{I}(S)$ are the subspaces generated by any two elements of S; incidence is containment.

THEOREM 2 (Segre [1964]). *For any geometric t-spread S of* $\mathrm{PG}(r(t+1)-1,q)$ *with* $r>2$, *the incidence structure* $\mathcal{I}(S)$ *is isomorphic to the incidence structure formed by all points and all lines of* $\mathrm{PG}(r-1,q^{t+1})$ *and S is regular.*

8.3. *t-spreads of* $\mathrm{PG}(2t+1,q)$

If S is a t-spread of $\mathrm{PG}(2t+1,q)$, $t \geqslant 1$, then an affine translation plane $\mathcal{A}(S)$ of order q^{t+1} can be constructed as follows. Embed $\mathrm{PG}(2t+1,q)$ as a hyperplane in a space $\mathrm{PG}(2t+2,q)$. Points of $\mathcal{A}(S)$ are the points of $\mathrm{PG}(2t+2,q)\setminus\mathrm{PG}(2t+1,q)$; lines of $\mathcal{A}(S)$ are the $(t+1)$-dimensional subspaces of $\mathrm{PG}(2t+2,q)$ containing an element of S but not contained in $\mathrm{PG}(2t+1,q)$; incidence in $\mathcal{A}(S)$ is the incidence of $\mathrm{PG}(2t+2,q)$. The plane $\mathcal{A}(S)$ is Desarguesian if and only if the spread S is regular. Also, every finite affine translation plane can be obtained in this way. It is fair to mention here André [1954] who was the first to describe the correspondence between spreads and translation planes. For a detailed study of these t-spreads and the corresponding translation planes, we refer to Chapter 5.

The previous construction applied to any t-spread of $\mathrm{PG}(n,q)$ gives a linear space with parallelism (see Chapter 6).

A *t-regulus* in $\mathrm{PG}(2t+1,q)$ is a set R of $q+1$ mutually skew t-dimensional subspaces with the property: if a line L of $\mathrm{PG}(2t+1,q)$ intersects three elements of R, then L intersects all elements of R. Such a line L is called a *transversal* of R; the set of all transversals of R is denoted by R^T. If $t=1$, then R^T is a 1-regulus of $\mathrm{PG}(3,q)$, too. It is well known (see, e.g., Hirschfeld and Thas [1991], §25.6) that for any three mutually skew t-dimensional subspaces π_1, π_2, π_3 of $\mathrm{PG}(2t+1,q)$ there is exactly one t-regulus $R=R(\pi_1,\pi_2,\pi_3)$ containing π_1, π_2, π_3, and that for any t-regulus R of $\mathrm{PG}(2t+1,q)$ the union of the elements of R (and R^T) is a Segre variety[2] $S_{1;n}$; for $t=1$ the union of the lines of R (and R^T) is a Segre variety $S_{1;1}$, i.e. a hyperbolic quadric of $\mathrm{PG}(3,q)$.

For the proof of the following theorem, we also refer to Hirschfeld and Thas [1991], §25.6.

[2] Segre varieties are treated in Hirschfeld and Thas [1991].

THEOREM 3. *The following are equivalent:*

(i) *if* π_1, π_2, π_3 *are any three distinct elements of the t-spread* S *of* $\mathrm{PG}(2t+1,q)$, *then the whole t-regulus* $R(\pi_1,\pi_2,\pi_3)$ *is contained in* S;

(ii) S *is a t-spread of* $\mathrm{PG}(2t+1,q)$ *such that the t-spaces of* S *meeting any line not in an element of* S *form a t-regulus.*

Finally, any regular t-spread S *of* $\mathrm{PG}(2t+1,q)$ *satisfies* (i) *and* (ii).

The following interesting theorem was proved by Bruck and Bose [1966].

THEOREM 4. *For* $q > 2$ *any t-spread* S *of* $\mathrm{PG}(2t+1,q)$ *which satisfies* (i) *or* (ii) *in the statement of Theorem* 3, *is regular.*

For $q = 2$ conditions (i) and (ii) are trivially satisfied. Many examples of nonregular t-spreads in $\mathrm{PG}(2t+1,2)$ are known (see, e.g., Chapter 5).

In Hirschfeld and Thas [1991], §25.6, the following constructions for t-spreads in $\mathrm{PG}(2t+1,q)$ are given.

Consider a projective space $\mathrm{PG}(2m,q^2)$, $m \geqslant 1$, and let P be a partition of it by projective $2m$-dimensional subspaces $\pi_{i,q}$ over $\mathrm{GF}(q)$, $i = 1,2,\ldots,(q^{2m+1}+1)/(q+1)$. By Hirschfeld [1979], §4.3, such a partition P exists. Embed $\mathrm{PG}(2m,q^2)$ in the extension $\mathrm{PG}(4m+1,q^2)$ of $\mathrm{PG}(4m+1,q)$, and assume that $\mathrm{PG}(2m,q^2)$ contains no point of $\mathrm{PG}(4m+1,q)$. If $x \in \pi_{i,q}$ and if $\tilde{x}$ is the conjugate point of x with respect to the extension $\mathrm{GF}(q^2)$ of $\mathrm{GF}(q)$, then the intersection of the line $x\tilde{x}$ and the space $\mathrm{PG}(4m+1,q)$ is a line L_x of $\mathrm{PG}(4m+1,q)$. Then it can be shown that $\{L_x\colon\ x \in \pi_{i,q}\}$ is a set R_i^T of transversals of a $2m$-regulus R_i in $\mathrm{PG}(4m+1,q)$. The $q^{2m+1}+1$ elements of these $2m$-reguli R_i, $i = 1,2,\ldots,(q^{2m+1}+1)/(q+1)$, form a $2m$-spread S of $\mathrm{PG}(4m+1,q)$. We conclude that each partition P of $\mathrm{PG}(2m,q^2)$ by $2m$-spaces over $\mathrm{GF}(q)$ defines a $2m$-spread S of $\mathrm{PG}(4m+1,q)$.

Next, consider a projective space $\mathrm{PG}(2m+1,q^2)$, $m \geqslant 0$. Let P be a partition of $\mathrm{PG}(2m+1,q^2)$ consisting of α spaces $\pi_{i,q}$ of dimension $2m+1$ over $\mathrm{GF}(q)$ and β spaces π_{j,q^2} of dimension m over $\mathrm{GF}(q^2)$; then $\alpha(q+1)+\beta = q^{2m+2}+1$. Embed $\mathrm{PG}(2m+1,q^2)$ in the extension $\mathrm{PG}(4m+3,q^2)$ of $\mathrm{PG}(4m+3,q)$, and assume that $\mathrm{PG}(2m+1,q^2)$ does not contain a point of $\mathrm{PG}(4m+3,q)$. As in the previous paragraph, the space $\pi_{i,q}$ defines a $(2m+1)$-regulus R_i of $\mathrm{PG}(4m+3,q)$. The m-dimensional space π_{j,q^2} and its conjugate $\tilde{\pi}_{j,q^2}$ generate a $(2m+1)$-dimensional space π'_{j,q^2} of $\mathrm{PG}(4m+3,q^2)$, and $\pi'_{j,q^2} \cap \mathrm{PG}(4m+3,q)$ is a $(2m+1)$-dimensional space $\pi'_{j,q}$ of $\mathrm{PG}(4m+3,q)$. Then the elements of the α $(2m+1)$-reguli R_i together with the β spaces $\pi'_{j,q}$ form a $(2m+1)$-spread S of $\mathrm{PG}(4m+3,q)$; if $m = 0$, then we can take the lines of either R_i or R_i^T, $i = 1,2,\ldots,\alpha$.

We conclude this section with two theorems on maximal partial spreads, due to Beutelspacher [1980] and Bruen [1980], respectively.

THEOREM 5. *Let* S *be a maximal partial t-spread of* $\mathrm{PG}(2t+1,q)$, $t \geqslant 1$, *which is not a spread. Then*

$$q + \frac{q^{t-1}\sqrt{q}+1}{q^{t-1}+q^{t-2}+\cdots+q+1} \leqslant |S| \leqslant q^{t+1} - \frac{q^{t-1}\sqrt{q}}{q^{t-1}+q^{t-2}+\cdots+q+1}.$$

Moreover, if $t > 1$, *then*

$$q + \frac{q^{t-1}\sqrt{q}+1}{q^{t-1}+q^{t-2}+\cdots+q+1} < |S| < q^{t+1} - \frac{q^{t-1}\sqrt{q}}{q^{t-1}+q^{t-2}+\cdots+q+1}.$$

THEOREM 6. *Let* S *be a maximal partial t-spread of* $\mathrm{PG}(2t+1,q)$, $t \geqslant 1$, *which is not a spread. Then the following bounds hold:*

(i) $|S| \geqslant q+\sqrt{q}+1$ *for* $q \geqslant 4$;

(ii) $|S| \leqslant q^{t+1}-\sqrt{q}$;

(iii) *if* q *is not a square, then* $(d-1)(d^3-d^2+d+2)/2 \geqslant q^{t+1}$, *with* $d = q^{t+1}+1-|S|$.

8.4. *Spreads and partial spreads of* $\mathrm{PG}(3,q)$

In $\mathrm{PG}(3,q)$ a 1-spread is simply called a *spread*; a partial 1-spread is called a *partial spread*. Let S be a spread of $\mathrm{PG}(3,q)$. Then it is easy to show that each plane of $\mathrm{PG}(3,q)$ contains exactly one line of S. Hence, if Q^+ is the hyperbolic quadric in $\mathrm{PG}(5,q)$ and if O is the subset of Q^+ corresponding to S by the Klein correspondence, then any plane of Q^+ has exactly one point in common with O. A subset of Q^+ having that property is called an *ovoid* of Q^+. The spread S is regular if and only if the ovoid O is an elliptic quadric, i.e. if and only if O is the intersection of Q^+ with some $\mathrm{PG}(3,q)$.

It is easy to show that any spread of $\mathrm{PG}(3,2)$ is regular. Let S be a regular spread of $\mathrm{PG}(3,q)$ and let R be a 1-regulus, in short a *regulus*, contained in S. Then it is clear that $(S \setminus R) \cup R^T = S'$ also is a spread of $\mathrm{PG}(3,q)$. For $q > 2$ the spread S' is not regular, see, e.g., Hirschfeld [1985], §17.1. Further, it was shown by Bruck [1969] that in $\mathrm{PG}(3,3)$ any spread S either is regular or can be obtained from a regular spread by reversing one regulus.

A *subregular sequence of spreads of length k* is a sequence $S_0, S_1, \ldots, S_k$ of spreads such that (i) S_0 is regular, (ii) $S_{i+1} = (S_i \setminus R) \cup R^T$, where R is a regulus of S_i. A spread S is *subregular of index k* if (i) there exists a subregular sequence $S_0, S_1, \ldots, S_k$ with $S = S_k$, (ii) there is no shorter sequence beginning with a regular spread and ending with S. It follows that a regular spread is subregular of index 0. The following interesting theorem on subregular spreads is due to Orr [1976].

THEOREM 7. *Every subregular spread in* $\mathrm{PG}(3,q)$ *is obtainable by reversing a set of disjoint reguli in some regular spread* S.

A *packing* $\mathcal{P}$ of $\mathrm{PG}(3,q)$ is a partition of the lines of the space into spreads. So $\mathcal{P}$ comprises q^2+q+1 spreads, no two of which have a line in common.

THEOREM 8 (Denniston [1972]). *If* $q > 2$, *then* $\mathrm{PG}(3,q)$ *has at least two projectively distinct packings consisting of one regular spread and* q^2+q *subregular spreads of index one.*

In connection with packings of $\mathrm{PG}(3,2)$, a problem first posed in 1850 turns out to be pertinent. Kirkman's fifteen schoolgirls problem is to find an arrangement whereby

15 schoolgirls go walking each day in five rows of three so that in a week each girl has walked in the same row as every other girl. Now let $\mathcal{P}$ be a packing of PG(3, 2). If each point of PG(3, 2) corresponds to a girl, each line to a row, and each spread to a day, then the packing provides a solution to the problem. In 1910 Conwell proved the following theorem.

THEOREM 9. *There are* 240 *packings of* PG(3, 2), *which fall into two orbits of* 120 *under the action of* PGL(4, 2); *a correlation interchanges these two classes.*

There are solutions to the schoolgirls problem which cannot be embedded in PG(3, 2). The full set of solutions was first given by Woolhouse in 1862 and 1863; for an account we also refer to Cole [1922]. Finally, for some history of the fifteen schoolgirls problem, see Biggs [1981].

For packings in higher dimensions we refer to Beutelspacher [1974], Fuji-Hara and Vanstone [1984].

Let S be a maximal partial spread of PG(3, q); the *deficiency* δ of S is the integer $q^2 + 1 - |S|$. In PG(3, 2) the only maximal partial spreads are the spreads.

THEOREM 10. *Let S be a maximal partial spread with deficiency $\delta > 0$.*

(i) (Mesner [1967].) *We have $\delta \geqslant \sqrt{q} + 1$. If $\delta = \sqrt{q} + 1$, then the points of* PG(3, q) *which are not contained in S form a subgeometry* PG(3, $\sqrt{q}$).

(ii) (Bruen [1975].) *For $q = p^r$ with r odd, we have $(\delta - 1)(\delta^3 - \delta^2 + \delta + 2)/2 \geqslant q^2$.*

(iii) (Heden [1986].) *If $q = p^r$, if p does not divide $\delta - 1$, and if $\delta < (q + 1)/2$, then $\delta > 1 + (1 + \sqrt{5})\sqrt{q}/2$.*

(iv) (Blokhuis, Brouwer and Wilbrink [1989].) *If $q = p^r$ is not a square and if $\delta < 1 + \sqrt{3q}$, then $\delta > \sqrt{pq} - p + 1$.*

(v) (Glynn [1982].) *We always have $|S| \geqslant 2q$.*

Mesner [1967] has constructed in PG(3, 4) a maximal partial spread S for which $|S| = q^2 - \sqrt{q} = 14$. In [1976] Bruen and Thas gave the following construction of such a maximal partial spread. In Σ = PG(3, 2) we form a packing $\mathcal{P}$ consisting of seven (regular) spreads $S_1, S_2, \ldots, S_7$. Embed Σ in Σ^* = PG(3, 4). For each regular spread S_i, $1 \leqslant i \leqslant 7$, its five lines are concurrent with two mutually skew lines of Σ^* which are conjugate with respect to the quadratic extension of GF(2). So there arise 14 lines of Σ^*, any of them having no point in Σ. Since the seven spreads S_i constitute a packing, these 14 lines are mutually skew. Consequently they constitute a partial spread S of Σ^*. Moreover, the set of points of Σ^* lying on lines of S is precisely the set of points of $\Sigma^* \setminus \Sigma$. Since a line of Σ^* cannot be entirely contained in Σ, we conclude that S is a maximal partial spread of size 14 in PG(3, 4) = Σ^*.

We conclude this section with an existence theorem on maximal partial spreads.

THEOREM 11.

(i) (Bruen [1971b], Bruen and Thas [1976], Freeman [1980].) *In* PG(3, q), *with $q > 3$, a maximal partial spread of size $q^2 - q + 2$ always exists.*

(ii) (Beutelspacher [1980].) *In* PG$(3,q)$ *there exists a maximal partial spread of size* q^2+1-mq *for every integer* m *such that* $0 \leqslant m \leqslant q/2-1$.

8.5. *Partition of* PG(n,q^k) *into subgeometries* PG(n,q)

Instead of considering partitions of PG(n,q^k) consisting of subspaces PG(t,q^k), we may consider partitions of PG(n,q^k) consisting of subgeometries PG(n,q). Such partitions were already used to construct spreads in Section 8.3.

THEOREM 12. *There exists a partition of* PG(n,q^k) *into subgeometries* PG(n,q) *if and only if* $(q^{n+1}-1)/(q-1)$ *divides* $(q^{k(n+1)}-1)/(q^k-1)$, *i.e. if and only if* $(k,n+1)=1$.

Theorem 12 was first proved by Yang [1949], although his proof of the latter equivalence in the statement is incorrect; a correct proof of it is due to Tyrrell.

8.6. *Open problems*

(a) Classify all t-spreads of PG(n,q); in particular, classify all spreads of PG$(3,q)$.

(b) What is the maximum (resp., minimum) size of a maximal partial t-spread of PG$(2t+1,q)$, $t \geqslant 1$, which is not a spread? In particular, what about the case $t=1$?

(c) Determine all packings of PG$(3,q)$.

(d) Is there a maximal partial spread of PG$(3,q)$, q a square and $q \neq 4$, with deficiency $\delta = \sqrt{q}+1$? Metsch [private communication] answered negatively.

(e) Determine all partitions of PG(n,q^k) into subgeometries PG(n,q); in particular, classify all partitions of PG$(2,q^2)$ into subplanes PG$(2,q)$.

9. Ovoids and spreads of classical polar spaces, hemisystems

9.1. *Finite classical polar spaces*

Polar spaces were introduced in Section 4 of Chapter 2. Let P be a finite classical polar space of rank r, with $r \geqslant 2$. We shall use the following notation:

$W_n(q)$: the polar space arising from a symplectic polarity of PG(n,q), n odd and $n \geqslant 3$: here $r=(n+1)/2$;

$Q(2n,q)$: the polar space arising from a nonsingular quadric in PG$(2n,q)$, $n \geqslant 2$: here $r=n$;

$Q^+(2n+1,q)$: the polar space arising from a nonsingular hyperbolic quadric in PG$(2n+1,q)$, $n \geqslant 1$: here $r=n+1$;

$Q^-(2n+1,q)$: the polar space arising from a nonsingular elliptic quadric in PG$(2n+1,q)$, $n \geqslant 2$: here $r=n$;

$H(n,q^2)$: the polar space arising from a nonsingular Hermitian variety H in PG(n,q^2), $n \geqslant 3$: for n odd $r=(n+1)/2$, for n even $r=n/2$.

Let $|P|$ denote the number of points of P, and let $\Sigma(P)$ be the set of all maximal totally isotropic subspaces or maximal totally singular subspaces of P; all elements of

$\Sigma(P)$ have dimension $r-1$. For a proof of the following theorems, we refer, e.g., to Hirschfeld and Thas [1991].

THEOREM 1. *The numbers of points of the finite classical polar spaces are given by the formulae*:

$$
\begin{aligned}
|W_n(q)| &= \left(q^{n+1}-1\right)/(q-1),\\
|Q(2n,q)| &= \left(q^{2n}-1\right)/(q-1),\\
|Q^{+}(2n+1,q)| &= \left(q^{n}+1\right)\left(q^{n+1}-1\right)/(q-1),\\
|Q^{-}(2n+1,q)| &= \left(q^{n}-1\right)\left(q^{n+1}+1\right)/(q-1),\\
\left|H\left(n,q^2\right)\right| &= \left(q^{n+1}+(-1)^n\right)\left(q^{n}-(-1)^n\right)/\left(q^2-1\right).
\end{aligned}
$$

THEOREM 2. *The numbers of maximal totally isotropic subspaces or maximal totally singular subspaces of the finite classical polar spaces are given by*:

$$
\begin{aligned}
\left|\Sigma\left(W_n(q)\right)\right| &= (q+1)\left(q^2+1\right)\cdots\left(q^{(n+1)/2}+1\right),\\
\left|\Sigma\left(Q(2n,q)\right)\right| &= (q+1)\left(q^2+1\right)\cdots\left(q^{n}+1\right),\\
\left|\Sigma\left(Q^{+}(2n+1,q)\right)\right| &= 2(q+1)\left(q^2+1\right)\cdots\left(q^{n}+1\right),\\
\left|\Sigma\left(Q^{-}(2n+1,q)\right)\right| &= \left(q^2+1\right)\left(q^3+1\right)\cdots\left(q^{n+1}+1\right),\\
\left|\Sigma\left(H\left(2n,q^2\right)\right)\right| &= \left(q^3+1\right)\left(q^5+1\right)\cdots\left(q^{2n+1}+1\right),\\
\left|\Sigma\left(H\left(2n+1,q^2\right)\right)\right| &= (q+1)\left(q^3+1\right)\cdots\left(q^{2n+1}+1\right).
\end{aligned}
$$

9.2. *Ovoids and spreads of polar spaces*

Let P be a finite classical polar space of rank $r \geqslant 2$. An *ovoid* O of P is a pointset of P, which has exactly one point in common with every maximal totally isotropic subspace or maximal totally singular subspace of P. A *spread* S of P is a set of maximal totally isotropic subspaces or maximal totally singular subspaces of P, which constitutes a partition of the pointset. The following theorem is easily proved, cf., e.g., Thas [1981a].

THEOREM 3. *Let O be an ovoid and let S be a spread of the finite classical polar* *space P. Then*

$$\text{for } P = W_n(q), \quad |O| = |S| = q^{(n+1)/2} + 1,$$
$$\text{for } P = Q(2n, q), \quad |O| = |S| = q^n + 1,$$
$$\text{for } P = Q^+(2n+1, q), \quad |O| = |S| = q^n + 1,$$
$$\text{for } P = Q^-(2n+1, q), \quad |O| = |S| = q^{n+1} + 1,$$
$$\text{for } P = H(2n, q^2), \quad |O| = |S| = q^{2n+1} + 1,$$
$$\text{for } P = H(2n+1, q^2), \quad |O| = |S| = q^{2n+1} + 1.$$

9.3. *Existence and nonexistence of spreads*

A spread of $W_n(q)$, $n = 2t+1$, is also a t-spread of PG(n, q). For every $n = 2t+1$ the polar space $W_n(q)$ has a spread which is also a regular t-spread of PG(n, q); for a proof see, e.g., Thas [1977]. Many other examples of spreads of $W_n(q)$ are known.

In particular, let $n = 3$. Using the Klein correspondence, it is shown in Thas [1972] that, for q even, to each spread of $W_3(q)$ there corresponds an ovoid of PG$(3, q)$, and conversely. The spread is regular if and only if the ovoid is an elliptic quadric. The spreads corresponding to the Tits ovoids were first discovered by Lüneburg [1965]. In Kantor [1982a], for any odd q with q not a prime, a nonregular spread of $W_3(q)$ is constructed; see also Thas and Payne [to appear].

Proofs of the following results on spreads of quadrics can be found in Conway, Kleidman and Wilson [1988], Dye [1977], Kantor [1982a,b,c], Payne and Thas [1984], Shult [1985] and Thas [1992b]. It is clear that $Q^+(4n+1, q)$ has no spread. For q even, $Q(2n, q)$, $Q^-(2n+1, q)$ and $Q^+(4n+3, q)$ always have a spread. For q odd, $Q^+(3, q)$ and $Q^-(5, q)$ have a spread; for $q = p$ an odd prime and for q odd with $q \equiv 0$ or 2 (mod 3), $Q^+(7, q)$ and $Q(6, q)$ have a spread; the polar space $Q(4n, q)$, with q odd, has no spread.

Concerning spreads of the polar spaces $H(n, q^2)$ the following results are known. They are respectively due to Thas [1992b] and Brouwer [1981]: the polar spaces $H(2n+1, q^2)$ and $H(4, 4)$ do not have a spread.

OPEN PROBLEMS. The existence or nonexistence of spreads in the following cases.

(a) $Q(6, q)$ for q odd, with $q \equiv 1 \pmod 3$ and q not a prime;
(b) $Q(4n+2, q)$ for $n > 1$ and q odd;
(c) $Q^+(7, q)$ for q odd, with $q \equiv 1 \pmod 3$ and q not a prime;
(d) $Q^+(4n+3, q)$ for $n > 1$ and q odd;
(e) $Q^-(2n+1, q)$ for $n > 2$ and q odd;
(f) $H(4, q^2)$ for $q > 2$;
(g) $H(2n, q^2)$ for $n > 2$.

For more on spreads and derived structures, such as strongly regular graphs, codes and translation planes, we refer to Conway, Kleidman and Wilson [1988], Dye [1977], Johnson [1991], Kantor [1982a,b,c,d, 1983a,b], Kleidman [1989], Payne and Thas [1984], Shult [1985], and Thas [1977, 1980, 1981a, 1983, 1989, 1992b]; see also Section 1 in Chapter 5.

9.4. *Existence and nonexistence of ovoids*

In Thas [1972] it is shown that $W_3(q)$ has an ovoid if and only if q is even. Moreover, any ovoid of $W_3(q)$, q even, is an ovoid of PG$(3,q)$. Conversely, any ovoid of PG$(3,q)$, q even, is an ovoid of some $W_3(q)$ (cf. 3.2). Further, Thas [1981a] proves that $W_n(q)$, $n = 2t+1$ with $t > 1$, has no ovoid.

In Thas [1981a] also the nonexistence of ovoids in $Q(2n,q)$, with q even and $n > 2$, and $Q^-(2n+1,q)$, with $n > 1$, is proved. Kantor [1982a] shows that there is no ovoid in $Q^+(2n+1,2)$, $n \geqslant 4$, and Shult [1989] proves that there is no ovoid in $Q^+(2n+1,3)$, $n \geqslant 4$. The polar space $Q(4,q)$ always has an ovoid, see, e.g., Payne and Thas [1984]. Clearly, $Q^+(3,q)$ has an ovoid and in Section 8.4 it was mentioned that for all q $Q^+(5,q)$ admits an ovoid. For $q = 3^h$ the polar space $Q(6,q)$ has an ovoid, see Kantor [1982a] and Thas [1980, 1992b]. Applying triality (see Section 8 of Chapter 2 and Section 3 of Chapter 9) to the results on spreads of $Q^+(7,q)$ in Section 9.3, we find that $Q^+(7,q)$ has an ovoid in at least the following cases: q even, q an odd prime, and q odd with $q \equiv 0$ or 2 (mod 3).

Concerning ovoids of the polar spaces $H(n,q^2)$, the following results are known: it is easy to show that $H(3,q^2)$ admits ovoids (see, e.g., Payne and Thas [1984] and Thas [1983]) and in Thas [1981a] it is proved that $H(n,q^2)$, with n even, has no ovoid.

OPEN PROBLEMS. The existence or nonexistence of ovoids in the following cases.

(a) $Q(6,q)$ for q odd with $q \neq 3^h$;
(b) $Q(2n,q)$ for $n > 3$ and q odd;
(c) $Q^+(7,q)$ for q odd, with $q \equiv 1$ (mod 3) and q not a prime;
(d) $Q^+(2n+1,q)$ for $n > 3$ and $q > 3$; Moorhouse (private communication) found that there is no ovoid for $p^n > \binom{2n+p}{p-1}$, where $q = p^h$ with p prime;
(e) $H(n,q^2)$ for n odd and $n > 3$.

For more on ovoids and derived structures, such as codes and translation planes, we refer to Conway, Kleidman and Wilson [1988], Cooperstein [1990], Johnson [1991], Kantor [1982a,b,c,d, 1983a,b], Kleidman [1988], Payne and Thas [1984], Shult [1985, 1989], and Thas [1972, 1980, 1981a, 1983, 1992b].

9.5. *Hemisystems*

A *regular system of order* m, $m \in \mathrm{N}$, of $H(3,q^2)$ is a subset K of the lineset of $H(3,q^2)$, such that through every point of $H(3,q^2)$ there pass exactly m lines of K. Segre [1965] shows that, if K exists, then either K is the set of all lines of $H(3,q^2)$ or $m = (q+1)/2$.

In the latter case K consists of $(q+1)(q^3+1)/2$ lines and is called a *hemisystem*. So, for q even, there are no regular systems on $H(3,q^2)$ other than the set of all lines. Another corollary is that $H(3,q^2)$ has no spread (since $m \neq 1$). In fact, the proof of Segre is restricted to q odd, but Bruen and Hirschfeld [1978] remark that, with their definition of a quadric permutable with a Hermitian variety, it also holds for q even.

A very short proof of Segre's result is given by Thas [1981a]. It goes as follows. First he shows that for any regular system K of order m, with $0 < m < q+1$, the graph with as vertices the lines of $H(3,q^2)$ not in K and as adjacency being concurrent, is strongly regular with parameters $v = (q^3+1)(q+1-m)$, $k = (q^2+1)(q-m)$, $\lambda = q-m-1$ and $\mu = q^2+1-m(q+1)$. Since any strongly regular graph satisfies $(v-k-1)\mu = k(k-\lambda-1)$, it follows that $m = (q+1)/2$.

Let K be a hemisystem of $H(3,q^2)$ and let S be the incidence structure with as points the points of $H(3,q^2)$, as lines the lines of K, and as incidence relation that of $H(3,q^2)$. Then by the preceding paragraph S is a dual partial quadrangle (for the definition, see Chapter 10) with parameters $s = q^2$, $t = (q-1)/2$ and $\mu = (q-1)^2/2$.

The only known example of a hemisystem of $H(3,q^2)$ occurs in the case $q = 3$. The corresponding strongly regular graph has parameters $v = 56$, $k = 10$, $\lambda = 0$ and $\mu = 2$, so it is the well-known graph of Gewirtz [1969].

The polar space $H(3,q^2)$ is the dual of the polar space $Q^-(5,q)$, a proof of which can be found in Payne and Thas [1984]. So with a hemisystem K of $H(3,q^2)$ there corresponds a pointset K' on $Q^-(5,q)$ with the property that any line of $Q^-(5,q)$ has exactly $(q+1)/2$ points in common with K'. For $q = 3$ there arises a pointset of size 56 on $Q^-(5,3)$, which is nothing else than the 56-cap of Hill discussed in Section 3.5.

CONJECTURE. A hemisystem of $H(3,q^2)$ occurs only for $q = 3$.

10. Flocks, partial flocks and maximal exterior sets

10.1. *Flocks*

Let O be an ovoid of PG$(3,q)$. A partition of all but two points of O into $q-1$ disjoint ovals is called a *flock* of O. The two remaining points are called the *carriers* of the flock. If L is a line of PG$(3,q)$ having no points in common with O, then the $q-1$ planes through L which are nontangent to O intersect O in the elements of a flock F. Such a flock is called *linear*.

Next, let $Q^+(3,q)$ be a hyperbolic quadric of PG$(3,q)$. A partition of $Q^+(3,q)$ into $q+1$ disjoint irreducible conics is called a *flock* of $Q^+(3,q)$. If L is a line of PG$(3,q)$ having no points in common with $Q^+(3,q)$, then the $q+1$ planes through L intersect $Q^+(3,q)$ in the elements of a flock F. Such a flock is called *linear*.

Finally, let O be either an oval or a hyperoval of PG$(2,q)$, and let K be the cone with vertex a point x of PG$(3,q) \setminus$ PG$(2,q)$ and base O. A partition of $K \setminus \{x\}$ into q disjoint ovals or hyperovals in the respective cases is called a *flock* of K. If L is a line of PG$(3,q)$ having no points in common with K, then the q planes through L but not through x intersect K in the elements of a flock F. Such a flock is called *linear*.

10.2. *Flocks, translation planes and generalized quadrangles*

Independently, Walker [1976] and Thas discovered that with each flock of an irreducible quadric of PG$(3, q)$ there corresponds a translation plane of order q^2 and dimension at most two over its kernel; see also Bader, Lunardon and Thas [1990], Fisher and Thas [1979], Johnson, Lunardon and Wilke [1991] and Thas [1987b].

Payne [1980] (see also Kantor [1986] and Payne [1985]) showed that with a set of q upper triangular 2×2 matrices over GF(q) of a certain type, there corresponds a generalized quadrangle of order (q^2, q); for the definition of generalized quadrangle see Chapter 9. In Thas [1987b] it is proved that with such a set of q matrices there corresponds a flock of the quadratic cone of PG$(3, q)$, and conversely that with each flock of the quadratic cone there corresponds such a set of matrices. Hence with each flock of the quadratic cone of PG$(3, q)$ there corresponds a generalized quadrangle of order (q^2, q). For a discussion on this interconnection we refer to Chapter 9.

Exploiting this relationship between flocks, translation planes and generalized quadrangles, several new infinite classes of each of these objects were discovered.

10.3. *Flocks of ovoids*

The following theorem was proved in the odd case by Orr [1973], in the even case by Thas [1973b]. A short proof of Orr's result can be found in Fisher and Thas [1979].

THEOREM 1. *Any flock of an ovoid O of* PG$(3, q)$ *is linear.*

REMARK. In Orr's proof of Theorem 7 in Section 8, this theorem plays a key role.

10.4. *Flocks of hyperbolic quadrics*

The next theorem is due to Thas [1975b].

THEOREM 2. *For q even any flock of a hyperbolic quadric of* PG$(3, q)$ *is linear.*

REMARK. This theorem was used by Prohaska and Walker [1977] to exclude several types in the Hering classification of finite inversive planes of even order; see also Section 5.23 in Chapter 6.

From now on assume that q is odd.

Let $Q^+(3, q)$ be a hyperbolic quadric of PG$(3, q)$, with q odd. In the set of all irreducible conics of $Q^+(3, q)$ we define the following equivalence relation (cf. Thas [1975b, 1981b]): two conics C_1 and C_2 are equivalent if and only if there is an irreducible conic C on $Q^+(3, q)$ which is tangent to both C_1 and C_2. There are two equivalence classes, denoted by I and II. Let L be a line having no point in common with $Q^+(3, q)$, and let L' be the polar line of L with respect to $Q^+(3, q)$. The set of all conics of class I (resp., II) containing L is denoted by V (resp., V'). For $q \equiv 1 \pmod 4$ the set of all conics of class II (resp., I) containing L' is denoted by W (resp., W'); for

$q \equiv -1 \pmod 4$ the set of all conics of class I (resp., II) containing L' is denoted by W (resp., W'). Then it was shown by Thas [1975b] that $V \cup W$ (resp., $V' \cup W'$) is a nonlinear flock. By most authors these flocks are called *Thas flocks*.

Further, Bader [1988] showed that for $q = 11, 23, 59$ the hyperbolic quadric $Q^+(3,q)$ of PG$(3,q)$ has a flock which is neither a linear nor a Thas flock. These flocks were independently discovered by Johnson [1989a], and for $q = 11, 23$ also by Baker and Ebert [1987]. Since these flocks were derived by Bader from exceptional nearfield planes, exploiting the relationship between flocks and translation planes mentioned in Section 10.2, Bader calls these flocks the *exceptional flocks*.

In Thas [1991b] the following nice description of the exceptional flock for $q = 11$ was given. Let $Q^+(3,11)$ be the hyperbolic quadric of PG(3, 11) with equation

$$X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0.$$

Further, let $F = \{C_1, C_2, \ldots, C_{12}\}$ be an exceptional flock of $Q^+(3,11)$, let C_i be contained in the plane π_i, let x_i be the pole of π_i with respect to $Q^+(3,11)$, and let $F' = \{x_1, x_2, \ldots, x_{12}\}$. Then up to a projectivity the points of F' have coordinates:

$$x_1(1,0,0,0), \quad x_2(0,1,0,0), \quad x_3(0,0,1,0), \quad x_4(0,0,0,1),$$

$$x_5(-1,1,1,1), \quad x_6(1,-1,1,1), \quad x_7(1,1,-1,1), \quad x_8(1,1,1,-1),$$

$$x_9(1,1,1,1), \quad x_{10}(1,1,-1,-1), \quad x_{11}(1,-1,1,-1), \quad x_{12}(1,-1,-1,1).$$

Let $V_1 = \{x_1, x_2, x_3, x_4\}$, $V_2 = \{x_5, x_6, x_7, x_8\}$, $V_3 = \{x_9, x_{10}, x_{11}, x_{12}\}$. Then V_1, V_2, V_3 are self-polar tetrahedra of $Q^+(3,11)$. Any line joining two points of V_i has exactly two points in common with F', $i = 1,2,3$; any line joining a point of V_i to a point of V_j contains exactly one point of V_k, $\{i,j,k\} = \{1,2,3\}$. A plane of PG(3, 11) contains at most six points of F', and there are exactly 12 of these planes, say $\bar\pi_1, \bar\pi_2, \ldots, \bar\pi_{12}$. If

$$\bar\pi_i \cap Q^+(3,11) = \overline{C}_i, \quad i = 1,2,\ldots,12,$$

then $\overline{F} = \{\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_{12}\}$ again is an exceptional flock of $Q^+(3,11)$. We have

$$\left|\pi_i \cap \bar\pi_j \cap Q^+(3,11)\right| \in \{0,2\} \quad \text{for all } i,j.$$

Moreover, either $F \subset I$ and $\overline{F} \subset II$, or $F \subset II$ and $\overline{F} \subset I$. The points of F' together with the 3-secants of F' form a partial geometry D with parameters $s = 2$, $t = 3$, $\alpha = 2$, hence D is a dual net; for the terminology used here we refer to Chapter 10. It is clear that D satisfies the axiom of Pasch, and so by De Clerck and Thas [1977], $D \cong H_2^3$ (H_2^3 is the dual net consisting of all points of PG(3, 2) not on a given line L of PG(3, 2), and all lines of PG(3, 2) skew to L). Finally we notice that the dual net D satisfies the following property: two lines of D are concurrent in D if and only if they are concurrent in PG(3, 11).

The following theorem classifying all flocks of $Q^+(3,q)$, with q odd, is a combination of results of Bader and Lunardon [1989] and Thas [1990b].

THEOREM 3. *Any flock of the hyperbolic quadric $Q^+(3,q)$ of* PG(3, q), *q odd, either is a linear flock, or a Thas flock, or an exceptional flock.*

REMARK. Theorem 3 was used to prove Theorem 4 of Section 3 on inversive planes of odd order n.

10.5. *Flocks of cones*

First of all we mention an easy theorem, a proof of which can be found in Thas [1987b].

THEOREM 4. *Let O be an oval of* PG(2, q), *$q = 2^h$, and let K be the cone which projects O from a point x of* PG(3, q) \ PG(2, q). *Further, assume that $F = \{C_1, C_2, \ldots, C_q\}$ is a flock of K. Let y be the nucleus (or kernel) of O, and let O^* be the hyperoval $O \cup \{y\}$. If K^* is the cone which projects O^* from x, then the planes $\pi_1, \pi_2, \ldots, \pi_q$, with $\pi_i \supset C_i$, define a flock $F^* = \{C_1^*, C_2^*, \ldots, C_q^*\}$, $C_i \subset C_i^*$, of K^*.*

Hence we can restrict ourselves to cones projecting ovals, and because of the applications we are mainly interested in quadratic cones. Concerning nonquadratic cones we only mention that in Fisher and Thas [1979] an infinite class of nonlinear flocks of such cones is constructed.

We now list all the known nonlinear flocks of the quadratic cone K. Let K be represented by the equation $X_0X_1 = X_2^2$, and let the planes of the elements of the flock be represented by

$$a_iX_0 + b_iX_1 + c_iX_2 + X_3 = 0, \quad i = 1, 2, \ldots, q.$$

(1) The flocks FTW of Fisher, Thas and Walker (see Thas [1987b]): here $q \equiv -1 \pmod 3$ and

$$\{(a_i, b_i, c_i)\colon\ i = 1, 2, \ldots, q\} = \{(t, 3t^3, 3t^2)\colon\ t \in \mathrm{GF}(q)\}.$$

This flock is linear if and only if $q = 2$.

(2) The flocks $K1$ of Kantor (see Thas [1987b]): here q is odd, m is a given nonsquare, σ is an automorphism of GF(q) and

$$\{(a_i, b_i, c_i)\colon\ i = 1, 2, \ldots, q\} = \{(t, -mt^\sigma, 0)\colon\ t \in \mathrm{GF}(q)\}.$$

This flock is linear if and only if $\sigma = 1$.

(3) The flocks $K2$ of Kantor (see Thas [1987b]): here q is odd with $q \equiv \pm 2 \pmod 5$, and

$$\{(a_i, b_i, c_i)\colon\ i = 1, 2, \ldots, q\} = \{(t, 5t^5, 5t^3)\colon\ t \in \mathrm{GF}(q)\}.$$

This flock is linear if and only if $q = 3$.

(4) The flocks $P1$ of Payne (see Thas [1987b]): here $q = 2^e$ with e odd, and

$$\{(a_i, b_i, c_i)\colon\ i = 1, 2, \ldots, q\} = \{(t, t^5, t^3)\colon\ t \in \mathrm{GF}(q)\}.$$

This flock is linear if and only if $q = 2$.

(5) The flocks $K3$ of Kantor (see Gevaert and Johnson [1988]): here $q = 5^h$, k is a given nonsquare and

$$\{(a_i, b_i, c_i)\colon\ i = 1, 2, \ldots, q\} = \{(t, k^{-1}t + 2t^3 + kt^5, t^2)\colon\ t \in \mathrm{GF}(q)\}.$$

This flock is always nonlinear.

(6) The flocks G of Ganley (see Gevaert and Johnson [1988] and Payne [1989]): here $q = 3^h$, n is a given nonsquare, and

$$\{(a_i, b_i, c_i)\colon\ i = 1, 2, \ldots, q\} = \{(t, -(nt + n^{-1}t^9), t^3)\colon\ t \in \mathrm{GF}(q)\}.$$

This flock is linear if and only if $q = 3$.

(7) The flocks Fi of Fisher (see Gevaert and Johnson [1988], Payne [1988] and Thas [1987b]): let q be odd, let ζ be a primitive element of $\mathrm{GF}(q^2)$, so $w = \zeta^{q+1}$ is a primitive element of $\mathrm{GF}(q)$ and hence a nonsquare in $\mathrm{GF}(q)$; put $i = \zeta^{(q+1)/2}$, so $i^2 = w$, $i^q = -i$; put $z = \zeta^{q-1} = a + bi$, so z has order $q + 1$ in the multiplicative group of $\mathrm{GF}(q^2)$; then the triples (a_i, b_i, c_i) are given by

$$(t, -wt, 0)$$

with $t \in \mathrm{GF}(q)$ and $t^2 - 2(1 + a)^{-1}$ a square in $\mathrm{GF}(q)$, and

$$(-a_{2j}, -wa_{2j}, 2b_{2j})$$

with $j = 0, 1, \ldots, (q - 1)/2$,

$$a_k = (z^{k+1} + z^{-k})/(z + 1) \quad \text{and} \quad b_k = i(z^{k+1} - z^{-k})/(z + 1).$$

This flock is linear if and only if $q = 3$.

(8) For $q = 11, 16, 17, 23$, the nonlinear flocks C discovered by De Clerck and Herssens [1993] with the aid of a computer; for $q = 11$ see also Thas, Herssens and De Clerck [1993].

(9) The nonlinear flocks BLT, discovered by Bader, Lunardon and Thas [1990]; see also Johnson [1992]. These flocks are *derived* from the flocks $K3$, where $q = 5^h$ with $h > 1$. The process of '*derivation*' will be described at the end of this section.

(10) The nonlinear flocks JP, discovered by Johnson [1992] and Payne [1992], applying derivation to flocks $K2$ with $q \geqslant 13$.

(11) The nonlinear flocks $PTJLW$, discovered by Payne and Thas [1991] and Johnson, Lunardon and Wilke [1991] (see also Johnson [1992]), applying derivation to flocks G with $q = 3^h$, $h > 2$.

(12) The nonlinear flocks $P2$ discovered by Payne [1992] 're-coordinatizing' the generalized quadrangles arising from the flocks $P1$ with $q = 2^e$, e odd and $e > 3$.

The following results on flocks are taken from Thas [1987b].

(i) If q is even and if the q planes of the elements of the flock F all contain a common point, then F is linear.

(ii) If q is odd and if the q planes of the elements of the flock F all contain a common interior point of the cone K, i.e. a point on no tangent planes of K, then F is linear.

(iii) If q is odd and if the q planes of the elements of the flock F all contain a common exterior point of the cone K, i.e. a point on exactly two tangent planes of K, then F is a flock $K1$ of Kantor. With the notations of (2) this common point is the point $(0,0,1,0)$.

Further, Payne and Thas [1991] show that the Fisher flock Fi is the only nonlinear flock for which at least $(q-1)/2$ of the planes associated with the flock contain a common line.

The examples (1) to (12) show that there exists at least one nonlinear flock for any odd prime power q with $q > 3$, and any $q = 2^{2e+1}$ with $e \geqslant 1$.

In Thas [1987b] it is shown that for $q = 2,3,4$ any flock of the quadratic cone is linear, in De Clerck, Gevaert and Thas [1988] it is proved that for any $q \in \{5,7,8\}$ there exists exactly one nonlinear flock, by De Clerck and Mylle (see Mylle [1991]) it is shown with the aid of a computer that for $q = 9$ there exist exactly two nonlinear flocks, and De Clerck and Herssens [1993], also with the aid of a computer, obtained that for $q = 11$ there are exactly three nonlinear flocks and that for $q = 16$ there is exactly one nonlinear flock.

We conclude this section on flocks of cones with a short description of the process of '*derivation*' introduced by Bader, Lunardon and Thas [1990]. Let $F = \{C_1, C_2, \ldots, C_q\}$ be a flock of the quadratic cone K with vertex x of PG$(3,q)$, with q odd. The plane of C_i is denoted by π_i, $i = 1,2,\ldots,q$. Let K be embedded in the nonsingular quadric Q of PG$(4,q)$. Let the polar line of π_i with respect to Q be denoted by L_i and let $L_i \cap Q = \{x, x_i\}$, $i = 1,2,\ldots,q$. If H_i is the tangent hyperplane of Q at x_i, then put

$$H_i \cap Q = K_i, \qquad H_i \cap H_j \cap Q = K_i \cap H_j = C_{ij}$$

and

$$C_{ii} = C_i, \quad i,j = 1,2,\ldots,q \text{ and } i \neq j.$$

Now it is possible to prove that $F_i = \{C_{i1}, C_{i2}, \ldots, C_{iq}\}$ is a flock of K_i, $i = 1,2,\ldots,q$. We say that the flocks $F_1, F_2, \ldots, F_q$ are *derived* from the given flock F.

10.6. *Maximal exterior sets*

An *exterior set* with respect to the hyperbolic quadric $Q^+(2n-1,q)$ of PG$(2n-1,q)$, $n \geqslant 2$, is a set X of points such that each line joining two distinct elements of X has no point in common with $Q^+(2n-1,q)$. It is easy to show that for an exterior set X

of $Q^+(2n-1,q)$ we have $|X| \leqslant (q^n-1)/(q-1)$; see De Clerck and Thas [1985]. If $|X| = (q^n-1)/(q-1)$, then X is called a *maximal exterior set* (MES).

Let X be a MES of $Q^+(3,q)$. Then $|X| = q+1$. If $X = \{x_1, x_2, \ldots, x_{q+1}\}$, then let π_i be the polar plane of x_i with respect to $Q^+(3,q)$ and let $\pi_i \cap Q^+(3,q) = C_i$. Then $F = \{C_1, C_2, \ldots, C_{q+1}\}$ is a flock of $Q^+(3,q)$. Conversely, each flock of $Q^+(3,q)$ defines a MES of $Q^+(3,q)$. If F is linear, i.e. if the corresponding MES is a line, then the MES is also called *linear*.

Rees [1985] proves that the existence of a MES of $Q^+(5,q)$ implies the existence of a nonbuilding C_3-geometry (cf. Chapters 11 and 12).

De Clerck and Thas [1985] show that the existence of a MES of $Q^+(2n-1,q)$ implies the existence of a MES of $Q^+(2m-1,q)$, $2 \leqslant m \leqslant n$.

Relying on the nonexistence of ovoids of $Q^+(2n+1,2)$ with $n \geqslant 4$ (see Section 9.4) De Clerck and Thas [1985] prove that $Q^+(2n-1,2)$ with $n \geqslant 4$ has no MES and, relying on Theorem 2, Thas (see Rees [1985] and De Clerck and Thas [1985]) proves that $Q^+(5,q)$, q even and $q > 2$, has no MES. Hence $Q^+(2n-1,q)$, $n \geqslant 3$, q even and $(q,n) \neq (2,3)$ has no MES. Further, Conwell [1910] proves that $Q^+(5,2)$ admits a unique (up to a projectivity) MES.

Finally, Thas [1991a] shows that $Q^+(2n-1,q)$, $n \geqslant 3$, with q odd, has no MES.

We summarize in the following theorem.

THEOREM 5. *The hyperbolic quadric* $Q^+(2n-1,q)$, *with* $n \geqslant 3$ *and* $(q,n) \neq (2,3)$, *has no* MES. *The quadric* $Q^+(5,2)$ *admits a unique* (*up to a projectivity*) MES.

10.7. *Partial flocks*

Let Q be an elliptic quadric, a hyperbolic quadric or a quadratic cone of PG$(3,q)$. A *partial flock* of Q is a set P consisting of β disjoint irreducible conics of Q. If $\beta = q-2$, q or $q-1$ according as Q is elliptic, hyperbolic or a cone, then Johnson [1989b] proves that from P a translation plane of order q^2, of dimension at most two over its kernel, can be constructed. If F is a flock of Q, then the translation plane corresponding to $F \backslash \{C\}$, with C any conic of F, is a derivation of the plane arising from F as mentioned in Section 10.2. Orr [1973] shows that the elliptic quadric of PG$(3,q)$, with $q \in \{5,9\}$, has a partial flock of size $q-2$ which cannot be completed to a flock. Johnson [1989b] shows that the hyperbolic quadric $Q^+(3,4)$ of PG$(3,4)$ admits a partial flock of size 4 which cannot be completed to a flock of $Q^+(3,4)$; an analogous result for the hyperbolic quadric of PG$(3,9)$ is proved by Johnson and Pomareda [1990]. Finally, Payne and Thas [1991] show that any partial flock of size $q-1$ of the quadratic cone K of PG$(3,q)$ can be completed to a flock.

10.8. *Open problems*

(a) Classify all flocks of the quadratic cone of PG$(3,q)$.

(b) Does there exist a nonlinear flock of the quadratic cone K of PG$(2,q)$, $q > 4$, with q any even power of 2? Cherowitzo, Penttila, Pinneri and Royle [private communication] answered positively. These flocks, which exist for any $q = 2^e$ with $e \geqslant 4$, define a new infinite class of generalized quadrangles, of projective planes and of hyperovals.

(c) For which values of q does there exist a partial flock of size $q-2$ (resp., q) of an elliptic (resp., hyperbolic) quadric of $\mathrm{PG}(3, q)$, which cannot be completed to a flock?

(d) Which flocks C are sporadic?

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CHAPTER 8

Block Designs

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Contents

1. Basic definitions 351
2. Eigenvalue techniques 356
3. Association schemes 359
4. Coding techniques 361
5. Constructions of designs 370
6. Characterizations of designs 373

References 376

HANDBOOK OF INCIDENCE GEOMETRY
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Design theory is the study of properties of collections of subsets of a set (or of subsets of an association scheme, etc.) when some global regularity property is imposed. Usually all sets are assumed to be finite. The techniques used include counting, linear algebra, eigenvalue methods, coding theory and group theory. From the theory one usually gets inequalities and other feasibility conditions on the parameters, and structural information or even complete classification when the inequalities are tight. The theory also has a constructive branch, that deals with the construction of designs with prescribed parameters. Basic references are Hall [1967], Dembowski [1968], Cameron and Van Lint [1980], Lander [1983] and Beth, Jungnickel and Lenz [1985]. (See also the chapter Block Designs in the Handbook of Combinatorics for a discussion of many topics omitted in the present chapter.)

The theory arose as the branch of statistics studying the design and analysis of experiments. Closer to the statistical origin are books like Raghavarao [1971], Dey [1986], Street and Street [1987] and Montgomery [1984].

1. Basic definitions

A *design* is a set X (the *point set*) provided with a distinguished collection of subsets $\boldsymbol{B}$ (the collection of *blocks*).

Of course this is also the definition of a hypergraph (or of a subbasis for a topological space, etc.), but one talks about a design when some numerical regularity is assumed. (As opposed to a geometry, which is an incidence structure where configuration axioms are imposed. Of course, sometimes one kind of assumptions implies the other, and there is no sharp boundary.)

Sometimes it is useful not to view the blocks as sets of points but as entities in their own right, and have an explicit incidence relation between them. For example, the *dual* $\boldsymbol{D}^*$ of a design $\boldsymbol{D}$ is gotten by interchanging the roles of points and blocks, i.e. the dual of the design $(X, \boldsymbol{B}, \in)$ is the design $(\boldsymbol{B}, X, \ni)$. A design is said to have *repeated blocks* when two of its blocks are incident with the same set of points; otherwise it is called *simple*.

There are many types of design, but the ones most frequently encountered are variations on the concept of *t-design*; a t-(v, k, λ) design is a design with v points, and a collection of blocks, all of size k, such that each t-set of points is covered by precisely λ blocks. When $\lambda = 1$ one calls such a design a *Steiner system* and writes its parameters $S(t, k, v)$. Some people use $S_\lambda(t, k, v)$ as a synonym for t-(v, k, λ).

Examples are projective or affine spaces provided with their subspaces of a given dimension (here $t = 2$, but affine spaces with lines of size two yield 3-designs), or provided with subfield subgeometries. More generally, whenever some group acts t-transitively on a set, this set, together with any union of orbits of this group on the k-sets, forms a t-design.

The BIBD (balanced incomplete block design) of the statisticians is the same as a 2-design. The 'incomplete' referred to the requirement $k < v$, but that is now usually forgotten.

A design is called *square* when it has as many points as blocks, i.e. when its *point-block incidence matrix* $A = (a_{xB})_{x\in X,\ B\in \mathbf{B}}$ (defined by $a_{xB} = 1$ if $x \in B$ and $a_{xB} = 0$ otherwise) is square. Square 2-designs are often called *symmetric designs*.

Examples are projective planes, or, more generally, the designs of points and hyperplanes in a projective space. From a Hadamard matrix of order $4n$ (see Dembowski [1968]) one can derive a square 2-$(4n-1, 2n-1, n-1)$ design. Any strongly regular graph with parameters (ν, k, λ, μ), where $\lambda = \mu$ or $\lambda = \mu - 2$ yields a square 2-(ν, k, μ) design.

A design is called *resolvable* when its collection of blocks can be partitioned into *parallel classes*, i.e. into partitions of the point set. A *resolution* is such a partition of the collection of blocks.

Examples are affine planes, or, more generally, the designs of points and i-flats[1] in an affine space. The Hermitian unitals (see Chapter 7) in a projective plane of order q are resolvable designs $S(2, q+1, q^3+1)$ – each tangent line yields a resolution. The exterior lines and the points off a hyperoval in PG$(2,q)$, q even, form a resolvable $S(2, q/2, q(q-1)/2)$ – each point of the hyperoval yields a resolution.

A *large set* of designs of block size k is a collection of such designs such that their collections of blocks partition the set of all k-subsets of the point set.

A *tactical decomposition* of a design is a partition $(X_i)_i$ of the point set and a partition $(\mathbf{B}_j)_j$ of the collection of blocks, such that all designs $(X_i, \mathbf{B}_j)$ are 1-designs. The partitions into G-orbits for any group G of automorphisms is a tactical decomposition.

Leaving the realm of 1-designs we have *t-wise balanced designs,* or t-(ν, K, λ) designs, where each t-set is covered precisely λ times by a block, but block sizes are not necessarily constant – they are members of the set K. Pairwise balanced designs have been much studied because they yield ingredients for recursive constructions of BIBDs. Pairwise balanced designs with $\lambda = 1$ are called *linear spaces* (see Chapter 6).

t-designs

A t-(ν, k, λ) design is also a j-(ν, k, λ_j) design for $0 \leqslant j \leqslant t-1$, where $\lambda_j = \lambda_{j+1}(\nu - j)/(k-j)$ (and $\lambda_t = \lambda$). One usually writes $b := \lambda_0$ for the number of blocks, and $r := \lambda_1$ for the number of blocks on a given point.

Given a t-(ν, k, λ) design $(X, \mathbf{B})$ and a point $x \in X$, we may form its *derived design* (at x)

$$\big(X\backslash\{x\}, \{B\backslash\{x\}\colon\ x \in B \in \mathbf{B}\}\big),$$

which is a $(t-1)$-$(\nu-1, k-1, \lambda)$ design. We may also form the *residual design* (at x) by taking

$$\big(X\backslash\{x\}, \{B\colon\ x \notin B \in \mathbf{B}\}\big),$$

which is a $(t-1)$-$(\nu-1, k, \lambda_{t-1} - \lambda)$ design. An *extension* of a design $\boldsymbol{D}$ is a design $\boldsymbol{E}$ such that $\boldsymbol{D}$ is a residual of $\boldsymbol{E}$. Sometimes one can guarantee that an extension exists

[1] Which are called i-subspaces in Chapter 2 (Editor's note).

(see Alltop [1975] and Dehon [1976]). The most obvious way of constructing t-designs is to consider a t-transitive action of a group G on a set X, and take the union of an arbitrary collection of G-orbits on the k-subsets of X. In the early days people thought that for large t this might be the only way to get t-designs and hence conjectured that no nontrivial such designs existed for $t > 5$. Nowadays many constructions are known (cf. the section on constructions below), and Teirlinck [1987] was the first to show that t-designs (even large sets of t-designs) exist with arbitrarily large t. (However, his designs have very large λ's, and for small λ's the problem is wide open. In particular, the only Steiner systems known with $t \geqslant 5$ are $S(5,6,\nu)$ for $\nu = 12, 24, 48, 72, 84, 108$, $S(5,7,28)$ and $S(5,8,24)$.)

2-designs
A design $(X, \boldsymbol{B})$ is a 1-(ν, k, r) design if and only if its point-block incidence matrix A satisfies $AJ = rJ$ and $JA = kJ$ where the J's are all-1 matrices of suitable sizes, and we see that $\nu r = bk$.

That a 1-design is a 2-(ν, k, λ) design is expressed by the equation

$$AA^{\mathsf{T}} = (r - \lambda)I + \lambda J. \qquad (*)$$

Thus, $\det AA^{\mathsf{T}} = (r - \lambda)^{\nu-1} rk$. This implies:

PROPOSITION ('Fisher's inequality'). *In a* 2-(ν, k, λ) *design with* $1 < k < \nu$ *we have* $b \geqslant \nu$ *(or, equivalently,* $r \geqslant k$*).*

PROOF. The rank ν of AA^{T} is at most the number of columns b of A. □

Many generalizations and variations exist. The best known are perhaps the result of De Bruijn and Erdős [1948] stating that a linear space with not all points on one line has at least as many lines as points, and the result of Ray-Chaudhuri and Wilson [1975] stating that if $\nu \geqslant k + s$, then any $2s$-(ν, k, λ) design has $b \geqslant \binom{\nu}{s}$.

Square 2-designs
For square 2-designs we find that $k = r$ and $AJ = JA$. Now A^{-1} is defined (for $0 < k < \nu$) and multiplying $(*)$ on the left by A^{-1} and on the right by A we find $A^{\mathsf{T}}A = (r - \lambda)I + \lambda J$, i.e. the following proposition is true.

PROPOSITION. *The dual of a square 2-design is again a square 2-design, with the same parameters. In other words, in a square* 2-(ν, k, λ) *design any two blocks meet in* λ *points.*

For square 2-designs the concepts of 'derived' and 'residual' design are customarily defined in a slightly different way from the above definition for arbitrary t-designs: here the Derived (or Residual) design of a design $\boldsymbol{D}$ is the dual of the derived (residual) of its dual $\boldsymbol{D}^*$. The advantage of this definition is that the designs thus obtained are again 2-designs.

For square 2-designs the following parameter restriction is known (and due to Bruck and Ryser [1949], Schutzenberger [1949], Shrikhande [1950] and Chowla and Ryser [1950]).

THEOREM ('BRUCK–CHOWLA–RYSER'). *If a square 2-design has parameters 2-(v, k, λ), then either v is even and $k - \lambda$ is a square, or v is odd and the equation*

$$z^2 = (k-\lambda)x^2 + (-1)^{(v-1)/2}\lambda y^2$$

has a nontrivial integer solution.

PROOF. The first part follows from $(\det A)^2 = k^2(k-\lambda)^{v-1}$; the second part expresses the fact that the quadratic forms

$$\sum_{i=1}^{v} x_i^2$$

and

$$\sum_{j=1}^{v}\left(\sum_{i=1}^{v} a_{ij}x_i\right)^2 = (k-\lambda)\sum_{i=1}^{v} x_i^2 + \lambda\left(\sum_{i=1}^{v} x_i\right)^2$$

are equivalent. In fact the conditions of this theorem are necessary and sufficient for the existence of a rational matrix A of order v satisfying the above mentioned equations (cf. Hall [1967]). □

For example, there are no designs with parameters 2-(22,7,2) or 2-(43,7,1) (i.e. no projective plane of order 6). There is precisely one parameter set for a square 2-design which passes the Bruck–Chowla–Ryser criterion (and Fisher's inequality), while it is known that no corresponding design exists, namely that for a projective plane of order 10 (Lam, Thiel and Swiercz [1989]).

Automorphisms

Let $\boldsymbol{D} = (X, \boldsymbol{B})$ be a 2-design and let G be a group of automorphisms of $\boldsymbol{D}$. Suppose g is an element of G. For the incidence matrix A of $\boldsymbol{D}$ this means that $A = PAQ^{\top}$ where $P \in \mathrm{Sym}(X)$ and $Q \in \mathrm{Sym}(\boldsymbol{B})$ represent the action of g on points and blocks respectively (i.e. $P_{x,y} = \delta_{g(x),y}$ and $Q_{B,C} = \delta_{g(B),C}$). If $\boldsymbol{D}$ is square, then A is nonsingular and it follows that $\operatorname{tr} P = \operatorname{tr} Q$, proving the following result.

An automorphism of a square 2-design has as many fixed points as fixed blocks.

(Notice that this result holds for any square design with nonsingular incidence matrix.) Applying the Frobenius–Burnside Lemma (cf. Neumann [1979]) to this result one immediately obtains

An automorphism group of a square 2-design has equally many point and block orbits (Brauer [1941]).

In the case of an arbitrary design with incidence matrix of rank ρ we have (Block [1967]): $\overline{\nu} \leqslant \overline{b} + \nu - \rho$ and $\overline{b} \leqslant \overline{\nu} + b - \rho$ where $\overline{\nu}$ and $\overline{b}$ are the number of point and block orbits, respectively. (This is a purely combinatorial result: it holds when $\overline{\nu}$ and $\overline{b}$ are the number of point and block classes in a tactical decomposition.) When A has rank ν (as is the case for 2-designs) we find $\overline{\nu} \leqslant \overline{b}$.

There is an analogue of the theorem of Bruck–Chowla–Ryser in case g has prime order p (Hughes [1957]):

Let g be an automorphism of prime order p of a square 2-design, with f fixed points. Then g has $m = f + (\nu - f)/p$ point and block orbits, and the equation

$$z^2 = (k - \lambda)x^2 + (-1)^{(m-1)/2} p^{f+1} \lambda y^2$$

has a nontrivial integer solution.

Lander [1983] (Theorem 3.20) gives the following rather strong condition on p and f.

Let $\boldsymbol{D}$ be a square 2-design, and let g be a nonidentity automorphism of $\boldsymbol{D}$ of odd order p with f fixed points. Suppose that $r^j \equiv -1 \pmod p$ for some j and for some prime r dividing the square free part of n. Then f is odd.

A bound on f is given by Wilbrink (cf. Lander [1983], (3.7)) (using Haemers' interlacing result, see below):

Let g be a nonidentity automorphism of a square 2-design with f fixed points. Then $f \leqslant \nu - 2n$ and $f \leqslant k + \sqrt{n}$, and if equality holds in either inequality then g must be an involution and every nonfixed block contains precisely λ fixed points.

An automorphism g of $\boldsymbol{D}$ is called *central* if there is a point c (the *centre* of g) such that g fixes c and all blocks on c. Dually, an automorphism g of $\boldsymbol{D}$ is called *axial* if there is a block A (the *axis* of g) such that g fixes all points on A. An automorphism can have many axes (as is the case with the point vs. s-subspace design of PG(d, q) where $s < d - 1$) but usually at most one centre. To see this, suppose that c is a centre of $g \neq 1$. Notice that if B is a block not on c then every point $x \in B \cap g(B)$ is fixed (otherwise $g(x)$ is a third point on the line xc so $c \in xg(x) \subseteq g(B)$, a contradiction). In particular every fixed block not on c is an axis of g. If d is another centre of g then we get the rather absurd situation that every nonfixed point x is on the line cd (if B is a block on x and d, then $c \in B$ since B is fixed). For example, for Steiner systems and square 2-designs ($|g(B) \cap B| = k - 1$ for every block $B \not\ni c, d$ on a nonfixed point) this gives a contradiction.

A *difference set* D in a group G (written additively) is a set such that

$$\big(G, \{D + g\colon\ g \in G\}\big)$$

is a square 2-design; equivalently, $|G| = \nu$, $|D| = k$, and in the list of differences $d_1 - d_2$ (d_1, $d_2 \in D$) each nonzero group element occurs precisely λ times. Notice that G acts as a regular group of automorphisms of this design. Conversely, any square 2-design with a regular group of automorphisms is of this form. (More generally, one considers *difference families* and designs where the block set is the union of several G-orbits. This is called Bose's method of symmetrically repeated differences, cf. Bose [1939]. One calls the given representatives for the orbits *base blocks* or *initial blocks* and constructing the G-orbits *developing* (mod G).)

An important example was given by Singer [1938] who showed that the design of points and hyperplanes in the projective space PG(d, q) has a cyclic difference set (i.e. a difference set in $G = \mathbb{Z}_\nu$). Indeed, the points of PG(d, q) can be identified with the elements of the cyclic group $\mathbb{F}^*_{q^{d+1}}/\mathbb{F}^*_q$.

The most important result on difference sets is the multiplier theorem.

THEOREM (Hall and Ryser [1951]). *Let D be a difference set in $G = \mathbb{Z}_\nu$ and suppose that p is a prime with $p \mid k - \lambda$, $(p, \nu) = 1$, $p > \lambda$. Then p is a multiplier of D, i.e. $pD = D + e$ for some $e \in G$.*

Many generalizations exist. For a discussion of multiplier theorems see Ryser [1963] (Chapter 9), Hall [1967] (Chapter 11), Dembowski [1968] (pp. 87–90), Lander [1983], Beth et al. [1985] (Chapter 6).

One may always choose D so as to be fixed by a given multiplier p (Mann [1965]) (for: p induces an automorphism of the design and fixes the point 0 hence must also fix some block D), and, in case G is Abelian, one may even find a D which is fixed by all (numerical) multipliers of D (McFarland and Rice [1978]). Standard references for difference sets are Baumert [1971], Hall [1974], and Lander [1983].

A lot has been done in the special case where $\lambda = 1$ ('planar difference sets'); for a survey, see the books mentioned; a recent result is Wilbrink [1985]. Very strong results are available when -1 is a multiplier, cf. Jungnickel [1982], Smit-Ghinelli [1987], Pott [1989], Ma [1989], Arasu, Jungnickel and Pott [1990].

2. Eigenvalue techniques

In this section we explain some of the eigenvalue techniques which are useful for studying designs. The standard reference is Haemers [1980].

Suppose A is a square complex matrix of size n with n real eigenvalues $\lambda_i(A)$, $i = 1, \ldots, n$. We shall always assume that the eigenvalues are ordered:

$$\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A).$$

If B is another square complex matrix of size $m \leqslant n$ with m real eigenvalues $\lambda_i(B)$, $i = 1, \ldots, m$, and if

$$\lambda_i(A) \geqslant \lambda_i(B) \geqslant \lambda_{n-m+i}(A)$$

for $i = 1, \ldots, m$, then we say that the eigenvalues of B *interlace* those of A. The interlacing is called *tight* if there exists an integer k, $0 \leqslant k \leqslant m$, such that

$$\lambda_i(A) = \lambda_i(B), \quad i = 1, \ldots, k,$$
$$\lambda_{n-m+i}(A) = \lambda_i(B), \quad i = k+1, \ldots, m.$$

The following two theorems on the interlacing of eigenvalues (taken from Haemers [1979]) provide a machine for producing results on substructures of graphs and designs.

2.1. THEOREM. *Suppose*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is a Hermitian matrix, with square principal submatrix A_{11}. Then:

(i) *The eigenvalues of A_{11} interlace the eigenvalues of A.*

(ii) *If the interlacing is tight, then $A_{12} = A_{21} = 0$.*

2.2. THEOREM. *Let A be a Hermitian matrix partitioned as follows:*

$$A = \begin{pmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mm} \end{pmatrix}$$

where A_{ii} is square for $i = 1, \ldots, m$. Let $B = (b_{ij})_{ij}$, where b_{ij} is the average row sum of A_{ij}, for $i, j = 1, \ldots, m$. Then:

(i) *The eigenvalues of B interlace the eigenvalues of A.*

(ii) *If the interlacing is tight, then A_{ij} has constant row and column sums for $i, j = 1, \ldots, m$.*

Clearly, these theorems can be applied to square symmetric matrices, and in particular to the adjacency matrix of a graph. Regarding a design as a bipartite graph (with a fixed ordered bipartition), one can also apply these theorems to designs.

(If the design has point-block incidence matrix N, then the corresponding bipartite graph has adjacency matrix

$$A = \begin{pmatrix} 0 & N \\ N^\top & 0 \end{pmatrix};$$

a nonzero number λ is eigenvalue of A with multiplicity f if and only if λ^2 is eigenvalue of $NN^\top$ (or of $N^\top N$) with multiplicity f.[2])

Thus, Theorem 2.2 yields for sub-1-designs of 1-designs:

[2] The reader interested in properties of eigenvalues of matrices is advised to read Brouwer, Cohen and Neumaier [1989] (Editor's note).

2.3. THEOREM. *Let D be a 1-(v,k,r) design with b blocks. Let D_1 be a 1-(v_1,k_1,r_1) subdesign of D with b_1 blocks. Then*

$$(vr_1 - b_1k)(bk_1 - v_1r) \leqslant \lambda_2(A)^2(v - v_1)(b - b_1)$$

and if equality holds, then each point (resp., block) not in D_1 is incident with a constant number of blocks (resp., points) of D_1.

For a 2-(v,k,λ) design we have $\lambda_2(A)^2 = r - \lambda$. Thus, Theorem 2.2 yields for square substructures of square 2-designs the following theorem.

2.4. THEOREM. *Let D be a square 2-(v,k,λ) design. Suppose that F is a substructure with v_1 points and blocks, and an average of k_1 points on a block. Then*

$$k - \lambda \geqslant \left(\frac{k_1 v - k v_1}{v - v_1}\right)^2.$$

Moreover, if equality holds, then every block has exactly k_1 points in F.

Applying this result to the substructure of nonfixed points and nonfixed blocks of an automorphism g of D, and using the inequality $k_1 \geqslant k - \lambda$ (which follows from the observation that, for a nonfixed block B, points in $B^g \setminus B$ are certainly not fixed), one easily deduces that $v - v_1 \leqslant k + \sqrt{k - \lambda}$. It follows that any nonidentity automorphism of a square 2-(v,k,λ) design has at most $k + \sqrt{k-\lambda}$ fixed points, as announced before.

Another example of the use of eigenvalues is in the study of polarities of square designs. Suppose D is a square 2-(v,k,λ) design admitting a polarity π. Then D has a symmetric incidence matrix N, and the number of absolute points of π is $\operatorname{tr}(N)$. The eigenvalues of N are k (with multiplicity 1) and $\pm\sqrt{k-\lambda}$. Thus, the number a of absolute points of π equals $k + j\sqrt{k-\lambda}$ for some integer j with the same parity as $v - 1$. In particular, if $k - \lambda$ is not a square, then $j = 0$ and $a = k$.

Suppose that π has the property that each absolute point is on exactly one absolute line (as is the case, e.g., for projective planes). Then N can be partitioned as

$$N = \begin{pmatrix} I & * \\ * & * \end{pmatrix}$$

and Theorem 2.4 yields

$$a \leqslant \frac{v(\sqrt{k-\lambda} + 1)}{k + \sqrt{k-\lambda}}$$

with equality implying that the absolute points and nonabsolute blocks form a 2-design. In particular one finds for a projective plane of order n that $a \leqslant n^{3/2} + 1$, and if $a = n^{3/2} + 1$, then the absolute points and nonabsolute lines form a unital (see Hughes and Piper [1973] and Seib [1970] for proofs using counting arguments).

Similar arguments work for correlations. For example, we have the following proposition

2.5. PROPOSITION. *Let D be a design with blocks of size k, and let σ be a correlation of D (i.e. an automorphism of its point-block incidence graph Γ, swapping the two parts of its bipartition). Let p be a prime such that* $\gcd(p,\theta) \neq 1$ *for each eigenvalue $\theta \neq \pm k$ of the adjacency matrix A of Γ. Then the number of points of D incident with their image under σ is congruent to k* (mod p).

PROOF. Let

$$A = \begin{pmatrix} 0 & N \\ N^{\top} & 0 \end{pmatrix},$$

and let σ have (permutation) matrix S, say

$$S = \begin{pmatrix} 0 & I \\ T & 0 \end{pmatrix}.$$

That σ is an automorphism of Γ is expressed by $AS = SA$ (i.e. $NT = N^{\top} = TN$), so A and S are commuting normal matrices, and can be diagonalized simultaneously. Now S has eigenvalue ± 1 for the eigenvectors for which A has eigenvalue $\pm k$, so if P is a prime ideal dividing p in an algebraic number field containing the eigenvalues of A and S, then all eigenvalues of AS vanish (mod P), except for two eigenvalues k, and the number of points x incident with σx equals $\operatorname{tr} N = (1/2) \operatorname{tr} AS \equiv k \pmod{P}$, so that $\operatorname{tr} N \equiv k \pmod{p}$. □

In particular this holds for $p \mid (k - \lambda)$ when D is a square 2-(ν, k, λ) design. Thus, it follows that any correlation of PG(d, q) has points incident with their image.

A very similar argument was used in Brouwer and Cohen [1983] to show that finite buildings do not have proper quotients.

3. Association schemes

Let X be a finite set provided with a number of binary relations $R_0, \ldots, R_d$, such that the corresponding 0–1 matrices A_i (defined by $(A_i)_{xy} = 1$ if $(x,y) \in R_i$ and $(A_i)_{xy} = 0$ otherwise) satisfy

(i) $A_0 = I$,

(ii) $A_i = A_i^{\top}$ for all i,

(iii) $\sum_{i=0}^{d} A_i = J$ and

(iv) $A_i A_j = \sum_{k=0}^{d} p_{ij}^{k} A_k$ for all i, j and certain integers p_{ij}^{k}.

Then $(X, \{R_0, \ldots, R_d\})$ is called an *association scheme*.

For example, a connected graph Γ with vertex set X is a distance-regular graph of diameter d if and only if $(X, \{R_0, \ldots, R_d\})$ is an association scheme, where R_j is the relation of having (graph-)distance j.

The algebra A (called the *Bose–Mesner algebra*) spanned (over $\mathbb{R}$ or $\mathbb{C}$, say) by the matrices A_i is commutative and self-adjoint, and hence semisimple. Let E_j ($0 \leqslant j \leqslant d$) be its basis of minimal idempotents. We can express the E_j in terms of the A_i (and thus define the *dual eigenmatrix* Q of the scheme):

$$E_j = |X|^{-1} \sum_i Q_{ij} A_i.$$

Now the E_j are positive semidefinite (since all their eigenvalues are 0 or 1).

Define, for a nonempty subset Y of X the *inner distribution* a of Y by

$$a_i = \big|((Y \times Y) \cap R_i)\big| / |Y|,$$

the average number of points in Y in relation i with some point of Y.

3.1. THEOREM (Delsarte [1973a]). $aQ \geqslant 0$.

PROOF. If χ is the characteristic vector of Y, then

$$|Y|(aQ)_j = |Y| \sum_i a_i Q_{ij} = \sum_i \chi^{\mathrm{T}} Q_{ij} A_i \chi = |X| \chi^{\mathrm{T}} E_j \chi \geqslant 0$$

since E_j is positive semidefinite. □

This set of inequalities (known as Delsarte's *linear programming bound*) provides a powerful tool for obtaining upper bounds on the size of cliques and lower bounds on the size of designs in various structures. Additional information is available in case of equality in some of these inequalities. For much more information, see Delsarte [1973a], MacWilliams and Sloane [1977] (Chapter 21), and Brouwer et al. [1989] (Chapter 2).

The usual design theory studies regularity properties of collections of k-subsets of a fixed v-set, i.e. of subsets of the *Johnson scheme* (obtained from the Johnson graph $J(v, k)$, where two k-sets are adjacent whenever they have a $(k-1)$-set in common). It turns out that such a collection Y is a t-design if and only if its inner distribution a satisfies $(aQ)_j = 0$ for $1 \leqslant j \leqslant t$. Taking this property as a definition, we may study t-designs in arbitrary association schemes.

For the theory of distance-regular graphs, and connections with design theory, see Biggs [1974] and Brouwer et al. [1989].

A *strongly regular* graph is a distance-regular graph of diameter 2. For the known constructions of strongly regular graphs, see the surveys by Hubaut [1975] and Brouwer and Van Lint [1982].

A design is called *block schematic* when its collection of blocks, with as relations the intersection sizes, is an association scheme. For example, a *quasi-symmetric* 2-design, i.e. a 2-design in which the number of points any two distinct blocks have in common takes exactly two values, is necessarily block schematic (Goethals and Seidel [1970]); in particular, if we fix one of the two intersection numbers s and call two blocks adjacent whenever they meet in s points, we obtain a strongly regular graph. Another example is provided by the Steiner system $S(5,8,24)$, with the block intersection sizes 8, 0, 4, 2, where we find a distance-regular graph of diameter 3.

G. Higman (cf. Cameron [1983], Proposition 5.4) gave the following condition on automorphisms of association schemes.

3.2. PROPOSITION. *Let σ be an automorphism of the association scheme (X, R) and put $s_i := \#\{x\colon\ (x, \sigma x) \in R_i\}$. Then for each j the number*

$$|X|^{-1} \sum_i s_i Q_{ij}$$

is an algebraic integer.

PROOF. Let σ have permutation matrix S. Then S commutes with all A_i, and $s_i = \operatorname{tr} SA_i$. It follows that the number mentioned equals $\operatorname{tr} SE_j$, and hence is a sum of eigenvalues of S. □

For an application, see Cameron's paper cited above.

There are close relations between design theory and coding theory. On the one hand, codes (say, spanned by the rows or columns of the incidence matrix of a design) can be used as a tool to investigate designs, and on the other hand, one may use codes as a means to construct designs (taking for example the supports of the vectors of a given weight in the code). The latter use is described in more detail in the section on design construction. Let us here have a look at the former. Standard references on (algebraic) coding theory are MacWilliams and Sloane [1977] and Van Lint [1982].

4. Coding techniques

In the section on eigenvalue techniques we looked at the incidence matrices of designs as matrices with real (or complex) entries. But of course any ring with unity can be used to investigate a $\{0,1\}$-matrix. In particular, the rings $\mathbb{Z}$ and $\mathbb{F}_p$ have been used successfully to study a wide range of incidence structures. Of primary interest is the *code*, i.e. the subspace (or submodule) spanned by the rows or columns of the incidence matrix and provided with the (Hamming-)distance defined using its canonical basis; here, one asks questions like 'What is the dimension?', 'What is the weight distribution?', and 'What are the vectors of minimal nonzero weight?'. The standard example is provided by the very close relation between the Steiner system $S(5,8,24)$ and its $\mathbb{F}_2$-code, the extended binary Golay code, where one cannot investigate one without looking at the other as well. The study of the $\mathbb{F}_2$-code of a putative projective plane of order 10 was instrumental in the proof (begun by MacWilliams, Sloane and Thompson [1973], finished by Lam et al. [1989]) that no such plane exists. The proof in Bagchi [1988(1991)] of the nonexistence of a 3-(57,12,3) design would have been a nice example of the use of the $\mathbb{F}_3$-code of a design, but it seems that the proof is wrong (even after the correction given). If we can find a *self-dual* code, then this usually yields parameter restrictions (see below).

Let us first look at the dimension of the code C spanned by (the characteristic vectors of) the blocks of a 2-(ν, k, λ) design, i.e. by the columns of its point-block incidence matrix A. Here the *order* $n := r - \lambda$ of the design plays a major role. Clearly, $\dim_p C = \mathrm{rk}_p A$.

4.1. Proposition. *Let A be the incidence matrix of a 2-(ν, k, λ) design of order $n := r - \lambda$, and suppose that $p \nmid n$. Then $\mathrm{rk}_p A = \nu - 1$ if $p \mid k$ and $\mathrm{rk}_p A = \nu$ otherwise.*

Proof. We have $AA^\top = \lambda J + nI$, so that $\mathrm{rk}_p A \geqslant \mathrm{rk}_p AA^\top \geqslant \nu - 1$. Now if $uA = 0$, we have $0 = uAA^\top = \lambda(u, \mathbf{1})\mathbf{1} + nu$, so that u is a multiple of $\mathbf{1}$. But $\mathbf{1}A = k\mathbf{1}$, so A has full rank unless $p \mid k$. □

Thus, interesting things (i.e. results that depend on the structure of a design, instead of only the parameters) happen only when $p \mid n$. Often, the p-rank of A is a useful invariant for distinguishing designs with the same parameters. Hamada [1973] gives the examples of the four nonisomorphic 2-(8,4,3) designs ($n = 4$), where the 2-ranks are 4, 5, 6 and 7, and of the three biplanes 2-(16,6,2) (again with $n = 4$), that are distinguished by their 2-ranks 6, 7 and 8.

A lower bound on the p-rank of the incidence matrix of a 2-$(\nu, k, 1)$ design is given by the following theorem (see Bruen and Ott [1990] and Hillebrandt [1990]).

4.2. Theorem. *The p-rank of the incidence matrix of a Steiner system $S(2, k, \nu)$ is at least $(k-1)\sqrt{(r-1)r/k}$ if $n (= r - 1) \equiv 0 \pmod{p}$.*

Self-dual codes

A code C in a vector space V (of dimension n over $\mathbb{F}_p$) provided with a nondegenerate symmetric bilinear form f is called *self-dual* when $C = C^\perp$, and *self-orthogonal* when $C \subseteq C^\perp$. (When no form is explicitly specified one takes $V = \mathbb{F}_p^n$ and $f(x, y) = \sum x_i y_i$.) Clearly, if C is self-dual we have $\dim C = n/2$, and if C is self-orthogonal then $\dim C \leqslant n/2$.

4.3. Proposition. *Let C be the code over $\mathbb{F}_p$ generated by the blocks of a design with the property that there is an s such that any two (not necessarily distinct) blocks meet in a number of points that is congruent to $s \pmod{p}$. Then $\dim C \leqslant (\nu + 1)/2$. If moreover $p \mid s$, then $\dim C \leqslant \nu/2$.*

Proof. If $p \mid s$, then C is self-orthogonal and hence $\dim C \leqslant \nu/2$. Otherwise, $C \cap C^\perp$ is of codimension 1 in C (it is spanned by the differences of pairs of blocks), so $\dim C \leqslant (\nu + 1)/2$. □

In particular, this holds (with $s = k$) for square 2-(ν, k, λ) designs where $p \mid n$ $(= k - \lambda)$. Other applications were given by Calderbank [1987, 1988], Calderbank and Frankl [1990] and Blokhuis and Calderbank [1992] to the theory of quasisymmetric designs. (See also Bagchi [1992].)

Any self-orthogonal code C is contained in some maximal such code D, say, and all such codes D have the same dimension, the *Witt index* of (V, f). When $p = 2$ we have $\dim D = [n/2]$, and when p is odd we have $\dim D = n/2 - 1, (n-1)/2, n/2$ according as to whether (V, f) is elliptic, parabolic or hyperbolic. In particular, if (V, f) contains a self-dual code, and f is given by $f(x, y) = x^{\top}Qy$, then n is even and $(-1)^{n/2}\det Q$ is a square (mod p). If $p = 2$ (and $Q = I$) then any self-orthogonal code C with all weights divisible by 4 is contained in some maximal such code D, say, and again all such codes D have the same dimension: $\dim D = (n-3)/2, n/2 - 1, (n-1)/2, n/2$ according to the type of the vector space of even weight vectors of length n (modulo $\langle \mathbf{1} \rangle$[3] in case $4 \mid n$) provided with the quadratic form

$$x \mapsto \frac{1}{2}wt(x) \ (\mathrm{mod}\ 2) \quad \Big(= \sum_{i<j} x_i x_j \Big),$$

i.e. when $n \equiv \pm 3 \pmod 8$, $n \equiv 2, 4, 6 \pmod 8$, $n \equiv \pm 1 \pmod 8$, and $n \equiv 0 \pmod 8$, respectively.

The self-dual codes of small word length n over fields of small characteristic have been classified. For the binary case, see (the references in) Conway and Sloane [1990].

The Smith normal form

A basic tool for a more detailed study is the theorem on the invariant factors of a matrix (cf. Smith [1861], and MacDuffee [1946], §27).

4.4. THEOREM. *Let A be a $\nu \times b$ matrix (with $\nu \leqslant b$, say) with entries in a principal ideal domain Z. Then there exist invertible matrices U and V such that $UAV =$ $\mathrm{diag}(d_1, d_2, \ldots, d_\nu)$ with $d_1 \mid d_2 \mid \cdots \mid d_\nu$. The d_i are unique (up to units in Z).*

In the situation of this theorem, the d_i $(1 \leqslant i \leqslant \nu)$ are called the *invariant factors* of the matrix A; the maximal prime power factors of the invariant factors are called the *elementary divisors* of A; and the diagonal matrix UAV is called the *Smith normal form* of A.

This theorem can be used, for example, to compute the dimension of the $\mathbb{F}_p$-code of a square 2-(ν, k, λ) design in case p divides the order $n = r - \lambda$ $(= k - \lambda)$ exactly once (see Hamada [1973], Sachar [1979], Bridges, Hall Jr and Hayden [1981], Lander [1983]).

4.5. THEOREM. *Let C be the $\mathbb{F}_p$-code spanned by the rows of the incidence matrix of a square 2-(ν, k, λ) design. If $p \mid n$ but $p^2 \nmid n$, we have:*

(i) *If $p \mid k$, then* $\dim C = (\nu - 1)/2$.

(ii) *If $p \nmid k$, then* $\dim C = (\nu + 1)/2$.

[3] $\langle \mathbf{1} \rangle$ represents the vector with every component equal to 1 (Editor's note).

Similar results hold for arbitrary designs, when information on the intersection numbers (mod p) is available.

As an example of how to prove such results, let us compute for odd primes p the p-rank of the $\{0, \pm 1\}$-matrix Q of order p with entries $Q_{xy} = \chi(y - x)$ ($x, y \in \mathbb{F}_p$, χ is the quadratic residue character, $\chi(0) = 0$). (A more general result is given below.) We have $QQ^\top = pI - J$, so $\mathbf{1}$ is an $\mathbb{F}_p$-linear combination of the rows of Q, and bordering Q we find a matrix

$$B = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & Q & \\ 1 & & & \end{pmatrix}$$

of order $p+1$ with $\mathrm{rk}_p B = \mathrm{rk}_p Q$ and $BB^\top = pI$, so that $\det B = p^{(p+1)/2}$. Since the $\mathbb{F}_p$-rowspace of B is self-orthogonal, we have $\mathrm{rk}_p B \leqslant (p+1)/2$. But if $\mathrm{rk}_p B < (p+1)/2$, then the Smith normal form of B contains more than $(p+1)/2$ invariant factors p, contradicting the value of $\det B$. Hence $\mathrm{rk}_p Q = \mathrm{rk}_p B = (p+1)/2$.

Lucas' theorem

In many cases the dimension of a code is directly related to the number of binomial or multinomial coefficients $\binom{m}{k}$ that are nonzero (mod p). Here very often the following theorem provides the required information.

4.6. THEOREM (Lucas). *Let*

$$m = \sum_i m_i p^i \quad \textit{and} \quad k = \sum_i k_i p^i$$

be the representations of m and k in base p, for some prime p. Then

$$\binom{m}{k} \equiv \prod_i \binom{m_i}{k_i} \pmod p.$$

PROOF. The left hand side is the coefficient of x^k in $(1 + x)^m$, while the right-hand side is the coefficient of x^k in

$$\prod_i \left(1 + x^{p^i}\right)^{m_i}.$$

But in $\mathbb{F}_p$ these polynomials coincide. □

More generally, one shows in the same way (considering the coefficient of $x_1^{e_1} \cdots x_t^{e_t}$ in $(x_1 + \cdots + x_t)^m$ for $m = e_1 + \cdots + e_t$) that the same product rule holds for multinomial coefficients:

$$\binom{m}{e_1, \quad \dots, \quad e_t} \equiv \prod_i \binom{m_i}{e_{1i}, \quad \dots, \quad e_{ti}} \pmod p,$$

where the e_{ki} are the digits of e_k in base p. (Here

$$\binom{m_i}{e_{1i}, \quad \dots, \quad e_{ti}} = 0$$

when $m_i \neq e_{1i} + \cdots + e_{ti}$.) Below we give two applications. First we need a small lemma.

4.7. LEMMA. *Let*

$$p(x,y) = \sum_{i=0}^{d-1}\sum_{j=0}^{e-1} c_{ij}x^i y^j$$

be a polynomial with coefficients in a field F. Let $A, B \subseteq F$, with $m := |A| \geqslant d$ and $n := |B| \geqslant e$. Consider the $m \times n$ matrix $P = (p(a,b))_{a\in A, b\in B}$ and the $d \times e$ matrix $C = (c_{ij})_{ij}$. Then $\mathrm{rk}_F(P) = \mathrm{rk}_F(C)$.

PROOF. For any integer s and subset X of F, let $Z(s, X)$ be the $|X| \times s$ matrix $(x^i)_{x\in X,\ 0\leqslant i\leqslant s-1}$. Note that if $|X| = s$ then this is a Vandermonde matrix and hence invertible. We have $P = Z(d,A)CZ(e,B)^\top$, so $\mathrm{rk}_F(P) \leqslant \mathrm{rk}_F(C)$, but P contains a submatrix $Z(d,A')CZ(e,B')$ with $A' \subseteq A$, $B' \subseteq B$, $|A'| = d$, $|B'| = e$, and this submatrix has the same rank as C. □

APPLICATION. *For odd prime powers $q = p^e$, p prime, let Q be the $\{0, \pm 1\}$-matrix Q of order q with entries $Q_{xy} = \chi(y-x)$ ($x, y \in \mathbb{F}_q$, χ is the quadratic residue character, $\chi(0) = 0$). Then* $\mathrm{rk}_p\, Q = ((p+1)/2)^e$.

PROOF. Applying the above lemma with

$$p(x,y) = \chi(y-x) = (y-x)^{(q-1)/2} = \sum_i (-1)^i \binom{(q-1)/2}{i} x^i y^{(q-1)/2-i},$$

we see that $\mathrm{rk}_p\, Q$ equals the number of binomial coefficients $\binom{(q-1)/2}{i}$ with $0 \leqslant i \leqslant (q-1)/2$ not divisible by p. But

$$\frac{1}{2}(q-1) = \sum_i \frac{1}{2}(p-1)p^i,$$

so for each digit of i there are $(p+1)/2$ possibilities, and the result follows. □

Note that this proof shows that each submatrix of Q of order at least $(q+1)/2$ has the same rank as Q.

Projective and affine geometries

The codes of the projective and affine geometries are of special interest. By Proposition 4.1, for the design of points and hyperplanes in PG(d, q) and AG(d, q) only the prime p dividing q is of interest.

(For primes p not dividing q, Frumkin and Yakir [1990] and Yakir [1991] determine the p-rank of the inclusion matrix of k-subspaces vs. l-subspaces in PG(d, q) and in AG(d, q). The characteristic zero case was done already in Kantor [1972].)

More generally, the dimension over $\mathbb{F}_p$ of the design of points and s-dimensional subspaces is known (cf. Goethals and Delsarte [1968], Delsarte, Goethals and MacWilliams [1970], Smith [1969] and Hamada [1968, 1973]; see also Assmus and Key [1992]).

4.8. THEOREM. *Let p be a prime and let $s, n, e \in \mathbb{N}$. Write $q = p^e$, and $\nu = (q^{n+1} - 1)/(q - 1)$. The dimension $r(n, q, s)$ of the $\mathbb{F}_p$-code of the design of points and (projectively) s-dimensional subspaces of* PG(n, q) *is the number of integers t, $1 \leqslant t \leqslant \nu$, such that some multinomial coefficient*

$$\frac{(t(q-1))!}{(l_0(q-1))!(l_1(q-1))!\cdots(l_s(q-1))!}$$

where

$$\sum_{i=0}^{s} l_i = t, \quad l_i \neq 0 \ (0 \leqslant i \leqslant s)$$

is not congruent to zero modulo p. This dimension is also equal to the number of integers t, $1 \leqslant t \leqslant \nu$, satisfying the inequalities

$$w_q\big(p^i t(q-1)\big) \geqslant (s+1)(q-1), \quad i = 0, \ldots, e-1$$

(where $w_q(x)$ is the sum of the 'digits' of x written in base q).

If $e = 1$ (i.e. if $q = p$), this dimension is

$$\sum_{i=0}^{n-s} \sum_{0 \leqslant j \leqslant i - i/p} (-1)^j \binom{n+1}{j} \binom{n + i(p-1) - jp}{n}$$

so in particular this dimension equals

$$\sum_{i=0}^{n-s} \binom{n+1}{i}$$

when $q = 2$.

If $s = n - 1$ (i.e. for the square design of points and hyperplanes), this dimension is

$$\binom{n+p-1}{n}^e + 1.$$

In all these cases Lemma 4.7 can be used for the computation of the dimensions. As an example we will show that

The dimension of the $\mathbb{F}_q$-code C spanned by the (characteristic vectors of the) complements of hyperplanes in PG(n, q) *is $\binom{n+p-1}{n}^e$ if $q = p^e$.*

(From this it follows immediately that the hyperplanes span a code C' of dimension one more, since $\mathbf{1} \in C' \setminus C$.) Note that the $\mathbb{F}_q$-rank of a $\{0, 1\}$-matrix equals its $\mathbb{F}_p$-rank.

PROOF. Let $V = \mathbb{F}_q^{n+1}$. Fixing a nondegenerate bilinear form on V, we see that $\dim_p C$ equals the p-rank of the matrix M with rows and columns indexed by V, where $M_{u\nu} = (u, \nu)^{q-1}$. In order to apply Lemma 4.7, view V as the field of order q^{n+1}, and let $(u, \nu) := \mathrm{tr}(u\nu)$, where tr: $\mathbb{F}_{q^{n+1}} \to \mathbb{F}_q$ is the trace function:

$$\mathrm{tr}(x) = x + x^q + \cdots + x^{q^n}.$$

Now our lemma says that $\mathrm{rk}_p M$ equals the p-rank of the coefficient matrix of the polynomial $(\mathrm{tr}(xy))^{q-1}$, but this matrix is diagonal with diagonal entries

$$c_{ii} = \binom{q-1}{i_0, i_1, \ldots, i_n} \quad \text{where } i = \sum_j i_j q^j \text{ is the representation of } i \text{ in base } q,$$

and

$$\sum_j i_j = q - 1.$$

By the multinomial version of Lucas' theorem we find that the number of c_{ii} that do not vanish (mod p) equals $\binom{n+p-1}{n}^e$. □

There is a close connection between the codes studied here and the Reed–Muller codes studied in coding theory. Let us restrict ourselves to the binary case. Let $0 \leqslant r \leqslant m$. The (binary) *Reed–Muller code* $R(r, m)$ of order r and length 2^m is by definition the code consisting of the vectors $(f(x))_{x \in \mathbb{F}_2^m}$ where f is a polynomial in m variables of degree at most r. Clearly, the

$$d_{r,m} := \sum_{i=0}^{r} \binom{m}{i}$$

monomials $1, x_1, x_2, \ldots, x_m, x_1x_2, \ldots, x_{m-r+1} \ldots x_m$, form a basis for $R(r, m)$ so the dimension of $R(r, m)$ is $d_{r,m}$. If we delete the coordinate corresponding to $x = 0$ we obtain the *punctured Reed–Muller code* $R(r, m)^*$ of length $2^m - 1$ and with the same dimension $d_{r,m}$ (if $r < m$). Let $C(n, s)$ be the code spanned by the incidence vectors of the s-dimensional subspaces of PG$(n, 2)$. We show that $C(n, s) = R(n - s, n + 1)^*$. Indeed, the $\subseteq$ part follows since an s-dimensional subspace S of PG$(n, 2)$ is the intersection of $n - s$ hyperplanes, so that the incidence vector of S can be represented as the product of $n - s$ linear functions. Conversely, we have $C(n, t) \subseteq C(n, s)$ for $t \geqslant s$ and

$$R(n - s, n + 1) \subseteq C(n, s) + R(n - s - 1, n + 1),$$

so that the reverse inclusion follows by induction on s. This proves the above dimension formula for $q = 2$.

Black and List [1990] have computed the invariant factors over $\mathbb{Z}$ of the incidence matrix of points and hyperplanes of the projective spaces PG(n, p), p prime.

4.9. THEOREM. *The p-rank of the incidence matrix of the design of points and s-dimensional subspaces of* AG(n,q), $q = p^e$, *equals* $r(n,q,s) - r(n-1,q,s)$ (*with* $r(n,q,s)$ *as defined in the previous theorem*).

PROOF. View the point set Y of AG(n,q) (in the natural way) as a subset of the point set X of PG(n,q), so that $X \setminus Y$ is the point set of PG$(n-1,q)$. The s-spaces in AG(n,q) are intersections with Y of s-spaces in PG(n,q). Let C_n, C_{n-1}, D_n be the $\mathbb{F}_p$-codes spanned by the s-spaces in PG(n,q), PG$(n-1,q)$ and AG(n,q), respectively, where we regard C_{n-1} as a subspace of C_n. We have to prove that C_{n-1} is the kernel of the restriction map $C_n \to D_n$, i.e. that if a code word $u \in C_n$ has support contained in $X \setminus Y$, then $u \in C_{n-1}$. Fix a point $x_0 \in X \setminus Y$. For any s-space S in PG(n,q) not containing x_0, its projection $\pi(S) = \langle S, x_0 \rangle \setminus Y$ is an s-space in PG$(n-1,q)$, and if $u = \sum a_S S$, then also $u = \sum a_S \pi(S)$, where the latter sum is over the S not containing x_0. □

For the codes associated with projective geometries it is also known what the minimum weight vectors are: the minimum weight vectors of the code generated by the s-dimensional subspaces are precisely (constant multiples of) these generators (see Delsarte et al. [1970]).

Hamada's conjecture

Hamada [1973] conjectured that the designs of points and s-flats in PG(n,q) and in AG(n,q) are the unique designs with minimum p-rank among all designs with the same parameters. Hamada and Ohmori [1975] proved this conjecture in the case of hyperplanes in PG$(n,2)$ or AG$(n,2)$. Doyen, Hubaut and Vandensavel [1978] proved this for the design of the lines in PG$(n,2)$ or AG$(n,3)$, and Teirlinck [1980] (cf. Dehon [1980]) proved it for the design of the planes in AG$(n,2)$. Of the four projective planes of order 9, the Desarguesian one has 3-rank 37, while the other three have 3-rank 41 (Sachar [1973]). Tonchev [1986] found several designs with the same parameters and 2-rank as that of the planes in PG$(4,2)$ or of the 3-spaces in AG$(5,2)$, showing that the 'unique' part of Hamada's conjecture is false.

More generally, 'Hamada's conjecture' is taken to be the statement that among designs with the same parameters the nicest design has the smallest rank. But it is not always clear which design should be called 'nicest': among the Steiner systems $S(2,4,28)$, the Hermitian unital has the largest group and 2-rank 21, while the Ree unital has a smaller group and 2-rank 19.

The Moorhouse conjecture

Moorhouse [1991] looks at the codes of k-nets. Here, a k-net of order n is a partial linear space of n^2 points and nk lines of size n such that for every nonincident point-line pair (p, L), there exists a unique line through p which has no point in common with L. From this property it follows that parallelism (the property of being equal or disjoint) is an equivalence relation. Clearly, a k-net of order n has k parallel classes consisting of n lines each. Notice that any k parallel classes of an affine plane of order n give rise to a k-net of order n. (In design theory, the dual of a k-net of order n is called a transversal design TD[$k;n$]; this structure is equivalent to $k-2$ mutually orthogonal Latin squares of order n.)

As usual, the p-code of a net is defined to be the code generated by the incidence vectors of the lines over the field $\mathbb{F}_p$. Moorhouse makes the following interesting conjecture.

CONJECTURE. *If N is a k-net of order n and N' is a $(k-1)$-subnet of N (i.e. N' consists of $k-1$ parallel classes of N) and p is a prime such that p^2 does not divide n, then* $\mathrm{rk}_p(N) - \mathrm{rk}_p(N') \geqslant n-k+1$.

To support his conjecture, Moorhouse gives ample computational evidence and shows that the conjecture is true for $k \leqslant 3$ and also that it holds with equality for Desarguesian nets of order p. The interesting thing about this conjecture is that its truth would imply that any projective plane of order congruent to 2 (mod 4) or of square-free order is in fact Desarguesian of prime order.

Explicit results and applications

For the 2-transitive square designs other than the projective spaces, the dimensions are as follows. The unique Hadamard design with $\nu = 11$ and $k = 6$ has 3-rank 5. The design with $\nu = 176$, $k = 50$, $\lambda = 14$ has 2-rank 22 and 3-rank 50, and the design with $\nu = 2^{2m}$ associated with 2^{2m}: Sp($2m$, 2) has 2-rank $2m+2$ (cf. Lander [1983], p. 117).

Wilson [1990] computes the p-rank of the $\binom{\nu}{t} \times \binom{\nu}{k}$ $\{0,1\}$-matrix M describing the inclusion between t- and k-subsets of a given ν-set. (The case $p = 2$ was done earlier by Linial and Rothschild [1981].) In fact, he proves that there are unimodular integral matrices P and Q such that $D = PMQ$ is diagonal (i.e. $D_{ij} = 0$ for $i \neq j$) with diagonal entries $\binom{k-i}{t-i}$ with multiplicity $\binom{\nu}{i} - \binom{\nu}{i-1}$ $(0 \leqslant i \leqslant t)$. In particular,

$$\mathrm{rk}_p\, M = \sum \binom{\nu}{i} - \binom{\nu}{i-1},$$

where the sum is over those i for which $\binom{k-i}{t-i}$ is nonzero (mod p).

Key and Mackenzie [1991] give an upper bound for the p-rank of a translation plane of order $q = p^e$.

In Blokhuis, Brouwer and Wilbrink [1991] it is shown that the characteristic vector of a unital of order q is in the code generated by the lines of PG($2, q^2$) if and only if the unital is Hermitian. Key [1991] shows that also in PG(n, q^2) the Hermitian varieties are in the code generated by the hyperplanes.

Delsarte [1971] looks at the binary code of the Miquelian inversive plane of odd order (to be more precise, the interesting code is that generated by 'half' the circles).

Bagchi and Narasimha Sastry [1987] make a detailed study of the binary codes associated with generalized quadrangles Sp($4, q$), q even, in their characterization of the inversive planes of even order. Their Theorem 3 seems to be of independent interest and can be formulated as follows.

4.10. THEOREM. *If A is a symmetric $\{0,1\}$-matrix, then the diagonal of A is in the binary code spanned by the rows of A.*

PROOF. If $Ax = 0$, then

$$\sum a_{ii}x_i = \sum a_{ii}x_i^2 = x^\top Ax = 0.$$

In other words, any vector orthogonal to all rows of A is also orthogonal to its diagonal. □

Notice how this applies to the Tits ovoid, which is the set of absolute points of a polarity of the Sp$(4,q)$ generalized quadrangle. It also shows that for q even, the Hermitian unital is in the code of the plane PG$(2,q^2)$.

Bagchi, Brouwer and Wilbrink [1991] consider the binary codes of the generalized quadrangles of odd order q. Among other things we obtained the following interesting

COROLLARY. *If one point of a generalized quadrangle of odd order q is antiregular, then all its points are antiregular* (see Chapter 9).

No coding-free proof of this purely geometric fact is known.

5. Constructions of designs

As mentioned before, the classical way of constructing a 2-(v,k,λ) design is by use of a difference set (for a square design) or a difference family (in the general case). For example, one may construct a Steiner triple system STS(15) (i.e. a $S(2,3,15)$) by taking the three orbits of $\{0,5,10\}$, $\{0,1,4\}$ and $\{0,2,8\}$ under $\mathbb{Z}_{15}$. If a group G acts t-transitively on a set X, then every orbit or union of orbits of G on the collection $\binom{X}{k}$ of k-subsets of X will be a t-design. For example, from the Mathieu groups M_{12} and M_{24} one obtains the famous Witt designs $S(5,6,12)$ and $S(5,8,24)$. (But there are not very many t-transitive groups with $t \geqslant 2$, and none with $t \geqslant 6$, except for the symmetric and alternating groups, which yield trivial designs.) If the group G is only s-transitive, for $s < t$, then one searches for G-invariant t-designs by the method of Kramer and Mesner [1976]: First construct the matrix M with rows indexed by the G-orbits on $\binom{X}{k}$ and columns by the G-orbits on $\binom{X}{t}$, where M_{KT} is the number of k-sets in K containing any fixed t-set in T. Then search for $\{0,1\}$-vectors x such that $xM = \lambda\mathbf{1}$. (If multiple blocks are allowed, search for vectors x with non-negative integral entries.) In case M is small (at most a few dozen rows and columns) this is easily done by backtracking, maybe assisted by congruence arguments. (For example, in the above example of a STS(15) invariant under $\mathbb{Z}_{15}$, since we must find 35 blocks, we have to choose an orbit of length not divisible by 15, but the only such orbit is that of $\{0,5,10\}$.) When M gets larger, simple backtracking becomes too time-consuming; various heuristics were developed to cut down the search time (see Kramer, Leavitt and Magliveras [1985]), and these made it possible to construct the first simple 6-design (a 6-(33,8,36), by Leavitt and Magliveras [1984]). Kreher and Radziszowski noticed that one can profitably use the lattice reduction technique of Lenstra, Lenstra and Lovasz [1982]: If we let

$$N = \begin{pmatrix} M & I \\ -\lambda\mathbf{1} & 0 \end{pmatrix}$$

then our equation $xM = \lambda\mathbf{1}$ becomes $(x, 1)N = (0, x)$. Let Λ be the lattice consisting of all integral linear combinations of the rows of N. By Lenstra, Lenstra and Lovasz, find a basis of short vectors for Λ. If we are lucky, the vector $(0, x)$ we are looking for is one of the base vectors. Several 6-designs (such as a 6-(14,7,4)) have been constructed in this way. (See Kreher and Radziszowski [1986a,b, 1987a,b].)

Assmus–Mattson type results

Other sources of designs are found in coding theory. There are results, variations on a theme by Assmus and Mattson [1969], which roughly say that if a code C is nice enough, then the collection of supports of code words in C of any given weight w will be a nice design. For a detailed exposition, see MacWilliams and Sloane [1977], Chapter 6, with some improvements in Delsarte [1973b], Dumer [1980], and Roos [1980].

For the following results, let us agree that in case $k < t$, the t-(v, k, λ) designs are precisely the multiples of the design $\binom{X}{k}$ of all k-subsets of X. Given a code C and an integer w, let C_w be the collection of supports of vectors of weight w in C. We call C a *t-design code* if C_w is a t-design for each w. Call a weight w *essential* when C contains some, but not all, words of weight w. (Thus, the weight 0 is never essential, and the weight n is never essential in binary codes of word length n.) Let C be a code with inner distribution a and dual distribution $b := aQ$, the *MacWilliams transform* of a. Define the *dual distance* e of C by $b_1 = \ldots = b_{e-1} = 0$, $b_e \neq 0$. Define the *dual degree* r of C to be the number of $i > 0$ such that $b_i \neq 0$. (If C is linear, then b is the inner distribution, and e the minimum distance, of the dual code $C^{\perp}$. But here we do not assume that C is linear, or that q is a prime power, or that $\mathbf{0} \in C$.)

5.1. THEOREM. *If C has s essential weights, and $s \leqslant e$, then C is an $(e-s)$-design code. (And moreover, if for any essential weight w the collection C_w is an $(e-s+1)$-design, then C is an $(e-s+1)$-code.)*

For a generalization to arbitrary Q-polynomial association schemes, see Delsarte [1977] and Brouwer et al. [1989], §2.8.

5.2. THEOREM. *If $\mathbf{0} \in C$, and C has minimum distance d, and $r \leqslant d$, then C is a $(d-r)$-design code. If moreover C is binary and only contains even weight vectors, then C is a $(d-r+1)$-design code.*

For a generalization to arbitrary P-polynomial association schemes, see Brouwer et al. [1989], §11.1, together with Roos [1980]. For example, the extended binary Golay code is self-dual and has parameters $d = e = 8$, $r = 4$, $s = 3$ and has all weights even, and hence yields 5-designs (by both theorems), in fact designs with parameters 5-(24,8,1), 5-(24,12,48) and 5-(24,16,78). Similarly, the extended binary quadratic residue code of length 48 is self-dual and has parameters $d = e = 12$, $r = 8$, $s = 7$ (the weights are 0, 12, 16, 20, 24, 28, 32, 36, 48) and both theorems tell us that this is a 5-design code.

5.3. THEOREM (Assmus and Mattson [1969]). *Let C be a linear code over $\mathbb{F}_q$ of word length n and minimum distance d. Suppose that for some integer t with $0 < t < d$ at most $d - t$ nonzero dual weights are $\leqslant n - t$. Then C is a t-design code.*

In the binary case this theorem is equivalent to the previous one, since no weights w with $n - t < w < n$ can occur. In general it is stronger. For example, the extended ternary Golay code has $d = e = 6$, $r = 3$, with nonzero dual weights 6, 9, 12. Thus, Theorem 5.2 guarantees 3-designs, while Theorem 5.3 guarantees 5-designs. The reason for this difference is that the hypotheses of Theorem 5.2 allow a stronger conclusion, namely that the code words of any given weight w form a q-ary t-design (there is a constant number of code words that have specified nonzero entries on any t given positions), which is true for $t = 3$ in case of the extended ternary Golay code, while for the Assmus–Mattson theorem it is essential to take supports. Similarly, the symmetry codes of Pless [1969, 1970, 1972, 1975] (certain self-dual ternary codes) yield many new 5-designs. For a strengthening of this theorem, see Calderbank, Delsarte and Sloane [1991] and Calderbank and Delsarte [1991a,b].

5.4. THEOREM. *If C is a linear t-design code, then so is $C^{\perp}$.*

The above theorems provide us with t-designs, possibly with repeated blocks. If repeated blocks are not desired, then observe:

(i) if C is a binary code, then no C_w has repeated blocks;

(ii) if C is a linear code with minimum distance d over an alphabet of size q, and $w - \lceil w/(q-1) \rceil < d$, then code words of weight w with the same support must be multiples of each other, so that in C_w each block is repeated precisely $q - 1$ times, and $(1/(q-1))C_w$ is a simple design.

Recursive constructions

Finally, one may construct designs from designs. We have already mentioned the derived and residual designs, and the fact that sometimes the existence of an extension can be guaranteed. Hanani [1960, 1975] found many recursive constructions for 2- and 3-designs and constructed designs for all feasible parameter sets in the cases $(t, k) = (2,4), (2,5), (3,4)$. Wilson [1972a,b, 1975] proved using such recursive constructions (starting from designs constructed with difference families) that given k and λ there is a ν_0 such that a 2-(ν, k, λ) exists whenever $\nu \geqslant \nu_0$ and r $(= \lambda(\nu - 1)/(k - 1))$ and b $(= \nu r/k)$ are integers. The class of t-(ν, k, λ) designs is too small to work in with recursive techniques. Instead one uses pairwise balanced designs and (*group*) *divisible designs* (designs with a partition Π of the point set such that a pair of points contained in the same part of Π is contained in λ_1 (often 0) blocks, while the remaining pairs of points are contained in λ_2 (λ) blocks each), in particular transversal designs. In the proof of Wilson's asymptotic existence result, the result of Chowla, Erdős and Straus [1960], affirming the existence of arbitrarily many mutually orthogonal Latin squares of sufficiently large order (itself proved by recursive methods) plays an important role. These same methods solve many similar problems on the partition of pairs (graph partitions, bridge tournaments etc.), and there exists an extensive literature. For 3-designs the general theory is still not much farther than in Hanani's time; there is some progress by Hartman [1990].[4] For t-designs with larger t only a few constructions are known.

[4] Note added in proof: Recently, John Blanchard has enormously strengthened the Chowla–Erdős–Straus result by proving that there exist transversal t-designs TD(t, k, n) for any fixed t, k and all sufficiently large n.

Indeed, it is only recently that Teirlinck [1987] showed that nontrivial simple t-designs exist for all t. (In fact, his constructions produce *large sets* of t-designs, and this has renewed interest in large sets of designs. Also in coding theory the maximum number of pairwise disjoint t-(ν, k, λ) designs plays some role, cf. Brouwer, Shearer, Sloane and Smith [1990].) A nice recursive construction of (large sets of) t-designs was given by Khosrovshahi and Ajoodani-Namini [1991].

Tables of the known t-designs on a small number of points are given in Chee, Colbourn and Kreher [1990].

6. Characterizations of designs

There are many results that provide characterizations of designs or classes of designs. Perhaps the most important type of such theorems deal with the situation where one assumes that a 'big' group of automorphisms is present. The classification of all finite simple groups has made many of the earlier results obsolete, and we will content ourselves with describing the post-classification results. The first three results are taken from Kantor [1985].

6.1. THEOREM. *Let $\boldsymbol{D}$ be a 2-design with $\lambda = 1$ admitting an automorphism group 2-transitive on points. Then $\boldsymbol{D}$ is one of the following designs.*

(i) $\mathrm{PG}(d, q)$,

(ii) $\mathrm{AG}(d, q)$,

(iii) *The design with $\nu = q^3 + 1$ and $k = q + 1$ associated with* $\mathrm{PSU}(3, q)$ *or* ${}^2\mathrm{G}_2(q)$,

(iv) *One of two affine planes having 3^4 or 3^6 points, or*

(v) *One of two designs having $\nu = 3^6$ and $k = 3^2$.*

6.2. THEOREM. *Let $\boldsymbol{L}$ be a finite geometric lattice of rank at least* 3 *such that* $\mathrm{Aut}\, L$ *is transitive on ordered bases. Then either*

(i) *$\boldsymbol{L}$ is a truncation of a Boolean lattice or a projective or affine geometry, or*

(ii) *$\boldsymbol{L}$ is a lattice associated with a Steiner system $S(3, 6, 22)$, $S(4, 7, 23)$ or $S(5, 8, 24)$, or*

(iii) *$\boldsymbol{L}$ is the lattice associated with the* 65*-point design for* $\mathrm{PSU}(3, 4)$.

6.3. THEOREM. *Let $\boldsymbol{D}$ be a Steiner system $S(t, k, \nu)$ with $k > t \geqslant 3$, and let $G \leqslant \mathrm{Aut}\, \boldsymbol{D}$ be t-transitive on the points of $\boldsymbol{D}$. Then we have one of*

(i) *$\boldsymbol{D}$ consists of the points and planes of* $\mathrm{AG}(d, 2)$ *for some d, and $G \cong 2^d : \mathrm{GL}(d, 2)$ or $d = 4$ and $G \cong 2^4 : \mathrm{Alt}(7)$,*

(ii) *$\boldsymbol{D}$ is the Steiner system $S(3, q + 1, q^e + 1)$ with as blocks the images of $\{\infty\} \cup \mathbb{F}_q$ under* $\mathrm{PGL}(2, q^e)$, *$e \geqslant 2$ and $G \trianglerighteq \mathrm{PSL}(2, q^e)$, or*

(iii) *$\boldsymbol{D}$ is a Steiner system $S(4, 5, 11)$, $S(5, 6, 12)$, $S(3, 6, 22)$, $S(4, 7, 23)$ or $S(5, 8, 24)$, and $G \trianglerighteq \mathrm{M}_\nu$.*

The square designs with a 2-transitive group of automorphisms have been classified in Kantor [1985]. The result is as follows.

6.4. THEOREM. *Let D be a square design with $\nu > 2k$ such that* $\operatorname{Aut} D$ *is 2-transitive on points. Then D is one of the following*:

(i) *a projective space,*

(ii) *the unique Hadamard design with $\nu = 11$ and $k = 5$,*

(iii) *a unique design with $\nu = 176$, $k = 50$ and $\lambda = 14$, or*

(iv) *a design with $\nu = 2^{2m}$, $k = 2^{m-1}(2^m - 1)$ and $\lambda = 2^{m-1}(2^{m-1} - 1)$, of which there is exactly one for each $m \geqslant 2$.*

The classification of linear spaces (Steiner 2-designs) with a flag-transitive group of automorphisms is almost complete. As so often happens, this classification can be divided into two cases. In Case 1 the group G is *almost simple*, i.e. there is a non-Abelian simple group N such that $N \trianglelefteq G \trianglelefteq \operatorname{Aut} N$, and in Case 2 G is *affine*, i.e. the point set can be identified with a vector space V of dimension n over $\mathbb{F}_p$ for some prime p, and $G = T : G_0$ where T is the group of translations of V and $G_0 \leqslant \Gamma L(d, q)$ where $q^d = p^n$. The following result is taken from Buekenhout, Delandtsheer and Doyen [1988], Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [1990].

6.5. THEOREM. *Suppose that G is a flag-transitive group of automorphisms of the non-trivial linear space S. Then either*

(i) *G is almost simple and then we have one of*

(a) *$S =$ PG(d, q) and* $\mathrm{PSL}(d+1, q) \leqslant G \leqslant \mathrm{P\Gamma L}(d+1, q)$ *or $S =$ PG(3, 2) and $G =$ Alt(7), or*

(b) *S is a Hermitian unital on $q^3 + 1$ points and* $\mathrm{PSU}(3, q) \leqslant G \leqslant \mathrm{P\Gamma U}(3, q)$, *or*

(c) *S is a Ree unital on $q^3 + 1$ points, $q = 3^{2e+1}$, and* ${}^2G_2(q) \leqslant G \leqslant \operatorname{Aut}({}^2G_2(q))$, *or*

(d) *S is the design with as point set the exterior lines of a hyperoval in* PG$(2, q)$ *($q = 2^n$, $n \geqslant 3$), and as blocks the points off the hyperoval, and* $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{P\Gamma L}(2, q)$, *or*

(ii) *G is affine, say $G = T : G_0$, $T \cong p^n$, $G_0 \leqslant \Gamma L(n, p)$, and we have one of*

(a) *S is AG(d, q) with $d \geqslant 2$, $q^d = p^n$ (with the following possibilities for G: (α) G is 2-transitive, (β) $d = 2$, $q = 11$ or 23 and G is one of three soluble groups, (γ) $d = 2$, $q \in \{9, 11, 19, 29, 59\}$, $G_0^{(\infty)} \cong 2.\mathrm{Alt}(5)$ (where $G_0^{(\infty)}$ is the last term in the derived series of G_0), (δ) $d = 4$, $q = 3$ and $G_0^{(\infty)} \cong 2.\mathrm{Alt}(5)$. (Details are in ref.* [9] *of* Buekenhout et al. [1990]).

(b) *S is a non-Desarguesian translation plane; more precisely, either S is a Lüneburg plane of order q^2, $q = 2^{2e+1}$ ($e \geqslant 1$) and* ${}^2B_2(q) \leqslant G_0 \leqslant \operatorname{Aut}({}^2B_2(q))$, *or S is a Hering plane of order 27 and $G_0 \cong$* SL(2, 13) *(and G is 2-transitive on the points of S), or S is a nearfield plane of order 9 (with seven possibilities for G).*

(c) *S is a Hering space ($G_0 \cong$* SL(2, 13) *and G is 2-transitive).*

(d) $G \leqslant \mathrm{AGL}(1, q)$.

This last case is not completely settled. The known examples include some translation planes and the generalized Netto systems.

For Möbius planes (i.e. Steiner systems $S(3, q+1, q^2+1)$) the situation is as follows. A Möbius plane of odd order admitting a 2-transitive automorphism group is necessarily Miquelian (Hering [1967]). For even order Möbius planes Bagchi and Narasimha Sastry [1987] proved that transitivity of the automorphism group implies that the Möbius plane is either Miquelian or associated with a Tits ovoid. These same authors proved also (1990) that a Möbius plane I of even order s is Miquelian if it has an automorphism of order $s+1$, or an automorphism of order $s-1$ that fixes all circles through two fixed points of I. Notice that these results do not depend on the classification of finite simple groups.

Combinatorial characterizations

Let us next discuss some combinatorial characterizations of designs. For this we need some terminology. Let $\boldsymbol{D} = (X, \boldsymbol{B})$ be a design. For distinct points x and y, define the *line* on x and y (denoted xy) to be the intersection of all blocks containing x and y. Notice that if every pair of distinct points is in the same number of blocks, then

$$u, \nu \in xy \Rightarrow u\nu = xy \quad \text{for all } x, y, u, \nu \in X,\ x \neq y,\ u \neq \nu.$$

Similarly, if every triple of noncollinear points is contained in the same number of blocks, it makes sense to define *planes* as the intersection of all blocks containing a given triple of noncollinear points. The following theorem characterizes the square design of points and hyperplanes of a projective space.

6.6. THEOREM (Dembowski and Wagner [1960]). *A square 2-(ν, k, λ) design $\boldsymbol{D}$ is the design of points and hyperplanes of a finite projective space if any one of the following conditions is satisfied.*

(i) *Every line meets every block.*
(ii) *Every line has at least $(b-\lambda)/(r-\lambda)$ points.*
(iii) *Every plane is contained in exactly $\lambda(\lambda-1)/(k-1)$ blocks.*
(iv) *Every triple of noncollinear points is contained in equally many blocks.*

There is a similar characterization for affine spaces due to Dembowski [1964, 1967].

6.7. THEOREM. *A resolvable design $\boldsymbol{D}$ with $q = \nu/k > 2$ is the design of points and hyperplanes of an affine geometry if any one of the following conditions is satisfied.*

(i) *Every line l meets every block not parallel to a block containing l.*
(ii) *Every line consists of q points.*
(iii) *Every plane is contained in exactly $\lambda(\lambda-1)/(r-1)$ blocks.*
(iv) *$\boldsymbol{D}$ is smooth and affine (i.e. there is a constant μ such that every two blocks from different parallel classes meet in μ points).*

Kantor [1969] has similar but slightly stronger results. Lefèvre-Percsy [1980] also characterizes the design of points and t-dimensional subspaces of projective and affine spaces for arbitrary t. Cameron [1974] gives the following local characterization.

6.8. THEOREM. *If D is a 2-(ν, k, λ) design in which*

(a) *every three noncollinear points are contained in t blocks,*

(b) *every line has $s+1$ points; and*

(c) *for every point p the lines and planes through p form a square 2-design;*

then exactly one of the following occurs:

(i) *D is the design of points and hyperplanes in a projective geometry of dimension at least 3 over $\mathbb{F}_s$;*

(ii) *D is the design of points and hyperplanes in an affine geometry of dimension at least 3 over $\mathbb{F}_{s+1}$, and $s > 1$;*

(iii) *D is a Hadamard 3-design (and $s = 1$, $\nu = 4(t+1)$, $k = 2(t+1)$, $\lambda = 2t+1$);*

(iv) $\nu = (1+st)(1+2s+2s^2+(3s+2)s^2t+s^4t^2)$, $k = (1+st)(1+s+s^2t)$, $\lambda = 1+(2s+1)t+s^2t^2$, *and if* $s > 1$ *and* $t > 1$, *then* $t > s+2$ *and* $s \mid t(t-1)$;

(v) $t = 1$, $\nu = (s+1)^4(s^3+2s^2+3s+1)$, $k = (s+1)^2(s^2+s+1)$, $\lambda = s^3+3s^2+4s+3$;

(vi) $s = 1$, $t = 3$, $\nu = 496$, $k = 40$, $\lambda = 39$.

In the same paper Cameron also shows that a design which is locally an affine design is an inversive plane.

There are also group-theoretic characterizations of the symmetric designs of points and hyperplanes of projective spaces which are not based on some high degree of transitivity of the automorphism group but which rely on the existence of sufficiently many *elations* (automorphisms fixing some block B pointwise and some point $p \in B$ blockwise). As a typical example we mention one result from Kantor [1970].

6.9. THEOREM. *Let D be a symmetric design with $\lambda > 1$ admitting an automorphism group G fixing a block B and such that for every block C and every point $c \in B \cap C$ there exists a nontrivial elation in G fixing every point on C and every block on c. Then D is the design of points and hyperplanes of a projective space.*

For other results in this direction, see Piper [1983], Kelly [1984], Jackson [1991].

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CHAPTER 9

Generalized Polygons

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Contents

Introduction ... 385
1. Finite generalized polygons ... 386
   1.1. Graphs ... 386
   1.2. Graphs defined by finite incidence structures ... 386
   1.3. Finite generalized polygons ... 387
2. Restrictions on the parameters ... 388
   2.1. The theorem of Feit and Higman ... 388
   2.2. The inequalities of Higman, and Haemers and Roos ... 388
3. The classical finite generalized polygons ... 389
   3.1. The classical finite projective planes ... 389
   3.2. The classical finite generalized quadrangles ... 389
   3.3. The classical finite generalized hexagons ... 391
   3.4. The classical finite generalized octagons ... 392
4. Nonclassical finite generalized quadrangles ... 393
   4.1. The trivial nonclassical generalized quadrangles, grids and dual grids ... 393
   4.2. The nonclassical examples of Tits ... 393
   4.3. The examples of Ahrens and Szekeres, Hall, Jr., and Payne ... 394
   4.4. Generalized quadrangles as group coset geometries ... 395
   4.5. Isomorphisms ... 400
   4.6. Open problems ... 400
5. Generalized polygons with small parameters ... 400
   5.1. Generalized quadrangles with small parameters ... 401
   5.2. Generalized hexagons with small parameters ... 402
   5.3. Open problems ... 402

HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

6. Ovoids, spreads, polarities and subpolygons 402
6.1. Ovoids and spreads 402
6.2. Ovoids and spreads of generalized quadrangles 403
6.3. Ovoids and spreads of generalized hexagons 404
6.4. Polarities 404
6.5. Subpolygons 405
6.6. Open problems 407
7. Generalized quadrangles in finite projective and affine spaces 407
7.1. Generalized quadrangles in finite projective spaces 407
7.2. Generalized quadrangles in finite affine spaces 408
8. Combinatorial characterizations of the finite classical generalized quadrangles and hexagons 411
Introduction 411
8.1. Characterizations of $W(q)$ and $Q(4,q)$ 411
8.2. Characterizations of $Q^-(5,q)$ and $H(3,q^2)$ 413
8.3. Characterizations of $H(4,q^2)$ 414
8.4. Theorems simultaneously characterizing several classical generalized quadrangles 415
8.5. Characterizations of all thick classical and dual classical generalized quadrangles 417
8.6. Characterizations of nonclassical generalized quadrangles 418
8.7. Characterizations of the finite classical generalized hexagons 418
8.8. A combinatorial characterization of all finite thick classical generalized n-gons, with $n \geqslant 4$, and their duals 420
8.9. Open problems 420
9. Automorphisms of generalized polygons 420
9.1. Elation generalized quadrangles and translation generalized quadrangles 420
9.2. The sets $O(n,m,q)$ and the generalized quadrangles $T(n,m,q)$ 421
9.3. The known translation generalized quadrangles 423
9.4. Moufang conditions for finite generalized quadrangles 423
9.5. Other characterizations of finite generalized quadrangles using automorphisms 424
9.6. Moufang generalized n-gons with $n > 4$ 424
9.7. Epimorphisms 425
9.8. Open problems 426
10. Infinite generalized polygons 426
References 427

Introduction

Generalized polygons were introduced by Tits [1959] in his celebrated work 'Sur la trialité et certains groupes qui s'en déduisent'. The classical examples arise from groups with a BN-pair of rank 2, for which the Weyl group is a dihedral group. For finite thick generalized n-gons Feit and Higman [1964] show that $n = 2, 3, 4, 6$ or 8. Moreover several restrictions on the parameters of finite generalized n-gons hold. In the infinite case we do not have a Feit and Higman theorem, and here 'free' generalized n-gons may be constructed in much the same way as free projective planes (see Chapter 13). Generalized 2-gons are trivial structures, and the thick generalized 3-gons are the projective planes. Since the projective planes are considered in Chapters 2, 4 and 5, we restrict ourselves to generalized n-gons with $n \geqslant 4$. As much of the literature concerns the finite case, most of this chapter is on finite generalized quadrangles, hexagons and octagons. Moreover, since only a few classes of finite thick generalized hexagons and octagons are known, most of the papers concern finite generalized quadrangles. The main results, up to 1983, on this subject are contained in Payne and Thas [1984].

In the first two sections we introduce some basic notions on finite generalized polygons and also state important restrictions on the parameters, including the Feit–Higman theorem. Section 3 contains a description of the classical finite generalized polygons, in particular the only known finite thick generalized hexagons and octagons. All known nonclassical finite generalized quadrangles are considered in the next section; here the basic constructions are due to Kantor, Payne, Thas and Tits. Also some isomorphisms between the known generalized quadrangles are discussed. Section 5 is on the uniqueness of generalized polygons with small parameters. In Section 6 we consider ovoids, spreads, polarities and subpolygons of finite generalized polygons. These objects will appear to be useful tools in some characterizations of generalized polygons. Next all generalized quadrangles embedded in finite projective and affine spaces are classified. In the projective case the complete classification is due to Buekenhout and Lefèvre, in the affine case to Thas. In contrast with the projective case, five nontrivial 'sporadic' examples arise in the finite affine case. In Section 8, the largest one, important combinatorial characterizations of the finite classical generalized quadrangles and hexagons are stated; in the quadrangle case most of these theorems are due to Thas, in the hexagon case to Ronan. Automorphisms of finite generalized polygons are considered in Section 9. In the first part elation and translation generalized quadrangles are introduced, the second part contains characterizations of finite classical generalized polygons by automorphisms. The main theorem states that a finite thick generalized n-gon, $n \geqslant 4$, is Moufang if and only if it is classical or dual classical. It was observed by Tits that this theorem easily follows from the classification of finite split BN-pairs of rank 2 by Fong and Seitz. The section concludes with a theorem of Pasini on epimorphisms between finite thick generalized n-gons, $n > 2$. In the last section we say some words about infinite generalized polygons. In particular we state the result of Tits on Moufang polygons, and mention that all infinite generalized quadrangles embedded in a projective space were determined by Dienst.

Since most of the demonstrations are long, complicated and technical, we decided to not include proofs.

1. Finite generalized polygons

1.1. *Graphs*

A finite *graph* $G = (X, E)$ consists of a finite set X ($\neq \varnothing$) with n elements and a set E of unordered pairs of X. The elements of X are called the *vertices* of the graph G, while the elements of E are called the *edges*. A sequence of distinct vertices $x_0, x_1, \dots, x_m$ of G is a *path* of *length* m between x_0 and x_m if $\{x_{i-1}, x_i\}$ is an edge for $i = 1, 2, \dots, m$. The *distance* $d(x, y)$ between two vertices x and y is the length of the shortest path between x and y ($d(x, y) = \infty$ if G is disconnected and x and y are in distinct components of G). The *diameter* of G is the largest distance in G. A sequence of vertices $x_0, x_1, \dots, x_m$ is a *circuit* of *length* m if $x_1, x_2, \dots, x_m$ are distinct, $x_0 = x_m$, $m > 2$ and $\{x_{i-1}, x_i\}$ is an edge for $i = 1, 2, \dots, m$. The *girth* of G is the length of the shortest circuit in G. The graph $G = (X, E)$ is *bipartite* if

$$X = X_1 \cup X_2, \quad X_1 \cap X_2 = \varnothing, \quad X_1 \neq \varnothing \neq X_2,$$

and every edge has one vertex in X_1 and one vertex in X_2.

1.2. *Graphs defined by finite incidence structures*

Let $S = (P, B, I)$ be a finite incidence structure with P ($\neq \varnothing$) the set of all *points* of S, B ($\neq \varnothing$) the set of all *blocks* of S, and I a symmetric point-block *incidence relation*[1]. If any two distinct points are incident with at most one block, then the blocks usually are called *lines*. The *point graph* of S is the graph with vertex set P, two vertices being adjacent if and only if they are incident with a common block; the *block graph* of S is the graph with vertex set B, two vertices being adjacent if and only if they are incident with a common point. Now assume that $P \cap B = \varnothing$. Then the *incidence graph* of S has vertex set $P \cup B$, where two vertices x and y are adjacent if and only if either $x \in P$ and $y \in B$, or $x \in B$ and $y \in P$, and xIy. Clearly the incidence graph of S is bipartite.

Assume that any two distinct points of S are incident with at most one common element of B, that any point of S is incident with $t + 1$ ($t \geqslant 0$) lines of S, that any line of S is incident with $s + 1$ ($s \geqslant 0$) points of S. Further, let $P = \{x_1, x_2, \dots, x_v\}$, $B = \{L_1, L_2, \dots, L_b\}$, and let N be the corresponding incidence matrix of S, where rows are indexed by points and columns by lines. Then

$$NN^T - (t + 1)I$$

is the corresponding adjacency matrix of the point graph of S,

$$N^T N - (s + 1)I$$

is the corresponding adjacency matrix of the line graph of S, and

$$\begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}$$

is the corresponding adjacency matrix of the incidence graph of S.

[1] In Chapter 3, this is called a *rank* 2 *geometry*.

1.3. *Finite generalized polygons*

A *generalized n-gon of order* (s,t), $n \geqslant 2$, $s > 0$, $t > 0$, is a 1-$(v, s+1, t+1)$ design $S = (P, B, I)$ whose incidence graph has girth $2n$ and diameter n. A *generalized polygon* is a generalized n-gon for some n. If $s = t$, S is said to have *order* s. Generalized polygons were introduced by Tits [1959] in his celebrated work on triality. We emphasize that the generalized polygons form a particular class of buildings; see Chapter 11.

There is a point-line duality for generalized polygons (of order (s,t)) for which in any definition or theorem the words 'point' and 'line' are interchanged and the parameters s and t are interchanged. Normally, assume without further notice that the dual of a given theorem or definition has also been given.

Clearly in a generalized 2-gon, i.e. a generalized *digon*, every point is incident with every line, and a generalized n-gon with $s = t = 1$ is an ordinary n-gon. Generalized n-gons with $s > 1$ and $t > 1$ are called *thick*. It is not difficult to verify that a generalized 3-gon, i.e. a *generalized triangle*, of order (s,t) is either a triangle or a projective plane of order s, $s > 1$. Thus for a generalized triangle one always has $s = t$.

A generalized 4-gon, i.e. a *generalized quadrangle*, of order (s,t) is easily seen to be any incidence structure of points and lines satisfying:

(i) each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $xIMIyIL$.

Generalized 6-gons and generalized 8-gons are usually called *generalized hexagons* and *generalized octagons*, respectively.

Let $S = (P, B, I)$ be a generalized n-gon of order s. Consider the incidence structure $S' = (P \cup B, F, I')$, where F is the set of incident point-line pairs and I' is the natural incidence relation. Then S' is a generalized $2n$-gon of order $(1, s)$. Conversely, it can be proved easily that all generalized n-gons of order $(1, s)$, $s > 1$, are of this form.

Let $S = (P, B, I)$ be a generalized polygon. Given two (not necessarily distinct) points x, y of S, we write $x \sim y$ and say that x and y are *collinear*, provided that there is some line L for which $xILIy$. And $x \not\sim y$ means that x and y are not collinear. Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or nonconcurrent respectively. If $x \sim y$ (respectively, $L \sim M$) we may also say that x (respectively, L) is *orthogonal* or *perpendicular* to y (respectively, M). The line (respectively, point) which is incident with distinct collinear points x, y (respectively, distinct concurrent lines L, M) is denoted by xy (respectively, LM or $L \cap M$).

For $x \in P$ put $x^\perp = \{y \in P\colon\ y \sim x\}$, and note that $x \in x^\perp$. More generally, if $A \subset P$, A 'perp' is defined by $A^\perp = \cap\{x^\perp\colon\ x \in A\}$.

Isomorphisms (or collineations), anti-isomorphisms (or correlations), automorphisms, anti-automorphisms, involutions and polarities of generalized polygons are defined in the usual way.

2. Restrictions on the parameters

2.1. *The theorem of Feit and Higman*

THEOREM 1 (Feit and Higman [1964]). *If $S = (P, B, I)$ is a generalized n-gon of order (s, t), which is not an ordinary n-gon ($s = t = 1$), then there are only the following possibilities*:

(i) $n = 2$;

(ii) $n = 3$ *and* $s = t$;

(iii) $n = 4$;

(iv) $n = 6$ *and* $s = 1$ *or* $t = 1$ *or* st *is a square*;

(v) $n = 8$ *and* $s = 1$ *or* $t = 1$ *or* $2st$ *is a square*;

(vi) $n = 12$ *and either* $s = 1$ *or* $t = 1$.

Let $|P| = v$ and $|B| = b$. Clearly $(s + 1)b = (t + 1)v$. In case (i) $b = t + 1$ and $v = s + 1$, and in case (ii) $v = b = s^2 + s + 1$. In the other cases v and b are given by the following theorem, a proof of which can be found in, e.g., Dembowski [1968].

THEOREM 2. *If $S = (P, B, I)$ is a generalized $2n$-gon of order (s, t), with $|P| = v$ and $|B| = b$, then*

$$v = (1 + s)\big(1 + st + (st)^2 + \cdots + (st)^{n-1}\big)$$

and

$$b = (1 + t)\big(1 + st + (st)^2 + \cdots + (st)^{n-1}\big).$$

Calculating the multiplicities of the eigenvalues of the adjacency matrix of the point graph of S, necessary conditions for the existence of a generalized n-gon of order (s, t) are obtained. Two examples: for $n = 4$, $s + t$ divides $st(s + 1)(t + 1)$, see, e.g., Payne and Thas [1984], for $n = 6$, $s^2 + st + t^2$ divides $s^3(s^2t^2 + st + 1)$, see, e.g., Haemers and Roos [1981].

2.2. *The inequalities of Higman, and Haemers and Roos*

THEOREM 3 (Higman [1974]). *If S is a thick generalized quadrangle or octagon of order (s, t), then $t \leqslant s^2$, and dually $s \leqslant t^2$.*

THEOREM 4 (Haemers and Roos [1981]). *If S is a thick generalized hexagon of order (s, t), then $t \leqslant s^3$, and dually $s \leqslant t^3$.*

For generalized quadrangles attaining the bound, we have the following theorem.

THEOREM 5 (Bose and Shrikhande [1972]). *Let S be a thick generalized quadrangle of order (s,t). Then $t = s^2$ if and only if for each triple $\{x,y,z\}$ of pairwise noncollinear points there is a constant number of points collinear with x, y and z. This constant number of points is $s+1$.*

Let $S = (P,B,I)$ be a generalized hexagon of order (s,t), and let d denote the distance in the incidence graph of S. For a line L and points x and y define

$$p_{ijk}(L,x,y) = \left|\{z \in P\colon\ d(z,L) = 2i+1,\ d(z,x) = 2j,\ d(z,y) = 2k\}\right|,$$

for $i = 0,1,2$ and $j,k = 0,1,2,3$. If $d(x,y) \leqslant 4$, then the configuration *induced* by L, x and y is the substructure of S formed by the points and the lines, which are on the shortest path between L and x, L and y, x and y; if $d(x,y) = 6$, then the configuration *induced* by L, x and y is the substructure of S formed by the points and the lines, which are on the shortest path between L and x, L and y. For generalized hexagons attaining the bound in Theorem 4, we have the following theorem.

THEOREM 6 (Haemers [1979]). *If a generalized hexagon has order (s,s^3), then*

$$p_{ijk}(L,x,y) = p_{ijk}(L',x',y') \quad \textit{for all } i,j,k$$

if there is an isomorphism θ of the configuration induced by L, x and y onto the configuration induced by L', x' and y', with $L^\theta = L'$, $x^\theta = x'$ and $y^\theta = y'$.

3. The classical finite generalized polygons

3.1. *The classical finite projective planes*

A projective plane of order s is called *classical* if it is the (Desarguesian) projective plane PG$(2,s)$ over the Galois field GF(s).

3.2. *The classical finite generalized quadrangles*

We give a brief description of three families of generalized quadrangles, known as the *classical generalized quadrangles*, all of which are associated with classical groups and were first recognized as generalized quadrangles by Tits; see, e.g., Dembowski [1968].

(i) Consider a nonsingular quadric Q^+ (respectively, Q and Q^-) of Witt index 2, i.e. of projective index 1, in the projective space PG$(3,q)$ (respectively, PG$(4,q)$ and PG$(5,q)$). Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters$Q^+(3,q)Q(4,q)$

$$s = q,\ t = 1,\ v = (q+1)^2,\ b = 2(q+1), \quad \text{for } Q^+(3,q),$$

$$s = t = q,\ v = b = (q+1)(q^2+1), \quad \text{for } Q(4,q),$$

$$s = q,\ t = q^2,\ v = (q+1)(q^3+1),\ b = (q^2+1)(q^3+1),\quad \text{for } Q^-(5,q).$$

$Q^-(5,q)$. Since $Q^+(3,q)$ has $t = 1$, its structure is trivial. Further, recall that these quadrics have the following canonical equations:

$$Q^+\colon\ X_0X_1 + X_2X_3 = 0,$$

$$Q\colon\ X_0^2 + X_1X_2 + X_3X_4 = 0,$$

$$Q^-\colon\ F(X_0, X_1) + X_2X_3 + X_4X_5 = 0,$$

where F is an irreducible homogeneous polynomial in X_0, X_1 over GF(q).

(ii) Let H be a nonsingular Hermitian variety of the projective space PG(d, q^2), $d = 3$ or 4. Then the points of H, together with the lines on H, form a generalized quadrangle $H(d, q^2)$ with parameters:

$$s = q^2,\ t = q,\ v = (q^2+1)(q^3+1),\ b = (q+1)(q^3+1),\quad \text{when } d = 3,$$

$$s = q^2,\ t = q^3,\ v = (q^2+1)(q^5+1),\ b = (q^3+1)(q^5+1),\quad \text{when } d = 4.$$

Remember that H has the canonical equation

$$X_0^{q+1} + X_1^{q+1} + \cdots + X_d^{q+1} = 0.$$

(iii) The points of PG($3, q$), together with the totally isotropic lines with respect to a symplectic polarity, form a generalized quadrangle $W_3(q)$, shortly $W(q)$, with parameters

$$s = t = q,\ v = b = (q+1)(q^2+1).$$

Remember that the lines of $W_3(q)$ are the elements of a general linear complex of lines of PG($3, q$), see, e.g., Hirschfeld [1985], and that a symplectic polarity of PG($3, q$) has the following canonical bilinear form:

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2 = 0.$$

THEOREM 1 (Payne and Thas [1984]).

(a) *$Q(4,q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4,q)$ (or $W(q)$) is self-dual if and only if q is even.*

(b) *$Q^-(5,q)$ is isomorphic to the dual of $H(3,q^2)$.*

The polarities of $W(q)$ are discussed in Section 3.2 of Chapter 7.

3.3. *The classical finite generalized hexagons*

Let Q^+ be a hyperbolic quadric in PG$(7,q)$ with systems U and V of generators. A *triality* is a permutation τ of order three of $Q^+ \cup U \cup V$ such that

$$(Q^+)^\tau = U, \qquad U^\tau = V, \qquad V^\tau = Q^+$$

and which preserves *incidence*, where incidence is defined as follows:
(a) a point is incident with a solid (3-space) if it lies in the solid,
(b) two points are incident if the line joining them lies on Q^+,
(c) two solids of the same system of Q^+ are incident if they meet in a line, and
(d) two solids of distinct systems of Q^+ are incident if they meet in a plane.
See also Section 8 of Chapter 2.

For x a point of Q^+, the solid x^τ is the *trial solid* of x. If xy is a line of Q^+, then so are $x^\tau \cap y^\tau$ and $x^{\tau^2} \cap y^{\tau^2}$; these are cases of generators of the same system intersecting in a line. The point x is *self-conjugate* if it lies in its own trial solid x^τ. If x and y are self-conjugate points and if $y \in x^\tau$, then $xy = x^\tau \cap y^\tau$ and is called a *fixed line.*

Tits [1959] showed that any triality with some self-conjugate point corresponds to a collineation of PG$(2,q)$, of order dividing three, of one of the following types:

I_σ: $x_0' = x_0^\sigma,\ x_1' = x_1^\sigma,\ x_2' = x_2^\sigma$;
I_{id}: $x_0' = x_0,\ x_1' = x_1,\ x_2' = x_2$;
II: $x_0' = x_1,\ x_1' = x_2,\ x_2' = x_0$.

Here, σ is a field automorphism of order three, and so exists only when q is a cube. Since the characteristic polynomial of II is $X^3 - 1$, its properties vary considerably according to whether $q \equiv 0, 1, -1 \pmod 3$.

Let τ be a triality with some self-conjugate point. The set of self-conjugate points is denoted by P and the set of fixed lines by B.

(1) If τ is of type I_{id}, the set P is a nonsingular quadric, the section of Q^+ by a hyperplane. Further, (P, B, I), with I the natural incidence, is a generalized hexagon of order q. It is denoted by $H(q)$.

(2) If τ is of type I_σ, then q is a cube and (P, B, I), with I the natural incidence, is a generalized hexagon of order $(q, \sqrt[3]{q})$. It is denoted by $H(q, \sqrt[3]{q})$.

(3) If τ is of type II and $q \equiv 1 \pmod 3$, then (P, B, I), with I the natural incidence, is a generalized hexagon of order $(q, 1)$.

(4) If τ is of type II and $q \equiv -1 \pmod 3$, then $|P| = q^3 + 1$ and $|B| = 0$. Each solid of $U \cup V$ has exactly one point in common with P, and so P is an ovoid of Q^+ (cf. Section 9 of Chapter 7).

(5) If τ is of type II and $q \equiv 0 \pmod 3$, then B consists of all lines of a generalized hexagon $H(q)$ of type (1) which meet a fixed line of $H(q)$, and P consists of all points on these lines. Here $|B| = q^2 + q + 1$ and $|P| = q^3 + q^2 + q + 1$.

The generalized hexagons of type (1) and (2) are called the *classical generalized hexagons*. Together with their duals these are the only known generalized hexagons of order (s, t) with $s > 1$ and $t > 1$. Dickson's group $\mathsf{G}_2(q)$ and the triality group ${}^3\mathsf{D}_4(\sqrt[3]{q})$ act as automorphism groups on $H(q)$ and $H(q, \sqrt[3]{q})$, respectively (see Tits [1959]).

Let Q be the nonsingular quadric of $\mathrm{PG}(6,q)$ with equation

$$X_3^2 = X_0X_4 + X_1X_5 + X_2X_6.$$

Tits [1959] showed that $H(q)$ is isomorphic to the incidence structure formed by all the points of Q and by those lines on Q whose *Grassmann coordinates* (for the definition of Grassmann coordinates, see Hodge and Pedoe [1947]), satisfy

$$\begin{aligned} &p_{34} = p_{12}, \quad &p_{35} = p_{20}, \quad &p_{36} = p_{01}, \\ &p_{03} = p_{56}, \quad &p_{13} = p_{64}, \quad &p_{23} = p_{45}. \end{aligned} \tag{1}$$

Further, these lines are also the lines of $\mathrm{PG}(6,q)$ whose Grassmann coordinates satisfy (1) together with

$$p_{04} + p_{15} + p_{26} = 0.$$

REMARK. We refer to Cameron and Kantor [1979] for another construction of the generalized hexagon $H(q)$. See also Bader and Lunardon [1993] for a purely geometrical and very elegant construction of $H(q)$, q odd and $q \neq 3^h$.

For a proof of the following theorem we refer to Tits [1960].

THEOREM 2. *The generalized hexagon $H(q)$ is self-dual if and only if q is a power of* 3; *it admits a polarity if and only if $q = 3^{2h+1}$, $h \geqslant 0$.*

3.4. *The classical finite generalized octagons*

Consider the Ree group ${}^2\mathrm{F}_4(q)$, with $q = 2^{2h+1}$, $h \geqslant 0$. Let M be a minimal parabolic subgroup of ${}^2\mathrm{F}_4(q)$ and let P_1, P_2 be the maximal parabolic subgroups of ${}^2\mathrm{F}_4(q)$, with $|P_1| > |P_2|$, containing M (for the definition of parabolic subgroup, see, e.g., Carter [1972]). Further, let

$$P = \{xP_1\colon\ x \in {}^2\mathrm{F}_4(q)\}, \qquad B = \{yP_2\colon\ y \in {}^2\mathrm{F}_4(q)\},$$

and call xP_1 and yP_2 incident if and only if $xP_1 \cap yP_2 \neq \varnothing$. Then by Tits [1976], (P, B, I) is a generalized octagon of order (q, q^2). This generalized octagon and its dual are called the *classical generalized octagons*. These are the only known generalized octagons of order (s,t) with $s > 1$ and $t > 1$. No purely geometrical construction for these octagons is known.

Finally we notice that a similar group-theoretical construction can also be given for the classical generalized quadrangles and hexagons; the groups involved here are classical groups and the groups $\mathrm{G}_2(q)$, ${}^3\mathrm{D}_4(\sqrt[3]{q})$, respectively.

4. Nonclassical finite generalized quadrangles

From now on a generalized quadrangle will be briefly denoted by GQ.

4.1. *The trivial nonclassical generalized quadrangles, grids and dual grids*

A *grid* is an incidence structure $S = (P, B, I)$ with

$$P = \{x_{ij}\colon\ i = 0, 1, \ldots, s_1,\ j = 0, 1, \ldots, s_2\}, \quad s_1 > 0,\ s_2 > 0,$$

$$B = \{L_0, \ldots, L_{s_1}, M_0, \ldots, M_{s_2}\},$$

$x_{ij} I L_k$ if and only if $i = k$, and $x_{ij} I M_k$ if and only if $j = k$. A grid with parameters s_1, s_2 is a generalized quadrangle if and only if $s_1 = s_2$. Clearly any generalized quadrangle with $t = 1$ is a grid. A *dual grid* is an incidence structure $S = (P, B, I)$ with

$$B = \{L_{ij}\colon\ i = 0, 1, \ldots, t_1,\ j = 0, 1, \ldots, t_2\}, \quad t_1 > 0,\ t_2 > 0,$$

$$P = \{x_0, \ldots, x_{t_1}, y_0, \ldots, y_{t_2}\},$$

$L_{ij} I x_k$ if and only if $i = k$, and $L_{ij} I y_k$ if and only if $j = k$. A dual grid with parameters t_1, t_2 is a generalized quadrangle if and only if $t_1 = t_2$. Clearly any generalized quadrangle with $s = 1$ is a dual grid.

4.2. *The nonclassical examples of Tits*

The earliest known nontrivial nonclassical examples of GQ were discovered by Tits and first appeared in Dembowski [1968].

Let $d = 2$ (respectively, $d = 3$) and let O be an oval (respectively, an ovoid) of $\mathrm{PG}(d, q)$; for the definitions of oval and ovoid, see Sections 1 and 3 of Chapter 7. Further, let $\mathrm{PG}(d, q)$ be embedded as a hyperplane in $\mathrm{PG}(d + 1, q)$. Define points as

(i) the points of $\mathrm{PG}(d + 1, q) \setminus \mathrm{PG}(d, q)$,
(ii) the hyperplanes X of $\mathrm{PG}(d + 1, q)$ for which $|X \cap O| = 1$, and
(iii) one new symbol (∞).

Lines are defined as

(a) the lines of $\mathrm{PG}(d + 1, q)$ which are not contained in $\mathrm{PG}(d, q)$ and meet O (necessarily in a unique point), and
(b) the points of O.

Incidence is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(d + 1, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of O in it. The point (∞) is incident with no line of type (a) and all lines of type (b). It is an easy exercise to show that the incidence structure so defined is a GQ with parameters

$$s = t = q,\ v = b = (q + 1)(q^2 + 1), \quad \text{when } d = 2,$$

$$s = q,\ t = q^2,\ v = (q + 1)(q^3 + 1),\ b = (q^2 + 1)(q^3 + 1), \quad \text{when } d = 3.$$

If $d = 2$, the GQ is denoted by $T_2(O)$; if $d = 3$, the GQ is denoted by $T_3(O)$. If no confusion is possible, these quadrangles are also denoted by $T(O)$.

4.3. *The examples of Ahrens and Szekeres, Hall, Jr., and Payne*

For each prime power q, Ahrens and Szekeres [1969] constructed a GQ with order $(q-1, q+1)$. For q even, these examples were found independently by Hall, Jr., [1971]. Then a construction was found by Payne [1971] which included all these examples and for q even produced some additional ones (see also Payne [1972, 1985a] and Payne and Thas [1984]). These examples yield the only known cases, with $s > 1$ and $t > 1$, in which s and t are not powers of the same prime.

4.3.1. *The construction of Ahrens and Szekeres [1969] and Hall, Jr., [1971], for q even*

Let O be a hyperoval (see Section 1 of Chapter 7), i.e. a $(q+2)$-arc, of the projective plane $\mathrm{PG}(2,q)$, $q = 2^h$, and let $\mathrm{PG}(2,q)$ be embedded as a plane in $\mathrm{PG}(3,q)$. Define an incidence structure $T_2^*(O)$ by taking for points just those points of $\mathrm{PG}(3,q) \setminus \mathrm{PG}(2,q)$ and for lines just those lines of $\mathrm{PG}(3,q)$ which are not contained in $\mathrm{PG}(2,q)$ and meet O (necessarily in a unique point). The incidence is that inherited from $\mathrm{PG}(3,q)$. It is evident that the incidence structure so defined is a GQ with parameters

$$s = q-1,\ t = q+1,\ v = q^3,\ b = q^2(q+2).$$

4.3.2. *The construction of Ahrens and Szekeres [1969], for q odd*

Let the elements of P be the points of the affine 3-space $\mathrm{AG}(3,q)$ over $\mathrm{GF}(q)$, q odd. Elements of B are the following curves of $\mathrm{AG}(3,q)$:

(i) $x = \sigma,\ y = a,\ z = b,$

(ii) $x = a,\ y = \sigma,\ z = b,$

(iii) $x = c\sigma^2 - b\sigma + a,\ y = -2c\sigma + b,\ z = \sigma.$

Here the parameter σ ranges over $\mathrm{GF}(q)$ and a, b, c are arbitrary elements of $\mathrm{GF}(q)$. The incidence I is the natural one. Then (P, B, I) is a GQ of order $(q-1, q+1)$. It will be denoted by $\mathrm{AS}(q)$.

4.3.3. *The construction of Payne [1971]*

Continuing with the same notation as in 1.3, it is clear that

$$|\{x,y\}^{\perp}| = t+1 \quad \text{and} \quad |\{x,y\}^{\perp\perp}| \leqslant t+1$$

for each pair of noncollinear points x, y, and

$$|\{x,y\}^{\perp}| = |\{x,y\}^{\perp\perp}| = s+1$$

for each pair of distinct collinear points x, y. The *trace* of a pair (x, y) of distinct points is defined to be the set

$$\mathrm{tr}(x,y) = x^{\perp} \cap y^{\perp} = \{x,y\}^{\perp};$$

the *span* of the pair (x, y) is the set

$$\mathrm{sp}(x,y) = \{x,y\}^{\perp\perp} = \{u \in P\colon\ u \in z^{\perp}\ \forall z \in x^{\perp} \cap y^{\perp}\}.$$

For $x \not\sim y$, $\mathrm{sp}(x,y)$ is also called the *hyperbolic line* defined by x and y. If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x,y\}^{\perp\perp}| = t+1$, we say that the pair (x,y) is *regular*. The point x is *regular* provided (x,y) is regular for all $y \in P$, $y \neq x$.

Let x be a regular point of the GQ $S = (P, B, I)$ of order s, $s > 1$. Then P' is defined to be the set $P \setminus x^{\perp}$. The elements of B' are of two types: the elements of type (a) are the lines of B which are not incident with x; the elements of type (b) are the hyperbolic lines $\{x,y\}^{\perp\perp}$, $y \not\sim x$. Now let us define the incidence I'. If $y \in P'$ and $L \in B'$ is of type (a), then $yI'L$ if and only if yIL; if $y \in P'$ and $L \in B'$ is of type (b), then $yI'L$ if and only if $y \in L'$. The incidence structure $P(S,x) = (P', B', I')$ so defined is a GQ of order $(s-1, s+1)$.

A quick look at the examples of order s in 3.2 and 4.2 reveals that regular points and regular lines arise in the following cases (see Payne and Thas [1984]): all lines of $Q(4,q)$ are regular; the points of $Q(4,q)$ are regular if and only if q is even; all points of $W(q)$ are regular; the lines of $W(q)$ are regular if and only if q is even; the unique point (∞) of type (iii) of $T_2(O)$ is regular if and only if q is even; all lines of type (b) of $T_2(O)$ are regular.

It is easily seen that for q even the GQ $P(T_2(O), (\infty))$ is the GQ $T_2^*(\overline{O})$, with $\overline{O}$ the hyperoval containing the oval O. Further, $P(W(q), x)$, with x any point of $\mathrm{PG}(3,q)$, has the following nice description: points are the points of $\mathrm{PG}(3,q)$ not in the plane $\pi_x = x^{\zeta}$, with ζ the symplectic polarity defining $W(q)$; lines are the lines of $W(q)$ not in π_x together with the lines of $\mathrm{PG}(3,q)$ through x but not in π_x; incidence is the natural one.

4.4. *Generalized quadrangles as group coset geometries*

The following construction method for GQ was introduced by Kantor [1980]; it was motivated by the parabolic subgroup construction of 3.4.

Let G be a finite group of order s^2t, $1 < s$, $1 < t$, together with a family

$$J = \{A_i\colon\ 0 \leqslant i \leqslant t\}$$

of $1+t$ subgroups of G, each of order s. Assume furthermore that for each $A_i \in J$, there exists a subgroup A_i^* of G, of order st, containing A_i. Put $J^* = \{A_i^*\colon\ 0 \leqslant i \leqslant t\}$ and define as follows a point-line geometry $S = (P, B, I) = S(G, J)$.

Points are of three kinds: (i) the elements of G; (ii) the right cosets A_i^*g, $A_i^* \in J^*$, $g \in G$; (iii) a symbol (∞).

Lines are of two kinds: (a) the right cosets A_ig, $A_i \in J$, $g \in G$; (b) the symbols $[A_i]$, $A_i \in J$.

A point g of type (i) is incident with each line A_ig, $A_i \in J$; a point A_i^*g of type (ii) is incident with $[A_i]$ and with each line A_ih contained in A_i^*g; the point (∞) is incident with each line $[A_i]$ of type (b).

Then Kantor [1980] proved that the following holds: $S(G,J)$ is a GQ of order (s,t) provided

K1: $A_iA_j \cap A_k = \{1\}$, for i, j, k distinct, and
K2: $A_i^* \cap A_j = \{1\}$, for $i \neq j$.

If the conditions K1 and K2 are satisfied, then one easily sees that

$$A_i^* = \bigcup\{A_ig\colon\ A_ig = A_i \text{ or } A_ig \cap A_j = \varnothing \text{ for all } A_j \in J\},$$

so that A_i^* is uniquely defined by A_i.

Suppose K1 and K2 are satisfied. For any $h \in G$ let us define θ_h by

$$g^{\theta_h} = gh, \quad (A_ig)^{\theta_h} = A_igh,$$

$$(A_i^*g)^{\theta_h} = A_i^*gh, \quad [A_i]^{\theta_h} = [A_i], \quad (\infty)^{\theta_h} = (\infty),$$

with $g \in G$, $A_i \in J$, $A_i^* \in J^*$. Then θ_h is an automorphism of $S(G,J)$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h\colon\ h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (i).

If K1 and K2 are satisfied, then J is called a *4-gonal family for G*.

Let $F = \mathrm{GF}(q)$, q any prime power. Let $f\colon\ F^2 \times F^2 \to F$ be a fixed symmetric, nonsingular biadditive map. Put $G = \{(\alpha, c, \beta)\colon\ \alpha, \beta \in F^2,\ c \in F\}$. Define a binary operation on G by:

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = \big(\alpha + \alpha', c + c' + f(\beta, \alpha'), \beta + \beta'\big).$$

This makes G into a group whose centre is $C = \{(0, c, 0) \in G\colon\ c \in F\}$. Suppose that for each $u \in F$ there is an additive map $\delta_u\colon\ F^2 \to F^2$ and a map $g_u\colon\ F^2 \to F$ for which

$$g_u(\alpha + \beta) - g_u(\alpha) - g_u(\beta) = f(\alpha^{\delta_u}, \beta) = f(\beta^{\delta_u}, \alpha),$$

for all $\alpha, \beta \in F^2$, $u \in F$. With such a setup, we can define a family of subgroups of G by:

$$A(u) = \big\{\big(\alpha, g_u(\alpha), \alpha^{\delta_u}\big)\colon\ \alpha \in F^2\big\}, \quad u \in F,$$

and

$$A(\infty) = \big\{(0, 0, \beta) \in G\colon\ \beta \in F^2\big\}.$$

Then put $J = \{A(u)\colon\ u \in F \cup \{\infty\}\}$ and $J^* = \{A^*(u)\colon\ u \in F \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \big\{\big(\alpha, c, \alpha^{\delta_u}\big)\colon\ \alpha \in F^2\big\}, \quad u \in F,$$

and

$$A^*(\infty) = \big\{(0, c, \beta) \in G\colon\ \beta \in F^2\big\}.$$

Necessary and sufficient conditions were worked out in Payne [1980] (or see Section 10.4 of Payne and Thas [1984]) for J to be a 4-gonal family.

THEOREM 1. *J is a 4-gonal family for G if and only if*

(i) $\delta(u,r)$: $\alpha \mapsto \alpha^{\delta_u} - \alpha^{\delta_r}$ *is bijective for* $u \neq r$,

(ii) $g_u(\alpha) = g_r(\alpha)$, $u \neq r$, *implies* $\alpha = 0$,

(iii) *if* u, r, *and* v *are distinct, then* $\gamma = 0$ *is the only solution to*

$$g_u\big(\gamma^{\delta^{-1}(u,v)}\big) - g_v\big(\gamma^{\delta^{-1}(u,v)}\big) + g_v\big(-\gamma^{\delta^{-1}(v,r)}\big) - g_r\big(-\gamma^{\delta^{-1}(v,r)}\big) = 0.$$

Let $\mathcal{C} = \{A_u\colon\ u \in F\}$ be a set of q distinct 2×2-matrices over F. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is anisotropic whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0,0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$, and then define $g_u(\alpha) = \alpha A_u \alpha^T$ and $\alpha^{\delta_u} = \alpha K_u$ for $\alpha \in F^2$. Then necessarily $f(\alpha,\beta) = \alpha\beta^T$ for all $\alpha, \beta \in F^2$. With G, $A(u)$, $A^*(u)$, J as above, the following theorem is a combination of results of Payne [1980, 1985b] and Kantor [1986].

THEOREM 2. *The set J is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan.*

In particular, let $\mathcal{C} = \{A_u\colon\ u \in F\}$ be a set of q upper triangular 2×2-matrices over F, with

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \qquad x_u, y_u, z_u, u \in F.$$

For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{2}$$

is a nonsquare of F whenever $r, u \in F$, $r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$\mathrm{tr}\,\big((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}\big) = 1 \tag{3}$$

whenever $r, u \in F$, $r \neq u$. (Here $\mathrm{tr}(w)$ is the trace of w over the prime subfield, so that $\mathrm{tr}(w) = 1$ if and only if $X^2 + X + w = 0$ has no solution in F.)

In Thas [1987] it was shown that (2) and (3) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathrm{PG}(3,q)$ to define a flock of the quadratic cone $X_0X_1 = X_2^2$ (for the definition of flock, see Section 10 of Chapter 7).

THEOREM 3. *To any flock of the quadratic cone of* $\mathrm{PG}(3,q)$ *corresponds a* GQ *of order* (q^2,q).

Now we consider all flocks listed in Section 10.5 of Chapter 7; see also the private communication in 10.8(b) of Chapter 7.

(0) To the linear flocks correspond the classical GQ $H(3,q^2)$. For a proof that these are classical we refer to Payne and Thas [1984].

(1) To the flocks *FTW* correspond GQ first discovered by Kantor [1980]. Kantor used the classical generalized hexagons $H(q)$ to construct these GQ.

(2) The flocks $K1$ were derived by Thas [1987] from q-clans discovered by Kantor [1986]. The GQ were first mentioned by Kantor [1986].

(3) The flocks $K2$ were derived by Thas [1987] from q-clans discovered by Kantor [1986]. The GQ were first mentioned by Kantor [1986].

(4) The flocks $P1$ were derived by Thas [1987] from q-clans discovered by Payne [1985b]. The GQ were first mentioned by Payne [1985b].

(5) The flocks $K3$ and the corresponding q-clans were derived by Gevaert and Johnson [1988] from Kantor's 'likeable' planes, using the connections between flocks and translation planes discovered by Thas and Walker (see Thas [1987]) and the connection between flocks and q-clans discovered by Thas [1987]. The GQ were first mentioned by Gevaert and Johnson [1988].

(6) The flocks G and the corresponding q-clans were derived by Gevaert and Johnson [1988] from some semifield planes of Ganley [1981], using the connections between flocks and translation planes discovered by Thas and Walker (see Thas [1987]) and the connection between flocks and q-clans discovered by Thas [1987]. The GQ were first mentioned by Gevaert and Johnson [1988].

(7) The q-clans and GQ derived from the flocks Fi were first mentioned by Thas [1987].

(8) The q-clans, $q \in \{11, 16, 17, 23\}$, and GQ derived from the flocks C were not yet studied in detail.

The following important theorem on derivation of flocks (see Section 10 of Chapter 7) is due to Payne and Rogers [1990].

THEOREM 4. *The process of derivation produces new flocks and new planes, but never new* GQ.

The last examples of 4-gonal families were deduced by Payne [1988, 1989] from the semifield planes of Ganley [1981], but more than just the ideas of Thas [1987] is required. These are the only examples of GQ known to arise from 4-gonal families but not from q-clans via the procedure just described. More about these examples will be said in Section 9.

Let $q = 3^h$ and let n be a nonsquare of GF(q). With the notations of this section, put

$$g_u(\gamma) = \gamma \begin{pmatrix} u & 0 \\ 0 & -nu \end{pmatrix} \gamma^T + \left(\gamma \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \gamma^T \right)^{1/3} + \left(\gamma \begin{pmatrix} 0 & 0 \\ 0 & -n^{-1}u \end{pmatrix} \gamma^T \right)^{1/9},$$

$$\alpha^{\delta_u} = u\alpha,$$

and

$$f(\alpha,\beta) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & -n \end{pmatrix} \beta^T + \left(\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \beta^T\right)^{1/3} + \left(\alpha \begin{pmatrix} 0 & 0 \\ 0 & n^{-1} \end{pmatrix} \beta^T\right)^{1/9}.$$

Then Payne [1988, 1989] shows that the corresponding set J is a 4-gonal family, thus giving a GQ of order (q^2, q). As these results were obtained while he was visiting the University of Rome, Payne calls these GQ the *Roman generalized quadrangles.*

The construction of Knarr

Let $F = \{C_1, C_2, \cdots, C_q\}$ be a flock of the quadratic cone K with vertex x_0 of PG(3, q), with q odd. The plane of C_i is denoted by $\pi_i, i = 1, 2, \ldots, q$. Let K be embedded in the nonsingular quadric Q of PG(4, q). The polar line of π_i with respect to Q is denoted by L_i; let

$$L_i \cap Q = \{x_0, x_i\}, \quad i = 1, 2, \ldots, q.$$

Then no point of Q is collinear with all three of x_0, x_i, x_j, $1 \leqslant i < j \leqslant q$. In Bader, Lunardon and Thas [1990] it is proved that it is also true that no point of Q is collinear with all three of x_i, x_j, x_k, $0 \leqslant i < j < k \leqslant q$ (see also the description of derivation in Section 10.5 of Chapter 7). Such a set U of $q+1$ points of Q will be called a BLT-*set* in Q, following a suggestion of Kantor [1991]. Since the GQ $Q(4, q)$ arising from Q is isomorphic to the dual of the GQ $W(q)$, to a BLT-set in Q corresponds a set V of $q+1$ lines of $W(q)$ with the property that no line of $W(q)$ is concurrent with three distinct lines of V; such a set V will also be called a BLT-*set.*

To F corresponds a GQ S of order (q^2, q). Knarr [1992] proves that S is isomorphic to the following incidence structure.

Start with a symplectic polarity ζ of PG(5, q). Let $p \in$ PG(5, q) and let PG(3, q) be a 3-dimensional subspace of PG(5, q) for which $p \notin$ PG(3, q) $\subset p^\zeta$. In PG(3, q) ζ induces a symplectic polarity ζ', and hence a GQ $W(q)$. Let V be a BLT-set of the GQ $W(q)$ and construct a geometry $S = (P, B, I)$ as follows.

Points: (i) p; (ii) lines of PG(5, q) not containing p but contained in one of the planes $\pi_t = pL_t$, with L_t a line of the BLT-set V; (iii) points of PG(5, q) not in p^ζ.

Lines: (a) planes $\pi_t = pL_t$, with $L_t \in V$; (b) totally isotropic planes of ζ not contained in p^ζ and meeting some π_t in a line (not through p).

The incidence relation I is just the natural incidence inherited from PG(5, q).

Then Knarr [1992] proves that S is a GQ of order (q^2, q) isomorphic to the GQ arising from the flock F.

4.5. *Isomorphisms*

For a proof of the following theorem we refer to Payne [1989] and Payne and Thas [1984].

THEOREM 5.

(a) *The* GQ $T_2(O)$ *is isomorphic to the classical* GQ $Q(4,q)$ *if and only if* O *is an irreducible conic; it is isomorphic to* $W(q)$ *if and only if* q *is even and* O *is a conic.*

(b) *The* GQ $T_2^*(O)$ *and* AS(q) *are isomorphic to the respective* GQ $P(T_2(O'), (\infty))$, *with* $O' = O \setminus \{x\}$ *and* $x \in O$, *and* $P(W(q), y)$, *with* y *any point of* $W(q)$.

(c) *The* GQ $T_3(O)$ *is isomorphic to* $Q^-(5,q)$ *if and only if* O *is an elliptic quadric of* PG$(3,q)$.

(d) *Apart from a few small values of* q, *no* GQ *in any of the nine classes described in* 4.4 *(excluding the classical* GQ *corresponding to the linear flocks) is isomorphic to a previously known* GQ.

REMARK. If q is odd, then the oval O is a conic (see Chapter 7), implying $T_2(O) \cong Q(4,q)$. In such a case $T_2(O)$ is not self-dual. If q is even and O is a conic, then $T_2(O)$, which is isomorphic to $Q(4,q)$, is self-dual. The problem of determining all ovals for which $T_2(O)$ is self-dual has been solved (cf. Eich and Payne [1972], and Payne and Thas [1976, 1984]).

If q is odd, then the ovoid O is an elliptic quadric (see Chapter 7), implying $T_3(O) \cong Q^-(5,q)$.

4.6. *Open problems*

(a) Are $Q(4,q)$ and $W(q)$ the only GQ of order q, q odd and $q \geqslant 3$?

(b) Is AS(q) the only GQ of order $(q-1, q+1)$, q odd?

(c) Is $H(4,q^2)$ the only GQ of order (q^2,q^3), $q \geqslant 3$?

(d) Are the Roman generalized quadrangles the only GQ arising from 4-gonal families but not from q-clans?

(e) Are some of the GQ arising from the flocks C sporadic?

(f) Let F be a flock of the quadratic cone K of PG$(3,q)$, with q even. Give a geometrical construction of the GQ of order (q^2,q) arising from F.

5. Generalized polygons with small parameters

Since generalized digons are trivial incidence structures and since projective planes were considered in great detail in Chapters 4 and 5, we will consider here only generalized n-gons with $n \geqslant 4$.

5.1. *Generalized quadrangles with small parameters*

Let $S = (P, B, I)$ be a finite GQ of order (s,t), $1 < s \leqslant t$.

5.1.1. $s = 2$
By 2.1 $s + t$ divides $st(s+1)(t+1)$, and by 2.2 $t \leqslant s^2$. Hence $t \in \{2,4\}$.

THEOREM 1. *Up to isomorphism there is only one* GQ *of order* 2 *and only one* GQ *of order* $(2,4)$.

COROLLARY. *The* GQ $W(2)$ *and* $Q(4,2)$ *are self-dual and mutually isomorphic. The* GQ AS(3) *is isomorphic to the* GQ $Q^-(5,2)$.

It is easy to show that the GQ of order 2 is unique. The uniqueness of the GQ of order (2,4) was proved independently at least five times, by Dixmier and Zara [1976], Seidel [1968], Shult [1972], Thas [1974] and Freudenthal [1975].

Two interesting models of the GQ *of order* 2
(1) Let O be a hyperoval, i.e. a 6-arc of PG(2,4). Points of the GQ are the points of PG(2,4) not on O, lines of the GQ are the lines of PG(2,4) intersecting O, incidence is the natural one.

(2) The following construction was apparently first discovered by Sylvester [1904]. A *duad* is an unordered pair $ij = ji$ of distinct integers from among $1,2,\ldots,6$. A *syntheme* is a set $\{ij, kl, mn\}$ of three duads for which i, j, k, l, m, n are distinct. It is routine to verify that Sylvester's syntheme-duad geometry with duads playing the role of points, synthemes playing the role of lines, and containment as the incidence relation, is the unique GQ of order 2.

Two interesting models of the GQ *of order* (2,4)
(1) In Payne and Thas [1984] the following construction of the GQ of order (2,4) is given. In addition to the duads and synthemes given above, let $1,2,\ldots,6$ and $1',2',\ldots,6'$ denote twelve additional points, and let $\{i, ij, j'\}$, $1 \leqslant i,j \leqslant 6$, $i \neq j$, denote thirty additional lines. It is easy to verify that the 27 points and 45 lines just constructed yield a representation of the unique GQ of order (2,4).

(2) Points of the GQ are the 27 lines on a general cubic surface V in PG(3,C), lines of the GQ are the 45 tritangent planes of V, and incidence is inclusion (for the properties of the general cubic surface in PG(3,C) we refer, e.g., to Baker [1921–1934]; see also Hirschfeld [1985] where a full chapter is devoted to cubic surfaces).

5.1.2. $s = 3$
Again by 2.1 and 2.2 we have $t \in \{3,5,6,9\}$.

THEOREM 2. *Any* GQ *of order* $(3,5)$ *must be isomorphic to the* GQ $T_2^*(O)$ *arising from a hyperoval in* PG(2,4), *any* GQ *of order* $(3,9)$ *must be isomorphic to* $Q^-(5,3)$, *and a* GQ *of order* 3 *is isomorphic to either* $W(3)$ *or to its dual* $Q(4,3)$. *Finally, there is no* GQ *of order* $(3,6)$.

The uniqueness of the GQ of order $(3,5)$ was proved by Dixmier and Zara [1976], the uniqueness of the GQ of order $(3,9)$ was proved independently by Dixmier and Zara [1976] and Cameron (see Payne and Thas [1976]), the determination of all GQ of order 3 is due independently to Dixmier and Zara [1976] and Payne [1975]. Dixmier and Zara [1976] proved that there is no GQ of order $(3,6)$. All these proofs, some of them simplified or streamlined, are also contained in Payne and Thas [1984].

5.1.3. $s = 4$

Using 2.1 and 2.2 it is easy to check that $t \in \{4,6,8,11,12,16\}$. Nothing is known about $t = 11$ or $t = 12$. In the other cases unique examples are known, but the uniqueness question is settled only in the case $t = 4$.

THEOREM 3. *A* GQ *of order* 4 *must be isomorphic to* $W(4)$.

The proof of this fact that appears in Payne and Thas [1984] is that of Payne [1977], with a gap filled in by Tits.

5.2. *Generalized hexagons with small parameters*

Let $S = (P,B,I)$ be a finite generalized hexagon of order (s,t), $s \leqslant t$. By 1.3 the generalized hexagons of order $(1,t)$, $t > 1$, correspond to the projective planes of order t. Now suppose $s = 2$. By 2.1 and 2.2 we have $t \in \{2,8\}$.

THEOREM 4 (Cohen and Tits [1985]). *Any generalized hexagon of order 2 must be isomorphic to either the generalized hexagon* $H(2)$ *or to its dual. Any generalized hexagon of order* $(2,8)$ *must be isomorphic to the dual of* $H(8,2)$.

5.3. *Open problems*

(a) Is there a unique GQ of order $(4,t)$, $t \in \{6,8,16\}$?
(b) Is there a unique generalized hexagon of order 3?
(c) Does there exist a GQ of order 6?

6. Ovoids, spreads, polarities and subpolygons

Again we will consider only finite generalized n-gons with $n \geqslant 4$.

6.1. *Ovoids and spreads*

An *ovoid* of the finite GQ $S = (P,B,I)$ is a set O of points of S such that each line of S is incident with a unique point of O. Dually, a *spread* of S is a set R of lines of S such that each point of S is incident with a unique line of R. It is trivial that a GQ with $s = 1$ or $t = 1$ has ovoids and spreads. The following theorem is easy to prove.

THEOREM 1. *If O is an ovoid of the* GQ S *of order* (s,t), *then* $|O| = 1 + st$; *dually, if* R *is a spread of the* GQ S *of order* (s,t), *then* $|R| = 1 + st$.

Let $S = (P, B, I)$ be a finite generalized hexagon, briefly denoted by GH, of order s. An *ovoid* of S is a set O of points of S, any two at distance 6 in the incidence graph of S, such that each point of $P \setminus O$ is collinear with a unique point of O. A *spread* of S is defined dually. The following theorem is easy to prove.

THEOREM 2. *If O is an ovoid of the* GH S *of order* s, *then* $|O| = s^3 + 1$; *dually, if* R *is a spread of the* GH S *of order* s, *then* $|R| = s^3 + 1$.

6.2. *Ovoids and spreads of generalized quadrangles*

THEOREM 3 (Payne and Thas [1984]).

(i) *The* GQ $Q(4,q)$ *always has ovoids. It has spreads if and only if* q *is even.*
(ii) *The* GQ $Q^-(5,q)$ *has spreads but no ovoids.*
(iii) *The* GQ $H(4,q^2)$ *has no ovoid.*

EXAMPLES.
(a) Consider the GQ $Q(4,q)$ in PG$(4,q)$, and let PG$(3,q)$ be a hyperplane of PG$(4,q)$ which intersects the quadric Q in an elliptic quadric O. Then O is an ovoid of $Q(4,q)$.
(b) Since $Q(4,q)$, with q even, is self-dual, any ovoid of $Q(4,q)$ defines a spread of $Q(4,q)$.
(c) Let U be a nonsingular Hermitian curve on the Hermitian variety defining the GQ $H(3,q^2)$. Then U is an ovoid of $H(3,q^2)$. Since $H(3,q^2)$ is isomorphic to the dual of $Q^-(5,q)$, it follows that $Q^-(5,q)$ admits a spread.

Concerning spreads of $H(4,q^2)$ we have just one partial result.

THEOREM 4 (Brouwer [1981]). $H(4,4)$ *has no spread.*

Using the Klein correspondence (see Section 7 of Chapter 2), it is shown in Thas [1972a] that for q even, to each spread of $W(q)$ (and so, to each ovoid of $Q(4,q)$) corresponds an ovoid of PG$(3,q)$, and conversely. The spread is regular if and only if the ovoid is an elliptic quadric. The spreads corresponding to the Tits ovoids were first discovered by Lüneburg [1965]. In Kantor [1982], for any odd q with q not a prime, a nonregular spread of $W(q)$ is constructed; see also Thas and Payne [to appear] and Thas [to appear].

In Thas [1972a] it is shown that any ovoid of $W(q)$, q even, is an ovoid of PG$(3,q)$. Conversely, any ovoid of PG$(3,q)$, q even, is an ovoid of some $W(q)$ (cf. Section 3.2 of Chapter 7).

Several constructions of spreads of $Q^-(5,q)$ are given in Thas [1983]. It is also shown there that to each spread of $Q^-(5,q)$ corresponds a semipartial geometry (cf. Chapter 10).

Parts (i), (ii), (iii) of the following theorem are due to Payne and Thas [1984]; part (iv) to Thas and Payne [to appear].

THEOREM 5.

(i) *The* GQ $T_2(O)$ *always has an ovoid.*

(ii) *The* GQ $T_3(O)$ *has no ovoid but always has spreads.*

(iii) *The* GQ $P(S,x)$ *always has spreads. It has an ovoid if and only if S has an ovoid containing x.*

(iv) *Each* GQ *arising from a flock has an ovoid.*

REMARK. Clearly the lines of type (b) of $P(S,x)$, i.e. the pointsets $\{x,y\}^{\perp\perp} \setminus \{x\}$ of S, form a spread of $P(S,x)$. Recently Payne and Thas, cf. Payne [1990], discovered a new class of spreads of the GQ $P(T_2(O),(\infty))$ of order $(q-1,q+1)$, with q even and O a conic.

In the construction of Knarr, see Section 4.4, let π be a plane in p^ζ through p, but with $\pi \cap \mathrm{PG}(3,q)$ skew to all the lines of the BLT-set V. Further, let $\overline{\mathrm{PG}(3,q)}$ be a 3-dimensional subspace of PG(5, q), with $\pi \subset \overline{\mathrm{PG}(3,q)} \not\subset p^\zeta$. Then $(\overline{\mathrm{PG}(3,q)} \setminus \pi) \cup \{p\}$ is an ovoid of the GQ S arising from the flock F defining V.

Finally, Thas and Payne [to appear] show that any Roman GQ has spreads.

6.3. *Ovoids and spreads of generalized hexagons*

Consider the classical GH $H(q)$ of order q embedded in the nonsingular quadric Q of PG(6, q). Let PG(5, q) be a hyperplane of PG(6, q) such that PG(5, q) $\cap\, Q$ is an elliptic quadric Q^-. Then it is shown in Thas [1980a] that the lines of $H(q)$ on Q^- constitute a spread of both the GQ $Q^-(5,q)$ and the GH $H(q)$. Further, it is shown in Thas [1981a] that O is an ovoid of $H(q)$ if and only if O is an ovoid of the polar space $Q(6,q)$. So we have the following theorem (cf. Section 9 of Chapter 7 for results on ovoids of $Q(6,q)$).

THEOREM 6. *The* GH $H(q)$ *always has a spread. It has an ovoid if and only if* $Q(6,q)$ *has an ovoid. In particular $H(q)$, with q even, has no ovoid, and $H(q)$, with $q = 3^h$, has an ovoid.*

REMARK. Let x be any point of the GH $H(q)$. Then the $q+1$ lines of $H(q)$ through x all lie in a plane π_x. Now let O be an ovoid of $H(q)$. It is clear that the q^3+1 planes π_x corresponding to the points of O are mutually skew. Hence they constitute a spread of the polar space $Q(6,q)$.

6.4. *Polarities*

THEOREM 7 (Payne [1968]). *If the* GQ $S = (P,B,I)$ *of order s admits a polarity, then either $s = 1$ or $2s$ is a square. Also, the set of all absolute points of a polarity θ of S is an ovoid of S, and the set of all absolute lines of θ is a spread of S.*

Tits [1962] shows that $W(q)$ admits a polarity if and only if $q = 2^{2e+1}$. The corresponding ovoids are the Tits ovoids and the corresponding spreads are the Lüneburg spreads. For more details, we refer to Section 3.2 of Chapter 7 and to Chapter 5.

The first part of the next theorem on polarities of GH is due to Cameron, Thas and Payne [1976], the second part to Ott [1981].

THEOREM 8.

(i) *If θ is a polarity of the* GH S *of order* s, *then the set of absolute points of* θ *is an ovoid of* S, *and the set of absolute lines of* θ *is a spread of* S.

(ii) *If the* GH S *of order* s *admits a polarity, then either* $s = 1$ *or* $3s$ *is a square.*

It is easy to show that the GH of order 1 admits a polarity, and in Section 3.3 it was mentioned that $H(q)$ admits a polarity if and only if $q = 3^{2h+1}$, $h \geqslant 0$.

Finally, Thas [1980a] proves that the spread of $H(q)$ described in the first paragraph of Section 6.3 is never a spread arising from a polarity of $H(q)$.

6.5. *Subpolygons*

The generalized n-gon $S' = (P', B', I')$ is called a *sub-n-gon* of the finite generalized n-gon $S = (P, B, I)$ if and only if $P' \subseteq P$, $B' \subseteq B$ and $I' = I \cap ((P' \times B') \cup (B' \times P'))$. If $S' \neq S$, we say that S' is a *proper* sub-n-gon of S.

THEOREM 9 (Payne and Thas [1984]). *Let* $S' = (P', B', I')$ *be a proper subquadrangle of order* (s', t') *of the finite* GQ $S = (P, B, I)$ *of order* (s, t). *Then either* $s = s'$ *or* $s \geqslant s't'$. *If* $s = s'$ *and if* $x \in P \setminus P'$, *then* x *is collinear with the* $1 + st'$ *points of an ovoid of* S'; *if* $s = s't'$ *and if* $x \in P$ *is incident with no line of* B', *then* x *is collinear with exactly* $1 + s'$ *points of* S'. *The dual holds, similarly.*

The next results are easy consequences of Theorem 9, although they first appeared in Thas [1972b].

THEOREM 10. *Let* $S' = (P', B', I')$ *be a proper subquadrangle of the* GQ $S = (P, B, I)$, *with* S *having order* (s, t) *and* S' *having order* (s, t'), *i.e.* $s = s'$ *and* $t > t'$. *Then we have*:

(i) $t \geqslant s$; *if* $t = s$, *then* $t' = 1$.

(ii) *If* $s > 1$, *then* $t' \leqslant s$; *if* $t' = s \geqslant 2$, *then* $t = s^2$.

(iii) *If* $s = 1$, *then* $1 \leqslant t' < t$ *is the only restriction on* t'.

(iv) *If* $s > 1$ *and* $t' > 1$, *then* $\sqrt{s} \leqslant t' \leqslant s$, *and* $s^{3/2} \leqslant t \leqslant s^2$.

(v) *If* $t = s^{3/2} > 1$ *and* $t' > 1$, *then* $t' = \sqrt{s}$.

(vi) *Let* S' *have a proper subquadrangle* S'' *of order* (s, t''), $s > 1$. *Then* $t'' = 1$, $t' = s$, *and* $t = s^2$.

EXAMPLES. Here we shall describe some of the known subquadrangles of the finite classical GQ. Examples of subquadrangles of nonclassical GQ are contained in Payne and Thas [1984].

(a) Consider $Q^-(5, q)$ and intersect Q^- with a nontangent hyperplane PG$(4, q)$. Then the points and lines of $Q' = Q^- \cap \mathrm{PG}(4, q)$ form the GQ $Q'(4, q)$. Here $s^2 = t = q^2$,

$s = s' = t'$, so that $t = s't'$. Since all lines of $Q'(4,q)$ are regular (cf. Section 4.3.3), $Q^-(5,q)$ and $Q'(4,q)$ have subquadrangles with $t'' = 1$ and $s'' = s' = s$.

(b) Similarly, consider $H(4,q^2)$, with H a nonsingular Hermitian variety of PG$(4,q^2)$. Intersect H with a nontangent hyperplane PG$(3,q^2)$. Then the points and lines of $H' = H \cap \mathrm{PG}(3,q^2)$ form the GQ $H'(3,q^2)$. Here $t = s^{3/2} = q^3$, $s = s'$, $t' = \sqrt{s}$, and again $t = s't'$. Since all points of $H'(3,q^2)$ are regular, $H'(3,q^2)$ has subquadrangles with $t'' = t' = \sqrt{s}$ and $s'' = 1$.

(c) Now consider $Q(4,q)$ and extend GF(q) to GF(q^2). Then Q extends to $\overline{Q}$ and $Q(4,q)$ to $\overline{Q}(4,q^2)$. Here $Q(4,q)$ is a subquadrangle of $\overline{Q}(4,q^2)$, and we have $t = s = q^2$ and $t' = s' = q$. Hence $t = s = s't'$.

A theorem which appears to be very useful for several characterization theorems is the following.

THEOREM 11 (Thas [1972b]). *Let* $S' = (P', B', I')$ *be a substructure of the* GQ $S = (P, B, I)$ *of order* (s,t) *for which the following conditions are satisfied*:

(i) *if* $x, y \in P'$, $x \neq y$, *and* $xILIy$, *then* $L \in B'$;

(ii) *each element of* B' *is incident with* $1 + s$ *elements of* P'.

Then there are four possibilities:

(a) S' *is a dual grid* (*and then* $s = 1$);

(b) *the elements of* B' *are lines which are incident with a distinguished point of* P, *and* P' *consists of those points of* P *which are incident with these lines*;

(c) $B' = \varnothing$ *and* P' *is a set of pairwise noncollinear points of* P;

(d) S' *is a subquadrangle of order* (s,t').

The following theorem gives all possibilities for the substructure S_θ of the fixed elements of an automorphism θ of the GQ S.

THEOREM 12 (Payne and Thas [1984]). *The substructure* $S_\theta = (P_\theta, B_\theta, I_\theta)$ *of the fixed elements of an automorphism* θ *of the* GQ $S = (P, B, I)$ *of order* (s,t), *is given by at least one of the following*:

(i) $B_\theta = \varnothing$ *and* P_θ *is a set of pairwise noncollinear points*;

(i)′ $P_\theta = \varnothing$ *and* B_θ *is a set of pairwise nonconcurrent lines*;

(ii) P_θ *contains a point* x *such that* $x \sim y$ *for every point* $y \in P_\theta$ *and each line of* B_θ *is incident with* x;

(ii)′ B_θ *contains a line* L *such that* $L \sim M$ *for every line* $M \in B_\theta$ *and each point of* P_θ *is incident with* L;

(iii) S_θ *is a grid*;

(iii)′ S_θ *is a dual grid*;

(iv) S_θ *is a subquadrangle of order* (s',t'), $s' \geqslant 2$ *and* $t' \geqslant 2$.

The basic inequality concerning subhexagons is due to Thas.

THEOREM 13 (Thas [1976a]). *If $S' = (P', B', I')$ is a proper subhexagon of order (s', t') of the generalized hexagon $S = (P, B, I)$ of order (s, t), then $st \geqslant s'^2t'^2$. If $s > s'$ and $st = s'^2t'^2$, and if $x \in P$ is collinear with no point of P', then there are exactly $1 + t'$ lines of B' at distance 3 (in the incidence graph of S) from x; the dual holds, similarly.*

Finally, we have the following theorem of Thas on suboctagons.

THEOREM 14 (Thas [1979]). *Let $S' = (P', B', I')$ be a proper suboctagon of order (s', t') of the generalized octagon $S = (P, B, I)$ of order (s, t). Then:*

(i) *either $t = t'$, or*

$$s^3(s+1)t^2 + s\big(s^2 + s'^4t'^3 - ss'(1+t')\big(1 + s't' + s'^2t'^2\big)\big)t + (s+1)s'^4t'^3 \geqslant 0;$$

(ii) *if $s = s'$, then there are the following possibilities*

(a) *$s = 1$ and $t \geqslant t'^2$,*

(b) *$t' = 1$ and $t > s$.*

The dual holds, similarly.

REMARK. In Thas [1979] the possibility $s = s'$, $t = s^2$, $s = t'^2$ also is mentioned. But as 2st is a square, see 2.1, this case cannot occur. This was pointed out to us by Van Maldeghem.

6.6. *Open problems*

(a) Classify all ovoids of $Q(4, q)$.
(b) Classify all spreads of $Q^-(5, q)$.
(c) Does $H(4, q^2), q > 2$, have a spread?
(d) Does $H(q), q$ odd and $q \neq 3^h$, have an ovoid?
(e) Is any spread of $H(q)$, $q \neq 3^{2h+1}$, a spread of some $Q^-(5, q)$?
(f) Improve (i) of Theorem 14.

7. Generalized quadrangles in finite projective and affine spaces

7.1. *Generalized quadrangles in finite projective spaces*

A (finite) *projective* GQ $S = (P, B, I)$ is a GQ for which P is a subset of the pointset of some projective space PG(d, q), B is a set of lines of PG(d, q), P is the union of all members of B, and the incidence relation I is the one induced by that of PG(d, q). We also say that S is *embedded* in PG(d, q). If PG(d', q) is the subspace of PG(d, q) generated by all points of P, then we say that PG(d', q) is the *ambient space* of S.

The following fundamental and beautiful theorem is due to Buekenhout and Lefèvre.

THEOREM 1 (Buekenhout and Lefèvre [1974]). *A projective* GQ $S = (P, B, I)$ *with ambient space* PG(d, q) *must be obtained in one of the following ways*:

(i) *there is a nonsingular quadric* Q *of Witt index* 2 *in* PG(d, q), $d = 3, 4$ *or* 5, *such that* P *is the set of points of* Q *and* B *is the set of lines on* Q;

(ii) *there is a nonsingular Hermitian variety* H *in* PG(d, q^2), $d = 3$ *or* 4, *such that* P *is the set of points of* H *and* B *is the set of lines on* H;

(iii) $d = 3$, P *is the set of all points of* PG$(3, q)$ *and* B *is the set of all totally isotropic lines with respect to some symplectic polarity of* PG$(3, q)$.

Hence S must be one of the classical examples described in 3.2.

REMARK. *Weak projective* GQ were considered by Lefèvre-Percsy [1981, 1982]. Here the lines of the GQ are subsets of the lines of PG(d, q).

7.2. *Generalized quadrangles in finite affine spaces*

We say that the GQ $S = (P, B, I)$ is *embedded* in the finite affine space AG(d, q) if P is a subset of the pointset of AG(d, q), B is a set of lines of AG(d, q), P is the union of all members of B, and the incidence relation I is the one induced by that of AG(d, q). If AG(d', q) is the subspace of AG(d, q) generated by all points of P, then we say that AG(d', q) is the *ambient space* of S. All GQ embedded in AG(d, q) were determined by Thas [1978a]; the theorem on the embedding in AG$(3, q)$ was proved independently by Bichara [1978].

We note that in contrast with the projective case, there arise five nontrivial 'sporadic' cases in the finite affine case.

THEOREM 2. *If the* GQ S *of order* (s, t) *is embedded in* AG$(2, s + 1)$, *then the lineset of* S *is the union of two parallel classes of the plane and the pointset of* S *is the pointset of the plane.*

THEOREM 3. *Suppose that the* GQ $S = (P, B, I)$ *of order* (s, t) *is embedded in* AG$(3, s + 1)$, *and that* P *is not contained in a plane of* AG$(3, s + 1)$. *Then one of the following cases must occur*:

(i) $s = 1$, $t = 2$ (*trivial case*);

(ii) $t = 1$ *and the elements of* S *are the affine points and affine lines of a hyperbolic quadric of* PG$(3, s + 1)$, *the projective completion of* AG$(3, s + 1)$, *which is tangent to the plane at infinity of* AG$(3, s + 1)$;

(iii) P *is the pointset of* AG$(3, s + 1)$ *and* B *is the set of all lines of* AG$(3, s + 1)$ *whose points at infinity are the points of a hyperoval* O *of the plane at infinity of* AG$(3, s + 1)$, *i.e.* $S = T_2^*(O)$ (*here* $s + 1 = 2^h$ *and* $t = s + 2$);

(iv) P *is the pointset of* AG$(3, s + 1)$ *and* $B = B_1 \cup B_2$, *where* B_1 *is the set of all affine totally isotropic lines with respect to a symplectic polarity* θ *of the projective completion* PG$(3, s + 1)$ *of* AG$(3, s + 1)$ *and where* B_2 *is the class of parallel* *lines defined by the pole* x *(the image with respect to* θ*) of the plane at infinity of* AG$(3, s+1)$, *i.e.* $S = P(W(s+1), x)$ *(here* $t = s+2$*)*;

(v) $s = t = 2$, *and up to a collineation of the space* AG$(3,3)$ *there is just one embedding of a* GQ *of order* 2 *in* AG$(3,3)$.

The embedding of the GQ *of order* 2 *in* AG$(3,3)$

Let ω be a plane of AG$(3,3)$ and let $\{L_0, L_1, L_2\}$ and $\{M_x, M_y, M_z\}$ be two classes of parallel lines of ω. Suppose that $\{x_i\} = M_x \cap L_i$, $\{y_i\} = M_y \cap L_i$, and $\{z_i\} = M_z \cap L_i$, $i = 0, 1, 2$. Further, let N_x, N_y, N_z be three lines containing x_0, y_0, z_0, respectively, such that $N_x \notin \{M_x, L_0\}$, $N_y \notin \{M_y, L_0\}$, $N_z \notin \{M_z, L_0\}$, such that the planes N_xM_x, N_yM_y, N_zM_z are parallel, and such that the planes ω, L_0N_x, L_0N_y, L_0N_z are distinct. The points of N_x are x_0, x_3, x_4; the points of N_y are y_0, y_3, y_4; and the points of N_z are z_0, z_3, z_4; where notation is chosen in such a way that x_3, y_3, z_3, and also x_4, y_4, z_4, are collinear. Then the points of the GQ are

$$x_0, \ldots, x_4, y_0, \ldots, y_4, z_0, \ldots, z_4,$$

and the lines are

$$L_0, L_1, L_2, M_x, M_y, M_z, N_x, N_y, N_z, x_3y_4, x_4y_3, x_3z_4, x_4z_3, y_3z_4, y_4z_3.$$

THEOREM 4. *Suppose that the* GQ $S = (P, B, I)$ *of order* (s, t) *is embedded in* AG$(4, s+1)$ *and that* P *is not contained in an* AG$(3, s+1)$. *Then one of the following cases must occur:*

(i) $s = 1$, $t \in \{2, 3, 4, 5, 6, 7\}$ *(trivial case)*;

(ii) $s = t = 2$, *and up to a collineation of the space* AG$(4,3)$ *there is just one embedding of the* GQ *with* 15 *points and* 15 *lines in* AG$(4,3)$ *(so that the ambient space is* AG$(4,3)$*)*;

(iii) $s = t = 3$, S *is isomorphic to the* GQ $Q(4,3)$, *and up to a collineation (whose companion automorphism is the identity) of the space* AG$(4,4)$ *there is just one embedding of a* GQ *of order* 3 *in* AG$(4,4)$;

(iv) $s = 2$, $t = 4$, *and up to a collineation of the space* AG$(4,3)$ *there is just one embedding of the* GQ *with* 27 *points and* 45 *lines in* AG$(4,3)$.

The embedding of the GQ *of order* 2 *in* AG$(4,3)$ *(with ambient space* AG$(4,3)$*)*

Let PG$(3,3)$ be the hyperplane at infinity of AG$(4,3)$; let ω_∞ be a plane of PG$(3,3)$, and let l be a point of PG$(3,3) \setminus \omega_\infty$. In ω_∞ choose points m_{01}, m_{02}, m_{11}, m_{12}, m_{21}, m_{22} in such a way that m_{01}, m_{21}, m_{11} are collinear, that m_{11}, m_{02}, m_{22} are collinear, that m_{21}, m_{02}, m_{12} are collinear, and that m_{01}, m_{22}, m_{12} are collinear. Let L be an affine line containing l, and let the affine points of L be denoted by p_0, p_1, p_2. The points of the GQ are the affine points of the lines

$$p_0m_{01}, p_0m_{02}, p_1m_{11}, p_1m_{12}, p_2m_{21}, p_2m_{22}.$$

The lines of the GQ are the affine lines of the (2-dimensional) hyperbolic quadric containing p_0m_{01}, p_1m_{11}, p_2m_{21}, respectively, p_0m_{02}, p_1m_{11}, p_2m_{22}, respectively, p_0m_{02}, p_1m_{12}, p_2m_{21}, and, respectively, p_0m_{01}, p_1m_{12}, p_2m_{22}.

The embedding of a GQ *of order* 3 *in* AG(4, 4)

Let PG(3, 4) be the hyperplane at infinity of AG(4, 4), let ω_∞ be a plane of PG(3, 4), let H be a Hermitian curve of ω_∞, and let l be a point of PG(3, 4) $\setminus \omega_\infty$. In ω_∞ there are exactly four triangles $m_{i1}m_{i2}m_{i3}$, $i = 0, 1, 2, 3$, whose vertices are exterior points of H and whose sides are secants (nontangents) of H. Any line $m_{0a}m_{1b}$, with $a, b \in \{1, 2, 3\}$, contains exactly one vertex m_{2c} of $m_{21}m_{22}m_{23}$ and one vertex m_{3d} of $m_{31}m_{32}m_{33}$, and the cross-ratio $\{m_{0a}, m_{1b}; m_{2c}, m_{3d}\}$ is independent of the choice of $a, b \in \{1, 2, 3\}$. Let L be an affine line through l, and let p_0, p_1, p_2, p_3 be the affine points of L, where notation is chosen in such a way that

$$\{p_0, p_1; p_2, p_3\} = \{m_{0a}, m_{1b}; m_{2c}, m_{3d}\}.$$

The points of the GQ are the 40 affine points of the lines $p_i m_{ij}$, $i = 0, 1, 2, 3$, $j = 1, 2, 3$. The lines of the GQ are the affine lines of the (2-dimensional) hyperbolic quadric containing p_0m_{0a}, p_1m_{1b}, p_2m_{2c}, p_3m_{3d}, with $a, b = 1, 2, 3$.

The embedding of the GQ *of order* (2, 4) *in* AG(4, 3)

Let PG(3, 3) be the hyperplane at infinity of AG(4, 3), let ω_∞ be a plane of PG(3, 3), and let l be a point of PG(3, 3)$\setminus\omega_\infty$. In ω_∞ choose points m, n_x, n_y, n_z, n'_x, n'_y, n'_z, n''_x, n''_y, n''_z, in such a way that m, n_x, n_y, n_z are collinear, that m, n'_x, n'_y, n'_z are collinear, that m, n''_x, n''_y, n''_z are collinear, and that n_a, n'_b, n''_c with $\{a, b, c\} = \{x, y, z\}$ are collinear. Let L be an affine line through l, and let x, y, z be the affine points of L. The plane defined by L and m is denoted by ω. The points of the GQ are the 27 affine points of the lines am, an_a, an'_a, an''_a, with $a = x, y, z$. The 45 lines of the GQ are the affine lines of ω having as point at infinity either l or m, the affine lines of the (2-dimensional) hyperbolic quadric containing am, bn_b, cn_c, respectively, am, bn'_b, cn'_c, respectively, am, bn''_b, cn''_c, and, respectively, an_a, bn'_b, cn''_c, always with $\{a, b, c\} = \{x, y, z\}$.

THEOREM 5. *Suppose that the* GQ $S = (P, B, I)$ *of order* (s, t) *is embedded in* AG(d, $s + 1$), $d \geqslant 5$, *and that* P *is not contained in any hyperplane* AG($d - 1, s + 1$). *Then one of the following cases must occur:*

- (i) $s = 1$ *and* $t \in \{[d/2], \ldots, 2^{d-1} - 1\}$, *with* $[d/2]$ *the greatest integer less than or equal to* $d/2$ (*trivial case*);
- (ii) $d = 5$, $s = 2$, $t = 4$, *and up to a collineation of the space* AG(5, 3) *there is just one embedding of the* GQ *with* 27 *points and* 45 *lines in* AG(5, 3) (*so that the ambient space is* AG(5, 3)).

The embedding of the GQ *of order* (2, 4) *in* AG(5, 3) (*with ambient space* AG(5, 3))

Let PG(4, 3) be the hyperplane at infinity of AG(5, 3), let H_∞ be a hyperplane of PG(4, 3) and let l be a point of PG(4, 3) $\setminus H_\infty$. In H_∞ choose points m_x, m_y, m_z, n_x, n_y, n_z, n'_x, n'_y, n'_z, n''_x, n''_y, n''_z in such a way that m_x, m_y, m_z are collinear, that m_x, m_y, m_z, n_x, n_y, n_z are in a plane ω_∞, that m_x, m_y, m_z, n'_x, n'_y, n'_z are in a plane ω'_∞, that m_x, m_y, m_z, n''_x, n''_y, n''_z are in a plane ω''_∞, that m_a, n_b, n_c are collinear, that m_a, n'_b, n'_c are collinear, that m_a, n''_b, n''_c are collinear, and that n_a, n'_b, n''_c are collinear, always with $\{a, b, c\} = \{x, y, z\}$. Let L be an affine line through l, and let x, y, z be the affine points of L. The points of the GQ are the 27 affine points of the lines am_a, an_a, an'_a, an''_a, with $a = x, y, z$. The 45 lines of the GQ are the affine lines of the (2-dimensional) hyperbolic quadric containing xm_x, ym_y, zm_z, respectively, am_a, bn_b, cn_c, respectively, am_a, bn'_b, cn'_c, respectively, am_a, bn''_b, cn''_c, and, respectively, an_a, bn'_b, cn''_c, always with $\{a, b, c\} = \{x, y, z\}$.

8. Combinatorial characterizations of the finite classical generalized quadrangles and hexagons

Introduction

In this section we review the most important combinatorial characterizations of the finite classical generalized polygons. Several of these theorems appeared to be very useful and were important tools in the proofs of certain results concerning strongly regular graphs, coding theory, the classification of collineation groups in projective spaces, etc.

In the first part characterizations of the classical GQ $W(q)$ and $Q(4,q)$ are given. The second part will contain characterizations of $Q^-(5,q)$ and $H(3,q^2)$. Next characterizations of $H(4,q^2)$ are given. Then there is a section with conditions characterizing several classical GQ at the same time. Next we have two characterizations of all classical GQ and their duals, and the section on GQ ends with references on combinatorial characterizations of nonclassical GQ. Detailed proofs of Theorems 1 to 23, and of Theorem 25, of this section can be found in Payne and Thas [1984].

Finally, important combinatorial characterizations of the finite classical generalized hexagons are given.

8.1. *Characterizations of $W(q)$ and $Q(4,q)$*

Let $S = (P, B, I)$ be a finite GQ of order (s,t). If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x,y\}^{\perp\perp}| = t+1$, we say the pair (x,y) is *regular*. The point x is *regular* provided (x,y) is regular for all $y \in P \setminus \{x\}$. A point x is *coregular* provided each line incident with x is regular. The pair (x,y), $x \not\sim y$, is *antiregular* provided

$$|z \cap \{x,y\}^{\perp}| \leqslant 2 \quad \text{for all} \quad z \in P \setminus \{x,y\}.$$

A point x is *antiregular* provided (x,y) is antiregular for all $y \in P \setminus x^{\perp}$.

A *triad* (of points) is a triple of pairwise noncollinear points. Given a triad T, a *centre* of T is just a point of $T^{\perp}$.

The *closure* of the pair (x,y) is $\mathrm{cl}(x,y) = \{z \in P\colon\ z^{\perp} \cap \{x,y\}^{\perp\perp} \neq \varnothing\}$.

THEOREM 1. *Let $S = (P, B, I)$ be a* GQ *of order $s > 1$.*

(a) *For a regular point x, the incidence structure π_x with pointset $x^{\perp}$, with lineset the set of spans $\{y,z\}^{\perp\perp}$, where $y, z \in x^{\perp}$ with $y \neq z$, and with the natural incidence, is a projective plane of order s.*

(b) *For an antiregular point x and a point y in $x^{\perp}\setminus\{x\}$, the incidence structure $\pi(x,y)$ with pointset $x^{\perp}\setminus\{x,y\}^{\perp}$, with lines the sets $\{x,z\}^{\perp\perp}\setminus\{x\}$ with $x \sim z \not\sim y$ and the sets $\{x,u\}^{\perp}\setminus\{y\}$ with $y \sim u \not\sim x$, and with the natural incidence, is an affine plane of order s.*

In 4.3.3 it was observed that all points of $W(q)$ are regular. Dually, all lines of $Q(4,q)$ are regular. For q even $W(q)$ is self-dual, and so for q even all lines of $W(q)$ are regular. Dually, all points of $Q(4,q)$, q even, are regular. Further, each point of $Q(4,q)$, q odd, is antiregular, and, dually, each line of $W(q)$, q odd, is antiregular. Finally, each point of the GQ $H(3,q^2)$ is regular and, dually, all lines of $Q^-(5,q)$ are regular.

Historically, the next result is probably the oldest combinatorial characterization of a class of GQ. A proof is essentially contained in a paper by Singleton [1966] (although he erroneously thought he had proved a stronger result), but the first satisfactory treatment may have been given by Benson [1970]. No doubt it was discovered independently by several authors, e.g., Tallini [1971].

THEOREM 2. *A GQ S of order s, $s \neq 1$, is isomorphic to $W(s)$ if and only if all its points are regular.*

The next result is a slight generalization of the preceding theorem.

THEOREM 3 (Thas [1977]). *A GQ S of order (s,t), $s \neq 1$, is isomorphic to $W(s)$ if and only if each hyperbolic line has at least $s+1$ points.*

THEOREM 4 (Thas [1973]). *A GQ S of order s, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has an ovoid O, each triad of which has at least one centre.*

THEOREM 5 (Thas [1973]). *A GQ S of order s, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has an ovoid O, each point of which is regular.*

THEOREM 6 (Payne and Thas [1976]). *A GQ S of order s, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has a regular pair (L_1, L_2) of nonconcurrent lines with the property that any triad of points lying on lines of $\{L_1, L_2\}^{\perp}$ has at least one centre.*

THEOREM 7 (Mazzocca [1973], Payne and Thas [1976]). *Let S be a GQ of order s, $s \neq 1$, having an antiregular point x. Then S is isomorphic to $Q(4,s)$ if and only if there is a point y, $y \in x^{\perp}\setminus\{x\}$, for which the associated affine plane $\pi(x,y)$ is Desarguesian.*

There is an easy corollary.

COROLLARY. *Let S be a GQ of order s, $s \neq 1$, having an antiregular point x. If $s \leqslant 8$, i.e. if $s \in \{3,5,7\}$, then S is isomorphic to $Q(4,s)$.*

8.2. *Characterizations of $Q^-(5,q)$ and $H(3,q^2)$*

Let $S=(P,B,I)$ be a GQ of order (s,t), with $s^2=t>1$. By Theorem 5 of Section 2 for any triad $\{x,y,z\}$ we have $|\{x,y,z\}^\perp|=s+1$. Clearly $|\{x,y,z\}^{\perp\perp}|\leqslant s+1$. We say $\{x,y,z\}$ is *3-regular* provided $|\{x,y,z\}^{\perp\perp}|=s+1$. The point x is called *3-regular* if and only if each triad containing x is 3-regular.

Consider the classical GQ $Q^-(5,q)$ and the corresponding orthogonal polarity ζ. If $T=\{x,y,z\}$ is a triad of $Q^-(5,q)$, then $T^\perp$ is the conic $Q^-\cap\pi$, where π is the polar plane of the plane xyz, and $T^{\perp\perp}$ is the conic $Q^-\cap xyz$. So $|T^{\perp\perp}|=q+1$, and consequently each point of $Q^-(5,q)$ is 3-regular. Dually, each line of $H(3,q^2)$ is 3-regular.

The following characterization theorem is very important, not only for the theory of GQ, but also for other areas in combinatorics.

THEOREM 8 (Thas [1978b]). *Let S be a* GQ *of order (s,s^2), $s\neq 1$.*

(i) *$S\cong Q^-(5,s)$ if and only if all points of S are 3-regular.*

(ii) *When s is odd, then $S\cong Q^-(5,s)$ if and only if it has a 3-regular point.*

(iii) *When s is even, then $S\cong Q^-(5,s)$ if and only if it has at least one 3-regular point not incident with some regular line.*

REMARK. Independently Mazzocca [1974] proved (i) for s odd.

Next we consider the role of subquadrangles in characterizing $Q^-(5,q)$.

THEOREM 9 (Thas [1978b]).

(i) *A* GQ *S of order (s,t), $s>1$, is isomorphic to $Q^-(5,s)$ if and only if every triad of lines with at least one centre is contained in a proper subquadrangle of order (s,t').*

(ii) *A* GQ *S of order (s,t), $s>1$ and $t>1$, is isomorphic to $Q^-(5,s)$ if and only if for each triad $\{u,u',u''\}$ with distinct centers x, x' the five points u,u',u'',x,x' are contained in a proper subquadrangle of order (s,t').*

Let S be a GQ of order (s,t), and let $\{L_1,L_2,L_3\}$ and $\{M_1,M_2,M_3\}$ be two triads of lines for which $L_i\not\sim M_j$ if and only if $\{i,j\}=\{1,2\}$. Let x_i be the point defined by $L_iIx_iIM_i$, $i=1,2$. This configuration Γ of seven distinct points and six distinct lines is called a *broken grid* with *carriers* x_1 and x_2. We say Γ satisfies *axiom* (D) *with respect to the pair* (L_1,L_2) provided the following holds: if $L_4\in\{M_1,M_2\}^\perp$ with $L_4\not\sim L_i$, $i=1,2,3$, then $\{L_1,L_2,L_4\}$ has at least one centre. Interchanging L_i and M_i gives the definition of axiom (D) for Γ with respect to the pair (M_1,M_2). Further, Γ is said to satisfy *axiom* (D) provided it satisfies axiom (D) with respect to both pairs (L_1,L_2) and (M_1,M_2).

Let x be any point of S. Then S is said to satisfy *axiom (D)$'_x$* if the broken grid Γ satisfies axiom (D) with respect to (L_1,L_2) whenever xIL_1; it satisfies *axiom* (D)$''_x$ if Γ satisfies axiom (D) with respect to (M_1,M_2) whenever xIL_1.

THEOREM 10 (Thas [1978b]). *Let S be a* GQ *of order (s,t), with $s \neq t$, $s > 1$, $t > 1$.*

(i) *If s is odd, then $S \cong Q^-(5,s)$ if and only if S contains a coregular point x for which* $(\mathrm{D})'_x$ *or* $(\mathrm{D})''_x$ *is satisfied.*

(ii) *If s is even, then $S \cong Q^-(5,s)$ if and only if all lines of S are regular and S contains a point x for which* $(\mathrm{D})'_x$ *or* $(\mathrm{D})''_x$ *is satisfied.*

In order to conclude this section dealing with characterizations of $Q^-(5,s)$, we introduce one more basic concept. Let $S = (P,B,I)$ be a GQ of order (s,t). If $B^{\perp\perp}$ is the set of all spans $\{x,y\}^{\perp\perp}$ with $x \not\sim y$, then let $S^{\perp\perp} = (P, B^{\perp\perp}, \in)$. For $x \in P$, say that S satisfies *property* $(\mathrm{A})_x$ if for any $M = \{y,z\}^{\perp\perp} \in B^{\perp\perp}$ with $x \in \{y,z\}^{\perp}$, and any $u \in \mathrm{cl}(y,z) \cap (x^{\perp} \setminus \{x\})$ with $u \notin M$, the substructure of $S^{\perp\perp}$ generated by M and u is a dual affine plane. The GQ S is said to satisfy *property* (A) if it satisfies $(\mathrm{A})_x$ for all $x \in P$. So S satisfies (A) if for any $M = \{y,z\}^{\perp\perp} \in B^{\perp\perp}$ and any $u \in \mathrm{cl}(y,z) \setminus (\{y,z\}^{\perp} \cup \{y,z\}^{\perp\perp})$, the substructure of $S^{\perp\perp}$ generated by M and u is a dual affine plane. The duals of $(\mathrm{A})_x$ and (A) are denoted by $(\widehat{\mathrm{A}})_L$ and $(\widehat{\mathrm{A}})$, respectively.

THEOREM 11 (Thas [1981b]). *Let S be a* GQ *of order (s,t), $s \neq t$, $t > 1$.*

(i) *If $s > 1$, s odd, then S is isomorphic to $Q^-(5,s)$ if and only if $(\widehat{\mathrm{A}})_L$ is satisfied for all lines L incident with some coregular point x.*

(ii) *If s is even, then S is isomorphic to $Q^-(5,s)$ if and only if all lines of S are regular and $(\widehat{\mathrm{A}})_L$ is satisfied for all lines L incident with some point x.*

Let $S = (P,B,I)$ be a GQ of order (s,t) and let

$$B^* = \{\{x,y\}^{\perp\perp} \colon\ x,y \in P,\ x \neq y\}.$$

Then $S^* = (P, B^*, \in)$ is a linear space (cf. Chapter 6). So as to have no confusion between collinearity in S and collinearity in S^*, points $x_1, x_2, \ldots$ of P which are on a line of S^* will be called *S^*-collinear*. A *linear variety* of S^* is a subset $P' \subseteq P$ such that $x,y \in P'$, $x \neq y$, implies $\{x,y\}^{\perp\perp} \subseteq P'$. If $P \neq P'$ and $|P'| > 1$, the linear variety is *proper*; if P' is generated by three points which are not S^*-collinear, P' is said to be a *plane* of S^*.

Now we state a fundamental characterization of the GQ $H(3,s)$.

THEOREM 12 (Tallini [1971]). *Let $S = (P,B,I)$ be a* GQ *of order (s,t), with $s \neq t$, $s > 1$ and $t > 1$. Then S is isomorphic to $H(3,s)$ if and only if*

(i) *all points of S are regular, and*

(ii) *if the lines L and L' of B^* are contained in a proper linear variety of S^*, then also the lines $L^{\perp}$ and $L'^{\perp}$ of B^* are contained in a proper linear variety of S^*.*

8.3. *Characterizations of $H(4,q^2)$*

An elegant characterization of $H(4,q^2)$ is the following theorem.

THEOREM 13 (Thas [1976b]). *A* GQ S *of order* (s,t), $s^3 = t^2$ *and* $s \neq 1$, *is isomorphic to the classical* GQ $H(4,s)$ *if and only if every hyperbolic line has at least* $\sqrt{s}+1$ *points.*

Relying on Theorem 13 one obtains the following characterization.

THEOREM 14 (Payne and Thas [1976]). *Let* S *have order* (s,t) *with* $1 < s^3 \leqslant t^2$. *Then* S *is isomorphic to* $H(4,s)$ *if and only if each trace* $\{x,y\}^{\perp}$, *with* $x \sim y$, *is a plane of* S^* *which is generated by any three non-*S^**-collinear points in it.*

8.4. *Theorems simultaneously characterizing several classical generalized quadrangles*

First two new definitions are required. A point u of S is called *semiregular* provided that $z \in \mathrm{cl}(x,y)$ whenever u is the unique centre of the triad $\{x,y,z\}$. And a point u has *property* (H) provided $z \in \mathrm{cl}(x,y)$ if and only if $x \in \mathrm{cl}(y,z)$, whenever $\{x,y,z\}$ is a triad consisting of points in $u^{\perp}$. It follows easily that any semiregular point has property (H).

We give some examples.

In $W(q)$, $Q(4,q)$, $Q^-(5,q)$ and $H(3,q^2)$ all points and lines are semiregular and have property (H). In $H(4,q^2)$ all points are semiregular and have property (H); all lines have property (H). No line of $H(4,q^2)$ is semiregular. So property (H) does not imply semiregularity.

THEOREM 15 (Thas [1977]). *Let* S *have order* (s,t) *with* $s \neq 1$. *Then* $|\{x,y\}^{\perp\perp}| \geqslant s^2/t + 1$ *for all* x,y, *with* $x \not\sim y$, *if and only if one of the following occurs*:

(i) $t = s^2$;

(ii) $S \cong W(s)$;

(iii) $S \cong H(4,s)$.

THEOREM 16 (Thas [1977], Thas and Payne [1976]). *In the* GQ S *of order* (s,t) *each point has property* (*H*) *if and only if one of the following holds*:

(i) *each point is regular*;

(ii) *each hyperbolic line has exactly two points*;

(iii) $S \cong H(4,s)$.

THEOREM 17 (Thas [1977], Thas and Payne [1976]). *Let* S *be a* GQ *of order* (s,t). *Then each point is semiregular if and only if one of the following occurs*:

(i) $s > t$ *and each point is regular*;

(ii) $s = t$ *and* $S \cong W(s)$;

(iii) $s = t$ *and each point is antiregular*;

(iv) $s < t$, *each hyperbolic line has exactly two points, and no triad of points has a unique centre*;

(v) $S \cong H(4,s)$.

THEOREM 18 (Thas [1977]). *In a* GQ S *of order* (s,t) *all triads* $\{x,y,z\}$ *with* $z \notin \mathrm{cl}(x,y)$ *have a constant number of centers if and only if one of the following occurs*:

(i) *all points are regular*;

(ii) $s^2 = t$;

(iii) $S \cong \dot{H}(4,s)$.

THEOREM 19 (Thas [1977]). *The* GQ S *of order* (s,t), $s > 1$, *is isomorphic to one of* $W(s)$, $Q^-(5,s)$ *or* $H(4,s)$ *if and only if for each triad* $\{x,y,z\}$ *with* $x \notin \mathrm{cl}(y,z)$ *the set* $\{x\} \cup \{y,z\}^\perp$ *is contained in a proper subquadrangle of order* (s,t').

THEOREM 20 (Thas [1977]). *Let* S *be a* GQ *of order* (s,t) *for which not all points are regular. Then* S *is isomorphic to* $Q(4,s)$, *with* s *odd, to* $Q^-(5,s)$ *or to* $H(4,s)$ *if and only if each set* $\{x\} \cup \{y,z\}^\perp$, *where* $\{x,y,z\}$ *is a triad with at least one centre and* $x \notin \mathrm{cl}(y,z)$, *is contained in a proper subquadrangle of order* (s,t').

Next, we give a characterization in terms of matroids.

A finite *matroid* (which, in Chapter 6, is called a dimensional linear space) is a pair (P,M) where P is a finite set of elements called *points* and M is a closure operator which associates to each subset X of P a subset $\overline{X}$ (the *closure* of X) of P, such that the following conditions are satisfied:

(i) $\overline{\varnothing} = \varnothing$, and $\overline{\{x\}} = \{x\}$ for all $x \in P$;

(ii) $X \subseteq \overline{X}$ for all $X \subseteq P$;

(iii) $X \subseteq \overline{Y} \Rightarrow \overline{X} \subseteq \overline{Y}$ for all $X, Y \subseteq P$;

(iv) $y \in \overline{X \cup \{x\}}$, $y \notin \overline{X} \Rightarrow x \in \overline{X \cup \{y\}}$ for all $x, y \in P$ and $X \subseteq P$.

The sets $\overline{X}$ are called the *closed sets* of the matroid (P,M). It is easy to prove that the intersection of closed sets is always closed. A closed set C has *dimension* h if $h+1$ is the minimum number of points of any subset of C whose closure coincides with C. The closed sets of dimension one are the *lines* of the matroid.

THEOREM 21 (Mazzocca and Olanda [1979]). *Suppose that* $S = (P,B,I)$ *is a* GQ *of order* (s,t), $s > 1$ *and* $t > 1$. *Then* P *is the pointset and*

$$B^* = \big\{\{x,y\}^{\perp\perp}\colon\ x,y \in P \ \text{ and } \ x \neq y\big\}$$

is the lineset of some matroid (P,M) *having all sets* $x^\perp$, $x \in P$, *as closed sets, if and only if one of the following occurs*:

(i) $S \cong W(s)$;

(ii) $S \cong Q(4,s)$;

(iii) $S \cong H(4,s)$;

(iv) $S \cong Q^-(5,s)$;

(v) *all points of* S *are regular,* $s = t^2$, *and every three non-*S^**-collinear points are contained in a proper linear variety of the linear space* $S^* = (P,B^*,\in)$.

REMARK. The original proof of Theorem 21 is contained in Mazzocca and Olanda [1979], but a very short proof is given by Payne and Thas [1984].

Let S be a thick GQ of order (s,t). A *quadrilateral* of S is just a subquadrangle of order $(1,1)$. A quadrilateral S' is said to be *opposite a line* L if the lines of S' are not concurrent with L. If S' is opposite L, the four lines incident with the points of S' and concurrent with L are called the *lines of perspectivity of* S' *from* L. Two quadrilaterals S_1 and S_2 are in *perspective from* L if either

(a) $S_1 = S_2$ and S_1 is opposite L; or

(b) (i) $S_1 \neq S_2$, (ii) S_1 and S_2 are both opposite L, (iii) the lines of perspectivity of S_1, and of S_2, from L are the same.

THEOREM 22 (Ronan [1980a]). *The* GQ $S = (P,B,I)$ *of order* (s,t), $s > 1$ *and* $t > 2$, *is isomorphic to* $Q(4,s)$ *or* $Q^-(5,s)$ *if and only if given a quadrilateral* S_1 *opposite a line* L *and a point* x', *not incident with* L *but incident with a line of perspectivity of* S_1 *from* L, *there is a quadrilateral* S_2 *containing* x' *and in perspective with* S_1 *from* L.

REMARK. If $t = 2$ and $s > 1$, then by Section 5 $S \cong Q(4,2)$ or $S \cong H(3,4)$. One can check that in these two cases the quadrilateral condition of the preceding theorem is satisfied.

8.5. *Characterizations of all thick classical and dual classical generalized quadrangles*

The reader is reminded of properties (A) and $(\widehat{A})$ introduced in 8.2. Let $B^{\perp\perp}$ be the set of all hyperbolic lines of the GQ $S = (P,B,I)$, and let $S^{\perp\perp} = (P, B^{\perp\perp}, \in)$. We say that S satisfies *property* (A) if for any $M = \{y,z\}^{\perp\perp} \in B^{\perp\perp}$ and any $u \in \mathrm{cl}(y,z) \setminus (\{y,z\}^{\perp} \cup \{y,z\}^{\perp\perp})$ the substructure of $S^{\perp\perp}$ generated by M and u is a dual affine plane. The dual of (A) is denoted by $(\widehat{A})$.

THEOREM 23 (Thas [1981b]). *Let* $S = (P,B,I)$ *be a thick* GQ *of order* (s,t). *Then* S *is a classical or a dual classical* GQ *if and only if it satisfies either condition* (A) *or* $(\widehat{A})$.

Let $S = (P,B,I)$ be a GQ of order (s,t) and let x, y be distinct points of the line L. Further, let S_i be a quadrilateral of S with points $z_1^i, \ldots, z_4^i$, lines $M_1^i, \ldots, M_4^i$, and assume

$$z_1^i I M_1^i I z_2^i I M_2^i I z_3^i I M_3^i I z_4^i I M_4^i I z_1^i, \quad \text{with } i = 1,2.$$

Then we call S_1 and S_2 in *perspective from* $\{x, L, y\}$ if for every $i \in \{1,2,3,4\}$, z_i^1 and z_i^2 are collinear with a same point on L, and M_i^1 and M_i^2 are concurrent with a same line through x, respectively y. A quadrilateral S' is called a $\{x, L, y\}$*-quadrilateral* if at least one of its points, say z', is collinear with x or y, say x, and if neither x nor y is incident with any of the lines of S'; in such a case the line $z'x$ is called a *base-line of the pair* $(\{x, L, y\}, S')$. We call S $\{x, L, y\}$*-Desarguesian* if for every $\{x, L, y\}$-quadrilateral S_1 and every point z_1^2, with $z_1^2 \neq x, y$, on any base-line, there exists a $\{x, L, y\}$-quadrilateral S_2 containing z_1^2 which is in perspective with S_1 from $\{x, L, y\}$.

THEOREM 24 (Thas and Van Maldeghem [1990]). *A finite thick* GQ *is classical or dual classical if and only if it is* $\{x, L, y\}$*-Desarguesian for all sets* $\{x, L, y\}$ *with* x, y *distinct points on the line* L.

REMARK. Theorem 22 by Ronan is an easy corollary of Theorem 24.

Let $\{p, L\}$ be a *flag* of the GQ $S = (P, B, I)$, i.e. let pIL. The quadrilateral S' is said to be *opposite the flag* $\{p, L\}$ if the lines of S' are not concurrent with L and if the points of S' are not collinear with p. Let S_i be a quadrilateral of S with points $z_1^i, \ldots, z_4^i$, lines $M_1^i, \ldots, M_4^i$, and assume

$$z_1^i I M_1^i I z_2^i I M_2^i I z_3^i I M_3^i I z_4^i I M_4^i I z_1^i, \quad \text{with } i = 1, 2.$$

If S_1 and S_2 are opposite a flag $\{p, L\}$, then we say that S_1 and S_2 are in *perspective from* $\{p, L\}$ provided z_i^1 and z_i^2 are collinear with a same point on L, and M_i^1 and M_i^2 are concurrent with a common line through p, $i = 1, 2, 3, 4$. Note that this definition is self-dual. The GQ S, with flag $\{p, L\}$ is said to be $\{p, L\}$*-Desarguesian* provided the following condition holds: for any quadrilateral S_1 opposite $\{p, L\}$ and containing a flag $\{z_1^1, M_1^1\}$ and any flag $\{z_1^2, M_1^2\}$ satisfying

(i) z_1^2 is not incident with L and p is not incident with M_1^2,
(ii) $z_1^1 \sim r_1 \sim z_1^2$ for some r_1 incident with L,
(iii) $M_1^1 \sim R_1 \sim M_1^2$ for some R_1 incident with p,

there is a quadrilateral S_2 opposite $\{p, L\}$, containing the flag $\{z_1^2, M_1^2\}$ and in perspective with S_1 from $\{p, L\}$.

THEOREM 25 (Van Maldeghem, Payne and Thas [1994]). *A finite thick* GQ S *is classical or dual classical if and only if* S *is* $\{p, L\}$*-Desarguesian for all flags* $\{p, L\}$.

8.6. *Characterizations of nonclassical generalized quadrangles*

There are many interesting characterizations of nonclassical GQ. As an illustration we just mention one of these theorems.

THEOREM 26 (Thas [1978b]). *A* GQ *of order* (s, s^2), $s > 1$, *is isomorphic to* $T_3(O)$ *if and only if it has a* 3*-regular point.*

For literature on the subject we refer to De Finis [1985], De Soete [1987a,b], De Soete and Thas [1984, 1986a,b, 1987], Payne [1985a], and Thas [1974, 1975, 1978b, 1981b].

8.7. *Characterizations of the finite classical generalized hexagons*

Let $S = (P, B, I)$ be a (finite) generalized hexagon of order (s, t), and let $d(\cdot, \cdot)$ denote distance in the incidence graph of S. So if x and y are distinct points, then $d(x, y) = 2, 4$, or 6. For $x \in P$, let

$$x^{\perp^*} = \{y \in P \colon\ d(x, y) \leqslant 4\},$$

and for distinct points x, y let

$$\{x,y\}^{\perp^*} = x^{\perp^*} \cap y^{\perp^*} \quad \text{and} \quad \{x,y\}^{\perp^*\perp^*} = \bigcap \{z^{\perp^*}\colon\ z \in x^{\perp^*} \cap y^{\perp^*}\}.$$

For $x \in P$, let

$$x^{\tilde{\perp}} = \{L \in B\colon\ d(x,L) \leqslant 3\},$$

and for distinct points x, y let

$$\{x,y\}^{\tilde{\perp}} = x^{\tilde{\perp}} \cap y^{\tilde{\perp}} \quad \text{and} \quad \{x,y\}^{\tilde{\perp}\tilde{\perp}} = \bigcap \{M^{\tilde{\perp}}\colon\ M \in x^{\tilde{\perp}} \cap y^{\tilde{\perp}}\}.$$

THEOREM 27 (Ronan [1980b]). *A thick finite generalized hexagon $S = (P, B, I)$ of order (s,t) is isomorphic to $H(s,t)$, with $s = t^3$, or $H(s)$ if and only if $|\{x,y\}^{\perp^*\perp^*}| = 1 + t$ for all pairs (x,y), $x, y \in P$, with $d(x,y) = 4$.*

REMARK. Yanushka [1976] proved the following weaker theorem: a thick finite generalized hexagon $S = (P, B, I)$ of order s is isomorphic to $H(s)$ if and only if $|\{x,y\}^{\perp^*\perp^*}| = 1 + s$ for all pairs (x,y), $x, y \in P$, with $d(x,y) = 4$.

Yanushka's theorem motivated Kantor to give Ronan Theorem 27 as a thesis subject.
The following theorem is an interesting corollary of the theorem of Yanushka and Ronan.

THEOREM 28 (Thas [1980b]). *Let $S = (P, B, I)$ be a generalized hexagon of order (s,t), with $2 \leqslant t \leqslant s$. Then S is isomorphic to the classical generalized hexagon $H(s)$ if and only if for any three points x, y, z we have*

$$\{u \in P\colon\ d(u,x) \leqslant 2,\ d(u,y) \leqslant 4 \ \text{ and } \ d(u,z) \leqslant 4\} \neq \varnothing.$$

In order to conclude this section dealing with characterizations of the known generalized hexagons, we introduce two more basic concepts. Let $S = (P, B, I)$ be a generalized hexagon of order (s,t). If for any two points x, y with $d(x,y) = 6$ we have $|\{x,y\}^{\tilde{\perp}\tilde{\perp}}| = 1 + s$, then S is said to satisfy the *regulus condition*. For $z, u \in P$ with $d(z,u) = 6$,

$$z^u = \{v \in P\colon\ d(v,z) = 2\} \cap \{w \in P\colon\ d(w,u) = 4\}.$$

Now let x, y, z be three points such that $d(x,y) = 4$, $d(x,z) = d(y,z) = 6$ and $d(z, x * y) = 4$ with $x * y$ the unique point for which $d(x, x * y) = d(y, x * y) = 2$. Then we call $z^x \cap z^y$ an *intersection set* if $z^x \neq z^y$.

THEOREM 29 (Ronan [1981]). *If S is a finite thick generalized hexagon of order (s,t) with the regulus condition, then all intersection sets have one point if and only if $S \cong H(s)$ or $S \cong H(s, \sqrt[3]{s})$, two points if and only if S is isomorphic to the dual of $H(s)$, and $s^2 + 1$ points if and only if S is isomorphic to the dual of $H(s^3, s)$.*

THEOREM 30 (Ronan [1980c]).

(a) *A finite thick generalized hexagon S of order (s,t), with $s=t^3$, satisfies the regulus condition if and only if it is isomorphic to the classical generalized hexagon $H(s,t)$.*

(b) *A finite thick generalized hexagon S of order (s,t), with $t=s^3$, satisfies the regulus condition if and only if it is isomorphic to the dual of the classical generalized hexagon $H(t,s)$.*

REMARK. In Cameron and Kantor [1979] the classical generalized hexagon $H(s)$ is characterized in terms of metrically regular graphs embedded in $\mathrm{PG}(n,s)$.

8.8. *A combinatorial characterization of all finite thick classical generalized n-gons, with $n \geqslant 4$, and their duals*

In Van Maldeghem [1990] a common combinatorial characterization of all finite thick classical generalized n-gons, $n \geqslant 4$, and their duals is given; it is a generalization of Theorem 24.

8.9. *Open problems*

(a) Is every GQ of order s, s odd and $s>1$, for which each point is antiregular, isomorphic to $Q(4,s)$?

(b) Are the planes in Theorem 1 always Desarguesian?

(c) Is every GQ of order (s,t), $1<s<t$, with all lines regular, isomorphic to $Q^-(5,s)$?

(d) Does there exist a GQ of order (s,t), with $s<t<s^2$, for which all lines are regular?

9. Automorphisms of generalized polygons

9.1. *Elation generalized quadrangles and translation generalized quadrangles*

Let $S=(P,B,I)$ be a GQ of order (s,t), $s \neq 1$, $t \neq 1$. A collineation θ of S is a *whorl* about the point p provided θ fixes each line incident with p. Let θ be a whorl about p. If $\theta = \mathrm{id}$ or if θ fixes no point of $P \setminus p^\perp$, then θ is an *elation* about p. If θ fixes each point of $p^\perp$, then θ is a *symmetry* about p. Any symmetry about p is automatically an elation about p (cf. Payne and Thas [1984]). Let $p,p' \in P$, $p \not\sim p'$. A *generalized homology* with *centers* p,p' is an automorphism θ of S which is a whorl about both p and p'. The group of all generalized homologies with centers p,p' is denoted $H(p,p')$.

If there is a group G of elations about p acting regularly on $P \setminus p^\perp$, we say that S is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* p. Briefly, we say that $(S^{(p)},G)$ or $S^{(p)}$ is an EGQ. Most known examples of GQ or their duals are EGQ, the notable exceptions being those of order $(s-1,s+1)$ and their duals. If the group G is Abelian, then we say that the EGQ $(S^{(p)},G)$ is a *translation* *generalized quadrangle* (TGQ) with *translation group* G and *base point* p. For any TGQ the parameters s and t satisfy $s \leqslant t$ (cf. Payne and Thas [1984]).

Let $(S^{(p)}, G)$ be an EGQ of order (s,t), and let y be a given point of $P \setminus p^{\perp}$. Let $L_0, L_1, \dots, L_t$ be the lines incident with p, and define z_i and M_i by $L_i I z_i I M_i I y$, $0 \leqslant i \leqslant t$. Put

$$A_i = \{\theta \in G\colon\ M_i^{\theta} = M_i\}, \quad A_i^* = \{\theta \in G\colon\ z_i^{\theta} = z_i\}, \quad 0 \leqslant i \leqslant t.$$

Then conditions K1 and K2 of 4.4 are satisfied. Conversely, if conditions K1 and K2 are satisfied then the corresponding GQ $S(G,J)$ is an EGQ with base point (∞). Moreover, it follows rather easily that G acts by right multiplication as a (maximal) group of elations about (∞).

Let $(S^{(p)}, G)$ be a TGQ with A_i and A_i^* as above. The *kernel* K of $S^{(p)}$ or $(S^{(p)}, G)$ or of the 4-gonal family $J = \{A_i\colon\ 0 \leqslant i \leqslant t\}$, is the set of all endomorphisms α of G for which $A_i^{\alpha} \subset A_i$, $0 \leqslant i \leqslant t$. The following basic results are due to Payne and Thas [1984].

THEOREM 1. *With the usual addition and multiplication of endomorphisms the kernel* K *is a field, so that* $A_i^{\alpha} = A_i$, $(A_i^*)^{\alpha} = A_i^*$ *for all* $i = 0, 1, \dots, t$ *and all* $\alpha \in K \setminus \{0\}$.

THEOREM 2. *The group* G *is elementary Abelian, and* s *and* t *must be powers of the same prime. If* $s < t$, *then there is a prime power* q *and an odd integer* a *for which* $s = q^a$ *and* $t = q^{a+1}$. *If* s *(or* t*) is even then either* $s = t$ *or* $s^2 = t$.

THEOREM 3. *If* $(S^{(p)}, G)$ *is a* TGQ *of order* (s,t), *then* G *is the complete set of all elations about* p.

THEOREM 4. *The multiplicative group of the kernel of the* TGQ $S^{(p)}$ *is isomorphic to the group* $H(p,p')$ *of generalized homologies about* p *and* p', *with* $p \not\sim p'$.

9.2. *The sets* $O(n,m,q)$ *and the generalized quadrangles* $T(n,m,q)$

In PG$(2n+m-1,q)$ consider a set $O(n,m,q)$ of $q^m + 1$ $(n-1)$-dimensional subspaces

$$\mathrm{PG}^{(0)}(n-1,q), \ \dots, \ \mathrm{PG}^{(q^m)}(n-1,q),$$

every three of which generate a PG$(3n-1,q)$, and such that each element $\mathrm{PG}^{(i)}(n-1,q)$ of $O(n,m,q)$ is contained in a $\mathrm{PG}^{(i)}(n+m-1,q)$ having no point in common with any $\mathrm{PG}^{(j)}(n-1,q)$ for $j \neq i$. It is easy to check that $\mathrm{PG}^{(i)}(n+m-1,q)$ is uniquely determined, $i = 0, \dots, q^m$. The space $\mathrm{PG}^{(i)}(n+m-1,q)$ is called the *tangent space* of $O(n,m,q)$ at $\mathrm{PG}^{(i)}(n-1,q)$. Embed PG$(2n+m-1,q)$ in a PG$(2n+m,q)$, and construct a point-line geometry $T(n,m,q)$ as follows.

Points are of three types:

(i) the points of PG$(2n+m,q) \setminus$ PG$(2n+m-1,q)$;

(ii) the $(n+m)$-dimensional subspaces of PG$(2n+m,q)$ which intersect PG$(2n+m-1,q)$ in one of the $\mathrm{PG}^{(i)}(n+m-1,q)$;

(iii) the symbol (∞).

Lines are of two types:

(a) the n-dimensional subspaces of $PG(2n+m,q)$ which intersect $PG(2n+m-1,q)$ in a $PG^{(i)}(n-1,q)$;

(b) the elements of $O(n,m,q)$.

Incidence in $T(n,m,q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $PG(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $O(n,m,q)$ contained in it. The point (∞) is incident with no line of type (a) and with all lines of type (b).

The following theorems are due to Payne and Thas [1984].

THEOREM 5. *$T(n,m,q)$ is a TGQ of order (q^n,q^m) with base point (∞) for which $GF(q)$ is a subfield of the kernel. Moreover, the translations of $T(n,m,q)$ induce the translations of the affine space $AG(2n+m,q) = PG(2n+m,q) \setminus PG(2n+m-1,q)$. Conversely, every TGQ for which $GF(q)$ is a subfield of the kernel is isomorphic to a $T(n,m,q)$. It follows that the theory of TGQ is equivalent to the theory of the sets $O(n,m,q)$.*

REMARK. For $n = m = 1$, $O(n,m,q) = O(1,1,q) = O$ is an oval of $PG(2,q)$ and $T(1,1,q)$ is the GQ $T_2(O)$ of Tits associated with O; for $n = 1$ and $m = 2$ $O(n,m,q) = O(1,2,q) = O$ is an ovoid of $PG(3,q)$ and $T(1,2,q)$ is the GQ $T_3(O)$ of Tits associated with O.

THEOREM 6. *The following hold for any $O(n,m,q)$:*

(i) *$n = m$ or $n(a+1) = ma$ with a odd;*

(ii) *if q is even, then $n = m$ or $m = 2n$;*

(iii) *if $n \neq m$, then each point of $PG(2n+m-1,q)$ which is not contained in an element of $O(n,m,q)$ belongs to 0 or $1+q^{m-n}$ tangent spaces of $O(n,m,q)$;*

(iv) *if $m = 2n$, then each point of $PG(4n-1,q)$ which is not contained in an element of $O(n,2n,q)$ belongs to exactly $1+q^n$ tangent spaces of $O(n,2n,q)$;*

(v) *if $m = n$ and q is odd, then each point of $PG(3n-1,q)$ which is not contained in an element of $O(n,n,q)$ belongs to 0 or 2 tangent spaces of $O(n,n,q)$; if $m = n$ and q is even, then each point of $PG(3n-1,q)$ which is not contained in an element of $O(n,n,q)$ belongs to 1 or q^n+1 tangent spaces of $O(n,n,q)$;*

(vi) *if $n \neq m$, then each hyperplane of $PG(2n+m-1,q)$ which does not contain a tangent space of $O(n,m,q)$ contains 0 or $q^{m-n}+1$ elements of $O(n,m,q)$;*

(vii) *if $m = 2n$, then each hyperplane of $PG(4n-1,q)$ which does not contain a tangent space of $O(n,2n,q)$ contains exactly $1+q^n$ elements of $O(n,2n,q)$;*

(viii) *if $n \neq m$, then the q^m+1 tangent spaces of $O(n,m,q)$ form an $O^*(n,m,q)$ in the dual space of $PG(2n+m-1,q)$ (the tangent spaces of $O^*(n,m,q)$ are the elements of $O(n,m,q)$); so in addition to $T(n,m,q)$ there arises a TGQ $T^*(n,m,q)$, called the translation dual of $T(n,m,q)$.*

THEOREM 7. *Let* $(S^{(p)}, G)$ *be a* TGQ *arising from the set* $O(n, 2n, q)$. *Then* $S \cong T_3(O')$ *for some ovoid* O' *if and only if one of the following holds*:

(i) $|H(p, p')| = s - 1 = q^n - 1$ *for any point* p', *with* $p' \not\sim p$;

(ii) *for each point* z *not contained in an element of* $O(n, 2n, q)$, *the* $q^n + 1$ *tangent spaces containing* z *have exactly* $(q^n - 1)/(q - 1)$ *points in common*;

(iii) *each* $\mathrm{PG}(3n - 1, q)$ *containing at least three elements of* $O(n, 2n, q)$ *contains exactly* $q^n + 1$ *elements of* $O(n, 2n, q)$.

9.3. *The known translation generalized quadrangles*

As already mentioned, the GQ $T_2(O)$ and $T_3(O)$ of Tits (of respective orders (q, q) and (q, q^2)) are TGQ. Here the kernel is isomorphic to GF(q), hence has maximal size.

The GQ arising from the flocks K1 (see 4.4 and Chapter 7, 10.5) are TGQ. The kernel is the subfield of GF(q) consisting of the elements fixed by the automorphism σ of GF(q). Any such TGQ is isomorphic to its translation dual.

The duals of the GQ arising from the flocks G (see 4.4 and Chapter 7, 10.5) are TGQ. The kernel of such a TGQ is always GF(3). The translation duals of these TGQ are the Roman GQ of Payne (see 4.4). Finally, it can be shown that for $q > 9$ the original TGQ is not isomorphic to its translation dual.

REMARK. For q even no GQ arising from a nonlinear flock is a TGQ.

For proofs and many other results on the subject we refer to Johnson [1987], Payne [1985c, 1988, 1989] and Rogers [1990].

9.4. *Moufang conditions for finite generalized quadrangles*

In this section we always assume that $S = (P, B, I)$ is a finite thick GQ of order (s, t).

For any chosen point p, let us define the following condition:

$(\mathrm{M})_p$: for any two lines A and A' of S incident with p, the group of collineations of S fixing A and A' pointwise and p linewise is transitive on the lines ($\neq A$) incident with a given point x on A ($x \neq p$).

The GQ S is said to satisfy *condition* (M) provided it satisfies $(\mathrm{M})_p$ for all points $p \in P$. For a fixed line $L \in B$ let $(\widehat{\mathrm{M}})_L$ be the condition that is the dual of $(\mathrm{M})_p$ and let *condition* $(\widehat{\mathrm{M}})$ be the dual of (M). If S satisfies both (M) and $(\widehat{\mathrm{M}})$, then it is said to be a *Moufang* GQ. It is, e.g., easy to show that any TGQ $S^{(p)}$ satisfies $(\mathrm{M})_p$.

Tits [1976a] shows that from a celebrated theorem of Fong and Seitz [1973, 1974] we have

THEOREM 8. *The* GQ S *is Moufang if and only if it is classical or dual classical.*

In a paper by Thas, Payne and Van Maldeghem [1991] the following result is proved.

THEOREM 9. *The* GQ S *satisfies* (M) *if and only if it satisfies* $(\widehat{\mathrm{M}})$.

As a corollary we have a considerable improvement of Theorem 8.

THEOREM 10. *The* GQ S *satisfies* (M) (*respectively*, ($\widehat{\mathrm{M}}$)) *if and only if it is classical or dual classical.*

9.5. *Other characterizations of finite generalized quadrangles using automorphisms*

In this section we state four theorems characterizing finite GQ by automorphisms.

Assume that $S = (P, B, I)$ is a finite thick GQ of order (s, t).

THEOREM 11 (Ealy, Jr., [1977]). *Let the group of symmetries about each point of* S *have even order. Then* s *is a power of* 2 *and one of the following must hold*: (i) $S \cong W(s)$, (ii) $S \cong H(3, s)$, (iii) $S \cong H(4, s)$.

THEOREM 12 (Walker [1977]). *Let* G *be a group of automorphisms of* S *leaving no point or line of* S *fixed. Suppose that* S *has a point* p *and a line* L *for which the group of symmetries about* p, *and the group of symmetries about* L, *has order at least* 3 *and is a subgroup of* G. *Then* S *contains a* G*-invariant subquadrangle* $S' \cong W(2^n)$ (*for some integer* $n \geqslant 2$) *such that the restriction of* G *to this subquadrangle contains* $\mathrm{PSp}(4, 2^n)$.

THEOREM 13 (Thas [1985, 1986]). *The* GQ S *is classical if and only if* $|H(p, p')| = s - 1$ *for all* $p, p' \in P$ *with* $p \not\sim p'$.

Given a flag $\{p, L\}$ of S, a $\{p, L\}$-collineation is a collineation of S which fixes each point on L and each line through p. For any line N incident with p, $N \neq L$, and any point u incident with L, $u \neq p$, the group $G(p, L)$ of all $\{p, L\}$-collineations acts semiregularly on the lines M concurrent with N, p not incident with M, and on the points w collinear with u, w not incident with L. If the group $G(p, L)$ is transitive on the lines M, or equivalently, on the points w, then we say that S is $\{p, L\}$*-transitive*.

THEOREM 14 (Van Maldeghem, Payne and Thas [1992]). *The* GQ S *is classical or dual classical if and only if* S *is* $\{p, L\}$*-transitive for all flags* $\{p, L\}$.

REMARK. In Van Maldeghem, Payne and Thas [1994] it is also shown that for a given flag $\{p, L\}$ the GQ S is $\{p, L\}$-transitive if and only if it is $\{p, L\}$-Desarguesian (cf. Theorem 25 of Section 8).

9.6. *Moufang generalized* n*-gons with* $n > 4$

Let $S = (P, B, I)$ be a finite thick generalized hexagon of order (s, t). The generalized hexagon is said to satisfy *condition* (M) if for all distinct $A, A', A'' \in B$ and all distinct $p, p' \in P$, with $AIpIA'Ip'IA''$, the group of collineations of S fixing A, A', A'' pointwise and p, p' linewise is transitive on the lines ($\neq A$) incident with a given point x on A ($x \neq p$). Further, let *condition* ($\widehat{\mathrm{M}}$) be the dual of (M). If S satisfies both (M) and ($\widehat{\mathrm{M}}$), then it is said to be a *Moufang generalized hexagon*.

Let $S = (P, B, I)$ be a finite thick generalized octagon of order (s, t). The generalized octagon is said to satisfy *condition* (M) if for all distinct $A, A', A'', A''' \in B$ and all distinct $p, p', p'' \in P$, with $AIpIA'Ip'IA''Ip''IA'''$, the group of collineations of S fixing A, A', A'', A''' pointwise and p, p', p'' linewise is transitive on the lines ($\neq A$) incident with a given point x on A ($x \neq p$). Further, let *condition* ($\widehat{\text{M}}$) be the dual of (M). If S satisfies both (M) and ($\widehat{\text{M}}$), then it is said to be a *Moufang generalized octagon*.

Tits [1976a] shows that from Fong and Seitz [1973, 1974] we get:

THEOREM 15. *The finite thick generalized n-gon S, $n \in \{6, 8\}$, is Moufang if and only if it is classical or dual classical.*

Let $S = (P, B, I)$ be a finite thick generalized n-gon, $n \in \{6, 8\}$, of order (s, t), and let $d(\cdot, \cdot)$ denote distance in the incidence graph of S. If p and p' are at distance n, then a collineation θ of S fixing all lines incident with p and p' is called a *generalized homology* with *centers* p and p'. The group of all generalized homologies with centers p and p' is denoted by $H(p, p')$.

Let $d(p, p') = n$ and let u be a point for which $d(u, p) = 2$ and $d(u, p') = n - 2$. Then S is called $\{p, p'\}$*-transitive* if $H(p, p')$ is transitive on the set of points, distinct from u and p, which are incident with the line pu. The generalized n-gon S is called $\{p, p'\}$*-quasi-transitive* if the group $H(p, p')$ is transitive on the set of lines, distinct from up and from the line incident with u and at distance $n - 3$ from p', which are incident with the point u.

THEOREM 16 (Van Maldeghem [1991a,b]). *Let S be a finite thick generalized n-gon, $n \in \{6, 8\}$, of order (s, t).*

(i) *If $n = 6$, then S is classical or dual classical if and only if S is $\{p, p'\}$-transitive for all points p, p' with $d(p, p') = 6$ and $\{L, L'\}$-transitive for all lines L, L' with $d(L, L') = 6$.*

(ii) *If $n = 8$ and $s \neq 2 \neq t$, then S is classical if and only if either S or its dual is $\{p, p'\}$-transitive for all points p, p' with $d(p, p') = 8$ and $\{L, L'\}$-quasi-transitive for all lines L, L' with $d(L, L') = 8$.*

REMARK. Other characterizations of classical generalized n-gons, $n \in \{6, 8\}$, are given by Cameron and Kantor [1979], Walker [1980, 1982a,b], Van Maldeghem [1990] and Van Maldeghem and Weiss [1992].

9.7. *Epimorphisms*

It is well known that an epimorphism between two finite thick generalized 3-gons, i.e. finite projective planes, must be an isomorphism (see Hughes and Piper [1973]). The following theorem generalizes this result to generalized n-gons, $n > 2$.

THEOREM 17 (Pasini [1984]). *Every epimorphism between two finite thick generalized n-gons, $n > 2$, is an isomorphism.*

9.8. *Open problems*

(a) Does there exist a TGQ of order (q^a, q^{a+1}), q odd and $a > 1$?
(b) Is every TGQ of order (q, q^2) with all lines regular isomorphic to $Q^-(5, q)$?
(c) Is every TGQ of order q isomorphic to a GQ $T_2(O)$ of Tits ?
(d) Does there exist a TGQ of order (q, q^2), q even, not isomorphic to a GQ $T_3(O)$ of Tits ?
(e) For q odd, classify the TGQ arising from flocks.
(f) For generalized hexagons, does $\{p, p'\}$-transitivity, with p, p' points for which $d(p, p') = 6$, imply $\{L, L'\}$-transitivity, with L, L' lines for which $d(L, L') = 6$?
(g) For generalized octagons, does $\{p, p'\}$-transitivity, with p, p' points for which $d(p, p') = 8$, imply $\{L, L'\}$-quasi-transitivity, with L, L' lines for which $d(L, L') = 8$?
(h) Does Theorem 16 hold for $n = 8$ and s and/or t equal to 2?
(i) Does the Moufang condition (M) imply the Moufang condition ($\widehat{\mathrm{M}}$) for finite thick generalized n-gons, $n > 4$?

10. Infinite generalized polygons

In this section we define generalized polygons, without the assumption of finiteness.

A *generalized n-gon*, $n \geqslant 2$, is an incidence structure $S = (P, B, I)$ of points and lines satisfying:

(i) $P \neq \varnothing$, $B \neq \varnothing$, each point is incident with at least two lines, each line is incident with at least two points;
(ii) the incidence graph of S has girth $2n$ and diameter n.

With such a definition grids and dual grids are GQ. To avoid this, one can consider, e.g., only thick generalized polygons.

For infinite generalized polygons there is no Feit–Higman theorem.

It is clear how to generalize conditions (M) and ($\widehat{\mathrm{M}}$) to generalized n-gons, $n \geqslant 4$. A generalized n-gon, $n \geqslant 4$, satisfying both (M) and ($\widehat{\mathrm{M}}$) is said to be a *Moufang generalized polygon*. In such a case, by Tits [1976b, 1979] and Weiss [1979], we have $n \in \{4, 6, 8\}$, and, by Tits [1976a, 1977a, 1983, 1990], all Moufang generalized n-gons, $n \geqslant 4$, are known.

Further, Theorem 22 of 8.4 and Theorems 27 and 29 of 8.7 are particular cases of theorems by Ronan on generalized quadrangles, respectively hexagons, without the finiteness condition. Also the main results of Thas and Van Maldeghem [1990] and Van Maldeghem [1990] hold in the infinite case.

The GQ embedded in an infinite projective space were completely determined by Dienst [1980a,b].

Work on topological GQ was done by Forst [1979] and Schroth [1988]. See also Section 6 of Chapter 23.

An interesting problem on generalized polygons is the following one: does there exist a generalized n-gon, $n \geqslant 4$, with a finite number of points on any line and an infinite number of lines through any point? For thick GQ it is not difficult to show that for $s = 2$ there is no such GQ. By a group theoretical argument Kantor [1984] proved that for $s = 3$ there is no such GQ either; Kantor described his proof as complicated and asked for a simpler combinatorial argument. Shortly afterwards Brouwer found indeed an ingenious combinatorial proof; see Brouwer [1991]. To my knowledge no other partial solutions to the problem are known.

Finally, an important reference on infinite polygons is Tits [1977b].

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CHAPTER 10

Some Classes of Rank 2 Geometries

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Contents

Introduction ... 435
1. Partial geometries ... 441
   1.1. Definitions ... 441
   1.2. Remarks ... 441
   1.3. The point graph of a partial geometry ... 441
   1.4. The known models of proper partial geometries ... 443
   1.5. Some characterization theorems for partial geometries ... 446
2. Semipartial geometries ... 448
   2.1. Definitions ... 448
   2.2. A first list of examples of proper semipartial geometries ... 450
   2.3. The linear representations of semipartial geometries ... 451
   2.4. Semipartial geometries and generalized quadrangles ... 453
   2.5. Semipartial geometries and SPG reguli ... 454
   2.6. Some characterization theorems for semipartial geometries ... 456
3. Copolar spaces ... 458
4. Near n-gons ... 458
   4.1. Definitions ... 458
   4.2. Classical and sporadic near n-gons ... 459
   4.3. Regular near n-gons ... 460
5. Moore geometries ... 462
   5.1. Moore graphs ... 462
   5.2. (Generalized) Moore geometries ... 463
6. (g, d_p, d_l)-gons ... 464
   6.1. Definitions ... 464
   6.2. Examples ... 464
   6.3. Characterizations by automorphisms ... 466
References ... 471

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Introduction

In the preceding chapters a lot of rank 2 geometries, such as projective and affine planes, designs, linear spaces, generalized polygons, ... are studied in detail. In this chapter we will discuss some more rank 2 geometries that are merely generalizations of these geometries. A treatment of all the known rank 2 geometries is impossible. We have therefore made a choice which is rather restrictive and subjective. Although a lot of the basic definitions are given in other chapters (especially in Chapter 3) we will, for the sake of completeness, recall the most important ones. This will also give us the opportunity to fix the notations used in this chapter.

1. *Generalities on geometries*

Referring to definitions given in Chapter 3 we can say that a *rank* 2 *geometry* S is a $\{0,1\}$-geometry. The elements of type 0 will be called the *points* while the elements of type 1 will be called *lines*. In some other chapters a rank 2 geometry is called an *incidence structure* $S = (P, B, \mathrm{I})$ with P ($\neq \varnothing$) the set of points, B ($\neq \varnothing$) the set of lines and a symmetric incidence relation $\mathrm{I} \subseteq (P \times B) \cup (B \times P)$. In this chapter both the sets P and B will be finite and the geometry will be connected. In a lot of cases, lines will be subsets of the point set P and the incidence I will be the natural incidence ($\in$).

The *dual* of a rank 2 geometry $S = (P, B, \mathrm{I})$ is the geometry $S^D = (P^D, B^D, \mathrm{I}^D)$ with $P^D = B$, $B^D = P$, and $\mathrm{I}^D = \mathrm{I}$.

A rank 2 geometry S is called a *partial linear space*, if each point is on at least 2 lines, if all lines have at least two points and if any two distinct points in P are incident with at most one line, or equivalently, if any two distinct lines are incident with at most one point. Some authors call this a *semilinear space*. Lines incident with only 2 points, are called *thin lines*. If all lines are thin lines, then S is called a *thin partial linear space*. If all lines are incident with at least 3 points and if every point is incident with at least 3 lines, the partial linear space is called *thick*. Two points are said to be *collinear* if they are incident with a common line. Note that a point is collinear with itself. Dually, two lines are said to be *concurrent* if they are incident with a common point. We will denote collinear points x and y (resp., concurrent lines L and M) by $x \sim y$ (resp., $L \sim M$). On the other hand, we will sometimes use the standard notation for the set of points collinear to a point x: $x^{\perp} = \{y \in P\colon\ y \sim x\}$.

If any two different points are collinear, then S is called a *linear space*. For more details on linear spaces we refer to Chapter 6.

In this chapter we will mainly deal with quite special partial linear spaces. They will have the next two properties:

(S_1): Each point is incident with $t + 1\,(t \geqslant 1)$ lines.
(S_2): Each line is incident with $s + 1\,(s \geqslant 1)$ points.

A partial linear space S satisfying these two properties will be called a partial linear space of *order* (s, t).

Let S be a connected partial linear space. Let (x, L) be an antiflag of S, i.e. x is a point and L is a line of S, such that x is not incident with L. We denote by $\alpha(x, L)$ the number of points on L collinear with x, or equivalently the number of lines through x

concurrent with L. We will sometimes call $\alpha(x, L)$ the *incidence number* of the antiflag (x, L).

In this chapter we will mainly deal with connected partial linear spaces in which $\alpha(x, L)$ can take only a few values. For instance, if $\alpha(x, L)$ can only have the values 0 and $\alpha \neq 0$, then the connected partial linear space is called a $(0, \alpha)$-geometry in De Clerck and Thas [1983] and Thas, Debroey and De Clerck [1984]. One can easily check that if S is a $(0, \alpha)$ geometry with $\alpha > 1$ then there exist two integers s ($\geqslant 1$) and t ($\geqslant 1$) such that S is of order (s, t). The dual of a $(0, \alpha)$-geometry is of course again a $(0, \alpha)$-geometry. There are a lot of examples of $(0, \alpha)$-geometries. We will restrict ourselves to some special classes with extra regularity conditions.

A lot of the examples we will encounter in this chapter have points and lines in a projective or affine space. To be more precise, a geometry $S = (P, B, \mathrm{I})$ is said to be *embedded* in a projective or an affine space if B is a subset of the set of lines of the space and if P is the set of all points of the space on these lines. We will always assume in what follows that the dimension of the space is the smallest possible dimension for an embedding. Some authors call this a *full embedding* or a *flat embedding*.

A special type of affine embedding is the so-called *linear representation* of a geometry of order (s, t) in $\mathrm{AG}(n+1, s+1)$. It is an embedding of $S = (P, B, \mathrm{I})$ in $\mathrm{AG}(n+1, s+1)$ such that the line set B of S is a union of parallel classes of lines of $\mathrm{AG}(n+1, s+1)$ hence the point set P of S is the point set of $\mathrm{AG}(n+1, s+1)$. These lines of S define in the hyperplane at infinity Π_∞ a set of points $\mathcal{K}$ of size $t+1$. If S is a $(0, \alpha)$-geometry, then every line of Π_∞ intersects $\mathcal{K}$ in either $0, 1$ or $\alpha + 1$ points. A line intersecting $\mathcal{K}$ in m points will be called an *m-secant*. A 1-secant will also be called a *tangent line*, while a line not intersecting $\mathcal{K}$ will be called a *passant*.

Using standard notations, the linear representation of a geometry S in $\mathrm{AG}(n+1, s+1)$ will be denoted by $T_n^*(\mathcal{K})$. We shall give several examples in the next sections.

However we first need some graph theoretical definitions.

2. *Graphs and rank 2 geometries*

A finite *graph* $\Gamma = (X, E)$ is a structure consisting of a set X ($\neq \varnothing$) with v elements and a set E of unordered pairs of X. The elements of X are called the *vertices* of the graph Γ, while the elements of E are called the *edges*. If x and y are two different vertices such that $\{x, y\} \in E$, then x and y are called *adjacent* and we write $x \sim y$; if $\{x, y\} \notin E$ then we denote this by $x \not\sim y$; remark that $x \not\sim x$. If E is the set of all unordered pairs of X then Γ is called the *complete graph* denoted by K_v. The *complement* Γ^C of a graph $\Gamma = (X, E)$ is the graph $\Gamma^C = (X^C, E^C)$ with $X^C = X$ and $E^C = X^{|2|} \setminus E$. The *line graph* $\mathcal{L}(\Gamma)$ of a graph Γ is the graph with vertices the edges of Γ, two edges being adjacent if and only if they have a common vertex.

A *path* of length m from x to y, is a set of vertices $x = x_0, x_1, x_2, \ldots, x_m = y$ such that $x_i \sim x_{i+1}$, $0 \leqslant i \leqslant m-1$. If $x = y$ then any such path with $x_i \neq x_{i+2}$ ($0 \leqslant i \leqslant m-2$) will be called a *circuit*. Two vertices x and y of a graph Γ are at *distance* $d(x, y)$, provided there exists a path of length $d(x, y)$ between these vertices and there exists no shorter one. A vertex has distance 0 from itself and distance 1 from all its adjacent vertices. We will denote by $\Gamma_i(x)$ the set of all vertices of Γ at distance i from x. For convenience we will use $\Gamma(x)$ for the set $\Gamma_1(x)$. A graph is *connected* if and only if for any two distinct vertices x and y, there is at least one path connecting these 2 vertices. The *diameter* of a graph Γ is the maximum value of the distance function $d(x, y)$. The *girth* of Γ is the length of its shortest circuit.

Given a partial linear space S, one may define the *point graph* or *collinearity graph* $\Gamma(S)$, by taking as vertices the points of S. Two different vertices are adjacent whenever they are collinear. Remark that we are using the same symbol ($\sim$) for the collinearity relation as for the adjacency relation, although a point x is collinear to itself but not adjacent to itself. A geometry is connected whenever its point graph is.

On the other hand, the *incidence graph* $\mathcal{I}(S)$ is the graph with vertices the elements of $P \cup B$, and 2 vertices are adjacent if and only if the corresponding elements are incident, hence edges of $\mathcal{I}(S)$ are the flags of S. Unlike the case of the collinearity graph, the geometry is completely determined by its incidence graph. Obviously, two vertices of the same type (i.e. either points or lines) in the incidence graph are connected by paths of even length. In particular, a circuit in an incidence graph has even length and hence the girth is an even positive integer, say $2g$. By definition, g is called the *gonality* of S, and $2g$ is called *the (geometric) girth* of S. Let x be a point or a line. A *geodesic (based at* x) is a path γ in the incidence graph starting in x and such that the length of γ is equal to the distance $d(x, y)$, where y is the last element of γ. A *maximal geodesic* is a geodesic that is not properly contained in another one. The *local diameter* $d(x)$ is the length of the longest geodesic based at x, whether x be a point or a line. The *point-diameter* d_p (resp., *line-diameter* d_l) of S is the greatest value taken by $d(x)$ for x a point (resp., a line). In Chapter 3, d_p is denoted by d_0 and is called the 0-diameter while d_l is denoted by d_1, the 1-diameter. The *diameter* d of a geometry S is the diameter of the incidence graph $\mathcal{I}(S)$, hence it is the largest of the two numbers d_p, d_l.

Finally, the *flag graph* of a rank 2 geometry S is the graph with vertices the maximal flags of S and 2 flags are adjacent whenever they share exactly one element. The *flag-diameter* d^* of S is the diameter of the flag graph.

A successful attempt to unify the study of all rank 2 geometries of this chapter (and some others) was made by Buekenhout [1982] (see also Chapter 3). He studied those geometries by considering their gonality g and the diameters d_p, d_l and d^*. He proved that a connected geometry S such that every element of S is incident with at least two other elements, and such that all points (resp., lines) have the same local diameter d_p (resp., d_l) has the property that d^* is the smallest of the two numbers d_p, d_l. Hence $\{d, d^*\} = \{d_p, d_l\}$ and we always have $g \leqslant d^* \leqslant d$. Moreover, $d - d^* \leqslant 1$. For the sequel we may assume that $d_p \leqslant d_l$, which is no loss of generality since one can always consider the dual geometry. With this terminology and under these conditions, S is called by Buekenhout [1982] a (g, d_p, d_l)-gon provided $d_p \leqslant g + 2$. The last condition is purely subjective and comes from the observation that almost all the 'nice' rank 2 geometries arising from finite simple groups – and especially the sporadic ones – satisfy this condition. Exceptions arise mainly from truncations of higher dimensional geometries. For instance, Suz acts on a $(3, 8, 8)$-gon which is a truncation of an extended generalized quadrangle (see Buekenhout [1985]).

In the last section we will return to the theory of the (g, d_p, d_l)-gons in order to explain how all the rank 2 geometries introduced in the other sections fit into this more global point of view.

3. *Distance regular graphs*

A graph Γ is called *regular* provided every vertex of Γ is adjacent to a constant number k of vertices, and this number k is called the *valency* or the *degree* of the graph.

A *distance regular graph* Γ with diameter d, is a regular and connected graph of valency k with the following property. There are natural numbers:

$$b_0 = k,\ b_1, \ldots, b_{d-1}; \qquad c_1 = 1,\ c_2, \ldots, c_d,$$

such that for each pair (x, y) of vertices at distance j, we have:

(1) $|\Gamma_{j-1}(y) \cap \Gamma_1(x)| = c_j$ $(1 \leqslant j \leqslant d)$;

(2) $|\Gamma_{j+1}(y) \cap \Gamma_1(x)| = b_j$ $(0 \leqslant j \leqslant d-1)$.

The *intersection array* of Γ is defined by

$$i(\Gamma) = \{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}.$$

For any graph Γ of diameter d, and vertex set $\{x_1, \ldots, x_v\}$, the distance matrices A_h, $h = 0, \ldots, d$, are the $v \times v$ matrices defined as follows:

$$(A_h)_{ij} = \begin{cases} 1 & \text{if } d(x_i, x_j) = h, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 1 (Damerell [1973]). *Let Γ be a distance regular graph with intersection array*

$$i(\Gamma) = \{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}.$$

For $1 \leqslant i \leqslant d-1$, put $a_i = k - b_i - c_i$; then

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (1 \leqslant i \leqslant d-1).$$

Moreover, the distance matrices form an algebra of dimension $d+1$, and $\{A_0 = I, A_1, \ldots, A_d\}$ is a basis for this algebra.

This implies that we can use a lot of techniques from linear algebra, such as eigenvalue techniques, to find for instance restrictions on the intersection array. For all information on distance regular graphs and an extensive bibliography, we refer to Brouwer, Cohen and Neumaier [1989].

A distance regular graph of diameter 2 is better known as a *strongly regular graph*. For reasons of convenience we will recall the definition in order to introduce the notations that are mainly used for these graphs.

A regular graph Γ is called a strongly regular graph (notation srg(v, k, λ, μ)) provided:

(1) any two vertices x and y, $x \sim y$, are both adjacent to a constant number λ of vertices (independent of the choice of the adjacent pair $\{x, y\}$);

(2) any two vertices x and y, $x \not\sim y$, are both adjacent to a constant number μ of vertices (independent of the choice of the nonadjacent pair $\{x, y\}$).

We exclude disconnected graphs and their complements, hence we assume $0 < \mu < k < v-1$. It is easy to check that the complement of a $\mathrm{srg}(v,k,\lambda,\mu)$ is a $\mathrm{srg}(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$ and that a $\mathrm{srg}(v,k,\lambda,\mu)$ is indeed equivalent to a distance regular graph with intersection array $\{k, k-1-\lambda; 1, \mu\}$.

The distance matrix $A_1 = A$ (or the (0, 1) adjacency matrix) of a $\mathrm{srg}(v,k,\lambda,\mu)$ satisfies

$$AJ = kJ, \qquad A^2 + (\mu-\lambda)A + (\mu-k)I = \mu J,$$

where J is the all-one matrix. Hence A has the valency k as an eigenvalue with multiplicity 1, and two other eigenvalues r and l ($r > 0$ and $l < 0$) with $r + l = \lambda - \mu$ and $rl = \mu - k$.

There are some known necessary conditions for the existence of a $\mathrm{srg}(v,k,\lambda,\mu)$. We will summarize the most important ones in the next theorem. For the proofs and more information on strongly regular graphs, we refer to Bose [1963], Van Lint and Seidel [1969], Cameron [1978], Seidel [1979], Cameron and Van Lint [1980], and Brouwer and Van Lint [1984].

THEOREM 2. *If Γ is a* $\mathrm{srg}(v,k,\lambda,\mu)$ *then the following is true.*

(1) $v - 2k + \mu - 2 \geqslant 0$.

(2) $k(k-\lambda-1) = \mu(v-k-1)$.

(3) *The multiplicities of the eigenvalues r and l of A are, respectively,*

$$f = \frac{-k(l+1)(k-l)}{(k+rl)(r-l)} \quad \textit{and} \quad g = \frac{k(r+1)(k-r)}{(k+rl)(r-l)}.$$

They clearly have to be integers.

(4) *The eigenvalues $r > 0$ and $l < 0$ are both integers, except for one family of graphs, the so-called conference graphs, which are* $\mathrm{srg}(2k+1, k, k/2-1, k/2)$. *In this case the number of vertices can be written as a sum of two squares, and the eigenvalues are* $(1 \pm \sqrt{v})/2$.

(5) *The Krein conditions*:

$$(r+1)(k+r+2rl) \leqslant (k+r)(l+1)^2,$$
$$(l+1)(k+l+2rl) \leqslant (k+l)(r+1)^2.$$

(6) *The absolute bound*:

$$v \leqslant \tfrac{1}{2} f(f+3), \qquad v \leqslant \tfrac{1}{2} g(g+3).$$

(7) *The claw bound*: *if* $\mu \neq l^2$, $\mu \neq l(l+1)$, *then* $2(r+1) \leqslant l(l+1)(\mu+1)$.

There are a lot of examples of strongly regular graphs known, see, e.g., Hubaut [1975] and Brouwer and Van Lint [1984]. We shall give here a short description of some examples which are important for the rest of this chapter. For information on the automorphism group of these graphs, we refer to Section 6.3 (see Table 1).

1. The pentagon Pn(5) is the unique srg$(5, 2, 0, 1)$.

2. The line graph of the complete graph K_n is called the triangular graph and is denoted by $T(n)$. This graph is a srg$(\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$. If $n \neq 8$ then every strongly regular graph Γ with these parameters is indeed a triangular graph. If $n = 8$ there are exactly three nonisomorphic graphs with the same parameters but not triangular, these graphs are known as the graphs of Chang [1959], see also Seidel [1967].

3. The strongly regular graph $T(5)^C$ is better known as the Petersen graph Pe(10); it is the unique srg$(10, 3, 0, 1)$. This graph can also be constructed by taking as vertices the 10 points of a Desargues configuration, two vertices being adjacent if they are not on a line of the Desargues configuration.

4. The Clebsch graph Cl(16) is a srg$(16, 5, 0, 2)$.

There is only one graph with these parameters (easy exercise). The graph can be constructed as follows. Take a set C of cardinality 5. The vertices of the graph are the set C and the subsets of cardinality 1 and 2. The vertex C is adjacent to the 5 singletons, a singleton is adjacent to C and to all the pairs containing it, a pair is adjacent to the 2 singletons it contains and to the 3 pairs of C that are disjoint from it. Other constructions of this graph are known. Another simple construction goes as follows. The vertices of the graph are the elements of GF(16), two vertices are adjacent whenever their difference is a 3rd power in GF(16). The name comes from the fact that this graph corresponds to the 16 lines on the Clebsch quartic surface (see Clebsch [1868] or Coxeter [1950]).

5. The graph HoS(50) (see Hoffman and Singleton [1960]).

A lot of constructions of this graph are known. This graph is a srg(50,7,0,1), and is uniquely defined by its parameters. We shall give only one construction which will be useful later on. It is commonly known that there is a bijection between the 35 unordered triples of a 7-set and the 35 lines of PG$(3, 2)$, such that lines intersect if and only if the corresponding triples have exactly one element in common. The graph HoS(50) is constructed as follows. The vertices are the 15 points together with the 35 lines of PG$(3, 2)$. Points are mutually nonadjacent. A point is adjacent to a line whenever the point lies on that line. Two lines are adjacent whenever the corresponding two triples are disjoint.

6. The Higman–Sims family (see, e.g., Hubaut [1975]).

These graphs are constructed using the Steiner system $S(3, 6, 22)$. This Steiner system is the uniquely defined extension of the projective plane PG$(2, 4)$. Hence this Steiner system has as other parameters $b = 77, r = 21, \lambda_2 = 5$ and two different blocks intersect in 0 or 2 points.

(a) The Higman–Sims graph HS(100).

Take as vertices of the graph a symbol ∞, together with the 22 points and the 77 blocks of $S(3, 6, 22)$. The symbol ∞ is adjacent to all the 22 points but to no block. Points are never adjacent and a point is adjacent to a block whenever it is contained in that block. Two blocks are adjacent whenever they are disjoint. This graph is a srg$(100, 22, 0, 6)$, and is uniquely defined by its parameters, see Gewirtz [1970].

(b) The Higman–Sims graph HS(77).

This graph is the subgraph defined on the set $\Gamma_2(\infty)$ of the vertices of HS(100) that are not adjacent to ∞. It is a srg$(77, 16, 0, 4)$, and is uniquely defined by its parameters, see Brouwer [1983].

(c) The graph Gew(56) of Gewirtz [1969].

Delete from $\bar{S}(3, 6, 22)$ all the 21 blocks through a fixed point. Take as vertices the 56 other blocks which are adjacent whenever they are disjoint. This graph is a srg(56,10,0,2), and is also uniquely defined by its parameters.

1. Partial geometries

1.1. *Definitions*

A (finite) *partial geometry* $S = (P, B, \mathrm{I})$ is a partial linear space of order (s, t) such that for all antiflags (x, L) the incidence number $\alpha(x, L)$ is a constant α ($\neq 0$). The numbers s, t and α are called the *parameters* of S. This incidence structure was introduced by Bose [1963].

1.2. *Remarks*

1. If $S = (P, B, \mathrm{I})$ is a partial geometry with parameters s, t, α, then the dual structure $S^D = (P^D, B^D, \mathrm{I}^D) = (B, P, \mathrm{I})$, is a partial geometry with parameters $s^D = t$, $t^D = s$ and $\alpha^D = \alpha$.
2. $|P| = v = (s+1)\dfrac{(st+\alpha)}{\alpha}$ and $|B| = b = (t+1)\dfrac{(st+\alpha)}{\alpha}$.
3. The partial geometries can be divided into four (nondisjoint) classes.
   (a) The partial geometries with $\alpha = 1$, the generalized quadrangles. See Chapter 9 and Payne and Thas [1984].
   (b) The partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$, i.e. the 2-$(v, s+1, 1)$ designs and their duals. See Chapter 8.
   (c) The partial geometries with $\alpha = s$ or dually $\alpha = t$. The partial geometries with $\alpha = t$ are the *Bruck nets of order* $s+1$ *and degree* $t+1$; Bruck [1963].
   (d) Finally, the so-called *proper* partial geometries with $1 < \alpha < \min(s, t)$. We shall mainly deal with this class of partial geometries.

1.3. *The point graph of a partial geometry*

THEOREM 3 (Bose [1963]). *The point graph $\Gamma(S)$ of a partial geometry S is a*

$$\mathrm{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

(If $\alpha = s+1$, the graph is a complete graph.) Each strongly regular graph Γ having parameters in this form with $t \geqslant 1$, $s \geqslant 1$, $1 \leqslant \alpha \leqslant s+1$ and $1 \leqslant \alpha \leqslant t+1$ is called a *pseudo-geometric* (t, s, α)-graph. If the graph Γ indeed is the point graph of at least one partial geometry, then Γ is called *geometric*. Given a pseudo-geometric (t, s, α)-graph Γ, the problem is to find a subset B of cliques of $s+1$ vertices of Γ, such that any two adjacent vertices of Γ are in exactly one element of B. In Bose [1963] the following condition for a pseudo-geometric (t, s, α)-graph to be geometric is proved.

THEOREM 4 (Bose [1963]). *A pseudo-geometric* (t, s, α)*-graph* Γ *is geometric if*

$$2(s+1) > t(t+1) + \alpha(t+2)(t^2+1).$$

This condition however is in general too strong in order to construct partial geometries from the graph Γ. In Cameron, Goethals and Seidel [1978] it is proved (using the Krein condition on the point graph of the dual geometry) that for a pseudo-geometric (t, s, α)-graph Γ satisfying the Bose inequality in the above theorem, $t \leqslant 2\alpha - 1$ holds.

Attempts to construct a partial geometry from a pseudo-geometric (t, s, α)-graph Γ were in most cases unsuccessful; we refer, e.g., to Spence [1992], De Clerck and Tonchev [1992], and De Clerck, Gevaert and Thas [1988], an exception however is the sporadic partial geometry of Haemers [1981] (see 1.4.5).

If we translate the necessary conditions for strongly regular graphs in Theorem 2 in terms of the parameters of a pseudo-geometric (t, s, α)-graph, then this yields the following theorem.

THEOREM 5. *If* Γ *is a pseudo-geometric* (t, s, α)*-graph, then* $r = s - \alpha$, $l = -t - 1$,

$$f = \frac{st(s+1)(t+1)}{\alpha(s+t+1-\alpha)}.$$

Moreover:

(1) v *is an integer, hence* $\alpha \mid (s+1)st$.

(2) *The multiplicities of the eigenvalues of the adjacency matrix are integers, hence*

$$\alpha(s+t+1-\alpha) \mid st(s+1)(t+1).$$

(3) *The Krein inequalities for strongly regular graphs are satisfied, hence*

$$(s+1-2\alpha)t \leqslant (s-1)(s+1-\alpha)^2.$$

REMARK. If $\alpha = 1$ (and $s \neq 1$), then the Krein inequality is better known as the Higman inequality $t \leqslant s^2$, Higman [1971]. Moreover, Cameron et al. [1978] proved that any pseudo-geometric $(s^2, s, 1)$-graph is geometric (see also Haemers [1980]). It is not known whether this theorem also holds for pseudo-geometric (t, s, α)-graphs satisfying the Krein equality in the case $\alpha > 1$.

OPEN QUESTION. There exists a pseudo-geometric $(27, 4, 2)$-graph, the McLaughlin graph, see, e.g., Goethals and Seidel [1975]. This graph does satisfy the Krein equality; however although several attempts have been made, e.g., by Van Lint [1984], it is not known whether this graph is geometric or not. For the moment, there is no known partial geometry with these parameters.

1.4. *The known models of proper partial geometries*

1.4.1. *The partial geometry $S(\mathcal{K})$*

This infinite family was constructed by Thas [1973, 1974] and independently by Wallis [1973]. Let $\mathcal{K}$ be a maximal arc of degree d in a projective plane π of order q, i.e. a $\{qd - q + d; d\}$-arc (see Chapter 7 for the definitions and examples). We define the incidence structure $S(\mathcal{K}) = (P, B, \mathrm{I})$. The points of $S(\mathcal{K})$ are the points of π that are not contained in $\mathcal{K}$. The lines of $S(\mathcal{K})$ are the lines of π that are incident with d points of $\mathcal{K}$. The incidence is the one of π. Then $S(\mathcal{K})$ is a partial geometry with parameters $t = q - q/d$, $s = q - d$, $\alpha = q - q/d - d + 1$.

REMARKS. As there exist $\{2^{h+m} - 2^h + 2^m; 2^m\}$-arcs, whenever $0 < m < h$, in $\mathrm{PG}(2, 2^h)$, there exists a class of partial geometries $S(\mathcal{K})$ with parameters

$$s = 2^h - 2^m,\ t = 2^h - 2^{h-m},\ \alpha = (2^m - 1)(2^{h-m} - 1).$$

This is a generalized quadrangle if and only if $h = 2$, and then it is the unique quadrangle of order 2.

Suppose $m = h - 1, h \geqslant 2$. Then the point graph of $S(\mathcal{K})$ is $T(2^h + 2)^C$, the complement of the triangular graph $T(2^h + 2)$. Hence $T(2^h + 2)^C$ is a geometric $(2^h - 2, 2^{h-1}, 2^{h-1} - 1)$-graph. Although these graphs are uniquely defined by their parameters, this does not imply that the geometry is unique. For instance, Mathon [1981] proved by computer that there exist exactly two partial geometries with parameters $t = 6, s = 4, \alpha = 3$ (and point graph $T(10)^C$). All the complements of the triangular graphs $T(2n)$ are pseudo-geometric $(2(n-2), n-1, n-2)$-graphs. However it is possible to prove that $T(8)^C$ and the complements of the Chang graphs (having the same parameters as $T(8)^C$) are not geometric (De Clerck [1979]). Moreover, Lam, Thiel, Swiercz, and McKay [1983] proved that $T(12)^C$ is not geometric.

1.4.2. *The partial geometry $T_2^*(\mathcal{K})$*

Let $\mathcal{K}$ be a maximal arc of degree d in the projective plane $\mathrm{PG}(2, q)$ over $\mathrm{GF}(q)$ ($q = p^h$, p prime). As $\mathcal{K}$ has only passants and d-secants, it will yield a linear representation of a partial geometry in $\mathrm{AG}(3, q)$. This partial geometry $T_2^*(\mathcal{K})$ has parameters $t = (q+1)(d-1), s = q - 1, \alpha = d - 1$. This infinite family was constructed for the first time by Thas [1973, 1974].

REMARK. The partial geometry $T_2^*(\mathcal{K})$ using a maximal arc of degree 2^m, $0 < m < h$, in $\mathrm{PG}(2, 2^h)$ has parameters $s = 2^h - 1$, $t = (2^h + 1)(2^m - 1)$, $\alpha = 2^m - 1$. This is a generalized quadrangle if and only if $m = 1$, i.e. if and only if $\mathcal{K}$ is a hyperoval. (See Chapter 9).

1.4.3. *The partial geometries* $\mathrm{PQ}^+(4n - 1, q)$, $q = 2$ *or* $q = 3$

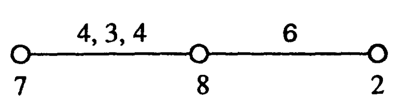
1. *Some properties of hyperbolic quadrics in* $\mathrm{PG}(2m - 1, q)$. Let $Q^+ = Q^+(2m - 1, q)$, $m \geqslant 2$, be the hyperbolic quadric in $\mathrm{PG}(2m - 1, q)$ (the quadric with projective index

$m-1$). The set of maximal totally isotropic or singular subspaces on a hyperbolic quadric Q^+ is divided into two disjoint families $\mathcal{D}_1$ and $\mathcal{D}_2$. Two maximal totally isotropic or singular subspaces on the quadric are in the same family if and only if the codimension of their intersection has the parity of $m-1$ (see Chapter 2 for more details on quadrics).

Assume q is odd, let x and y be two points of $\mathrm{PG}(2m-1,q)\setminus Q^+$. Then x and y are called equivalent if and only if there exists a point $z \in \mathrm{PG}(2m-1,q)\setminus Q^+$ such that the lines xz and yz are tangent lines of Q^+. This relation can also be defined as follows. Embed Q^+ in the nonsingular hyperquadric Q of $\mathrm{PG}(2m,q)$. The pole of $\mathrm{PG}(2m-1,q)$ with respect to Q is denoted by p. Then x and y are equivalent if and only if the lines xp and yp are both secants or are both exterior lines of Q. The proof that this relation indeed is an equivalence relation was given by Thas [1981b]. There are two equivalence classes E_1 and E_2. For some i, $Q^+ \cup E_i$ is the projection of the nonsingular hyperquadric Q of $\mathrm{PG}(2m,q)$, from the point p onto $\mathrm{PG}(2m-1,q)$.

2. *The partial geometry* $\mathrm{PQ}^+(4n-1,2)$. De Clerck, Dye and Thas [1980] constructed an infinite class of partial geometries as follows. Define a spread Σ of the nonsingular hyperbolic quadric $Q^+ = Q^+(4n-1,2)$, $n \geqslant 2$, in $\mathrm{PG}(4n-1,2)$ to be a (maximal) set of $2^{2n-1}+1$ disjoint $(2n-1)$-dimensional spaces on Q^+. Let Σ be a spread of $Q^+ = Q^+(4n-1,2)$ and let Ω be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1,2) = (P,B,\mathrm{I})$ with P the set of points of $\mathrm{PG}(4n-1,2)$ not on the quadric, $B = \Omega$ and $x\mathrm{I}L$, $x \in P$ and $L \in B$, if and only if x is contained in the polar space L^* of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1,2)$ is a partial geometry with parameters $s = 2^{2n-1}-1$, $t = 2^{2n-1}$, $\alpha = 2^{2n-2}$.

If $n=2$, then the parameters of $\mathrm{PQ}^+(7,2)$ are $s=7$, $t=8$, $\alpha=4$. Cohen [1981b] was the first to construct a partial geometry with these parameters using the root system E_8. In Haemers and Van Lint [1982] a partial geometry with parameters $s=8$, $t=7$, $\alpha=4$, was constructed using coding theory. Kantor [1982a] proved that $\mathrm{PQ}^+(7,2)$ and the dual of the geometry of Haemers–Van Lint are isomorphic. Later on Tonchev [1984] showed with the help of a computer that the model of Cohen and the dual of the geometry of Haemers–Van Lint are isomorphic. In De Clerck et al. [1988] this isomorphism is proved without the use of a computer. Actually, L. Soicher (private communication) has checked by computer that $\mathrm{PQ}^+(7,2)$ is uniquely determined by its point graph, as is its dual. Note that this partial geometry also appears as a residue of an element of type 2 in a rank 3 geometry for the Thompson group $\mathsf{F}_3 = \mathsf{Th}$ (see Buekenhout [1985]) with diagram



(see also Chapter 22).

The residue of an element of type 0 in this rank 3 geometry is isomorphic to the (unique) generalized hexagon of order $(8,2)$ (see Chapter 9).

Remark that nonisomorphic spreads of the quadric $\mathrm{PQ}^+(4n-1,2)$ will produce nonisomorphic partial geometries. If $2n-1$ is composite then $\mathrm{PQ}^+(4n-1,2)$ has nonisomorphic spreads, and probably this is true for all $n > 2$ (see Kantor [1982b]).

3. *The partial geometry* $\mathrm{PQ}^+(4n-1,3)$. For $q=3$ an analogous construction is given by Thas [1981b]. Again let Σ be a spread of $Q^+ = Q^+(4n-1,3)$ and let Ω be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1,3) = (P,B,\mathrm{I})$ with P one of the sets E_i, $B=\Omega$ and with $x\mathrm{I}L$, $x\in P$ and $L\in B$, if and only if x is contained in the polar space L^* of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1,3)$ is a partial geometry with parameters $s=3^{2n-1}-1$, $t=3^{2n-1}$, $\alpha = 2\cdot 3^{2n-2}$.

Up to now it is only known that $Q^+(7,3)$ has a spread. This yields a geometry $\mathrm{PQ}^+(7,3)$ with parameters $s=26$, $t=27$, $\alpha=18$ ($v=1080$, $b=1120$).

OPEN QUESTION. Do there exist partial geometries of type $\mathrm{PQ}^+(4n-1,3)$, $n>2$?

1.4.4. *The sporadic partial geometry of Van Lint–Schrijver*
Van Lint and Schrijver [1981] constructed the following sporadic proper partial geometry. We will sketch two constructions of this geometry. Let β be a primitive element of GF(3^4). Then $\gamma=\beta^{16}$ is a primitive 5-th root of the unity. Let $P=\mathrm{GF}(81)$, let B be the set

$$\{(b, 1+b, \gamma+b, \gamma^2+b, \gamma^3+b, \gamma^4+b)\colon\ b\in \mathrm{GF}(81)\},$$

I is the natural incidence, namely inclusion. Then $S=(P,B,\mathrm{I})$ is a partial geometry with $s=t=5$ and $\alpha=2$. The point graph of this geometry has parameters $v=81$, $k=30$, $\lambda=9$, $\mu=12$, and is a graph which was not known before.

Another construction of this geometry is given by Cameron and Van Lint [1982]. Let C be the ternary repetition code of length 6, i.e.

$$C=\{(0,0,0,0,0,0),(1,1,1,1,1,1),(2,2,2,2,2,2)\}.$$

Any coset of C in GF(3)^6 has a well-defined type i in GF(3), i.e. the sum i of the coordinates of any vector in the coset. Let $\mathcal{A}_i$ be the set of cosets of type i. Define a tripartite graph Γ by joining the coset $C+v$ to the coset $C+v+w$ for each vector w of weight 1. Any element in $\mathcal{A}_i$ has 6 neighbours in $\mathcal{A}_{i+1}$ and 6 in $\mathcal{A}_{i+2}$ (indices taken mod 3).

Consider the incidence structure with point set $\mathcal{A}_i$, and line set $\mathcal{A}_{i+1}$, in which incidence is defined by adjacency in Γ. Then this incidence structure is the partial geometry of Van Lint–Schrijver.

For example, suppose $i=0$, and p is the point with coset representative $(0,0,0,0,0,0)$. The six lines incident with p have representatives of the form $(1,0,0,0,0,0)$, hence the 30 points ($\neq p$) collinear with p have representatives of the form $(1,2,0,0,0,0)$. It is immediately clear that two points are incident with at most one line. A line not incident with p has a representative of the form $(2,2,0,0,0,0)$ or $(2,1,1,0,0,0)$; in both cases, it is incident with two points collinear with p.

REMARK. Assume S is a proper symmetric partial geometry with $\alpha=2$, then the numerical conditions of Theorem 2 yield $s=5$. It is not known whether this geometry is unique. There is a great doubt that this geometry is a member of an infinite family. However the point graph is a member of an infinite family of so-called cyclotomic type (see Calderbank and Kantor [1986]).

1.4.5. *The sporadic partial geometry of W. Haemers*

Haemers [1981] constructed another sporadic proper partial geometry. It has parameters $s = 4$, $t = 17$, $\alpha = 2$. The point graph Γ however was known before (see, e.g., Hubaut [1975]). This graph Γ is constructed as follows. The vertices of Γ are the 175 edges of the Hoffman–Singleton graph HoS(50). Two vertices of Γ are adjacent whenever the corresponding edges of HoS(50) have distance two (i.e. the two edges are disjoint and there exists an edge connecting both). One can prove that this graph is a srg$(175, 72, 20, 36)$, moreover Γ is a pseudo-geometric $(17, 4, 2)$-graph. Haemers proved that Γ is indeed geometric. First of all we remark that a line of the partial geometry will be a set of 5 disjoint edges pairwise at distance two in the Hoffman–Singleton graph HoS(50). It is easy to see that in a Petersen graph there are 6 such sets. If we can find 105 Petersen graphs in the Hoffman–Singleton graph, then we have the right number of lines. However there are more than 105 Petersen graphs in HoS(50). W. Haemers was able to find a good subset of 105 special Petersen graphs in the Hoffman–Singleton graph, such that every pentagon of HoS(50) is contained in exactly one such special Petersen graph. Note that any two edges at distance two in HoS(50) are in a unique pentagon, so in a unique special Petersen graph, hence they define a unique set of 5 disjoint edges pairwise at distance two. In other words, the incidence structure of the 175 vertices of Γ and the 630 so-called 1-factors of the special Petersen graphs of HoS(50) has the property that any two adjacent vertices define a unique line. This is enough to conclude that the pseudo-geometric graph Γ indeed is geometric. The geometry is the unique one with this point graph (L. Soicher, private communication).

REMARKS.

1. The point graph of the dual of this geometry has parameters $v = 630$, $k = 85$, $\lambda = 20$, $\mu = 10$ and was not known before. This graph has exactly 175 cliques of size 18, and so this graph uniquely determines the dual of the Haemers geometry (L. Soicher, private communication).
2. Another construction of this geometry is related to the Steiner system $S(5, 8, 24)$ and was given by Calderbank and Wales [1984].
3. There are reasons enough to conjecture that this partial geometry is not a member of an infinite family.
4. Haemers [1991] proved that this partial geometry has a one point extension to a rank 3 geometry for the Mathieu group M_{22}.

1.5. *Some characterization theorems for partial geometries*

1.5.1. *The graphs with $\alpha = s - 1$*

If S is a partial geometry with $\alpha = s - 1$, then the complement of its point graph has a negative eigenvalue -2. All the strongly regular graphs with smallest eigenvalue -2 however are classified by Seidel [1968]. Using this theorem a classification of the pseudo-geometric $(t, s, s - 1)$-graphs can be given. Moreover a lot of information on geometric $(t, s, s - 1)$-graphs is known.

THEOREM 6 (De Clerck [1979]). *If Γ is a pseudo-geometric $(t, s, s-1)$-graph, then one of the following cases occurs.*

(1) $s = 2$ *and* $t = 1,\ 2$ *or* 4. *Then Γ is geometric and the corresponding generalized quadrangles are unique.*

(2) $s = 3$ *and* $t = 1, 2$ *or* 4.

(a) *If $t = 1$, then Γ is geometric and the corresponding partial geometry is the unique dual 2-design with these parameters.*

(b) *If $t = 2$, then Γ is geometric and there are exactly two possibilities, i.e. the corresponding nets are the two nets of order* 4 *and degree* 3.

(c) *If $t = 4$, then Γ never is geometric.*

(3) $s > 3$, $t = s - 1$ *and Γ is geometric if and only if there exists an affine plane of order* $s + 1$. *The corresponding net is obtained by deleting two parallel classes from the affine plane.*

(4) $s > 3$, $t = 2(s-1)$. *If there exists a hyperoval $\mathcal{O}$ in some projective plane of order* $2s$ *then Γ is geometric, the geometry being the dual of $S(\mathcal{O})$.*

1.5.2. *Partial geometries and the axiom of Pasch*

Let us first introduce the *axiom of Pasch* ($\mathcal{P}$), also called the *axiom of Veblen and Young*.

If $L_1 \mathrm{I} x \mathrm{I} L_2$, $L_1 \neq L_2$, $M_1 \not\mathrm{I} x \not\mathrm{I} M_2$, $L_i \sim M_j$ for all $i, j \in \{1, 2\}$, then $M_1 \sim M_2$.

We remark that the dual axiom is called the diagonal axiom ($\mathcal{D}$). For a generalized quadrangle both ($\mathcal{P}$) and ($\mathcal{D}$) are satisfied in a trivial way. Evidently a 2-$(v, s+1, 1)$ design satisfies ($\mathcal{D}$). A 2-$(v, s+1, 1)$ design with $s > 1$ satisfying ($\mathcal{P}$), is an n-dimensional projective space ($n \geqslant 2$). The only known partial geometry with $\alpha \notin \{1, s+1, t+1\}$ and satisfying the axiom of Pasch is the dual net H_q^{n+1}. This dual net is constructed as follows. Let $H \cong \mathrm{PG}(n-1, q)$ be a subspace of a projective geometry $\Sigma \cong \mathrm{PG}(n+1, q)$. Then H_q^{n+1} is the incidence structure of points of $\Sigma \setminus H$ and lines of Σ skew to H, the incidence being the one of Σ. The parameters are $s = q$, $t = q^n - 1$, $\alpha = q$. In Thas and De Clerck [1977] it was proved that this dual net is the only one that satisfies the axiom of Pasch.

THEOREM 7 (Thas and De Clerck [1977]). *Let S be a dual net of order $s+1$ and degree $t+1$ $(t+1 > s)$. If S satisfies ($\mathcal{P}$), then S is isomorphic to H_q^{n+1} (hence $q = s$, $t+1 = q^n$).*

REMARKS.

1. Note that H_q^{n+1} may be seen as the complement of a singular symplectic geometry in $\Sigma \cong \mathrm{PG}(n+1, q)$ with radical H. Moreover the dual, $(H_q^{n+1})^D$, is also known as a *regulus net*, see De Clerck and Johnson [1992].

2. For a more general characterization theorem for partial geometries satisfying the axiom of Pasch, we refer to Thas and De Clerck [1977]. Moreover we remark that partial linear spaces satisfying both ($\mathcal{P}$) and ($\mathcal{D}$) are classified by Sprague [1981], see also Chapter 3.

1.5.3. *Partial geometries embedded in projective and affine spaces*
There exists a complete classification of partial geometries embedded in a projective space.

THEOREM 8 (De Clerck and Thas [1978]). *If $S = (P, B, \mathrm{I})$ is a partial geometry with parameters s, t, α, which is embedded in a projective space* PG(n, s), *but not in a* PG(n', s), *with $n' < n$, then the following cases may occur.*

(1) $\alpha = s + 1$, *and S is the design of points and lines of* PG(n, s).

(2) $\alpha = 1$, *and S is a classical generalized quadrangle* (see Buekenhout and Lefèvre [1974]).

(3) $\alpha = t + 1, n = 2$ *and S is a dual design in* PG$(2, s)$.

(4) $\alpha = s$ *and* $S = H_s^n (n \geqslant 3)$.

There also exists a complete classification of partial geometries embedded in an affine space by Thas [1978]. For the case of a generalized quadrangle, we refer to Chapter 9, in this case some sporadic embeddings can occur. We will however restrict ourselves here to the case of proper partial geometries.

THEOREM 9 (Thas [1978]). *If S is a proper partial geometry embedded in an affine space* AG$(n, s+1)$, *but not in an* AG$(n', s+1)$ *with $n' < n$, then $n = 3$ and $S = T_2^*(\mathcal{K})$ with $\mathcal{K}$ a maximal arc in the plane at infinity.*

COROLLARY. *If we combine the results on the affine embedding of generalized quadrangles in Chapter 9 and the above theorem we can conclude that if $T_n^*(\mathcal{K})$ $(n > 1)$ is a linear representation of a partial geometry of order (s, t), then either $\mathcal{K}$ is the complement of a hyperplane (hence $\alpha = s$), or $n = 2$.*

For more characterization theorems, especially regarding geometries of type $S(\mathcal{K})$ and of type $T_2^*(\mathcal{K})$, we refer to De Clerck, De Soete and Gevaert [1987], Gevaert [1987] and De Clerck et al. [1988].

2. Semipartial geometries

2.1. *Definitions*

A *semipartial geometry* (Debroey and Thas [1978a]) with parameters s, t, α, μ is a partial linear space $S = (P, B, \mathrm{I})$ of order (s, t), such that for each antiflag (x, L), the incidence number $\alpha(x, L)$ equals 0 or a constant $\alpha(> 0)$ and such that for any two points which are not collinear, there are μ $(\mu > 0)$ points collinear with both (μ-condition).

REMARKS.

1. A semipartial geometry is a $(0, \alpha)$-geometry such that, because of the μ-condition, the point graph is strongly regular. Besides the parameter μ, the other parameters of the

graph are

$$v = 1 + \frac{(t+1)s(\mu + t(s-\alpha+1))}{\mu}, \ k = (t+1)s, \ \lambda = s - 1 + t(\alpha - 1).$$

2. A semipartial geometry with $\alpha = 1$ is called a *partial quadrangle* and was introduced by Cameron [1974] as a generalization of the generalized quadrangles. Semipartial geometries generalize at the same time the partial quadrangles and the partial geometries. It is immediately clear that a semipartial geometry is a partial geometry if and only if $\mu = (t+1)\alpha$. If we want to exclude the partial geometries we will speak about *proper* semipartial geometries. In any case, for the rest of this section we will suppose, unless the contrary is stated, that S is not a 2-design, hence that $\alpha \leqslant \min(t+1, s)$.

3. The dual of a semipartial geometry again is a semipartial geometry if and only if either $s = t$ or S is a partial geometry (see Debroey and Thas [1978a]).

Using the fact that the point graph is strongly regular, and using other counting arguments, one can deduce a lot of conditions between the parameters of a semipartial geometry. We will give a summary of the most important ones in the next theorem.

THEOREM 10. *Let $S = (P, B, \mathrm{I})$ be a proper semipartial geometry with parameters s, t, α, μ, then*

(1) $t \geqslant s$, *hence* $|B| = b = \dfrac{v(t+1)}{s+1} \geqslant v$;

(2) $D = (t(\alpha-1) + s - 1 - \mu)^2 + 4((t+1)s - \mu)$ *is either a square or equals* 5 (*then* S *is isomorphic to the pentagon*) *and*

$$\frac{2(t+1)s + (v-1)(t(\alpha-1) + s - 1 - \mu + \sqrt{D})}{2\sqrt{D}}$$

is an integer;

(3) $\alpha^2 \leqslant \mu \leqslant (t+1)\alpha$ *and* $\alpha \mid \mu$;

(4) $\mu \mid (t+1)st(s+1-\alpha)$;

(5) $\alpha \mid ts(t+1)$ *and* $\alpha \mid ts(s+1)$;

(6) $\alpha^2 \mid \mu st$;

(7) $\alpha^2 \mid t((t+1)\alpha - \mu)$;

(8) $2 \mid v(t+1)s$;

(9) $3 \mid v(t+1)s(s-1)$ *and* $3 \mid v(t+1)st(\alpha-1)$;

(10) $8 \mid v(t+1)s(s-1)(s-2)$;

(11) $8 \mid v(t+1)s(t(\alpha-1)((t-1)(\alpha-1) - (\alpha-2)) + t(s+1-\alpha)(\mu - 2\alpha + 1))$.

REMARK. The Krein inequalities for strongly regular graphs also yield some extra conditions, but these are rather complicated formulae.

2.2. *A first list of examples of proper semipartial geometries*

2.2.1. *The thin partial quadrangles*
Let Γ be a strongly regular graph with $\lambda = 0$. Then this graph is a partial quadrangle with $s = 1$ and $t = k-1$, hence a thin geometry. Up to now the only known examples of such graphs are the pentagon Pn(5), the Petersen graph Pe(10), the Clebsch graph Cl(16), the Hoffman–Singleton graph HoS(50), and the graphs from the Higman–Sims family (i.e. Gew(56), HS(77) and HS(100)). The parameter sets (v, k, μ) for these graphs are, resp., equal to $(5,2,1)$, $(10,3,1)$, $(16,5,2)$, $(50,7,1)$, $(56,10,2)$, $(77,16,4)$, $(100,22,6)$. All these graphs are uniquely defined by their parameters.

2.2.2. *The semipartial geometries* $\overline{M(k)}$, $k \in \{2,3,7,57\}$
The three thin partial quadrangles with $\mu = 1$ are better known as *Moore graphs*. These graphs are the graphs with valency $k > 1$, girth 5 (i.e. they have no 3-cycles nor 4-cycles but they do have 5-cycles) and with the minimum number of vertices, which is $k^2 + 1$. It is known that necessarily $k \in \{2,3,7,57\}$. However a Moore graph with $k = 57$ is not known to exist.

With each Moore graph Γ there is associated another semipartial geometry, which we will denote by $\overline{M(k)}$. The point set P is the set of vertices Γ, the line set B is the set $\{\Gamma(x)\colon\ x \in P\}$, with $\Gamma(x)$ the set of vertices adjacent to x, I is the natural incidence relation. Then $\overline{M(k)} = (P, B, \mathrm{I})$ is a semipartial geometry with parameters $s = t = \alpha = k - 1$, $\mu = (k-1)^2$ (Debroey and Thas [1978a]).

2.2.3. *The semipartial geometries* $U_{2,3}(n)$
Let U be a set of cardinality n. Let P be the set of pairs, let B be the set of unordered triples of U, and let I be the inclusion relation. Then $U_{2,3}(n) = (P, B, \mathrm{I})$ is a semipartial geometry with parameters $s = \alpha = 2$, $t = n - 3$, $\mu = 4$ (Debroey and Thas [1978a]). The point graph of this geometry is the triangular graph $T(n)$.

2.2.4. *The semipartial geometries* $\mathrm{LP}(n, q)$
Define P as the set of lines of $\mathrm{PG}(n,q)$ $(n \geqslant 4)$, B as the set of planes of $\mathrm{PG}(n,q)$, and I as the inclusion relation. Then (P, B, I) is a semipartial geometry with parameters

$$s = q(q+1),\ t = \frac{q^{n-1}-1}{q-1} - 1,\ \alpha = q+1,\ \mu = (q+1)^2$$

(Debroey and Thas [1978a]). Remark that for $n = 3$ this construction yields the dual design of lines and planes of $\mathrm{PG}(3,q)$.

2.2.5. *The semipartial geometries* $\overline{W(2n+1,q)}$
Let σ be a symplectic polarity of $\mathrm{PG}(2n+1,q)$, $n \geqslant 1$. Let P be the point set of $\mathrm{PG}(2n+1,q)$, B the set of lines which are not totally isotropic (i.e. hyperbolic) with respect to σ, and I the incidence relation of $\mathrm{PG}(2n+1,q)$. Then $\overline{W(2n+1,q)} = (P, B, \mathrm{I})$ is a semipartial geometry with parameters

$$s = q,\ t = q^{2n} - 1,\ \alpha = q,\ \mu = q^{2n}(q-1)$$

(Debroey and Thas [1978a]).

2.2.6. *The semipartial geometries* $\mathrm{NQ}^{\pm}(2n-1,2)$

Let Q be a (nonsingular) hyperquadric in $\mathrm{PG}(2n-1,2)$. Let P be the set of points off the quadric, let B be the set of nonintersecting lines of Q, and let I be the incidence of $\mathrm{PG}(2n-1,2)$. Then (P,B,I) is a semipartial geometry with parameters

$$s=\alpha=2,\ t=2^{2n-3}-\varepsilon 2^{n-2}-1,\ \mu=2^{2n-3}-\varepsilon 2^{n-1},$$

where $\varepsilon=+1$ for the hyperbolic quadric and $\varepsilon=-1$ for the elliptic quadric (we will denote these geometries by $\mathrm{NQ}^{+}(2n-1,2)$ and $\mathrm{NQ}^{-}(2n-1,2)$, respectively). This was first remarked by H. Wilbrink (private communication).

2.2.7. *The semipartial geometries* $H_q^{(n+1)*}$

This semipartial geometry is defined by taking as point set P the set of lines of a projective space $\Sigma\cong\mathrm{PG}(n+1,q)$ skew to a fixed projective space $H\cong\mathrm{PG}(n-1,q)$ and as line set B the set of the planes of Σ which intersect H in exactly one point. This semipartial geometry has parameters

$$s=q^2-1,\ t=\frac{q^n-1}{q-1}-1,\ \alpha=q,\ \mu=q(q+1).$$

REMARK. It is known that a (semi)partial geometry S satisfying the diagonal axiom, gives rise to a semipartial geometry $\overline{S}$ satisfying the diagonal axiom (see De Clerck and Thas [1978] and Debroey [1979]). Indeed, let x and y be two collinear points. We denote by $D^1_{x,y}$ the set of points collinear with x and y but not on the line $L_{x,y}$ joining x and y and by $D^2_{x,y}$ the set of points of $L_{x,y}$ which are collinear with a point (hence with all points) of $D^1_{x,y}$. Then $D_{x,y}=D^1_{x,y}\cup D^2_{x,y}$ is a maximal set of pairwise collinear points, any such a set $D_{x,y}$ is called a *diagonal clique*. The incidence structure $\overline{S}$ with the same point set as S and with line set, the set of diagonal cliques of S is a semipartial geometry with parameters

$$\bar{t}=s/(\alpha-1)-1,\ \bar{s}=(t+1)(\alpha-1),\ \bar{\alpha}=\alpha,\ \bar{\mu}=\mu$$

which satisfies the diagonal axiom. Note that $\overline{S}$ has the same point graph as S and that $\overline{\overline{S}}\cong S$. In this way the dual of H_q^{n+1} is related to the semipartial geometry $H_q^{(n+1)*}$.

2.3. *The linear representations of semipartial geometries*

If $T_n^*(\mathcal{K})$ is a linear representation of a semipartial geometry, then one easily proves that $\mathcal{K}$ has to be a set of points in $\mathrm{PG}(n,q)$ such that each line of the projective space is either a passant, a tangent or an $(\alpha+1)$-secant and such that, because of the μ-condition, each point of $\mathrm{PG}(n,q)\setminus\mathcal{K}$ is on $\mu(\alpha(\alpha+1))^{-1}$ $(\alpha+1)$-secants. The following examples are known.

2.3.1. *Linear representations of proper partial quadrangles*
In this case the set $\mathcal{K}$ is a $(t+1)$-cap with the property that each point not in $\mathcal{K}$ is on $t+1-\mu$ tangents. Calderbank [1982] has given an almost complete classification of partial quadrangles with a linear representation. His proof is a number-theoretic proof. He lists the possible parameter values of the associated strongly regular graph.

The following cases occur.

1. $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid of the projective space $\mathrm{PG}(3,q)$. It is a partial quadrangle with parameters $s=q-1$, $t=q^2$, $\mu=q(q-1)$ and was first constructed by Cameron [1974].

2. Suppose $q=3$ and assume that $\mathcal{K}$ is not an ovoid. Then $\mathcal{K}$ is either an 11-cap in $\mathrm{PG}(4,3)$, see, e.g., Coxeter [1958] and Pellegrino [1974] for a description, the partial quadrangle $T_4^*(\mathcal{K})$ has parameters $s=2$, $t=10$, $\mu=2$, or $\mathcal{K}$ is the unique 56-cap in $\mathrm{PG}(5,3)$ in which case the partial quadrangle has parameters $s=2$, $t=55$, $\mu=20$. This 56-cap was first constructed by Segre [1965] but was also studied by several other authors, e.g., by Berlekamp, Van Lint and Seidel [1973], Bruen and Hirschfeld [1978], Hill [1973], McLaughlin [1969], and Thas [1981a].

3. Suppose $q=4$. Then either $\mathcal{K}$ is an ovoid in $\mathrm{PG}(3,4)$ or it is a 78-cap in $\mathrm{PG}(5,4)$ such that each external point is on 7 secants, or a 430-cap in $\mathrm{PG}(6,4)$ such that each external point is on 55 secants. If $\mathcal{K}$ is a 78-cap, the partial quadrangle $T_5^*(\mathcal{K})$ has parameters $s=3$, $t=77$, $\mu=14$. At least one example exists and was discovered by Hill [1976]. If $\mathcal{K}$ is a 430-cap then the partial quadrangle has parameters $s=3$, $t=429$, $\mu=110$. Up to now however, the existence of such a cap is not known.

4. Suppose $q\geqslant 5$. Then it was proved by Tzanakis and Wolfskill [1987] that the partial quadrangle has to be $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid.

REMARKS.

1. If $T_n^*(\mathcal{K})$ is a linear representation of a partial quadrangle with $q=2$, then the partial quadrangle coincides with its point graph, and Calderbank [1982] proved that there is only one solution, the strongly regular graph $\mathrm{srg}(v=16,k=5,\lambda=0,\mu=2)$ which is the Clebsch graph Cl(16) and is of type $T_3^*(\mathcal{O})$ with $\mathcal{O}$ the elliptic quadric in $\mathrm{PG}(3,2)$.

2. The existence of a cap $\mathcal{K}$ in $\mathrm{PG}(n,q)$, such that every exterior point is on a constant number of tangents, implies the existence of a uniformly packed $[|\mathcal{K}|,|\mathcal{K}|-n-1,4]$ code C, which means that the dual code $C^{\perp}$ is a $[|\mathcal{K}|,n+1]$ code over $\mathrm{GF}(q)$ with exactly 2 weights (see Calderbank [1982] and Calderbank and Kantor [1986]).

2.3.2. *Linear representations of proper semipartial geometries with $\alpha>1$*
In this case the following models are known.

1. The set $\mathcal{K}$ is a unital $\mathcal{U}$ in the projective plane $\Pi_\infty=\mathrm{PG}(2,q^2)$ at infinity, and $T_2^*(\mathcal{U})$ has parameters $s=q^2-1$, $t=q^3$, $\alpha=q$, $\mu=q^2(q^2-1)$.

2. If $\mathcal{K}$ is a Baer subspace $\mathcal{B}$ of the projective space $\Pi_\infty=\mathrm{PG}(n,q^2)$ at infinity, then $T_n^*(\mathcal{B})$ has parameters

$$s=q^2-1,\ t=\frac{q^{n+1}-1}{q-1}-1,\ \alpha=q,\ \mu=q(q+1).$$

Note that this geometry is isomorphic to $H_q^{(n+2)*}$.

2.4. *Semipartial geometries and generalized quadrangles*

It is known that if $S = (P, B, \mathrm{I})$ is a generalized quadrangle, then one can construct in the following way a (0, 1)-geometry. Let p be any point of S, let $p^\perp$ be the set of all points of S collinear with p (the trace of p) and let $B(p)$ be the set of lines of S through p, then the incidence structure $S_p = (P_p, B_p, \mathrm{I}_p)$ with $P_p = P \setminus p^\perp$, $B_p = B \setminus B(p)$, and with $\mathrm{I}_p = \mathrm{I} \cap (P_p \times B_p)$ is clearly a (0, 1)-geometry of order $(s-1, t)$. Moreover S_p satisfies the following property.

(∗) If L and M are two disjoint lines of S_p then there are either 0, $s-1$, or s lines of S_p concurrent to both L and M.

Note that this property of course is trivial in the case $s = 2$. The point graph $\Gamma(S_p)$ of S_p will be a strongly regular graph with parameter μ if and only if for any 2 noncollinear points x and y in P_p, the set $\{p, x, y\}^\perp$ of points in S collinear with p, x and y has cardinality $t+1-\mu$. It is known (see Bose and Shrikhande [1972] and Cameron [1974]) that in a generalized quadrangle S, $|\{x, y, z\}^\perp|$ is a constant for any triad $\{x, y, z\}$ of noncollinear points, if and only if S has order (s, s^2), moreover in this case $|\{x, y, z\}^\perp| = s+1$. Hence the only partial quadrangles of type S_p have parameters $(s-1, s^2, s(s-1))$.

There are a lot of generalized quadrangles of order (s, s^2) known. In all of them s is a prime power q and we will therefore in the sequel use q instead of s.

First of all there is the semiclassical example $T_3(\mathcal{O})$, constructed by Tits [1959], see Chapter 9. If p is the special point ∞ in $T_3(\mathcal{O})$ then the resulting partial quadrangle has a linear representation in AG(4, q); it is the partial quadrangle $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid in the hyperplane Π_∞. If p is any other point of $T_3(\mathcal{O})$ then the resulting partial quadrangle might be nonisomorphic to $T_3^*(\mathcal{O})$. On the other hand, any flock of a cone in PG(3, q) implies the existence of a generalized quadrangle of order (q, q^2) (see Chapter 9) and these generalized quadrangles give rise to a lot of nonisomorphic partial quadrangles with parameters $(q-1, q^2, q(q-1))$.

Ivanov and Shpectorov [1991] prove that every partial quadrangle with parameters $(q-1, q^2, q(q-1))$ is of type S_p and is uniquely extendible to a generalized quadrangle S. For this they prove that such a partial quadrangle always satisfies property (∗). In fact they even prove a more general result: every strongly regular graph

$$\mathrm{srg}\left(q^4, (q^2+1)(q-1), q-2, q(q-1)\right),$$

such that every 2 adjacent vertices are contained in a clique of order q, is the point graph of a partial quadrangle of type S_p, and this partial quadrangle is uniquely extendible to a generalized quadrangle of order (q, q^2). Remark that this implies that the srg(81, 20, 1, 6) is unique; see, e.g., Brouwer and Haemers [1992] where another proof of the result of Ivanov and Shpectorov is given.

Anyhow, it follows from the theorem of Calderbank that the linear representation of a partial quadrangle with parameters $(q-1, q^2, q(q-1))$ should be in AG(4, q), hence it should be $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an ovoid in Π_∞.

In De Clerck and Van Maldeghem [1994] the following theorem is proved.

THEOREM 11. *Let $T_n^*(\mathcal{K})$ $(n \geqslant 3)$ be a linear representation of a $(0,1)$-geometry, of order $(q-1,t)$, $q > 2$, that satisfies $(*)$. If $\mathcal{K}$ spans the hyperplane Π_∞, then $T_n^*(\mathcal{K})$ is the partial quadrangle $T_3^*(\mathcal{O})$.*

REMARK. It is clear that the $(0,1)$-geometries of order $(q-1,t)$ of type $T_1^*(\mathcal{K})$ are the grids of order $q-1$, i.e. the generalized quadrangles of order $(q-1,1)$. If S is a $(0,1)$-geometry of type $T_2^*(\mathcal{K})$, then $\mathcal{K}$ is a set of points in the plane Π_∞ such that every line intersects in 0,1 or 2 points, i.e. $\mathcal{K}$ is an arc in Π_∞. $T_2^*(\mathcal{K})$ satisfies $(*)$ if and only if $|\mathcal{K}|$ is $q+1$ or $q+2$.

There is another way to construct semipartial geometries from generalized quadrangles. Suppose that $\overline{S}$ is a generalized quadrangle embedded in a projective space $\mathrm{PG}(n,q)$, hence $\overline{S}$ is classical. Suppose that p is a point of $\mathrm{PG}(n,q)$ and that Π is a hyperplane of $\mathrm{PG}(n,q)$ not containing p. Let $\overline{P}_1$ be the projection of the point set of $\overline{S}$ from p on Π and let $\overline{P}_2$ be the set of points of Π on a tangent through p at $\overline{S}$. Let S be the geometry with point set P the set $\overline{P}_1 \setminus \overline{P}_2$, whereas the line set B is the set of lines of Π which intersect $\overline{P}_1$ in at least two points. The incidence is the one of the projective space. It turns out that if $\overline{S}$ is $Q^-(5,q)$ or $H(4,q^2)$ one gets semipartial geometries. Of course it depends on whether p is a point on $\overline{S}$ or not.

If $\overline{S} = Q^-(5,q)$ and p is a point on the quadric, the semipartial geometry S is $T_3^*(\mathcal{O})$, with $\mathcal{O}$ the elliptic quadric in $\mathrm{PG}(3,q)$. However if p is not on the quadric, it yields a semipartial geometry with parameters

$$s = q-1,\ t = q^2,\ \alpha = 2,\ \mu = 2q(q-1).$$

This construction is due to Hirschfeld and Thas [1980]. Another construction of this partial geometry was given by R. Metz (private communication). Let Q be a nonsingular hyperquadric of the projective space $\mathrm{PG}(4,q)$. If we define P as the set of 2-dimensional elliptic quadrics on Q, B as the set of bundles of such elliptic quadrics which are tangent to each other in a common point, and I as the natural incidence relation, then $S = (P, B, \mathrm{I})$ is isomorphic to the one we just described.

If $\overline{S} = H(4,q^2)$ and p is a point on $H(4,q^2)$, the semipartial geometry S is $T_2^*(\mathcal{U})$, with $\mathcal{U}$ the Hermitian unital in $\mathrm{PG}(2,q^2)$. However if p is not on $H(4,q^2)$, it yields a semipartial geometry with parameters

$$s = q^2-1,\ t = q^3,\ \alpha = q+1,\ \mu = q(q+1)(q^2-1).$$

This example is due to J.A. Thas (private communication).

2.5. *Semipartial geometries and SPG reguli*

In Thas [1983] a new construction method for semipartial geometries is introduced. We will give here a brief description of this construction but refer to Thas [1983] for the proofs and more details.

An SPG *regulus* is a set R of m-dimensional subspaces $\mathrm{PG}^{(1)}(m,q), \ldots, \mathrm{PG}^{(r)}(m,q)$ of $\mathrm{PG}(n,q)$, satisfying:

(1) $\mathrm{PG}^{(i)}(m,q) \cap \mathrm{PG}^{(j)}(m,q) = \varnothing$ for all $i \neq j$.

(2) If $\mathrm{PG}(m+1,q)$ contains $\mathrm{PG}^{(i)}(m,q)$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $R \setminus \{\mathrm{PG}^{(i)}(m,q)\}$. If this $\mathrm{PG}(m+1,q)$ has no point in common with $\mathrm{PG}^{(j)}(m,q)$ for all $j \neq i$, then it is called a tangent $(m+1)$-space of R at $\mathrm{PG}^{(i)}(m,q)$.

(3) If the point x of $\mathrm{PG}(n,q)$ is not contained in an element of R, then it is contained in a constant number θ ($\theta \geqslant 0$) of tangent $(m+1)$-spaces of R.

By considering all the $(m+1)$-dimensional spaces through $\mathrm{PG}^{(i)}(m,q)$ we obtain that $\alpha(q-1)$ has to divide $(r-1)(q^{m+1}-1)$, and we see that the number of tangent $(m+1)$-spaces of R at $\mathrm{PG}^{(i)}(m,q)$ equals

$$\frac{q^{n-m}-1}{q-1} - \frac{r-1}{\alpha} \cdot \frac{q^{m+1}-1}{q-1}.$$

By counting the number of ordered pairs (M,x), with M a tangent $(m+1)$-space $\mathrm{PG}(m+1,q)$ of R, and x a point of M which is not in an element of R, we obtain:

$$\theta = \frac{\big(\alpha(q^{n-m}-1) - (r-1)(q^{m+1}-1)\big) r q^{m+1}}{\alpha\big((q^{n+1}-1) - r(q^{m+1}-1)\big)}.$$

Note that by $r > 1$ and by the first condition in the definition of R we have $n \geqslant 2m+1$. If $n = 2m+1$, then there are no tangent $(m+1)$-spaces, and $\alpha = r-1$. If $n = 2m+2$, then any two tangent $(m+1)$-spaces at distinct elements of R intersect.

Given an SPG regulus R, with $r > 1$, one can construct a semipartial geometry $S = (P, B, \mathrm{I})$ as follows. Embed $\mathrm{PG}(n,q)$ as a hyperplane in $\mathrm{PG}(n+1,q)$. The points of S are the points in $\mathrm{PG}(n+1,q) \setminus \mathrm{PG}(n,q)$. The lines of S are the $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1,q)$ which contain an element of R but are not contained in $\mathrm{PG}(n,q)$. Incidence is that of $\mathrm{PG}(n+1,q)$. Then S is a semipartial geometry with parameters

$$s = q^{m+1}-1,\ t = r-1,\ \alpha = \alpha,\ \mu = (r-\theta)\alpha,$$

see Thas [1983].

REMARKS.

1. This geometry is a partial geometry if and only if $\theta = 0$, hence if $\theta \neq 0$, which implies that $r \geqslant q^{m+1}$, then S is a proper semipartial geometry.
2. If $n = 2m+1$, then S is a net of order $s+1 = q^{m+1}$ and degree $t+1 = r$.

SPG reguli and polar spaces

A spread R of the nonsingular elliptic quadric $Q^-(2m+3,q)$ ($m \geqslant 0$) contains $q^{m+2}+1$ elements (of dimension m) and is always an SPG regulus. The parameters of the corresponding semipartial geometry are

$$s = q^{m+1}-1,\ t = q^{m+2},\ \alpha = q^m,\ \mu = q^{m+1}(q^{m+1}-1).$$

For $m = 0$, this is the partial quadrangle $T_3^*(\mathcal{O})$. For $m = 1$, the semipartial geometry has parameters

$$s = q^2-1,\ t = q^3,\ \alpha = q,\ \mu = q^2(q^2-1)$$

which also are the parameters of the semipartial geometry $T_2^\star(\mathcal{U})$. Indeed $T_2^\star(\mathcal{U})$ is isomorphic to the semipartial geometry arising from a regular spread R (see Chapter 7) of $Q^-(5,q)$. However if the spread is nonregular, then the associated semipartial geometry is not isomorphic to $T_2^\star(\mathcal{U})$. If $m > 1$, and q is even, then the quadric $Q^-(2m+3,q)$ has spreads, hence this yields new semipartial geometries. If q is odd, no spread of the quadric $Q^-(2m+3,q)$ $(m > 1)$ is known.

If the nonsingular quadric $Q(2m+2,q)$ (of PG$(2m+2,q)$), $m \geqslant 0$, has a spread R, then it is not an SPG regulus.

If R is a spread of the quadric $Q^+(2m+1,q)$, $m \geqslant 1$, then necessarily m is odd, moreover this spread is an SPG regulus, but the associated semipartial geometry is a net.

Let $H(n,q^2)$ be a nonsingular Hermitian variety of PG(n,q^2), $n \geqslant 2$. If n is odd, the Hermitian variety has no spread (see Bruen and Thas [1976] for the case $n = 3$ and Thas [1989] for $n \geqslant 5$). Assume that n is even. Then R is always an SPG regulus with $m = (n/2) - 1$ and $|R| = q^{n+1} + 1$. Hence there corresponds a semipartial geometry S with parameters

$$s = q^n - 1,\ t = q^{n+1},\ \alpha = q^{n-1},\ \mu = q^n(q^n - 1).$$

However if $n = 2$ then this semipartial geometry is $T_2^\star(\mathcal{U})$. Unfortunately for $n > 2$ no spread of $H(n,q^2)$, n even, is known. Brouwer (private communication, 1981) proved that $H(4,4)$ has no spread. For more details on spreads of polar spaces, we refer to Chapter 7.

2.6. *Some characterization theorems for semipartial geometries*

THEOREM 12 (Debroey and Thas [1978a]). *If S is a proper semipartial geometry with $\alpha = t$, then $S \cong \overline{M(t+1)}$, hence $s = t$, $\mu = \alpha^2$ and $t \in \{1,2,6,56\}$.*

THEOREM 13 (Debroey [1979], Wilbrink and Brouwer [1984]). *Let S be a proper semipartial geometry with $\mu = \alpha^2$.*

(1) *If $\alpha = 2$, then $S \cong U_{2,3}(n)$.*
(2) *If $2 < \alpha = s$ then $\alpha = t \in \{1,2,6,56\}$ and $S \cong \overline{M(t+1)}$.*
(3) *If $2 < \alpha < s$ then $S \cong$ LP(n,q).*

THEOREM 14 (Debroey [1979]). *Let S be a proper semipartial geometry with parameters s, t, α (> 1) and $\mu = \alpha(\alpha+1)$. If S satisfies the diagonal axiom $(\mathcal{D})$, then S is isomorphic to a semipartial geometry of type $H_q^{(n+1)\star}$.*

REMARK. In Wilbrink and Brouwer [1984] it is proved that all proper semipartial geometries with $\mu = \alpha^2$ and $2 \leqslant \alpha < s$ satisfy the diagonal axiom. Moreover, they proved that up to possibly a finite number of exceptions, all proper semipartial geometries with $\mu = \alpha(\alpha+1)$ satisfy the diagonal axiom. Cuypers [1992] observed that by adding some extra combinatorial conditions, their proofs can even be generalized to $(0,\alpha)$-geometries.

THEOREM 15 (Cuypers [1992]). *Let S be a finite $(0,\alpha)$-geometry with $\alpha \notin \{1,3\}$. Suppose that S satisfies the following conditions.*

(1) *If $\alpha \neq 2$ then $s > f(\alpha)$ where $f(4) = 12$, $f(5) = 6$, $f(6) = f(7) = 17$, $f(8) = 18$, $f(9) = 19$, $f(10) = 21$, $f(11) = 23$ and $f(\alpha) = 2\alpha$ for $\alpha \geqslant 12$;*

(2) $t \geqslant \max(s+1, \alpha(\alpha+1))$;

(3) *Two noncollinear points have either 0, α^2 or $\alpha(\alpha+1)$ common neighbours, and the last two cases both do occur;*

(4) *Let (x, L) be an antiflag, such that $\alpha(x, L) = \alpha$. Then for every two points y and z on $L \setminus x^\perp$, $|x^\perp \cap y^\perp| = |x^\perp \cap z^\perp|$.*

Then $\alpha = 2$ and S satisfies the diagonal axiom.

On the embedding of semipartial geometries in projective and affine spaces the following results are known.

THEOREM 16. *If S is a proper semipartial geometry with parameters s, t, α (> 1), μ, embedded in* PG(n, s), *$n \geqslant 3$ and $s > 2$, but not in* PG(n', s), *$n' < n$, then n is odd and S is the semipartial geometry $\overline{W(n, s)}$.*

REMARK. This theorem was proved by Debroey and Thas [1978b] for the case $n = 3$ and by Thas et al. [1984] for $n > 3$. If S is any semipartial geometry with $\alpha = s = 2$, then S is a cotriangle space and those are classified (see Theorem 19). A complete classification of the embedded cotriangle spaces exists for $n = 3$ (Debroey and Thas [1978b]) and for $n = 4$ (Thas et al. [1984]). In Lefèvre-Percsy [1983] an embedding of $U_{2,3}(n+2)$ in PG(n,2) is given. The lines of this geometry are hyperbolic lines, i.e. lines which are not totally isotropic, for some symplectic polarity. Also an embedding of $U_{2,3}(n+3)$ in PG(n,2) is described. The lines of this geometry are hyperbolic for some symplectic polarity if and only if n is odd. The problem of determining all embeddings of $U_{2,3}(n)$ in PG$(d, 2)$ is equivalent to determining (up to equivalence) all binary codes of length n with all weights even and minimum weight greater than 4, see Hall [1983].

THEOREM 17 (De Clerck and Thas [1983]). *If S^D is the dual of a semipartial geometry S with $\alpha > 1$, and if S^D is embedded in a projective space* PG(n, s), *$n \geqslant 3$, but not in* PG(n', s), *$n' < n$, then $n = 3$ and S^D is the design of points and lines in* PG$(3, q)$, *or $S^D = H_s^3$ or $S^D =$* NQ$^-(3, 2)$ (see 2.2.6).

OPEN QUESTION. Let $\mathcal{H}$ be a nonsingular Hermitian variety in PG$(3, q^2)$. The incidence structure $S = (P, B, \mathrm{I})$, defined by taking as point set P the point set of $\mathcal{H}$ and as line set B the set of lines of $\mathcal{H}$ minus all the lines concurrent with a given line L, is a dual partial quadrangle embedded in PG$(3, q^2)$. One can prove that the dual of this geometry is isomorphic to $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an elliptic quadric. It is not known whether this is the only proper dual partial quadrangle embedded in a projective space.

A complete classification of all proper semipartial geometries embedded in affine spaces is still open. However the problem is solved by Debroey and Thas [1977] for dimensions 2 and 3.

THEOREM 18. *A proper semipartial geometry S with parameters s, t, α, μ is not embeddable in an affine plane* AG$(2, s+1)$. *If S is embedded in* AG$(3, s+1)$, *then S is either the pentagon embedded in* AG$(3, 2)$ (*trivial case*) *or a linear representation and $S = T_2^\star(\mathcal{U})$ or $S = T_2^\star(\mathcal{B})$ (hence $s+1$ is a square).*

3. Copolar spaces

A *copolar space* (see Hall [1982]) is a partial linear space $S = (P, B, \mathrm{I})$ such that for each antiflag (x, L), the incidence number $\alpha(x, L)$ equals 0 or $|L| - 1$.

A partial linear space with this property has been called a *proper Δ-space* by Higman [1979]. He observed that the above property is more or less the converse of the defining property of a polar space. This is the reason why J.I. Hall calls it a copolar space.

It is easily seen that the copolar spaces of order (1,t) are precisely those graphs which contain no triangles. A copolar space of order (2,t) is better known as a *cotriangle space.*

The copolar space S is called *indecomposable* if and only if S is not the union of two or more copolar spaces on disjoint point sets. A *reduced* copolar space is an indecomposable copolar space such that for all vertices x and y in the point graph $\Gamma(S)$, $\Gamma(x) = \Gamma(y)$ implies $x = y$.

Remark that a semipartial geometry with parameters s, t, $\alpha = s$ is indeed a copolar space of order (s, t). Of course the dual of a net is a copolar space, and since there is no hope to classify them, we assume from now on that there exists at least one antiflag (x, L) such that $\alpha(x, L) = 0$.

In Hall [1982] the finite reduced copolar spaces of order (s, t), $s \geqslant 2$, are classified up to isomorphism. It turns out that the reduced copolar space of order (s, t) is a (proper) semipartial geometry.

We summarize the results in the next theorem.

THEOREM 19 (Hall [1982]). *If $S = (P, B, \mathrm{I})$, is a finite reduced copolar space of order (s, t) $s \geqslant 2$, then S is isomorphic to one of the following semipartial geometries*:

(1) $\overline{M(k)}$, $k \in \{2, 3, 7, 57\}$,

(2) $U_{2,3}(n)$,

(3) $\overline{W(2n+1, q)}$,

(4) NQ$^{\pm}(2n-1, 2)$.

REMARK. The cotriangle spaces were in fact classified by Shult [1975], an earlier version of which was proved by Seidel [1973].

4. Near n-gons

4.1. *Definitions*

A partial linear space S is called a *near n-gon* if and only if the following axioms hold:

(1) n is an even integer and the diameter of the point graph is at most $n/2$,

(2) given any point p and any line L, L contains a unique point nearest to p.

This incidence structure was introduced by Shult and Yanushka [1980] because of their interest in line systems with few angles. See also Shad and Shult [1979].

Any generalized n-gon with n even can easily be seen to be a near n-gon. In the case $n = 4$, the converse is true: any near 4-gon is a generalized quadrangle. This is not true for $n > 4$.

A thin near n-gon (i.e. all lines have size 2) is just a connected bipartite graph. We will not discuss them here (see for instance Shad and Shult [1979] and Shad [1984]). We will assume from now on that the near n-gon is thick. However some of the next theorems still hold (possibly under some extra conditions) if one assumes that there are thin lines.

4.2. *Classical and sporadic near n-gons*

A subset Y of the point set P is called *geodetically closed* if for any two points $y_1, y_2 \in Y$ all the shortest paths between y_1 and y_2 are contained in Y. A *quad* is a geodetically closed subset of P of diameter 2 such that not all its points are adjacent to one fixed point. This quad is a near 4-gon, hence a generalized quadrangle. In Shult and Yanushka [1980] it is proved that for any thick near n-gon, any two points x and y at distance 2 with at least two common neighbours determine a unique quad $Q(x, y)$ containing them. Even more can be said.

THEOREM 20 (Brouwer and Wilbrink [1983a]). *If x and y are two points at distance i of a (thick) near n-gon, then they are contained in a unique geodetically closed sub-$2i$-gon.*

The existence of those geodetically closed sub-$2i$-gons has been very important for the characterization theorems, as we will illustrate by the next theorems.

THEOREM 21 (Shult and Yanushka [1980]). *Let Q be a quad in a thick near n-gon $S = (P, B, \mathrm{I})$. Then for any point x not lying in Q, either*

(1) *x has distance d to exactly one point y in Q, distance $d + 1$ to all points of Q collinear with y, and distance $d + 2$ to all points of Q at distance 2 from y;*

(2) *x has distance d to all points of an ovoid in Q and distance $d + 1$ to all remaining points.*

In the first case the pair (x, Q) is said to be of *classical* type, in the second case the pair (x, Q) is said to be of *ovoidal* type. A near n-gon is called *classical* if all its (nonincident) point-quad pairs are classical, otherwise it is called *sporadic*.

THEOREM 22 (Cameron [1982]). *A classical (thick) near n-gon with quads is a dual polar space (i.e. a partial linear space whose points and lines are respectively the maximal and second-maximal singular subspaces of a polar space of rank $n/2$, reverse containment signifying incidence).*

The embedding problem for near n-gons is settled by the next theorem.

THEOREM 23 (Cameron [1981]). *A (thick) near n-gon which is not a generalized n-gon, embedded in a projective space of order q, is classical and is of type $O_{n+1}(q)$ $(n > 4)$ (i.e. the polar space is the quadric $Q(n,q)$).*

REMARK. For a detailed discussion on polar spaces, and dual polar spaces, we refer to Chapter 12.

4.3. *Regular near n-gons*

A near n-gon will be called *regular* with parameters $(s, t_2, t_3, \ldots, t_n = t)$ if and only if (i) it is a thick near n-gon of order (s,t) and (ii) whenever two points x and y are at distance $d > 1$, exactly $t_d + 1$ lines through y contain points at distance $d-1$ from x. The point graph $\Gamma(S)$ of a regular near n-gon is distance regular. Hence a lot of graph-theoretical results can be used in this case. For more details, we refer to Brouwer and Wilbrink [1983a] and Brouwer et al. [1989].

THEOREM 24 (Shult and Yanushka [1980]). *Let S be a (thick) regular near n-gon and let $\Gamma_d(x)$ be the set of points at distance d from x. Then*

$$|\Gamma_1(x)| = s(t+1),$$

$$|\Gamma_d(x)|\big(s(t-t_d)\big) = |\Gamma_{d+1}(x)|(t_{d+1}+1).$$

Hence

$$|\Gamma_d(x)| = \frac{s^d(1+t)t(t-t_2)(t-t_3)\cdots(t-t_{d-1})}{(1+t_2)\cdots(1+t_d)}, \quad d \geqslant 2.$$

If $t_2 \neq 0$, then $t_d > t_{d-1}$, $d = 3,4,\ldots,n$, each point lies in $t(t+1)(t_2(t_2+1))^{-1}$ quads of order (s,t_2) and each line lies in t/t_2 quads (hence $t(t+1)(t_2(t_2+1))^{-1}$ and t/t_2 are integers). The set of all lines through a point p together with the set of all quads containing p form a 2-$(t+1,\ t_2+1, 1)$-design.

The regular near hexagons (i.e. 6-gons) with $s = 2$, i.e. with parameters $(2,t_2,t)$ are completely classified. By the above theorem, if $t_2 \neq 0$, the near hexagons contain generalized quadrangles of order $(2,t_2)$, and hence (see Chapter 9), $t_2 = 1$, 2, or 4 and in each of the cases the generalized quadrangle is unique up to isomorphism.

We summarize the results in the next theorem.

THEOREM 25 (Shult and Yanushka [1980], Brouwer [1982a]). *If S is a regular near hexagon with $s = 2$ then the following cases occur.*

(1) *S is a generalized quadrangle, hence $t = t_2 = 1, 2$ or 4 (v is equal to $9, 15$ and 27, resp.).*

(2) *S is a generalized hexagon, $t_2 = 0$, and $t = 1, 2$ or 8 (v is equal to $21, 63$ and 819, resp.).*

(3) *S is a proper regular near hexagon and the following cases occur:*

(a) $(t, t_2) = (2, 1)$, $v = 27$.

(b) $(t, t_2) = (11, 1)$, $v = 729$.

(c) $(t, t_2) = (6, 2)$, $v = 135$.

(d) $(t, t_2) = (14, 2)$, $v = 759$.

(e) $(t, t_2) = (20, 4)$, $v = 891$.

REMARKS. These regular near hexagons with 3 points on a line are uniquely defined by their parameters. For the generalized quadrangles and hexagons with $s = 2$, we refer to Chapter 9. The proper regular near hexagons with $(t, t_2) = (6, 2)$, $v = 135$ and $(t, t_2) = (20, 4)$, $v = 891$ both are of classical type, they are the dual of the polar spaces Sp(6, 2) and U(6, 4). Each regular near hexagon with parameters $(s, t_2, t) = (s, 1, 2)$, for instance the one with $v = 27$, is of Hamming type (or a generalized cube): the point set is the set of all ordered triples from a set X, $|X| = s + 1$, the lines are the maximal cliques in the Hamming graph on X, i.e. two triples are collinear if and only if they differ in only one coordinate. More generally one can define in the same way near n-gons of Hamming type by taking the ordered n-tuples from a set X; they are uniquely defined by their parameters (see Shult and Yanushka [1980], Egawa [1981], Brouwer and Wilbrink [1983a]). The near hexagon with 729 points is derived from the extended ternary Golay code, its uniqueness is proved by Brouwer [1982a]. The uniqueness of the near hexagon on 759 points is also proved by Brouwer [1982b]. The points of this near hexagon are the 759 octads (i.e. the blocks of size 8) of the unique Steiner system $S(5, 8, 24)$. Two blocks are called collinear if they are disjoint. Since the complement of the union of two disjoint octads is an octad, a line of the near hexagon will be a set of 3 pairwise disjoint octads. This near hexagon has also a nice combinatorial characterization.

THEOREM 26 (Brouwer and Wilbrink [1983a]). *If a regular near hexagon satisfies $s > 1$, $t_2 > 0$, and $1 + t = (1 + t_2)(1 + st_2)$ then it is the unique regular near hexagon with $s = t = 2$, $v = 759$.*

This near hexagon cannot be a (geodetically closed) sub-hexagon of a regular near n-gon ($n \geqslant 8$), from which it follows that any sporadic regular near n-gon will have $1 + t_3 > (1 + t_2)(1 + st_2)$. For the regular near octagons one might even conjecture that they almost never exist (see Brouwer and Wilbrink [1983a] and Brouwer et al. [1989]). A sporadic regular near octagon with parameters $(s, t_2, t_3, t) = (2, 0, 3, 4)$ is constructed by Cohen [1981a]. The Hall–Janko group J_2 acts on this geometry, the point graph is a distance regular graph, which is uniquely defined by its parameters (see Cohen and Tits [1985]). This near octagon contains no quads but does contain generalized hexagons of type $G_2(2)$. It has 315 points and 525 lines. It has generalized octagons of order (2,1) and generalized hexagons of order (2,1) and (2,2) as subgeometries. It is itself a subgeometry of the dual of the classical generalized hexagon of order (4,4) arising from $G_2(4)$ (see Chapter 9).

REMARKS. Regular near hexagons with 4 or 5 points on each line and regular near octagons with 3 points on each line are discussed in Shad and Shult [1979]. For more constructions of regular near n-gons and their distance regular graphs, we refer to Brouwer et al. [1989].

We finally remark that nonregular near n-gons do exist. For instance, consider the graph $T(2n)^C$; the maximal cliques in this graph have n vertices. If one takes as points these maximal cliques and as lines the cliques with $n-2$ vertices, then the geometry with respect to the natural incidence is a near $2(n-1)$-gon of order $(2, n(n-1)/2-1)$ and is nonregular for $n \geqslant 4$ (see Brouwer and Wilbrink [1983b]).

In Theorem 25 the classification of the regular near-hexagons with line size 3 is given. However the nonregular hexagons with line size 3 having quads are also completely classified.

THEOREM 27 (Brouwer, Cohen and Wilbrink [1983], Brouwer [1985]). *Let S be a nonregular near hexagon with lines of size 3 having quads, then the following cases occur, and in each case the near hexagon is unique for given v, t, t_2.*

(1) $(v, t, t_2) = (45, 3, 1$ *or* $2)$,
(2) $(v, t, t_2) = (81, 5, 1$ *or* $4)$,
(3) $(v, t, t_2) = (105, 5, 1$ *or* $2)$,
(4) $(v, t, t_2) = (243, 8, 1$ *or* $4)$,
(5) $(v, t, t_2) = (405, 11, 1$ *or* 2 *or* $4)$,
(6) $(v, t, t_2) = (567, 14, 2$ *or* $4)$.

REMARK. The classification of the near hexagons with lines of size 3 has been republished by Brouwer, Cohen, Hall and Wilbrink [1994].

5. Moore geometries

5.1. *Moore graphs*

In Section 1 we have introduced the Moore graphs of diameter 2. In fact they can be defined for any diameter d. Indeed, for a regular graph of valency k and diameter d one has the inequality

$$v \leqslant 1 + k + k(k-1) + \cdots + k(k-1)^{d-1}$$

(proved by Moore, see Hoffman and Singleton [1960]), and graphs for which equality holds are called Moore graphs. The girth of a Moore graph is odd and satisfies $g = 2d+1$. Singleton [1968] proved that a connected graph with diameter d and girth $2d+1$ is necessarily regular and moreover is a Moore graph. As we have done in Section 1 for the special case of girth 5, one can also define Moore graphs with respect to a lower bound. Indeed, the number of vertices v of a regular graph of valency k and odd girth g satisfies

$$v \geqslant 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2},$$

and graphs for which the equality hold are the Moore graphs.

Remark that a $(2d+1)$-gon is a Moore graph of diameter d and valency 2. The fact that we restricted ourselves in Section 1 to the diameter 2 case, is not that restrictive, as we can notice in the next theorem.

THEOREM 28 (Damerell [1973]). *A Moore graph with valency $k = 2$ is a polygon. A Moore graph with valency $k \geqslant 3$ has diameter 2 and $k \in \{3, 7, 57\}$.*

As we already remarked in Section 2.2, no example with $k = 57$ is known, and if $k = 3$ the graph is the Petersen graph, whereas for $k = 7$, the graph is the Hoffman–Singleton graph.

5.2. *(Generalized) Moore geometries*

The concept of a Moore graph was generalized by Bose and Dowling [1971], they defined a Moore geometry of diameter d. This was even more generalized by Roos and Van Zanten [1982]: they introduced the concept of generalized Moore geometries.

A generalized Moore geometry of type $\mathrm{GM}_d(s,t,c)$ is a (finite) partial linear space of order (s,t), such that the point graph has diameter d, any two points at distance $i < d$ are joined by a unique shortest path, and any two points at distance d are joined by exactly c shortest paths. In order to exclude various trivial structures, $st > 1$ is assumed. These geometries include as special cases the Moore graphs ($s = c = 1$), the Moore geometries ($c = 1$), and the generalized $2d$-gons ($c = t+1$). Another subfamily, namely, those with $c = s+1$, is proved to exist only for small values of the diameter d, in a series of papers, the last of which by Damerell, Roos and Van Zanten [1989].

THEOREM 29. *A generalized Moore geometry of type $\mathrm{GM}_d(s,t,s+1)$ with $st > 1$, cannot exist for diameter $d > 3$.*

The proof is by using the fact, that if the geometry does exist, then the point graph is distance regular, and the eigenvalues of the adjacency algebra have to be rational.

Known examples of generalized Moore geometries of type $\mathrm{GM}_d(s,t,s+1)$ with $st > 1$ and $d \leqslant 3$ are the Clebsch graph ($d = 2,\ s = 1,\ t = 4$), the Gewirtz graph ($d = 2$, $s = 1, t = 9$), the odd graph O_4 ($d = 3,\ s = 1,\ t = 3$) (the vertices are the 35 unordered triples from a set X of cardinality 7, two triples being adjacent if and only if they are disjoint) and the generalized $2d$-gons, $d = 2$ or 3, with $s = t$.

Also, for other types of generalized Moore geometries it is proved that the diameter of the point graph (which is distance regular) should be small, but a discussion of these theorems would bring us too far. For more information and references we refer to Brouwer et al. [1989].

We only will state the theorem for the case $c = 1$, i.e. the Moore geometries as they were defined by Bose and Dowling [1971]. The proof again is a combination of several papers by Fuglister, Damerell and Georgiacodis, references of which can be found in Brouwer et al. [1989].

THEOREM 30. *A Moore geometry of diameter d is either a $(2d+1)$-gon ($st = 1$) or $d \leqslant 2$.*

Note that if $s = 1$ we have the Moore graphs. Moore geometries of diameter 1 are the Steiner systems $S(2, s+1, v)$, there is no example of a nontrivial Moore geometry of diameter 2. We finally remark that also another generalization of the Moore graphs exists, see Kantor [1977].

6. (g, d_p, d_l)-gons

6.1. *Definitions*

Recall that one of the main motivations for studying geometries is provided by the fact that it gives ways to study groups by their flag-transitive action on geometries. This group action implies a certain regularity in the geometry S, such as the number of points on a line is constant, etc. If we have a group transitive on longer geodesics then we also have more regularity properties. Note that a flag is here considered as a geodesic of length 1.

From now on assume that S is a (g, d_p, d_l)-gon. Denote by Γ the incidence graph of S. Buekenhout and Van Maldeghem [1992] call S a *regular* (g, d_p, d_l)-gon if

$$|\Gamma_i(x) \cap \Gamma_j(y)| = |\Gamma_i(z) \cap \Gamma_j(u)|$$

for all positive integers i, j and all elements x, y, z, u whenever $d(x, y) = d(z, u)$ and x and z are either both points or both lines.

6.2. *Examples*

We first show how all this fits into the geometries of the preceding sections. Afterwards, we will give some other notable examples.

Some classical examples
Probably the most important class of (g, d_p, d_l)-gons is the class of *generalized polygons*, see Chapter 9. A generalized n-gon is an (n, n, n)-gon.

Another large class is the class of *linear spaces*; these are either projective planes or $(3, 3, 4)$-gons. Let us just mention a trivial example: every set P is the set of points of a linear space $L(P)$ by declaring all pairs of points to be the lines. In fact, this is a *circle* geometry in the sense of Buekenhout [1979].

A symmetric 2-(v, k, λ)-design with $1 < \lambda < k$ is a regular $(2, 3, 3)$-gon. If $\lambda = k$ it is a generalized digon. If $\lambda = 1$, then it is a projective plane, hence a regular $(3, 3, 3)$-gon.

In general, every design which is not a linear space, can be regarded as a $(2,3,d)$-gon with $d \in \{3,4\}$. The case $d=3$ corresponds exactly to the symmetric designs.

The diameter of a *partial geometry* S is at most 4 and we have the following possibilities.

(1) S is a regular $(3,3,3)$-gon, that is, a generalized triangle or a projective plane, hence S has parameters (s,s,s) for some positive integer s.

(2) S is a regular $(3,3,4)$-gon or its dual, i.e. S is a regular *proper* linear space or a regular proper dual linear space.

(3) S is a regular $(3,4,4)$-gon. Among these, we have the nets and the dual nets. The other members in this class are the proper partial geometries.

(4) S is a generalized quadrangle.

A partial quadrangle S with parameters s,t,μ is in general a $(4,5,6)$-gon, but if $\mu = t+1$, then we have a generalized quadrangle, hence a $(4,4,4)$-gon; if $\mu = 1$, then we have a $(5,5,6)$ or $(5,5,5)$-gon; if S is also a dual partial quadrangle, then it is a regular $(4,5,5)$-gon.

A *proper* Moore geometry is a $(g,g,g+1)$-gon for $g \geqslant 3$ and g odd. By Theorem 30, $g=3$ or 5. A generalized Moore geometry of type $\mathrm{GM}_d(s,t,c)$ which is not a Moore geometry or a generalized $2d$-gon can be a regular $(2d,2d+1,2d+1)$-gon (if $s=t$) or a regular $(2d,2d+1,2d+2)$-gon (in the other cases).

A *near n-gon* is in general a $(4,n,n)$-gon.

Some more examples

The near hexagon on 759 points (see Theorem 25 and its remarks) provides three examples of (g,d_p,d_l)-gons. The geometry itself is a $(4,6,6)$-gon (as mentioned above). If we take the quads as new lines and remove the old lines, then one obtains a $(3,4,4)$-gon of order $(14,34)$. We can also keep the old lines, remove the points and take as new points the quads. This constitutes a $(3,5,5)$-gon of order $(6,14)$.

The sporadic group J_1 of Janko acts on a regular graph of valency 11 with 266 vertices. Take as points the vertices of this graph. Define the lines to be the pairs of opposite vertices. Then we obtain a $(5,7,8)$-gon of order (1,11).

Consider the Steiner system $S(5,6,12)$. Take as points of a geometry the triads and as lines the *linked threes* (i.e. 4 triads every 2 of which form a hexad); incidence is the natural one. We obtain a (nonregular) $(5,6,6)$-gon of order (3,3) with the group M_{12} acting as an automorphism group. The geometry is self-dual (the outer automorphisms of M_{12} interchange points and lines).

The Hall–Janko group J_2 acts on the sporadic regular near octagon of Cohen–Tits which is a $(6,8,8)$-gon.

The group McL of McLaughlin acts on $U_3(5)$ as a rank 5 group. The induced graph has triangles, but no 4-cliques with one edge removed. If we take as points the vertices of that graph and as lines the triangles, we obtain a (4,6,6)-gon of order (2,125) of 7128 points. A similar construction with Co_3 acting on HS yields a (4,6,6)-gon of order (2,175) consisting of 11178 points (due to Soicher, private communication).

6.3. *Characterizations by automorphisms*

Let G be a (type preserving) automorphism group of the (g, d_p, d_l)-gon S. We shall use the following terminology.

(1) Suppose G acts transitively on the set of pairs (x, y) of points at distance i from each other, for all even positive integers i. We call (S, G) a *point distance transitive* (g, d_p, d_l)-pair, dually a *line distance transitive* (g, d_p, d_l)-pair. If (S, G) is both point distance transitive and line distance transitive, then we call (S, G) a *weakly distance transitive* (g, d_p, d_l)-pair. If G acts transitively on each set of pairs of elements at distance j from each other and having fixed type, for all positive integers j, then (S, G) is called a *distance transitive* (g, d_p, d_l)-pair.

(2) Suppose G acts transitively on each set of geodesics based at some point x of S and ending in a point y at maximal distance, for all points $x \in P$, then we call (S, G) a *point geodesic transitive* (g, d_p, d_l)-pair. Similarly as above we can define *line geodesic transitive* (g, d_p, d_l)-pairs, respectively *weakly geodesic transitive* and *geodesic transitive*.

(3) If G acts transitively on each set of geodesics of length i based at some fixed variety x, for all varieties x, then (S, G) is called a *locally i-arc transitive* (g, d_p, d_l)-pair.

It is easy to see that, if S is a (g, d_p, d_l)-gon and if $2 \leqslant g \leqslant d_p \leqslant d_l \leqslant g + 1$, then each of the above assumptions on G implies that S is regular. Hence from now on we assume that all geometries are regular.

With this terminology, one can classify large classes of geometries with groups acting transitively on sets of relatively long geodesics. The following results are proved in Buekenhout and Van Maldeghem [1992, 1993], using the classification of finite single groups. The symbol q denotes a prime power and we follow the ATLAS (Conway, Curtis, Norton, Parker and Wilson [1985]) for the notation of the groups.

THEOREM 31. *Let (S, G) be a finite geodesic transitive (g, d_p, d_l)-pair, $2 \leqslant g \leqslant d_p \leqslant d_l \leqslant g + 1$; then one of the following holds.*

(1) *S is a generalized polygon related to an irreducible finite adjoint or twisted adjoint Chevalley group or Ree group of type ${}^2\mathsf{F}_4$ and $X_n(q) \leqslant G \leqslant \mathrm{Aut}(X_n(q))$, where $X_n(q)$ is the corresponding Chevalley group, or $G \cong A_6$ and S is the unique generalized quadrangle of order $(2,2)$, or S is the flag complex of a self-dual classical generalized polygon and G is as above extended by a graph automorphism, or S is an ordinary polygon.*

(2) *S can be identified with the Petersen graph* Pe(10), *resp., the Hoffman–Singleton graph* HoS(50); *the lines are the edges of the graph and $G \cong S_5$, resp., $\mathrm{U}_3(5) \trianglelefteq G \leqslant \mathrm{U}_3(5) : 2$. Here, S is considered as a Moore geometry, in particular a $(5,5,6)$-gon of order $(1,2)$, resp., $(1,6)$.*

(3) *S is a $(3,4,4)$-gon. The following cases occur.*

(3.1) *S is a net of order q and degree q obtained from the Desarguesian projective plane $\mathrm{PG}(2,q)$ by deleting a flag (x,l) and all varieties incident with one of x, l, and G contains the stabilizer in $\mathrm{PGL}_3(q)$ of the flag (x,l) in $\mathrm{PG}(2,q)$.*

(3.2) *S is the net $(H_q^{n+1})^D$ of order q^n and degree $q+1$ and G contains a group isomorphic to the semidirect product of an elementary Abelian group q^{2n} with a group isomorphic to*

(a) $(\mathrm{SL}_2(q) \times \mathrm{SL}_n(q))/Z(\mathrm{SL}_2(q) \times \mathrm{SL}_n(q))$ *if* $n > 2$, *or*

(b) $(\mathrm{SL}_2(q) \times \mathrm{GL}_2(q))/Z(\mathrm{SL}_2(q) \times \mathrm{GL}_2(q))$ *if* $n = 2$, *or*

(c) $\mathrm{SL}_2(2) \times A_7$ *if* $(n, q) = (4, 2)$.

(3.3) *S is the dual of* 3.2.

(3.4) *S is a net of order* 16 *and degree* 9 *whose points can be identified with the points of an affine space* AG(8, 2) *and whose lines are the affine* 4*-subspaces whose* 3*-spaces at infinity constitute a* 2*-transitive spread of a hyperbolic quadric in* PG(7, 2); *G contains the full translation group of* AG(8, 2), *and its kernel 'at infinity' is A_9.*

(3.5) *S is the dual of* 3.4.

(4) *S is a* (3, 3, 4)*-gon. Three cases occur.*

(4.1) *S is the linear space consisting of the points and lines of* $\mathrm{PG}(d, q)$, $q \geqslant 3$, *and* $\mathrm{L}_{d+1}(q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_{d+1}(q)$.

(4.2) *S is the Desarguesian affine plane* AG(2, q); *G contains all translations and its kernel 'at infinity' contains* $\mathrm{L}_2(q)$.

(4.3) *G is a group acting* 4*-transitively on the set of points of S and the lines of S can be identified with the pairs of points.*

(5) *S is a* (2, 3, 3)*-gon. Here, S is a symmetric* 2*-design with* $\lambda > 1$ *and four cases occur* (see also Chapter 8).

(5.1) *S can be identified with* $\mathrm{PG}(d, q), d \geqslant 3$, *the blocks are either the hyperplanes or their complements and* $\mathrm{L}_{d+1}(q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_{d+1}(q) : 2$ *or* $G \cong A_7$ *or* S_7 (*if* $(d, q) = (3, 2)$ *and blocks are the hyperplanes*).

(5.2) *S is the* (*unique*) *Paley* (*or Hadamard*) *design on* 11 *points and* $\mathrm{L}_2(11) \trianglelefteq G \leqslant \mathrm{L}_2(11) : 2$.

(5.3) *S is isomorphic to one of Kantor's designs* $S^{\pm}(n)$ *and* $G \geqslant 2^{2n} : \mathrm{Sp}_{2n}(2)$ (see Kantor [1975]).

(5.4) *S is the design with some point set P and with blocks the complements of the singletons* $\{p\}$ *in P. The group G acts* 3*-transitively on P.*

(6) *S is a generalized quadrangle of order* $(1, s)$ *or* $(s, 1)$ *and there is an almost simple* 2*-transitive group* T_O *of degree* $s+1$ *with socle T such that* $T \times T \trianglelefteq G \leqslant T_O \,\mathrm{wr}\, S_2$.

(7) *S is a generalized digon.*

For various subclasses of geometries, they obtain more general results by weakening the hypothesis on G. We start with the class of generalized polygons.

THEOREM 32. *Let S be a thick generalized n-gon,* $n \geqslant 3$, *and suppose G is a group of automorphisms acting point distance transitively on S; then (S, G) is one of the thick examples in Theorem* 31(1) *above and G is the corresponding Chevalley group or its derived group, or S is the unique generalized quadrangle of order* (3, 5) *and G contains a group isomorphic to* $2^6 : 3 : A_6$.

We say that the pair (S,G) has the *Tits property* if G acts transitively on the set of ordered circuits of minimal length (in which case the length is twice the girth). For a definition of the *Moufang property* and the *half Moufang property* for generalized polygons, we refer to Chapter 9. The next result is a corollary to Theorem 31.

THEOREM 33. *Let S be a finite thick generalized polygon and let $G \leqslant \mathrm{Aut}(S)$. Then the following conditions are equivalent.*

(1) *(S,G) has the Tits property.*

(2) *(S,G) has the Moufang property.*

For generalized hexagons and octagons every distance transitive group induces both the Tits and the Moufang property; for point distance transitive groups and for generalized quadrangles there are a few exceptions (namely, the smallest ones).

The following result is also proved by Buekenhout and Van Maldeghem [1993].

THEOREM 34. *Let S be a thick finite generalized hexagon or octagon and G an automorphism group of S. The pair (S,G) is half Moufang if and only if it is point distance transitive or line distance transitive, depending on the type of Moufang roots in S. In particular, half Moufang implies Moufang whenever $(s,t) \neq (2,2)$ (for generalized hexagons) or $(s,t) \neq (2,4),(4,2)$ (for generalized octagons). Also, S is Moufang with respect to some automorphism group if and only if it is half Moufang with respect to some (possibly other) automorphism group. Finally, (S,G) is half Moufang if and only if G is flag-transitive on S.*

Note that for generalized quadrangles this result already was proved in Thas, Payne and Van Maldeghem [1991], without the classification of the finite simple groups, see also Chapter 9.

We now consider partial geometries and partial quadrangles. For proofs, see Buekenhout and Van Maldeghem [1992].

THEOREM 35. *Let S be a proper partial geometry and suppose that $G \leqslant \mathrm{Aut}(S)$ acts weakly distance transitively on S. Then the points of S can be identified with the points of an affine line $\mathrm{AG}(1,q)$ and $G \leqslant \mathrm{A\Gamma L}_1(q)$.*

No examples satisfying the hypothesis of the above theorem are known though.

THEOREM 36. *Let S be a partial quadrangle which is not a generalized quadrangle and let $G \leqslant \mathrm{Aut}(S)$ act point distance transitively on S. Then either the point set of S can be identified with the affine line $\mathrm{AG}(1,q)$ and $G \leqslant \mathrm{A\Gamma L}_1(q)$, or $\mathrm{Aut}(S)$ acts point geodesic transitively and one of the following possibilities occurs.*

(1) *$s = 1$ and S is a rank 3 strongly regular graph. The possibilities for (S,G) are listed in Table 1.*

(2) *S has a linear representation in the affine space $\mathrm{AG}(n,q)$, G acts point geodesic transitively, it contains the full translation group of $\mathrm{AG}(n,q)$ and the centre of*

the stabilizer of a point is an almost simple group M, (i.e. M contains a normal simple group and is included in its automorphism group) where the possibilities for S, n, q, M are given in Table 2.

Table 1
Point distance transitive partial quadrangles with $s = 1$

S	G	(s,t,μ)	Remarks
Pn(5)	D_{10}	$(1,1,1)$	G is geodesic transitive;
Pe(10)	$A_5 \trianglelefteq G \leqslant S_5$	$(1,2,1)$	G is point geodesic transitive; S_5 is geodesic transitive
HoS(50)	$\mathrm{U}_3(5) \trianglelefteq G \leqslant \mathrm{U}_3(5):2$	$(1,6,1)$	G is point geodesic transitive and geodesic transitive
Gew(56)	$\mathrm{L}_3(4) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_3(4)$	$(1,9,2)$	G is point geodesic transitive but *not* geodesic transitive
HS(77)	$\mathrm{M}_{22} \trianglelefteq G \leqslant \mathrm{M}_{22}:2$	$(1,15,4)$	G is point geodesic transitive but *not* geodesic transitive
HS(100)	$\mathrm{HS} \trianglelefteq G \leqslant \mathrm{HS}:2$	$(1,21,6)$	G is point geodesic transitive and geodesic transitive
Cl(16)	$2^4 : D_{10} \leqslant G \leqslant 2^4 : S_5$	$(1,4,2)$	$2^4 : (5:4)$ is point geodesic transitive; $2^4 : A_5$ is geodesic transitive

Table 2
Point geodesic transitive partial quadrangles with $s > 1$

S	AG(n,q)	M	Restrictions
$T_3^*(\mathcal{Q})$	AG$(4,q)$	$\mathrm{L}_2(q^2)$	$\mathcal{Q}$ an elliptic quadric in PG$(3,q)$
$T_3^*(\mathcal{O})$	AG$(4,q)$	$\mathrm{Sz}(q)$	$\mathcal{O}$ the Suzuki–Tits ovoid in PG$(3,q)$, $q = 2^{2e+1}$
$T_5^*(\mathcal{K})$	AG$(5,3)$	M_{11}	$\mathcal{K}$ the 11-cap in PG$(5,3)$ arising from M_{11}

This has the following immediate consequence:

THEOREM 37. *No locally 4-arc transitive $(4,5,5)$-pair exists.*

Note that from Theorem 31 it follows that there do not exist geodesic transitive $(g, g+1, g+1)$-pairs with $g \geqslant 4$. It is a conjecture that there exist no such geometries at all.

It is appropriate also to mention in this context a result which follows almost immediately from the classification of flag-transitive linear spaces by Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [1990].

THEOREM 38. *Every linear space listed in Table 3 gives rise to a locally 3-arc transitive $(3,3,4)$-pair. Conversely, if (S,G) is a locally 3-arc transitive $(3,3,4)$-pair with G type* *preserving, then it is one of the examples of Table 3. If (S, G) is a geodesic transitive (or equivalently a weakly geodesic transitive) $(3,3,4)$-pair with G type preserving, then it is one of the first 3 examples in Table 3.*

Finally, we mention a characterization of a class of Moore geometries.

THEOREM 39. *If (S, G) is a point distance transitive $(g, g, g+1)$-pair, $g \geqslant 5$, with G type preserving, then it is one of the two examples of Table 4. Moreover, if S is* HoS(50) *then G acts geodesic transitively, if S is* Pe(10) *then G acts geodesic transitively if and only if $G \cong S_5$.*

Table 3
Geodesic transitive and locally 3-arc transitive linear spaces

S	G	Restrictions and remarks
$\mathrm{PG}(n, q)$	$\mathrm{L}_{n+1}(q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_{n+1}(q)$	$n \geqslant 3$
$\mathrm{AG}(2, q)$	$\mathrm{L}_2(q) \trianglelefteq G_0 \leqslant \Gamma\mathrm{L}_2(q)$	G contains all translations
$S(P)$	G	G is almost simple and acts 4-transitively on P
$\mathrm{PG}(3, q)$	A_7	G contains all translations
$\mathrm{U}_H(q)$	$\mathrm{PGU}_3(q) \trianglelefteq G \leqslant \mathrm{P\Gamma U}_3(q)$	Hermitian unital in $\mathrm{PG}(3, q^2)$
$\mathrm{AG}(n, q)$	$\mathrm{L}_n(q) \trianglelefteq G_0 \leqslant \Gamma\mathrm{L}_n(q)$	$n \geqslant 3$ and G contains all translations
$\mathrm{AG}(4, 2)$	$G \cong 2^4 : A_7$	
$S(P)$	G	G is almost simple and acts 3-transitively on P

Table 4
Point distance transitive Moore geometries with $d \geqslant 2$

S	G	Remarks
Pe(10)	$A_5 \trianglelefteq G \leqslant S_5$	S_5 is geodesic transitive
HoS(50)	$\mathrm{U}_3(5) \trianglelefteq G \leqslant \mathrm{U}_3(5) : 2$	G is geodesic transitive

THEOREM 40. *If we allow diameter 1 for Moore geometries, then a point distance transitive $(g, g, g+1)$-pair is one of the examples in Tables 3 and 4 or it is one of the following linear spaces:*

- *the Hermitian unital with $G \cong \mathrm{U}_3(q)$;*
- *the Ree unital and G is an automorphism group of the corresponding Ree group;*
- *the affine space $\mathrm{AG}(n, q)$ with $\mathrm{AL}_1(q^n) \leqslant G \leqslant \mathrm{A\Gamma L}_1(q^n)$.*
- *the Hering plane of order 27 or the nearfield plane of order 9* (see Chapter 5);
- *a Hering space* (see Chapter 22, 1.9.8);
- *a circle geometry with a 2-transitive almost simple group acting.*

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CHAPTER 11

Buildings

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Contents

Introduction . . . 479
1. Numbered complexes and chamber systems . . . 482
   Introduction . . . 482
   1.1. Numbered complexes: definitions and basic notions . . . 482
   1.2. Chamber systems . . . 488
   1.3. The relationship between numbered complexes and chamber systems . . . 490
   1.4. Numbered complexes and chamber systems with a group action . . . 494
   Notes to Section 1 . . . 497
2. Coxeter–Tits complexes . . . 500
   Introduction . . . 500
   2.1. A survey of Coxeter groups . . . 502
   2.2. Thin complexes and reflections . . . 504
   2.3. Foldings and convexity . . . 506
   2.4. Characterizations of Coxeter–Tits complexes . . . 509
   2.5. Some calculations in Coxeter groups . . . 512
   Notes to Section 2 . . . 517
3. The axiomatics of buildings . . . 520
   Introduction . . . 520
   3.1. Complexes belonging to a Coxeter diagram . . . 520
   3.2. Systems of apartments and buildings . . . 526
   3.3. The 'First Main Characterization of Buildings' . . . 531
   3.4. Some applications of the 'First Main Characterization' . . . 538
   Notes to Section 3 . . . 546
4. Some important classes of buildings . . . 549
   Introduction . . . 549
   4.1. Buildings of type A_n . . . 550
   4.2. Buildings of type C_n and D_n . . . 553
   4.3. Tits systems and buildings with a transitive group action . . . 562
   4.4. The building of a semisimple algebraic group . . . 567
   Notes to Section 4 . . . 572

HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

5. On the geometry of buildings 573
Introduction 573
5.1. Roots, convexity and projection maps 574
5.2. Opposite chambers and apartments in spherical buildings 578
5.3. Roots in spherical buildings and the Moufang property 583
Notes to Section 5 589
6. Buildings and covering theory 592
Introduction 592
6.1. Coverings of chamber systems 592
6.2. The universal cover of a chamber system of Coxeter type 595
6.3. The 'Second Main Characterization of Buildings' 598
6.4. Some applications, I: Geometrical axioms 604
6.5. Some applications, II: Group amalgamations 609
Notes to Section 6 615
7. The classification of buildings of irreducible spherical type and rank at least 3 618
Introduction 618
7.1. Isomorphisms of spherical buildings 619
7.2. Constructing buildings from blueprints 624
7.3. The classification 634
Notes to Section 7 637
References 639

Introduction

The 'Tits buildings' or just 'buildings' are combinatorial objects whose theory was developed by Jacques Tits since the mid 50's for purposes of geometry and group theory. One historical background of this development was the classification of simple Lie groups, due to Elie Cartan in the 20's. In this work, the exceptional Lie groups, belonging to the diagrams usually called G_2, F_4, E_6, E_7, E_8 were established as equal counterparts to the 'classical', i.e. linear, symplectic, orthogonal, and unitary, groups. Cartan's classification was essentially from an axiomatic point of view, giving some structural information, but in a sense no 'direct' definition. Tits' intention was to interpret the exceptional Lie groups from a geometrical point of view, that is, to realize them as automorphism groups of certain geometries as the classical groups are for projective spaces or hyperquadrics. Of course, some results in this direction were known before Tits, in particular in the context of Cayley (octave) algebras. The point was the search for a unified treatment, related to the structure theory of the simple Lie groups.

Tits associated a building to any simple Lie group, later to Chevalley groups, and then to arbitrary simple algebraic groups. The key to the construction is the presence of certain subgroups, the so-called parabolic subgroups, whose properties were axiomatically developed in the theory of BN-pairs, or 'Tits systems'. Here, B and N are subgroups of some group G satisfying certain axioms which imply that the 'Weyl group' $W = N/(B \cap N)$ is a Coxeter group, and that all subgroups of G containing B, the parabolic subgroups, can be constructed from W and B. In the case of a classical group, the parabolics are the stabilizers of flags of subspaces of the corresponding geometry; the maximal parabolics are the stabilizers of single subspaces, whose dimension, however, may vary. Roughly speaking, it is this symmetric role of all subspaces which distinguishes Tits' point of view from the older and of course still equally important view of geometry, which starts from points as particular objects, and then treats subspaces in a derived manner.

Despite the close connection to Tits systems, the formal definition of a building does not refer to a group. Buildings are a class of objects which generalize the just mentioned families of all flags of subspaces of some geometry. Such a set of flags has the obvious structure of a (combinatorial) simplicial complex, a notion known since the 30's in combinatorial topology. In general, buildings are defined to be certain simplicial complexes with additional properties. The main condition is the existence of a 'sufficiently large' family of subcomplexes called apartments which are supposed to be all isomorphic to each other and to have a simple combinatorial description. For instance, typical apartments are the barycentric subdivision of the simplex, or of the n-dimensional cube. The idea is that the apartments control a large part of the geometric properties of the whole complex, and that the totality of buildings is divided into families (each specified by a 'Coxeter diagram') such that all members of one family have the same apartment.

If a building Δ comes from a Tits system (B, N) in G, then G acts transitively on the maximal simplices, called chambers, B being a typical stabilizer. It even acts transitively on the pairs (Σ, C), where C is a chamber, and Σ an apartment containing C. Conversely, if an abstract building admits such a transitive group action, and in addition satisfies a certain axiom of thickness, it gives rise to a Tits system in the group in question.

A survey of the theory of buildings could be conveniently subdivided as follows:
Axiomatic foundations
Structure theory
Classification
Geometrical characterizations
Applications.

The axiomatic foundations and the structure theory (which are not strictly separated topics) will be developed in Sections 1 to 6, and here lies the main emphasis of our report. There seems to be a particular demand for a unified presentation of this work, which, up to now, has only appeared scattered in the literature. The classification of buildings of spherical type and the principal methods to achieve this classification (those methods that go beyond the general theory developed before) are presented in Section 7. Geometrical characterizations of buildings (by their incidence properties) are not our main subject although some information is contained in Sections 3.4, 4 and 6.4. Further results in the case of the exceptional spherical diagrams are contained in Chapter 12. Applications of the theory of buildings are the subject of Chapter 20; see also Ronan [1992], Part II. We only mention in Section 6.5 some applications to presentations of certain groups.

We now say a few more words about the contents of the individual sections.

In Section 1, we provide the axiomatic technical background which is grouped around the basic notions of a numbered (simplicial) complex and a chamber system. Connectedness properties and their consequences are treated, and the 'basic graph' of a complex or chamber system (which is a weak form of a Coxeter diagram) is introduced.

In Section 2, we treat in some detail the Coxeter complexes which are the apartments in buildings. (We shall throughout use the terminology 'Coxeter–Tits complex' which to our opinion is more adequate.) As it was remarked above, the apartments already carry a large part of the geometry involved in general buildings. The theory of Coxeter–Tits complexes can be considered as a geometric version of the theory of Coxeter groups as treated in Bourbaki [1968]. In view of the great importance of these groups in several areas of mathematics, not only in relation to buildings, Section 2 could be of some independent interest.

In Section 3, the notion of a building is introduced, several distinct approaches (apartments, 'types' of galleries, Weyl-group-valued metric, and others) are discussed and eventually shown to be equivalent. These considerations are continued in Section 6. In Section 3.4, we give some 'internal' applications of the results obtained so far: incidence properties for certain buildings, formulae for counting chambers.

The purpose of Section 4 is two-fold. In Sections 4.1 and 4.2, we treat in detail the flag complexes of projective spaces, resp., polar spaces, as buildings of type A_n, resp., C_n. This illustrates how the general concepts of the previous sections look like in the case of some classes of familiar geometrical objects. In Section 4.3, we report on the important notion of a Tits system or BN-pair in a group, whose axiomatics reformulate the axiomatics of a thick building with a 'strongly transitive' group action. Such Tits systems exist in semisimple algebraic groups. Section 4.4, the relevant part of the structure theory and the classification of such groups is recalled.

In Section 5, we develop the main 'internal' geometric properties of buildings. These properties can best be visualized as properties of the topological (or even metric) space underlying the abstract simplicial complex. They are crucial for the applications as well as for the possibility of classifying certain classes of buildings. Of particular importance in this respect is the so-called gate property (existence of projection maps). In the spherical case, an important additional concept is the Moufang property; it generalizes the property with the same name of projective planes.

In Section 6, we continue to develop the axiomatic theory by giving a detailed report on the so-called 2-coverings of chamber systems and a characterization of the buildings among the 2-simply-connected chamber systems. These results have applications to the characterization of buildings in terms of geometrical axioms, depending on the diagram (Section 6.4), to group amalgamations, and to the classification of certain complexes belonging to a Coxeter diagram which are not buildings (Section 6.5).

In Section 7 we finally come to what could be considered as the most important result of the whole theory, namely the classification of all buildings of irreducible 'spherical type' and rank at least 3. A report on the classification of a certain class of buildings (for the totality of buildings, it is out of question) naturally consists of two parts: Firstly, a description of all 'known' buildings of that class, secondly an outline of a proof of the fact that any building in the particular class is isomorphic to one of the 'known ones'. The principal technical tool for the second step is a very difficult theorem of Tits [1974], dealing with the extension of certain locally defined morphisms between buildings of spherical type. It relies on the techniques developed in Section 5 and will be presented in Section 7.1. The first step is to some extent already contained in Section 4; a more complete description of the buildings of spherical type is given in Section 7.3, treating the various diagrams case by case. In the intermediate Section 7.2, we report in some detail on the important paper Ronan and Tits [1987], where a rather general construction of buildings out of certain local data (the so-called blueprints and foundations) is given. This construction heavily relies on the 'First Main Characterization of Buildings' from Section 3.3 as well as the results from Section 6. Although the technique of blueprints applies in principle to buildings of general type, it has turned out particularly useful for simplifying the proof of the classification in the case of spherical type by making a systematic and extensive use of the basic Extension Theorem from Section 7.1. In the whole Section 7, we shall be considerably more brief than in the other sections by giving only a few proofs. We refer the reader to the original monograph Tits [1974] and to the recent text book Ronan [1989] for a full treatment.

We do not treat buildings of affine type in this survey although they are certainly the most important ones for applications to other areas of current research, notably for the investigation of various classes of (infinite, discrete) groups. They were introduced in the papers Iwahori and Matsumoto [1965], Bruhat and Tits [1966b,c,d, 1966/67] and Iwahori [1965/66]. For an easily accessible introduction to affine buildings, see the book Brown [1989]. A brief sketch is also contained in Chapter 20 of this book, or in Ronan [1992], Part I. A condensed presentation of the classification, without proofs, is given in Tits [1977/79]. The paper Tits [1984/86] and the book Ronan [1989] are references for the methods behind the classification. The unique source for the particular axiomatics of affine buildings, for a detailed description of the known buildings (those of semisimple groups over local fields), and for proofs of their properties still is the monograph Bruhat and Tits [1972], which is complemented by the papers Bruhat and Tits [1984, 1987].

1. Numbered complexes and chamber systems

Introduction

In this section, we develop in some detail the technical foundations for the theory of buildings.

The numbered complexes (simplicial complexes with a 'type' function) to be treated in the first subsection are axiomatic generalizations of the flag complexes of the subspaces of classical projective or polar geometries. The point of view is more or less the same as for the notion of incidence geometry in the sense of Tits and Buekenhout as developed in Chapter 3 of this book. Incidence geometries can be naturally considered as a subclass of numbered complexes (under some mild restrictions), and the general notion of a numbered complex is more or less of technical nature. One justification for this more general concept is the fact that for buildings, the particular property of 'being' a geometry is not presupposed, but is a consequence of the other axioms (which are needed anyway). The chamber systems introduced in Section 1.2 are a more recent development, first introduced in Tits [1981a]. They can be considered as a reformulation (and generalization) of those complexes satisfying certain connectedness properties and are particularly useful for the covering theory treated in Section 6. In Section 1.3 we discuss in some detail the relationship between the two concepts of a numbered complex and a chamber system. Although these matters are rather straightforward and more or less well known, a fully systematic and complete account cannot be found in the literature. Therefore, we present full proofs here. Section 1.4 contains the description of complexes and chamber systems with a transitive group action as certain 'coset spaces'. These objects are doubtlessly the most important ones among buildings and related structures. The classical origin of the theory (axiomatic treatment of parabolic subgroups of semisimple Lie groups) deals with this class of geometries as well as do the recent geometrical developments around the classification of finite simple groups (cf. the notes to Section 6).

The entire first section of our report on buildings is rather technical, and we have to apologize for not having attempted to emphasize the geometrical spirit right from the start and to develop the geometrical intuition parallel to the technical notions. The reader who wishes to have at once a more complete and clear impression of buildings as mathematical objects, rather than of their axiomatics, is advised to proceed immediately to Sections 3.1, 3.2, 4.1, 4.2 and 7, then back to Section 2, and only occasionally back to Section 1. For a more elaborate study of Sections 3.3, 3.4, 5 and 6, a detailed familiarity with Section 1 however seems to be unavoidable.

1.1. *Numbered complexes: definitions and basic notions*

1.1.1. *Basic definitions.* A (simplicial) *complex* is a partially ordered set $(\Delta, \subseteq)$ such that there exists a smallest element $\varnothing$, there exists a greatest lower bound $A \cap B$, for any two elements $A, B \in \Delta$, and all sets $\mathcal{P}(A) = \mathcal{P}_\Delta(A) := \{X \in \Delta\colon\ X \subset A\}$, $A \in \Delta$ are isomorphic to a full power set. The elements of Δ are often called *simplices*, $A \subseteq B$ is read 'A is a *face* of B', and $\varnothing$ is called the empty simplex or empty face.

A *morphism* of a complex Δ into a complex Δ' is a map $\varphi\colon\ \Delta \to \Delta'$ which induces an isomorphism of ordered sets $\mathcal{P}_\Delta(A) \to \mathcal{P}_{\Delta'}(A')$ for all $A \in \Delta$. The minimal nonempty faces are called *vertices*. If $\widetilde{A}$ denotes the set of vertices contained in a simplex A, then the map $A \mapsto \widetilde{A}$ is injective. In fact, it is an isomorphism of ordered sets of Δ onto a certain set of sets $\widetilde{\Delta}$ (thus the notations $\subseteq, \cap, \varnothing$). This $\widetilde{\Delta}$ contains all subsets of any of its elements, and conversely, any set of sets with this property is a complex if ordered by inclusion. In particular, a full power set $P(I)$, where I is any set, is a complex.

Our discussion shows that we may redefine a simplicial complex as a pair (X, Δ), where X is some set (whose elements are called vertices), and Δ is a set of subsets of X such that $A \subseteq B \in \Delta$ implies $A \in \Delta$, and $\{x\} \in \Delta$ for all $x \in X$. This definition is slightly more concrete, but if we want the category of simplicial complexes to be closed under isomorphisms of ordered sets, we have to adopt the first definition. Whenever convenient, we shall switch between the two points of view. In particular, we will often not distinguish between a vertex x and the one-element set $\{x\}$. In this spirit, we denote by $\mathcal{P}(A)$ the simplicial complex consisting of all faces of the simplex A. The smallest upper bound of $A, B \in \Delta$ (if it exists) is denoted by $A \cup B$. If we consider A and B as sets of vertices, $A \cup B$ really is the set-theoretic union.

A maximal simplex of Δ is called a *chamber*. The *rank* of Δ is the maximum, possibly ∞, of the cardinalities $|\widetilde{A}|$, as A ranges over Δ. Sometimes $|\widetilde{A}|$ is also called the rank of A; if $A \subseteq B \in \Delta$, we set $|\widetilde{B} \setminus \widetilde{A}| =: \operatorname{cod}_B A$, the *codimension* of A in B. If $\operatorname{rank} \Delta = \infty$, it is a nontrivial condition that every simplex be contained in a chamber.

The *join* $\Delta \times \Delta'$ of two complexes Δ, Δ' is the Cartesian product with the product ordering; its set of chambers is the product of the chamber sets of Δ and Δ', its vertex set can be identified with the disjoint union of the vertex sets of Δ and Δ'. Two chambers C and C' are called *adjacent* if

$$\operatorname{cod}_C (C \cap C') = \operatorname{cod}_{C'} (C \cap C') = 1,$$

and a face of codimension 1 in C is called a *panel of C*. A *gallery* of length m in Δ is a finite sequence $(C_0, \ldots, C_m)$ of chambers such that any two consecutive elements are adjacent or equal. The tuple $(C_0, \ldots, C_m)$ is typically abbreviated by $\boldsymbol{C}$. The gallery is called *simple* if $C_{t-1} \neq C_t$ for all $t = 1, \ldots, m$. The chambers C_0, C_m are called the *extremities* of $\boldsymbol{C}$, C_0 is the *origin* and of course C_m the *endpoint* of $\boldsymbol{C}$. (This slightly deviates from Tits' terminology who calls C_m the extremity.) We say that $\boldsymbol{C}$ is *closed* if $C_0 = C_m$. If any two chambers can be joined by a gallery, Δ is called *connected*.

A connected complex is automatically *pure*, that is any two chambers have the same rank. If Δ is pure and $\operatorname{rank} \Delta = n < \infty$, we define the *codimension* $\operatorname{cod} A$ of $A \in \Delta$ as $\operatorname{cod} A := n - \operatorname{rank} A$. It equals $\operatorname{cod}_C A$ (as defined above), for any chamber $C \supseteq A$. A pure complex of finite rank is called *thick* (resp. *thin*) if every panel is contained in at least 3 (resp. exactly 2) chambers.

For $A \in \Delta$, we define the *star* of A in Δ as

$$\operatorname{St}_\Delta A = \{B \in \Delta\colon\ B \supseteq A\}.$$

With the induced ordering, this is a complex. Its empty simplex equals A. If each $\mathrm{St}_\Delta A$, $A \in \Delta$, is connected, then Δ is called *strongly connected.*

A *numbering* (type function) of a complex Δ is a morphism (mapping chambers onto chambers)

$$\text{type: } \Delta \to \mathcal{P}(I)$$

of Δ onto some simplex $\mathcal{P}(I)$. We shall also say that the complex Δ is numbered over I, or even that Δ is a *complex over I.* Of course, numberings of Δ can be identified with maps from the vertex set of Δ to some set I such that the restriction to (the vertex set of) any chamber is bijective. In particular, $\operatorname{rank} \Delta = |I|$. It is easily shown that a numbering is essentially unique (in the obvious sense) provided Δ is connected. If a connected complex Δ admits a numbering, then there is a '*canonical numbering*' type: $\Delta \to \text{type}(\Delta)$. Here, type($\Delta$) is defined as the quotient of Δ by the equivalence relation 'being of the same type'. The vertex set of the simplex type(Δ) is called the *canonical set of types* and denoted by $I(\Delta)$.

In a more down-to-earth way, a numbered complex can be defined as a simplicial complex Δ together with a partitioning[1]

$$X = \dot{\bigcup_{i \in I}} X_i$$

of its vertex set X such that each chamber contains precisely one vertex of each X_i. Every simplex is of the form $A = \{x_j \colon j \in J\}$ for some $J \subseteq I$ and $x_j \in X_j$, and J is the type of A. For $A \in \Delta$, set

$$\operatorname{cotype} A := I \setminus \operatorname{type} A.$$

If Δ is a numbered complex over I, and if C and C' are adjacent chambers, we say that C, C' are *i-adjacent* if $\text{cotype}(C \cap C') = i \in I$. In this way, we associate to any simple gallery $(C_0, C_1, \ldots, C_m)$ a sequence $(i_1, \ldots, i_m)$ in I, where C_{t-1} and C_t are i_t-adjacent. We say that $(i_1, \ldots, i_m)$ is the *type of the gallery* $(C_0, \ldots C_m)$, and use the notation

$$\boldsymbol{C} = (C_0, C_1, \ldots, C_m; i_1, \ldots, i_m).$$

We often write $i_1 \ldots i_m$ instead of $(i_1, \ldots, i_m)$ and think of $i_1 \ldots i_m$ as a word over the alphabet I.

The join of two numbered complexes is naturally numbered over the disjoint union of the two type sets.

For any $A \in \Delta$, the restriction of type makes $\mathrm{St}_\Delta A$ into a complex over the set $\operatorname{cotype} A$. A numbered complex of rank 2 'is' a bipartite graph (more precisely, a bipartite graph with a fixed partitioning of its vertex set). The chambers are the edges if regarded as two-element subsets of the vertex set.

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[1] The symbol $\dot{\bigcup}$ means 'disjoint union' (Editor's note).

1.1.2. *Example: partially ordered sets.* After all the preceding general notions whose relations to geometry are perhaps not obvious, we want to introduce one important class of examples which hopefully will make the numbered complexes more concrete.

A *poset* is a set X together with a partial ordering, that is, a transitive and antisymmetric relation $\leqslant$. It is called *pure* if any two maximal flags (totally ordered subsets) have the same finite cardinality $n+1$ (or 'length' n). This number n is called the *dimension* of X. If X is pure and $x \in X$, and

$$x_0 < x_1 < \cdots < x_d = x < \cdots < x_n$$

is a maximal flag, then d, the *dimension of x in X*, is independent of the choice of the flag. If $(X, \leqslant)$ is a pure poset, then the set of flags in X is a simplicial complex. It is called the *flag complex* of $(X, \leqslant)$ and denoted by $\mathrm{Flag}(X, \leqslant)$. The dimension function makes $\mathrm{Flag}(X, \leqslant)$ into a numbered complex with type set $\{0, \ldots, n\}$. Often, $d+1$ is called the *rank* of x, and $(X, \leqslant)$ is said to be of *rank* $n+1$.

We now introduce the 'basic graph' of a numbered complex, which is a weak form of a diagram in the sense of Buekenhout and Tits. In the case of a Coxeter diagram, which will be treated later, the basic graph is just the underlying graph in the usual sense, that is $i, j \in I$ are connected if and only if $m_{ij} \geqslant 3$, where (m_{ij}) is the Coxeter matrix. See Sections 2.2, 3.3.

1.1.3. DEFINITION. Let Δ be a numbered complex with type set I. The *basic graph* of Δ has the vertex set I, and $\{i, j\} \subseteq I$ is not an edge if for each $A \in \Delta$ of cotype $\{i, j\}$, the star $\mathrm{St}\, A$ is a 'generalized digon', that is, each vertex of type i in $\mathrm{St}\, A$ is incident with each vertex of type j.

If Δ is the flag complex of a pure poset X of dimension n, then $i, j \in \{0, \ldots, n\}$ are not connected if $|i-j| \geqslant 2$. The basic graph of each star $\mathrm{St}\, A$, $A \in \Delta$, is a not necessarily induced subgraph of the basic graph of Δ. That is, if $i, j \in \operatorname{cotype} A$ are connected with respect to $\mathrm{St}\, A$, then they are also connected with respect to Δ.

If some graph with vertex set I is given and $J, K \subset I$, we say that J and K are *separated* from each other if there exists no path in the graph starting inside J and ending inside K. Equivalently, the graph is the disjoint union of two subgraphs having vertex sets J', K' such that $J \subseteq J'$ and $K \subseteq K'$. If L is a third vertex set, then J and K are *separated by L* if $J \setminus L$ and $K \setminus L$ are separated from each other in the subgraph induced on $I \setminus L$. This means that there exist $\widehat{J} \supseteq J$ and $\widehat{K} \supseteq K$ such that $I \setminus \widehat{J}$ and $I \setminus \widehat{K}$ are separated from each other, and

$$I \setminus L = (I \setminus \widehat{J}) \mathbin{\dot{\cup}} (I \setminus \widehat{K}).$$

The following 'Main theorem on the basic graph' is due to Tits for the case of buildings, and has been proved for general incidence geometries in Buekenhout [1979].

1.1.4. PROPOSITION. *Let Δ be a strongly connected numbered complex and let A_1, A_2, A_3 be simplices of Δ; set $J_t :=$* type A_t. *If $A_1 \cup A_2$ and $A_2 \cup A_3$ are simplices, and J_1 and J_3 are separated by J_2 in the basic graph, then $A_1 \cup A_2 \cup A_3$ is a simplex.*

PROOF. By strong connectedness, it is sufficient to prove the claim inside $\mathrm{St}\, A_2$. Therefore consider two simplices A, B such that, for $J := \mathrm{type}\, A$, $K := \mathrm{type}\, B$, the whole type set I equals $J \dot\cup K$, and J, K are separated from each other. We have to show that $A \cup B$ is a simplex. Set $A_0 = A$ and choose B_0 and D such that $A_0 \dot\cup B_0$ and $D \dot\cup B$ are chambers. Connect $A_0 \cup B_0$ and $D \cup B$ by some gallery, and write this gallery in the form

$$(A_0 \cup B_0, A_1 \cup B_1, \ldots, A_m \cup B_m = D \cup B), \tag{$*$}$$

where $\mathrm{type}\, A_t = J$, $\mathrm{type}\, B_t = K$ for all t. Consider three consecutive chambers

$$A_{t-1} \cup B_{t-1}, \quad A_t \cup B_t, \quad A_{t+1} \cup B_{t+1} \tag{$**$}$$

and assume that in the first step a vertex a_1 such that $\mathrm{type}\, a_1 \in J$ and in the second step a vertex b_1 such that $\mathrm{type}\, b_1 \in K$ is exchanged:

$$A' \cup a_1 \cup B' \cup b_1, \ A' \cup a_2 \cup B' \cup b_1, \ A' \cup a_2 \cup B' \cup b_2.$$

Considering $\mathrm{St}(A' \cup B')$ whose basic graph consists of two isolated vertices, we see that also $A' \cup a_1 \cup B' \cup b_2$ is a simplex. Hence we can modify ($**$) such that the change takes place at first in K and then in J, and not conversely. In this way the gallery ($*$) can be modified in such a way that $B_0 \neq B_1 \neq \cdots \neq B_r = B_{r+1} = \cdots = B_m$ for some $r < m$. So $A \cup B = A_0 \cup B_m = A_r \cup B_r$ is a simplex, as desired. □

1.1.5. COROLLARY. *A strongly connected numbered complex Δ is naturally the join of J-numbered (sub-)complexes Δ_J, where J runs over the connected components of the basic graph.*

1.1.6. EXAMPLE. *Linear diagrams.* We present a sort of converse to 1.1.2 in the case of strongly connected numbered complexes. Let Δ be such a complex and assume that the basic graph is *linear*. By this we mean that the type set can be totally ordered in such a way that $i, j \in I$ are not connected if there exists an element strictly between them. That is, we may identify $I = \{1, \ldots, n\}$, and i and j are connected only if $|i - j| = 1$. We want to 'realize' Δ as a flag complex as in 1.1.2. The construction of the appropriate partially ordered set depends on specifying some total order on I, that is some identification with $\{1, \ldots, n\}$ as above. Of course we are mainly interested in the case of a connected linear basic graph (often called a *string*) where there are precisely two choices (if $n > 1$). The construction now is as follows. Define a relation '$\leqslant$' on the set of vertices X_Δ of Δ by setting

$$x \leqslant y \Leftrightarrow x \cup y \in \Delta \text{ and type}\, x \leqslant \text{type}\, y.$$

It readily follows from 1.1.4 that this relation is transitive, and thus clearly a partial ordering. Again by 1.1.4, this partial ordering is pure: any flag is contained in a flag of $|I|$ elements. The flag complex $\mathrm{Flag}(X_\Delta, \leqslant)$ can be identified with the original complex Δ. This construction is 'canonical' in the usual sense: any type preserving isomorphism

$\Delta \to \Delta'$ induces an isomorphism $(X_\Delta, \leqslant) \to (X_{\Delta'}, \leqslant)$. Combining these considerations with 1.1.2, we get bijective, canonical correspondence between all strongly connected numbered complexes with as basic graph a (fixed) string of cardinality $n+1$ and certain pure posets of dimension n. We leave it to the reader to characterize this class of posets by appropriate 'connectedness properties'.

The separation in the basic graph is of particular importance if the first of the three type sets involved is fixed. If we consider any graph on a set I and fix a subset $I_0 \subseteq I$, then the relation

$$J \preceq K \;:\Leftrightarrow\; I_0 \text{ and } K \text{ are separated by } J$$

between subsets $J, K \subseteq I$ is reflexive and readily seen also to be transitive. Thus, the relation '$J \preceq K \preceq J$' is an equivalence relation, and $\preceq$ induces a partial ordering on the equivalence classes. This will be of particular importance to describe the inclusion of the so-called *shadows* of arbitrary simplices on the simplices of the fixed type I_0. See the Notes below and 6.4.2 for more details. Here, we shall restrict ourselves to a proposition which shows that the partial ordering associated to I_0 and the separation relation has some particular nice properties.

1.1.7. PROPOSITION. *For a finite graph with vertex set I and a distinguished subset $I_0 \subseteq I$, the separation relation '$\preceq$' and the associated equivalence relation '$J \preceq K \preceq J$' on subsets of I have the following properties*:

(a) *The equivalence class of each $J \subseteq I$ contains a smallest set J_{red} (called reduced).*

(b) *The equivalence class of each $J \subseteq I$ contains a largest set $\bar{J}$.*

(c) *The mapping $J \mapsto \bar{J}$ is a monotonous closure operator*:

$$J \subseteq \bar{J} = \bar{\bar{J}}; \; J \subseteq K \Rightarrow \bar{J} \subseteq \overline{K}.$$

(d) $J \preceq K \;\Leftrightarrow\; \bar{J} \supseteq \overline{K}.$

The easy proof is left to the reader; for (a), one shows that $J_1 \preceq J$, $J_2 \preceq J$ implies $J_1 \cap J_2 \preceq J$.

We close this subsection with an easy criterion for the existence of a numbering of a simplicial complex. Let Δ be a connected simplicial complex. Consider for any two adjacent chambers C, C' the unique morphism $\mathcal{P}(C) \to \mathcal{P}(C')$ which is the identity on $\mathcal{P}(C \cap C')$. By composition, this gives a morphism

$$\alpha_{\boldsymbol{C}}\colon \mathcal{P}(C) \to \mathcal{P}(D)$$

for any gallery $\boldsymbol{C} = (C = C_0, C_1, \ldots, C_n = D)$.

1.1.8. LEMMA. *Let Δ be a connected complex in which all stars of vertices are connected. If $\alpha_{\boldsymbol{C}}$ is the identity for all closed galleries, then Δ possesses a numbering.*

PROOF. Fix any chamber D. We want to define a retraction $\rho\colon \Delta \to \mathcal{P}(D)$ by setting $\rho(A) = \alpha_C(A)$, where C is an arbitrarily chosen gallery joining D to some chamber $C \supseteq A$. It is readily checked from the assumption of the lemma that this is well defined. □

1.2. *Chamber systems*

Historically, chamber systems occurred in the work of Tits as a technical reformulation and generalization of the strongly connected numbered complexes. For the reader who already is familiar with the subject, we mention some results where chamber systems turned out to be more flexible than complexes, and we make a few remarks about the different point of view in those cases where the concepts of a complex and a chamber system finally coincide.

1. The 'First Main Characterization of Buildings' (by a kind of distance function with values in the Weyl group) is naturally formulated in terms of chamber systems by Tits [1981a].

2. If one wants to characterize buildings as far as possible by 'local properties', one has to develop a theory of coverings, and this is most easily done via chamber systems by Tits [1981a], Ronan [1980].

3. Independently of the theory of buildings, chamber systems admit 'geometrical realizations' which are essentially more general than the 'standard realization' of a simplicial complex.

4. If one considers objects with a transitive group action, then in the case of classical or algebraic groups, the complex is determined by the system of maximal parabolic subgroups, whereas the chamber system is determined by the system of minimal parabolics. The latter is often more desirable for group theoretic applications and generalizations.

We now come to the definition of chamber systems which is very simple.

1.2.1. DEFINITION. A *chamber system* over some (index) set I is an object $(\mathcal{C}, \overset{i}{\sim}, i \in I)$, where $\mathcal{C}$ is a set and the $\overset{i}{\sim}$ are equivalence relations on $\mathcal{C}$.

The elements of $\mathcal{C}$ are called *chambers*; we write $C \overset{i}{—} D$ if

$$C, D \in \mathcal{C}, \qquad C \overset{i}{\sim} D, \qquad C \neq D,$$

and say that C and D are i-*adjacent* in this case. The *rank* of $\mathcal{C}$ is by definition the cardinality of I.

The principal examples are the sets $\mathcal{C}(\Delta)$ of chambers of an I-numbered complex. Adjacency is as defined previously, i.e. for C, D chambers of Δ we set

$$\begin{aligned} C \overset{i}{\sim} D :&\Leftrightarrow C \text{ is } i\text{-adjacent to } D, \text{ or } C = D \\ &\Leftrightarrow \operatorname{cotype}(C \cap D) \subseteq \{i\}. \end{aligned}$$

In passing, we remark that a chamber system $\mathcal{C}(W)$ is naturally associated to any discrete reflection group W on a 'reasonable' space, for example the n-sphere, Euclidean n-space, or hyperbolic n-space. If the fundamental domain of W is not a simplex, which in the hyperbolic case occurs very often, then $\mathcal{C}(W)$ is not of the form $\mathcal{C}(\Delta)$, for a complex Δ.

A *morphism* from a chamber system $\mathcal{C}$ over I to a chamber system $\mathcal{C}'$ over I' is a map $\varphi\colon \mathcal{C} \to \mathcal{C}'$, together with a bijection $\varphi_*\colon I \xrightarrow{\simeq} I'$ such that

$$C \overset{i}{\sim} D \Rightarrow \varphi C \overset{\varphi_* i}{\sim} \varphi D \quad \text{for all } C, D \in \mathcal{C}.$$

Sometimes φ is called a morphism 'relative to φ_*'. In most cases, φ_* is determined by φ (and the above condition). If $I = I'$ and $\varphi_* = \mathrm{id}$, we say that φ is *special* or *type preserving*. Often, the type preserving morphisms are the only ones that matter.

The map $\Delta \mapsto \mathcal{C}(\Delta)$ is obviously functorial: If $\varphi\colon \Delta \to \Delta'$ is a morphism of numbered complexes with induced map $\varphi_*\colon I \to I'$ on the type sets, then $\varphi_{|\mathcal{C}(\Delta)}$ gives a morphism of chamber systems relative to the same φ_*.

We now continue the discussion of chamber systems without going back to the complexes every time a new notion occurs. The reader who wishes to see the connections right away is asked to switch back and forth between this and the following subsection.

Let $(\mathcal{C}, \overset{i}{\sim}, i \in I)$ be a chamber system. A *gallery* in $\mathcal{C}$ is defined to be a sequence

$$\boldsymbol{C} = (C_0, C_1, \ldots, C_n; i_1, \ldots, i_n)$$

such that $C_\nu \in \mathcal{C}$, $i_\nu \in I$ for all ν, and $C_{\nu-1} \overset{i_\nu}{\sim} C_\nu$ for $\nu = 1, \ldots, n$. As before, it is called *simple* if $C_{\nu-1} \neq C_\nu$ for $\nu = 1, \ldots, n$.

The *length* of $\boldsymbol{C}$ is n, the word $i_1 \ldots i_n$ is called the *type* of $\boldsymbol{C}$. A chamber system is *connected* if any two chambers can be joined by a gallery.

An equivalence relation $\overset{J}{\sim}$ is defined for all subsets $J \subseteq I$ as follows:

$$C \overset{J}{\sim} D \Leftrightarrow \text{there exists a gallery } (C = C_0, C_1, \ldots, C_n = D; i_1, \ldots, i_n)$$
$$\text{s.t. } i_\nu \in J \text{ for all } \nu = 1, \ldots, n.$$

Thus, $\overset{J}{\sim}$ is the equivalence relation 'generated by' the $\overset{j}{\sim}$, $j \in J$. More formally, $\overset{J}{\sim}$ is the supremum of the $\overset{j}{\sim}$, $j \in J$, in the lattice of all equivalence relations on $\mathcal{C}$. This is the intersection of all equivalence relations containing all $\overset{j}{\sim}$, $j \in J$, considered as subsets of $\mathcal{C} \times \mathcal{C}$.

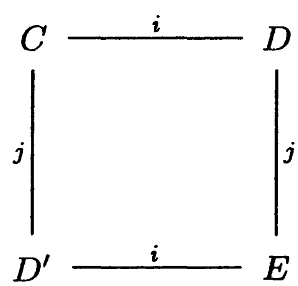
An equivalence class $\mathcal{A}$ for the relation $\overset{J}{\sim}$ is called a *J-star* for short; it can be considered as a chamber system over J, just by restricting the $\overset{j}{\sim}$, $j \in J$, to $\mathcal{A}$. Notice that a J-star as a chamber system is always connected. A J-star is often also called a *J-residue*.

A chamber system is *strongly connected* if it is connected, and if the following holds for any two subsets $J, K \subseteq I$, and any two chambers $C, D \in \mathcal{C}$:

$$C \overset{J}{\sim} D, C \overset{K}{\sim} D \Rightarrow C \overset{J \cap K}{\sim} D.$$

Strong connectedness means that the relation $\overset{J\cap K}{\sim}$ is the infimum of $\overset{J}{\sim}$ and $\overset{K}{\sim}$ in the lattice of equivalence relations. The results of the following subsection will show that this is the 'right' definition. In addition to the 'technical use' of strong connectedness in the following subsection, the notion also has some significance when dealing with 'geometrical realizations'.

We finally define the *basic graph* of a chamber system $\mathcal{C}$ over I. It has the vertex set I, and i and j are not connected if the corresponding adjacency relations commute, that is, if for every gallery of the form $(C, D, E; i, j)$ with $C \neq D \neq E$ there exists a chamber $D' \neq C, E$ such that $(C, D', E; j, i)$ is a gallery:



The following 'main theorem on the basic graph' for chamber systems has a trivial proof.

1.2.2. PROPOSITION. *Let $(\mathcal{C}, \overset{i}{\sim}, i \in I)$ be a chamber system, $\mathcal{A} \subseteq \mathcal{C}$ a J-star for some $J \subseteq I$. Let $\mathcal{A}_1 \subseteq \mathcal{A}$, $\mathcal{A}_2 \subseteq \mathcal{A}$ be a J_1-star and a J_2-star, respectively, and suppose that $J = J_1 \dot\cup J_2$, where J_1, J_2 are not connected in the basic graph. Then $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \varnothing$.*

We shall see in the next subsection that this proposition gives a new proof of the corresponding result for numbered complexes.

1.3. *The relationship between numbered complexes and chamber systems*

In this subsection we treat the connections between numbered complexes and chamber systems in some detail. We use the notation $\mathcal{C}(\Delta)$ for the chamber system of an I-numbered complex Δ. This is the set of chambers of Δ together with the adjacency relations $\overset{i}{\sim}$, $i \in I$. The main result of this subsection, which is certainly known, but cannot be found in detailed form in the literature, is the following.

1.3.1. THEOREM. *The construction $\Delta \mapsto \mathcal{C}(\Delta)$, together with the construction $\mathcal{C} \mapsto \Delta(\mathcal{C})$ defined below, induce equivalences essentially inverse to each other, between the category of strongly connected I-numbered complexes and the category of strongly connected chamber systems over I.*

If $\mathcal{C}$ is any chamber system over I, and $C, D \in \mathcal{C}, J \subseteq I$ we recall the definition

$$C \overset{J}{\sim} D \Leftrightarrow \text{there is a gallery of type contained in } J, \text{ joining } C \text{ and } D.$$

We set

$$\mathcal{S}(C,J) := \{D \in \mathcal{C}\colon\ C \overset{J}{\sim} D\}, \quad C \in \mathcal{C}, J \subseteq I$$

the J-star or J-residue of C.

We shall obtain this theorem by proving a couple of easy propositions which hold under slightly weaker conditions than strong connectedness. The main goal is to characterize chamber systems of shape $\mathcal{C}(\Delta)$, at least for a reasonably large class of complexes. The easiest way to achieve this is via a functor $\mathcal{C} \mapsto \Delta(\mathcal{C})$ in the opposite direction. Perhaps it is not quite obvious how to define this construction, but once the definition is given, it is easy to determine the conditions under which the two functors are inverse to each other. Before we go into the details which are a bit lengthy and require still further notation, we want to emphasize that the main idea of how to reconstruct a complex from its chamber system is very simple.

Indeed, suppose that Δ is *firm* (for every chamber C and every $i \in I$ there is a chamber C' i-adjacent to C). Then every simplex A can be reconstructed from the set $\mathcal{C}(\operatorname{St} A)$ of chambers containing A; just take the intersection of all members of $\mathcal{C}(\operatorname{St} A)$. If furthermore Δ is strongly connected, then by definition $\mathcal{C}(\operatorname{St} A)$ is precisely the J-class of any of its members, where $J = \operatorname{cotype} A$:

$$\mathcal{C}(\operatorname{St} A) = \mathcal{S}(C, \operatorname{cotype} A),$$

where C is an arbitrary chamber containing A. All equivalence classes $\mathcal{S}(C,J)$ occur as stars of simplices.

This observation suggests that one should define, for a given chamber system $\mathcal{C}$, the simplices of the desired complex $\Delta(\mathcal{C})$ just as the J-classes, for all $J \subseteq I$. Unfortunately, in general this is not the 'right' definition, since it does not always give a complex. A definition which works for all $\mathcal{C}$ is the following.

1.3.2. CONSTRUCTION. If $\mathcal{C}$ is any chamber system over I, denote by $\Delta(\mathcal{C})$ the following I-numbered complex: The set of vertices is

$$X(\mathcal{C}) = \dot{\bigcup_{i\in I}} X_i(\mathcal{C})$$

where

$$X_i(\mathcal{C}) = \{(\mathcal{A}, i)\colon\ \mathcal{A} \text{ is an } (I \setminus i)\text{-class in } \mathcal{C}\}.$$

A subset $A = \{(\mathcal{A}_1, i_1), \ldots, (\mathcal{A}_r, i_r)\}$ of $X(\mathcal{C})$ is a simplex if the i_ν are pairwise distinct, and if

$$\bigcap_{\nu=1}^{r} \mathcal{A}_\nu \neq \varnothing.$$

It is obvious that $\Delta(\mathcal{C})$ is a simplicial complex. Since the various $(I\setminus j)$-stars in a simplex have a common representative, every simplex is of the form

$$C[J] := \{(\mathcal{S}(C, I \setminus j), j)\colon\ j \in J\}.$$

This also shows that every simplex is contained in a simplex of type I, namely $C[J] \subseteq C[I]$, and therefore $\Delta(\mathcal{C})$ indeed is a numbered complex.

The construction is functorial since a morphism of chamber systems $\varphi\colon \mathcal{C} \to \mathcal{C}'$ relative to some bijection $\varphi_*\colon I \to I'$ of the corresponding type sets associates an $(I' \setminus \varphi_* i)$-class to any $(I \setminus i)$-class in $\mathcal{C}$. This extended map which we denote by $\widetilde{\varphi}$ satisfies the rule

$$\widetilde{\varphi}C[J] = (\varphi C)[\varphi_* J],$$

in particular $\widetilde{\varphi}$ is a morphism of numbered complexes. The map

$$\mathcal{C} \to \mathcal{C}(\Delta(\mathcal{C})), \quad C \mapsto C[I]$$

is surjective, and it is a morphism of chamber systems. For $C, C' \in \mathcal{C}'$, the equality $C[I] = C'[I]$ holds if and only if $C \overset{I\setminus i}{\sim} C'$ for all $i \in I$. Similarly, for fixed i we have $C[I] \overset{i}{\sim} C'[I]$ if and only if $C[I] \overset{I\setminus j}{\sim} C'[I]$ for all $j \neq i$. This proves the implication (iii) $\Rightarrow$ (ii) of the following proposition. The implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) are obvious.

1.3.3. PROPOSITION. *For a chamber system $\mathcal{C}$, the following three properties are equivalent.*

(i) $\mathcal{C} \simeq \mathcal{C}(\Delta)$, *for a numbered complex Δ over I.*

(ii) *The canonical morphism $\mathcal{C} \to \mathcal{C}(\Delta(\mathcal{C}))$ is an isomorphism.*

(iii) $C, D \in \mathcal{C}$, $C \overset{I\setminus i}{\sim} D$ *for all $i \in I$ implies $C = D$, and $C, D \in \mathcal{C}$, $C \overset{I\setminus j}{\sim} D$ for all $j \neq i$ implies $C \overset{i}{\sim} D$, for all $i \in I$.*

In order to recover Δ from $\mathcal{C}(\Delta)$, we start with an arbitrary numbered complex Δ, and we study the relationship between Δ and $\Delta(\mathcal{C}(\Delta))$. There is an obvious canonical morphism

$$\Delta(\mathcal{C}(\Delta)) \to \Delta$$

defined as follows. Take any simplex in $\Delta(\mathcal{C}(\Delta))$, write it in the form $C[J]$ for some $C \in \mathcal{C}(\Delta)$, $J \subseteq I$. As earlier, denote by $C(J) \subseteq C$ the face of type J in C. We claim that

$$C[J] \mapsto C(J)$$

is well defined; it is then clear that this is a type preserving morphism of numbered complexes. If $C[J] = D[J]$, $C, D \in \mathcal{C}(\Delta)$, then $C \overset{I\setminus j}{\sim} D$ for all $j \in J$. But this implies that the vertices $C(j), D(j)$ of type j in C, resp., D, coincide, and thus $C(J) = D(J)$, as desired. The injectivity of the map $C[J] \mapsto C(J)$ is guaranteed once we know that any two chambers C, D such that $C(j) = D(j)$ for some $j \in I$ are indeed $(I \setminus i)$-equivalent, that is, can be connected by a gallery inside $\mathrm{St}\, C(j)$. We finally note that the connectedness of $\mathrm{St}\, x$, for all vertices x, automatically holds in complexes of the shape $\Delta(\mathcal{C})$. Indeed, the chambers in $\Delta(\mathcal{C})$ are of the form $C[I] = \{\mathcal{S}(C, I \setminus i)\colon\ i \in I\}, C \in \mathcal{C}$, and for fixed $x = (\mathcal{S}(C_0, I \setminus j), j)$, the $C[I]$'s containing x are precisely the $C[I]$ with $C \in \mathcal{S}(C_0, I \setminus j)$, i.e. $C \overset{I\setminus j}{\sim} C_0$. Applying the morphism $C \mapsto C[I]$, $\mathcal{C}(\Delta(\mathcal{C})) \to \mathcal{C}$, we see that $C[I] \overset{I\setminus j}{\sim} C_0[I]$ for any two chambers $C_0[I], C[I] \in \mathrm{St}\, x$, that is $C_0[I], C[I]$ are connected inside $\mathrm{St}\, x$.

Altogether, we have proved the following proposition.

1.3.4. PROPOSITION. *For a numbered complex Δ, the following three properties are equivalent:*

(i) $\Delta \simeq \Delta(\mathcal{C})$, *for some chamber system $\mathcal{C}$.*

(ii) *The canonical morphism $\Delta(\mathcal{C}(\Delta)) \to \Delta$ is an isomorphism.*

(iii) *All stars of vertices in Δ are connected.*

We shall finally show that the two functors $\Delta \mapsto \mathcal{C}(\Delta)$ and $\mathcal{C} \mapsto \Delta(\mathcal{C})$, preserve the property of strong connectedness. For this, we have however to assume that the conditions of Propositions 1.3.3 and 1.3.4 hold.

1.3.5. PROPOSITION.

(a) *A numbered complex Δ is strongly connected if and only if each* $\mathrm{St}_\Delta\, x, x$ *a vertex, is connected, and its chamber system $\mathcal{C}(\Delta)$ is strongly connected.*

(b) *A chamber system $\mathcal{C}$ is strongly connected if and only if the conditions of* 1.3.3 (iii) *hold, and if its associated complex $\Delta(\mathcal{C})$ is strongly connected.*

PROOF. Let Δ be strongly connected. Then

$$C \overset{J}{\sim} D \Rightarrow C(I \setminus J) = D(I \setminus J),$$

for any two chambers $C, D \in \mathcal{C}(\Delta)$, and any $J \subseteq I$. From this, the desired implication

$$C \overset{J}{\sim} D, C \overset{K}{\sim} D \Rightarrow C \overset{J\cap K}{\sim} D, \quad C, D \in \mathcal{C}(\Delta),\ J, K \subseteq I,$$

immediately follows. The second condition for strong connectedness is just the ordinary connectedness, that is the connectedness of $\Delta = \mathrm{St}\, \varnothing$.

Next, start with a strongly connected chamber system $\mathcal{C}$, and consider any simplex $C_0[J]$, $C_0 \in \mathcal{C}$, $J \subseteq I$ in $\Delta(\mathcal{C})$. If $C, D \in \mathcal{C}$ are such that $C[I], D[I]$ both contain some chamber $C_0[J]$ of $\mathcal{C}(\Delta)$, then $C[J] = C_0[J] = D[J]$, which by definition means that $C \overset{I\setminus j}{\sim} C_0 \overset{I\setminus j}{\sim} D$ for all $j \in J$. Now the strong connectedness of $\mathcal{C}$ implies that

$C \overset{I\setminus J}{\sim} D$, and therefore also $C[I] \overset{I\setminus J}{\sim} D[I]$. Thus $C[I]$ is connected to $D[I]$ inside $\operatorname{St} C_0[J]$, as desired.

For the converse direction in (a), notice that $\Delta \simeq \Delta(\mathcal{C}(\Delta))$, by Proposition 1.3.4, and $\Delta(\mathcal{C}(\Delta))$ is strongly connected by (b). Analogously, the converse direction in (b) follows from (a) and Proposition 1.3.3. □

Combining all the propositions of this subsection in the strongly connected case finally gives Theorem 1.3.1.

1.4. *Numbered complexes and chamber systems with a group action*

Let Δ be an I-numbered complex and G a group acting on Δ by type preserving automorphisms and transitively on chambers. Fix a chamber $\{x_i\colon i \in I\}$ of Δ and consider the stabilizers $G^i := G_{x_i} \subseteq G$ of its vertices. The complex Δ can be recovered from G and the G^i as follows. Notice first that G acts transitively on the simplices of any fixed type, in particular, transitive on the vertices of any type i. Let vertices $g_j x_j,\ j \in J \subseteq I,\ g_j \in G$, be given. They form a simplex if and only if there exists a $g \in G$ such that

$$g\{x_j\colon j \in J\} = \{g_j x_j\colon j \in J\},$$

that is $g x_j = g_j x_j$ for all $j \in J$, or equivalently $gG^j = g_j G^j$ for all $j \in J$. Setting

$$G^J := \bigcap_{j \in J} G^j,$$

this means that the mapping

$$G/G^J \to \Delta, \quad gG^J \mapsto \{g x_j\colon j \in J\},$$

is a bijection of G/G^J onto the simplices of type J. This observation leads to the following

1.4.1. Definition. Let G be any group, and $G^i,\ i \in I$, a family of subgroups. Let

$$\Delta(G, G^i, i \in I) := \coprod_{J \subseteq I} G/G^J$$

and define a relation $\leqslant$ on $\Delta(G, G^i, i \in I)$ by

$$\left(J, gG^J\right) \leqslant (K, hG^K) \Leftrightarrow J \subseteq K \text{ and } gG^J \cap hG^K \neq \varnothing. \tag{$*$}$$

Notice that under the assumption $J \subseteq K$, it actually follows from $gG^J \cap hG^K \neq \varnothing$ that $gG^J \supseteq hG^K$. In particular, $\leqslant$ indeed is a partial ordering.

1.4.2. PROPOSITION. *For a group G and subgroups G^i, $i \in I$, consider*

$$\Delta = \Delta\big(G, G^i, i \in I\big)$$

with the partial ordering defined by (∗).

(a) *$(\Delta, \leqslant)$ is a numbered complex with vertex set $\{(i, xG^i)\colon\ i \in I,\ x \in G\}$, chamber set $\{(I, xG^I)\colon\ x \in G\} \cong G/G^I$, and type function $(i, xG^i) \mapsto i$.*

(b) *G operates type preservingly and chamber transitively on Δ by left translations.*

(c) *The star in Δ of the 'typical simplex of type J' equals the corresponding complex for the group G^J:*

$$\mathrm{St}_\Delta\big(J, G^J\big) = \Delta\big(G^J, G^{J\cup i},\ i \in I \setminus J\big).$$

(d) *Any numbered complex with a chamber transitive group G of special automorphisms is G-isomorphic to a complex of the shape $\Delta(G, G^i, i \in I)$.*

PROOF. (a), (b) and (c) are trivial, and (d) has been observed before. □

In Section 1.1, we have introduced a couple of elementary concepts for complexes like chambers, connectedness, and the basic graph. For transitive complexes, these concepts will now be translated into the group G and its system of subgroups G^i, $i \in I$.

Let $\Delta = \Delta(G, G^i, i \in I)$ be as above, set $C := (I, G^I)$ the 'standard' chamber,

$$G_i := G^{I\setminus i}, \quad i \in I,$$

and for $g \in \langle G_i, i \in I\rangle$, let $l(g)$ be the smallest l such that g is a product of l elements in the generating set $\bigcup G_i$.

1.4.3. PROPOSITION.

(a) *$gC \overset{i}{\sim} hC \Leftrightarrow gG_i = hG_i$, for any two elements $g, h \in G$.*

(b) *For $g \in G$, the mapping*

$$(s_1, \ldots, s_q) \mapsto (C, s_1C, \ldots, s_1s_2\ldots s_iC, \ldots, gC)$$

is a surjection of the set of words over $\bigcup G_i$ representing g onto the set of galleries joining C and gC. It is a bijection if and only if $G^I = \{1\}$, and in this case

$$d(C, gC) = l(g) \quad \text{for all } g \in G.$$

(c) *Δ is connected if and only if G is generated by the G_i.*

(d) *Δ is strongly connected if and only if*

$$G^J = \langle G_i\colon\ i \in I \setminus J\rangle \quad \text{for all } J \subset I.$$

1.4.4. PROPOSITION. *Under the assumptions of Propositions* 1.4.2 *and* 1.4.3, *two elements* $i, j \in I$ *are not joined in the basic graph of* Δ *if and only if*

$$G_i G_j = G^{I \setminus \{i,j\}}.$$

PROOF. By transitivity and Proposition 1.4.2(c), a typical star of type $\{i, j\}$ is a generalized digon if and only if $xG_i \cap G_j \neq \varnothing$ for all elements $x \in G^{I \setminus \{i,j\}}$. □

We finally come to transitive actions on chamber systems. Let $\mathcal{C}$ be a chamber system over I and G a group acting transitively and type preservingly on $\mathcal{C}$. Fix a chamber $C \in \mathcal{C}$, and let $B := G_C$ be its stabilizer. It is clear that

$$G_i := \{g \in G\colon\ gC \overset{i}{\sim} C\}$$

is a subgroup of G containing B, and that two arbitrary chambers gC and hC are i-adjacent if and only if $gG_i = hG_i$. This proves part (d) of the following analogue of Proposition 1.4.2, the other statements are equally trivial.

1.4.5. PROPOSITION. *Let* G *be a group,* $B \subseteq G$ *a subgroup and* G_i, $i \in I$, *a family of subgroups containing* B. *Let*

$$\mathcal{C}(G, B, G_i, i \in I) := G/B$$

together with the family of relations

$$xB \overset{i}{\sim} yB \Leftrightarrow xG_i = yG_i.$$

Then the following is true.

(a) $\mathcal{C}(G, B, G_i, i \in I)$ *is a chamber system over* I.

(b) G *operates type preservingly and transitively on* $\mathcal{C}(G, B, G_i, i \in I)$ *by left translations.*

(c) *For* $J \subseteq I$, *the* J-*star of* C *equals the corresponding chamber system for the subgroup* $G_J := \langle G_j\colon\ j \in J\rangle$:

$$\mathcal{S}(C, J) = \mathcal{C}(G_J, B, G_j, j \in J).$$

(d) $\mathcal{C}(G, B, G_i, i \in I)$ *is strongly connected if and only if*

$$G_J \cap G_K = G_{J \cap K}$$

for any two subsets $J, K \subseteq I$.

(e) *Any chamber system with a transitive group* G *of special automorphisms is* G-*isomorphic to a chamber system of the form* $\mathcal{C}(G, B, G_i, i \in I)$.

Finally, we give the description of the basic graph of a transitive chamber system which is the same as in the case of complexes.

1.4.6. PROPOSITION. *Under the assumptions of Proposition* 1.4.5, *the vertices* $i, j \in I$ *are not joined in the basic graph of* $C(G, B, G_i, i \in I)$ *if and only if* $G_i G_j = G_{\{i,j\}}$.

Notes to Section 1

This first section of our report is intended to be a unified and reasonably complete account of the basic concepts of the theory, including both numbered complexes and chamber systems in an equally complete fashion, without favouring either point of view. There is no direct reference in the literature for this kind of presentation, except that part of it, in condensed form, is contained in Scharlau [1990]; in particular, Proposition 1.1.4 is Proposition 1.1 of *loc. cit.*

Our treatment of numbered complexes goes back to unpublished lecture notes of the author (University of Bielefeld, 1982) and has emerged as a detailed exposition of the first chapter of the fundamental text Tits [1974] combined with earlier work by Tits, in particular Tits [1956]. In those lectures, we had tried to make explicit and precise various definitions and easy consequences which were only sketched in Tits' papers. Our treatment is to a certain extent parallel to parts of Buekenhout's work on the foundations of incidence geometries which was inspired by more or less the same sources. The Bielefeld lectures just mentioned were continued in 1982/83 jointly with Andreas Dress; the emphasis of this course was on the 'second fundamental paper' Tits [1981a]. Our treatment of chamber systems in the present article is strongly influenced by this collaboration with A. Dress who, in the context of tessellations of manifolds, had developed the essentials of the concept of a chamber system independently of Tits; cf. Dress [1987].

It would be possible to give explicit references for almost all results of this section. We shall not do this since the only essential and original references to the best of our knowledge are those to the work of Tits already given. This first section is to be considered as the first chapter of a textbook rather than a complete report on existing literature. Our main aim is to prepare the ground for the following subsections where we wish to present some of the advanced results of the theory of buildings.

We now want to mention three important topics which we did not treat so far since they are somewhat beyond the main stream of our exposition.

Incidence geometries. If $(X, *)$ is an incidence geometry with set of types I and set of objects $X = \bigcup_{i \in I} X_i$, then we can consider the *incidence complex* or *flag complex* $\mathrm{Flag}(X, *)$ which is defined as follows. Its vertex set is X, and a subset $A \subseteq X$ is a simplex if it is a flag of the geometry, i.e. its elements are pairwise incident. This is a numbered complex if the standard axiom holds that any flag should be contained in a flag of type I. The geometry can be reconstructed from its (numbered) complex in the obvious way: the set of objects is the vertex set of the complex, two objects a, b are incident if and only if $\{a, b\}$ is a simplex. Two incidence geometries are isomorphic if and only if their flag complexes are isomorphic. A numbered complex is (isomorphic to) an incidence complex if and only if the following two equivalent conditions are satisfied:

(I) If A is a set of vertices such that any two of them form an edge, then A is a simplex.

(I′) If A, B, C are simplices such that any two of them are contained in a common simplex, then all three are contained in a common simplex.

These remarks show that, formally, the theory of incidence geometries can be embedded into the theory of numbered complexes. In the author's opinion, one loses nothing if one uses throughout the more general notion. For most of what is done with incidence geometries, the additional property (I) is not used.

At this place, a remark on strong connectedness is in order. The connectedness of an incidence geometry is usually defined in terms of its incidence graph. This of course is different from the chamberwise connectedness as defined for complexes. It is however easily shown that the residual (strong) connectedness of a geometry is equivalent to the strong connectedness of its flag complex. The proof is most easily achieved if one shows at the same time the strong connectedness of the corresponding chamber system.

We shall occasionally use parts of the common terminology for incidence geometries also for numbered complexes: the vertices are sometimes called *objects*, the simplices are *flags* (also in the case when the incidence relation is not given by a partial ordering of the objects), and the stars are often called *residues*. The latter terminology is also quite common for stars in chamber systems.

Geometrical realizations. Every simplicial complex Δ as defined in this text has a standard geometrical realization, where each simplex becomes an actual simplex in real Euclidean space, i.e. its vertices are considered as affinely independent points and one takes their full convex hull. See, e.g., Spanier [1966], Chapter 3 for a formal definition and a description of the topology on the space $\|\Delta\|$ thus derived from Δ. (It is the weak topology with respect to the simplices.) In fact, it was precisely for this purpose that the notion of a (combinatorial) simplicial complex was invented in the 30's; see, e.g., Seifert and Threlfall [1934/80], Kapitel 2. Although the geometrical realization of a building is formally not needed for the purposes of this article, we would like to emphasize the fact that it is nevertheless an important aspect of buildings. For certain applications, in particular of affine buildings, it is even the only thing which really matters: A building is viewed as a certain topological space on which a certain group acts; it is not important how this space was constructed and what its combinatorial axioms are. See Chapter 20 for more details. Even if one stays inside the combinatorial framework of the present chapter, the geometrical realization is sometimes helpful to visualize certain concepts. This remark in particular applies to the theory of Coxeter–Tits complexes (Section 2) and to Section 5.

If $\mathcal{C}$ is a chamber system, one can easily and naturally define a geometrical realization $\|\mathcal{C}\|$ in a direct way, not using the relation between chamber systems and complexes. To construct $\|\mathcal{C}\|$, one takes for each chamber C a copy $\|C\|$ of the standard simplex $\|I\|$ (with vertex set the type set I of the chamber system), and one identifies the codimension-one-faces of the various $\|C\|$ according to the equivalence relation of the chamber system. In this way, one obtains a topological space with a 'cell complex structure' (which in the strongly connected case is identical to the geometrical realization of the simplicial complex $\Delta(\mathcal{C})$ introduced before). See, i.e. Dress [1987], Section 1, or Ronan [1989], Chapter 1.1. In the non-strongly-connected case, $\|\mathcal{C}\|$ is not necessarily the geometrical realization of a simplicial complex. But this is not the only reason why geometrical realizations of chamber systems are of independent interest. The essential point is that in the case of chamber systems one can replace the standard simplex over I by an arbitrary 'space with panels', where the panels are subspaces indexed by the elements of I. See, e.g., Dress [1987], Section 4 (e), or Scharlau [1990], Section 2. Such more general chambers naturally occur for reducible reflection groups on Euclidean space, and thus for affine buildings of semisimple groups which are not simple, and for reflection groups acting on hyperbolic space. Cf. the notes of Section 2.

Shadows. At the beginning of these notes, we have mentioned the paper Tits [1956] as one of the early sources for today's formalism of incidence geometries, and thus for the axiomatic background of building theory. At about the same time Tits [1957] also proposed a slightly different geometrical interpretation of the simple Lie groups (notably the exceptional ones). This approach was based on the notion of an 'R-space' which by definition is a set of 'points' together with certain subsets called 'subspaces'. Later, after the formal definition of a building had been introduced, it became clear that the point of view of R-spaces is in a sense subordinate to the concept of a numbered complex (though perhaps geometrically more intuitive). Using today's point of view, the precise connection between the two approaches is achieved via the notion of shadows and 'shadow geometries'. To any numbered complex Δ together with a distinguished subset $I_0 \subseteq I$ of its type set, one can associate a shadow geometry which is defined as follows: the 'set of points' is the set $S(\Delta, I_0)$ of all simplices of type I_0, the subspaces are the 'shadows' $\mathrm{Sh}_{I_0}(B)$ of simplices $B \in \Delta$, where $\mathrm{Sh}_{I_0}(B)$ is defined as the set of all 'points' incident with B:

$$\mathrm{Sh}_{I_0}(B) = \{A \in S(\Delta, I_0)\colon\ A \cup B \in \Delta\}.$$

One important consequence of 1.1.4 is the fact that one obtains all shadows $\mathrm{Sh}_{I_0}(B)$ if one restricts B to the simplices whose type is I_0-reduced in the sense of Proposition 1.1.7(a). Often, one takes for I_0 a one-element subset of I, preferably one of the end nodes of the diagram of Δ if this is a tree, in particular, if it is a 'chain' like the spherical Coxeter diagrams A_n or C_n. In the latter case, the reduced types are just the one-element subsets of I.

Formally, one could consider a shadow geometry as a partially ordered set, namely the set of all subspaces (including single points) ordered by inclusion. In Scharlau [1990] we have proposed a slightly different definition of a partially ordered set $\mathcal{S}(\Delta, I_0)$ which behaves better with respect to various properties of the complex Δ and which, in important cases like buildings, is canonically isomorphic to the partially ordered set of subspaces. The definition of $\mathcal{S}(\Delta, I_0)$ and its partial order relies on the fundamental theorem on the basic graph 1.1.4, and on Proposition 1.1.7 which is part of Proposition 1.7 of *loc. cit.* As a set, $\mathcal{S}(\Delta, I_0)$ just consists of all simplices of reduced type. See the notes of Section 6 for the definition of the partial ordering. The main result of *loc. cit.*

says that the geometrical realization of any shadow poset of a strongly connected numbered complex Δ is canonically homeomorphic to the geometrical realization of Δ itself. This explains why many properties of Δ are inherited by all of its shadow geometries. Looking once more at the case where the basic graph is a chain and I_0 consists of one end node, we observe that the shadow poset $\mathcal{S}(\Delta, I_0)$ is identical to the partially ordered set introduced in Example 1.1.6.

The main properties of shadow geometries in case that Δ is a building have been derived in the Appendix 2 of Tits [1974]; we will come back to this in Section 6.4.

2. Coxeter–Tits complexes

Introduction

The finite irreducible Coxeter–Tits complexes are parametrized by the so called spherical Coxeter diagrams shown in the following Figure 2.1. Here n denotes the number of nodes.

The Coxeter–Tits complex belonging to a diagram M will be denoted by $\Sigma(M)$. Here, M can be any graph on a finite vertex set I with 'labels' $\in \{3, 4, \ldots, \infty\}$ for the edges (see Chapter 3). Equivalently, M can be a so-called *Coxeter matrix*

$$M = (m_{ij})_{i,j \in I}, \quad m_{ij} \in \{2, 3, \ldots, \infty\} \text{ if } i \neq j,$$

$$m_{ij} = m_{ji} \text{ for all } i, j \in I,$$

$$m_{ii} = 1 \text{ for all } i \in I.$$

The edges of the diagram are the $\{i, j\}$ such that $m_{ij} \geqslant 3$. A formal definition of $\Sigma(M)$ will be given below in 2.1.6.

The complexes $\Sigma(M)$, where M is a 'linear' diagram in the above list, are the barycentric subdivisions of the classical Platonic solids: for A_n the n-simplex, B_n the n-cube or n-cross-polytope, F_4 the 24-cell in 4-space, H_3 the dodecahedron or icosahedron, H_4 the 120-cell or 600-cell in 4-space, cf. Coxeter [1963]. The complex $\Sigma(\mathsf{I}_2(m))$ is simply the ordinary $2m$-gon, considered as a bipartite graph. All these barycentric subdivisions are considered as numbered complexes in the sense of Section 1: the type of an original vertex of the solid is 1, the type of the centre of an edge is 2, the type of the centre of a 2-face is 3, etc. $\Sigma(\mathsf{D}_n)$ comes from the n-cross-polytope (n-octahedron) by partitioning the maximal faces into two types $n-1$, n in such a way that any two maximal faces of the same type intersect in a face of even codimension. The faces of dimension $i = 0, 1, \ldots, n-3$ are the objects of type $i+1$ as in the C_n-case. The complexes $\Sigma(\mathsf{E}_n)$, $n = 6, 7, 8$, are related to the so called Gosset's polytopes in n-space; see Coxeter [1963], Chapter 11.8.

The technically most efficient way of introducing Coxeter–Tits complexes is via Coxeter groups. Indeed, the above spherical Coxeter diagrams occur in the classification

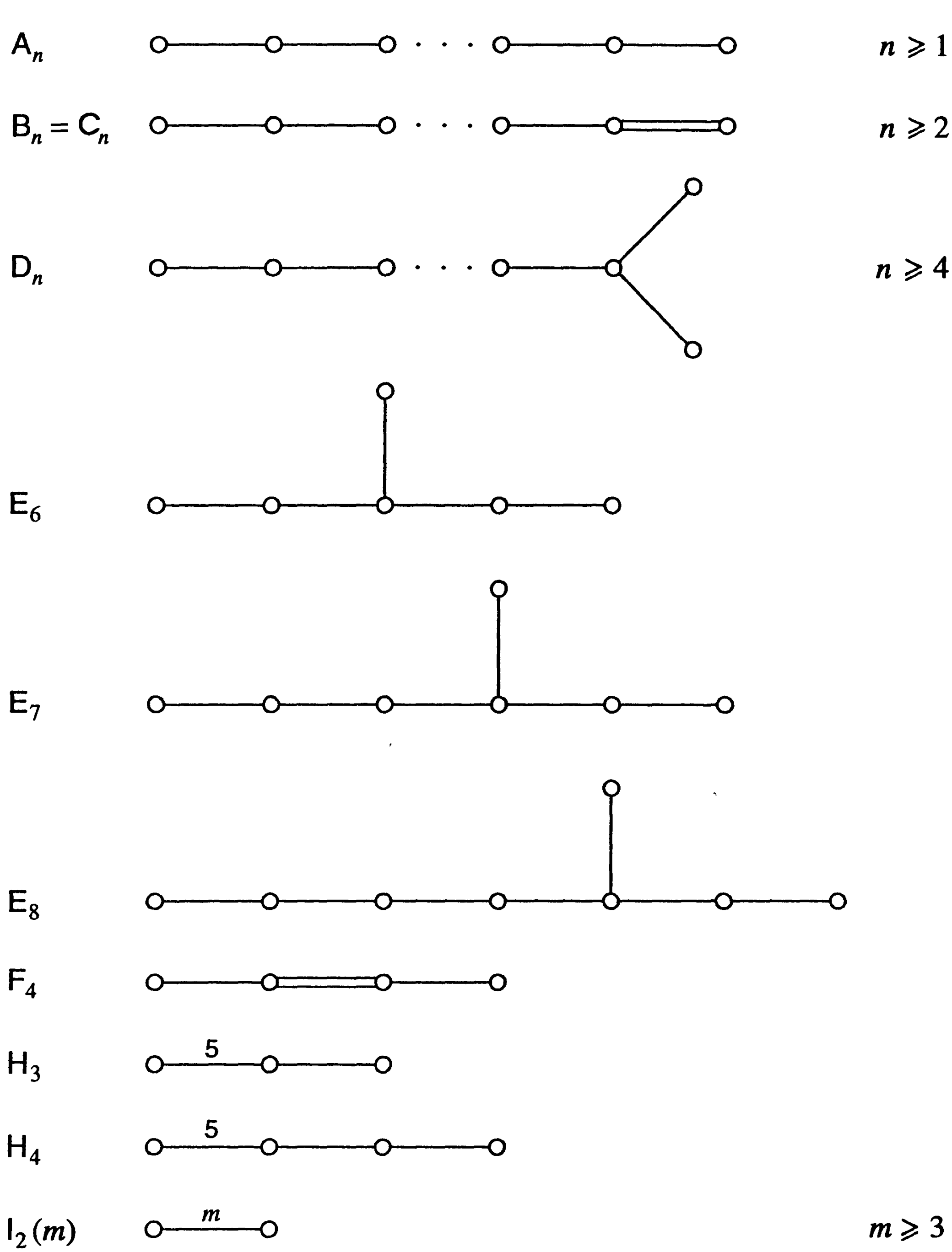


Figure 2.1. The irreducible spherical Coxetér diagrams.

of the finite reflection groups in Euclidean space (i.e. finite groups generated by reflections in hyperplanes Coxeter [1935]). The Coxeter groups are an 'abstract' version of the reflection groups. We shall report briefly on these in the following Section 2.1. In Sections 2.2, 2.3, and 2.4 we shall present a more geometrical approach to Coxeter–Tits complexes based on the notion of a folding. In the context of buildings this is most useful because the characterization by foldings will later allow us to prove the basic theorem by Tits that the apartments in a thick building automatically are Coxeter–Tits complexes.

2.1. *A survey of Coxeter groups*

DEFINITION. Let $M = (m_{ij})_{i,j\in I}$ be a Coxeter matrix as defined in the introduction. The corresponding *Coxeter group* $W(M)$ is defined by generators and relations as follows:

$$W(M) = \langle i \in I\colon\ (ij)^{m_{ij}} = 1 \text{ if } m_{ij} \neq \infty\rangle.$$

The group element corresponding to i will usually be denoted by s_i and is called a reflection. If $i_1i_2i_3\ldots$ is a word over I, i.e. an element of the free monoid over I, we denote by $s(i_1i_2i_3\ldots)$ its image in $W(M)$. More generally, if $(W, s_i, i \in I)$ is a system consisting of a group W and a family of generating involutions indexed by I, we say that $(W, s_i, i \in I)$ is a *Coxeter system* (or, loosely speaking, that W is a Coxeter group) with Coxeter diagram M if m_{ij} is an exponent of s_is_j, and if the canonical epimorphism $W(M) \to W$ is an isomorphism. The definition of a Coxeter group and the terminology of reflections are motivated by the fact that every discrete group of isometries of Euclidean space generated by reflections in hyperplanes is a Coxeter group, the generating reflections being the reflections in the walls of a fixed 'geometrical chamber'. Here, a geometrical chamber is defined as a connected component of the complement of the union of all hyperplanes whose orthogonal reflection belongs to W. See Bourbaki [1968], Chapter 5, §3.2.

To give a purely combinatorial example, we consider the symmetric group $\mathbb{S}_{n+1}$ on the set $\{1,\ldots,n+1\}$ and for $i = 1,\ldots,n$ the transposition $\sigma_i = (i, i+1) \in \mathbb{S}_{n+1}$. If we look at the diagram A_n above (numbering the nodes $1,\ldots,n$ in natural order), we see that there is a surjective homomorphism $W(\mathsf{A}_n) \to \mathbb{S}_{n+1}$ defined by $s_i \mapsto \sigma_i$. This is in fact an isomorphism, and thus $\mathbb{S}_{n+1}$ is a Coxeter group.

The following basic theorem due to Witt and Tits says that an arbitrary Coxeter diagram can be recovered from its group, i.e. there is a one-to-one correspondence between Coxeter diagrams and Coxeter groups.

2.1.2. THEOREM. *Let M be any Coxeter diagram. Then m_{ij} is the precise order of s_is_j in the corresponding Coxeter group $W(M)$. In particular, the s_i are involutions and remain distinct in $W(M)$.*

This theorem is proved by showing that an arbitrary Coxeter group possesses a 'geometrical representation' by linear transformations on a vector space such that the reflections correspond to (not necessarily Euclidean) reflections in hyperplanes. For details and proofs, the reader is referred to Bourbaki [1968], Chapter 5, §4, in particular Section 4.3.

The following property of Coxeter groups is crucial for the application of Coxeter groups to the geometrical properties of buildings.

2.1.3. DEFINITION. Let W be a group, together with a generating set S consisting of involutions. For $w \in W$, let

$$l(w) = \min\{q\colon\ \exists s_1,\ldots,s_q \in S \text{ such that } s_1\ldots s_q = w\}$$

be the length with respect to S. We say that (W, S) satisfies the *exchange condition* if, for every *reduced expression*

$$w = s_1 \dots s_q, \quad s_i \in S,\ q = l(w),$$

and for every $s \in S$ such that $l(sw) \leqslant q$ there exists a $t \in \{1, \dots q\}$ such that

$$ss_1 \dots s_q = s_1 \dots \hat{s}_t \dots s_q.$$

(Here, $\widehat{\ }$ means that the corresponding term is omitted.)

2.1.4. THEOREM (Matsumoto [1964]). *Let W be a group and $S \subseteq W$ a set of involutions which generates W. The exchange condition is satisfied for (W, S) if and only if (W, S) is a Coxeter system.*

For a proof, see Bourbaki [1968], Chapter 4, §1.6.

The following results about Coxeter groups are well known and easily proved using the exchange condition, see *loc. cit.* §1.8.

2.1.5. PROPOSITION. *Let (W, S) be a Coxeter system.*

(a) *If $s_1, \dots, s_q, s'_1, \dots, s'_q \in S$, $s_1 \dots s_q = s'_1 \dots s'_q = w$, and $l(w) = q$, then $\{s_1, \dots, s_q\} = \{s'_1, \dots, s'_q\}$.*

(b) *For $X \subseteq S$, set $W_X := \langle X \rangle \subseteq W$. For any family (X_i) of subsets of S we have*

$$W_{\bigcap X_i} = \bigcap W_{X_i}.$$

In particular,

$$\bigcap_{s \in X} W^s = W_{S \setminus X} =: W^X, \quad \text{where } W^s := W_{S \setminus s}.$$

For $w \in W$, we denote by S_w the unique set $\{s_1, \dots, s_q\}$ of generators in S occurring in any reduced expression $w = s_1 \dots s_q$. Notice that (b) is an immediate consequence of (a) since obviously

$$W_X = \{w \in W\colon\ S_w \subseteq X\}.$$

If $W = W(M)$, $S = \{s_i\colon\ i \in I\}$, for a Coxeter diagram M with index set I, and $J \subseteq I$ we set $W_J := W_{\{s_j\colon\ i \in J\}} = \langle s_j\colon\ j \in J \rangle$, and $W^J := \langle s_i\colon\ i \in I \setminus J \rangle$.

2.1.6. DEFINITION. Let M be a Coxeter diagram over I. The *Coxeter–Tits complex of type M* or *belonging to M* is defined to be

$$\Sigma(M) := \Delta\big(W(M), W^i, i \in I\big)$$

(see Section 1.4).

Of course, any numbered complex isomorphic to some $\Sigma(M)$ is also called a Coxeter–Tits complex. From Proposition 2.1.5 it immediately follows that the condition of Proposition 1.4.3(a) holds. Therefore, any Coxeter–Tits complex is strongly connected.

The following proposition will later be reinterpreted geometrically for the Coxeter–Tits complexes. For $X = \varnothing$ it is a special case of Theorem 5.2.1.

2.1.7. PROPOSITION. *Let (W, S) be a Coxeter system, and $X, Y \subseteq S$. Let w_0 be a shortest element in its double coset $W_X w_0 W_Y$. Then, for any $w \in W_X w_0 W_Y$, there exist $x \in W_X$, $y \in W_Y$ such that*

$$w = xw_0y \quad \text{and} \quad l(w) = l(x) + l(w_0) + l(y).$$

In particular, w_0 is uniquely determined by its (double) coset.

For purposes of later reference, we close this subsection with a proposition which generalizes the universal property defining a Coxeter group.

2.1.8. PROPOSITION. *Let M be a Coxeter diagram over I. Let a_i, $i \in I$, be elements of a monoid A such that*

$$a_i a_j a_i \ldots = a_j a_i a_j \ldots \quad m_{ij} \text{ factors},$$

for all $i, j \in I$, $i \neq j$. Then there exists a map (obviously unique) from $W(M)$ into A mapping $s(i_1 i_2 i_3 \ldots)$ to $a_{i_1} a_{i_2} a_{i_3} \ldots$, for any reduced word $i_1 i_2 i_3 \ldots$ over I.

This is Proposition 5 in Chapter 4, §1 of Bourbaki [1968], where the reader will find a proof. It is also an immediate consequence of Theorem 2.5.2 below.

2.2. *Thin complexes and reflections*

In this subsection, all complexes are 'chamber complexes', that is, pure and of finite rank and all morphisms are supposed to map chambers onto chambers. We shall not assume that the complexes are numbered, unless this is stated explicitly. We recall the notation

$$\mathcal{P}(A) = \{B \in \Delta \colon\ B \subseteq A\},$$

for any simplex A in a complex Δ.

A complex is called *thin* (*thin-with-boundary*) if every panel is contained in exactly 2 (at most 2) chambers. The *boundary* is the subcomplex of all panels contained in precisely one chamber, together with all their faces. If

$$\Sigma = \Sigma(M) = \Delta(W, W^i, i \in I)$$

is a Coxeter–Tits complex, the stabilizer W_i of a typical panel of cotype i has order 2, and thus Σ is thin (without boundary, cf. Proposition 1.4.2(c)).

Remark that a triangulation of a manifold is known to be thin-with-boundary, and the boundary then corresponds to the topological boundary. In topology, a thin complex is therefore also called a pseudomanifold.

The following easy lemma is fundamental for the study of thin complexes. We recall that a gallery $(C_0, \ldots, C_m)$ is simple if $C_{\nu-1} \neq C_\nu$ for all ν.

2.2.1. LEMMA. *Let Σ, Σ' be thin-with-boundary, $\varphi, \psi\colon \Sigma \to \Sigma'$ two morphisms, $C = (C_0, \ldots, C_m)$ a simple gallery, suppose that $\varphi|_{\mathcal{P}(C_0)} = \psi|_{\mathcal{P}(C_0)}$, consider the chamber subcomplex*

$$\Gamma := \bigcup_{\nu=0}^{m} \mathcal{P}(C_\nu).$$

(a) *Suppose that ψC is simple. Then either φC is not simple, or φ and ψ coincide on all of Γ.*

(b) *Suppose that $\Sigma = \Sigma'$, and $\varphi|_{\mathcal{P}(C_0)} = \mathrm{id}$. Then either φC is not simple, or φ is the identity on all of Γ.*

(c) *If Σ is connected, and φ, ψ restricted to the set of chambers $\mathcal{C}(\Sigma)$ are injective, then $\varphi = \psi$. In particular, if $\Sigma = \Sigma'$, $\varphi|_{\mathcal{P}(C_0)} = \mathrm{id}$, and φ is injective on the chambers, then φ is the identity.*

PROOF. (a) Assume that φC is simple, and that $\varphi|_{\mathcal{P}(C_\nu)} = \psi|_{\mathcal{P}(C_\nu)}$ is already shown for some ν. From

$$\varphi C_{\nu+1} \supseteq \varphi(C_\nu \cap C_{\nu+1}) = \psi(C_\nu \cap C_{\nu+1})$$

it follows that $\varphi C_{\nu+1} = \psi C_{\nu+1}$, since ψC_ν, $\psi C_{\nu+1}$ are the only chambers containing $\psi(C_\nu \cap C_{\nu+1})$, and $\varphi C_{\nu+1} \neq \psi C_\nu = \varphi C_\nu$, by assumption.

(b) is a specialization of (a) and (c) follows from (b). □

If Σ is a thin connected complex, an automorphism of Σ is called a *reflection* if it fixes all faces of some panel F, and interchanges the two chambers containing F. For a given F, there is at most one automorphism with this property, by Lemma 2.2.1. If it exists, we call it the *reflection in* F and denote it by σ_F. Again by Lemma 2.2.1 we have $\sigma^2 = \mathrm{id}$ for any reflection, i.e. reflections are involutions.

2.2.2. PROPOSITION. *Let Σ be a connected thin complex, suppose that the reflection in F exists for each panel F of some fixed chamber C. Then the group W generated by these reflections acts transitively on the chambers of Σ. If Σ is numbered, then W is equal to the group $W(\Sigma)$ of all type preserving automorphisms of Σ, and W operates sharply transitively on the chambers.*

PROOF. The orbit WC contains all chambers adjacent to any of its members, and therefore is all of $\mathcal{C}(\Sigma)$. If Σ is numbered, a reflection is necessarily type-preserving, i.e. $W \subseteq W(\Sigma)$. By Lemma 2.2.1(c), no nonidentity element of $W(\Sigma)$ fixes a chamber, and therefore $W = W(\Sigma)$, and this group acts sharply transitively on $\mathcal{C}(\Sigma)$. □

The ordinary k-gon, for odd k, is a non-numbered complex satisfying the assumptions of the first half of the proposition.

In the situation of 2.2.2, the group $W(\Sigma)$ is sometimes called the *Weyl group* of Σ. The barycentric subdivisions of the Platonic solids are examples of complexes having the properties stated in Proposition 2.2.2. More generally, all Coxeter–Tits complexes as defined in the previous subsection are of this kind. The results of the following subsections will give a characterization of the Coxeter–Tits complexes among all thin, flag-transitive complexes.

2.2.3. The general results of Section 1.4 immediately give the following description of a thin, strongly connected flag-transitive complex Σ over I as a coset complex: Choose a chamber C, denote by σ_i, $i \in I$, the reflection in the panel $C(i) \subseteq C$ of cotype i, and set $W^i = \langle \sigma_j \colon j \neq i \rangle$. Then

$$\Sigma \simeq \Delta(W(\Sigma), W^i, i \in I)$$

canonically. The groups $W_J := \langle \sigma_j \colon j \in J \rangle$ satisfy the conditions

$$W_I = W(\Sigma),$$
$$W_{J \cap K} = W_J \cap W_K \quad \text{for any two subsets } J, K \subseteq I.$$

Recall that the last property comes from the strong connectedness of Σ. If, conversely, $W = \langle \sigma_i \colon i \in I \rangle$ is a group generated by a family of involutions such that $W_{J \cap K} = W_J \cap W_K$ always holds, the W^i being defined as above, then $\Delta(W, W^i, i \in I)$ is a thin, flag-transitive complex.

Notice that stars in thin, flag-transitive complexes are always thin, flag-transitive. In the above notation, the Weyl group of $\mathrm{St}(C(I \setminus J))$ is to be identified with W_J.

2.3. *Foldings and convexity*

In this subsection we develop a geometrical property, the existence of 'enough foldings' that distinguishes the Coxeter–Tits complexes among all thin, flag-transitive complexes. This property roughly says that the fixed point set of any reflection separates the complex into two disjoint 'halves' which are interchanged by the reflection. Such a fixed point set is called a *wall*, the half-complexes usually being called *roots*. These roots will be axiomatized later. The reader should think of the two halfspaces determined by a hyperplane reflection in the n-sphere, Euclidean n-space, or hyperbolic n-space. More generally, it is one of the crucial axioms for 'topological reflection' groups that the complement of the fixed point set of a reflection should have precisely two connected components. See Koszul [1965], Chapter 3, §3, and also Straume [1981], Davis [1983]. This property is closely related to the simple connectedness of the 'space' on which the reflection group acts. If one adopts a certain combinatorial notion of simple connectedness (in the sense of '2-coverings' of chamber systems defined in 2.4.1 below) it is indeed true that the Coxeter complexes are the simply connected thin, flag-transitive complexes. Technically, the existence of 'enough foldings' of a thin complex automatically implies that Σ is flag-transitive; we do not have to assume the existence of reflections in advance.

Let Σ be thin complex. An endomorphism $\pi \colon \Sigma \to \Sigma$ is called a *folding* if it is idempotent ($\pi^2 = \pi$), and if every chamber in the image of π has precisely 2 preimages.

For a given folding π, define a map $\bar{\pi}$: $\mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma)$ as follows:

$$\bar{\pi}C = C \quad \text{if } C \notin \pi\Sigma,$$
$$\pi^{-1}C = \{C, \bar{\pi}C\} \quad \text{if } C \in \pi\Sigma.$$

The first basic fact about foldings is that $\bar{\pi}$ is always adjacency preserving; more precisely

2.3.1. LEMMA. *Let π be a folding, $C, D \in \mathcal{C}(\Sigma)$ be adjacent chambers. Then $\bar{\pi}C$ and $\bar{\pi}D$ are adjacent or coincide.*

PROOF. We may assume that $C \in \pi\Sigma$. Let $C' := \bar{\pi}C$, $F := C \cap D$, define $F' \subseteq C'$ by $\pi F' = F$. Let D' be the unique chamber such that $D' \cap C' = F'$. Now consider $\pi D'$. This equals C or D, since $\pi D' \supseteq \pi F' = F$. If $\pi D' = C$, then $D' = C$ (since $D' \neq C'$), in particular $F' \subseteq C$, and therefore $F' = F$. But then $\{D, C\} = \{D', C'\}$, that is $C' = D$, $\bar{\pi}D = C' = \bar{\pi}C$. If $\pi D' = D$, then $D' \neq D$, for otherwise $F = F'$, again $\{D, C\} = \{D', C'\}$, and $C = C'$, a contradiction. Thus $\bar{\pi}D = D'$ is adjacent to $\bar{\pi}C = C'$. □

2.3.2. REMARK. If the assumptions of Lemma 2.3.1 hold and in addition Σ is numbered, then π necessarily is type preserving, and if $C \overset{i}{-} D$, the proof just given shows that $\bar{\pi}C = \bar{\pi}D$ or $\bar{\pi}C \overset{i}{-} \bar{\pi}D$. That is, $\bar{\pi}$ is a type preserving morphism of chamber systems. If in addition all stars of vertices are connected, then by Proposition 1.3.4, $\bar{\pi}$ induces a morphism π': $\Sigma \to \Sigma'$. Obviously, π' is a folding as well. This gives the result of Guillitte and Lefèvre-Percsy [1981].

The last observation does not contribute to the theory of Coxeter–Tits complexes as presented here, since we shall conversely assume the existence of 'enough foldings' in order to conclude that Σ admits a numbering.

We now come to the notion of convexity, which is basic for many properties and applications of buildings, and whose importance is by no means restricted to the present section on the foundations of Coxeter–Tits complexes. Convexity is primarily defined for sets of chambers: A set of chambers $\mathcal{A}$ of some complex is called *convex* if it is connected and if, for any two chambers $C, D \in \mathcal{A}$, any geodesic (shortest gallery) joining C and D is contained in $\mathcal{A}$. Thus, we have the usual notion of convexity in a graph, where the graph structure on $\mathcal{C}(\Sigma)$ is the adjacency relation. Of course, this definition applies to subsets of arbitrary chamber systems.

2.3.3. PROPOSITION. *Let Σ be a thin connected complex, π a folding of Σ, set $\mathcal{D} = \pi\mathcal{C}(\Sigma)$, $\mathcal{D}' = \mathcal{C}(\Sigma) \setminus \mathcal{D} = \bar{\pi}\mathcal{C}(\Sigma)$.*

(a) *$\mathcal{D}, \mathcal{D}'$ are convex.*

Let $C \in \mathcal{D}$, $C' \in \mathcal{D}'$ be adjacent.

(b) *If D is an arbitrary chamber, then*

$$d(C', D) = d(C, D) + 1 \quad \textit{if } D \in \mathcal{D},$$

$$d(C', D) = d(C, D) - 1 \quad \textit{if } D \in \mathcal{D}'.$$

(c) *π is the unique folding mapping C' onto C.*

PROOF. (a) $\mathcal{D}$ and $\mathcal{D}'$ are connected, since $\mathcal{C}(\Sigma)$ is connected (for $\mathcal{D}'$, use Lemma 2.3.1). Consider a geodesic $\boldsymbol{C} = (C_0, \ldots, C_m)$ such that $C_0, C_m \in \mathcal{D}$. Suppose that $C_{\nu-1} \in \mathcal{D}$, $C_\nu \notin \mathcal{D}$ for some index ν. Then $\pi C_\nu = C_{\nu-1}$, therefore $\pi\boldsymbol{C}$ gives rise to a shorter gallery joining C_0 and C_m, a contradiction. For $C_0, C_m \in \mathcal{D}'$, we can apply the same argument, replacing π by $\bar{\pi}$. Remember that by Lemma 2.3.1, also $\bar{\pi}\boldsymbol{C}$ is a gallery.

(b) First consider the case $D \in \mathcal{D}$. Choose a geodesic $\boldsymbol{C} = (C', \ldots, D)$. As in (a), the gallery $\pi\boldsymbol{C} = (C, \ldots, D)$ is not simple, therefore $d(C, D) < d(C', D)$, necessarily $d(C, D) = d(C', D) - 1$. If $D \in \mathcal{D}$, again replace π by $\bar{\pi}$.

(c) Consider any other folding φ such that $\varphi C' = C$. From (b) it follows that $\varphi\mathcal{C}(\Sigma) = \mathcal{D}$, $\bar{\varphi}\mathcal{C}(\Sigma) = \mathcal{D}'$. Applying Lemma 2.2.1(c) to π, φ and to the subcomplex

$$\Pi = \bigcup_{D \in \mathcal{D}} \mathcal{P}(D)$$

gives $\varphi|_\Pi = \pi|_\Pi$, and similarly for

$$\Pi' = \bigcup_{D' \in \mathcal{D}'} \mathcal{P}(D').$$

But $\Pi \cup \Pi' = \Sigma$. □

2.3.4. PROPOSITION. *Let Σ be a thin, connected complex, and C, C' be adjacent chambers in Σ. Suppose that there exist foldings π and π' such that $\pi C' = C$ and $\pi' C = C'$. Then the reflection in the panel $C \cap C'$ exists. It coincides on $\pi\Sigma$ with π', and on $\pi'\Sigma$ with π.*

PROOF. If $\Pi := \pi\Sigma$, and $\Pi' := \pi'\Sigma$, then $\pi|_{\Pi\cap\Pi'} = \mathrm{id} = \pi'|_{\Pi\cap\Pi'}$, therefore $\rho|_\Pi = \pi'|_{\Pi'}$, $\rho|_{\Pi'} = \pi|_{\Pi'}$ defines an endomorphism ρ of Σ. By Lemma 2.2.1(c), $\rho^2 = \mathrm{id}$ (in particular, ρ is an automorphism). □

If π, π' are as in Proposition 2.3.4, they are called *opposite*. We emphasize once more the fact that, by Proposition 2.3.3(b), $\Pi = \pi\Sigma$, $\Pi' = \pi'\Sigma$ are determined by the pair (C, C') alone, together with the distance function on $\mathcal{C}(\Sigma)$. We say that Π, Π' are *opposite half-complexes* or *opposite roots*. They are connected, even convex in Σ (a chamber subcomplex by definition is convex if its set of chambers is convex), and their boundaries (in the sense of Section 2.1) coincide. The common boundary is the fixed point set of ρ.

Recall that even the pair of opposite foldings (π, π') is determined by (C, C') and the unordered pair $\{\pi, \pi'\}$ is determined by the panel $F = C \cap C'$. We say that $\{\pi, \pi'\}$ are the foldings *belonging to the panel F*. This additional structure for the reflections ρ_F makes precise some of the remarks at the beginning of this subsection.

2.4. *Characterizations of Coxeter–Tits complexes*

After the introductory remarks and the technical preparations of Section 2.3, we want to proceed immediately to the main theorem which shows the equivalence of the various notions of a Coxeter–Tits complex. In order to collect all important characterizations in one place, we must however first introduce one additional notion which will be treated in more detail in Section 6.

2.4.1. DEFINITION. An *m-covering* of a chamber system $\mathcal{C}$ over I, where $0 \leqslant m < \operatorname{rank}\mathcal{C}$, is a morphism π: $\widetilde{\mathcal{C}} \to \mathcal{C}$ such that, for every star $\widetilde{\mathcal{A}} \subseteq \widetilde{\mathcal{C}}$ of $\operatorname{rank}\widetilde{\mathcal{A}} \leqslant m$, the induced map π: $\widetilde{\mathcal{A}} \to \pi\widetilde{\mathcal{A}}$ is an isomorphism. A chamber system $\mathcal{C}$ is *simply m-connected* provided it is connected and every m-covering of $\mathcal{C}$ is an isomorphism.

In this subsection, we shall only need the following simple property of m-coverings, which holds for all $m \geqslant 1$: If φ: $\widetilde{\mathcal{C}} \to \mathcal{C}$ is an m-covering, and if $\boldsymbol{C} = (C_0, \ldots, C_q; i_1, \ldots, i_q)$ is a gallery in $\mathcal{C}$, then for any preimage $\widetilde{C}_0$ of C_0, there is a unique *lifting* with base point $\widetilde{C}_0$, i.e. a gallery $\widetilde{\boldsymbol{C}} = (\widetilde{C}_0, \ldots)$ such that $\varphi\widetilde{\boldsymbol{C}} = \boldsymbol{C}$.

We recall the notation $W(\Sigma)$ for the Weyl group of a thin, flag-transitive complex; if C is a chamber, we denote by $S(C) \subseteq W(\Sigma)$ the set of reflections in the panels of C.

2.4.2. THEOREM. *For a thin connected complex Σ, the following properties are equivalent.*

- (i) *Σ is a Coxeter–Tits complex.*
- (ii) *Σ is flag-transitive, strongly connected, numbered, and $(W(\Sigma), S(C))$ is a Coxeter system (for some or every chamber C).*
- (iii) *Σ is flag-transitive, strongly connected, numbered, and its chamber system $\mathcal{C}(\Sigma)$ is simply 2-connected.*
- (iv) *For every panel $F \in \Sigma$, there is a pair of foldings belonging to F (in the sense of the last subsection).*

The rest of this subsection is devoted to a proof of this theorem. For convenience, we shall say that 'Σ has sufficiently many foldings' if condition (iv) holds.

2.4.3. LEMMA. *Assume that Σ has sufficiently many foldings, and consider $A \in \Sigma$. The set of chambers $\mathcal{C}(\operatorname{St} A)$ is convex, in particular connected. Furthermore, $\operatorname{St} A$ also has sufficiently many foldings.*

PROOF. Consider chambers $C, C' \supseteq A$ and any geodesic $(C = C_0, C_1, \ldots, C_q = C')$ in Σ. By induction it suffices to show $A \subseteq C_1$. Consider the folding π such that $\pi C_0 = C_1$. By Proposition 2.3.3(b), all C_ν, $\nu \geqslant 1$, are contained in $\pi\Sigma$. Since $A \subseteq C_q$ it follows that $A = \pi A \subseteq \pi C_0 = C_1$, as desired. □

PROOF of Theorem 2.4.2. (iv) $\Rightarrow$ (iii). By Lemma 2.4.3, Σ is strongly connected, and by Proposition 2.3.4, the reflection ρ_F exists for all panels F. In order to show that Σ possesses a numbering, we use the criterion of Lemma 1.1.8. Consider a closed gallery $\boldsymbol{C} = (C_0, C_1, \dots, C_q = C_0)$, assume that $\alpha_{\boldsymbol{C}} \neq \mathrm{id}$, and assume q minimal under these conditions. Now we use the folding π such that $\pi C_1 = C_0$. The gallery $\boldsymbol{C}' = (C_0 = \pi C_0, \pi C_2, \dots, C_0)$ is shorter, and $\alpha_{\boldsymbol{C}'} = \alpha_{\pi \boldsymbol{C}} = \alpha_{\boldsymbol{C}} \neq \mathrm{id}$, a contradiction.

Now consider any 2-covering $\varphi\colon \widetilde{\mathcal{C}} \to \mathcal{C}(\Sigma)$, where $\widetilde{\mathcal{C}}$ is connected. Assume that φ is not injective, and let $D, D' \in \widetilde{\mathcal{C}}$ be such that $D \neq D'$, $\varphi D = \varphi D'$. Choose a geodesic $\boldsymbol{D} = (D = D_0, D_1, \dots, D_q = D'; i_1, \dots, i_q)$. Then

$$\varphi \boldsymbol{D} = \boldsymbol{C} = (C_0, \dots, C_q = C_0; i_1, \dots, i_q)$$

is a closed gallery. Since φ is a covering, $\boldsymbol{C}$ is simple, in particular $C_{q-1} \neq C_0$. Now we make use of the folding π with $\pi C_{q-1} = C_0$. Lift the gallery

$$\begin{aligned} &\left(C_0, C'_1, \dots, C'_{q-2}, C_0; i_1, \dots, i_{q-2}, i_q\right) \\ &\quad := (C_0, \pi C_1, \dots, \pi C_{q-2}, C_0; i_1, \dots, i_{q-2}, i_q) \end{aligned}$$

to a gallery $\boldsymbol{D}' = (D'_0, D'_1, \dots, D'_{q-2}, D'_q; i_1, \dots i_{q-2}, i_q)$ with initial chamber $D'_0 = D_0$. From the thinness of $\widetilde{\mathcal{C}}$ it follows that $D'_1 = D_1$, $\dots$, $D'_{q-2} = D_{q-2}$. Now consider the configuration

$$\begin{array}{ccccc} D_{q-2} & \xrightarrow{\quad i_{q-1} \quad} & D_{q-1} & \xrightarrow{\quad i_q \quad} & D_q \\ \Big| {\scriptstyle i_q} & & & & \\ D'_q & & & & \end{array} ,$$

which under φ maps onto

$$\begin{array}{ccccc} C_{q-2} & \xrightarrow{\quad i_{q-1} \quad} & C_{q-1} & \xrightarrow{\quad i_q \quad} & C_q = C_0 \\ \Big| {\scriptstyle i_q} & & & & \\ C_0 & & & & \end{array} .$$

But this configuration is contained in a 2-star, and therefore mapped injectively. In particular $D_q = D'_q$, and therefore $d(D_0, D_q) \leqslant q - 1$, a contradiction.

(iii) $\Rightarrow$ (ii). Let $(W(M), s_i, i \in I)$ be the Coxeter system defined by the diagram $M = (m_{ij})_{i,j \in I}$ of Σ. We want to show that the canonical homomorphism

$$W(M) \to W(\Sigma), \quad s_i \mapsto \sigma_i,$$

(σ_i is the reflection in the panel $C(i)$) is an isomorphism. But when we consider both $W(M)$ and $W(\Sigma)$ as chamber systems (recall Section 2.2), then, by the very definition of M, this map is a 2-covering.

(ii) $\Rightarrow$ (i) follows from the isomorphism $\Sigma \simeq \Delta(W(\Sigma), S(C))$ (see 2.2.3).

(i) $\Rightarrow$ (iv). Let $\Sigma = \Delta(W, S)$ for a Coxeter system (W, S). Denote by C the 'base chamber' $C = 1 \in W$. By transitivity, it is enough to construct a folding π such that $\pi sC = C$ for every sC adjacent to C. It is clear that

$$\pi wC = \begin{cases} wC & \text{if } wC \in \mathcal{C}^+, \\ swC & \text{if } wC \in \mathcal{C}^-, \end{cases}$$

if $\mathcal{C}^+$ is the set of chambers of $\pi\Sigma$, ad $\mathcal{C}^- = \mathcal{C} \setminus \mathcal{C}^+$, where $\mathcal{C} = \mathcal{C}(\Sigma)$.

Thus, we have to define $\mathcal{C}^+$, $\mathcal{C}^-$, and by Proposition 2.3.3(b) we know how we must do this:

$$\mathcal{C}^+ := \{D \in \mathcal{C}\colon\ d(D, sC) > d(D, C)\} = \{wC\colon\ w \in W,\ l(sw) > l(w)\},$$

$$\mathcal{C}^- := \{wC\colon\ w \in W,\ l(sw) < l(w)\}.$$

Now we have to show that π is a morphism of chamber systems. Consider any two adjacent chambers $wC, ws'C$, $w \in W$, $s' \in S$. According to the definition of π, there are now 4 cases to be considered:

			$\pi(wC)$	$\pi(ws'C)$
1.	$l(sw) > l(w)$	$l(sws') > l(ws')$	wC	$\pi(ws'C)$
2.	$>$	$<$	wC	$sws'C$
3.	$<$	$>$	swC	$ws'C$
4.	$<$	$<$	swC	$sws'C$

In the cases 1 and 4, it is obvious that $\pi(wC)$ and $\pi(ws'C)$ are s'-adjacent. For the cases 2 and 3, one uses the following well known lemma on Coxeter groups, which is easily proved using the exchange condition:

2.4.4. LEMMA. *For a Coxeter system (W, S) and $w \in W$, $s, s' \in S$ suppose that $l(sw) > l(w)$, $l(sws') < l(ws')$. Then $w = sws'$.* □

2.4.5. REMARK. In order to convince the reader that the existence of sufficiently many foldings really is a geometrical translation of the exchange condition, we also give a direct proof of (iv) $\Rightarrow$ (ii) (already assuming that Σ is flag-transitive). To this end, let

$s, s_1, \ldots, s_q \in S(C)$ be given, set $w = ss_1 \ldots s_q$, and assume that $l(w) \leqslant l(sw)$. This means that $d(C, wC) \leqslant d(sC, wC)$. As usual, this has to be interpreted as $wC \in \pi\Sigma$, where π is the folding such that $\pi sC = C$. Now consider the gallery $(C_0, C_1, \ldots, C_q)$, where $C_0 = sC$, $C_\nu = ss_1 \ldots s_\nu C$. Since $C_0 \notin \pi\Sigma$, $C_q = wC \in \pi\Sigma$, there is an index ν such that $C_{\nu-1} \notin \pi\Sigma$, $C_\nu \in \pi\Sigma$, and therefore $\pi C_{\nu-1} = C_\nu$. But $\pi D = sD$ for all chambers $D \notin \pi\Sigma$, in particular $\pi C_{\nu-1} = sC_{\nu-1}$. The equation $sC_{\nu-1} = C_\nu$ reads $sss_1 \ldots s_{\nu-1} = ss_1 \ldots s_\nu$, which proves the exchange property.

2.5. *Some calculations in Coxeter groups*

Let (W, S) be a Coxeter system. We write $S = \{s_i\colon\ i \in I\}$ for some index set I (which, in later chapters, will be the type set of a numbered complex). As in Section 2.1, we denote by W_J, for a subset $J \subseteq I$, the subgroup generated by the s_j, $j \in J$. In Proposition 2.1.7, we introduced the unique shortest representatives for cosets wW_J. In this section, we want to explain how one actually calculates these representatives. For technical reasons, we shall treat cosets $W_J w$ instead. $(W, s_i, i \in I)$ and the Coxeter diagram $M = (m_{ij})_{i,j \in I}$ being fixed, we denote by A_J the set of shortest elements in cosets $W_J w$. It follows easily from the exchange condition that this set can be described as

$$A_J = \{w \in W\colon\ l(s_j w) > l(w) \text{ for all } j \in J\}.$$

2.5.1. EXAMPLE. $M = \mathsf{A}_n$, $I = \{1, \ldots, n\}$, $J = \{2, \ldots, n\}$. Then

$$A_J = \{(), 1, 12, \ldots, 12 \ldots n\}.$$

One could prove this in an *ad hoc* way by using the isomorphism $W \cong \mathbb{S}_{n+1}$ (symmetric group), with the transpositions $(i, i+1)$ as generators. In the general case, it is however necessary (and already for concrete diagrams with a well known group W much more efficient) to make use of the following theorem from Tits [1968].

We first have to introduce some notations and notions which will also be important in the following Sections 3.3, 3.4, and in Section 6. Given M and $i, j \in I$, we set

$$p_{ij} = iji \ldots \quad (m_{ij} \text{ factors}).$$

Two words f and g over I are said to be *elementary homotopic* if g is obtained from f by replacing a subword p_{ij} by p_{ji} (elementary homotopy). Two words are called *homotopic* if they can be joined by a chain of elementary homotopies. It is clear that two homotopic words represent the same element in the Weyl group. The following basic theorem of Tits [1968] states that the converse also holds.

2.5.2. THEOREM.

(a) *Any nonreduced word is homotopic to a word containing a repetition ii.*

(b) *Any two reduced words representing the same group element are homotopic.*

Notice that this theorem in particular gives a solution of the word problem for Coxeter groups. We now come back to the problem of calculating shortest coset representatives.

2.5.3. PROCEDURE. When writing down all words over I in form of a 'rooted graph', where the distance from the root is the length of the word, it is obvious by the last theorem how to recognize the nonreduced words which can then be deleted, proceeding by induction over the length. In this way, one could obtain a list of all group elements (up to a given length) with all reduced representations of each. Now recall that the elements of A_J are precisely the elements w such that $l(s_j w) > l(w)$ for all $j \in J'$, that is, the elements w such that $i_1 \notin J$ for any reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_l}$. Therefore, one only lists words $i_1 \dots i_l$ such that $i_1 \notin J$, and one only keeps words in the list such that any other word representing the same group element also starts with a symbol not in J. We leave it to the reader to make a formal algorithm out of the procedure just outlined. Instead of this, we discuss some examples which are used for the calculation of the 'weight enumerator' in Section 3.4, and also for other purposes later on.

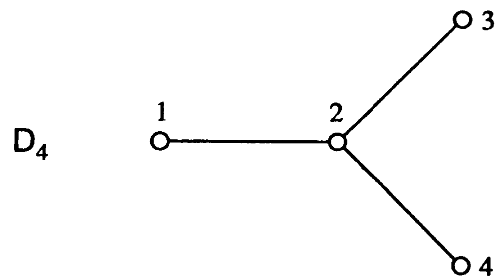
Let us start with $M = \mathsf{A}_n$, $I \setminus J = \{1\}$, as above. The 'left W^1-reduced' words which we are looking for have to start with $i_1 = 1$. Suppose that for some $l < n$, we already know that $12 \dots l$ is the only left W^1-reduced word of length l. Trivially, a left W^1-reduced word of length $l+1$ then is of the shape $12 \dots li$ for some $i \in I$. For $i = l$, this word is not even reduced. For $i \geqslant l+2$ (if $l+2 \leqslant n$), we have $12 \dots li \equiv i12 \dots l$, which shows that this element is not W^1-reduced. (We write $f \equiv g$ for short, if $s(f) = s(g)$, where f, g are words over I.) For $i < l$, say $i = l-1$, we use the relation $(l-1)l(l-1) \equiv l(l-1)l$. We may replace $1 \dots (l-2)(l-1)l(l-1)$ by $1 \dots (l-2)l(l-1)l$, and already $1 \dots (l-2)l$ is not W^1-reduced, as was seen before. Therefore, $i = l+1$ is the only possibility. Furthermore, for $l = n$ there is no possibility at all for $i = i_{n+1}$, thus $1 \dots n$ is the longest W^1-reduced word. We now have proved that A_J is as claimed in Example 2.5.1.

If we replace the diagram A_n by C_n, the same discussion of W^1-reduced words applies as long as $l \leqslant n$. However, now also the word $12 \dots n(n-1)$ is possible, since $m_{n-1,n} = 4$ and therefore $(n-1)n(n-1) \not\equiv n(n-1)n$. A word $12 \dots ni$, $i < n-1$, is not possible, since replacing ni by in would give a subword $12 \dots (n-1)i$ which already was not allowed at an earlier stage of the procedure. Continuing one sees that all words $12 \dots n(n-1) \dots (n-i)$, $i < n$, are reduced and unique for the respective group element, in particular, W^1-reduced. The word $12 \dots n(n-1) \dots 1$ is the longest W^1-reduced word, since any choice i for the next symbol would immediately lead to a reduced word or a nonadmissible left subword. For instance, for $i = 2$ one would replace the last three symbols 212 by 121, and $12 \dots n \dots 31$ is not allowed. We have proved the following:

2.5.4. EXAMPLE. $M = \mathsf{C}_n$, $I = \{1, \dots, n\}$, $J = \{2, \dots, n\}$. Then

$$A_J = \{(),\ 1,\ 12,\ \dots,\ 12 \dots n,\ 12 \dots n(n-1),\ \dots,\ 12 \dots n \dots 21\}.$$

We next look at the diagram



The W^1-reduced words are as follows:

$$(),\ 1,\ 12,\ 123,\ 124,\ 1234 \equiv 1243,\ 12342,\ 123421.$$

It is convenient to represent this list in form of a graph with edges labelled by $i \in I$ (i.e. a chamber system), where the nodes • are the group elements and an edge $\stackrel{i}{—}$ means right multiplication by i. In this way, one avoids writing down all the words, but one can read them off from the paths in the graph, going from left to right:

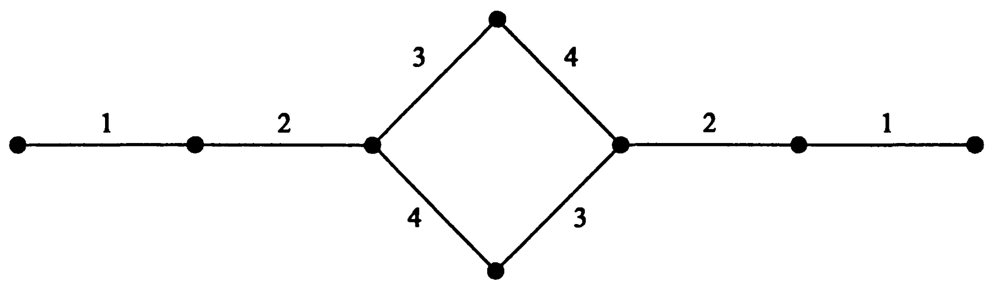


Figure 2.2. Coset representatives for D_4/A_3.

Extending this scheme in the obvious way to D_n, $n \geqslant 4$, we see the following.

2.5.5. EXAMPLE. $M = D_n$, $I = \{1, \ldots, n\}$, $J = \{2, \ldots, n\}$. Then

$$A_J = \{(),\ 1,\ 12,\ \ldots,\ 12\ldots(n-2)(n-1),\ 12\ldots(n-2)n,$$
$$12\ldots(n-2)(n-1)n,\ 12\ldots n(n-2),\ \ldots,\ 12\ldots n(n-2)\ldots 21\}.$$

2.5.6. REMARK. Before coming to a geometrical interpretation of the last results, we want to mention that the above procedure for listing shortest coset representatives also gives, as a by-product, the list of the shortest representatives for double cosets $W_J w W_K$ (see Proposition 2.1.7). One simply has to omit from the tree of left W_J-reduced words all group elements which have at least one representative ending with a symbol $k \in K$; i.e. one has to omit all nodes which have at least one edge $\stackrel{k}{—}\!\bullet$ coming from the left.

2.5.7. EXAMPLE. Let $M = \mathsf{A}_n$, C_n or D_n, $n \geqslant 4$, $I = \{1, \dots, n\}$ as usual and $j \in I$. The shortest representatives for the double cosets $W^1 w W^j$ are as follows:

$$\mathsf{A}_n\text{: } (),\ 12 \dots j,$$

$$\mathsf{C}_n\text{: } (),\ 12 \dots j,\ 12 \dots (n-1)n(n-1) \dots (j+1)j,$$

$$\mathsf{D}_n\text{: } (),\ 12 \dots j,\ 12 \dots (n-1)n(n-2) \dots (j+1)j \quad \text{if } j \leqslant n-2.$$

2.5.8. *Geometrical interpretation.* We finally wish to explain that statements as in the last example have an immediate translation into geometrical properties of Coxeter–Tits complexes as proved in Tits [1957]. To make this precise, we have to introduce some terminology from that paper. For the moment, let Δ be any I-numbered complex with vertex set X; we also consider Δ as an incidence geometry with set of objects X. A *chain* in Δ (or X) is a path in the incidence graph, i.e. a finite sequence of objects $(x_0, x_1, \dots, x_s)$ such that x_{t-1} is incident to x_t for all $t = 1, \dots, s$. Following Tits, we shall often denote a chain as follows, to give a typical example:

$$i - j - k - j' - k' - j' - i'.$$

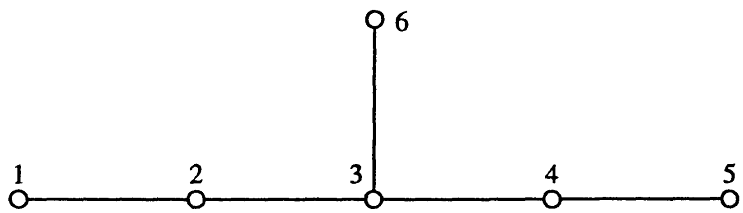
Here, i, j, k are symbols from the type set I, which stand for objects of the respective type, j' is an object of type distinct from j etc., and $i - j$ of course denotes incidence. In addition to such chains, we shall also use *galleries* joining an object x to an object y, more generally joining any two simplices A, B. By definition, this is an ordinary gallery of chambers $(C_0, \dots, C_d)$ such that $C_0 \supseteq A$, $C_d \supseteq B$. (This notion is taken from Tits [1974] and will be treated in more detail in Section 5.) Now observe that the existence of galleries of a certain 'simple' type also trivially implies the existence of particularly 'simple' chains. For instance, if two objects i, j can be joined by a gallery not involving the type k, then this means that there is an object of type k incident to both i and j, i.e. a chain $i - k - j$. Just take for k the common vertex of type k of all chambers $C_0, \dots, C_d$. Now let us specialize to the case that Δ is the Coxeter–Tits complex of a Coxeter group $(W, s_i, i \in I)$. The basic observation is that the existence of galleries of a certain type between two simplices vW^J, wW^K, without loss of generality $v = e$, is just a question of the double coset $W^J w W^K$. To see this, consider any word representing w and write it in the form fgh, where f is a word over J and h a word over K. If we interpret all words as galleries based at the distinguished chamber $e \in W$, we see that already the subword g gives a gallery joining A to B. Indeed, the face of type J does not change along f, and the face of type K does not change along h.

For instance, in the case $M = \mathsf{A}_n$ it follows from the last example that any two objects of type 1 are incident with a common object of type 2 ('any two points are incident with a common line'). Of course this is completely obvious from the explicit description of the Coxeter–Tits complex of type A_n as the full power set of an $(n+1)$-element set. The point is that the statement will immediately carry over to an arbitrary complex 'of type A_n' as defined in the next section. (We shall in fact see that any such complex is the flag complex of subspaces of a projective space.)

Similarly, if $M = \mathsf{C}_n$, we conclude from the last example with $j = 2$ that an object of type 1 and an object of type 2 are either incident or there exists a chain $1 — 2' — 1' — 2$. That is, a point 1 and a line 2 are either incident or there exists a line $2'$ through 1 having a common point $1'$ with the line 2.

We close this section with two propositions similar to Example 2.5.7, but for the diagram E_6, resp., F_4. They will be needed later in Section 3.4.

2.5.9. LEMMA. *Consider the diagram* E_6 *with the following labelling of its nodes:*



(a) *The words* (), 1, 12346321 *represent the shortest elements in double cosets* $W^1 w W^1$.

(b) *Any two objects of type* 1 *in the Coxeter–Tits complex are incident with a common object of type* 5.

PROOF. Figure 2.3 below shows the result of the procedure 2.5.3, with $J = \{2, \ldots, 6\}$. Now (a) follows immediately from Remark 2.5.6. Part (b) is immediate from (a), since none of the three representatives for the $W^1 w W^1$ involves the type 5. □

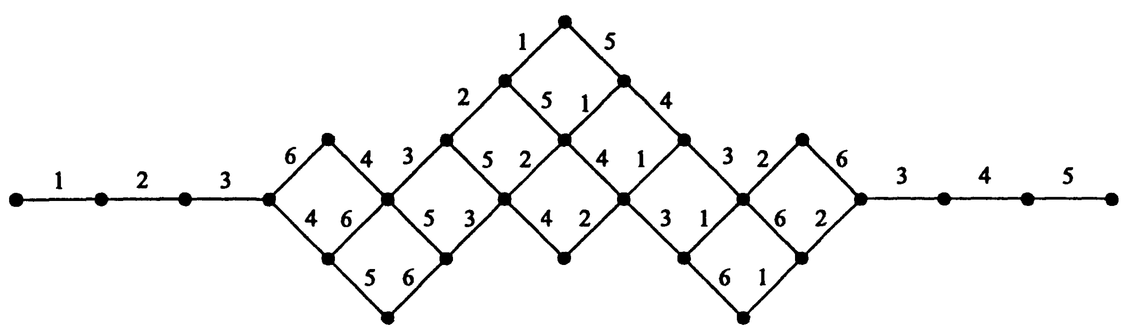
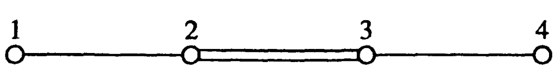


Figure 2.3. Coset representatives for $\mathsf{E}_6/\mathsf{D}_5$.

2.5.10. LEMMA. *Consider the diagram* F_4 *with the following labelling of its nodes:*



(a) *The words* (), 1234 *and* 12321 43234 *represent the shortest elements in double cosets* $W^1 w W^4$.

(b) *Given an object of type* 1 *and an object of type* 4 *in the Coxeter–Tits complex, they are incident or there exists a chain* $1 — 4' — 1' — 4$.

As before, (a) is read off from Figure 2.3, showing all left-$\{2,3,4\}$-reduced words; (b) follows from (a).

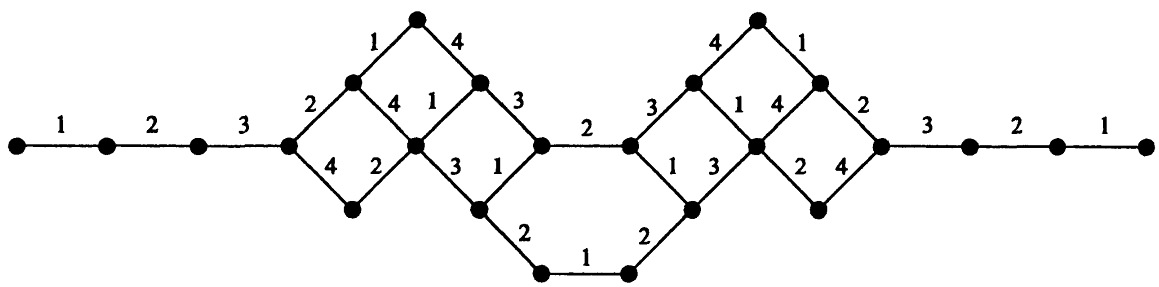


Figure 2.4. Coset representatives for F_4/C_3.

2.5.11. REMARK. There is a partial ordering on the set A_J of shortest coset representatives, and on W/W^J, canonically associated to the rooted graphs as shown in the previous figures. Formally, $w \leqslant w'$ if w is given by a 'left subword' $i_1 i_2 \ldots i_l$ of a reduced (and left W^J-reduced) expression $i_1 i_2 \ldots i_l \ldots i_{l'}$ for w'. Pictorially, there is a path in A_J, going 'from left to right' and connecting w to w'. This is the so-called *weak ordering* on A_J, in particular on $A_\varnothing = W$. Its combinatorial properties are studied in Björner [1984]. Clearly, A_J with this ordering is a pure poset whose dimension function equals the length function on W. The following property is less obvious: If $J \subseteq K$, then the weak ordering of A_K is the ordering induced by the weak ordering of A_J. See the Notes below for the related, widely studied topics of so-called Bruhat order, and for references for Figures 2.3 and 2.4 in that context.

Notes to Section 2

Coxeter groups. The complete list of all finite Coxeter groups in terms of spherical Coxeter diagrams as reproduced at the beginning of this section was given for the first time in Coxeter [1934]. To be precise, Coxeter classified in this paper all finite groups of Euclidean isometries generated by reflections; in the subsequent paper Coxeter [1935] he showed that every abstract finite Coxeter group possesses such a geometrical realization, and thus the previous list applies. Although this is not the place to give an extensive historical account of Coxeter groups and reflection groups, we would like to mention at least the following. The list of finite reflection groups is essentially already contained in Cartan [1927]. This work was restricted to groups stabilizing a lattice, i.e. Weyl groups of root systems. In the 3- and 4-dimensional case the results are much older; see the Notes Historiques of Bourbaki [1968]. Notice that the result from Coxeter [1935] just quoted contains the result of Theorem 2.1.2 in the particular case of spherical diagrams. As was mentioned above, the general case of Theorem 2.1.2 is treated in a similar way, namely by realizing the abstract presentation of the group by linear isometries preserving an appropriate 'canonical bilinear form' B_M. The underlying vector space is $\mathbb{R}^{(I)}$, i.e. has a basis $(e_i)_{i \in I}$ indexed by the indices of the Coxeter matrix M, and the form is defined by $B(e_i, e_j) = -\cos(\pi/m_{ij})$. This form was introduced in full generality in Witt [1941] where the results of Coxeter [1934, 1935] were extended to reflection groups on Euclidean space having a compact fundamental domain or, equivalently, to affine Weyl groups of root systems. The list of the corresponding Coxeter diagrams, the so-called Euclidean or affine diagrams is also given in Bourbaki [1968], Chapter 6, §4, Théorème 4. These are the Coxeter diagrams underlying the extended Dynkin diagrams of root systems. It was only 20 years later that Tits proved in general that the canonical representation of an abstract Coxeter group by isometries of B_M is injective. Theorem 2.1.2 is an obvious consequence of this fact. This result of Tits is given in Tits [1961a], a paper which is widely quoted but was never published officially. Its contents however have been incorporated into Bourbaki [1968], partly as exercises. In Tits [1961a] the canonical linear representation of a Coxeter group is also used to realize the Coxeter–Tits complex as a simplicial subdivision of a convex cone. In fact, this was the way that the Coxeter–Tits complex was introduced for the first time. See Vinberg [1971] for other linear representations of Coxeter groups, and Koszul [1965, 1967], Straume [1981], Davis [1983], Vinberg [1985] for Coxeter groups acting on other spaces.

The exchange condition for Coxeter groups (the easier half of Theorem 2.1.4) is an easy consequence of the action of the Coxeter groups on the halfspaces of the Coxeter–Tits complex. The proof of this result given in Bourbaki [1968], Chapter 4, §1, Lemme 3, is a combinatorial copy of the original geometric proof. Matsumoto's part is the converse, more difficult direction of Theorem 2.1.4. In Matsumoto [1964] even a bit more is proved: it is shown that under the assumption of the exchange condition, the conclusion of Proposition 2.1.8 holds.

We have used the book Bourbaki [1968] as a standard and in a sense 'the original' reference for the theory of Coxeter groups. Recently, there has appeared another, beautiful text book on Coxeter groups Humphreys [1990]. This book covers everything we need from the theory of Coxeter groups, and much more. It may very well serve as a motivating and precise introduction for the reader not familiar with the subject, but it is also a valuable source for the expert looking for a unified presentation of some advanced topics (see below). Another text which we suggest for further reading on Coxeter groups is the survey article Cohen [1991]. It treats some basic aspects in text book form with full proofs. Its Theorem 4.1 collects essentially all known characterizations of Coxeter groups.

Coxeter–Tits complexes. The contents of Sections 2.2 to 2.4 have been taken without much change from the first half (2.1 to 2.17) of Chapter 2 of Tits [1974], except that the notion of a 2-covering used in 2.4.1 and 2.4.2 was formally only introduced in Tits [1981a]. The study of roots and convexity will be continued in Section 5.1, and the notion of oppositeness will be extended and treated in more detail in Section 5.4. See Abels [1991b] for an application of foldings in a different context. We have changed Tits' presentation a little bit by devoting a separate Section 2.2 to those 'elementary' properties of Coxeter–Tits complexes and their groups which only rely on the thinness and of course the assumption of a flag-transitive group (and not on some sort of universality or simple connectedness). The study of such 'regular' complexes has been an area of active research in the context of regular polytopes and tilings, independently of buildings; see, e.g., McMullen [1967], Dress [1981] or Danzer and Schulte [1982], Schulte [1983], McMullen and Schulte [1990].

Combinatorics of Coxeter groups. We cannot give an explicit reference for the procedure 2.5.3 for calculating one-sided and double cosets and their graphical representation as in Examples 2.5.4, 9 and 10. This method has grown out of unpublished joint work with A. Dress, but we do not claim any originality for this, since it is after all a straightforward application of Tits' basic Theorem 2.5.2. Brouwer and Cohen [1985], §4, point out the possibility of calculating the length function and the cosets wW_J using the linear representation of W (more precisely, using the action on roots and co-roots). Furthermore, they remark that one can simplify the procedure if one is only interested in double cosets. For the (somewhat related) problem of calculating a reduced expression for a group element given by an arbitrary word in the generators, an algorithm is described in Cohen [1991], 8.2.

At this place, it is appropriate to mention some further topics from the combinatorial theory of Coxeter groups which we did not touch upon in our context of buildings. First of all, there is the widely studied concept of Bruhat ordering on a Coxeter group W, and its coset spaces W/W_J. This ordering comes from algebraic geometry: if the Coxeter group W is the Weyl group of a simple algebraic group with Borel subgroup B, the Bruhat ordering describes the inclusion of the Zariski closures of the Schubert (Bruhat) cells BwB, $w \in W$ (cf. Section 4.3). See Humphreys [1990], Chapters 5 and 8, for a purely combinatorial definition of the Bruhat order, and for further results. The weak ordering studied above is really a weakening of the Bruhat ordering. See Björner [1983/84] for a combinatorial study and comparison of the two orderings.

There is still a third class of partially ordered sets related to Coxeter groups and their orderings. They are associated to (ordinary finite dimensional) representations of the Lie algebra (say over the complex numbers) with Weyl group W. The partially ordered set in question is the set of weights of the representation, ordered by the standard notion of a positive weight. Using the action of the Weyl group on the set of weights, the Bruhat order is related to the ordering of weights. In certain special cases (microweights or minuscule weights), the two orderings even coincide, and also agree with the weak ordering (cf. Proctor [1984], Lemma 3.1, Vavilov [1990/91], pp. 236, 238, Hiller [1982], Chapter 5). This explains why exactly the same graph as our Figure 2.3 appears in Vavilov [1990/91], p. 259, and Proctor [1984], p. 335. Both papers also contain the analogous diagram for $\mathsf{E}_7/\mathsf{E}_6$. The survey article Vavilov [1990/91] contains an exhaustive bibliography on weight posets. An early reference for the subject is Curtis, Iwahori and Kilmoyer [1971]. On page 90 of *loc. cit.*, a method for our above problem of calculating double cosets $W_J\backslash W/W_{J'}$ is described. It seems, however, to require greater effort than the procedure described above in 2.5.3.

It is a remarkable fact that, in the context of geometrical applications as in 2.5.8 (finding a short chain between two objects of a geometry), the problem of double cosets seems never to have been investigated thoroughly. An exception to the last statement are §§72 and 73 of Freudenthal and De Vries [1969] to which we shall come back in the Notes to Section 5.1 in the context of convex hulls.

The investigation of certain combinatorial aspects of Coxeter groups will be continued in Section 3.4 when we deal with the weight enumerator of a Coxeter group.

3. The axiomatics of buildings

Introduction

In this section, we finally come to the central topic of our report by introducing the notion of a building. We have chosen the 'classical' approach of Tits [1974], using apartments. It has the great advantage of being closely related to the structure theory of simple algebraic groups, that is to their root systems, Weyl groups, and parabolic subgroups. We present this definition, together with a few easy consequences, in Section 3.2. In Section 3.1, we introduce the more general class of complexes belonging to a Coxeter diagram. This is supposed to serve the reader's intuition; Section 3.1 is not needed for Section 3.2, which contains the really basic definition. One purpose of Section 3.1 is to show at an early stage that buildings really are a special case of incidence geometries belonging to a diagram, in the sense of Buekenhout and Tits, as developed in Chapter 3. This larger class of geometries is defined more easily, and is more intuitive, but unfortunately it is a rather delicate matter to characterize buildings among all geometries belonging to a Coxeter diagram. Such a characterization is the subject of Theorem 3.3.1. It is in terms of certain relations between chambers which are indexed by the elements of the Weyl group. That theorem leads to a sort of metric on the chambers of a building which takes values in the Weyl group. Buildings can be characterized in terms of such metrics (Proposition 3.3.12). In Section 3.4, we present two kinds of applications of the characterization of buildings given in 3.3.1. The first application is that the number of chambers of a finite building is obtained by evaluating the 'combinatorial polynomial' or 'weight enumerator' of the Weyl group. The second application is to the characterization of buildings by certain geometrical axioms. Further results in this direction can only be obtained by deeper methods which will be developed later in Section 6.

3.1. *Complexes belonging to a Coxeter diagram*

We first define a certain class of numbered complexes of rank 2, equivalently, of bipartite graphs, the so-called *generalized polygons*. We introduce these rather briefly, for details the reader is referred to Tits [1959], Appendice, Tits [1976], and to Chapter 9. We start with some terminology on graphs which for our present purpose is *ad hoc*.

A path of length n in a graph Γ is a sequence of vertices $x_0, \dots, x_n$ such that $\{x_{t-1}, x_t\}$ is an edge for all $t = 1, \dots, n$. The path is called *reduced* if $x_t \neq x_{t+2}$ for all t. The *distance* $d(x, y)$ of any two vertices x, y in the same connected component of Γ is the length of a shortest path joining x and y. The *diameter* $\operatorname{diam}\Gamma$ is the maximum distance between two points, possibly infinite. A *circuit* is a closed path $x_0, \dots, x_n = x_0$ such that $x_0, \dots, x_{n-1}$ are pairwise distinct. If Γ is bipartite, then n is necessarily even. For the remainder of this subsection, all graphs are connected. The following definition is the one given in Tits [1976].

3.1.1. DEFINITION. A *generalized m-gon*, for a natural number $m \geqslant 2$, is a numbered complex of rank 2 with the following two properties:

(C_m) Any nontrivial circuit is of length $\geqslant 2m$.
(D_m) Any two edges are contained in some circuit of length $2m$.

For the convenience of the reader, we briefly recall a couple of reformulations of these properties. It is readily seen that (C_m) is equivalent to the following property:

For any two vertices x and y, there is at most one reduced path of length $< m$ joining x and y.

Obviously, (D_m) implies that $\operatorname{diam} \Gamma \leqslant m$. The following proposition shows that the converse implication holds if one assumes (C_m) (which then implies $\operatorname{diam} \Gamma = m$) and one further little property (F).

(F) Any vertex has at least two neighbours.

3.1.2. PROPOSITION. *For a numbered complex Γ of rank* 2, *and an integer $m \geqslant 2$, the following are equivalent*:

(i) *Γ is a generalized m-gon*;

(ii) $\operatorname{diam} \Gamma \leqslant m$, *and properties* ($\mathrm{C}_m$) *and* (F) *hold.*

Property (ii) is used as the definition in Tits [1981a]. The essential step in the proof of the implication (ii) $\Rightarrow$ (i) is to show that any geodesic path is contained in a geodesic path of length m. In particular, to any vertex x there exists a vertex y such that $d(x, y) = m = \operatorname{diam} \Gamma$, i.e. y is 'opposite' to x.

The most important values of m are $m = 2, 3, 4$.

$m = 2$. A bipartite graph is a generalized digon if and only if any two vertices of different types form an edge, i.e. it is a 'complete bipartite graph'.

$m = 3$. The generalized triangles are precisely the flag complexes (incidence graphs) of projective planes.

The proof is obvious if one uses the standard definition of a projective plane; it is not required that any line contain at least three points.

$m = 4$. Take any (skew) field K with an involution and a vector space V over K with a nondegenerate Hermitian or (pseudo-) quadratic form (see, e.g., Tits [1974]) of Witt index 2. Consider the 1-dimensional isotropic subspaces as 'points' and the 2-dimensional totally isotropic subspaces as 'lines'. The incidence graph of this geometry is a generalized quadrangle.

We once more refer the reader to Chapter 9 for more information, in particular about the classification of generalized quadrangles.

The definition of a generalized m-gon is naturally extended to the case $m = \infty$ if one defines a generalized ∞-gon to be a bipartite tree satisfying the axiom (F) (i.e. having no finite ends). If one interprets a circuit of length ∞ as a doubly infinite injective path, then it is still true that ∞-gons can be characterized by (C_∞), (D_∞) or by $\operatorname{diam} \Gamma = \infty$, ($\mathrm{C}_\infty$) and (F).

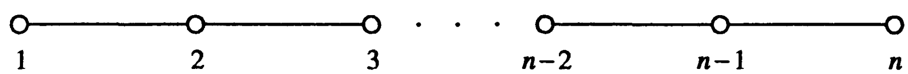
3.1.3. DEFINITION. Let M be a Coxeter diagram over the index set I. An I-numbered complex Δ is *of type M* or *belongs to M* if it is strongly connected and if, for any two element subset $\{i, j\} \subseteq I$ and any simplex $A \in \Delta$ such that $\operatorname{cotype} A = \{i, j\}$, the star $\mathrm{St}_\Delta A$ is a generalized m_{ij}-gon.

Sometimes we say that Δ is *of Coxeter type* if it belongs to some Coxeter diagram M (which is not specified further).

If Δ belongs to M, then the graph underlying M in the usual sense (i and j are connected if and only if $m_{ij} \geqslant 3$) is the basic graph of Δ as defined in Section 1. This follows immediately from the fact that a generalized m-gon is the complete bipartite graph over its vertex set $X = X_i \,\dot{\longrightarrow}\, X_j$ if and only if $m = 2$.

3.1.4. EXAMPLE. If M is arbitrary and if Σ denotes the corresponding Coxeter–Tits complex $\Sigma(M)$, then Σ belongs to M. Each $\{i, j\}$-star is isomorphic to (the incidence graph of) an ordinary m_{ij}-gon.

3.1.5. EXAMPLE. Let Δ be the flag complex of (subspaces of) an n-dimensional projective space, $n \geqslant 2$, with its natural type set $\{1, \ldots, n\}$. Then Δ belongs to the diagram A_n with its natural labelling



Indeed, the $\{i, i+1\}$-stars are projective planes, i.e. generalized triangles.

3.1.6. EXAMPLE. Let V be a vector space over a (skew) field with a nondegenerate Hermitian or (pseudo-) quadratic form of Witt index n. The complex Δ of flags of totally isotropic subspaces belongs to the diagram C_n

(The type of a subspace is its projective dimension $+1$.) To verify the diagram, notice that the star of a maximal totally isotropic subspace (vertex of type n) is the flag complex of an $(n-1)$-dimensional projective space and thus belongs to the diagram A_{n-1}. It is easily seen that the star of a 'point' (one-dimensional isotropic linear subspace) has an analogous description for a form of Witt index $n-1$, or it is checked directly that the star of a flag $\{x_0, \ldots, x_{n-3}\}$, where the x_i are totally isotropic of dimension $i+1$, is a generalized quadrangle (see above).

We shall see later that the complexes in Examples 3.1.4–6 are in fact buildings. Before presenting further examples, we make a general remark about the class of complexes of type M, for a fixed M. Consider a 2-covering φ: $\Delta \to \Delta'$ as introduced in Section 2.4, i.e. for any $A \in \Delta$ of codimension 2 the induced map $\mathrm{St}_\Delta A \to \mathrm{St}_{\Delta'} \varphi A$ is an isomorphism. Then by definition Δ is of type M if and only if Δ' is of type M. The next two examples are based on this remark.

3.1.7. EXAMPLE. Consider the Coxeter–Tits complex $\Sigma(\mathsf{C}_n)$, $n \geqslant 3$, and let w_0 be the longest element (nontrivial central element) in the Weyl group W (C_n). The quotient $\Sigma(\mathsf{C}_n)/\{1, w_0\}$ is a numbered complex in a natural way and the projection

$$\Sigma(\mathsf{C}_n) \to \Sigma(\mathsf{C}_n)/\{1, w_0\}$$

satisfies the above condition (is a 2-covering). Thus $\Sigma(\mathsf{C}_n)/\{1, w_0\}$ again is of type C_n.

Recall that w_0 acts as -1 in the natural geometric representation of $W(\mathsf{C}_n)$ and $\Sigma(\mathsf{C}_n)$ on the $(n-1)$-sphere. That is, $\Sigma(\mathsf{C}_n)/\{1, w_0\}$ is realized as a regular triangulation of real projective $(n-1)$-space.

3.1.8. Example (Tits [1978/80]). Let f be a positive definite quadratic form on a real vector space V of dimension $2n$ or $2n+1$, $n \geqslant 3$. Consider the numbered complex $\Delta = \Delta(f, \mathbb{C})$ of totally isotropic subspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$, which is of type C_n. Complex conjugation extends to an automorphism α of Δ. From the positive definiteness of f over $\mathbb{R}$ it readily follows that α has the properties analogous to w_0 in Example 3.1.7. Thus $\Delta/\{1, \alpha\}$ again is a numbered complex of type C_n.

We shall see later that $\Sigma(\mathsf{C}_n)/\{1, w_0\}$ and $\Delta(f, \mathbb{C})/\{1, \alpha\}$ are not buildings. In fact, buildings are 'simply connected', they do not occur as proper quotients in the above sense. More precisely, if $\varphi\colon \Delta \to \Delta'$ is a 2-covering as above, and Δ' is assumed to be a building, then φ has to be an isomorphism (Proposition 3.3.11). There exist examples of nonbuildings of type M, for certain diagrams M, which do not occur as such quotients of buildings. A famous example for $M = \mathsf{C}_3$ is the following one which was first found by A. Neumaier.

3.1.9. Example (The Alt_7-geometry belonging to C_3). Set

$$X = \{1, 2, \ldots, 7\}, \quad \text{'points'},$$

$$Y = \binom{X}{3}, \quad \text{the 3-element subsets of } X, \text{ 'lines'},$$

$$Z = \mathrm{Alt}_7 \cdot z_0 \subset 2^X, \quad \text{'planes'},$$

where Alt_7 is the alternating group of degree 7, and $z_0 = \{123, 145, 167, 246, 257, 347, 356\}$ is the set of lines of a fixed Fano plane structure on X. Trivially, the full symmetric group Sym_7 acts transitively on the set of all Fano plane structures on X, therefore there are $7!/168 = 30$ such structures. Since the stabilizer of any of these is contained in Alt_7, there are 2 orbits under Alt_7, that is $|Z| = 15$. Define incidence on $X \times Y$ by $x \in y$ and on $Y \times Z$ by $y \in z$, and extend this to $X \times Z$ by transitivity. (In fact, any point is incident with any plane.) Since Alt_7 acts transitively on Z, and the stabilizer of any $z \in Z$ acts flag-transitively on $\mathrm{St}\, z$, the group Alt_7 acts chamber-transitively on $\Delta = \Delta(X \cup Y \cup Z)$. It is not difficult to see that the star of any $x \in X$ is a generalized quadrangle. In fact, the stabilizer of x is $\mathrm{Alt}_6 \simeq \mathrm{Sp}(4,2)'$, and the quadrangle is the one belonging to the alternating form of rank 4 over $\mathbb{F}_2$. Thus Δ belongs to the diagram C_3. The Alt_7-geometry is 'simply connected' in the sense of 2-coverings (see Section 6.1), but it is not a building.

We now come to the description of the chamber systems corresponding to complexes of type M. The following proposition is not difficult to verify.

3.1.10. PROPOSITION. *Let m be an integer $\geqslant 2$ or ∞. A connected chamber system $\mathcal{C}$ of rank 2 over $I = \{i, j\}$ corresponds to a generalized m-gon* (cf. Section 1.4) *if and only if it possesses the following three properties*:

(CS1) *For any $C \in \mathcal{C}$ and $i \in I$ there exists $C' \in \mathcal{C}$ such that $C' \overset{i}{\sim} C$, $C' \neq C$.*

(CS_m2) *$\mathcal{C}$ contains no circuit of length strictly less than $2m$.*

(CS_m3) *If C and C' can be joined by a gallery of type $iji\ldots$ (m factors), they can also be joined by a gallery of type $jij\ldots$ (m factors).*

(When $m = \infty$, (CS_m2) means that $\mathcal{C}$ has no circuits, and (CS_m3) is vacuous.)

In view of the criteria given in Section 1.4, we remark that the strong connectedness of a chamber system over $\{i, j\}$ follows from (CS_m2). Indeed, in the rank 2 case, strong connectedness reduces to

$$C \overset{i}{\sim} D,\ C \overset{j}{\sim} D \Rightarrow C = D.$$

As in the case of generalized m-gons, the corresponding properties (CS1), (CS_m2), (CS_m3) of chamber systems also admit a couple of reformulations in terms of the following property

(CS_m4) For any two chambers C and D, there exists a circuit of length $2m$ containing C and D.

Here, it is supposed that m is finite. The following proposition holds also for $m = \infty$ if one replaces ($CS_\infty4$) by (CS1).

3.1.11. PROPOSITION. *For fixed $m \geqslant 2$, the following sets of conditions on a chamber system of rank 2 are equivalent*:

(i) (CS1), (CS_m2), (CS_m3).

(ii) (CS_m2), (CS_m4).

(iii) (CS_m4), *and any minimal circuit has length $2m$.*

The following characterization of the chamber systems of complexes of type M now immediately follows from Theorem 1.3.1, together with the two previous propositions.

3.1.12. PROPOSITION. *Let M be a Coxeter diagram over I. A chamber system $\mathcal{C}$ over I corresponds to a complex of type M if and only if it is strongly connected, and for any 2-element subset $\{i, j\}$ of I, every $\{i, j\}$-star in $\mathcal{C}$ satisfies the conditions* (i), (ii), *or* (iii) *of Proposition* 3.1.11, *with $m = m_{ij}$.*

In this situation we say that $\mathcal{C}$ *belongs to M*, or *is of type M*, and if M is not specified, that $\mathcal{C}$ *is of Coxeter type*.

For the remainder of this subsection, we fix a Coxeter diagram $M = (m_{ij})_{i,j \in I}$, denote by

$$W = W(M) = \langle i \in I \mid (ij)^{m_{ij}} = 1 \rangle$$

the corresponding Coxeter group, and by $f \mapsto s(f)$ the canonical map from the set of words over I onto W. We recall from Section 2.1 the length function on W and the notion of a reduced (with respect to M) word over I. We consider an I-numbered complex Δ belonging to M, and its chamber system $\mathcal{C} = \mathcal{C}(\Delta)$. We shall now have a closer look at the galleries in $\mathcal{C}$ and explain the 'universal' role of the group W for all complexes of type M. The basic result in this direction is the following:

3.1.13. PROPOSITION. *Let $\mathcal{C}$ be a chamber system belonging to M. For each $w \in W$, there is a relation $\overset{w}{\sim}$ on $\mathcal{C}$ which is characterized as follows:*

$C \overset{w}{\sim} D \Leftrightarrow$ *for any reduced word f representing w there is a gallery of type f with origin C and extremity D.*

PROOF. This immediately follows from Proposition 2.1.8, applied to the map $i \mapsto \overset{i}{-}$ from I into the monoid of all relations on $\mathcal{C}$. (Recall that $C \overset{i}{-} D$ means $C \overset{i}{\sim} D$, $C \neq D$.) The property (CS$_m$3), for the various $\{i, j\}$-stars and $m = m_{ij}$ is precisely the assumption of 2.1.8, and the conclusion then is that the extension of the above map to all reduced words factors over W. □

Notice that the following fact about composition of the $\overset{w}{\longrightarrow}$-relations is an immediate consequence of the definition:

$$C \overset{v}{\longrightarrow} D \overset{w}{\longrightarrow} E,\ l(vw) = l(v) + l(w) \Rightarrow C \overset{vw}{\longrightarrow} E.$$

Because of the importance of Proposition 3.1.13, we also give a slightly different proof which does not use such formal things as the monoid of all relations on $\mathcal{C}$. To that end, we have to recall from Section 2.5 the notion of homotopy of words (with respect to a Coxeter diagram M), and the fact that any two reduced words representing the same group element are homotopic (Theorem 2.5.2). We shall now generalize this kind of homotopy to arbitrary galleries in chamber systems belonging to a Coxeter diagram.

3.1.14. DEFINITION. An *elementary Weyl homotopy* in a chamber system $\mathcal{C}$ of type M is a replacement of a subgallery of type p_{ij} of some gallery $\boldsymbol{C}$ by a subgallery of type p_{ji} (with the same extremities). A *Weyl homotopy* of $\boldsymbol{C}$ to some gallery $\boldsymbol{D}$ is a sequence $\boldsymbol{C} = \boldsymbol{C}_0, \boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_s = \boldsymbol{D}$ of galleries such that $\boldsymbol{C}_{t+1}$ is elementary Weyl homotopic to $\boldsymbol{C}_t$ for all t. Necessarily, $\boldsymbol{D}$ has the same extremities as $\boldsymbol{C}$.

In Section 6, we shall introduce the notion of 2-homotopy of galleries which applies to arbitrary chamber systems and in the case of chamber systems belonging to a Coxeter diagram is in general weaker than Weyl homotopy. When the intended meaning is clear, we shall sometimes speak simply of (elementary) homotopy.

In view of (CS$_m$3), the following is clear.

3.1.15. LEMMA. *Let C be a gallery of type f in a chamber system belonging to the Coxeter diagram M. Any homotopy of words ($f = f_0, f_1, f_2, \ldots, f_s = g$) from f to some word g can be lifted uniquely to a Weyl homotopy $\boldsymbol{C} = \boldsymbol{C}_0, \boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_s = \boldsymbol{D}$ from $\boldsymbol{C}$ to a gallery $\boldsymbol{D}$ of type g with the same extremities.*

'Lifted' of course means that type $C_t = f_t$ for all t.

Theorem 2.5.2(b) together with the last lemma immediately implies that if two chambers C and D can be joined by a gallery of a particular reduced type, they can also be joined by a gallery of any other reduced type representing the same group element. This again proves Proposition 3.1.13. An immediate consequence of part (a) of Theorem 2.5.2 is the following:

3.1.16. COROLLARY. *In a chamber system of type M, the type of any geodesic gallery is reduced with respect to M.*

PROOF. Assume the contrary. Then there also exists a gallery of the same length and with the same extremities whose type contains a repetition ii. But such a gallery is obviously nongeodesic. □

3.1.17. EXAMPLE. In the Alt_7-geometry of type C_3 (Example 3.1.9 above), it is readily seen that there exists a closed gallery of length 9 (in particular, non-geodesic) of type 123123123, which is reduced.

Such closed galleries of reduced type, however, do not exist in buildings, and we shall see in Theorem 3.3.1 that this property characterizes the buildings among all complexes of type M.

3.2. *Systems of apartments and buildings*

If one considers the buildings to be defined in this subsection as geometrical objects in the spirit of the previous subsection, the apartments correspond to the (unordered) bases of the geometry. In the case of a projective space, the bases are the usual ones ($n + 1$ independent points, where n is the dimension); for a polar space, the bases are the polar frames. In the case of a general simplicial complex, the notion is axiomatized as follows. In the definition, it is not presupposed that Δ is numbered, not even that Δ is a chamber complex.

3.2.1. DEFINITION. A *structured complex* is a complex Δ together with a family $\boldsymbol{\Sigma}$ of subcomplexes such that the following holds.

(S1) The elements $\Sigma \in \boldsymbol{\Sigma}$ are thin, connected chamber complexes.

(S2) For any two $A, B \in \Delta$, there is a $\Sigma \in \boldsymbol{\Sigma}$ containing A and B.

(S3) For any $\Sigma, \Sigma' \in \boldsymbol{\Sigma}$, $A, B \in \Sigma \cap \Sigma'$, there exists an isomorphism $\Sigma \to \Sigma'$ which is the identity on A, B and all their faces.

The elements of $\boldsymbol{\Sigma}$ are called *apartments*, and any $\boldsymbol{\Sigma}$ subject to (S1), (S2), (S3) is called a *system of apartments*. If the apartments are Coxeter–Tits complexes, then (Δ, $\boldsymbol{\Sigma}$) is called a *building*.

The first observation is that (S2) and (S3) imply that the chambers of the apartments of a structured complex (Δ, Σ) are maximal in all of Δ:

3.2.2. PROPOSITION. *A structured complex is a connected chamber complex.*

In the next lemma and proposition we record an improvement of the property (S3).

3.2.3. LEMMA. *Suppose Σ, Σ' are apartments of a structured complex having at least one chamber in common. Then there is a unique isomorphism $\Sigma \to \Sigma'$ which is the identity on the faces of at least one chamber of $\Sigma \cap \Sigma'$. Moreover, it is the identity on all of $\Sigma \cap \Sigma'$.*

PROOF. If C is a chamber in $\Sigma \cap \Sigma'$, and $A \in \Sigma \cap \Sigma'$ arbitrary, then axiom (S3) gives an isomorphism $\phi_{C,A}$: $\Sigma \to \Sigma'$ which is the identity on C, A and all their faces. By Lemma 2.2.1, $\phi_{C,A}$ does not depend on A. The existence and uniqueness of the desired isomorphism is obvious. □

3.2.4. PROPOSITION. *Let Σ be an apartment of a structured complex Δ and $C \in \Sigma$ be a chamber. Then there exists a unique idempotent morphism*

$$\rho_{\Sigma,C}\colon \Delta \to \Sigma$$

which induces an isomorphism $\Sigma' \to \Sigma$ for any apartment $\Sigma' \ni C$. It has the property $\rho_{\Sigma,C}^{-1}(B) = \{B\}$ for all $B \subseteq C$. It is called the retraction of Δ onto Σ with centre C.

PROOF. If Σ' is another apartment containing C, then $\rho_{\Sigma,C}|_{\Sigma'}$: $\Sigma' \to \Sigma$ necessarily equals the unique isomorphism $\phi_{\Sigma',\Sigma}$ given by Lemma 3.2.3. The uniqueness of $\phi_{\Sigma',\Sigma}$ also easily implies that the various $\phi_{\Sigma',\Sigma}$ can be glued together into a map defined on all of Δ. □

3.2.5. COROLLARY. *Every building possesses a numbering.*

PROOF. Recall that a numbering is the same thing as a morphism onto a simplex $\mathcal{P}(I)$. By Proposition 3.2.4, this exists if it exists for an apartment. □

We recall the canonical numbering type: $\Delta \to \text{type}(\Delta)$, and the canonical set of types $I(\Delta)$.

In the following proposition, we list some important consequences of the existence of retractions.

3.2.6. PROPOSITION. *Let Σ be an apartment of the structured complex Δ.*

(a) *The distance function on $\mathcal{C}(\Sigma)$ is the restriction of the distance function on $\mathcal{C}(\Delta)$. Every geodesic in Σ remains geodesic in Δ.*

(b) *$d(C, \rho_{\Sigma,C}D) = d(C, D)$ for all $C \in \mathcal{C}(\Sigma)$, $D \in \mathcal{C}(\Delta)$.*

(c) *$\mathcal{C}(\Sigma)$ is convex in $\mathcal{C}(\Delta)$; i.e. any geodesic in Δ whose extremities are in Σ is completely contained in Σ.*

PROOF. (a) If $C, D \in \mathcal{C}(\Sigma)$, then trivially $d_\Delta(C,D) \leqslant d_\Sigma(C,D)$. On the other hand, if ρ: $\Delta \to \Sigma$ is any retraction, then $d_\Sigma(C,D) = d_\Sigma(\rho C, \rho D) \leqslant d_\Delta(C,D)$.

(b) Choose an apartment Σ' such that $C, D \in \Sigma'$. Then $\rho_{\Sigma,C}$ restricted to Σ' is an isomorphism, and therefore $d_{\Sigma'}(C,D) = d_\Sigma(C, \rho_{\Sigma,C}D)$. But $d_{\Sigma'}(C,D) = d_\Delta(C,D)$, by part (a).

(c) Consider a geodesic $\boldsymbol{C} = (C_0, C_1, \ldots, C_m)$ in Δ such that $C_0, C_m \in \Sigma$. Assuming that $\boldsymbol{C}$ is not contained in Σ, let the index ν be such that

$$C_\nu \notin \Sigma, \qquad C_{\nu+1}, \ldots, C_m \in \Sigma.$$

Let $C \in \mathcal{C}(\Sigma)$ be the unique chamber of Σ containing $C_\nu \cap C_{\nu+1}$ and distinct from $C_{\nu+1}$. For $\rho_{\Sigma,C}C_\nu$, there are only two possibilities, C or $C_{\nu+1}$, being the only chambers in Σ containing $C_\nu \cap C_{\nu+1}$. Since, by Proposition 3.2.4, $\rho^{-1}_{\Sigma,C}(C) = \{C\}$, the first possibility drops out, and necessarily $\rho_{\Sigma,C}C_\nu = C_{\nu+1} = \rho_{\Sigma,C}C_{\nu+1}$. But then $\rho_{\Sigma,C}\boldsymbol{C}$, after deleting this repetition, gives a shorter gallery joining C_0 to C_m, a contradiction.□

We want to emphasize the following consequence of Proposition 3.2.6(a) and the axiom (S2). If a building Δ is of *spherical type*, i.e. has finite apartments, then its *diameter*

$$\operatorname{diam} \Delta := \sup_{C,D \in \mathcal{C}(\Delta)} d(C,D)$$

is finite and equal to the diameter of any of its apartments Σ:

$$\operatorname{diam} \Delta = \operatorname{diam} \Sigma.$$

We now come to the converse of Corollary 3.1.16, announced earlier.

3.2.7. PROPOSITION. *Any gallery of reduced type in a building is geodesic.*

PROOF. Let $\boldsymbol{C} = (C_0, C_1, \ldots, C_m)$ be the gallery in question. We proceed by induction on m and thus may assume that $d(C_1, C_m) = m - 1$. Let Σ be any apartment containing C_1 and let $\rho = \rho_{\Sigma,C_1}$ be the retraction onto Σ with centre C_1. By Proposition 3.2.6(b), the image $(\rho C_1, \ldots, \rho C_m)$ in Σ is geodesic and thus in particular simple (without repetitions). Necessarily, it has the same type as $(C_1, \ldots, C_m)$, and since $\rho C_0 \neq \rho C_1 = C_1$, the whole image $\rho\boldsymbol{C}$ has the same type as $\boldsymbol{C}$. Since type $\boldsymbol{C}$ is reduced, $\rho\boldsymbol{C}$ is clearly geodesic in Σ, and by Proposition 3.2.6(a) also in the whole building. Thus $m = d(\rho C_0, \rho C_m) \leqslant d(C_0, C_m)$, necessarily $d(C_0, C_m) = m$, as desired. □

It will later turn out convenient to extend the notion of a gallery, a geodesic, and the function $d(C,D)$ as follows. Let A, B arbitrary simplices in a connected complex. By a *gallery joining* A to B we mean an ordinary gallery $\boldsymbol{C}$ whose origin contains A and whose extremity contains B; we sometimes write

$$\boldsymbol{C} = (A \subseteq C_0, C_1, \ldots, C_m \supseteq B).$$

A gallery of this kind of shortest possible length, for given A, B, is called a *geodesic joining A to B*; its length is denoted by $d(A, B)$. Of course, d is not a distance function on the simplices. Rather it is the usual distance of the subsets $\mathcal{C}(\operatorname{St} A)$ and $\mathcal{C}(\operatorname{St} B)$ in the metric space $\mathcal{C}(\Delta)$:

$$d(A, B) = \min\{d(C, D)\colon\ C \in \mathcal{C}(\operatorname{St} A),\ D \in \mathcal{C}(\operatorname{St} B)\}.$$

The set of all geodesics between A and B will occasionally be denoted by $\mathcal{G}(A, B)$, or $\mathcal{G}_\Delta(A, B)$, if the reference to Δ is important. Also, we write $\mathcal{G}(A)$ for the set of all geodesics starting at A and ending somewhere:

$$\mathcal{G}(A) := \bigcup_{C \in \mathcal{C}(\Delta)} \mathcal{G}(A, C).$$

Using these concepts, Proposition 3.2.6 can be restated and slightly generalized as follows:

$$d_\Delta(A, B) = d_\Sigma(A, B) \quad \text{for all } A, B \in \Sigma,$$

$$\mathcal{G}_\Sigma(A) \subseteq \mathcal{G}_\Delta(A) \quad \text{for all } A \in \Sigma,$$

$$\mathcal{G}_\Delta(A, D) = \mathcal{G}_\Sigma(A, D) \quad \text{for all } A \in \Sigma,\ D \in \mathcal{C}(\Sigma).$$

Furthermore, arguing as in the proof of (b), the following is readily checked:

$$\rho_{\Sigma, C}\mathcal{G}_\Delta(A) = \mathcal{G}_\Sigma(A) \quad \text{if } A \subseteq C \in \Sigma. \tag{1}$$

3.2.8. THEOREM. *Every thick structured complex is a building. That is, its apartments are Coxeter–Tits complexes.*

PROOF. The proof is based on the characterization of Coxeter–Tits complexes by the existence of enough foldings (Theorem 2.4.2). If Σ is an apartment of the structured complex Δ, and $C, C' \in \Sigma$ are adjacent chambers, the desired folding π of Σ mapping C to C' is constructed with the aid of a third chamber $D \in \Delta$ containing $F := C \cap C'$ and distinct from C and C', and of an apartment $\Sigma_1 \supseteq \{C, D\}$. Define π to be the retraction $\rho_{\Sigma_1, C}$, followed by the retraction $\rho_{\Sigma, C'}$, restricted to Σ:

$$\pi = \rho_{\Sigma, C'} \circ \rho_{\Sigma_1, C}\big|_\Sigma.$$

Symmetrically, we set

$$\pi' = \rho_{\Sigma, C} \circ \rho_{\Sigma_1, C'}\big|_\Sigma.$$

Since π and π' act as the identity on $\Sigma \cap \Sigma_1$, we have

$$\pi A = \pi' A = A \quad \text{for all } A \subseteq F = C \cap C'. \tag{2}$$

Furthermore, it is clear that

$$\pi C' = \pi C = C, \qquad \pi' C = \pi' C' = C'. \tag{3}$$

Indeed $\rho_{\Sigma_1,C}C'$ is in Σ_1, contains F, and is distinct from C, therefore $\rho_{\Sigma_1,C}C' = D$. Similarly, $\rho_{\Sigma,C'}D$ is in Σ, contains F, and is distinct from C', therefore $\rho_{\Sigma,C'}D = C$. From the formula (1) stated above it finally follows that π maps geodesics starting at F again to geodesics starting at F, the same for π':

$$\pi\mathcal{G}_\Sigma(F) \subseteq \mathcal{G}_\Sigma(F) \supseteq \pi'\mathcal{G}_\Sigma(F). \tag{4}$$

We shall now show that, given two adjacent chambers C, C' in any thin connected complex, a pair of endomorphisms π and π' of Σ which satisfy (2), (3), and (4) is a pair of opposite foldings. We first have to recall Lemma 2.2.1(b), which we shall use several times. Consider the set of galleries

$$\mathcal{H} := \{\boldsymbol{C} = (C_0, \ldots)\colon\ C_0 = C,\ \boldsymbol{C} \in \mathcal{G}(F)\},$$
$$\mathcal{H}' := \{\boldsymbol{C} = (C_0, \ldots)\colon\ C_0 = C',\ \boldsymbol{C} \in \mathcal{G}(F)\},$$

and the sets $\mathcal{C}$ and $\mathcal{C}'$ of chambers which are the ends of some geodesic in $\mathcal{H}$, resp., $\mathcal{H}'$. In the course of the proof, $\mathcal{C}$ and $\mathcal{C}'$ will turn out to be the half-complexes determined by C and C', i.e. the images of π, π'. Notice first that $\mathcal{C}(\Sigma) = \mathcal{C} \cup \mathcal{C}'$. By assumption (4), $\pi\mathcal{C}(\Sigma) = \mathcal{C}$. From the lemma recalled above, it follows that $\pi D = D$ for all $D \in \mathcal{C}$. Indeed, if $\boldsymbol{C} = (C, \ldots, D) \in \mathcal{H}$, is a geodesic, then $\pi\boldsymbol{C}$ is again geodesic, by (4), in particular it has no repetitions, and by the lemma, π is the identity on all of $\boldsymbol{C}$. Thus $\pi^2 = \pi$. In order to calculate $\pi^{-1}(D)$, for $D \in \mathcal{C} = \pi\mathcal{C}(\Sigma)$, we apply the same argument to $\pi\pi'$ instead of π. It follows that $\pi'\colon \mathcal{C} \to \mathcal{C}'$ and $\pi\colon \mathcal{C}' \to \mathcal{C}$ are inverse to each other. Since $\pi|_{\mathcal{C}} = \mathrm{id}$ and $\mathcal{C}(\Sigma) = \mathcal{C} \cup \mathcal{C}'$, it follows that $\pi^{-1}(D) = \{D, \pi' D\}$. The last thing to show is that $D \neq \pi' D$ for all $D \in \mathcal{C}$. Assume the contrary and choose a geodesic $\boldsymbol{C} = (F \subseteq C, \ldots, D) \in \mathcal{G}(F)$. Then $\pi'\boldsymbol{C} = (F \subseteq C', \ldots, D)$ is again a geodesic, in particular has no repetitions, furthermore $\pi' B = B$ for all $B \subseteq D \in \mathcal{C}'$. The standard lemma shows that π' is the identity on all of $\boldsymbol{C}$, contradicting the fact that $\pi' C = C'$.□

3.2.9. PROPOSITION.

(a) *If $(\Delta_1, \boldsymbol{\Sigma}_1)$ and $(\Delta_2, \boldsymbol{\Sigma}_2)$ are buildings, then the join*

$$(\Delta_1 \times \Delta_2,\ \{\Sigma_1 \times \Sigma_2\colon\ \Sigma_1 \in \boldsymbol{\Sigma}_1,\ \Sigma_2 \in \boldsymbol{\Sigma}_2\})$$

is a building.

(b) *If $(\Delta, \boldsymbol{\Sigma})$ is a building and $A \in \Delta$, then the star*

$$(\mathrm{St}_\Delta A,\ \{\mathrm{St}_\Sigma A\colon\ \Sigma \in \boldsymbol{\Sigma},\ A \in \Sigma\})$$

is a building.

PROOF. The statement holds for Coxeter–Tits complexes, and (S2) and (S3) obviously carry over to joins and stars. □

3.2.10. PROPOSITION.

(a) *If (Δ, Σ) is a building of rank 2, then Δ is a generalized m-gon, and Σ is the set of all circuits of length $2m$, where $\circ \overset{m}{—} \circ$ is the Coxeter diagram of (Δ, Σ). Conversely, the set Σ of all circuits of length $2m$ in a generalized m-gon Δ is a system of apartments for Δ.*

(b) *A building of type M, for some Coxeter diagram M, belongs to the diagram M (in the sense of Section* 3.1).

PROOF. (a) is straightforward to prove; (b) is an immediate consequence of (a) and Proposition 3.2.9(b). □

3.3. *The 'First Main Characterization of Buildings'*

In this subsection, we maintain the following notation:

M is a Coxeter diagram over I,

$W = W(M)$ is the Weyl group belonging to M,

$\mathcal{C}$ is a chamber system of type M.

Recall that the assumption on $\mathcal{C}$ implies the following. If

$$\boldsymbol{C} = (C_0, \ldots, C_q;\ i_1, \ldots, i_q)$$

is a gallery such that the word $i_1 \ldots i_q$ is reduced with respect to M (a gallery of reduced type), then for any other reduced word $j_1 \ldots j_q$ representing the same group element,

$$s(i_1 \ldots i_q) = s(j_1 \ldots j_q),$$

there also exists a gallery

$$\boldsymbol{D} = (C_0 = D_0, D_1, \ldots, D_q = C_q; j_1 \ldots j_q)$$

of type $j_1 \ldots j_q$ having the same extremities. In 3.1.13, we have introduced the notation

$$C \xrightarrow{w} D, \quad C, D \in \mathcal{C},\ w \in W,$$

to denote the existence of a reduced gallery $(C = C_0, \ldots, C_q = D;\ i_1, \ldots, i_q)$ such that $s(i_1 \ldots i_q) = w$.

If $\mathcal{C}$ is a building, the element $w = s(i_1 \ldots i_q)$ is the same for given C, D and all choices of reduced galleries joining C to D. To prove this, let Σ be an apartment containing C, D. By Proposition 3.2.7 (and 3.2.6), any reduced gallery joining C to D is contained in Σ. Thus it suffices to verify the claim for Coxeter complexes where it is obvious: if Σ is the coset complex of W with set of chambers W and without loss of generality $C = e \in W$, then $C \xrightarrow{w} D$ just means $D = w$.

The purpose of this subsection is to prove that buildings are even characterized by this property.

3.3.1. THEOREM ('First Main Characterization of Buildings'). *The chamber system $\mathcal{C}$ of type M is a building if and only if the following property* (P_C) *holds for all chambers C:*

(P_C) $C \xrightarrow{w} D,\ C \xrightarrow{w'} D \Rightarrow w = w'$ *for all* $D \in \mathcal{C},\ w, w' \in W$.

It is sufficient to assume the property (P_C) *for just one chamber C. It is even sufficient to assume the following weaker property:*

(P_C^0) $C \xrightarrow{w} C \Rightarrow w = e$ *for all* $w \in W$.

That is, $\mathcal{C}$ is a building if and only if any closed reduced gallery based at C is trivial, for at least one chamber C.

Under the assumption of (P_C) for all C, the proof of the theorem will be contained in the following lemmata. Independently of this main part of the proof, one shows that (P_C^0) implies (P_C), and that (P_C) for one C implies (P_C) for all C. We shall not reproduce the proof of the latter statement: see Tits [1981a], p. 532 or Ronan [1989], proof of Theorem 4.2.

3.3.2. PROOF of (P_C^0) $\Rightarrow$ (P_C). Let

$$C \xrightarrow{w} D,\ C \xrightarrow{w'} D, \quad w \neq w' \neq e$$

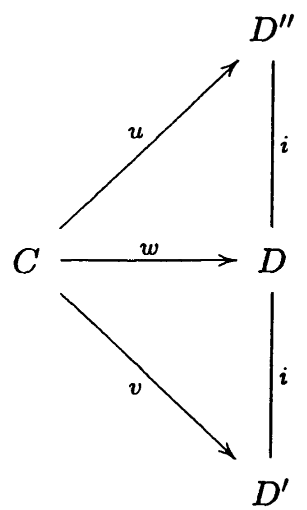
be a counterexample to (P_C) with minimal length $l = l(w') \leqslant l(w)$, write

$$w' = vs_i, \qquad l(v) = l(w) - 1, \quad i \in I,$$

and let $D' \in \mathcal{C}$ be such that $C \xrightarrow{v} D' \xrightarrow{i} D$. From $l(ws_i) > l(w)$ it would follow that

$$C \xrightarrow{ws_i} D',\ C \xrightarrow{v} D'$$

is a smaller counterexample. Thus $l(ws_i) < l(w)$, and from the exchange property for the Weyl group we get $w = us_i$ for some $u \in W$ of length $l(w) - 1$. Let D' be a chamber with $C \xrightarrow{u} D' \xrightarrow{v} D$.



If $D = D''$, then $C \xrightarrow{v} D',\ C \xrightarrow{u} D'$ is a smaller counterexample. If $D \neq D'$, then $C \xrightarrow{u} D'' \xrightarrow{i} D'$ and thus $C \xrightarrow{w} D'$. Since also $C \xrightarrow{v} D'$ and certainly $v \neq w$ (even $l(v) < l(w') \leqslant l(w)$), we again arrive at a smaller counterexample. Thus in either case we get a contradiction.

We now prepare the main part of the proof of Theorem 3.3.1. For $w, v \in W$, we denote by $\xrightarrow{w} \circ \xrightarrow{v}$ the usual composite of the relations $\xrightarrow{w}$ and $\xrightarrow{v}$, that is $C \xrightarrow{w} \circ \xrightarrow{v} D$ if there exists a chamber E such that $C \xrightarrow{w} E$ and $E \xrightarrow{v} D$.

3.3.3. LEMMA. *Let $w, v \in W$, and choose a reduced expression $i_1 \ldots i_m$ for v.*
If $C \xrightarrow{w} \circ \xrightarrow{v} D$, $C, D \in \mathcal{C}$, then there exists a subword $i_{t_1} i_{t_2} \ldots i_{t_k}$ such that

$$C \overset{ws(i_{t_1} \ldots i_{t_k})}{\longrightarrow} D$$

PROOF. By induction on $m = l(v)$, it is enough to show that

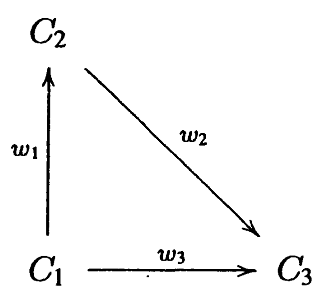
$$C \xrightarrow{w} \circ \xrightarrow{i} D \Rightarrow C \xrightarrow{w} D \text{ or } C \xrightarrow{ws_i} D.$$

Choose a reduced expression $j_1 \ldots j_q$ for w. If $l(ws_i) > q$, the claim is clear by definition. If $l(ws_i) \leqslant q$, then there exists an index p such that

$$w = s(j_1 \ldots j_q) = s(j_1 \ldots \widehat{j_p} \ldots j_q i),$$

by the exchange condition. Let E be such that $C \xrightarrow{w} E$, $E \xrightarrow{i} D$, choose a gallery $(C = C_0, C_1, \ldots, C_{q-1}, C_q = E; g, i)$, where $g := j_1 \ldots \widehat{j_p} \ldots j_q$. We have $C \xrightarrow{s(g)} C_{q-1} \xrightarrow{i} E$. If $C_{q-1} = D$, we have $C \xrightarrow{s(g)} D$, i.e. $C \xrightarrow{ss_i} D$. Otherwise $C_{q-1} \xrightarrow{i} D$, and $(C, \ldots, C_{q-1}, D; g, i)$ is a gallery of reduced type, thus $C \xrightarrow{w} D$. □

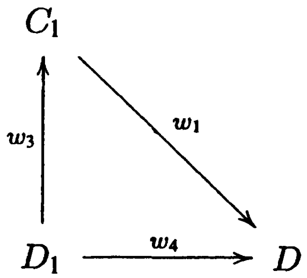
A subset $\mathcal{A} \subseteq \mathcal{C}$ is called *flat* if



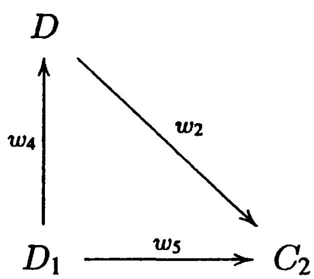
implies $w_3 = w_1 w_2$, for any $C_1, C_2, C_3 \in \mathcal{A}$, $w_1, w_2, w_3 \in W$. If $\mathcal{A}$ is an apartment in a building $\mathcal{C}$, then $\mathcal{A}$ is obviously flat. Conversely, we shall obtain the desired apartments in the proof of the theorem as the maximal flat subsets of $\mathcal{C}$.

3.3.4. LEMMA. *If $\mathcal{A}$ is flat, then the convex hull $\bar{\mathcal{A}}$ also is flat.*

PROOF. There exists a maximal flat subset $\mathcal{A}' \subseteq \bar{\mathcal{A}}$. If $\mathcal{A}'$ is convex, then $\mathcal{A}' = \bar{\mathcal{A}}$. Now suppose that $\mathcal{A}'$ is not convex. Then there exists $C_1, C_2 \in \mathcal{A}'$ and $D \in \bar{\mathcal{A}} \setminus \mathcal{A}'$ such that $d(C_1, C_2) = d(C_1, D) + d(D, C_2)$. We shall show that $\mathcal{A}' \cup \{D\}$ is flat, thus having a contradiction. We first look at a 'triangle' of the form



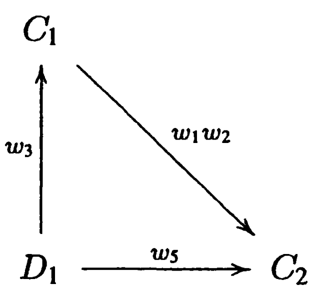
By Lemma 3.3.3, there exists $\overline{w}_1$ coming from a subexpression of w_1 such that $D_1 \stackrel{w_3\overline{w}_1}{\longrightarrow} D$. By the general assumption, $w_3\overline{w}_1 = w_4$. Now look at



Similarly, $w_4\overline{w}_2 = w_5$. Thus $w_5 = w_3\overline{w}_1\overline{w}_2$. Because of

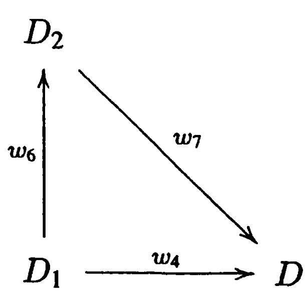
$$d(C_1, C_2) = d(C_1, D) + d(D, C_2),$$

we have $C_1 \stackrel{w_1w_2}{\longrightarrow} C_2$. On the other hand,

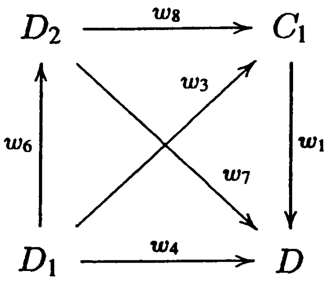


lies inside the flat set $\mathcal{A}'$, thus $w_3w_1w_2 = w_5$, and finally $\overline{w}_1\overline{w}_2 = w_1w_2$. From $l(w_1w_2) = l(w_1) + l(w_2)$ it follows that $\overline{w}_1 = w_1$, $\overline{w}_2 = w_2$, and $w_3w_1 = w_4$.

Now consider the general case of a triangle in $\mathcal{A}' \cup \{D\}$ involving D



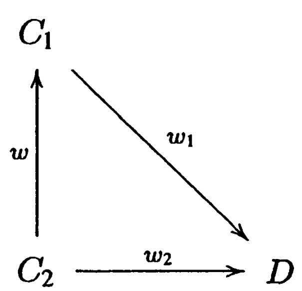
Consider the completed picture



By the above special case, $w_7 = w_8 w_1$, and $w_6 = w_3 w_8^{-1}$, by the flatness of $\mathcal{A}$. Thus $w_6 w_7 = w_3 w_1$, and this equals w_4, as was shown above. □

3.3.5. Lemma. *If $\mathcal{A}$ is maximal flat, $C \in \mathcal{A}$, $i \in I$, then there exists $D \in \mathcal{A}$ such that $C \xrightarrow{i} D$.*

Proof. Assume that C, i are such that $C \xrightarrow{i} D$ implies $D \notin \mathcal{A}$. We shall show that $\mathcal{A} \cup \{D\}$ is flat for any $D \in \mathcal{C}$ such that $C \xrightarrow{i} D$, which then contradicts the maximality of $\mathcal{A}$. Consider a triangle



, $C_1, C_2 \in \mathcal{A}$.

By Lemma 3.3.3 we have

$$C_1 \xrightarrow{w_1} C \quad \text{or} \quad C_1 \xrightarrow{w_1 s_i} C,$$

and in the first case $l(w_1 s_i) < l(w_1)$, and there exists an E such that

$$C_1 \xrightarrow{w_1 s_i} E \xrightarrow{i} C.$$

But then $E \in \operatorname{conv}\{C_1, C\} \subseteq \mathcal{A}$, by Lemma 3.3.4, which contradicts the assumption on i. Therefore $C_1 \xrightarrow{w_1 s_i} C$, and analogously $C_2 \xrightarrow{w_2 s_i} C$. From the flatness of $\mathcal{A}$ it follows that $w w_1 s_i = w_2 s_i$, therefore $w w_1 = w_2$, as desired. □

Proof of Theorem 3.3.1. Let $\underline{\mathcal{A}} := \{\mathcal{A} \subseteq \mathcal{C}\colon \mathcal{A} \text{ is maximal flat}\}$. Using the above lemmata, we shall show that $\underline{\mathcal{A}}$ is a system of apartments for $\mathcal{C}$. First of all, the assumption of the theorem precisely says that all 2-element subsets of $\mathcal{C}$ are flat, in particular, are contained in at least one member of $\underline{\mathcal{A}}$. Now let $C \in \mathcal{A} \in \underline{\mathcal{A}}$, and let $w \in W$. From Lemma 3.3.4 it follows by induction on $l(w)$ that there exists an $E \in \mathcal{A}$ such that $C \xrightarrow{w} E$. By the flatness of $\mathcal{A}$, and the assumption of the theorem, it follows that $E =: \varphi_{\mathcal{A},C}(w)$ is uniquely determined by w. This map $\varphi_{\mathcal{A},C}\colon W \to \mathcal{A}$ is an isomorphism of chamber systems. If $\mathcal{B}$ is another apartment containing C, the isomorphism $\varphi_{\mathcal{B},C} \circ \varphi_{\mathcal{A},C}^{-1}\colon \mathcal{A} \to \mathcal{B}$ obviously fixes all elements in $\mathcal{A} \cap \mathcal{B}$. □

3.3.6. COROLLARY. *In a building, a gallery is geodesic if and only if its type is reduced.*

PROOF. Suppose that $(C_0, C_1, \ldots, C_n; f)$ is a gallery of reduced type, and

$$(C_0 = D_0, D_1, \ldots, D_m = C_n; g)$$

is a geodesic of length $m < n$. Since g automatically is reduced, Theorem 3.3.1 says that $s(f) = s(g)$. Thus we have reduced words of different length representing the same element of the Weyl group, which is absurd. □

We shall now describe a general application of Theorem 3.3.1 to the 'internal' geometrical properties of buildings which will be in more detail the subject of Section 5. Consider the following:

3.3.7. *Exchange condition for geodesics.* Let $(C_0, \ldots, C_n; i_1, \ldots, i_n)$ be a geodesic (in an arbitrary chamber system), D a chamber, and $i \in I$ such that $C_n \stackrel{i}{-} D$. If $(C_0, \ldots, C_n, D; i_1, \ldots, i_n, i)$ is not geodesic, then there exists a gallery of the form

$$\left(C_0 = C_0', C_1', \ldots, C_{n-1}';\ i_1, \ldots, \widehat{i_\nu}, \ldots, i_n\right) \quad \text{(with } i_\nu \text{ omitted)}$$

such that $C_{n-1}' \stackrel{i}{-} C_n$.

Notice that the new gallery necessarily is geodesic, since $C_n \stackrel{i}{-} C_{n-1}' \stackrel{i}{-} C_n$, and therefore

$$\left(C_0 = C_0', C_1', \ldots, C_{n-1}', C_n; i_1, \ldots, \widehat{i_\nu}, \ldots, i_n, i\right)$$

is a gallery, of length $n = d(C_0, C_n)$. In the thin case, one necessarily has $C_1' = C_1, \ldots, C_{\nu-1}' = C_\nu$; we shall see in a moment that this holds also in the general case. If furthermore $\Delta = \Delta(W, W^i, i \in I)$ is thin, flag-transitive (where W is the group of type preserving automorphisms of Δ, the subgroups $\{1, s_i\}$, $i \in I$, are the stabilizers of the panels of a fixed chamber, and $W^i = \langle s_j\colon j \neq i\rangle$ are the stabilizers of its vertices, see Section 2.2), then necessarily $s_{i_1} \ldots s_{i_n} = s_{i_1} \ldots \widehat{s_{i_\nu}} \ldots s_{i_n} s_i$, and the exchange condition for geodesics reduces to the ordinary exchange condition to the group generated by involutions $(W, s_i, i \in I)$. Thus, a thin, flag-transitive complex satisfies the exchange condition if and only if it is a Coxeter–Tits complex; cf. also the remark at the end of Section 2.4.

We shall now see how this generalizes to arbitrary buildings, i.e. to the not necessarily thin case.

3.3.8. THEOREM. *A chamber system of Coxeter type is a building if and only if it satisfies the exchange condition for geodesics.*

PROOF. We shall not prove the 'if' part here (which is the difficult one). The reader may consult Theorem 5.1.7 below whose proof shows that the exchange property for geodesics implies the so-called gate property for stars. After this, one can immediately apply Theorem 5.1.11 below.

The 'only if' part readily follows from Corollary 3.3.6: under the assumptions of the exchange condition, $i_1 \ldots i_n i$ is not reduced. Now apply the ordinary exchange condition for the Weyl group; there exists a index ν such that

$$s(i_1 \ldots \widehat{i_\nu} \ldots i_n i) = s(i_1 \ldots i_n),$$

and therefore there exists a gallery

$$\left(C_0 = C_0', C_1', \ldots, C_{n-1}', C_n' = C_n; i_1, \ldots, \widehat{i_\nu}, \ldots, i_n, i\right),$$

as claimed. □

3.3.9. PROPOSITION. *Let C be a building, or more generally, any chamber system satisfying the exchange property for geodesics. (C is not necessarily of Coxeter type.) Then the following holds:*

If $(C_0, C_1, \ldots, C_n; i_1, \ldots, i_n)$ and $(C_0', C_1', \ldots, C_n'; i_1', i_2', \ldots, i_n')$ are two geodesics such that $C_0 = C_0'$, $C_n = C_n'$, and $i_\nu = i_\nu'$, $i_{\nu+1} = i_{\nu+1}'$, $\ldots$, $i_n = i_n'$, for some ν, then $C_{\nu-1} = C_{\nu-1}'$, $C_\nu = C_\nu', \ldots$. In particular, two geodesics with the same origin and endpoint and the same type coincide.

PROOF. It suffices to show that $C_{n-1} = C_{n-1}'$. Assume the contrary, and apply the exchange property for geodesics to $(C_0, \ldots, C_{n-1}; i_1, \ldots, i_{n-1})$ and $C_{n-1} \overset{i_n}{—} C_{n-1}'$. The conclusion is that there exists a gallery of shape

$$(C_0 = D_0, \ldots D_{n-2}, C_{n-1};\ i_1, \ldots, \widehat{i_\mu}, \ldots i_n).$$

Replacing C_{n-1} by C_n, which is also i_n-adjacent to D_{n-2}, we see that $d(C_0, C_n) < n$, a contradiction. □

Taking into account Corollary 3.3.6, the last proposition has the following corollary.

3.3.10. COROLLARY. *In a building, the following property* (Q_C) *holds for all chambers C:*

(Q_C) *If two simple galleries have the same origin C, the same extremity, and the same reduced type, they coincide.*

(Recall that 'simple' means that any two consecutive chambers of the gallery are distinct.) This property will be used later in Section 6.

The following proposition is an easy application of the 'First main characterization' which is preparatory for the results on covering theory developed in Section 6. Recall the notion of 2-simple-connectedness introduced in Section 2.4.

3.3.11. PROPOSITION. *Any (chamber system of a) building is 2-simply-connected.*

PROOF. Let π: $\widetilde{\mathcal{C}} \to \mathcal{C}$ be a 2-covering, where $\mathcal{C}$ is a building, and $\mathcal{C}$ is connected. Consider chambers $C, D \in \widetilde{\mathcal{C}}$ such that $\pi C = \pi D$, and choose a geodesic

$$\mathcal{C} = (C = C_0, C_1, \dots, C_m = D; f).$$

By Corollary 3.1.16, its type f is reduced. The gallery $\pi\mathcal{C}$ is a simple gallery of reduced type in the building $\mathcal{C}$. By the theorem, f must be the trivial word, i.e. $C = D$. □

We close this section with another characterization of buildings which, again relies on the First Main Characterization 3.3.1. By this theorem, one has for every building $\mathcal{C}$ with diagram M a well defined function δ: $\mathcal{C} \times \mathcal{C} \to W(M)$ given by

$$\delta(C, D) = w \Leftrightarrow C \xrightarrow{w} D.$$

This function trivially determines the chamber system structure on $\mathcal{C}$, since $C \xrightarrow{i} D$ is equivalent to $\delta(C, D) = s_i$. Therefore it is a natural idea to characterize $\mathcal{C}$ in terms of properties of δ. This is done in the following proposition which is stated without proof in Tits [1989], §1.4.

3.3.12. PROPOSITION. *Let (W, S) be a Coxeter system. A function δ: $\mathcal{C} \times \mathcal{C} \to W(M)$ (where $\mathcal{C}$ is any set) defines a building structure on $\mathcal{C}$ if and only if it has the following properties*:

(D1) $\delta(C, D) = 1$ *if and only if* $C = D$;

(D2) *If* $C, D, E \in \mathcal{C}$, $\delta(D, E) = s \in S$, *then* $\delta(C, E) \in \{w, ws\}$, *where* $\delta(C, D) = w$; *if in addition* $l(ws) > l(w)$, *then* $\delta(C, E) = ws$;

(D3) *If* $C, D \in \mathcal{C}$, $s \in S$, *then there exists* $E \in \mathcal{C}$ *such that* $\delta(D, E) = s$ *and* $\delta(C, E) = \delta(C, D)s$.

The proof of this results is not difficult: once one has shown that the chamber system structure on $\mathcal{C}$ which is defined by δ actually is of type M, one can immediately apply the First Main Characterization of Buildings.

3.4. *Some applications of the 'First Main Characterization'*

We present two different kinds of 'internal' applications. The first are certain formulas for the number of chambers, and more generally, of simplices of any type, in finite buildings. The second application is a characterization by 'geometrical axioms' of the buildings of types C_n and D_n among all complexes belonging to those diagrams, and a proof of the fact that for the diagram A_n, no additional axioms are needed. For both applications, certain properties of the Weyl group have to be investigated. More precisely, one has to calculate shortest representatives of cosets wW^J and double cosets $W^K wW^J$, where K and J are subsets of the type set and W^J, W^K as earlier denote the 'standard Coxeter subgroups', $W^J = \langle s_i\colon\ i \notin J\rangle$. These calculations have been carried out in Section 2.5.

3.4.1. DEFINITION. An I-numbered complex is *locally finite* if each panel is contained in only finitely many chambers. It *admits parameters* $(q_i)_{i\in I}$, if $\operatorname{St} F$ consists of precisely $q_i + 1$ chambers, for each panel F of cotype i. Here, the q_i are positive integers, the *parameter system* of the complex.

As an example, the flag complex of a projective n-space over the finite field $\mathbb{F}_q$ with q elements admits parameters $q_i = q$ for all $i \in \{0, \ldots, n-1\}$. Indeed, $q+1$ is the cardinality of the projective line over $\mathbb{F}_q$. This example is a very special case of the following general result.

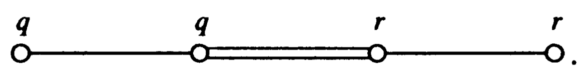
3.4.2. PROPOSITION. *Every thick finite building admits a parameter system $(q_i)_{i\in I}$ which is of the following shape.*

For the diagrams A_n, D_n or E_n, $n = 6, 7, 8$, all q_i are equal to each other.

For C_n with its natural labelling, all q_i, $i = 1, \ldots, n-1$ are equal to each other

C_n 

For F_4 with its natural labelling, $q_1 = q_2$ and $q_3 = q_4$:

F_4 

We did not include the diagrams $\mathsf{I}_2(m)$, m odd, $m \geqslant 5$, and H_3, H_4 in the proposition since thick finite buildings for these diagrams do not exist by the famous theorem of Feit–Higman (see Chapter 9). In fact, thick buildings of type H_3 or H_4 do not exist at all, as will be seen in Theorem 5.3.8.

The proof of Proposition 3.4.2 will be postponed until the end of Section 5.2; it will turn out to be an easy consequence of the results of that subsection.

The more difficult part of the proposition is the fact that any building admits parameters. The particular shape of the parameter system then is an immediate consequence of the following lemma.

3.4.3. LEMMA. *Let Δ be any locally finite building admitting a parameter system $(q_i)_{i\in I}$. If i and j are types such that m_{ij} is finite and odd, then $q_i = q_j$.*

Notice that this lemma actually reduces to a fact about generalized polygons: in a generalized m-gon, for odd m, the stars of two vertices of different type have the same cardinality. The proof of this fact does not really need the results of Section 4.3, but can be given more directly, see Tits [1959], §11.2. The basic idea, however, is the same as in Section 5.2. The lemma leads us to introduce the graph structure on the type set I whose edges are the two-element subsets $\{i, j\}$ such that m_{ij} is odd. The parameter system, if it exists, is constant on the connected components of this graph. This in particular gives the special shape of the parameter systems for the spherical diagrams as claimed in Proposition 3.4.2.

After Proposition 3.2.6, we observed that a building of spherical type has finite diameter. If Δ is any complex of finite diameter which furthermore is locally finite, then Δ actually is finite. Indeed, if C_0 is some fixed chamber, and d any natural number, then, by induction on d, there are only finitely many chambers C such that $d(C_0, C) \leqslant d$. For buildings, we shall now present a quantitative version of this result. The number of chambers of a finite building will turn out to be a certain polynomial function in the parameters. We shall now define this polynomial in terms of the Weyl group.

To this end, we start with an arbitrary Coxeter diagram $M = (m_{ij})_{i,j \in I}$ over a finite index set I. Fix a group element $w \in W = W(M)$, and consider some reduced representation $i_1 \dots i_r$ of w. For each $i \in I$, we are interested in the number of indices ν such that $i_\nu = i$. More precisely, we want to formalize to what extent this number does not depend on the particular representation $i_1 \dots i_r$. By Theorem 2.5.2, any other reduced representation is obtained from $i_1 \dots i_r$ by a sequence of elementary homotopies. Thus assume that $j_1 \dots j_r$ is elementary homotopic to $i_1 \dots i_r$, i.e. obtained from $i_1 \dots i_r$ by replacing a subword

$$p_{ij} := iji \dots \quad (m_{ij} \text{ factors})$$

by p_{ji}. If m_{ij} is even, then the number of occurrences of any $k \in I$ in the word does not change at all; if m_{ij} is odd, the number of i's decreases by one, and the number of j's increases by 1. This again leads to considering the graph structure on I whose edges are the $\{i, j\}$ such that m_{ij} is odd. Denote by $\mathcal{I} = \mathcal{I}(M)$ the set of connected components of this graph. (The fact that, in the most important case of spherical, irreducible diagrams, $\mathcal{I}$ consists of at most two elements, has already been observed in Proposition 3.4.2.) If $J \in \mathcal{I}$ is such a connected component, we just saw that the total number of types $j \in J$ occurring in the reduced word $i_1 \dots i_r$ is independent of the particular choice of the word representing the given group element w. We denote it by $l_J(w)$. In this way, we have represented the length function of W as a sum of 'partial length functions' running over the set $\mathcal{I}$ of connected components:

$$l(w) = \sum_{J \in \mathcal{I}} l_J(w).$$

For $M = \mathsf{C}_n$, we write

$$l(w) = l_1(w) + l_2(w),$$

where $l_2(w)$ is the number of occurrences of $i = n$, in the natural labelling of $I = \{1, 2, \dots, n\}$, similarly for F_4. We now introduce the formal power series in variables X_J, $J \in \mathcal{I}$,

$$P_M := \sum_{w \in W} \prod_{J \in \mathcal{I}} X_J^{l_J(w)} \in \mathbb{Z}^{\mathcal{I}} = \mathbb{Z}[X_J; J \in \mathcal{I}],$$

which we call the *weight enumerator* of M or $W(M)$. Often, the term *Poincaré series* of W is used. For C_n and F_4, we of course write

$$P_M(X_1, X_2) = \sum_{w \in W} X_1^{l_1(w)} X_2^{l_2(w)}.$$

If we set all X_J equal to one variable X, we have

$$P_M(X,\dots,X) = \sum_{w\in W} X^{l(w)}.$$

If (and only if) M is spherical, P_M reduces to a polynomial. If M is arbitrary and $d \in \mathbb{N}$ some constant, we denote by $P_M^{\leqslant d}$ the polynomial consisting of the terms of total degree at most d in P_M:

$$P_M^{\leqslant d} = \sum_{\substack{w\in W,\\ l(w)\leqslant d}} \prod_{J\in\mathcal{I}} X_J^{l_J(w)}.$$

Now we return to buildings and assume that Δ is a building of type M admitting a parameter system $(q_i)_{i\in I}$. Recall that the q_i are constant on the connected components of the graph considered above: $q_i = q_j$ if $i, j \in J \in \mathcal{I}$. Thus we may write the parameter system as $(q_J)_{J\in\mathcal{I}}$.

3.4.4. PROPOSITION. *Let Δ be a building of type M admitting a parameter system $\mathcal{Q} = (q_J)_{J\in\mathcal{I}}$, and $d \in \mathbb{N}$ some constant. The number of chambers in Δ at distance at most d from a fixed chamber C_0 is obtained by substituting the q_J into the polynomial $P_M^{\leqslant d}$:*

$$\#\{C \in \mathcal{C}(\Delta)\colon\ d(C_0, C) \leqslant d\} = P_M^{\leqslant d}(\mathcal{Q}).$$

In particular, if M is spherical, then the total number of chambers is $P_M(\mathcal{Q})$.

PROOF. By Theorem 3.3.1 and its Corollary 3.3.6, every chamber $C \in \Delta$ satisfies the relation $C_0 \xrightarrow{w} C$ for a unique element $w \in W$ of length $d(C_0, C)$. Therefore it is sufficient to show that the number of such C, for a given w, equals

$$\prod_{J\in\mathcal{I}} q_J^{l_J(w)}.$$

Fix a reduced word $f = i_1 i_2 \dots i_l$ representing w. Obviously, the number of distinct galleries of type f and origin C_0 equals

$$\prod_{J\in\mathcal{I}} q_J^{l_J(w)}.$$

But by Proposition 3.3.9, any two distinct such galleries have different endpoints. This proves our claim. □

We now say a few words about the calculation of the power series P_M and give some examples. By making use of shortest coset representatives, one can reduce the question to Coxeter diagrams of smaller rank; in particular, one can calculate P_M inductively for sequences of diagrams like A_n, $n = 1,2,3,\dots$, or C_n, $n = 1,2,3,\dots$. To this end, choose a subset $I' \subset I$ (preferably $|I \setminus I'| = 1$), denote by $W_{I'}$ the corresponding Coxeter subgroup $W_{I'} = \langle s_j \in W\colon\ j \in I' \rangle \cong W(M')$, where M' is the restriction of the diagram to I', and by $A_{I'}$ the set of shortest elements in the cosets $W_{I'}w$. From Proposition 2.1.7 (unique decomposition) it is obvious that

$$P_M = P_{M'} \cdot R_{I'},$$

where

$$R_{I'} := \sum_{w \in A_{I'}} \prod_{J \in \mathcal{I}} X_J^{l_J(w)}$$

is the power series analogous to P_M for the subset $A_{I'} \subseteq W$. We shall now illustrate the use of this formula by calculating the weight enumerator P_M for the spherical diagrams $M = \mathsf{A}_n$, C_n, and D_n.

3.4.5. PROPOSITION. *For the spherical Coxeter diagrams* A_n, $n \geqslant 1$, C_n, $n \geqslant 2$, D_n, $n \geqslant 3$, *the weight enumerators are as follows*:

$$P_{\mathsf{A}_n}(X) = \prod_{j=1}^{n} (1 + X + \cdots + X^j),$$

$$P_{\mathsf{C}_n}(X_1, X_2) = \prod_{j=1}^{n} (1 + X_1 + \cdots + X_1^{j-1} + X_1^{j-1} X_2 + \cdots + X_1^{2j-2} X_2)$$

(*the first factor is* $1 + X_2$), *in particular*

$$P_{\mathsf{C}_n}(X) = P_{\mathsf{C}_n}(X, X) = \prod_{j=1}^{n} (1 + X + \cdots + X^{2j-1}),$$

$$P_{\mathsf{D}_n}(X) = \prod_{j=2}^{n} (1 + \cdots + X^{j-2} + 2X^{j-1} + X^j + \cdots + X^{2j-2})$$

(*the first factor is* $1 + 2X + X^2 = (1 + X)^2$).

PROOF. Recall from Examples 2.5.1, 2.5.4 and 2.5.5 the list $A_{I'}$ of shortest coset representatives for $I' = \{2, \ldots, n\} \subset I = \{1, \ldots, n\}$. It shows that $R_{I'}$ is as follows:

$$\mathsf{A}_n\colon\ R_{I'} = 1 + X + X^2 + \cdots + X^n,$$

$$\mathsf{C}_n\colon\ R_{I'} = 1 + X_1 + \cdots + X_1^{n-1} + X_1^{n-1} X_2 + \cdots + X_1^{2n-2} X_2,$$

$$\mathsf{D}_n\colon\ R_{I'} = 1 + \cdots + X^{n-2} + 2X^{n-1} + X^n + \cdots + X^{2n-2}.$$

Now the proposition immediately follows by induction from the formula

$$P_M = P_{M'} \cdot R_{I'},$$

taking into account that

$$P_{\mathsf{A}_1}(X) = P_{\mathsf{C}_1}(X) = 1 + X, \qquad P_{\mathsf{D}_2}(X) = P_{\mathsf{A}_1 \times \mathsf{A}_1}(X) = (1+X)^2.$$

□

Notice that for $n = 3$, the formulae for A_n and D_n indeed give the same polynomial

$$P_{\mathsf{A}_3}(X) = (1+X)^2(1+X^2)(1+X+X^2).$$

3.4.6. REMARK. It is a general fact that for any spherical diagram M of rank n,

$$P_M(X) = \prod_{j=1}^{n} \left(1 + X + \cdots + X^{k_j}\right),$$

where $k_1 = 1, k_2, \ldots, k_n$ are the so called exponents of the Coxeter group $W(M)$. See Bourbaki [1968], Chapter 5, §6.2, for a definition of the exponents and *loc. cit.* Chapter 6, Exercise 4 of §4, or Solomon [1966] for the formula. For the exceptional groups of type E_6, E_7, E_8, the exponents are

E_6: 1, 4, 5, 7, 8, 11,

E_7: 1, 5, 7, 9, 11, 13, 17,

E_8: 1, 7, 11, 13, 17, 19, 23, 29.

The reader is invited to check the polynomials $P_{\mathsf{E}_6}, P_{\mathsf{E}_7}$ and P_{E_8} by the above method.

We now come to the second line of consequences of Theorem 3.3.1, dealing with 'geometrical properties' of buildings involving points, lines, ..., subspaces, and other incidence properties of these. The following proposition improves Example 3.1.5 above. We show that any strongly connected complex of type A_n is a building, not assuming *a priori* that it is the flag complex of a projective space. (If one already knows that a complex of type A_n is a building, it is easy to verify the axioms for subspaces of a projective space.)

3.4.7. PROPOSITION.

(a) *Let $\mathcal{C}$ be a connected chamber system of type A_n such that*

$$C \overset{i}{\sim} D,\; C \overset{\{i+1,\ldots,n\}}{\sim} D \Rightarrow C = D$$

for all $C, D \in \mathcal{C}$, $i \in I = \{1, \ldots, n\}$, $i < n$ (e.g., $\mathcal{C}$ strongly connected). Then $\mathcal{C}$ is a building.

(b) *Let Δ be a connected complex of type A_n such that $\mathrm{St}_\Delta\, x$ is connected for all vertices x of Δ (e.g., Δ strongly connected). Then Δ is a building.*

PROOF. (a) We use induction on n. For $n = 2$, the claim equals the assumption. According to Theorem 3.3.1, we have to show that $C \xrightarrow{w} C$ for some $C \in \mathcal{C}$ and $w \in W = W(\mathsf{A}_n)$ implies $w = e$. We use the fact that W^1 and $W^1 s_1 W^1$ are the only double cosets of the subgroups $W^1 = \langle s_2, \ldots, s_n \rangle \subseteq W$. This is a trivial consequence of the fact used above in Example 3.1.4 that $e, 1, 12, \ldots, 12\ldots n$ are a full set of representatives of the left cosets $W^1 w$, and $1, 12, \ldots, 12\ldots n$ all belong to the same right coset wW^1. If $w \in W^1$, we can look at the relation '$C \xrightarrow{w} C$' inside the star $\mathcal{C}(C, \{2, \ldots, n\})$. This is a connected chamber system of rank $n-1$ satisfying our assumptions. By the induction hypothesis, $w = e$. If $w \in W^1 s_1 W^1$, choose any reduced expression $f = i_1 i_2 \ldots i_l$ of w such that $i_m = 1$ for precisely one m. By the definition of $\xrightarrow{w}$, there exists a closed gallery $(C = C_0, C_1, \ldots, C_e = C)$ of type f satisfying $C_{t-1} \neq C_t$ for all t. But the chambers C_{m-1} and C_m are simultaneously 1-equivalent and $\{2, \ldots, n\}$-equivalent, therefore $C_{m-1} = C_m$ by assumption. This is a contradiction.

(b) Consider the chamber system $\mathcal{C}(\Delta)$. Any chamber system coming from a numbered complex satisfies the assumption of part (a). Furthermore, $\mathcal{C}(\Delta)$ is of type A_n. Therefore, $\mathcal{C}(\Delta)$ is a building, by part (a). Thus, by Proposition 1.3.4, $\Delta \cong \Delta(\mathcal{C}(\Delta))$ is a building as well. □

We next characterize buildings of type C_n or D_n. To this end, we introduce the following axiom of 'linearity'.

3.4.8. AXIOM (Lin). Let x_i, y_i be vertices of type i, and x_{i+1}, y_{i+1} of type $i+1$, where $i+1 \leqslant n-1$ in the case C_n, $i+1 \leqslant n-2$ in the case D_n. If x_i and y_i are both incident to x_{i+1}, y_{i+1}, then $x_i = y_i$ or $x_{i+1} = y_{i+1}$.

If one interprets vertices of type 1 as points, vertices of type 2 as lines, the axiom says that two distinct lines share at most one point, and similarly for higher dimensional subspaces of two consecutive dimensions.

3.4.9. PROPOSITION. *A strongly connected numbered complex of type C_n or D_n is a building if and only if it satisfies the linearity axiom* (Lin).

PROOF. We shall only proof the if-part here, by essentially the same technique as before in the A_n case. The only-if-part will be treated in Section 4.2 below. For $n = 2$, resp., 3, nothing is to be shown. Recall from Example 2.5.7 that the double cosets $W^1 w W^1$ are given by the following shortest representatives:

C_n: (), 1, $123\ldots n\ldots 21$,

D_n: (), 1, $12\ldots(n-1)n(n-2)(n-3)\ldots 21$.

Now consider a nontrivial relation $C \xrightarrow{w} C$, $w \in W$, $w \neq \mathrm{id}$. If $w \in W^1$, this is a contradiction, by the inductive hypothesis. If $w \in W^1 s_1 W^1$, choose a word f

representing w such that 1 occurs precisely once. Then, in any gallery of type f, the vertices of type 1 in the first and last chamber are distinct. In particular, such a gallery cannot be closed. (This is the same contradiction as before in the A_n case.) We shall show that also in a gallery $(C_0, C_n, \ldots, C_{2n-1})$ of type $12\ldots n\ldots 21$, the vertex of type 1 in C_0 is distinct from the one in C_{2n-1}.

Write the gallery in the form

$$\begin{aligned}
C_0 &= (x_1, \ldots, x_n),\\
C_1 &= (y_1, x_2, \ldots, x_n),\\
C_2 &= (y_1, y_2, x_3, \ldots, x_n),\\
&\ldots\\
C_{n-1} &= (y_1, y_2, \ldots, y_{n-1}, x_n),\\
C_n &= (y_1, \ldots, y_n),\\
C_{n+1} &= (y_1, \ldots, y_{n-2}, z_{n-1}, y_n),\\
&\ldots\\
C_{2n-2} &= (y_1, z_2, \ldots, z_{n-1}, y_n),\\
C_{2n-1} &= (z_1, z_2, \ldots, z_{n-1}, y_n),
\end{aligned}$$

where $x_t \neq y_t$ for all t and $y_t \neq z_t$ for all $t \leqslant n-1$. Now assume that $z_1 = x_1$. Then x_1 and y_1 are incident with both x_2 and z_2 (look at $C_0, C_1, C_{2n-2}, C_{2n-1}$). By the property (Lin), $x_2 = z_2$. Now we can again proceed by induction on the gallery $(C_1, \ldots, C_{2n-2}; 23\ldots n\ldots 32)$.

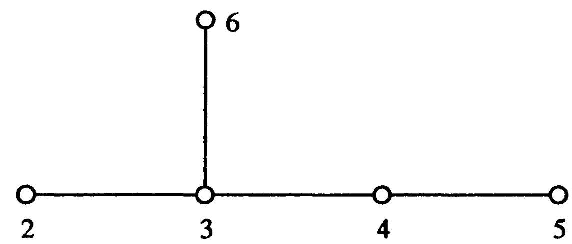
For the diagram D_n, the proof is similar. □

3.4.10. REMARK. For D_n, Th. Meixner has observed that (Lin) automatically holds if one assumes the complex to be a flag complex of an incidence geometry; see Timmesfeld [1983]. Brouwer and Cohen [1983] have reproved this statement and extended it to the diagram E_6: An incidence geometry of type E_6 automatically is a building. For general complexes of type E_6, the result is as follows.

3.4.11. PROPOSITION. *A strongly connected numbered complex of type* E_6 *is a building if and only if it satisfies the linearity Axiom* 3.4.8 *for the types* $i = 1, 2$.

The proof will be analogous to the proof of Proposition 3.4.9 and will rely on Lemma 2.5.8 and on the corresponding result for the diagram D_5 of rank one less. The distinguished node of the D_5 occurring here is however different from the node 1 used in defining the linearity Axiom. Therefore, we still need another lemma.

3.4.12. LEMMA. *Let the diagram* D_5 *be given in the following form:*



A strongly connected numbered complex of type D_5 *is a building if the linearity Axiom holds in the following form:*

> *If two objects* $2, 2'$ *are simultaneously incident to two objects* $3, 3'$, *then* $2 = 2'$ *or* $3 = 3'$.

PROOF. Following the proof of Proposition 3.4.9, we first calculate the shortest representatives for the double cosets W^2wW^2. These turn out to be ( ), 2, 234632. If a closed gallery of reduced type does not involve the type 2, it lies in a star with diagram A_4 which is a building by 3.4.8, and thus the gallery is trivial. If its type (or rather the corresponding group element) is in the double coset of 234632, the gallery gives rise to a gallery of type 234632 in a St 5 with diagram D_4, and having the initial vertex of type 2 equal to the terminal vertex of type 2. By the proof of Proposition 3.4.9, such a gallery does not exist. □

The remainder of the proof of Proposition 3.4.11 is completely analogous, using the shortest double coset representatives ( ), 1, 12346321 from Lemma 2.5.9.

Notes to Section 3

The notion of a building. We first give a couple of historical remarks on the subject of Section 3.2, the definition of buildings as structured complexes whose apartments are Coxeter–Tits complexes. The standard reference for this, which was also the main source of our treatment, is Tits [1974], going back to an Oberwolfach workshop in 1968. We would like to emphasize that precisely the same definition as the one adopted in our text had been presented for the first time already in Tits [1961b]. (The only restriction of *loc. cit.* is that the complex is *a priori* assumed to be the flag complex of an incidence geometry.) The result 3.2.8 that in the thick case the apartments are automatically Coxeter–Tits complexes, had been obtained in Tits [1965]. In Tits [1974], this was used to weaken the axiomatics accordingly, avoiding in particular the numbering. The earlier papers on the subject did not contain an explicit axiomatic definition of a 'building'. They rather gave a definition and geometrical description of the particular buildings of simple Lie groups, resp. simple algebraic groups. This in particular contained more or less explicitly the definition of a general incidence geometry and of a geometry belonging to a Coxeter diagram, as defined in our Section 3.1. The additional 'global' property or axiom used in the early papers was the 'intersection property' for shadows. See the Notes of Section 6 for more about this. Here we only want to remark that it was only in Tits [1981a] that the equivalence of the two approaches was proved; it is restricted to spherical type (which was the only type considered in the early papers). The contents of the fundamental paper Tits [1981a] can be subdivided into three parts: The first part is what we have presented here in Section 3.3 under the name of the 'First Main Characterization of Buildings'. Recall that this is a quite general and in a sense very abstract characterization of buildings among all geometries of Coxeter type by forbidding the existence of certain closed galleries. The second part relates the criterion from the first part to coverings of complexes (and chamber systems) as introduced above in Section 2.2. It will be treated in detail in Section 6. The third part gives applications of the general theorems to 'concrete characterizations' of buildings by particular 'geometrical' axioms, depending on the diagram. Examples of such results have been given above in the second half of Section 3.4. The intersection property mentioned above is a strong version of such axioms, valid for all diagrams.

Geometries of Coxeter type. We come to some remarks on the contents of this section. For our purposes concerning generalized polygons, we used Tits [1959, Appendice; 1974, 1976, 1981a] and solved the exercise to show that all the definitions are equivalent to each other (Proposition 3.1.2). (Here is not the place to give a complete account of the history of generalized polygons.) The results 3.1.10 to 3.1.12 on chamber systems of generalized polygons have been taken from Scharlau [1985a] and can be considered as standard exercises as well. Cf. also Cohen [1991], 7.1. Concerning Example 3.1.8, we should mention the note Tits [1983b] where it is discussed to what extent this example carries over to other diagrams. Neumaier's Example 3.1.9 of the Alt_7-geometry with diagram C_3 is of particular significance since Aschbacher [1984] has shown, under certain additional assumptions of essentially group theoretical nature, that this is the only finite nonbuilding C_3-geometry with a flag-transitive automorphism group (Theorem 6.5.8 below). It should be mentioned that Neumaier had already found this example in the late seventies, although he published it only in Neumaier [1984].

We now make some brief indications about known results on the structure and classification of geometries of spherical type and rank at least 3 which are not buildings. For complete information, we refer the reader to Lunardon and Pasini [1990], Pasini [1988a,b], and to Chapter 22. Existing work mainly deals with the finite case; a general classification seems to be out of reach at present. Essentially, only the diagrams C_n and F_4 are of interest (cf. the Notes to Section 6). One further condition which is often imposed is the existence of parameters (in the sense of 3.4.1). Clearly, this is fulfilled if the geometry admits a chamber-transitive automorphism group. Apparently, one major distinction has to be made between the nonthick and the thick case. Various constructions are known which produce nonthick (nonbuilding) geometries of Coxeter type, starting from certain simpler objects (e.g., geometries of smaller rank). The general picture is analogous to the situation with weak buildings (see Tits [1977], Scharlau [1987]). One could say that one does not obtain objects which are truly new. In the thick case, the picture looks different: the only known finite nonbuilding is the Alt_7-geometry with diagram C_3. Therefore, the existing literature mainly concentrates on deriving restrictions on such geometries, and proving conditional nonexistence theorems.

For instance, it is known that every geometry of type F_4 (finite, thick) with chamber-transitive automorphism group is a building. So far, the only general 'deep' method for attacking the nonexistence question appears to be the spectral theory of graphs and its higher-dimensional analogue, involving Hecke algebras (see below). Despite the efforts of various people, these methods did not turn out sufficient to solve the C_3 case, which of course is the crucial one.

The 'First Main Characterization of Buildings'. The principal result of Section 3.3 is of course Theorem 3.3.1 which is Theorem 2 of the celebrated paper 'A Local Approach to Buildings' Tits [1981a]. It is certainly one of the highlights of the era of buildings after Tits [1974]. The point of this theorem is not so much that it deals with chamber systems instead of complexes; (a direct translation of the classical definition into chamber systems is given in Section 2.5 of Tits [1981a]). The result rather is that one gets the apartments for free if the chamber system, which must be assumed to be of type M in advance, has 'no more relations' than the Weyl group itself. Notice that this result does not touch upon the previous Theorem 3.2.8 saying that one does not need the Coxeter group in advance once one assumes apartments and thickness. A complement to Theorem 3.3.1, or rather to 3.3.12, involving the assumption of thickness but not using apartments, has been given by Abels [1990/91]. He shows that, if a group valued 'distance' δ on a set $\mathcal{C}$ satisfies the conditions of Proposition 3.3.12 and if the underlying chamber system of $(\mathcal{C}, \delta)$ is thick, then the group of values automatically is a Coxeter group and thus $(\mathcal{C}, \delta)$ is a building. See Abels [1991a,b] for details about the function δ in the case of buildings of type A_n, and for a generalization to other posets.

The proofs given in Section 3.3 above are original; they have been taken from the unpublished work of A. Dress already mentioned in the notes to Section 1. In particular, the concept of a flat set of chambers and the exchange condition for galleries are due to him.

Parameters and the weight enumerator. The weight enumerator P_M and its use for counting chambers in essence go back (at least) to Solomon [1966]. A natural generalization of his method leads to finite M-geometries with parameters. To our knowledge, they have been systematically used for the first time by Kilmoyer and Solomon [1973] in the context of Hecke algebras and their action on the chambers. The object of that paper was a new proof of the famous theorem by Feit and Higman on finite generalized polygons; later Ott [1981, 1985] worked out their general set-up and gave some further applications. The above method of calculating P_M using double cosets is taken from Bourbaki [1968], Chapter 4, §1, Exercise 25c) (where only the series $P_M(X, \ldots, X)$ in one variable is considered). A different, full proof of the product formula 3.4.6 is given in the book Carter [1972], Section 9.4. This formula goes back to Bott (reference in *loc. cit.*). Brouwer and Cohen [1985] give a formula for counting arbitrary simplices of specified type J in a finite building. Instead of summing over the entire Weyl group W, one merely has to sum over all coset representatives for the subgroup W^J. The main result of *loc. cit.* is the determination of certain parameters for the incidence graph of a point-line geometry of a finite building.

Geometrical axioms, I. The geometrical Propositions 3.4.7, 3.4.9 and 3.4.11 are essentially taken from Tits [1981a], although we use slightly weaker assumptions. The proofs that we give for these results are original. They are elementary and as straightforward as possible in the sense that they directly go back to the First Main Characterization and use neither the axiomatics of projective, respectively polar spaces nor advanced general techniques like covering theory, i.e. the 'Second Main Characterization' to be treated later. Proofs avoiding coverings have also been given by Lunardon and Pasini [1990] (in case F_4, not in case E_6).

4. Some important classes of buildings

Introduction

The purpose of this section is to present some particular classes of buildings. This will illustrate the concepts introduced so far, and will help preparing the ground for the refined structure theory to be dealt with in the following sections.

In Section 4.1, we consider projective spaces (always assumed to be finite-dimensional). These are quite classical objects which are often used as examples in the literature on buildings. First of all, we state the principal result that projective spaces correspond bijectively to buildings of type A_n. Besides this, our main goal is to give an explicit description of the apartments.

The 'polar spaces' treated in Section 4.2 axiomatize the incidence properties of totally isotropic subspaces of a vector space with respect to some quadratic or Hermitian form. In most situations, one could also deal in more geometrical terms with totally isotropic subspaces of a projective space with respect to a so-called polarity. Although such structures are quite classical as well, it seems that it was only in the context of the theory of buildings that a systematic axiomatic treatment had been given. We collect here the basic definitions concerning polar spaces, mention some fundamental consequences of the axioms, and formulate the relation to buildings. Like in the case of projective spaces, we put some emphasis on presenting a precise and detailed model for the apartments.

Finally in Section 4.3, we treat the relationship between Tits systems (often called BN-pairs) in a group and buildings. These structures correspond to buildings together with a 'strongly transitive' group of automorphisms. Thus, unlike in the first two subsections, we are not dealing with a particular diagram or a particular class of diagrams. The subject rather is a kind of 'dictionary' translating the axioms of buildings into the properties of a system of subgroups of the strongly transitive group in question. Using this correspondence, the fundamental result that the Weyl group of a Tits system is a Coxeter group becomes a special case of the theorem that the apartments of a thick building automatically are Coxeter–Tits complexes.

Many interesting buildings admit a strongly transitive automorphism group, that is, they come from a Tits system. This is true particular in the case of spherical diagrams, where the groups often are groups of rational points of semisimple algebraic groups. It would be far beyond the scope of this article to develop in detail the relevant part of the structure theory of algebraic groups. For this, we refer the reader to the text book Borel [1991].

There is however one demand which is not fulfilled by this or other general texts. If one wants to give the classification of spherical buildings over some specified 'ground field' in form of an explicit list, it is convenient to have the corresponding information concerning algebraic groups at one's disposal. The classification of algebraic groups over non-algebraically closed fields is not available in text book form. The standard reference for the classification is Tits [1965/66]. In that paper Tits introduces 'relative Dynkin diagrams' for semisimple groups over arbitrary fields. For well-behaving ground fields like finite fields, p-adic fields or number fields, these diagrams lead to an explicit list of all groups. In order to prepare for Section 7.3 below, we give in the concluding Section 4.4 a brief introduction to these 'Tits diagrams' for semisimple groups.

4.1. *Buildings of type* A_n

In this subsection, we shall describe one of the standard facts of the theory in some detail: the correspondence between (thick) buildings of type A_n and projective spaces. We refer to Chapter 2 for basic results about projective spaces: in particular for the notion of a Desarguesian projective space, and for the theorem that any thick (see below) Desarguesian space P is isomorphic to the space of subspaces of a vector space over a (not necessarily commutative) field. Here, the points of P are the 'lines' (one-dimensional subspaces) of P.

We consider the diagram A_n with the following numbering:

Essentially this means that one of the two orientations of the graph A_n, equivalently one of the end nodes, is distinguished.

4.1.1. Definition. The Coxeter–Tits complex $\Sigma(\mathsf{A}_n)$ is defined as follows:

$$S := \{1, \ldots, n+1\}, \quad \text{'points'},$$

$$X := \{x \subset S \colon \varnothing \neq x \neq S\}, \quad \text{'subspaces'},$$

$$\Sigma(\mathsf{A}_n) := \mathrm{Flag}(X, \subseteq), \quad \text{'flag complex'},$$

$$\text{type: } X \to \{1, \ldots, n\} =: I, \quad x \mapsto |x|,$$

$$X_i := \{x \in X \colon |x| = i\}.$$

More generally, let $(X, \leqslant)$ be any pure (see 1.1.2) partially ordered set of rank n. We recall from 1.1.2 that the flag complex is $\{1, \ldots, n\}$-numbered in a natural way, and from 1.1.3 (cf. 1.1.6) that the basic graph is linear. It is readily checked in our particular case that the above complex actually belongs to the Coxeter diagram A_n, in the sense of Definition 3.1.3. Of course, more is true.

4.1.2. Proposition. *The complex* $\Sigma(\mathsf{A}_n)$ *as defined above is a Coxeter–Tits complex.*

PROOF. Clearly, the group $W = \mathrm{Sym}_{n+1}$ (symmetric group) acts chamber-transitively on $\Sigma = \Sigma(\mathsf{A}_n)$. Consider the typical chamber

$$C = \{\{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n\}\}$$

and its panels

$$F_i = C \setminus \{1, \ldots, i\}.$$

There is precisely one further chamber containing F_i, namely

$$F_i \cup \{\{1, \ldots, i-1, i+1\}\}.$$

Therefore Σ is thin. The reflection in F_i clearly is

$$\sigma_i := (i, i+1) \quad \text{(transposition in } \mathbb{S}_{n+1}).$$

The only nontrivial property is that $(W, \sigma_i, i \in I)$ is a Coxeter system, equivalently that each σ_i is associated with a pair of opposite foldings. Concerning the first property, we refer the reader to Bourbaki [1968]. Alternatively, we leave it as an exercise to prove that the two sets of chambers

$$C_i^+ = \{wC\colon\ w \in W,\ w^{-1}(i) < w^{-1}(i+1)\},$$

$$C_i^- = \{wC\colon\ w \in W,\ w^{-1}(i) > w^{-1}(i+1)\}$$

determine opposite roots (with F_i in their common boundary, of course). □

4.1.3. *Projective spaces.* Remember that a projective space is a set of 'points' $S = X_1$, together with a set X_2 of subsets of S, called 'lines', such that the following hold:

(P1) For any two distinct points $p, q \in S$, there exists a unique line $l \in X_2$ such that $p \in l$, $q \in l$. Notation: $l =: \overline{pq}$.
(P2) (Veblen–Young axiom). For any pairwise distinct points p, q, r, s such that $\overline{pq} \cap \overline{rs} \neq \varnothing$, it follows that also $\overline{pr} \cap \overline{qs} \neq \varnothing$.
(P3) All lines have cardinality at least two (see Chapters 2 and 4).

A projective space is called *thick* if all lines have cardinality at least three. A trivial example of a nonthick projective space is obtained by taking for X_2 all 2-element subsets of S (here S is arbitrary, and (P2) is void). Such a space is called *thin.* A subset $U \subseteq S$ of a projective space is called a *subspace* if $\overline{pq} \subseteq U$ for any two distinct points $p, q \in U$. Clearly, an arbitrary intersection of subspaces is a subspace, and thus we may define the subspace $[M]$ generated by an arbitrary subset $M \subseteq P$ as the intersection of all subspaces containing M. The *sum* $U_1 + U_2$ of two subspaces is defined as the subspace generated by $U_1 \cup U_2$, and more generally

$$\mathop{+}_{j \in J} U_i := \Big[\bigcup_{j \in J} U_j\Big]$$

for an arbitrary family $(U_j)_{j\in J}$ of subspaces. This is the least upper bound of the U_j in the partially ordered set of all subspaces, which in fact is a lattice. A subset $M \subseteq S$ is called *independent* if $p \notin [M \setminus \{p\}]$ for all $p \in M$. A *basis* is an independent generating set. Like for vector spaces, one shows that bases exist and all have the same cardinality. If P is finitely generated, we set $\operatorname{rank} P$, the *rank* of P equal to $|B|$, where B is any basis. Otherwise, $\operatorname{rank} P := \infty$. From now on, we assume that $n = \dim P := \operatorname{rank} P - 1$ is finite. Set

$$X = X(P) := \{U \subset P\colon\ U \text{ a subspace}, \varnothing \neq U \neq P\}.$$

Then X, ordered by inclusion, is pure of rank equal to n. If $U \subseteq P$ is a subspace, it is a projective space in its own right, and $\operatorname{rank} U$ is equal to the rank or height of U as a member of the partially ordered set $\{\varnothing\} \cup X(P)$. In particular,

$$\operatorname{Flag} P := \operatorname{Flag}(X(P), \subseteq)$$

is a numbered complex with natural type set $\{1, \dots, n\}$ and $\operatorname{type} x = \operatorname{rank} x$.

It is a nontrivial fact, relying on the Veblen–Young axiom, that the lattice $(X, \subseteq)$ is *modular*:

$$U_1 \subseteq U_3 \Rightarrow U_3 \cap (U_1 + U_2) = U_1 + (U_3 \cap U_2)$$

for any three subspaces U_1, U_2, U_3. A general consequence of this is the modular equation for the rank-function:

$$\operatorname{rank}(U_1 + U_2) + \operatorname{rank}(U_1 \cap U_2) = \operatorname{rank} U_1 + \operatorname{rank} U_2.$$

The lattice $(X, \subseteq)$ is clearly *atomic*: every element is the join (smallest upper bound) of a set of atoms, i.e. minimal nonzero elements.

4.1.4. THEOREM. *Let Δ be a strongly connected I-numbered complex, where $I = \{1, \dots, n\}$. The following properties are equivalent.*

(i) *Δ is of type A_n.*
(ii) *Δ is a building of type A_n.*
(iii) *Δ is the flag complex of an n-dimensional projective space.*
(iv) *Δ is the flag complex of a pure, atomic, modular lattice of dimension n.*

As one might guess, this is one of the basic theorems of the theory of buildings, having obvious consequences also for the other spherical diagrams. It is surprising that no full proof is contained in the literature. The proof is most conveniently organized according to the scheme

$$\text{(i)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}.$$

Chapter 21 deals thoroughly with the use of lattices in the theory of projective spaces. In 3.4.7, we have already seen a direct, short proof of the implication (i) $\Rightarrow$ (ii), which used the more sophisticated parts of the theory. The equivalence of (iii) and (iv) is a standard fact about projective spaces, completely independent of the theory of buildings.

We shall now say a few words about the proof of the implication (i) $\Rightarrow$ (iv) (or (i) $\Rightarrow$ (iii)). This is important since one actually shows more than we have stated. Before formulating this precisely, we recall that our diagram(s) A_n have a fixed orientation. Thus we may speak of type preserving isomorphisms not only between complexes of the form $\operatorname{Flag} P$, but between arbitrary complexes of type A_n. In the proof of the theorem, one associates to an arbitrary complex Δ of type A_n in a canonical way a partially ordered set $(X_\Delta, \leqslant)$, and a projective space P_Δ. Recall from 1.1.6 the construction of the poset: X_Δ is the set of vertices of Δ, in accordance with general notation of Section 1. The partial ordering is defined by setting

$$x \leqslant y \Leftrightarrow x \cup y \in \Delta \text{ and } \operatorname{type} x \leqslant \operatorname{type} y.$$

The definition of P_Δ is also obvious: set $P := X_1$ (vertices of type 1 = left end node of the diagram). A subset $l \subset P$ is a line if it is of the form $S_x := \{p \in P\colon\ p \cup x \in \Delta\}$ (the 'shadow of x'), for a vertex $x \in \Delta$ of type 2. In the course of the proof of the theorem, one shows that this actually is a projective space, and that its partially ordered set of subspaces can be identified with X_Δ with the above ordering. This identification is again obtained by taking 'shadows' of vertices on the set of points. Cf. the Notes in Section 1.

From the preceding discussion, we explicitly record the important fact that a projective space can be canonically reconstructed from its flag complex.

4.1.5. PROPOSITION. *Let $\Delta = \operatorname{Flag} P$ and $\Delta' = \operatorname{Flag} P'$ be flag complexes of projective spaces P and P'. Then any isomorphism from Δ onto Δ' is induced by a unique isomorphism from P onto P'.*

4.2. *Buildings of type C_n and D_n*

We know from Example 3.1.6 that complexes of type C_n can be obtained as flag complexes of totally isotropic subspaces with respect to a Hermitian or quadratic form of Witt index n over a field. These complexes have an 'algebraic origin' like the complexes of type A_n, $n \geqslant 3$. Of course, the situation in case C_n is more involved from the point of view of classification. Even if a (skew) field k and a (possibly trivial) involution σ on k is specified, the classification of σ-Hermitian forms (quadratic forms) is for 'general k' complicated and even unsolved. There is yet another difference to the A_n case: for $n = 3$, the buildings of Hermitian and quadratic forms do not exhaust all buildings of type C_n. Therefore we shall in a moment introduce Tits' concept of a *polar space*. It is a sort of coordinate-free axiomatic version of the geometry of isotropic subspaces, like projective spaces are a coordinate-free version of the (projective spaces of) vector spaces. Furthermore, there is a one-to-one correspondence between polar spaces and C_n-buildings also for $n = 2$ and $n = 3$. Before coming to the details, we first have a look at the Coxeter complex of type C_n. We shall at the same time treat the closely related case of the diagram D_n.

4.2.1. DEFINITION. The Coxeter–Tits complex $\Sigma(\mathsf{C}_n)$:

$$S := \{\pm 1, \pm 2, \ldots, \pm n\}, \quad \text{'points'},$$
$$X := \{x \subset S\colon \{i, -i\} \not\subseteq x \text{ for all } i\},$$
$$\Sigma(\mathsf{C}_n) := \mathrm{Flag}(X, \subseteq), \quad \text{flag complex},$$
$$\text{type: } X \to \{1, \ldots, n\} = I, \quad x \mapsto |x|,$$
$$X_i := \{x \in X\colon |x| = i\}.$$

It is clear that this type-function makes $\Sigma(\mathsf{C}_n)$ into a numbered complex. A typical chamber, its vertices and panels are the following:

$$C = \{x_1 \subset x_2 \subset \ldots \subset x_n\},$$
$$x_i = \{1, \ldots, i\},$$
$$F_i = C \setminus \{x_i\}.$$

The group

$$W := \{\sigma \in \mathrm{Sym}(S)\colon \sigma(-i) = -\sigma(i) \text{ for all } i\} \cong Z_2^n \rtimes \mathrm{Sym}_n$$

acts type preservingly and chamber transitively on Σ. We have $\mathrm{St}_\Sigma x_n = \Sigma(\mathsf{A}_{n-1})$ as defined above. In order to check the rest of the diagram C_n, we have to look at $\mathrm{St}_\Sigma\{1, \ldots, n-2\}$ which can be identified with $\Sigma(\mathsf{C}_2)$. This complex clearly is identical to the flag complex of a square (cf. 4.2.3 below). The proof has also shown that $\Sigma(\mathsf{C}_n)$ is thin. The reflection σ_i at F_i is

$$\sigma_i = (i, i+1)(-i, -(i+1)), \quad 1 \leqslant i \leqslant n-1,$$
$$\sigma_n = (n, -n).$$

Notice the 4-cycle

$$\sigma_{n-1} \circ \sigma_n = (n-1, n, -(n-1), -n).$$

4.2.2. PROPOSITION. *The complex $\Sigma(\mathsf{C}_n)$ as defined above is a Coxeter–Tits complex.*

As earlier, the only nontrivial part of the proof is the verification of the equivalent properties of 2.4.2.

4.2.3. *A remark on geometrical realization.* The geometrical realization of a finite Coxeter–Tits complex is homeomorphic to the $(n-1)$-sphere. In the case of the irreducible, linear diagrams $\mathsf{A}_n, \mathsf{C}_n, \mathsf{F}_4, \mathsf{H}_3, \mathsf{H}_4, \mathsf{I}_2(m)$ it can more precisely be identified with the barycentric subdivision of the corresponding platonic solid. We now want to make this correspondence explicit in the case of the diagram C_n. In fact, the reader will see in a moment that our concrete model $\Sigma(\mathsf{C}_n)$ defined above can be derived immediately from the standard cube in affine space $\mathbb{R}^n$.

Let

$$K = \{p = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\colon\ |\xi_i| \leqslant 1 \text{ for all } i\}$$

be the standard *hypercube* in Euclidean n-space $\mathbb{R}^n$, i.e. the convex hull of all points of the form $(\pm 1, \ldots, \pm 1)$. To any vertex $x \in X$ of $\Sigma(\mathsf{C}_n)$ we associate the following vector

$$e_x = (\xi_1, \ldots, \xi_n)\colon\ \xi_i = 0 \text{ if } i \notin x,\ \xi_i = \pm 1 \text{ if } \pm i \in x.$$

Each e_x is a barycenter of a uniquely determined face A_x of K. The dimension of A_x is equal to $n - |x|$. The map $x \mapsto A_x$ is an inclusion-reversing bijection from X onto the set of all proper faces of K. Remember that the flag complex of the faces of K can be identified with the barycentric subdivision K' of (the surface of) K. The simplex corresponding to a maximal flag $A_{x_n} \subset A_{x_{n-1}} \subset \cdots \subset A_{x_1}$ of faces is the convex hull of the point set $\{e_{x_1}, e_{x_2}, \ldots, e_{x_n}\}$. In this way we get an isomorphism of (combinatorial) simplicial complexes $\Sigma(\mathsf{C}_n) \xrightarrow{\cong} K'$. Of course, this extends to a piecewise linear homeomorphism of the standard topological realization of $\Sigma(\mathsf{C}_n)$ onto K.

Clearly, we could also use the *cross-polytope* instead of K which by definition is the convex hull of the $2n$ unit vectors $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$. Here, the bijection from X onto the set of faces is inclusion preserving, but the 'face vectors' require some renormalizations.

We now come to the Coxeter–Tits complex of type D_n. It is closely related to $\Sigma(\mathsf{C}_n)$: in the geometrical realization, its chambers are unions of pairs of adjacent chambers of $\Sigma(\mathsf{C}_n)$, so the Weyl group $W(\mathsf{D}_n)$ can be considered as a subgroup of index 2 in $W(\mathsf{C}_n)$.

4.2.4. Definition. The Coxeter–Tits complex $\Sigma(\mathsf{D}_n)$. Let $S, X, x_1, \ldots, x_n$ be as in 4.2.1. Define a new vertex set as follows:

$$Y := \{y \in X\colon\ |y| \neq n-1\}.$$

Define an incidence relation on Y by

$$y|y' \Leftrightarrow y \subseteq y' \text{ or } y' \subseteq y \text{ or } |y| = |y'| = n,\ |y \cap y'| = n-1.$$

Now we set

$$\Sigma(\mathsf{D}_n) := \operatorname{Flag}(Y, |), \quad \text{flag complex},$$

$$\varepsilon\colon Y \to \{\pm 1\}, \quad \varepsilon(y) = \prod_{j\in y} \operatorname{sgn} j,$$

$$\text{type}\colon Y \to \{1, \dots, n\},$$

$$\text{type}\, y := \begin{cases} |y| & \text{if } |y| \leqslant n-2, \\ n-1 & \text{if } |y| = n,\ \varepsilon(y) = 1, \\ n & \text{if } |y| = n,\ \varepsilon(y) = -1, \end{cases}$$

$$C = \{y_1, \dots, y_n\} \quad \text{where } y_i = x_i \text{ if } i \leqslant n-2,$$

$$y_{n-1} = \{1, \dots, n-1, -n\},\ y_n = x_n,$$

reflections in the panels $F_i = C \setminus \{y_i\}$:

$$\sigma_i,\ i = 1, \dots, n-1, \text{ as in 4.2.1}, \quad \sigma_n = (n-1, -n)(-(n-1), n).$$

We now come to the definition of a polar space as a set of 'points' together with a collection of subsets called 'subspaces', subject to appropriate axioms. As usual, the minimal subspaces with more than one point are called lines. All higher dimensional subspaces are determined by the lines in the following standard way: a subset U is a subspace if and only if it contains every line having more than one point in common with itself. We mention that according to a fundamental result of Buekenhout and Shult [1974], there are rather simple axioms on a collection of 'points' and 'lines' in order that the corresponding 'space with subspaces' actually be a polar space; see Chapter 12 for details. Despite this result, we use Tits' definition of polar spaces: we want to avoid the thickness axiom, and we want to be close to buildings already by definition. Indeed, the star of an object of type n in a C_n-building is of type A_{n-1}, thus a projective $(n-1)$-space. Therefore, the first of the following axioms has to be expected.

4.2.5. DEFINITION. A *polar space* of rank n, for some integer $n \geqslant 1$, is a set S of 'points' together with a set of subsets, called (linear) *subspaces*, such that the following axioms hold.

(P1) A subspace L, together with the subspaces it contains, is a projective space of dimension less than n.

(P2) The intersection of any two subspaces is a subspace.

(P3) Given a subspace L of dimension $n-1$ and a point $p \in S \setminus L$, there is a unique subspace M such that
(a) $p \in M$, $\dim(M \cap L) = n-2$.
It has the further property
(b) If M' is an arbitrary subspace with $p \in M'$, then $M' \cap L \subseteq M$.

(P4) There exist two disjoint subspaces of dimension $n-1$.

EXAMPLE. If $S = \{\pm 1, \ldots, \pm n\}$ and $X \subseteq P(S)$ are as in 4.2.1, then (S, X) is a polar space of rank n. Such a polar space is called *thin*.

Before formulating the correspondence between polar spaces and C_n-buildings, we collect a few easy properties and further definitions that we have to use in the proof.

4.2.6. *Further definitions and properties.* A subset $U \subseteq S$ is called *prelinear* if its points are pairwise collinear. It follows from (P3) that any prelinear set is contained in a subspace. By (P2), there is a smallest such subspace which we denote by $[U]$. At the same time, one proves that the maximal subspaces are precisely the $(n-1)$-dimensional ones.

If L and L' are subspaces, then the set

$$\pi_L(L') = \{p \in L\colon\ L' \cup \{p\} \text{ is prelinear}\}$$

is a subspace, the union $\pi_L(L') \cup L'$ is prelinear and

$$\dim[\pi_L(L') \cup L'] \geqslant \dim L.$$

If L' is maximal, then $\pi_L(L') = L \cap L'$.

4.2.6. PROPOSITION. *If L and L' are disjoint maximal subspaces, more generally if $\dim L = \dim L'$ and $\pi_L(L') = \varnothing$, then the restriction of π_L to the set of subspaces of L' is an isomorphism $\pi_{L,L'}\colon\ L' \to L^*$ (L^* the dual of L).*

The proof is obtained by considering dimensions. Notice that $\pi_{L'} \circ \pi_L$ acts as the identity on the set of subspaces of L'.

4.2.7. DEFINITION. A (polar) *frame* B in a polar space S of rank n is a set B of $2n$ points such that any of them is collinear with all others but one.

Notice that S induces on B a structure of a thin polar space. In particular, we can speak about the flag complex $\operatorname{Flag} B$. The mapping $U \mapsto [U]$ from subspaces of B to subspaces of S is injective and extends to an injection

$$\operatorname{Flag} B \hookrightarrow \operatorname{Flag} S.$$

We shall usually identify $\operatorname{Flag} B$ with its image in $\operatorname{Flag} S$.

4.2.8. THEOREM. *Let S be a polar space of rank n. Then the flag complex* $\operatorname{Flag} S$ *of subspaces of S is a building of type C_n. The apartments are the subcomplexes* $\operatorname{Flag} S(B)$ *where $B \subset S$ is a polar frame.*

Conversely, for a building Δ of type C_n, with vertex set X, set $S = X_1$ (vertices of type 1) and call subspaces of S the sets

$$S_x := \{p \in S\colon\ p \cup x \in \Delta\},$$

where x runs over X. This defines a polar space, and the mapping $x \mapsto S_x$ extends to a type-preserving (i.e. dimension-preserving) isomorphism $\Delta \xrightarrow{\simeq} \operatorname{Flag} S$.

If one writes down all details, the proof of this theorem is long. The main points are given in Tits [1974], 7.6.

4.2.9. REMARK. Let S be a polar space of rank n such that every line has at least 3 points.

(a) If every subspace of rank $n-1$ is contained in at least 3 maximal subspaces, then S is thick.

(b) If S is not thick, then every subspace of rank 2 is contained in exactly 2 maximal subspaces.

PROOF. By assumption and by Proposition 4.1.5 we know that all subspaces, i.e. all residues of objects of type n are thick. Part (a) immediately follows. Part (b) is a general fact about 'weak buildings'; see Scharlau [1987]. One can also give a direct *ad hoc* proof, reducing to the case $n = 2$. Such a proof can be found in Tits [1974], Proposition 7.14. □

4.2.10. DEFINITION. The *oriflamme complex* Orifl(S) of a polar space of rank n is the flag complex of the following incidence relation: the objects are the subspaces of rank $\neq n-1$ of S; two objects L, L' are incident if $L \subseteq L'$, or $L' \subseteq L$, or $\operatorname{rank}(L \cap L') = n-1$.

It is clear that the oriflamme complex of a thin polar space, corresponding to the Coxeter–Tits complex $\Sigma(\mathsf{C}_n)$, is equal to the Coxeter–Tits complex $\Sigma(\mathsf{D}_n)$ as defined above.

The following theorem describes all buildings of type D_n.

4.2.11. THEOREM.

(a) *Let S be a polar space of rank $n \geqslant 2$ such that every subspace of rank $n-1$ is contained in exactly two maximal subspaces. Then* Orifl(S) *is a building of type D_n. The numbering of this complex is as follows. The vertices of type i, where $i \leqslant n-2$, are the subspaces of rank i. Two maximal subspaces M, M' have the same type* ($n-1$ *or* n) *if and only if* $\operatorname{codim}(M \cap M')$ *is even.*

(b) *Conversely, let Δ be a building of type D_n. Let S be the set of vertices of type* 1, *and call subspaces all sets*

$$S_A := \{x \in S\colon\ x \cup A \text{ exists}\},$$

where A runs over all simplices of Δ. Then S is a polar space of rank n such that every subspace of rank $n-1$ is contained in precisely two maximal subspaces. Every subspace is of the form S_A, with A a vertex or $\operatorname{type} A = \{n-1, n\}$. *The rank of S_A is equal to i if* $\operatorname{type} A = i \leqslant n-2$, *equal to $n-1$ if* $\operatorname{type} A = \{n-1, n\}$, *and equal to n if* $\operatorname{type} A = n-1$ *or n.*

4.2.12. COROLLARY (the 'Klein quadric'). *Let P be a 3-dimensional projective space and S the set of its lines. The following collection of 'subspaces' makes S into a polar space of rank* 3: *the set of all lines containing some point p of P, or contained in a plane z of P, or the intersection of two such sets (with $p \in z$). Every polar space of rank* 3 *with each line contained in exactly two planes can be obtained in this way.*

We now come to an algebraic description of polar spaces and thus of buildings of type C_n or D_n.

4.2.13. *Hermitian forms.* We collect some definitions and general facts concerning Hermitian forms. Let k be a skew-field (the general concepts also apply to the case where k is any ring with unit), and $\sigma\colon\ k \to k$ an *anti-automorphism.* This means that $(a+b)^\sigma = a^\sigma + b^\sigma$ and $(ab)^\sigma = b^\sigma a^\sigma$ for all $a, b \in k$. A *σ-sesquilinear form* is a mapping $h\colon\ V \times V \to k$ for some right k-module V which is biadditive and satisfies

$$h(xa, yb) = a^\sigma k(x, y)b \quad \text{for all } x, y \in V,\ a, b \in k.$$

Equivalently,

$$\hat{h}\colon\ V \to V^* = \mathrm{Hom}_k(V, k), \quad y \mapsto h(-, y),$$

is a homomorphism of right-k modules from V into its dual $V^* = \mathrm{Hom}_k(V, k)$. Here, V^* is considered as a right k-module by $f.b := b^\sigma.f$, where $f \in V^*$, $b \in k$, and the right-hand side refers to the canonical left-module structure of V^*.

If h is a sesquilinear form and $\delta \in k^*$ (units of k), then $\delta.h$ is sesquilinear with respect to the anti-automorphism $t \mapsto \delta t^\sigma \delta^{-1}$. The two forms h and $\delta.h$ are then called proportional. A σ-sesquilinear form is called *reflexive* if there exists an $\varepsilon \in k^*$ such that

$$h(y, x) = h(x, y)^\sigma \varepsilon. \tag{H}$$

This condition is sufficient, and in all interesting cases also necessary for the relation

$$\text{‘}h(x, y) = 0 \quad (x, y \in V)\text{’}$$

to be symmetric; cf. Bourbaki [1959], §3, Exercise 1. The term ‘reflexive’ is in general use, although a term like ‘quasi-symmetric’ would perhaps be more adequate. A form satisfying (H) is usually called a *(σ, ε)-Hermitian form.* If σ and ε are such that a nonzero (σ, ε)-Hermitian form exists, then

$$\varepsilon^\sigma = \varepsilon^{-1},$$

$$t^{\sigma^2} = \varepsilon t \varepsilon^{-1} \quad \text{for all } t \in k.$$

If one replaces a reflexive form h by a proportional one $\delta.f$, then ε changes to $\delta\delta^{-\sigma}\varepsilon$. Using these formulae, one shows the following:

4.2.14. PROPOSITION. *Any reflexive sesquilinear form is proportional to a (σ, ε)-Hermitian form, where $\sigma^2 = \mathrm{id}$, and $\varepsilon = \pm 1$. If $\sigma^2 = \mathrm{id} \neq \sigma$, then ε can even be prescribed to be* 1 (*or* -1).

In view of this proposition, the following three cases cover all possibilities (up to proportionality):

$$\text{Hermitian forms: } \sigma \neq \text{id}, \ \varepsilon = 1,$$

$$\text{symmetric bilinear forms: } \sigma = \text{id}, \ \varepsilon = 1,$$

$$\text{anti-symmetric bilinear forms: } \sigma = \text{id}, \ \varepsilon = -1.$$

If one wants to work with a specific involution, it often is convenient to admit also the case of

$$\text{anti-Hermitian forms: } \sigma \neq \text{id}, \ \varepsilon = -1.$$

For instance, if k is noncommutative, the most important case is that of a quaternion algebra. Then there are precisely two involutions, up to inner automorphisms, distinguished by the dimension (1 or 3) over the centre of the fixed point set. Changing the involution means passing from Hermitian to anti-Hermitian forms, or conversely.

A Hermitian form h is called *trace valued* if $h(x,x)$ lies in the set $\{t + t^\sigma \varepsilon \colon \ t \in k\}$. The relevance of this notion will become clear in the context of quadratic forms. See also the Notes at the end of this section. For symmetric bilinear forms in characteristic 2 'trace-valued' means 'alternating': $h(x,x) = 0$ for all $x \in V$.

4.2.15. *Quadratic forms.* The 'quadratic forms' which we shall introduce in a moment are essentially the 'pseudo-quadratic forms' of Tits [1974], Chapter 8.2. Let k be a skew field, σ an anti-automorphism of k with $\sigma^2 = \text{id}$, and let $\varepsilon = \pm 1$. We set

$$k_{\sigma,\varepsilon} := \{t - t^\sigma \varepsilon \colon \ t \in k\},$$

$$k^{(\sigma,\varepsilon)} := k / k_{\sigma,\varepsilon},$$

$$k^{\sigma,\varepsilon} := \{a \in k \colon \ a + a^\sigma \varepsilon = 0\}.$$

If $a \in k^*$, then the map $b \mapsto a^\sigma b a$ from k into itself maps $k_{\sigma,\varepsilon}$ into itself. Thus, we get a well-defined right action of k^* on $k^{(\sigma,\varepsilon)}$ by defining

$$\bar{b} \cdot a := \overline{a^\sigma b a} \qquad (\bar{b} = b + k_{\sigma,\varepsilon}).$$

When dealing with quadratic forms, we make the following

GENERAL ASSUMPTION. $(\sigma,\varepsilon) \neq (\text{id}, -1)$ if $\text{char}(k) \neq 2$.

Then $k_{\sigma,\varepsilon} \neq k$ and thus $k^{(\sigma,\varepsilon)} \neq 0$.

DEFINITION. A (σ,ε)-quadratic form on a k-vector space V is a function $q\colon V \to k^{(\sigma,\varepsilon)}$ such that

(i) $q(xa) = q(x)\cdot a$ for all $x \in V$, $a \in k$, and

(ii) there exists a (σ,ε)-Hermitian form h on V such that

$$q(x+y) = q(x) + q(y) + (h(x,y) + k_{\sigma,\varepsilon}) \quad \text{for all } x, y \in V.$$

One can show that this holds if and only if there exists a σ-sesquilinear form $g\colon V \times V \to k$ such that

$$q(x) = g(x,x) + k_{\sigma,\varepsilon} \quad \text{for all } x \in V.$$

The associated (σ,ε)-Hermitian form h is uniquely determined by q and is denoted by h_q.

The following proposition shows that under quite general assumptions, a quadratic form is determined by its associated Hermitian form. The most prominent exceptions to this rule are of course the ordinary ($\sigma = \mathrm{id}$) quadratic forms over commutative fields of characteristic 2.

4.2.16. PROPOSITION. *Let k, σ, ε be as before. Assume that $k_{\sigma,-\varepsilon} = k^{\sigma,-\varepsilon}$. Then every (σ,ε)-Hermitian form is trace-valued, and any (σ,ε)-quadratic form is uniquely determined by its associated (σ,ε)-Hermitian form.*

The second statement is understood in a strict sense: every isometry $h_q \to h_{q'}$ is induced by an isometry $q \to q'$.

Notice that the assumption $k_{\sigma,-\varepsilon} = k^{\sigma,-\varepsilon}$ is in particular fulfilled if $\mathrm{char}(k) \neq 2$ or the restriction of σ to the centre of k is not the identity.

4.2.17. Let k, σ, ε be as before, V a right k-vector space, possibly of infinite dimension, and f a (σ,ε)-Hermitian or (σ,ε)-quadratic form on V. A subspace $L \subset V$ is called *totally isotropic* with respect to f if f vanishes identically on L. If this holds for a (σ,ε)-quadratic form, then also the associated (σ,ε)-Hermitian form vanishes on L. A nonzero vector $x \in V$ is called *isotropic* if $f(v,v) = 0$, resp., $f(v) = 0$. The space (V,f) is called *anisotropic* if it contains no isotropic vector. To unify the notation for the moment, set $h = f$ if f is Hermitian and $h = h_f$ if f is quadratic. We say that (V,f) is *regular* if the induced map $\widehat{h}$ (see 4.2.13) is an isomorphism from V onto its dual. The space $U^\perp$ orthogonal to a subspace $U \subseteq V$ is defined as

$$U^\perp := \{x \in V\colon\ h(x,y) = 0 \text{ for all } y \in U\}.$$

A Hermitian space $(V,f) = (V,h)$ is called *nondegenerate* if $V^\perp = \{0\}$. A quadratic space (V,f) is called *nondegenerate* if $V^\perp \cap f^{-1}(0) = \{0\}$. (The latter terminology is not standard, but it is the terminology of Tits [1974].)

WITT'S THEOREM. *Let (V, f) be a finite-dimensional trace-valued (σ, ε)-Hermitian or (σ, ε)-quadratic space.*

(a) *Any two maximal totally isotropic subspaces L and L' of V have the same dimension, called the Witt index of f. Any isomorphism of vector spaces $L \to L'$ preserving $V^\perp$ extends to an automorphism of (V, f).*

(b) *There exists an orthogonal decomposition $V = V_0 \perp V_1 \perp V_{an}$ such that $f|_{V_0} = 0$, $f|_{V_{an}}$ is anisotropic, $f|_{V_1}$ is regular, and V_1 is the direct sum of two totally isotropic subspaces. The anisotropic part $(V_{an}, f|_{V_{an}})$ is determined up to isometry by (V, f).*

We finally come back to polar spaces.

4.2.18. PROPOSITION. *Let (V, f) be a nondegenerate (σ, ε)-Hermitian or (σ, ε)-quadratic space of finite Witt index n. Let S_f be the set of one-dimensional totally isotropic subspaces of V. Call a subset of S_f a subspace if it consists of all members of S_f contained in some totally isotropic subspace. This makes S_f into a polar space. Its flag complex can be canonically identified with the flag complex $\Delta(f)$ of all totally isotropic subspaces of V. The building $\Delta(f)$ is thick except for the case $\sigma = \mathrm{id}$, $\varepsilon = 1$, $\mathrm{char}(k) \neq 2$, $\dim V = 2n$.*

The proof is not difficult. If L, M are arbitrary subspaces, the subspace $\pi_L(M)$ defined in 4.2.6 is equal to $L \cap M^\perp$.

4.2.19. REMARK. Although we shall not use it in the rest of this article, we briefly recall the notion of a *polarity* in a projective space. A polarity is a symmetric relation '$\perp$' between pairs of points of P such that the set $p^\perp = \{q \in P\colon\ p \perp q\}$ is of codimension 1 or equal to P, for all points $p \in P$. If the latter case does not occur, the polarity is called *nondegenerate*. Then the mapping $p \mapsto p^\perp$ is in fact an isomorphism of P onto its dual space P^*. It is an immediate corollary of this fact, combined with the 'fundamental theorem of projective geometry', that any nondegenerate polarity on the projective space of a vector space comes from a nondegenerate sesquilinear form. In view of the symmetry assumption on the $\perp$-relation, this form has to be reflexive (except for certain uninteresting cases, cf. 4.2.13). It may be replaced by a proportional form, in particular a $(\sigma, \pm 1)$-Hermitian form, $\sigma^2 = \mathrm{id}$. The proportionality class is uniquely determined. We wish to emphasize that in characteristic 2, the polarities are not sufficient to describe all polar spaces. This is so even if one restricts to spaces which can be 'embedded' into the polar space of a polarity. The polar spaces of (pseudo-)quadratic forms in characteristic 2 are not of this form.

4.3. *Tits systems and buildings with a transitive group action*

In Section 3, we have developed in some detail the axiomatic theory of buildings as simplicial complexes with various additional structures, in particular a system of apartments and a W-valued 'metric' on the set of chambers (W the Weyl group). In this subsection we specialize to the case that a group G acts in a strong way transitively on the building

Δ and thus finally come to the classical origin of the notion, namely its relationship to algebraic (and related) groups.

4.3.1. DEFINITION. A group G acts *strongly transitively* on a building Δ if it acts transitively on the pairs (C, Σ), where C is a chamber of Δ and Σ an apartment containing C.

Equivalently, the stabilizer of C, which is usually denoted by $B := G_C$ (the *Borel subgroup*), acts transitively on the apartments containing C. Alternatively, the (set-wise) stabilizer N of Σ induces a chamber-transitive group on Σ. If this holds, then, by the results of Section 2, $W := N/(B \cap N)$ is a Coxeter group with as distinguished generators the (residue classes of the) reflections in the panels of C. Following Tits (in particular Tits [1974]) and the treatment given in Bourbaki [1968], we shall now axiomatically treat the triples of groups (G, B, N) occurring in such a way. Before doing so, we (once more) look at the standard example of buildings of type A_n, $n \geqslant 3$.

4.3.2. EXAMPLE. Let V be a vector space of finite dimension $n+1 \geqslant 4$ over a skew field, and $G = \mathrm{GL}(V), \mathrm{SL}(V), \mathrm{PGL}(V), \mathrm{PSL}(V)$ – the general or special linear group, respectively, the corresponding projective group. Denote by Δ the building of flags of (proper, nonempty) subspaces of V. Then G acts strongly transitively on Δ. If $(v_1, \ldots, v_{n+1})$ is an ordered basis of V,

$$C = \{\langle v_1 \rangle, \langle v_1, v_2 \rangle, \ldots, \langle v_1, \ldots, v_n \rangle\}$$

is the corresponding chamber, and $\Sigma \ni C$ is the apartment consisting of all chambers obtained by permuting $(v_1, \ldots, v_{n+1})$, and if we identify G with $\mathrm{GL}_{n+1}(K)$ (resp., $\mathrm{SL}_{n+1}(K)$ etc.), acting on column vectors, then B is the group of all upper triangular matrices and N the group of all monomial matrices (having only one nonzero entry in each row and column).

Notice that strong transitivity as claimed in this example immediately follows from the given description of Σ and $N := G_\Sigma$. This group induces the symmetric group Sym_{n+1} on $(v_1, \ldots, v_{n+1})$ which acts chamber-transitively on $\Sigma = \Sigma(\mathsf{A}_n)$. Cf. Section 4.1 above.

In this example, W can be identified with the subgroup of all permutation matrices in G, and N is a semidirect product $N = W \cdot T$, where $T := B \cap N$ are the diagonal matrices. Similar descriptions hold for other classical groups.

4.3.3. DEFINITION. Let G be a group and B and N subgroups of G, set $T := B \cap N$. The pair (B, N) is called a *Tits system* or a *BN-pair* in G if the following axioms hold.

(T0) T is normal in N.

(T1) $B \cup N$ generates G.

There exists a system S of involutions generating $W := N/T$ such that

(T2) $sBw \subseteq BwB \cup BswB$ for all $s \in S$, $w \in W$,

(T3) $sBs \neq B$ for all $s \in S$.

One also says that (G, B, N) is a Tits system. The main result of this subsection is that a Tits system gives rise to a thick building with a strongly transitive group action, and conversely. This correspondence is bijective if one restricts to 'saturated' (see 4.3.9) Tits systems. Before formulating this correspondence precisely, we shall forget buildings for a moment and shall derive internal group theoretical consequences of axioms (T0) to (T3). Before doing so, we give some explanations on expressions like sBw or BwB occurring in axioms (T2) and (T3). If H and H' are subgroups of a group G, we denote by $H\backslash G/H'$ the set of all double cosets

$$HgH' := \{hgh' \colon\ h \in H,\ h' \in H'\}.$$

They are the orbits of an obvious (left) action of $H \times H'^{\circ}$ (H'° is the opposite group of H') on G; in particular, as g varies, two double cosets are either disjoint or equal. $H\backslash G/H'$ can also be considered as the set of orbits of the canonical left action of H on $G/H' = \{gH' \colon\ g \in G\}$, or as the set of orbits of the canonical right action of H' on $H\backslash G = \{Hg \colon\ g \in G\}$. Returning to the groups G, B, N, W above, we notice that an expression BwB, $w \in W$, makes sense since w is a subset of G. The subset BwB of G is a double coset, namely equal to BnB, where n is any representative of $w = nT$. (This follows from $T \subseteq B$.) In the following, l denotes (as earlier) the length function with respect to S on W.

For the proofs of the following propositions, we refer the reader to Bourbaki [1968].

4.3.4. PROPOSITION (Bruhat decomposition). *Assume that* (G, B, N) *satisfies* (T0), (T1), (T2). *For* $X \subseteq S$, *set* $W_X := \langle X \rangle \subseteq W$ *and* $G_X = BW_XB$.

(a) *The* G_X *are subgroups of* G, *and* $G = G_S = BWB$.

(b) *The mapping* $w \mapsto BwB$, $W \to B\backslash G/B$, *is a bijection.*

(c) *For* $X, Y \subseteq S$, *the mapping* $w \mapsto G_XwG_Y$, $W \to G_X\backslash G/G_Y$, *induces a bijection*

$$W_X\backslash W/W_Y \cong G_X\backslash G/G_Y.$$

The groups G_X are called *parabolic subgroups* of G (with respect to the specified Tits system). We recall that the proof of Proposition 4.3.4 relies on the following lemma.

4.3.5. LEMMA. *For* $s_1, \ldots, s_q \in S$, $w \in W$, *one has*

$$s_1 \ldots s_q Bw \subseteq \bigcup_{1 \leqslant i_1 < \ldots < i_p \leqslant q} Bs_{i_1} \ldots s_{i_p} wB.$$

In particular $W_X BwB \subseteq BW_X wB$ *for any* $X \subseteq S$, $w \in W$.

The following proposition and its corollary describe in more detail the situation of axiom (T2). They are preparatory for the important Theorem 4.3.8 below.

4.3.6. PROPOSITION.

(a) *Suppose that* (G, B, N) *satisfies* (T0), (T1), (T2), *and let* $s \in S$ *and* $w \in W$. *Then*

$$l(sw) \geqslant l(w) \Rightarrow sBw \subseteq BswB.$$

(b) *Let* (G, B, N) *be a Tits system,* $s \in S$, $w \in W$. *Then*

$$l(sw) \leqslant l(w) \Leftrightarrow sBw \nsubseteq BswB,$$

$$sBw \nsubseteq BswB \Leftrightarrow sBw \cap BwB \neq \varnothing$$

$$\Leftrightarrow s \subseteq BwBw^{-1}B.$$

4.3.7. COROLLARY. *Let* (G, B, N) *be a Tits system.*

(a) *If* $w = s_1 \dots s_l$, $s_t \in S$, $l = l(w)$, *then all* s_t *lie in* $\langle B, w \rangle$.

(b) *The set* S *is uniquely determined as the set of all* $w \in W$ *such that* $B \cup BwB$ *is a group.*

(c) S *is a minimal set of generators of* W.

PROOF. Part (a) follows by induction from 4.3.6(b). Part (b) follows from (a) and 4.3.4(a). Part (c) is an obvious consequence of (b). □

The next theorem shows that the existence of a Tits system (B, N) in G has strong implications for the structure of G: the overgroups of B are restricted to a class of groups explicitly given in terms of B and the generators of W.

4.3.8. THEOREM (Parabolic subgroups). *Let* (G, B, N) *be a Tits system.*

(a) *Any subgroup* $P \subseteq G$ *containing* B *('parabolic subgroup') is equal to* G_X, *for some* $X \subseteq S$, *namely* $X = X_P := \{s \in S\colon\ s \subseteq P\}$.

(b) $G_X = G_Y$ *only holds if* $X = Y$.

(c) $G_{\cap X_i} = \bigcap G_{X_i}$ *holds for any family* (X_i) *of subsets of* S.

(d) $W_X = W_Y$ *only holds if* $X = Y$.

(e) $W_{\cap X_i} = \bigcap W_{X_i}$.

PROOF. (a) Notice that P is a union of double cosets BwB. Given such a w, choose a reduced decomposition $w = s_1 \dots s_l$. By Corollary 4.3.7, all s_i are contained in P, as desired.

(b) We show that $X(G_X) = X$ for all $X \subseteq S$. If $s \in X(G_X)$, the $s \in W_X$ by Proposition 4.3.4(b). Assume $s \notin X$. Then $S \setminus \{s\}$ still generates W, contradicting Corollary 4.3.7(c).

Statements (c), (d) and (e) are obvious consequences of (a) and (b). □

The next two theorems are the main results of this subsection. For a given group G, they state a bijective correspondence between thick buildings with a strongly transitive G-action and Tits systems (B, N) in G. Before formulating the results, we remind the reader of the chamber transitive complexes $\Delta(G, G^i, i \in I)$ described in Section 1.5. The reason for introducing the following definition will become clear in a moment.

4.3.9. DEFINITION. A system (G, B, N) of groups satisfying (T0), (T1), (T2) is called *saturated* if

$$T = \widetilde{T} := \bigcap_{n \in N} nBn^{-1}.$$

4.3.10. THEOREM. *Let (B, N) be a Tits system in the group G, with Weyl group W and generating system $S \subseteq W$ as above. For $s \in S$, set*

$$W^s := W_{S \setminus \{s\}}, \qquad G^s := G_{S \setminus \{s\}},$$

where W_X, G_X, $X \subseteq S$, are as above. Consider the S-numbered complex

$$\Delta := \Delta(G, G^s, s \in S) = \bigcup_{X \subseteq S} G/G_X$$

and its subcomplex

$$\Sigma := \{nG_X\colon\ n \in N,\ X \subseteq S\}.$$

Let $\boldsymbol{\Sigma} := \{g\Sigma\colon\ g \in G\}$. Then

(a) *$(\Delta, \boldsymbol{\Sigma})$ is a thick building.*
(b) *G acts strongly transitively on Δ.*
(c) *$\Sigma \cong \Delta(W, W^s, s \in S)$ canonically. In particular, $W(\Sigma) \cong W$.*
(d) *(W, S) is a Coxeter system.*
(e) *B is the stabilizer G_B of the chamber B. The set-wise stabilizer G_Σ equals $\widetilde{N} := N\widetilde{T}$ where $\widetilde{T}$ is as in* 4.3.9.

The difficult part of this theorem is part (a). The other statements are easy. Notice that part (e) is trivial. We have stated it explicitly to set in evidence that a saturated system (G, B, N) can be recovered in a canonical way from its associated complex.

4.3.11. THEOREM. *Let $(\Delta, \boldsymbol{\Sigma})$ be a building and G a group acting strongly transitively on $(\Delta, \boldsymbol{\Sigma})$. Fix a chamber C and an apartment $\Sigma \ni C$, set $B = G_C$, $N = G_\Sigma$.*

(a) *The system of subgroups (B, N) in G has properties* (T0), (T1), (T2). *If Δ is thick, then* (T3) *holds as well.*
(b) *(G, B, N) is saturated.*
(c) *The restriction mapping $N \to W(\Sigma)$ induces an isomorphism of Weyl groups*

$$W = N/(B \cap N) \to W(\Sigma).$$

(d) *If Δ is thick, then $(\Delta, \boldsymbol{\Sigma})$ is canonically isomorphic to the building $\Delta(G, G^i, i \in I)$ belonging to (G, B, N), where $I = I(\Delta)$.*

For (d), notice that the set S of generators of W is canonically indexed by I.

The following proposition reduces all questions about parabolic subgroups to saturated Tits systems. It is an immediate consequence of Theorem 4.3.11 (and 4.3.10). One could easily give a direct proof.

4.3.12. PROPOSITION. *Suppose that* (G, B, N) *satisfies* (T0), (T1), (T2). *Set*

$$\widetilde{T} = \bigcap_{n \in N} nBn^{-1}$$

and $\widetilde{N} = N\widetilde{T}$ *as before. Then also* $(G, B, \widetilde{N})$ *satisfies* (T0), (T1), (T2). *If* (G, B, N) *is a Tits system, then* $(G, B, \widetilde{N})$ *also is. The Weyl groups of* (G, B, N) *and* $(G, B, \widetilde{N})$ *are canonically isomorphic, the parabolic subgroups* G_X, $X \subseteq S$, *are the same with respect to* N *or* $\widetilde{N}$.

4.4. *The building of a semisimple algebraic group*

4.4.1. *On the structure of semisimple algebraic groups.* After the axiomatic theory of the preceding paragraphs, we now come to the construction (or description, depending on one's point of view) of a large class of Tits systems which gives rise to most of the thick buildings of spherical type. The description of this class of Tits systems depends on the structure theory of algebraic groups: Borel subgroups, maximal tori, roots, semisimple groups. A general reference for this theory is the book Borel [1991], in particular Chapter 4 and Chapter 5, §§20, 21 and 24A. In fact, if we really assume full knowledge of this theory, not much is left to be said about the Tits systems and buildings of the algebraic groups in question. We recall a small number of basic definitions and facts. Let k be a commutative field. An algebraic group G over k, or *k-group* for short, is a representable group-valued functor $R \mapsto G(R)$ defined on the category of all commutative k-algebras R. The group $G(R)$ is called the group of R-rational points. All our algebraic groups are linear algebraic groups, that is $G(R)$ 'is' a subgroup (defined by polynomial identities) of the matrix group $\mathrm{GL}_n(R)$, for some n. If k is algebraically closed, we may identify G with its group of points $G(k)$. In the following, fix an algebraic closure $\overline{k}$ of k. Every k-group can be viewed as a $\overline{k}$-group (whereby part of the structure gets lost).

A k-group T is called a *torus* if, for some n, it is isomorphic over $\overline{k}$ to the product D_n of n copies of the 'multiplicative group' G_m, where $G_m(R) = R^*$ (units of R). A torus T is said to be *k-split* if it is isomorphic over k to D_n. If this holds, T acts diagonalizably over k in every k-representation of T (for instance on the Lie algebra of an algebraic overgroup $G \supset T$). Assuming $k = \overline{k}$ for the moment, a *Borel subgroup* B of G is defined as a maximal connected solvable subgroup. All Borel subgroups are conjugate. The identity component of the intersection of all Borel subgroups is called the *radical of* G. If its radical is trivial, then G is called *semisimple*. G is called *reductive* if its radical is a torus. In this case, a Borel subgroup is in fact the 'B' of a Tits system in G (= $G(\overline{k})$); see below. A *parabolic subgroup* of G is an algebraic subgroup containing a Borel subgroup.

Now suppose that G is a connected, reductive k-group, with k arbitrary. All maximal k-split tori in G are conjugate; see Borel [1991], 20.9. If T is maximal k-split, and $N_G(T)$, $Z_G(T)$ denote its normalizer and centralizer in G, then

$$W_k = W_k(T, G) := N_G(T)(k)/Z_G(T)(k)$$

is finite and is called 'the' (*relative*) *Weyl group* of G. It is a Coxeter group, and even the Weyl group of the root system of $T(k)$ acting on the (k-points of) the Lie algebra of G. The Weyl group W of $G(\overline{k})$ (which usually comes from a torus of larger dimension) is often called the *absolute Weyl group* of G. One can identify W_k with a subgroup of W. The parabolic subgroups of $G(\overline{k})$ which are definable over k are of particular importance. They are called *k-parabolic* subgroups of G. The minimal ones among them are all conjugate over k. If the minimal parabolic subgroups are in fact Borel subgroups, equivalently, if G possesses a Borel subgroup defined over k, then G is called *quasisplit* (over k). If G possesses a maximal k-torus which is k-split, then G is called *split* (over k). For a better understanding of this definition, one should know that $G(\overline{k})$ always possesses a maximal torus which is defined over k; cf. Borel [1991], 18.2(i), 18.6, 18.7. We remark that 'split' implies 'quasi-split'.

4.4.2. *The Tits system.* For the next theorem, we fix the following notation:
k is a commutative field;
G is a reductive algebraic group defined over k;
T is a maximal k-split torus in G;
$N = N_T$ is the normalizer of T in G;
B is a minimal k-parabolic subgroup of G with $B \supseteq T$.

THEOREM. *$(B(k), N(k))$ is a Tits system in the group of k-rational points $G(k)$.*

The building associated with this Tits system is called the *building of G over k*, or the k-building of G, and is denoted by $\Delta(G, k)$. For a related description of this building, see Tits [1974], Theorem 5.2. The proof follows from 5.15 and 5.16 of Borel and Tits [1965]. See also Borel [1991], Theorem 21.15.

Notice that the Weyl group of $\Delta(G, k)$ can be identified with the relative Weyl group of G over k. In particular, the Coxeter diagram of $\Delta(G, k)$ is underlying the relative Dynkin diagram of G over k.

4.4.3. *Diagrams for reductive groups and their buildings.* When one wants to use the structure theorem in 4.4.2 for the explicit classification and tabulation of certain classes of buildings, one must be able to classify reductive algebraic groups in a appropriate way, and to derive from their invariants the Coxeter diagram and possible further invariants of the associated building. It is explained in Tits [1965/66] how to achieve these goals. The classification of the groups G is worked out in terms of two main invariants, the *index* and the *anisotropic kernel* (the first of which is itself a collection of several data). Practically speaking, the index consists of a 'decorated Dynkin diagram' of the shape shown in Figure 4.1 below and of an extension field K/k (of degree $\leqslant 6$, for type $\neq \mathsf{D}_4$

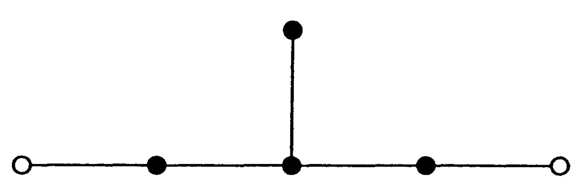
even of degree $\leqslant 2$). The index is the easy invariant, whereas the anisotropic kernel is, at least in the general case, the difficult one. Luckily, the Coxeter diagram of $\Delta(G, k)$ depends on the index alone, and can in fact be derived by a simple formula which we explain now. Formally, the data of the index additional to the underlying Dynkin diagram consist of a subset I_{an} of the index set $\bar{I}$ of the diagram, a Galois extension K/k, and an effective action of the Galois group $\Gamma = \Gamma(K/k)$ by diagram automorphisms respecting I_{an}. If we neglect the D_4 case, Γ thus must be of order $\leqslant 2$ and in fact trivial for diagrams B_n, C_n, F_4, E_7, E_8, G_2. It furthermore suffices to specify the 2-element orbits of the Γ-action. In the figures (case ${}^2\mathsf{E}_6$), we achieve this by drawing nodes in one orbit 'close to each other'. The black nodes in the diagrams are the ones in I_{an}. The Coxeter diagram of $\Delta(G, k)$ is now derived as follows: Its set of types consists of the Γ-orbits on the 'hollow nodes' $\bar{I} \setminus I_{\text{an}}$ ('distinguished types' in Tits [1965/66]). If $\mathcal{O}_a, \mathcal{O}_b$ are two such orbits, with representatives $a, b \in \bar{I}$, then

$$m_{ab} = 2(f_{ab} - f_0)/(f_a - f_0 + f_b - f_0),$$

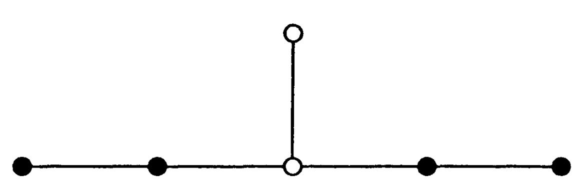
where f_{ab}, f_a, f_b, f_0 are the numbers of roots of the root systems generated by I_{an}, $I_{\text{an}} \cup \mathcal{O}_\alpha$, $I_{\text{an}} \cup \mathcal{O}_\beta$, $I_{\text{an}} \cup \mathcal{O}_\alpha \cup \mathcal{O}_\beta$, respectively. See Tits [1965/66], 2.5.5, for an example with nontrivial group Γ. The index of a k-group is subject to various restrictions of an inductive nature, and the list of all possible indices is smaller than one would expect at a first guess (and still smaller for particular ground fields like $\mathbb{F}_p, \mathbb{Q}_p, \mathbb{R}$). All possible indices are collected in Table II of Tits [1965/66]. The result is of course particularly important for the exceptional groups. But also for the classical groups it is interesting to see which types of quadratic or (skew-)Hermitian forms correspond to which indices.

In the following, we reproduce the list of all indices for the groups of absolute type E_n such that the rank of the building is at least two. We use the notation $({}^i\mathsf{X}_n^m, \mathsf{Y}_r)$, where ${}^i\mathsf{X}_n^m$ is the index as in Tits [1965/66] (X_n is the underlying Dynkin diagram of rank n, i is the order of Γ which of course is omitted for E_7, E_8, F_4, G_2), and Y_r is the Coxeter diagram derived from the index by the above method.

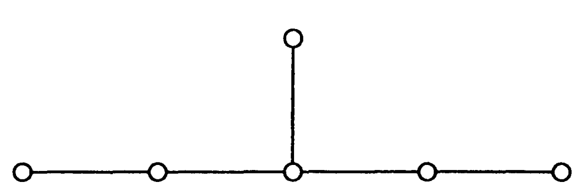
$({}^1\mathsf{E}_6^{28}, \mathsf{A}_2)$



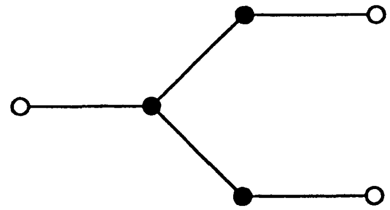
$({}^1\mathsf{E}_6^{16}, \mathsf{G}_2)$



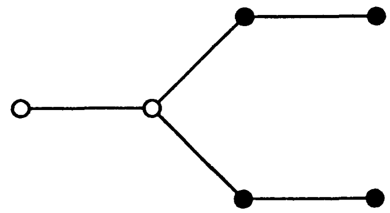
$({}^1\mathsf{E}_6^{0}, \mathsf{E}_6)$



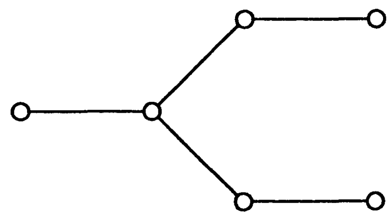
$({}^2E_6^{16'}, C_2)$



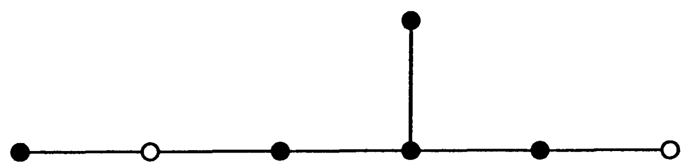
$({}^2E_6^{16''}, G_2)$



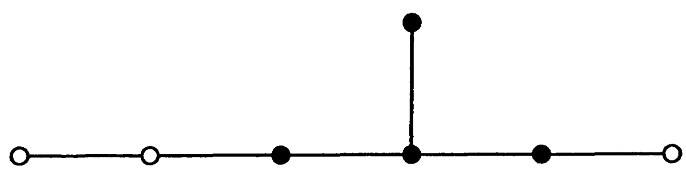
$({}^2E_6^{2}, F_4)$



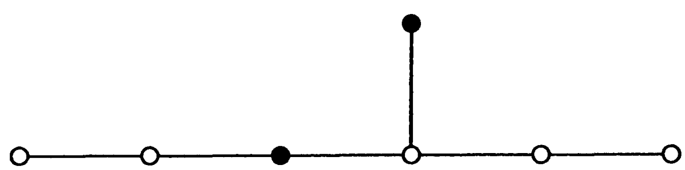
(E_7^{31}, C_2)



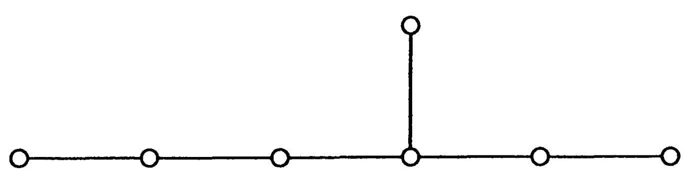
(E_7^{28}, C_3)



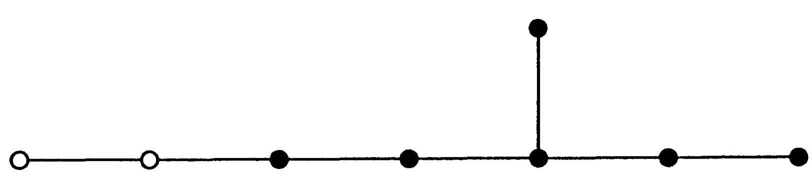
(E_7^{9}, F_4)



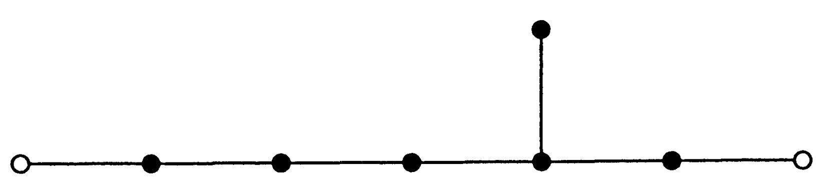
(E_7^{0}, E_7)



(E_8^{78}, G_2)



(E_8^{66}, C_2)



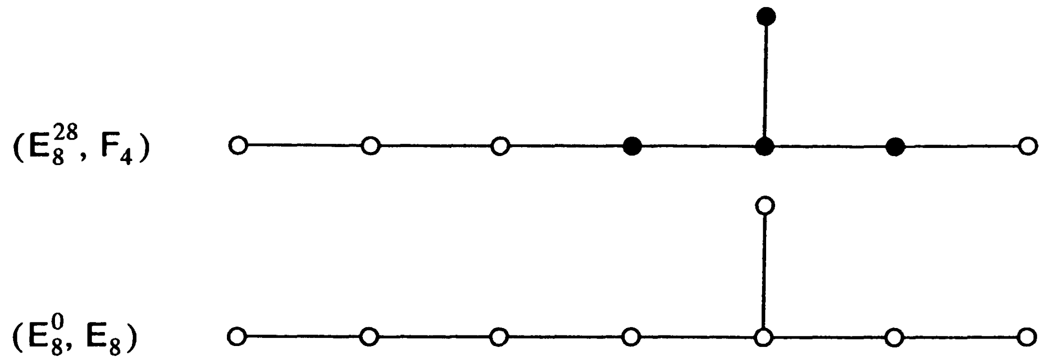


Figure 4.1. Tits diagrams for semisimple groups of absolute type E.

Here are some brief indications about the theory which is behind the recipe just explained. First of all, the semisimple groups over an algebraically closed field (it suffices to assume separably closed) are classified by ordinary reduced root systems, like the semisimple complex Lie groups, and thus by Dynkin diagrams. The Dynkin diagram underlying the index simply comes from considering G as a group over the separable closure k_s of k. The nodes of the diagram correspond to conjugacy classes of maximal parabolic subgroups of G. These can be defined over k_s (but not necessarily over k), and are permuted by the Galois group $G(k_s/k)$ by 'twisting' the structure. The above field K is the fixed field of the kernel of the resulting action on the Dynkin diagram. The hollow nodes in the index correspond by definition to the parabolic subgroups which are defined over the ground field k.

The anisotropic kernel, which we did not explain so far, is the semisimple part of the centralizer of a maximal k-split torus. This subgroup itself is 'anisotropic' in the sense that it has no nontrivial k-split torus, and hence no proper parabolic subgroup defined over k (cf. Borel [1991], 20.6(ii)). Its Dynkin diagram is precisely the subdiagram of 'black nodes'. It also has a direct significance for the building $\Delta(G, k)$ by acting chamber transitively on certain rank-1-residues. This fact makes it plausible (but does not prove) that the following theorem should hold.

4.4.4. THEOREM. *The building $\Delta(G, k)$ of a semisimple algebraic group over a field k determines the field k uniquely and the group G over k up to isogeny.*

This is stated in Tits [1974], 5.8. It follows from deep and difficult results by Borel and Tits [1973].

The next proposition shows that residues in buildings $\Delta(K, k)$ are of the same shape, and its diagrams are the obvious ones.

4.4.5. PROPOSITION. *Let G be a semisimple k-group and $\bar{I}$ be the (underlying type set) of its index. Let $J \subseteq \bar{I}$ be a subset of the distinguished types ('hollow nodes') which is a union of orbits. Then, a typical residue of type J in $\Delta(K, k)$ is canonically isomorphic to a building of the form $\Delta(H, k)$, where H is a semisimple k-group whose index is obtained from the index of G by removing J.*

Notes to Section 4

A_n*-buildings.* The implication (i) $\Rightarrow$ (iii) of Theorem 4.1.4 goes essentially back to Tits [1956], Section 7, (see in particular 7.2). There, the general spirit of the proof is explained. An explicit statement of the result was only given much later in Tits [1981a], Proposition 6, where also (most of) the proof is given. The easier result (ii) $\Rightarrow$ (iii) is stated in Theorem 6.3 of Tits [1974], with a sketch of a proof.

C_n*-buildings.* Our definition of $\Sigma(\mathsf{C}_n)$ in 4.2.1 is adapted from Tits' description of a thin polar space in Tits [1974], 7.3; similarly for D_n. The general facts about polar spaces and buildings 4.2.5–4.2.12 have been taken explicitly from *loc. it* Chapter 7; in particular, 4.2.8 comes from 7.4, and 4.2.11 from 7.12.

The remainder of Section 4.2, dealing with Hermitian and quadratic forms, has been taken from Tits [1974], Chapter 8. Tits' concept of a (σ, ε)-(pseudo-)quadratic form has also turned out to be the right notion (say for purposes of Hermitian K-theory) for forms over an arbitrary ring with involution. The additional concept of a 'form parameter' is needed. This is an additive group lying between $k_{\sigma,\varepsilon}$ and $k^{\sigma,\varepsilon}$. See, e.g., the text books W. Scharlau [1985], Chapter 7, or Knus [1991] for more about this, and for a general treatment of Witt's theorem. Proposition 4.2.18, dealing with polar spaces of Hermitian or quadratic forms, combines 8.3.2, 8.3.4 and 8.4.2 of Tits [1974].

Tits systems. The concept of a Tits system or BN-pair and the results 4.3.3 to 4.3.9 go back to Tits [1962a]. We have followed Bourbaki [1968], but the material is covered in many other texts as well, e.g., Carter [1972], Sections 8.2, 8.3. The relation to buildings (Theorems 4.3.10 and 11) can be found in convenient form in Tits [1974], 3.2.6 and 3.11. Of course, the result is as old as the notions involved are. It goes back at least to Tits [1965].

One further topic to be mentioned in connection with BN-pairs is the simplicity of certain groups. In Tits [1964a] it is shown how Tits systems can be used to unify the proofs for the simplicity of classical groups, and to generalize the result to groups of k-rational points of algebraic groups. See also Bourbaki [1968], Chapter 4, §2, no. 7, and Carter [1972], Section 11.1.

Algebraic groups and their geometries. The basic reference for the structure theory of reductive groups over nonalgebraically closed fields, containing most general and most complete results, is Borel and Tits [1965]. Most of the results are also contained in the lecture notes Satake [1971], which also deal with the classification, and which contain more complete proofs than the survey Tits [1965/66]. In particular, the classification over p-adic fields is treated in some detail. The paper Tits [1990b] contains complements to Tits [1965/66].

Almost all special cases of the classification for fixed diagram, in terms of the anisotropic kernel, had been treated, by various authors in various separate papers, before the general theory was worked out completely. For instance, the case $({}^1\mathsf{E}_6^{28}, \mathsf{A}_2)$ essentially describes the 'classical' relation between projective planes over a Cayley algebra, their automorphism groups, and the norm-forms of such algebras. See Freudenthal [1951, 1954-63], Springer [1960, 1962], and Tits [1953, 1955a,b, 1958, 1959] for other, more involved cases.

5. On the geometry of buildings

Introduction

In this section, we develop some of the main 'internal' geometrical properties of buildings, describing how the whole simplicial complex is built up from its simplices and apartments, how the stars of simplices are embedded, what certain convex sets look like, etc. Thus the term 'geometry' in the title of this section refers to the 'geometry' of triangulated topological spaces, and not to incidence geometry.

The first half of Subsection 5.1 complements the results of Subsection 2.3 on convexity in Coxeter–Tits complexes. In the second half of Subsection 5.1, a basic geometrical property of (the chamber system of) a building is described, the so-called 'gate property'. It deals with the way the stars are embedded into the whole chamber system. The gate property is equivalent to the existence of certain projection maps from the whole chamber system onto its stars. The projections are defined in metrical terms; an easy consequence of their definition is that they are idempotent and distance-non-increasing. Thus they can be thought of as some sort of retraction onto a star, or as a particular kind of convexity. The following topological properties are easy consequences of the gate property: A spherical building has the homotopy type of a bouquet of spheres, an affine building is contractible. The former property gives rise to applications of buildings to representation theory starting with Solomon [1969], the latter is the key to results on discrete subgroups of p-adic groups, Serre [1971]. See Chapter 20 for more details. Our presentation of projection maps differs from the original one in Tits [1974]. Following Dress and Scharlau [1987], we give an axiomatic treatment which, to some extent, just deals with metric spaces. This makes the proofs easier, and may help to develop modifications and extensions of the theory.

In Subsection 5.2, we derive some particular consequences from the results of Subsection 5.1 which hold in the spherical case only. The stars of any two 'opposite' simplices are shown to be canonically isomorphic; the main result of this subsection says that in the thick case, any two stars of the same type are (noncanonically) isomorphic.

The final Subsection 5.3 describes in greater detail the totality of all half-complexes (half-apartments, roots) in a spherical building. The so called 'Moufang property' is introduced, dealing with the existence of geometrically defined 'root groups' in the automorphism group of a building which generalize the root groups known from the structure theory of Chevalley groups. Using a basic Extension Theorem of Tits [1974], Chapter 4, one shows that this property automatically holds for large classes of spherical buildings. See also Subsection 7.1 below. The Moufang property will be of great importance for the classification of spherical buildings. Another application of the techniques around half-complexes is the analysis of the structure of 'weak', that is nonthick, buildings made in Scharlau [1987]. As a result in itself, this may be of minor importance, but it illustrates very well many of the methods developed in this section.

5.1. *Roots, convexity and projection maps*

Recall from 2.3.2 that a chamber subcomplex Δ' of a chamber complex is convex if its chamber set $\mathcal{C}(\Delta')$ is convex; that is, for $C, D \in \mathcal{C}(\Delta')$, any shortest gallery joining C to D is completely contained in Δ'.

5.1.1. DEFINITION. An arbitrary subcomplex of some chamber complex is called *convex* if it is the intersection of convex chamber subcomplexes. The convex hull $\Gamma(\Lambda)$ of an arbitrary subset $\Lambda \subseteq \Delta$ is the intersection of all convex chamber subcomplexes containing Λ.

Remember that a root in a Coxeter complex is the image of a folding, and that roots come in pairs of opposite roots. A root or, more generally, any image of an idempotent (chamber) endomorphism is convex.

5.1.2. LEMMA. *Let Σ be a thin connected complex, C, C' be adjacent chambers, suppose that there is a folding π such that $\pi C' = C$. If Φ is a convex chamber subcomplex such that $C \in \Phi$, $C' \notin \Phi$, then $\Phi \subseteq \pi\Sigma$.*

PROOF. Let $D \in \Phi$. By Proposition 2.3.3, it is sufficient to show that $d(D, C) = d(D, C') + 1$ does not hold. But this equality would imply that C' is a member of some geodesic joining C and D, that is $C' \in \Phi$, a contradiction. □

5.1.3. PROPOSITION. *In a Coxeter–Tits complex, every proper convex chamber subcomplex is an intersection of roots.*

PROOF. Let Φ be the subcomplex in question, and C any chamber not contained in Φ. Choose a geodesic $\boldsymbol{C} = (C = C_0, C_1, \ldots, C_q)$ such that $C_q \in \Phi$, and let the index ν be such that $C_{\nu-1} \notin \Phi$, $C_\nu \in \Phi$. If π is the folding such that $\pi C_{\nu-1} = C_\nu$, then $\Phi \subseteq \pi\Sigma$, by Lemma 5.1.2. But $C \notin \pi\Sigma$, for otherwise $\pi\boldsymbol{C}$ would give a shorter gallery joining C and C_q. □

5.1.4. LEMMA. *For any two chambers C, D of a Coxeter–Tits complex, $d(C, D)$ is the number of roots containing C but not D.*

PROOF. Choose a geodesic $(C = C_0, C_1, \ldots, D_d = D)$. If $\boldsymbol{\Pi}(C, D)$ is the set of roots in question, then for any $\Pi \in \boldsymbol{\Pi}(C, D)$ there is a unique index ν such that $C_{\nu-1} \in \Pi$, $C_\nu \notin \Pi$, and the map $\boldsymbol{\Pi}(C, D) \to \{1, \ldots, d\}$, $\Pi \mapsto \nu$ is bijective. □

5.1.5. PROPOSITION. *If C, D are chambers in a Coxeter–Tits complex Σ, then the convex hull conv$\{C, D\}$ in $\mathcal{C}(\Sigma)$ is equal to the 'interval'*

$$[C, D] := \{E \in \mathcal{C}(\Sigma)\colon\ d(C, D) = d(C, E) + d(E, D)\}.$$

PROOF. The inclusion

$$[C, D] \subseteq \operatorname{conv}\{C, D\}$$

is obvious. Conversely, let $E \in \operatorname{conv}\{C, D\}$, and consider the sets $\Pi(C,D)$, $\Pi(C,E)$ and $\Pi(E,D)$, in the notation of the last proof. From $E \in \operatorname{conv}\{C,D\}$ it follows that $\Pi(C,E) \subseteq \Pi(C,D)$ and $\Pi(E,D) \subseteq \Pi(C,D)$. But trivially

$$\Pi(C,D) \subseteq \Pi(C,E) \cup \Pi(E,D) \quad \text{and} \quad \Pi(C,E) \cap \Pi(E,D) = \varnothing.$$

Thus $\Pi(C,D)$ is the disjoint union of $\Pi(C,E)$ and $\Pi(E,D)$, and the desired equality $d(C,D) = d(C,E) + d(E,D)$ follows from Lemma 5.1.4. □

As a general comment on Proposition 5.1.5, we point out that the notion of convexity is naturally defined in terms of the notion of an interval $[C,D]$, and therefore applies to any metric space. That is, a subset of a metric space is convex if it contains, with any two points C and D, the whole interval $[C,D]$. In general, it is not true that $[C,D]$ is all of $\operatorname{conv}\{C,D\}$.

We now continue along the lines of Propositions 5.1.3 and 5.1.5 by showing that the complete structure of a Coxeter–Tits complex, that is, its simplices and their inclusion relation, can be described in terms of roots.

5.1.6. PROPOSITION. *Let Σ be a Coxeter–Tits complex and $A, B \in \Sigma$.*

(a) *$A \cup B$ is a member of Σ if and only if, for any pair Π, Π' of opposite roots, at least one contains A and B.*

(b) *$A \subseteq B$ holds if and only if $A \in \Pi$ for all roots $\Pi \ni B$.*

(c) *$\mathcal{P}(A)$ is the intersection of all roots containing A.*

PROOF. (b) and (c) are immediate consequences of (a). For (a), suppose that $A \cup B$ does not exist, and take chambers $C \supseteq A$, $D \supseteq B$ of minimum distance $d = d(C,D) > 0$. Choose any geodesic $(C = C_0, C_1, \ldots, C_d = D)$, and consider the foldings π, π' such that $\pi C_1 = C_0$, $\pi' C_0 = C_1$. Then $A \in \pi\Sigma$, $B \in \pi'\Sigma$, and, as usual $A \notin \pi'\Sigma$ (otherwise $\pi' C \supseteq A$, $\pi' D = D \supseteq B$ would have distance less than d), and symmetrically $B \notin \pi\Sigma$. □

Proposition 5.1.6 can be visualized by saying that a Coxeter–Tits complex is 'cut out' by the system of its roots. For the 'concrete' complex of a reflection group W on n-space (with simplicial fundamental domain), in particular, for the symmetry groups of the Platonic solids, this is true, simply by the definition of the complex of W. Since the 'abstract' theory of general Coxeter–Tits complexes has been modeled by J. Tits after the example of 'concrete' reflection groups (see the Notes of Section 2), Proposition 5.1.6 is in a sense the most basic property to expect for general Coxeter–Tits complexes.

We now turn to the notion of projection maps which is the second important topic of this subsection.

5.1.7. THEOREM. *Let Δ be a building. Consider any star $\operatorname{St}_\Delta A$, $A \in \Delta$. For each chamber C, there exists a chamber $C' \in \mathcal{A} := \mathcal{C}(\operatorname{St}_\Delta A)$ such that*

$$d(C,D) = d(C,C') + d(C',D) \quad \textit{for all} \quad D \in \mathcal{A}.$$

The chamber C' is obviously unique (if it exists), since $d(C', C') = 0$ for any two chambers C', C' with the above property. It is called the *projection of C onto $\mathcal{A}$*, or onto $\operatorname{St} A$, and is denoted by $C' := \operatorname{pr}_{\mathcal{A}} C = \operatorname{pr}_A C$.

The existence of such a projection map $\operatorname{pr}_{\mathcal{A}}\colon \mathcal{C} \to \mathcal{A}$ is a property that makes sense for any subset $\mathcal{A}$ of any metric space $\mathcal{C}$. A subset $\mathcal{A}$ of $\mathcal{C}$ is called *gated* if the property of Theorem 5.1.7 holds, that is, if $\operatorname{pr}_{\mathcal{A}}$ exists. Even if one is only interested in the properties of stars in buildings, the general idea is very helpful in organizing the various little properties and facts about projections occurring in the literature.

PROOF of Theorem 5.1.7. Let $C' \in \mathcal{A}$ be such that $d(C, C')$ is minimal, and let $D \in \mathcal{A}$ be arbitrary. Choose geodesics

$$\big(C = C_0, C_1, \ldots, C_q = C'; i_1, \ldots, i_q\big)$$

and

$$\big(C' = D_0, D_1, \ldots, D_r = D; j_1, \ldots, j_r\big),$$

$j_\mu \in J = \operatorname{cotype} A$ for all $\mu = 1, \ldots, r$. Assume that the juxtaposition of these two geodesics is not geodesic. Let s between 0 and $r - 1$ be such that

$$\big(C_0, \ldots, C_q, D_1, \ldots, D_s; i_1, \ldots i_q, j_1, \ldots, j_s\big)$$

is geodesic, but $(C_0, \ldots, D_{s+1}; \ldots)$ is not. Now apply the exchange property for geodesics 3.3.7. If the omitted index is among $j_1, \ldots, j_s$, then the new gallery looks like

$$\big(C, C_1, \ldots, C_q = C', C'_{q+1}, \ldots, C'_{q+s} = D_s; i_1, \ldots, i_q, j_1, \ldots, \widehat{j_\mu}, j_s\big).$$

Now $(C', C'_{q+1}, \ldots, C'_{q+s-1}, D_{s+1}; j_1, \ldots, \widehat{j_\mu}, \ldots, j_s, j_{s+1})$ is a gallery of length s, contradicting the fact that $d(C', D_{s+1}) = s + 1$. (In the case $s = 1$, the last gallery just reads $(C', D_2; j_2)$).

If, on the other hand, the omitted index is among $i_1, \ldots, i_q$, then the new gallery looks like

$$\big(C = C_0, C'_1, \ldots, C'_{q-1}, D'_0, D'_1, \ldots, D'_s, D_s; i_1, \ldots, \widehat{i_\nu}, \ldots, i_q, j_1, \ldots, j_{s+1}\big).$$

But then $C'_{q-1} \overset{J}{\sim} D_s$, therefore $C'_{q-1} \in \mathcal{A}$, contradicting the fact that q was the minimum of all $d(C, D')$, $D' \in \mathcal{A}$. □

5.1.8. REMARK. The following two properties of a projection map pr_A are immediate consequences of the definition.

1. pr_A is idempotent: $\operatorname{pr}_A^2 = \operatorname{pr}_A$.
2. pr_A decreases distances: $d(\operatorname{pr}_A C, \operatorname{pr}_A C) \leqslant d(C, D)$.

The second property in particular says:

3. If C is adjacent to D, then $\operatorname{pr}_A C = \operatorname{pr}_A D$, or $\operatorname{pr}_A C$ and $\operatorname{pr}_A D$ are adjacent.

The last property immediately raises the question to what extent can projections be considered as morphisms. The following proposition gives two answers at the same time. First, it shows that between certain pairs of stars in $\mathcal{C}$, the projections indeed induce isomorphims of chamber systems. In general, these stars may be small. But in spherical buildings the situation frequently occurs that any pr_A restricted to some 'opposite' $\operatorname{St} B$ is an isomorphism of $\operatorname{St} B$ onto all of $\operatorname{St} A$. At the same time, 5.1.9 says how to extend pr_A, although not being a morphism, to all simplices of Δ in a meaningful way.

5.1.9. PROPOSITION. *Let $\mathcal{A} = \mathcal{C}(\mathrm{St}\,A)$, $\mathcal{B} = \mathcal{C}(\mathrm{St}\,B)$ be stars in a building, $p := \mathrm{pr}_{\mathcal{A}}$, $q := \mathrm{pr}_{\mathcal{B}}$ be the projections. Consider the images $\mathcal{A}' := p\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}' := q\mathcal{A} \subseteq \mathcal{B}$. Then $\mathcal{A}', \mathcal{B}'$ also are stars, i.e. $\mathcal{A}' = \mathcal{C}(\mathrm{St}\,A')$, $\mathcal{B}' = \mathcal{C}(\mathrm{St}\,B')$ for some $A', B' \in \Delta$, and q, p induce isomorphisms, inverse to each other, between $\mathcal{A}'$ and $\mathcal{B}'$, relative to some bijection between the type sets* $\mathrm{cotype}\,A'$, $\mathrm{cotype}\,B'$.

This proposition is contained in the Theorem and Proposition 3 of Dress and Scharlau [1987]; we shall not reproduce the proof here. The fact that q, p induce isometries between $\mathcal{A}'$ and $\mathcal{B}'$ holds for arbitrary gated subsets $\mathcal{A}, \mathcal{B}$ of a metric space. The fact that they are isomorphisms of chamber systems uses the exchange condition for geodesics and is therefore more or less restricted to buildings. The type sets $\mathrm{type}\,\mathcal{A}' = \mathrm{cotype}\,A'$, $\mathrm{type}\,\mathcal{B}' = \mathrm{cotype}\,B'$ and the bijection $\mathrm{type}\,\mathcal{A}' \cong \mathrm{type}\,\mathcal{B}'$ are given by a certain element $w = w(A, B)$ in the Weyl group:

$$\mathrm{type}\,\mathcal{A}' = \mathrm{type}\,\mathcal{A} \cap w(\mathrm{type}\,\mathcal{B})w^{-1},$$

$$\mathrm{type}\,\mathcal{B}' = w^{-1}(\mathrm{type}\,\mathcal{A}')w = w^{-1}(\mathrm{type}\,\mathcal{A})w \cap \mathrm{type}\,\mathcal{B},$$

and the bijection is $j \mapsto w^{-1}jw$ ($j \in \mathrm{type}\,\mathcal{A}'$).

The last proposition has immediate implications about the structure of the convex hull (in the sense of Definition 5.1.1) of a two-element subset of a building. In the following proposition, we formulate them explicitly. Since any two elements are contained in an apartment which is convex, we may restrict to Coxeter–Tits complexes.

5.1.10. PROPOSITION. *Let Σ be a Coxeter–Tits complex with automorphism group W and $A, B \in \Sigma$.*

(a) *The intersection $W_A \cap W_B$ of the stabilizers of A and B in W is the stabilizer of* $\mathrm{pr}_A B$ *(or* $\mathrm{pr}_B A$*). In particular, $W_A \cap W_B$ is a Coxeter subgroup of W.*

(b) *Choose a chamber $C \supseteq \mathrm{pr}_A B$, and let $D := \mathrm{pr}_B C$ be the corresponding chamber containing* $\mathrm{pr}_B A$ (cf. 5.1.9). *The convex hull $\Gamma(A, B)$ (in the sense of Definition* 5.1.1) *is the fixed point set of $W_A \cap W_B$ in the chamber subcomplex $\Gamma(C, D)$. It is a pure subcomplex of rank equal to the rank of* $\mathrm{pr}_A B$.

Part (a) of this proposition is 12.5 in Tits [1974]. Notice that the nontrivial inclusion $W_{\mathrm{pr}_A B} \subseteq W_B$ readily follows from what we have said about the proof of 5.1.9. Part (b) was inspired by p. 188 of Tits [1959/62]. In the form as stated here, it cannot be found in the literature. For the proof, one first observes that $\Gamma(A, B)$ is the intersection of all $\Gamma(C, D)$, where C runs over all chambers containing $\mathrm{pr}_A B$ and $D = \mathrm{pr}_B C$. Then one uses part (a).

In the notes at the end of this section, we shall make a few comments about the significance of convex hulls $\Gamma(A, B)$ for the incidence properties of buildings.

Before deriving further, particular consequences of the gate property in the spherical case, we close this subsection with the main theorem from Scharlau [1985a] which shows the fundamental character of this property even for general Coxeter diagrams.

5.1.11. THEOREM. *Let $\mathcal{C}$ be a chamber system of Coxeter type such that all stars of rank 1 and 2 are gated inside $\mathcal{C}$. Then $\mathcal{C}$ corresponds to a building.*

The proof uses the First Main Characterization of Buildings.

5.2. *Opposite chambers and apartments in spherical buildings*

In this subsection, for the first time we have to impose a substantial restriction on the diagrams of our buildings. A building is called of *spherical type*, or *spherical* for short, if its apartments are finite. This holds if and only if the Weyl group is finite. Remember that this is the case if and only if the connected components of the Coxeter diagram are in the list

$$\mathsf{A}_n\ (n \geqslant 1), \qquad \mathsf{C}_n\ (n \geqslant 2), \qquad \mathsf{D}_n\ (n \geqslant 4),$$

$$\mathsf{I}_2(m)\ (m \geqslant 5), \qquad \mathsf{H}_3,\ \mathsf{F}_4,\ \mathsf{H}_4,\ \mathsf{E}_6,\ \mathsf{E}_7,\ \mathsf{E}_8,$$

given at the beginning of Section 2.

If Σ is any finite connected complex, then its diameter

$$\operatorname{diam} \Sigma := \max_{C,D \in \mathcal{C}(\Sigma)} d(C, D)$$

is finite. We remark that, conversely, if Σ has finite diameter and if all the stars of panels are finite, then Σ is finite. Indeed, by induction on d, there are only finitely many chambers such that $d(C_0, C) \leqslant d$, for every d, where C_0 is some fixed chamber.

5.2.1. PROPOSITION. *Let Σ be a finite Coxeter–Tits complex.*

(a) *For any two chambers $C, D \in \mathcal{C}(\Sigma)$, the distance is maximal, $d(C, D) = \operatorname{diam} \Sigma$, if and only if $\operatorname{conv}\{C, D\} = \mathcal{C}(\Sigma)$.*

(b) *For given C, there is a unique chamber $D = \operatorname{op}_\Sigma C$ such that $d(C, D) = \operatorname{diam} \Sigma$.*

(c) *The mapping op_Σ is a (not necessarily type-preserving) automorphism of Σ.*

(d) $\operatorname{op}_\Sigma^2 = \operatorname{id}$; $\operatorname{op}_{\Sigma'} \circ \varphi = \varphi \circ \operatorname{op}_\Sigma$, *for any isomorphism $\varphi\colon \Sigma \to \Sigma'$.*

In particular, op_Σ centralizes the automorphism group of Σ.

PROOF. Suppose that $d(C, D)$ is maximal. By Proposition 5.1.3, we have to show that no root contains C and D. Suppose on the contrary that there exists a folding π such that $C, D \in \pi\Sigma$, consider the opposite folding π', and choose a geodesic $\boldsymbol{C} = (C, \ldots, \pi' D)$. Then $\pi \boldsymbol{C}$ has repetitions, therefore

$$d(C, D) = d(\pi C, \pi\pi' D) < d(C, \pi' D),$$

a contradiction. □

For the converse, we use the fact that $\operatorname{conv}\{C, D\} = [C, D]$ (Proposition 5.1.5). That is,

$$d(C, D) = d(C, E) + d(E, D) \geqslant d(C, E)$$

for every chamber E. By transitivity, $d(C, D) \geqslant d(C', E')$ for any pair C', E' of chambers. Furthermore, if E is such that $d(C, E) = d(C, D)$, then $d(E, D) = 0$, which proves the uniqueness of D, for a given C. The relation $\operatorname{op}_\Sigma^2 = \operatorname{id}$ and the canonicity stated in (d) are obvious from the uniqueness.

In order to show that op_Σ is a morphism (of chamber systems), assume that $C \overset{i}{—} C'$, for some $i \in E$. From the equations

$$d(C, \operatorname{op}_\Sigma C) = d(C, C') + d(C', \operatorname{op}_\Sigma C),$$
$$d(C', \operatorname{op}_\Sigma C') = d(C', \operatorname{op}_\Sigma C) + d(\operatorname{op}_\Sigma C, \operatorname{op}_\Sigma C')$$

it follows that $d(\operatorname{op}_\Sigma C, \operatorname{op}_\Sigma C') = 1$, that is $\operatorname{op}_\Sigma C \overset{i_*}{—} \operatorname{op}_\Sigma C'$ for some $i_* \in I$. What is left to show is that the map $i \mapsto i_*$ does not depend on the choice of C. But if σ_i, $i \in I$, are the reflections in the panels of C, then $w\sigma_i w^{-1}$, $i \in I$, are the reflections in the panels of wC, and our claim immediately follows from the fact that op_Σ centralizes all $w \in W$.

For certain purposes, it is useful to rephrase the last proposition in explicit form for the coset complex of a finite Coxeter group. This is done in the next proposition which is basically taken from Bourbaki [1968], Chapter 4, §1, Exercise 22.

5.2.2. PROPOSITION. *Let (W, S) be a finite Coxeter group and $\Sigma = \Delta(W, W^s, s \in S)$ be its Coxeter–Tits complex.*

(a) *There is a unique element $w_0 \in W$ of maximal length. It is characterized by the property that $l(sw_0) < l(w_0)$ for all $s \in S$ and satisfies $l(ww_0) = l(w_0) - l(w)$ for all $w \in W$.*

(b) *The opposition involution on the vertices of Σ is given by*

$$wW^s \mapsto ww_0W^{w_0sw_0}.$$

The last proposition in particular shows that op_Σ is type preserving if and only if w_0 is central. By Bourbaki [1968], Chapter 5, §4, Exercise 3, this is the case if and only if $w_0 = -1$ in the canonical linear representation. For the dihedral groups $W(\mathsf{I}_2(m))$, either of these properties obviously holds if and only if m is even. For H_3, H_4 the opposition must be type preserving since there is no diagram automorphism. The other finite Coxeter groups are Weyl groups of root systems, and their longest elements w_0 are tabulated in Bourbaki [1968]. One can also use Exercise 7 (b) in Chapter 6, §4, of *loc. cit.* which says that $-1 \in W(R)$ for an irreducible root system R if and only if $Q(R) \supseteq 2P(R)$. The following proposition is an immediate consequence of these facts.

5.2.3. Proposition. *If Σ is spherical of irreducible type, then* $(\mathrm{op}_\Sigma)_*$ *is trivial if the diagram is* A_1, C_n, D_n *(n even),* $\mathsf{I}_2(m)$, *(m even),* H_3, F_4, H_4, E_7, E_8. *It is the nontrivial diagram automorphism for* A_n *($n \geqslant 2$),* D_n *(n odd), and* E_6.

Let us go back to Proposition 5.2.2 for a moment. It is clear that if an element w_0 of maximal length exists, then $l(w_0 s) < l(w_0)$ for all $s \in S$. It is not so obvious but true that if an element w_0 in a Coxeter group (W, S) satisfies this last property, then W is necessarily finite (and, as a consequence, w_0 its unique longest element). This is the content of the next proposition.

5.2.4. Proposition.

(a) *Let Σ be a Coxeter–Tits complex and C, D chambers whose distance is a local maximum, i.e. $d(C', D) < d(C, D)$ for any chamber $C' \in \Sigma$ adjacent to C. Then Σ is finite, and $d(C, D) = \operatorname{diam} \Sigma$.*

(b) *Let (W,S) be a Coxeter group and $w_0 \in W$ such that $l(w_0 s) < l(w_0)$ for all $s \in S$. Then W is finite and w_0 its longest element.*

Part (b) of this proposition is of course an immediate reformulation of (a). For (a), one shows by induction on $d(C, E)$ that any chamber E lies in $[C, D]$ (and consequently Σ is of finite diameter $d(C, D)$). In the induction step, the projection property 5.1.7 of an appropriate rank 2 star is used. For a full proof, see Ronan [1989], Theorem 2.16.

For later purposes, we draw the following consequence from 5.2.4

5.2.5. Lemma. *Let $(W, s_i, i \in I)$ be an arbitrary Coxeter group, $w \in W$ and $J \subseteq I$ be such that $l(ws_j) < l(w)$ for all $j \in J$. Then J is spherical (i.e. W_J is finite), and if w_J denotes its longest element, then $l(w) = l(ww_J^{-1}) + l(w_J)$.*

Proof. Write $w = w_1 w_2$, where w_1 is the shortest element in wW_J and $w_2 \in W_J$ (Proposition 2.1.7). From the assumption and 2.1.7, again, it follows that $l(w_2 s_j) < l(w_2)$ for all $j \in J$. Now apply 5.2.4 to W_J and w_2. □

We now turn from finite Coxeter–Tits complexes to arbitrary buildings Δ of spherical type, i.e. whose apartments are spherical. Since any two chambers C, D are contained in an apartment Σ, and since $d_\Sigma(C, D) = d_\Delta(C, D)$, we see that $\operatorname{diam} \Delta = \operatorname{diam} \Sigma$. We say that two chambers C and D are *opposite*, and use the notation $C \leftrightarrow D$, if $d(C, D) = \operatorname{diam} \Delta$. Now recall that $\operatorname{conv}_\Delta\{C, D\} = \operatorname{conv}_\Sigma\{C, D\}$. Therefore the following proposition immediately follows from Proposition 5.2.1.

5.2.6. Proposition. *In a spherical building, any two opposite chambers C, D are contained in a unique apartment, whose set of chambers equals $\{C, D\}$. The system of apartments Σ of a spherical building (Δ, Σ) is uniquely determined by the complex Δ.*

Now consider any two simplices $A, B \in \Delta$, choose an apartment $\Sigma \supseteq \{A, B\}$. Extending the above definition, we say that A and B are *opposite*, $A \leftrightarrow B$, if $A = \mathrm{op}_\Sigma B$.

This does not depend on the choice of Σ. (If $\Sigma' \supseteq \{A, B\}$, take an isomorphism $\varphi\colon \Sigma \to \Sigma'$ fixing A and B, and use Proposition 5.2.1(d).)

The involutions $(\mathrm{op}_\Sigma)_*\colon I(\Sigma) \to I(\Sigma)$ all induce the same involution

$$\mathrm{op}_*\colon I(\Delta) \to I(\Delta)$$

on the diagram of Δ. Two type sets J, K are called *opposite* if $J = \mathrm{op}_* K$. If C, D are opposite chambers, $A \subseteq C$, $B \subseteq D$, then A and B are opposite if and only if their types are. This describes the oppositeness of simplices independently of the choice of an apartment.

We now relate the notion of opposite elements to projection maps. Although it is completely straightforward how to do this (both notions being defined by simple equalities, only involving the distance function), it has the striking application that in a thick spherical building, any two stars of the same type are actually isomorphic.

5.2.7. LEMMA. *Suppose that A, B are opposite simplices in a spherical building, consider chambers $C \supseteq A, D \supseteq B$. They are opposite if and only if C and $\mathrm{pr}_A D$ are opposite in $\mathrm{St}\,A$. In particular, in a Coxeter–Tits complex,*

$$\mathrm{pr}_A|_{\mathrm{St}\,B} = \mathrm{op}_{\mathrm{St}\,A} \circ \mathrm{op}_\Sigma|_{\mathrm{St}\,B}.$$

PROOF. Since A, B are opposite, there exists in $\mathrm{St}\,A$ some chamber opposite to D. Therefore, $C \leftrightarrow D$ is equivalent to the maximality of $d(C, D)$, where C ranges over $\mathrm{St}\,A$. But

$$d(C, D) = d(C, \mathrm{pr}_A D) + d(\mathrm{pr}_A D, D).$$

Therefore this maximality indeed is equivalent to the maximality of $d(C, \mathrm{pr}_A D)$, $C \in \mathrm{St}\,A$. □

5.2.8. THEOREM. *If A and B are opposite simplices in a spherical building, then pr_B and pr_A induce isomorphisms between $\mathrm{St}\,A$ and $\mathrm{St}\,B$ that are inverse to each other.*

PROOF. By 5.2.7, this is true for the restriction to any apartment containing A and B. The bijectivity then immediately carries over to all of $\mathrm{St}\,A$, $\mathrm{St}\,B$. The fact that pr_A is a morphism on all of $\mathrm{St}_\Delta B$ now follows from the general result Proposition 5.1.9. Alternatively, it suffices to show that for any two chambers $D, D' \supseteq B$, there exists an apartment $\Sigma \supseteq \{A, D, D'\}$. Take any apartment Γ in $\mathrm{St}\,B$, containing D and D', consider the opposite D° of D in Γ, and write $D^\circ = \mathrm{pr}_B C$, $C \in \mathrm{St}\,A$ (i.e. $C = \mathrm{pr}_A D^\circ$). Then any apartment Σ containing $\{C, D\}$ satisfies our claim. Indeed,

$$D' \in \mathcal{C}(\Gamma) = \mathrm{conv}\{D, D^\circ\} \subseteq \mathrm{conv}\{C, D\} \subseteq \Sigma.$$

(Incidentally, Σ is unique, by Lemma 5.2.7.) □

The last theorem gives an isomorphism $\mathrm{St}_\Delta A \xrightarrow{\cong} \mathrm{St}_\Delta B$, for two simplices A, B of the same type, provided A and B have a common opposite. This is in fact true for any two simplices of the same type in any thick building. More generally, the following holds:

5.2.9. THEOREM. *Let Δ be a building such that every panel is contained in at least $q+1$ chambers, for some q, and let A, B be opposite simplices (possibly $A = B = \varnothing$). Any q chambers $C_1, C_2, \ldots, C_q \in \mathrm{St}\, A$ have a common opposite in $\mathrm{St}\, B$.*

PROOF. By Lemma 5.2.7, one easily reduces the claim to the case $A = B = \varnothing$.

Now suppose that for some $\nu < q$, we have already found a chamber D which is opposite to $C_1, \ldots, C_\nu$. Assume that, among all such D, the distance $d(D, C_{\nu+1})$ is maximal. We claim that then D is automatically opposite to $C_{\nu+1}$. Suppose the contrary. Then it is easily seen that there exists a panel F of D such that $\mathrm{pr}_F\, C_{\nu+1} = D$. (To see this, choose an apartment $\Sigma \supseteq \{C_{\nu+1}, D\}$, and a geodesic $(C_{\nu+1} = D_0, D_1, \ldots, D_\mu = D, D_{\mu+1}, \ldots, \mathrm{op}_\Sigma\, C_{\nu+1})$, and set $F = D_\mu \cap D_{\mu+1}$.)

Now by assumption, there exists a chamber $E \supset F$ distinct from $\mathrm{pr}_F\, C_1, \ldots, \mathrm{pr}_F\, C_{\nu+1}$. Now observe that, for all $\mu = 1, \ldots, \nu+1$

$$d(E, C_\mu) = d(E, \mathrm{pr}_F\, C_\mu) + d(\mathrm{pr}_F\, C_\mu, C_\mu) = 1 + d(\mathrm{pr}_F\, C_\mu, C_\mu).$$

For $\mu \leqslant \nu$, we estimate the right hand side by

$$d(\mathrm{pr}_F\, C_\mu, C_\mu) \geqslant d(D, C_\mu) - 1 = \mathrm{diam}\, \Sigma - 1,$$

i.e. E is opposite to C_μ. For $\mu = \nu + 1$, we see that

$$d(E, C_{\nu+1}) = 1 + d(D, C_{\nu+1}).$$

But this contradicts the maximality assumption on D. □

We finally state the following consequence of Theorems 5.2.8 and 5.2.9, which has already been mentioned several times:

5.2.10. THEOREM. *If A, B are two simplices of the same type in a thick, spherical building Δ, then there exists a type preserving isomorphism $\mathrm{St}_\Delta A \xrightarrow{\cong} \mathrm{St}_\Delta B$.*

As a corollary of the results of this subsection, we shall now give the proof of the proposition, stated earlier, that a thick finite building admits a parameter system.

PROOF of Proposition 3.4.2. The first part is an immediate consequence of Theorem 5.2.10: the stars of any two panels of the same type are isomorphic. Now we want to relate q_i and q_j, for two arbitrary types i and j. Consider any simplex A of cotype $\{i, j\}$, and any pair F, F' of opposite panels in $\mathrm{St}\, A$. If m_{ij} is odd, then F and F' are of different cotype. By 5.2.10, their stars are isomorphic, and therefore $q_i = q_j$. It follows that, more generally $q_i = q_j$ if there is a sequence of types $i = i_0, i_1, \ldots, i_l = j$ such that $m_{i_{\nu-1} i_\nu}$ is odd for all $\nu = 1, \ldots, l$. The claim of the proposition about the parameters of the irreducible spherical diagrams is now obvious. □

The following theorem is a good illustration of the strength of preceding results. It states that an isomorphism of a thick, spherical building is determined by its restriction to a surprisingly small subset. This makes precise the intuitive statement that spherical buildings are rigid objects.

If Δ is any chamber complex, $C \in \mathcal{C}(\Delta)$, and t a natural number, we set

$$\mathcal{E}_t(\Delta,\ C_0) = \mathcal{E}_t(C_0) := \{C \in \mathcal{C}(\Delta)\colon\ \operatorname{cod}(C \cap C_0) \leqslant t\}.$$

In particular, $\mathcal{E}_1(C_0)$ consists of C_0 and all chambers adjacent to C_0.

5.2.11. THEOREM. *Let Δ be a thick building of spherical type, $\Sigma \subset \Delta$ an apartment, C_0 a chamber of Σ. Any isomorphism $\varphi\colon\ \Delta \to \Delta'$ is determined by its restriction to $\Sigma \cup \mathcal{E}_1(C_0)$.*

PROOF. Without loss of generality we may assume that $\Delta = \Delta'$ and φ restricted to $\Sigma \cup \mathcal{E}_1(C_0)$ is the identity. Consider the morphism $\omega := \mathrm{op}_\Sigma \circ \rho_{\Sigma, C_0}\colon\ \Delta \to \Sigma$. As a preliminary step to the proof, we claim that $A \leftrightarrow \omega A$ for all simplices A. Since the types are opposite, it suffices to verify the claim for chambers. But

$$d(C, \omega C) \geqslant d(\rho C, \rho\omega C) = d(\rho C, \omega C) = d(\rho C, \mathrm{op}_\Sigma \rho C) = \operatorname{diam} \Delta.$$

We now show by induction on $d(C_0, C)$ that $\varphi|_{\mathcal{E}_1(C)} = \mathrm{id}$. Choose a chamber C' adjacent to C such that $d(C_0, C') = d(C_0, C) - 1$, and set $F = C \cap C'$. By Theorem 5.2.9, there exists a chamber D in $\operatorname{St} \omega F$ such that $C' \leftrightarrow D \leftrightarrow C$. By the induction hypothesis, $\varphi|_{\operatorname{St} F} = \mathrm{id}$. It follows that

$$\varphi \circ (\mathrm{pr}_{\omega F}|_{\operatorname{St} F}) = \mathrm{pr}_{\varphi\omega F} \circ (\varphi|_{\operatorname{St} F}) = \mathrm{pr}_{\omega F} \circ (\varphi|_{\operatorname{St} F}) = \mathrm{pr}_{\omega F}|_{\operatorname{St} F}.$$

Since $\mathrm{pr}_{\omega F}\colon \operatorname{St} F \to \operatorname{St} \omega F$ is bijective, by 5.2.8, it follows that $\varphi|_{\operatorname{St} \omega F} = \mathrm{id}$; in particular, $\varphi D = D$. Now let F_C be an arbitrary panel of C. Let $F_{C'}, F_D$ be the panels of C', resp., D of the same type, so that $F_C \leftrightarrow F_D \leftrightarrow F_{C'}$. Since $\varphi|_{\operatorname{St} F_{C'}} = \mathrm{id}$, we conclude, by applying the last conclusion twice, first that $\varphi|_{\operatorname{St} F_D} = \mathrm{id}$, and second that $\varphi|_{\operatorname{St} F_C} = \mathrm{id}$, as desired. □

5.3. *Roots in spherical buildings and the Moufang property*

In this subsection, we only consider buildings of spherical type. We recall the notation $\operatorname{conv} \mathcal{A}$ for the convex hull of a subset $\mathcal{A}$ of a chamber system, and $\Gamma(\Lambda) = \Gamma_\Delta(\Lambda)$ for the intersection of all convex chamber subcomplexes containing the subset Λ of the chamber complex Δ.

In Sections 2.3 and 2.4, we have defined a *root* in a Coxeter–Tits complex to be the image of a folding. We have made a detailed study of foldings, their relationship to convexity, and their use for characterizing Coxeter complexes. For the purposes of this subsection, we only have to recall the following description of roots: If Σ is a Coxeter–Tits complex, and C, C' are adjacent chambers, then the set of chambers

$$\{D \in \mathcal{C}(\Sigma)\colon\ d(C, D) < d(C', D)\},$$

together with all their faces, form a root Φ, and every root is obtained in this way. If s is the reflection interchanging C and C', then the fixed point set of s is precisely the set of all panels which are contained in exactly one chamber of Φ; this is the boundary $\partial\Phi$ of Φ. Remember that a root in a building is by definition a root of one of its apartments.

5.3.1. PROPOSITION. *Let Φ be a root in a spherical building, F a panel in $\partial\Phi$, and C a chamber containing F, not contained in Φ. Then $\Gamma(\Phi \cup \{C\})$ is an apartment. The mapping*

$$\mathcal{C}^{\#}(F) \to \{\Sigma\colon\ \Sigma \text{ an apartment},\ \Sigma \supset \Phi\}, \quad C \mapsto \Gamma(\Phi \cup \{C\}),$$

where $\mathcal{C}^{\#}(F) = \{C \in \mathcal{C}(\Delta)\colon\ C \supset F,\ C \notin \Phi\}$, is a bijection.

PROOF. We first show that $\Phi = \Gamma(F, D)$ for a certain chamber $D \in \mathcal{C}(\Phi)$ (which is in fact unique). To this end, denote by C_F the unique chamber of Φ containing F, choose any apartment $\Sigma \supset \Phi$, let C'_F be the other chamber of Σ containing F, and set $D := \operatorname{op}_\Sigma C'_F$. Then Φ is the root determined by the pair (C_F, C'_F), and in particular $D \in \Phi$ since

$$d(C'_F, D) = \operatorname{diam}\Delta > d(C_F, D).$$

Notice that $\Gamma(F, D) = \Gamma(C_F, D)$ since $C_F = \operatorname{pr}_F D$ (cf. Subsection 5.2). Now let $E \in \Phi$ be an arbitrary chamber. Then

$$\operatorname{diam}\Delta = d(C'_F, D) = d(C'_F, E) + d(E, D).$$

Furthermore $d(C'_F, E) = d(C_F, E) + 1$, since $E \in \Phi$. Thus

$$d(C_F, E) + d(E, D) = \operatorname{diam}\Delta - 1 = d(C_F, D),$$

and therefore $E \in \operatorname{conv}\{C_F, D\}$. That is, $\mathcal{C}(\Phi) = \operatorname{conv}\{C_F, D\}$, as claimed.

Now let $C \in \mathcal{C}^{\#}(F)$ be given. Then $d(C, D) > d(C_F, D)$, that is $d(C, D) = \operatorname{diam}\Delta$, since $C_F = \operatorname{pr}_F D$, the unique chamber containing F and closest to D in the whole complex Δ. Thus $\operatorname{conv}\{C, D\}$ is the set of chambers of an apartment $\Gamma(C, D)$ which contains $\operatorname{conv}\{C_F, D\}$ and thus contains Φ. In particular, $\Gamma(C, D) = \Gamma(\Phi \cup \{C\})$. We have already seen that every apartment $\Sigma \supset \Phi$ is of the form $\Gamma(\Phi \cup \{C\})$.

Finally, C can be recovered from Σ as the unique chamber in Σ, containing F, and distinct from F. □

5.3.2. DEFINITION. Let Φ be a root in a spherical building Δ. The *root group* U_Φ is defined as the group of all automorphisms of Δ fixing all chambers having at least one panel in $\Phi \setminus \partial\Phi$:

$$U_\Phi = \{g \in \operatorname{Aut}\Delta\colon\ gC = C \text{ for all } C \in \operatorname{St}_\Delta F,\ F \in \Phi \setminus \partial\Phi\}.$$

5.3.3. PROPOSITION. *Suppose that the diagram M has no isolated node* (*no factor* A_1). *Then any root group U_Φ operates freely on the apartments Σ containing Φ. That is,*

$$g \in U_\Phi,\ \Sigma \supset \Phi,\ g\Sigma = \Sigma \Rightarrow g = \mathrm{id}.$$

PROOF. The assumption on M implies that Σ possesses a chamber C having no panel in $\partial\Phi$, as we shall see in a moment. An element g in U_Φ by definition fixes all neighbours of C; in the notation used earlier it fixes $\mathcal{E}_1(C)$. If g furthermore fixes Σ, then it in fact is the identity on Σ, and from Proposition 5.2.11 it follows that $g = \mathrm{id}_\Delta$. The claim about the existence of C is verified as follows. Choose any panel $F_0 \in \partial\Phi$ and consider the chamber $C_0 \supset F_0$, $C_0 \in \Phi$. Let $s_j = s_j(C_0)$, $j \in I$, the reflection in the j-panel of C_0. Let i be the cotype of F_0. Choose $j \neq i$ in I which is connected to i, and set $C := s_j(C_0)$. Assume that the k-panel of C is in $\partial\Phi = \operatorname{Fix}(s_i)$, for some $k \in I$. Then $s_i s_j\langle s_k\rangle = s_j\langle s_k\rangle$, that is $s_i s_j = s_j s_k$, necessarily $i = k$, a contradiction. □

5.3.4. DEFINITION. Let Δ be a building of spherical type M, where M has no isolated nodes. Δ is called *Moufang* if, for each root Φ, the root group U_Φ acts transitively (and hence sharply transitively) on the apartments containing Φ.

It is a basic fact about spherical buildings that the Moufang property automatically holds if Δ is thick and has no factors of rank $\leqslant 2$.

5.3.5. THEOREM. *Let Δ be a thick building of spherical type M; assume that M has no components of rank* 1 *or* 2. *Then Δ is Moufang.*

PROOF. Let Φ be a root and Σ, Σ' two apartments containing Φ. From the assumption on the diagram M it follows that there exists a chamber $C \in \Phi$ with $\mathrm{cod}(C \cap \partial\Phi) \geqslant 3$. That is, no codimension 2 face of C is in $\partial\Phi$. The argument is similar to the proof of 5.3.3: start with a chamber $C_0 \in \Phi$ having a panel, say of type i, in $\partial\Phi$, and consider $C' := s_j s_k C_0$ in some $\Sigma \supset \Phi$, where j is connected to i, and k is connected to i or j. Assuming that the face of cotype $\{l, m\}$ of C' is fixed by s_i gives an equality $s_i s_j s_k \langle s_l, s_m\rangle = s_j s_k \langle s_l, s_m\rangle$ of cosets in W, that is, $s_i s_j s_k \in s_j s_k \langle s_l, s_m\rangle$. A straightforward reasoning, using shortest coset representatives and Tits' solution of the word problem for Coxeter groups, Theorem 2.5.2, shows that this is contradictory.

Now define an isomorphism of chamber systems

$$\varphi\colon\ \mathcal{E}_2(C) \cup \Sigma \to \mathcal{E}_2(C) \cup \Sigma'$$

as follows: $\varphi|_{\mathcal{E}_2(C)} = \mathrm{id}$, $\varphi|_\Sigma = \alpha_{C,C,\Sigma,\Sigma'}$, the unique isomorphism $\Sigma \to \Sigma'$ fixing C. We know that $\alpha_{C,C,\Sigma,\Sigma'}$ is the identity on all of $\Sigma \cap \Sigma'$, in particular on Φ. Therefore, in order to see that φ is well defined, it is sufficient to know that $\mathcal{E}_2(C) \cap \Sigma \subseteq \Phi$. If not, then there exists a chamber $D \in \Sigma \setminus \Phi$ with $\mathrm{cod}(C \cap D) \leqslant 2$. But then

$$C \cap D \in \Phi \cap \Phi' = \partial\Phi,$$

which contradicts $\mathrm{cod}(C \cap \partial\Phi) \geqslant 3$. By the Extension Theorem 7.1.1, φ extends to an automorphism of Δ. It now has to be shown that φ is in U_Φ, that is, φ fixes the stars of all panels in $\Phi \setminus \partial\Phi$. We shall not do this here, but refer the reader to Tits [1977], Lemma 4. We wish to point out that this is the most complicated part of the proof (though much less complicated than the proof of the Extension Theorem itself). □

The last theorem is completely wrong in the rank two case, that is, for generalized m-gons. This is already seen for $m = 3$, i.e. in the case of projective planes. Although generalized polygons are not our proper subject, we want to at least state the following famous theorem by Tits which says that the diameter of a Moufang polygon is far from being arbitrary.

5.3.6. THEOREM. *Thick Moufang generalized m-gons exist only for* $m = 2, 3, 4, 6, 8$.

For a proof, see Tits [1976, 1979] or Weiss [1979].

In the case of projective spaces, the complete freedom of dimension 2 disappears in higher dimensions: spaces of dimension $\geqslant 3$ automatically are Desarguesian and thus defined over a division ring. The following proposition in a sense generalizes this behaviour to arbitrary spherical buildings in an a priori way, i.e. without assuming the full classification.

5.3.7. PROPOSITION. *Any star in a Moufang building also is Moufang.*

For a proof, see Tits [1977], Lemma 5.

The following theorem which was the principal objective of Tits [1977] is an immediate consequence of the last two results.

5.3.8. THEOREM. *There exists no thick building of type* H_3 *or* H_4.

Of course this uses only the case $m = 5$ of Theorem 5.3.6 for which a special, comparatively short proof has been given in Section 4.6 of Tits [1973/76].

For the remainder of this section, Δ is supposed to be a spherical Moufang building. We fix an apartment Σ, and denote by $R = R(\Sigma)$ the set of all roots of Σ. We shall now consider in more detail the system of root groups $(U_\Phi, \Phi \in R)$. This will lead to a certain refinement of the Bruhat decomposition, and will prepare the ground for a certain class of so-called 'labellings' of Moufang buildings which will be used in Section 7. For roots $\Phi, \Psi \in R$ such that $\Phi \neq \pm\Psi$, we set

$$[\Phi, \Psi] := \{\Pi \in R\colon\ \Pi \supseteq \Phi \cap \Psi\},$$

$$]\Phi, \Psi[\ := [\Phi, \Psi] \setminus \{\Phi, \Psi\}.$$

The following lemma reduces the determination of $[\Phi, \Psi]$ to the case of buildings of rank 2, and in particular clarifies why $[\Phi, \Psi]$ should be considered as an 'interval', as is suggested by the notation.

5.3.9. LEMMA. *Let* $\Phi \neq \pm\Psi$, *and let* A *be any simplex of codimension* 2 *in* $\partial\Phi \cap \partial\Psi$. *Then* $\Phi \cap \mathrm{St}_\Sigma A$, $\Psi \cap \mathrm{St}_\Sigma A$ *are nonopposite roots in* $\mathrm{St}_\Sigma A$, *and*

$$\Pi \mapsto \Pi \cap \mathrm{St}_\Sigma A$$

is a bijection of $[\Phi, \Psi]$ *onto* $[\Phi \cap \mathrm{St}_\Sigma A, \Psi \cap \mathrm{St}_\Sigma A]$.

Since $\mathrm{St}_\Sigma A$ is an ordinary polygon, $[\Phi \cap \mathrm{St}_\Sigma A, \Psi \cap \mathrm{St}_\Sigma A]$ is an interval in an obvious way; and we can write

$$[\Phi, \Psi] = \{\Phi_1 = \Phi, \Phi_2, \ldots, \Phi_k = \Psi\}$$

such that Φ_t is the root containing C_{t-1}, and not containing C_t, for a unique geodesic $C_0, C_1, \ldots, C_{k+1}$ in $\mathrm{St}\, A$ (in particular, $k \leqslant \operatorname{diam} \mathrm{St}\, A$).

In the following theorem, $[U_\Phi, U_\Psi]$ as usual denotes the commutator of the two groups U_Φ, U_Ψ.

5.3.10. THEOREM. *The root groups in a Moufang building of spherical type have the following properties, where Φ, Ψ are any two roots with $\Phi \neq \pm\Psi$:*

(a) $[U_\Phi, U_\Psi] \subseteq \langle U_\Pi\colon \Pi \in]\Phi, \Psi[\rangle$,

(b) $\langle U_\Pi\colon \Pi \in [\Phi, \Psi]\rangle = U_{\Phi_1} U_{\Phi_2} \dots U_{\Phi_k}$, *where* $\Phi_1 = \Phi, \Phi_2, \dots, \Phi_k = \Psi$ *is as above. In particular,* $U_{\Phi_1} U_{\Phi_2} \dots U_{\Phi_k} = U_{\Phi_k} U_{\Phi_{k-1}} \dots U_{\Phi_1}$.

This is Theorem 6.12 in Ronan [1989], where the reader may also find a proof. Notice that (b) is an easy consequence of (a).

5.3.11. DEFINITION. Fix a chamber $C_0 \in \Sigma$. For $w \in W$, define $U_w := U_{\Phi_1} U_{\Phi_2} \dots U_{\Phi_l}$, where $f = i_1 i_2 \dots i_l$ is any reduced word representing w, and Φ_t is the unique root containing C_{t-1}, but not C_t, where $(C_0, \dots, C_l)$ is the unique gallery of type f in Σ, starting at C_0.

From part (b) of the last theorem it follows that U_w is indeed independent of the choice of f, using Theorem 2.5.2 or Proposition 2.1.8 as usual.

The next theorem can be considered as a refinement of the Bruhat decomposition. In the present axiomatic theory, it is, conversely, used as the main lemma for the following Theorem 5.3.13 which says that the group generated by all root groups actually admits a Tits system. Before formulating the theorem, we recall that we have fixed an apartment Σ containing C_0; the chamber wC_0 occurring in part (a) is understood to be an element of this Σ, and is defined via the identification of W with $\operatorname{Aut}\Sigma$ defined by C_0 (or the action of W on Σ 'by galleries'). The Φ_t's are as in the last definition. Part (a) of the following theorem is proved using the commutator relation in part (a) of the previous theorem. Part (b) is an immediate 'continuation along geodesics' of the sharp transitivity of root groups U_Φ on the stars of panels in the boundary of Φ (Proposition 5.3.3); it will also follow from the proof of 5.3.16(b) below.

5.3.12. THEOREM.

(a) *The U_w are subgroups of* $\operatorname{Aut}\Delta$.

(b) *Given a chamber $C \in \Delta$ and any reduced expression $i_1 \dots i_l$ of $w := \Delta(C_0, C)$, there are unique elements $v_t \in U_{\Phi_t}$, $t = 1, \dots, l$, with $C = v_1 v_2 \dots v_l(wC_0)$. That is, for given f, the representation of an element of U_w as a product over the U_{Φ_t} is unique, and U_w acts sharply transitively on the set of chambers at δ-distance w from C_0.*

Notice that if our building is already known to come from a Tits system (B, N) in a group G, then the last theorem can be considered as a refinement of the Bruhat decomposition: each double coset BwB is a union of cosets uwB, for unique group elements $u \in U_w$.

The next theorem, which is an easy consequence of the results stated so far, says that any thick, spherical building of irreducible type and rank $\geqslant 3$ is associated with a Tits system in a group. Historically, this result had first been obtained as a consequence of the full classification of such buildings.

5.3.13. THEOREM. *Consider a thick Moufang building Δ with a fixed apartment Σ. The group $G = \langle U_\Phi\colon\ \Phi \in R\rangle$ generated by all root groups of Σ acts strongly transitively on Δ and thus admits a Tits system.*

PROOF. In view of the transitivity properties of the $U_w \subseteq G$ (Theorem 5.3.12(b)) it suffices to show that G_Σ induces the full automorphism group of Σ. This follows from the following lemma. □

5.3.14. LEMMA. *For any root Φ and any element $u \in U_\Phi \setminus \{1\}$, there exists an element $m(u) \in U_{-\Phi}uU_{-\Phi}$ mapping Σ onto itself, and inducing the reflection in $\partial\Phi$.*

For a proof, see Ronan [1989], Chapter 6.4.

Theorem 5.3.12 can be considered as a sort of parametrization of the building in terms of the root groups U_Φ. This aspect will be further pursued in Section 7.2, when we shall introduce the so-called labellings and blueprints. In order to prepare this, we shall now draw another consequence of the last lemma which will allow us to replace the Weyl group elements w occurring in the refined Bruhat decomposition by certain representatives $n(w)$ which actually act on the whole building and not only on the specified apartment. As a consequence, we may replace each root group occurring in 5.3.12(b) by a conjugate 'fundamental root group' $U_i := U_{s_i}$ and arrive at a parametrization by the fundamental root groups alone.

5.3.15. LEMMA. *Keep the assumptions and notations of* 5.3.12 *and* 5.3.14. *For each $i \in I$, select an element $e_i \in U_i \setminus \{1\}$, set $n_i = m(e_i)$. For any two types i, j, the equality $n_in_jn_i \ldots = n_jn_in_j \ldots$ (m_{ij} factors in each case) holds. Consequently, for any $w \in W$, there is a unique element $n(w) \in G$ such that $n(w) = n_{i_1} \ldots n_{i_l}$ for any reduced representation $i_1 \ldots i_l$ of w.*

For a proof, see Ronan [1989], Appendix 1.

For purposes of reference, we explicitly record the parametrization of a Moufang building as given by 5.3.12 combined with 5.3.15.

5.3.16. PROPOSITION. *Keep the assumptions and notation of* 5.3.12 *to* 5.3.15.

(a) *Any chamber of Δ can be written in the form $un(w)C_0$, for unique elements $w \in W$ and $u \in U_w$.*

(b) *For any geodesic $(C_0, C_1, \ldots, C_l; i_1, \ldots, i_l)$, there are unique elements $u_t \in U_{i_t}$ such that $C_t = u_1n_{i_1}u_2n_{i_2} \ldots u_tn_{i_t}C_0$, $t = 1, \ldots, l$.*

PROOF. The proof is completely straightforward. We give the proof of (b) mainly to make explicit the relation between the elements u_t in the 'fundamental root groups' U_{i_t} used here and the elements v_t used in the previous Theorem 5.3.12(b). For $t = 1, \ldots, l$, set $w_t := s_{i_1}s_{i_2} \ldots s_{i_t}$; then $n(w_t) = n_{i_1}n_{i_2} \ldots n_{i_t}$. Find the desired elements $u_t \in U_{i_t}$ inductively as follows: $u_1 \in U_{i_1}$ is uniquely defined by $u_1n_{i_1}C_0 = C_1$. (Remember that U_{i_1} acts sharply transitively on the i_t-panel of C_0.) If $g_t := u_1n_{i_1}u_2n_{i_2} \ldots u_tn_{i_t}$ is already defined, consider $C' := g_t^{-1}(C_{t+1})$ which is i_{t+1}-adjacent to C_0 and define

$u_{t+1} \in U_{i_{t+1}}$ by $u_{t+1}n_{i_{t+1}}C_0 = C'$. Notice that the roots Φ_t of 5.3.12 are given by $\Phi_t = w_{t-1}\Phi_{i_t}$ (where $w_0 = \mathrm{id}$), and thus $U_{\Phi_t} = n(w_{t-1})U_{i_t}n(w_{t-1})^{-1}$. Therefore, the elements

$$v_1 = u_1, \ldots, v_t = n(w_{t-1})u_t n(w_{t-1})^{-1}$$

meet the requirements of 5.3.12: $u_1 u_2 \ldots u_t w_t C_0 = C_t$. □

The parametrization by the fundamental root groups $(U_i)_{i \in I}$ as given in 5.3.16(b) is identical to the 'natural labelling of a Moufang building' used in Section 7.2.

Notes to Section 5

The gate property. The first half of Subsection 5.1 is a direct continuation of 2.3; it is basically taken from Tits [1974], 2.18 to 2.28. The second half, dealing with projections and the gate property, covers the results of *loc. cit.*, 2.29 to 2.35, 3.18 to 3.20 and basically also 12.12 to 12.15. As mentioned above, our treatment here is different from the one by Tits, leading to perhaps more straightforward proofs and, in the case of Theorem 5.1.9 to a more precise statement.

The assumptions in Theorem 5.1.11 can be considerably weakened; see the original paper Scharlau [1985a]. B. Mühlherr [1994] has observed that there are finite gated chamber systems which are not buildings, as a consequence of constructions in Grünbaum [1971]. This improves the class of examples given at the end of Scharlau [1985a], where infiniteness cannot be avoided. It is well known (and obvious if one looks at the proof) that the gate property is the key to the theorem of Solomon and Tits on the homotopy type of a spherical building (see Chapter 20). It even implies the property of 'shellability' which has been widely studied for (flag complexes of) partially ordered sets, and also for general simplicial complexes, in the context of the Stanley–Reisner ring and the Cohen–Macaulay property. Stanley [1983] (among many others) is a general reference for these topics, Björner [1984] treats the relation to buildings. In Scharlau [1985b], a short axiomatic treatment of the relation between shellings and projections is given, avoiding any particular properties of buildings.

The gate property, or rather its topological consequences also are of interest for certain subcomplexes of affine buildings. Such subcomplexes are used by Abels and Brown [1987], Brown [1987], Abels and Abramenko [1993], Abels [1991c] and Abramenko [1994a] to study the finiteness properties of certain discrete subgroups of p-adic groups. Here, finiteness properties are understood in the sense of the hierarchy of finiteness conditions (F_n), $n = 1, 2, \ldots$, introduced for the first time in Serre [1971], based on earlier topological work by C.T.C. Wall.

Convex hulls. From the modern, simplicial point of view of buildings, the definition of the convex hull of a set of chambers, and also the slightly more complicated definition of the full convex hull of an arbitrary subset, are very natural as part of a technical machinery for analyzing the structure of the whole complex. We know that this aspect is indeed important, in connection with stars, the gate property, and opposites. But this is not the only aspect of convex hulls. We take the opportunity to point out that this concept (at least for 2-element subsets of a building) goes back to the early days of the theory, and thus is historically primary related to the incidence-geometrical interpretation of buildings. Recall Tits [1956], dealing with 'short' chains between two objects, more generally two flags, of a building (see Section 3.4). In that paper, the task of analyzing 'shortest' chains, their types and uniqueness was not pursued. In Tits [1959/62], however, the subgeometry of elements 'determined by' two elements A and B was defined rigorously as the fixed point set of the stabilizer group of A and B in a chamber-transitive group G acting on the building. It is easily seen that this set coincides with the analogous set formed in an apartment containing A and B, thus replacing G by the Weyl group W. (Cf. also Tits [1974], 12.7.) Theorem 5.1.10(b) now shows that the subgeometry 'determined by' A and B actually is equal to the full convex hull $\Gamma(A, B)$. The following example indicates a characterization of the convex hull involving only the elementary notion of a chain. Let x, y be objects of a building of type A_n, that is, subspaces of some projective space P. Short and particularly natural chains between x and y are

$$x — x + y — y \quad \text{and} \quad x — x \cap y — y$$

(assuming that $x + y \neq P$, resp., $x \cap y \neq \varnothing$ so that the middle term actually is a member of the building). On the basis of 5.1.10, it is readily worked out that $\Gamma(x, y)$ has precisely the vertices $x,\ y,\ x + y,\ x \cap y$. The join $z = x + y$ and the intersection $z = x \cap y$ are uniquely 'determined by' x and y also in another sense: The chain $x — z — y$ is the only chain of its type, joining x to y. This last example is a special case of the following general result (Property (D1) on p. 188 of Tits [1959/62]): All vertices of the convex hull $\Gamma(x, y)$ of any two vertices in a building are obtained as the vertices occurring in the 'minimal' chains joining x to y. Here, a minimal chain is a chain of flags which is uniquely determined by its type, origin, and endpoint.

The subject of convex hulls in the spirit of Tits [1956, 1959/62] has been taken up in the book Freudenthal and De Vries [1969], §§72–74 (but, to the best of our knowledge, in no other publication). They use the above definition from Tits [1959/62] and introduce the term 'covariant' of A and B for the set of flags 'determined by' A and B. Complementing the last section of Tits [1959/62], they present a detailed work-out of the classification of pairs $\{x, y\}$ of objects in a F_4-building, and of the structure of their convex hulls. The classification of all pairs of course amounts to listing all double cosets $W_J w W_K$. (Work inside an apartment, assume x is a vertex of the base chamber and y a vertex of the chamber corresponding to $w \in W$, set $J = \operatorname{cotype} x$, $K = \operatorname{cotype} y$.) The structure of $\Gamma(x, y)$ could be obtained on the basis of Proposition 5.1.10, extending the algorithm from 2.5.3 for listing the double cosets. Freudenthal and De Vries use a different method. Namely, they compute the convex hull as an intersection of half-spaces and hyperplanes inside the vector-space-realization of the Coxeter–Tits complex. It follows from 5.1.3 that $\Gamma(A, B)$ (as defined above) can actually be obtained in such a way.

We have seen that full convex hulls of pair of elements are a relatively well understood class of subcomplexes (usually of strictly smaller dimension) of Coxeter–Tits complexes. By work of Abramenko [1994b], one can say the same about walls in spherical Coxeter–Tits complexes. For more general results about 'gated subcomplexes', see Mühlherr [1994].

Geometry of spherical buildings. Our treatment of opposite objects in spherical buildings again follows Tits [1974], 2.37 to 2.42 and 3.22 to 3.36. As we have already seen in Theorems 5.2.10 and 5.2.11, the existence of opposite simplices together with the gate property of stars (existence of projection maps) has strong implications on how a spherical building is built up from its lower-dimensional (e.g., two-dimensional) parts. The Extension Theorem to be treated in Subsection 7.1 will be a direct continuation of these considerations which in this way become an essential ingredient of the final classification of all thick irreducible buildings of spherical type. The notion of opposite chambers with all its consequences carries over to the so-called *twin buildings* which are pairs of buildings Δ, Δ' together with a 'co-distance function' $\delta^*\colon \mathcal{C}(\Delta) \times \mathcal{C}(\Delta') \to W$. Such objects arise from Kac–Moody groups; they have pairs of 'opposite' Borel subgroups which are nonconjugate. See Tits [1984/85, 1987, 1989, 1990a, 1990/92].

Moufang buildings. The notion of a (spherical, of course) Moufang building is briefly introduced in the Addenda of Tits [1974]; the fundamental Theorem 5.3.5 is stated there without proof. This theorem is the main objective of Tits [1977] where details and full proofs are given. In particular, the reader finds there a complete proof of a lemma which says that the automorphism φ whose existence easily follows from the Extension Theorem actually lies in the root group. This Lemma 4 of *loc. cit.* relies on another rather tricky and special Lemma 2 which studies the numerical function of chambers $C \mapsto \operatorname{cod}(C \cap \partial\Phi)$ along certain galleries. A further nontrivial result which Tits uses in his proof is that in an irreducible building every root Φ contains chambers having no vertices in the boundary $\partial\Phi$. In our (sketch of a) proof of 5.3.5, we avoid using this general result by observing that a chamber C with $\operatorname{cod}(C \cap \partial\Phi) \geqslant 3$ suffices and is readily found by a direct argument. A similar remark already applies to our proof of 5.3.3. Proposition 5.3.1 is already contained in Tits [1974], 3.27. The second half of Subsection 5.3 dealing with the system of all root groups of one apartment, their commutator properties etc. is in a sense historically anterior to the notion of a Moufang building. Already in the theory of Chevalley groups G (see also the Notes to Section 4) one has a certain family of 'root subgroups' $(U_\alpha)_{\alpha \in R}$, where this time R is the root system of the simple Lie algebra defining G. Their commutator properties and certain consequences there of are quite classical; see, e.g., Carter [1972], Chapters 4, 5, 6 and 12. See Bruhat and Tits [1972] for an axiomatic treatment of so called 'root data' in a group. It is not obvious from the definitions (even, say, in the general linear group) that these 'classical' root groups are the same as the ones defined above. This will only be seen after one has also derived from the axioms of root data the analogue of Theorem 5.3.9 (refined Bruhat decomposition). For general reductive algebraic groups and their buildings, it is proved in Tits [1974], 5.6, that the 'structural' root groups coincide with the geometrical ones.

The Moufang property in a building corresponds to the notion of a split Tits system or split BN-pair in a group. In this language the results 5.3.9 to 5.3.13 have been treated in Ronan and Tits [1987]. It should be mentioned that in the rank 2 case, the notion of a split BN-pair is older (and in a sense more important, since it is a substantial restriction). We mention in particular the famous papers Fong and Seitz [1973/74] where the finite Moufang polygons are classified. More recently, Tits [1987] has given a set of axioms for root groups also in the case of non-spherical diagrams; they in particular cover the properties of 'split Kac–Moody groups' and are closely related to the subject of twin buildings mentioned above. For these, one can also naturally define a Moufang property, and the basic result that this property automatically holds in the irreducible case of rank $\geqslant 3$ carries over with a similar proof as in the spherical case; see Tits [1990a].

6. Buildings and covering theory

Introduction

This section deals with the question, already mentioned in the introduction to Section 3, of characterizing buildings among all geometries of Coxeter type. The main results on this problem are contained in the paper Tits [1981a] 'A Local Approach to Buildings'. This article was already the basis for Section 3.3 above. It is of fundamental importance for the theory and applications of buildings for at least two different reasons. First, it contains more or less concluding results in a direction which was sketched but never fully worked out in the early papers Tits [1956, 1957, 1959/60, 1959/62]. Secondly, it was the starting point and the principal technical background for the study of building-like geometries initiated by Buekenhout, Kantor, Timmesfeld, and others in connection with finite group theory. Some of the technically involved parts of the 'Local Approach' are written in a very condensed form. To our best knowledge an easily accessible and fully worked out presentation of this material has not appeared so far. Therefore, we have chosen to present here a detailed work-out of parts of Tits' paper (of those parts which have not already been covered in Section 3). The principal concepts and results consist of a covering theory for chamber systems, in particular, chamber systems of Coxeter type, and a characterization of buildings among all simply connected chamber systems of Coxeter type.

In the last two subsections we deal with two more specialized topics where the results can partly be considered as applications of the general theory developed before. In Section 6.4, we finish the characterization of spherical buildings by geometrical axioms (cf. Section 3.4). In Section 6.5, we deal with certain '*exceptional buildings*' (of non-spherical type, but admitting finite, flag-transitive quotients), and with the question of amalgamating groups acting transitively on a chamber system from the corresponding parabolic subgroups.

6.1. *Coverings of chamber systems*

6.1.1. DEFINITION. Let $\mathcal{C}$ and $\mathcal{C}'$ be chamber systems over the same type set I, and m be an integer, $0 < m < |I|$. A type preserving morphism α: $\mathcal{C}' \to \mathcal{C}$ is called an *m-covering* if it is surjective and if for every $J \subseteq I$ of cardinality $\leqslant m$, each J-star in $\mathcal{C}'$ is mapped bijectively onto a J-star in $\mathcal{C}$.

6.1.2. LEMMA. *Let α be an m-covering for some m (i.e. a 1-covering). Fix chambers $C_0 \in \mathcal{C}$, $C_0' \in \mathcal{C}'$ such that $\alpha C_0' = C_0$. Then for any gallery $\mathbf{C}$ in $\mathcal{C}$ starting at C_0 there is a single lifting $\mathbf{C}'$ of $\mathbf{C}$ to $\mathcal{C}'$, starting at C_0'.*

'Lifting' of course means that $\alpha C' = C$. This lemma immediately implies the following: If $\mathcal{C}$ is connected, and if β: $\mathcal{C}'' \to \mathcal{C}$ is another m-covering, together with an element $C_0'' \in \mathcal{C}''$ such that $\beta C_0'' = C_0$, then there is at most one factorization of α through β, i.e. a morphism φ: $\mathcal{C}' \to \mathcal{C}''$ such that $\varphi C_0' = C_0''$ and $\alpha = \beta \circ \varphi$. If $\mathcal{C}'$ is connected, and if a factorization exists for every such β, then α: $\mathcal{C}' \to \mathcal{C}$ or just $\mathcal{C}'$ is called a *universal m-covering* of $\mathcal{C}$. It is unique up to (a unique) isomorphism in the category of 'pointed chamber systems' ($\mathcal{C}$, $C_0 \in \mathcal{C}$).

Before we can prove the existence of universal m-coverings, we first have to talk about homotopy of galleries and about the fundamental group. Two galleries $\boldsymbol{C}$ and $\boldsymbol{D}$ in a chamber system $\mathcal{C}$ are called *elementary m-homotopic* if $\boldsymbol{D}$ is obtained from $\boldsymbol{C}$ by replacing a subgallery which is entirely contained in a star of rank $\leqslant m$ by another subgallery (necessarily with the same origin and endpoint) contained in the same star. $\boldsymbol{C}$ and $\boldsymbol{D}$ are called *m-homotopic* if they can be connected by a *m-homotopy* which is by definition a sequence of elementary m-homotopies. The following lemma is trivial, but fundamental.

6.1.3. LEMMA. *If α: $\mathcal{C}' \to \mathcal{C}$ is an m-covering, and if $\boldsymbol{C}'$ and $\boldsymbol{D}'$ are liftings to $\mathcal{C}'$ of galleries $\boldsymbol{C}$ and $\boldsymbol{D}$ in $\mathcal{C}$, with common origin $C_0' \in \mathcal{C}'$, then any m-homotopy from $\boldsymbol{C}$ to $\boldsymbol{D}$ can be 'lifted' to a single m-homotopy from $\boldsymbol{C}'$ to $\boldsymbol{D}'$. In particular, $\boldsymbol{C}'$ and $\boldsymbol{D}'$ have the same endpoint.*

6.1.4. DEFINITION. The *homotopy group* $\pi(\mathcal{C}, C_0)$ of a pointed chamber system $(\mathcal{C}, C_0)$ is the group of all m-homotopy classes $[\boldsymbol{C}]$ of closed galleries in $\mathcal{C}$, based at C_0.

Of course, the homotopy group depends on m, but we shall suppress this in the notation. The composition in $\pi(\mathcal{C}, C_0)$ comes from composing galleries, and the existence of inverses relies on the fact that $\boldsymbol{C}^{-1}\boldsymbol{C}$ obviously is null-homotopic (for any m, i.e. for $m = 1$). If $\mathcal{C}$ is connected, then any two homotopy groups $\pi(\mathcal{C}, C_1)$ and $\pi(\mathcal{C}, C_2)$ are isomorphic. A morphism of pointed chamber systems α: $(\mathcal{C}', C_0') \to (\mathcal{C}, C_0)$ induces a homomorphism $\alpha_\star$: $\pi(\mathcal{C}', C_0') \to \pi(\mathcal{C}, C_0)$. The following lemma, which is well known from the covering theory of topological spaces, is the basis for constructing universal covers.

6.1.5. LEMMA. *Let α: $(\mathcal{C}', C_0') \to (\mathcal{C}, C_0)$ and β: $(\mathcal{C}'', C_0'') \to (\mathcal{C}, C_0)$ be m-coverings. Then a factorization φ: $(\mathcal{C}', C_0') \to (\mathcal{C}'', C_0'')$, $\alpha = \beta \circ \varphi$, exists if and only if* $\operatorname{Im}\alpha_\star \subseteq \operatorname{Im}\beta_\star \subseteq \pi(\mathcal{C}, C_0)$.

PROOF. The 'only if' part is obvious. The 'if' part follows readily from the unique lifting of galleries based at C_0 and of homotopies of such (6.1.2 and 6.1.3). □

Now we can prove

6.1.6. PROPOSITION. *Any connected chamber system $(\mathcal{C}, C_0)$ possesses a universal m-covering.*

PROOF. By the last lemma, it is sufficient to construct an m-covering

$$\alpha\colon (\widehat{\mathcal{C}}, \widehat{C}_0) \to (\mathcal{C}, C_0) \quad \text{such that} \quad \pi(\widehat{\mathcal{C}}, \widehat{C}_0) = 1.$$

In the following, we denote galleries in $\mathcal{C}$ by $G, H, \ldots$, by ΩG the endpoint and by $[G]$ the homotopy class of G. Let $\widehat{\mathcal{C}}$ be the set of all homotopy classes of galleries based at C_0, and define a chamber system structure on $\widehat{\mathcal{C}}$ by

$$[G] \overset{i}{\sim} [H] \Leftrightarrow \Omega[G] =: C \overset{i}{\sim} D := \Omega[H] \text{ and } [G] = [H \circ (C, D; i)].$$

The map $\alpha\colon \widehat{\mathcal{C}} \to \mathcal{C}$, $[G] \mapsto \Omega G$, is obviously a morphism. If H is of the form $G \circ K$, where K is contained in a star of rank $\leqslant m$, and furthermore $\Omega G = \Omega H$, then $[G] = [H]$. This shows that α is an m-covering. If G is any gallery of $\mathcal{C}$ based at C_0 and $\widehat{G}$ its unique lifting based at $\widehat{C}_0 := [(C_0)]$, then obviously $\Omega\widehat{G} = [G]$. If Γ is a closed gallery of $\widehat{\mathcal{C}}$ at $\widehat{C}_0$, and $G := \alpha\Gamma$, then $\widehat{G} = \Gamma$ and consequently $[G] = \Omega\Gamma = [(C_0)]$, i.e. G is null-homotopic. By 6.1.3, any homotopy from G to (C_0) can be lifted to a homotopy from Γ to $(\widehat{C}_0)$, and thus Γ is null-homotopic, as desired. □

We close this subsection with some standard results about lifting automorphisms to covers, in particular to the universal cover. The fundamental group is interpreted as a group of 'deck transformations'. These results are straightforward consequences of Lemma 6.1.5, and are familiar from the covering theory of topological spaces.

6.1.7. PROPOSITION. *Let $\varphi\colon (\widetilde{\mathcal{C}}, \widetilde{C}_0) \to (\mathcal{C}, C_0)$ be an m-covering, let g be an automorphism of $\mathcal{C}$ and $\widetilde{C} \in \varphi^{-1}(gC_0)$. There exists a* lifting *$\widetilde{g}$ of g to $\mathcal{C}$ (an automorphism $\widetilde{g} \in \mathrm{Aut}(\widetilde{\mathcal{C}})$ with $\varphi \circ \widetilde{g} = g \circ \varphi$) such that $\widetilde{g}\widetilde{C}_0 = \widetilde{C}$ if and only if*

$$\varphi_\star\pi(\widetilde{\mathcal{C}}, \widetilde{C}) = g_\star\varphi_\star\pi(\widetilde{\mathcal{C}}, \widetilde{C}_0).$$

In this case, the lifting is unique (for given $\widetilde{C}$).

In particular, the lifting of g always exists if $\widetilde{\mathcal{C}}$ is simply connected. We record this most important special case in a separate proposition.

6.1.8. PROPOSITION. *Let $\varphi\colon (\widetilde{\mathcal{C}}, \widetilde{C}_0) \to (\mathcal{C}, C_0)$ be a universal m-covering.*

(a) *The group of* deck transformations

$$\mathrm{Aut}(\widetilde{\mathcal{C}}, \varphi) := \{\widetilde{g} \in \mathrm{Aut}(\widetilde{\mathcal{C}})\colon \varphi \circ \widetilde{g} = \varphi\}$$

acts sharply transitively on the fibers of φ (inverse images of single chambers). It is canonically isomorphic to the fundamental group $\pi(\mathcal{C}, C_0)$.

(b) *For any subgroup $G \subseteq \mathrm{Aut}(\mathcal{C})$ the set of all liftings*

$$\widetilde{G} := \{\widetilde{g} \in \mathrm{Aut}(\mathcal{C})\colon \varphi \circ \widetilde{g} = g \circ \varphi \text{ for some } g \in G\}$$

is a subgroup of $\mathrm{Aut}(\widetilde{\mathcal{C}})$, and the map $\widetilde{G} \to G$, $\widetilde{g} \mapsto g$, is a surjective homomorphism with kernel $\mathrm{Aut}(\widetilde{\mathcal{C}}, \varphi)$. For any chamber $\widetilde{C} \in \widetilde{\mathcal{C}}$, the stabilizer $\widetilde{G}_{\widetilde{C}}$ is mapped isomorphically onto the stabilizer $G_{\varphi\widetilde{C}}$.

The isomorphism from $\pi(\mathcal{C}, C_0)$ onto $\mathrm{Aut}(\widetilde{\mathcal{C}}, \varphi)$ is obtained as follows. Let $[\boldsymbol{C}]$ be any element in $\pi(\mathcal{C}, C_0)$, represented by the closed gallery $\boldsymbol{C}$ based at C_0. Let $\widetilde{C}$ be the endpoint of the unique lifting of $\boldsymbol{C}$ to $\widetilde{\mathcal{C}}$ based at $\widetilde{C}_0$. Then $\widetilde{C} \in \varphi^{-1}(C_0)$, and the image of $[\boldsymbol{C}]$ is the single $\widetilde{g} \in \mathrm{Aut}(\widetilde{\mathcal{C}}, \varphi)$ with $\widetilde{g}\widetilde{C} = \widetilde{C}_0$.

6.2. *The universal cover of a chamber system of Coxeter type*

For the convenience of the reader, we recall some notation from previous subsections, and fix some assumptions which hold throughout this subsection. Let $M = (m_{ij})$ be a Coxeter diagram over the finite index set I, and $\mathcal{C}$ a chamber system of type M. This means that I is the type set of $\mathcal{C}$, and any $\{i,j\}$-star in $\mathcal{C}$ is a generalized m_{ij}-gon. Weakening the condition from 3.1.12, we assume only that $\mathcal{C}$ is connected (but not necessarily strongly connected). The Weyl group of M is

$$W = W(M) := \langle i \in I\colon\ (ij)^{m_{ij}} = 1 \rangle.$$

The canonical map from words over I to W is denoted by $f \mapsto s(f)$. The relations

$$C \xrightarrow{w} D, \quad C, D \in \mathcal{C},\ w \in W,$$

have been defined as follows: there exists a gallery $(C, \ldots, D;\ f)$ of reduced type with origin C and extremity D such that $s(f) = w$. If such a gallery exists for at least one f representing w, then it exists for all f representing w. The 'First Main Characterization of Buildings' proved above states that $\mathcal{C}$ is a building if and only if, for given chambers $C, D \in \mathcal{C}$, a group element w with $C \xrightarrow{w} D$ is uniquely determined. This property is equivalent to the nonexistence of certain closed galleries, namely nontrivial closed galleries of reduced type, and thus has some resemblance to the property of 2-simple-connectedness. Indeed, we have observed at the end of Section 3.3 that buildings are 2-simply-connected. The starting point for the theory developed in this subsection is the fact that the converse conclusion does not hold. In the following, we will exhibit conditions which imply that a 2-simply-connected chamber system actually is a building. The property (Q_C) of the following definition has already been introduced in Corollary 3.3.10.

6.2.1. DEFINITION. Consider the following properties of a chamber system $\mathcal{C}$ with a 'base chamber' C:

(Q_C) If two simple galleries have the same origin C, the same extremity, and the same reduced type, they coincide.

(Q'_C) Let $D, D', E \in \mathcal{C},\ w \in W,\ i \in I$ such that $D \neq E \neq D'$, $l(ws_i) > l(\omega)$ and

$$\begin{array}{ccc} C & \xrightarrow{\quad w \quad} & D \\ {\scriptstyle w}\Big\downarrow & & \Big|{\scriptstyle i} \\ D' & \text{―――――} & E \end{array}$$

Then $D = D'$. It is obvious that $(\mathrm{Q}_C) \Leftrightarrow (\mathrm{Q}'_C)$.

In this subsection, the following weak version of Property (Q_C) will be crucial:

(R_C) If two simple galleries have the same origin C, the same extremity, the same reduced type, and furthermore are Weyl-homotopic (see 3.1.14), then they coincide.

Before formulating the main result of this section, we want to emphasize that Property (R_C) is invariant under 2-coverings.

6.2.2. LEMMA. *Let π: $\mathcal{C}' \to \mathcal{C}$ be a 2-covering of chamber systems of type M and $C_0' \in \mathcal{C}$, $C_0 := \pi C_0'$. Then $\mathcal{C}'$ satisfies ($R_{C_0'}$) if and only if $\mathcal{C}$ satisfies (R_{C_0}).*

PROOF. Assume that ($R_{C_0'}$) holds. Consider two simple galleries

$$\boldsymbol{C} = (C_0, C_1, \ldots, C_m; f), \qquad \boldsymbol{D} = (C_0 = D_0, D_1, \ldots, D_m = C_m; f)$$

of reduced type f in $\mathcal{C}$ which are Weyl-homotopic. Denote by $\boldsymbol{C}', \boldsymbol{D}'$ their respective unique liftings with origin C_0'. Since π is a 2-covering, a Weyl homotopy from $\boldsymbol{C}$ into $\boldsymbol{D}$ can be lifted to a Weyl homotopy from $\boldsymbol{C}'$ into $\boldsymbol{D}'$ (in particular, $\boldsymbol{C}'$ and $\boldsymbol{D}'$ have the same extremity). Applying Property ($R_{C_0'}$) of $\mathcal{C}'$ to $\boldsymbol{C}'$ and $\boldsymbol{D}'$ shows that $\boldsymbol{C}' = \boldsymbol{D}'$, and therefore $\boldsymbol{C} = \pi\boldsymbol{C}' = \pi\boldsymbol{D}' = \boldsymbol{D}$.

Assume conversely that $\mathcal{C}$ satisfies property (R_{C_0}). Consider simple galleries and $\boldsymbol{C}'$ and $\boldsymbol{D}'$ in $\mathcal{C}'$ with common origin C_0' and common extremity, and of reduced type, which are Weyl-homotopic to each other. Then $\boldsymbol{C} := \pi\boldsymbol{C}'$ and $\boldsymbol{D} := \pi\boldsymbol{D}'$ have common origin C_0, common extremity, and are Weyl-homotopic as well. From (R_C) it follows that $\boldsymbol{C} = \boldsymbol{D}$. But then $\boldsymbol{C}'$ and $\boldsymbol{D}'$ have to coincide as well, being the unique liftings of $\boldsymbol{C}$, resp., $\boldsymbol{D}$ with origin C_0'. □

We can now formulate the theorem whose proof is the goal of this subsection.

6.2.3. THEOREM. *Let $\mathcal{C}$ be a chamber system of type M, for some Coxeter diagram M. The universal 2-cover $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ is a building if and only if $\mathcal{C}$ satisfies property (R_C) for at least one chamber $C \in \mathcal{C}$. If this is the case, then (R_C) holds for all C.*

PROOF. If $\widetilde{\mathcal{C}}$ is a building, it satisfies Property ($Q_{\widetilde{C}}$) for all $\widetilde{C} \in \widetilde{\mathcal{C}}$ (see Corollary 3.3.10), and *a fortiori* satisfies ($R_{\widetilde{C}}$). By 6.2.2, $\mathcal{C}$ satisfies (R_C) for all $C \in \mathcal{C}$.

Now assume that, conversely, Property (R_{C_0}) holds in $\mathcal{C}$ for some chamber $C_0 \in \mathcal{C}$. By the remark, we may replace $\mathcal{C}$ by $\widehat{\mathcal{C}}$ and thus assume that $\mathcal{C} = \widehat{\mathcal{C}}$. In this proof, we denote galleries in $\mathcal{C}$ by $G, H, \ldots$, and AG is the origin and ΩG the extremity of G. Furthermore, if G is a gallery of reduced type, then $[G]_w$ denotes its Weyl homotopy class. We set

$$\overline{\mathcal{C}} := \{[G]_w\colon\ AG = C_0,\ \text{type}\, G \text{ reduced}\},$$

$$\pi\colon\ \overline{\mathcal{C}} \to \mathcal{C}, \quad \pi[G]_w = \Omega G.$$

The aim of the proof is to define a chamber system structure on $\overline{\mathcal{C}}$ such that π becomes a 2-covering, and $\overline{\mathcal{C}}$ satisfies Property ($P_{\overline{C}_0}$) for $\overline{C}_0 := [(C_0)]$ and thus is a building. Notice first that there is an obvious map

$$\sigma\colon\ \overline{\mathcal{C}} \to W, \quad \sigma[G]_w := s(\text{type}\, G).$$

(In fact, σ is defined on arbitrary Weyl homotopy classes of galleries of reduced type.)

Fix a subset $J \subseteq I$. We shall define a certain map $\rho_J\colon \overline{\mathcal{C}} \to \overline{\mathcal{C}}$ which will later turn out to be given by $\rho_J(\overline{C}) = \mathrm{pr}_{\mathcal{C}(\overline{C},J)} \overline{C}_0$, the projection of $\overline{C}_0$ onto the J-star in $\overline{\mathcal{C}}$ containing $\overline{C}$. The construction is as follows. Given $\overline{C} \in \overline{\mathcal{C}}$, write $w := \sigma(\overline{C})$ in the form $w = w'w'$, where w' is the shortest element in the coset wW_J, and $w' \in W_J$. Choose a representative $G' \cdot G'$ of $\overline{C}$ such that $s(\operatorname{type} G') = w'$, $s(\operatorname{type} G') = w'$. The crucial observation now is the following consequence of the assumption (R_{C_0}):

CLAIM. *The homotopy class $[G']_w$ is independent of the choice of G', G''.*

Indeed, suppose that also $\overline{C} = [H' \cdot H'']_w$, where H', H'' are of reduced type and $s(\operatorname{type} H') = w'$, $s(\operatorname{type} H'') = w''$. There exist galleries K', K'' which are Weyl-homotopic to H', H'', respectively (in particular, $\Omega H' = \Omega K'$), and such that $\operatorname{type} K' = \operatorname{type} G'$, $\operatorname{type} K'' = \operatorname{type} G''$. The galleries $K' \cdot K''$ and $G' \cdot G''$ both start at C_0, have the same reduced type, and are Weyl-homotopic. By (R_{C_0}), they coincide, and thus $G' = K'$ is Weyl-homotopic to H', as claimed.

The map ρ_J is now defined by $\rho_J(\overline{C}) = [G']_w$, where $\overline{C} = [G' \cdot G'']_w$ as above. The following two properties are readily checked from the definition.

(1) $\rho_J \circ \rho_{J'} = \rho_J$ if $J \supseteq J'$, in particular, $\rho_J^2 = \rho_J$.

(2) If $j_1 j_2 \ldots j_r$, $j_\nu \in J$, is a reduced expression for w', then $\rho_J = \rho_{j_1} \circ \cdots \circ \rho_{j_r}$.

The chamber system structure on $\overline{\mathcal{C}}$ is now defined by

$$\overline{C} \overset{i}{\sim} \overline{D} \;:\Leftrightarrow\; \rho_i(\overline{C}) = \rho_i(\overline{D}).$$

Given $\overline{C}, \overline{D} \in \overline{\mathcal{C}}$, write

$$\overline{C} = [G' \cdot G''], \qquad \overline{D} = [H' \cdot H''],$$

$$\operatorname{type} G'' \in \{(), i\}, \qquad \operatorname{type} H'' \in \{(), i\}$$

as in the definition of ρ_i. If $\overline{C} \overset{i}{\sim} \overline{D}$, then $\Omega G' = \Omega H'$, and thus

$$\pi\overline{C} = \Omega G'' \overset{i}{\sim} \Omega G' = \Omega H' \overset{i}{\sim} \Omega H'' = \pi\overline{D}.$$

Therefore, π is a morphism. More generally, a decomposition of w'' and ρ_J as in (2) gives a gallery of type $j_r j_{r-1} \ldots j_1$ joining $\overline{C}$ to $\rho_J(\overline{C})$. Therefore, we have

(3) $\overline{C} \overset{J}{\sim} \rho_J(\overline{C})$ for all $\overline{C} \in \overline{\mathcal{C}}$, $J \subseteq I$.

In particular, if $\rho_J(\overline{C}) = \rho_J(\overline{D})$, then $\overline{C} \overset{J}{\sim} \overline{D}$. If, conversely, $\overline{C} \overset{J}{\sim} \overline{D}$, then $\overline{C}$ and $\overline{D}$ can be joined by a gallery of the form

$$(\overline{C}, \rho_{j_1}(\overline{C}), \rho_{j_2}\rho_{j_1}(\overline{C}), \ldots, \rho_{j_r}\rho_{j_{r-1}} \cdots \rho_{j_1}(\overline{C}))$$

such that $j_\nu \in J$ for all ν. But then (1) immediately shows that $\rho_J(\overline{C}) = \rho_J(\overline{D})$. Summing up our discussion, we see that $\overline{C}$ and $\overline{D}$ are in the same J-star of $\overline{\mathcal{C}}$ if and only if $\rho_J(\overline{C}) = \rho_J(\overline{D})$. If $\overline{C} = [G' \cdot G'']$, type $G'' \subseteq J$ as before, then we can describe the J-star of $\overline{C}$ as follows:

$$\mathcal{C}(\overline{C}, J) = \{[G' \cdot H]\colon\ H \in \mathcal{H}(\pi\rho_1\overline{C}, J)\},$$

where $\mathcal{H}(C, J)$ denotes the set of all galleries H in $\mathcal{C}$ of reduced type contained in J, and starting at C. Now assume that $|J| = 2$. Then the J-star of any $C \in \mathcal{C}$ is a building, and therefore the Weyl homotopy class of any $H \in \mathcal{H}(C, J)$ is determined by its extremity ΩH. But this means that π restricted to $\mathcal{C}(\overline{C}, J)$ is bijective. Applying the same argument to the rank 1 stars contained in $\mathcal{C}(\overline{C}, J)$, we see that π restricted to $\mathcal{C}(\overline{C}, J)$ even is an isomorphism. Altogether, we have shown that $\pi\colon \overline{\mathcal{C}} \to \mathcal{C}$ is a 2-covering.

Finally, if $(\overline{C}_0, \overline{C}_1, \ldots, \overline{C}_m;\ i_1, \ldots, i_m)$ is any gallery of reduced type in $\overline{\mathcal{C}}$, based at the distinguished chamber $\overline{C}_0$, then it is readily checked that $s(i_1, \ldots i_m) = \sigma(\overline{C}_m)$. Thus, $s(i_1 \ldots i_m)$ is determined by the extremity $\overline{C}_m$ alone, and Property $(\mathrm{P}_{\overline{C}_0})$ holds. By the First Main Characterization of Buildings, $\overline{\mathcal{C}}$ is a building.

6.3. *The 'Second Main Characterization of Buildings'*

This subsection is an immediate continuation of the previous one, and we keep our general assumptions and notations $M, W(M), \mathcal{C}$. A subset J of the type set I is called *spherical* if the corresponding subgroup $W(J) := \langle s_j\colon\ j \in J\rangle \subseteq W$ of the Weyl group is finite.

We now give another criterion for the universal 2-cover of $\mathcal{C}$ to be a building. It is more suitable for applications and seems to be the best known and most widely quoted result from the 'Local Approach'.

6.3.1. THEOREM ('Second Main Characterization of Buildings'). *The universal 2-covering of $\mathcal{C}$ is a building if and only if, for every subset $J \subseteq I$ such that the induced diagram M_J is C_3 or H_3, the universal 2-covering of any J-star in $\mathcal{C}$ is a building.*

6.3.2. COROLLARY. *If $\mathcal{C}$ is 2-simply connected, and M contains no subdiagram $\mathsf{C}_3, \mathsf{H}_3$, then $\mathcal{C}$ is a building.*

If M is as in the corollary and furthermore contains no subdiagram A_3, it is indeed true that there exists an abundance of buildings belonging to M, by results of Ronan and Tits. Cf. Section 7.2 and Ronan [1984/86].

The 'only if' part of the theorem is clear. For the 'if' part, we may assume that the universal 2-covering of any J-star, with J spherical of cardinality 3, is a building. Indeed, if the induced diagram on J is not one of $\mathsf{A}_3, \mathsf{C}_3, \mathsf{H}_3$, it is disconnected, and the corresponding star is the join of chamber systems of rank 1 or 2. But a chamber system of rank 1 or 2 and of Coxeter type automatically is a building. If the induced subdiagram is A_3, then also the corresponding stars automatically are buildings, by Proposition 3.4.7.

Theorem 6.3.1 now will be proved by showing that condition (R_C) of Theorem 6.2.3 is satisfied if it is satisfied for all rank 3 stars of spherical type. Considering the particular nature of this condition, the reduction to rank 3 stars will purely be a matter of the Cayley graph of the Coxeter group of M. More precisely, we have to investigate the subgraph consisting of all reduced representations of one group element and its homotopy properties. All work actually dealing with chamber systems has already been done in the Section 6.2.

We denote by

$\overline{F}$ the free monoid over the type set I,

$s\colon \overline{F} \to W$ the canonical map; as before,

$\overline{F}_w := s^{-1}(w)$ for $w \in W$,

$F_w \subset \overline{F}_w$ the set of all reduced words representing w,

$p_{ij} = ijij\ldots \in F$ (of length m_{ij} if $m_{ij} < \infty$).

We make each F_w into a graph by introducing the edges

$$\{fp_{ij}g, fp_{ji}g\}, \quad f, g \in F,\ i, j \in I, \text{ such that } i \neq j \text{ and}$$
$$m_{ij} < \infty,\ s(fp_{ij}g) = w$$

and each $\overline{F}_w$ by introducing in addition the edges

$$\{fiig, fg\}, \quad f, g \in F,\ i \in I,\ s(fg) = w.$$

A *path* in a graph F (undirected, without loops) for us is a sequence

$$\chi = (f_0, f_1, \ldots, f_r)$$

of vertices such that $\{f_{\nu-1}, f_\nu\}$ is an edge, or $f_{\nu-1} = f_\nu$, for all $\nu = 1, \ldots, r$. We set

$$f_0 =: a(\chi), \qquad f_r =: z(\chi).$$

The paths in F_w and in $\overline{F}_w$ are the Weyl homotopies of words introduced in 3.1.14. The *composition* $\chi \cdot \eta$ of two paths is defined in the obvious way if $z(\chi) = a(\eta)$, or $\{z(\chi), a(\eta)\}$ is an edge. Two paths are called *elementary-homotopic* if χ can be obtained from η or η from χ by one of the following processes:

replacing a sub-path (f, f) by (f),

replacing a sub-path (f, g, f) by (f). Two paths are *homotopic* if they can be connected by a sequence of elementary homotopies; this defines an equivalence relation. Notice that homotopic paths have the same extremities a and z. By $[\chi]$, we denote the homotopy class of χ. The composition of paths induces a 'multiplication' of homotopy classes. It is associative, and for the *inverse path* $\chi^{-1} := (f_r, \ldots, f_1)$ we have

$$[\chi] \cdot [\chi^{-1}] = [(a(\chi))].$$

In particular, for any vertex f, the homotopy classes of all closed paths at f,

$$\pi(F, f) := \{[\alpha]\colon\ \alpha \text{ a path, } a[\alpha] = z[\alpha] = f\}$$

is a group with neutral element $[(f)]$ and inverse $[\alpha]^{-1} = [\alpha^{-1}]$. It is called the *fundamental group* of F af f and can be identified with the topological fundamental group of the standard geometrical realization of the connected component of f in F. If F is connected, then all groups $\pi(F, f)$ are isomorphic to each other and are called 'the' fundamental group of F.

Now assume that F is connected. If A is a set of closed paths, we say that A *generates the fundamental group* of F if, for $f \in F$, the set

$$\{[\chi\alpha\chi^{-1}]\colon\ \alpha \in A,\ \chi \text{ a path such that } a(\chi) = f,\ z(\chi) = a(\alpha)\}$$

generates $\pi(F, f)$. If this holds for one f, it holds for all f.

If we specialize to the above graphs $F_w, \overline{F}_w$ of (reduced) representations of a group element w, the paths are by definition the homotopies of words introduced before. Theorem 2.5.2 (Tits' theorem about the word problem in Coxeter groups) can be rephrased by just saying that all F_w and $\overline{F}_w$ are connected.

The application of the fundamental group of F_w to the proof of Theorem 6.3.1 is via Property (R_C) occurring in Theorem 6.2.3. To see this connection, consider two galleries $\boldsymbol{C}$ and $\boldsymbol{D}$ of the same reduced type f which have the same extremities and are Weyl-homotopic. A Weyl homotopy $\gamma = (\boldsymbol{C} = \boldsymbol{C}^{(0)}, \boldsymbol{C}^{(1)}, \dots, \boldsymbol{C}^{(r)} = \boldsymbol{D})$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ gives rise to a homotopy from type $\boldsymbol{C}$ to type $\boldsymbol{D}$, i.e. a closed path $\alpha = (f = f_0, f_1, \dots, f_r = f)$ in F_w. Moreover, γ is the unique lifting of α, starting at $\boldsymbol{C}$ (see Lemma 3.1.15). The question now is whether or not this unique lifting based at $\boldsymbol{C}$ is closed for all α. Obviously, this only depends on the homotopy class $[\alpha]$. Furthermore, it is clear that all liftings are closed if the liftings of a generating set are. Proposition 6.3.4 below shows that, under the assumption of Theorem 6.3.1, this is indeed true.

The statement and the proof of the following lemma are in principle straightforward, but complicated to write down.

6.3.3. LEMMA. *Let F, F' be connected graphs, $\varphi\colon F \to F'$ a morphism, assume that $\varphi^{-1}(f')$ is connected, for all $f' \in F'$. Consider the following three assumptions on a set A of closed paths in F.*

(i) *φA generates the fundamental group of F'.*

(ii) *$A_{f'} := \{\beta = \chi\alpha\chi^{-1}\colon\ \alpha \in A,\ \beta \subseteq \varphi^{-1}(f')\}$ generates the fundamental group of $\varphi^{-1}(f')$, for all $f' \in F'$. Here, χ runs over all paths such that $z(\chi) = a(\alpha)$, for given α.*

(iii) *For any edge $\{g', h'\}$ in F', and any two preimages*

$$\{g_1, h_1\}, \quad \{g_2, h_2\} \quad (\varphi g_i = g',\ \varphi h_i = h'),$$

there exists a path γ in $\varphi^{-1}(g')$, $a(\gamma) = g_1$, $z(\gamma) = g_2$, and a path δ in $\varphi^{-1}(h')$, $a(\delta) = h_1$, $z(\delta) = h_2$, such that $\gamma \cdot \delta^{-1}(g_1) \in A$.

If (i), (ii), *and* (iii) *hold, then A generates the fundamental group of F.*

PROOF. Whether A generates the fundamental group or not does not change if one enlarges A by all paths $\chi\alpha\chi^{-1}$ as above, furthermore by all paths homotopic to a path in A, and finally by all paths inverse to a path in A. If A is 'closed' in this sense which we shall assume from now on, then the following two properties hold:

(1) If α and β are closed paths at f and $\alpha \in A$, $\alpha\beta \in A$, then $\beta \in A$.

(2) If α is in A, then every cyclic permutation of α is in A.

Assumption (ii) now takes the simpler form

(ii′) All closed paths contained in a fibre $\varphi^{-1}(f')$ belong to A.

Under the assumptions of (iii), the following holds, by (iii) and (2):

$$(h_1)\cdot\gamma\cdot\delta^{-1} \in A, \qquad (g_2)\cdot\delta^{-1}\cdot\gamma \in A, \qquad \delta^{-1}\cdot\gamma\cdot(h_2) \in A.$$

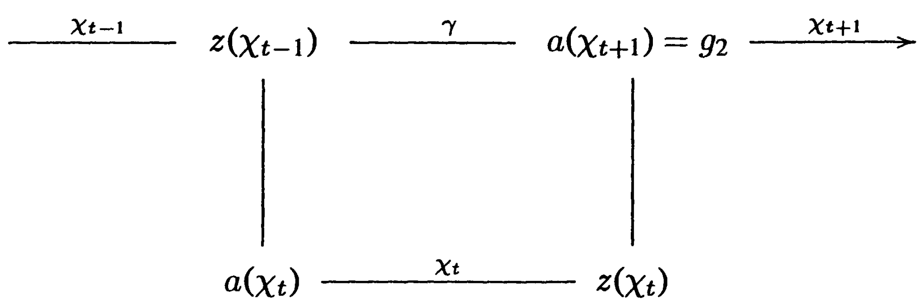
Assumption (iii) now can be sharpened to

(iii′) If $g', h', g_1, h_1, g_2, h_2$ are as in (iii), and $\delta = (h_1, \ldots, h_2)$ is a path in $\varphi^{-1}(h')$, then there exists a path $\gamma = (g_1, \ldots, g_2)$ in $\varphi^{-1}(g')$ such that $\gamma\cdot\delta^{-1}\cdot(g_1) \in A$.

To see this, choose γ and δ_1 according to (iii), in particular $\delta_1^{-1}\cdot\gamma\cdot(h_2) \in A$. It follows that $\delta\cdot\delta_1^{-1}\cdot\gamma\cdot\delta^{-1} \in A$, and $\delta\cdot\delta_1^{-1} \in A$, by (ii′). Now from (1) it follows that $(h_1)\cdot\gamma\cdot\delta^{-1} \in A$.

After these preliminary steps, we come to the proof of the lemma. Let $f \in F$, $f' := \varphi f$, set $A_f := \{\alpha \in A\colon a(\alpha) = f\}$. By i), it is sufficient to show that the kernel of the induced map $\varphi_*\colon \pi(F, f) \to \pi(F', f')$ is contained in the subgroup generated by $[A_f]$. Let $[\chi]$ be a member of this kernel. We decompose $\chi = \chi_1\cdot\chi_2\cdot\ldots\cdot\chi_r$ in such a way that each χ_ν is fully contained in one fibre $\varphi^{-1}(f'_\nu)$, and $f_{\nu-1} \neq f_\nu$ for $\nu = 2, \ldots, r$. If $r = 1$, the desired conclusion $\chi \in A$ holds, by (iii′). Thus let $r > 1$, and suppose that the claim holds for paths in the kernel having a smaller value of r.

From $[\varphi\chi] = []$ it follows that there exists a t such that $0 < t < r$ and $f'_{t-1} = f'_{t+1}$. Now apply property (iii′) to $z(\chi_{t-1}), a(\chi_t), a(\chi_{t+1}), z(\chi_t)$, and $\chi_t = \delta$.



There exists a path γ with $a(\gamma) = z(\chi_{t-1})$, $z(\gamma) = a(\chi_{t+1})$ such that $(g_2)\cdot\chi_t^{-1}\cdot\gamma \in A$. Then

$$\eta := (\chi_{t+1}\cdot\ldots\cdot\chi_r)^{-1}\cdot\chi_t^{-1}\cdot\gamma(\chi_{t+1}\cdot\ldots\cdot\chi_r)$$

is in A_0. Furthermore, $[\eta]$ is in the kernel of the map induced by φ on homotopy classes, since $\varphi((g_2)\cdot\chi_t^{-1}\cdot\gamma)$ equals $(f'_{t+1}, f'_t, f'_{t+1})$, up to repetitions. Now replace χ by $\chi\cdot\eta$. This path is homotopic to

$$\chi_1\cdot\ldots\cdot\chi_{t-1}\cdot\gamma\cdot\chi_{t+1}\cdot\ldots\cdot\chi_r,$$

and $\chi_{t-1}\cdot\gamma\cdot\chi_{t+1}$ is inside the fibre of $f'_{t-1} = f'_{t+1}$. The new path has a smaller value of r, and by induction is in the group generated by $[A_0]$. □

We now return to the graphs F_w of reduced representations of a Weyl group element and derive from Proposition 6.3.3 a generating set of the fundamental group of F_w.

6.3.4. PROPOSITION. *For $w \in W$, let A_w consist of the following closed paths.*

(a)

$$\alpha = \alpha(i,j,f,f',\delta) := \delta p_{ij} f' \cdot \delta^{-1} p_{ji} f' \cdot (f p_{ij} f')$$
$$= (f_0 p_{ij} f', f_1 p_{ij} f', \ldots, f_r p_{ij} f', f_r p_{ji} f', f_{r-1} p_{ji} f', \ldots, f_0 p_{ji} f', f_0 p_{ij} f'),$$

where i, j runs over all pairs such that $m_{ij} \neq \infty$, f, f' over all words such that $f p_{ij} f' \in F_w$, and δ over all paths in $F_{s(f)}$, starting at f.

(b) $\alpha = \alpha(J,f,f',\delta) := f\delta f'$, *where J runs over all spherical subsets with $|J| \leqslant 3$, g over all words over J, f, f' over all words such that $fgf' \in F_w$, and δ over all closed paths in $F_{s(g)}$ at g.*

This set A_w generates the fundamental group of F_w.

PROOF. We use induction on the length $l(w)$. Suppose $l(w) \geqslant 2$. Consider the set $J_w := \{i \in I\colon l(ws_i) < l(w)\}$, and the mapping $\varphi\colon F_w \to J_w$, $i_1 \ldots i_l \mapsto i_l$. By the Exchange Property (recall 2.1.3 and 2.1.4), φ is surjective. We consider J_w as a complete graph; then φ is certainly a morphism. We shall show that φ and A_w satisfy the assumptions of Lemma 6.3.3.

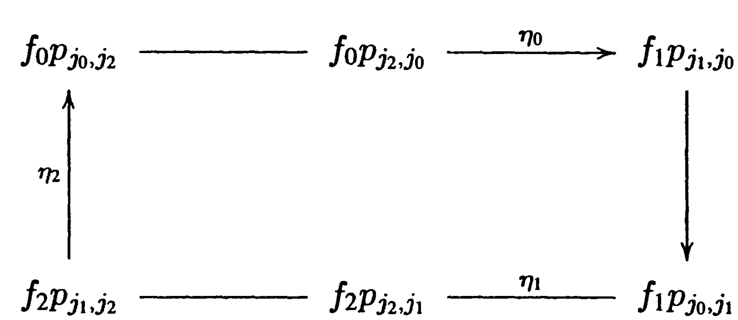
First we remark that for $j \in J_w$, the preimage $\varphi^{-1}(j)$ is isomorphic as a graph to F_{ws_j}, by the map $f' \mapsto f'j$. In particular, $\varphi^{-1}(j)$ is connected. We now show that the conditions (i), (ii) and (iii) of Proposition 6.3.3 hold.

Proof of (i). Consider a 'triangle' (j_0, j_1, j_2, j_0), i.e. $J = \{j_0, j_1, j_2\} \subseteq J_w$, of cardinality 3.

We shall see that this triangle is, up to repetitions, the image of a path as in (b). By Lemma 5.2.5, $J \subseteq J_w$ is spherical, and if w_J is the longest element in W_J and $w' := ww_J^{-1}$, then $l(w) = l(w') + l(w_J)$. In the following, the index ν is to be taken modulo 3. Consider $w_\nu = s(p_{i_{\nu-1},i_\nu})$, the longest element of $W_{\{j_{\nu-1},j_\nu\}}$. Again by 5.2.5, there exists a reduced representation of w_J of the form $f_\nu p_{i_{\nu-1},i_\nu}$, $f_\nu \in F(J)$. There exists a homotopy

$$\eta_\nu = (f_\nu p_{j_{\nu-1},j_\nu}, \ldots, f_{\nu+1} p_{j_{\nu+1},j_\nu})$$

all whose terms end with j_ν. Consider the closed path $\delta := \eta_0 \cdot \eta_1 \cdot \eta_2 \cdot (f_0 p_{j_2,j_0})$



Choose an arbitrary element f' in $F_{w'}$. Then $f'\delta$ has the shape as in (b), and its image under φ is (j_0, j_1, j_2, j_0), up to repetitions.

Proof of (ii). Let $j \in J_w$. By the induction hypothesis, $\pi(F_{ws_j})$ is generated by A_{ws_j}, and thus $\pi(\varphi^{-1}(i))$ by the paths α_j, $\alpha \in A_{ws_j}$. But these are all members of A_w.

Proof of (iii). Let $g_1, h_1 \in F_w$ be elementary homotopic and such that

$$\varphi g_1 =: i \neq j := \varphi h_1,$$

similarly g_2, h_2. The elementary homotopy in question necessarily is of the following shape: $m_{ij} < \infty$, and there exists a word k_1 such that $g_1 = k_1 p_{ij}$, $h_1 = k_1 p_{ji}$, similarly $g_2 = k_2 p_{ij}$, $h_2 = k_2 p_{ji}$. If we join k_1 and k_2 by any homotopy χ, then χp_{ij} and χp_{ji} are the desired paths joining g_1 and g_2 in $\varphi^{-1}(i)$, and h_1 and h_2 in $\varphi^{-1}(j)$, respectively. □

The self-homotopies described in part (a) of the proposition are called *inessential*.

6.3.5. Proof of Theorem 6.3.1. We return to the discussion preceding Proposition 6.3.3. According to Proposition 6.3.4, we have to show that the liftings of all generators α as described in Proposition 6.3.4 to homotopies of galleries are closed. For α as in (a), this is obvious (without any assumption on $\mathcal{C}$). For α as in (b), it is enough to consider the case that all types involved are elements in J, where J is spherical and $|J| \leqslant 3$. But, by the assumption of Theorem 6.3.1, the universal cover of the appropriate J-star is a building, therefore, this J-star satisfies (R_C) for all C, and the lifting of α is indeed closed. □

The last proposition (and its application to Theorem 6.3.1) shows that the fundamental group $\pi(F_w)$ is of particular interest in the case of spherical Coxeter diagrams of rank 3. The following proposition shows how $\pi(F_w)$ looks like in these cases.

6.3.6. Proposition. *Let M be a spherical Coxeter diagram of rank* 3, *denote by $w_0 \in W(M)$ the longest element in the Weyl group, and let $w \in W(M)$ be arbitrary.*

The fundamental group $\pi(F_w)$ modulo the normal subgroup generated by the inessential self-homotopies of w is trivial if $w \neq w_0$ and infinite cyclic if $w = w_0$. For $\mathsf{A}_3, \mathsf{C}_3$ *the following self-homotopies of w_0 are generators of the factor group*:

A_3 123121_1 121321_2 212321_3 213231_4 213213_5 231213_6 232123_7 323123_8 321323_9 321232_{10} 312132_{11} 132132_{12} 132312_{13} 123212_{14} 123121_1.

C_3 123123123_1 121323123_2 212323123_3 213232123_4 213231213_5 213213213_6 231213213_7 232123213_8 232123231_9 232132321_{10} 232312321_{11} 323212321_{12} 323121321_{13} 323123121_{14} 323123212_{15} 321323212_{16} 321232312_{17} 312132312_{18} 312132132_{19} 312312132_{20} 312321232_{21} 132321232_{22} 123231232_{23} 123213232_{24} 123212323_{25} 123121323_{26} 123123123_1

We omit the proof except for an indication of the geometrical significance of the explicit generators. Consider the following two-dimensional cell complex $\Sigma(w)$: the vertices are all group elements v represented by a 'left subword' of a word representing

w, that is, $l(w) = l(v) + l(v^{-1}w)$. Two vertices v_1, v_2 are joined by an edge if $v_1^{-1}v_2 \in S$, the set $\{s_i\}$ of generating involutions. The two-cells in $\Sigma(w)$ are obtained by 'filling' all $2m_{ij}$-gons with vertices $\{vx\colon x \in W_{ij}\}$, where $\{i, j\}$ is one of the three two-element subsets of I and $W_{ij} \subset W(M)$ the corresponding dihedral subgroup (and v is of course such that all vx are represented by subwords of w). If $w = w_0$, then all group elements are vertices of $\Sigma(w_0)$ (by Proposition 5.2.2), and the 1-skeleton of $\Sigma(w_0)$ is the Cayley graph of W with respect to the generating set S. Since the rank of W is 3, $\Sigma(w_0)$ is the dual of the Coxeter–Tits complex of W, and the underlying topological space $\|\Sigma(w_0)\|$ is therefore homeomorphic to the 2-sphere $\mathbb{S}^2$. If $w \neq w_0$, then $\|\Sigma(w)\|$ is contractible. To each elementary Weyl homotopy (edge in F_w), one can naturally associate an oriented 2-cell of $\Sigma(w)$. In the case $w = w_0$, the desired generating circuit has the property that the associated sequence of 2-faces simply covers the sphere.

In order to write it down explicitly observe that the cell complex $\Sigma(w_0)$ can be identified with one of the Archimedean (i.e. vertex transitive) solids (considered as a tessellation of the sphere), namely the unique Archimedean solid for which the given group W in its natural linear representation on $\mathbb{S}^2$ acts sharply transitively on the vertices. For $M = \mathsf{A}_3$, this is the truncated octahedron, consisting of 8 hexagons and 6 squares, and the generator for $\pi(F_{w_0})$ given in the proposition is read off from this geometrical figure. In the case C_3, the group W is the symmetry group of the cube, and $\Sigma(w_0)$ is the so-called great rhombicuboctahedron, consisting of 6 octagons, 8 hexagons and 12 squares. We leave it to the reader to check the above generator, and to treat also the other spherical diagrams of rank 3.

6.4. *Some applications, I: Geometrical axioms*

In this subsection, we finish the characterization of irreducible spherical buildings by geometrical axioms. Recall that the cases A_n, C_n, D_n and E_6 have already been treated in Section 3.4. In view of Theorem 5.3.6, we shall exclude the cases H_3 and H_4, and thus it remains to treat the cases E_7, E_8, F_4.

The necessity of our axioms in the case of buildings is a consequence of two quite general, important theorems about shadows in buildings which we present first. Recall that the concept of shadows or of a 'shadow space' was already mentioned in the notes to Section 1. We state once more the definition.

6.4.1. DEFINITION. Let Δ be a strongly connected, I-numbered complex and $I_0 \subseteq I$ be any subset of its type set. The *I_0-shadow* $\mathrm{Sh}_{I_0}(B)$ of a flag B is defined as the set of all flags of type I_0 incident with B:

$$\mathrm{Sh}_{I_0}(B) := \{x \in \Delta\colon \operatorname{type} x = I_0,\ x \cup B \in \Delta\}.$$

Now we have to use the definitions and results from 1.1.7: the transitive relation $J \preceq K$ on the subsets of I, the notion of a reduced set of types, and the unique reduced subset $J_{\mathrm{red}} \subseteq J$ of any set of types J. All these notions depend on the specified type set I_0 defining our 'space of shadows'. A flag is called reduced if its type is reduced. If B is an arbitrary flag, its reduction B_{red} is defined via reduction of its type:

$$B_{\mathrm{red}} \subseteq B, \quad \operatorname{type} B_{\mathrm{red}} := (\operatorname{type} B)_{\mathrm{red}} \quad \text{for any } B \in \Delta.$$

The following theorem describes, in the case of buildings, all shadows and their inclusion in terms of the relation $\preceq$ and of the incidence of reduced flags.

6.4.2. THEOREM. *Let Δ be a building, fix a set of types $I_0 \subseteq I$, consider shadows and reduced flags with respect to I_0.*

(a) *Any shadow is the shadow of a uniquely determined reduced flag; more precisely:* $\mathrm{Sh}(B) = \mathrm{Sh}(B_{\mathrm{red}})$ *for all flags, and if A and B are reduced, then* $\mathrm{Sh}(A) = \mathrm{Sh}(B)$ *holds only if $A = B$.*

(b) *Let A and B be reduced. Then*

$$\mathrm{Sh}(A) \subseteq \mathrm{Sh}(B) \;\Leftrightarrow\; A \cup B \in \Delta \textit{ and } \operatorname{type} A \preceq \operatorname{type} B.$$

For a proof, see Tits [1974], 12.15, or Scharlau [1990], 1.4 and 4.4.

We next deal with the intersection of shadows.

6.4.3. DEFINITION. A numbered complex Δ satisfies the *Intersection Property* for flags, resp., for objects (vertices) if the following holds for Δ and all its residues:

(Int-F) For any I_0, the intersection $\mathrm{Sh}(A) \cap \mathrm{Sh}(A')$ of any two I_0-shadows is either empty or the shadow of a flag B incident with both A and A'.

(Int-O) For any one-element subset $\{i\} \subseteq I$, the intersection $\mathrm{Sh}(a) \cap \mathrm{Sh}(a')$ of the i-shadows of any two objects $a, a' \in X_\Delta$ is either empty or the shadow of a flag B incident with both a and a'.

6.4.4. THEOREM. *Every building satisfies the Intersection Property for flags.*

In view of 6.4.2(b), this theorem is an immediate consequence of the following result for which we have to recall the projection $\mathrm{pr}_A\, B$ of an arbitrary flag B onto a star $\mathrm{St}\, A$ which was defined in 5.1.9.

6.4.5. PROPOSITION. *Let Δ be a building, fix a set of types $I_0 \subseteq I$.*

(a) *If the intersection of two shadows* $\mathrm{Sh}(A)$ *and* $\mathrm{Sh}(A')$ *is nonempty, then*

$$\mathrm{Sh}(A) \cap \mathrm{Sh}(A') = \mathrm{Sh}(\mathrm{pr}_A\, A').$$

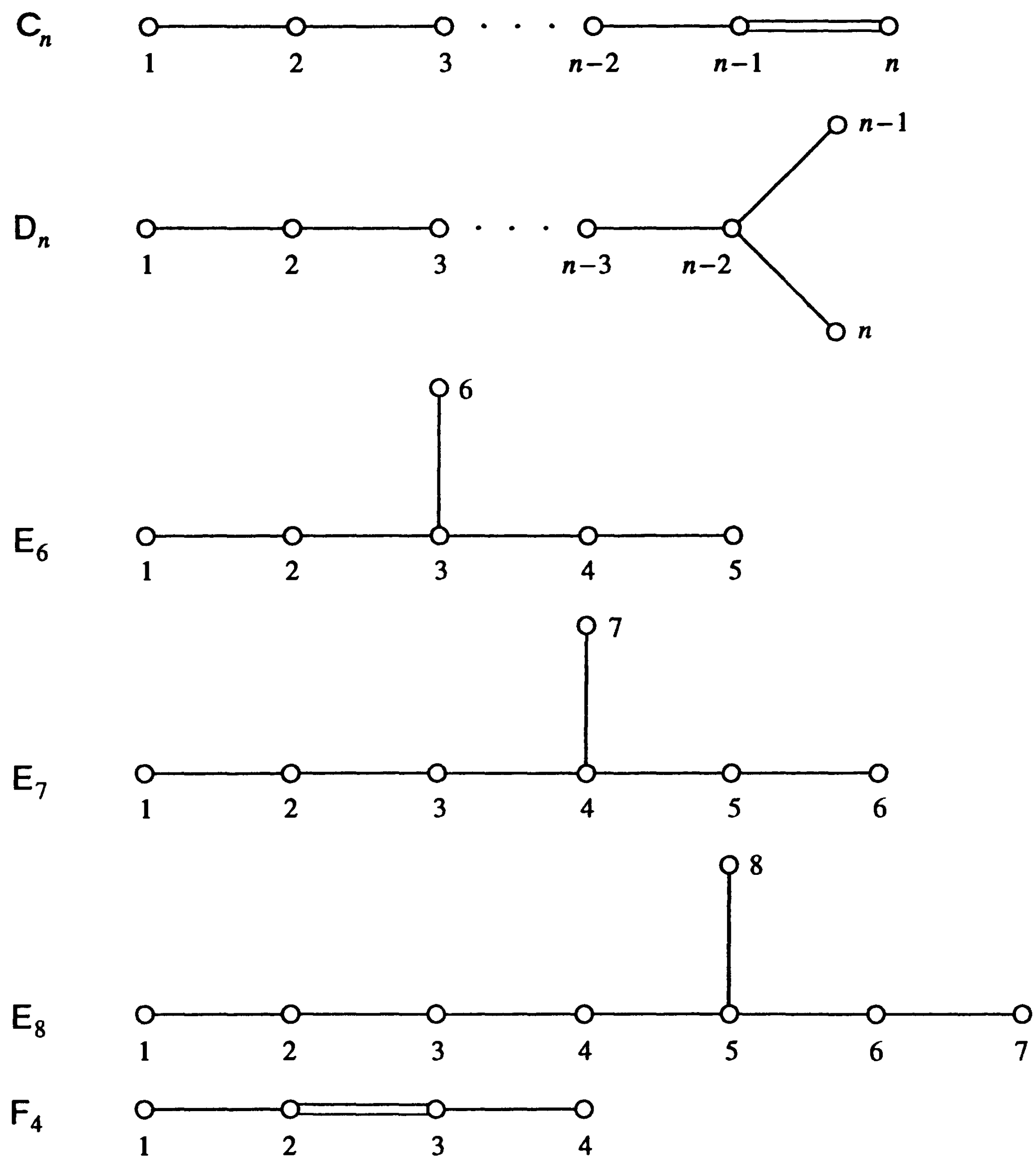
(b) *Under the assumptions of part* (a), $(\mathrm{pr}_A\, A')_{\mathrm{red}}$ *is incident with both A and A'.*

For a proof, see Tits [1974], 12.9. Further explanations on 6.4.4 and 6.4.5 will be given in the Notes below.

We now come to the main topic of this first part of the present subsection, namely a converse of Theorem 6.4.4 for spherical diagrams.

6.4.6. THEOREM. *Let Δ be a strongly connected complex of type M, where M is an irreducible spherical Coxeter diagram $\neq \mathsf{H}_3, \mathsf{H}_4$. If Δ satisfies the Intersection Property* (Int-O) *for vertices, then Δ is a building.*

As we have already mentioned, the proof of this result goes case by case, inductively using the result for subdiagrams. Notice that Property (Int-O) actually is an entire collection of properties parametrized by the types $I_0 = \{i\}$, type $a =: j$, type $a' =: k$. In each case, only a part of them is used. In particular, we shall see in a moment that the axiom (Lin) previously used in cases C_n, D_n and E_6 is a consequence of the intersection property and thus the theorem is already proved for these diagrams. Following Tits [1981a], we shall now list some further axioms. For convenience, and for purposes of induction, we shall include once more diagrams C_n, D_n and E_6. We shall use the following labelling of the diagrams and call the objects of type 1 *points*, the objects of type 2 *lines* and the objects corresponding to the right end node of the diagram (n, resp., 5, 6, 7, 4) *hyperlines*.



6.4.7. Axioms.

(Lin) If two objects a, a' of type $j-1$ are incident with two objects b, b' of type j, then $a = a'$ or $b = b'$. This is required for all $j > 1$, $j \leqslant n-1$ in case C_n, $j \leqslant n-2$

in case D_n, $j = 2, 3$ in case E_6, $j \leqslant 4$ in case E_7, $j \leqslant 5$ in case E_8, $j = 2$ in case F_4.

(O) If two objects of type j have the same shadow on the set of points, they coincide. Here j assumes the same values as in (Lin).

(LL) If two lines are both incident with two distinct points, they coincide.

(LH) If a line and a hyperline are both incident to two distinct points, they are incident.

(HH) If two distinct hyperlines are both incident to two distinct points, the latter are both incident to a line.

Notice that (Lin) includes (LL).

In the following two propositions, the complex is assumed to satisfy the general assumptions of 6.4.6.

6.4.8. PROPOSITION. *If a strongly connected complex has the Intersection Property, it satisfies all properties* (Lin), (O), (LL), (LH), (HH).

The proof is easily achieved inductively by reducing to the residue of an appropriate point and using the following lemma which generalizes (LH).

6.4.9. LEMMA. *Assume the Intersection Property. If two distinct points are incident with an object a of type j, where j is as in* (Lin), *and with an object b of type $k > j$, then a and b are incident.*

In view of Proposition 6.4.8, Theorem 6.4.6 follows from 3.4.7, 3.4.9, 3.4.11 and the following theorem.

6.4.10. THEOREM.

(a) *A strongly connected complex of type* E_7 *is a building if (and only if) it has the Properties* (O), (LL) *and* (LH), *or the Properties* (Lin) *and* (LH).

(b) *A strongly connected complex of type* E_8 *or* F_4 *is a building if (and only if) it has the Properties* (O), (LL), (LH) *and* (HH), *or the Properties* (Lin), (LH) *and* (HH).

The proof of this result will make essential use of covering theory and, more specifically, of the Second Main Characterization 6.3.1 of buildings. In a first step we shall show that the universal cover $\widetilde{C}$ of $C = C(\Delta)$ is a building (this is obvious for $\mathsf{E}_7, \mathsf{E}_8$). If $\widetilde{\Delta} := \Delta(\widetilde{C})$ is the complex belonging to $\widetilde{C}$ (which is strongly connected since it is a building) we have an induced 2-covering $\alpha\colon \widetilde{\Delta} \to \Delta$ of simplicial complexes. The principle of the remainder of the proof is contained in the following obvious lemma.

6.4.11. LEMMA. *Let $\alpha\colon \widetilde{\Delta} \to \Delta$ be a 2-covering of simplicial complexes. Assume that, for some i, it is injective on the set of vertices of type i and on all residues* $\operatorname{St} a$, *where* $\operatorname{type} a = i$. *Then α is an isomorphism.*

We shall give the proof of 6.4.10 only in cases E_7 and F_4; case E_8 is similar to E_7 though slightly more complicated. See the original paper Tits [1981a]. We shall make use of the techniques from Section 2.5. Recall in particular the notation for chains of in a numbered complex as introduced in 2.5.8. By p, q etc. we shall always denote points, by l we denote lines and by h hyperlines.

6.4.12. LEMMA. *In a strongly connected complex of type* E_7, *the following holds*:

(a) *For any* p, h *there exist* q, l *such that* $p - l - q - h$.

(b) *For any* p, l *there exist* q, h *such that* $p - h - q - l$.

PROOF. (a) There exists a chain

$$h - p_1 - l_1 - p_2 - l_2 - \dots - p_m - l_m - p.$$

We show that it can be shortened if $m \geqslant 2$. Lemma 2.5.9(b) applied to $\mathrm{St}\, p_1$ yields

$$h - p_1 - l_1 - h' - l_2 - \dots - p,$$

$$h - p_1 - h' - p_3 - l_3 - \dots - p.$$

Applying the same lemma to $\mathrm{St}\, p_1$, but this time reversing the roles of lines and hyperlines etc. (using the symmetry of diagram E_6) gives a new chain

$$h - l' - h' - p_3 - l_3 - \dots - p.$$

Now we apply the last statement of 2.5.8 to the D_n-complex $\mathrm{St}\, h'$ and get

$$h - l' - p' - l'' - p_3 - l_3 - \dots - p,$$

$$h - p' - l'' - p_3 - l_3 - \dots - p,$$

as desired.

Part (b) is readily derived from (a): Choose h' with $l - h'$ and using part (a) choose l' and q' such that

$$p - l' - q' - h' - l.$$

Inside $\mathrm{St}\, h'$ find l'', q such that

$$p - l' - q' - l'' - q - l.$$

Finally, apply Lemma 2.5.9(b) once more to $l, l' \in \mathrm{St}\, q'$ and find

$$p - l' - h - l'' - q - l,$$

$$p - h - q - l. \qquad \square$$

PROOF of Theorem 6.4.10 in case E_7. Let $\alpha\colon \widetilde{\Delta} \to \Delta$ be the universal 2-cover of Δ. Since there is no C_3-subdiagram, $\widetilde{\Delta}$ is a building, by Theorem 6.3.1. From the E_6-case (Proposition 3.4.11) we know that all stars $\mathrm{St}\,p$ of points are buildings. In particular, $\alpha|_{\mathrm{St}\,p}$ is injective. According to Lemma 6.4.11, it remains to show that α is injective on the set of points. Assume the existence of points p, p', $p \neq p'$ with $\alpha p = \alpha p'$. By the D_6-case (Proposition 3.4.9) all $\mathrm{St}\,h$ are buildings. In particular, $\alpha|_{\mathrm{St}\,h}$ is injective. Thus p and p' cannot be contained in a common $\mathrm{St}\,h$, and a fortiori cannot be incident with a common line. Choose a line l' incident with p', and by the last lemma, part (b) choose q and h such that

$$p - h - q - l' - p'.$$

Notice that $p \neq q$, since p and p' are not collinear. It follows that $\alpha p \neq \alpha q$, since $p, q \in \mathrm{St}\,h$. Now apply Axiom (LH) to $\alpha p, \alpha q, \alpha l', \alpha h$. It follows that $\alpha l' - \alpha h$. Since α: $\mathrm{St}\,h$ is an isomorphism, there exists an l incident with h, p, q and $\alpha l' = \alpha l$. But α is injective on $\mathrm{St}\,q$ and thus $l = l'$. This means that p and p' are both incident with h, contradictory to the observation above. □

PROOF of Theorem 6.4.10 in case F_4. Again denote by $\alpha\colon \widetilde{\Delta} \to \Delta$ be the universal 2-cover of Δ. We first show that all residues of type C_3 are buildings; it follows then in particular that $\widetilde{\Delta}$ is a building. For stars of hyperlines it follows directly from 3.4.9 that they are buildings. For the stars of points this is a little bit more difficult, since we have not assumed (Lin) in its dualized form. We have to show that if two distinct objects x, x' of type 3 are incident with h and h', then $h = h'$. In the C_3-building $\mathrm{St}\,h$, we can find points p, p' on x, resp., x' which are noncollinear. (This is a general fact about any two distinct planes in any polar space of rank 3.) But by (LH), p and p' are noncollinear in the whole complex Δ, and from (HH) we then conclude $h = h'$.

Now we know that $\widetilde{\Delta}$ is a building and that α is injective on all stars of points and hyperlines. As above, we want to show that it is injective on the set of points. Assume the existence of points $p \neq p'$ with $\alpha p = \alpha p'$. Choose an h' with $p' - h'$, and according to Lemma 2.5.10(b) q, h with

$$p - h - q - h' - p'.$$

We have $h \neq h'$, since p and p' are not incident with a common h. Inside $\mathrm{St}\,q$ we conclude that also $\alpha h \neq \alpha h'$, and inside $\mathrm{St}\,h$ we see that $\alpha p \neq \alpha q$. Thus we can apply (HH) to $\alpha p, \alpha q, \alpha h, \alpha h'$ and find a line $\bar{l}$ incident with $\alpha p \neq \alpha q$. By (LH), it is incident to both $\alpha h, \alpha h'$. Take inverse images l, l' of $\bar{l}$ in $\mathrm{St}\,h, \mathrm{St}\,h'$. They are in fact equal since α is injective on $\mathrm{St}\,q$. But then p, p' are both incident with h, a contradiction. □

6.5. *Some applications, II: Group amalgamations*

We now come to applications of the theory of buildings to group amalgamations. One important result in this direction is the fact that a group G with a Tits system (B, N) is generated by the minimal parabolic subgroups $P_i \supset B$ in such a way that a set of defining relations can already be given inside the rank 2 parabolics P_{ij}. This result 6.5.2 is an immediate consequence of the 2-simple-connectedness of buildings (Proposition 3.3.9). Using the results of 6.1 to 6.3, one can attack the in a sense converse problem: given a collection of groups P_{ij} with Tits systems of rank 2, and specified identifications of the subgroups P_i, can the P_{ij} be embedded into one common overgroup G? Specifically, we discuss some complexes (belonging to nonspherical Coxeter diagrams) which provide a tool for investigating certain amalgams of *finite* groups and their possible quotients. This topic can be treated here only in a very incomplete fashion; our interest is to illustrate general methods which (at least partly) rely on Theorem 6.3.1 rather than to give specific results.

In order to make the problems and results precise, we first have to recall the notion of an amalgamated sum (of certain groups with respect to certain homomorphisms).

6.5.1. DEFINITION. Let $\mathcal{G}$ be a family of groups and Φ be a family of homomorphisms φ: $A_\varphi \to B_\varphi$, where $A_\varphi, B_\varphi \in \mathcal{G}$. The *amalgamated sum* of the system $(\mathcal{G}, \Phi)$ is a group G, together with a family of homomorphisms $\varkappa_A$: $A \to G$ satisfying $\varkappa_B \circ \varphi = \varkappa_A$ for all φ: $A \to B$ in Φ, which is universal in the sense that any other such system $(H, \lambda_A$: $A \to H, A \in \mathcal{G})$ is obtained from a unique homomorphism α: $G \to H$ by $\lambda_A = \alpha \circ \varkappa_A$ for all $A \in \mathcal{G}$. A group G is said to be the *amalgamated sum of a family $\mathcal{G}$ of subgroups $A \subseteq G$* if the family of inclusions ι_A: $A \hookrightarrow G$ satisfies this property for Φ the collection of inclusions $\iota_{A,B}$, where (A, B) runs over all pairs in $\mathcal{G}$ such that $A \subseteq B$.

The amalgamated sum of a general system $(\mathcal{G}, \Phi)$ is obviously unique up to a unique isomorphism. It always exists: take for G the group freely generated by the members of $\mathcal{G}$ and factor out all relations $ab = c$, where a, b, c lie in one common $A \in \mathcal{G}$, and the relations $\varphi x = x$, for all φ: $A \to B$ in Φ and all $x \in A = A_\varphi$.

The following easy proposition relates the connectedness properties of a transitive chamber system $\mathcal{C}(G, B, P_i, i \in I)$ (see 1.4.5) to the question of amalgamating G from the P_j.

6.5.2. PROPOSITION. *Let G be a group, B and P_i, $i \in I$, subgroups of G such that $B \subset P_i$ for all $i \in I$. Assume that G is generated by the P_i. Set $P_J := \langle P_j$: $j \in J\rangle$ for all nonempty subsets $J \subseteq I$, as earlier, and $P_\varnothing := B$. The following are equivalent, for a natural number $m \leqslant |I|$:*

(i) *the chamber system $\mathcal{C}(G, P_i, i \in I)$ is m-simply-connected;*

(ii) *G is the amalgamated sum of the P_J, $|J| \leqslant m$.*

PROOF. (i) $\Rightarrow$ (ii). Let

$$(\widetilde{G}, \varkappa_J\colon\ P_J \to \widetilde{G}, J \subseteq I, |J| \leqslant m)$$

be the amalgamated sum and α: $\widetilde{G} \to G$ be the canonical homomorphism such that $\alpha \circ \varkappa_J = \iota_J$ for all $J \subseteq I$, $|J| \leqslant m$, where ι_J: $P_J \to G$ denotes the inclusion map. α is surjective since the P_i generate G. Setting $\widehat{P}_J := \varkappa_J P_J$, we have an induced morphism of chamber systems

$$\bar{\alpha}\colon\ \widetilde{\mathcal{C}} := \mathcal{C}(\widetilde{G}, \widetilde{B}, \widetilde{P}_i) \to \mathcal{C}(G, B, P_i).$$

Since, for $|J| \leqslant m$, the J-star of the base chamber $\widetilde{B}$ in $\widetilde{C}$ equals $C(\widetilde{P}_J, \widetilde{B}, P_j, j \in J)$, and since α: $\widetilde{P}_J \to P_J$ is bijective, $\overline{\alpha}$ is an m-covering. By assumption, $\bar{\alpha}$ is an isomorphism. Thus $\operatorname{Ker}\alpha \subseteq \widetilde{B}$; but $\alpha|_{\widetilde{B}}$ is injective and hence α is injective, as desired.

(ii) $\Rightarrow$ (i). Let φ: $\widetilde{C} \to C := C(G, B, P_i, i \in I)$ be the universal m-covering. By Proposition 6.1.8, G lifts to a chamber transitive automorphism group

$$\widetilde{G} := \{\widetilde{g} \in \operatorname{Aut}(\widetilde{C})\colon\ \exists\ g \in G \text{ such that } \varphi \circ \widetilde{g} = g \circ \varphi\},$$

so α: $\widetilde{G} \to G$, $\widetilde{g} \mapsto g$, is a surjective homomorphism. Fixing some chamber $\widetilde{C}_0 \in \varphi^{-1}(C_0)$, $C_0 = B \in C$, we have an isomorphism

$$C(\widetilde{G}, \widetilde{B}, \widetilde{P}_i, i \in I) \to \widetilde{C}, \quad \widetilde{g}\widetilde{B} \mapsto \widetilde{g}\widetilde{C}_0,$$

where $\widetilde{B} \subset \widetilde{G}$ is the stabilizer of $\widetilde{C}_0$, and $\widetilde{P}_i$ the stabilizer of its i-panel. Using this identification, we see that

$$\varphi(\widetilde{g}\widetilde{C}_0) = (\alpha\widetilde{g})C_0.$$

Since φ is an m-covering, each $\widetilde{P}_J$, $|J| \leqslant m$, is mapped bijectively by α onto P_J. Since G is the amalgamated sum of the P_J, $|J| \leqslant m$, there is a unique homomorphism β: $G \to \widetilde{G}$ such that $\beta \circ \iota_J = (\alpha|_{\widetilde{P}_J})^{-1}$: $P_J \to G$. That is, $\beta \circ \alpha|_{\widetilde{P}_J} = \mathrm{id}$, and thus $\beta \circ \alpha$ is the identity on all of $\widetilde{G}$. Hence α is an isomorphism, and necessarily φ is bijective, as desired. □

In view of the 2-simple-connectedness of buildings (Proposition 3.3.11), the last proposition implies the following corollary.

6.5.3. Corollary. *If (G, B, N) is a Tits system, then G is the amalgamated sum of the parabolic subgroups P_J, $|J| \leqslant 2$.*

Let $(\mathcal{G}, \Phi)$ be a system of group homomorphisms as in 6.5.1. We say that $(\mathcal{G}, \Phi)$ *does not collapse*, or loosely speaking, that the amalgam G of $(\mathcal{G}, \Phi)$ *does not collapse* if all the $\varkappa_A$: $A \to G$ $(A \in \mathcal{G})$ are injective. In general, it is difficult to decide whether an amalgam collapses or not. The following theorem, which is a translation of the Second Main Characterization of buildings in the chamber transitive case, gives a positive result under particular assumptions.

6.5.4. Theorem. *Let M be a Coxeter diagram over I, and denote by $\mathcal{J}$ the set of all spherical subsets $J \subseteq I$ of cardinality $\leqslant 2$ (i.e. $J = \varnothing$, $J = \{i\}$, or $J = \{i, j\}$, $i \neq j$, $m_{ij} < \infty$). Let there be given a family $\mathcal{P}$ of groups P_J, $J \in \mathcal{J}$, together with a family of injective homomorphisms $\iota_{J,J'}$, $J \subseteq J' \in \mathcal{J}$, such that $\iota_{J',J''} \circ \iota_{J,J'} = \iota_{J,J''}$ whenever $J \subseteq J' \subseteq J'' \in \mathcal{J}$. For $K \subseteq I$, denote by $\mathcal{P}_K$ the subfamily consisting of all P_i, $i \in K$, together with the $\iota_{J,J'}$, $J \subseteq J' \subseteq K$, $J' \in \mathcal{J}$.*

If, for all spherical subsets J of cardinality $\leqslant 3$, the amalgam $G(\mathcal{P}_J)$ does not collapse and the chamber system $C(G(\mathcal{P}_J), P_\varnothing, P_j, j \in J)$ is a building of type M_J, then the whole amalgam $G(\mathcal{P})$ does not collapse, $C(G(\mathcal{P}), P_\varnothing, P_i, i \in I)$ is a building of type M, and the canonical map $G(\mathcal{P}_K) \to G(\mathcal{P})$ is injective for any subset $K \subseteq I$.

In the notation of the conclusion of the theorem, $P_\varnothing$ and all P_i's are considered as subgroups of all $G(\mathcal{P}_J)$, $J \ni i$, in view of the assumption (resp., conclusion) of noncollapsing.

6.5.5. *The locally finite case.* It has to be admitted that in general it appears to be difficult to produce the assumptions of the last theorem without assuming *a priori* that all P_J, $J \in \mathcal{J}$, are already subgroups of one common group $G = \langle P_i \rangle$, (in which case the noncollapsing is assumed in advance). Nevertheless, the second conclusion that $\widetilde{\mathcal{C}} = \mathcal{C}(\widetilde{G}, P_i)$, where $\widetilde{G} = G(\mathcal{P})$ denotes the amalgam, is a building might still be interesting in view of the canonical surjective homomorphism α: $\widetilde{G} \to G$ mapping a chamber stabilizer $\widetilde{B}$ of $\widetilde{G}$ in $\widetilde{\mathcal{C}}$ onto the given subgroup $B = P_\varnothing \subset G$.

Thus for given diagram M, the investigation of groups G containing a system P_J, $J \in \mathcal{J}$, of subgroups as above leads one to considering pairs $(\widetilde{\mathcal{C}}, \widetilde{G})$, where $\widetilde{\mathcal{C}}$ is a building of type M and $\widetilde{G}$ a chamber transitive subgroup of $\operatorname{Aut}\widetilde{\mathcal{C}}$ with a prescribed chamber stabilizer B. In view of the search for possible *finite* groups G occurring in such a situation, most of the literature deals with the problem under the following two additional assumptions:

(a) the building $\widetilde{\mathcal{C}}$ is locally finite (i.e. the stars of all panels are finite);

(b) the stabilizer B in $\widetilde{G}$ of a chamber is finite.

In the case where M is spherical, $\widetilde{\mathcal{C}}$ itself is finite, and is one of the known buildings if M is irreducible of rank $\geqslant 3$. In this situation (and also for the 'classical' rank 2 cases), all possible $\widetilde{G}$ have been classified by Seitz [1973]; except for a few individual cases $\widetilde{G}$ must be close to the full automorphism group of $\widetilde{\mathcal{C}}$.

An important further class of diagrams where $\widetilde{\mathcal{C}}$ is known (i.e. the classification of all buildings of type M is known), are the irreducible affine diagrams of rank $\geqslant 4$. Here, the building is the Bruhat–Tits building of a simple algebraic group over a local field F (i.e. F is complete with respect to a discrete valuation), and (a) means that the residue class field of the valuation is finite (i.e. F is a non-Archimedean locally compact field).

In this situation, the chamber stabilizer in $\operatorname{Aut}(\widetilde{\mathcal{C}})$ is infinite, and condition (b) says that $\widetilde{G}$ should be a 'small' subgroup of $\operatorname{Aut}(\widetilde{\mathcal{C}})$ (in fact, a discrete subgroup of $\operatorname{Aut}(\widetilde{\mathcal{C}})$ with its natural topology). The following theorem announced in Kantor, Liebler and Tits [1987] or Tits [1985/86] says that there is only a finite number of in a sense 'exceptional' buildings $\widetilde{\mathcal{C}}$ admitting such a discrete chamber-transitive automorphism group. The result also includes Bruhat–Tits buildings of rank 3 which however are not the only buildings belonging to an irreducible affine diagram of rank 3.

6.5.6. THEOREM. *Let Δ be the Bruhat–Tits building of a simple algebraic group G of relative rank $\geqslant 2$ over a locally compact local field F. Suppose that Δ admits a chamber transitive automorphism group with finite chamber stabilizer. Then (G, F) is one of the following*:

- $F = \mathbb{Q}_2$ *and G is the split group of type (i.e. belonging to one of the Dynkin diagrams)* A_2, C_2, G_2, A_3, B_3, D_4;
- $F = \mathbb{Q}_2$, *G is quasi-split of type* A_3, *and splits over* $\mathbb{Q}_2(\sqrt{-1})$ *or* $\mathbb{Q}_2(\sqrt{-3})$;
- $F = \mathbb{Q}_3$ *and G is split of type* C_2 *or quasi-split of type* A_3, *splitting over* $\mathbb{Q}_3(\sqrt{-3})$;
- $F = \mathbb{F}_2((x))$ *or* $\mathbb{F}_8((x))$ *and G is split of type* A_2.

See 4.4.1 for the notions 'split' and 'quasi-split' occurring in the theorem. The theorem of Kantor, Liebler and Tits says that buildings admitting finite chamber transitive quotients (in the sense of 2-coverings) are extremely rare in the case of affine Coxeter diagrams (irreducible, of rank $\geqslant 4$). One might suspect that this relies on particular properties of this restricted class of diagrams. Work of Timmesfeld, Stroth, Meixner and others shows that in a sense this is not the case. It is not the diagram, but essentially the restrictions on the rank 2 residues and the condition of a finite chamber stabilizer which make it possible to prove results of the kind of Theorem 6.5.6.

To make this a little bit more precise, let us fix the following assumptions: M is an arbitrary Coxeter diagram, without loss of generality irreducible, of rank $\geqslant 3$, and $\mathcal{C} = \mathcal{C}(\widetilde{G}, B, P_i, i \in I)$ is a transitive chamber system of type M satisfying (a) and (b) above. Since generalized polygons are not classified even in the finite chamber transitive case, it is necessary to explicitly require certain restrictions on the rank 2 residues. (In the case where M is spherical and $\mathcal{C}$ is a building, such a restriction is given by the Moufang Property, but no such theorem is known in the general case.) Following Timmesfeld and others, we make the following assumption:

(c) for all $i, j \in I$ with $m_{ij} > 2$, the generalized polygon $\mathcal{C}_{ij} = \mathcal{C}(P_{ij}, B, P_i, P_j)$ is classical, i.e. it belongs to one of the finite groups of Lie type or to the Ree group ${}^2\mathsf{F}_4(2^{2h+1})$ (cf. Chapter 9; the Ree groups are the ones with $m = 8$, i.e. those leading to generalized octagons).

This assumption in particular says that $m_{ij} \in \{2, 3, 4, 6, 8\}$ whenever $i \neq j$. Furthermore, by Seitz' result mentioned above, not only the generalized polygons, but also the possible factor groups $\overline{P}_{ij}$ of the P_{ij} acting effectively on $\mathcal{C}_{ij}$ are known. For the remainder of this section, we shall use the *ad hoc* terminology *classical amalgam of type* M for such a system of groups (B, P_i, P_{ij}) over M occurring in some group G. For the results on which we shall report now, it is only assumed that $\mathcal{C}$ is of type M, not necessarily a building. It should be mentioned that there are other chamber-transitive generalized polygons (for $m = 3, 4$) besides the classical ones; thus, assumption (c) really is of a group theoretical nature. The problem of classifying such systems and if possible even all finite groups containing a given system was initiated and partly solved in the paper Timmesfeld [1983]. Today a full classification of all classical amalgams is known by combining results of Timmesfeld [1983, 1984/86, 1987], Stroth [1988, 1990] and Meixner [1990]. There are various such systems in addition to those given by the finite buildings; in particular, some infinite families of diagrams occur, namely complete graphs and complete bipartite graphs. On the other hand, it is a remarkable fact that only finitely many residues occur if we exclude the obvious cases where $\widetilde{\mathcal{C}}$ is a finite building. We refer to the survey article Meixner [1990] for a complete list of all cases, for a precise statement of the assumptions and conclusions, and for references. See also Chapter 22, Section 3. We only state one result which makes the 'negative side' of the classification more precise. It is a very weak form of the main result of Timmesfeld [1987]. This paper makes essential use of Niles [1982], from which the weak version in fact immediately follows.

6.5.7. THEOREM. *Let (B, P_i, P_{ij}) be a classical amalgam such that at least one rank 1 residue P_i/B has more than 9 elements. Then $\mathcal{C} = \mathcal{C}(\widetilde{G}, B, P_i, i \in I)$ is a finite building and $\widetilde{G}$ contains the simple group belonging to $\mathcal{C}$.*

If the diagram is spherical, one may replace the number 9 occurring in the last theorem by 2, and only one chamber system in a addition to the finite buildings occurs. We now state the corresponding theorem.

6.5.8. THEOREM. *Let (B, P_i, P_{ij}) be a classical amalgam of spherical type in G. Then one of the following holds*:

(a) $\mathcal{C} = \mathcal{C}(G, B, P_i)$ *is a finite building and* G *contains the corresponding simple group of Lie type, or* $G = \mathrm{Alt}_7$ *(alternating group) acting on projective* 3*-space over* $\mathbb{F}_2$;

(b) $M = \mathsf{C}_3$, $(\mathcal{C}, G = \mathrm{Alt}_7)$ *is as described in Example* 3.1.9.

In view of the Second Main Characterization of Buildings, this theorem answers in every concrete situation the question whether the universal 2-cover of a chamber system belonging to a classical amalgam is a building. For $M = \mathsf{C}_3$, the last theorem makes precise the result of Aschbacher mentioned in the Notes to Section 3.

Although we did not list the classical amalgams we finally at least want to mention that they can be roughly subdivided into (at least) two classes (in the nonspherical case). Many of them are closely related to (in particular, have the same universal cover as) so-called parabolic systems in a group. Such systems are classical amalgams subject to certain group-theoretical hypotheses, in particular the parabolics share a common p-Sylow subgroup, for some prime p. This result which is stated in precise form in Timmesfeld [1987] and Timmesfeld [1984/86], 3.2, in particular shows that hypothesis (c) above really is a group theoretical one (of course, modulo Seitz' theorem). The classical amalgams which do not come from parabolic systems in particular include those where all P_{ij} are among the exceptionally small chamber transitive groups admitted by Seitz' theorem. In these cases, B may even be trivial. This in particular eliminates the difficulty mentioned at the beginning of 6.5.5 of how to specify an amalgam without knowing at least one group in which it is contained. We reproduce one well known example where one can apply directly Theorem 6.5.4. Details are given in Tits [1985/86] or Ronan [1989]

6.5.9. EXAMPLE. Let M be the complete graph on I, i.e. $m_{ij} = 3$ for all $i \neq j$, let $B = \{1\}$, $P_i = Z_3$ cyclic of order 3, for all i, and P_{ij} be the Frobenius group of order 21, for all $i \neq j$. Fix two distinct subgroups $P_{i,j}, P_{j,i}$ of order 3 in P_{ij}. Then it is first of all readily checked (or well known) that $\mathcal{C}(P_{ij}, B, P_{i,j}, P_{j,i})$ is the projective plane of order 3. There are two possible embeddings $\iota_{i,j}\colon P_i \to P_{i,j} \subset P_{ij}$ which are invariantly distinguished by the action on the subgroup of order 7 in P_{ij}: a distinguished generator of $P_i = Z_3$ can act by squaring or by raising to the 4th power. Specifying one of these two cases, for each ordered pair (i, j), defines a system Φ as 6.5.1, more specifically a classical amalgam. Since there are no spherical subdiagrams of rank 3, the 2-simply-connected chamber system $\widetilde{\mathcal{C}}$ in any case is a building. For instance, for $|I| = 3$, i.e. $M = \widetilde{\mathsf{A}}_2$, there are essentially 4 such amalgams and thus 4 different buildings. Two of them occur in Theorem 6.5.6 and had been described earlier in Köhler, Meixner and Wester [1983, 1984]. They admit infinitely may finite chamber transitive quotients, in particular an infinite series of finite groups of Lie type (with fixed diagram) acts on such chamber systems. For the third amalgam, at least one finite quotient is known, with group the alternating group Alt_7; see Ronan [1989], Chapter 4, Exercise 12. For the last one, the existence of a finite quotient is not known.

Notes to Section 6

Coverings and the Second Main Characterization of Buildings. There are almost no bibliographical notes on Sections 6.1 to 6.3, since – as we pointed out earlier – everything is taken from Tits [1981a], and there are almost no other papers dealing with this subject. Our treatment of covering theory in 6.1 is a little bit more explicit than that of *loc. cit.* We have made use of an unpublished manuscript of A. Dress; the same information is also contained in Ronan [1980, 1989]. The results are anyway precisely what one expects from the analogy to covering theory of topological spaces. We are not aware of any further publication dealing with the important Theorem 6.2.3. This is surprising since the proof of this theorem raises quite a few questions. Vaguely spoken, a better understanding of the construction of $\overline{\mathcal{C}}$ is desirable, and of the conditions under which it works. More precisely, one could expect $\overline{\mathcal{C}}$ to be a 'universal cover' in some more general covering theory, related to Weyl homotopies. If one had formulated such a covering theory, one could ask for an 'if and only if' statement dealing with the building property of $\overline{\mathcal{C}}$. Finally, it might turn out to be fruitful to relate the proof of 6.2.3 in an explicit way to the gate property 5.1.7 and the exchange property for geodesics 3.3.9. Observe that the gate property for the Coxeter group (the existence of shortest coset representatives) is used in an apparently essential way in the above proof of 6.2.3. From Scharlau [1985a] (see 5.1.11) it is known that the gate property together with 'minor' additional assumptions implies the building property. But the natural question whether this could contribute to Tits' proof does not seem to have been answered yet.

The results 6.3.6 and 6.3.7 seem to be new (at least in our explicit and precise form). Although the 'essential generating homotopy' in the A_3-case has already been given in Ronan and Tits [1987], there have been no indications about the topological explanation of the formula. Remark 6.3.7 is intended to make precise the remark at the beginning of 4.3.3 in Tits [1981a]: there is not only an analogy with topological notions, but simple topological facts can actually be used in the present context. See Ronan [1981] for a vaguely related result about the second homotopy group of more general complexes.

Shadows in buildings. We first of all want to emphasize the fact that the theory of shadows in buildings (Theorem 6.4.2 and 6.4.5) is a subject in its own right whose significance goes beyond the use of these results in the present context. Lack of space prevents us from treating this topic in full detail. Complementing what we said in the Notes to Section 1 about shadows in general, we give now some comments about the particular properties of 'shadow geometries' and shadow posets of buildings.

In the Notes to Section 1, we already mentioned the 'shadow poset' $\mathcal{S} = \mathcal{S}(\Delta, I_0)$ introduced in Scharlau [1990]. It consists by definition of all I_0-reduced simplices. Its partial ordering '$\preceq$' is given exactly by the condition occurring in Theorem 6.4.2(b):

$$A \preceq B :\Leftrightarrow A \cup B \text{ and } \operatorname{type} A \preceq \operatorname{type} B.$$

This definition works for any strongly connected numbered complex, and the implication '$\Leftarrow$' of 6.4.2(b) holds in general, as well as does the equality $\mathrm{Sh}(B) = \mathrm{Sh}(B_{\mathrm{red}})$ of 6.4.2(a). That is, one has a surjective morphism from the shadow poset $\mathcal{S}$ onto the set of subspaces of the 'point set' $S = S(\Delta, I_0)$ introduced earlier. Notice that, once one has shown that the above relation '$\preceq$' on reduced simplices is antisymmetric, the second part of 6.4.2(a) is a consequence of 6.4.2(b). In Proposition 6.4.5(a), the inclusion '$\supseteq$' is the nontrivial and interesting one. It is easily reduced to the case of a Coxeter complex. Analysing Tits' proof, one sees that the formula $W_A \cap W_B = W_{\mathrm{pr}_A B}$ of Proposition 5.1.10(a) is the essential ingredient. For more about this formula, see also Cohen [1991], 5.2. The reader will have no difficulty in deriving the inclusion '$\supseteq$' of 6.4.5(a) directly from 5.1.10(a), without following Tits' way through 12.7 of Tits [1974]. In view of the basic importance of Theorem 6.4.5(a) for the geometrical interpretation of buildings one might ask whether one can also give a 'modern' proof of this result in the spirit of the 'First Main Characterization of Buildings', i.e. avoiding the reduction to apartments and avoiding explicit calculations in the Weyl group. This is indeed possible: the desired inclusion of shadows is a straightforward consequence of Theorem 5.1.9 (or rather its more complete version stated in Dress and Scharlau [1987]).

We have no reference for result 6.4.5(b). It was suggested by Tits [1981a], 6.2.2, though in essence it also goes back to Tits [1974], Appendix I: Shadows. Trivial examples show that the result really holds only for $(\mathrm{pr}_A A')_{\mathrm{red}}$, and not for $\mathrm{pr}_A A'$.

Geometrical axioms, II. The results 6.4.6 to 6.4.12 have been taken without changes from Tits [1981a]. For further results on geometries belonging to the diagram F_4 and on the characterization of the buildings among them, see Pasini [1988a,b] and the references quoted there. In the case of finite geometries belonging to one of the diagrams E_6, E_7, E_8, a remarkable improvement of Tits' results has been given in Brouwer and Cohen [1983]: buildings are the only finite, thick complexes belonging to one of these diagrams. This is derived from the Second Main Characterization of Buildings and the quite general result that thick finite buildings do not admit proper quotients (see the proof of Proposition 8 of *loc. cit.*). The paper contains also some sharpenings of Tits' results in the general case, but under the additional hypothesis that the complex Δ is the flag complex of an incidence geometry; recall 3.4.10 dealing with E_6.

The particular role that 'points' and 'lines' play in the above characterizations of buildings is the subject of a general theory of 'point-line-spaces' (leading to buildings). This theory is mainly due to Buekenhout, Shult, Cooperstein, Cohen, Hanssens, Van Maldeghem, and is presented in Chapter 12. Notice that, though our focus here is on exceptional diagrams, the origin of the point-line approach is the Veblen–Young axiomatics of projective spaces and the Buekenhout–Shult theory of polar spaces. The point-line theory in particular shows that a building of type E_6, E_7 or E_y is determined by its point-line space. (For F_4 this is more or less clear by Example 1.1.6, since here the points correspond to an end node of a linear diagram.) A sort of generalization of this fact is the following result which is proved, though not stated in fully explicit form, in Tits [1974], 12.18–12.23: a building Δ is determined up to isomorphism by any of its shadow spaces (i.e. the distinguished type set I_0 is arbitrary), together with the partitioning of the vertices of Δ given by the type function.

Exceptional geometries for certain Coxeter diagrams. Section 6.5 contains only a brief and partial sketch of an area of particularly active research starting about 1980 (with earlier predecessors) and coming into some definite status in about 1987. The initial question was the search for systems of subgroups P_i, $i \in I$, in a finite (close to simple) group G such that the chamber system $\mathcal{C}(G, P_i, i \in I)$ is of Coxeter type, but not one of the known buildings. (Of course, other diagrams were considered as well, but this is not our topic here.) A particular motivation for pursuing this question was the discovery, by Buekenhout and Kantor, of a geometry with diagram $\widetilde{\mathsf{G}}_2$ for the Lyons sporadic group Ly (Kantor [1981]). Other examples were found in groups of Lie type, say over $\mathbb{F}_3$, but with $\mathbb{F}_2$ as the field of definition of the residues; see Ronan and Smith [1979/80], Kantor [1981, 1985], Timmesfeld [1983]. The question for a description or 'identification' of the universal cover $\widetilde{\mathcal{C}}$ of the geometry (or chamber system) in question immediately arose. By Theorem 6.3.1 (which was quite new at that time), one knew that in many cases $\widetilde{\mathcal{C}}$ had to be a building. Of special interest was the case of an affine Coxeter diagram which often occurred in the examples. Here, a big list of candidates for $\widetilde{\mathcal{C}}$ was available, namely the Bruhat–Tits buildings of semisimple groups over local fields. In fact, the main theorem of Tits [1984/86] says that any affine building of rank $\geqslant 4$ is necessarily of this kind. The algebraic identification of certain universal covers and the construction of further examples, starting from an infinite building and not from some finite group, is the subject of Kantor [1985], Köhler, Meixner and Wester [1983, 1984], Meixner [1984/86, 1986], Meixner and Wester [1986], Kantor, Meixner and Wester [1990], and some further papers. The theorem from Kantor, Liebler and Tits [1987] quoted above can be considered as the final result in this direction of research. Its full proof is still unpublished. Some further information about the buildings and groups obtained in this theorem is given in Kantor [1988]; indications about a closer connection to the theory of arithmetic groups are made in Kantor [1988/90].

Returning to the question of parabolic systems, we finally want to mention that geometries of Coxeter type alone certainly are not sufficient for a 'geometrical treatment' (in whatever precise sense) of the sporadic simple groups. Other, 'sporadic', rank-2 residues must be admitted. This is clearly shown, e.g., by the paper Ronan and Stroth [1984]. In this paper, 'minimal parabolic subgroups' of a finite group are defined in an appropriate way, starting from the normalizer of a p-Sylow subgroup, and the chamber systems of minimal generating families of such subgroups are determined. As one example for a more general, though not really 'sporadic', rank-2 constituent we mention the natural geometry of the group $3\,\mathrm{Sp}_4(2)$ which is a 3-fold cover of the generalized quadrangle belonging to $\mathrm{Sp}_4(2)$. See also Buekenhout [1984/86] and Chapter 22 of this handbook for geometries for sporadic groups.

Amalgams. Corollary 6.5.3 and essentially also Proposition 6.5.2 have been taken from Tits [1974], Appendix 2 'Generators and relations'. We cannot give an explicit reference for 6.5.2 in its above simple and general form. Tits also gives a variation of 6.5.3 which says that a group G with a Tits system (B, N) can be obtained by amalgamating the P_i and N over B, and is thus determined by B, the inclusions $B \hookrightarrow P_i$ and the $N_i := N \cap P_i$. In Tits [1981b], the systems $(B, P_i, N_i,\ i \in I)$ which actually lead to a Tits system such that the P_i embed into G are characterized. Applications of this result include an 'elementary' existence proof for Kac–Moody groups. A general theory, which contains the rank-2 case of 6.5.3 as a special case, is developed in the book Serre [1977/80]. The main result describes how to obtain, for an arbitrary group G acting on a connected graph, a presentation of G in terms of certain isotropy groups and the fundamental group of the graph.

Despite its general title, the remainder of Section 6.5 deals almost exclusively with certain exceptional geometries of Coxeter type. In conclusion, we want to mention some further results which fit under the theme 'Geometries and group amalgamations'. In Tits [1986], group actions on partially ordered sets are studied in connection with a certain notion of homotopy and covering in the category of posets. A proposition is proved (Corollary 1 of *loc. cit.*) which generalizes both Serre's theorem for groups acting on trees and Proposition 6.5.2 above. The greater flexibility of this result allows Tits to attack also the problem of analyzing, for certain Tits systems (G, B, N), the structure of the group B itself. If the Tits system comes from a 'root datum' (cf. the Notes to Section 5), like in the case of Chevalley groups or Kac–Moody groups, the greatest 'positive unipotent' subgroup, usually called U, can often be shown to be the amalgamation of its subgroups U_{ab} belonging to sub-root-systems of rank 2, over the root groups U_a (Section 16 of *loc. cit.*). Often, the problem of restricting the common subgroup B of a family P_i, $i \in I$, of groups which one wants to amalgamate, is set under the assumption that a certain part of the structure of the P_i is specified. Geometrically, some 'local structure' of the action of the (desired) amalgam G on the chamber system $\mathcal{C}(G, B, P_i, i \in I)$ is given in advance. This question has found particular attention in the case where the P_i are finite (i.e. where B is finite and the chamber system has finite valencies); recall the above notion of a parabolic system. (In the finite case, usually the factor groups $P_i/O_p(P_i)$ are restricted in advance; here $O_p(H)$ denotes the intersection of all p-Sylow subgroups of H.) The so-called 'amalgam-method' is a general method for attacking such problems. It originated from the paper Goldschmidt [1980], and developed into a proper theory in the monograph Delgado and Stellmacher [1985]; they solve completely a certain rank-2 classification problem with rather general assumptions. The amalgam method is also one of the tools used in the solution of the classification problem for parabolic systems. See Stellmacher and Timmesfeld [1989] for a classification in rank 3 with more general rank-2 residues than generalized polygons, obtained by the amalgam method. The amalgam method has also applications to so-called 'pushing-up' problems in finite groups; see, e.g., Stellmacher [1986], Meierfrankenfeld [1988a,b].

7. The classification of buildings of irreducible spherical type and rank at least 3

Introduction

In this section, we shall report on the classification of buildings of irreducible, spherical type and rank $\geqslant 3$, i.e. belonging to one of the Coxeter diagrams A_n, C_n, D_n, $n \geqslant 3$, F_4, E_6, E_7, E_8. This classification was achieved by Tits around 1968 and is the main result of Tits [1974]. The proof roughly consists of two parts: a description of all 'known' buildings (known in terms of quadratic forms, algebraic groups, or the like, and mostly known long before the proof of the final result), and a proof of the fact that any abstractly given building (of irreducible, spherical type and rank $\geqslant 3$) is isomorphic to one of the known buildings. The techniques for the second step also provide a description of the automorphism groups of the buildings in question which in most cases turns out to be an algebraic or classical simple group. These two aspects of the classification are not strictly separate from each other. For instance, the existence of a building of type D_n, $n \geqslant 4$, or E_6, E_7, E_8 for each field k is most easily proved in terms of the corresponding groups (easily if one assumes the theory of Chevalley groups for these diagrams). Roughly speaking, one could say that the existence part of the result is close to the theory of algebraic and related groups and to 'classical' geometric algebra, whereas the isomorphism part is one of the highlights of the abstract, axiomatic theory of buildings as developed in the preceding sections of this survey. At least this was the situation with the classical text Tits [1974]; the more recent general notions and techniques of Tits [1981a] and Ronan and Tits [1987] allow to treat some of the existence questions in an 'elementary' fashion, independent of the theory of algebraic groups.

This section is subdivided into three subsections as follows. Subsection 7.1 reports on a basic theorem due to Tits which says that a thick spherical building is determined by the sub-chamber system consisting of all chambers having a panel of codimension 2 in common with some fixed chamber. This is the content of Chapter 4 of Tits [1974] which is entitled 'Reduction'. I suggest the name 'Extension Theorem' for the main result. Remember that the Extension Theorem has already been used above in an essential way for the proof of the Moufang property. The proof of the Extension Theorem is long, complicated, and technical. We cannot give it in this article.

Subsection 7.2 describes a technique of constructing buildings due to Ronan and Tits [1987] which is known under the notions of foundations, labellings, and blueprints. The abstract part of this technique is based on the First and Second Main Characterization of Buildings, whereas the concrete application of the method in the case of spherical diagrams heavily uses the Moufang property.

In the concluding Subsection 7.3, we state a series of theorems which, altogether, can be considered as a classification of all thick, spherical, irreducible buildings of rank $\geqslant 3$. We have tried to give complete and precise formulations which are as self-contained as possible. Lack of space prevents us from presenting other parts of the proof than those already covered previously. For further results, the reader is referred to Tits [1974]. One main result whose proof is missing here is the classification of C_3-buildings (or polar spaces of rank 3).

7.1. *Isomorphisms of spherical buildings*

This subsection is devoted to one single theorem which ensures the existence of isomorphisms between spherical buildings which extend certain maps defined only locally. The precise formulation of this theorem is as follows.

7.1.1. EXTENSION THEOREM (strongest form). *Let Δ, Δ' be thick buildings of spherical type, fix chambers $C_0 \in \Delta$, $C_0' \in \Delta'$. Consider the sets of chambers*

$$\mathcal{E}_2(C_0) := \{C \in \mathcal{C}_\Delta\colon\ \mathrm{cod}(C \cap C_0) \leqslant 2\},$$

and $\mathcal{E}_2'(C_0')$ *defined analogously. Any adjacency preserving map*

$$\varphi\colon\ \mathcal{E}_2(C_0) \to \mathcal{E}_2'(C_0'),$$

i.e.

$$C - D \Leftrightarrow \varphi C - \varphi D \text{ for all } C, D \in \mathcal{E}_2(C_0),$$

extends to an isomorphism from Δ *onto* Δ'.

This theorem is the essential tool for the classification of thick, spherical buildings. Roughly speaking, it allows to 'identify' spherical buildings with a fixed diagram and fixed rank 2 residues. A precise version of what 'identification' could mean is given by the technique of blueprints explained in the next subsection. The concrete application of this machinery (which is partly formulated in very general terms) relies on the Moufang Property of (spherical) buildings. Recall from Subsection 5.4 that the Moufang property itself is a consequence of the Extension Theorem.

Notice that in Theorem 7.1.1 we have used the notion of an adjacency preserving map between chamber systems. *A priori* this is much weaker than the notion of a morphism: it is not required that the types of adjacencies, i.e. the various $\overset{i}{\sim}$ are respected. In a first step of the proof one shows that the map φ actually is a morphism of chamber systems with respect to an appropriate bijection between the two type sets. At the same time it turns out that this bijection between types actually is an isomorphism of Coxeter diagrams. In other words, Δ and Δ' have isomorphic apartments. These considerations reduce 7.1.1 to the following weaker version of the Extension Theorem.

7.1.2. EXTENSION THEOREM. *Let* Δ, Δ', $C_0 \in \Delta$, $C_0' \in \Delta'$ *be as in* 7.1.1, *suppose further that* Δ *and* Δ' *have the same diagram, and let* $\Sigma \ni C_0$, $\Sigma' \ni C_0'$ *be apartments. Then any isomorphism*

$$\varphi\colon\ \mathcal{E}_2(C_0) \cup \Sigma \to \mathcal{E}_2'(C_0') \cup \Sigma'$$

extends to a unique isomorphism from Δ *onto* Δ'.

This form of the theorem is, in fact, sufficient for the essential applications. Remember that the uniqueness of the extension is already known from Theorem 5.2.11 (with a weaker hypothesis, $\mathcal{E}_1$ instead of $\mathcal{E}_2$). The uniqueness proof gives an initial indication of how the existence proof runs: the crucial tools are opposite chambers and projections onto stars.

7.1.3. EXAMPLE. In this example, we want to illustrate the nonuniqueness of the extension of φ in 7.1.1. Let k be a field, $n \geqslant 3$, and Δ the building of type A_n over k in its standard form, i.e. the flag complex of (nonempty, proper) subspaces of the vector space k^{n+1}. Let C be its standard chamber (coming from the ordered basis of the unit vectors.)

Let U be the subgroup of $\mathrm{GL}_{n+1}(k)$ fixing pointwise $\mathcal{E}_2(\Delta, C)$. Then U consists of the following matrices, in particular U is highly nontrivial:

$$\begin{pmatrix} a & 0 & 0 & a_{1,4} & \dots & a_{1,n-3} & \dots & \dots & a_{1,n} \\ 0 & a & 0 & 0 & \dots & a_{2,n-3} & \dots & \dots & a_{2,n} \\ 0 & 0 & a & 0 & \dots & \dots & \dots & \dots & a_{3,n} \\ 0 & 0 & 0 & a & \dots & \dots & \dots & \dots & a_{4,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & a & 0 & 0 & a_{n-4,n} \\ 0 & \dots & \dots & \dots & \dots & 0 & a & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & a \end{pmatrix}$$

for arbitrary elements $a_{i,j} \in k$ and nonzero $a \in k$. For $n = 3$ and $C = (p, l, \pi)$, we observe that the group fixing pointwise $\mathcal{E}_2(\Delta, C)$ is the set of elations with centre p and axis π.

7.1.4. *Applications to polar spaces.* We now present two results about isomorphisms and automorphisms of polar spaces which easily follow from the general extension theorems. The first one actually uses the strong form 7.1.1. (This is just for convenience, though; one could also give a proof using 7.1.2 only.)

If S is a polar space and $p \in S$ a point, we set

$$S(p) = \{q \in S\colon\ p, q \text{ collinear}\}.$$

7.1.4a. THEOREM. *Let S and S' be thick polar spaces of rank $\geqslant 3$ and $p \in S,\ p' \in S'$. If there exists a collinearity preserving bijection between $S(p)$ and $S'(p')$, then S is isomorphic to S'.*

PROOF. Let Y be the set of all subspaces L of S such that $L \subseteq S(p)$, and define similarly Y'. Clearly, a collinearity preserving bijection $\varphi\colon S(p) \to S'(p')$ extends to an isomorphism of complexes

$$\operatorname{Flag} Y \xrightarrow{\cong} \operatorname{Flag} Y'.$$

Fix a chamber $C = \{L_1, \dots, L_n\}$ of S such that $L_1 = p$ and $L_i \subset S(p)$ for all i. If $D = \{K_1, \dots, K_n\}$ is a chamber of S such that $K_i \neq L_i$ for at most two indices i, then $K_n \supset p$ and thus $K_i \subset S(p)$ for all i. Thus

$$\mathcal{E}_2(C) \subset \operatorname{Flag} Y, \quad \mathcal{E}_2(C') \subset \operatorname{Flag} Y',$$

and Theorem 7.1.1 can be applied to the restriction to $\mathcal{E}_2(C)$ of the extended map. □

For the next theorem, we fix two noncollinear points p_1, p_1' of a polar space S of rank n, and we set

$$S_\infty = S_\infty(p_1, p_1') := S(p_1) \cap S(p_1').$$

Note that S_∞ is a polar space of rank $n - 1$.

7.1.4b. THEOREM. *Let S be a thick polar space of rank $n \geqslant 3$ whose subspaces are Desarguesian, as it is the case when $n \geqslant 4$. Let*

$$H := \{\alpha \in \mathrm{Aut}(S)\colon\ \alpha \textit{ fixes } p_1, p_1' \textit{ and } S_\infty \textit{ pointwise}\}.$$

Then for any line l with $l \ni p_1$, the group H acts sharply transitively on $l \setminus \{p_1, l_\infty\}$, where l_∞ denotes the unique point in $l \cap S_\infty$.

PROOF. For given l, choose a polar frame $\{p_2, \dots, p_n, p_2', \dots, p_n'\}$ of S_∞ with $p_2 = l_\infty$. Then $B = \{p_1, \dots, p_n, p_1', \dots, p_n'\}$ is a polar frame of S. It is a well known property of Desarguesian projective spaces that for given points $x, y \in l \setminus \{p_1, p_2\}$, there exists an automorphism β of the projective space $M := [p_1, \dots, p_n]$ fixing p_1 and $M_\infty := M \cap S_\infty$ pointwise and mapping x to y. (For the projective space of a vector space, β is given by a diagonal matrix with the 2nd to n-th entry equal to 1.) Using Theorem 7.1.2, we want to extend β to all of S. To this end, let Σ be the apartment belonging to frame B and $C_0 = \{[p_1], [p_1, p_2], \dots, [p_1, \dots, p_n] = M\}$ its standard chamber. We have to deal with the following three sets of vertices of $\Delta(S)$, i.e. subspaces L of S:

$$X_M := \{L \subset S\colon\ L \subseteq M\}, \qquad X_p := \{L \subset S\colon\ L \ni p_1\},$$
$$X_\Sigma := \{L \subset S\colon\ L \in \Sigma\}.$$

First of all we show that if C is a chamber in $\mathcal{E}_2(C_0)$, then actually

$$C \in \mathrm{Flag}(X_M \cup X_p).$$

Indeed, if $C = \{L_1, L_2, \dots, L_n\}$ with $L_i \neq [p_1, \dots, p_i]$ for at most two indices i, it readily follows that $L_i \ni p_1$ or $L_i \subseteq M$, for each i. We now claim that it makes sense to extend β from X_M to $X_M \cup X_p \cup X_\sigma$, in particular to $\mathcal{E}_2(C_0) \cup \Sigma$, by requiring $\widetilde{\beta} L = L$ for $L \in X_p \cup X_\Sigma$. This is well defined because, if $L \in X_M \cap X_p$ or $L \in X_M \cap X_\Sigma$, then $L = [p_1, L \cap M_\infty]$ and thus $\beta L = L$, by the choice of β. Let α be the extension of $\widetilde{\beta}$ to all of $\Delta(S)$, according to Theorem 7.1.2. We have to show that $\alpha \in H$, i.e. α fixes S_∞ pointwise. But $M_\infty = [p_2, \dots, p_n]$ and $M_\infty' = [p_2', \dots, p_n']$ are two complementary maximal subspaces of S_∞ which are preserved by α and such that the first one is even fixed pointwise. Thus, Corollary 4.2.6(a) implies our claim.

The proof that α acts sharply transitively is given by the same method. If α fixes one further point $x \in l \setminus \{p_1, p_2\}$, it is the identity on l and on M_∞ and thus on M. Applying 4.2.6(b) to M and $M' = [p_1', \dots, p_n']$ gives $\alpha = \mathrm{id}$. □

We devote the remainder of this subsection to a 'weak version' of the Extension Theorem, where φ is already supposed to be defined globally. This theorem is the main step in the reduction of the strong Extension Theorem 7.1.1 to Theorem 7.1.2. It also is of some independent interest, since it illustrates how strong the notion of a building is even if one does not use types and the diagram explicitly.

7.1.5. THEOREM. *Let Δ, Δ' be buildings for a Coxeter diagram M with all $m_{ij} < \infty$. Let*

$$\varphi\colon \mathcal{C}(\Delta) \to \mathcal{C}(\Delta')$$

be an adjacency preserving bijection. Then there exists an unique isomorphism of diagrams

$$\varphi_*\colon \operatorname{diag} \Delta \to \operatorname{diag} \Delta'$$

such that φ is an isomorphism with respect to φ_*, i.e.

$$C \overset{i}{\sim} D \Rightarrow \varphi C \overset{\varphi_* i}{\sim} \varphi D \quad \text{for all } C, D \in \mathcal{C}(\Delta).$$

By the general results of Subsection 1.3, φ uniquely extends to an isomorphism from Δ onto Δ'.

7.1.6. PROOF of 7.1.5. Here are some observations which will be used in the construction of φ_*.

7.1.6a. LEMMA. *Let Δ be a numbered complex, $C, C_1, C_2 \in \mathcal{C}_\Delta$.*

(a) *If C_1, C_2 are adjacent and* $\operatorname{cod}(C \cap C_1) \leqslant 2$, $\operatorname{cod}(C \cap C_2) \leqslant 2$, *then*

$$\operatorname{cod}(C \cap C_1 \cap C_2) \leqslant 2.$$

(b) *If C, C_1, C_2 are pairwise adjacent, then they have a panel in common.*

The proof is straightforward.

7.1.6b. COROLLARY. *Let Δ, Δ' be numbered complexes, $C \in \mathcal{C}(\Delta)$, and let $\varphi\colon \mathcal{E}_2(C) \to \mathcal{C}(\Delta')$ be an injection whose restriction to all* $\operatorname{St} A$, $A \subseteq C$, $\operatorname{cod} A \leqslant 2$ *is adjacency preserving. Then φ is adjacency preserving.*

Notice that part (b) of Lemma 7.1.6a can be rephrased by saying that the stars of panels are precisely the maximal sets of pairwise adjacent chambers. Thus the following is clear:

7.1.6c. COROLLARY. *If φ is as in the Extension Theorem, then it uniquely extends to the panels F with* $\operatorname{cod}(F \cap C) \leqslant 2$, *by the formula*

$$\varphi\mathcal{C}(\operatorname{St} F) = \mathcal{C}(\operatorname{St} \varphi F).$$

If φ is defined on all of $\mathcal{C}(\Delta)$, then the extension is defined for all panels.

Corollary 7.1.6c already gives the desired map $\varphi_*\colon I(\Delta) \to I(\Delta')$ between the type sets. In order to show that this actually is an isomorphism of diagrams, we must go one step further, namely extend φ to elements of codimension 2. This is done as follows.

7.1.6d. LEMMA. *Let Σ be a Coxeter–Tits complex, $C, D \in \mathcal{C}(\Sigma)$. Assume that there are precisely 2 geodesics joining C to D. Then* $\operatorname{cod}(C \cap D) = 2$, *and the two galleries exhaust all of* $\mathcal{C}(\operatorname{St}(C \cap D))$.

For the proof, consider the two convex chamber subcomplexes defined by the two galleries minus the initial chamber C. □

7.1.6e. LEMMA. *Under the assumptions of the Extension Theorem, φ induces a bijection*

$$\mathcal{C}(\operatorname{St}(F_1 \cap F_2)) \to \mathcal{C}(\operatorname{St}(\varphi F_1 \cap \varphi F_2))$$

for all panels $F_1, F_2 \subseteq C_0$.

Remember that φF_i was defined in Corollary 7.1.6c.

PROOF. Let $A := F_1 \cap F_2$, $A' := \varphi F_1 \cap \varphi F_2$, let $D \in \mathcal{C}_{\operatorname{St} A}$. Choose an apartment Σ containing C_0 and D, and apply Lemma 7.1.6d to the intersection $\Sigma \cap \operatorname{St} A$. The whole situation is mapped by φ into Δ', and that lemma shows that $A' \subseteq \varphi D$, as desired. □

Lemma 7.1.6e can be rephrased by saying that φ induces an isomorphism $\operatorname{St}_\Delta A \to \operatorname{St}_{\Delta'} A'$ for any $A \subseteq C_0$, $\operatorname{cod} A = 2$, where A' is defined by $A' \subseteq C_0'$, $\operatorname{cotype} A' = \varphi_* \operatorname{cotype} A$. In particular:

7.1.6f. PROPOSITION. *Under the assumptions of the Extension Theorem* 7.1.1, *φ_*, as defined after Corollary* 7.1.6c, *is an isomorphism of diagrams*

$$\varphi_*\colon \operatorname{diagr} \Delta \xrightarrow{\simeq} \operatorname{diagr} \Delta',$$

and $\varphi\colon \mathcal{E}_2(C_0) \to \mathcal{E}_2(C_0')$ is an isomorphism of chamber systems with respect to φ_*.

END OF THE PROOF OF THEOREM 7.1.5. For any $C \in \mathcal{C}(\Delta)$, consider $\varphi_C := \varphi|_{\mathcal{E}_2(C)}$. Apply the results obtained so far to the various φ_C. An obvious argument using connectedness shows that $(\varphi_C)_*$ is independent of C. Thus, the theorem follows from Proposition 7.1.6f. □

7.2. *Constructing buildings from blueprints*

In the previous subsection, we have seen that a building of spherical type is determined up to isomorphism by the chamber system $\mathcal{E}_2(C_0)$ of all chambers having a face of codimension at most two in common with some specified chamber C_0. In this subsection we introduce chamber systems looking like some $\mathcal{E}_2(C_0)$, i.e. unions of generalized polygons all containing one fixed chamber, as objects in their own right. These are called foundations. We know that an isomorphism φ between two chamber systems $\mathcal{E}_2(C_0, \Delta)$, $\mathcal{E}_2(C_0', \Delta')$ cannot be extend in a canonical way to an isomorphism from Δ onto Δ' (see 7.1.3). Therefore the crucial problem of this subsection is the following:

under which additionally imposed structure can one actually (re)construct a building from a foundation? It has turned out to be fruitful to consider labellings of a foundation. A labelled foundation is also called a *blueprint*. A blueprint $\mathcal{B}$ (belonging to some fixed Coxeter diagram M) is an object to which one can canonically associate a chamber system $\mathcal{C}(\mathcal{B})$ such that $\mathcal{C}(\mathcal{B})$ is isomorphic to Δ (or rather $\mathcal{C}(\Delta)$) in the case where $\mathcal{B}$ already comes from a labelled building. When $\mathcal{B}$ is an abstractly given blueprint, the question as to whether or not $\mathcal{C}(\mathcal{B})$ is a building reduces to the rank 3 stars of spherical type. The proof of this result (Theorem 7.2.3 below) relies on the Second Main Characterization of buildings.

The results of this subsection are all contained in the paper Ronan and Tits [1987]. See also Ronan [1989], Chapter 7.

In the following, we fix a Coxeter diagram $M = (m_{ij})_{i,j \in I}$ over some type set I. We recall the notation $\mathcal{S}(\mathcal{C}, C, J)$ for the star of type J, containing the chamber C, in a chamber system $\mathcal{C}$. A foundation of type M is a union of generalized m_{ij}-gons with a common chamber, identified over the rank 1 stars containing that chamber. More precisely:

7.2.1. DEFINITION. A *foundation* of type M, or simply an M-foundation, is a pointed chamber system $(\mathcal{F}, C_0)$ over I such that the following hold:

(a) The $\{i, j\}$-star $S_{ij} := \mathcal{S}(\mathcal{F}, C_0, \{i, j\})$ is a generalized m_{ij}-gon, for all 2 element subsets $\{i, j\} \subseteq I$.

(b) $\mathcal{F}$ is the union of the S_{ij}.

(c) $S_{ij} \cap S_{kl} = \{C_0\}$ if $\{i, j\} \cap \{k, l\} = \varnothing$; $S_{ij} \cap S_{ik} = S_i := \mathcal{S}(\mathcal{F}, C_0, i)$, the i-panel of C_0, if $i \neq j \neq k \neq i$.

If $(\mathcal{C}, C_0)$ is a chamber system of type M, e.g., a building with distinguished chamber C_0, then the set of chambers

$$\mathcal{E}_2(\mathcal{C}, C_0) = \{C \in \mathcal{C} \colon \operatorname{cod}(C \cap C_0) \leqslant 2\},$$

introduced earlier, is a foundation. Notice that condition (c) is a special case of the strong connectedness of $\mathcal{C}$. In passing, we remark that if $\mathcal{F}$ is a foundation, then property (c) automatically sharpens to the strong connectedness of $\mathcal{F}$. The structure of a foundation reduces to the pointed generalized polygons building it up plus the identification of the rank 1 stars containing the distinguished chamber. This is formalized as follows:

7.2.2. By a *parameter system over* I, we mean a collection $(S_i, \infty_i, i \in I)$ of sets S_i with a distinguished element $\infty_i \in S_i$, indexed by I. Suppose that in addition a family $(S_{ij}, \infty_{ij}, \{i, j\} \subseteq I)$ of generalized m_{ij}-gons is given, each with a distinguished chamber ∞_{ij}. Here, $\{i,\ j\}$ runs over all 2-element subsets of I. Finally, we give ourselves a family of bijections

$$\varphi_{i,j} \colon S_i \to \mathcal{S}(S_{ij}, \infty_{ij}, i) \subset S_{ij}$$

such that $\varphi_{i,j}(\infty_i) = \infty_{ij}$, where i,j runs over all pairs of distinct elements in I. Consider the disjoint union of the S_{ij}, together with the equivalence relation given by $\varphi_{i,j}(C) \equiv \varphi_{i,k}(C)$, for all $i \in I$, $C \in S_i$, and $j,k \in I$, $j \neq i \neq k$. The quotient set

$$\mathcal{F} := \mathcal{F}(S_i, S_{ij}, \varphi_{i,j}) := \coprod_{\{i,j\}} S_{ij}/\langle \varphi_{i,j}(C) \equiv \varphi_{i,k}(C)\rangle$$

is a foundation of type M in a natural way, with distinguished chamber C_0 the common image of all ∞_{ij}. Indeed, the S_{ij}'s and S_i's embed into $\mathcal{F}$ and are the $\{i,j\}$-stars, respectively i-stars of C_0.

If one wants to actually construct a foundation from a given set of generalized polygons S_{ij}, the parameter sets S_i will often not be given in advance. Rather it will be necessary to specify bijective 'gluing maps'

$$\alpha_i^{j,k}\colon\ S_i^j \xrightarrow{\cong} S_i^k,$$

where $S_i^j = \mathcal{S}(S_{ij}, \infty_{ij}, i)$ denotes the distinguished i-star inside S_{ij}. In the above setting, we have $\alpha_i^{j,k} = \varphi_{i,k} \circ \varphi_{i,j}^{-1}$. In order to be able to canonically identify the various S_i^j (i fixed) with each other, it is of course necessary that the $\alpha_i^{j,k}$ be compatible in the following sense:

$$\alpha_i^{k,l} \circ \alpha_i^{j,k} \circ \alpha_i^{l,j} = \mathrm{id}$$

for any four pairwise distinct types i,j,k,l. Thus, for fixed i essentially $n-2$ gluing maps, where $n = |I|$, must be specified.

We say that a foundation $\mathcal{F}$ *supports* a building $\mathcal{C}$ if there exists an isomorphism of $\mathcal{F}$ onto $\mathcal{E}_2(\mathcal{C}, C)$, for some chamber $C \in \mathcal{C}$. Using the Second Main Characterization of Buildings 6.3.1, one can prove the following 'reduction theorem'.

7.2.3. THEOREM. *Suppose that M contains no* H_3*-subdiagram, let $\mathcal{F}$ be a foundation of type M. If each residue of $\mathcal{F}$ of type* A_3 *or* C_3 *supports a building, then $\mathcal{F}$ supports a building.*

This is Theorem 2 in Ronan and Tits [1987]. We shall sketch a proof later in 7.2.13.

We now turn to the subject of labellings of buildings and of blueprints. They actually allow to construct buildings.

7.2.4. DEFINITION. Let $(S_i, \infty_i, i \in I)$ be a parameter system. A *labelling* of a pointed building or of a foundation $(\mathcal{C}, C_0)$ over I, *using the parameter system* (S_i), is a family $\mathcal{L}$ of bijections

$$\varphi_F\colon\ S_i \to F,$$

where F runs over all nontrivial i-panels, such that $\varphi_F(\infty_i) = \mathrm{pr}_F\, C_0$, for all panels F. A foundation together with a labelling is called a *blueprint*.

If $\mathcal{F} = \mathcal{F}(S_i, S_{ij}, \varphi_{ij})$ is a foundation described in terms of S_i, S_{ij} as before, and if $\mathcal{L}$ is a labelling of $\mathcal{F}$, we shall often suppress the base chamber in the notation, thus writing $\mathcal{B} = (S_i, S_{ij}, \mathcal{L})$, or $\mathcal{B} = (\mathcal{F}, \mathcal{L})$.

7.2.5. DEFINITION. (a) Let $\mathcal{B} = (S_i, S_{ij}, \Phi)$ be a blueprint, and $(\mathcal{C}, C_0, S_i, \Psi)$ be a labelled building, using the same parameter system. A *realization* of $\mathcal{B}$ by $(\mathcal{C}, \Psi)$ is a family of isomorphisms

$$\varphi_A\colon\ S_{ij} \to A$$

of generalized polygons which preserve the labellings. Here, A runs over all rank 2 stars in $\mathcal{C}$, of all types $\{i, j\} \subseteq I$, and A carries the obvious labelling obtained by restriction from the one on $\mathcal{C}$, based at $\mathrm{pr}_A\, C_0$. In this situation we also say that $(\mathcal{C}, C_0, S_i, \varphi_F)$ *conforms to* $\mathcal{B}$.

(b) A building $\mathcal{C}$ *conforms to* a blueprint $\mathcal{B}$ if there exists some labelling (S_i, φ_F) of $\mathcal{C}$, based at some C_0, which conforms to $\mathcal{B}$.

(c) A labelling $\mathcal{L}$ of a pointed building $(\mathcal{C}, C_0)$ is called *consistent* if it conforms to the blueprint obtained by restricting $\mathcal{L}$ to the foundation $\mathcal{E}_2(C_0)$.

Of course, a blueprint is called *realizable* if it admits some realization, i.e. if there exists a building which conforms to it. If this holds, the underlying foundation supports the building.

For a better understanding of what realizability actually amounts to, see Proposition 7.2.8.

We now come to the question of constructing a chamber system $\mathcal{S}(\mathcal{B})$ (which in the realizable case will turn out to be a building) from a given blueprint. In order to prepare the construction of $\mathcal{S}(\mathcal{B})$, we first take a closer look at the rank 2 case.

7.2.6. DEFINITION ('elementary homotopy of labels'). Let S be a generalized m-gon over $\{i, j\}$, and $C_0 \in \mathcal{C}(S)$ a fixed chamber. For every chamber C, there exists a geodesic $\boldsymbol{C} = (C_0, C_1, \ldots, C_d = C)$. If $d < m$, this geodesic is unique, if $d = m$, there are precisely two such $\boldsymbol{C}$. To every geodesic $\boldsymbol{C}$ based at C_0, we can associate a unique 'label' which is a sequence of labels of chambers

$$(u_1, \ldots, u_d) \in \mathcal{S}_d := (S_i' \times S_j' \times S_i' \times \cdots) \,\dot{\cup}\, (S_j' \times S_i' \times S_j' \times \cdots)$$

where $S_i' := S_i \setminus \{\infty_i\}$, defined by

$$\varphi_{F_t}(u_t) = C_t \quad \text{where } F_t = C_{t-1} \cap C_t. \tag{$*$}$$

Conversely, for any sequence $(u_1, \ldots, u_d)$, $d \leqslant m$, there is a geodesic $\boldsymbol{C}$ of length d, based at C_0, such that $(*)$ holds. That is, we have a bijection

$$\bigcup_{d=1}^{m} \mathcal{S}_d \xrightarrow{\ \cong\ } \mathcal{G}_S(C_0).$$

Here, $\mathcal{G}(C_0)$ denotes, as in Section 3.2, the set of all geodesics based at C_0. The composite map

$$\bigcup_{d=1}^{m} \mathcal{S}_d \to \mathcal{C}(S),$$

obtained by mapping each geodesic to its extremity, is surjective, injective on

$$\bigcup_{d<m} \mathcal{S}_d,$$

and every chamber C opposite to C_0 has exactly two preimages which we call *elementary homotopic*.

7.2.7. *Construction.* Let $\mathcal{B} = (S_i, S_{ij}, (\varphi_F))$ be a blueprint, $S_i' = S_i \setminus \{\infty_i\}$ as usual. Remember that a Coxeter diagram M is fixed. For any reduced word $f = i_1 \ldots i_d$ over I, consider

$$\widetilde{\mathcal{S}}_f := \{\boldsymbol{u} = (u_1, \ldots, u_d)\colon\ u_t \in S'_{i_t}\},$$

and let $\widetilde{\mathcal{S}}$ be the union of the $\widetilde{\mathcal{S}}_f$'s, where f ranges over all reduced words. $\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}(M, S_i)$ is a chamber system over I if we define

$$\boldsymbol{u} \overset{i}{\sim} \boldsymbol{v} \Leftrightarrow \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ are both of type } f \text{ or } fi, \text{ for some } f,$$
$$\text{and the first } l(f) \text{ symbols coincide.}$$

Two sequences $\boldsymbol{u}$ of type $p(i,j) = iji\ldots$ and $\boldsymbol{v}$ of type $p(j,i)$ are called *elementary homotopic* if they are elementary homotopic as defined in 7.2.6 in the generalized polygon S_{ij}. This notion is naturally extended to sequences $\boldsymbol{u}$ and $\boldsymbol{v}$ whose type has the shape $f_1 p(i,j) f_2$, resp., $f_1 p(j,i) f_2$. Finally, $\boldsymbol{u}$ and $\boldsymbol{v}$ are called *homotopic*, $\boldsymbol{u} \simeq \boldsymbol{v}$, if they can be connected by a sequence of elementary homotopies. Denote by

$$\mathcal{S}(\mathcal{B}) := \widetilde{\mathcal{S}}(\mathcal{B})/\simeq$$

the set of all homotopy classes of elements in $\widetilde{\mathcal{S}}(\mathcal{B})$, and by $\boldsymbol{u} \mapsto [\boldsymbol{u}]$ the canonical map. Since $s(\text{type}\,\boldsymbol{u}) = s(\text{type}\,\boldsymbol{v}) \in W$ if $\boldsymbol{u} \simeq \boldsymbol{v}$, we have a well defined map

$$\rho\colon\ \mathcal{S}(\mathcal{B}) \to W, \quad [\boldsymbol{u}] \mapsto s(\text{type}\,\boldsymbol{u}),$$

which of course is our candidate for the W-valued distance $\Delta(C_0, -)$ (see 3.3.11). For $i \in I$, we define a relation $x \overset{i}{\sim} y$ on $\mathcal{S}(\mathcal{B})$ by requiring that there exist representatives $\boldsymbol{u}, \boldsymbol{v}$, $[\boldsymbol{u}] = x$, $[\boldsymbol{v}] = y$, such that $\boldsymbol{u} \overset{i}{\sim} \boldsymbol{v}$. In general, this need not be an equivalence relation. The construction of $\mathcal{S}(\mathcal{B})$ is motivated by the following fact. Let $\mathcal{F}$ be the foundation underlying $\mathcal{B}$, and suppose that $\mathcal{F}$ is already of the form $\mathcal{E}_2(\mathcal{C}, C_0)$, for a building $\mathcal{C}$ of type M, suppose furthermore that the labelling φ_F is given in a consistent way (in the sense of Definition 7.2.5(c)) for all panels F of $\mathcal{C}$. Then there is an isomorphism of $\mathcal{S}(\mathcal{B})$ onto $\mathcal{C}$ 'defined in terms of geodesics'. More precisely, map the class of $(u_1, \ldots, u_d)$ onto the chamber $C = C_d$ defined by

$$C_{t+1} = \varphi_{F_t}(u_{t+1}), \quad t = 0, \ldots, d-1,$$

where F_t is the i_{t+1}-panel of C_t. We record this explicitly.

7.2.8. PROPOSITION. *A blueprint $\mathcal{B}$ is realizable if and only if $\mathcal{S}(\mathcal{B})$ is a building. In this case, $\mathcal{S}(\mathcal{B})$ conforms to $\mathcal{B}$.*

The 'only if' part and the last statement are obvious. The following result will be the basis for the construction of a building from a blueprint, for a given diagram and given rank-2 residues.

7.2.9. MAIN LEMMA. *Let $\mathcal{B}$ be a blueprint. The chamber system $\mathcal{S}(\mathcal{B})$ is a building if and only if for any two sequences $\boldsymbol{u}, \boldsymbol{v} \in \widetilde{\mathcal{S}}$ of the same reduced type, $\boldsymbol{u} \simeq \boldsymbol{v}$ implies $\boldsymbol{u} = \boldsymbol{v}$.*

SKETCH OF PROOF. Any sequence $\boldsymbol{u} = (u_1, \dots, u_d) \in \widetilde{\mathcal{S}}_f$ gives rise to a gallery $(C_0, \dots, C_d)$ of type f in $\mathcal{S}(\mathcal{B})$, where $C_t = [u_1, \dots, u_t]$. It is readily checked that every gallery of type f in $\mathcal{S}(\mathcal{B})$, based at $C_0 = [\varnothing]$, is of this form, for unique $(u_1, \dots, u_d)$. If $\mathcal{S}(\mathcal{B})$ is a building, then property (Q_{C_0}) from Subsection 6.2.1 holds, that is, $(C_0, \dots, C_d)$ and thus $\boldsymbol{u} = (u_1, \dots, u_d)$ (supposed to be of type f) is uniquely determined by its endpoint $[u]$. If, conversely, $\boldsymbol{u} \simeq \boldsymbol{v}$ implies $\boldsymbol{u} = \boldsymbol{v}$, then one can prove that every $\{i, j\}$-star in $\mathcal{S}(\mathcal{B})$ is isomorphic to S_{ij} (Ronan [1989], proof of Theorem 7.1), in particular, $\mathcal{S}(\mathcal{B})$ is of type M. Now it immediately follows from the description of the reduced galleries in $\mathcal{S}(\mathcal{B})$ that the assumption of the 'First main characterization of buildings', Theorem 3.3.1, is satisfied: If $(C_0, \dots, C_d)$, $(D_0 = C_0, \dots, D_d)$ are two galleries of reduced type with $C_d = D_d$, and $\boldsymbol{u}$, $\boldsymbol{v}$ as above the corresponding sequences, then $[\boldsymbol{u}] = C_d = D_d = [\boldsymbol{v}]$, i.e. $\boldsymbol{u} \simeq \boldsymbol{v}$, and *a fortiori* type $\boldsymbol{u}$ and type $\boldsymbol{v}$ are homotopic, and thus represent the same element in the Weyl group, as desired. □

The next theorem is an immediate consequence of the Main Lemma, using the result 6.3.4 on the generation of self homotopies of reduced words.

7.2.10. THEOREM. *A blueprint is realizable if (and only if) its restriction to any spherical rank* 3 *subdiagram of M is realizable.*

PROOF. The 'only if' part is clear. For the converse, we show that $\mathcal{S}(\mathcal{B})$ satisfies the assumption of 7.2.9. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be of common type f. A homotopy from $\boldsymbol{u}$ to $\boldsymbol{v}$ induces by definition a self-homotopy of f in the sense of Subsection 6.3. By 6.3.4, we only have to consider two cases: If the homotopy is inessential it is clear that $\boldsymbol{u} = \boldsymbol{v}$. If the homotopy lives in a spherical rank 3-star, we apply the 'only if' part of Lemma 7.2.9 (taking into account 7.2.8) and conclude again that $\boldsymbol{u} = \boldsymbol{v}$. □

We now come to introducing the natural labelling of a Moufang building. Since the stars of the panels of a fixed base chamber C are in one-to-one correspondence with the fundamental root groups U_i (as defined in Subsection 5.3.11) it is natural to take these as the parameter sets. Technically, we use the label set $S_i = U_i \cup \{\infty_i\}$. The idea now is to use the sharply transitive action of the groups U_w on the chambers at distance w from C. More precisely, the 'refined Bruhat decomposition' by the cosets $un(w)B$ as given in Theorem 5.3.12 and Lemma 5.3.15 will be used to extend the labelling of the stars of panels of C to the whole building.

7.2.11. *Construction (the natural labelling of a Moufang building).* Let $\mathcal{C}$ be the chamber system of a Moufang building. Fix a base chamber C_0 and an apartment $\Sigma \ni C_0$, and as in 5.4.16 denote by Φ_i, $i \in I$, the roots determined by (the panels of) C_0 and by U_i the corresponding 'fundamental' root groups. For any 2-element subset $\{i, j\} \subseteq I$, let $S_{ij} := \mathcal{S}(\mathcal{C}, C_0, \{i, j\})$ be the $\{i, j\}$-star containing C_0. Identify S_i (as introduced above) with the i-star of C_0 in S_{ij}, via the simply transitive action of the U_i on that star (5.3.1, 5.3.3). 'The' natural labelling

$$\mathrm{N}(e_i, i \in I) = (\nu_F = \nu_F(e_i, i \in I))$$

of $\mathcal{C}$ depends on the choice of nonidentity elements $e_i \in U_i$ and is defined as follows. Recall from 5.3.14 that for each i, there is an automorphism $n_i = n(e_i)$ of $\mathcal{C}$, stabilizing Σ and inducing the reflection in the panel of cotype i of C_0. This element becomes unique by requiring $u_i \in U_{-\Phi_i} e_i U_{-\Phi_i}$. The map $e_i \mapsto n_i$ extends to a section $w \mapsto n(w)$, $W \to \mathrm{Aut}(\Sigma)_C$. Now let $F \subseteq \mathcal{C}$ be any i-star, write $F = \mathcal{S}(\mathcal{C}, C, i)$, where $C = \mathrm{pr}_F\, C_0$. According to Proposition 5.3.16, write

$$C = un(w)C_0 \quad \text{where } w = \Delta(C_0, C),\ u \in U_w.$$

The label function ν_F: $S_i \to F$ is now defined by $\infty_i \mapsto C$ and $v \mapsto un(w)vn_iC$. It is an immediate consequence of 5.3.12 and 5.3.15 (in fact, an obvious modification of 5.3.16(a)) that ν_F is indeed bijective.

7.2.12. THEOREM. *Every Moufang building conforms to a blueprint.*

PROOF. In view of Proposition 7.2.8, it suffices to show that every natural labelling of a Moufang building is consistent. Let $\mathcal{A}$ be any rank-2 star, of type $\{i, j\}$ say, and $C = \mathrm{pr}_{\mathcal{A}}\, C_0$. Write $C = un(w)C_0$ where $w = \Delta(C_0, C)$ and $u \in U_w$. It is readily checked that $un(w)$ induces the desired label preserving isomorphism from S_{ij} onto $\mathcal{A}$. □

We remark that in 7.2.12, we have proved more than we stated: if $\mathcal{F}$ is a foundation of the form $\mathcal{F} = \mathcal{E}_2(\Delta, C_0)$, where Δ is a Moufang building, then any natural labelling of $\mathcal{F}$ (see 7.2.13 below) is realizable. In fact, it is realizable by Δ.

For the remainder of this section it seems convenient to have a precise notion of a (natural) Moufang blueprint.

7.2.13. DEFINITION. A blueprint $(\mathcal{B}, \Phi)$ is called a *Moufang blueprint* if all S_{ij} are Moufang polygons and Φ is a 'natural labelling' of the form $\mathrm{N}(e_i,\ i \in I)$. This by definition means that Φ restricted to S_{ij} is of the form $\mathrm{N}(e_i, e_j)$ (see 7.2.11), for all $\{i, j\} \subseteq I$ and a choice of elements $e_i \in U_i, i \in I$. Here, for fixed i all root groups U_i inside the various S_{ij} are identified with each other via S_i.

Notice that there is an ambiguity of what we consider as the parameter sets of such a natural labelling of a Moufang blueprint. The general convention is that the parameter sets of a blueprint are the rank 1 stars S_i of the underlying foundation. On the other hand, the parameter sets of a natural labelling $\mathrm{N}(e_i, e_j)$ of S_{ij} are the two fundamental root groups of S_{ij} which rigorously should be denoted, e.g., by U_i^j, U_j^i. We however want to avoid introducing still another notation for the canonical bijection $g \mapsto gC_0$, $U_i^j \to S_i^j \leftrightarrow S_i$.

7.2.14. PROOF of 7.2.3. Let $\mathcal{F} = \mathcal{F}(S_i, S_{ij})$ be the foundation in question. Let $\mathcal{I}$ be the set of $\{i, j\} \subseteq I$ which are contained in an A_3- or C_3-subdiagram, and with $m_{ij} \in \{3, 4\}$. For a type i occurring in an $\{i, j\} \in \mathcal{I}$, the fundamental root group U_i^j is defined and in canonical bijection with S_i. Choose elements $e_i \in U_i^\#$ for these i. Define a labelling $\mathcal{L}$ of $\mathcal{F}$ as follows: $\mathcal{L}|_{S_{ij}}$ equals $\mathrm{N}(e_i, e_j)$ if $\{i, j\} \in \mathcal{I}$, it equals the product labelling if $m_{ij} = 2$, and is arbitrary otherwise. Now we want to apply Theorem 7.2.10 and thus have to show that $\mathcal{L}_J$ is realizable for spherical $J \subseteq I$ of rank 3. This is clear by construction if M_J is disconnected. For $M_J \cong \mathsf{A}_3$ or C_3, the realizability follows from the remark after 7.2.12. □

After Theorem 7.2.12 we know that a considerable class of buildings (including all irreducible spherical buildings of rank $\geqslant 3$) actually come from blueprints. In the remainder of this subsection, we want to describe these Moufang blueprints in more concrete terms. This in particular will allow us to reduce the condition of the fundamental existence theorem 7.2.10 to a very simple statement which is easily verified in concrete cases.

Naturally, we have to start with the rank 2 case. Since our main interest will be in spherical diagrams, we shall assume that our diagram M has only single and double bonds: $m_{ij} \in \{2, 3, 4\}$. For the moment, we shall concentrate on the case of single bonds. The first thing is to set up a description of projective planes which is appropriate for our purposes. Classically, Moufang planes are coordinatized by 'alternative fields'. These fields are octonion algebras in the case when the plane is not Desarguesian, and ordinary (skew) fields otherwise. In accordance with the definition of a natural labelling, we use a coordinatization by the root groups. Actually, this amounts to the same thing, but we shall not formally refer to the classical theory. Rather, a proof of the following proposition can be given on the basis of the Main Lemma 7.2.9.

7.2.15. PROPOSITION. *Let S be a Moufang projective plane with a distinguished chamber, let U_1, U_2 be its fundamental root groups. After choosing nonidentity elements $e_i \in U_i^\#$, they acquire structures of alternative fields $[U_i, e_i]$ with multiplicative unit element e_i. They are canonically anti-isomorphic to each other.*

The additive structure of the field is the group structure of U_i, the multiplication comes from the commutator map $U_1 \times U_2 \to U_{12}$, where '12' is the third positive root determined by the fundamental roots '1' and '2'. The next proposition says that 'the' natural labelling of a Moufang plane is essentially unique. Technically, it will later be used to verify the assumptions of 7.2.10 for particular diagrams.

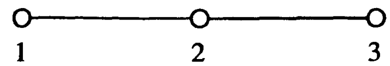
7.2.16. PROPOSITION. *Any two natural labellings of a Moufang plane can be transformed into each other by an automorphism fixing the base chamber.*

For a proof, see Ronan and Tits [1987], Section 3. It should be mentioned that this proof does not quite stay inside our framework of 'abstract' building theory since it uses a result from 'classical' coordinatization theory of projective planes: Any two isotopic alternative division rings are isomorphic.

The next proposition describes 'the' natural labelling of an A_3-building or, equivalently, the realizability of Moufang blueprints over A_3. In view of the reduction theorem 7.2.10, an analogous result for arbitrary diagrams having only single bonds immediately follows. The result can be roughly expressed by saying that a Moufang blueprint is realizable if and only if the structures on the rank 1 residues S_i induced by the various polygons S_{ij} containing S_i are all compatible in a certain sense. In this vague form the result also carries over to double bonds; see below.

7.2.17. PROPOSITION. *Let* $\mathcal{B} = (\mathcal{F}, \mathrm{N}(e_1, e_2, e_3))$ *be a Moufang blueprint for the diagram*

A_3:



Suppose that the underlying foundation $\mathcal{F}$ *supports a building. Then*:

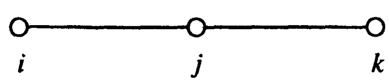
(a) *The alternative fields* $[U_i^j, e_i, e_j]$, $\{i,j\} \neq \{1,3\}$, *are fields* (*not necessarily commutative*).

(b) *The given labelling* $\mathrm{N}(e_1, e_2, e_3)$ *is realizable if and only if the field structures* $[U_2^3, e_2, e_3]$ *and* $[U_2^1, e_2, e_1]$, *induced on* U_2 *by* S_{23} *and* S_{21}, *respectively, are opposite to each other.*

Now let M be an arbitrary diagram having only single bonds. In view of 7.2.10, the last proposition completely answers the question of realizability of Moufang blueprints for M. If we rephrase this criterion in terms of gluing maps $\alpha_i^{j,k}$ of generalized polygons (see 7.2.3 above), the criterion reads as follows: in order that a system

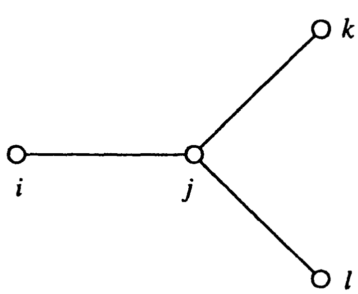
$$\alpha_i^{j,k}\colon\ S_i^j \to S_i^k$$

defines a realizable Moufang blueprint, it is necessary and sufficient that for all A_3-subdiagrams



the gluing map $\alpha_j^{i,k}$ is an isomorphism of the skew field S_j^i onto the opposite of S_j^k. Visibly, this is possible in an essentially unique way if the diagram is connected, has only single bonds, and has no ramification point (so that $M \simeq A_n$ or $M \simeq \widetilde{A}_n$, see Chapter 15, 4.3).

On the other hand, if M contains D_4 as a subdiagram, say



and $\mathcal{B}$ is a Moufang blueprint over M, then the skew field $S_j \cong U_j$ is commutative. In fact, the structures on S_j^k, S_j^l, S_j^i are pairwise opposite to each other which is possible only if they all coincide.

We formulate these results in the following two theorems.

7.2.18. THEOREM. *Let $M = \mathsf{A}_n$ or $\widetilde{\mathsf{A}}_n$ over $I = \{1, \ldots, n\}$, resp., $\{0, \ldots, n\}, n \geqslant 3$, with the canonical ordering of the diagram, let k be a skew field. There exists a unique Moufang blueprint (S_{ij}) for M such that each $S_{i,i+1}$ (where i runs modulo $n+1$ if $M = \widetilde{\mathsf{A}}_n$) is the projective plane over k.*

7.2.19. THEOREM. *Let M be a connected diagram having only single bonds and at least one D_4-subdiagram, and let k be a commutative field. Then there exists a realizable Moufang blueprint such that all S_{ij} ($m_{ij} = 3$) are projective planes over k. It is unique up to isomorphism if M has no $\widetilde{\mathsf{A}}_2$-subdiagram.*

We now come to diagrams involving double bonds. To describe Moufang blueprints for such diagrams, we first of all need a description of Moufang quadrangles, i.e. Moufang buildings for diagram C_2. We consider this diagram with a fixed labelling (or rather with a fixed ordering of its two nodes). The respective objects are called 'points' and 'lines'. Here, an isomorphism of generalized quadrangles is supposed to be type preserving.

The following theorem is a consequence of the classification of all C_n-buildings, $n \geqslant 3$, as obtained in Tits [1974]. Recall the notion of an ε-Hermitian and of a (pseudo-)quadratic form as introduced in Section 4.2.

7.2.20. THEOREM. *Let S be a (Moufang) quadrangle occurring as a residue in a C_3-building. Then S is of one of the following two types.*

(a) *S is 'classical' in the sense that it consists of the totally isotropic 1- and 2-spaces of an (σ, ε)-Hermitian or (σ, ε)-quadratic form of Witt index 2. In particular, the root group U_1 (and the residue S_1) acquires the structure of a field.*

(b) *S is the dual of a classical generalized quadrangle given by a quadratic form q of the following shape: k is commutative, $\sigma = \mathrm{id}$, $\varepsilon = 1$,*

$$q\colon\ k^4 \times K \to k, \quad q(x_1, x_2, x_3, x_4, y) = x_1x_2 + x_3x_4 + n_{K/k}(y),$$

where $n_{K/k}$ is the norm-form of a Cayley algebra K over k.

In particular, the root U_1 group and the residue S_1 carry the structure of an 8-dimensional vector space over a commutative field k, together with a proportionality class of anisotropic forms.

Now we can formulate the realizability condition for C_3-blueprints. After the preparatory statements in 7.2.20, it looks completely analogous to the A_3-case.

7.2.21. THEOREM. *A C_3-foundation is realizable if and only if its A_2- and C_2-residue induce the same structure (field, or 8-dimensional quadratic space) on their common rank-1 residue.*

The following theorem is an obvious consequence of 7.2.10 and 7.2.17.

7.2.22. THEOREM. *A C_n-building, $n \geqslant 4$, is determined by its typical C_3-star. This C_3-star can be any C_3-building whose planes are Desarguesian.*

7.3. *The classification*

In this final subsection, we want to collect at one single place of this article the classification of thick buildings of irreducible, spherical type and rank $\geqslant 3$. Many partial results have been stated before at various places of this article and will now be repeated just for the reader's convenience. For other parts of the classification which have not been treated so far, we have to refer to other texts, in particular Tits [1974]. Some further information is also contained in Ronan and Tits [1987], Ronan [1989], and Chapter 12. For the rank 2 case, see Tits [1973/76, 1983a], Ronan [1989], Appendix 2, and Chapter 9 of this Handbook.

Diagram A_n
Remember that the diagram A_n is supposed to have a fixed orientation (e.g., by identifying the type set with $\{1, \ldots, n\}$).

7.3.1. THEOREM. *Every building of type A_n is isomorphic to the flag complex of a projective space. Two such buildings are isomorphic if and only if the projective spaces from which they are derived are isomorphic. The isomorphism classes of buildings of type A_n are in one-to-one correspondence with the isomorphism classes of skew fields.*

If we look at the isomorphism class belonging to the reversed orientation of the diagram, we obtain the dual projective space which is defined by the opposite of the skew field.

Diagrams D_n, $n \geqslant 4$, E_6, E_7, E_8
The following theorem is a special case of 7.2.19.

7.3.2. THEOREM. *Let M be one of the Coxeter diagrams D_n, $n \geqslant 4$, E_6, E_7, E_8. The isomorphism classes of buildings of type M are in one-to-one correspondence with the isomorphism classes of commutative fields. More precisely, if Δ is a building of type M, then all rank-2 stars of type A_2 in Δ are isomorphic to each other, and are projective planes $\mathbb{P}_2(K)$ over a commutative field K. Conversely, for a given commutative field K, there exists a building $\Delta = \Delta(M, K)$, uniquely determined up to isomorphism, such that the A_2-stars in Δ are isomorphic to $\mathbb{P}_2(K)$.*

7.3.3. REMARK. The building $\Delta(M, K)$ of the preceding theorem is the building of the natural Tits system (B, N) in the group $G_M(K)$ of K-rational points of the split algebraic group (Chevalley group) G_M with diagram M. Cf. Section 4.4.

Diagram C_n, $n \geqslant 3$
Recall from Section 4.2 that the buildings of type C_n, $n \geqslant 2$, are precisely the flag complexes of polar spaces of rank n. Remember furthermore that polar spaces can be obtained as flag complexes of totally isotropic subspaces of a vector space with a non-degenerate Hermitian or quadratic form. In the following classification theorem, we precisely state which forms give rise to C_n-buildings, and which C_n-buildings arise in that way. For $n \geqslant 4$, these are all buildings.

7.3.4. THEOREM. *Every* C_3*-building whose typical plane is Desarguesian, and every* C_n*-building, for* $n \geqslant 4$*, is the flag complex of totally isotropic subspaces with respect to a nondegenerate trace valued Hermitian form (including the case of symmetric bilinear forms in characteristic* $\neq 2$*, and of alternating forms), or a nondegenerate (pseudo-)quadratic form. This form is determined up to proportionality by the building.*

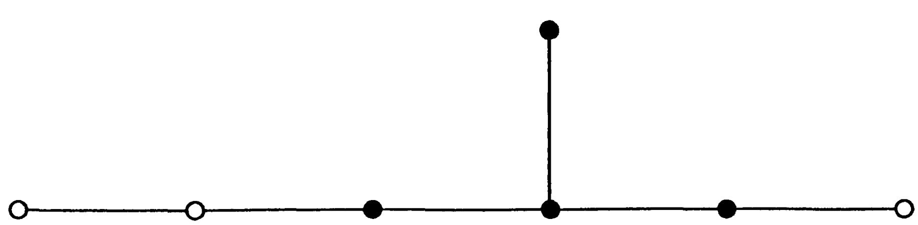
This theorem is due to Veldkamp (in characteristic $\neq 2$) and Tits; see the Notes below. Tits' proof uses, among other ideas, the results from 7.1.4.

We now come to those polar spaces which are not covered by the last theorem. Remember that the projective planes occurring as A_2-stars in a C_3-building are Moufang planes, i.e. either Desarguesian or planes over a Cayley (octave) algebra. Let K be a Cayley algebra with centre k; then K has dimension 8 over k, and the norm $n_K\colon K \to k$ is an anisotropic quadratic form. Denote by $\mathbb{H} = \mathbb{H}_k$ the hyperbolic plane over k. This is the vector space $k \times k$ with the quadratic form xy.

7.3.5. THEOREM. *For any Cayley algebra* K *over* k*, there exists a* C_3*-building, unique up to isomorphism, whose* A_2*-stars are projective planes over* k*.*

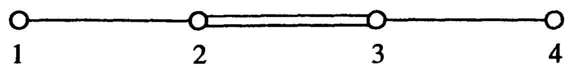
Its C_2*-stars are dual to the generalized quadrangle belonging to the quadratic space* $(K, n_k) \perp \mathbb{H} \perp \mathbb{H}$.

This is Theorem 9.1 of Tits [1974], where one also finds some hints on an 'elementary' existence proof, using the 'classical' theory of polar spaces of H. Freudenthal (see the references in *loc. cit.*). A very short existence proof which uses the existence of certain algebraic groups of exceptional type E_7 is given in *loc. cit.*, 10.3: given the Cayley algebra K/k, there is an associated algebraic group of k-index $E^{28}_{7,3}$

$E^{28}_{7,3}$ 

(cf. 4.4.4), whose anisotropic kernel is the spin group of the norm form $n_{K/k}$. The k-building of this group is the desired one. (See, e.g., Scharlau [1985], Chapter 9, §3.4, for the 'elementary' definition of the spin group of a quadratic form f, and, e.g., Borel [1991] for its relevance in the theory of algebraic groups: it is the 'simply-connected cover' of the special orthogonal group of f. The only thing that concerns us here is the fact that the spin group has the same building as the orthogonal group.) For another existence proof, using the Extension Theorem 7.1.1, see Mühlherr [1990].

Diagram F_4
As with A_n and C_2, we consider diagram F_4 with a fixed type set $1, 2, 3, 4$, essentially with a fixed orientation.



As before, isomorphisms of complexes are supposed to be type preserving. This means that we distinguish a complex Δ of type F_4 from its *dual* Δ° which by definition is obtained by reversing the order of the type set.

Similar to the description of C_n-buildings as flag complexes of polar spaces, there exists a description of buildings of type F_4 as flag complexes of so-called *metasymplectic spaces*. These are incidence geometries consisting of 'points', 'lines', 'planes', and 'symplecta', where the symplecta are polar spaces of rank 3 (as is imposed by the diagram F_4). These metasymplectic spaces basically go back to the work of H. Freudenthal, cf. Tits [1974], and are treated in detail in Chapter 12 of this book. They will not be used in the following. The classification of F_4-buildings is given by the next theorem. Like in the C_n-case, such a building is determined by a typical C_2-residue; the possibilities for this residue are however much more restricted.

NOTATION. If Δ is a building of type F_4, we denote by $k(\Delta)$ the division rings (skew field or Cayley algebra) such that a typical residue over $\{1,2\}$, resp., $\{3,4\}$ is a projective plane over $k(\Delta)$, resp., $K(\Delta)$. By $S(\Delta)$ we denote a typical C_2-residue, i.e. over $\{2,3\}$.

7.3.6. THEOREM. (a) *Let Δ be a building of type F_4. After possibly replacing Δ by its dual Δ°, the division ring $k = k(\Delta)$ is a commutative field, and $K = K(\Delta)$ has the structure of a k-algebra such that one of the following holds:*

(i) $K = k$ *and* $\operatorname{char}(k) \neq 2$;

(ii) *K is a separable quadratic extension of k;*

(iii) *K is a quaternion algebra over k;*

(iv) *K is a Cayley algebra over k;*

(v) $\operatorname{char}(K) = 2$ *and* $k \subseteq K \subseteq k^{1/2}$.

(b) *Under the assumptions of* (a), *the polar space $S(\Delta)$ is determined from the pair $(k(\Delta), K(\Delta))$ in the following way: $S(\Delta)$ is isomorphic to the polar space $S(k, K)$ associated to the quadratic space $(K, n_{K/k}) \perp \mathbb{H} \perp \mathbb{H}$ over k, where $n_{K/k}\colon K \to k$ is the function $x \mapsto x^2$ in cases* (i) *and* (v), *and the norm form in cases* (ii), (iii), (iv), *and $\mathbb{H}$ denotes the hyperbolic plane over k.*

(c) *Conversely, if a field k and a k-algebra K are given subject to one of the conditions* (i) *to* (v), *there exists a building Δ of type* F_4 *such that $S(\Delta)$ is isomorphic to $S(k, K)$. It is uniquely determined by this property, and has the further property that the pair (k, K) is isomorphic to $(k, (\Delta), K(\Delta))$ in the obvious sense. Denote this building by $\Delta(k, K)$.*

(d) *The dual of a building* $\Delta(k,K)$ *as in* (c) *is never of this type except for the following cases*: *case* (i), *or case* (v) *with* $k = K$, *in which* $\Delta(k,k)$ *is self-dual*; *case* (v) *with* $k \subsetneqq K$, *in which case* $\Delta(k,K)^{\circ} \cong \Delta(K, k^{1/2})$ *(and may be self-dual or not).*

A proof of the existence is obtained from the results of the previous subsection: it is easy to write down F_4-foundations with the prescribed A_2- and C_2-residues, and to check the realizability criterion 7.2.21 for the C_3-residues. For details, see Ronan [1989], Chapter 8.5.

The existence also is an immediate consequence of the (structure theory and) the classification of algebraic groups of (absolute) types F_4, E_6, E_7, E_8. The relevant groups from the list in 4.4.3 (Figure 4.1) are F_4 (split), $({}^2\mathsf{E}_6^2, \mathsf{F}_4)$, $(\mathsf{E}_7^9, \mathsf{F}_4)$, $(\mathsf{E}_8^{28}, \mathsf{F}_4)$ for cases (i) to (iv), respectively. In the case of nonperfect ground fields of characteristic 2, some extra work is needed to treat case (v). In Theorem 10.4 of Tits [1974], also the automorphism group of each F_4-building is determined.

The proof of the fact that the F_4-buildings described in the theorem are the only ones is easier. In fact, the description of the $\{2,3\}$-residue $S(\Delta)$ and thus the relationship between $k(\Delta)$ and $K(\Delta)$ is a straightforward consequence of the classification of C_3-buildings. One merely has to check which Moufang quadrangles of classical type have the property that both the quadrangle and its dual come from an ε-Hermitian or pseudo-quadratic form of Witt index 2.

Notes to Section 7

The Extension Theorem. Everything in Section 7.1 except for 7.1.3 and 7.1.4 is taken from Tits [1974], Chapter 4. The Extension Theorem had been announced earlier in Tits [1963/64], Théorème 4.1. (Remember that the lecture notes Tits [1974] go back essentially to the year 1968.) Although the Extension Theorem is probably the most important and certainly the most difficult theorem in the 'basic' theory of buildings, not a single subsequent publication deals with this subject. This is surprising since it is not impossible that an easier or more conceptual proof of the theorem might exist. The only additional reference to be mentioned is the 'Résumé de cours' Tits [1990a]. Here, a sketch of the analogous Extension Theorem for twin buildings is given. The proof is essentially the same as in the spherical case, with minor 'local' improvements. Understanding the general lines of the proof is even easier in the twin case than in the spherical case, due to the fact that in a twin building, the roles of a chamber and any of its opposites are clearly separated (since they live in different buildings).

The applications to polar spaces in 7.1.4 are taken from Chapter 8 of Tits [1974]. They are part of the proof of the Tits–Veldkamp theorem *loc. cit.*, 8.21 (which is essentially already our Classification Theorem 7.3.4 below). More precisely, 7.1.4b is 8.21.3 of *loc. cit.*, and 7.1.4a makes explicit the key argument of *loc. cit.*, 8.21.15.

Blueprints and the classification. Section 7.2 again comes from a unique source: Ronan and Tits [1987]. We have also made use of the presentation as given in Ronan [1990], Chapters 7 and 8, and Appendix 1. Concerning the basic notions around blueprints, we have tried to be a little bit more explicit than Ronan and Tits are. But our report is no substitute for a detailed study of the later parts of the original source. We want to emphasize the fact that, despite its elegance, the blueprint approach to spherical buildings still presents open problems. One challenge for future research appears to be the fact that so far only for the simply-laced diagrams D_n, E_n, does the method give an independent, new proof of the classification. The application to buildings of type C_n, $n \geqslant 4$, or F_4 substantially relies (via 7.2.21) on Theorem 7.2.20 and thus on an *a priori* knowledge of the classification of C_3-buildings. This classification has to be taken from the classical source Tits [1974], and its proof seems to be of essentially the same degree of difficulty as in the case of general $n \geqslant 3$. One could ask whether there exists a weaker form of 7.2.20 which still would allow to formulate the gluing criterion 7.2.21. In a similar direction, we suggest that it should be possible to derive, in the spirit of Ronan and Tits [1987], pp. 298–300, also in the C_3-case the commutator properties of the root groups and finally the axiomatics of alternative fields, directly from the Main Lemma 7.2.9 (using as in the A_3-case the explicit information of 6.3.6). One algebraic ingredient of the proof which seems to be indispensable is the Bruck–Kleinfeld theorem which derives the concrete shape of alternative fields as 'Cayley numbers', determined by their norm forms.

History of the classification. We want to draw attention to the fact that the classification of buildings in Section 7.3 considerably simplifies in the finite case. A finite field admits only few quadratic or Hermitian forms, finite noncommutative fields or Cayley algebras do not exist. The classification of finite buildings was announced, long before the appearance of Tits [1974], in Théorème 2.1 of Tits [1963/64]. The corresponding classification theorem for finite groups with a Tits system (B, N) was announced in the same paper, Théorème 2.3. It is stated in precise form and proved in Tits [1974], 11.7. The finite buildings and their parameters (see Section 3.4) are conveniently tabulated in Appendix 6 of Ronan [1989].

Also in the general (i.e. not necessarily finite) case, most of the results on the classification of buildings were already stated in Tits [1963/64]. This is so in particular for the diagrams D_n or E_n (Théorème 7.1 in *loc. cit.*). (Remember that the Extension Theorem is already contained in *loc. cit.* (see above), and recall from 7.3.3 that the classification of buildings of type D_n or E_n is an immediate consequence of this theorem.) For buildings of type C_n, $n \geqslant 4$, over ground fields of characteristic not equal to 2, one could say that their classification was already known before their formal definition had been given. Indeed, the classification of the corresponding polar spaces goes back to Veldkamp [1959] (an Utrecht doctoral dissertation guided by H. Freudenthal), and one may suspect that the equivalence between polar spaces in the sense of Veldkamp and C_n-buildings was clear to the experts immediately after the appearance of Tits [1961/62]. We do not know of any published reference before Tits [1974]. In the case of characteristic 2, Veldkamp already showed that polar spaces can be embedded into spaces coming from a polarity of a projective space P. But it was only recognized around 1968, by Tits, that his generalized quadratic forms were the appropriate tool to describe the image in P of such an embedding. In the C_3-case, the crucial result which is needed in addition says that the projective planes occurring in such a polar space are Desarguesian or Moufang. Today, this is a special case of the general result that all irreducible, spherical buildings and all of their residues are Moufang (5.3.5, 5.3.7, Tits [1977]). In the C_3-case, the result is, again, much older than Tits [1977]. It is stated (without proof) in Tits [1963/64], Théorème 6.2. For the classification in the Desarguesian case, our historical remarks concerning the case $n \geqslant 4$ apply. For the octave case, the first reference for Theorem 7.3.5 above is Tits [1974], Chapter 9. But again, the essential ingredients of the result were essentially known around the appearance of Tits [1963/64]. The existence of the polar space, given a Cayley algebra, was known from the theory of algebraic groups (see Section 4.4, basically already from earlier work of Freudenthal), and the uniqueness is a consequence of the Extension theorem.

Theorem 7.3.6 on the classification of buildings of type F_4 was already conjectured in 8.1 of Tits [1963/64], except that case (v) was missing. In Tits [1974], 10.3.1, it is indicated that one can give an 'elementary' proof for the existence of $\Delta(k, K)$ by direct construction of the corresponding metasymplectic space, along the lines of Freudenthal [1954–63], Chapters 8 to 9. Parts of the arguments can be obtained from Chapter 12, Section 8.4.

We refer to the last section of Chapter 9 for bibliographical remarks about the classification of generalized polygons, which we could not treat in this chapter.

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CHAPTER 12

Point-Line Spaces Related to Buildings

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Contents

Introduction 649
1. Standard notions for spaces 650
2. Embeddings in projective spaces 653
3. Polar spaces 662
4. Shadow spaces of buildings 678
5. Models 689
6. Characterisations 717
7. Related structures 723
References 730

HANDBOOK OF INCIDENCE GEOMETRY
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Introduction

In a way, this chapter is a natural continuation of Chapter 2 entitled 'Projective and Affine Geometry over a Division Ring'. The geometries we are interested in here have in common with projective and affine geometries that they naturally come from the *buildings of spherical type*, and hence (cf. Chapter 11, Section 7.3), for rank at least 3, from groups with Tits systems. The main object of this chapter is to survey synthetic descriptions of these geometries in terms of points and lines.

Much of the material in this chapter is closely related to parts of the book by Buekenhout and Cohen [1992]. I am grateful to Francis Buekenhout not only for allowing me to use it, but also for inspiring discussions regarding the content of this chapter. I also gratefully acknowledge improvements of an earlier version of this text suggested to me by Peter Cameron, Hans Cuypers, Peter Johnson, William Kantor, Marc Van Leeuwen, and Ernest Shult.

The basic concept in point-line geometry is that of a space. A *space* S is a pair $(P, \mathcal{L})$, also denoted by $(\mathcal{P}(S), \mathcal{L}(S))$, consisting of a nonempty set P, whose members are called *points*, and a collection $\mathcal{L}$ of subsets of P of cardinality at least two, whose members are called *lines*. Thus, in the extreme case where all lines have cardinality two, a space is just a graph (in fact, we can take this to be the definition of a graph). The cardinality of a line will also be referred to as its *size* or its *length*. The concept of space is used to gain insight into structures like algebras, quadratic forms, and groups by drawing pictures with points and lines. An essential aspect of this part of geometry is concerned with axioms. Given a set $\mathcal{S}$ of algebraic structures (e.g., vector spaces), the goal is to list a (preferably 'nice' and succinct) set of properties of spaces such that any space with these properties corresponds to a member of $\mathcal{S}$. If $\mathcal{S}$ is the set of all vector spaces, Pasch's Axiom for projective spaces (see §2.1 below) is a good example of the kind of property that plays a role here.

The first examples studied after projective spaces come from vector spaces with a quadratic, unitary, or symplectic form. These spaces are *polar spaces*. In Section 3, we show how they are classified. Prior to this, in Section 2, we treat some criteria for embedding a space into a projective space. This study is a basic ingredient of our approach to the classification of polar spaces.

In Section 4, we introduce the central topic of this chapter: shadow spaces of buildings. We relate polar spaces to buildings and indicate how a classification of buildings of type B_n and D_n follows from the classification of polar spaces.

In Section 5, several constructions of shadow spaces of buildings of spherical type are given. In view of Section 4, they provide constructions of the corresponding buildings as well.

Section 6 is devoted to the theme of finding axioms that suffice for characterisations of shadow spaces of buildings.

Finally, in Section 7, some attention is paid to further structural results such as embedding theorems, related group theory, hyperplanes, affine spaces, etc. We begin however with a section entirely devoted to basic concepts.

1. Standard notions for spaces

In this section we collect most of the standard notation and definitions needed for the rest of the chapter. At first reading, the reader is well advised to use this section to refer back to rather than to peruse it.

A *morphism* α: $(P,\mathcal{L}) \to (P',\mathcal{L}')$ of spaces is a map α: $P \to P'$ such that the image under α of every line in $\mathcal{L}$ is contained in a line of $\mathcal{L}'$. *Isomorphism* and *automorphism* are defined in the obvious way.

Let $x, y \in P$. We shall write $x \perp y$, and say x is *collinear with* y, to indicate the existence of a line containing both x and y. The set of all points in P collinear with x, including x itself, will be denoted by $x^\perp$.

The definition of space is very general in that any collection of nonempty, nonsingleton subsets of any set can be the set of lines of a space. A stricter class of spaces is formed by the *partial linear spaces*, i.e. spaces in which no two different points are on two different lines. For two distinct collinear points x, y of such a space, we shall write xy to denote the line containing them.

A space $S = (P, \mathcal{L})$ is called *nondegenerate* if no point is collinear with all other points (i.e. there is no $x \in P$ with $P = x^\perp$). At the other extreme, S is called *singular* (or *complete*) if $P = x^\perp$ for each $x \in P$, i.e. if any two of its points are collinear. Singular partial linear spaces are also called *linear*; thus, S is linear if and only if every pair of points is on a unique line.

For $X \subseteq P$, we denote by $\mathcal{L}(X)$ the set $\{L \cap X\colon\ L \in \mathcal{L} \text{ and } |L \cap X| > 1\}$. The pair $(X, \mathcal{L}(X))$ is again a space. We say that X is a *subspace* of S if every line $L \in \mathcal{L}$ meeting X in at least two points entirely belongs to X. Thus, if X is a subspace, then $\mathcal{L}(X) \subseteq \mathcal{L}$. If S is a partial linear space, the converse is also true: then the subset X of P is a subspace if $\mathcal{L}(X) \subseteq \mathcal{L}$. Instead of the subspace X of S, we shall also refer to the pair $(X, \mathcal{L}(X))$ as being a subspace of S. The set P itself is a subspace. A space is partial linear if and only if all of its lines are subspaces. The intersection of a collection of subspaces of S is again a subspace. Therefore, for any subset $Y \subseteq P$, the notion of subspace *generated by* Y is well defined: it is the intersection of all subspaces of S containing Y; it is denoted by $\langle Y\rangle$. For $Y, Z \subseteq P$ and $p, q \in P$, we shall usually write $\langle Y, Z\rangle$ instead of $\langle Y \cup Z\rangle$ and $\langle p, q\rangle$ instead of $\langle\{p, q\}\rangle$, and so on.

Let $X \subseteq P$. We say that X is *singular* if every two points of X are collinear. If X is a subspace of S, this is equivalent to saying that the space $(X, \mathcal{L}(X))$ is singular. We set

$$X^\perp = \bigcap_{x \in X} x^\perp.$$

The *radical* of $X \subseteq P$ is the subset $\operatorname{Rad} X = X \cap X^\perp$. Thus saying that P has empty radical amounts to asserting that S is nondegenerate. The lemma below lists some properties of the operator $X \mapsto X^\perp$.

1.1. CLOSURE LEMMA. *Let S be a space, and $X, Y \subseteq \mathcal{P}(S)$. Then*

(i) *if $X \subseteq Y$ then $Y^\perp \subseteq X^\perp$;*

(ii) $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$;
(iii) $X \subseteq X^{\perp\perp}$;
(iv) $X^{\perp\perp\perp} = X^{\perp}$;
(v) *X is singular if and only if $X \subseteq X^{\perp}$;*
(vi) *for each singular set X of points of S we have* $\mathrm{Rad}(X^{\perp}) = X^{\perp\perp}$.

In full generality, there is little one can say about partial linear spaces. The spaces of main interest in this chapter are so-called *gamma spaces* (cf. Higman [1983]); this means that they satisfy the following property: for any $p \in P$ the subset $p^{\perp}$ is a subspace. Thus $(P, \mathcal{L})$ is a gamma space if and only if, for any $p \in P$ and $L \in \mathcal{L}$, the set $p^{\perp} \cap L$ is empty, consists of a single point, or coincides with L.

The space $(P, \mathcal{P}_3(P))$, in which every subset of P of size 3 is a line, is an example of a gamma space that is not partial linear provided P has cardinality at least 4.

1.2. GAMMA SPACE LEMMA. *Let S be a gamma space. Put $P = \mathcal{P}(S)$.*

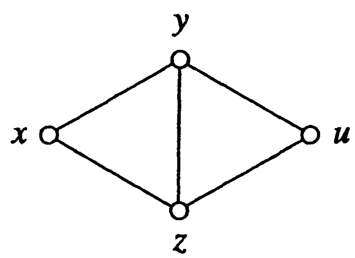
(i) *For any subset X of P, the subset $X^{\perp}$ is a subspace of S and coincides with $\langle X\rangle^{\perp}$.*
(ii) *If X is a singular subset of P, then $\langle X\rangle$ is a singular subspace of S and $X^{\perp}$ is a subspace of S. Moreover, $X^{\perp\perp}$ is singular subspace of S containing X.*
(iii) *Every singular subset is contained in a maximal singular subset; every maximal singular subset X of P satisfies $X = X^{\perp} = X^{\perp\perp}$.*
(iv) *If X is a singular subset of P, then $X^{\perp\perp}$ is the intersection of all maximal singular subsets of $\mathcal{P}(S)$ containing X.*
(v) *If $x, y \in P$ are distinct and collinear, then $\{x, y\}^{\perp\perp}$ is a singular subspace containing every line on x and y.*

1.3. *Graphs and spaces.* Recall that *graphs* are spaces (V, E) whose lines (members of E) have size two; usually points are called *vertices* and elements are called *edges*. They are partial linear gamma spaces. Singular subsets of Γ are cliques.

In general, lines with more than two points are called *thick*, those with exactly two points are called *thin*.

If Γ denotes a graph, then $V\Gamma$ and $E\Gamma$ are understood to be its vertex set and edge set, respectively, so that $\Gamma = (V\Gamma, E\Gamma)$. We shall often write $\gamma \in \Gamma$ rather than $\gamma \in V\Gamma$. Moreover, for $\gamma, \delta \in \Gamma$, we usually write $\gamma \sim \delta$ to abbreviate $\{\gamma, \delta\} \in E\Gamma$.

The *collinearity graph* of a space S is the graph whose vertex set is $\mathcal{P}(S)$ and in which two points are adjacent whenever they are collinear (but distinct). Many spaces that we shall consider – important exceptions being projective and affine spaces – can be reconstructed from their collinearity graphs by means of the following construction, suggested by Lemma 1.2(v) above. Let Γ be a graph. Denote by $\mathcal{L}(\Gamma)$ the collection of all $\{\gamma, \delta\}^{\perp\perp}$ for $\{\gamma, \delta\} \in E\Gamma$. Then, under some mild regularity conditions, $(V\Gamma, \mathcal{L}(\Gamma))$ is a partial linear space. First, we provide an example showing that this is not true in general. Start with the graph Γ with vertex set $\{x, y, z, u\}$ and edge set consisting of all pairs except for $\{x, u\}$.



Then $\{x,y\}^{\perp\perp} = \{x,y,z\} \supset \{y,z\} = \{y,z\}^{\perp\perp}$, so there are at least two lines on $\{y,z\}$ in $(V\Gamma, \mathcal{L}(\Gamma))$.

1.4. LEMMA. *Suppose Γ is a graph. If the space $(V\Gamma, \mathcal{L}(\Gamma))$ is partial linear, then it is a gamma space. Furthermore, $(V\Gamma, \mathcal{L}(\Gamma))$ is partial linear provided, for every singular subset $\{x,y,z\}$ of size 3, the inclusion $\{x,y\}^{\perp} \subseteq \{y,z\}^{\perp}$ implies equality.*

Sufficient conditions for the lemma to apply are:
(i) there is a natural number $\lambda < \infty$ such that $\lambda = |\{\gamma,\delta\}^{\perp}|$ for all $\{\gamma,\delta\} \in E\Gamma$,
or, in the presence of a big group G of automorphisms of Γ, by:
(ii) G induces the full symmetric group Sym_3 on every 3-clique of Γ.

1.5. *Further definitions.* Let S_1 and S_2 be two spaces. The *join* of S_1 and S_2 is the space whose point set is the disjoint union of $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ and whose lines are the lines of each S_j and the pairs $\{x_1, x_2\}$ with each $x_j \in \mathcal{P}(S_j)$. The *direct product* of S_1 and S_2 is the pair $(\mathcal{P}(S_1) \times \mathcal{P}(S_2), \mathcal{L})$, where $\mathcal{L}$ consists of the sets of the form $\{(p_1, x)\colon x \in \ell_2\}$ or $\{(x, p_2)\colon x \in \ell_1\}$ for $p_j \in \mathcal{P}(S_j)$ and $\ell_j \in \mathcal{L}(S_j)$.

The *singular rank* of a space S is the maximal number n (possibly ∞) for which there exists a chain of distinct subspaces $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n$ such that X_i is a singular subspace for each i ($0 \leqslant i \leqslant n$). Thus, the singular rank of the empty space, a point, a line is -1, 0, 1, respectively. Also, the singular rank of a projective space is just its rank or projective dimension (for in this case all subspaces are singular).

A subset X of the point set of a space S is called *geodetically closed* if, for all $x, y \in X$, each point on a path of minimal length from x to y in the collinearity graph of S is also contained in X.

Much like in various other structure theories, spaces (or graphs) can be thought of as being built up of smaller spaces (or graphs, respectively). An example that we shall come across is the case of a nontrivial surjective morphism all of whose fibres are similar (for instance singular spaces or graphs). We shall call a graph Γ *reduced* when $\gamma^{\perp} = \delta^{\perp}$ implies $\gamma = \delta$ for any two vertices γ, δ of Γ. The relation $x^{\perp} = y^{\perp}$ is an equivalence relation $\equiv$ whose classes are cliques. The quotient set $V\Gamma/\equiv$ can be given the structure of a graph by letting the class of x be adjacent to the class of y if and only if x, y are adjacent in Γ. (This definition is independent on the choice of x and y from their equivalence classes.) Denote this quotient graph by $\Gamma/\equiv$. It gives a method to mod out the $\equiv$-equivalence classes.

1.6. PROPOSITION (cf. Buekenhout and Shult [1974], Johnson and Shult [1989]). *Let Γ be a graph. Then the natural map $\Gamma \to \Gamma/\equiv$ is a surjective morphism with the property* *that the images of two vertices γ, δ of Γ in $\Gamma/\equiv$ are adjacent if and only if $\gamma \sim \delta$. Moreover, $\Gamma/\equiv$ is reduced.*

See Aschbacher [1976], Blass [1978], Hall [1992] for more results of this kind.

1.7. *The Grassmannians.* Let $k \geqslant 1$, let $\boldsymbol{D}$ be a division ring and denote by $\boldsymbol{P}$ a projective space over $\boldsymbol{D}$. Let P be the set of all $(k-1)$-dimensional (projective) subspaces of $\boldsymbol{P}$ and let $\mathcal{L}$ be the collection of all subsets of P of the form

$$\{X \in P \colon Y \subset X \subset Z\},$$

where Y, Z is an incident pair of subspaces of $\boldsymbol{P}$ of rank $k-2$ and k, respectively. Then $(P, \mathcal{L})$ is a partial linear gamma space, called the *Grassmannian of k-spaces* of $\boldsymbol{P}$. If $\boldsymbol{P}$ has dimension (=singular rank) $n \geqslant k$, it will also be denoted by $A_{n,k}(\boldsymbol{D})$, for reasons that will become clear in §4. Thus $A_{n,1}(\boldsymbol{D}) = \boldsymbol{P}(\boldsymbol{D}^{n+1})$. Suppose $1 < k < n$. Then $A_{n,k}(\boldsymbol{D})$ is a nondegenerate, partial linear gamma space of singular rank $\max\{k, n+1-k\}$. The subspaces of $\boldsymbol{P}$ of dimension $k-2$ and k induce maximal singular subspaces in the Grassmannian. The collinearity graph of $A_{n,1}(\boldsymbol{D})$ is given by:

$$X \sim X' \text{ iff } \dim(X + X') = k \quad \text{for any } X, X' \in P.$$

The subspace $X^{\perp}$ has radical $\{X\}$, and $X^{\perp}/\equiv$, where $\equiv$ is defined on $x^{\perp}$ as in §1.5, is the join of X and the direct product of two projective spaces of dimension $k-1$ and $n-k$, respectively. The subspaces of $(P, \mathcal{L})$ consisting of all subspaces contained (or containing) a given subspace of $\boldsymbol{P}$ are geodetically closed.

If $\boldsymbol{D}$ is a field, $A_{n,k}(\boldsymbol{D})$ can be thought of as the projective points and lines consisting of pure wedges in $\bigwedge^k \boldsymbol{D}^{n+1}$. The (projective) point set is the intersection of quadrics given by the Plücker relations, see Theorem 5.21.

2. Embeddings in projective spaces

For the definition of (reducible) projective spaces, we refer to Chapter 2. The aim of synthetic geometry is to axiomatically describe relevant and/or 'classical' geometries (such as projective spaces) in such a way that, on the basis of simple properties (axioms), the geometry can be fully analysed and recognized. An early result of this kind is:

2.1. VEBLEN AND YOUNG THEOREM (cf. Birkhoff [1935], Menger [1936]). *Let S be a linear space. Suppose that Pasch's Axiom holds:*

If $\ell, \ell' \in \mathcal{L}$ with $\ell \cap \ell' = \varnothing$ and $x \in \mathcal{P}(S) \setminus (\ell \cup \ell')$, then there is at most one line on x meeting both ℓ and ℓ'.

Then S is a (possibly reducible) projective space.

Consequently, if S is a linear space satisfying Pasch's Axiom, having no lines of size 2, and having singular rank $n \geqslant 3$, then there is a division ring $\boldsymbol{D}$ such that $S \cong \boldsymbol{P}(\boldsymbol{D}^n)$. Thus, a fully algebraic description of the space S has been obtained. This example will be used to deal with spaces related to the classical groups. These spaces (certain polar spaces, see below) will be embedded in a projective space in the following sense.

2.2. *Embeddings.* Suppose S and T are spaces. We call ϕ: $S \to T$ an *embedding* (of spaces) if ϕ is an injective morphism of spaces mapping lines onto lines. We then say that ϕS is *embedded* in T. Note that ϕ may be the identity on S, in which case S itself would be embedded in T.

In most cases we shall consider, T will be a projective space. In particular, T will be a space constructed from an axiomatically defined space S, and, by use of the Veblen and Young theorem, T will be shown to be a projective space. Before pursuing this further, we consider arbitrary spaces embedded in a projective space.

In general, a partial linear space that can be embedded in a projective space $\boldsymbol{P}$ is isomorphic to a subspace of $\boldsymbol{P}$ only if it is singular.

A gamma space in which every singular subspace is a (possibly reducible) projective space, will be called a *paraprojective space.* Thus, every projective space is a paraprojective space, and every paraprojective space is a partial linear gamma space.

2.3. *Hyperplanes.* A *hyperplane*[1] of a space S is a proper subspace H of S, such that any line of S has at least one point in H. Thus, if S is a projective space, its hyperplanes are the usual maximal proper subspaces. In the literature, hyperplanes appear in various guises and under various names. In Buekenhout and Hubaut [1977] they do not even have a name! In Teirlinck [1980], they are called projective hyperplanes, in Ronan [1987] they are called geometric hyperplanes. Hyperplanes will be used as a tool for analysing the structure of embedded spaces (cf. Veldkamp [1959]). Another nice property of hyperplanes is their inductive character: if U is a subspace of S not contained in the hyperplane H, then $H \cap U$ is a hyperplane of U.

The following result helps to reduce the study of arbitrary paraprojective spaces to those without radicals. It is stated in a form suitable for application to polar spaces.

2.4. THEOREM. *Let S be a gamma space all of whose lines are thick. Suppose X is a singular subspace of S with the property that any singular subspace Y containing X and for every $p \in Y^\perp \setminus Y$, the subset Y is a hyperplane of $\langle Y,p\rangle$. Consider the assignment of the subspace $\rho_X(p) = \langle X,p\rangle$ of $X^\perp$ to $p \in X^\perp \setminus X$, and the resulting space $\rho_X S = (\rho_X P, \rho_X \mathcal{L})$ where*

$$\rho_X P = \{\rho_X(p)\colon\ p \in X^\perp \setminus X\}$$

and

$$\rho_X \mathcal{L} = \{\{\rho_X(p)\colon\ p \in \ell\}\colon\ \ell \in \mathcal{L},\ \ell \subset X^\perp,\ \ell \cap X = \varnothing\}.$$

Then $\rho_X S$ is a space with radical $\rho_X(\mathrm{Rad}(X^\perp) \setminus X)$, and the canonical surjective map $X^\perp \setminus X \to \rho_X S$ is a morphism defined on $(X^\perp \setminus X, \mathcal{L}(X^\perp \setminus X))$ with the property that points x, y of $\rho_X S$ are collinear if and only if each $p \in \rho_X^{-1}(x)$ and $q \in \rho_X^{-1}(y)$ are collinear.

[1] In Chapter 3, this is called a projective hyperplane (Editor's note).

PROOF. First note that $\langle X, p\rangle$, being the span of the hyperplane X and p, is the union of all lines on p meeting X.

Now suppose x, y are distinct points of $\rho_X P$ and let $p \in \rho_X^{-1}(x)$ and $q \in \rho_X^{-1}(y)$.

If x and y are collinear, there is a line $\ell \in \mathcal{L}$ disjoint from X such that $\{\langle X, z\rangle\colon z \in \ell\}$ is a line containing $\rho_X(x)$ and $\rho_X(y)$. Then $\langle X, p, q\rangle$ is a singular subspace of S. Since $\ell \subseteq \langle X \cup \{p, q\}\rangle$ and $\ell \cap X = \varnothing$, the subspace X is not a hyperplane of $\langle X \cup \{p, q\}\rangle$. If any line through p and q met X, then $\langle X, p, q\rangle = \langle X, p\rangle$, contradicting the fact that X is not a hyperplane of $\langle X, p, q\rangle$. Hence there is a line on p and q in $X^{\perp} \setminus X$.

Conversely, if p and q are collinear, then, as $\rho_X p = x$ and $\rho_X q = y$ are distinct, p does not lie in $\langle X, q\rangle$, yielding that there is a line ℓ on p and q disjoint from X, so that x, y are both on the line $\{\langle X, z\rangle\colon z \in pq\}$ of $\rho_X S$.

Finally, assume $x \in \operatorname{Rad} \rho_X S$. Then $x = \langle X, z\rangle$ for some $z \in X^{\perp} \setminus X$, and, by the previous paragraph, z is collinear to all of $X^{\perp} \setminus X$, whence all of $X^{\perp}$, so that $z \in \operatorname{Rad}(X^{\perp}) \setminus X$. □

We shall abbreviate ρ_X to ρ if $X = \operatorname{Rad} S$. For x a point of $A_{n,k}(D)$ as in §1.7, the space $\rho(x^{\perp})$ is a direct product of $A_{k-1,1}(D)$ and $A_{n-k,1}(D)$.

If S is embedded in a projective space $\boldsymbol{P}$, then any hyperplane H of $\boldsymbol{P}$ gives rise to a hyperplane of S by intersecting it with $\mathcal{P}(S)$, unless $\mathcal{P}(S) \subseteq H$. If we assume that S spans $\boldsymbol{P}$, then the exceptional case does not occur. Now the collection $\mathcal{H}$ of all hyperplanes of S thus obtained, provides a tomography of S, which might enable us to reconstruct $\boldsymbol{P}$ intrinsically from S. The first step in this strategy will be the analysis of $\mathcal{H}$ as resulting from a given embedding.

2.5. LEMMA. *Suppose S is a space embedded in a projective space $\boldsymbol{P}$ which is spanned by $\mathcal{P}(S)$. If H is a maximal subspace of S, then the subspace $\langle H\rangle_{\boldsymbol{P}}$ of $\boldsymbol{P}$ spanned by H contains a hyperplane of $\boldsymbol{P}$. If, moreover, $\langle H\rangle_{\boldsymbol{P}} \neq \boldsymbol{P}$, then $H = \langle H\rangle_{\boldsymbol{P}} \cap \mathcal{P}(S)$.*

PROOF. Write $H' = \langle H\rangle_{\boldsymbol{P}}$. For $p \in \mathcal{P}(S) \setminus H$, we have

$$\langle H', p\rangle_{\boldsymbol{P}} = \langle H, p\rangle_{\boldsymbol{P}} = \langle \langle H, p\rangle_S\rangle_{\boldsymbol{P}} = \langle \mathcal{P}(S)\rangle_{\boldsymbol{P}} = \boldsymbol{P},$$

showing that H' contains a hyperplane of $\boldsymbol{P}$. The remainder of the proof is straightforward. □

2.6. PROPOSITION. *Suppose S is a space embedded in a projective space $\boldsymbol{P}$ which is spanned by $\mathcal{P}(S)$. Set*

$$\mathcal{H} = \{H' \cap \mathcal{P}(S)\colon\ H' \text{ hyperplane of } \boldsymbol{P}\}.$$

(a) *For any two distinct points x, y of S, there is a member of $\mathcal{H}$ containing x but not y.*

(b) *If H_1 and H_2 are distinct members of $\mathcal{H}$, then, for each point p of S, there is a member of $\mathcal{H}$ containing p and $H_1 \cap H_2$.*

PROOF. This is immediate from the properties of $\boldsymbol{P}$. □

2.7. *Ample connectedness.* In general, the collection $\mathcal{H}$ need not be enough to reconstruct $\boldsymbol{P}$. For example, if we take S to be a set of n noncollinear points in an $(n-1)$-dimensional projective space $\boldsymbol{P}$ in general position, then $\mathcal{H}$ is the collection of all subsets of $\mathcal{P}(S)$, which is not enough to reflect all hyperplanes of $\boldsymbol{P}$. We need to ensure that S is sufficiently rich to capture the collections of hyperplanes of $\boldsymbol{P}$. This richness of S will be formulated in terms of ample connectedness, which we shall now introduce.

We call a space S *amply connected* if, for any proper subspace U and points $p, q \in \mathcal{P}(S) \setminus U$, there is a sequence $p = p_1, p_2, \ldots, p_n = q$ with each $p_i \in \mathcal{P}(S) \setminus U$ and such that any two consecutive members p_i, p_{i+1} are collinear. The following up to §2.12 is a choice of results in Veldkamp [1959] and in Shult [1992b, 1993b]. For ease of presentation, we have not aimed for greatest generality. The proofs given here are similar in spirit to those in Buekenhout and Cohen [1992], which in turn are inspired by arguments occurring in Teirlinck [1980].

2.8. LEMMA. *Let S be an amply connected space and H a hyperplane in S. Then H is a maximal proper subspace of S.*

PROOF. If x, y are distinct points in $\mathcal{P}(S) \setminus H$, we must show that $\langle H, x\rangle$ contains y. Consider a chain $x = x_1, \ldots, x_n = y$ from x to y in which consecutive members are distinct, collinear and not in H. Then a line ℓ on x_1 and x_2 intersects H in a point p and so $\langle H, x\rangle$ contains p and x, hence ℓ. The same argument applies inductively to lines containing x_i and x_{i+1} for $i = 2, \ldots, n-1$. Therefore $y \in \langle H, x\rangle$. □

2.9. *Projective sets of hyperplanes.* We next consider one more axiom regarding a collection $\mathcal{H}$ of hyperplanes of a space S.

(c) If A, B, C, D are in $\mathcal{H}$, then $D \supseteq A \cap B \cap C$ implies either $D \supseteq A \cap B$ or $A \cap B \cap C = A \cap B \cap D$.

This condition holds if, for any two (not necessarily) distinct members A, B of $\mathcal{H}$, hyperplanes of $A \cap B$ are maximal subspaces of $A \cap B$. In turn, this follows, by the above lemma, from ample connectedness of such intersections.

For future use, we record the following consequence of (c):

(d) If A, B, C are in $\mathcal{H}$, then $C \supseteq A \cap B$ implies either $C \supseteq A$ or $A \cap C = A \cap B$.

In general, for a space S, we shall turn such a collection $\mathcal{H}$ of hyperplanes of S into a singular space by taking for lines all subsets of the form

$$H_1H_2 = \{H \in \mathcal{H}\colon\ H \supseteq H_1 \cap H_2\},$$

where H_1, $H_2 \in \mathcal{H}$ are distinct. This space will be called the *Veldkamp space*[2] of S relative to $\mathcal{H}$, and denoted by $\mathcal{V}_{\mathcal{H}}(S)$. If $\mathcal{H}$ is the collection of all hyperplanes of S, we shall occasionally drop $\mathcal{H}$ as an index, and call the result the *Veldkamp space* of S. Observe that (d) implies that $\mathcal{V}_{\mathcal{H}}(S)$ is a linear space.

[2] This definition differs from Veldkamp's definition for a V-space in Chapter 19 (Editor's note).

A general strategy for embedding S is to try and find conditions on $\mathcal{H}$ which guarantee that the Veldkamp space $\mathcal{V}_{\mathcal{H}}(S)$ is a projective space and that S embeds in $\mathcal{V}(\mathcal{V}_{\mathcal{H}}(S))$ via the map $x \mapsto \{H \in \mathcal{H}\colon\ x \in H\}$ $(x \in S)$. This principle is laid down in the proposition below.

A collection $\mathcal{H}$ will be called *projective* if it arises from an embedding of S as described in Proposition 2.6. It will be called *quasiprojective* if it satisfies the conditions of Proposition 2.6:

(a) for any two points x, y of S, there is a member of $\mathcal{H}$ containing x but not y; and

(b) if H_1 and H_2 are distinct members of $\mathcal{H}$, then, for each point p of S, there is a member of $\mathcal{H}$ containing p and $H_1 \cap H_2$.

If, in addition, it satisfies (c), it will be called *fully projective*. By the proposition, $\mathcal{H}$ is quasiprojective if it is projective. Below we shall see that $\mathcal{H}$ is projective if it is fully projective.

2.10. PROPOSITION. *Let S be an amply connected space. If $\mathcal{H}$ is a quasiprojective collection of amply connected hyperplanes, then condition* (d) *is satisfied. In particular, $\mathcal{V}_{\mathcal{H}}(S)$ is a linear space.*

2.11. LEMMA. *Singular spaces are amply connected. The Veldkamp space of a singular space is a linear space.*

PROOF. Let S be a singular space and X a proper subspace. Any two distinct points x, $y \in \mathcal{P}(S) \setminus X$ are connected by a line joining them, so the collinearity graph of $\mathcal{P}(S) \setminus X$ is connected.

Since subspaces of linear spaces are again linear spaces, Lemma 3.8 yields that condition (d) is satisfied, so the Veldkamp space of S is linear. □

If S is an affine space of singular rank at least 2, then a hyperplane of S is not an affine hyperplane in the classical sense (i.e. the point set of an affine subspace of codimension 1), as there are lines parallel to affine hyperplanes. But affine hyperplanes are maximal subspaces. Since there are no other maximal subspaces, we have $\mathcal{V}(S) = \varnothing$. Thus, no embeddings of affine spaces into projective spaces exist. (Of course, contrary to what we are considering here, in the more common interpretation of an embedding of an affine space in a projective space lines do not necessarily map onto lines.)

On the other hand, if S is a projective space then $\mathcal{V}(S)$ is readily seen to be the dual projective space S^*.

For $x \in \mathcal{P}(S)$, set $x^{\mathcal{H}} = \{H \in \mathcal{H}\colon\ x \in H\}$.

2.12. THEOREM. *Suppose S is a space having a fully projective set of hyperplanes $\mathcal{H}$. Then*

(i) *for each $x \in \mathcal{P}(S)$, the set $x^{\mathcal{H}}$ is a hyperplane of $\mathcal{V}_{\mathcal{H}}(S)$;*

(ii) *the map $x \mapsto x^{\mathcal{H}}$ $(x \in S)$ is an embedding of S into $\mathcal{V}(\mathcal{V}_{\mathcal{H}}(S))$;*

(iii) *$\mathcal{V}_{\mathcal{H}}(S)$ is a projective space, so the mapping in* (ii) *leads to an embedding of S in a projective space;*

(iv) *The projective embedding in* (iii) *into the subspace of* $\boldsymbol{P}$ *spanned by* $\mathcal{P}(S)$ *is, up to isomorphism, the unique such embedding yielding* $\mathcal{H}$.

PROOF. (i) Suppose H_1H_2 is a line of $\mathcal{V}_{\mathcal{H}}(S)$. If $x \in H_1 \cap H_2$, then each member of the line H_1H_2 obviously contains x, whence belongs to $x^{\mathcal{H}}$. Suppose therefore $x \notin H_1 \cap H_2$, then, by properties (b), (c) and §2.8 above for $\mathcal{H}$, there is a unique member of the line H_1H_2 containing x. In other words, there is a unique member of the line H_1H_2 belonging to $x^{\mathcal{H}}$. By (a) we cannot have $x^{\mathcal{H}} = \mathcal{H}$, so $x^{\mathcal{H}}$ is indeed a hyperplane of $\mathcal{V}_{\mathcal{H}}(S)$.

(ii) It is straightforward from property (a) of full projectiveness for $\mathcal{H}$ that the map $x \mapsto x^{\mathcal{H}}$ is injective.

Now suppose ℓ is a line of S containing the points x and y. If $z \in \ell$ then, as $x^{\mathcal{H}} \cap y^{\mathcal{H}}$ consists of all hyperplanes of S containing both x and y, each of its members contains z, proving that $z^{\mathcal{H}}$ contains $x^{\mathcal{H}} \cap y^{\mathcal{H}}$. Hence $z^{\mathcal{H}}$ is on the line $x^{\mathcal{H}}y^{\mathcal{H}}$ of $\mathcal{V}_{\mathcal{H}}(S)$, establishing that we have a morphism.

It remains to show that this morphism sends lines onto lines. Suppose xy is a line of S and $D \in x^{\mathcal{H}}y^{\mathcal{H}}$ satisfies $D \neq x^{\mathcal{H}}, y^{\mathcal{H}}$. Thus $D \in \mathcal{V}(\mathcal{V}_{\mathcal{H}}(S))$ and

$$x^{\mathcal{H}} \cap y^{\mathcal{H}} = \{H' \in \mathcal{H}\colon\ x, y \in H'\} \subseteq D.$$

Take $H'' \in D$ with $x \notin H''$. It exists because otherwise we would have $D \subseteq x^{\mathcal{H}}$, whence, by Lemma 2.8 (as both are hyperplanes of the linear space $\mathcal{V}_{\mathcal{H}}(S)$ which is amply connected) $D = x^{\mathcal{H}}$, a contradiction. Then $H'' \cap xy$ must be a point, say w, of S. Since $\mathcal{V}_{\mathcal{H}}(S)$ is a linear space, so is its hyperplane $x^{\mathcal{H}}$; thus, $x^{\mathcal{H}}$ is amply connected and (again by Lemma 2.8) $x^{\mathcal{H}} \cap y^{\mathcal{H}} = x^{\mathcal{H}} \cap D$. Now, from $x^{\mathcal{H}} \cap y^{\mathcal{H}} \subseteq w^{\mathcal{H}}$ (due to the previous paragraph), we find that $w^{\mathcal{H}}$ contains $D \cap x^{\mathcal{H}}$ and H'', whence D, so $w^{\mathcal{H}} = D$. Thus the map sends lines onto lines. As it is injective, we conclude it is an embedding.

(iii) We show Pasch's Axiom for $\mathcal{V}_{\mathcal{H}}(S)$. Let $A, B, C, P, Q \in \mathcal{H}$ all be distinct with $P \in AB$, and $Q \in AC$. We need to establish the existence of a member of $BC \cap PQ$. Then $A \cap B \subseteq P$ and $A \cap C \subseteq Q$, so $A \cap B \cap C \subseteq P \cap Q$. If $A \cap B \cap C = P \cap Q$, the picture collapses to a line and there is nothing to prove. So assume the contrary: take $p \in P \cap Q \setminus A \cap B \cap C$. Then, as $\mathcal{H}$ is fully projective, using property (b), there exists $H \in \mathcal{H}$ containing p and $B \cap C$. Since $A \cap B \cap C$ and p are in $P \cap Q$ and the former is a hyperplane and $P \cap Q$ is linear, we have, by (c),

$$P \cap Q = \langle P \cap Q \cap A, p\rangle = \langle A \cap B \cap C, p\rangle \subseteq H.$$

Now H contains both $P \cap Q$ and $B \cap C$, so H is the point of intersection of the lines BC and PQ of $\mathcal{H}$. Hence Pasch's Axiom holds for $\mathcal{V}_{\mathcal{H}}(S)$, proving that it is a (possibly reducible) projective space. Now $\mathcal{V}(\mathcal{V}_{\mathcal{H}}(S))$ coincides with the dual projective space of $\mathcal{V}_{\mathcal{H}}(S)$, and so it also is a (possibly reducible) projective space. By (ii), S embeds in it. □

2.13. DEFINITION. A *polarity* of a space S is a symmetric relation π on $\mathcal{P}(S)$ with the property that $x^{\pi} := \{y \in \mathcal{P}(S)\colon\ y\,\pi\,x\}$ is a hyperplane of S for each $x \in \mathcal{P}(S)$.

In §3.4, a more general notion of polarity is introduced under the name of quasi-polarity.

A warning is in order: in the literature, the notion of polarity of a space is sometimes used for a map (preserving incidence) interchanging points and lines. For a generalised quadrangle (see Chapter 9), the map $x \mapsto x^{\perp}$ (for notation, cf. Section 2) defined on the point set would not be a polarity in that sense, but it is a polarity in our sense.

2.14. PROPOSITION (Anne Parmentier, cf. Buekenhout [1994]). *Suppose S is a singular space, and π is a polarity of S. Then S is a (possibly reducible) projective space.*

PROOF. Take $\mathcal{H} = \{x^\pi\colon x \in \mathcal{P}(S)\}$. We show that $\mathcal{H}$ is fully projective.

Suppose x, y are distinct points of S. If $z \in \mathcal{P}(S) \setminus x^\pi \cap y^\pi$, then there is a unique point u in $xy \cap z^\pi$. Now u^π is a member of $\mathcal{H}$ containing both $x^\pi \cap y^\pi$ and z. This shows that $x^\pi \not\subseteq y^\pi$, for otherwise, maximality of x^π (cf. Lemma 3.8) would lead to $u^\pi = \mathcal{P}(S)$, contradicting that u^π is a hyperplane. It also shows that condition (b) of §2.6 holds.

Since $x^\pi \not\subseteq y^\pi$ there exists a point $z \in x^\pi \setminus y^\pi$. Then z^π is a member of $\mathcal{H}$ containing x but not y. This proves the separation property (a).

We now check property (c) for $\mathcal{H}$ to be fully projective. Let $a, b, c, d \in \mathcal{P}(S)$ with

$$d^\pi \supseteq a^\pi \cap b^\pi \cap c^\pi,$$

and assume $d^\pi \not\supseteq a^\pi \cap b^\pi$, so that $d \notin ab$. We have to show

$$a^\pi \cap b^\pi \cap c^\pi = a^\pi \cap b^\pi \cap d^\pi.$$

As $d^\pi \cap a^\pi \cap b^\pi$ is a hyperplane of the linear space $a^\pi \cap b^\pi$ containing the hyperplane $a^\pi \cap b^\pi \cap c^\pi$ (thereof), Lemma 2.11 and Proposition 2.10 give property (d).

Thus, $\mathcal{H}$ is fully projective, and so, by the above lemma, $\mathcal{V}_{\mathcal{H}}(S)$ is a projective space. We derive that the map $S \to \mathcal{V}_{\mathcal{H}}(S)$ given by $x \mapsto x^\pi$ is an embedding. By hypothesis, the map is injective. Let $x, y \in S$ be distinct. If $z \in xy$ then $z^\pi \supseteq x^\pi \cap y^\pi$, so lines are mapped into lines. Conversely, if $z \in \mathcal{P}(S)$ is such that $z^\pi \supseteq x^\pi \cap y^\pi$ and $z \neq x, y$, then there exists $u \in z^\pi \setminus x^\pi \cap y^\pi$. As before, this implies that there is a unique point $z' \in xy \cap u^\pi$. From $z'^\pi \supseteq x^\pi \cap y^\pi$ we find

$$z^\pi = \langle y^\pi \cap z^\pi, u\rangle = \langle y^\pi \cap x^\pi, u\rangle \subseteq z'^\pi,$$

so $z^\pi = z'^\pi$, whence $z = z'$. We conclude that $x \mapsto x^\pi$ maps lines onto lines, and so is indeed an embedding of S. Finally, the embedded space S^π is a singular subspace of the projective space $\mathcal{V}_{\mathcal{H}}(S)$, whence projective. □

2.15. EXAMPLE. Consider the generalised quadrangle S whose points are the 2-sets from $\{1, \ldots, 6\}$ and whose lines are the triples of 2-sets whose union coincides with $\{1, \ldots, 6\}$. Let H be a hyperplane. Then it is easy to check that H is of one of the following:

– a set $p^\perp$ for some point p, consisting of 7 points and 3 lines. There are 15 such hyperplanes;

– a grid, consisting of 9 points and 6 lines. There are 10 such hyperplanes;

– an ovoid, consisting of 5 points and containing no line. There are 6 such hyperplanes.

Hence $\mathcal{V}(S)$ has $15+10+6=31$ points. But the collection of all hyperplanes is not fully projective. Indeed, taking $A=\{12,13,14,15,16\}$ and $B=\{12,23,24,25,26\}$ and $C=(34)^\perp$, we find $C\supseteq\{12\}=A\cap B$, but $A\cap C=\{12,15,16\}\neq A\cap B$, proving that (d) is not satisfied. Accordingly, the sets

$$\{H\in\mathcal{V}(S)\colon\ H\supseteq H_1\cap H_2\},$$

for H_1,H_2 distinct points of $\mathcal{V}(S)$, do not always represent lines of cardinality 3: for two distinct collinear points a,b of S, the Veldkamp line on $a^\perp$ and $b^\perp$ in $\mathcal{V}(S)$ has a point of the form $c^\perp$ (namely for c with $ab=\{a,b,c\}$), but also four grids containing ab, which are hyperplanes. Taking $\mathcal{H}$ to be the set of 15 hyperplanes of the form $p^\perp$ for $p\in S$, we do obtain a fully projective collection of hyperplanes. Then, $\mathcal{V}_{\mathcal{H}}(S)$ is the projective space $\boldsymbol{P}(\boldsymbol{F}_2^4)$. The image of the resulting embedding (cf. Theorem 2.12) is the Sp(4, 2) generalised quadrangle.

If the lines have size three, there is an easier way to define lines on the collection of hyperplanes.

2.16. THEOREM. *Let S be a partial linear gamma space all of whose lines have three points. Then, for any two distinct hyperplanes A,B in $\mathcal{H}$, the subset*

$$A*B=(A\cap B)\cup\bigl(S\setminus(A\cup B)\bigr)$$

of $\mathcal{P}(S)$ is also a hyperplane. If $\mathcal{H}$ is a collection of hyperplanes of S which is closed under $*$ (i.e. $A*B\in\mathcal{H}$ whenever A, B are distinct members of $\mathcal{H}$), then $\mathcal{H}$, together with all lines of the form $\{A,B,A*B\}$, is a projective space. If, in addition, $\mathcal{H}$ satisfies 2.6(a), *the space S embeds in the dual of $\mathcal{H}$.*

Of course, the set $\mathcal{H}$ of all hyperplanes of S is always $*$-closed.

PROOF. Let A,B be distinct hyperplanes. Consider $C=A*B$. It is a projective hyperplane containing $A\cap B$, as can be straightforwardly checked.

We check that $\mathcal{H}$ satisfies Pasch's Axiom. Suppose that A,B,C,P,Q are hyperplanes of S such that A,B,P and A,C,Q are collinear on distinct lines of $\mathcal{H}$. Then $P=A*B$, $Q=A*C$. Straightforward set operations lead to

$$P\cap Q=(A\cap B\cap C)\cup\bigl(S\setminus(A\cup B\cup C)\bigr),$$

$$S\setminus(P\cup Q)=\bigl((B\cap C)\setminus A\bigr)\cup\bigl(A\setminus(B\cup C)\bigr)$$

whence

$$P*Q=(B\cap C)\cup\bigl(S\setminus(B\cup C)\bigr)=B*C$$

and so the lines BC and PQ of $\mathcal{H}$ have a common point. The proof of the remainder is straightforward. □

Let S be as in the theorem. Consider the vector space V, i.e. the quotient of the free vector space with basis $\mathcal{P}(S)$ by the set of relations $x+y+z=0$ for all lines $\{x,y,z\}$ of S. Then, to a hyperplane H of S, we can assign the linear functional $f_H\colon V \to \mathbb{Z}/2$ determined (and well-defined) by

$$f_H(x) = \begin{cases} 1 & \text{if } x \in \mathcal{P}(S) \setminus H, \\ 0 & \text{if } x \in H. \end{cases}$$

Thus, for hyperplanes A, B of S, the hyperplane $A * B$ is characterised by $f_{A*B} = f_A + f_B$. This gives another way of looking at the embedding of the theorem.

For a finite space S all of whose lines have size 3, there is an effective method of finding an embedding (in fact the universal one) if it exists. Take the formal vector space over $\mathbb{F}_2$ with basis $P = \mathcal{P}(S)$ and form its quotient V by the subspace generated by all vectors of the form $x + y + z$ for $\{x,y,z\} \in \mathcal{L}(S)$. Denote the image of $x \in P$ in V by $\overline{x}$. It may happen that $\overline{x} = 0$, in fact $V = 0$ may occur (e.g., when S is the affine plane of order 3). But if $\overline{x} \neq 0$ for all $x \in P$, the natural morphism $x \mapsto \mathbb{F}_2 x$ from S to $\boldsymbol{P}(V)$ is an embedding (and conversely, if S has a projective embedding, then this map is one). For several point-lines spaces, the dimension of V is known. If S is the generalized octagon of order $(2,4)$ related to ${}^2\mathsf{F}_4(2)$ (cf. Chapter 9 and §5.16) then $\dim V = 80$. If S is the generalized hexagon related to ${}^3\mathsf{D}_4(2)$ (cf. Chapter 9 and §5.1), $\dim V = 28$, see Frohardt and Smith [1992] (where the same dimension is found for the near octagon related to the second sporadic Janko group J_2 (cf. Conway, Wilson, Curtis, Norton and Parker [1985])).

See Ronan [1987] for a further discussion of hyperplanes and embeddings, including examples.

2.17. EXAMPLE. Consider once more the above example of the generalized quadrangle S of 15 points and 15 lines. Take $\mathcal{H}$ to be the collection of all 31 hyperplanes of S. In Example 2.15, we have seen that the usual Veldkamp space $\mathcal{V}_{\mathcal{H}}(S)$ is not isomorphic to $\boldsymbol{P}(\mathbb{F}_2^5)$. But $\mathcal{H}$, together with the lines $\{A, B, A*B\}$ of Theorem 2.16, is isomorphic to $\boldsymbol{P}(\mathbb{F}_2^5)$, and the embedding of S maps onto the quadric $\mathrm{PO}(5,2)$. By Chapter 9 (or by a direct check), we know that S is isomorphic to this quadric.

To end the section on embeddings, we mention the existence of a relation between embeddings developed in Wells [1983]. The proof is easy.

2.18. LEMMA. *Let S be a space embedded in the projective space $\boldsymbol{P}$. Denote by $S+S$ the set of all points of $\boldsymbol{P}$ on lines of $\boldsymbol{P}$ spanned by points of S, and let $\phi\colon \boldsymbol{P} \to \boldsymbol{P}'$ be a morphism of projective spaces with projective kernel K. (That is, ϕ is a nontrivial semilinear map on the corresponding vector spaces whose kernel is the linear subspace K.) Then the following three statements are equivalent.*

(i) *$\phi|_S$ is defined (i.e. $\mathcal{P}(S) \cap K = \varnothing$) and is an embedding of S in $\boldsymbol{P}'$;*

(ii) *no point of K is a point of S or lies on a projective line joining two points of S;*

(iii) *$(S+S) \cap K = \varnothing$.*

Coming back to the generalized quadrangle S of order $(2,2)$ of §2.12 embedded in the projective space $\boldsymbol{P}$ of dimension 5 as a quadric, we can take K to be the unique point perpendicular to all embedded points with respect to the bilinear form determined by the quadratic form defining S. Then (iii) above is satisfied, so that $\boldsymbol{P}/K$ is also an embedding of S. It coincides with the embedding discussed in §2.15.

3. Polar spaces

A *polar space* is a space in which, for every point x, the subset $x^\perp$ is either a hyperplane or the whole point set. As a consequence of the definition, polar spaces are gamma spaces of diameter at most 2. Singular spaces are degenerate examples of polar spaces. A polar space is said to be *of rank* n if its singular rank is $n-1$. The origin of this definition is in Tits [1974], see also Buekenhout and Shult [1974] for 'finite rank'. The extensions of their results to arbitrary rank are discussed in Johnson [1990], Buekenhout [1990] and Percsy [1989], and also in the present section.

The join (cf. §1.5) of n copies of the empty graph $\overline{K}_2$ on two vertices is a graph on $2n$ vertices which is a polar space of rank n. It is an example in which all lines are thin. The join of two polar spaces is again a polar space. Lines joining the two constituents have size two. If a polar space has both thick and thin lines, it can be reconstructed from polar spaces with thick lines only. The reconstruction goes by means of two well understood procedures, one of which is the join. The papers Buekenhout [1990], Buekenhout and Sprague [1982] contain reductions of the case in which lines of length 2 occur, to polar spaces without thin lines. Consequently, it is not essential restriction in the study of polar spaces to restrict to those all of whose lines are thick.

3.1. THEOREM. *Let S be a polar space. The following assertions hold.*

(i) *Any subspace of S is a polar space.*

(ii) *The quotient space ρS, where ρ is as in* §2.4, *is a nondegenerate polar space.*

(iii) *If S is nondegenerate, then, for any two noncollinear points points x, y, the subspace $x^\perp \cap y^\perp$ is a nondegenerate polar space; moreover, $x^{\perp\perp} = \{x\}$ and, for any line ℓ, also $\ell^{\perp\perp} = \ell$.*

(iv) *If S is nondegenerate, then it is paraprojective and all of its singular subspaces of rank* 2 *are Moufang planes* (cf. Chapter 11, §3).

The last statement is only meaningful for the case where S has rank 3, for if the rank is higher, each plane occurs in a singular subspace of higher rank, which, by (iv), is a projective space and so is Desarguesian.

PROOF. See Buekenhout and Shult [1974] for (i)–(iii) (under finite rank assumption) and Tits [1974] for (iv). A newer treatment is in Johnson [1990] and Buekenhout and Cohen [1992]. □

The polar spaces motivating the definition come from polarities of projective spaces. Veldkamp [1959] first studied these examples from a synthetic point of view in the context of what he called polar geometries.

3.2. EXAMPLES. A polar space of rank 1 is a space with points but no lines. A polar space of rank 2 is a generalized quadrangle, see Chapters 9 and 10. The Grassmannian $A_{3,2}(D)$ (cf. §1.7) is a nondegenerate polar space of rank 3.

The following result shows that nondegenerate polar spaces admit automorphisms and shows some hope for classification by means of induction on the rank.

3.3. THEOREM. *Suppose S and S' are nondegenerate polar spaces of rank $r \geqslant 3$ (possibly $r = \infty$), all of whose lines have at least three points. Let p, q, respectively p', q' be noncollinear points of S, respectively S'.*

(i) *Any (collinearity graph) isomorphism from $p^\perp \cup \{q\}$ onto $p'^\perp \cup \{q'\}$ extends uniquely to an isomorphism from S onto S'.*

(ii) *If $p^\perp$ and $p'^\perp$ are isomorphic, then S and S' are isomorphic.*

PROOF. The proof is entirely synthetic. It is essentially due to Tits and can be found in Buekenhout and Cohen [1992]. □

This result can be used to derive that the automorphism group of a nondegenerate polar of rank at least 3 with thick lines has a Tits system (cf. Chapter 11, 5.3). If p, q, q' are points of such a polar space S with $q, q' \notin p^\perp$ and $q^\perp \cap p^\perp = q'^\perp \cap p^\perp$, then, by Theorem 3.3(i), there is a unique automorphism fixing $p^\perp$ pointwise and mapping q to q'. Such an automorphism will be called a *polar transvection* of S.

3.4. *Polarities.* See Tits [1974]. A *quasi-polarity* of a projective space $\boldsymbol{P}$ is a symmetric relation $\pi \subseteq \boldsymbol{P} \times \boldsymbol{P}$ such that, for each $x \in \boldsymbol{P}$, the set $\pi(x)$ of all points of $\boldsymbol{P}$ in relation π to x is either a hyperplane of $\boldsymbol{P}$ or equal to $\boldsymbol{P}$. The points $x \in \boldsymbol{P}$ with $x \in \pi(x)$ are called the *absolute points* of $\boldsymbol{P}$ with respect to π. More generally, a subspace X of $\boldsymbol{P}$ is called *absolute* with respect to π if $a \in \pi(b)$ for all $a, b \in X$. We shall write $\pi(X)$ to denote the subspace $\bigcap_{x\in X} \pi(x)$ of $\boldsymbol{P}$. Thus X is absolute if and only if $X \subseteq \pi(X)$. The *absolute space of* π, denoted by $\boldsymbol{P}_\pi$, is the space whose points are the absolute points of $\boldsymbol{P}$ with respect to π and whose lines are the absolute lines of $\boldsymbol{P}$ with respect to π. Thus, $\boldsymbol{P}_\pi$ is a space embedded in $\boldsymbol{P}$.

Let D be the intersection of all $\pi(x)$ for x varying over all points of $\boldsymbol{P}$, i.e. $D = \pi(\boldsymbol{P})$. Then $x \in D$ if and only if $\pi(x) = \boldsymbol{P}$. A polarity (as defined in §2.13) is a quasipolarity π of $\boldsymbol{P}$ such that $\pi(x)$ is a hyperplane for each point $x \in \boldsymbol{P}$ (equivalently: such that $D = \varnothing$). A quasipolarity π on $\boldsymbol{P}$ induces a polarity on the projective quotient space $\boldsymbol{P}/D$.

A polarity can be viewed as a mapping from $\boldsymbol{P}$ to its dual $\boldsymbol{P}^*$ with the property that $a \in \pi(b)$ implies $b \in \pi(a)$ for all points a, b of $\boldsymbol{P}$.

3.5. THEOREM. *Suppose $\boldsymbol{P}$ is a projective space of dimension at least 2. Let π be a quasipolarity of $\boldsymbol{P}$ and let $\boldsymbol{P}_\pi$ be its absolute space. Then the following hold.*

(i) *A line ab of* $\boldsymbol{P}$ *is absolute if and only if* a *and* b *belong to* $\pi(a) \cap \pi(b)$.

(ii) *A subspace* X *of* $\boldsymbol{P}$ *is absolute if and only if all points and lines contained in* X *are absolute.*

(iii) $\boldsymbol{P}_\pi$ *is a polar space.*

(iv) *If the points of* $\boldsymbol{P}_\pi$ *span* $\boldsymbol{P}$, *then the space* $\boldsymbol{P}_\pi$ *is degenerate if and only if there exists* $x \in \mathcal{P}(\boldsymbol{P})$ *with* $\pi(x) = \mathcal{P}(\boldsymbol{P})$.

(v) *The map* π *is a morphism of projective spaces.*

PROOF. The proof is not hard. Part (v) can be found in Lenz [1954]. □

The algebraic counterpart of polarities are reflexive sesquilinear forms.

3.6. *Sesquilinear forms.* See Dieudonné [1963], Gross [1979], Tits [1974]. Let $\boldsymbol{D}$ be a division ring, σ an anti-automorphism of $\boldsymbol{D}$, and V a *right vector space* over $\boldsymbol{D}$. A *σ-sesquilinear form on* V, or *sesquilinear form relative to* σ, is a mapping $f\colon V \times V \to \boldsymbol{D}$ such that

$$f(a+b,c) = f(a,c) + f(b,c), \qquad f(a,b+c) = f(a,b) + f(a,c),$$
$$f(a\lambda,b) = \lambda^\sigma f(a,b), \qquad f(a,b\mu) = f(a,b)\mu,$$

for all $a,b,c \in V$ and all $\lambda,\mu \in \boldsymbol{D}$. Here λ^σ denotes the image of λ under σ.

Given a σ-sesquilinear form f on V, a relation π_f on $\boldsymbol{P} = \boldsymbol{P}(V)$ can be defined by setting $a\,\pi_f\,b$ if and only if $f(a,b) = 0$. (Observe that $\pi_f(a) := \{b \in \mathcal{P}(P)\colon\ b\pi_f a\}$ depends only on the choice of $\langle a \rangle \in \boldsymbol{P}(V)$.) Since f is linear in b, the set $\pi_f(a)$ of all points $\langle x \rangle$ of $\boldsymbol{P}$ for which $f(a,x) = 0$ is either a hyperplane of $\boldsymbol{P}$ or all of $\boldsymbol{P}$. Observe that π_f need not be a symmetric relation.

We say that a σ-sesquilinear form f on a vector space V is *reflexive* if π_f is a quasipolarity, or, equivalently, if for each pair $x,y \in V$ the relation $f(x,y) = 0$ implies $f(y,x) = 0$. The *radical* of f, denoted by $\operatorname{Rad} f$, is the subspace $\{x \in V\colon\ f(x,V) = 0\}$ of V. The form f is called *nondegenerate* if $\operatorname{Rad} f = 0$.

Suppose $\dim V \geqslant 2$ and f is a nondegenerate σ-sesquilinear form on V. Then f is reflexive if and only if there is a scalar $\varepsilon \in \boldsymbol{D} \setminus \{0\}$ such that $f(y,x) = f(x,y)^\sigma \varepsilon$ for all x,y in V. If, in addition, $f \neq 0$, then

$$\varepsilon^\sigma = \varepsilon^{-1} \quad \text{and} \quad t^{\sigma^2} = \varepsilon t \varepsilon^{-1} \quad \text{for all } t \in \boldsymbol{D}. \tag{1}$$

See Tits [1974] for proofs. A pair σ, ε of an anti-automorphism σ of $\boldsymbol{D}$ and an element $\varepsilon \in \boldsymbol{D}$ satisfying these relations will be called *admissible*. If the form f is nonzero, it determines σ, ε uniquely.

3.7. PROPOSITION. *Let* σ, ε *be an admissible pair for* $\boldsymbol{D}$. *If* $\{e_i\}_{i\in I}$ *is a basis of* V *and if* σ *is an anti-automorphism of* $\boldsymbol{D}$, *then every* σ*-sesquilinear form on* V *is a mapping* f *such that*

$$f(x,y) = \sum_{i,j\in I} x_i^\sigma f_{ij} y_j,$$

where $[f_{ij}]$ *is the matrix over* $\boldsymbol{D}$ *defined by* $f_{ij} = f(e_i,e_j)$. *Conversely, any such mapping is a* σ*-sesquilinear form.*

3.8. THEOREM (Tits [1974]). *Let V be a right vector space over some division ring $\boldsymbol{D}$ and set $\boldsymbol{P} = \boldsymbol{P}(V)$.*

(i) *If f is a reflexive σ-sesquilinear form on V, then π_f determines a quasipolarity in $\boldsymbol{P}(V)$. The resulting absolute space is nondegenerate if and only if f is nondegenerate, which, in turn, is equivalent to π_f being a polarity.*

(ii) *Suppose $\boldsymbol{P}$ has dimension at least 2 and π is a polarity on $\boldsymbol{P}$. Then there is a nondegenerate reflexive σ-sesquilinear form f on V, relative to some anti-automorphism σ of $\boldsymbol{D}$, such that $\pi = \pi_f$.*

If f is as in (ii), we shall also write $\boldsymbol{P}_f$ instead of $\boldsymbol{P}_\pi$.

3.9. DEFINITIONS. If f is a σ-sesquilinear form on V and $\lambda \in \boldsymbol{D}\backslash\{0\}$, then the mapping λf is a sesquilinear form relative to the anti-automorphism ρ given by $\alpha^\rho = \lambda\alpha^\sigma\lambda^{-1}$ ($\alpha \in \boldsymbol{D}$). The forms f and λf are called *proportional.* Since π_f and $\pi_{\lambda f}$ are the same relation, a polarity can be expected to determine the sesquilinear form only up to proportionality.

We assume $f \neq 0$. A σ-sesquilinear form f such that $f(y,x) = f(x,y)^\sigma\varepsilon$ for all $x, y \in V$, is called *(σ,ε)-Hermitian.* Then (σ,ε) is admissible. If $\varepsilon = 1$, then f is called *σ-Hermitian.* If $\varepsilon = -1$, then f is called *σ-anti-Hermitian.* If $\sigma = 1$, then $\boldsymbol{D}$ is a field and $\varepsilon^2 = 1$, hence $\varepsilon = 1$ or -1. If $\sigma = \text{id}$ and $\varepsilon = 1$, the form f is called *symmetric.* An *alternating form* is a σ-sesquilinear form f on V such that $f(x,x) = 0$ for all $x \in V$. Then, for all x, y in V, we have $0 = f(x+y,x+y) = f(x,y)+f(y,x)$ and so f is $(\text{id}, -1)$-Hermitian (in the literature, this is also called *antisymmetric*). Finally, *Hermitian* means σ-Hermitian for some anti-automorphism σ, and similarly for *anti-Hermitian.* Note that, in both cases, we have $\sigma^2 = 1$.

3.10. THEOREM (cf. Tits [1974]). *Suppose $\boldsymbol{P}$ has dimension at least* 1. *Every nondegenerate reflexive sesquilinear form is proportional to a Hermitian form or to an alternating form.*

Thus, by Theorems 3.8(ii) and 3.10, every polarity of $\boldsymbol{P}$ (of dimension at least 2) is of the form π_f for an admissible pair (σ,ε) with $\sigma^2 = 1$ and $\varepsilon = \pm 1$.

We now find an algebraic criterion for $\mathcal{P}(\boldsymbol{P}_\pi)$ spanning $\boldsymbol{P}$ (cf. 3.5). To this end, we introduce the *trace set* $T(\sigma,\varepsilon) = \{t + t^\sigma\varepsilon\colon\ t \in \boldsymbol{D}\}$. This is obviously an additive subgroup of $\boldsymbol{D}$. By use of the Axiom of Choice (applied to find and well-order a basis of V, which is needed for the implications (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i)) we have the following result.

3.11. PROPOSITION. *Let V be a right vector space over $\boldsymbol{D}$ and f a nondegenerate (σ,ε)-Hermitian form on V. Write $\boldsymbol{P} = \boldsymbol{P}(V)$. If the absolute geometry $\boldsymbol{P}_f$ is nonempty, then the following properties are equivalent:*

(i) $f(x,y) = g(x,y) + g(y,x)^\sigma\varepsilon$ *for some σ-sesquilinear form g and all $x, y \in V$;*

(ii) $f(x,x) \in T(\sigma,\varepsilon)$ *for all $x \in V$;*

(iii) $\mathcal{P}(\boldsymbol{P}_\pi)$ *spans $\boldsymbol{P}$;*

(iv) *there is a basis $(e_i)_{i\in I}$ of V such that $f(e_i,e_i) \in T(\sigma,\varepsilon)$ for all $i \in I$.*

3.12. *Trace-valued forms.* A reflexive σ-sesquilinear form f is called *trace-valued* if it has the equivalent properties of the proposition.

For a (σ, ε)-Hermitian form f, it only happens that the corresponding absolute space $\boldsymbol{P}_f$ does not span the whole space in special cases with $\boldsymbol{D}$ of characteristic 2. It can be shown (cf. Tits [1974]) that if the characteristic of $\boldsymbol{D}$ is not equal to 2, or if $\boldsymbol{D}$ is of characteristic 2 and σ does not fix all elements of the centre $Z(\boldsymbol{D})$, then every (σ, ε)-Hermitian form is trace-valued.

Let f be a nondegenerate (σ, ε)-Hermitian form on a vector space V. Consider the set W of all vectors x such that $f(x, x) \in T(\sigma, \varepsilon)$. Then W is a subspace of V and, of course, the restriction f' of f to W is trace-valued. So, non-trace-valued forms have a canonical trace-valued restriction. Moreover, $\mathcal{P}(\boldsymbol{P}(V)_f) \subseteq \mathcal{P}(\boldsymbol{P}(W))$, so $\boldsymbol{P}(V)_f = \boldsymbol{P}(W)_{f'}$ embeds in $\boldsymbol{P}(W)$. In view of these observations, we can restrict attention to trace-valued forms.

3.13. EXAMPLE. Let $\boldsymbol{D}$ be a field of characteristic 2, $(\sigma, \varepsilon) = (1, 1)$ and let f be a symmetric form on a vector space V over $\boldsymbol{D}$ such that $f(x, x) \neq 0$ for some $x \in V$. Then $T(\sigma, \varepsilon) = 0$, so f cannot be trace-valued. An explicit example is the form f on $\boldsymbol{D}^3$ defined by

$$f(x, y) = x_1 y_1 + x_2 y_3 + x_3 y_2$$

for $x = (x_i)_i$, $y = (y_i)_i$ in $\boldsymbol{D}^3$. Now

$$W = \left\{ (x_i)_i \in \boldsymbol{D}^3 \colon \sum_i x_i^2 = 0 \right\}$$

so $\boldsymbol{P}(\boldsymbol{D}^3)_f = \boldsymbol{P}(W) \cong \boldsymbol{P}(\boldsymbol{D}^2)$.

3.14. *Classical groups.* Putting together Theorems 3.8(ii) and 3.10, we find that every polarity in a projective space $\boldsymbol{P}(V)$ the point set of whose absolute space spans $\boldsymbol{P}(V)$ falls into one of the three following mutually exclusive cases:

– *unitary* polarity determined by a σ-Hermitian form such that $\sigma \neq 1$;

– *orthogonal* polarity determined by a symmetric bilinear form over a field of characteristic $\neq 2$;

– *symplectic* polarity determined by an alternating form over a field.

Here, *orthogonal* means 1-Hermitian. The above classification is of importance for the connection with classical groups. These groups are of the form $\mathrm{PSL}(V)_f$, the subgroup of the projective special linear group consisting of all transformations preserving (projectively) the form f related to the polarity.

Let $n = \dim V \geqslant 1$ and suppose $n < \infty$. If $\boldsymbol{D}$ is a finite field of order q, the nondegenerate examples give rise to:

– in the unitary case a unique group for each n (and q is a square). The group is usually denoted by $\mathrm{PSU}(n, q)$ or $\mathrm{PSU}(n, \sqrt{q})$ (note the confusion!);

– in the orthogonal case, two groups for each even dimension n, denoted by $\mathrm{PSO}^{\varepsilon}(n, q)$ where $\varepsilon = +$ if the underlying form is of Witt index $n/2$ (cf. Chapter 17, 1.18) and

$\varepsilon = -$ otherwise (in which case the Witt index is $n/2 - 1$); and a unique group for each odd n;

– in the symplectic case, a unique group for each even dimension n, denoted by $\mathrm{PSp}(n, q)$.

The orthogonal groups can be defined for arbitrary characteristic. However, for characteristic 2 and even n, they cannot be defined by use of a polarity; they are usually defined as the stabiliser of an orthogonal form. Later, in §3.29(ii), we shall see a geometric explanation.

The totality of finite groups found so far are summarised in the table below. We have added the special linear groups, because the resulting set of groups forms all so-called *classical groups*. (They are groups of automorphisms of the 'degenerate polar space' $P(\mathbb{F}_q^n)$.)

Classical groups notation

Classical name	Lie name	Order
$\mathrm{PSL}(n, q)$	$\mathsf{A}_{n-1}(q)$	$q^{\binom{n}{2}} \prod_{j=2}^{n}(q^j - 1)/(n, q - 1)$
$\mathrm{PSU}(n, q)$	${}^2\mathsf{A}_{n-1}(q)$	$q^{\binom{n}{2}} \prod_{j=2}^{n}(q^j - (-1)^j)/(n, q + 1)$
$\mathrm{PSO}^+(2n, q)$	$\mathsf{D}_n(q)$	$q^{n(n-1)}(q^n - 1) \prod_{j=1}^{n-1}(q^{2j} - 1)/(4, q^n - 1)$
$\mathrm{PSO}^-(2n, q)$	${}^2\mathsf{D}_n(q)$	$q^{n(n-1)}(q^n + 1) \prod_{j=1}^{n-1}(q^{2j} - 1)/(4, q^n + 1)$
$\mathrm{PSO}(2n + 1, q)$	$\mathsf{B}_n(q)$	$q^{n^2} \prod_{j=1}^{n}(q^{2j} - 1)/(2, q - 1)$
$\mathrm{PSp}(2n, q)$	$\mathsf{C}_n(q)$	$q^{n^2} \prod_{j=1}^{n}(q^{2j} - 1)/(2, q - 1)$

Often, the full group of automorphisms of the polar space is somewhat larger: diagonal automorphisms (corresponding to linear transformations having determinant different from 1) and field automorphisms (corresponding to Galois automorphisms of the underlying field D) are usually the only extra ones needed to generate the full group. Most of the classical groups are *almost simple*, i.e. they lie between a simple group and its automorphism group (with exceptions for $n \leqslant 2$ or small fields). Good references for these groups are Aschbacher [1984b], Carter [1972], Dieudonné [1963], Taylor [1992].

For general fields, there are many more different forms. The determination of the related absolute spaces and their automorphism groups (Tits [1974]) and the normal structure of these groups (Artin [1957], Dieudonné [1963], Gross [1979], James [1979], Scharlau [1985]) have been studied quite intensively, and still present some open problems.

We now turn our attention to the embedding of an arbitrary polar space in a projective space. In order to embed a thick polar space, we need to analyse whether it is amply connected. If S is a polar space for which the rank of ρS (cf. §2.4) is equal to 1, then

$\operatorname{Rad} S$ is a hyperplane of S such that $\mathcal{P}(S) \setminus \operatorname{Rad} S$ is disconnected. Thus, $\operatorname{Rad} S$ is a hyperplane of S which is not a maximal subspace (because it is contained in the proper subspace $\langle \operatorname{Rad} S, x \rangle$ for any point x off $\operatorname{Rad} S$). In particular, S is not amply connected. For ρS of higher rank, fortunately, we have a positive result.

3.15. THEOREM. *Let S be a polar space whose lines are thick.*

(i) *If $\operatorname{rk}(\rho(S)) \geqslant 2$, then S is amply connected.*

(ii) *If S is nondegenerate and the rank of S is at least 3, then $\mathcal{V}(S)$ is a linear space and the map $S \to \mathcal{V}(S)$ defined by $x \mapsto x^\perp$ is an embedding.*

PROOF. (i) Suppose X is a subspace of S and $x, y \in \mathcal{P}(S) \setminus X$ are not connected by a path of length at most 2 in the subgraph of the collinearity graph of S induced on $\mathcal{P}(S) \setminus X$. Then $x^\perp \cap y^\perp$ is entirely contained in X. Take two noncollinear points, u, v of $x^\perp \cap y^\perp$. (They exist, see Theorem 3.1(iii).) As lines are thick, there is a third point, p, on xu. It belongs to $\mathcal{P}(S) \setminus X$ since the line xu has only one point in X, namely u. Now there is a unique point $q \in yv \cap p^\perp$, which also belongs to $\mathcal{P}(S) \setminus X$, yielding a path x, p, q, y of length 3 from x to y inside $\mathcal{P}(S) \setminus X$. Hence the diameter of the induced graph on the complement of X is at most 3, yielding (i).

(ii) Nondegeneracy of S guarantees that the mapping is injective. (For $x^\perp = y^\perp$ does not occur, see Theorem 3.1(iii).) If H is a hyperplane of S in the Veldkamp line $(x^\perp)(y^\perp)$ for two distinct collinear points x, y of S, take a line ℓ not contained in $x^\perp \cup y^\perp$. (Such a line exists, because there is a point $p \in \mathcal{P}(S) \setminus x^\perp$ and, as $\operatorname{Rad} p^\perp = \{p\}$ and lines have at least 3 points, there is a point $q \in p^\perp \setminus y^\perp$ distinct from p, so we can take $\ell = pq$.) Because $x^\perp \cap y^\perp = x^\perp \cap H$ (by ample connectedness of $x^\perp$, cf. (i)), the subspace $\ell \cap H$ of ℓ must be a singleton, say $\{w\}$. Moreover, the point z with $\{z\} = w^\perp \cap xy$ satisfies $z \in xy \subset H$, and $z^\perp = \langle x^\perp \cap y^\perp, w \rangle = H$ (by ample connectedness of $z^\perp$). Hence the line xy maps onto the Veldkamp line $(x^\perp)(y^\perp)$. □

3.16. THEOREM. *Let S be a nondegenerate polar space of rank at least 3 all of whose lines are thick and all of whose singular subspaces of rank 2 are Desarguesian projective spaces. If the rank of S equals 3, assume, in addition, that each line is on at least 3 planes. Then the Veldkamp subspace $\mathcal{V}(S)$ is a projective space. In particular, S has a projective embedding.*

PROOF. In view of Theorem 3.15(ii), the last conclusion is a direct consequence of the first one. The first conclusion is one of the hardest steps in the entire classification of polar spaces, especially when the rank of S is 3. See Veldkamp [1959] for the first proof. The second one, due to Tits [1974], works via automorphisms and embeds the space directly into a space with a polarity. Veldkamp's proof has been reworked by Buekenhout and Cohen [1992]. Also in the same book, a relatively short proof for rank at least 4 is given. In Cuypers, Johnson and Pasini [1992], a slick argument is given proving that the Veldkamp subspace generated by all $p^\perp$ for $p \in \mathcal{P}(S)$ (and S of rank at least 4) is projective. □

3.17. REMARK. Suppose S is as in the hypotheses, with rank 3, but with the exception that there is a line which lies on precisely two planes. Then, consider the maximal cliques in the collinearity graph of S. It is relatively easy to derive that meeting in a point is an equivalence relation on singular planes, and has only two classes. By taking for points the members of one class and for lines all singular planes of S meeting some fixed plane from the other equivalence class in a line, a projective space is obtained. This yields that S is isomorphic to the Grassmannian $A_{3,2}(D)$ defined in §1.7, for some division ring D.

3.18. THEOREM. *Let S be a nondegenerate polar space of rank at least 2 embedded in a projective space $\boldsymbol{P}$. Suppose that $\mathcal{P}(S)$ spans the projective space $\boldsymbol{P}$. Then*

(i) *there is a unique quasipolarity π such that S is a subspace of $\boldsymbol{P}_\pi$;*

(ii) *for π as in* (i) *and for each collinear triple $a, b, c \in \boldsymbol{P}$ with $b, c \in \mathcal{P}(S) \setminus \pi(a)$, the perspectivity of $\boldsymbol{P}$ with centre a, axis $\pi(a)$ sending b to c leaves S invariant.*

PROOF. See Buekenhout and Cohen [1992], Buekenhout and Lefèvre-Percsy [1974, 1976], Dienst [1980], Johnson [1993] for proofs of this theorem. In most of these papers, the quasipolarity comes about as follows. A *tangent* to S at p is a line of $\boldsymbol{P}$ whose intersection with $\mathcal{P}(S)$ either comprises the whole line or coincides with $\{p\}$. For any $x \in \boldsymbol{P}$, define the *collar of* x to be the set

$$S_x = \{p \in \mathcal{P}(S)\colon\ px \text{ is a tangent to } S\},$$

and set $\pi(x) = \langle S_x \rangle_{\boldsymbol{P}}$. It is relatively easy to establish that $\pi(x)$ contains a hyperplane of $\boldsymbol{P}$ for each $x \in \boldsymbol{P}$, and that it is a hyperplane of $\boldsymbol{P}$ if $x \in \mathcal{P}(S)$. Let D be the subspace $\bigcap_{x \in \mathcal{P}(S)} \pi(x)$ of $\boldsymbol{P}$. If $\dim \boldsymbol{P}/D \geqslant 4$, it can be shown synthetically that $\pi(x)$ is a hyperplane of $\boldsymbol{P}$ for all $x \in \boldsymbol{P} \setminus D$. If $\dim \boldsymbol{P}/D = 3$ (and S has rank 2), however, the proof of this conclusion given in Dienst [1980] is still essentially the only one available. It makes use of coordinates, and so uses a translation of the problem into an algebraic one. □

Since a nondegenerate polar space S may be properly embedded in $\boldsymbol{P}_\pi$, the above theorem still does not fully determine nondegenerate polar spaces. We now construct examples of such S. They arise from a generalization of quadratic forms.

Let D be a division ring and V a right vector space over D and let (σ, ε) be an admissible pair for D. Write

$$D_{\sigma,\varepsilon} = \{t - t^\sigma \varepsilon\colon\ t \in D\} \quad \text{and} \quad D^{\sigma,\varepsilon} = \{x \in D\colon\ x^\sigma \varepsilon = -x\}.$$

3.19. LEMMA. *The following properties hold for $D_{\sigma,\varepsilon}$:*

(i) *$D_{\sigma,\varepsilon}$ and $D^{\sigma,\varepsilon}$ are additive subgroups of D;*

(ii) $D_{\sigma,\varepsilon} \subseteq D^{\sigma,\varepsilon}$;

(iii) *for each $\lambda \in D$ we have $\lambda^\sigma D_{\sigma,\varepsilon} \lambda \subseteq D_{\sigma,\varepsilon}$;*

(iv) *if $D_{\sigma,\varepsilon} = D$, then $\sigma = 1$, $\varepsilon = -1$ and D is of characteristic $\neq 2$;*

(v) *if D has characteristic distinct from 2, then $D_{\sigma,\varepsilon} = D^{\sigma,\varepsilon}$.*

PROOF. We prove (v); the rest are equally direct. If D has characteristic distinct from 2, then, for each $x \in D^{\sigma,\varepsilon}$, the equality $x^\sigma\varepsilon = -x$ implies $x = (x - x^\sigma\varepsilon)/2 \in D_{\sigma,\varepsilon}$, so $D^{\sigma,\varepsilon} \subseteq D_{\sigma,\varepsilon}$. □

We shall make use of the additive quotient group $D/D_{\sigma,\varepsilon}$. For $a \in D$, write $\bar{a} = a + D_{\sigma,\varepsilon}$. Consider the product

$$D/D_{\sigma,\varepsilon} \times D \to D/D_{\sigma,\varepsilon} \quad \text{given by} \quad \bar{a} * b = b^\sigma ab + D_{\sigma,\varepsilon} \quad (a, b \in D).$$

In view of part (iii) of the lemma, $\bar{a} * b$ does not depend on the particular choice of $a \in \bar{a}$. It also implies that $*$ can be viewed as right scalar multiplication on the additive subspace $D^{\sigma,\varepsilon}/D_{\sigma,\varepsilon}$ of $D/D_{\sigma,\varepsilon}$, giving the latter the structure of a right vector space over D.

3.20. EXAMPLES. In the orthogonal case, $(\sigma, \varepsilon) = (1, 1)$, we have $D_{\sigma,\varepsilon} = \{0\}$ and $a * b = b^2 a$. As $D^{\sigma,\varepsilon} = \{x \in D\colon 2x = 0\}$, the vector space on $D/D_{\sigma,\varepsilon}$ with scalar multiplication $*$ is trivial if D has characteristic distinct from 2 and a twisted version of D otherwise.

In the symplectic case, $(\sigma, \varepsilon) = (1, -1)$, with D of characteristic $\neq 2$, we have $D_{\sigma,\varepsilon} = D$, i.e. the converse of Lemma 3.19(iv) holds. Now $D/D_{\sigma,\varepsilon} = \{0\}$ so $*$ is the trivial scalar multiplication.

3.21. PROPOSITION (Tits [1974]). *The following properties concerning a map* $q\colon V \to D/D_{\sigma,\varepsilon}$ *are equivalent*:

(i) *there exists a* σ*-sesquilinear form* $g\colon V \times V \to D$ *such that*

$$q(x) = g(x, x) + D_{\sigma,\varepsilon} \quad \textit{for all } x \in V;$$

(ii) $q(x\lambda) = q(x) * \lambda$ *for all* $x \in V$, $\lambda \in D$, *and there exists a trace-valued* (σ, ε)*-Hermitian form* $f\colon V \times V \to D$ *such that*

$$q(x + y) = q(x) + q(y) + \big(f(x, y) + D_{\sigma,\varepsilon}\big) \quad \textit{for all } x, y \in V.$$

Moreover, if g *is as in* (i) *then the form* f *of* (ii) *can be chosen in such a way that* $f(x, y) = g(x, y) + g(y, x)^\sigma\varepsilon$.

3.22. *Pseudoquadratic forms.* A function q with the properties (i) and (ii) of the proposition is called a (σ, ε)*-quadratic form* or a *pseudoquadratic form relative to* σ and ε. We call the form f with

$$f(x, y) = g(x, y) + g(y, x)^\sigma\varepsilon \quad (x, y \in V)$$

a *Hermitian form of* q. By a σ*-quadratic* form we mean a $(\sigma, 1)$-quadratic and *quadratic* stands for 1-quadratic.

If $D_{\sigma,\varepsilon} \neq D$, the form f is uniquely determined by q. For, by linearity, it suffices to check this for $q = 0$, in which case f takes its values in $D_{\sigma,\varepsilon}$; but then, if $f(x,y)$ is a nonzero value of f, its value set contains $f(x, y\lambda) = f(x,y)\lambda$ for each $\lambda \in D$, whence D, contradicting $D \neq D_{\sigma,\varepsilon}$; thus $f = 0$.

Let q be a (σ,ε)-quadratic form on the right vector space V over the division ring D and let g, f be as in the above proposition with f a Hermitian form of q. Write $\pi = \pi_f$ and consider the polar space P_π. We define the *pseudoquadric* P_q as the space whose points are those points of $P(V)$ on which q vanishes, and whose lines are the lines of P fully contained in the point set of P_q. Observe that q vanishes at the point $\langle x\rangle$ $(x \in V,\ x \neq 0)$ if and only if $q(x) = 0 \in D/D_{\sigma,\varepsilon}$ (or, equivalently, $g(x,x) \in D_{\sigma,\varepsilon}$).

Fix $\lambda \in D$ and consider the pair (σ', ε'), where $t^{\sigma'} = \lambda t^\sigma \lambda^{-1}$ and $\varepsilon' = \lambda(\lambda^\sigma)^{-1}\varepsilon$. Then $\lambda D_{\sigma,\varepsilon} = D_{\sigma',\varepsilon'}$, and λq: $V \to D/D_{\sigma',\varepsilon'}$, given by $(\lambda q)(x) = \lambda(q(x))$, is (well-defined and) a (σ',ε')-quadratic form. If $\lambda \neq 0$, we say that q and λq are *proportional* (see Tits [1974], 7.2.2). Clearly, $P_q = P_{\lambda q}$.

3.23. THEOREM. *Suppose q is a (σ,ε)-quadratic form on a right vector space V and f is a Hermitian form of q. Put $P = P(V)$. If f is alternating and the characteristic of D is distinct from 2, then $q = 0$, so $P_q = P$ (as spaces). Suppose this condition is not satisfied. Then*

(i) *P_q is a subspace of P_f;*

(ii) *if $\mathcal{P}(P_q)$ spans P, then* $\operatorname{Rad} P_q = \mathcal{P}(P_q) \cap \operatorname{Rad} P_f$;

(iii) *if $D_{\sigma,\varepsilon} = D^{\sigma,\varepsilon}$, then $P_q = P_f$.*

PROOF. (i) For g as in Proposition 3.21, $\langle x\rangle \in P_q$ implies $g(x,x) \in D_{\sigma,\varepsilon}$, so, by Lemma 3.19(ii),

$$f(x,x) = g(x,x)^\sigma \varepsilon + g(x,x) = 0,$$

yielding $\langle x\rangle \in P_f$, proving $\mathcal{P}(P_q) \subseteq \mathcal{P}(P_f)$.

The alternating form case follows from the discussion in §3.20. Thus, let either $(\sigma,\varepsilon) \neq (1,-1)$ or D be of characteristic 2.

Let x and y be points of P_q. If xy is a line of P_q, then $q(x + y\lambda) = D_{\sigma,\varepsilon}$ for all $\lambda \in D$, by 3.21(ii), $f(x,y)\lambda \in D_{\sigma,\varepsilon}$ for all $\lambda \in D$. But then $f(x,y) = 0$, for otherwise $D = f(x,y)D \subseteq D_{\sigma,\varepsilon}$, contradicting Lemma 3.19(iv). This establishes that xy is a line of P_f.

Conversely, if xy is a line of P_f, then $f(x,y) = 0$, so $q(x + y\lambda) = 0$ for all $\lambda \in D$, proving that xy is a line of P_f. Hence P_q is a subspace of P_f.

(iii) By the reverse of the argument in the first paragraph of this proof, the assumption $D_{\sigma,\varepsilon} = D^{\sigma,\varepsilon}$ implies that the point sets of P_f and P_q coincide, so that $P_q = P_f$.

(ii) is easy. □

By (i) and a reduction to nondegenerate P_q, we can embed P_q in a unique polar space of the type P_f. We can apply Wells' Lemma 2.18 to the pseudoquadric P_q, using $K = \operatorname{Rad} f$ as the kernel, for

$$\mathcal{P}(P_q) \cap P(K) = \mathcal{P}(P_q) \cap \operatorname{Rad} P_f = \operatorname{Rad} P_q = \varnothing.$$

Thus, P_q embeds in the nondegenerate polar space $P_{\bar{f}}$ induced by f on $P/\operatorname{Rad} f$. Below, we shall actually make an inverse construction in order to pinpoint the embedded polar space under consideration.

There are two ways of finding pseudo-quadrics corresponding to a given (σ, ε)-Hermitian form.

3.24. PROPOSITION. *Suppose f is a (σ, ε)-Hermitian form on the right D-vector space V.*

(i) *If D is of characteristic distinct from 2, consider the map q: $V \to D/D_{\sigma,\varepsilon}$ given by*

$$q(x) = \frac{1}{2} f(x, x) + D_{\sigma,\varepsilon};$$

(ii) *If f is trace-valued, let g be as in* 3.11(i), *and consider the map q: $V \to D/D_{\sigma,\varepsilon}$ given by $q(x) = g(x, x) + D_{\sigma,\varepsilon}$ $(x \in V)$.*

In each of these cases, q is a (σ, ε)-quadratic form and f is the Hermitian form of q.

3.25. EXAMPLES.

(i) Let $V = D^2$ and (σ, ε)-admissible. For $x = (x_1, x_2) \in V$, put $q(x) = x_1^\sigma x_2 + D_{\sigma,\varepsilon}$. Then q is a pseudoquadratic form relative to σ and ε. The Hermitian form f of q is

$$f(x, y) = x_1^\sigma y_2 + x_2^\sigma \varepsilon y_1 \quad (x, y \in V).$$

The point $(0, 1)D$ belongs to P_q. All other points of P_q are of the form $(1, a)D$ with $a \in D_{\sigma,\varepsilon}$. The points of P_f are $(0, 1)D$ and those of the form $(1, a)D$ with $a = -a^\sigma \varepsilon$. Thus, we see again from Lemma 3.19(ii) that P_q is contained in P_f. If we take $\sigma = 1$, $\varepsilon = 1$ and D of characteristic 2, then $D^{\sigma,\varepsilon} = D$ and, as we have seen in §3.20, $D_{\sigma,\varepsilon} = 0$, so P_q is a proper subspace of P_f.

(ii) See Buekenhout and Cohen [1992]. Let D be a 4-dimensional associative algebra over the field k, with unity 1 and basis $1, e_1, e_2, e_3$ whose multiplication table is given below:

	e_1	e_2	e_3
e_1	$e_1 + a$	e_3	$e_3 + ae_2$
e_2	$e_2 - e_3$	b	$b - be_1$
e_3	$-ae_2$	be_1	$-ab$

for certain $a, b \in k$. Then D can be supplied with a quadratic form Q: $D \to k$ given by

$$x_0^2 + x_0 x_1 - a x_1^2 - \left(x_2^2 + x_2 x_3 - a x_3^2\right) b$$

for $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$, and

$$Q(xy) = Q(x) Q(y) \quad \text{for all } x, y \in D.$$

If $Q(x) \neq 0$ then x has inverse $\overline{x}/Q(x)$, where $\overline{x} = x_1 + 2x_0 - x$. The map σ: $x \mapsto \overline{x}$ defines an anti-automorphism of D. If the inverse exists for each element of D, the composition algebra is called a *quaternion division ring*. Now, D is a division ring if and only if $Q(x) \neq 0$ for all $x \neq 0$. This condition is satisfied, e.g., if we take $k = F_2(t)$ to be the field of rational functions over F_2 and $a = 1$, $b = t$.

Assume now that D is a division ring and that k has characteristic 2. Consider σ: $x \mapsto \overline{x}$ and $\varepsilon = 1$. Then $\sigma^2 = 1$ and $D_{\sigma,\varepsilon} = k$. Take $V = D^2$ and put

$$g(v,w) = \overline{v}_1 w_2 \quad \text{for } v = (v_1, v_2),\ w = (w_1, w_2) \in V.$$

The form f defined by

$$f(v,w) = g(v,w) + g(w,v)^\sigma = \overline{v}_1 w_2 + \overline{v}_2 w_1$$

is nondegenerate and trace-valued. The elements v of V with $f(v,v) = 0$ are those for which $\overline{v}_1 v_2 = \overline{\overline{v}_1 v_2}$. Since

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 = \overline{x} \quad \text{if and only if} \quad x_1 = 0,$$

there are elements v of V such that $f(v,v) = 0$ while $g(v,v) = \overline{v}_1 v_2$ is not in k. Let q: $V \to D/D_{\sigma,\varepsilon}$ be defined by g as in Proposition 3.21(i). Then P_q spans $P(V)$ and $\mathcal{P}(P_q)$ is properly contained in $\mathcal{P}(P_f)$.

Let f be a Hermitian form of a (σ,ε)-quadratic form q on V. From Theorem 3.23 and the above example, we see that the possibility for a pseudoquadric to be embedded in and distinct from P_π is related to the fact that $D_{\sigma,\varepsilon}$ is contained in and distinct from $D^{\sigma,\varepsilon}$. Recall, from the discussion in §3.19, the right D-vector space $D^{\sigma,\varepsilon}/D_{\sigma,\varepsilon}$ in which scalar multiplication is given by

$$v * \lambda = \lambda^\sigma v \lambda + D_{\sigma,\varepsilon} \quad \big(v \in D^{\sigma,\varepsilon}/D_{\sigma,\varepsilon},\ \lambda \in D\big).$$

It will be used to 'lift' the ambient projective space P and to reduce the study of (non-symplectic) polarities (if D does not have characteristic 2) to the study of pseudoquadrics.

3.26. LEMMA. *Suppose σ is an involutory anti-automorphism of D. The inclusion map ϕ_q: $D^{\sigma,1}/D_{\sigma,1} \to D/D_{\sigma,1}$ is a nonzero σ-pseudoquadratic form and its Hermitian form f as in Proposition* 3.21(ii) *is the zero form.*

The proof is easy. The following result is also not hard to prove.

3.27. PROPOSITION. *Suppose q is a nontrivial (σ,ε)-quadratic form on the projective space P over D and the polar space P_q does not coincide with the ambient polar space P_f for the Hermitian form f of q. Then* char $D = 2$. *Set*

$$\overline{V} = V \oplus D^{\sigma,1}/D_{\sigma,1}$$

and let $\overline{q} = q \oplus \phi_q$, where ϕ_q is as in the above lemma. Then the canonical projection $\overline{V} \to V$ onto the first factor induces an isomorphism of the pseudoquadric $\overline{P}_{\overline{q}}$, where $\overline{P} = P(\overline{V})$, onto the polar space P_f. If $\overline{f}$ is the Hermitian form of $\overline{q}$ and f is nondegenerate, then the radical of $P_{\overline{f}}$ coincides with $P(D^{\sigma,1}/D_{\sigma,1})$.

If $\overline{f}$ denotes a Hermitian form of $\overline{q}$ as in the proposition, then $(S+S)\cap\overline{\boldsymbol{P}}_{\overline{f}} = \varnothing$, so the projection satisfies Wells' conditions 2.18. In fact, if rk $\boldsymbol{P}_q \geqslant 3$ and $\boldsymbol{P}_q$ is nondegenerate, all of its embeddings lie between the two extremes V and $\overline{V}$.

Putting together Theorems 3.18(i), 3.23, and Proposition 3.27 we obtain that any embeddable nondegenerate polar space of rank at least 2 can be embedded in a pseudoquadric or in the absolute space of a symplectic polarity over a field of characteristic distinct from 2. In order to finish the classification, we identify S with the ambient polar space by use of the strong result formulated in Theorem 3.18(ii).

3.28. DEFINITION. Let $\boldsymbol{P}$ be a projective space with polarity π. A subset X of $\mathcal{P}(\boldsymbol{P})$ will be called *perspectively closed with respect to* π if it satisfies the property that, for each collinear triple a, b, $c \in \boldsymbol{P}$ with $b, c \in X \setminus \pi(a)$, the perspectivity of $\boldsymbol{P}$ with centre a, axis $\pi(a)$ sending b to c leaves X invariant. Thus 3.18(ii) states that any polar space embedded in the absolute space of a polarity is perspectively closed.

3.29. LEMMA (cf. Buekenhout [1975]). *Suppose* $V = \boldsymbol{D}^2$. *Let* f *be a nondegenerate* (σ, ε)*-sesquilinear form on* V.

(i) *If* f *is an alternating form on* V *and* $\boldsymbol{D}$ *has characteristic distinct from* 2, *then, for any two points* $p, p' \in \boldsymbol{P}$, *the absolute space* $\boldsymbol{P}_f$ *is the smallest perspectively closed set of points from* $\boldsymbol{P}_f$ *containing* p *and* p'.

(ii) *If* q *is a nontrivial* (σ, ε)*-quadratic form on* V *such that* f *is a Hermitian form of* q, *and if* p, p' *are two points of* $\boldsymbol{P}_q$, *then the pseudoquadric* $\boldsymbol{P}_q$ *is the smallest perspectively closed subset of* $\boldsymbol{P}_f$ *containing* p *and* p'.

PROOF. Without loss of generality, we choose a basis of V so that $p = (1,0)\boldsymbol{D}$ and $p' = (0,1)\boldsymbol{D}$ in both (i) and (ii) and $f((0,1),(1,0)) = 1$, so

$$f(x,y) = x_1^\sigma y_2 + x_2^\sigma \varepsilon y_1, \quad x = (x_1, x_2),\ y = (y_1, y_2).$$

Now let X be a perspectively closed subset of $\boldsymbol{P}_f$ containing p and p'. The perspectivity with centre $(a:1)$ and axis $\pi_f(a:1)$ sending p to p' is given by

$$x \mapsto x - \big(1, a^{-1}\big)\varepsilon^{-1} f\big((a,1), x\big) \quad (x \in V).$$

Consequently, it maps p' to $(1, a^{-1} - (a^{-1})^\sigma \varepsilon)\boldsymbol{D}$, and so the latter point belongs to X.

(i) In this case, $(\sigma, \varepsilon) = (1, -1)$, so the image of p' is $(1, 2a^{-1})\boldsymbol{D}$. Thus, by varying $a \in \boldsymbol{D}$, we obtain that X all contains all points of $\boldsymbol{P}$.

(ii) With f as above, q is given by $q(x) = x_1^\sigma x_2 + \boldsymbol{D}_{\sigma,\varepsilon}$ $(x \in V)$. Letting a vary we obtain all points of the form $(1,b)\boldsymbol{D}$ with $b \in \boldsymbol{D}_{\sigma,\varepsilon}$. As we have seen in Example 3.25(i), this gives $\mathcal{P}(\boldsymbol{P}_q) \subseteq X$. Moreover, $\mathcal{P}(\boldsymbol{P}_q)$ is perspectively closed. □

By taking intersections with appropriate lines, the more general case can be reduced to the above lemma, with the following result.

3.30. PROPOSITION. *Let $\boldsymbol{P}$ be a Desarguesian projective space over $\boldsymbol{D}$. Suppose S is a nondegenerate polar space of rank at least 2 embedded in $\boldsymbol{P}_f$, for a nondegenerate (σ, ε)-Hermitian form f, and $\mathcal{P}(S)$ spans $\boldsymbol{P}$.*

(i) *If f is proportional to an alternating form and $\boldsymbol{D}$ has characteristic distinct from 2, then $S = \boldsymbol{P}_f$;*

(ii) *If S is a subspace of $\boldsymbol{P}_q$ for a nondegenerate pseudoquadratic form q such that f is its Hermitian form, then $S = \boldsymbol{P}_q$.*

PROOF. (i) Assume $x \in \mathcal{P}(\boldsymbol{P}_f) \setminus \mathcal{P}(S)$. If there is a line ℓ of $\boldsymbol{P}$ on x meeting $\mathcal{P}(S)$ in at least two points, say y and z, then y and z are not collinear in $\boldsymbol{P}_f$ (for otherwise, since S is a subspace of $\boldsymbol{P}_f$ by Theorem 3.18(i), ℓ would be a line of S, contradicting $x \in \ell$). Hence f induces a nondegenerate symplectic polarity on ℓ. Theorem 3.18(ii) and the above lemma then imply that ℓ is contained in $\mathcal{P}(S)$, leading to the same contradiction.

Thus, every line of $\boldsymbol{P}$ on x meets S in at most one point, that is $x \in \pi_f(\mathcal{P}(S))$. Since $\mathcal{P}(S)$ spans $\boldsymbol{P}$, this implies $x \in \operatorname{Rad} f$, contradicting the fact that the latter is empty. Hence (i).

(ii) The proof is similar to the first case. The final contradiction is derived from

$$x \in \mathcal{P}(S) \cap \operatorname{Rad} f \subseteq \mathcal{P}(\boldsymbol{P}_q) \cap \operatorname{Rad} f = \varnothing. \qquad \square$$

3.31. *Classification of embeddable polar spaces.* A polar space will be called *classical* if it is isomorphic to $\boldsymbol{P}_f$ for some (σ, ε)-Hermitian form f or to $\boldsymbol{P}_q$ for some pseudoquadratic form q.

By Proposition 3.30, observing that, in its proof, all that was used of $\mathcal{P}(S)$ is the fact that it is perspectively closed with respect to π_f, and by using Theorems 3.15, 3.16 in case the rank of S is at least 3, and Theorem 3.18 in case the rank of S is at least 2, we obtain the following classification.

3.32. COROLLARY. *Let S be a nondegenerate polar space with thick lines. If S has a singular subspace, i.e. a Desarguesian plane, or has rank at least 2 and is embedded in a projective space, then S is a classical polar space or $A_{3,2}(\boldsymbol{D})$.*

PROOF. It remains to show that a subspace X of $\boldsymbol{P}_q$ which is a nondegenerate polar space and which spans $\boldsymbol{P}$ coincides with $\boldsymbol{P}_q$. By Theorem 3.18(ii), X is perspectively closed. By Lemma 3.29, for every line ℓ of $\boldsymbol{P}$ having two points in $\ell \cap X$ has the property that $\ell \cap X$ coincides with $\ell \cap \boldsymbol{P}_q$. A standard connectivity argument then gives $X = \boldsymbol{P}_q$. $\square$

3.33. *Moufang polar spaces.* There are more polar spaces of rank 3 than we have discussed so far. Let k be a field affording an anisotropic quadratic form κ: $k^8 \to k$ on the 8-dimensional vector space k^8 over k. Then, up to isomorphism, there is a unique Cayley division ring C with norm form κ. There exists a rank 3 polar space S_κ

(i) whose singular planes are Moufang projective planes over C, and

(ii) whose residues $\rho(x^\perp)$ ($x \in \mathcal{P}(S)$) are isomorphic to the dual of the generalized quadrangle $\boldsymbol{P}(k^4 \oplus C)_{\overline{\kappa}}$, where

$$\overline{\kappa}(x) = \kappa(\xi) + x_1x_3 - x_2x_4 \quad \text{if } x = (x_1, x_2, x_3, x_4, \xi) \in k^4 \oplus C.$$

The existence of S_κ follows from algebraic group constructions, see §5.5 and Chapter 11, 7.3.6. By use of automorphism extensions such as in Theorem 3.3 (cf. Tits [1974]), it can be shown that S_κ is uniquely determined (up to isomorphism) by the properties (i) and (ii).

From the embedding of S into $\mathcal{V}(S)$, the Veldkamp space $\mathcal{V}(S)$ cannot be a projective space if S contains a non-Desarguesian singular plane. The structure of $\mathcal{V}(S)$ is given in Cohen and Shult [1990]: $\mathcal{V}(S)$ and S coincide as point sets via the embedding map. The Veldkamp lines are either obtained from the embedded space S or of the form

$$\{z^\perp\colon\ z \in \{x, y\}^{\perp\perp}\}$$

for noncollinear points x and y of S.

Let S be a nondegenerate polar space of rank at least 3. By Theorem 3.1(v), its singular planes are Moufang. If the singular planes are non-Desarguesian, no singular spaces of higher rank can occur, so the rank of S must be 3. Moreover, there exists a Cayley division ring C coordinatizing the singular planes and it is shown in Tits [1974] that S is isomorphic to S_κ where κ is the norm form of C. If the singular planes are Desarguesian, then §§3.16, 3.18 and 3.32 imply that S is classical. Thus we arrive at the following ground-breaking result.

3.34. THEOREM. *Suppose S is a nondegenerate polar space of rank at least* 3 *with thick lines. Then exactly one of the following statements holds*:

(i) *there is a vector space V and a pseudoquadratic form q on V such that $S \cong \boldsymbol{P}(V)_q$*;

(ii) *there is a vector space V over a field of characteristic distinct from* 2 *and an alternating form f on V such that $S \cong \boldsymbol{P}(V)_f$*;

(iii) *the rank of S is equal to* 3, *and there is a division ring D such that $S \cong A_{3,2}(D)$*;

(iv) *the rank of S is equal to* 3, *and there is a field k affording an anisotropic normform κ: $k^8 \to k$ such that $S \cong S_\kappa$.*

In this guise the theorem is due to Buekenhout and Shult [1974], Tits [1974], Veldkamp [1959] and, for an extension to infinite rank, Johnson [1990].

3.35. *Dual polar spaces.* In a nondegenerate polar space S, two maximal singular subspaces A, B are called *adjacent* when they have a common hyperplane. Notation: $A \sim B$. We are interested in the graph induced by $\sim$ on the set $\mathcal{M}$ of all maximal singular subspaces of S. This graph is called the *dual polar graph* of S.

A *near polygon* is a space $(P, \mathcal{L})$ in which, for every vertex x and every $\ell \in \mathcal{L}$, there is a unique vertex on ℓ at minimal distance to x. If its diameter is d then the near polygon is called a *near* $2d$*-gon*. The notion was developed in the study Shult and Yanushka [1980] of the dual polar graph of a polar space. Near 4-gons are the same as generalized quadrangles.

3.36. THEOREM (Cameron [1982]). *Let S be a nondegenerate polar space of finite rank n. Then its dual polar graph $(\mathcal{M}, \sim)$ is the collinearity graph of a near $2n$-gon.*

The converse does not hold: e.g., there exist a near hexagon on 759 points with automorphism group the sporadic Mathieu group M_{24} and one on 729 points admitting the sporadic Mathieu group M_{12} as group of automorphisms, both due to Shult and Yanushka [1980], and a near octagon on 315 points (already mentioned in §2.16) with automorphism group $J_2 \cdot 2$ due to Cohen [1981]. (See Conway et al. [1985] for the sporadic groups.) These examples all have three points per line. For near polygons with lines of this length, there is an interesting connection with *Fischer spaces* (discussed in §6.8), see Brouwer, Cohen, Hall and Wilbrink [1993]. In Brouwer, Cohen and Neumaier [1989] and Chapter 10 more examples of near polygons are discussed.

There is another way of looking at the near polygon axiom. Polar spaces can be viewed as gamma spaces $(P, \mathcal{L})$ in which the possibility $x^\perp \cap \ell = \varnothing$ for $x \in P$ and $\ell \in \mathcal{L}$, does not occur. Here, the possibility that this intersection is all of ℓ is ruled out whenever x does not belong to ℓ. In other words, lines are maximal singular subspaces. Let S be a space and $j \in N$. If S has diameter d and, for each $x \in \mathcal{P}(S)$ and $\ell \in \mathcal{L}(S)$ with mutual distance $d(x, \ell) \leqslant j$ in the collinearity graph, the set $\pi_x(\ell)$ of all points of ℓ at minimal distance to x is a singleton, then it is called a *near $2d$-gon of depth j*. Thus, a near $2d$-gon of depth $d-1$ is a near $2d$-gon. For $j \geqslant 1$, the space S is partial linear and lines are just maximal cliques of its collinearity graph, so S is uniquely determined by its collinearity graph.

In the absence of further assumptions, the following result can be seen as the best approximation of a converse to Theorem 3.36.

3.37. THEOREM (Brouwer and Wilbrink [1983], Shult and Yanushka [1980]). *Let S be a near $2d$-gon of depth j (for some $j \geqslant 2$) in which lines have at least three points and each pair of points at distance two has at least two common neighbours. Then each pair of points at mutual distance j can be embedded in a unique geodetically closed subspace isomorphic to a near $2j$-gon.*

For a proof, see Shult and Yanushka [1980] for the case where $j = 2$, and Brouwer and Wilbrink [1983] for the remaining cases.

Applying the above lemma with $j = 2$, we obtain subspaces isomorphic to generalized quadrangles. They form the rank 2 counterpart of the symplecta in Theorem 4.1, and are called *quads*.

EXAMPLE. The third Janko group J_3 (cf. Conway et al. [1985]) has a subgroup, L say, isomorphic to $L(2, 16) : 2$. Calling two conjugates L^g and L^h adjacent whenever their intersection contains a subgroup isomorphic to Alt_5, we obtain a graph on the vertex set $P = \{L^g\colon\ g \in J_3\}$ in which every vertex lies on 17 maximal cliques, each of size 6. (See, e.g., Tits [1980].) Letting $\mathcal{L}$ be the family of all maximal cliques, we obtain a near 6-gon $(P, \mathcal{L})$ of depth 2, but not of depth 3, all of whose quads are 6×6-grids.

A *quadrangle* in a space S is a set of 4 points, each of which is collinear with exactly two others, for which the four resulting lines are distinct.

3.38. THEOREM (Brouwer and Cohen [1986], Cameron [1982], Ronan [1986], Shult and Yanushka [1980]). *Suppose S is a near $2d$-gon of depth 3 such that lines have size at least three and each pair of points at mutual distance 2 is contained in a quadrangle. If, for each point x and each quad Q with $d(x, Q) \leqslant 2$, the set $\pi_x(Q)$ is a singleton, then, for each $x \in \mathcal{P}(S)$, the space of the lines and quads on x is a projective space. If, moreover, this space has finite rank (for some $x \in \mathcal{P}(S)$), then S is the quotient of a dual polar space by a group of automorphisms whose nontrivial elements map each point to a point at distance $\geqslant 8$.*

In the finite case, a polar space does not admit nontrivial groups of automorphisms with the property specified at the end of the theorem, so S is a dual polar space itself (cf. Brouwer and Cohen [1983]).

The dual polar spaces of type O_{2n} are thin in the sense that lines consist of precisely two points. From the axioms, it is then clear that the collinearity graph is a bipartite graph. Two vertices belong to the same part if their intersection (as maximal singular subspaces of the underlying polar space) has even codimension in one of the two.

Taking the induced graph on one of the two parts, in which adjacency is having distance 2 in the original graph, we arrive at a *half dual polar graph*. The *half dual polar space* can be defined as the space obtained from this graph by the procedure of §1.3.

4. Shadow spaces of buildings

Buildings are the central topic of Chapter 11 in this Handbook. Buildings can be viewed either as geometries or as chamber systems. These two viewpoints do not essentially differ because of the equivalence of categories of residually connected geometries and residually connected chamber systems, see Chapter 11 and Theorem 4.1 below.

We first recall some basic notation. A *chamber system* over an index set I is a pair $\mathcal{C} = (C, \{\stackrel{i}{\sim}: i \in I\})$ consisting of a set C, whose members are called *chambers*, and a collection of equivalence relations $\stackrel{i}{\sim}$ on C indexed by $i \in I$. These relations are interpreted as graph structures on C. For each $i \in I$, the graph $(C, \stackrel{i}{\sim})$ is a disjoint union of cliques. Two chambers c, d are called *i-adjacent* if $c \stackrel{i}{\sim} d$ holds. The *rank of* $\mathcal{C}$ is $|I|$.

Let Γ be a geometry over I. Denote by $\mathcal{C}(\Gamma)$ the set of chambers (maximal flags) of Γ, together with, for each $i \in I$, the relation $\stackrel{i}{\sim}$ given by $c \stackrel{i}{\sim} d$ for two chambers $c, d \in \mathcal{C}(\Gamma)$ whenever c and d have exactly the same element of type j for all $j \neq i$. Then $\mathcal{C}(\Gamma)$ is a chamber system, which is said to be the *chamber system of* Γ.

For a subset J of I, we shall write J' to abbreviate its complement $I \setminus J$. For singletons $\{j\}$, we also abbreviate $\{j\}'$ to j'. A J-component of a chamber system $\mathcal{C}$ will be understood to be a connected component of the graph with vertex set C and adjacency $\bigcup_{j \in J} \stackrel{j}{\sim}$.

To a chamber system $\mathcal{C}$ over I, we assign the pair $\Gamma(\mathcal{C}) = ((\mathcal{C}_i)_{i \in I}, *)$ in which $\mathcal{C}_i$ is the collection of all i'-components of $\mathcal{C}$ and $*$ is the relation on the union of these collections given by $A * B$ if and only if $A \cap B \neq \varnothing$. Thus, $\Gamma(\mathcal{C})$ is a geometry over I.

Recall from Chapter 3 the definitions of residual connectedness for geometries and chamber systems. We say that the chamber system $\mathcal{C} = (C, \{\stackrel{i}{\sim}: i \in I\})$ is *residually connected* if, for every subset J of I and every family $\{Z_j: j \in J\}$ where Z_j is a j'-component, with the property that any two have a nonempty intersection, it follows that $\bigcap_{j\in J} Z_j$ is a nonempty J'-component.

Applying this with $J = \varnothing$ we see that C is nonempty and that $(C, \sim)$ is connected. For $J = I$, we see that chambers which are in the same j'-components for all $j \in I$, are equal.

If c is a chamber, we denote by cJ^* the J-component containing c. More generally, for a subset X of $\mathcal{C}$, we shall denote by XJ^* the union of all xJ^* for $x \in X$.

4.1. THEOREM (Chamber System Correspondence, cf. Chapter 11, 1.3). *Let I be a finite set of types. For each residually connected chamber system $\mathcal{C}$ over I, the map $\psi_{\mathcal{C}}$: $\mathcal{C} \to \mathcal{C}(\Gamma(\mathcal{C}))$ given by*

$$\psi_{\mathcal{C}}(c) = \{c(i')^*: i \in I\}$$

is an isomorphism of chamber systems over I. Conversely, for each residually connected geometry Γ over I, the map ϕ_Γ: $\Gamma \to \Gamma(\mathcal{C}(\Gamma))$ given by $\phi_\Gamma(x) = ci^*$ whenever c is a chamber of Γ containing x and i is the type of x, is an isomorphism of geometries over I.

In other words, there is a bijective morphism-preserving correspondence. To each residually connected geometry there corresponds a unique residually connected chamber system over I (up to isomorphism) with the same automorphism group as Γ, and vice versa. Corollary 4.5 below shows that the correspondence is valid for buildings.

4.2. *Coxeter groups.* We shall need a number of properties of Coxeter groups. For introductions to Coxeter groups, see Bourbaki [1968], Cohen [1991], Humphreys [1990], or Chapter 11, 2.1.

Let M be a symmetric $n \times n$ matrix with diagonal entries 1 and off-diagonal entries from $(\mathbb{N} \setminus \{0, 1\}) \cup \{\infty\}$. Fix a Coxeter system (W, R) of type M. That is, W is a group generated by $R = \{r_1, \ldots, r_n\}$ and, if the (i, j)-entry of M is denoted by $m_{i,j}$, defining relations for W are $(r_i r_j)^{m_{i,j}} = 1$ $(1 \leqslant i, j \leqslant n$ whenever $m_{ij} < \infty)$. The length function l: $W \to \mathbb{N}$ assigns to $w \in W$ the minimal length of an expression of w as a product of elements of R. An expression of minimal length is called a *reduced expression*. Note that $r_i^2 = 1$ for each i.

Let J, K be subsets of R. The subgroup $\langle J \rangle$ of W, also denoted by W_J, is the subgroup generated by J. It is again a Coxeter group, with Coxeter system $(\langle J \rangle, J)$ and length function $l|_{\langle J \rangle}$. Clearly, $W_\varnothing = \{1\}$ and $W_R = W$, while $J \subseteq K$ implies $W_J \subseteq W_K$. It can be shown that each (W_J, W_K) double coset in W contains a unique element of minimal length. Set

$$D_{J,K} = \{w \in W: l(rw) > l(w) \text{ and } l(ws) > l(w) \text{ for all } s \in J \text{ and } r \in K\}.$$

Then the set $D_{J,K}$ is the system of double (W_J, W_K)-coset representatives of minimal length.

For $S \subseteq R$ write

$$S^{\perp} = \{r \in R\colon\ m_{r,s} = 2 \text{ for all } s \in S\},$$

and for $r \in R$ write $r^{\perp} = \{r\}^{\perp}$. For $w \in W$, denote by R_w the set of all elements of R occurring in a reduced expression of w as a product of elements of R. (This occurrence does not depend on the choice of the reduced expression.) For $x \in W$ and $V \subseteq W$ write ${}^{x}V = xVx^{-1}$.

4.3. LEMMA. *Let (W, R) be a Coxeter system. Suppose J and K are subsets of R and $d \in D_{J,K}$. Then*

(i) $\langle J\rangle \cap {}^{d}\langle K\rangle = \langle J \cap {}^{d}K\rangle$;

(ii) $\langle J\rangle \cap {}^{d}\langle K\rangle \subseteq \langle R_d \cup R_d^{\perp}\rangle = \langle R_d\rangle\langle R_d^{\perp}\rangle$;

(iii) $\langle J\rangle \cap {}^{r}\langle J\rangle = \langle J \cap r^{\perp}\rangle$ *for each* $r \in R \setminus J$.

PROOF. (i) The case $d = 1$ is known from Chapter 11. Clearly,

$$\langle J \cap {}^{d}K\rangle \subseteq \langle J\rangle \cap {}^{d}\langle K\rangle.$$

For the opposite inclusion, suppose $w \in \langle J\rangle \cap {}^{d}\langle K\rangle$. If $l(w) = 0$, then $w = 1 \in \langle J \cap {}^{d}K\rangle$. We shall reason by induction on $l(w)$. Assume $l(w) \geqslant 1$. Then there is $s \in J$ with $l(sw) < l(w)$. Since $w \in {}^{d}\langle K\rangle$ there is $v \in \langle K\rangle$ with $dv = wd$. Now, since $d \in D_{J,K}$, we have

$$l(sdv) = l(swd) = l(sw) + l(d) < l(w) + l(d) = l(wd) = l(dv),$$

so in view of the Exchange Condition (cf. Chapter 11) we have $sdv \in d\langle K\rangle$. In particular, $d^{-1}sd \in \langle K\rangle v^{-1} = \langle K\rangle$. Again since $d \in D_{J,K}$, this yields

$$1 + l(d) = l(sd) = l\bigl(d\bigl(d^{-1}sd\bigr)\bigr) = l(d) + l\bigl(d^{-1}sd\bigr).$$

Hence $l(d^{-1}sd) = 1$. Now $d^{-1}sd \in \langle K\rangle \cap R = K$. Consequently, $s \in J \cap {}^{d}K$. Finally, $sw \in \langle J\rangle \cap {}^{d}\langle K\rangle$, and $l(sw) < l(w)$ so, by induction, $sw \in \langle J \cap {}^{d}K\rangle$, whence $w = s(sw) \in \langle J \cap {}^{d}K\rangle$. This proves (i).

(ii) Let $t \in J \cap {}^{d}K$. According to (i), for the proof of the inclusion, it suffices to show $t \in R_d \cup R_d^{\perp}$. By assumption on t, there is $s \in K$ such that $td = ds$. As $t \in J$, $s \in K$ and $d \in D_{J,K}$, we have $l(td) = l(ds) = 1 + l(d)$, whence

$$\{t\} \cup R_d = R_{td} = R_{ds} = \{s\} \cup R_d.$$

Assume now that $t \notin R_d$. Then $s = t$, so

$$d = tdt^{-1} \in \langle R_d\rangle \cap {}^{t}\langle R_d\rangle = \langle R_d \cap {}^{t}R_d\rangle$$

in view of (i) and $t \in D_{R_d,R_d}$. But then $R_d = tR_dt^{-1}$. Choose any $r \in R_d$. Then there is $r' \in R_d$ with $r = tr't^{-1}$. Since $t \notin R_d$, we have $r \neq t$, so $r' \neq t$. From

$$\{r\} = R_r = R_{tr't} \subseteq \{t, r'\},$$

it follows that $r = r'$ and $t \in r^{\perp}$. The conclusion is that t belongs to

$$\bigcap_{r \in R_d} r^{\perp} = R_d^{\perp}.$$

This proves the inclusion. Since the equality in (ii) is a trivial consequence of the fact that R_d and $R_d^{\perp}$ commute, this ends the proof of (ii).

(iii) By (i) and (ii) and, we have

$$\langle J\rangle \cap {}^r\langle J\rangle = \langle J \cap {}^rJ\rangle \cap \langle r\rangle\langle r^{\perp}\rangle = \langle J \cap {}^rJ \cap \{r\}\rangle\langle J \cap {}^rJ \cap r^{\perp}\rangle = \langle J \cap r^{\perp}\rangle.$$

□

The lemma shows that the stabiliser of the ordered pair of two flags xW_J and yW_K of $\Gamma(W)$ of type $R \setminus J$ and $R \setminus K$, respectively, can be determined as follows. Let $d \in D_{J,K}$ be in the same double coset as $x^{-1}y$. Then the stabiliser in $W = \mathrm{Aut}(\Gamma(W))$ of xW_J and yW_K is

$${}^xW_J \cap {}^yW_K = {}^{xu}\big(W_J \cap {}^{dv}W_K\big) = {}^{xu}W_J \cap {}^dK,$$

where $y = xudv$ with $u \in W_J$, $d \in D_{J,K}$, and $v \in W_K$, cf. Chapter 11, 5.1.10. But here is another important consequence of the lemma. See also Tits [1974], 12.9.

4.4. Proposition. *If the chamber system C is a building over I, then, for any three subsets J, K, L of I and any two chambers c, d of C, we have either*

$$cJ^*L^* \cap dK^*L^* = \varnothing$$

or there exist a chamber $e \in cJ^*$ and $w \in D_{J,K} \cap \langle L\rangle$ such that

$$cJ^*L^* \cap dK^*L^* = eM^*L^*,$$

where $M = J \cap {}^wK$. In the latter case, there exist $f \in dK^*$ and $w \in \langle L\rangle \cap D_{J,K}$ and $x \in \langle J\rangle$, $y \in \langle K\rangle$ such that xwy represents a reduced expression of the type of a gallery from c to d and

$$c \xrightarrow{x} e \xrightarrow{w} f \xrightarrow{y} d.$$

Proof. Suppose a is a chamber in $cJ^*L^* \cap dK^*L^*$. By the general theory on Coxeter groups (cf. Bourbaki [1968]), there are $x \in \langle J\rangle$, $y \in \langle K\rangle$, such that $c \xrightarrow{xwy} d$ for some $w \in D_{J,K} \cap \langle L\rangle$. Consequently, there are $e \in cJ^*$ and $f \in dK^*$ with

$$c \xrightarrow{x} e \xrightarrow{w} f \xrightarrow{y} d.$$

We need to show $aL^* \subseteq eM^*L^*$ (for $cJ^* = eJ^*$ and $dK^* = fK^*$). Replacing a by a suitable element from aL^*, we may assume that $a \in fK^* \cap eJ^*L^*$. Thus, there are $x' \in \langle J\rangle$ and $w' \in \langle L\rangle \cap D_{J,\varnothing}$ such that $e \xrightarrow{x'} e' \xrightarrow{w'} a$ for some chamber $e' \in eJ^*$.

Let y' be the type of a shortest gallery from f to a. Considering the two galleries e, e', a and e, f, a from e to a, we find that $wy' = x'w'$, whence

$$w' = x'^{-1}wy' \in \langle L\rangle.$$

As $w \in D_{J,K}$, we can find $x'' \in \langle L \cap J\rangle$ and $y'' \in \langle L \cap K\rangle$ such that $w' = x''^{-1}wy''$ is a reduced expression for w'. Since $w' \in \langle L\rangle$, we have $x'', y'' \in \langle L\rangle$. Furthermore, by the above lemma,

$$x'x''^{-1} = wy'y''^{-1}w^{-1} \in \langle J\rangle \cap {}^w\langle K\rangle = \langle M\rangle,$$

so $x'w' \in \langle M\rangle x''w' \subseteq \langle M\rangle\langle L\rangle$, proving $a \in eM^*L^*$ as required. □

Recall from Chapter 3 that a geometry is said to have the *intersection property* if, for each flag F, element x and type J such that the intersection of the J-shadows (see §4.6) of F and x are nonempty, there exists a flag G incident to F with the property that the J-shadows intersection coincides with the J-shadow of G.

4.5. COROLLARY. *Buildings are residually connected. The geometry of a building satisfies the intersection property.*

PROOF. Residual connectedness of the chamber system follows from Proposition 4.4 with $L = \varnothing$ and use of the chamber system correspondence, Theorem 4.1. Similarly, the intersection property is the special case of Proposition 4.4 in which K (the cotype of x) is the complement of a singleton. □

The above statements largely coincide with Chapter 11, 6.4.4 and 1.3.1.

4.6. *Shadow spaces.* To make the connection with spaces, we need the notion of shadow space of a geometry. See Tits [1974] for the original definitions. See also Scharlau [1990] and Pasini [1983].

By I we denote a finite index set. Let Γ be a residually connected geometry of type I and let F be a flag of Γ of type $K \subseteq I$. For any $J \subseteq I$, we define the *J-shadow* of a flag F of Γ as the collection of all J-flags incident to F. In terms of chamber systems, if c is a chamber on F, the J-shadow of F is the set $c(I \setminus K)^*(I \setminus J)^*$ of chambers of $\mathcal{C}(\Gamma)$.

The *shadow space* of Γ of type (J, K) is the pair $(P, \mathcal{L})$ where P is the set of all flags of Γ of type J and $\mathcal{L}$ consists of all shadows on P (i.e. J-shadows) of flags of Γ of type K. If the geometry is firm (cf. Chapter 3), then $(P, \mathcal{L})$ is a space. The most important special case occurs for $J = \{j\}$ and $K = I \setminus J$. The resulting shadow space (of type $(\{j\}, I \setminus \{j\})$) is said to be the *shadow space over j*.

At the other extreme, we can take $J = I$. The lines of the shadow space of type (I, K) are just the $I \setminus K$-cells of the chamber system; they partition the point set of this space, i.e. the set of all chambers.

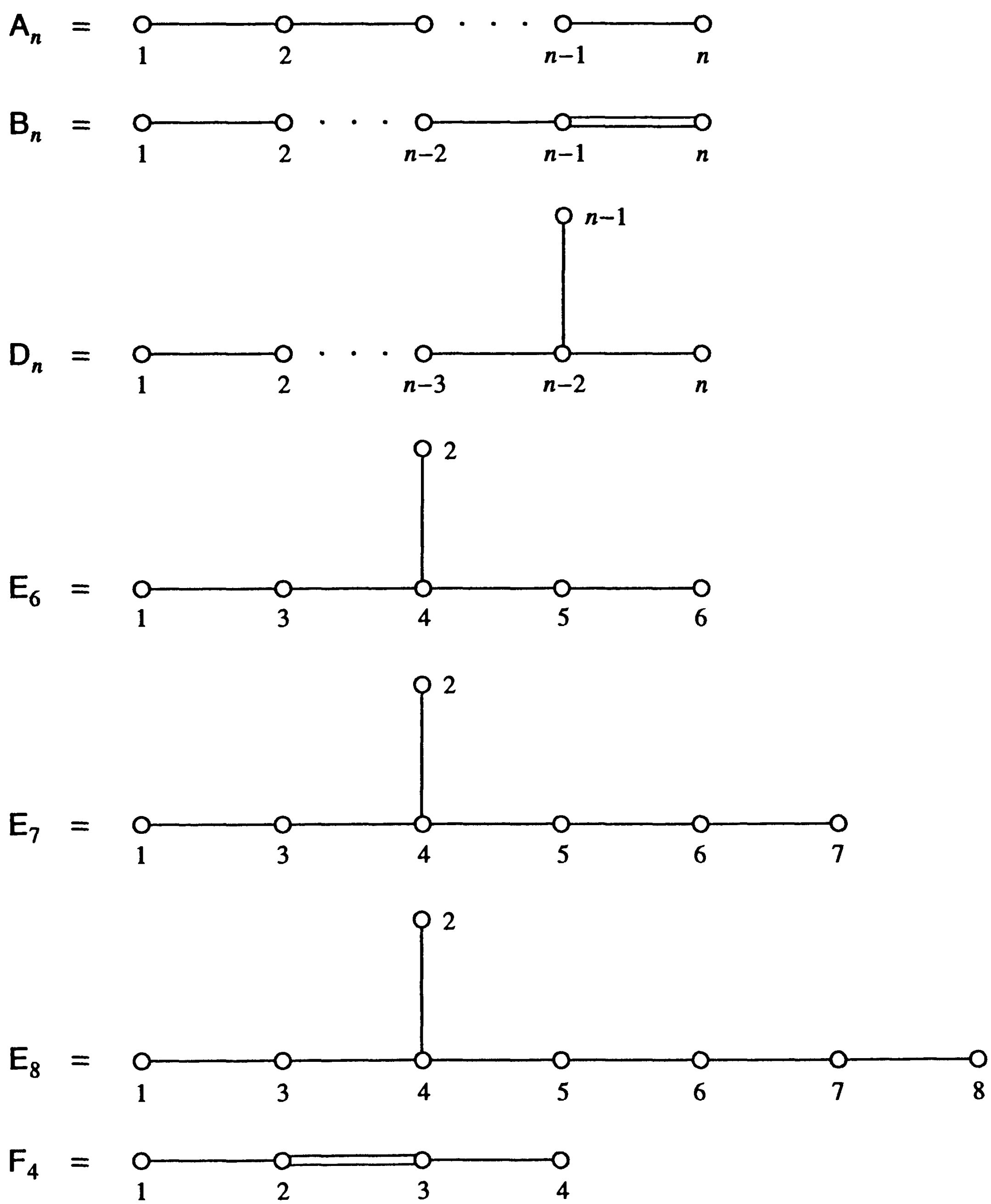
For convenience of notation, the abbreviations K' of $I \setminus K$ for $K \subseteq I$ and j' of $I \setminus \{j\}$ for $j \in I$ are quite standard.

Occasionally, the *shadow spaces* of Γ of type $(J, \{K_1, \ldots, K_r\})$, where each $K_j \subseteq I$, is needed. Such a space is a pair $(P, \mathcal{L})$ where P is the set of all flags of Γ of type J and $\mathcal{L}$ consists of all shadows on P of flags of Γ of type K_j for some $j \in \{1, \ldots, r\}$. For example, consider the set of all incident point-hyperplane pairs of a projective space of rank $n < \infty$. Its elements are the $\{1, n\}$-shadows of the corresponding projective geometry. There are two natural definitions of lines, corresponding to $1'$-shadows and to n'-shadows. Allowing both to be lines, we obtain the shadow space of type $(\{1, n\}, \{1', n'\})$, which can alternatively be described as the subspace of the direct product of the shadow space over 1 with the shadow space over n consisting of all incident pairs.

4.7. *Shadow spaces of buildings.* Let B be a building B over I whose type is a finite connected diagram. Given the shadow space S of B over j, the objects of the building geometry are uniquely determined by their shadows on $\mathcal{P}(S)$. This follows from the fact that buildings satisfy the intersection property (Corollary 4.5) and that the intersection property implies property (O), cf. Chapter 3.

For simplicity of presentation, we shall mostly restrict attention to shadow spaces over j. If Y_n is a Coxeter diagram and j one of its n nodes, a space is said to be of *type* $\mathsf{Y}_{n,j}$ if it is the shadow space over j of a building of type Y_n.

From Chapter 11, 2.1, we recall the classification of the finite irreducible Coxeter groups. We shall use Bourbaki's labelling throughout. Apart from those of rank at most 2 and those for H_3 and H_4 (excluded because there do not exist thick buildings of these types, see Chapter 11), here is the full list.



Thus, e.g., the space of type $\mathsf{A}_{n,k}$ is the Grassmannian of $(k-1)$-spaces in a projective n-space.

4.8. *Spaces and rank 2 geometries.* There is a slight difference between the notions of space and of rank 2 geometry, also called *incidence system.*

Suppose the triple $(X_1, X_2, *)$ is a geometry over $\{1,2\}$. The shadow space over 1 is the pair $(X_1, \mathcal{L})$, where $\mathcal{L}$ is the collection of all $\{x\colon\ x*\ell\}$ for ℓ running over all elements of X_2. Thus, the shadow space over X_1 is a space if every object in X_2 belongs to at least two maximal flags.

Conversely, suppose $(P, \mathcal{L})$ is a space. Consider the rank 2 geometry $(P, \mathcal{L}, *)$, where $* \subseteq P \times \mathcal{L}$ stands for containment, and objects from P, $\mathcal{L}$ have type 1, 2, respectively. This geometry has the property that every object ℓ of type 2 is in at least two chambers (but need not be firm as a point in P need not lie on more than one line), and is uniquely determined by its shadow $\{x \in P\colon\ \ell * x\}$.

Moreover, every geometry over $\{1,2\}$ with these properties can be reconstructed, up to isomorphism, from its shadow space on the variety of type 1.

Now, in the above construction of a shadow space of type $\mathsf{Y}_{n,j}$, suppose j is an end node of the Coxeter diagram Y_n with neighbour k (i.e. k is the only node to which j is connected in Y_n). Then the shadow space can be obtained by first truncating the building to a geometry over $\{j, k\}$ and next taking the shadow space over j, see §4.13.

The next result exhibits some of the more important properties shadow spaces over j have.

4.9. THEOREM (cf. Brouwer et al. [1989], Cooperstein [1976], Tits [1974]). *Suppose B is a building of type $M = (m_{j,k})_{j,k\in I}$ over the finite index I. Let $i \in I$, and let S be the shadow space of B over i. The following statements hold for S.*

(i) *The space S is paraprojective. Either there is a unique $j \in I$ with $m_{i,j} = 3$ and, in the diagram with node i removed, the connected component I' containing j has the property that the diagram induced on $\{k \in I'\colon\ m_{i,k} = 2 \text{ or } 3\}$ is isomorphic to A_m (for some $m \in I$), with j as end node, or $\{x,y\}^{\perp\perp}$ is the unique line containing x, y for every pair x, y of distinct collinear points.*

(ii) *Every shadow X is a geodetically closed subspace which is isomorphic to the shadow space over i of a building isomorphic to the residue of a flag of B. The flag can be taken to be maximal with respect to the property that its shadow on $\mathcal{P}(S)$ coincides with X.*

(iii) *For each shadow X and each point x the subset*

$$\pi_x(X) := \left\{y \in X\colon\ d(x,y) = \min_{z\in X} d(x,z)\right\}$$

of $\mathcal{P}(S)$ is a subspace of S. In particular, S is a partial linear gamma space.

(iv) *If $x, y \in \mathcal{P}(S)$ satisfy $d(x,y) = 2$, then either $\{x,y\}^{\perp}$ is a subspace isomorphic to a projective space (necessarily of rank 0 if M is spherical) or $\{x,y\}^{\perp} \cup \langle x,y\rangle$ is contained in a shadow subspace of type $\mathsf{B}_{m,1}$ $(m > 1)$ or $\mathsf{D}_{m,1}$ $(m > 2)$, whence isomorphic to a polar space.*

(v) *The shadow of an apartment of B (i.e. the set of points belonging to an apartment) induces a subgraph of* $(\mathcal{P}(S), \perp)$ *which is isomorphic to the shadow space of the thin building of type M over i.*

PROOF. See Brouwer et al. [1989], Cohen [1982a, 1986] for various parts of the proof. Here, we only prove that shadows X are subspaces of S. Put $L = I \setminus \{j\}$. Suppose ℓ is a line of S and X is a j-shadow with $|\ell \cap X| \geqslant 2$. Thus there are chambers c and d and there is a subset K of I such that $\ell = cj^*L^*$ and $X = dK^*L^*$. Suppose there is a point in $\ell \cap X$. Without loss of generality, we may take this point to be cL^*.

By Proposition 4.4, we have $\ell \cap X = e(\{j\} \cap {}^wK)^*L^*$ for some $e \in cj^*$ and some $w \in D_{\{j\},K}$ such that $c \xrightarrow{xwy} d$ for certain $x \in \langle j \rangle$ and $y \in \langle K \rangle$.

If $j \in {}^wK$ then

$$\ell = cj^*L^* \subseteq e\big(\{j\} \cap {}^wK\big)^*L^* = \ell \cap X$$

and we are done. Otherwise, $\{j\} \cap {}^wK = \varnothing$, so $cL^* \subseteq \ell \cap X = eL^*$, proving that $\ell \cap X$ is the single point cL^*. Hence X is a subspace of S. □

4.10. *Mutual positions of shadows.* Proposition 4.4 gives information about the intersections of j-shadows of different types of flags. The strategy to determine the possible ways in which two shadows of type J' and K', respectively, may intersect is first to determine the set $D_{J,K}$ of distinguished double coset representatives, and next, for each $w \in D_{J,K} \cap \langle L \rangle$, the intersection $J \cap {}^wK$. The latter index set informs us about the type of a feasible intersection of shadows.

For example, let us consider a building of type E_6. The relative positions of cells of type $J = K = 6'$ are indexed by the members of $D_{J,K} = \{\varnothing, 6, 65423456\}$. Thus, if we take $L = 1'$, then each $w \in D_{J,K}$ belongs to $\langle L \rangle$ and so realises an instance of intersection. In particular, in a space of type $\mathsf{E}_{6,1}$, any two symplecta (cf. §4.19), being 1-shadows of flags of type 6, meet (nonemptily). In the respective cases we find $J \cap {}^wK = 6'$, $\{5,6\}'$, and $\{1,6\}'$, so that the intersections are, respectively, the whole symplecton, a maximal singular subspace of projective dimension 4, and a point.

But if $L = 2'$ (yielding the space of type $\mathsf{E}_{6,2}$), two out of the three $w \in D_{J,K}$ lie in $\langle L \rangle$, showing that empty intersections of subspaces of flags of type 6 occur. For the two nonempty intersection patterns, corresponding to $w = \varnothing, 6$, we know $J \cap {}^wK$ from the previous paragraph. The result is that two distinct shadows of flags of the type 6 are either disjoint or meet in a maximal singular subspace (isomorphic to a space of type $\mathsf{A}_{4,1}$) of the space of type $\mathsf{E}_{6,2}$.

4.11. *Incidence from the shadow space.* As we have seen above, the intersection property takes care of recovering the objects of the building geometry from the shadow space. To retrieve the complete picture, we need to be able to recognise incidence of objects from their shadows within the shadow space.

However, before setting up incidence of the building under construction, one would have to specify its diagram. Polar spaces of rank $n \geqslant 3$ that are 'thin from above' (i.e. such that every singular subspace of rank 2 is contained in precisely 2 maximal singular subspaces, cf. §3.37) are examples of spaces of type $\mathsf{B}_{n,1}$ as well as of type $\mathsf{D}_{n,1}$ (see §4.15 below). So there may be a choice involved.

Let $S = (P, \mathcal{L})$ be a space of type $Y_{n,j}$. We distinguish classes $\mathcal{L}_L$ of subspaces, one class for each nonempty subset L of I, such that $\mathcal{L}_{\{j\}} = P$ and $\mathcal{L}_{I\setminus\{j\}} = \mathcal{L}$. Now two subspaces $U \in \mathcal{L}_L$ and $V \in \mathcal{L}_M$ of S are called *incident*, denoted by $U * V$, if and only if $U \cap V \in \mathcal{L}_{L\cup M}$. Thus, from S, given a diagram Y over I, we can construct the geometry

$$\Gamma(S) = \big((\mathcal{L}_i)_{i\in I}, *\big),$$

for which the following holds.

4.12. PROPOSITION (cf. Tits [1974], 12.15). *If $\mathcal{B}$ is a building of type Y_n over I and S is its shadow space over j for some $j \in I$, then $\Gamma(S)$, where $\mathcal{L}_J$ is the collection of shadows on S of flags of type J for each $J \subseteq I$, is isomorphic to the geometry corresponding to $\mathcal{B}$.*

If we start with a space we would like to establish comes from a building, recognising the geometry thus defined (where, so far, we have not addressed the question how to recover the subspaces which will function as shadow subspaces) as a building can be done by means of the important characterisations in Tits [1981b].

4.13. REMARK. Some classes of shadows of flags of different types often coincide. If $K \subset L$ is a separating set (cf. Chapter 11, 1.1) for some subset L of I with respect to J, then the J-shadows of flags of type L coincide with the J-shadows of flags of type K. Thus, economy leads to taking minimal separating sets. For example, if $L = \{\ell\}$ and $M = \{m\}$ and m separates j from ℓ in Y_n, then a subspace $U \in \mathcal{L}_\ell$ will be incident to a subspace $V \in \mathcal{L}_m$ if and only if $U \cap V = V$. Thus, $\mathcal{L}_{\{\ell,m\}} = \mathcal{L}_{\{m\}}$, and U, V are incident if and only if $V \subseteq U$.

4.14. *Projective geometries.* Set $I = \{1, \ldots, n\}$. By Theorem 5.9, the shadow space of type $\mathsf{A}_{n,1}$ (thus: over 1 of a building of type A_n) is a (possibly reducible) projective space. Shadows of objects of type i ($1 \leqslant i \leqslant n$) are subspaces of the shadow space of singular rank $i - 1$.

Now, conversely, suppose S is a (reducible) projective space of rank n. For $i \in I$, take $\mathcal{L}_i$ to be the collection of subspaces of singular rank $i-1$. Thus, indeed, $\mathcal{L}_1 = \mathcal{P}(S)$ and $\mathcal{L}_2 = \mathcal{L}(S)$. For nonempty $J \subseteq I$, set $\mathcal{L}_J = \mathcal{L}_j$ where $j = \min J$. Then the geometry $\Gamma(S)$ coincides with $(\mathcal{L}_1, \ldots, \mathcal{L}_n, *)$ where $x * y$ is *symmetrised inclusion* of x and y, i.e. $x \subseteq y$ or $y \subseteq x$. It is a residually connected geometry of type A_n, whence a building of this type, cf. Chapter 11.

Now consider a space $S = (P, \mathcal{L})$ of type $\mathsf{A}_{n,2}$ for $n \geqslant 4$. Set $\mathcal{L}_2 = P$. Put $\mathcal{L}_1$ for the set of all maximal singular subspaces of S having rank $n-1$. Then $(\mathcal{L}_1, \mathcal{L}_2, *)$, where $x * y$ for $x \in \mathcal{L}_1$ and $y \in \mathcal{L}_2$ stands for membership of y in x, is a projective space (or, to be precise, the rank 2 geometry corresponding to the projective space as in §4.8). By the previous paragraph, this suffices for the reconstruction of the building of type A_n from S. Here a pattern emerges for axiomatically characterising spaces of type $\mathsf{A}_{n,2}$: find a system of axioms leading to the collection $\mathcal{L}_2$ producing the projective space $(\mathcal{L}_1, \mathcal{L}_2, *)$. Thus, for $k = 2$, the recognition of spaces of type $\mathsf{A}_{n,k}$ is based on the recognition of spaces of type $\mathsf{A}_{n,k-1}$. The procedure for higher $k \leqslant n/2$ is similar. An example can be found in Theorem 6.3 below.

4.15. *Polar geometries and a related example.* Given a polar space S of rank n, the most natural way to turn it into a geometry over $I = \{1, \ldots, n\}$ is to take

$$\mathcal{L}_i = \{\text{singular subspaces of rank } i - 1\}$$

for each $i \in I$ and, as for A_n (and indeed, as for any linear diagram), with the usual total ordering of nodes. For nonempty $J \subseteq I$, set $\mathcal{L}_J = \mathcal{L}_j$ where $j = \min J$. If S is nondegenerate, $\Gamma(S)$ is a building of type B_n.

Now let us consider the 'thin from above' case for polar spaces of rank n. Then there is an alternative way to construct a building over $I = \{1, \ldots, n\}$, which runs as follows. For $i = 0, \ldots, n-3$, define $\mathcal{L}_i$ as before. The maximal singular subspaces come in two classes, cf. §3.37; let $\mathcal{L}_{n-2}$ be one of them and denote the other by $\mathcal{L}_{n-1}$. Now take $\mathcal{L}_{n-1,n-2}$ to be the collection of singular subspaces of rank $n-2$. For further nonempty subsets J of I, we define $\mathcal{L}_J = \mathcal{L}_j$ where $j = \min J$. Then $\Gamma(S)$ is a building of type D_n.

Having performed these identifications, we obtain that a dual polar space is in fact a space of type $\mathsf{B}_{n,n}$ and a half dual polar space is a space of type $\mathsf{D}_{n,n}$.

Not all residually connected geometries of type Y_n for some Coxeter diagram Y_n are buildings, cf. Chapter 11, 3.1.7–9, the remark on (O) and (LL), where the famous example appears which shows that two more axioms are needed: the geometry related to Alt_7, obtained as follows: take X_1 to be a set of cardinality 7, let X_2 be the collection of all triples from X_1, and let X_3 be a single Alt_7-orbit of projective plane structures on X_1. (There are two such orbits.) Now, for $1 \leqslant i < j \leqslant 3$ and $x_i \in X_i$, we have $x_i * x_j$ if and only if either $i = 1$ and x_1 is contained in x_2, or $i = 2$ and x_2 is a line of x_3. Then $(X_1, X_2, X_3, *)$ is a geometry of type B_3 whose shadow space over 1 is a singular space on seven points with lines all $\binom{7}{3}$ triples of points. Under the assumption that the geometry is flag-transitive and that rank 2 residues are 'classical', the Alt_7 geometry is known to be the only 'sporadic' example, see Lunardon and Pasini [1989] and Aschbacher [1984b].

4.16. REMARK.

(i) One can push the reduction from building to space a little further in restricting oneself to collinearity graphs of shadow spaces. This does not work for $\mathsf{A}_{n,1}$, since these graphs are singular. Although some other complications arise (due to the fact that $\{x, y\}^{\perp\perp}$ strictly contains the line on x and y in such cases as $\mathsf{F}_{4,2}$ (isomorphic to $\mathsf{F}_{4,3}$) and $\mathsf{B}_{n,n-2}$), these are essentially the only spherical counterexamples. Thus, apart from $\mathsf{A}_{n,1}$ and its isomorph $\mathsf{A}_{n,n}$, a shadow space of spherical type $\mathsf{Y}_{n,j}$ is uniquely determined by its collinearity graph. See Cooperstein [1976] and Cohen [1986].

(ii) There is a more general technique of constructing a chamber system of a given (suitable) type from a space. It is based on the idea that, for a chamber system, all we need to reconstruct it from a shadow space is the local picture of the flag, i.e. of the subspaces related to the flag. Thus, we need not always construct the whole geometry from global knowledge of a space S, but rather can construct chambers of the form $(x_i)_{i\in I}$ where x_1 is a point, x_2 a line containing the point x_1, and x_7, say, a subspace of the residue $x_1^{\perp}/x_1$ of x_1, rather than a globally defined subspace of S. This generalization has been made possible by Tits' chamber system approach to buildings in Tits [1981b].

4.17. *On the diameter.* If the shadow space over j of a building of finite rank has finite diameter, then the type of the building is spherical. This follows from one of the many characterisations of finite Coxeter groups, found in Deodhar [1982], cf. Cohen [1991].

The diameters of all shadow spaces of type $\mathsf{Y}_{n,j}$ with Y_n spherical are known, see Brouwer et al. [1989]. The number equals the maximum over all minimal numbers of occurrences of r_j in reduced expressions for elements of $D_{I\setminus\{j\},I\setminus\{j\}}$. (Here $I = \{1,\ldots,n\}$.)

Theorem 4.9 also raises the question of classifying geodetically closed subspaces of a shadow space by means of the diameter. A first case is the following

4.18. LEMMA. *Suppose S is a shadow space over j. Then all of its singular subspaces are shadows of flags whose residues have type A_m with j occurring as an end node.*

The proof is left as an exercise based on Theorem 10.2.10 of Brouwer et al. [1989].

The list of diameter 2 shadow spaces is also rather restricted. However, if 'singular' in the above lemma is replaced by 'diameter 2', the conclusion can no longer be that the subspace is the shadow of a flag.

4.19. *Parapolar spaces.* A *parapolar space* is a connected partial linear gamma space possessing a collection of geodetically closed subspaces $\mathcal{S}$, called *symplecta* (singular: *symplecton*), isomorphic to nondegenerate polar spaces of rank at least 2, with the properties that each line is contained in a symplecton and each quadrangle is contained in a unique symplecton. The nondegeneracy assumptions in this definition are slightly stronger than those found in the literature regarding parapolar spaces (Buekenhout [1982], Cohen and Cooperstein [1983], Hanssens [1988], Shult [1987]); this does not harm the results to be discussed, but simplifies their presentations. If all symplecta have (polar) rank k, or rank at least k the space is said to have *polar rank k*, respectively, polar rank at least k. If S is a parapolar space in which every pair of points at distance 2 belongs to a symplecton, it is also called *strongly parapolar*.

Suppose S is a shadow space of spherical type $\mathsf{Y}_{n,j} \neq \mathsf{A}_{n,1}, \mathsf{A}_{n,n}$. Taking $\mathcal{S}$ to be the collection of all shadows of flags whose residues induce nondegenerate polar spaces on S, we see from Theorem 4.9 that S becomes a parapolar space.

By Theorem 3.37, a near $2d$-gon of depth 2 with thick lines is parapolar of polar rank 2.

4.20. *Shadows from a Tits system.* Particular examples of buildings, and in fact all buildings of spherical type and rank at least 3, are obtained from Tits systems (also called BN pairs, cf. Chapter 11) (B, N, W, R) in groups G. Here we explain how to view shadows in terms of subgroups in case the building comes from a Tits system. For $L \subseteq I$, write $P_L = B\langle L\rangle B$. These are the *standard parabolic* subgroups of G. The shadow space over j of the building $\mathcal{B}$ corresponding to the Tits system is

$$\bigl(G/P_L, \{gP_{\{j\}}P_L/P_L\colon\ g \in G\}\bigr) \quad \text{where } L = I \setminus \{j\}.$$

Thus, G is transitive on the set of all pairs (p, ℓ) of a point p and a line ℓ incident to it. Since $N_G(P_L) = P_L$, the G-sets G/P_L and $X = \{gP_Lg^{-1}\colon\ g \in G\}$ are equivalent. Here G acts on X by conjugation. Thus, another description of the shadow space over j of $\mathcal{B}$ is as the space with point set X in which the lines are G-conjugates of $\{gP_Lg^{-1}\colon\ g \in P_j\}$.

In the case of $A_{n,d}(D)$, the points are the d-dimensional subspaces of D^{n+1}, whose stabilisers are precisely the maximal parabolic subgroups of cotype d. In the case of B_n, consider the full classical group G with natural module V (i.e. G is the full stabiliser of a pseudoquadratic, Hermitian or alternating form). Witt's Theorem, see Gross [1979], shows that any two singular subspaces of V of given dimension are equivalent under G. The stabilisers of these singular spaces are the maximal parabolic subgroups. In the case of D_n, the maximal parabolic subgroups are again stabilisers of singular subspaces (here singular is what is classically called totally isotropic). But the groups related to D_n have two orbits on the set of all maximal singular subspaces. For further examples of BN pairs, see §5.13.

5. Models

The section following this one deals with characterisations of shadow spaces of buildings. Despite their intrinsic beauty and characterising power, axiomatic approaches are not always the best setting for studying projective geometries. Certain results have simpler proofs and are more easily seen in terms of an explicit model, that is a space defined as $P(V)$ for a vector space V. Similar observations hold for the other shadow spaces of buildings. This is an important reason for devoting attention to explicit models of the more important shadow spaces. Another reason is of course that the explicit models furnish very concrete existence proofs of the geometries and buildings involved.

The absolute space of a polarity given by a sesquilinear form as treated in §§3.4–8 is an example of a model. We shall first discuss some ad hoc examples, and then deal with Lie algebras to provide a shadow for each spherical building (of rank at least 3) over an algebraically closed field.

5.1. *Generalized hexagons.* We briefly indicate a construction of the known generalized hexagons. Let k be a field and let O be the set of matrices

$$x = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$$

where $\alpha, \beta \in k$ and $a, b \in k^3$. Then the usual entry-wise scalar multiplication and addition turn O into a vector space. We provide O with the following multiplication:

$$xy = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \alpha' & a' \\ b' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' - a\cdot b' & \alpha a' + \beta' a + b \times b' \\ \alpha' b + \beta b' + a \times a' & \beta\beta' - b\cdot a' \end{pmatrix},$$

where $a \cdot b$ and $a \times b$ are the usual standard inner and exterior product on k^3. It is clear that O is a nonassociative algebra over k with unity

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The elements of O are sometimes called *octonions*, or *octaves*, hence the letter O. The quadratic form Q: $O \to k$ given by

$$Q(x) = \alpha\beta + a \cdot b$$

is obviously nondegenerate and of Witt index 4. It turns O into a composition algebra (cf. §3.25), i.e. O has the following property: $Q(xy) = Q(x)Q(y)$ for all $x, y \in O$.

The algebra O is uniquely determined up to isomorphism by its dimension 8, the Witt index 4 of the quadratic form, and the above identity, cf. Springer [1963]. There is an anti-automorphism of O given by

$$\overline{x} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix},$$

and satisfies $Q(x)1 = x\overline{x}$ for all $x \in O$.

We shall work with the projective space $\mathcal{P}(O)$ of O. If $x \in O$, $x \neq 0$, let $\langle x \rangle$ denote the point of $\mathcal{P}(O)$ determined by x, i.e. the subspace kx of the vector space O.

Let H be the space whose points are all $\langle x \rangle$ of $\mathcal{P}(O)$ such that $x^2 = 0$. Then $\mathcal{P}(H)$ coincides with the intersection of $1^\perp$ and the quadric determined by Q. The lines of H are defined to be all lines of the form $\langle x, y \rangle$ for $\langle x \rangle, \langle y \rangle \in \mathcal{P}(H)$ with $xy = 0$. Then H is a generalized hexagon. On the following six points, a hexagon (an apartment) is induced:

$$\begin{pmatrix} 0 & \varepsilon_i \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ \varepsilon_i & 0 \end{pmatrix}, \quad \varepsilon_i \text{ is standard basis of } k^3,\ i = 1, 2, 3.$$

If k has order q, then H has order (q, q). The lines of H are maximal cliques of the collinearity graph. Details of this construction are to be found in Schellekens [1962]. The automorphism group of O, i.e. the subgroup of GL(O) of multiplication preserving transformations, is isomorphic to the group $G_2(k)$ as defined in §5.10 below. It preserves the quadratic form Q on O and so embeds in an orthogonal group $\mathrm{GL}(O)_Q$.

There is a twisted version in *loc. cit.*, which we explain for finite fields $k = \mathrm{GF}(q^3)$. It leads to the generalized hexagons of order (q^3, q), related to the twisted Chevalley group ${}^3\mathrm{D}_4(q)$ (cf. §5.14 below). Suppose σ is a generator of the Galois group of k over GF(q). Consider the algebra O_σ whose underlying vector space coincides with that of O, and whose multiplication $\circ$ is given by

$$x \circ y = \overline{(x^\sigma)(y^{\sigma^2})} = \overline{\left((y^\sigma x)^\sigma\right)} \quad (x, y \in O_\sigma).$$

Here σ is an automorphism of O obtained by extending the above Galois automorphism via action on the coefficients with respect to a standard basis of O over k. It will be indicated by the same symbol, and written as an exponent, and $\overline{x}$ is the anti-automorphism of O defined above.

Now let H_σ be the set of all $\langle x\rangle \in P$ with $x \circ x = 0$, and call $\langle x\rangle$ and $\langle y\rangle$ adjacent if $x \circ y = 0$. Then adjacency is a symmetric relation and the resulting graph is the collinearity graph of a uniquely determined generalized hexagon of order (q^3, q). Given distinct adjacent $\langle x\rangle, \langle y\rangle \in H_\sigma$, the line on these two points is the maximal clique $\{\langle z\rangle\colon\ z \in \langle x\rangle + \langle y\rangle,\ z \neq 0\}$ of H_σ.

See Cohen and Cooperstein [1992], Kantor [1986] for elementary constructions of the classical generalized hexagons. A more general treatment of triality, lying at the origin of the notion of generalized hexagon, is found in Tits [1959].

Consider the form

$$(x, y, z) \mapsto f(x, y, z) = x\big(\overline{yz}\big) + (yz)\overline{x}$$

on O. Its restriction to $1^\perp$ is a trilinear alternating form, whose stabiliser in GL(O) coincides with Aut O. For $g \in \mathrm{GL}(O)_Q$, Springer [1963] shows the existence of g' and g'' in $\mathrm{GL}(O)_Q$, unique up to a central element of the group, such that

$$f\big(gx, g'y, g''z\big) = f(x, y, z) \quad \text{for all } x, y, z \in O.$$

The map $g \mapsto g'$ thus obtained leads to a 'triality' automorphism of $\mathrm{GL}(O)_Q$. The fixed elements in $\mathrm{GL}(O)_Q$ of this automorphism form the group Aut O.

5.2. *The* E_6 *example.* We next provide a more elaborate example, namely groups of type E_6 in a 27-dimensional representation with space $\boldsymbol{K}$. Detailed studies of this module appear in Aschbacher [1988b], Buekenhout and Cohen [1992]. The group $\mathsf{E}_6(k)$ will be introduced as the set of all linear transformations preserving a cubic form. The shadow space of the building appears as an embedded space in $\boldsymbol{P}(\boldsymbol{K})$.

For the duration of this section, k denotes a field, $\boldsymbol{K}$ denotes the 27-dimensional vector space over k consisting of all ordered triples $x = [x^{(1)}, x^{(2)}, x^{(3)}]$ of 3×3-matrices $x^{(i)}$, $1 \leqslant i \leqslant 3$ (addition and scalar multiplication are entry-wise). The vector space $\boldsymbol{K}$ is supplied with the cubic form $D\colon\ \boldsymbol{K} \to k$ given by

$$D(x) = \det x^{(1)} + \det x^{(2)} + \det x^{(3)} - \operatorname{trace} x^{(1)}x^{(2)}x^{(3)} \quad (x \in \boldsymbol{K}).$$

The study of the pair $\boldsymbol{K}, D$ is facilitated by the bilinear form $(\cdot,\cdot)$ on $\boldsymbol{K}$ defined as follows:

$$(x, y) = \operatorname{trace}\big(x^{(1)}y^{(1)} + x^{(2)}y^{(3)} + x^{(3)}y^{(2)}\big) \quad (x, y \in \boldsymbol{K}).$$

It gives rise to the map $x \mapsto x^\#$ on $\boldsymbol{K}$ determined by the fact that $(x^\#, y)$ is the part of $D(x + y)$ that is homogeneous and quadratic as a polynomial in x and linear in y. Explicitly, for $x \in \boldsymbol{K}$ the map is given by

$$x^\# = \big[\big(x^{(1)}\big)^\# - x^{(2)}x^{(3)},\ \big(x^{(3)}\big)^\# - x^{(1)}x^{(2)},\ \big(x^{(2)}\big)^\# - x^{(3)}x^{(1)}\big],$$

where $(x^{(i)})^{\#}$ is the matrix whose k,j-entry is the j,k-minor (with sign) of $x^{(i)}$. Thus,

$$D(x+y) = D(x) + \big(x^{\#}, y\big) + \big(x, y^{\#}\big) + D(y) \quad (x, y \in \boldsymbol{K}).$$

The following identity is very basic. All others can be derived from it. See Jacobson [1981], Springer [1962] for axiomatic approaches taking it as a starting point.

$$x^{\#\#} = D(x)x \quad (x \in \boldsymbol{K}).$$

The linearization of $x \mapsto x^{\#}$ is denoted by $\times$, and the linearization of D by $(\cdot,\cdot,\cdot)$. Thus, for all $x, y, z \in \boldsymbol{K}$,

$$x \times y = (x+y)^{\#} - x^{\#} - y^{\#}$$

and

$$(x,y,z) = D(x+y+z) - D(x) - D(y) - D(z) + D(x+y) + D(x+z) + D(y+z).$$

Here are a series of useful identities, valid for all $x, y, z \in \boldsymbol{K}$.

$$\begin{aligned} (x \times y, z) &= (x, y, z) \quad \text{(symmetric in } x, y, z\text{)},\\ x^{\#} \times (x \times y) &= D(x)y + \big(x^{\#}, y\big)x,\\ x \times \big(x^{\#} \times y\big) &= D(x)y + (x, y)x^{\#},\\ (x \times y)^{\#} + x^{\#} \times y^{\#} &= \big(x^{\#}, y\big)y + \big(x, y^{\#}\big)x. \end{aligned}$$

The point set R of the space S we wish to construct consists of all projective points $\langle x\rangle$ of $\boldsymbol{K}$ with $x^{\#} = 0$; its line set consists of all projective lines l with $x^{\#} = 0$ for each $\langle x\rangle \in l$. Thus, the collinearity graph $(\mathcal{P}(S), \times)$ of S is given by $\langle x\rangle \sim \langle y\rangle$ if and only if $x \times y = 0$.

Writing out $x^{\#}$ on the standard basis, one can give a very straightforward description of S. For $x \in \boldsymbol{K}$, write $x = (x_i)_{1 \leqslant i \leqslant 27}$, so that x_i is the coordinate function attached to the standard basis element labelled i in the following scheme:

$$\left[\begin{pmatrix} 6 & 4 & 11\\ 7 & 5 & 12\\ 25 & 26 & 27 \end{pmatrix}, \begin{pmatrix} 14 & 13 & 15\\ 17 & 16 & 18\\ 9 & 8 & 10 \end{pmatrix}, \begin{pmatrix} 23 & 20 & 2\\ 24 & 21 & 3\\ 22 & 19 & 1 \end{pmatrix}\right].$$

Thus, e.g., $x_6 = x^{(1)}_{1,1}$. An explanation of the ordering of the indices can be found at the end of §5.21 below. Then $\mathcal{P}(S)$ is the projective zero set of the following 27 homogeneous quadratic polynomials, each of which corresponds to a coordinate of $x^{\#}$:

$$x_{10}x_{19} - x_{25}x_4 + x_{26}x_6 + x_9x_{20} + x_8x_{21},$$

$$x_{15}x_{22} - x_5x_{27} + x_{12}x_{26} + x_{14}x_{23} + x_{13}x_{24},$$

$$x_{22}x_{20} - x_{19}x_{23} - x_{25}x_{13} - x_{26}x_{16} - x_{27}x_8,$$

$$x_{11}x_9 - x_{21}x_1 + x_3x_{19} + x_6x_{14} + x_4x_{17},$$

$$x_9x_{13} - x_8x_{14} - x_{22}x_4 - x_{19}x_5 - x_1x_{26},$$

$$x_{16}x_{10} - x_{18}x_8 - x_{23}x_6 - x_{20}x_7 - x_2x_{25},$$

$$x_{11}x_5 - x_4x_{12} + x_{14}x_2 + x_{13}x_3 + x_{15}x_1,$$

$$x_{18}x_{22} - x_{12}x_{25} + x_7x_{27} + x_{17}x_{23} + x_{16}x_{24},$$

$$x_{11}x_{10} - x_{20}x_3 + x_2x_{21} + x_6x_{15} + x_4x_{18},$$

$$x_{12}x_9 - x_3x_{22} + x_{24}x_1 + x_7x_{14} + x_5x_{17},$$

$$x_{15}x_{16} - x_{13}x_{18} + x_{23}x_{11} + x_{20}x_{12} + x_2x_{27},$$

$$x_{17}x_{10} - x_{18}x_9 + x_{24}x_6 + x_{21}x_7 + x_3x_{25},$$

$$x_{11}x_7 - x_6x_{12} - x_{17}x_2 - x_{16}x_3 - x_{18}x_1,$$

$$x_{10}x_{22} - x_7x_{26} + x_5x_{25} + x_9x_{23} + x_8x_{24},$$

$$x_{12}x_{10} - x_2x_{24} + x_{23}x_3 + x_7x_{15} + x_5x_{18},$$

$$x_{21}x_{22} - x_{24}x_{19} + x_{25}x_{14} + x_{26}x_{17} + x_{27}x_9,$$

$$x_{15}x_{17} - x_{14}x_{18} - x_{24}x_{11} - x_{21}x_{12} - x_3x_{27},$$

$$x_{16}x_9 - x_{17}x_8 + x_{22}x_6 + x_{19}x_7 + x_1x_{25},$$

$$x_6x_5 - x_4x_7 - x_9x_2 - x_8x_3 - x_{10}x_1,$$

$$x_{15}x_{19} - x_{26}x_{11} + x_{27}x_4 + x_{14}x_{20} + x_{13}x_{21},$$

$$x_{23}x_{21} - x_{20}x_{24} - x_{25}x_{15} - x_{26}x_{18} - x_{27}x_{10},$$

$$x_{11}x_8 - x_{19}x_2 + x_1x_{20} + x_6x_{13} + x_4x_{16},$$

$$x_{14}x_{16} - x_{13}x_{17} - x_{22}x_{11} - x_{19}x_{12} - x_1x_{27},$$

$$x_{10}x_{13} - x_8x_{15} + x_{23}x_4 + x_{20}x_5 + x_2x_{26},$$

$$x_{18}x_{19} - x_{27}x_6 + x_{25}x_{11} + x_{17}x_{20} + x_{16}x_{21},$$

$$x_{12}x_8 - x_1x_{23} + x_{22}x_2 + x_7x_{13} + x_5x_{16},$$

$$x_{10}x_{14} - x_9x_{15} - x_{24}x_4 - x_{21}x_5 - x_3x_{26}.$$

Furthermore, $\mathcal{L}(S)$ is the set of lines of $\boldsymbol{P}(\boldsymbol{K})$ entirely contained in $\mathcal{P}(S)$.

The space S is a parapolar space satisfying the following properties.

(i) Its symplecta are of the form $z \times \boldsymbol{K}$ for $\langle z\rangle \in R$. If $\langle x\rangle$ and $\langle y\rangle$ are two noncollinear points of S, they belong to the unique symplecton $Q(x, y) = (x \times y) \times \boldsymbol{K}$. In particular, S has diameter 2.

(ii) Each maximal singular subspace has rank 4 or 5, and both ranks occur.

(iii) Each symplecton Q is a subspace isomorphic to a quadric of polar rank 5 in $P(F^{10})$. Its stabiliser in G satisfies $N_G(Q)/C_G(Q) \cong \mathrm{PO}(10, F)$.
(iv) Each symplecton Q of S corresponds to a unique point $\langle z\rangle \in S$ such that the points of Q are precisely the points $\langle z \times v\rangle$ with $(z, v^{\#}) = 0$.
(v) The intersection of two distinct symplecta is either empty, a point or a maximal singular subspace of either symplecton.
(vi) If $\langle x\rangle \in S$ and Q is a symplecton with $\langle x\rangle \notin Q$, then $\langle x\rangle^{\times} \cap Q$ is either empty or a singular subspace of rank 4.
(vii) The map $\langle x\rangle \mapsto x \times \boldsymbol{K}$ from R to the set S of symplecta is bijective. Moreover $\langle x\rangle \times \langle y\rangle = 0$ holds if and only if $x \times \boldsymbol{K}$ and $y \times \boldsymbol{K}$ meet in a singular subspace of rank 4.

Rather than proving these statements, we pay some attention to the construction of automorphisms of S and their use in a proof. To this end, consider the group $G = \mathrm{GL}(\boldsymbol{K})_D$ of (invertible) linear transformations g of $\boldsymbol{K}$ such that $D(g(x)) = D(x)$ for all $x \in \boldsymbol{K}$. This group has a centre of order 1 or 3, depending on the nonexistence or existence of a nonidentity element ω in k with $\omega^3 = 1$. The quotient $G/Z(G)$ is the Chevalley group of type E_6 defined over k, and G is its universal cover. The group was studied first for arbitrary fields by Dickson [1901]. Later, the group was studied in Freudenthal [1985] and in Springer [1963].

G contains a subgroup isomorphic to the central product

$$\mathrm{SL}(3, k) \circ \mathrm{SL}(3, k) \circ \mathrm{SL}(3, k),$$

as follows. For $g_1, g_2, g_3 \in \mathrm{SL}(3, k)$ define s_{g_1,g_2,g_3} by

$$s_{g_1,g_2,g_3}(x) = \left[g_1 x^{(1)} g_2^{-1}, g_2 x^{(2)} g_3^{-1}, g_3 x^{(3)} g_1^{-1}\right] \quad (x \in \boldsymbol{K}).$$

Clearly, each s_{g_1,g_2,g_3} belongs to G. If k contains a nontrivial cube root of unity ω, the central element of G is $s_{1,\omega,\omega^2} = s_{\omega,1,\omega^2}$.

The three factors $\mathrm{SL}(3, k)$ in the product are permuted by a subgroup $\langle \rho, \tau\rangle \cong \mathrm{Sym}_3$ of G, where, for $x \in \boldsymbol{K}$,

$$\rho(x) = \left[x^{(3)}, x^{(1)}, x^{(2)}\right] \quad \text{and} \quad \tau(x) = \left[{x^{(1)}}^{\mathsf{T}}, {x^{(3)}}^{\mathsf{T}}, {x^{(2)}}^{\mathsf{T}}\right].$$

A *maximal torus* of G is a maximal Abelian subgroup of G all of whose elements are semisimple (i.e. diagonizable as elements of $\mathrm{GL}(\boldsymbol{K})$). The following maximal torus T consists of all diagonal transformations of G with respect to the standard basis. An arbitrary element $h \in T$ has the form

$$h(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) = \left[\begin{pmatrix} \alpha\gamma^{-1}\delta^{-1} & \alpha\beta\gamma & \alpha\delta \\ \alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1} & \alpha^{-1}\gamma & \alpha^{-1}\beta^{-1}\delta \\ \gamma^{-1}\delta^{-1} & \beta\gamma & \delta \end{pmatrix},\right.$$

$$\begin{pmatrix} \gamma\delta\zeta^{-1}\varepsilon^{-1} & \beta\gamma\delta\varepsilon & \gamma\delta\zeta \\ \beta^{-1}\gamma^{-1}\zeta^{-1}\varepsilon^{-1} & \gamma^{-1}\varepsilon & \beta^{-1}\gamma^{-1}\zeta \\ \delta^{-1}\zeta^{-1}\varepsilon^{-1} & \beta\delta^{-1}\varepsilon & \delta^{-1}\zeta \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{-1}\zeta\varepsilon & \alpha\beta\zeta\varepsilon & \varepsilon\zeta \\ \alpha^{-1}\beta^{-1}\varepsilon^{-1} & \alpha\varepsilon^{-1} & \beta^{-1}\varepsilon^{-1} \\ \alpha^{-1}\zeta^{-1} & \alpha\beta\zeta^{-1} & \zeta^{-1} \end{pmatrix}\Bigg] .$$

Here, each entry represents the scalar by which the corresponding basis element of $\boldsymbol{K}$ is multiplied in the action of h. Consider the following 6 transformations $n_1, \ldots, n_6$ contained in N given by

$$n_1 = s_{1,-(13),1}, \qquad n_2 = s_{-(12),1,1}, \qquad n_3 = s_{1,-(12),1},$$

$$n_4(x) = \begin{pmatrix} x^{(i)}_{1,1} & -x^{(i+2)}_{2,3} & x^{(i)}_{1,3} \\ -x^{(i+1)}_{3,2} & x^{(i)}_{2,2} & x^{(i+1)}_{1,2} \\ x^{(i)}_{3,1} & x^{(i+2)}_{2,1} & x^{(i)}_{3,3} \end{pmatrix}^{(i)}_{i=1,2,3} \qquad (x \in \boldsymbol{K}),$$

$$n_5 = s_{1,1,-(12)}, \qquad n_6 = s_{1,1,-(13)}.$$

It is not hard to derive that $N_G(T) = T.\langle n_1, \ldots, n_6\rangle$, that $T \cap \langle n_1, \ldots, n_6\rangle \cong \boldsymbol{Z}_2^6$ and that $N_G(T)/T \cong W(\mathsf{E}_6)$, the Weyl group of type E_6.

For $x, y \in \boldsymbol{K} \setminus \{0\}$ with $x^\# = y^\# = 0$ and $(x, y) = 0$ the linear map $t_{x,y}$ on $\boldsymbol{K}$ given by

$$t_{x,y}(v) = v + y \times (x \times v) - (y, v)x \quad (v \in \boldsymbol{K})$$

also is a transformation belonging to G.

Using the above members of G, it is relatively easy to prove that G is highly transitive. Put $\mathcal{L}_1 = \mathcal{P}(S)$, $\mathcal{L}_3 = \mathcal{L}(S)$ and write $\mathcal{L}_2$ for the collection of all maximal singular subspaces of rank 5, $\mathcal{L}_6$ for the collection of all symplecta, and $\mathcal{L}_5$ for the collection of all maximal singular subspaces of rank 4. Furthermore, let $\mathcal{L}_4$ be the set of all singular subspaces of S of rank 2 and $\mathcal{L}_{\{2,5\}}$ the set of all singular subspaces of rank 3. Finally, set $\mathcal{L}_{\{2,6\}}$ to be the collection of all nonmaximal singular subspaces of S of rank 4. Then G is transitive on each of these collections. Representative elements are:

$\langle e^{(3)}_{1,2}\rangle$ for $\mathcal{L}_1$;

$\langle e^{(3)}_{1,2}, e^{(3)}_{2,2}\rangle$ for $\mathcal{L}_3$;

$\langle e^{(3)}_{1,2}, e^{(3)}_{2,2}, e^{(3)}_{3,2}\rangle$ for $\mathcal{L}_4$;

$\langle e^{(1)}_{1,1}, e^{(3)}_{1,2}, e^{(3)}_{2,2}, e^{(3)}_{3,2}\rangle$ for $\mathcal{L}_5$;

$\langle e^{(1)}_{1,1}, e^{(1)}_{1,2}, e^{(1)}_{1,3}, e^{(3)}_{1,2}, e^{(3)}_{2,2}, e^{(3)}_{3,2}\rangle$ for $\mathcal{L}_2$;

$\langle e^{(1)}_{1,1}, e^{(3)}_{1,2}, e^{(3)}_{2,2}, e^{(3)}_{3,2}\rangle$ for $\mathcal{L}_{\{2,5\}}$;

$\langle e^{(1)}_{1,1}, e^{(1)}_{1,3}, e^{(3)}_{1,2}, e^{(3)}_{2,2}, e^{(3)}_{3,2}\rangle$ for $\mathcal{L}_{\{2,6\}}$;

$\langle e^{(1)}_{1,1}, e^{(1)}_{1,3}, e^{(1)}_{3,1}, e^{(1)}_{3,3}, e^{(2)}_{2,1}, e^{(2)}_{2,2}, e^{(2)}_{2,3}, e^{(3)}_{1,2}, e^{(3)}_{2,2}, e^{(3)}_{3,2}\rangle$ for $\mathcal{L}_6$.

To a large extent, the statements (i)–(vii) can be shown to hold by reducing their verifications to the G-orbit representatives just found.

Now, consider the geometry $\Gamma = (\mathcal{L}_1, \dots, \mathcal{L}_6, *)$ where $x * y$ stands for symmetrised containment, except for the following two cases (up to an interchange of x and y):

$x \in \mathcal{L}_2$, $y \in \mathcal{L}_5$, in which case $x * y$ if and only if $x \cap y \in \mathcal{L}_{\{2,5\}}$;

$x \in \mathcal{L}_2$, $y \in \mathcal{L}_6$, in which case $x * y$ if and only if $x \cap y \in \mathcal{L}_{\{2,6\}}$.

Thus, the representatives listed above correspond to a maximal flag of Γ.

Then Γ is a building of type E_6, whose shadow space over 1 is naturally isomorphic to S. (Note that the above definition of $*$ aligns with §4.9.) Again, using the automorphisms given above, it can be shown that G is transitive on the set of maximal flags. Let B be the stabiliser of the maximal flag given above. Then B and $N = N_G(T)$ form a Tits system (B, N, W, R) in G, where $W = N/T$ and $R = \{n_iT\colon\ i = 1, \dots, 6\}$. In particular, the fact that the building Γ is of type E_6 is reflected by the isomorphism $W \cong W(\mathsf{E}_6)$.

We have achieved more than a construction of the building and Tits system of type E_6 for a given field k: we also have an embedding of its shadow space S in the projective space $\boldsymbol{P}(\boldsymbol{K})$. The G-orbits on $\boldsymbol{P}(\boldsymbol{K})$ have been extensively studied, cf. Aschbacher [1987, 1988b, 1990b, 1991], Cohen and Cooperstein [1988], Mars [1966] for further details.

Using nondegeneracy of the form $(\cdot,\cdot)$ on $\boldsymbol{K}$, we can define $g^\# \in \mathrm{GL}(\boldsymbol{K})$ by demanding that $(gx, g^\# y) = (x, y)$ for all $x, y \in \boldsymbol{K}$. It can be shown that $g \mapsto g^\#$ is an involutory automorphism of G. It induces a non-type-preserving automorphism of the building associated with the above Tits system.

5.3. *Metasymplectic spaces.* The geometries of type F_4 can be approached via metasymplectic spaces. Consider once more the G-module $\boldsymbol{K}$ of the previous section. There is a single G-orbit of projective points on which the cubic form D does not vanish. The point $\langle e \rangle$ with $e = e^{(1)}_{1,1} + e^{(1)}_{2,2} + e^{(1)}_{3,3}$ has this property: $D(e) = 1$. Its stabiliser $F = G_e$ has a Tits system of type F_4. The group preserves the inner product $(\cdot,\cdot)$ (and can actually be defined as the subgroup of G preserving it). Except when the characteristic is 3, the orthoplement $e^\perp$ in $\boldsymbol{K}$ with respect to this bilinear form is a 26-dimensional irreducible F-module (in characteristic 3 it is indecomposable with irreducible factors of degree 1 and 25). By use of the F-invariant multiplication $\times$, an F-invariant Jordan algebra structure with unit e can be put on $\boldsymbol{K}$.

Let us indicate the construction of the shadow space S'. Its points are the projective points $\langle x \rangle$ with $x^\# = 0$ and $(1, x) = 0$. Its lines are the lines of $\boldsymbol{P}(\boldsymbol{K})$ consisting entirely of points of S'. Again using the group, it can be shown that maximal singular subspaces have rank 2 and that there is only one F-orbit of these. Moreover, S' is a parapolar space with symplecta isomorphic to polar spaces of type $\mathsf{B}_{3,1}$. Writing

$$\mathcal{L}_1 = \mathcal{P}(S'), \quad \mathcal{L}_3 = \{\text{rank 2 singular subspaces}\},$$

$$\mathcal{L}_2 = \mathcal{L}(S'), \quad \mathcal{L}_4 = \{\text{symplecta of } S'\},$$

and letting $*$ be symmetrised inclusion, we obtain a geometry $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, *)$, which is a building of type F_4.

Other groups with a Tits system of type F_4 exist. The subgroup F of G can be viewed as the set of elements fixed under the involutory automorphism $g \mapsto g^{\#}$ of G defined at the end of §5.2. (This follows from the characterisation of F as the stabiliser in G of $(\cdot,\cdot)$: $g \in F$ if and only if $(gx, gy) = (x, y) = (gx, g^{\#}y)$ for all $x, y \in \boldsymbol{K}$, which is equivalent to $g = g^{\#}$.) By composing this involution with a field automorphism of order 2, and considering the resulting subgroup of G of all fixed elements, one obtains groups ${}^2\mathsf{E}_6(k)$ with a Tits system of type F_4.

Further examples of metasymplectic spaces are described in §5.14.

5.4. *The* E_7 *geometries.* The groups of type E_7 can be studied in a fashion similar to, but more complicated than the treatment of E_6 in §5.2. Here, a 56-dimensional vector space V over a field k comes into play, on which a nondegenerate alternating form $\langle\cdot,\cdot\rangle$ and a quartic form R are defined. If k has characteristic different from 2, 3, by linearization a nonzero trilinear map $(\cdot,\cdot,\cdot)$: $V \times V \times V \to V$ can be found satisfying the following identities for $x, y, z \in V$.

$$R(x) = \langle x, (xxx)\rangle;$$

$$\langle (xyz), w\rangle \quad \text{is symmetric in all 4 variables } x, y, z, w;$$

$$\big((xxx)xy\big) = \langle y, (xxx)\rangle x - \langle x, y\rangle (xxx);$$

$$\big((xxx)xx\big) = R(x)x; \quad \text{and}$$

$$3\big((xxy)xx\big) = R(x)y + 2\langle y, (xxx)\rangle x + 2\langle x, y\rangle (xxx).$$

These identities can be taken as a starting point for an axiomatic treatment (Brown [1969], Mars (private communication, 1964)). In Baily [1970], Freudenthal [1953], explicit formulae for the quartic and alternating forms are given, which are based on §5.2. To this end, V is viewed as the vector space $k^2 \oplus \boldsymbol{K}^2$ on which the group $\mathrm{GL}(\boldsymbol{K})_D$ acts trivially on the component k^2, and via $(x, y) \mapsto (gx, g^{\#}y)$ on the component $\boldsymbol{K}^2$. The subgroup of $\mathrm{GL}(V)$ stabilising R and $\langle\cdot,\cdot\rangle$ is a group of type E_7. See Brown [1969], Cooperstein [1988], Faulkner and Ferrar [1977], Haris [1971] for other explicit constructions. A construction that is also valid if the characteristic of k is 2 or 3 can be found in Aschbacher [1988a].

Modules similar to this one for E_8 have not been studied. An explanation is the absence of nontrivial modules of degree less than 248 (the dimension of the Lie algebra module, see §5.9). The next degree following 248 of an irreducible nontrivial representation for $\mathsf{E}_8(\mathbb{C})$ is 3875. On the latter's underlying vector space, there are unique (up to scalar multiplication) nonzero group-invariant quadratic and cubic forms (this can easily be verified using a software package like LiE, cf. Van Leeuwen, Cohen and Lisser [1992]), and so the module affords a unique (up to scalar multiplication) nonzero group-invariant commutative multiplication (not necessarily associative). It would be of interest to find a description of this multiplication.

5.5. *Moufang polar spaces.* Let k be a field affording an 8-dimensional division Cayley division ring C with norm form κ. By Proposition 7 of Tits [1966], there is an algebraic k-group $\mathcal{G}$ of adjoint type E_7 whose anisotropic kernel is isogenous to $\Omega(n)$, the commutator subgroup of the group of all linear transformations of the 8-dimensional k-vector space C that leave invariant the quadratic form κ. By G we denote the group of all k-rational points of $\mathcal{G}$. Let P_1 be a maximal parabolic corresponding to the first node (see §4.7 for the labelling of the nodes of diagram E_7). Then P_1 has a unipotent radical of dimension 33 with centre of dimension 1. Let ζ_1 denote this normal subgroup of P_1 isomorphic to k^+, the additive groups of the field k. Now take P to be the set of all conjugates in G of ζ_1. Two such centres of unipotent radicals commute if and only if there is a parabolic of type 6 containing them both. Take $\mathcal{L}$ to be the collection of all subsets of P of the form $\{\zeta \in P\colon \zeta \subseteq Y\}$ for some maximal parabolic Y of type 6. Then $S_k = (P, \mathcal{L})$ is a nondegenerate polar space of rank 3. Its maximal singular subspaces are Moufang, but non-Desarguesian (and correspond to k-subgroups of type E_6).

Recently, a simpler description of the construction is given by Mühlherr [1990]. It uses an involutory automorphism σ of the building $\mathsf{E}_7(k)$ such that

– σ fixes two opposite objects h, h' of type 1;

– σ fixes no vertex of type $2, 3, 4, 5$ in $\operatorname{Res} h$;

– the incidence relation induces a generalized quadrangle structure on the set of all vertices of type 6 and 7 in $\operatorname{Res} h$.

Then it can be synthetically shown that, taking points and lines to be the σ-fixed vertices of type 1 and 6, respectively, one obtains a rank 3 polar space, whose planes correspond to the σ-fixed vertices of type 7.

5.6. *Dual and half-dual polar spaces.* These spaces are constructed by use of the Clifford algebra based on V, q, where V is a vector space with quadratic form q. See, e.g., Cameron [1992], Chevalley [1954].

5.7. *Lie algebras.* The most generic method of producing examples of buildings of spherical type employs Lie algebras. A *Lie algebra* over a field k is a vector space $\boldsymbol{g}$ over k with a bilinear product map $\boldsymbol{g} \times \boldsymbol{g} \to \boldsymbol{g}$, usually denoted by $x, y \mapsto [x, y]$ satisfying the following two rules for all $x, y, z \in \boldsymbol{g}$:

$$[x, x] = 0,$$

$$\big[[x, y], z\big] + \big[[y, z], x\big] + \big[[z, x], y\big] = 0. \qquad \text{(Jacobi identity)}$$

By linearization, the first identity yields:

$$[x, y] + [y, x] = 0. \qquad \text{(Anticommutativity)}$$

In the case of characteristic different from 2, the latter identity is equivalent to $[x, x] = 0$.

A *subalgebra* of $\boldsymbol{g}$ is a subspace closed under multiplication. An *ideal* in $\boldsymbol{g}$ is a subspace closed under multiplication by an arbitrary element from $\boldsymbol{g}$.

Important examples of Lie algebras are obtained as follows. Let V be a vector space. Then $\mathbf{gl}(V)$, the vector space of all linear maps $V \to V$ endowed with the multiplication

$$[x, y] = xy - yx \quad \big(x, y \in \mathbf{gl}(V)\big),$$

is a Lie algebra. The linear subspace $\mathbf{sl}(V)$ of $\mathbf{gl}(V)$ consisting of all linear maps with zero trace is a Lie subalgebra of $\mathbf{gl}(V)$. Further Lie subalgebras of $\mathbf{gl}(V)$ may be obtained as annihilators of multilinear forms. If $f\colon \otimes^d V \to k$ is a d-linear form on V, then the image $(\operatorname{ad} g)f$ of f under $g \in \mathbf{gl}(V)$ by the derivation action of $\mathbf{gl}(V)$ on $\otimes^d V$ may be computed by working over the ring $k[\varepsilon]$ of dual numbers, where $\varepsilon^2 = 0$, by calculating the coefficient of ε in $(1 + \varepsilon g) \cdot f$, where $\cdot$ stands for the natural action of $\mathrm{GL}(V \otimes k[\varepsilon])$ on $\otimes^d(V \otimes k[\varepsilon])$. The subspace

$$\mathbf{gl}(V)_f = \{g \in \mathbf{gl}(V)\colon\ (\operatorname{ad} g)f = 0\}$$

is a Lie subalgebra of $\mathbf{gl}(V)$. If $f \in \bigwedge^n V$, $f \neq 0$, where $n = \dim V$, we obtain $\mathbf{gl}(V)_f = \mathbf{sl}(V)$.

A Lie algebra is called *simple* if g and 0 are its only ideals. If k is algebraically closed of characteristic 0 and g is simple, then it is isomorphic to one of the Lie algebras that we shall construct in §5.9.

5.8. *Root systems.* Before being able to describe the Lie algebras of importance to us, we need the notion of a root system of a Coxeter group. Let (W, R) be a Coxeter system with $n = |R|$. Denote by ρ its reflection representation in $V = \mathbb{R}^n$ and by $(\cdot, \cdot)$ the corresponding $\rho(W)$-invariant bilinear form on $\mathbb{R}^n$ (cf. Chapter 11). A *fundamental reflection* of W is an element of R. A *reflection* of W is an element of W conjugate to a fundamental reflection. A *root* of a reflection is its -1 eigenvector in V (which is unique up to a nonzero scalar multiple). A (*reduced*) *root system* is a subset Φ of $\mathbb{R}^n$ such that

(i) For each $r \in R$ there is a distinguished root $\alpha_r \in \Phi$;
(ii) $\Phi = \{(\rho w)\alpha_r\colon\ w \in W, r \in R\}$;
(iii) if $a \in \mathbb{R}$ and $v \in \Phi$ then $av \in \Phi$ implies $a = \pm 1$.

Each Coxeter group has a root system. For, take a hyperplane of V not through any root of a fundamental reflection, and fix H^+ to be one of its half-spaces. For each $r \in R$, choose the root $\alpha_r \in H^+$ that satisfies $(\alpha_r, \alpha_r) = 2$. Then the union Φ over all $W\alpha_r$ for $r \in R$ is a root system. If the lengths (α_r, α_r) of the roots α_r are chosen differently, other root systems for W emerge.

The root system Φ can be partitioned into two sets Φ^+ and Φ^- where $\alpha \in \Phi^+$ if and only if it belongs to H^+. This partitioning of Φ has the remarkable property that all $\alpha \in \Phi^+$ belong to

$$\sum_{r \in R} \mathbb{R}_{\geqslant 0}\, \alpha_r,$$

and all $\alpha \in \Phi^-$ belong to $-\Phi^+$.

If r_α denotes the reflection in W with root $\alpha \in \Phi$, we have

$$(\rho r_\alpha)v = v - \langle v, \alpha \rangle \alpha \quad (v \in V),$$

where $\langle v, \alpha \rangle = 2(v, \alpha)/(\alpha, \alpha)$. Thus, the Coxeter group W can be recovered from the root system Φ and the bilinear form $(\cdot, \cdot)$ as the group generated by these linear transformations.

W is called a *Weyl group* if it is a finite Coxeter group and it has a root system Φ with $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. (The requirement for a finite Coxeter group W to be a Weyl group is equivalent to $m_{ij} \in \{2, 3, 4, 6\}$ for all $1 \leqslant i < j \leqslant n$.) Then $\mathbb{Z}\Phi$ is a W-invariant integral lattice, called the *root lattice* of W.

The lattice of all $v \in \mathbb{R}^n$ with $\langle v, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$ is called the *weight lattice* of W. Its elements are called *weights*. Each root, in fact, each element of the form $2\alpha/(\alpha, \alpha)$ for a root $\alpha \in \Phi$ is a weight.

Let $S \subseteq R$ be a set of W-orbit representatives of reflections of W. Two elements of R are in the same orbit if the Coxeter diagram has a path joining them all of whose edges have odd labels. Hence, for irreducible Weyl groups, we have $|S| \leqslant 2$. If $|S| = 1$ then $\mathbb{Z}\Phi$ is unique up to homotheties. If $|S| = 2$, the root system need not be unique up to homotheties. This distinction is crucial for Coxeter groups of type B_n. The corresponding root systems, realised with $(\cdot, \cdot)$ the standard Euclidean inner product on $\mathbb{R}^n$ and $(\varepsilon_i)_{1 \leqslant i \leqslant n}$ the standard orthonormal basis, are

$$\Phi_m = \{\pm m\varepsilon_i,\ \varepsilon_i \pm \varepsilon_j\colon\ 1 \leqslant i, j \leqslant n,\ i \neq j\}.$$

The root system is referred to as B_n if $m = 1$ and as C_n if $m = 2$. Thus $\mathsf{C}_2 \cong \mathsf{B}_2$, but $\mathsf{C}_n \ncong \mathsf{B}_n$ for $n > 2$. For this reason, the Coxeter diagrams do not suffice to parametrize root systems. But when supplied with a $<$ sign on the edges joining roots of different lengths, pointing towards the nodes representing the shorter roots, the diagrams suffice to distinguish essentially different root systems. The resulting diagrams are known as *Dynkin diagrams*, see Bourbaki [1968].

5.9. *Simple Lie algebras.* Let (W, R) be the Coxeter system of an irreducible Weyl group with root system Φ, and set $n = |R|$. Denote by L the sublattice of the weight lattice of W spanned by all $2\alpha/(\alpha, \alpha)$ for $\alpha \in \Phi$.

We fix the fundamental roots $\alpha_1, \ldots, \alpha_n$, so that Φ can be partitioned into two sets Φ^+ and Φ^- (as indicated in §5.8) indicating whether or not the coefficients of its members, when written as a linear combination of $\alpha_1, \ldots, \alpha_n$, are all $\geqslant 0$ or all $\leqslant 0$, respectively.

Consider the free $\mathbb{Z}$-module

$$g_{\mathbb{Z}} = L \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z} X_\alpha,$$

where X_α are formal basis elements. Thus, the rank of this $\mathbb{Z}$-module is $n + |\Phi|$. On $g_{\mathbb{Z}}$, an antisymmetric bilinear map $[\cdot, \cdot]\colon\ g_{\mathbb{Z}} \times g_{\mathbb{Z}} \to g_{\mathbb{Z}}$ is defined as follows, where

$\alpha, \beta \in \Phi$ and $\lambda, \mu \in L$.

$$[\lambda, \mu] = 0,$$

$$[\mu, X_\beta] = (\mu, \beta) X_\beta,$$

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 2\beta/(\beta, \beta) & \text{if } \alpha + \beta = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here, the structure constants $N_{\alpha,\beta} \in \mathbb{Z}$ should satisfy the following requirements (see Chevalley [1955], Gilkey and Seitz [1988]). Given $\alpha, \beta \in \Phi$, let $t_{\alpha,\beta}$ be the maximal number t for which $-t\alpha + \beta \in \Phi$.

$$N_{\alpha,\beta} = -N_{\beta,\alpha}, \tag{1}$$

$$N_{\alpha,-\beta} = N_{\beta,-\alpha}, \tag{2}$$

$$N_{\alpha,\beta} = \pm(t_{\alpha,\beta} + 1), \tag{3}$$

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = -(t_{\alpha,\beta} + 1)^2, \tag{4}$$

$$N_{\alpha,\beta}/(\gamma, \gamma) = N_{\beta,\gamma}/(\alpha, \alpha) \quad \text{if } \alpha + \beta + \gamma = 0, \tag{5}$$

$$\begin{aligned} &N_{\alpha,\beta} N_{\gamma,\delta}/(\alpha+\beta, \alpha+\beta) + N_{\beta,\gamma} N_{\alpha,\delta}/(\beta+\gamma, \beta+\gamma) \\ &\quad + N_{\gamma,\alpha} N_{\beta,\delta}/(\gamma+\alpha, \gamma+\alpha) = 0 \quad \text{if } \alpha+\beta+\gamma+\delta = 0, \end{aligned} \tag{6}$$

$$N_{\alpha,\beta} = 0 \quad \text{if } \alpha, \beta \in \Phi^+ \text{ but } \alpha + \beta \notin \Phi^+. \tag{7}$$

For a function N: $\Phi \times \Phi \to \mathbb{Z}$ with these properties, $[\cdot, \cdot]$ defines a Lie algebra structure on the vector space g obtained from $g_{\mathbb{Z}}$ by tensoring it with a field k.

The recursive nature of these relations implies that the structure of the Lie algebra is fully determined once we know the coefficients $N_{\alpha,\beta}$ for a restricted set of pairs $\alpha, \beta \in \Phi$. For example, for E_8, we can specify N_{α_i,α_j} for $i, j = 1, \ldots, 8$ by the following matrix (cf. Cohen, Griess Jr and Lisser [1993]) with respect to the fundamental roots $\alpha_1, \ldots, \alpha_8$:

$$(\eta_{\alpha_i,\alpha_j})_{1 \leqslant i,j \leqslant 8} = \begin{pmatrix} -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

Now $N_{\alpha_i,\alpha_j} = \eta_{\alpha_i,\beta_j}$ whenever $\alpha_i + \alpha_j \in \Phi$. Extending the matrix η to a bilinear map η: $L \times L \to \{\pm 1\}$ we obtain a much faster way of computing $N_{\alpha,\beta}$ than the recursive method based on (1)–(7). Suppose $\alpha, \beta, \alpha+\beta \in \Phi^+$. Then, $N_{\alpha,\beta} = \eta_{\alpha,\beta}$. Similar matrices η can be found for other types than E_8, see Kac [1985], Exercise 14.5, or Segal [1981], Springer [1981]. By $\mathrm{Aut}\,\boldsymbol{g}$ we mean the group consisting of all k-linear transformations of $\boldsymbol{g}$ preserving the Lie algebra multiplication. The Lie subalgebra $\boldsymbol{h} = L \otimes k$ of $\boldsymbol{g}$ is a maximal commutative subalgebra with the property that $x \mapsto [x, h]$ ($x \in \boldsymbol{g}$) is a diagonal linear transformation of $\boldsymbol{g}$. This observation helps to find diagonal transformations $h_\alpha(t)$ ($t \in k^*$, $\alpha \in \Phi$) belonging to $\mathrm{Aut}\,\boldsymbol{g}$:

$$h_\alpha(t)v = \begin{cases} v & \text{if } v \in \boldsymbol{h}, \\ t^{\langle\beta,\alpha\rangle} v & \text{if } v = X_\beta, \end{cases}$$

where, as before $\langle\alpha,\beta\rangle = 2(\alpha,\beta)/(\beta,\beta)$. In fact, these are all diagonal elements of $\mathrm{GL}(\boldsymbol{g})$ with respect to the Chevalley basis belonging to $\mathrm{Aut}\,\boldsymbol{g}$. The subgroup of $\mathrm{Aut}\,\boldsymbol{g}$ generated by all $h_\alpha(t)$ is denoted by H. Also, by adjustment of scalar multiples from ± 1, for each $w \in W$, there is an automorphism $\widetilde{w} \in \mathrm{Aut}\,\boldsymbol{g}$ given by

$$\widetilde{w}v = \begin{cases} wv & \text{if } v \in \boldsymbol{h}, \\ \pm X_{w\beta} & \text{if } v = X_\beta. \end{cases}$$

The subgroup of $\mathrm{Aut}\,\boldsymbol{g}$ generated by all $\widetilde{w}$ for $w \in W$ and H is readily seen to be the full normaliser of H in $\mathrm{Aut}\,\boldsymbol{g}$. It is denoted by N. Observe that $N/H \cong W$. Thus far, in addition to (W, R), we have found the subgroup N of the Tits system (B, N, W, R) in the group we are about to describe – only B is still lacking.

If we join to $\{X_\alpha\colon \alpha \in \Phi\}$ the L-basis $\alpha_1, \ldots, \alpha_n$, we obtain a basis of $\boldsymbol{g}$ which is known as a *Chevalley basis* of $\boldsymbol{g}$.

5.10. *The Chevalley groups.* Let $\boldsymbol{g}$ be a Lie algebra over a field k. For $x \in \boldsymbol{g}$, define the map $\mathrm{ad}\,x\colon \boldsymbol{g} \to \boldsymbol{g}$ by putting

$$(\mathrm{ad}\,x)v = [x, v] \quad (v \in \boldsymbol{g}).$$

Due to the Jacobi identity (§5.7), $\mathrm{ad}\,x$ is an element of $\mathbf{gl}(\boldsymbol{g})$ which is a derivation of the Lie algebra multiplication, i.e.

$$(\mathrm{ad}\,x)[v, w] = \big[(\mathrm{ad}\,x)v, w\big] + \big[v, (\mathrm{ad}\,x)w\big].$$

Thus ad is a Lie algebra morphism $\boldsymbol{g} \to \mathbf{gl}(\boldsymbol{g})$. An element $x \in \boldsymbol{g}$ is called *nilpotent* if $\mathrm{ad}\,x$ is a nilpotent linear transformation of $\mathbf{gl}(V)$. In this case, the following map is well defined for each $t \in k$:

$$\exp(tx) := \sum_{j=0}^{\infty} (t\,\mathrm{ad}\,x)^j / j!,$$

provided the coefficients of the nonvanishing powers of $t\,\mathrm{ad}\,x$ are defined in k.

Now suppose g is as in §5.9. If $x = X_\alpha$ for some $\alpha \in \Phi$, then no denominators appear in the coefficients of $\operatorname{ad} x$ on the Chevalley basis. Thus, we have transformations

$$x_\alpha(t) = \exp(tX_\alpha) \in \operatorname{Aut} g.$$

Let Y_n denote the type of root system. The *Chevalley group* $\mathsf{Y}_n(k)$ is now defined as the subgroup of $\operatorname{Aut} g$ generated by all $x_\alpha(t)$ for $\alpha \in \Phi$ and $t \in k$. The name of this group refers to the celebrated paper Chevalley [1955] where this construction appeared for the first time. If k is the finite field of order q, we also write $\mathsf{Y}_n(q)$ instead of $\mathsf{Y}_n(\boldsymbol{F}_q)$. For $n \geqslant 2$, the Chevalley groups $\mathsf{B}_n(q)$ and $\mathsf{C}_n(q)$ are isomorphic only if q is a power of 2 or $n = 2$.

Conjugates in $\mathsf{Y}_n(k)$ of the subgroup

$$U_\alpha = \langle x_\alpha(t)\colon\ t \in k\rangle = \{x_\alpha(t)\colon\ t \in k\}$$

are called *root groups*. They are abstractly isomorphic to the additive group k^+ of the field k.

5.11. *Steinberg presentation.* There are very explicit presentations for Chevalley groups by means of generators and relations. Below we give the so-called Steinberg relations determined by the root system Φ (cf. Steinberg [1968a], p. 30, or Carter [1972]). If Φ is the root system of the group $\mathsf{Y}_n(k)$, the group $\widetilde{\mathsf{Y}}_n(k)$ with generators (the formal symbols) $x_\alpha(t)$ (one for each $\alpha \in \Phi$ and $t \in k$) and relations indicated below is a central extension of the Chevalley group $\mathsf{Y}_n(k)$ (cf. Steinberg [1968a]). In fact, it is the universal central extension of $\mathsf{Y}_n(k)$.

Write $[x, y] = xyx^{-1}y^{-1}$ for elements x, y from the same group. Let $\alpha, \beta \in \Phi$, let $\alpha_1, \ldots, \alpha_n$ be a set of fundamental roots of Φ, and denote by c, c_{ij}, t, u elements of the field k. The *Steinberg relations* are of the following form.

$$x_\alpha(t)x_\alpha(u) = x_\alpha(t+u),$$

$$[x_\alpha(t), x_\beta(u)] = \prod_{i,j \in \mathbf{Z}_{\geqslant 0}} x_{i\alpha+j\beta}\big(c_{ij}t^iu^j\big) \quad (\text{for } \alpha + \beta \neq 0),$$

$$\text{where } c_{ij} = 0 \text{ if } i\alpha + j\beta \notin \Phi$$

$$\text{and } c_{11} = N_{\alpha,\beta},$$

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}\big(-t^{-1}\big)x_\alpha(t),$$

$$h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1},$$

$$n_\alpha = w_\alpha(1),$$

$$n_\alpha h_\beta(t) n_\alpha^{-1} = h_{w_\alpha\beta}(t),$$

$$
\begin{aligned}
h_\alpha(t) &= \prod_{i=1}^{n} h_{\alpha_i}\left(t^{a_i(\alpha_i,\alpha_i)/(\alpha,\alpha)}\right) \quad \text{where } \alpha = \sum_{i=1}^{n} a_i\alpha_i,\\
h_\alpha(t)h_\alpha(u) &= h_\alpha(tu),\\
h_\alpha(t)h_\beta(u) &= h_\beta(u)h_\alpha(t),\\
n_\alpha x_\beta(t)n_\alpha^{-1} &= x_{w_\alpha\beta}(ct) \quad \text{where } c = \pm 1,\\
h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} &= x_\beta\left(t^{\langle\beta,\alpha\rangle}u\right).
\end{aligned}
$$

The precise values of the coefficients c and c_{ij} can be determined by computing them for $\mathsf{Y}_n(k)$. The group $\widetilde{\mathsf{Y}}_n(k)$ thus defined acts on all finite-dimensional representation spaces for the Lie algebra g.

Similar, but slightly different presentations have been given in Curtis [1965] and Tits [1981a].

The simple Chevalley groups define connected algebraic group schemes. This can be shown by observing that x_α: $k^+ \to \mathrm{GL}(g)$ and h_α: $\mathrm{GL}(1,k) \to \mathrm{GL}(g)$ are algebraic group morphisms and using the fact that a subgroup of $\mathrm{GL}(g)$ generated by connected algebraic subgroups is again connected algebraic, see Springer [1981], Steinberg [1968a] for details. Apart from very few exceptions (such as $\mathsf{G}_2(k)$ if k has characteristic 2), the Chevalley groups are almost the full automorphism groups of the underlying Lie algebras (see Steinberg [1961]).

Good introductions to the theory of algebraic groups are Borel [1991], Humphreys [1975], Springer [1981].

5.12. *The Lie algebra* $\mathbf{sl}_n$. The Lie algebra of type A_n over the field k is isomorphic with $\mathbf{sl}(k^{n+1})$. We shall give an explicit isomorphism. The root system Φ is usually realised in $\mathbb{R}^{n+1}$, with standard inner product $(\cdot,\cdot)$, as the set of vectors $\varepsilon_i - \varepsilon_j$ for all distinct $i, j \in \{1,\ldots,n+1\}$. The fundamental roots are $\varepsilon_i - \varepsilon_{i+1}$ for $i = 1,\ldots,n$. For $i, j \in \{1,\ldots,n+1\}$, write $E_{i,j}$ for the $(n+1)\times(n+1)$ matrix all of whose entries are zero except for i, j, which entry equals 1. Then the isomorphism $g \to \mathbf{sl}(k^{n+1})$ is given in terms of the following map on basis elements

$$
\begin{aligned}
&\varepsilon_i \mapsto E_{i,i} - E_{i+1,i+1},\\
&X_{\varepsilon_i-\varepsilon_j} \mapsto E_{i,j}.
\end{aligned}
$$

Using the isomorphism, there are easy rules for the coefficients $N_{\alpha,\beta} = N_{ij,kl}$:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}.$$

Fixing a root $\alpha = \varepsilon_i - \varepsilon_j$, computations with α, X_α and $X_{-\alpha}$ can be done inside $\mathbf{sl}(k\varepsilon_i + k\varepsilon_j)$. Also, for other types of Lie algebras, the Lie subalgebra generated by these three elements is isomorphic to $\mathbf{sl}(k^2)$ and it is convenient to analyse the subalgebra in this guise. Thus, taking $i, j = 1, 2$ and $n = 1$, we obtain the following representation of the Steinberg generators as elements of $\mathrm{SL}(k^2) = \widetilde{\mathsf{A}}_1(k)$.

$$x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

$$w_\alpha(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}, \quad h_\alpha(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

This shows the role of the individual elements: $x_\alpha(t)$ and $x_{-\alpha}(t)$ are opposite unipotent elements, $h_\alpha(t) \in H$ is a diagonal element with respect to the standard basis which coincides with the one given in §5.9, and $n_\alpha \in N$. In particular, for any Dynkin diagram Y_n, the subgroup N of $\mathrm{Aut}\, g$ defined in §5.9 is a subgroup of $\mathsf{Y}_n(k)$.

Similar explicit isomorphisms can be written down for the classical groups $\mathsf{B}_n(F)$ (isomorphic to $\mathrm{PSO}(2n+1, F)$), $\mathsf{C}_n(F)$ (isomorphic to $\mathrm{PSp}(2n, F)$), and $\mathsf{D}_n(F)$ (isomorphic to $\mathrm{PSO}(2n, F)$). See Carter [1972], Freudenthal and De Vries [1969], Varadarajan [1984] for various aspects.

For $\alpha = \varepsilon_1 - \varepsilon_{n+1}$ in the root system Φ of type A_n, the root group U_α is the group of all transvections of $P(k^{n+1})$ with centre $k\varepsilon_1$ and axis $k\varepsilon_1 + \cdots + k\varepsilon_n$. This viewpoint extends to examples of type D_n, C_n and B_n, where the root groups consist of polar transvections as defined in §3.3.

The Lie algebras and their corresponding groups can also be explicitly identified in the models given for G_2 in §5.1 (cf. Humphreys [1972]) and for E_6 in §5.2 (cf. Aschbacher [1987], Buekenhout and Cohen [1992], Freudenthal [1959], Mars [1966]).

5.13. *Tits systems of Chevalley groups.* A Tits system of the group $\mathsf{Y}_n(k)$ can be obtained by taking

$$U = \langle U_\alpha\colon\ \alpha \in \Phi^+ \rangle, \qquad H = \langle h_\alpha(t)\colon\ \alpha \in \Phi,\ t \in k^* \rangle,$$

$$B = UH, \qquad N = \langle H n_\alpha\colon\ \alpha \in \Phi \rangle,$$

and (W, R) the Coxeter system underlying the construction of g in §5.9. Similarly, a Tits system is found for $\widetilde{\mathsf{Y}}_n(k)$ instead of $\mathsf{Y}_n(k)$.

By Chapter 11, Section 4.3, the Tits system of $\mathsf{Y}_n(k)$ gives rise to a building of type Y_n on which $\mathsf{Y}_n(k)$ acts as a group of automorphisms. If $j \in I$, we write $\mathsf{Y}_{n,j}(k)$ to denote the shadow space of type $\mathsf{Y}_{n,j}$ arising as the shadow over j of this building.

The overload of notation $\mathsf{A}_{n,j}(k)$ emanating from this definition and §1.7 is justified because, for $D = k$, the two spaces involved are isomorphic. Indeed, $\mathsf{A}_n(k) = \mathrm{PSL}(n+1, k)$ and, with the notation of §5.12,

$$U = \langle I_{n+1} + E_{i,j}\colon\ 1 \leqslant i < j \leqslant n+1 \rangle,$$

so B is the group of upper triangular matrices, i.e. the stabiliser of the maximal flag

$$\{k\varepsilon_1,\ k\varepsilon_1 + k\varepsilon_2,\ \ldots,\ k\varepsilon_1 + k\varepsilon_2 + \cdots + k\varepsilon_n\}.$$

Now the maximal parabolic subgroup of $\mathsf{Y}_n(k)$ containing B and corresponding to node j of the Coxeter diagram is easily seen to be the stabiliser of the subspace $k\varepsilon_1 + \ldots + k\varepsilon_j$.

For $J \subseteq \{1, \ldots, n\}$, put

$$\Phi_J = \Phi \cap \Big(\sum_{j \in J} \mathbb{Z}\alpha_j\Big).$$

Then the parabolic subgroup of $\mathsf{Y}_n(k)$ of type J containing B is a semidirect product $P_J = U_J L_J$ of a unipotent normal subgroup

$$U_J = \langle U_\alpha\colon\ \alpha \in \Phi^+ \setminus \Phi_J \rangle$$

and the *Levi subgroup*

$$L_J = \langle HU_\alpha\colon\ \alpha \in \Phi_J \rangle.$$

Up to a centre (coming from $H \cap L_J$), the Levi subgroup L_J is itself a Chevalley group whose type is the restriction to J of Y_n. The quotient group $U_J/[U_J, U_J]$ can be viewed as an L_J-module. It has remarkably nice properties. For instance, the number of L_J-orbits is finite. See Richardson, Röhrle and Steinberg [1992] for a parametrization of these orbits in case U_J is Abelian.

5.14. *Twisted Chevalley groups.* Let A be a set of automorphisms of the Chevalley group $\mathsf{Y}_n(k)$. Under suitable conditions on A, the subgroup ${}^A\mathsf{Y}_n(k)$ of $\mathsf{Y}_n(k)$ consisting of all elements fixed by A often inherits some of the building structure from $\mathsf{Y}_n(k)$. For instance, if $k = \overline{F}_p$ and A is the Frobenius automorphism of order q on each coordinate with respect to the Chevalley basis of g, one obtains that ${}^A\mathsf{Y}_n(k)$ contains $\mathsf{Y}_n(F_q)$, with a small index due to the difference between $\mathsf{Y}_n(k)$ and $\widetilde{\mathsf{Y}}_n(k)$. To appreciate why equality does not always hold, consider PSL$(2, \overline{F}_p)$ for $p \equiv 3 \bmod 4$. For A the group generated by the automorphism of $\mathsf{A}_1(\overline{F}_p)$ inducing the Frobenius automorphism $x \mapsto x^p$ on each matrix entry, the projective element corresponding to

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{where } i \in F_{p^2} \text{ satisfies } i^2 + 1 = 0,$$

belongs to ${}^A\mathsf{A}_1(\overline{F}_p)$ but not to $\mathsf{A}_1(F_p)$. (The discrepancy vanishes if we replace ${}^A\widetilde{\mathsf{A}}_1$ by ${}^A\mathsf{A}_1$.)

If A has a normal subgroup B, the fixed point subgroup $C_G(A)$ actually is the subgroup $C_D(\overline{A})$ of $D = C_G(B)$ and $\overline{A}$ is the automorphism group induced on D by A. Thus, the study of fixed point subgroups can be reduced to the case where A is simple. In practice, A is almost always cyclic of prime order. (This can be understood for the finite Chevalley groups by noticing that they are almost simple and that the validity of Schreier's conjecture implies that their outer automorphism groups are solvable.)

For a description of the automorphisms of Chevalley groups, see Carter [1972], Steinberg [1968b]. An account of involutory automorphisms is given in Satake [1971], Springer [1987]. They are of importance because of their correspondence with real forms of the algebraic groups (and hence the Lie groups) involved.

Both the finite Chevalley groups and their twisted versions give rise to series of finite simple groups. The finite simple groups arising as twisted Chevalley groups are described in detail in Carter [1972], Steinberg [1968a]. They admit Tits systems that can suitably be described in terms of 'twisted' versions of the root groups U_α. A rough description of the procedure runs as follows. Set $G = \mathsf{Y}_n(k)$, and suppose $A = \langle a \rangle$ for some $a \in \operatorname{Aut} \mathsf{Y}_n(k)$ of prime order. Since the conjugacy class of H consists of subgroups which are unique with the properties of being Abelian, consisting of diagonalizable elements and having $N_G(H) \cong W$, we have that $a = ga'$, where a' is an automorphism normalising H and g represents conjugation by $g \in G$. Now, often, after suitably extending the field k, an element $g' \in G$ can be found so that $ga' = g'a'g'^{-1}$. Thus, replacing a by a G-conjugate, we may assume that a normalises H. Then a interchanges the root groups U_α for $\alpha \in \Phi$. A further reduction of the same kind leads to the case where a leaves invariant the set of fundamental roots, so that a induces an action on diagram Y_n. For ${}^aG$ to have a Tits system of type different from Y_n, the automorphism should not preserve all types, i.e. a should induce a nontrivial permutation on the nodes of Y_n. The nodes of the diagram for ${}^A\mathsf{Y}_n(k)$ are the A-orbits of the Y_n-nodes. The root group corresponding to the node $\overline{\alpha}_i = \{a^j\alpha_i\colon j \in \mathbb{Z}\}$ will be the set of A-fixed elements in $\langle U_\alpha\colon \alpha \in \overline{\alpha}\rangle$.

It may be of interest to note that, for each Dynkin diagram with a multiple bond, the corresponding Weyl group W can be obtained as the subgroup of a Weyl group $\overline{W}$ of a single bonded diagram consisting of all fixed points of a given group A of automorphisms of $\overline{W}$, and similarly for the corresponding Chevalley groups. Indeed, the group $\mathsf{C}_n(k)$ arises as a twisted Chevalley group ${}^\pi\mathsf{A}_{2n-1}(k)$ for a symplectic polarity π of $\boldsymbol{P}(k^{2n})$, viewed as an automorphism of $\mathsf{A}_{2n-1}(k)$, and likewise for B_n, the group $\mathsf{F}_4(k)$ arises from $\mathsf{E}_6(k)$ as the fixed point subgroup of an automorphism related to the map # in §5.2, and $\mathsf{G}_2(k)$ arises from $\mathsf{D}_4(k)$ as the fixed point subgroup of the triality map indicated at the end of §5.1.

For $\mathsf{Y}_n = \mathsf{E}_6$, by taking fixed point subgroups of involutory automorphisms of $\mathsf{E}_6(k)$, we find twisted groups with a Tits system of type F_4, but also of type C_4, cf. Cohen, Liebeck, Saxl and Seitz [1992]. For suitable fields, namely those for which a totally anisotropic quadratic form on k^8 can be found, there is an involution a such that ${}^a\mathsf{E}_6(k)$ is the group of automorphisms of the Cayley projective plane. This automorphism can be extended to an automorphism of $\mathsf{E}_7(k)$ conjugate to the automorphism σ discussed in §5.5, and even further to an involutory automorphism of $\mathsf{E}_8(k)$ whose fixed elements act on a 'Moufang metasymplectic space'. For $k = \mathbb{R}$, this space corresponds to the south-east entry of the 'magic square' of geometries, presented in Freudenthal [1962].

5.15. *Root group spaces.* Let k be a field and set $G = \mathsf{Y}_n(k)$ and retain the notation for the root system and various subgroups of G. The standard parabolic subgroups of G

(i.e. those containing B) are obtained as $N_G(X)$ for certain subgroups X of U which are normal in B. An example of particular interest is the space $S = (P, \mathcal{L})$ with

$$P = \{U_{\alpha_0}^g \colon g \in G\},$$

$$\mathcal{L} = \{\{U \in P \colon U \subseteq U_\beta U_\gamma\}^g \colon \beta, \gamma \in \Phi,\ (\beta,\gamma) = 0,\ g \in G\},$$

where $\alpha_0 \in \Phi$ is the high weight of the adjoint representation of G (i.e. the highest root; see also §5.18 below) and U_α ($\alpha \in \Phi$) are the root groups defined in §5.10. This is the so-called *root group space*. It is discussed in Kantor [1978].

For $\mathsf{Y}_n = \mathsf{A}_n$, the root group space S is isomorphic to the shadow space of type $(\{1, n\}, \{1', n'\})$ (see §4.6 for notation). To see this, first observe that a matrix $X \in \mathbf{gl}(k^{n+1})$ of rank 1 can be written as $pq^\top$ for two nonzero column vectors $p, q \in k^{n+1}$, which gives rise to the point $(kp, \{kx \colon q^\top x = 0\})$ of the direct product space of $\boldsymbol{P}(k^{n+1})$ with its dual. Let S_2 denote the subspace of this direct product space induced on the set incident point-hyperplane pairs. Using that $\mathsf{E}_{1,n+1} = \varepsilon_1 \varepsilon_{n+1}^\top$ corresponds to the point (p, H) of S with $p = k\varepsilon_1$ in $H = \{x \in \boldsymbol{P}(k^{n+1}) \colon \varepsilon_{n+1}^\top x = 0\}$ it is readily seen that the $\mathsf{A}_n(k)$-orbit of $k\mathsf{E}_{1,n+1}$ coincides with the point set of S_2. The high weight of the adjoint representation (in $\mathbf{gl}(k^{n+1})$) is $\alpha_0 = \varepsilon_1 - \varepsilon_{n+1}$, the corresponding eigenvector is $X_{\alpha_0} = \mathsf{E}_{1,n+1}$, and the corresponding root group is $U_{\alpha_0} = I_{n+1} + k\mathsf{E}_{1,n+1}$. As we have seen in §5.13, the root group U_{α_0} determines X_{α_0} uniquely (up to scalar multiples) as the centre in $\boldsymbol{P}(k^{n+1})$ of the transvections it contains. Thus the point set P corresponds bijectively to the $\mathsf{A}_n(k)$-orbit of $\mathsf{E}_{1,n+1}$, which is the point set of S_2. Using that lines of S_2 'are' 2-dimensional subspaces all of whose nonzero members have rank 1, the map from P to the point set of S_2 is readily seen to extend to an isomorphism $S \to S_2$ of spaces.

Let $\alpha, \beta \in W\alpha_0$. Then, the Lie subalgebra generated by X_α and X_β, and the subgroup of $Y_n(k)$ generated by U_α and U_β for various mutual positions of roots $\alpha, \beta \in \Phi$ are as described by the following table.

(α, β)	$\langle X_\alpha, X_\beta \rangle$	$\langle U_\alpha, U_\beta \rangle$	Pair description
2	kX_α	k^+	identical
-2	$kX_\alpha + k\alpha + kX_\beta$	(P) $\mathrm{SL}(k^2)$	opposite
0	$kX_\alpha + kX_\beta$	k^2	polar
1	$kX_\alpha + kX_\beta$	k^2	collinear
-1	$kX_\alpha + kX_\beta + kX_{\alpha+\beta}$	k^{1+2}	special

The Lie subalgebras generated by X_α and X_β are $\mathbf{sl}_2$ for opposite pairs, commutative (i.e. the multiplication is the zero map) for collinear and polar pairs, and the Lie subalgebra of upper triangular matrices with zero entries on the diagonal of $\mathbf{sl}_3(k)$ (an extension of a 1-dimensional central Lie algebra with a commutative 2-dimensional Lie algebra, sometimes called *special*) for special pairs.

The two cases where a commutative Lie subalgebra Q is generated can be distinguished by the subalgebras of Q which are $\mathsf{Y}_n(k)$-conjugate to X_α. In the collinear case, they are all 1-dimensional subspaces of Q, whereas in the polar case there are only two, viz., kX_α and kX_β.

If the Dynkin diagram Y_n has single bonds only, the Weyl group W is transitive on Φ and on each of the sets of pairs given by a line of the above table. Moreover, these are all orbits on $\Phi \times \Phi$. Then, by the Bruhat decomposition

$$G = \bigcup_{w \in W} B\widetilde{w}B$$

(see Chapter 11), the group G is transitive on P and on $\mathcal{L}$. There are at most five orbits of G on $P \times P$. Each orbit contains a representative pair X_α, X_β as in the above table.

5.16. *A generalized octagon.* By way of example of how a space corresponding to a twisted Chevalley group arises, we present the generalized octagon of order $(2, 4)$, a fixed point subspace of the metasymplectic space $\mathsf{F}_{4,1}(2)$ under an involutory automorphism. Of all explicit descriptions of the known generalized octagons (cf. Sarli [1988], Tits [1983]), the one following from a group presentation is the easiest. For the smallest example, we present a construction originating from Tits [1964]. It involves the following presentation of the associated group $F = {}^2\mathsf{F}_4(2)$ of order 35942400. Its commutator subgroup, the so called Tits group, has index 2 in F and is simple. Here, $[x, y]$ is as in §5.11 above.

Generators:

$$u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, r_1, r_8.$$

Relations:

$$u_1^4 = u_3^4 = u_5^4 = u_7^4 = v_1^4 = v_3^4 = v_5^4 = v_7^4 = 1,$$

$$u_2^2 = u_4^2 = u_6^2 = u_8^2 = v_2^2 = v_4^2 = v_6^2 = v_8^2 = 1,$$

$$[u_1, u_2] = [u_1, u_5] = [u_2, u_4] = [u_2, u_6] = 1,$$

$$u_2 = [u_1^{-1}, u_3], \quad u_3^2 = [u_1^{-1}, u_4], \quad u_4u_6 = \left[u_2^{-1}, u_8\right],$$

$$u_3^2u_4u_5^2 = \left[u_1^{-1}, u_6\right], \quad u_2u_3^3u_5 = \left[u_1^{-1}, u_7\right],$$

$$u_2u_3^2u_4u_5^3u_6u_7 = \left[u_1^{-1}, u_8\right],$$

$$r_1 = u_1v_1^2u_1^{-1}, \quad r_8 = u_8v_8u_8^{-1}, \quad (r_1r_8)^8 = 1,$$

$$r_1u_1r_1 = v_1, \quad r_1u_2r_1 = u_8, \quad r_1u_3r_1 = u_7,$$

$$r_1u_4r_1 = u_6, \quad r_1u_5r_1 = u_5, \quad r_1v_2r_1 = v_8,$$

$$r_1v_3r_1 = v_7, \quad r_1v_4r_1 = v_6, \quad r_1v_5r_1 = v_5,$$
$$r_8u_1r_8 = u_7, \quad r_8u_2r_8 = u_6, \quad r_8u_3r_8 = u_5,$$
$$r_8u_4r_8 = u_4, \quad r_8u_8r_8 = v_8, \quad r_8v_1r_8 = v_7,$$
$$r_8v_2r_8 = v_6, \quad r_8v_3r_8 = v_5, \quad r_8v_4r_8 = v_4.$$

The points and lines of the generalized octagon are the involutions that are F-conjugate to u_{2i+1}^2 and u_{2i}, respectively. A point x is incident to a line y if the pair x, y is F-conjugate to u_1^2, u_2; this is equivalent to y lying in $C_F(x)'''$, the third commutator subgroup of the centraliser of x in F. The chain

$$u_1^2, u_2, u_3^2, \ldots, u_8, v_1^2, v_2, \ldots, v_8, u_1^2$$

forms a 16-circuit in the incidence graph – an apartment in the generalized octagon.

5.17. *The finite case.* If a 'thick' shadow space is finite, it has finite diameter and so comes from a spherical building. Thus, for rank at least 3, we know that it is realised by a Tits system. There are numerous questions that can be (and have been) posed (and answered) regarding the numbers of points in a given position. A basic one is of course the determination of the total number of points of the shadow space over j for $G = \widetilde{\mathsf{Y}}_n(k)$ with $k = \mathrm{GF}(q)$. This is $|G/P|$, where P is the corresponding maximal parabolic. For $|G|$ there is a product formula as well as a sum formula. The latter comes from the Bruhat decomposition. Recall from Chapter 11, 5.2.2, that w_0 stands for the longest element in W. The sum formula reads:

$$|G| = \sum_{w \in W} |U_w wUH| = \sum_{w \in W} |U_w| \cdot |U| \cdot |H| = \sum_{w \in W} q^{l(w_0)-l(w)} q^N (q-1)^n,$$

where N is the number of roots $|\Phi|$ and n is the Lie rank of G. The product formula for the universal Chevalley group $\widetilde{\mathsf{Y}}_n(k)$ is

$$|G| = q^N \prod_{i=1}^{n} \left(q^{d_i} - 1\right),$$

where $d_1, \ldots, d_n$ are the degrees (cf. Bourbaki [1968]) of the Weyl group $W(Y_n)$.

For finite fields $k = F_q$, the Chevalley groups $\mathsf{Y}_n(q)$ corresponding to classical groups are listed in §3.14. The degrees can be read off from the exponents i of the factors $q^i - 1$ in the product formula expressing their orders. The table below gives the Chevalley groups with Y_n of exceptional type, together with the degrees.

Untwisted finite Chevalley groups of exceptional type

Y_n	$\lvert Z(\widetilde{\mathsf{Y}}_n(q))\rvert$	Degrees	Number of roots
E_6	$(3, q-1)$	2,5,6,8,9,12	36
E_7	$(2, q-1)$	2,6,8,10,12,14,18	63
E_8	1	2,8,12,14,18,20,24,30	120
F_4	1	2,6,8,12	24
G_2	1	2,6	6

The twisted versions of the finite Chevalley groups that are not classical are listed in the theorem below. The symbol I_2^8 stands for the rank 2 diagram with a bond of valency 8 (for octagon).

Twisted Chevalley groups with Tits systems of rank 1 or exceptional type

Group	Type	Restrictions	Order
$^2B_2(q)$	A_1	$q = 2^{2m+1}$	$q^2(q-1)(q^2+1)$
$^3D_4(q)$	G_2		$q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$
$^2E_6(q)$	F_4		$q^{36}(q^2-1)(q^5+1)(q^6-1)$ $\times(q^8-1)(q^9+1)(q^{12}-1)/(3,q+1)$
$^2F_4(q)$	I_2^8	$q = 2^{2m-1}$	$q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$
$^2G_2(q)$	A_1	$q = 3^{2m-1}$	$q^3(q-1)(q^3+1)$

Here $q = |k|$ is a prime power. Thus, e.g.,

$$|^2F_4(2)| = 2^{12}(2-1)\left(2^3+1\right)\left(2^4-1\right)\left(2^6+1\right) = 35942400$$

as claimed in §5.16. Moreover, the notation $^2E_6(GF(q^2))$ is 'abbreviated' to $^2E_6(q)$, and likewise for $^3D_4$, $^2A_{n-1}$ and ^2D_n (the latter two in §3.14).

As $|U| = q^N$ and $|H| = (q-1)^n$, each formula for $|G|$ can be rewritten so as to provide a formula for the number of points in the flag variety $|G/B|$. Since $P = U_J L_J$ (semidirect product), where L_J and U_J are as in §5.13, induction tells us how to compute $|P| = |U_J| \cdot |L|$, so that $|G/P|$ follows. For instance, if J is the complement of j, then L_J corresponds, up to a central factor, to the Chevalley group over k whose Dynkin diagram is the one obtained from Y_n by removal of the node j. Thus, if its degrees are $e_1, \ldots, e_{n-1}$, we put $e_n = 1$ and obtain

$$|G/P| = \prod_{i=1}^{n} \frac{q^{d_i}-1}{q^{e_i}-1}.$$

Next fix the point P in G/P. In view of the Bruhat decomposition

$$G = \bigcup_{w \in W} BwB = \bigcup_{w \in D_{J,J}} PwP,$$

the number of points in relation w to P for some $w \in D_{J,J}$ is

$$|PwP/P| = \sum_{x \in D_{\varnothing,J} \cap W_J w} q^{l(x)}.$$

Summing over all $w \in D_{J,J}$, this gives an additive formula for $|G/P|$.

Pushing this one step further to information regarding the so-called coherent configuration (cf. Higman [1975]) underlying the shadow space, one might want to know about the parameters

$$p_{uv}^{w} := |\{x \in G/P\colon\ x \text{ is in relation } u \text{ to } y \text{ and relation } v \text{ to } z\}|$$

whenever z and y in relation w are given. By the Bruhat lemma, this number is independent of the particular choice of z and y. In certain cases, such as when w coincides with the j-th fundamental reflection, closed expressions can be given, see Brouwer et al. [1989]. In Brouwer and Cohen [1985] some explicit results are given.

5.18. *Representations of algebraic groups.* Let k be a field, and let G be an algebraic group defined over k. The algebraic representation theory over k of G is highly interesting for geometry. See Jantzen [1987] for a thorough treatment of this theory. For Lie groups, the representation theory is dealt with in a vast number of books, e.g., Freudenthal and De Vries [1969], Varadarajan [1984]. Fix a split Borel subgroup $B = UH$ of G and an algebraic representation $\rho\colon B \to \mathrm{GL}(V)$ of B into a finite dimensional vector space V over k. If ρ is absolutely irreducible, it is a 1-dimensional representation, i.e. a linear character, which is trivial on the unipotent radical U and so comes from a linear character of H. Such a linear character of H is called a *weight*. The weights form a lattice in the reflection representation space $\mathbb{R}\Phi$ of the Weyl group W of G, which can be identified with the weight lattice as defined in §5.8, hence the same terminology. As a basis of the weight lattice, we take the *fundamental weights* ω_j $(j = 1, \ldots, n)$ defined by $\langle \omega_j, \alpha_i \rangle = \delta_{ij}$. A weight λ is called *higher* than another weight μ if $\lambda - \mu$ is a linear combination of the α_j with all coefficients non-negative integers. A weight λ is called *dominant* if it is a linear combination of the ω_j with all coefficients $\geqslant 0$. It is called *antidominant* if $-\lambda$ is dominant (equivalently, if $w_0\lambda$ is dominant).

The representation V_ρ of G induced from ρ can be constructed as follows: take the vector bundle $G \times^B V$ over G/B to be the quotient of the trivial bundle $G \times V$ by the B-action

$$(g, v) \mapsto \big(gb^{-1}, \rho(b)v\big) \quad (g \in G,\ v \in V,\ b \in B),$$

with the map $G \times^B V \to G/B$ obtained from projection onto the first factor. Then V_ρ is the vector space of all sections $G/B \to G \times^B V$ of this bundle.

For k algebraically closed and of characteristic 0, each irreducible representation of G is of the form V_ρ where ρ is an antidominant weight of B.

5.19. *Embeddings of shadow spaces in group modules.* Suppose ρ is a weight of B. Then $G \times^B V$ is a line bundle $\mathcal{L}_\rho$, and a common notation for V_ρ is $H^0(G/B, \mathcal{L}_\rho)$. Each member $\sigma \in V_\rho$ can be viewed as a polynomial map $\sigma\colon\ G \to k$ with

$$\sigma(gb) = \rho(b)^{-1}\sigma(g) \quad \text{for all } g \in G,\ b \in B.$$

For, such a map σ can be identified with the map $G \to G \times k$ given by $g \mapsto (g, \sigma(g))$ which, in turn, determines a unique map $G/B \to G \times^B k$ belonging to $H^0(G/B, \mathcal{L}_\rho)$.

The action of G on $G \times V$ given by $g'(g,v) = (g'g,v)$ $(g', g \in G,\ v \in V)$ commutes with the above defined B-action and so induces an action on $G \times^B V$. In terms of the map σ, the action of G on V_ρ is given by

$$(g \cdot \sigma)(h) = \sigma\left(g^{-1}h\right) \quad (g, h \in G).$$

Let $\widetilde{w}_0$ be an element of N corresponding to w_0 as in §5.9, let ρ be an antidominant weight, and let $\sigma_0 \in V_\rho$ be the map $G \to k$ determined by

$$\sigma_0\left(b_1\widetilde{w}_0b_2\right) = \rho^{w_0}(b_1)^{-1}\rho(b_2)^{-1} \quad (b_1, b_2 \in B)$$

on the (Zariski-)open dense subset $B\widetilde{w}_0B$ of G. Observe that σ_0 is well defined since, for $h \in H$, we have $\sigma_0(h\widetilde{w}_0) = \rho^{w_0}(h)^{-1} = \sigma_0(\widetilde{w}_0h^{w_0})$. Then, for $b \in B$ and $g = b_1\widetilde{w}_0b_2$ with $b_1, b_2 \in B$, we have

$$(b \cdot \sigma_0)(g) = \sigma_0\left(b^{-1}g\right) = \rho^{w_0}(b)\sigma_0(g),$$

so σ_0 is an eigenvector of B with weight ρ^{w_0}. This eigenvector characterises the module V_ρ; it is called the *module with high weight* ρ^{w_0}, because ρ^{w_0} is higher than any other weight of H on this module. Since w_0 interchanges dominant and antidominant weights, we can parametrize the (inequivalent) high weight representations of G by dominant weights λ, by setting $V(\lambda) = V_{\lambda^{w_0}}$.

For a dominant weight λ, let P_λ be the parabolic subgroup $BW_\lambda B$ of G containing B, to which λ can be extended as a linear character. Then P_λ stabilises kv_λ, so the $V(\lambda)$ yields an embedding of G/P_λ into $\boldsymbol{P}(V(\lambda))$ via $gP_\lambda \mapsto k(gv)$ for $g \in G$, where $v \in V(\lambda)$ is a high weight vector. In particular, if λ is a weight which is trivial on all but one fundamental root, say α_j, then P_λ is the standard maximal parabolic subgroup corresponding to the j-th node of the Coxeter diagram: $P_\lambda = P_{j'}$. Take $\lambda = m\omega_j$ (i.e. λ is the extension of the weight $m\omega_j$ to P_λ). We have embedded the point set G/P_λ of the j-shadow space of the building of G in $\boldsymbol{P}(V(\lambda))$. For this to be an embedding of the shadow space itself, we need that the orbit $P_j(kv)$ of the subgroup $P_j = B\langle n_{\alpha_j}\rangle B$ of G on kv span a line. This is equivalent to requiring that the Levi subgroup of P_j isomorphic to (P) SL(2, k) has a 2-dimensional representation, which in turns comes down to $m = 1$, i.e. $\lambda = \omega_j$. Thus, the j-shadow space of the building for G embeds in the projective space of $V(\lambda_j)$, the *j-th fundamental representation* of G.

This construction explains the general phenomenon of which we have already seen several instances. For example, if we take G of type E_6 and $\lambda = \omega_1$, we obtain the 27-dimensional module for the group of type E_6 which leads to $\boldsymbol{K}$ as in §5.2. If we take G of type E_7 and $\lambda = \omega_7$, we obtain the 56-dimensional module of §5.4. See also Ronan and Smith [1985, 1986].

There exist embeddings of shadow spaces in projective spaces that do not come from these group-theoretic embeddings. A clear example is the generalized octagon of order (2,4) given in §5.16, which (as already mentioned in §2.16) has an embedding in the projective space of rank 79 over $\boldsymbol{F}_2$, whereas the representation theory gives an embedding in the projective space of the high weight module $V(\omega_1)$ for $\mathsf{F}_4(2)$, which is of rank 25.

For $G = \mathrm{SL}(n,\mathbb{C})$ the representation $V(\lambda)$ can be obtained from the standard n-dimensional module by use of the symmetrised power of the standard representation V with respect to the partition

$$(\lambda_1 + \cdots + \lambda_{n-1}, \lambda_2 + \cdots + \lambda_{n-1}, \ldots, \lambda_{n-1})$$

of $\sum_i i\lambda_i$. Thus, e.g., $V(\omega_3)$ and $V(3\omega_1)$ are the third exterior and the third symmetric power of V, respectively.

5.20. *Algebraic sets.* Suppose k is an algebraically closed field. Let j be a fundamental node of the Dynkin diagram of a Chevalley group $G = \mathsf{Y}_n(k)$. As we have seen, the j-shadow space of the building defined by means of the Tits system of G can be embedded in $\boldsymbol{P} = \boldsymbol{P}(V(\omega_j))$. Viewing G as the k-points of an algebraic group, we know that G/P is a projective subvariety of $\boldsymbol{P}$. In fact, the point set of this space is the projective zero set of a family of homogeneous quadratic polynomials, as we shall see.

Fix a Borel subgroup B of G, a decomposition $B = UH$ with U unipotent and normal in B and H an Abelian complement consisting of semisimple elements. Suppose λ is a dominant weight of H with respect to B. Choose a high weight vector v_λ (unique up to scalar multiples) in $V(\lambda)$. The stabiliser of the projective point $\langle v_\lambda \rangle$ is the parabolic subgroup $P = BW_JB$ containing B of type J, where J is a subset of the diagram nodes for which $W_J = W_\lambda$ (= the stabiliser in W of the weight λ). As an H-module, $V(\lambda)$ has a basis of eigenvectors. We shall call such a basis of V an *H-frame.*

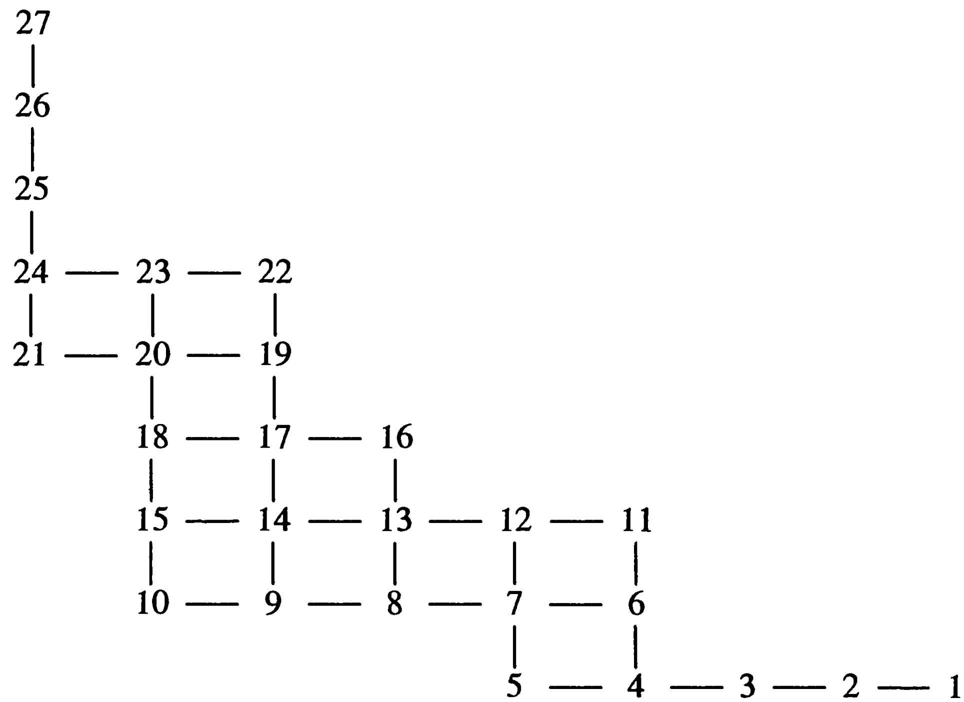
For any dominant weight λ, the weight $\lambda^* = \lambda^{-w_0}$ is the dominant weight corresponding to the dual G-representation of $V(\lambda)$. Observe that λ is a fundamental weight if and only if λ^* is. By $S^2V(\lambda)$ we denote the vector space of all homogeneous polynomials of degree 2 on $V(\lambda)$.

5.21. THEOREM (cf. Brion [1985], Lichtenstein [1982]). *Let k be an algebraically closed field of characteristic* 0. *Suppose G is a simple split algebraic group over k with Tits system (B, N, W, R), and set $H = B \cap N$. Let λ be a dominant weight, and let $P = BW_\lambda B$ be the corresponding parabolic subgroup of G. Then the G-module $S^2V(\lambda)$ contains the highest weight module $V(2\lambda^*)$ with multiplicity* 1 *and has a G-invariant complement M. The ideal I in $k[V(\lambda)]$ of the highest weight orbit $G(kv_\lambda) \cong G/P$ of G in $\boldsymbol{P}(V(\lambda))$ is generated by the polynomial quadratic maps of an H-frame of M.*

If there exists an H-frame in $V(\lambda)$ of the form $(v_\mu)_{\mu \in W_\lambda}$, then λ is called *minuscule.* In this case λ is a fundamental weight ω_j and $\mathsf{Y}_{n,j}$ is one of

$$\mathsf{A}_{n,j},\ \mathsf{B}_{n,n},\ \mathsf{C}_{n,1},\ \mathsf{D}_{n,1},\ \mathsf{D}_{n,n-1},\ \mathsf{D}_{n,n},\ \mathsf{E}_{6,1},\ \mathsf{E}_{6,6},\ \mathsf{E}_{7,7}.$$

(cf. Hiller [1982]). For minuscule weights λ, the set of generators for the ideal I of the theorem can be chosen so that I, viewed as a linear subspace of the vector space $k[V(\lambda)]$ of all polynomials on $V(\lambda)$, has a complement spanned by all monomials whose index sets, when viewed as labels from W/W_λ, form a chain in the Bruhat order induced on W/W_λ. These monomials are called *standard monomials.* For instance, for $\mathsf{E}_{6,1}$, with the numbering of $W/W_{1'}$ from §5.2 (where $1' = \{2,3,4,5,6\}$), the Hasse diagram of the Bruhat poset is depicted below:



A node in this diagram represents a coset higher (in the Bruhat order) than any coset represented by a node that can be reached by a south/east bound path. The indexing of the 27 polynomials of §5.2 is chosen so as to refine the Bruhat ordering of $W/W_{1'}$. There are 27 incomparable pairs in the Bruhat poset $W/W_{1'}$, each occurring as indices of a monomial in a unique quadratic equation in §5.2. Using these 27 equations, each monomial in which an incomparable pair of indices occurs can be rewritten mod I as a linear combinations of standard monomials.

This then leads to a solution of the word problem for the quotient ring $k[V(\lambda)]/I$. The result is due to Seshadri [1978], and led to several generalizations and more specific details. The topic discussed here is known as *standard monomial theory*. See Cohen and Cushman [1993] for a more detailed description and a way to implement computations in this quotient ring.

In the case of A_n in its representation $V(\varepsilon_j)$, the variables from an H-frame of $V(\varepsilon_j)$ are the *Plücker coordinates*, and the special generators of I are the so-called *Plücker relations*. The procedure which, for a given polynomial in $k[V(\varepsilon_j)]$, finds the unique linear combination of standard monomials to which it is equal mod I is known as *straightening*.

5.22. *Lines of embedded shadow spaces.* According to §5.19 the space $S = Y_{n,j}(k)$ embeds in $\boldsymbol{P}(V(\omega_j))$. We have seen that the point set can be defined as the zero set of certain (explicitly computable) polynomials. But what about the lines of S? The set of lines of S does not always coincide with the set of lines of $\boldsymbol{P}$ that are entirely on the variety. In most cases, including the Grassmannians, it does. In the case $\mathsf{Y}_{n,j} = \mathsf{G}_{2,1}$, for instance, the set of all lines on the variety gives the quadric $\mathsf{B}_{3,1}(k)$ rather than $\mathsf{G}_{2,1}(k)$. Tits has shown how the lines of $\mathsf{G}_{2,1}$ can be defined algebraically by introducing polynomial maps $P^7 \to P^7$ (coming from triality) whose 'absolute' lines are those of S.

5.23. *Forms of higher degree.* One may also wonder why the degree 2 of forms is so special. Part of the explanation comes from dimension arguments. Let V be a vector space over an algebraically closed field k. If $\dim V = n$, then $\dim \mathrm{GL}(V) = n^2$ and the vector space $S^d(V)$ of homogeneous forms of degree d has dimension $\binom{n+d-1}{d}$. If the latter dimension is strictly less than n^2, then the stabiliser $\mathrm{GL}(V)_f$ in $\mathrm{GL}(V)$ of any form f has strictly positive dimension. This happens if $d = 2$, in which case the forms give rise to polar spaces and classical groups.

On the other hand, if $d > 2$ and d is large enough, then $n^2 < \dim S^d(V)$, so no $\mathrm{GL}(V)$-orbit can be dense in $S^d(V)$; hence there are infinitely many inequivalent homogeneous forms of degree d.

A further explanation may be given by a theorem of Jordan [1880] that has been generalized and rediscovered several times (Borovik [1989a], Orlik and Solomon [1978], Sah [1974], Schneider [1973]), and whose principal statement is that, if f is a homogeneous polynomial of degree at least 3 on a vector space V, then the singular locus of the zero set of f in V is nonempty, or the stabiliser $\mathrm{GL}(V)_f$ of f in $\mathrm{GL}(V)$ is a finite group. For example, if f is such a form, and if $\mathrm{GL}(V)_f$ has strictly positive dimension, then its Lie algebra will be a set of derivations annihilating f in the sense that for g in the Lie algebra we have

$$\big((\mathrm{ad}\, g)f\big)(x) = (\mathrm{grad}_x\, f, gx) = 0 \quad \text{for all } x \in V,$$

where $\mathrm{ad}\, g$ is as in §5.7. This is a set of linear equations in $g \in \mathbf{gl}(V)$ and so can be solved. For example, if $V = k^n$ and $d > 2$, then

$$f = x_1^d + x_2^d + \cdots + x_n^d$$

leads to

$$\mathrm{grad}_x\, f = d\big(x_1^{d-1}, \ldots, x_n^{d-1}\big)$$

so

$$(\mathrm{grad}_x\, f, gx) = d \sum_{i,j} g_{ij} x_i x_j^{d-1}$$

is identically equal to zero only if $g = 0$. Thus, for k of characteristic 0, the connected component of $\mathrm{GL}(V)_f$ is trivial, and so $\mathrm{GL}(V)_f$ is finite. But, for $d = 2$, the form f is quadratic and the solutions g are the elements of the Lie algebra of the orthogonal group $\mathrm{GL}(V)_f$.

6. Characterisations

The general pattern of the results of this section runs as follows. Let $I = \{1, \ldots, n\}$, $j \in I$ and let Y_n be a spherical diagram. We ask which axioms for spaces characterise the shadow spaces of type $\mathsf{Y}_{n,j}$ among all spaces. A starting point for most of the characterisations to be discussed is the parapolar space property. Using it and adding axioms so as to enable us to construct more shadows, we invoke the construction of §4.11 to arrive at a geometry. Then Tits [1981b] is used to conclude the geometry is a building of type Y_n. Methods such as those discussed in the previous section enable us to find axioms in abundance.

It should be noted that the pattern just described indicates how to go back and forth between two abstract settings (spaces and buildings) but does not provide a classification of these structures. The harder part of the classification of buildings, however, is the classification of buildings of type B_n and D_n, and is fully covered by the classification of polar spaces as discussed in §4. The classification of buildings of type A_n, of course, reduces to that of spaces of type $\mathsf{A}_{n,1}$, the projective spaces; see, e.g., Artin [1940].

Rank 2 buildings are generalized polygons, discussed in Chapter 9. Let us pass to higher rank. A general way of obtaining geodetically closed 'polar' subspaces of partial linear gamma spaces has originated from the work Cooperstein [1977], see also Buekenhout [1983], Cohen [1982b].

6.1. THEOREM. *Suppose S is a gamma space with the property that, for each pair x, y of points at distance 2, the subspace $\{x, y\}^\perp$ is either a singleton or a nondegenerate polar space of rank at least 2. If each line is contained in a quadrangle, then S is a parapolar space whose symplecta are the geodesic closures of $\{x, y\} \cup \{x, y\}^\perp$ for x, y points of S at distance 2 such that $|\{x, y\}^\perp| > 1$.*

If, in what follows, the symplecta of a parapolar space are not mentioned explicitly, they are understood to be as described in this theorem.

Since a parapolar space S is a gamma space, there is a natural notion of the local space $x^\perp/x$ at a point x. In terms of incidence systems (cf. §4.8), its points are the lines on x, and its lines are the planes on x. Its collinearity graph corresponds to $x^\perp \setminus \{x\}/\equiv$ for $\equiv$ as in §1.5 (compare also with $\rho(x^\perp)$ for ρ as in §2.4).

6.2. PROPERTIES OF PARAPOLAR SPACES. *Parapolar spaces are paraprojective. If x is a point and Q, R are two distinct symplecta of a parapolar space with $x \notin Q$, then $x^\perp \cap Q$ and $Q \cap R$ are singular subspaces. If its polar rank is at least 3, a parapolar space is locally strong parapolar.*

A typical axiom of the kind that will be used is the following.

(A) *For $x \in P$ and $\ell \in \mathcal{L}$ such that $x^\perp \cap \ell = \varnothing$ but $x^\perp \cap \ell^\perp \neq \varnothing$, we have $x^\perp \cap \ell^\perp \in \mathcal{L}$.*

It will be applied to recognise Grassmannians, the shadow spaces of type $\mathsf{A}_{n,j}$, cf. §1.7.

6.3. GRASSMANNIAN CHARACTERISATION (Cooperstein [1977], Cohen [1983]). *Let S be a strongly parapolar space of polar rank 2 and of finite singular rank, whose lines are thick. If S satisfies* (A), *then one of the following holds for j the minimal singular rank of a maximal singular subspace of S.*

(i) *S is a nondegenerate polar space of rank 3 and $j = 2$.*

(ii) *There are $n \geqslant 4$ and a division ring $\boldsymbol{D}$ such that $S \cong A_{n,j}(\boldsymbol{D})$; moreover, $j \leqslant (n+1)/2$.*

(iii) *There are an infinite division ring $\boldsymbol{D}$, and an involutory automorphism a of $A_{2j-1,j}(\boldsymbol{D})$ such that $S \cong A_{n,j}(\boldsymbol{D})/\langle a \rangle$. Moreover, $j \leqslant 5$. The automorphism a is induced by a polarity of the underlying projective space $A_{2j-1,1}(\boldsymbol{D})$ and satisfies $d(x, x^a) \geqslant 5$ for all points x of $A_{2j-1,j}(\boldsymbol{D})$.*

A general technique here is to proceed by induction on j. A major task is to select the collection $\mathcal{M}$ of maximal singular subspaces corresponding to the node $j-1$ in the Coxeter diagram of the building of the projective space sought for. Once found, this collection $\mathcal{M}$ will serve to recognise a readily defined graph with vertex set $\mathcal{M}$ as the collinearity graph of a space of type $\mathsf{A}_{n,j-1}$.

Many more characterisations of Grassmannians are known. We shall only discuss a few that involve the partition of maximal singular subspaces of the Grassmannian into those corresponding to node $j-1$ and those corresponding to node $j+1$. Shult [1983], Proposition 6.1, and Bichara and Tallini [1983], Tallini [1981] have exploited these in their characterisations.

6.4. THEOREM. *Suppose S is a gamma space in which each point is on at least one line. Assume that there are two classes $\mathcal{M}$, $\mathcal{M}'$ of maximal singular subspaces of S with the properties*

(i) *if $M \in \mathcal{M}$ and $M' \in \mathcal{M}'$ then $M \cap M'$ is either empty or a line;*

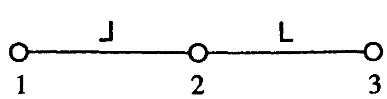
(ii) *each line is on a unique member of each class;*

(iii) *if $K_1, K_2 \in \mathcal{M}$ and $K_1', K_2' \in \mathcal{M}'$ satisfy $K_i \cap K_{i'}' \neq \varnothing$ for all $i, i' \in \{1, 2\}$, then $K_1 \cap K_2 \neq \varnothing$;*

(iv) *there exist $j \geqslant 2$ and a member of $\mathcal{M}'$ that has singular rank j.*

Then S is a space of type $\mathsf{A}_{n,j}$ for $n \in \mathbb{N} \cup \{\infty\}$, $n > j$.

This theorem is quite close to an interpretation of a Grassmannian as a truncated geometry of type



Sprague's theorem in Sprague [1981] takes this way of looking at Grassmannians as a starting point. The relation with the above theorem becomes clear if we let the objects of type 1, 2, 3 be the members of $\mathcal{M}$, P, $\mathcal{M}'$, respectively.

6.5. *Metasymplectic spaces.* Shadow spaces of type $\mathsf{F}_{4,1}$ are called *metasymplectic spaces.* Such a space satisfies the following axiom with $J = \{-1, 1\}$.

(F)$_J$ For each symplecton Y and each point $x \in \mathcal{P}(S) \setminus Y$, the rank of $x^{\perp} \cap Y$ is a member of J.

The proof is as sketched in §4.10.

A *pentagon* is a set $x_0, \ldots, x_4$ of five distinct points such that x_i is collinear with x_{i+1} but not with x_{i+2} for each i (indices mod 5).

(P) Each pentagon contains a point x such that $x^{\perp} \cap \ell \neq \varnothing$, where ℓ is the line spanned by the two points of the pentagon noncollinear with x.

We give two characterisations, valid for arbitrary metasymplectic spaces. See Cohen and Cooperstein [1989] for another characterisation, valid for finite ones only.

6.6. THEOREM (Cohen [1982b]). *Let S be a parapolar space of polar rank k ($k \geqslant 3$). Suppose S is a parapolar space in which property* (P) *holds and in which, for each point-symplecton pair x, A the intersection $x^{\perp} \cap A$ is either empty or a line (i.e.* (F)$_{\{-1,1\}}$ *holds). Then either S is a polar space of rank k or $k = 3$ and S is a metasymplectic space.*

6.7. THEOREM (Shult [1989]). *Let S be a parapolar space of polar rank at least* 3. *If property* (P) *holds in $x \cap x^{\perp}$ for each point x of S and* (F)$_{\{-1,1\}}$ *holds in S, then S is one of the following spaces:*

(i) *a nondegenerate polar space of rank at least* 3;

(ii) *a space of type $\mathsf{B}_{n,2}$ ($n \geqslant 3$) (a polar Grassmannian)*;

(iii) *a space of type $\mathsf{F}_{4,1}$ (a metasymplectic space); or*

(iv) *a Grassmann space $A_{n,2}(D)$ for some division ring D.*

Dual polar spaces have been discussed in §3.27–28. A space of type $\mathsf{D}_{n,n}$ is a half-dual polar space. They are also known as half-spinor spaces or half-spin geometries.

6.8. HALF DUAL POLAR SPACE CHARACTERISATION (Shult [1991]). *Let S be a parapolar space with thick lines having the property that, for each maximal singular subspace M and all points x, y not incident to M:*

– *if $y^{\perp} \cap M \subset x^{\perp} \cap M$ (proper inclusion), then $y^{\perp} \cap M = \varnothing$;*

– *the set $x^{\perp} \cap M$ is not a singleton.*

Then S is one of

(i) *a space of type $\mathsf{B}_{n,1}$ for $n \geqslant 3$;*

(ii) *a space of type $\mathsf{A}_{n,d}$ with $2d < n$;*

(iii) *the quotient of a space of type $\mathsf{A}_{2d+1,d}$ by a group of automorphisms generated by a polarity of Witt index at most ($d - 5$); or*

(iv) *a half-dual polar space $\mathsf{D}_{n,n}$ for $n \geqslant 4$.*

Its proof is based on a reduction to Theorem 1 of Cohen and Cooperstein [1983]. The result improves Cooperstein [1983] and extends §6.3. Here, as indicated in §4.16, not all objects of the building 'under recognition' have been recovered as subspaces of S. The techniques used are described in Brouwer and Cohen [1986], Ellard and Shult [1988], Ronan [1986].

6.9. THEOREM (Hanssens and Thas [1987]). *Suppose S is a finite parapolar space of polar rank 3 with thick lines such that*

- *if H and K are symplecta containing a line ℓ, then either both are of type D_3 and there exist planes $\alpha \subset H$ and $\beta \subset K$ containing ℓ with $\alpha \perp \beta$, or the intersection $H \cap K$ contains a plane;*
- *the graph induced on $x^\perp \setminus \{x\}$ is connected of diameter at most 3 for each point x;*
- *if each line is contained in exactly three maximal singular spaces of rank i, j, k, respectively, with $i \geqslant 2$ and $j, k \geqslant 3$, then $i = 2$, $j = 3$ and $k \in \{4, 5\}$, up to an interchange of j and k.*

Then the type of S is one of $\mathsf{A}_{n,j}$ ($n \geqslant 3$, $2 \leqslant j < n$), $\mathsf{B}_{n,n-2}$ ($n \geqslant 3$), $\mathsf{D}_{n,n-2}$ ($n \geqslant 4$), $\mathsf{E}_{7,4}$, $\mathsf{E}_{8,4}$, $\mathsf{F}_{4,1}$.

6.10. THEOREM (Hanssens [1986]). *Let S be a parapolar space of polar rank $r \geqslant 3$, all of whose lines have at least 3 points, such that*

- *the graph induced on $x^\perp \setminus \{x\}$ is connected of diameter at most 3 for each point x;*
- *for any two symplecta H and K of polar rank r the rank of $H \cap K$ is different from $r - 1$.*

If S has finite singular rank, then it is a space of type $\mathsf{A}_{n,2}$ ($n \geqslant 3$), $\mathsf{B}_{r,1}$, $\mathsf{D}_{r,1}$, $\mathsf{D}_{n,n}$ (or a quotient of the latter; $n \geqslant 4$), $\mathsf{E}_{n,1}$ ($n \geqslant 6$, extending the E series beyond $\mathsf{E}_{8,1}$ to shadow spaces of nonspherical buildings), $\mathsf{E}_{7,7}$, $\mathsf{E}_{8,8}$, $\mathsf{F}_{4,1}$.

If S is the root group space of a Chevalley group with a Dynkin diagram with simple bonds only (cf. §5.15), then it is a parapolar space of polar rank $k + 1$ (for some $k \in \mathbb{N}$) satisfying $(\mathrm{F})_{0,k}$.

6.11. EXCEPTIONAL ROOT GROUP SPACE CHARACTERISATION (see Cohen and Cooperstein [1983]). *Suppose S is a parapolar space of polar rank $k \geqslant 3$, all of whose lines are thick and of finite singular rank, satisfying $(\mathrm{F})_{0,k-1}$. Then S is of type $\mathsf{B}_{k,1}$, $\mathsf{D}_{5,5}$, $\mathsf{E}_{6,1}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, or $\mathsf{E}_{8,8}$.*

The proof uses the fact that point residues $x^\perp/x$ for $x \in \mathcal{P}(S)$ are strong parapolar spaces of rank $k - 1$ satisfying $(\mathrm{F})_{-1,k-2}$, for which a separate characterisation theorem is supplied. This implies that the point residues are spaces of type $\mathsf{B}_{k-1,1}$, $\mathsf{A}_{n,d}$, $\mathsf{D}_{n,n}$, $\mathsf{E}_{6,1}$, $\mathsf{E}_{7,7}$, or quotients thereof by groups mapping points to points at distance at least 5 (see also Brouwer and Cohen [1983]). This result, in turn, allows for a construction of subspaces which play the role of shadows of objects of the building under reconstruction that have not become apparent yet. For instance, in the course of recognising $\mathsf{E}_{7,1}$, when $k = 5$ and the singular rank is 6, the role of objects in the building of types 1, 3, 2, 6 are played by subspaces which are singletons, lines, maximal subspaces of rank 6, and symplecta, respectively, and the objects of types 4, 5 are also easily recovered. The problem, however, is the determination of objects of type 7, which, ideally, correspond to subspaces of type $\mathsf{E}_{6,1}$. A construction of such subspaces is then based on the fact that the point residues are of type $\mathsf{D}_{5,5}$ and so contain subspaces of type $\mathsf{D}_{4,4}$. A subspace of S of type $\mathsf{E}_{6,1}$ then arises by patching suitable choices of points $x \in \mathcal{P}(S)$ and subspaces Y_x of $x^\perp$ containing x with Y_x/x of type $\mathsf{D}_{4,4}$.

Another difficulty arises in the course of recognising $\mathsf{E}_{6,2}$, where we are in the case $k = 3$ and singular rank 4, while the point residue is of type $\mathsf{A}_{5,3}$. Rather than constructing new types of objects, existing classes, such as the one of all symplecta, have to be separated into two classes, such as the ones of objects of types 1 and 6, respectively. This is done by setting up a suitable equivalence relation.

6.12. *Further characterisations.* The Grassmannians of polar spaces have also been characterised in terms of parapolar spaces S whose local structure $x^\perp/x$ (cf. §6.1) for a point x is the direct product of spaces of type $\mathsf{A}_{m,1}$ and $\mathsf{B}_{n,1}$ ($m \geqslant 1$ and $n \geqslant 3$). In Hanssens [1987], globally defined subspaces of type $\mathsf{A}_{n,2}$ are constructed; from this Hanssens derived that S is a suitable quotient of a space of type $\mathsf{B}_{m+n+1,m+1}$. See Ellard and Shult [1988] for an extension of this technique.

Another characterisation of shadow spaces encompassing quite a few types is given in Hanssens [1988]. To summarise, we present a table of characterisations of spaces of spherical type $\mathsf{Y}_{n,k}$ for rank $n \geqslant 3$.

Very little has been done with regard to the shadow spaces of affine type. We mention Shult [1989] for $\widetilde{\mathsf{F}}_4$ (the diagram has rank 5 and maximal singular subspaces have rank 2), Cohen [1986] for $\widetilde{\mathsf{E}}_8$, and Hanssens and Van Maldeghem [1991] for $\widetilde{\mathsf{B}}_2$.

6.13. *Local recognition.* Let Δ be a graph. A graph Γ is said to be *locally* Δ if, for each vertex γ of Γ, the subgraph $\gamma^\perp \setminus \{\gamma\}$ is isomorphic to Δ. More generally, when $\mathcal{X}$ is a class of graphs, we say that Γ is *locally* $\mathcal{X}$ if, for each vertex γ of Γ, the subgraph $\gamma^\perp \setminus \{\gamma\}$ is isomorphic to a member of $\mathcal{X}$. In this vein, by saying that a graph is *locally a grid* (a grid being a graph that can be written as a product of two cliques), we mean that it is locally $\mathcal{X}$, where $\mathcal{X}$ stands for the class of all grids. An intriguing problem of merely graph-theoretic nature is: given an 'interesting' class $\mathcal{X}$, determine all graphs which are locally $\mathcal{X}$ (cf. Blass, Harary and Miller [1980], Buekenhout and Hubaut [1977], Hall [1980], Hall and Shult [1985], Higman [1983]). The theory of buildings also led to local characterisation questions for spaces, which can be reduced to local recognition problems for graphs.

In Cohen [1986, 1990], characterisations of shadow spaces of spherical buildings are interpreted as providing information on the minimal $h \in \mathrm{N}$ such that the shadow space is uniquely determined by the subgraphs induced on the sphere of points at distance at most h to a given point. For polar spaces, we have $h = 1$ due to the following consequence of a result in Johnson and Shult [1989] (cf. Buekenhout and Hubaut [1977]).

Type	Kind of space	Reference
$A_{n,1}$	projective	Veblen and Young [1918]
$A_{n,k}$	Grassmann	Cohen [1983], Cooperstein [1977]
$B_{n,1}$	polar	Tits [1974], Veldkamp [1959], Buekenhout and Shult [1974], Johnson and Shult [1989]
$B_{n,n}$	dual polar	Cameron [1982], Shult and Yanushka [1980], Brouwer and Wilbrink [1983], Brouwer and Cohen [1986]
$B_{n,k}$ $(k < n-1)$	polar Grassmann	Hanssens [1987], Ellard and Shult [1988]
$B_{n,n-2}$		Hanssens and Thas [1987]
$D_{n,n-2}$		Hanssens and Thas [1987]
$D_{n,n}$	half dual polar	Cooperstein [1983], Shult [1991]
$E_{6,1}$	Schläfli/Dickson	Cohen and Cooperstein [1983]
$E_{6,2}$	root group	Cohen and Cooperstein [1983],
$E_{7,4}$		Hanssens and Thas [1987]
$E_{7,7}$	root group	Cohen and Cooperstein [1983]
$E_{7,1}$	Gosset/Mars	Cohen and Cooperstein [1983]
$E_{8,1}$	root group	Cohen and Cooperstein [1983],
$E_{8,4}$		Hanssens and Thas [1987]
$F_{4,1}$	metasymplectic	Cohen [1982b], Shult [1989] Cohen and Cooperstein [1989]

6.14. THEOREM. *Suppose S is a partial linear gamma space all of whose subspaces $x^\perp$ $(x \in \mathcal{P}(S))$ are polar spaces. If S contains a triangle, then it is a polar space.*

Part of the explanation why such a local characterisation is at all possible, lies in the fact that the girth of the collinearity graph of a polar space is small. It should be contrasted with the following result.

6.15. THEOREM (Weetman [1993]). *Let H be a regular graph of valency at least 2 and girth at least 6. Then there is an infinite graph G which is locally H.*

For H as in the hypothesis, by Ronan [1981], any graph which is locally H has a universal covering. Often, there are many possible graphs G for a fixed H. The construction given by Weetman [1993] yields a graph that is its own universal cover. For example, if H is a hexagon, the universal covering G is obtained from a tessellation of the plane by hexagons.

In general, for a finite graph H, the problem of deciding whether a finite graph which is locally H exists or not is (recursively) undecidable; cf. Bulitko [1973], Winkler [1983].

6.16. AFFINE HALF DUAL POLAR SPACES (Munemasa and Shpectorov [1992]). *Suppose $q > 2$ is a prime power, and $n \geqslant 5$. Suppose Γ is a graph with the property that, for each vertex x, the graph $\Gamma(x)$ induced on the neighbours of x has reduced graph $\Gamma(x)/\equiv$ (cf. §1.5) isomorphic to the collinearity graph of $\mathsf{A}_{n,2}(q)$ and reduction fibres of size $q-1$. If, for any two vertices at distance 2 in Γ, the set of common neighbours has size $q^2(q^2+1)$, then Γ is a quotient of the alternating forms graph* $\mathrm{Alt}(n,q)$.

Here $\mathrm{Alt}(n,q)$ stands for the graph whose vertices are the alternating forms on F_q^n and in which two forms are adjacent whenever their difference is a form of rank 2, see Brouwer et al. [1989], §9.5. For $q = 2$, a characterisation is given under somewhat stronger conditions; the graphs appearing in the conclusion of the characterisation are quotients of $\mathrm{Alt}(n,2)$ or the quadratic forms graphs $\mathrm{Quad}(n-1,2)$ (cf. *loc. cit.*, §9.6).

The graph $\mathrm{Alt}(n,q)$ can be obtained by removal of a vertex and all points at non-maximal distance to it from a half dual polar space $\mathsf{D}_{2n,2n}(q)$. By removing other (kinds of) hyperplanes from $\mathsf{D}_{2n,2n}(q)$, other examples of graphs Γ emerge that satisfy all conditions of the theorem but for the last cardinality requirement (regarding the common neighbour set). As observed by Buekenhout, yet other examples can be constructed by embedding $\mathsf{A}_{n,2}(q)$ in a hyperplane H of a projective space $\boldsymbol{P}$, taking for point set the affine set $\mathcal{P}(\boldsymbol{P}) \setminus H$, and taking for lines the affine lines whose projective counterparts meet H in a point of the embedded $\mathsf{A}_{n,2}(q)$.

7. Related structures

In this section, we discuss hyperplanes, affine structures, some relations with group theory and nonparabolic geometry.

7.1. *Hyperplanes and embeddings.* Let S be a shadow space of the building related to $\mathsf{Y}_n(k)$. Suppose S can be embedded in the projective space $\boldsymbol{P}$ such that its point set spans $\boldsymbol{P}$. Suppose also that $\mathsf{Y}_n(k)$ acts on $\boldsymbol{P}$ as a group of linear transformations preserving S. Then every hyperplane of $\boldsymbol{P}$ gives a hyperplane of S. Thus, a description of all $\mathsf{Y}_n(k)$-orbits of hyperplanes may be related, in complexity, to a description of all orbits of $\mathsf{Y}_n(k)$ on the dual of $\boldsymbol{P}$. Thus, a description of all hyperplanes within the framework of S may be too much to ask for. A better description of the hyperplanes would be in terms of the hyperplanes of $\boldsymbol{P}$. In this vein, for the following shadow spaces S with embedding in a projective space $\boldsymbol{P}$, it has been established that all hyperplanes of S arise from intersections of hyperplanes of $\boldsymbol{P}$ with $\mathcal{P}(S)$. Here, the rank n is assumed to be at least 3.

Space	Embedded as	Reference
$\mathsf{A}_{n,1}$	$\mathsf{A}_{n,1}$	Veblen and Young [1918]
$\mathsf{B}_{n,1}$	quasiquadric	Tits [1974]
$\mathsf{A}_{n,k}$	Grassmannian	Hall and Shult [1993], Cooperstein and Shult [1991]
$\mathsf{D}_{n,n}$	half-spin	Shult [1992a]
$\mathsf{B}_{n,n}(q)$	spin (q odd)	Shult and Thas [1991]
$\mathsf{E}_{6,1}$	cf. §5.2	Cooperstein and Shult [1991]

For other shadow spaces, the hyperplane determination is an open problem.

7.2. *Affine spaces.* Consider the algebraic structure of a vector space V over a division ring F. Letting P be the set V of vectors, and $\mathcal{L}$ the set of (*affine*) *lines*, so that a typical line has the form $\{u + \lambda v\colon\ \lambda \in F\}$ for certain u, v, we obtain the *affine space* $(P, \mathcal{L})$ *of* V, denoted by $A(V)$. Such a space $(P, \mathcal{L})$ has the following properties:

(i) it is linear, i.e. every pair of points is on a unique line;

(ii) there is an equivalence relation $\|$, called *parallelism*, on $\mathcal{L}$ such that each equivalence class forms a partitioning of P.

(iii) Given any three points, there is a subspace containing them which is isomorphic to an affine plane.

The relation $\|$ is the usual parallelism in planes. Now define an *affine space* to be a triple $(P, \mathcal{L}, \|)$ where $(P, \mathcal{L})$ is a linear space satisfying (i), (ii) with the equivalence relation $\|$ and (iii). Then, obviously, any affine space of a vector space is an affine space. The converse is also easily shown to hold. However the following generalization holds.

7.3. Theorem (Buekenhout [1969]). *If A is a space with lines of size at least* 4, *in which* (i) *and* (iii) *of the definition of affine space hold, then there is a vector space* (*unique up to isomorphism*) *V such that $A = A(V)$.*

The condition on the size of lines is crucial, as can be seen from the existence (Hall Jr [1960]) of Moufang loops of exponent 3 that do not come from affine spaces.

Removing a hyperplane H from a shadow space S of type $\mathsf{Y}_{n,j}$ leads to the partial linear space on $\mathcal{P}(S)\setminus H$ with line set $\mathcal{L}(S) = \{\ell\cap\mathcal{P}(S)\colon\ \ell \in \mathcal{L}(S')\}$ (cf. the introduction to §2) with the same local structure as S (i.e. for $x \in \mathcal{P}(S) \setminus H$, the lines and planes of $S \setminus H$ on x form a space isomorphic to the space of lines and planes of S on x). We call such a space an *affine space* of type $Y_{n,j}$ because if S is a projective space, the usual affine space appears, and (hence) if S is arbitrary, every (singular) plane of S not contained in H becomes a 'classical' affine plane in the space $S \setminus H$.

There is an analogue to the above theorem for polar spaces.

7.4. Characterisation of affine polar spaces (Cohen and Shult [1990]). *Suppose S is a connected space with thick lines satisfying the following axioms*:

(i) *S is a partial linear space; any three pairwise collinear points lie in a plane, i.e. a singular subspace of singular rank* 2;

(ii) *the points and lines incident with (i.e. contained in) any fixed plane form an affine plane; there exists a plane; and*

(iii) *if $p \in \mathcal{P}(S)$ and π is a plane such that p is not contained in π, then $p^{\perp} \cap \pi$ is either empty, is the set of points on a line, or coincides with π. Moreover, $x^{\perp} \subseteq y^{\perp}$ implies $x = y$ for any two points x and y.*

Then there is a nondegenerate polar space S' of rank at least 3 *such that S is obtained from S' by removal of a hyperplane H. Thus S' is an affine space of type $\mathsf{B}_{n,1}$ for $n \geqslant 3$.*

(Actually, *loc. cit.* contains a version which is valid for lines of arbitrary length.) All hyperplanes of nonembeddable polar spaces have also been determined, see Cohen and Shult [1990]. All hyperplanes of the nonembeddable polar spaces of rank 3 related to E_7 are of the form $x^{\perp}$ for some point x.

The relation 'having distance 3 in the collinearity graph' of the complement of the hyperplane H in a nondegenerate polar space S of rank at least 3 is an equivalence relation on $\mathcal{P}(S) \setminus H$. The quotient space of $S \setminus H$ by the equivalence relation is again a space N which is locally polar, but no longer a gamma space. Recently, Cuypers and Pasini (see Cuypers [1992a], Cuypers and Pasini [1992]) have extended the methods of Cohen and Shult [1990] to obtain a characterisation of the space N and intermediate quotients between $S \setminus H$ and N.

The affine picture can be of help in obtaining an elementary construction of the polar spaces related to E_7. To illustrate the idea, we first describe the projective planes related to E_6. Let C be a Cayley division ring. Set

$$P = \{(x, y)\colon\ x, y \in C\},$$

$$\mathcal{L} = \big\{\{a + (x, dx)\colon\ x \in C\}\colon\ a \in P,\ d \in C\big\}$$

$$\cup \big\{\{a + (0, y)\colon\ y \in C\}\colon\ a \in P\big\}.$$

Then $C^2 = (P, \mathcal{L})$ is the affine plane over C. The projective plane over C is obtained by the usual transition from affine to projective spaces. Thus, another construction of certain twisted versions of groups of type E_6 results from a description of the automorphism group of the corresponding projective plane. Note that, as a translation group, $P \cong k^8$.

As for E_7, consider the affine polar space $\mathcal{P}(S_k) \setminus x^{\perp}$ for x a point of S_k and S_k as discussed in §5.5. In Tits [1954], it is outlined how to construct it in an elementary fashion. The point set is $P_7 = k \oplus C^2 \oplus C^2$, where each C^2 represents a copy of the affine plane over C. The translation group has structure k^{1+32}, i.e. it has centre k^+ and is isomorphic to k^{32} modulo its centre. In fact this is the full unipotent radical of the stabiliser of a point of the Cayley polar space. The points at distance 3 in the affine polar space to the unit element 0 of the group P_7 are the members of $k^+ \setminus \{0\}$. The Levi group of type D_6 (with anisotropic kernel of type D_4) in the stabilizer of x acts on P_7 and can be used to define the lines on P_7.

It would probably be even harder to present a similar description of the metasymplectic space related to E_8.

7.5. *Group theory.* The automorphism groups of the finite buildings are close to being finite simple groups with Tits systems, cf. §5.14 and Chapter 11. Except for the cyclic groups of prime order, the alternating groups, and 26 sporadic groups, all finite simple groups have Tits systems. This interaction with the study of finite simple groups, and to a certain extent, its use for the classification of finite simple groups, is one of the big sources of inspiration for the synthetic approach of these shadow spaces. An example is the classification of polar spaces as used in Aschbacher [1877]. A more recent example is in Timmesfeld [1991, 1992], discussed below. In fact, attempts to classify groups in this manner go back to Fischer, cf. Aschbacher [1980]. The general setting here is a conjugacy class (or union of classes) Σ of subgroups of a given group G, for which the isomorphism type(s) of subgroups $\langle A, B\rangle$ generated by pairs A, B from Σ is known.

7.6. *Fischer groups.* Fischer's first attempt in this direction concerned the case where Σ consists of a single conjugacy class of subgroups of order 2, or, equivalently, of involutions of G, and the group generated by any two different involutions is a dihedral group of order 4 or 6 (equivalently, the order of the product of the involutions is either 2 or 3). Such a class is called a class of 3-*transpositions.* The symmetric groups provide standard examples with D the set of all transpositions. Fischer [1966] treated nearly simple groups generated by 3-transpositions, and discovered that three exceptional examples exist, nowadays referred to as Fi_{22}, Fi_{23}, Fi_{24}. See Van Bon and Weiss [1992], Conway et al. [1985], for descriptions. The groups Fi_{22}, Fi_{23}, Fi'_{24} are sporadic simple groups.

Fischer began a classification of (finite) 3-transposition groups under suitable nondegeneracy conditions, which has now been redone in greater generality and with a more extensive use of geometry. In the following version all finiteness assumptions have been removed; it is taken from Cuypers and Hall [1992].

7.7. THEOREM. *Let G be a group having a class D of* 3-*transpositions. Assume that G has no nontrivial solvable normal subgroup. Then D is one of:*

(i) *the class of transpositions of a symmetric group;*

(ii) *the class of transvections of a nondegenerate quadric over F_2 with centre off the quadric;*

(iii) *the class of polar transvections of the absolute space of a nondegenerate symplectic polarity over F_2;*

(iv) *the class of polar transvections of the absolute space of a nondegenerate Hermitian polarity over F_4;*

(v) *a class of reflections of the isometry group of a nondegenerate quadratic form over F_3;*

(v) *a unique class of involutions in one of the five groups* $\mathrm{D}_4(2):\mathrm{Sym}_3$, $\mathrm{D}_4(3):\mathrm{Sym}_3$, Fi_{22}, Fi_{23}, Fi_{24}.

Here, the notation $A{:}B$ stands for the semidirect product of the (normal) subgroup A with the group B.

In Cuypers and Hall [1992] it is also mentioned that, at the cost of a longer list of conclusions, the restriction on the normal subgroups of G can be weakened to $Z(G) = 1$. For more details on Fischer groups, including their construction, see also Aschbacher [1980], Van Bon and Weiss [1992], Fischer [1966], Norton [1988], Zara [1986].

In group theory, Fischer's work has lead to several generalizations, such as odd transpositions (Aschbacher [1972, 1973], Timmesfeld [1975]) and classes of elements of order 3 with prescribed 2-generator subgroups (Aschbacher and Hall [1973], Stellmacher [1974]). In the conclusion of the latter work, the sporadic groups J_2, Sz and Co_1 appear. The characterisation of locally Petersen graphs in Hall [1980] is a geometric interpretation of a special instance of this classification, from which the group-theoretic hypotheses have been completely removed.

7.8. *Fischer spaces.* A geometric development of these groups beginning with Buekenhout [1972] tried the geometric approach to a classification. Suppose G is a group having a class D of 3-transpositions. Let S be the space (D, L), where L is the collection of all triples $\{e, d, ede\} \subseteq D$ whenever $|de| = 3$. Then it is straightforward to show that any two intersecting lines span a subspace isomorphic to either the dual affine plane of order 2 or the affine plane of order 3. Taking this property as an axiom in a space all of whose lines are of size 3, we obtain the definition of a *Fischer space.*

Conversely, it is readily seen that a Fischer space S admits a set D of involutory automorphisms generating a group G such that (G, D) is a 3-transposition group (except that D need not consist of a single conjugacy class of G) whose associated space (D, L) coincides with S.

7.9. *Root groups.* The starting point of the work Timmesfeld [1990, 1991] is the notion of a *class of k-root subgroups of G.* Here G is a group, k is a field, and Σ is such a class if it consists of Abelian subgroups of G such that, if $A, B \in \Sigma$, then either $[A, B] = 1$, or A and B generate a subgroup of G isomorphic to $\mathrm{SL}(2, k)$ or $\mathrm{PSL}(2, k)$ and both A and B are full unipotent subgroups, or A and B are 'special'. The latter is understood to mean that they generate a subgroup S of G with derived group $[S, S]$ contained in the centre $Z(S)$, and, for all nontrivial $a \in A$, $b \in B$,

$$[a, B] = [A, b] = [A, B] \in \Sigma.$$

For $A, B \in \Sigma$, write $A \perp B$ to denote that A and B commute. Define lines AB as in §1.3. Thus $AB = \{A, B\}^{\perp\perp}$.

A look at the subgroups U_α of the Chevalley groups described in §5.15 shows that they form classes of k-root subgroups of $\mathrm{Y}_n(k)$. Note that the two cases where A and B commute in that section are not distinguished in the definition of Σ.

If no special pairs occur, the class Σ is also called a *class of k-transvections of G.*

Put together with Cuypers' removal (see Cuypers [1992b]) of a finiteness condition on ascending chains of subgroups of G generated by elements of Σ (coming with a more geometric proof), Timmesfeld's work gives the following group-theoretic characterisation of polar spaces.

7.10. POLAR TRANSVECTION GROUP CHARACTERISATION. *Let k be a field and Σ a class of k-transvection subgroups of G containing a commuting pair A, B with $A^\perp \not\subseteq B^\perp$. Suppose G has no nontrivial nilpotent normal subgroups. If $|k| > 3$, then $(\Sigma, \perp)$ is the collinearity graph of a nondegenerate polar space of rank at least 2 whose building is Moufang. Each $A \in \Sigma$ acts on the polar space as a group of polar transvections with respect to A* (cf. §4.3).

Observe that the case $|k| = 2$ is covered by Theorem 7.7. This shows how the sporadic Fischer groups can be interpreted as small field exceptions of a characterisation of shadow spaces of buildings.

The case $|k| = 3$ is dealt with by Theorem 1.4 in Cuypers [1993b], where, apart from the expected examples, the class of (pseudo-)reflections (of order 3) in the isometry group of a nondegenerate Hermitian form over $\boldsymbol{F}_4$ appears.

7.11. EXCEPTIONAL ROOT GROUP CHARACTERISATION (Timmesfeld [1991]). *Let Σ be a class of k-root subgroups of G for which every ascending chain of subgroups of G generated by elements of Σ is finite. Suppose G has no nontrivial nilpotent normal subgroup. Then G is quasisimple and there is a collection $\mathcal{L}$ of subsets of pairwise commuting elements of Σ such that $(\Sigma, \mathcal{L})$ is the root group space of a (twisted) Chevalley group on which Σ acts as a set of root subgroups.*

The proof of this theorem rests on the characterisation of root group spaces described in §7.13.

7.12. *Generalized Fischer spaces.* A *generalized Fischer space* is a partial linear space in which all subspaces generated by two intersecting lines are either affine planes or duals of affine planes (cf. Cuypers and Shult [1990]). Thus, a Fischer space is a generalized Fischer space in which all lines have cardinality 3.

7.13. THEOREM (Cuypers [1993a]). *Let S be a generalized Fischer space. Suppose both the collinearity graph and its complement are connected and reduced* (cf. §1.5). *Then S is isomorphic to a Fischer space (known from* §7.7*) or to one of the following*:

(i) *the space with point set a nondegenerate symplectic space and line set the hyperbolic lines*; *or*

(ii) *the space whose point set consists of the nonisotropic points of a nondegenerate unitary space over $\boldsymbol{F}_4$ and whose lines are all tangent lines.*

Observe that, in (ii), we have the geometric version of the exceptional $|k| = 3$ conclusion alluded to in the previous section.

7.14. *Nonparabolic geometry.* According to a result of Seitz [1974], the permutation representations on the point sets of the shadow spaces are the only (primitive) permutation representations of the Chevalley groups $\mathsf{Y}_n(k)$ whose permutation ranks, when viewed as a function of k, do not depend on $|k|$. One step beyond these are the transitive permutation representations of $\mathsf{Y}_n(k)$ having a permutation rank that is linear as a function of $|k|$.

In various cases, a coherent configuration (cf. Brouwer et al. [1989], Higman [1975] for a definition) can be found whose number of classes is independent of the order of k. An example is furnished by the permutation representation of $\mathsf{A}_n(q) \cong \mathrm{PSL}(n+1, q)$ on the set P of antiflags (= nonincident point-hyperplane pairs) in the associated projective space. The permutation rank of the full group $\mathrm{Aut}\,\mathsf{A}_n(q)$ (graph automorphisms included) on P is $q+3$ (cf. Darafsheh [1986]). Nevertheless, the projective geometry naturally produces an association scheme of 6 classes (one class is empty if the field has order 2). Other examples come from association schemes on the nonisotropic points in an orthogonal or unitary space. See Bannai [1990] for more on this theme.

Taking two antiflags to be adjacent when the point of each of them lies in the hyperplane of the other, turns P into a 'commuting tori' graph, i.e. a graph whose vertex set is a conjugacy class of 1-parameter subgroups consisting of semi-simple elements of $\mathsf{Y}_n(k)$ in which two vertices are adjacent when they commute. The classification of finite simple groups uses 'component type' theorems which are related to these commuting tori graphs; see, e.g., Aschbacher [1986], where involutions take over the role of the tori in Chevalley groups defined over fields of odd characteristic. In the example just given, the tori centralisers have a 'simple component' of type A_{n-1}.

7.15. *Maximal subgroups.* Taking the point of view that group-related geometry starts with group actions on point sets, we see from the correspondence between transitive permutation representations and subgroups of a given group G that it is relevant to know the maximal subgroups of G. Over the last few years, considerable progress has been made on this problem for G a finite (twisted) Chevalley group. Here we only give a few pointers to the literature.

For subgroups of classical groups, Aschbacher [1984a] has provided a structure theorem. The subsequent analysis in Kleidman and Liebeck [1990] is a good reference.

Borovik [1989b, 1992] describes a striking maximal subgroup in groups of type E_8 whose socle is isomorphic to $\mathrm{Alt}_5 \times \mathrm{Alt}_6$. It is unique as a subgroup of a simple algebraic group over an algebraically closed field k in that it is finite, a maximal closed algebraic subgroup, normalises no Abelian subgroup, and has a socle consisting of more than one simple factor. For specific finite fields k, it occurs as a maximal subgroup of $\mathsf{E}_8(k)$ as well. The local subgroups of exceptional algebraic groups have been determined in Cohen et al. [1992]. The remaining maximal subgroups have a simple socle, and so the classification of finite simple groups can be of use to their determination. See Seitz [1992] for a survey, from which it is clear that the smaller rank exceptional groups have been fully dealt with. Also E_6 and F_4 (Aschbacher [1987, 1988b], Cohen and Wales [1992], Magaard [1990], Testerman [1989]) are close to being fully analysed. For E_7 and E_8 a lot of work remains to be done.

Recently, some small finite subgroups of the exceptional groups G have been found, which lead to a case-by-case proof of Kostant's conjecture that, for h the Coxeter number (cf. Bourbaki [1968]) of $G = \mathsf{Y}_n(k)$, the finite group $\mathrm{PSL}(2, 2h+1)$ embeds in G (for k sufficiently large), see Cohen et al. [1993], Cohen and Wales [1992, 1993], Griess Jr and Ryba [1991], Kleidman and Ryba [1992].

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CHAPTER 13

Free Constructions

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Contents

Introduction ... 741
1. Free extensions: model-theoretic point of view ... 742
   1.1. Preliminaries from model theory ... 742
   1.2. Examples ... 744
   1.3. General theory of (hyper-)free extensions ... 748
   1.4. Model-theoretic construction of free extensions ... 758
   1.5. Degenerate models of Th(Σ) ... 760
2. Survey of further results on free models ... 761
   2.1. Canonical free models and free equivalence ... 761
   2.2. Free projective and affine planes ... 762
   2.3. Other classes of (free) geometries ... 764
3. Groups of projectivities in free geometries ... 765
   3.1. Unifying treatment: model-theoretic point of view ... 765
   3.2. Examples ... 766
   3.3. Theorems about groups of projectivities in free models ... 769
4. Automorphism groups of free geometries ... 771
5. Free extensions and amalgamation techniques ... 772

References ... 777

HANDBOOK OF INCIDENCE GEOMETRY
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Introduction

Free extensions are often used in geometry to show the existence of models for a given theory and to construct examples whose properties are contrary to those of common models. Ever since Hall [1943] introduced free extensions for projective planes, the increasing interest for other classes of incidence geometries has led to repeated variations of Hall's ideas, where the constructions themselves only had to be adapted according to the different situations given by the language, and the axioms of the geometry under consideration.

Besides a survey on significant results obtained thus far, the main purpose of this article is the development of a unifying treatment including all classes of incidence geometries which can be characterized by a set Σ of axioms formulated in a first-order language $\mathcal{L}$ (e.g., projective planes, affine planes, generalized n-gons, Benz planes, etc.). By using rather simple model-theoretic tools, we can define the notions of *(hyper-)free*, *open*, *confined*, *closed*, *(hyper-)free extensions*, and *degenerate geometries* without knowing Σ and $\mathcal{L}$ explicitly. To a very vast extent, we succeed in reformulating and proving the main results concerning free extensions within this general frame. The application of these results reduces to an easy verification of some model-theoretic conditions on the axioms. Thus it becomes clear that the real nature of free extensions in fact lies beyond geometry. We hope that this insight will contribute to stop splitting research on that subject. On the other hand, our treatment yields new results, in particular, on generalized n-gons.

Surprisingly, groups of projectivities also prove themselves to be sensitive to our unifying point of view, as far as their algebraic structure as a free group or connection with the group of automorphisms is concerned (cf. Theorems 10, 11, and Theorem 15). In addition, repercussions arise about traditional definitions for the group of projectivities. So, for affine planes, central perspectivities with parallel carriers reveal themselves as natural as the usual parallel perspectivities (cf. Theorem 13).

In the last section, in order to illustrate applicability of hyperfree extensions, we combine them with certain amalgamation techniques and broach three rather archetypical questions: Is every group G the full automorphism group of some model in a given class of geometries? May every model be embedded into some homogeneous model enjoying nice transitivity properties? For which classes of geometries do there exist highly transitive groups of projectivities?

Free extensions lead to 'full' models

$$\mathfrak{M}_\lambda = \bigcup_{\nu<\lambda} \mathfrak{I}_\nu$$

by union of an ascending sequence $\cdots \subseteq \mathfrak{I}_\nu \subseteq \mathfrak{I}_{\nu+1} \subseteq \mathfrak{I}_{\nu+2} \subseteq \cdots$ of partial models where $\nu < \lambda$ for some limit ordinal λ. Usually, the sequence starts with $\mathfrak{I}_0$ and one has $\lambda = \omega$. These processes can easily be generalized for an arbitrary limit ordinal μ by adding suitable hyperfree elements to $\mathfrak{M}_\lambda$ for each limit ordinal $\lambda < \mu$ (to ensure that the extension process does not stop).

1. Free extensions: model-theoretic point of view

1.1. *Preliminaries from model theory*

Each class of geometries, for which some kind of *free extensions* has been introduced so far, can be seen as a class of *models* satisfying some set Σ of axioms. Σ is formulated in terms of a (first-order) *language* $\mathcal{L}$ consisting of relation and function symbols. (Usually there is no need of constant symbols when dealing with geometry.) One has, however, two or more *sorts of variables*, e.g., one sort $p, q, r, \ldots$ over points and a different sort $l, m, \ldots$ over lines.

An $\mathcal{L}$-structure $\mathfrak{A}$ consists of a *universe* A, a nonempty set, and an *interpretation function*[1] distinguishing the elements of A sort by sort and mapping the symbols of $\mathcal{L}$ to appropriate relations and functions in A. An *$\mathcal{L}$-substructure* $\mathfrak{B}$ of $\mathfrak{A}$ has a subset $B \subseteq A$ as its universe such that the interpretation of each relation or function symbol is the restriction of the corresponding interpretation in $\mathfrak{A}$. We denote an $\mathcal{L}$-structure $\mathfrak{A}$ by $[x_1, \ldots, x_n\colon \mathcal{R}]$ if $\mathfrak{A}$ has a universe consisting of the distinct elements $x_1, \ldots, x_n$ satisfying precisely the list $\mathcal{R}$ of relations such that the negation of any other possible relation between elements in $\{x_1, \ldots, x_n\}$ holds true in $\mathfrak{A}$.

From the set of all finite sequences, each element of which is either a logic building block $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $=$, $\forall$, $\exists$,), (, or an element of $\mathcal{L}$, the meaningful ones are singled out in the following way: *terms* of $\mathcal{L}$ are variables $x, y, z, \ldots$ and function values $f(z_1, z_2, \ldots)$ where $f \in \mathcal{L}$ and $z_1, z_2, \ldots$ are again terms; *atomic formulas* of $\mathcal{L}$ consist of equalities or relations between terms (e.g., $t_1 = t_2$, $R(t_1, t_2, \ldots)$ where $R \in \mathcal{L}$); (first-order) *formulas* are built up from (atomic) formulas using the logic building blocks in the usual way (e.g., $\varphi \wedge \psi, (\forall x)\varphi$, where φ, ψ are formulas and x appears among the variables of φ); finally, a *sentence* of $\mathcal{L}$ is a formula without any free (i.e. nonquantified) variable; cf. Barwise [1977], pp. 18–20.

So far, terms, formulas, and sentences of $\mathcal{L}$ are simply finite strings of symbols. To give them the intended meaning with respect to an $\mathcal{L}$-structure $\mathfrak{A}$, one proceeds as follows.

An *assignment* in $\mathfrak{A}$ is a function s with domain being the set of variables of $\mathcal{L}$ and range a subset of A. Each assignment s can be extended in a natural way to terms. Then one defines the *satisfaction relation* $\mathfrak{A} \models \varphi[s]$ (read: the assignment s satisfies the formula φ in $\mathfrak{A}$) between $\mathcal{L}$-structures on the one hand and formulas on the other hand by induction on the complexity of formulas, starting with atomic ones:

$$\mathfrak{A} \models (t_1 = t_2)[s] \quad \text{if and only if } s(t_1) = s(t_2),$$
$$\mathfrak{A} \models R(t_1, t_2, \ldots)[s] \quad \text{if and only if } R(s(t_1), s(t_2), \ldots),$$

cf. Barwise [1977], Definition 3.8, p. 21.

Note that the truth or falsity of $\mathfrak{A} \models \varphi[s]$ depends only on the values of $s(x)$ for variables x which are actually free in φ. Hence, if φ is a sentence, the truth or falsity of $\mathfrak{A} \models \varphi[s]$ is completely independent of s and we can write $\mathfrak{A} \models \varphi$ (read: $\mathfrak{A}$ is a model of φ). An $\mathcal{L}$-structure $\mathfrak{M}$ is a *model* of a set T of sentences if $\mathfrak{M} \models \varphi$ for all $\varphi \in T$.

[1] Which, in geometrical terms, is the type function, as defined in Chapter 3 (Editor's note).

The class of all models satisfying the set T of sentences will be denoted by Mod(T). If a sentence φ holds in each member of Mod(T), we write $T \models \varphi$.

A *proof* of a sentence φ from a set T of sentences is a finite sequence $\psi_1, \ldots, \psi_n$ of formulas with $\psi_n = \varphi$, each of which is either a member of T, a logical axiom (see also Chang and Keisler [1973], pp. 24, 25), or else follows from earlier ψ_i by one of the rules of inference. If there is a proof of φ from T, we write $T \vdash \varphi$.

Gödel's *Completeness Theorem* says that a sentence φ has a proof from T if and only if φ is true in each model of T, i.e.

$$T \models \varphi \quad \text{if and only if} \quad T \vdash \varphi;$$

cf., e.g., Barwise [1977], p. 35.

A set T of sentences of $\mathcal{L}$ is called a theory in $\mathcal{L}$. Two theories T_1 and T_2 are said to be *equivalent* if they have precisely the same class of models. A theory T in $\mathcal{L}$ is said to be *closed* if it is closed under the relation $\models$, i.e. closed under the relation $\vdash$ by the above theorem. A set Σ of sentences which is equivalent to some closed theory T is called a *set* of *axioms* for T. Usually Σ consists only of a small (finite) number of sentences characterizing T:

$$\mathrm{Th}(\Sigma) := \{\varphi \in \mathcal{L}\colon\ \Sigma \vdash \varphi\} = T.$$

A formula of type $(\forall x_1)\ldots(\forall x_n)\psi$, where ψ has no quantifiers is called a *universal* formula of $\mathcal{L}$. A theory is named *universal* if it is equivalent to a set of universal sentences. The Łoś–Tarski theorem asserts that a theory T is universal if and only if every $\mathcal{L}$-substructure of a model of T is again a model of T; cf. Barwise [1977], p. 53 and p. 62.

Let Π be the set of all universal formulas of $\mathcal{L}$. Then, for any theory T formulated in $\mathcal{L}$, the *universal part* $T_\forall$ of T consists of all sentences of $\mathcal{L}$ which have a proof from $T \cap \Pi$. The last mentioned theorem makes sure that one has

$$T_\forall = \{\varphi \in \mathcal{L}\colon\ \mathfrak{B} \models \varphi \text{ for all } \mathfrak{B} \subseteq \mathfrak{A} \text{ with } \mathfrak{A} \models T\}.$$

When dealing with geometries, models of $T_\forall$ are usually called *partial* models.

A theory T is said to be *inductive* if T is equivalent to a set of sentences of type $(\forall x_1)\ldots(\forall x_n)(\exists y_1)\ldots(\exists y_m)\psi$, where ψ has no quantifiers. The Chang–Łoś–Suszko theorem claims that T is inductive if and only if the union of any ascending chain $\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M}_2 \subseteq \cdots$ of models of T again is a model of T; cf. Barwise [1977], p. 55 and p. 63.

For $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}(\mathrm{Th}(\Sigma))$, a homomorphism $\varepsilon\colon \mathfrak{A} \to \mathfrak{B}$ is a mapping such that for each relation symbol R of $\mathcal{L}$, any set of elements in $\mathfrak{A}$ satisfying R in $\mathfrak{A}$ is sent into a set of elements satisfying R in $\mathfrak{B}$ and for each m-ary function symbol F of $\mathcal{L}$ one has

$$\varepsilon(F(x_1, \ldots, x_m)) = F(\varepsilon(x_1), \ldots, \varepsilon(x_m))$$

for all $x_1, \ldots, x_m$ in $\mathfrak{A}$. A *monomorphism* $\varepsilon\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ is called an *embedding* if $\varepsilon^{-1}\colon \varepsilon(\mathfrak{A}) \to \mathfrak{A}$ is still a homomorphism.

Finally, let $\mathfrak{A}$ be a model Th(Σ) and R and n-ary relation symbol of the language $\mathcal{L}$. We introduce a new $(n-s)$-ary relation $R'_{a_1,\ldots,a_s}$ in $\mathfrak{A}$ by defining for all $x_1, \ldots, x_{n-s} \in \mathfrak{A}$:

$$R'_{a_1,\ldots,a_s}(x_1, \ldots, x_{n-s}) \quad \text{if, and only if,} \quad R\big(\sigma(a_1, \ldots, a_s, x_1, \ldots,\ x_{n-s})\big)$$

for fixed elements $a_1, \ldots, a_s$ in $\mathfrak{A}$ and a suitable permutation σ of the n elements $a_1, \ldots, a_s, x_1, \ldots, x_{n-s}$. The relation $R'_{a_1,\ldots,a_s}$ will be referred to as the *localized relation R* (with respect to $a_1, \ldots, a_s$).

1.2. *Examples*

This section should illustrate that the common classes of incidence geometries are natural examples of Mod(Th(Σ)) satisfying all the model-theoretic hypotheses mentioned before. For each class we fix a set Σ of axioms in terms of a k-sorted first-order language $\mathcal{L}$ (for some $k \in \mathbb{N}$) consisting of a finite number of relation symbols.

1.2.1. *A projective plane* consists of a set of points (denoted by $p_0, p_1, \ldots$), a set of lines $(l_0, l_1, \ldots)$, and an incidence relation $|$ between points and lines such that the following set Σ of axioms is satisfied.

'Join': $(\forall p_1)(\forall p_2)(\exists l_0)\ (p_1|l_0 \wedge p_2|l_0) \vee (p_1 = p_2)$.

'Meet': $(\forall l_1)(\forall l_2)(\exists p_0)\ (p_0|l_1 \wedge p_0|l_2) \vee (l_1 = l_2)$.

'Uniqueness of join and meet':

$$(\forall p_1)(\forall p_2)(\forall l_1)(\forall l_2)\left(\bigwedge_{i,j=1,2} p_i|l_j\right) \Rightarrow (p_1 = p_2 \vee l_1 = l_2).$$

Note that Mod(Th(Σ)$_\forall$) coincides with the class of all *incidence structures* in the sense of Pickert [1975] which are characterized by one and the only axiom 'Uniqueness of join and meet'.

1.2.2. *An affine plane* consists of a set of points (denoted by $p_0, p_1, \ldots$), a set of lines $(l_0, l_1, \ldots)$, an incidence relation $|$ between points and lines, and a parallelism relation $\|$ between lines such that the following set Σ of axioms is satisfied.

'Join': $(\forall p_1)(\forall p_2)(\exists l_0)\ (p_1|l_0 \wedge p_2|l_0) \vee (p_1 = p_2)$.

'Meet': $(\forall l_1)(\forall l_2)(\exists p_0)\ (l_1\|l_2) \vee (p_0|l_1 \wedge p_0|l_2)$.

'Euclid': $(\forall p_0)(\forall l_0)(\exists l_1)\ (p_0 \mid l_0) \vee (p_0|l_1 \wedge l_0\|l_1)$.

'Uniqueness of join and meet':

$$(\forall p_1)(\forall p_2)(\forall l_1)(\forall l_2)\left(\bigwedge_{i,j=1,2} p_i|l_j\right) \Rightarrow (p_1 = p_2 \wedge l_1 = l_2).$$

'Uniqueness of parallel lines':

$$(\forall p_0)(\forall l_1)(\forall l_2)\left(\bigwedge_{i=1,2} p_0|l_i \wedge l_1\|l_2\right) \Rightarrow (l_1 = l_2).$$

'Parallelism is an equivalence relation':

$$(\forall l_0) l_0\|l_0,$$

$$(\forall l_0)(\forall l_1)(l_0\|l_1) \Rightarrow (l_1\|l_0),$$

$$(\forall l_0)(\forall l_1)(\forall l_2)(l_0\|l_1 \wedge l_1\|l_2) \Rightarrow (l_0\|l_2).$$

Obviously, the class Mod(Th(Σ)$_\forall$) is characterized by the last five (purely universal) axioms; note that Mod(Th(Σ)$_\forall$), i.e. the class of partial affine planes, coincides with the class of all *affine incidence structures* in the sense of Schleiermacher and Strambach [1969].

1.2.3. Let $n \geqslant 4$ be some natural number. A *generalized n-gon* consists of a set of points, a set of lines, and an incidence relation $|$ such that the following set Σ of axioms is satisfied (cf. Dembowski [1968], pp. 300, 301, Funk, Kegel and Strambach [1985], p. 61).
'Consistency': $(\forall x)\neg(x \text{ is point } \wedge\ x \text{ is line})$.
'Incidence between points and lines':

$$(\forall x)(\forall y)\ (x \mid y) \Rightarrow ((x \text{ is point } \wedge y \text{ is line}) \vee\ (x \text{ is line } \wedge y \text{ is point})).$$

'Any two elements are contained in some ordinary n-gon':

$$(\forall x)(\forall y)(\exists z_1)\ldots(\exists z_{2n})(z_1|z_2|\cdots|z_{2n}|z_1) \wedge (z_1 = x)$$
$$\wedge \left(\bigvee_{i=1,\ldots,2n} z_i = y \right) \wedge \left(\bigwedge_{\substack{1\leqslant i<j\leqslant 2n \\ j-i\notin\{1,2n-1\}}} \neg(z_i \mid z_j) \right).$$

'There are no m-gons for $m < n$':

$$(\forall z_1)\ldots(\forall z_{2k})(z_1|z_2|\cdots|z_{2k}|z_1) \Rightarrow \left(\bigvee_{1\leqslant i<j\leqslant 2k} z_i = z_j \right)$$

for $k = 1, \ldots, n-1$.

NOTE. The first two axioms render the relation $|$ into a symmetrical incidence relation between points and lines; this makes possible the very concise formulation of the essential axioms for generalized n-gons. In the other examples, similar (but weaker) axioms are hidden in the claim that the incidence relation $|$ is a relation between points and lines.

Partial models are characterized by the first axiom and the last (group of) axiom(s) claiming the nonexistence of sub-m-gons for $m < n$. Substructures of type $[x_1, \ldots, x_h\colon x_1|x_2|\cdots|x_h]$ are usually called *irreducible chains*, and *closed irreducible chains* are submodels of type $[x_1, \ldots, x_h\colon x_1|\cdots|x_h|x_1]$.

1.2.4. Let $k \geqslant 3$ be some natural number. A *k-net* consists of a set of points (denoted by $p_0, p_1, \ldots$), and k sets of lines (denoted by $l_0^j, l_1^j, \ldots$, where $j \in \{1, \ldots, k\}$ indicates the sort), as well as an incidence relation $|$ between points and lines of any sort such that the following set Σ of axioms is satisfied.
'Drawing lines': $(\forall p_0)(\exists l_0^j)p_0|l_0^j$ for $j = 1, \ldots, k$.
'Uniqueness of drawing':

$$(\forall p_0)(\forall l_0^j)(\forall l_1^j)\big(p_0|l_0^j \wedge p_0 \mid l_1^j\big) \Rightarrow \big(l_0^j = l_1^j\big)$$

for $j = 1, \ldots, k$.

'Meet': $(\forall l_0^i)(\forall l_1^j)(\exists p_0)p_0|l_0^i \wedge p_0|l_1^j$ for $1 \leqslant i < j \leqslant k$.
'Uniqueness of meet':

$$(\forall p_0)(\forall p_1)(\forall l_0^i)(\forall l_1^j)\big(p_0|l_0^i \wedge p_0|l_1^j \wedge p_1|l_0^i \wedge p_1|l_1^j\big) \Rightarrow (p_0 = p_1)$$

for $1 \leqslant i < j \leqslant k$.

Partial k-nets are characterized by the groups of axioms 'Uniqueness of drawing' and 'Uniqueness of meet'.

1.2.5. A *Möbius*, *Laguerre*, or *Minkowski plane*[2] consists of a set of points (denoted by $p_0, p_1, \ldots$), a set of blocks ($b_0, b_1, \ldots$), an incidence relation $|$ between points and blocks, a ternary tangency relation ρ between blocks in a point, and, respectively, zero, one ($\|_1$), and two ($\|_1, \|_2$) parallelisms between points such that the following set Σ of axioms is satisfied.
'Join':

$$(\forall p_1)(\forall p_2)(\forall p_3)(\exists b_0)\left(\bigvee_{1\leqslant\mu<\nu\leqslant 3} p_\mu = p_\nu\right) \vee \left(\bigvee_{\substack{1\leqslant\mu<\nu\leqslant 3\\ 1\leqslant i\leqslant 2}} p_\mu \|_i p_\nu\right)$$
$$\vee \left(\bigwedge_{1\leqslant\lambda\leqslant 3} p_\lambda | b_0\right).$$

'Uniqueness of join':

$$(\forall p_1)(\forall p_2)(\forall p_3)(\forall b_1)(\forall b_2)\left(\bigwedge_{\substack{1\leqslant\mu\leqslant 3\\ 1\leqslant\nu\leqslant 2}} p_\mu | b_\nu\right)$$
$$\Rightarrow \left(\left(\bigvee_{1\leqslant\mu<\lambda\leqslant 3} p_\mu = p_\lambda\right) \vee b_1 = b_2\right).$$

'Touching implies incidence':

$$(\forall p_0)(\forall b_1)(\forall b_2)\rho(b_1, b_2, p_0) \Rightarrow \left(\bigwedge_{i=1,2} p_0 | b_i\right).$$

'Touching block':

$$(\forall p_1)(\forall p_2)(\forall b_1)(\exists b_2)\neg(p_1|b_1) \vee (p_2|b_1) \vee (\rho(b_1, b_2, p_1) \wedge p_2|b_2).$$

[2] See also Chapters 6 and 24.

'Uniqueness of the touching block':

$$(\forall p_1)(\forall p_2)(\forall b_1)(\forall b_2)\left(\bigwedge_{\mu,\nu=1,2} p_\mu|b_\nu \wedge \rho(b_1,b_2,p_1) \wedge \rho(b_1,b_2,p_2)\right)$$
$$\Rightarrow (p_1 = p_2 \vee b_1 = b_2).$$

'Touching is locally an equivalence relation':

$$(\forall p_0)(\forall b_0)p_0|b_0 \Rightarrow \rho(b_0,b_0,p_0),$$
$$(\forall p_0)(\forall b_1)(\forall b_2)\rho(b_1,b_2,p_0) \Rightarrow \rho(b_2,b_1,p_0),$$
$$(\forall p_0)(\forall b_1)(\forall b_2)(\forall b_3)(\rho(b_1,b_2,p_0) \wedge \rho(b_2,b_3,p_0)) \Rightarrow \rho(b_1,b_3,p_0).$$

Möbius planes are defined by these axioms. For Laguerre planes, we change the axiom 'Touching block' and impose additional requirements.
'Touching block':

$$(\forall p_1)(\forall p_2)(\forall b_1)(\exists b_2)\neg(p_1|b_1) \vee (p_2|b_1) \vee (p_1\|_1 p_2) \vee (\rho(b_1,b_2,p_1) \wedge p_2|b_2).$$

'Parallel projection':

$$(\forall p_1)(\forall b_1)(\exists p_2)(p_1|b_1) \vee (p_1\|_i p_2 \wedge p_2|b_1) \quad \text{for } i = 1.$$

'Uniqueness of parallel projection':

$$(\forall p_1)(\forall p_2)(\forall b_1)(p_1|b_1 \wedge p_2|b_1 \wedge p_1\|_i p_2) \Rightarrow (p_1 = p_2) \quad \text{for } i = 1.$$

'Parallelism is an equivalence relation': for $i = 1$

$$(\forall p_1)p_1\|_i p_1,$$
$$(\forall p_1)(\forall p_2)(p_1\|_i p_2) \Rightarrow (p_2\|_i p_1),$$
$$(\forall p_1)(\forall p_2)(\forall p_3)(p_1\|_i p_2 \wedge p_2\|_i p_3) \Rightarrow (p_1\|_i p_3).$$

For Minkowski planes we substitute the axiom 'Touching block' by the following (second) variant and additionally require the following axioms.
'Touching block':

$$(\forall p_1)(\forall p_2)(\forall b_1)(\exists b_2)\neg(p_1|b_1)\vee(p_2|b_1)\vee(p_1\|_1 p_2)\vee(p_1\|_2 p_2)\vee(\rho(b_1,b_2,p_1)\wedge p_2|b_2).$$

'Parallel projection' for $i = 1,2$.
'Uniqueness of parallel projection' for $i = 1,2$.

'Parallelism is an equivalence relation' for $i = 1, 2$.
'Meet of parallel projections': $(\forall p_1)(\forall p_2)(\exists p_0)\ p_1 \|_1 p_0 \wedge p_2 \|_2 p_0$.
'Uniqueness': $(\forall p_1)(\forall p_2)(p_1 \|_1 p_2 \wedge p_1 \|_2 p_2) \Rightarrow p_1 = p_2$.

Möbius, Laguerre, and Minkowski planes are also called *Benz planes*.

A Benz plane is named *projective* if the following axiom also holds.

'Meet of blocks':

$$(\forall b_1)(\forall b_2)(\exists p_1)(\exists p_2)\left(\bigvee_{i=1,2} \rho(b_1, b_2, p_i)\right) \vee \left(\bigwedge_{\mu,\nu=1,2} p_\mu | b_\nu\right).$$

Note that in each of the three cases $\mathrm{Mod}(\mathrm{Th}(\Sigma)_\forall)$ coincides, respectively, with the classes of partial Möbius, Laguerre, and Minkowski planes introduced in Schleiermacher and Strambach [1969], Heise and Sörensen [1973].

1.2.6. Let n, k be natural numbers such that $2 \leqslant k < n$. A (k, n)-*Steiner system* consists of a set of points (denoted by $p_0, p_1, \ldots$), a set of blocks ($b_0, b_1, \ldots$), and an incidence relation $|$ between points and blocks such that the following set Σ of axioms is satisfied.

'Join':

$$(\forall p_1)\ldots(\forall p_k)(\exists b_0)\left(\bigvee_{1\leqslant\mu<\nu\leqslant k} p_\mu = p_\nu\right) \vee \left(\bigwedge_{1\leqslant\lambda\leqslant k} p_\lambda | b_0\right).$$

'Uniqueness of join':

$$(\forall p_1)\ldots(\forall p_k)(\forall b_0)(\forall b_1)$$

$$\left(\bigwedge_{\substack{1\leqslant\nu\leqslant k\\ 1\leqslant\lambda\leqslant 2}} p_\nu | b_\lambda\right) \Rightarrow \left(\left(\bigvee_{1\leqslant\mu<\lambda\leqslant k} p_\mu = p_\nu\right) \vee (b_1 = b_2)\right).$$

'Cardinality of blocks':

$$(\forall b_0)(\exists p_1)\ldots(\exists p_n)\left(\bigwedge_{1\leqslant\lambda\leqslant n} p_\lambda | b_0\right) \wedge \left(\bigwedge_{1\leqslant\mu<\nu\leqslant n} \neg(p_\mu = p_\nu)\right),$$

$$(\forall p_1)\ldots(\forall p_{n+1})(\forall b_0)\left(\bigwedge_{1\leqslant\lambda\leqslant n+1} p_\lambda | b_0\right) \Rightarrow \left(\bigvee_{1\leqslant\mu<\nu\leqslant n+1} p_\mu = p_\nu\right).$$

Partial (k, n)-Steiner systems are characterized by the axiom 'Uniqueness of join'.

1.3. *General theory of (hyper-)free extensions*

Let T be a theory formulated in a many-sorted first-order language $\mathcal{L}$ without any function or constant symbol. Let Σ be a set of axioms for $\mathrm{Mod}(T)$.

DEFINITION 1. Suppose that $\mathcal{S}$ is a subset of the set $\mathcal{R}$ of all relation symbols of the language $\mathcal{L}$. With any model $\mathfrak{T}$ of $\mathrm{Th}(\Sigma)_\forall$ we associate an (undirected) *graph* $\Delta_{\mathcal{S}}(\mathfrak{T})$, without loops and without multiple edges, in the following way. The vertices of $\Delta_{\mathcal{S}}(\mathfrak{T})$ are the elements of $\mathfrak{T}$; distinct vertices $a_1, \dots, a_n$ are contained in a complete subgraph of $\Delta_{\mathcal{S}}(\mathfrak{T})$ provided there exists an m-ary relation symbol R in $\mathcal{S}$ with $m \geqslant n$ such that one has $R(\sigma(a_1, \dots, a_n))$ in $\mathfrak{T}$ for some m-tuple $\sigma(a_1, \dots, a_n)$ which contains each a_i at least once. In particular, if $\mathcal{S} = \mathcal{R}$, we simply write $\Delta(\mathfrak{T})$ instead of $\Delta_{\mathcal{R}}(\mathfrak{T})$.

In the case of projective planes or generalized n-gons, the graph $\Delta(\mathfrak{T})$ is just the incidence graph.

DEFINITION 2.

(i) Let $\mathfrak{I}$ be a model of $\mathrm{Th}(\Sigma)_\forall$ and $\mathfrak{J}$ a finite submodel (i.e. $\mathcal{L}$-substructure) of $\mathfrak{I}$. Then we call the submodel $\mathfrak{J}_\mathfrak{I} \subseteq \mathfrak{I}$ with the universe

$$J_\mathfrak{I} := \{x \in \mathfrak{I}\colon \ \text{distance}(x, \mathfrak{J}) \leqslant 1 \text{ in } \Delta(\mathfrak{I})\}$$

interior construction of $\mathfrak{J}$ in $\mathfrak{I}$. Replacing $\Delta(\mathfrak{I})$ by $\Delta_{\mathcal{S}}(\mathfrak{I})$ for some subset $\mathcal{S}$ of the set $\mathcal{R}$ of all relation symbols of $\mathcal{L}$, we yield a substructure $\mathfrak{J}_\mathfrak{I}^{\mathcal{S}}$ of $\mathfrak{J}_\mathfrak{I}$ which we call *interior construction of* $\mathfrak{J}$ *in* $\mathfrak{I}$ *with respect to* $\mathcal{S}$.

(ii) If Σ renders some binary relation symbol R of $\mathcal{L}$ into a (localized) binary reflexive and symmetric relation between elements of the same sort, interior constructions can become quite large. We therefore introduce the notion of a *restricted interior construction* $\langle\mathfrak{J}\rangle_\mathfrak{I}$ of $\mathfrak{J}$ in $\mathfrak{I}$, whose universe is obtained from $J_\mathfrak{I}$ in the following way: For each (localized) binary reflexive and symmetric relation R, consider the orbits of transitivity of R, i.e. maximum subsets $O(R)$ of elements contained in $\mathfrak{J}_\mathfrak{I} \setminus \mathfrak{J}$ and related in pairs by R. Call an element y in $\mathfrak{J}_\mathfrak{I} \setminus \mathfrak{J}$ exceptional if y is contained in at least two orbits of transitivity arising from distinct relations; now, if the orbit $O(R)$ does not contain any exceptional element, reduce $O(R)$ to just one representative; otherwise reduce $O(R)$ to its subset of exceptional elements. Obviously, for each restricted interior construction one has $\mathfrak{J} \subseteq \langle\mathfrak{J}\rangle_\mathfrak{I} \subseteq \mathfrak{J}_\mathfrak{I} \subseteq \mathfrak{I}$.

(iii) A pair $(\mathfrak{I}, \mathfrak{J})$ of models of $\mathrm{Th}(\Sigma)_\forall$ with $\varnothing \neq \mathfrak{J} \subsetneqq \mathfrak{I}$ is called a *Σ-construction* provided every embedding α of $\mathfrak{I} \setminus \mathfrak{J}$ into any model $\mathfrak{G}$ of $\mathrm{Th}(\Sigma)$ can be extended to an embedding $\alpha'\colon \mathfrak{I} \hookrightarrow \mathfrak{G}$ in a unique way, up to automorphisms of $\mathfrak{I}$ inducing the identity on $\mathfrak{I} \setminus \mathfrak{J}$.

(iv) A Σ-construction $(\mathfrak{I}, \mathfrak{J})$ is called *reducible* if one of the following conditions holds:

(a) There exists a further Σ-construction $(\mathfrak{K}, \mathfrak{L}) \not\cong (\mathfrak{I}, \mathfrak{J})$ with an embedding $\varepsilon\colon \mathfrak{K} \setminus \mathfrak{L} \hookrightarrow \mathfrak{I} \setminus \mathfrak{J}$ and an isomorphism $\eta\colon \mathfrak{K} \to \mathfrak{I}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{K} \setminus \mathfrak{L} & \xhookrightarrow{\ \varepsilon\ } & \mathfrak{I} \setminus \mathfrak{J} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{K} & \xrightarrow{\ \eta\ } & \mathfrak{I} \end{array}$$

commutes, where $\mathfrak{K} \setminus \mathfrak{L}$ and $\mathfrak{I} \setminus \mathfrak{J}$ are embedded into $\mathfrak{K}$ and $\mathfrak{I}$, respectively.

(b) There exists a further Σ-construction $(\mathfrak{K}, \mathfrak{L}) \not\cong (\mathfrak{I}, \mathfrak{J})$ with embeddings η: $\mathfrak{L} \hookrightarrow \mathfrak{J}$ and ϑ: $\mathfrak{K} \hookrightarrow \mathfrak{I}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{L} & \overset{\eta}{\hookrightarrow} & \mathfrak{J} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{K} & \overset{\vartheta}{\hookrightarrow} & \mathfrak{I} \end{array}$$

commutes, where $\mathfrak{L}$ and $\mathfrak{J}$ are embedded into $\mathfrak{K}$ and $\mathfrak{I}$, respectively.

DEFINITION 3. Irreducible Σ-constructions $(\mathfrak{M}, \mathfrak{N})$ are called *Σ-free constructions* and $\mathfrak{N}$ will be named a *Σ-brick.*

In order to illustrate the notion of Σ-free constructions, we shall list all of them for the geometries presented in Section 1.2.

Projective planes. Up to isomorphism, there exist precisely two Σ-free constructions in Mod(Th$(\Sigma)_\forall$), cf. Funk, Kegel and Strambach [1985], p. 34:

$$\mathfrak{M}_1 = [p_1, p_2, l_0\colon\ p_1|l_0,\ p_2|l_0], \quad \mathfrak{N}_1 = [l_0],$$

$$\mathfrak{M}_2 = [l_1, l_2, p_0\colon\ p_0|l_1,\ p_0|l_2], \quad \mathfrak{N}_2 = [p_0].$$

Affine planes. Up to isomorphism, there exist precisely three Σ-free constructions in Mod(Th$(\Sigma)_\forall$), cf. Funk et al. [1985], p. 39:

$$\mathfrak{M}_1 = [p_1, p_2, l_0\colon\ p_1|l_0,\ p_2|l_0], \quad \mathfrak{N}_1 = [l_0],$$

$$\mathfrak{M}_2 = [p_0, l_1, l_2\colon\ p_0|l_1,\ p_0|l_2], \quad \mathfrak{N}_2 = [p_0],$$

$$\mathfrak{M}_3 = [p_0, l_1, l_2\colon\ p_0|l_2,\ l_1\|l_2], \quad \mathfrak{N}_3 = [l_2].$$

Generalized n-gons. If $n \geqslant 5$, then, up to isomorphism, there exist, for n even, exactly one, and for n odd, exactly two Σ-free constructions in Mod(Th$(\Sigma)_\forall$), cf. Funk et al. [1985], p. 62:

$$\mathfrak{M} = [z_1, \ldots, z_{2n}\colon\ z_1|z_2|\cdots|z_{2n}|z_1],$$

$$\mathfrak{N} = [z_{n+3}, \ldots, z_{2n}\colon\ z_{n+3}|\cdots|z_{2n}].$$

Obviously, for n odd, the elements z_{n+3} and z_{2n} are either both points or both lines; hence one has two isomorphism types of Σ-free constructions. If $n = 4$, the pair

$(\mathfrak{M}, \mathfrak{N})$ is at least a Σ-construction, but reducible with respect to the following Σ-free construction:

$$\mathfrak{M}' = [z_1, z_6, z_7, z_8\colon\ z_6|z_7|z_8|z_1], \quad \mathfrak{N}' = [z_7, z_8\colon\ z_7|z_8].$$

This is a consequence of the fact that any nonincident point-line pair in a *generalized 4-gon* $\mathfrak{I}$ has always distance 3 in the graph $\Delta(\mathfrak{I})$, whereas in generalized n-gons with $n \geqslant 5$ the elements $z_2, \ldots, z_{n+1}$ serve to fix the right distance between z_1 and z_{n+2} in $\Delta(\mathfrak{M})$.

k-nets. Up to isomorphism, there exist precisely $k + \frac{1}{2}k(k-1)$ Σ-free constructions in $\mathrm{Mod}(\mathrm{Th}(\Sigma)_\forall)$, see Funk et al. [1985], p. 72:

$$\mathfrak{M}_\lambda = \left[p_0, l_1^i, l_2^j\colon\ p_0|l_1^i, p_0|l_2^j\right], \quad \mathfrak{N}_\lambda = [p_0] \quad \text{for } \lambda = (i, j)$$

with $1 \leqslant i < j \leqslant k$;

$$\mathfrak{M}_j = \left[p_0, l_0^j\colon\ p_0|l_0^j\right], \quad \mathfrak{N}_j = \left[l_0^j\right] \quad \text{for } j = 1, \ldots, k.$$

Benz planes. Up to isomorphism, one has the following three Σ-free constructions for all (projective and nonprojective) Benz planes:

$$\mathfrak{M}_1 = \left[p_1, p_2, p_3, b_0\colon\ p_i|b_0 \text{ for } 1 \leqslant i \leqslant 3\right], \quad \mathfrak{N}_1 = [b_0];$$

$$\mathfrak{M}_2 = \left[p_1, p_2, b_1, b_2\colon\ p_1|b_1, p_1|b_2, p_2|b_2, \rho(b_1, b_2, p_1)\right], \quad \mathfrak{N}_2 = [b_2];$$

$$\mathfrak{M}_3 = \left[p_1, p_2, b_1, b_2\colon\ p_\mu|b_\nu \text{ for } \mu, \nu = 1, 2\right],$$

$$\mathfrak{N}_3 = [p_1] \quad \text{in the case of a nonprojective Benz plane,}$$

$$\mathfrak{N}_3' = [p_1, p_2] \quad \text{in the case of a projective Benz plane.}$$

Note that, for projective Benz planes, $(\mathfrak{M}_3, \mathfrak{N}_3)$ is a Σ-construction, too, but reducible with respect to the Σ-construction $(\mathfrak{M}_3, \mathfrak{N}_3')$. In the case of Möbius planes, these are the only ones. In the case of Laguerre planes, up to isomorphism there exists one further Σ-free construction:

$$\mathfrak{M}_4 = \left[p_1, p_2, b_1\colon\ p_1|b_1, p_1\|_1 p_2\right], \quad \mathfrak{N}_4 = [p_1].$$

Finally, in the case of Minkowski planes, up to isomorphism, there exist two additional Σ-free constructions besides $(\mathfrak{M}_4, \mathfrak{N}_4)$:

$$\mathfrak{M}_5 = \left[p_1, p_2, b_1\colon\ p_1|b_1, p_1\|_2 p_2\right], \quad \mathfrak{N}_5 = [p_1];$$

$$\mathfrak{M}_6 = \left[p_1, p_2, p_0\colon\ p_1\|_1 p_0, p_2\|_2 p_0\right], \quad \mathfrak{N}_6 = [p_0].$$

For the proof, see Funk et al. [1985], pp. 56, 57.

(k,n)-Steiner systems. Up to isomorphism, there exist precisely two Σ-free constructions in Mod(Th(Σ)$_\forall$):

$$\mathfrak{M}_1 = \left[p_1,\dots,p_k,b_0\colon\ p_\lambda | b_0 \text{ for } 1 \leqslant \lambda \leqslant k\right], \quad \mathfrak{N}_1 = [b_0];$$

$$\mathfrak{M}_2 = \left[p_1,\dots,p_n,b_0\colon\ p_\lambda | b_0 \text{ for } 1 \leqslant \lambda \leqslant n\right],$$

$$\mathfrak{N}_2 = [p_1,\dots,p_n].$$

DEFINITION 4. Let $\mathfrak{I}$ be a model of Th(Σ)$_\forall$. A finite nonempty submodel $\mathfrak{J}$ of $\mathfrak{I}$ is called *Σ-hyperfree* in $\mathfrak{I}$ provided that for each restricted interior construction $\langle\mathfrak{J}\rangle_\mathfrak{I}$ there exists a Σ-free construction $(\mathfrak{M},\mathfrak{N})$ with embeddings $\mathfrak{J} \hookrightarrow \mathfrak{N}$ and $\langle\mathfrak{J}\rangle_\mathfrak{I} \hookrightarrow \mathfrak{M}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{J} & \hookrightarrow & \mathfrak{N} \\ \downarrow & & \downarrow \\ \langle\mathfrak{J}\rangle_\mathfrak{I} & \hookrightarrow & \mathfrak{M} \end{array}$$

commutes where $\mathfrak{J}$ and $\mathfrak{N}$ are embedded into $\langle\mathfrak{J}\rangle_\mathfrak{I}$ and $\mathfrak{M}$, respectively. A finite nonempty model $\mathfrak{I}$ of Th(Σ)$_\forall$ is called *Σ-confined* if it does not contain any Σ-hyperfree submodel. Models of Th(Σ)$_\forall$ which can be seen as unions of finite Σ-confined submodels are still named *Σ-confined.* A model of Th(Σ)$_\forall$ is called *Σ-open* if it does not contain any Σ-confined submodel.

Concerning the geometries presented in Section 1.2 we yield the following.

Σ-hyperfree elements in a partial *projective plane* are just those elements which are incident with at most two elements of the other sort. Hence a finite nonempty partial projective plane is Σ-confined if and only if each of its elements is incident with at least three distinct elements of the other sort (cf. Funk et al. [1985], p. 35).

Σ-hyperfree elements in a partial *affine plane* $\mathfrak{I}$ are those points of $\mathfrak{I}$ which are incident in $\mathfrak{I}$ with at most two lines and those lines of $\mathfrak{I}$ which are incident with at most one point of $\mathfrak{I}$, or with two points of $\mathfrak{I}$ while forming a trivial parallel class in $\mathfrak{I}$ (cf. Funk et al. [1985], p. 40). Hence a nonvoid finite partial affine plane is Σ-confined if and only if for each element $a \in \mathfrak{I}$ there exist at least three further elements $a_1, a_2, a_3 \in \mathfrak{I}$ such that aR_ia_i, $i = 1,2,3$, where $\|$ occurs at most once among the relation symbols R_i (cf. Schleiermacher and Strambach [1969], p. 26).

For $n \geqslant 4$, an irreducible chain $z_1|z_2|\cdots|z_{n-2}$ is Σ-hyperfree in a partial *generalized n-gon* if $z_2,\dots,z_{n-3}$ have no further incidences and one of the elements z_1, z_{n-2} has at most two incidences. Thus Σ-hyperfree elements in a partial generalized n-gon are those elements which are incident with at most one element of the other sort or lie in a Σ-hyperfree irreducible chain. Hence a finite nonvoid partial generalized n-gon is

Σ-confined if each of its elements is incident with at least three distinct elements of the other sort (cf. Funk et al. [1985], p. 63).

Σ-hyperfree elements in a partial *k-net* $\mathfrak{J}$ are those points of $\mathfrak{J}$ which are incident with at most two lines and those lines which are incident with at most one point. Hence a finite nonvoid partial k-net $\mathfrak{J}$ is Σ-confined if, and only if, each point of $\mathfrak{J}$ is incident with at least three distinct and each line is incident with at least two distinct points.

In the case of a *Benz plane*, for the sake of brevity, define the *valency* of an element x in a finite partial Benz plane to be the sum of the number of distinct elements incident with x and the number of distinct nontrivial parallel classes or tangent pencils containing x (where nontrivial means with at least two distinct elements). Then a point p and a block b of a finite partial Benz plane $\mathfrak{J}$ are Σ-hyperfree in $\mathfrak{J}$ if and only if their valency is at most 2 and 3, respectively. Hence a finite nonvoid partial Benz plane is Σ-confined if and only if the valency of each point and each block of $\mathfrak{J}$ is at least 3 and 4, respectively (cf. Funk et al. [1985], p. 54, Schleiermacher and Strambach [1969], p. 212; Heiše and Sörensen [1973], Iden [1984a].

Σ-hyperfree elements of a partial (k, n)-*Steiner system* $\mathfrak{J}$ are precisely those points of $\mathfrak{J}$ which are incident with at most one block of $\mathfrak{J}$ and those blocks of $\mathfrak{J}$ which are incident with at most k points of $\mathfrak{J}$. Hence a finite nonempty partial (k, n)-Steiner system is Σ-confined if and only if each of its points is incident with at least two distinct blocks and each of its blocks is incident with at least $k + 1$ distinct points.

DEFINITION 5.

(i) Let $\mathfrak{J}$ be a model of $\mathrm{Th}(\Sigma)_\forall$. An *extension series* is an ascending sequence E: $\mathfrak{J} = \mathfrak{J}_0 \subseteq \mathfrak{J}_1 \subseteq \mathfrak{J}_2 \subseteq \cdots$ of models of $\mathrm{Th}(\Sigma)_\forall$. If the *extension*

$$E(\mathfrak{J}) := \bigcup_{\nu=0}^{\infty} \mathfrak{J}_\nu$$

is a model of the 'full' theory $T = \mathrm{Th}(\Sigma)$, then E is called an *extension process*.

(ii) We will say that $E(\mathfrak{J})$ is *generated over* $\mathfrak{J}$ if one has for all $\nu \in \mathrm{N}_0$:

(a) for each embedding ε of a Σ-brick $\mathfrak{N}$ into $\mathfrak{J}_{\nu+1} \setminus \mathfrak{J}_\nu$ and each restricted interior construction $\langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}}$ with $\langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}} \setminus \mathfrak{N}^\varepsilon \subseteq \mathfrak{J}_\nu$, there exists a Σ-free construction $(\mathfrak{M}, \mathfrak{N})$ with an embedding $\mathfrak{M} \hookrightarrow \langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{N} & \longrightarrow & \mathfrak{N}^\varepsilon \\ \cap\downarrow & & \cap\downarrow \\ \mathfrak{M} & \hookrightarrow & \langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}} \end{array}$$

commutes;

(b) each element x of $\mathfrak{J}_{\nu+1} \setminus \mathfrak{J}_\nu$ is contained in such a Σ-brick $\mathfrak{N}^\varepsilon$ embedded in $\mathfrak{J}_{\nu+1} \setminus \mathfrak{J}_\nu$.

(iii) We call an extension series *Σ-hyperfree* if for all $\nu \in \mathrm{N}_0$ each submodel $\mathfrak{J}$ which either is a Σ-brick or consists only of one element, and which lies in $\mathfrak{J}_{\nu+1} \setminus \mathfrak{J}_\nu$, is Σ-hyperfree in $\mathfrak{J}_{\nu+1}$.

(iv) If $E(\mathfrak{J})$ is generated over $\mathfrak{J}$ via a Σ-hyperfree extension series, then $E(\mathfrak{J})$ is named *Σ-freely generated* over $\mathfrak{J}$. If $\mathfrak{J}$ is a Σ-open model of $\mathrm{Th}(\Sigma)_\forall$ and if E is an extension process, then $E(\mathfrak{J})$ is called a *Σ-free model* of $\mathrm{Th}(\Sigma)$.

Concerning the geometries presented in Section 1.2 we remark the following.

The free extension process for *projective planes* has been introduced by Hall [1943].

The partial plane $\mathfrak{J}_{2\nu+1}$ extends the partial plane $\mathfrak{J}_{2\nu}$ in such a way that, for each unordered pair of distinct points p_1, p_2 in $\mathfrak{J}_{2\nu}$ not joined by any line of $\mathfrak{J}_{2\nu}$, a new line is added which in $\mathfrak{J}_{2\nu+1}$ is only incident with the points p_1, p_2.

The partial plane $\mathfrak{J}_{2\nu+2}$ extends the partial plane $\mathfrak{J}_{2\nu+1}$ in such a way that, for each unordered pair of distinct lines l_1, l_2 in $\mathfrak{J}_{2\nu+1}$ not meeting at any point of $\mathfrak{J}_{2\nu+1}$, a new point is added which in $\mathfrak{J}_{2\nu+2}$ is only incident with the lines l_1, l_2.

The free extension

$$F(\mathfrak{J}_0) = \bigcup_{\nu=0}^{\infty} \mathfrak{J}_\nu$$

is a nondegenerate projective plane (i.e. it contains a quadrangle) if and only if $\mathfrak{J}_0$ is a nondegenerate partial plane in the sense of Pickert [1975], p. 13.

The free extension process for *affine planes* can be described as follows (see Schleiermacher and Strambach [1969], p. 28).

The partial affine plane $\mathfrak{J}_{3\nu+1}$ extends the partial affine plane $\mathfrak{J}_{3\nu}$ in such a way that, for each unordered pair of distinct points p_1, p_2 in $\mathfrak{J}_{3\nu}$ not joined by any line of $\mathfrak{J}_{3\nu}$, a new line is added which in $\mathfrak{J}_{3\nu+1}$ is only incident with the points p_1, p_2 and which is not parallel to any other line of $\mathfrak{J}_{3\nu+1}$.

The partial affine plane $\mathfrak{J}_{3\nu+2}$ extends $\mathfrak{J}_{3\nu+1}$ in such a way that, for each unordered pair of lines l_1, l_2 in $\mathfrak{J}_{3\nu+1}$ neither parallel to each other nor meeting at any point of $\mathfrak{J}_{3\nu+1}$, a new point is added which in $\mathfrak{J}_{3\nu+2}$ is only incident with the lines l_1, l_2.

The partial affine plane $\mathfrak{J}_{3\nu+3}$ extends $\mathfrak{J}_{3\nu+2}$ in such a way that, for each nonvoid parallel class P of lines and each point p_0 in $\mathfrak{J}_{3\nu+2}$ not incident with any line of P, a new line is added which in $\mathfrak{J}_{3\nu+3}$ is only incident with the point p_0 and belongs exclusively to P.

The free extension

$$F(\mathfrak{J}_0) = \bigcup_{\nu=0}^{\infty} \mathfrak{J}_\nu$$

is a nondegenerate affine plane (i.e. it contains a triangle) if and only if $\mathfrak{J}_0$ is a nondegenerate partial affine plane in the sense of Schleiermacher and Strambach [1969], p. 29.

The free extension process for *generalized n-gons*, given below, can be found in Tits [1977].

The partial generalized n-gon $\mathfrak{I}_{\nu+1}$ extends the partial generalized n-gon $\mathfrak{I}_\nu$ in such a way that, for each unordered pair of elements x, y in $\mathfrak{I}_\nu$ with distance $n+1$ in the graph $\Delta(\mathfrak{I}_\nu)$, $n-2$ new distinct elements $z_1, \ldots, z_{n-2}$ are added such that in $\mathfrak{I}_{\nu+1}$ these new elements occur in precisely the following irreducible chain:

$$x|z_1|z_2|\cdots|z_{n-2}|y.$$

The free extension

$$F(\mathfrak{I}_0) = \bigcup_{\nu=0}^{\infty} \mathfrak{I}_\nu$$

is a nondegenerate generalized n-gon if $\mathfrak{I}_0$ satisfies the following conditions:

(i) the graph $\Delta(\mathfrak{I}_0)$ is connected if $n \geqslant 5$;

(ii) $\mathfrak{I}_0$ is not too poor (e.g., $\mathfrak{I}_0$ contains an $(n+1)$-gon, or the graph $\Delta(\mathfrak{I}_0)$ has diameter $\geqslant n+2$).

The free extension process for *k-nets* can be described as follows (cf. Barlotti and Strambach [1983], p. 12).

The partial k-net $\mathfrak{I}_{2\nu+1}$ extends the partial k-net $\mathfrak{I}_{2\nu}$ in such a way that, for each unordered pair of lines l_0^i, l_1^j with $i \neq j$ having no point of $\mathfrak{I}_{2\nu}$ in common, a new point is added which in $\mathfrak{I}_{2\nu+1}$ is only incident with the lines l_0^i and l_1^j. The partial k-net $\mathfrak{I}_{2\nu+2}$ extends $\mathfrak{I}_{2\nu+1}$ in such a way that, for each point p_0 incident in $\mathfrak{I}_{2\nu+1}$ with less than k distinct lines, for each missing sort j, a new line l^j is added which in $\mathfrak{I}_{2\nu+2}$ is only incident with the point p_0. The free extension

$$F(\mathfrak{I}_0) = \bigcup_{\nu=0}^{\infty} \mathfrak{I}_\nu$$

is a nondegenerate k-net (i.e. it contains k nonconfluent lines of pairwise distinct sorts) if and only if $\mathfrak{I}_0$ is a nondegenerate partial k-net in the sense of Barlotti and Strambach [1983], p. 13.

The free extension processes for *Benz planes* can be found in Funk et al. [1985], p. 54; Schleiermacher and Strambach [1969], p. 212; Heise and Sörensen [1973]; Iden [1984a].

The partial Benz plane $\mathfrak{I}_{\nu+1}$ extends the partial Benz plane $\mathfrak{I}_\nu$ in such a way that, for each unordered triple of points, nonparallel in pairs, and without any joining block in $\mathfrak{I}_\nu$, a new block is added which in $\mathfrak{I}_{\nu+1}$ is incident with exactly these three points and does not touch any other block of $\mathfrak{I}_{\nu+1}$.

The partial Benz plane $\mathfrak{I}_{\nu+2}$ extends $\mathfrak{I}_{\nu+1}$ in such a way that, for each incident point-block pair (p, b) and for each point q neither incident with b nor parallel to p and such that no block through p and q is touching b in p, a new block is added which in $\mathfrak{I}_{\nu+2}$ is only incident with p, q and touches b in p.

The partial Benz plane $\mathfrak{I}_{\nu+3}$ extends $\mathfrak{I}_{\nu+2}$ in such a way that, for each unordered pair of nontangent blocks incident with only one point of $\mathfrak{I}_{\nu+2}$, a new point is added which in $\mathfrak{I}_{\nu+3}$ is only incident with these two blocks and is not parallel to any other point of $\mathfrak{I}_{\nu+3}$.

The partial Laguerre or Minkowski plane $\mathfrak{I}_{\nu+4}$ extends $\mathfrak{I}_{\nu+3}$ in such a way that, for each nonincident point-block pair (p, b) and each parallel class $\bar{p}$ of p without an element incident with b, a new point is added which in $\mathfrak{I}_{\nu+4}$ is only incident with b and belongs exclusively to $\bar{p}$.

The partial Minkowski plane $\mathfrak{I}_{\nu+5}$ extends $\mathfrak{I}_{\nu+4}$ in such a way that, for each ordered pair (p_1, p_2) of points for which there is no point q in $\mathfrak{I}_{\nu+4}$ with $p_1 \|_1 q \|_2 p_2$, a new point q is added which in $\mathfrak{I}_{\nu+5}$ exactly fulfills those two relations.

In the case of a projective Benz plane, the following additional step has to be inserted.

The partial Benz plane $\mathfrak{I}_{\nu+6}$ extends $\mathfrak{I}_{\nu+5}$ in such a way that, for each unordered pair of distinct blocks without any intersection point in $\mathfrak{I}_{\nu+5}$, two new points are added which in $\mathfrak{I}_{\nu+6}$ are only incident with these two blocks and are not parallel to any other point of $\mathfrak{I}_{\nu+6}$.

The free extension

$$F(\mathfrak{I}_0) = \bigcup_{\nu=0}^{\infty} \mathfrak{I}_\nu$$

is a nondegenerate Benz plane (i.e. there exist, in the case of Möbius and Laguerre planes, at least one block b and three pairwise nonparallel points exactly two of which are incident with b, and, in the case of Minkowski planes, at least one block b and two nonparallel points exactly one of which is incident with b) if and only if $\mathfrak{I}_0$ is nondegenerate in the sense of Schleiermacher and Strambach [1969], p. 217; Artzy [1980]; Iden [1984a].

The free extension process for (k, n)-*Steiner systems* can be described as follows.

The partial (k, n)-Steiner system $\mathfrak{I}_{2\nu+1}$ extends the partial (k, n)-Steiner system $\mathfrak{I}_{2\nu}$ in such a way that, for each unordered k-tuple of distinct points not incident with one and the same block in $\mathfrak{I}_{2\nu}$, a new block is added which in $\mathfrak{I}_{2\nu+1}$ is only incident with those k points.

The partial (k, n)-Steiner system $\mathfrak{I}_{2\nu+2}$ extends $\mathfrak{I}_{2\nu+1}$ in such a way that, for each block incident in $\mathfrak{I}_{2\nu+1}$ with $m < n$ distinct points, $n - m$ new points are added which in $\mathfrak{I}_{2\nu+2}$ are only incident with that block.

The free extension

$$F(\mathfrak{I}_0) = \bigcup_{\nu=0}^{\infty} \mathfrak{I}_\nu$$

is a nondegenerate (k, n)-Steiner system (i.e. it contains more than one block) if and only if either $\mathfrak{I}_0$ contains more than one block or more than k distinct points not incident with just one block.

REMARK. The free extension processes F: $\mathfrak{I}_0 \subseteq \mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq \cdots$ described above can be modified inserting additional steps of the following kind.

The partial model $\mathfrak{I}'_\nu$ extends the partial model $\mathfrak{I}_\nu$ in such a way that new elements are added which are Σ-hyperfree in $\mathfrak{I}'_\nu$, where, for generalized n-gons with $n \geqslant 5$, the graph $\Delta(\mathfrak{I}'_\nu)$ must not lose its connectedness. Then

$$F'(\mathfrak{I}_0) = \bigcup_{\nu=0}^{\infty} \mathfrak{I}'_\nu$$

is a Σ-hyperfree extension and each Σ-hyperfree extension process for the above theories can be obtained in this way.

But it may happen that one has $F'(\mathfrak{J}_0) = F(\mathfrak{J})$ for a suitable $\mathfrak{J} \in \mathrm{Mod}(\mathrm{Th}(\Sigma)_\forall)$. Actually, this is a rather intrinsic problem if F stands for the model-theoretic construction of the free extension process given in 1.4. In the beginning of 2.1, we shall survey all what is known about this problem thus far.

PROPOSITION 1. *Let E: $\mathfrak{J} = \mathfrak{J}_0 \subseteq \mathfrak{J}_1 \subseteq \mathfrak{J}_2 \subseteq \cdots$ be a Σ-hyperfree extension process. Then one has*:

(i) *each Σ-confined submodel $\mathfrak{L}$ of $E(\mathfrak{J})$ is already contained in $\mathfrak{J}$*;

(ii) *in particular, if $\mathfrak{J}$ is Σ-confined, then every automorphism of $E(\mathfrak{J})$ induces an automorphism of $\mathfrak{J}$.*

PROOF. See Funk and Strambach [1991], Proposition 1. □

LEMMA 1. *Let E: $\mathfrak{J} = \mathfrak{J}_0 \subseteq \mathfrak{J}_1 \subseteq \mathfrak{J}_2 \subseteq \cdots$ be an extension series. Suppose that for every $\nu \in \mathrm{N}_0$ one has*:

(i) *for each embedding ε of a Σ-brick $\mathfrak{N}$ into $\mathfrak{J}_{\nu+1} \setminus \mathfrak{J}_\nu$, every restricted interior construction $\langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}}$ with $\langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}} \setminus \mathfrak{N}^\varepsilon \subseteq \mathfrak{J}_\nu$, together with $\mathfrak{N}^\varepsilon$ forms a Σ-free construction*;

(ii) *each element of $\mathfrak{J}_{\nu+1} \setminus \mathfrak{J}_\nu$ is contained in such a Σ-brick $\mathfrak{N}^\varepsilon$.*

Then $E(\mathfrak{J})$ is Σ-freely generated over $\mathfrak{J}$.

PROOF. See Funk and Strambach [1991], Lemma 1. □

THEOREM 1. *Submodels of Σ-free models $E(\mathfrak{J})$ can be written as extensions of Σ-open models of* $\mathrm{Th}(\Sigma)_\forall$ *obtained by Σ-hyperfree extension processes.*

PROOF. See Funk and Strambach [1991], Theorem 1. □

DEFINITION 6. A Σ-free construction $(\mathfrak{M}, \mathfrak{N})$ is called *tied* if for each Σ-brick $\mathfrak{N}' \subseteq \mathfrak{N}$ every automorphism of $\mathfrak{M} \setminus \mathfrak{N}'$ which induces the identity on $\mathfrak{M} \setminus \mathfrak{N}$ can be extended to an automorphism of $\mathfrak{M}$. A theory $\mathrm{Th}(\Sigma)$ is named *tied* if each Σ-free construction is tied.

REMARK. A sufficient condition for a theory $\mathrm{Th}(\Sigma)$ to be tied is that every Σ-brick consists only of at most two elements or that the Σ-bricks are pairwise nonembeddable into each other. The geometries 1.2.1, 1.2.2, 1.2.4–1.2.6 (as well as generalized 3-gons and 4-gons) are examples for the first situation, whereas generalized n-gons with $n \geqslant 5$ fulfil the second situation. In general, however, theories need not be tied; cf. Funk and Strambach [1991], p. 356.

LEMMA 2. *Let $E(\mathfrak{J})$ be a Σ-freely generated model of a tied theory* $\mathrm{Th}(\Sigma)$. *Then for each embedding ε of a Σ-brick $\mathfrak{N}$ into $\mathfrak{J}_{\nu+1} \setminus \mathfrak{J}_\nu$ for some $\nu \in \mathrm{N}_0$ with $\langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}} \setminus \mathfrak{N}^\varepsilon \subseteq \mathfrak{J}_\nu$ each restricted interior construction $\langle \mathfrak{N}^\varepsilon \rangle_{\mathfrak{J}_{\nu+1}}$, together with $\mathfrak{N}^\varepsilon$, forms a Σ-free construction.*

PROOF. See Funk and Strambach [1991], Lemma 2. □

DEFINITION 7. A Σ-free construction $(\mathfrak{M}, \mathfrak{N})$ is called *compatible* with respect to homomorphisms if for each homomorphism α: $\mathfrak{M} \setminus \mathfrak{N} \to \mathfrak{G}$ into some model $\mathfrak{G}$ of the 'full' theory Th(Σ) there exists an extension α': $\mathfrak{M} \to \mathfrak{G}$.

REMARK. All the Σ-free constructions listed above turn out to be compatible with respect to homomorphisms, which can immediately be verified. In general, however, compatibility does not hold true. For a counterexample, see Funk and Strambach [1991], p. 358.

THEOREM 2. *Suppose that the theory* Th(Σ) *is tied and that all Σ-free constructions are compatible with respect to homomorphisms. Let $E(\mathfrak{J})$ and $\widehat{E}(\mathfrak{J})$ be models of the 'full' theory* Th(Σ) *such that $E(\mathfrak{J})$ is generated and $\widehat{E}(\mathfrak{J})$ is freely generated over $\mathfrak{J}$. Then there exists an epimorphism α: $\widehat{E}(\mathfrak{J}) \to E(\mathfrak{J})$ such that $\alpha|_{\mathfrak{J}}$ is the identity.*

PROOF. See Funk and Strambach [1991], Theorem 2.

Note that, in general, α is not unique; this already happens for homomorphisms of the free projective plane $\mathfrak{F}_4 = F(\mathfrak{J}(4))$ (cf. 2.1) onto the projective plane over GF(2) generated by $\mathfrak{J}(4)$. □

1.4. *Model-theoretic construction of free extensions*

HYPOTHESES. Let T be a theory formulated in a many-sorted first-order language $\mathcal{L}$ without any function or constant symbol. Let Σ be a set of axioms for Mod(T). To confine our attention to the essential situation, let us assume that $\mathcal{L}$ consists of a finite number of relation symbols (in particular, there is only a finite number of sorts $s_1, \ldots, s_k$), and that there exists only a finite family of isomorphism types of Σ-free constructions $(\mathfrak{M}_0, \mathfrak{N}_0), \ldots, (\mathfrak{M}_{t-1}, \mathfrak{N}_{t-1})$. As an indispensable condition, however, we suppose:

(*) Each sort of variables is represented in at least one Σ-brick $\mathfrak{N}_i$ for some $i \in \{0, \ldots, t-1\}$.

Finally, we assume that the disjoint union of partial models of T is again a partial model of T. (In general, this need not be true; cf. Funk and Strambach [1991], p. 358.)

CONSTRUCTION. Let $\mathfrak{J}$ be a model of Th(Σ)$_\forall$. We construct an ascending sequence $\mathfrak{J} = \mathfrak{J}_0 \subseteq \mathfrak{J}_1 \subseteq \mathfrak{J}_2 \subseteq \cdots$ of models of Th(Σ)$_\forall$ by the following recursive definition in t steps.

Suppose that $\mathfrak{J}_{t\nu+\mu}$ has been defined for some $\nu \in \mathrm{N}_0$ and some μ with $0 \leqslant \mu < t-1$. Let $N_{\mu+1}$ be the universe of $\mathfrak{N}_{\mu+1}$. Let $\mathcal{I}$ be the subset of all *essential* relation symbols occurring in $\mathfrak{M}_{\mu+1}$, i.e. all relation symbols R such that

$$\Delta_{\mathcal{R}\setminus\{R\}}(\mathfrak{M}_{\mu+1}) \neq \Delta(\mathfrak{M}_{\mu+1})$$

(cf. Definition 1). Consider the model $\mathfrak{M}_{\mu+1} \setminus \mathfrak{N}_{\mu+1}$ of Th(Σ)$_\forall$ and all those embeddings η into $\mathfrak{J}_{t\nu+\mu}$ which cannot be extended to embeddings of $\mathfrak{M}_{\mu+1}$ into $\mathfrak{J}_{t\nu+\mu}$. Any two such embeddings

$$\eta_1, \eta_2\colon \mathfrak{M}_{\mu+1} \setminus \mathfrak{N}_{\mu+1} \hookrightarrow \mathfrak{I}_{t\nu+\mu}$$

are said to be equivalent if the interior constructions $(\mathfrak{M}_{\mu+1} \setminus \mathfrak{N}_{\mu+1})^{\eta_i}$ in $\mathfrak{I}_{t\nu+\mu}$ with respect to $\mathcal{I}$ are equal, i.e.

$$\big((\mathfrak{M}_{\mu+1} \setminus \mathfrak{N}_{\mu+1})^{\eta_1}\big)^{\mathcal{I}}_{\mathfrak{I}_{t\nu+\mu}} = \big((\mathfrak{M}_{\mu+1} \setminus \mathfrak{N}_{\mu+1})^{\eta_2}\big)^{\mathcal{I}}_{\mathfrak{I}_{t\nu+\mu}}.$$

Then, for every equivalence class $\bar{\eta}$ of such embeddings, label by $\bar{\eta}$ a copy of the universe $N_{\mu+1}$, say $N^{\bar{\eta}}_{\mu+1}$. Put

$$I_{t\nu+\mu+1} := \bigcup_{\bar{\eta}} N^{\bar{\eta}}_{\mu+1} \cup I_{t\nu+\mu}.$$

Extend the interpretation functions from $I_{t\nu+\mu}$ and $N^{\bar{\eta}}_{\mu+1}$ in the natural way. (Thus far, $\mathfrak{I}_{t\nu+\mu+1}$ would consist of the disjoint union of the models $\mathfrak{I}_{t\nu+\mu}$ and the copies of $\mathfrak{N}_{\mu+1}$.) Finally, for each $\bar{\eta}$, let $V^{\bar{\eta}}$ be the union of all universes U_η of the submodels $(\mathfrak{M}_{\mu+1} \setminus \mathfrak{N}_{\mu+1})^{\eta}$ where η ranges over $\bar{\eta}$; define additional relations between elements of $V^{\bar{\eta}} \cup N^{\bar{\eta}}_{\mu+1}$ such that, for each $\eta \in \bar{\eta}$, every subset $U_\eta \cup N^{\bar{\eta}}_{\mu+1}$ gives rise to a submodel isomorphic to $\mathfrak{M}_{\mu+1}$ where the submodel $\mathfrak{N}_{\mu+1}$ of $\mathfrak{M}_{\mu+1}$ comes from the subuniverse $N^{\bar{\eta}}_{\mu+1}$ contained in $U_\eta \cup N^{\bar{\eta}}_{\mu+1}$. Then $\mathfrak{I}_{t\nu+\mu+1}$ turns out to be again a model of $\mathrm{Th}(\Sigma)_\forall$.

Define

$$F(\mathfrak{I}) := \bigcup_{\nu=0}^{\infty} \mathfrak{I}_\nu.$$

Since $\mathrm{Th}(\Sigma)_\forall$ is always inductive, $F(\mathfrak{I})$ is at least a model of $\mathrm{Th}(\Sigma)_\forall$. Moreover, $F(\mathfrak{I})$ is Σ-freely generated over $\mathfrak{I}$, cf. Lemma 1. We call $F\colon \mathfrak{I} = \mathfrak{I}_0 \subseteq \mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq \cdots$ the free *extension series* and $F(\mathfrak{I})$ the *free extension* of $\mathfrak{I}$.

With each Σ-free construction $(\mathfrak{M}_i, \mathfrak{N}_i)$, $i = 0, \ldots, t-1$, we associate two $\mathcal{L}$-sentences φ_i, ψ_i in the following way.

Let x_μ, $\mu = 1, \ldots, m(i)$, and y_ν, $\nu = 1, \ldots, n(i)$, be the elements of $\mathfrak{M}_i \setminus \mathfrak{N}_i$ and $\mathfrak{N}_i$, respectively. Let $S_i(x_\mu, y_\nu)$ be the list of all unary relations $s_\varkappa(x_\mu)$ and $s_\lambda(y_\nu)$ where $s_\varkappa$ and s_λ are the sorts of x_μ and y_ν, respectively. Let $R_i(x_\mu, y_\nu)$ be the list of all relations, all negations of relations, and all negations of equalities between elements, which hold true in $\mathfrak{M}_i$. Then we interpret x_μ and y_ν as variables with $S_i(x_\mu, y_\nu)$, which implies that $R_i(x_\mu, y_\nu)$ becomes a list of atomic formulas. Let $\wedge R_i(x_\mu, y_\nu)$ be the formula obtained by connecting all atomic formulas in $R_i(x_\mu, y_\nu)$ by $\wedge$. Write:

$$\varphi_i := (\forall x_1)\ldots(\forall x_{m(i)})(\exists y_1)\ldots(\exists y_{n(i)}) \wedge R_i(x_\mu, y_\nu),$$

$$\psi_i := (\forall x_1)\ldots(\forall x_{m(i)})(\forall y_1)\ldots(\forall y_{n(i)})(\forall y'_1)\ldots(\forall y'_{n(i)})$$
$$(\wedge R_i(x_\mu, y_\nu)) \wedge (\wedge R_i(x_\mu, y'_\nu)) \Rightarrow (y_1 = y'_1 \wedge \cdots \wedge y_{n(i)} = y'_{n(i)}).$$

Let $\mathrm{Th}(\Sigma')$ be the theory for which $\Sigma' := \{\varphi_i, \psi_i\colon\ i = 0, \ldots, t-1\}$ is a set of axioms.

DEFINITION 8. The theory $\mathrm{Th}(\Sigma)$ is called *reproducible* with respect to its Σ-free constructions if $\mathrm{Th}(\Sigma)$ is equivalent to $\mathrm{Th}(\Sigma')$.

This definition leads immediately to the following

THEOREM 3. *Let* $\mathrm{Th}(\Sigma)$ *be a theory satisfying the hypotheses of Section* 1.3. *If* $\mathrm{Th}(\Sigma)$ *is reproducible with respect to its* Σ*-free constructions, then, for each model* $\mathfrak{J}$ *of* $\mathrm{Th}(\Sigma)_\forall$, *the free extension* $F(\mathfrak{J})$ *is a model of the 'full' theory* $\mathrm{Th}(\Sigma)$, *hence* F *is an extension process.*

Evidently this theorem allows generalizations avoiding finiteness assumptions in the hypotheses.

REMARK. The theories considered in 1.2.1, 1.2.2, 1.2.4, 1.2.5, and 1.2.6 are obviously reproducible with respect to their Σ-free constructions. Concerning generalized n-gons, however, there are some troubles (perhaps due to the language and set of axioms presented in 1.2.3).

If $n = 4$, then the system Σ' of axioms derived from the Σ-free constructions by the above introduced procedure consists of the claim that for every noincident point-line pair $\{z_6, z_1\}$ there exists precisely one irreducible chain $z_7|z_8$ such that one has $z_6|z_7|z_8|z_1$. This is the traditional axiom for generalized 4-gons (disregarding requirements on non-degeneracy), see Payne and Thas [1984]. Hence the theory of generalized 4-gons is reproducible with respect to its Σ-free construction.

If, however, $n \geqslant 5$, then the system Σ' of axioms characterizes its models by the fact that each irreducible, nonclosed, chain with $n+2$ distinct elements lies in precisely one closed irreducible chain with $2n$ distinct elements. Since no ordinary m-gon with $m \leqslant (n+3)/2$ is excluded as a model for $\mathrm{Th}(\Sigma')$, the class $\mathrm{Mod}(\mathrm{Th}(\Sigma'))$ is larger than the class $\mathrm{Mod}(\mathrm{Th}(\Sigma))$. Hence, unfortunately, $\mathrm{Th}(\Sigma)$ is not reproducible with respect to its Σ-free constructions. The theory $\mathrm{Th}(\Sigma')$, however, *is* reproducible with respect to its Σ'-free constructions, which coincide with the Σ-free constructions.

The extension process for generalized n-gons described above applies for the theory $\mathrm{Th}(\Sigma')$, too. In order to get a 'full' model of $\mathrm{Th}(\Sigma)$, we must suppose that $\mathfrak{J}_0$ is a model of $\mathrm{Th}(\Sigma)_\forall$ and that the graph $\Delta(\mathfrak{J}_0)$ is connected. If $F(\mathfrak{J}_0)$ were not a model of $\mathrm{Th}(\Sigma)$, then it would contain a closed irreducible chain with less than $2n$ elements; but this chain would already lie in $\mathfrak{J}_0$, which contradicts the choice of $\mathfrak{J}_0$.

1.5. *Degenerate models of* $\mathrm{Th}(\Sigma)$

DEFINITION 9. A partial model $\mathfrak{J}$ for $\mathrm{Th}(\Sigma)$ is said to be *degenerate with respect to some extension process* F if for each substructure $\mathfrak{Z} \subseteq \mathfrak{J}$ there exists a non-negative integer n such that one has:

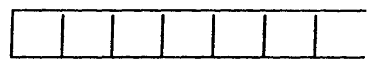
$$F\colon\ \mathfrak{Z} = \mathfrak{Z}_0 \subseteq \mathfrak{Z}_1 \subseteq \cdots \subseteq \mathfrak{Z}_n = \mathfrak{Z}_{n+\nu} \quad \text{for all } \nu \in \mathrm{N}.$$

In particular, if F is the canonic free extension process defined in 1.4, then $\mathfrak{J}$ is called Σ*-degenerate.*

REMARK.

(i) Concerning the theories of projective planes, affine planes, k-nets, Benz planes, and (k, n)-Steiner systems, one immediately verifies the equivalence of the notions 'Σ-degenerate' and 'degenerate', as mentioned in the presentation of the canonical free extension processes in 1.2.

(ii) Any Σ-degenerate generalized quadrangle can be obtained from the following generalized quadrangle (or its dual) with a suitable large cardinality $\varkappa$



$\varkappa$

in perfect analogy to the behaviour of projective planes.

2. Survey of further results on free models

2.1. *Canonical free models and free equivalence*

First of all, the apparently large variety of Σ-hyperfree extension processes can reduce to the free extension processes only: it turns out that, if $F'(\mathfrak{J})$ is a proper Σ-hyperfree extension, $F'(\mathfrak{J})$ is isomorphic to a free extension $F(\mathfrak{J})$ of a suitable (eventually infinite) partial model $\mathfrak{J}$. This holds true for projective planes (Siebenmann [1965]), affine planes (Schleiermacher and Strambach [1969], p. 30), k-nets (Barlotti and Strambach [1983], p. 15), (projective and nonprojective) Benz planes (Schleiermacher and Strambach [1969], p. 212; Iden [1984a, 1992]).

As a consequence of this we get the following strengthening of Theorem 1.

THEOREM 4. *Subgeometries of Σ-free models are Σ-free where 'geometries' stands for projective planes, affine planes, k-nets, as well as for projective and nonprojective Benz planes.*

Next let us survey canonical representatives for the isomorphism classes of Σ-free models $F(\mathfrak{J})$ which are free extensions of finite partial models $\mathfrak{J}$. This can be done in terms of the following standard partial models $\mathfrak{J}(n)$ or the corresponding free extensions $\mathfrak{F}_n := F(\mathfrak{J}(n))$, where $n \geqslant n_\Sigma$ for some suitable minimum number n_Σ.

DEFINITION 10. In the case of projective planes, $\mathfrak{J}(n)$ consists of a line l, two distinct points not incident with l, and $n-2$ distinct points incident with l; the minimum number n_Σ equals 4. In the case of affine planes (or projective and nonprojective Benz planes), $\mathfrak{J}(n)$ consists of a line (or block) l, one point not incident with l, and $n - 1$ (pairwise nonparallel) points incident with l; the minimum number n_Σ equals 2 for Minkowski planes, else $n_\Sigma = 3$. In the case of k-nets, $\mathfrak{J}(n)$ consists of $n - 1$ distinct lines of the first sort and one line of the second sort; the minimum number n_Σ equals 3.

COROLLARY 1. *Every $\mathfrak{F}_n$ contains subgeometries isomorphic to $\mathfrak{F}_m$ for each $m \geqslant n_\Sigma$.*

PROOF. In every $\mathfrak{F}_n$ there exist submodels isomorphic to $\mathfrak{I}(m)$ for each $m \geqslant n_\Sigma$. Consider the subgeometry of $\mathfrak{F}_n$ generated over $\mathfrak{I}(m)$ and apply Theorem 4 which says that the extension within $\mathfrak{F}_n$ is isomorphic to the free extension $F(\mathfrak{I}(m))$. □

THEOREM 5. *Let $F(\mathfrak{I}_0)$ be a nondegenerate Σ-free model generated over a finite partial model $\mathfrak{I}_0$. Then $F(\mathfrak{I}_0)$ proves to be isomorphic to $\mathfrak{F}_n$ for one precise natural number n where* Mod(Th(Σ)) *ranges over the classes of projective planes, affine planes, k-nets and projective Benz planes.*

For the proofs of Theorem 5, see Pickert [1975], Hall [1943], Schleiermacher and Strambach [1969], Barlotti and Strambach [1988], Iden [1984a].

REMARK. Slightly varying the definitions of *confined* and *open*, Iden [1994] proves a theorem for nonprojective Benz planes which, disregarding one exceptional case, has the same wording as Theorem 5. The exception is the Laguerre plane $\mathfrak{G}$ generated by a block l and two nonparallel points which are not incident with l.

DEFINITION 11. Let $\mathfrak{I}$ be a finite partial model and denote by π, λ, ι, and (ρ, σ) the number of points, lines or blocks, incidences (and tangent pencils and parallel classes, if any) of $\mathfrak{I}$, respectively; recall that a tangent pencil may consist of only one incident point-block pair and that a parallel class may contain only one point or line. Then the number

$$r(\mathfrak{I}) := \begin{cases} 2(\pi+\lambda)-\iota & \text{for projective planes,} \\ 2\pi+\lambda-\iota+\sigma & \text{for affine planes,} \\ 2\pi+\lambda-\iota & \text{for } k\text{-nets,} \\ 2\pi+3\lambda-2\iota+\rho & \text{for Möbius planes,} \\ \pi+3\lambda-2\iota+\rho+\sigma & \text{for Laguerre planes} \end{cases}$$

is called the *rank* of $\mathfrak{I}$. Usually one also says that $F(\mathfrak{I})$ has rank $r(\mathfrak{I})$. Theorem 5 implies that the rank is the only invariant for free extensions of finite partial models.

THEOREM 6. *Let* Mod(Th(Σ)) *be the class of projective planes, affine planes, k-nets, projective Möbius and Laguerre planes, and suppose that $\mathfrak{I}_1, \mathfrak{I}_2$ are finite partial Σ-open non-Σ-degenerate models. Then $F(\mathfrak{I}_1)$ is isomorphic to $F(\mathfrak{I}_2)$ if and only if $\mathfrak{I}_1$ and $\mathfrak{I}_2$ have the same rank. In particular, the rank of the standard models $\mathfrak{F}_n$ equals $n+4 \geqslant 8$, $n+4 \geqslant 7$, $n \geqslant 3$, $n+4 \geqslant 7$, and $n+4 \geqslant 7$, respectively.*

2.2. *Free projective and affine planes*

Thus far, the classes of projective and affine planes have been studied more thoroughly and intensively. This has lead to further rather specific results which lie outside of our model-theoretic approach. We would like to mention some important ones in the sequel.

THEOREM 7. *In a free projective or affine plane $\mathfrak{F}$ any ascending sequence of subplanes $\mathfrak{E}_0 \subsetneqq \mathfrak{E}_1 \subsetneqq \mathfrak{E}_2 \subsetneqq \cdots$ of bounded rank $r(\mathfrak{E}_i) \leqslant m$ is finite. If $\mathfrak{F}$ contains a subplane $\mathfrak{G}$ of finite rank r which is contained in no subplane of smaller rank, then each descending sequence $\mathfrak{E}_0 \supsetneqq \mathfrak{E}_1 \supsetneqq \mathfrak{E}_2 \supsetneqq \cdots \supsetneqq \mathfrak{G}$ of subplanes of rank r is finite.*

PROOF. In the case of projective planes the assertion follows from Sandler [1964], Theorems 3 and 4; cf. also Iden [1974], Kelly [1978], Kopejkina [1945]. Now assume that $\mathfrak{F}$ is an affine plane. Then the projective closure $\overline{\mathfrak{F}}$ of $\mathfrak{F}$, as well as the projective closures $\overline{\mathfrak{E}}_0, \overline{\mathfrak{E}}_1, \overline{\mathfrak{E}}_2, \ldots$, are free projective planes, see Schleiermacher and Strambach [1969], Theorem B11, and the above results may be applied. □

THEOREM 8. *Let* Mod(Th(Σ)) *be the class of projective or affine planes. Then the intersection of two subplanes both isomorphic to $\mathfrak{F}_{n_\Sigma}$ in a free plane $\mathfrak{F}$ is either degenerate or one subplane is contained in the other. The intersection of infinitely many subplanes all isomorphic to $\mathfrak{F}_{n_\Sigma}$ is always degenerate. Hence the set $\mathcal{I}$ of all subplanes of $\mathfrak{F}_{n_\Sigma}$ which are either isomorphic to $\mathfrak{F}_{n_\Sigma}$ or degenerate forms a complete lattice* $(\mathcal{I}, \subseteq)$.

PROOF. In the projective case the assertion has been shown in Joussen and Liersch [1981], cf. also Iden [1984b]. Now assume that Mod(Th(Σ)) is the class of affine planes. Let $\mathfrak{E}_1, \mathfrak{E}_2$ be two subplanes both isomorphic to $\mathfrak{F}_3$. By Schleiermacher and Strambach [1969], Theorem B11 and Lemma B13, the rank of the projective closures $\overline{\mathfrak{E}}_1$ and $\overline{\mathfrak{E}}_2$ is 8, i.e. they are isomorphic to the projective standard model $\mathfrak{F}_4$. Applying the projective version of this Theorem we obtain that $\overline{\mathfrak{E}}_1 \cap \overline{\mathfrak{E}}_2$ either is a degenerate projective plane or, say $\overline{\mathfrak{E}}_1 \subseteq \overline{\mathfrak{E}}_2$, i.e. $\mathfrak{E}_1 \subseteq \mathfrak{E}_2$. In the case that $\overline{\mathfrak{E}}_1 \cap \overline{\mathfrak{E}}_2$ is degenerate, the line at infinity l_∞ of the projective closure $\overline{\mathfrak{E}}$ is contained in $\overline{\mathfrak{E}}_1 \cap \overline{\mathfrak{E}}_2$. Removing l_∞, together with all points of $\overline{\mathfrak{E}}_1 \cap \overline{\mathfrak{E}}_2$ incident with l_∞, we obtain an affine plane without any triangle. Hence $\mathfrak{E}_1 \cap \mathfrak{E}_2$ is degenerate.

This, in turn, implies that, given an infinite family $\mathcal{F} = \{\mathfrak{E}_i\}_{i \in I}$ of distinct affine subplanes $\mathfrak{E}_i$ isomorphic to the affine standard plane $\mathfrak{E}_3$, with nondegenerate intersection $\bigcap_{i \in I} \mathfrak{E}_i$, any two members of $\mathfrak{E}$ have a nondegenerate intersection, too. Hence, $\mathcal{F}$ is linearly ordered by inclusion. Since the members of $\mathcal{F}$ are distinct in pairs, $\mathcal{F}$ can be seen as an infinite ascending sequence

$$\cdots \mathfrak{E}_\nu \subsetneqq \mathfrak{E}_{\nu+1} \subsetneqq \mathfrak{E}_{\nu+2} \subsetneqq \cdots$$

of affine planes isomorphic to $\mathfrak{F}_3$. This contradicts Theorem 8; hence $\bigcap_{i \in I} \mathfrak{E}_i$ must be degenerate. □

THEOREM 9. *A projective or affine plane is Σ-open if and only if every nondegenerate finite partial subplane of it generates a free subplane.*

PROOF. In the projective case the assertion has been shown in Dembowski [1960]. Obviously, an affine plane is Σ-open if every nondegenerate finite submodel generates a free affine subplane. To the other end, let $\mathfrak{F}$ be a Σ-open affine plane and $\mathfrak{I}$ be a partial subplane of $\mathfrak{F}$. Then $\mathfrak{I}$ is Σ-open and its projective closure $\overline{\mathfrak{I}}$ remains a projective Σ-open partial subplane of the projective closure $\overline{\mathfrak{F}}$ of $\mathfrak{F}$, which is Σ-open, too, cf. Schleiermacher and Strambach [1969], Remark B9. Hence $\overline{\mathfrak{I}}$ generates a free projective subplane $\overline{\mathfrak{E}}$ of $\overline{\mathfrak{F}}$. By Schleiermacher and Strambach [1969], Theorem B26, removing the line at infinity, $\mathfrak{F}$ remains a free affine plane, since $\mathfrak{I}$ has been an affine Σ-open partial model. □

Projective planes have also been considered as universal algebras (cf. Dembowski [1960], Appendix). For a more algebraic description of free extension processes, in terms of Universal Algebra, see Shirshov and Nikitin [1981], Nikitin [1981]. Finally, Iden [1992] gives a new description of free extensions (for projective, affine, and Benz planes) in terms of partial order: for a partial model $\mathfrak{I}$, consider the set $S(\mathfrak{I})$ of all partial models obtained from $\mathfrak{I}$ as the result of a sequence of *simple* extensions each of which extends the former one by just one free element; sequences of simple extensions define a relation between the initial and final members; this relation partially orders $S(\mathfrak{I})$; the maximal objects in $S(\mathfrak{I})$ are 'full' models by construction and prove to be order isomorphic: the resulting structure is the free extension of $\mathfrak{I}$.

2.3. *Other classes of (free) geometries*

2.3.1. All generalizations of Benz planes introduced thus far fall within the notion of *(k, m, n)-geometries* (see Funk et al. [1985], p. 52 ff.) which form a family of geometries whose languages and axioms differ only accordingly to the values of the parameters k, m, n. Also for these geometries the notions *Σ-hyperfree*, *Σ-confined*, *Σ-open*, and *Σ-degenerate*, as well as corresponding free extension processes are established (cf. Funk et al. [1985], pp. 54–56). We would like to mention that Proposition 1, Lemma 1, Theorem 1, Lemma 2, and Theorem 2 hold as well.

2.3.2. A two-parameter family of generalized affine planes are *(m, n)-planes*, see Schreiber [1979], Funk et al. [1985], p. 45. Our knowledge on free (m, n)-planes reaches the level concerning free affine planes; all results on affine planes discussed in Section 2.1 are valid for (m, n)-planes, too. In particular, the rank formula reads:

$$r = (2 - n)\pi + (2 - m)\lambda - \iota + \sigma_\pi + \sigma_\lambda,$$

where σ_π and σ_λ denote the numbers of parallel classes of points and lines, respectively.

2.3.3. Another class of free geometries, for which all results on k-nets discussed in Section 2.1 hold, is the class of *double nets*, see Barlotti and Strambach [1988], pp. 7–12; here the rank formula reads:

$$r = 2\pi + \lambda - \iota.$$

2.3.4. The large variety of classes Mod(Th(Σ)) of geometries which are reproducible with respect to their Σ-free constructions should not lead to the assumption that this holds true for every nice class of geometries. An instance is the class of *noncommutative geometries* (*LP-spaces* in the sense of André [1976, 1981, 1988]) where there is a lack of Σ-free constructions (see Funk et al. [1985], p. 83).

2.3.5. For classes of geometries Mod(Th(Σ)) whose sets Σ of axioms contain a relatively large number of *identities* (i.e. purely universal axioms), one might think of 'hyperfree' extensions like the following. In a recursive construction one first adds new elements to $\mathfrak{I}_{2\nu}$ in order to satisfy the existential requirements of Σ; at least this yields a model, say $\mathfrak{I}_{2\nu+1}$, for the underlying language; secondly one introduces some kind of 'congruence relation' $\equiv$ in $\mathfrak{I}_{2\nu+1}$ induced by the identities of Σ; thirdly one defines $\mathfrak{I}_{2\nu+2} := \mathfrak{I}_{2\nu+1}/\equiv$. The main problem is to keep this kind of extension under control. For instance, the existence of free Fano planes is still an open question.

A first successful approach in this direction concerning buildings whose diagrams do not contain subdiagrams of type H_3 can be found in Ronan and Tits [1987]. Concerning 'free constructions' of buildings with no rank 3 residues of spherical type, which have some features of hyperfree extensions, see Ronan [1986].

3. Groups of projectivities in free geometries

3.1. *Unifying treatment: model-theoretic point of view*

HYPOTHESES FOR THIS CHAPTER. Given a class Mod(Th(Σ)) of geometries, we consider such models $\mathfrak{I}$ where each block may be identified with the set of all points incident with it. Thus the incidence relation | between points and blocks becomes the usual inclusion and each block b can be seen as a *pencil of points* carried by b. Moreover, we suppose that each equivalence class of points which is available from the language $\mathcal{L}$ intersects every block in at most one point.

DEFINITION 12.

(i) In order to introduce *projection pencils*, consider a Σ-free construction $(\mathfrak{M}, \mathfrak{N})$ such that $\mathfrak{M} \setminus \mathfrak{N}$ contains at least one point p and $\mathfrak{N}$ does not consist of only one point. Let ε be an embedding of $(\mathfrak{M} \setminus \mathfrak{N}) \setminus [p]$ into a fixed model $\mathfrak{I}$ of Th(Σ). Let $\mathcal{E}$ be the family of all possible extensions ε' of ε to homomorphisms of $\mathfrak{M}$ into $\mathfrak{I}$ such that $\varepsilon'|_{\mathfrak{N}}$ is an embedding. For each $\varepsilon' \in \mathcal{E}$, we define the *projection set* $B(\varepsilon')$ as the set of all blocks in $\mathfrak{N}^{\varepsilon'}$ and of all points incident with any such block. The blocks in $\mathfrak{N}^{\varepsilon'}$ will be called *carriers* of $B(\varepsilon')$; the family

$$\mathcal{B} := \mathcal{B}\big((\mathfrak{M} \setminus \mathfrak{N}) \setminus [p]\big)^{\varepsilon} := \big\{B(\varepsilon'):\ \varepsilon' \in \mathcal{E}\big\}$$

is called *projection pencil with centre* $Z(\mathcal{B}) := ((\mathfrak{M} \setminus \mathfrak{N})) \setminus [p])^{\varepsilon}$.

(ii) Suppose that R is a binary relation symbol of $\mathcal{L}$ which becomes an equivalence relation between points by virtue of Σ. If $\{x_1, \ldots, x_n\}$ is a set of distinct points of $\mathfrak{I}$, then the union of $\{x_1, \ldots, x_n\}$ and of all R-equivalence classes containing some x_i for $i \in \{1, \ldots, n\}$ is called *pseudo-projection set* (*generated by* $\{x_1, \ldots, x_n\}$ *with respect to* R). If we add to a pencil

$$\mathcal{B} = \mathcal{B}\big((\mathfrak{M} \setminus \mathfrak{N}) \setminus [p]\big)^{\varepsilon}$$

all pseudo-projection sets generated by the points of the centre $((\mathfrak{M} \setminus \mathfrak{N}) \setminus [p])^{\varepsilon}$, with respect to all available equivalence relations arising from $\mathcal{L}$ and Σ, we obtain the *projection pseudo-pencil* denoted also by $\mathcal{B}$.

PROPOSITION 2. *Let $\mathfrak{J}$ be a model of* Th(Σ). *Suppose that, for some Σ-free construction* $(\mathfrak{M}, \mathfrak{N})$, *the graph $\Delta(\mathfrak{M} \setminus \mathfrak{N})$ has a component which corresponds to a single point, say p. Then through each point x of $\mathfrak{J}$ not incident with any element in $((\mathfrak{M} \setminus \mathfrak{N}) \setminus [p])^{\varepsilon}$, passes at most one element $B(\varepsilon')$ of the pencil $\mathcal{B}((\mathfrak{M} \setminus \mathfrak{N}) \setminus [p])^{\varepsilon}$.*

PROOF. See Funk and Strambach [1991], Proposition 2. □

REMARK. Proposition 2 holds true for projective planes, affine planes, Benz planes, and generalized 4-gons, but not for generalized n-gons with $n \geqslant 5$.

DEFINITION 13. Let $(b, \mathcal{B})$ be a pair consisting of a pencil of points carried by some block b of $\mathfrak{J}$ and of a (pseudo-)pencil $\mathcal{B} = \mathcal{B}((\mathfrak{M} \setminus \mathfrak{N}) \setminus [p])^{\varepsilon}$ such that for each point $p \in b \setminus Z(\mathcal{B})$ there exists one and only one (pseudo-)projection set $B \in \mathcal{B}$ with $p \in B$. Suppose that the mapping $\varphi = (p \mapsto B)$: $b \setminus Z(\mathcal{B}) \to \mathcal{B}$ is injective. Assume that every point $z \in Z(\mathcal{B}) \cap b$ is contained in at least one projection set Z of $\mathcal{B} \setminus (b \setminus Z(\mathcal{B}))^{\varphi}$ such that φ can be extended to an injective mapping $(b, \mathcal{B})$: $b \hookrightarrow \mathcal{B}$ (sending z into some Z). Denote the image of $(b, \mathcal{B})$ by $\mathcal{B}^b$. We shall call $(b, \mathcal{B})$ a *(pseudo-)perspectivity* from b onto $\mathcal{B}^b$. The inverse mapping from the image $\mathcal{B}^b$ onto b is denoted by $(\mathcal{B}, b)$ and named (pseudo-)perspectivity from $\mathcal{B}^b$ onto the pencil of points b. If $\mathcal{B}^b = \mathcal{B}$, we call $(b, \mathcal{B})$ and $(\mathcal{B}, b)$ exhausting.

Products

$$\prod_{i=0}^{n-1} (b_i, \mathcal{B}_i)(\mathcal{B}_i, b_{i+1}) \tag{$*$}$$

of perspectivities and pseudo-perspectivities (arising in general from different Σ-free constructions) with $\mathcal{B}_i^{b_i} = \mathcal{B}_i^{b_{i+1}}$ are called *projectivities* from b_0 onto b_n. The set of all projectivities in $\mathfrak{J}$ forms a halfgroupoid in the sense of Bruck [1971], p. 1. For each pencil of points b_0, the subset of all projectivities ($*$) with $b_0 = b_n$ forms a group $\Pi(b_0)$ of permutations acting on the points of b_0 (with the composition as multiplication). If for any two pencils of points b_1, b_2 there exists a projectivity π from b_1 onto b_2, the corresponding groups $\Pi(b_1)$ and $\Pi(b_2)$ of projectivities are even isomorphic as permutation groups (since $\Pi(b_1) = \pi \Pi(b_2) \pi^{-1}$). In this case the isomorphism type of $\Pi(b)$ does not depend on the choice of b and we may associated to any model of Th(Σ) its group Π of projectivities. Natural subgroups of Π arise by using only certain types of (pseudo-)perspectivities in ($*$), where the differentiation depends on the choice of the underlying Σ-free constructions, on special positions of the centres, and on restriction to (or exclusion of) exhausting (pseudo-)perspectivities.

3.2. *Examples*

We discuss possible types of (pseudo-)perspectivities as well as the degree of transitivity for the corresponding groups of projectivities in nondegenerate models of some traditional classes of geometries (cf. Examples 1.2.1–1.2.5). For a complete description of (pseudo-)perspectivities, it is sufficient to indicate their carriers.

In *projective planes* the only projection pencils $\mathcal{B}$ are the sets $\mathcal{B}(z)$ of all lines passing through a point z which forms the centre of $\mathcal{B}$. Hence perspectivities $(b, \mathcal{B})$ exist for all nonincident line-point pairs (b, z). The group $\Pi(b)$ of projectivities acts 3-transitively.

In *affine planes* there exist two different types of projection pencils: pencils of type $\mathcal{B}(z)$ (arising from the Σ-free construction $(\mathfrak{M}_1, \mathfrak{N}_1)$), as well as classes $\bar{l} = \mathcal{B}(l)$ of lines parallel to some line l (coming from the Σ-free construction $(\mathfrak{M}_3, \mathfrak{N}_3)$). Note that the centre of $\mathcal{B}(l)$ is l, which, in its turn, is a projection set, too, arising from a proper homomorphism ε' of $\mathfrak{M}_3$ identifying $\mathfrak{N}_3$ with $[l]$. Clearly, (exhausting) perspectivities $(b, \bar{l})$ exist for all nonparallel line pairs (b, l). For any nonincident line-point pair (b, z), there exists one further type of nonexhausting perspectivities $(b, \mathcal{B}(z))$ where $\mathcal{B}(z)^b$ consists of all lines passing through z and not parallel to b. Two perspectivities of this type, say $(b, \mathcal{B}(z))$ and $(\mathcal{B}(z), c)$, can be composed if and only if $\mathcal{B}(z)^b = \mathcal{B}(z)^c$, i.e. the lines b and c are parallel. Often one considers only the subgroup Π_{aff} generated by all exhausting perspectivities as the *group of affine projectivities*, which operates 2-transitively, but not 3-transitively in general (cf. Freudenthal and Strambach [1975], Barlotti, Schreiber and Strambach [1978]). The same holds true even for the full group $\Pi(b)$ (since, e.g., the stabilizer Ω of any point in the group of projectivities of a Pappian projective plane acts sharply 2-transitively and $\Pi(b)$ can be seen as a subgroup of a suitable Ω).

In *k-nets* there are precisely k projection pencils, namely the classes $\bar{l}^j$ of line of the same sort j. Note that in this case the centre is the void set. Hence perspectivities $(b, \bar{l}^j)$ exist for each block b and each class $\bar{l}^j$ of lines with $b \notin \bar{l}^j$. The group $\Pi(b)$ acts simply transitively, but not 2-transitively, cf. Barlotti and Strambach [1983], Corollary 7.4.

In *Möbius planes* there exists only one type of pencils leading to perspectivities, namely sets $\mathcal{B}(z_1, z_2)$ of all blocks passing through distinct points z_1 and z_2, which form the centre of $\mathcal{B}(z_1, z_2)$. In *Laguerre* and *Minkowski planes* there exists analogous pseudo-pencils $\mathcal{B}(z_1, z_2)$, namely sets consisting of all blocks passing through nonparallel points z_1, z_2 and of the pseudo-projection sets generated by $\{z_1, z_2\}$ with respect to the relations $\|_i$. In all Benz planes $\mathcal{B}(z_1, z_2)$ arises from the Σ-free construction $(\mathfrak{M}_1, \mathfrak{N}_1)$. The corresponding (pseudo-)perspectivities $(b, \mathcal{B}(z_1, z_2))$ exist for each block b and each (pseudo-)pencil $\mathcal{B}(z_1, z_2)$ such that b passes through precisely one of the two points z_i, say z_1; note that the image of $z_1 \in b$ is the unique block contained in $\mathcal{B}(z_1, z_2)$ touching b in z_1. The group $\Pi_1(b)$ generated by these (pseudo-)perspectivities operates 3-transitively on b.

In *Benz planes* there is a second type of pseudo-pencils (coming from the Σ-free construction $(\mathfrak{M}_2, \mathfrak{N}_2)$), namely, for any incident point-block pair (z, c), the set $\mathcal{B}(z, c)$ consisting of all blocks touching c in z and of the pseudo-projection set generated by z. Note that the centre of $\mathcal{B}(z, c)$ is $[z, c]$ where c is a projection set, too, arising from a proper homomorphism ε' from $\mathfrak{M}_2$ identifying $\mathfrak{N}_2$ with $[c]$. The corresponding pseudo-perspectivities $(b, \mathcal{B}(z, c))$ exist for each block b passing through z and each pseudo-pencil $\mathcal{B}(z, c)$ not containing b. In the case of Minkowski planes, these pseudo-perspectivities are no longer exhausting since there exist two pseudo-projection sets generated by z with respect to $\|_1$ or $\|_2$. The group $\Pi_2(b)$ generated by these pseudo-perspectivities operates 3-transitively on b. The group $\Pi(b)$ generated by $\Pi_1(b)$ and $\Pi_2(b)$ acts 3-transitively, too, but not 4-transitively in general (cf. Freudenthal and Strambach [1975], Karzel and Kroll [1981]).

In a *generalized quadrangle* $\mathfrak{Q}$ the only projection pencils $\mathcal{B}$ are the sets $\mathcal{B}(z)$ consisting of all lines $l \neq z$ intersecting the line z, which forms the centre. For any nonintersecting line-line pair (b, z), there exists a nonexhausting perspectivity $(b, \mathcal{B}(z))$ where $\mathcal{B}(z)^b$ consists of all lines l in $\mathfrak{Q}$ intersecting both b and z. Hence we may write $\mathcal{B}(z)^b =: \mathcal{B}(b, z) = \mathcal{B}(z, b)$, which, in turn, implies that $(b, \mathcal{B}(z))$ and $(z, \mathcal{B}(b))$ have the same image $\mathcal{B}(z)^b = \mathcal{B}(b)^z$. Hence two perspectivities, say $(b, \mathcal{B}(z))$ and $(\mathcal{B}(u), c)$ can be composed if and only if $\mathcal{B}(z)^b = \mathcal{B}(u)^c$, i.e. $\mathcal{B}(b, z) = \mathcal{B}(u, c)$. Clearly, one has $b \neq z$ and $u \neq c$; on the other hand, the case $b = c$ and $z = u$ is trivial; thus either $u = b$ and $c = z$ or $\{b, z, u, c\}$ contains at least three distinct lines. In the first case, the composition $(b, \mathcal{B}(b, z))(\mathcal{B}(b, z), z)$ is admissible; in the second case, $\mathcal{B}(b, z) = \mathcal{B}(u, c) =: \mathcal{B}$ consists of all lines l which intersect each line of $\{b, z, u, c\}$, and one has:

$$\big(b, \mathcal{B}(z)\big)\big(\mathcal{B}(u), c\big) = (b, \mathcal{B})(\mathcal{B}, z)(z, \mathcal{B})(\mathcal{B}, u)(u, \mathcal{B})(\mathcal{B}, c).$$

Hence the group $\Pi(b)$ of projectivities is generated by products $(b, \mathcal{B}(b, z))(\mathcal{B}(b, z), z)$ of perspectivities where b, z are lines of distance 4 in the graph $\Delta(\mathfrak{Q})$.

For $n \geqslant 5$, in *generalized n-gons* there exists only one type of projection pencils $\mathcal{B}$. Suppose that $\mathfrak{I}$ is a generalized n-gon, and let $z_1|z_2|\cdots|z_{n+1}$ be an irreducible chain; then the pencil $\mathcal{B} = \mathcal{B}(z_1, \ldots, z_{n+1})$ with the centre $[z_1, \ldots, z_{n+1}]$ consists of all irreducible chains $x_0|\cdots|x_{n-2}$ for which there exists elements $x_0|z_1$ such that

$$x_0|x_1|\cdots|x_{n-2}|z_{n+1}|z_n|\cdots|z_1|x_0$$

is a closed irreducible chain, as well as of the irreducible chain $z_2|\cdots|z_n$ arising from a proper homomorphism ε' of $\mathfrak{M}$ identifying z_i with z_{2n+2-i} for $i = 2, \ldots, n$ (see examples after Definition 3). Note that the elements z_1, z_{n+1} of the centre have distance n in the graph $\Delta(\mathfrak{I})$; hence $\mathcal{B}$ consists of all irreducible chains joining them and we may write $\mathcal{B} = \mathcal{B}(z_1, z_{n+1})$. Thus perspectivities $(x, \mathcal{B}(x, y))$ exist for all pairs of elements x, y with distance n in $\Delta(\mathfrak{I})$. This, in turn, implies that for n odd we must formally accept stars of lines carried by one point as *pencils of points.*

Concerning the transitivity of the group of projectivities one has the following

PROPOSITION 3. *Let $\mathfrak{I}$ be a generalized n-gon with $n \geqslant 4$. Then the group $\Pi(x)$ of projectivities operates 2-transitively on x; in generalized 4-gons, the action is not always 3-transitive.*

PROOF. See Funk and Strambach [1991], Proposition 3. □

REMARK. In a free projective plane and a free affine plane, the group $\Pi(b)$ of projectivities has respectively no 4-transitive and no 3-transitive action. This follows from the fact, that free projective planes can be ordered, cf. Joussen [1966], and that the projective closure of a free affine plane is a free projective plane (cf. Schleiermacher and Strambach [1969], Theorem B11). It is very plausible to conjecture that even every Benz plane can be ordered; this would immediately imply that $\Pi(b)$ does not act 4-transitively in free Benz planes.

3.3. *Theorems about groups of projectivities in free models*

DEFINITION. To each (pseudo-)perspectivity $(b, \mathcal{B})$ and $(\mathcal{B}, b)$ we associate the symbols $[b, \mathcal{B}]$ and $[\mathcal{B}, b]$, respectively, and denote by $\mathcal{I}$ and $\mathcal{I}^{-1}$ the set of all symbols $[b, \mathcal{B}]$ and $[\mathcal{B}, b]$. Let Φ be the free group generated by $\mathcal{I}$ (cf. Hall [1965], p. 91 ff.). The elements of Φ are equivalence classes of *words*; each such class contains precisely one *reduced word*, i.e. either the empty set (which represents the unit element) or products of symbols of $\mathcal{I} \cup \mathcal{I}^{-1}$ without pairs of type $[b, \mathcal{B}][\mathcal{B}, b]$ and $[\mathcal{B}, b][b, \mathcal{B}]$. Then every subset $\widehat{\Pi}(b_0)$, consisting of equivalence classes whose words are of type

$$\prod_{i=0}^{n-1} [b_i, \mathcal{B}_i][\mathcal{B}_i, b_{i+1}], \tag{$**$}$$

where $b_0 = b_n$, $n \in \mathrm{N}$, and $\mathcal{B}_i^{b_i} = \mathcal{B}_i^{b_{i+1}}$ for all $i = 0, \ldots, n-1$, is a subgroup of Φ and hence a free group, too (see Hall [1965], Theorem 7.2.1). The rank of $\widehat{\Pi}(b_0)$ is countable whenever the rank of Φ is $\aleph_0$. There exists a canonical surjection α from $\widehat{\Pi}(b_0)$ onto $\Pi(b_0)$, namely:

$$\prod_{i=0}^{n-1} [b_i, \mathcal{B}][\mathcal{B}_i, b_{i+1}] \mapsto \prod_{i=0}^{n-1} (b_i, \mathcal{B}_i)(\mathcal{B}_i, b_{i+1}).$$

Since equivalent words of $\widehat{\Pi}(b_0)$ are mapped into different representations (as products of (pseudo-)perspectivities) of the same projectivity, α is in fact an epimorphism. The kernel of α consists of all words whose images are the identity on b_0. A representation

$$\pi = \prod_{i=0}^{n-1} (b_i, \mathcal{B}_i)(\mathcal{B}_i, b_{i+1}) \in \Pi(b_0)$$

is called *irreducible* if

$$\prod_{i=0}^{n-1} [b_i, \mathcal{B}_i][\mathcal{B}_i, b_{i+1}]$$

is a reduced word of $\widehat{\Pi}(b_0)$; in this case, the number n is named the *length* of the irreducible representation.

In general, even distinct reduced word of $\widehat{\Pi}(b_0)$ may have the same image under α. Hence α becomes a monomorphism if and only if the only irreducible representation of the identity id_{b_0} is the empty set.

THEOREM 10. *Suppose that* $\mathrm{Th}(\Sigma)$ *satisfies the hypotheses of Section* 3.1 *and let* $\mathfrak{F}$ *be a free model of* $\mathrm{Th}(\Sigma)$. *Then any subgroup* Θ *of its group* $\Pi(b_0)$ *of projectivities is a free group if and only if the identity on* b_0 *has no irreducible representation of positive length.*

THEOREM 11. *Let $\mathfrak{F}$ be a free model of* Th(Σ) *where* Mod(Th(Σ)) *ranges over the classes of projective planes, affine planes, k-nets, Benz planes, and generalized n-gons. Then the identity in the group $\Pi(b)$ of all projectivities has no irreducible representation (as product of perspectivities) of positive length.*

PROOF. See Funk and Strambach [1991], Theorem 5. □

THEOREM 12. *Let $\mathfrak{F}$ be a free model of* Th(Σ) *and let Θ be a subgroup of the group $\Pi(b)$ of projectivities in $\mathfrak{F}$. Then the pointwise stabilizer $\Gamma \leqslant \Theta$ on s_Σ distinct points consists only of the identity, whereas there exist $s_\Sigma - 1$ distinct points such that the pointwise stabilizer on them in Θ is a free group of rank $\aleph_0$ where:*

$$s_\Sigma = \begin{cases} 6 & \textit{for } \Theta = \Pi(b) \textit{ in projective planes,} \\ 4 & \textit{for } \Theta = \Pi_{\text{aff}} \textit{ or } \Theta = \Pi(b) \textit{ in affine planes,} \\ 4 & \textit{for } \Theta = \Pi(b) \textit{ in k-nets,} \\ 6 & \textit{for } \Theta = \Pi_1(b) \textit{ or } \Theta = \Pi(b) \textit{ in Benz planes,} \\ 5 & \textit{for } \Theta = \Pi_2(b) \textit{ in Benz planes.} \end{cases}$$

PROOF. These results can be found in Barlotti [1964] and Schleiermacher and Strambach [1967] for projective planes, and in Barlotti and Strambach [1983], Theorems 3.6, for k-nets; for Benz planes, see Funk [1982a], Theorems 1, 2 and 4. If $\mathfrak{F}$ is a free affine plane, the case $\Theta = \Pi_{\text{aff}}$ has been solved in Barlotti et al. [1978], Lemma 1.1, and the case $\Theta = \Pi(b)$ has to be treated in an analogous way. □

This theorem has an interesting consequence. For some subgroup Θ of the group of projectivities $\Pi(b)$, consider the following *regularity* condition (R_s, Θ): 'Each projectivity of Θ fixing s distinct points is the identity'. Obviously, for $s \geqslant s_\Sigma$, this condition cannot imply any Σ-confined configuration. On the other hand, if t_0 denotes the degree of transitivity of Θ, each regularity condition (R_s, Θ) with $t_0 \leqslant s < s_\Sigma$ forces strong Σ-confined configurations to hold true.

THEOREM 13. *Let $\mathfrak{I}$ be a projective plane, a k-net, or a Minkowski plane. If the group $\Pi = \Pi(b)$ of all projectivities respectively satisfies the condition* (R_5, Π), (R_3, Π), (R_5, Π), *then $\mathfrak{I}$ is a Pappian projective plane, a k-net satisfying the Thomsen condition or a Miquelian Minkowski plane. If an affine plane $\mathfrak{I}$ fulfills the condition* $(\mathrm{R}_3, \Pi_{\text{aff}})$, *then $\mathfrak{I}$ is a Desarguesian affine plane or an infinite translation plane with kernel* GF(2). *If $\mathfrak{I}$ satisfies* $(\mathrm{R}_3, \Pi(b))$, *and if $\mathfrak{I}$ is not an infinite translation plane with kernel* GF(2), *then $\mathfrak{I}$ is a Pappian affine plane.*

PROOF. The first four assertions have been proved in Schleiermacher [1967], Fritsch [1978], Funk [1981, 1982b, 1985]. If an affine plane $\mathfrak{I}$ satisfies $(\mathrm{R}_3, \Pi(b))$, then $\mathfrak{I}$ is at least a Desarguesian affine plane, since the second possibility is excluded by hypothesis. Hence $\mathfrak{I}$ can be coordinatized by a suitable (skew-)field Q. In Schleiermacher [1967], for each $k \in Q \setminus \{0, \pm 1\}$, a projectivity $\pi_k \in \Pi(b)$ has been given such that b is the ordinate of the coordinate system and such that π_k induces the mapping $x \mapsto k^{-1}xk$ on Q. Obviously, π_k has at least three distinct fixed points, namely 0, 1, and k; hence π_k is the identity for each such k, i.e. Q is commutative. □

REMARK. At the end of this section, we would like to mention that, by Σ-hyperfree extensions of suitable (not necessarily Σ-open) partial models, one yields the following results for the classes of geometries mentioned in Theorem 12 (Barlotti et al. [1978], Barlotti and Strambach [1983], Section 5). For each $t_0 \leqslant s < \aleph_0$, models of Th($\Sigma$) can be constructed such that the pointwise stabilizer in $\Pi(b)$ on s distinct points of b consists only of the identity, but there exist $s-1$ distinct points on b such that the pointwise stabilizer on them differs from the identity.

4. Automorphism groups of free geometries

Concerning automorphism groups of free models, major emphasis has been laid on the study of their algebraic structure and their behaviour as transformation groups. The algebraic structure, however, could be determined thus far only for some minimum standard free models $\mathfrak{F}_i$ and the exceptional Laguerre plane $\mathfrak{G}$ (cf. Section 2.1) by means of free products of groups with amalgamated subgroups. For a unifying treatment working in different classes of geometries, see Iden [1992].

If A, B are groups with isomorphic subgroups $H \leqslant A$ and $K \leqslant B$ then any element of the free product $A *_H B$ with amalgamated subgroup $H \cong K$ can be represented uniquely as a product $hc_1c_2 \ldots c_q$, where $h \in H$ and $1 \neq c_i$ are representatives of A mod H or B mod K such that c_i, c_{i+1} are not both in A or both in B, see Magnus, Karrass and Solitar [1976], Corollary 4.4.1. Now let S_i, D_i, and F_i be respectively the symmetric group on i elements, the dihedral group of order $2i$, and the free group of rank i.

THEOREM 14. *The automorphism group* Aut($\mathfrak{F}$) *of the free model* $\mathfrak{F}$ *is isomorphic to the following groups*:

$S_4 *_{D_4} S_4$ *for the projective plane* $\mathfrak{F} = \mathfrak{F}_4$ (Sandler [1965], Iden [1971, 1992]);

$D_3 *_{S_2} D_4$ *for the affine plane* $\mathfrak{F} = \mathfrak{F}_3$ (Artzy [1978, 1985], Iden [1992]);

$S_2 * (S_2 \times S_2)$ *for the nonprojective Möbius plane* $\mathfrak{F} = \mathfrak{F}_3$ (Iden and Moe [1978]);

$A * (S_2 \times S_2)$ *where (in terms of generators and relations)*

$$A \cong \langle \alpha_1, \alpha_2, \beta_1, \beta_2\colon\ \alpha_1^2, \alpha_2^2, \alpha_1\alpha_2\alpha_1\alpha_2, \beta_1^2, \beta_2^2, \beta_1\beta_2\beta_1\beta_2, \alpha_1\beta_1\alpha_1\beta_1, \alpha_1\beta_2\alpha_1\beta_2\rangle$$

for the nonprojective Laguerre plane $\mathfrak{F} = \mathfrak{F}_3$ (Iden and Moe [1978], Artzy [1980], Iden [1992]);

$(S_2 * S_2) \rtimes S_2$ *for the nonprojective exceptional Laguerre plane* $\mathfrak{F} = \mathfrak{G}$ (Iden [1992]);

$(F_2 * (S_3 \times S_3)) *_{S_2} S_2$ *where the right factor is identified with a diagonal subgroup in* $S_3 \times S_3$, *for the nonprojective free Minkowski plane generated by three points nonparallel in pairs* (Iden [1971]; *for an other equivalent representation, see* Artzy [1980]).

Once again this shows that the free nonprojective Laguerre planes $\mathfrak{F}_3$ and $\mathfrak{G}$ are not isomorphic (even though both have rank 7).

The only significant result concerning the orbits says that the collineation group of any free projective plane has infinitely many orbits of points and lines, each of infinite length (cf. Dembowski [1960]). In the standard free k-net $\mathfrak{F}_3$ the group of automorphisms neither acts transitively on the points nor on the lines (cf. Barlotti and Strambach [1983], Corollary 11.5).

For traditional classes of geometries there exist common models (in particular those with a nice description in algebraic terms), where the group $\Pi(b)$ of projectivities is induced by a subgroup of automorphisms leaving the block b invariant. In free models, however, the only projectivity induced by some automorphism is the identity.

THEOREM 15. *Let* Th(Σ) *be a theory satisfying the hypotheses of Section* 3.1. *Suppose that in a free model* $\mathfrak{F}$ *the identity has no irreducible representation*

$$\omega = \prod_{i=0}^{h-1} (b_i, \mathcal{B}_i)(\mathcal{B}_i, b_{i+1})$$

of positive length $h \geqslant 2$. *Furthermore assume that for each block* b *there exist at least five distinct pencils* $\mathcal{B}_i$ *such that* $(b, \mathcal{B}_i)$ *are distinct perspectivities. Then the identity is the only projectivity or* $\Pi(b)$ *induced by an automorphism of* $\mathfrak{F}$.

PROOF. See Funk and Strambach [1991], Theorem 6. □

REMARK. This theorem holds for the classes of projective planes (Schleiermacher and Strambach [1967]), affine and Benz planes (Barlotti et al. [1978]), k-nets with $k \geqslant 5$ (Barlotti and Strambach [1983], Theorem 10.28), and for generalized n-gons (which is a new result).

In classes of self-dual geometries, the group containing automorphisms and correlations is of interest, too. In projective planes, particular emphasis has been laid on polarities, i.e. correlations of order two. In free projective planes the only significant results deals with the configurations of the self-polar elements. In Glock [1969] (cf. also Abbiw-Jackson [1965]) it is shown that the number j of self-polar points of a polarity in a free plane of rank r satisfies $j \equiv r \bmod 2$ and $0 \leqslant j \leqslant r-6$. In free planes of rank $r \geqslant 10$, for each such number $j > 0$, there exists $\aleph_0$ types of polarities with precisely j self-polar points. Moreover, a free plane of rank 8, 9, or $\geqslant 10$ admits one further type of polarities which has respectively two, one, and no self-polar point. A free plane of rank 9 admits $\aleph_0$ further types of polarities with precisely three self-polar points. These are all of them.

5. Free extensions and amalgamation techniques

In the first four sections we have dealt exclusively with various aspects of Σ-hyperfree extensions in their own right. From a slightly different point of view, Σ-hyperfree extensions are considered just as a tool used to complete partial models into full models. The very impact of this instrument comes to light if Σ-hyperfree extension processes are combined with certain amalgamation techniques explained below. Thereby the free extension processes may even loose their dominant role within the family of all Σ-hyperfree extension processes (cf. Section 2.1) and other properties turn out to be quite valuable: A Σ-hyperfree extension process F is called *faithful* if each automorphism of any partial model $\mathfrak{I}$ can be extended to an automorphism of $F(\mathfrak{I})$ in a unique way.

THEOREM 16 (Group universality). *Let* Mod(Th(Σ)) *be a class of geometries formulated in some first-order many-sorted language* $\mathcal{L}$. *Assume that* Th(Σ) *admits faithful* Σ-*hyperfree extensions and has the amalgamation property for partial models. Further assume that there exists a family* $\{\mathfrak{J}_\nu\}_{\nu\in\mathrm{N}}$ *of pairwise nonembeddable* Σ-*confined partial models. Then for any given (abstract) group G and any given (partial) model* $\mathfrak{J}$ *of* Th(Σ) *there exists a model* $\mathfrak{L}$ *of* Th(Σ) *containing a submodel isomorphic to* $\mathfrak{J}$ *with* Aut($\mathfrak{L}$) $\cong G$.

In order to explain the amalgamation property for partial models, let us first discuss the underlying amalgamation techniques.

Let $\Gamma = (V, E)$ be a graph, $E \subseteq \binom{V}{2}$, and let $\{\mathfrak{J}_v\}_{v\in V}$ and $\{\mathfrak{J}_e\}_{e\in E}$ be two families of partial models indexed by the vertices and the edges of Γ, respectively. Assume further that for each edge $e = \{u, v\}$ of Γ two embeddings $\varepsilon_u^e\colon \mathfrak{J}_e \hookrightarrow \mathfrak{J}_u$ and $\varepsilon_v^e\colon \mathfrak{J}_e \hookrightarrow \mathfrak{J}_v$ are given. With these data we shall define a new $\mathcal{L}$-structure $\mathfrak{A}_\Gamma$ (cf. Section 1.1), the *amalgam* of the *supports* $\mathfrak{J}_v$, $v \in V$, with respect to the *joint*[3] *embeddings* $\varepsilon_u^e, \varepsilon_v^e, e = \{u, v\} \in E$. For elements x, x' in supports $\mathfrak{J}_v, v \in V$, we shall write $x \sim x'$ if either $x = x'$, or if for $x \in \mathfrak{J}_v$, $x' \in \mathfrak{J}_v$, there is a path $v = v_0, v_1, \ldots, v_{n-1}, v_n = v'$ in Γ such that

$$x' \in x^{\left(\varepsilon_{v_0}^{\{v_0,v_1\}}\right)^{-1}\left(\varepsilon_{v_1}^{\{v_0,v_1\}}\right)\cdots\left(\varepsilon_{v_{n-1}}^{\{v_{n-1},v_n\}}\right)^{-1}\left(\varepsilon_{v_n}^{\{v_{n-1},v_n\}}\right)}.$$

Clearly, this relation is an equivalence relation on the sets of elements belonging to the same sort. Take the set of all $\sim$-equivalence classes as universe of $\mathfrak{A}_\Gamma$. Finally, if R is an n-ary relation symbol of the language $\mathcal{L}$, define the relation R between elements $x_1, \ldots, x_n \in \mathfrak{A}_\Gamma$ if and only if there exist a vertex $v \in V$ and representatives $x_1^v, \ldots, x_n^v \in \mathfrak{J}_v$ for the equivalence classes $x_1, \ldots, x_n$ such that $R(x_1^v, \ldots, x_n^v)$ holds in $\mathfrak{J}_v$ (cf. Kegel and Schleiermacher [1973], Funk et al. [1985], Section 1.3). We may exchange the roles played by vertices and edges; the new $\mathcal{L}$-structure $\mathfrak{A}_\Gamma$ is then called a *co-amalgam*. In general, the (co-)amalgam $\mathfrak{A}_\Gamma$ of a family of partial models need not be again a partial model (cf., e.g., Kegel and Schleiermacher [1973], p. 383). In certain favourable situations, however, we can attack the question of whether $\mathfrak{A}_\Gamma \in$ Mod(Th(Σ)$_\forall$).

A theory Th(Σ) can be expanded to a theory Th(Σ^S) having the same class of models Mod(Th(Σ)) = Mod(Th(Σ^S)) using the procedure called *Skolemization* Σ^S of Σ and Σ-*Skolemization* $\mathcal{L}^\Sigma$ of $\mathcal{L}$, which is very often used in model theory (see Chang and Keisler [1973], Section 3.3, or Barwise [1977], pp. 185–187). Starting with $\Sigma = \Sigma^S$ and $\mathcal{L} = \mathcal{L}^\Sigma$, enlarge Σ^S and $\mathcal{L}^\Sigma$ recursively by the following clause: If $\psi = (\exists x)\varphi$ is a formula of $\mathcal{L}^\Sigma$ with free variables $x_1, \ldots, x_n$ such that

$$(\forall x_1)\ldots(\forall x_n)(\exists x)\varphi(x_1, \ldots, x_n, x)$$

is a sentence of Σ^S, then add a new n-ary function symbol F_ψ to $\mathcal{L}^\Sigma$ and define

$$(\forall x_1)\ldots(\forall x_n)\varphi\big(x_1, \ldots, x_n, F_\psi(x_1, \ldots, x_n)\big) \in \Sigma^S.$$

For instance, in projective planes, F_ψ is the function assigning to any two distinct points (lines) their joining line (meeting point).

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[3] Here 'joint' is an illustrative noun to be understood in the sense of 'welding' (cf. German 'Nahteinbettung', Funk et al. [1985], p. 15).

DEFINITION 15. A partial submodel $\mathfrak{J}$ of a partial model $\mathfrak{I}$ is called *Σ-closed* in $\mathfrak{I}$ if the images of the functions F_ψ, applied to elements of $\mathfrak{J}$, already lie in $\mathfrak{J}$, as far as they do exist in $\mathfrak{I}$. Now we say that Th(Σ) has the *amalgamation property for partial models* if

(a) Σ has only positive $\forall\exists$-axioms and

(b) each (co-)amalgam of *two* partial models with *at least one* Σ-closed joint embedding again is a partial model.

PROPOSITION 4. *Assume that* Th(Σ) *has the amalgamation property for partial models and let* $\mathfrak{A}_\Gamma$ *be the (co-)amalgam of a family of partial models with Σ-closed joint embeddings. Then* $\mathfrak{A}_\Gamma$ *again is a partial model if*

(i) *Γ is a tree, or if*

(ii) *for each support $\mathfrak{I}$ of $\mathfrak{A}_\Gamma$, the images of any two joint embeddings into $\mathfrak{I}$ do not intersect and the union of the images of all joint embeddings into $\mathfrak{I}$ is Σ-closed in $\mathfrak{I}$, too.*

The proof of this proposition is inductive, where the basis of the induction follows immediately from property (b) of Definition 15, whereas the inductive inference also needs property (a); cf. Funk et al. [1985], Lemma 1.3.6 and Lemma 1.3.10.

Proposition 4 turns out to be sufficient for our purpose, since all the constructions we perform are based on (co-)amalgams of that kind. For many classes of geometries, the amalgamation property for partial models holds true.

It may happen that, for some class $\mathcal{C}$ of geometries, the traditional languages force us to include nonpositive $\forall\exists$-axioms in a set Σ of axioms for $\mathcal{C} = \text{Mod(Th}(\Sigma))$; instances are generalized n-gons (cf. 1.2.3) or Benz planes (cf. 1.2.5). Hence, in order to save condition (a) of the amalgamation property for $\mathcal{C}$, we look for a more suitable language, a shift which, however, can have repercussions on the Σ-free constructions. We may try to shift the language as follows: add new relation symbols N and N_i writing $N(x_1, x_2)$ and $N_i(x_1, x_2, \ldots)$, respectively, in each $\forall\exists$-axiom instead of $x_1 \neq x_2$ and $\neg R_i(x_1, x_2, \ldots)$; then add new axioms:

$$(\forall x_1)(\forall x_2)\big(N(x_1, x_2) \Leftrightarrow (x_1 \neq x_2)\big)$$

$$(\forall x_1)(\forall x_2)\ldots\big(N_i(x_1, x_2, \ldots) \Leftrightarrow \neg R_i(x_1, x_2, \ldots)\big) \text{ for all } i.$$

This trick has already been used for generalized n-gons in Funk et al. [1985], p. 60, and Lemma 6.1.6, and works well also for Benz planes, which can be seen using the approach of Lemma 5.1.6 in Funk et al. [1985]. In both cases this language shift yields the same list of Σ-free constructions without altering their universes.

The condition (b) of the amalgamation property is not just a problem of choosing the right language. For instance, it does not hold true for generalized n-gons with $n \geqslant 5$ (cf. Funk et al. [1985], Lemma 6.1.6), neither for nonprojective Benz planes: take two copies of the classical Laguerre plane $\mathfrak{L}_\mathbb{C}$ over the complex numbers $\mathbb{C}$ and choose the canonical embedding of the real Laguerre plane $\mathfrak{L}_\mathbf{R}$ into $\mathfrak{L}_\mathbb{C}$ as joint embeddings, which of course are Σ-closed when Σ characterizes the class of nonprojective Laguerre planes.

Then every pair of blocks not intersecting one another in $\mathfrak{L}_{\mathbf{R}}$ has four distinct intersection points in the corresponding amalgam $\mathfrak{A}_\Gamma$. Thus $\mathfrak{A}_\Gamma$ no longer is a partial Laguerre plane.

To prove group universality under the hypothesis of Theorem 16, we first deal with the case $G = \{1\}$, i.e. we construct an embedding of a given (partial) model $\mathfrak{J}$ into a rigid one. To confine our attention to the essential situation, let us assume that $|\mathfrak{J}| \leqslant \aleph_0$ and that joint embeddings of partial models consisting of just one element are always Σ-closed.

Partition the family $S = \{\mathfrak{J}_\nu\}_{\nu \in \mathrm{N}}$ into $\aleph_0$ distinct subfamilies $S_\mu = \{\mathfrak{J}_{\mu,\nu}\}_{\nu \in \mathrm{N}}$, with $\mu \in \mathrm{N}$, such that each S_μ has cardinality $\aleph_0$, too. Let us start with $\mathfrak{A}_0 := \mathfrak{J}$. Then a model $\mathfrak{A}_{\mu+1}$ is obtained from the (partial) model $\mathfrak{A}_\mu$ by the following amalgamations: fix some injection $\mathfrak{A}_\mu \to \mathrm{N}$, say $x \mapsto \nu(x)$; then, starting with $\mathfrak{A}_\mu^0 := \mathfrak{A}_\mu$, for each $\nu(x)$, amalgamate the supports $\mathfrak{A}_\mu^{\nu(x)}$ and $\mathfrak{J}_{\mu,\nu(x)}$ with respect to some suitable joint embedding $[x] \hookrightarrow \mathfrak{J}_{\mu,\nu(x)}$ and the canonical embedding $[x] \hookrightarrow \mathfrak{A}_\mu^{\nu(x)}$, in order to obtain $\mathfrak{A}_\mu^{\nu(x)+1}$. Finally, define $\mathfrak{A}_{\mu+1}$ to be a faithful Σ-hyperfree extension of the partial model

$$\bigcup_{\nu(x)} \mathfrak{A}_\mu^{\nu(x)}.$$

By these means we get an ascending sequence of model $\mathfrak{A}_\mu$ such that for $\lambda \leqslant \mu$ one has: $\mathfrak{A}_\lambda \subseteq \mathfrak{A}_\mu$ and that every automorphism of $\mathfrak{A}_\mu$ induces the identity in $\mathfrak{A}_\lambda$. The rigid model we want is

$$\mathfrak{L} := \bigcup_{\mu=0}^{\infty} \mathfrak{A}_\mu;$$

in particular, $\mathfrak{L}$ turns out to be Σ-confined.

For a group $G \neq \{1\}$ we start with a graph Γ such that $\mathrm{Aut}(\Gamma) \cong G$ (cf. Frucht [1938], Sabidussi [1960]). Then we use some rigid Σ-confined model $\mathfrak{L}$ containing $\mathfrak{J}$ to blow up the edges of Γ: first we choose two disjoint partial submodels $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{L}$ (mostly just two points) and amalgamate two copies of $\mathfrak{L}$ with common joint $\mathfrak{B}$ to obtain a Σ-confined partial model $\mathfrak{R}$ which has precisely one nontrivial involution exchanging the partial submodels isomorphic to $\mathfrak{L} \setminus \mathfrak{B}$ and fixing $\mathfrak{B}$. Then we build the co-amalgam $\mathfrak{A}_\Gamma$ over Γ with copies of $\mathfrak{R}$ as supports and the induced embeddings of $\mathfrak{A}$ into $\mathfrak{R}$ as joint embeddings. $\mathfrak{A}_\Gamma$ is a Σ-confined partial model with $\mathrm{Aut}(\mathfrak{A}_\Gamma) \cong G$. Finally, the faithful Σ-hyperfree extension $F(\mathfrak{A}_\Gamma)$ yields a model of $\mathrm{Th}(\Sigma)$ such that $\mathrm{Aut}(F(\mathfrak{A}_\Gamma)) \cong G$, by Proposition 1.

For the class of projective planes, group universality has been shown for the first time in Mendelsohn [1972], where a special finite Σ-confined configuration is used instead of $\mathfrak{R}$. In Funk et al. [1985] the above method has been used to prove group universality for the following classes of geometries: projective planes, affine planes, (m, n)-planes, projective Benz planes, generalized 4-gons, k-nets, (k, n)-Steiner system, and LP-spaces. Group universality holds true even for nonprojective Benz planes. The troubles illustrated in the above counter-example can be avoided if the amalgamation is performed using only single points as joints. As a rigid Σ-confined configuration we can use the configuration in Mendelsohn [1972], interpreting the lines as blocks passing through one further point (cf. Strambach [1980]).

Recursive combination of the described amalgamation technique and arbitrary Σ-hyperfree extension processes (used to embed partial models into 'full' models) leads to the following

THEOREM 17 (Homogeneity). *Let* Mod(Th(Σ)) *be a class of geometries. Assume that* Th(Σ) *admits Σ-hyperfree extensions and has the amalgamation property for partial models. Then for every family $\{\mathfrak{K}_\iota\}_{\iota\in I}$ of pairwise nonisomorphic models $\mathfrak{K}_\iota$ of* Th(Σ) *there exists a model $\mathfrak{K}$ of* Th(Σ) *containing submodels isomorphic to $\mathfrak{K}_\iota$ for each $\iota \in I$ and such that for any two embeddings ζ, ζ' of $\mathfrak{K}_\iota$ into $\mathfrak{K}$ there is an automorphism η of $\mathfrak{K}$ with $\zeta\eta = \zeta'$.*

Instead of rewriting the whole proof which can be found in Funk et al. [1985], Section 1.6, we shall restrict ourselves to illustrate the basic construction which furnishes a transitive action on a certain set of partial submodels.

PROPOSITION 5. *Let $\mathfrak{I}$ be a partial model of* Th(Σ) *and $\mathfrak{J}$ a Σ-closed submodel of $\mathfrak{I}$. There exists a partial model $\mathfrak{K}$ containing $\mathfrak{I}$ as a Σ-closed submodel such that the automorphism group of $\mathfrak{I}$ is a subgroup of the automorphism group of $\mathfrak{K}$ and such that for each pair ζ, ζ' of Σ-closed embeddings of $\mathfrak{J}$ into $\mathfrak{I}$ there is an automorphism η of $\mathfrak{K}$ with $\zeta\eta = \zeta'$.*

PROOF. Let X be the set of all ordered pairs of distinct Σ-closed embeddings of $\mathfrak{J}$ into $\mathfrak{I}$. Partition the set X arbitrarily into two subsets Y, Y' such that $(\zeta, \zeta') \in Y$ implies $(\zeta', \zeta) \in Y'$. Denote by Φ the free group generated by the set Y, and interpret Y' as the subset Y^{-1} of Φ. As vertices of a graph Γ we choose the elements of Φ. By definition, the pair $\{f, f'\}$ of vertices of Γ represents an edge of Γ if there is an element $x \in X$ such that, in Φ, one has $fx = f'$ (or, equivalently, $f = f'x^{-1}$). It is easy to check that Γ is a tree.

To each vertex v of Γ let us associate a copy $\mathfrak{I}_v$ of $\mathfrak{I}$, together with a fixed isomorphism β_v: $\mathfrak{I} \to \mathfrak{I}_v$. For every edge $e = (f, f')$ of Γ put $\mathfrak{J}_e = \mathfrak{J}$ and $\varepsilon^e_f = \zeta'\beta_f$, where $fx = f'$ for some $x = (\zeta, \zeta') \in X$. Then the amalgam $\mathfrak{K}$ of the family $\{\mathfrak{I}_f\}_{f\in\Phi}$ of supports with the joint embeddings ε^e_f, ε^e_f is again a partial model and there exist canonical embeddings of $\mathfrak{I}_f$, $f \in \Phi$, onto Σ-closed submodels of $\mathfrak{K}$. The group Φ acts as a regular group of automorphisms on the graph Γ by $v^f = f^{-1}v$ for every f, $v \in \Phi$. This action of Φ on Γ carries over to an action of Φ on the family of supports if one puts $\beta^f_v = \beta_{f^{-1}v}$ and $(\varepsilon^e_v)^f = \varepsilon^{f^{-1}e}_{f^{-1}v}$. Thus, for any edge $\{f, f'\}$ of Γ given by $fx = f'$ with $x = (\zeta, \zeta')$, one has

$$\left(\varepsilon_f^{\{f,f'\}}\right)^x = \varepsilon_{x^{-1}f}^{\{x^{-1}f,x^{-1}f'\}} = \zeta'\beta_{x^{-1}f}$$

as well as

$$\left(\varepsilon_{f'}^{\{f,f'\}}\right)^x = \varepsilon_{x^{-1}f'}^{\{x^{-1}f,x^{-1}f'\}} = \zeta\beta_{x^{-1}f'}.$$

Identifying $\mathfrak{J}_v$ with its canonical image in $\mathfrak{K}$, it is apparent from the definition of amalgam that f in fact acts as an automorphism of $\mathfrak{K}$. Identifying $\mathfrak{J}$ with the Σ-closed submodel $\mathfrak{J}_1$ of $\mathfrak{K}$ via β_1, the pair ζ, ζ' of distinct Σ-closed embeddings of $\mathfrak{J}$ into $\mathfrak{J}_1$ satisfies $\zeta x = \zeta'$ where $x = \eta = (\zeta, \zeta')$ is an automorphism of $\mathfrak{K}$ as required. □

REMARK. In Funk et al. [1985], homogeneity has been proved for the following classes of geometries: projective planes (cf. also Kegel and Schleiermacher [1973]), affine planes, (m, n)-planes, all types of Benz planes (in general, all (k, m, n)-geometries), generalized n-gons, k-nets, (k, n)-Steiner systems, and LP-spaces.

A suitable combination of amalgamation techniques and free extension processes applies also in order to construct models of geometries with highly transitive groups of projectivities, as we have shown in Funk and Strambach [1991], Theorem 7, given below.

THEOREM 18. *Let* Mod(Th(Σ)) *be a class of geometries satisfying the hypotheses of Section* 3.1. *Assume that* Th(Σ) *admits free extensions* (cf. Section 1.3) *and has the amalgamation property for partial models. If, for each* $m \in \mathbb{N}$, *in some group* $\Pi(l)$ *of projectivities of a free model* $\mathfrak{F}$ *of* Th(Σ) *there exist elements mapping at least one* m*-tuple of distinct points on* l *onto a disjoint one, then, for each* $t \leqslant \aleph_0$, *there exists a model* $\mathfrak{T}$ *of* Th(Σ) *and a block* b *of* $\mathfrak{T}$ *such that the group* $\Pi(b)$ *of projectivities operates* t*-transitively on* b.

REMARK. This theorem holds true for projective planes, affine planes, (m, n)-planes, projective Benz planes, generalized quadrangles, and k-nets, since the corresponding theories have the amalgamation property for partial models (see Funk et al. [1985], pp. 35, 41, 50, 58, 64, 73). As far as nonprojective Benz planes or generalized n-gons with $n \geqslant 5$ are concerned, an inspection of the performed amalgams $\mathfrak{A}_\nu$ shows that they are still partial models; hence Theorem 18 is valid in these classes of geometries as well.

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CHAPTER 14

Chain Geometries

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Contents

Introduction 783

1. The projective line over a ring R 784
   1.1. Free modules over a ring 784
   1.2. The projective line over a ring 785
   1.3. Transitivity properties of the group of projectivities 786
   1.4. The projective line over special rings 787
2. The chain geometry $\Sigma(K, R)$ 788
   2.1. Algebras 788
   2.2. Chains on the projective line 789
   2.3. Properties of the chain geometry $\Sigma(K, R)$ 790
   2.4. Parallelism of points 791
   2.5. Geometries of Möbius, Laguerre and Minkowski 792
3. The affine chain geometry $A(K, R)$ 793
   3.1. Weak chain spaces 793
   3.2. Residual spaces 793
   3.3. Chain spaces 794
   3.4. Contact spaces 795
   3.5. Chain geometries are chain spaces 796
   3.6. Investigation of the affine chains in $A(K, R)$ 799
   3.7. Cremonian geometries 803
4. The chain geometry $\Sigma_K(U, R)$ 805
5. The chain geometry $G_K(U, R)$ 808
   5.1. Generalized reguli 809
   5.2. Intrinsic characterization of the distance on G 810
   5.3. Representation of chain geometries on the Grassmannian 810

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HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

6. Rational representations of chain geometries 812
6.1. The general case 812
6.2. Chain geometries over quadratic algebras 813
6.3. Chain spaces associated to quadratic sets 816
6.4. Stereographic projection 817
7. Chain geometries over commutative algebras 818
7.1. Using determinants 818
7.2. Angles between chains 819
7.3. Two $(8_3, 6_4)$-configurations 820
8. Characterizations and direct products of chain geometries 823
8.1. Möbius geometries 823
8.2. Strong chain spaces 825
8.3. Minkowski geometries 827
8.4. Direct products of chain geometries 829
9. Isomorphisms of chain geometries 832
9.1. Jordan homomorphisms 832
9.2. The structure theorem for chain geometries 832
10. n-chain geometries 834
10.1. Short historical review 835
10.2. The general concept 835
10.3. Burau geometries 836
11. Bibliographical remarks 838
References 838

Introduction

In a projective geometry over a ring R, as treated in Chapter 19, it makes no sense to intrinsically consider a projective line. Nevertheless, if R is a K-algebra, then on the projective line $P_1(R)$ we have an interesting incidence structure, the blocks of which are the K-sublines embedded into $P_1(R)$ in a canonical way, and this gives the chain geometry over R.

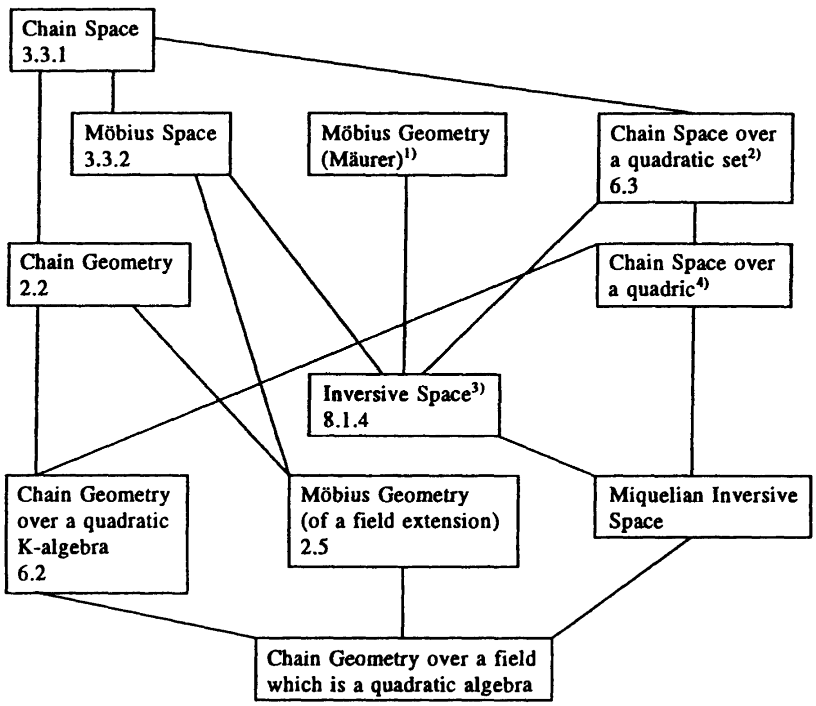


Figure 1. [1] Chapter 6, Mäurer [1968], Heise [1970]; [2] Chapter 6; [3] connected with inversive Geometry ('Kugelgeometrie'), compare Wilker [1982]; [4] Chapter 17: 'Quadric circle geometry'.

The notion of chain geometries arose from efforts to unify the treatment of such different geometries as the geometry of Möbius (lines and circles of the Euclidean plane), of Laguerre–Lie (spears and cycles), and of Minkowski (the pseudo-Euclidean plane with its hyperbolas as 'circles'). Of course, the classical geometries of these kinds were defined over $\mathbb{R}$ or $\mathbb{C}$. Then, Van der Waerden and Smid [1935] presented an axiomatic description of the geometries of Möbius and Laguerre over arbitrary fields. After contributions of several authors, a common view of all these different geometries was given in the famous book Benz [1973a] using the concept of projective line over a (commutative) K-algebra (where the chains are K-sublines such that the projective group acts transitively on this set of chains). Thus, the old concept of chains in Von Staudt [1856] was resumed. (See the short historical remark in Section 10.1. For more information, the interested reader should consult Karzel and Kroll [1988].)

In this chapter Benz's theory will be extended to general K-algebras. We present the development of this theory in the last 15 years. The main problems were:

(1) to find or to investigate affine or projective representations (Sections 3.6 to 6),

(2) to give characterizations (Sections 3.1 to 3.5, and 8),

(3) to determine all isomorphisms or even morphisms (Section 9),
(4) to develop higher dimensional analogs (Section 10).

The diagram (Figure 1) gives the relationships between the most important concepts of this chapter, and with related topics. Some remarks give the main sources for such concepts which could not be treated in this chapter (for Benz planes see Chapter 6 and for Miquelian Benz planes see Chapter 17).

Basic concepts of incidence geometry and algebraic geometry are used without definition and can be found in Chapters 1 to 3.

1. The projective line over a ring R

1.1. *Free modules over a ring*

Let R be any ring, associative and with $1 \neq 0$, and let its group of units (regular elements) be denoted by R^*. Let M be a (unitary) left R-module and $v_i \in M$. We call $v_1, \ldots, v_n$ a *basis* of M provided the mapping

$$R^n \to M, \quad (x_1, \ldots x_n) \mapsto \sum_{i=1}^{n} x_i v_i,$$

is bijective. Then M is called *free of rank n*. A submodule of the form Rm for $m \in M$ is called *cyclic*. Hence a free cyclic submodule is a free module of rank 1.

For the following, let M be a free left R-module of rank 2. If u, w is a fixed basis of M, we call (x, y) the *coordinates* of $xu + yw$; then (1,0) and (0,1) form the canonical basis of M. We write 0 for the zero-module $\{(0,0)\}$.

For an R-endomorphism (R-linear map) of M we have the matrix-representation

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

provided $(x, y)^\sigma = (xa + yc, xb + yd)$ holds. The group of R-automorphisms of M is denoted by $\mathrm{GL}_2(R)$.

The following remarks will be used later.

1.1.1. REMARK. Let u, w be any basis of M.

(a) $s_1 u + t_1 w, s_2 u + t_2 w$ form a basis of M iff

$$\begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix}$$

is an automorphism.

(b) Let σ be an R-endomorphism. Then u^σ, w^σ form a basis of M iff σ is an automorphism. Conversely, for any basis u', w' of M there is a unique R-automorphism σ with $u^\sigma = u'$ and $w^\sigma = w'$.

(c) The R-endomorphism γ of M is an automorphism iff the following hold:

(i) $\forall (x, y) \in M$: $(x, y)^\gamma = 0 \Rightarrow x = 0$ and $y = 0$.

(ii) There exist $u, w \in M$, such that u^γ, w^γ generate M as left R-module.

1.1.2. Definition. The element $v = (a, b)$ of M is called *unimodular* if there exist $a', b' \in R$ such that $aa' + bb' = 1$. (Clearly this is independent of the basis.)

The ring R is of *stable rank* 2 provided for every unimodular (a, b) there exists $c \in R$ with $ac + b \in R^*$. As for commutative rings, for R of stable rank 2, the rank of a free left R-module is uniquely determined, $ab = 1$ implies $ba = 1$ and v belongs to a basis iff v is unimodular (see Chapter 19).

1.1.3. Proposition. *The ring R is of stable rank 2 iff the following holds: For any $u, w \in M$, u and w unimodular, there is $v \in M$ such that u, v is a basis and v, w is a basis.*

Proof. Let R be of stable rank 2. Without loss of generality, for suitable $u' \in M$, let u', u be the fixed basis for the coordinates, hence $u = (0, 1)$. Since $w = (a, b)$ is unimodular, there exists $c \in R$ with $ac + b \in R^*$. Then $v = (1, -c)$ is as claimed, by 1.1.1. The converse direction uses analogous arguments. □

1.2. *The projective line over a ring*

The following definition deviates from the one given in Chapter 19, but is more adequate for our purpose since we admit a wider class of rings. For rings of stable rank 2 it is equivalent to the definition in Chapter 19.

1.2.1. Definition. The *projective line over R* is the set $P(R)$ of all free cyclic submodules X, such that there is a free cyclic submodule Y with $X \oplus Y = M$. We may write

$$P(R) = \{Ru\colon \text{ there exists } w \in M \text{ such that } u, w \text{ is a basis of } M\},$$

or using coordinates and following 1.1.1:

$$P(R) = \left\{R(a, b)\colon \exists c, d \in R\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)\right\}.$$

The elements of $P(R)$ are called *points*. Two points X and Y are called *distant*, provided $M = X \oplus Y$ holds: there exists a basis u, w of M with $X = Ru$ and $Y = Rw$, or in coordinates (for a fixed basis), there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

with $X = R(a, b)$ and $Y = R(c, d)$.

We have the three distinguished mutually distant points

$$U = R(1, 0), \quad V = R(1, 1), \quad W = R(0, 1)^1.$$

[1] This notation is only valid for the first three sections. Later on the letters U, V, W may have a different meaning. Recall that u, v, w may be chosen freely.

1.2.2. A *projectivity* of $P(R)$ is a map induced on $P(R)$ by an R-automorphism of M; and $\Gamma(R) = \mathrm{PGL}_2(R)$ is the group of projectivities: given any $\gamma \in \Gamma(R)$ there exists $\sigma \in \mathrm{GL}_2(R)$ with $(Rv)^\gamma = Rv^\sigma$ for every $Rv \in P(R)$. In fact, any R-automorphism of M maps bases onto bases, and therefore any projectivity operates on $P(R)$ and maps distant points onto distant points. Hence: *$\Gamma(R)$ is a group of distant-preserving permutations of $P(R)$ acting transitively on $P(R)$.*

1.3. *Transitivity properties of the group of projectivities*

In this section we generalize a number of results on projective lines over fields or division algebras. The proofs are straightforward generalizations of those given in Chapter 2.

1.3.1. LEMMA. *Let X, Y, Z be mutually distant points. Then there is a basis u, w of M with $X = Ru$, $Y = R(u+w)$, $Z = Rw$.*

We shall use the following notations: For $c \in R^*$ the *dilatation* $\delta(c)$ is the R-automorphism of M defined by $(x, z)^{\delta(c)} = (xc, yc)$, i.e.

$$\delta(c) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

Let Δ be the group of dilatations. By $C(R)$ we denote the *centre* of R, namely $C(R) = \{c \in R\colon\ cx = xc \text{ for every } x \in R\}$. Further, Z is the subgroup of Δ given by $Z = \{\delta(c)\colon\ c \in R^* \cap C(R)\}$. Remember that $\mathrm{GL}_2(R)_{UVW}$, the stabilizer of U, V, W, is the group of all $\sigma \in \mathrm{GL}_2(R)$ fixing each of the points U, V, W.

1.3.2. LEMMA.

(a) $\mathrm{GL}_2(R)_{UVW} = \Delta$.

(b) *Z is the group of all R-automorphisms fixing every point of $P(R)$.*

It is possible to show that Z is the centre of $\mathrm{GL}_2(R)$. On the other hand, according to 1.3.2(b), the group Z is the kernel of the epimorphism $\mathrm{GL}_2(R) \to \Gamma(R)$. Hence, as in the case of a field, we obtain

$$\Gamma(R) \cong \mathrm{GL}_2(R)/Z\big(\mathrm{GL}_2(R)\big).$$

1.3.3. PROPOSITION. *For any mutually distant points A, B, C and mutually distant points A', B', C' there exists $\gamma \in \Gamma(R)$ with $A^\gamma = A'$, $B^\gamma = B'$ and $C^\gamma = C'$, i.e. $\Gamma(R)$ is transitive on the set of triples of mutually distant points.*

PROOF. If A, B, C are mutually distant points, there is an R-automorphism

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $A = R(a, b)$, $B = R(a + c, b + d)$, $C = R(c, d)$ by 1.3.1 and 1.1.1. Let $\alpha \in \Gamma(R)$ be induced by σ, then $U^\alpha = A$, $V^\alpha = B$, $W^\alpha = C$. In the same way we find $\beta \in \Gamma(R)$ with $U^\beta = A'$, $V^\beta = B'$, $W^\beta = C'$. Then $\gamma = \alpha^{-1}\beta$ is as asserted. □

1.3.4. Proposition. *The following statements are equivalent*:

(a) $\Gamma(R)$ *is sharply transitive on the set of triples of mutually distant points.*

(b) *Any projectivity interchanging two distant points is an involution.*

(c) $R^* \subseteq C(R)$.

Proof. (a) $\Leftrightarrow$ (c) follows from 1.3.3 and from $Z = \Delta$ using 1.3.2. (b) $\Leftrightarrow$ (c): With respect to a suitable basis u, w, let $\sigma \in \mathrm{GL}_2(R)$ be such that $Ru^\sigma = Rw$ and $Rw^\sigma = Ru$, hence

$$\sigma = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad a, b \in R^*.$$

From $R^* \subseteq C(R)$ it follows that

$$\sigma^2 = \begin{pmatrix} ab & 0 \\ 0 & ba \end{pmatrix} \in Z.$$

Conversely for $a \in R^*$ consider

$$\sigma = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix},$$

then $\sigma^2 \in Z$ implies $a \in C(R)$. □

1.3.5. For $A, B, C, D \in P(R)$, where A, B, C are mutually distant and A, D are distant, we define the *cross ratio* $(ABCD)$ as a subset of R in the following way:

$$d \in (ABCD) \Leftrightarrow \exists \gamma \in \Gamma\colon\ \left(A^\gamma, B^\gamma, C^\gamma, D^\gamma\right) = \left(U, W, V, R(d,1)\right),$$

(so that there is a basis u, w of M, such that $A = Ru$, $B = Rw$, $C = R(u + w)$ and $D = R(du + w)$ hold. For the case of a skew field, see Chapter 2.) From 1.3.2(a) we derive that any cross ratio is a class of conjugates under R^*:

$$d \in (ABCD) \Leftrightarrow (ABCD) = \left\{z^{-1}dz\colon\ z \in R^*\right\},$$

and that the cross ratio is invariant under $\Gamma(R)$:

$$(ABCD) = (A^\gamma B^\gamma C^\gamma D^\gamma) \quad \text{for all } \gamma \in \Gamma.$$

If $(ABCD)$ consists of only one element r, by abuse of notation, we write $(ABCD) = r$.

1.4. *The projective line over special rings*

For commutative rings R we can simplify a lot of things by using the determinant or equivalent methods.

1.4.1. PROPOSITION. *If R is commutative, (a,b) is unimodular iff*

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^*$$

for some (c,d), so $R(a,b) \in P(R)$ iff (a,b) is unimodular. Two points $R(a,b)$ and $R(c,d)$ are distant iff

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^*.$$

Moreover $\Gamma(R)$ is sharply transitive on the set of triples of mutually distant points, and any projectivity interchanging two distant points is an involution, by 1.3.4.

Also for rings of stable rank 2 the situation becomes more simple.

1.4.2. PROPOSITION. *For R of stable rank 2 the projective line over R is of the form $P(R) = \{Rv\colon\ v \in M,\ v \text{ unimodular}\}$. For any two points P and Q there is some point S which is distant both from P and Q.*

This follows directly from 1.1.2 and 1.1.3.

2. The chain geometry $\Sigma(K,R)$

2.1. *Algebras*

Let K be a fixed subring of R contained in $C(R)$ and having the same identity element as R; thus R is a K-algebra. (The most important case is when K is a field. This is always supposed in Sections 3 to 9.[2]) We have a natural embedding $\mathrm{GL}_2(K) \to \mathrm{GL}_2(R)$. For $\sigma \in \mathrm{GL}_2(K)$,

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $v \in M$, $v = (x,y)$, $x, y \in R$ we define $v^\sigma = (xa + yc,\ xb + yd)$. A ring/algebra R is called a *local* ring/algebra provided $R \setminus R^*$ is a two-sided ideal of R. For K a field, the K-algebra R is called a *Laguerre algebra* provided there is a two-sided ideal N of R with $R^* = R \setminus N$ and $R = K \oplus N$. Hence a Laguerre algebra is a local algebra.

A simple example of a Laguerre algebra $R = K \oplus N$ is given by trivial multiplication on N as follows:

$$(k_1, n_1)(k_2, n_2) = (k_1 k_2, k_2 n_1 + k_1 n_2) \quad \text{for all } k_i \in K,\ n_i \in N.$$

[2] In Seier [1977], recently followed by Lang [1986] and Keppens [1987], K is a Hjelmslev ring. Compare also Melchior [1968] and Blunck and Stroppel [1994].

Here N is regarded as subspace of the K-vector space R. A 2-dimensional Laguerre algebra necessarily is of this form.

Another interesting class of rings are the semi-primary rings. For a ring R, the Jacobson radical[3] is denoted by $J(R)$. The ring/algebra R is called *semiprimary* if $R/J(R)$ is Artinian, i.e. $R/J(R)$ is a ring-direct sum of full matrix rings. The finite dimensional K-algebras, with K a field, are semi-primary, and the semi-primary rings are of stable rank 2. It should be observed that the local rings are semi-primary, hence of stable rank 2.

For $a \in R$ let $\bar{a}$ be the image of a under the natural epimorphism $R \to R/J(R)$. Then a is a unit iff $\bar{a}$ is a unit. The following lemma is useful for many purposes.

2.1.1. LEMMA. *Let R be a semi-primary K-algebra. Then for any $c \in R$ there exists $a \in R^*$ such that $ac - t$ is a unit for every $t \in K$ with $t \neq 0, 1$ (provided K is a field).*

PROOF. There exists $a \in R^*$ such that the image $\overline{ac}$ of ac in $R/J(R)$ is an idempotent: namely in a full matrix ring, for any element $\bar{c}$ one can find a regular matrix $\bar{a}$ such that $\overline{ac}$ is a projection.

But for $p^2 = p$ and $t \in K$ one computes

$$(p - t)\big(p + (t - 1)\big) = -t(t - 1),$$

i.e. $p - t \in R^*$ for $t \neq 0, 1$. □

2.2. *Chains on the projective line*

Let R be a K-algebra in the sense of Section 2.1. Remember that the projective line over K is given by

$$P(K) = \left\{ K(s,t)\colon \text{ there are } s', t' \in K \text{ such that } \begin{pmatrix} s & t \\ s' & t' \end{pmatrix} \in \mathrm{GL}_2(K) \right\}.$$

Using 1.1.1, we define for any basis u, w the mapping

$$\beta_{u,w}\colon\ P(K) \to P(R), \quad K(s,t) \mapsto R(su + tw),$$

called a *canonical embedding* of $P(K)$ into $P(R)$. The image of $\beta_{u,w}$ is called a *K-chain* denoted by $C(u, w)$, i.e.

$$C(u,w) = \left\{ R(su + tw)\colon\ K(s,t) \in P(K) \right\}.$$

By definition we have $C(u, w) = C(s_1u + t_1w, s_2u + t_2w)$ for

$$\begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} \in \mathrm{GL}_2(K),$$

[3] That is, the intersection of all maximal right ideals.

and $C(u,w) = C(zu, zw)$ for $z \in R^*$. For $u = (a,b)$ and $w = (c,d)$ we will also use the notation

$$C(u,w) = C\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{R(sa+tc, sb+td)\colon\ K(s,t) \in P(K)\}.$$

The set of all K-chains is denoted by $\boldsymbol{C}(R)$, hence

$$\boldsymbol{C}(R) = \{C(u,w)\colon\ u, w \text{ is a basis of } M\}$$
$$= \left\{C\begin{pmatrix} a & b \\ c & d \end{pmatrix}\colon\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)\right\}.$$

We have the distinguished K-chain

$$\boldsymbol{p} = C\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \{R(s,t)\colon\ K(s,t) \in P(K)\}$$

containing U, V, W. Since

$$\boldsymbol{p}^\gamma = C\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

if γ is induced by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R),$$

we immediately see

2.2.1. LEMMA. $\boldsymbol{C}(R) = \boldsymbol{p}^{\Gamma(R)} = \{\boldsymbol{p}^\gamma\colon\ \gamma \in \Gamma(R)\}$.

Next we state our main definition.

2.2.2. DEFINITION. The *chain geometry* $\Sigma(K,R)$ is the incidence structure with as set of points the projective line over R, and as blocks the K-chains:

$$\Sigma(K,R) = (P(R), \boldsymbol{C}(R)).$$

The blocks will be called *chains*; points are said to be *cocatenal* provided they belong to a common chain.

2.3. *Properties of the chain geometry* $\Sigma(K,R)$

2.3.1. PROPOSITION. *Let γ be a projectivity such that $U^\gamma, V^\gamma, W^\gamma$ belong to $\boldsymbol{p}$ again. Then γ fixes $\boldsymbol{p}$.*

PROOF. Let γ be induced by $\sigma \in \mathrm{GL}_2(R)$. By 1.3.3 applied to $P(K)$ there exists $\tau \in \mathrm{GL}_2(K) \leqslant \mathrm{GL}_2(R)$ such that $\sigma\tau$ fixes U, V, W. From 1.3.2(a) we get that $\sigma\tau$ is a dilatation $\delta(c)$. Since $K \subseteq C(R)$ we have $R(s,t)^{\sigma\tau} = R(s,t)^{\delta(c)} = R(s,t)$ for any $R(s,t) \in \boldsymbol{p}$. On the other hand, τ leaves $\boldsymbol{p}$ invariant, and we conclude

$$\boldsymbol{p}^\gamma = \boldsymbol{p}^{(\sigma\tau)\tau^{-1}} = \boldsymbol{p}^{\tau^{-1}} = \boldsymbol{p}. \qquad \square$$

2.3.2. THEOREM. *Any three mutually distant points are contained in exactly one chain.*

PROOF. Given three mutually distant points A, B, C, there exists $\gamma \in \Gamma(R)$ with $U^\gamma = A$, $V^\gamma = B$ and $W^\gamma = C$. Then the chain $\boldsymbol{c} = \boldsymbol{p}^\gamma$ contains A, B, C. Let $\boldsymbol{d}$ be any chain containing A, B, C. Then $\boldsymbol{d} = \boldsymbol{p}^\delta$ for a suitable $\delta \in \Gamma(R)$. Hence the points $U^{\gamma\delta^{-1}} = A^{\delta^{-1}}, V^{\gamma\delta^{-1}} = B^{\delta^{-1}}, W^{\gamma\delta^{-1}} = C^{\delta^{-1}}$ belong to $\boldsymbol{p}$, which by 2.3.1 implies that $\gamma\delta^{-1}$ fixes $\boldsymbol{p}$. Therefore,

$$\boldsymbol{d} = \boldsymbol{p}^\delta = \left(\boldsymbol{p}^{\gamma\delta^{-1}}\right)^\delta = \boldsymbol{p}^\gamma = \boldsymbol{c}. \qquad \square$$

As an easy consequence of 1.3.3, 2.2.1, and 2.3.2 we obtain:

2.3.3. THEOREM. *$\Gamma(R)$ is a flag-transitive group of automorphisms of $\Sigma(K, R)$.*

This gives to $\Sigma(K, R)$ a certain homogeneity.
From the properties of the cross ratio 1.3.5 we obtain:

2.3.4. THEOREM.
(a) *Four mutually distant points A, B, C, D are cocatenal iff $(ABCD) \in K^*$.*
(b) *Let A, B, C be mutually distant points. The chain containing A, B, C is the set $\{A\} \cup \{X \in P(R)\colon X \text{ distant from } A \text{ and } (ABCX) \in K\}$.*

2.4. *Parallelism of points*

We call points A, B of $P(R)$ *parallel*, written $A \parallel B$, provided they are *not* distant. Obviously $\parallel$ is a symmetric and reflexive relation on $P(R)$.

2.4.1. PROPOSITION.
(i) *In $\Sigma(K, R)$ the relation $\parallel$ on $P(R)$ is transitive (i.e. $\parallel$ is an equivalence relation) iff R is a local ring.*
(ii) *In that case every equivalence class of $\parallel$ has the cardinality of $R \setminus R^*$.*

PROOF. (i) By 1.1.1 we obtain

$$\begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix} \in \mathrm{GL}_2(R) \Leftrightarrow a - b \in R^*.$$

Let $N = R \setminus R^*$. Assuming $\parallel$ to be an equivalence relation we have to show $a, b \in N \Rightarrow a - b \in N$. From

$$\begin{pmatrix} 1 & 0 \\ 1 & a \end{pmatrix} \notin \mathrm{GL}_2(R) \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & b \end{pmatrix} \notin \mathrm{GL}_2(R)$$

we conclude $R(1,a) \parallel U$ and $U \parallel R(1,b)$. By transitivity of $\parallel$ now $R(1,a) \parallel R(1,b)$ follows, and this implies

$$\begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix} \notin \mathrm{GL}_2(R),$$

hence $a - b \in N$.

To prove the converse assertion we use the fact that for a local ring R every point of $P(R)$ is of the form $R(1,b)$ or $R(a,1)$ for suitable $a, b \in R$.

(ii) Assume R to be a local ring. Hence $\parallel$ is an equivalence relation on $P(R)$, the classes of which are called *parallel classes*. Using

$$\begin{pmatrix} 1 & c \\ n & 1 \end{pmatrix} \in \mathrm{GL}_2(R) \quad \text{for any } n \in N,$$

from the proof of (i) we conclude that the parallel class of $R(1,c)$ is the set $S = \{R(1, c+x)\colon\ x \in N\}$. But the mapping $N \to S,\ x \mapsto R(1, c+x)$, is bijective. □

From now on we assume the subring K of R to be a field. The dimension of R over K may be arbitrary unless we explicitly restrict it. Under this assumption we have a better motivation for the relation 'nondistant' to be called 'parallel'.

2.4.2. PROPOSITION. *Let K be a field. Then any two distinct points of a chain are distant. Therefore points are parallel iff they are not joined by any chain.*

To prove this, investigate the chain $\boldsymbol{p}$. If Q_1, Q_2 are distinct points of $\boldsymbol{p}$, then with $Q_i = R(s_i, t_i)$ for suitable $s_i, t_i \in K$, we have

$$\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} = s_1 t_2 - t_1 s_2 \neq 0,$$

hence

$$\begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} \in \mathrm{GL}_2(K) \subseteq \mathrm{GL}_2(R).$$

2.5. *Geometries of Möbius, Laguerre and Minkowski*

Let K be a field. The chain geometry $\Sigma(K,R)$ is called a *Möbius geometry* provided the parallel relation is the equality relation. Thus any three distinct points are incident with exactly one chain. Since skew fields are exactly those local rings R where $R \setminus R^*$ consists of a single element, from 2.4.1 we obtain:

2.5.1. COROLLARY. *The chain geometry $\Sigma(K,R)$ is a Möbius geometry iff R is a skew field.*

The chain geometry $\Sigma(K,R)$ is called *Laguerre geometry* provided the parallel relation is an equivalence relation on $P(R)$ and every chain meets every parallel class of points. The reader will easily convince himself that the following result holds.

2.5.2. PROPOSITION. *The chain geometry $\Sigma(K, R)$ is a Laguerre geometry iff R is a Laguerre algebra.*

The chain geometry $\Sigma(K, R)$ is called *Minkowski geometry* of dimension n provided $R = K^{(n)} = K \times \cdots \times K$ (n-times), where addition and multiplication are defined component-wise. (For a geometric characterization, see Section 8.3.)

The classical examples of the above three special chain geometries are the planes of Möbius, Laguerre and Minkowski, which together are the *Miquelian Benz planes*. Here the corresponding K-algebras are 2-dimensional. For $K = \mathbb{R}$ these algebras are of the form $\mathbb{R}[i]$ with $i^2 = -1, 0, 1$, respectively.

3. The affine chain geometry $A(K, R)$

3.1. *Weak chain spaces*

In this part, we start an axiomatic approach of the chain geometries $\Sigma(K, R)$ in terms of incidence. This will later allow for a characterization.

3.1.1. DEFINITION. A *weak chain space* is an incidence structure $\Sigma = (P, C)$. The blocks of Σ are called chains and two different points are called *distant* provided they are incident with a common chain. Moreover, the following postulates are fulfilled.

(i) Any three pairwise distant points are contained in exactly one chain.

(ii) Any chain contains at least 3 points and any point lies on at least one chain.

The weak chain space Σ is *nontrivial* iff there exists a chain c and a point outside of c, and Σ is of *order* m iff any chain consists of $m + 1$ points.

An arbitrary number of points are called *cocatenal* provided they are incident with a common chain. Two points p, q are *parallel*, written $p \parallel q$, iff they are either equal or not cocatenal. Hence 'distant' and 'nonparallel' (denoted as $\nparallel$) coincide. The weak chain space Σ is *connected* provided its incidence graph is connected.

3.1.2. LEMMA. *Let R be a K-algebra over a field K.*

(a) *$\Sigma(K, R)$ is a weak chain space of order $|K|$, which is nontrivial if* $\dim_K R > 1$.

(b) *If R is of stable rank* 2, *then $\Sigma(K, R)$ is connected.*

PROOF. (a) follows from 2.3.2 and 2.4.2; (b) follows from 1.4.2. □

3.2. *Residual spaces*

To facilitate the axiomatic description of chain geometries, one introduces the concept of residual space, which is well known in diagram geometries, cf. Chapter 3.

3.2.1. DEFINITION. Let $\Sigma = (P, C)$ be a weak chain space. For any point p define D_p to be the set of all points that are distant from p and let

$$C_p = \{c \setminus \{p\}\colon\ p \in c \in C\}$$

and

$$C \cap D_p = \{c \cap D_p\colon\ c \in C \text{ such that } |c \cap D_p| > 1\}.$$

Clearly, $C_p \subseteq C \cap D_p$. Then the *residual space* Σ_p is the incidence structure (D_p, C_p), and the *residual trace* of Σ with respect to p is $\Sigma \cap D_p = (D_p, C \cap D_p)$.

For the chain geometries $\Sigma(K, L)$, K a field, it turns out that the residual spaces are partial affine spaces.

3.2.2. DEFINITION. A *partial affine space* $S = (A, L, \mathrm{I}, \|)$ is the following incidence structure with parallelism. There is an affine space A (see Chapter 2, Section 1) with the natural parallel relation $\|$. The points of S are the points of A and the set L of blocks is any nonempty union of parallel classes of lines of A. Therefore in S the following properties hold:

(i) Two distinct points are incident with at most one common line.
(ii) Given a line l and a point P, there is exactly one line m with $P \mathrm{I} m$ and $l \parallel m$ (Euclid's parallel postulate).
(iii) If the lines l, m, x form a triangle then any line z parallel to x meeting l also meets m (Lenz' axiom).

As an example we may consider the partial affine planes. Here instead of (iii) the following stronger condition is fulfilled:

(iii)$'$ Any two nonparallel lines have a point in common.

Incidence structures with parallelism fulfilling (i), (ii), (iii)$'$ are called *nets*.

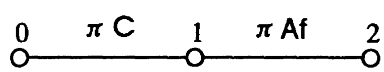
3.3. *Chain spaces*

Here we will axiomatically describe the incidence structure $\Sigma(K, R)$ for a field K.

3.3.1. DEFINITION. A weak chain space $\Sigma = (P, C)$ is a *chain space* provided the following additional postulate is fulfilled:

(iii) For every $p \in P$ the residual space Σ_p is a partial affine space.

The diagram for the geometry Σ involves only points, lines and planes: lines are the 2-sets of distant points and planes are the chains. Therefore a chain space has diagram



where π denotes a partial geometry.

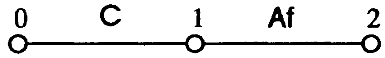
3.3.2. As a special instance of chain spaces (or weak chain spaces) we define Möbius spaces (or circular spaces, respectively) where the parallel relation on the point set is the equality relation: a *circular space* Σ is a weak chain space where (i) is replaced by the stronger postulate

(i)$'$ Any three distinct points are incident with exactly one chain.

A *Möbius space* is a chain space, which is a circular space. Hence, instead of (iii) it fulfils the stronger postulate

(iii)′ For every $p \in P$ the residual space Σ_p is an affine space.

Therefore a Möbius space is connected. A Möbius space is a 'locally affine' circular space, hence with diagram



3.3.3. We say that the chain space Σ is of *dimension* n provided the underlying affine space of Σ_p is uniquely determined by Σ_p and is of dimension n, for every point p. By definition, the circle planes are the chain spaces of dimension 2. Here one has to mention the *Benz planes*: The residual space is an affine plane where i parallel classes of lines are omitted, namely, $i = 0, 1, 2$ for the planes of Möbius (= 'inversive planes'), Laguerre, Minkowski, respectively. For a slightly different description of Benz planes, see Chapter 6, Section 5.

3.4. *Contact spaces*

Here a concept is given which describes what we know as contacting of circles. The axioms become weaker than the axioms of chain spaces. So it has a certain interest of its own.

Let $\Sigma = (P, C)$ be a weak chain space. A *contact relation* on Σ is a family (ρ_p), $p \in P$, where each ρ_p is an equivalence relation on $(p) = \{c \in C\colon p \in c\}$ with the following properties (we write apb instead of $a\rho_p b$):

(i) apb and $a \neq \mathbf{b}$ implies $|a \cap \mathbf{b}| = 1$;

(ii) for $a \in (p)$ and $q \in P$ with $p \nparallel q$ there is exactly one $b \in (p)$ with $q \in b$ and apb.

A weak chain space together with a contact relation is called *contact space*. If Σ is a contact space with contact relation (ρ_p), then ρ_p induces on the partial linear space Σ_p a parallelism such that 3.2.2 (i), (ii) hold. One can transfer the condition 3.2.2(iii) in Σ. This corresponds to a configuration in terms of chains (e.g., circles), points and contact, which in Benz [1973a], p. 109, is called beetle figure ('Käferfigur'). We have the following problem: which chain structures with beetle figure already are chain spaces? This reduces to the problem on the residual space: which incidence structures with parallelism fulfilling 3.2.2 (i), (ii), (iii) are partial affine spaces?

Khalifah [1982] considers Möbius contact spaces where any residual space is a *Sperner space* (weak affine space), i.e. 3.2.2(iii) is not postulated. In every chain space $\Sigma = (P, C)$ we have the following *natural contact relation*:

$$\forall p \in P\colon\ apb \Leftrightarrow a \setminus \{p\} \text{ and } b \setminus \{p\} \text{ are parallel lines in the residual space } \Sigma_p.$$

Note that a chain space together with the natural contact relation is a contact space. On an affine space of dimension $\geqslant 3$ one can define parallelisms which are not the natural ones (see, e.g., Herzer [1979a]). Hence in a Möbius geometry of dimension $\geqslant 3$ one can define contact relations which are not the natural ones.

Our next aim is to prove that chain geometries are chain spaces.

3.5. *Chain geometries are chain spaces*[4]

From now until the end of Section 9 we shall assume that K is a field.

3.5.1. For

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

we define

$$\overline{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{(sb+td)^{-1}(sa+tc)\colon\ K(s,t) \in P(K),\ sb+td \in R^*\}.$$

$$\overline{\mathcal{C}} = \left\{\overline{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) \text{ such that there exist two distinct } K(s_i,t_i) \in P(K) \text{ with } s_ib+t_id \in R^*\right\}.$$

We now define the *affine chain geometry* belonging to the K-algebra R as the incidence structure $A(K,R) = (R, \overline{\mathcal{C}})$, where the blocks are called *affine chains*, and R has to be considered as affine space over K. Since $\Sigma(K,R)$ possesses a point transitive group of automorphisms, i.e. $\Gamma(R)$, all residual spaces of $\Sigma(K,R)$ are isomorphic, and the same holds for all residual traces.

3.5.2. PROPOSITION. *$A(K,R)$ is isomorphic to the residual trace of $\Sigma(K,R)$ under the map $\iota\colon\ R \to D_U,\ c \mapsto R(c,1)$.*

PROOF. For $U = R(1,0)$, clearly $D_U = \{R(a,b)\colon\ b \in R^*\}$, since

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \in \mathrm{GL}_2(R) \quad \text{iff} \quad b \in R^*.$$

Hence the mapping $\iota\colon\ R \to D_U,\ c \mapsto R(c,1)$, is bijective. From the definition of $A(K,R)$ we infer that the mapping ι induces an isomorphism of $A(K,R)$ onto $\Sigma(K,R) \cap D_U$. □

3.5.3. PROPOSITION. *The residual space of $\Sigma(K,R)$ is isomorphic to the incidence structure $(R, \mathbf{L})$, where $\mathbf{L} := \{Ka+b\colon\ a \in R^*,\ b \in R\}$.*

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[4] For more general results, see Blunck [1994].

PROOF. We show that $\iota: R \to D_U$ induces an isomorphism $(R, L) \to \Sigma(K, R)_U$. A chain containing U can be written in the form

$$C\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \quad \text{with} \quad a \in R^*,$$

hence

$$\overline{C}\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} = aK + b.$$

Conversely, for $a \in R^*$ we obtain

$$(aK + b)^\iota = C\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \setminus \{U\}.$$ □

We define two points $x, y \in R$ to be *distant* provided x^ι, y^ι are distant. Using

$$\begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \in \mathrm{GL}_2(R) \Leftrightarrow a - b \in R^*,$$

one obtains the following:

3.5.4. REMARK.

(a) Two points a and b of R are distant iff $a - b \in R^*$.

(b) For any distinct parallel points a and b of R, the line of R (as affine space over K) joining a and b consists of pairwise parallel points. Such lines are called *singular lines*, the other lines, i.e. the lines of L, are the *regular lines*.

(c) The set $R \setminus R^*$ is the union of all singular lines which pass through the origin. $R \setminus R^*$ is called *cone of singularity* in (R, L). For $\dim_K R = n < \infty$ the cone of singularity is an algebraic set, since R can be embedded as a subalgebra in a matrix algebra $M_n(K)$, and then $R \setminus R^*$ consists of all zeros in R of the homogeneous polynomial det on $M_n(K)$.

3.5.5. LEMMA. *Let R be a semiprimary K-algebra. Then any singular line lies in a plane the lines of which are regular except for at most two directions*[5].

PROOF. Using a translation we may consider a singular line containing o. Then this line is of the form Kc. By 2.1.1 there is $a \in R^*$ such that $ac - t$ is a unit for every $t \in K$, $t \neq 0, 1$. Hence the plane E spanned by K and Kac has the property of the lemma, thus also the plane $a^{-1}E$ which contains the singular line Kc. □

3.5.6. THEOREM. *Let R be a K-algebra with* $\dim_K R = n > 1$, *where n may be infinite.*

(a) *$\Sigma(K, R)$ is a chain space. It is of dimension n provided the residual space uniquely determines the underlying affine space.*

(b) *If R is semiprimary and $|K| > 3$, then $\Sigma(K, R)$ possesses four pairwise distant points which are not cocatenal. Moreover, the underlying affine space of the residual space is uniquely determined by its regular lines.*

[5] Directions: parallel classes.

PROOF. Part (a) follows directly from 3.5.3 since L consists of whole parallel classes in R (considered as affine space over K).

(b) We consider the residual space in U. Let L be a singular line joining two points p and q. By 3.5.5 the line L is contained in a plane E, containing singular lines for at most two directions and regular lines for at least three directions since $|K| > 3$. So there are three noncollinear points of E which are pairwise distant, and hence together with U give four pairwise distant points which are not cocatenal.

Moreover E is uniquely determined by regular lines, and so is L: namely $L \setminus \{p, q\}$ is the set of those points of E which are not contained in a regular line through p or through q. □

3.5.7. LEMMA. *The stabilizer $\Gamma(R)_U$ operates on D_U and consists of the transformations whose matrices are of the form*

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \quad \text{where } a, d \in R^*, \ c \in R.$$

Via ι this group induces a group of affinities on R:

$$A(R) = \iota\Gamma(R)_U\iota^{-1} = \left\{x \mapsto d^{-1}xa + b\colon\ a, d \in R^*,\ b \in R\right\}.$$

$A(R)$ contains the full group of affine dilatations, namely,

$$\Delta = \left\{x \mapsto kx + b\colon\ k \in K^*,\ b \in R\right\}.$$

The matrices of the form

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad c \in R,$$

induce the translations.

PROOF. Let $\gamma \in \Gamma(R)_U$, γ induced by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $U^\gamma = U$ is equivalent to $b = 0$ and $a \in R^*$, hence $d \in R^*$. Let be $x \in R$. Then

$$\begin{aligned} x^{\iota\gamma\iota^{-1}} &= R(x, 1)^{\gamma\iota^{-1}} = R(xa + c, d)^{\iota^{-1}} \\ &= R(d^{-1}xa + d^{-1}c, 1)^{\iota^{-1}} = d^{-1}xa + b, \end{aligned}$$

where $b = d^{-1}c$. □

3.5.8. PROPOSITION. *For $|K| > 2$ let $(\rho_A)_{A\in P}$ be a contact relation on $\Sigma(K,R)$ which is preserved under $\Gamma(R)$:*

$$\forall \boldsymbol{a}, \boldsymbol{b} \in (A)\ \forall\gamma \in \Gamma(R) \quad \mathbf{a}\,\rho_A\,\boldsymbol{b} \Leftrightarrow \mathbf{a}^\gamma\,\rho_{A^\gamma}\,\boldsymbol{b}^\gamma.$$

Then $(\rho_A)_{A\in P}$ is the natural contact relation.

PROOF. We consider the residual space $\Sigma(K,R)_U$ in its isomorphic representation $(R, \boldsymbol{L})$, see 3.5.3. All dilatations are contained in $A(R)$, hence are induced by $\Gamma(R)_U$, compare 3.5.7.

For lines $x, y \in \boldsymbol{L}$ we define ρ by

$$x\rho y \Leftrightarrow x^\iota \cup \{U\}\,\rho_U\,y^\iota \cup \{U\}\,.$$

Suppose $x \neq y$ and $x\rho y$ so $x \cap y = \varnothing$. Then there are distinct points A, B, C such that A is a point of x and B is a point of y and A, B, C are collinear. There exists a line z containing C such that $x\rho z$ holds, and there is a dilatation δ with centre C and $B^\delta = A$. It follows that $y^\delta\rho z^\delta = z$, $x \cap y^\delta \neq \varnothing$ and $x\rho y^\delta$ which implies $x = y^\delta$, whence $y = x^{\delta^{-1}} \parallel x$. □

3.5.9. REMARK. Let R be a semiprimary K-algebra and $|K| > 3$. Then any automorphism of the incidence structure $\Sigma(K,L)$ preserves the natural contact relation. This follows directly from 3.5.6(b) since collineations of affine spaces of order > 2 preserve their natural parallelisms.

3.6. *Investigation of the affine chains in $A(K,R)$*

Two affine chains are called *affinely equivalent* provided they belong to the same orbit under $A(R)$. For $a \in R$, we set

$$c(a) := \overline{C}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \{0\} \cup \{(a-t)^{-1}\colon\ t \in K,\ a - t \in R^*\}.$$

Notice that $c(a) = c(b)$ iff $a - b \in K$, and that $c(a) \cap c(b) = \{0\}$ if $c(a) \neq c(b)$.

3.6.1. LEMMA. *Any affine chain is, up to a translation, of the form*

$$\{0\} \cup \{(a-dt)^{-1}\colon\ t \in K,\ a - dt \in R^*\}, \quad a \in R,\ d \in R^*.$$

In particular, any affine chain is affinely equivalent to $c(a)$ for a suitable $a \in R$.

PROOF. If

$$\overline{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an affine chain, we may assume that $R(c,d)$ is distant from U, hence $d \in R^*$. Now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d \end{pmatrix}, \quad a' \in R^*.$$

But

$$\overline{C}\begin{pmatrix} a' & b' \\ 0 & d \end{pmatrix} = \overline{C}\begin{pmatrix} 1 & b'' \\ 0 & d' \end{pmatrix}$$
$$= \{0\} \cup \{(b'' - d't)^{-1} \colon t \in K,\ b'' - d't \in R^*\},$$

for $b'' = {a'}^{-1}b'$, $d' = {a'}^{-1}d$. By multiplication with d' we obtain a chain $c(a'')$. □

For $a, b \in R^*$ with $a - b \in R^*$ the affine chain through $0, a, b$ has the form

$$\{0\} \cup \{(sa^{-1} + tb^{-1})^{-1} \colon s + t = 1,\ ta + sb \in R^*\}.$$

For an affine chain c and $p \in c$ the *tangent line* of c in p is defined as the unique chain containing U and contacting c in p, if we identify $A(K,R)$ with D_U under the isomorphism ι as in 3.5.2. In fact, this is the tangent line in the usual differential geometric sense. It is not hard to verify:

3.6.2. LEMMA. *The tangent line of* $c(a)$ *in* 0 *is* K, *and for* $a \in R^*$ *the tangent line of* $c(a)$ *in* a^{-1} *is* $a^{-1} + a^{-2}K$.

We define the *restricted affine chain geometry* $A^*(K,R) = (R^*, \overline{C})$, and divide $\overline{C}$ in the following disjoint classes $C_1, \ldots, C_4$.

C_1: set of lines of $\overline{C}$ containing 0.

C_2: set of all lines of $\overline{C}$ which do not contain 0.

C_3: set of all affine chains which are not lines and contain 0.

C_4: the rest.

The contact relation on $A(K,R)$ is inherited from $\Sigma(K,R)$ via ι in an obvious way.

3.6.3. PROPOSITION. *The 'inversion'* $x \mapsto x^{-1}$ *on* R^* *induces on* $\overline{C}$ *an involutory permutation leaving* C_1 *and* C_4 *invariant and interchanging* C_2 *and* C_3. *Any parallel class of lines contained in* $\overline{C}$ *is mapped onto a bundle of affine chains, mutually contacting in* 0 (*and vice versa*).

This is true because the map $x \mapsto x^{-1}$ is induced by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(operating on $P(R)$), compare also 3.7.2. (The inversion in the classical sense should fix a certain chain pointwise. For this situation, compare, e.g., Benz [1968].)

Let $K[X,Y]_h^r$ be the K-vector space of homogeneous polynomials of degree r in two indeterminates, which has dimension $r+1$ and natural basis $X^{r-i}Y^i$, $i=0,\dots,r$. For $F \in K[X,Y]_h^r$ we have

$$F = \sum_{i=0}^{r} a_i X^{r-i} Y^i \quad \text{for suitable } a_i \in K.$$

Then for $s,t \in K$ let

$$F(s,t) = \sum_{i=0}^{r} a_i s^{r-i} t^i.$$

In $P^n(K)$ a *rational curve of degree* r is given by

$$N = \{K(F_0(s,t),\dots,F_n(s,t)) \colon K(s,t) \in P^1(K)\},$$

where $F_0,\dots,F_n$ are in $K[X,Y]_h^r$ and have greatest common divisor 1. Then N spans a projective subspace of dimension $\leqslant r$. If $F_0,\dots F_n$ generate $K[X,Y]_h^r$ as a vector space over K, then N is called a *rational normal curve of degree* r (shortly: V_1^r). Any two V_1^r in $P^n(K)$ are projectively equivalent. These curves have many interesting properties, similar to the more general Veronese manifolds V_m^r, compare, e.g., Burau [1961]. For $|K| \geqslant r$ a V_1^r spans a subspace of dimension r.

As usual for a set M of points $K(x_0,\dots,x_n)$ of $P^n(K)$, the affine part of M in $A^n(K)$ is the set of the points

$$\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right) \quad \text{for all } K(x_0,\dots,x_n) \in M \text{ with } x_0 \neq 0.$$

An affine V_1^r is the affine part of a V_1^r, etc.

3.6.4. Lemma. *For $i=0,\dots,r$ let f_i be a polynomial in one indeterminate of degree $r-i$. Then*

$$\overline{N} = \{0\} \cup \left\{\left(\frac{f_1(t)}{f_0(t)},\dots,\frac{f_r(t)}{f_0(t)}\right) \colon t \in K,\ f_0(t) \neq 0\right\}$$

is an affine V_1^r in $A^r(K)$.

Proof. For

$$f_i = \sum_{j=0}^{r-i} k_j^{(i)} Y^j$$

define

$$F_i = \sum_{j=0}^{r-i} k_j^{(i)} X^{r-j} Y^j.$$

Then $F_0,\dots,F_r$ are a basis of $K[X,Y]_h^r$ and therefore

$$N = \{K(F_0(s,t),\dots,F_r(s,t)) \colon K(s,t) \in P^1(K)\}$$

is a V_1^r. Moreover, for $t \neq 0$ we have $F_0(0,t) \neq 0$, $F_i(0,t) = 0$, $1 \leqslant i \leqslant r$, and for $t \in K$ we get $F_i(1,t) = f_i(t)$. Hence

$$N = \{K(1,0,\dots,0)\} \cup \{K(f_0(t),\dots,f_r(t)) \colon t \in K\}$$

and $\overline{N}$ is the affine part of N and therefore an affine V_1^r. □

The following theorem has been proved earlier in the commutative case by Havlicek [1983] (see also the remark after 7.2.3).

3.6.5. THEOREM. *Let $a \in R$ have minimal polynomial of degree r over K. Then $c(a)$ is an affine V_1^r.*

PROOF. Let

$$m = \sum_{i=0}^{r} k_i Y^i$$

be the minimal polynomial of $a \in R$ with $k_r = 1$. Taking

$$f_i = \sum_{j=0}^{r-i} k_{i+j} Y^j$$

we obtain the equation

$$f_0(t) + (Y - t) \sum_{i=0}^{r-1} f_{i+1}(t) Y^i = m,$$

using $f_i(t) - t f_{i+1}(t) = k_i$.

Let $t \in K$ with $f_0(t) \neq 0$. Then substituting a for Y we obtain $m(a) = 0$ and therefore

$$(a - t)^{-1} = -\frac{1}{f_0(t)} \sum_{i=0}^{r-1} f_{i+1}(t) a^i.$$

Now from Lemma 3.6.4 the assertion follows, since the polynomial f_i has degree $r - i$ and $1 = a^0, a^1, \ldots, a^{r-1}$ are linearly independent. □

3.6.6. EXAMPLES.

(i) Consider the case $\dim_K R = 2,\ R = K + Ki$. Then for $a \in R \setminus K$, $a = x + iy$ with $x, y \in K,\ y \neq 0$ we have

$$c(a) = c(a - x) = c(iy) = y^{-1} c(i).$$

Since any V_1^2 is a conic, we find: the chains of $A(K, R)$ which are not lines are affinely equivalent to conics, say, ellipses, parabolas or hyperbolas. Thus we arrive at the 3 possible cases for $A(K, R)$ to be a Miquelian Benz plane, i.e. a Möbius, Laguerre, or Minkowski plane, respectively. In case $K = \mathbb{R}$, we may choose $i \in R$ with $i^2 = \varepsilon$ where $\varepsilon = -1, 0, 1$, respectively. We have

$$(i - t)^{-1} = \frac{i + t}{i^2 - t^2} = \frac{i + t}{\varepsilon - t^2}.$$

Using coordinates with respect to $1, i$ we get

$$c_0 = -2c(i) = \{(0,0)\} \cup \left\{ \left(\frac{2t}{t^2 - \varepsilon}, \frac{2}{t^2 - \varepsilon} \right) :\ t \in K,\ t^2 - \varepsilon \neq 0 \right\}.$$

If $\varepsilon = 0$, c_0 is the parabola $\{(x, y) \colon\ 2y = x^2\}$. For $\varepsilon = -1, 1$, consider

$$c_1 = c_0 + (0, \varepsilon) = \{(0, \varepsilon)\} \cup \left\{ \left(\frac{2t}{t^2 - \varepsilon}, \frac{\varepsilon t^2 + 1}{t^2 - \varepsilon} \right) :\ t \in K,\ t^2 - \varepsilon \neq 0 \right\};$$

this is a curve with equation $-\varepsilon x^2 + y^2 = 1$. For $\varepsilon = -1$, we thus find the unit circle, and for $\varepsilon = 1$, unit hyperbola with perpendicular asymptotes.

(ii) Examples with rational normal curves V_1^r of degree $r > 2$ as affine chains have been given in Havlicek [1983], Herzer [1985b, 1986a], Samaga [1976]. Interesting properties of special affine Möbius geometries can be found in Bruck [1973a,b].

3.7. *Cremonian geometries*

3.7.1. Let $e_1, \ldots, e_n$ be a K-basis of R and

$$1 = \sum_{i=1}^{n} \varepsilon_i e_i.$$

We have bilinear forms φ_k on R given by

$$x \cdot y = \sum_k \varphi_k(x, y) e_k.$$

They are determined by $\varphi_k(e_i, e_j) = \Gamma_{ij}^k$, the structure constants of R. Furthermore, we need the linear forms φ_{kj}, given by $\varphi_{kj}(x) = \varphi_k(x, e_j)$, i.e.

$$\varphi_{kj}(x) = \sum_i x_i \Gamma_{ij}^k \quad \text{for } x = \sum_i x_i e_i.$$

Writing out the condition $xy = 1$ for

$$x = \sum_i x_i e_i, \quad y = \sum_j y_j e_j,$$

we obtain a system of equations

$$\varphi_k(x, y) = \sum_j \varphi_{kj}(x) y_j = \varepsilon_k, \quad k = 1, \ldots, n.$$

By Cramer's rule we find homogeneous polynomials F_j and N in n indeterminates of degree $n - 1$ or n, respectively, with $N(x) = \det(\varphi_{ij}(x))$ and

$$y_j = \frac{F_j(x)}{N(x)}, \quad j = 1, \ldots, n,$$

for $x \in R^*$. Defining

$$\bar{x} = \sum_j F_j(x) e_j,$$

we obtain

$$\begin{aligned} N(x) &= x\bar{x} = \bar{x}x \neq 0 \quad \text{iff} \quad x \in R^*, \\ N(xy) &= N(x)N(y), \quad \text{hence } N(1) = 1, \\ \overline{xy} &= \bar{y}\bar{x}, \\ N(kx) &= k^n x, \\ \overline{kx} &= k^{n-1}\bar{x}. \end{aligned}$$

3.7.2. Assume $\dim_K R = n < \infty$. Let $\gamma \in \Gamma(R)$ be induced by

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R).$$

Define polynomials $G_i^{(\sigma)}, g^{(\sigma)}$ in n indeterminates by

$$g^{(\sigma)}(x) = N(xb + d), \quad \overline{(xb + d)}(xa + c) = \sum_i G_i^{(\sigma)}(x) e_i.$$

Then for $x \in R$ with $xb + d \in R^*$:

$$\begin{aligned} x^{\iota\gamma\iota^{-1}} &= R(x, 1)^{\gamma\iota^{-1}} = (xb + d)^{-1}(xa + c) \\ &= N(xb + d)^{-1}\overline{(xb + d)}(xa + c) \\ &= \sum_i \frac{G_i^{(\sigma)}(x)}{g^{(\sigma)}(x)} e_i. \end{aligned}$$

Since γ is invertible, $\iota\gamma\iota^{-1}$ describes a birational transformation (Cremona transformation, see Gröbner [1970]) and $\widetilde{\Gamma} = \iota\Gamma(R)\iota^{-1}$ is a group of birational transformations on the affine space R, which is transitive on the set $\overline{C}$ of affine chains. In general, lines are not mapped onto lines. A geometry having such a group of birational transformations is called a *Cremonian geometry*.

In this sense the affine chain geometry $A(K, R)$ is a Cremonian geometry, see Benz [1977], Benz, Samaga and Schaeffer [1981].

4. The chain geometry $\Sigma_K(U, R)$

In this section K is a field and R is a K-algebra of finite K-dimension n. We will interpret the points of the projective line over R as certain subspaces of a projective space over K and the chains as so-called 'reguli'. To distinguish these two concepts, we shall in this section denote the projective lines over R and K by $P^1(R)$ and $P^1(K)$, respectively, and the projective space of a vector space V over K by $P(V, K)$. So at first we need some definitions for such a projective space over K.

We consider a right vector space V over K of dimension $2r$. Let $L_r(V)$ denote the set of all subspaces of V of dimension r. We shall interpret $L_r(V)$ as the set of all projective $(r-1)$-subspaces in $P(V, K)$. However, in this section we shall always understand by *dimension* the dimension in the vector space sense.

A set $\boldsymbol{R} \subseteq L_r(V)$ is called *regulus* (in the literature usually called $(r-1)$-regulus) provided the following hold:

(i) The elements of $\boldsymbol{R}$ are mutually skew, i.e. disjoint, and $|\boldsymbol{R}| \geqslant 3$.

(ii) Let l be a line intersecting three elements of $\boldsymbol{R}$. Then every element of $\boldsymbol{R}$ meets l, and every point of l is contained in some element of $\boldsymbol{R}$.

Such a line l is called *transversal line* of $\boldsymbol{R}$.

4.1. PROPOSITION.

(i) *Let be* $V = U \oplus W$ *with* $\dim U = r = \dim W$ *and* $\alpha\colon U \to W$ *be a* K-*isomorphism,*

$$Z_\alpha(s,t) = \{us + (u\alpha)t\colon\ u \in U\} \quad \textit{for } s,t \in K,$$

$$\boldsymbol{R} = \{Z_\alpha(s,t)\colon\ K(s,t) \in P(K)\}.$$

Then $\boldsymbol{R} \subseteq L_r(V)$ *and* $\boldsymbol{R}$ *is a regulus. The mapping* $P^1(K) \to \boldsymbol{R}$, $K(s,t) \mapsto Z_\alpha(s,t)$, *is bijective.*

(ii) *Every regulus* $\boldsymbol{R} \subseteq L_r(V)$ *can be obtained in this way, starting from any* $U, W \in \boldsymbol{R}$ *such that* $V = U \oplus W$.

(iii) *Given* $X, Y, Z \in L_r(V)$, X, Y, Z *mutually skew, there is exactly one regulus in* $L_r(V)$ *containing* X, Y, Z.

PROOF. Most assertions follow by easy computations, e.g.,

$$\dim Z(s,t) = r,\ Z(s,t) \cap Z(s',t') \neq 0 \Rightarrow K(s,t) = K(s',t').$$

The transversal lines are given by $\langle u, u\alpha\rangle$, $0 \neq u \in U$. □

We recall that a right R-module U is called *faithful* provided $Ur = 0$ for some $r \in R$ implies $r = 0$. Now let U and U' be isomorphic faithful right R-modules of K-dimension r with R-isomorphism $'\colon\ U \to U'$, $u \mapsto u'$, i.e. for $a \in R$ we have $(ua)' = u'a$. Further, let $V = U \oplus U'$ with the usual structure of right R-module. For $R(a,b) \in P^1(R)$ define

$$[a,b] = \{(ua, u'b)\colon\ u \in U\}.$$

Then $[a,b]$ is well defined and $[a,b] \in L_r(V)$, where V has to be considered as vector space over K of dimension $2r$.

We define

$$P_R(U) = \{[a,b]\colon\ R(a,b) \in P^1(R)\},$$

$$C\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \{[sa+tc, sb+td]\colon\ K(s,t) \in P^1(K)\},$$

$$C(U) = \left\{ C\begin{bmatrix} a & b \\ c & d \end{bmatrix}\colon\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,R) \right\},$$

$$\Sigma_K(U,R) = \big(P_R(U), C(U)\big).$$

The elements of $P_R(U)$ and $C(U)$ will also be called *points* and *chains* of $\Sigma_K(U,R)$, respectively. For every

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

we have a linear K-automorphism $\widetilde{\sigma}$ of V defined by

$$(u,w')^{\widetilde{\sigma}} = (ua + wc, u'b + w'd) \quad \text{for } u, w \in U.$$

An easy computation shows

$$C\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{\widetilde{\sigma}} = C\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Hence we have the following proposition.

4.2. Proposition. *Let $\widetilde{\Gamma}(U)$ be the group of projectivities on $P(V,K)$ induced by all $\widetilde{\sigma}$ for $\sigma \in \mathrm{GL}_2(R)$. Then $\widetilde{\Gamma}$ is a group of automorphisms of $\Sigma_K(U,R)$.*

4.3. Theorem. *The mapping $\zeta\colon\ P^1(R) \to P(V,K)$, $R(a,b) \mapsto [a,b]$, induces an isomorphism of $\Sigma(K,R)$ onto $\Sigma_K(U,R)$, and we have*

$$\widetilde{\Gamma}(U)|_{P_R(U)} = \zeta^{-1}\Gamma(R)\zeta,$$

i.e. $\widetilde{\Gamma}(U)$ operates on $P_R(U)$ and $C(U)$ as a permutation group similar to $\Gamma(R)$ operating on $P(R)$ and $C(R)$, respectively.

The chains of $\Sigma_K(U,R)$ are exactly the reguli contained in $P_R(U) \subseteq L_r(V)$.

To prove the last assertion, it suffices to consider

$$C\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let α: $U \to U'$ be defined by $u \mapsto u'$. Then $Z_\alpha(s,t) = \{(us, u't)\colon\ u \in U\}$, and

$$C\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \big\{[s,t]\colon\ K(s,t) \in P(K)\big\} = \big\{Z_\alpha(s,t)\colon\ K(s,t) \in P(K)\big\}$$

in fact is a regulus in $L_r(V)$. Now the assertion is obvious, since any three pairwise distant points determine a unique chain. □

4.4. REMARK. On $L_r(V)$ a distance function d can be defined by

$$d(X,Y) = \dim(X+Y/X) = \dim(Y/X \cap Y),$$

whose values can be $0, 1, \ldots, r$.

To prove the triangle inequality:

$$\begin{aligned} d(X,Z) &\leqslant \dim\big((X+Y+Z)/X\big) \\ &= \dim\big((X+Y)/X\big) + \dim\big((X+Y+Z)/(X+Y)\big) \\ &\leqslant \dim\big((X+Y)/X\big) + \dim\big((Y+Z)/Y\big) = d(X,Y) + d(Y,Z). \end{aligned}$$

This distance function is a refinement of the distant-relation between points, for in $P_R(U)$ two points X, Y are distant if $d(X,Y) = r$ holds, otherwise they are parallel.

4.5. EXAMPLES.

(1) R itself is a faithful right R-module, and for $U = R$ we obtain $[a,b] = R(a,b)$. Therefore we can identify $\Sigma_K(R,R)$ and $\Sigma(K,R)$ considering the points of $\Sigma(K,R)$ as subspaces over K of dimension $n = \dim_K R$.

The next two cases are of this type.

(2) Let R be a skew field. Then $\Sigma(K,R)$ is a Möbius geometry. Any three distinct points are incident with exactly one chain. For $\Sigma_K(R,R)$ this means that any three distinct points of $P_R(R) \subseteq L_n(V)$ are contained in exactly one regulus. Since on the other hand any point $K(a,b)$ of $P(V,K)$ belongs to some member of $P_R(R)$, i.e. to $[a,b]$, it turns out that $P_R(R)$ is a *planar regular spread*.

(3) Let $R = K^{(n)}$, so $\Sigma(K,R)$ is a Minkowski geometry. The subspaces representing the points of $\Sigma_K(R,R)$ can be described in the following way: $V = R^2 \cong K^{2n}$ is the direct sum of n subspaces $L_1, \ldots, L_n$ of dimension 2, and so

$$P_R(R) = \big\{P_1 + \cdots + P_n\colon\ P_i \text{ is point of } L_i\big\}.$$

For $n = 2$, the set $P_R(R)$ is the hyperbolic congruence of lines in the 3-space over K.

(4) Let U be a vector space of dimension r over K, and let $R = \mathrm{End}_K(U)$ be the ring of K-endomorphisms of U. Then R is a K-algebra of dimension $n = r^2$ (isomorphic to the algebra of $(r \times r)$-matrices over K) and U is a faithful right R-module. We will show that $P_R(U) = L_r(V)$.

Let $X \in L_r(V)$ with $\dim([1,0] \cap X) = s$. We choose a basis $x_1, \ldots, x_r$ of X, $x_i = (u_i, w_i')$ such that $w_i = 0$ for $i = 1, \ldots, s$, (where $'$ again denotes the canonical isomorphism $U \to U'$).

Then $u_1, \ldots, u_s$ are independent and $w_{s+1}, \ldots, w_r$ are independent.

For a basis $v_1, \ldots, v_r$ of U there are $a, b \in R$ defined by $v_i a = u_i$, $v_i b = w_i$, $i = 1, \ldots, r$. Then $X = [a, b]$. On the other hand, there are $c, d \in R$ with

$$u_i c = v_i \quad \text{for } i = 1, \ldots, s,$$

$$w_i d = v_i - u_i c \quad \text{for } i = s+1, \ldots, r,$$

hence

$$v_i(ac + bd) = v_i \quad \text{for } i = 1, \ldots, r.$$

We conclude that $ac + bd = 1$, i.e. (a, b) is unimodular, from which it follows that $X = [a, b] \in P_R(U)$.

4.6. OPEN PROBLEMS. Let V be a vector space over K of dimension $2r$.

(1) Classify the sets $P \subseteq L_r(V)$ which are closed under reguli: for any $X, Y, Z \in P$ mutually skew, the regulus defined by X, Y, Z (following 4.1(iii)) is contained in P. Any such set P gives rise to a weak chain space $\Sigma = (P, C)$, where C is the set of reguli contained in P.

(2) Characterize those sets $P \subseteq L_r(V)$, closed under reguli, where $\Sigma = (P, C)$ is a chain geometry for C the set of all reguli contained in P (e.g., by a transitive automorphism group).[6]

4.7. REMARK. A construction of Möbius spaces whose points are the elements of a planar spread $\subseteq L_r(V)$ is presented in Hölz [1980].

5. The chain geometry $G_K(U, R)$

In this section we present a general method to represent $\Sigma(K, R)$ in such a way that its points and chains are points and curves in a projective space. We keep notations as in the previous section. We denote by G the Grassmannian $G_{2r-1,r-1}$ of V defined in the projective geometry $P(\wedge^r V, K)$ over the r-fold exterior product over V:

$$G = \{Kv_1 \wedge \cdots \wedge v_r \colon v_i \in V,\ v_1 \wedge \cdots \wedge v_r \neq 0\}.$$

[6] A solution has been given in the meantime by Herzer [1992a]; see also Kroll [1991, 1992].

As is known, the Grassmann map

$$\gamma\colon L_r(V) \to \boldsymbol{G}, \quad \langle v_1, \dots, v_r \rangle \mapsto K v_1 \wedge \dots \wedge v_r,$$

is well defined and bijective. We transfer the distance d on $L_r(V)$ to $\boldsymbol{G}$ by

$$\widehat{d}\bigl(X^\gamma, Y^\gamma\bigr) := d(X,Y) \quad \text{for } X, Y \in L_r(V).$$

5.1. *Generalized reguli*

To give an intrinsic definition of $\widehat{d}$ on $\boldsymbol{G}$ only referring to the normal rational curves $V_1^{\mathfrak{l}}$ on $\boldsymbol{G}$, we will use the concept of *generalized regulus of type* $(r, \mathfrak{l})$ (abbreviation: $g(r,\mathfrak{l})$), $1 \leqslant \mathfrak{l} \leqslant r$.

5.1.1. DEFINITION (Metz [1981]). Let $\boldsymbol{R}$ be a subset of $L_r(V)$ with $|\boldsymbol{R}| \geqslant 3$. Then $\boldsymbol{R}$ is a $g(r,\mathfrak{l})$ provided the following conditions hold.

(i) There exist subspaces A and B of dimension $r-\mathfrak{l}$ and $r+\mathfrak{l}$, respectively, such that for any two distinct members X, Y of $\boldsymbol{R}$ we have $X \cap Y = A$ and $X + Y = B$.

(ii) Any line $\mathfrak{l}$ skew to A which intersects three members of $\boldsymbol{R}$ meets every member of $\boldsymbol{R}$, and any point on $\mathfrak{l}$ is contained in some member of $\boldsymbol{R}$. (Hence a $g(r,r)$ is a regulus in $L_r(V)$, and a $g(2,1)$ is a pencil of lines in a plane.)

5.1.2. LEMMA. *A subset $\boldsymbol{R}$ of $L_r(V)$ is a $g(r,\mathfrak{l})$ iff the following is true. Let U, W be two different members of $\boldsymbol{R}$ and $A = U \cap W$. Then there exist $u_1, \dots, u_{\mathfrak{l}} \in U$ which form a basis of a complement of A in U and $w_1, \dots, w_{\mathfrak{l}} \in W$ which form a basis of a complement of A in W such that for*

$$Z(s,t) := \langle u_1 s + w_1 t, \dots, u_{\mathfrak{l}} s + w_{\mathfrak{l}} t \rangle + A$$

we have

$$\boldsymbol{R} = \bigl\{Z(s,t)\colon\ K(s,t) \in P(K)\bigr\}.$$

Remember the abbreviation $V_1^{\mathfrak{l}}$ for a normal rational curve of order $\mathfrak{l}$, see Section 3.

5.1.3. LEMMA. *If $\boldsymbol{R}$ is a $g(r,\mathfrak{l})$ then $\boldsymbol{R}^\gamma$ is a $V_1^{\mathfrak{l}}$.*

PROOF. Let $v_1, \dots, v_{r-\mathfrak{l}}$ be a basis of A. Then

$$K \cdot z(s,t) := Z(s,t)^\gamma = K\bigl(u_1 s + w_1 t\bigr) \wedge \dots \wedge \bigl(u_{\mathfrak{l}} s + w_{\mathfrak{l}} t\bigr) \wedge v_1 \wedge \dots \wedge v_{r-\mathfrak{l}}$$

$$= K \sum_{i=0}^{\mathfrak{l}} v^{(i)} s^{\mathfrak{l}-i} t^i, \quad v^{(i)} \in \wedge^r V.$$

After having checked that $v^{(0)}, \dots, v^{(\mathfrak{l})}$ are independent (from the properties of the exterior product), one sees immediately that

$$\boldsymbol{R}^\gamma = \bigl\{K \cdot z(s,t)\colon\ K(s,t) \in P(K)\bigr\}$$

is a $V_1^{\mathfrak{l}}$ on the Grassmannian $\boldsymbol{G}$. □

The converse of 5.1.3 is not true in general. In Herzer [1984a] examples are given of curves $V_1^{\mathfrak{l}}$ on G which are not the image of a $g(r, \mathfrak{l})$. To characterize those $V_1^{\mathfrak{l}}$ on G which come from a $g(r, \mathfrak{l})$ we will use the distance $\widehat{d}$ on G.

5.2. *Intrinsic characterization of the distance on G*

The proofs of the following results can be found in Herzer [1984], see also Lunardon [1984]. From now on let $|K| > r + 2$.

5.2.1. PROPOSITION. *$\widehat{d}(Q, P) = \mathfrak{l}$ for $P, Q \in G$ holds iff P and Q are joined by a $V_1^{\mathfrak{l}}$ on G but not by any V_1^m on G for $m < \mathfrak{l}$.*

5.2.2. DEFINITION. An *$\mathfrak{l}$-distance line* is a $V_1^{\mathfrak{l}}$ on G any two distinct points of which have distance $\mathfrak{l}$.

5.2.3. LEMMA. *If a rational curve v on G of order $\leqslant \mathfrak{l}$ possesses two points of distance $\mathfrak{l}$, then it is an $\mathfrak{l}$-distance line.*

5.2.4. LEMMA. *Every $\mathfrak{l}$-distance line is the image of some $g(r, \mathfrak{l})$.*

For instance, let $r = 2$. Then G is the Klein quadric $G_{3,1}$. In the case $\mathfrak{l} = 1$ every straight line is the image of a pencil of lines.

5.2.5. PROPOSITION. *The Grassmann map γ induces a bijection from the set of all $g(r, \mathfrak{l})$ in $L_r(V)$ onto the set of all $\mathfrak{l}$-distance lines of G.*

5.3. *Representation of chain geometries on the Grassmannian*

Following ideas of Hubaut [1965], Werner [1982f] we obtain

5.3.1. DEFINITION. $G_K(U, R) = (\boldsymbol{V}, \widehat{\mathcal{C}})$ is the chain geometry which is the image under γ of $\Sigma_K(U, R)$, namely, $\boldsymbol{V} = P_R(U)^\gamma$, $\widehat{\mathcal{C}} = \mathcal{C}(U)^\gamma = \{\boldsymbol{c}^\gamma\colon\ \boldsymbol{c} \in \mathcal{C}(U)\}$.

5.3.2. THEOREM. *There is a projective subspace T of $P(\wedge^r V, K)$ and a closed algebraic subset $\boldsymbol{W}$ of $\boldsymbol{G} \cap T$ such that $\boldsymbol{V} = (\boldsymbol{G} \cap T) \setminus \boldsymbol{W}$, in particular $\boldsymbol{V}$ is a quasiprojective variety of dimension r. The chains of $G_K(U, R)$ are the r-distance lines contained in $\boldsymbol{V}$: any 3 mutually distant points of $\boldsymbol{V}$ (i.e. points of distance r in $\boldsymbol{G}$) belong to exactly one r-distance line $\boldsymbol{v}$ in $\boldsymbol{G}$, and then $\boldsymbol{v}$ is entirely contained in $\boldsymbol{V}$. There is a projective subspace X of T of dimension[7] r such that $\boldsymbol{v} = \boldsymbol{V} \cap X$, namely, $X = \langle \boldsymbol{v} \rangle$.*

The intrinsic definition of distance according to 5.2.1 also works for $\boldsymbol{V}$ instead of $\boldsymbol{G}$ (with slight variations, see Herzer [1984a]). There exists a group $\widehat{\Gamma}(R)$ of projective collineations of T operating on $\boldsymbol{V}$ as group of automorphisms of $G_K(U, R)$, and $\widehat{\Gamma}(R)$ is similar to $\Gamma(R)$ as permutation group on points (or chains).

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[7] Here and in the following, dimension means projective dimension.

5.3.3. PROPOSITION (Herzer [1985c], see also 8.4.5(c)). *Let $R = K[a]$ be of K-dimension n. Then the quasiprojective variety $\boldsymbol{V}$ spans a projective space of dimension $2^n - 1$.*

5.3.4. PROPOSITION (Herzer [1986b, 1987b]). *$\boldsymbol{V}$ is a projective variety (i.e. $\boldsymbol{V} = \boldsymbol{G} \cap T$) iff R is semisimple.*

Since $\wedge^{2r} V$ has K-dimension 1 the mapping

$$\wedge^r V \times \wedge^r V \to \wedge^{2r} V, \quad (u, w) \mapsto u \wedge w$$

gives rise to a nondegenerate bilinear form on $\wedge^r V$ which is symmetric or antisymmetric according to whether r is even or odd. So in $P(\wedge^r V, K)$ the Grassmannian $\boldsymbol{G}$ is accompanied in a canonical way by its *fundamental polarity* $\varkappa$ induced by the above mentioned bilinear form.

5.3.5. PROPOSITION (Werner [1982f]). *Two points of $G_K(U, R)$ are parallel iff they are conjugate under $\varkappa$.*

Further, the natural contact relation of $G_K(U, R)$ in the sense of Section 3.3 can be described in geometric terms using the concept of tangent lines of the V_1^r. Two curves in a projective space are said to satisfy the *geometric contact relation* in a point p provided they have a common tangent line.

5.3.6. PROPOSITION (Werner [1982f]). *In $G_K(U, R)$ the natural contact relation and the geometric contact relation coincide.*

5.3.7. EXAMPLES.

(1) Using Example 4.5(4) one sees that $\boldsymbol{G}$ itself with all its r-distance lines is a chain geometry $G_K(U, R)$.

(2) The Minkowski geometry $G_K(R, R)$ for $R = K^{(n)}$ is a Segre manifold $S_{(1)^n}$. The chains are the V_1^n intersecting all the lines of all the shears of the $S_{(1)^n}$ exactly once.

(3) The Möbius geometry $G_K(R, R)$, where R is a skewfield of K-dimension n. Here we call $\boldsymbol{V}$ an *algebraic n-sphere* (Lunardon [1984]: 't-sfera'). Any $n + 1$ points of $\boldsymbol{V}$ are independent and any 3 points of $\boldsymbol{V}$ are contained in exactly one V_1^n lying on $\boldsymbol{V}$. Given a point p on $\boldsymbol{V}$, the tangent space to $\boldsymbol{V}$ at p is the osculating hyperplane of an arbitrary V_1^n through p. For (2) and (3), compare Herzer [1978].

5.3.8. OPEN PROBLEMS. 1. A set $\boldsymbol{V} \subseteq \boldsymbol{G}$ is called *closed* with respect to r-distance lines provided for any 3 distant points of $\boldsymbol{V}$ the unique r-distance line of $\boldsymbol{G}$ containing these 3 points is contained in $\boldsymbol{V}$.

Classify the closed sets $\boldsymbol{V}$, e.g., by giving an algebraic description. Characterize the closed $\boldsymbol{V} \subseteq \boldsymbol{G}$ belonging to a chain geometry $G_K(U, R)$, e.g., by transitivity conditions on their automorphism group. (These problems are essentially the same as the problems stated in 4.6.)

2. Same questions for an (abstract) algebraic n-sphere are described in 5.3.7(3).

6. Rational representations of chain geometries

6.1. *The general case*

For a field K with $|K| \geqslant r+2$ we shall say that the chain space Σ possesses a *rational representation of degree* r (*and dimension* m) provided Σ is isomorphic to an incidence structure $(\boldsymbol{V} \setminus \boldsymbol{W}, \boldsymbol{C})$, where $\boldsymbol{V}$ is a projective variety (spanning a $P^m(K)$), $\boldsymbol{W}$ is a (Zariski-)closed subset of $\boldsymbol{V}$ and $\boldsymbol{C}$ consists of rational curves of order r[8] lying on $\boldsymbol{V} \setminus \boldsymbol{W}$, and moreover two points of $\boldsymbol{V} \setminus \boldsymbol{W}$ are parallel whenever they are joined by a rational curve on $\boldsymbol{V}$ of order $l < r$. The rational representation is called *smooth*, if each chain is a smooth rational curve.

A standard example of smooth rational representations for chain geometries $\Sigma(K, R)$ are the geometries $G_K(U, R)$. They have a group $\widehat{\Gamma}$ operating transitively on the set of triplets of mutually distant points and on the set of chains.

If we ask for low-dimension representations, then in general we cannot expect such a large and transitive group of automorphisms. Using appropriate projections of $G_K(U, R)$ we have the following proposition (see also 8.4.4 and 8.4.5(a)):

6.1.1. PROPOSITION. *Let U be a faithful right R-module of dimension r. Then $\Sigma(K, R)$ possesses a smooth rational representation of degree r and projective dimension m, for all m with $2r + 1 \leqslant m \leqslant \overline{m}$, where $\overline{m}$ is the projective dimension of the representation* $G_K(U, R)$ (Herzer [1985a]).

6.1.2. If R is a local algebra of K-dimension n one can give an explicit representation of dimension $2n+1$ using the functions $\overline{}$ and N defined in 3.7.1. For this purpose we consider the mapping

$$P(R) \to P(K \oplus R \oplus R \oplus K, K), \quad R(a, b) \mapsto K\big(N(a), \bar{a}b, \bar{b}a, N(b)\big).$$

This mapping is well defined since for $c \in R^*$ we have the equations $N(ca) = N(c)N(a)$ and

$$\overline{ca}\, cb = \bar{a}\bar{c}cb = N(c)\bar{a}b \quad \text{etc.}$$

The map is injective since we may assume that either $a = 1$ or $b = 1$ if R is local, and

$$R(1, b) \mapsto K\big(1, b, \bar{b}, N(b)\big),$$
$$R(a, 1) \mapsto K\big(N(a), \bar{a}, a, 1\big)$$

hold.

Using coordinates as in Section 3.7, we see that homogeneous polynomials of degree n are used, and therefore chains become rational curves of order n. We mention without proof that this mapping can also be obtained by a suitable projection of $G_K(U, R)$; a proof will be given elsewhere.

[8] That is appropriate projections of a V_1^r.

6.1.3. If R is a skew field of K-dimension n we even get a representation of dimension $n+1$ (Werner [1983b]), using the mapping

$$P(R) \to P(K \oplus R \oplus K, K), \quad R(a,b) \mapsto K\big(N(a), \bar{b}a, N(b)\big).$$

Indeed, the only point which cannot be written in the form $R(a,1)$ is $R(1,0)$. The image of $P(R)$ is an algebraic hypersurface $\boldsymbol{V}$ of order n, given by the equation $xz^{n-1} = N(y)$ (for $(x,y,z) = (N(a), \bar{b}a, N(b))$). Further, $\boldsymbol{V}$ has $K(1,0,0)$ as its only singular point, provided $n > 2$.

Other examples of such rational representations are given in Benz et al. [1981], §5 (compare the last example).

6.2. *Chain geometries over quadratic algebras*

There is another class of K-algebras R which allow a smooth rational representation of dimension $n+1$ (and degree 2), namely the (associative) quadratic algebras. In this situation there also exists a large group of automorphisms (similar to $\widehat{\Gamma}$ in the representation $G_K(U,R)$).

Here we follow the lines of Hotje [1974, 1976].[9]

6.2.1. DEFINITION. A K-algebra R is *quadratic* provided for any $a \in R \backslash K$ the minimal polynomial m_a is quadratic. We define functions Tr, N from R to K in the following way: For $k \in K$ let $\mathrm{Tr}(k) = 2k$ and $N(k) = k^2$. For $a \in R\backslash K$ with $m_a = x^2 - k_a x + l_a$ define $\mathrm{Tr}(a) = k_a$ and $N(a) = l_a$.

Examples are the algebra $M_2(K)$ of two-by-two matrices with entries in K, the quaternion skewfields with centre K, the K-algebras of dimension 2 over K (they lead to the Miquelian Benz planes) and for $\operatorname{char} K = 2$ certain purely inseparable field-extensions of K.

Another example is provided by the Laguerre algebras $K \oplus N$ with $x^2 = 0$ for every $x \in N$, for instance that given in Section 2.1.

Quadratic algebras are semiprimary, whence of stable rank 2. This readily follows from the classification given in Bröcker [1972], Karzel [1973, 1974] (where quadratic are named *kinematic algebras*).

6.2.2. LEMMA (Hotje [1974]). *Let R be a quadratic K-algebra. The mapping $\bar{\ }$ of R onto R given by*

$$\bar{a} = \mathrm{Tr}(a) - a \quad \textit{for any } a \in R,$$

is an involutory K-linear anti-automorphism of R. Further,

(a) *$\bar{a}$ has the same minimal polynomial as a, since* $\mathrm{Tr}(a) = a + \bar{a} = \mathrm{Tr}(\bar{a})$ *and* $N(a) = a\bar{a} = \bar{a}a = N(\bar{a})$,

(b) *$N(ab) = N(a)N(b)$ and N is a quadratic form,* Tr *is a K-linear form,*

(c) $a \in R^* \Leftrightarrow N(a) \neq 0$.

Let R be a quadratic algebra different from $M_2(K)$. Then:

(d) *$a, b \in R \setminus R^*$ and (a,b) unimodular $\Rightarrow$ there is $c \in R^*$ with $\overline{cb} = ca$,*

(e) *the set $J = \{x \in R\colon\ x^2 = 0\}$ is the Jacobson radical of R.*

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[9] These ideas are used in a wider context by Blunck [1994b].

From now on let R be any quadratic algebra. We define $V = K \oplus R \oplus K$ as vector space over K and consider the mapping

$$\alpha\colon\ P(R) \to P(V,K), \quad R(a,b) \mapsto K\big(N(a), \bar{a}b, N(b)\big).$$

This mapping is well defined since for $c \in R^*$ we have

$$\big(N(ca), \bar{c}\bar{a}cb, N(cb)\big) = N(c)\big(N(a), \bar{a}b, N(b)\big).$$

On $V = K \oplus R \oplus K$ we define a quadratic form Q by means of

$$Q(x,y,z) = xz - N(y).$$

Let $\boldsymbol{Q}$ be the corresponding quadric. From 6.2.2 we immediately see that α maps $P(R)$ into $\boldsymbol{Q}$.

6.2.3. Examples. For $R = M_2(K)$ and $x \in R$,

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we obtain

$$\bar{x} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then one may see that the image of α is the Klein quadric. We know this from Example 4.5(4) using the Grassmann map γ. The quadrics belonging to the two-dimensional algebras are the elliptic quadric, the quadratic cone, the hyperbolic quadric corresponding to the Möbius, Laguerre, Minkowski plane, respectively. Here α even agrees with γ, using $\Sigma_K(R,R)$ in Section 4 (Werner [1982f]).

6.2.4. Proposition. *Let S be the set of double points of $\boldsymbol{Q}$. The map α is injective, and* $\mathrm{im}(\alpha) = \boldsymbol{Q} \setminus S$.

Proof. In view of 6.2.3 we omit the case $R = M_2(K)$.

(1) α is injective. Since $R(a,1)^\alpha = K(N(a), \bar{a}, 1)$ and $R(1,b)^\alpha = K(1, b, N(b))$ it suffices to investigate the case $R(a,b)^\alpha = K(0,e,0) = R(c,d)^\alpha$. From 6.2.2(d) we may assume $a = \bar{b}$ and $c = \bar{d}$, hence $Kb^2 = Ke = Kd^2$. Since here $m_b = x^2 - kx$ and $m_d = x^2 - lx$ for $k, l \in K^*$ we have $b^2 = kb$ and $d^2 = ld$. Thus there exists $s \in K^*$ with $d = sb$ and hence $\bar{d} = s\bar{b}$. Therefore, $R(a,b) = R(\bar{b}, b) = R(\bar{d}, d) = R(c,d)$.

(2) $K(x,y,z) \in S$ iff $x = 0 = z$ and $y^2 = 0$. We have the following equivalent statements:

$$\begin{aligned} K(x,y,z) \in S \Leftrightarrow\ & Q(x+x', y+y', z+z') \\ &= Q(x',y',z') \text{ for every } (x',y',z') \in V \\ \Leftrightarrow\ & x = 0 = z \text{ and } N(y+y') = N(y') \text{ for every } y' \in R \\ \Leftrightarrow\ & x = 0 = z \text{ and } N(y) = 0 = \mathrm{Tr}(y\bar{y}') \text{ for every } y' \in R. \end{aligned}$$

Hence $K(x, y, z) \in S$ implies $x = 0 = z$ and $N(y) = 0 = \mathrm{Tr}(y)$ from which follows $y^2 = 0$.

Conversely, let $y \in R$ with $y^2 = 0$. Then $N(y) = 0$ and $y \in J$ by 6.2.2 implies $y\bar{y}' \in J$ hence $\mathrm{Tr}(y\bar{y}') = 0$, therefore $K(0, y, 0) \in S$.

(3) $\operatorname{im}\alpha = Q \setminus S$. Let $P \in Q \setminus S$. Then $P = K(1, y, z)$ implies $z = N(y)$ and therefore $P = R(1, y)^\alpha$. In the same way, from $P = K(x, y, 1)$, we get $x = N(\bar{y})$ and therefore $P = R(\bar{y}, 1)^\alpha$. If $P = K(0, y, 0)$, then clearly $N(y) = 0$ and $y^2 = ky$ for some $k \in K$. From (2) we obtain $k \neq 0$, hence $P = R(\bar{y}, y)^\alpha$. □

6.2.5. PROPOSITION. *The permutation group $\alpha^{-1}\Gamma(R)\alpha$ on $Q \setminus S$ can be extended to a group $\widehat{\Gamma}$ of projective collineations of $P(V, K)$ leaving Q invariant.*

PROOF. Since R is of stable rank 2, the general linear group $\mathrm{GL}_2(R)$ is generated by matrices of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}, \quad \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } c \in R^* \text{ and } d \in R$$

(see Chapter 19).

Now the following list tells us what $\alpha^{-1}\gamma\alpha$ looks if $\gamma \in \Gamma(R)$ is induced by one of these generators. Here $R(a, b) \in P(R)$.

The map $(a, b) \mapsto$	is transformed into $(N(a), \bar{a}b, N(b)) \mapsto$	which is induced by the map $(x, y, z) \mapsto$
(b, a)	$(N(b), \bar{b}a, N(a))$	$(z, \bar{y}, x)$
(ac, b)	$(N(c)N(a), \bar{c}\bar{a}b, N(b))$	$(N(c)x, \bar{c}y, z)$
$(a + bd, b)$	$(N(a) + N(b)N(d) + \mathrm{Tr}(\bar{a}bd),$ $\bar{a}b + \bar{d}N(b), N(b))$	$(x + zN(d) + \mathrm{Tr}(yd), y + \bar{d}z, z)$

Read: 'The map $(a, b) \mapsto (b, a)$ is transformed into $(N(a), \bar{a}b, N(b)) \mapsto (N(b), \bar{b}a, N(a))$, which is induced by the map $(x, y, z) \mapsto (z, \bar{y}, x)$,' etc.

All these inducing maps are K-linear and invertible.

Remember that two points Ku, Kw of Q are *conjugate* provided they lie on a common line of Q, i.e. whenever $f(u, w) = 0$, where f is the bilinear form belonging to Q.

6.2.6. PROPOSITION. *Two points P, S of $P(R)$ are parallel iff P^α, S^α are conjugate.*

PROOF. We have $f((x_1, y_1, z_1), (x_2, y_2, z_2)) = x_1z_2 + x_2z_1 - \mathrm{Tr}(\bar{y}_1y_2)$. In view of 6.2.5 we may assume $P = R(1, 0)$. Then for $S = R(c, d)$ we obtain

$$\begin{aligned} P \parallel S &\Leftrightarrow \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \notin \mathrm{GL}_2(R) \Leftrightarrow d \in R \setminus R^* \\ &\Leftrightarrow N(d) = 0 \\ &\Leftrightarrow f\big((1,0,0), (N(c), \bar{c}d, N(d))\big) = 0. \end{aligned}$$

□

6.2.7. PROPOSITION. *For any chain* $\mathbf{c}$ *of* $\Sigma(K,R)$ *its image* $\mathbf{c}^\alpha$ *is a conic hence a plane section of* $\boldsymbol{Q}$. *Conversely, for any three points* P,Q,R *of* $\boldsymbol{Q}$ *which are mutually nonconjugate, whence* P,Q,R *span a plane* E, *the intersection* $\boldsymbol{Q}\cap E$ *is the image of a chain.*

PROOF. From the definition of α we infer that the image of a chain is a rational curve of order $\leqslant 2$ which by 6.2.6 contains no three collinear points, whence it is a conic. For example,

$$C\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^\alpha = \{K(s^2, st, t^2)\colon\ K(s,t)\in P(K)\}.$$

The second assertion follows using 6.2.6 again. □

6.3. *Chain spaces associated to quadratic sets*

Let $\Pi=(P,\boldsymbol{L})$ be a projective space, where P is the set of points and $\boldsymbol{L}$ the set of lines (considered as subsets of P). For $M\subseteq P$, call a line L *tangent* to M provided $|M\cap L|=1$ or $L\subseteq M$. The union of all tangent lines at $p\in M$ is called the *tangent space*. A subset $Q\subseteq P$ is a *quadratic set* provided

(i) any line which has more then two points in common with Q is contained in Q, and

(ii) for any $p\in Q$ the tangent space Q_p is a hyperplane or the whole space.

If Q_p is the whole space, we call p a *double point* of Q. The quadratic set Q is *degenerate* provided Q is contained in the union of two hyperplanes. An *ovoid* is a nondegenerate quadratic set containing no lines. We observe that quadrics are quadratic sets. In general it is provided that the order of Π is greater than 3.

6.3.1. DEFINITION. Let Q be a nondegenerate quadratic set in a projective space Π of dimension at least 3. A plane E is called *Q-admissible* provided $Q\cap E$ contains no line but more than one point. Let $\boldsymbol{E}$ be the set of all Q-admissible planes and S the set of all double points. Then we define $I(Q)=(Q\setminus S,\boldsymbol{E})$ where incidence is the obvious one. The following is easily to be seen.

6.3.2. PROPOSITION. $I(Q)$ *is a chain space. It is a Möbius space iff* Q *is an ovoid.*

We call $I(Q)$ the chain space *associated* to the quadratic set Q. (Compare Chapter 17, Section 4.2, Quadric circle geometry.)

For $p\in Q\setminus S$ the projection π_p with centre p maps $Q\setminus Q_p$ onto an affine space A and induces an embedding of $I(Q)\cap D_p$ into A.[10] The chains through p become lines of A, whereas the other chains become ovals minus 0, 1 or 2 points ('stereographic projection'). Compare also Chapter 6 and Chapter 17, Section 4.3.

[10] This is the residual trace as defined in 3.2.1.

6.3.3. THEOREM. *Let R be a quadratic algebra of dimension $n > 1$, $\boldsymbol{Q}$ the corresponding quadric in $V = K \oplus R \oplus K$, and α the mapping described after* 6.2.2. *Then α induces an isomorphism of $\Sigma(K,R)$ onto $I(\boldsymbol{Q})$. In particular, $\Sigma(K,R)$ has a smooth rational representation of order* 2 *and dimension $n+1$. For $P \in \boldsymbol{Q} \setminus S$ the image of the stereographic projection π_P is, up to an affinity, the affine chain geometry $A(K,R)$.*

PROOF of the last assertion. We may assume $P = K(0,0,1)$. Then the hyperplane $\boldsymbol{Q}_P$ is given by $x = 0$ and π_P is defined by $K(1,y,z) \to y$. Moreover,

$$X \in C\begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

implies

$$X^{\alpha} = K\big(N(bs+dt), (\overline{bs+dt})(as+ct), N(as+ct)\big) = K\big(1, z, N(z)\big)$$

for $z = (bs+dt)^{-1}(as+ct)$ provided $bs+dt \in R^*$. (If $bs+dt \in R \setminus R^*$ then $X^{\alpha} \in \boldsymbol{Q}_P$.) Hence we get

$$\left(C\begin{pmatrix} b & a \\ d & c \end{pmatrix} \cap D_U\right)^{\alpha\pi_P} = \overline{C}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where D_U is defined as in Section 3. □

Conversely, we have the following

6.3.4. THEOREM (Hotje [1978]). *If $\Sigma(K,R)$ is isomorphic to a chain space associated to a quadratic set, then R is a quadratic algebra.*[11]

Hotje [1977] gives a construction of the kinematic space belonging to a quadratic algebra R where he uses only the chain space $I(\boldsymbol{Q})$ of the corresponding quadric $\boldsymbol{Q}$ and certain properties of its automorphism group.

6.3.5. OPEN PROBLEM. Which chain spaces associated to a quadric are isomorphic to a chain geometry?

6.4. *Stereographic projection*

For $G_K(U,R)$ we have the following analog of 6.3.3.

6.4.1. THEOREM (Hotje [1985]). *For $G_K(U,R) = (\boldsymbol{V}, \widehat{\mathcal{C}})$ let Π be the projective space spanned by $\boldsymbol{V}$. There is a hyperplane H of Π, a subspace J of H and a point P of $\boldsymbol{V}$ such that the projection π_J maps $\boldsymbol{V} \setminus H$ onto an affine space A inducing an embedding of the residual trace $G_K(U,R) \cap D_P$ into A. The image of this embedding is, up to an affinity, isomorphic to $A(K,R)$.*

[11] Another proof can be found by using the remark given after 7.3.9.

The papers Herzer [1984b, 1985b] help to understand better how these mappings work. We have a chain of mappings

$$\Sigma(K,R)\cap D_x \to \Sigma_K(U,R)\cap D_Y \to \Sigma_K(U,R)\cap D_P \to A(K,R).$$

We can describe H and J geometrically in function of P. Namely, if H is the polar of P with respect to the fundamental polarity associated to G, then $H = \widehat{H}\cap\langle V\rangle$. Moreover, H is spanned by the points of V with distance from P at most $r-1$ and J is spanned by the points of V with distance from P at most $r-2$.

This stereographic projection also seems to work in suitable cases of more general rational representations. Thus, e.g., looking for the representations given in 6.1.2 and 6.1.3 we obtain a chain of maps, the second of which again is a stereographic projection (in the second case with centre $K(1,0,0)$, which is the only singular point):

$$\begin{array}{ll} R(a,1)\mapsto K\big(N(a),\bar{a},a,1\big) & \searrow \\ & \qquad K(a,1)\mapsto a. \\ R(a,1)\mapsto K\big(N(a),a,1\big) & \nearrow \end{array}$$

7. Chain geometries over commutative algebras

Benz [1973a] mainly investigates chain geometries over commutative algebras. We shortly mention three special aspects. Throughout this section let R be commutative.

7.1. *Using determinants*

We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) \Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in R^*.$$

For $a,b\in R^{(2)}$, $a=(a_1,a_2)$, $b=(b_1,b_2)$ we define

$$ab = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

which is in fact the exterior product. We see that Ra and Rb are distant points iff ab is a unit. Let also $c=(c_1,c_2)$, $d=(d_1,d_2)$ and let A,B,C,D be the points of $P(R)$ such that $A=Ra,\ \ldots,\ D=Rd$. If $B \not\parallel C$ and $A \not\parallel D$, the cross ratio may be defined by

$$(ABCD) = (ac)(bc)^{-1}(bd)(ad)^{-1}.$$

As in the case of a projective line over a commutative field (Chapter 2), one immediately sees that this expression is independent of the choice of representatives of the points and invariant under $\Gamma(R)$.

7.2. *Angles between chains*

Here the concept of angles is somewhat more general than the concept presented in Benz [1973a], p. 123 ff. An *angle* of $\Sigma(K,R)$ is given by an expression $(\boldsymbol{a}\,\boldsymbol{b}; P)$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are chains with $\boldsymbol{a} \cap \boldsymbol{b} \neq \varnothing$ and $P \in \boldsymbol{a} \cap \boldsymbol{b}$. By $\boldsymbol{a}P\boldsymbol{b}$ we will denote that $\boldsymbol{a}$ and $\boldsymbol{b}$ are in natural contact relation with respect to P.

We define an equivalence relation $\equiv$, *congruent*, on the set of angles. Let $(\boldsymbol{a}\,\boldsymbol{b}; P)$ and $(\boldsymbol{c}\,\boldsymbol{d}; Q)$ be angles. There are chains $\boldsymbol{b}_0$ and $\boldsymbol{d}_0$ with $\boldsymbol{b}P\boldsymbol{b}_0$ and $|\boldsymbol{a} \cap \boldsymbol{b}_0| \geqslant 2$ as well as $\boldsymbol{d}Q\boldsymbol{d}_0$ and $|\boldsymbol{c} \cap \boldsymbol{d}_0| \geqslant 2$. Now we define

$$(\boldsymbol{ab}; P) \equiv (\boldsymbol{cd}; Q) \Leftrightarrow \exists \gamma \in \Gamma(R)\colon\ \boldsymbol{c} = \boldsymbol{a}^{\gamma} \text{ and } \boldsymbol{d}_0 = \boldsymbol{b}_0^{\gamma} \text{ and } Q = P^{\gamma}.$$

This definition is independent of the choice of $\boldsymbol{b}_0$ and $\boldsymbol{d}_0$. The equivalence classes of $\equiv$ are called the *free angles* of $\Sigma(K,R)$. The free angle represented by $(\boldsymbol{ab}; P)$ is written as $[\boldsymbol{ab}; P]$. By 0 we denote the special free angle consisting of all angles $(\boldsymbol{ab}; P)$ with $\boldsymbol{a}P\boldsymbol{b}$.

On the set $\boldsymbol{W}$ of all free angles of $\Sigma(K,R)$ we define an *addition* $+$ in the following way. Given two elements of $\boldsymbol{W}$, we may represent these, in view of the transitivity properties of $\Gamma(R)$, by $(\boldsymbol{ab}; P)$ and $(\boldsymbol{bc}; P)$. Then

$$[\boldsymbol{ab}; P] + [\boldsymbol{bc}; P] := [\boldsymbol{ac}; P].$$

This addition is well defined and $(\boldsymbol{W}, +)$ is an Abelian group with neutral element 0.

Now consider the residual space of Σ in the form $(R, \boldsymbol{L})$ (compare 3.5.3) together with the isomorphism $\iota\colon (R,\boldsymbol{L}) \to \Sigma_U$. We define the mapping

$$w\colon\ \boldsymbol{L} \times \boldsymbol{L} \to R^*/K^*, \quad (Ka + a', Kb + b') \mapsto K^* a^{-1} b,$$

and see that w is preserved by the group

$$\iota\Gamma(R)_U\iota^{-1} = A(R) = \{x \mapsto ax + b\colon\ a \in R^*,\ b \in R\}.$$

(Here the commutativity of the multiplication in R is crucial.)

Therefore the mapping

$$\measuredangle\colon\ \boldsymbol{W} \to R^*/K^*, \quad \left[\left(l^{\iota} \cup \{U\}\right)\left(m^{\iota} \cup \{U\}\right); U\right] \mapsto w(l,m),$$

is well defined. (Any free angle can be represented by some $(\boldsymbol{ab}, U)$ with $U \in \boldsymbol{a} \cap \boldsymbol{b}$.)

For any angle $\alpha = (\boldsymbol{ab}; P)$ we call $\measuredangle\alpha = \measuredangle[\alpha]$ the *size* of α.

7.2.1. PROPOSITION. *The mapping $\measuredangle\colon W \to R^*/K^*$ is an isomorphism of groups. Two angles have the same size iff they are congruent.*

As a well-known example consider the representation of the Euclidean plane by the field of complex numbers $\mathbb{C}$. Here the group of sizes of angles is the unit circle $\{\cos\varphi + \mathrm{i}\sin\varphi\colon\ 0 \leqslant \varphi < 2\pi\}$, taken modulo $\{1, -1\}$, and this is isomorphic to $\mathbb{C}^*/\mathbb{R}^*$.

7.2.2. LEMMA. *Let $(\boldsymbol{ab}; P)$ be an angle with $\boldsymbol{a} \cap \boldsymbol{b} = \{P, Q\}$ for $P \neq Q$. For $A \in \boldsymbol{a} \setminus \boldsymbol{b}$ and $B \in \boldsymbol{b} \setminus \boldsymbol{a}$ we have*

$$\measuredangle(\boldsymbol{ab}; P) = (ABPQ)K^*.$$

For three mutually distant points X, Y, Z let (XYZ) be the unique chain containing X, Y and Z.

7.2.3. LEMMA. *Let A, B, P, Q, S be mutually distant points, let $\mathbf{c} = (ABP)$ and $S \notin \mathbf{c}$. Then Q is a point of $\mathbf{c}$ iff*

$$[(PAS)(PBS); P] = [(QAS)(QBS); Q].$$

(In a Miquelian Möbius plane, considering the affine trace with respect to S we obtain a well-known property for angles on a circle, compare Chapter 17, 3.2.6 ff.)

PROOF. For $A = Ra$ etc. we have by 7.2.2

$$\begin{aligned}(ap)(bp)^{-1}(bs)(as)^{-1}K^* &= (ABPS)K^* = (ABQS)K^* \\ &= (aq)(bq)^{-1}(bs)(as)^{-1}K^* \\ &\Leftrightarrow (ap)(bp)^{-1}(bq)(aq)^{-1} = (ABPQ) \in K^* \\ &\Leftrightarrow Q \in \boldsymbol{c} \quad \text{(by 2.3.4).}\end{aligned}$$

(In Havlicek [1983], this lemma is used to prove that affine chains are affine V_1^r.)

There are many other applications of 7.2.2 to obtain theorems about *angles* which generalize the case of classical geometries; see Benz [1983].

We conclude this section with the following

7.2.4. REMARK. From 7.2.3 we may derive a certain correspondence between affine chain geometries and partial affine spaces with angles, a weak form of metric space where the corresponding group of transformations is the group of 'similarities', i.e. equiforme mappings (cf. Chapter 17 and Schröder [1974]), and not only the group of motions.

For the residual space $(R, \boldsymbol{L})$, any 'angle preserving' affinity σ will be called a *similarity*, i.e. σ satisfies $w(l, m) = w(l^\sigma, m^\sigma)$ for all $l, m \in \boldsymbol{L}$.

Then we have the result that a similarity is even an automorphism of $A(K, R)$. Conversely, the group $A(R)$ (well known as a group of automorphisms of $A(K, R)$) is a group of similarities of $(R, \boldsymbol{L})$. From this it is more easily seen that $A(R)$ is a so-called *Süss group*, i.e. any $\sigma \in A(R)$ fixing a line of $\boldsymbol{L}$ and a point on it is a dilatation. In particular, if $\sigma \in A(R)$ has two distant fixed points, then $\sigma = 1$.

For angles in chain spaces $I(Q)$, for Q a quadric, compare Chapter 17, Section 4.4.9.

7.3. *Two* $(8_3, 6_4)$*-configurations*

These are tactical configurations with parameters $(v, r, b, k) = (8, 3, 6, 4)$ considered as substructures of a chain space. (Compare Chapter 6, Section 5, and Chapter 17.)

7.3.1. Let Σ be a chain space (contact space) where the contact relation for chains $\boldsymbol{c}, \boldsymbol{d}$ with respect to p is expressed by $\boldsymbol{c}p\boldsymbol{d}$. We call Σ *Miquelian* provided the following configurational theorem ('Miquel's condition') is fulfilled.

(M) Let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_4$ be chains, no three of which have a common point but $\boldsymbol{c}_i \cap \boldsymbol{c}_{i+1} = \{p_i, q_i\}$ for every i (subscripts taken modulo 4), where $p_i = q_i$ implies $\boldsymbol{c}_i p \boldsymbol{c}_{i+1}$. Moreover, let the p_i be pairwise distant and the q_i be pairwise distant and $p_i = q_i$ for at most one i. Then the four points p_i are cocatenal iff the four points q_i are cocatenal. (If p_i, q_i are 8 different points this gives a $(8_3, 6_4)$-configuration as mentioned above.)

We have the following famous result (see Chapter 6, Section 5.8).

7.3.2. A Benz plane Σ is Miquelian iff Σ is a chain geometry $\Sigma(K, R)$. (Clearly then $\dim_K R = 2$.)

We need the following rather obvious

7.3.3. Definition. Let $\Sigma = (P, \boldsymbol{C})$ be a chain space. We call $\Sigma' = (P', \boldsymbol{C}')$ with $P' \subseteq P$ and $\boldsymbol{C}' \subseteq \boldsymbol{C}$ a *subspace* of Σ provided Σ' is a chain space itself. Therefore the subset P' of P gives rise to a subspace of Σ (or, by abuse of language, 'is' a subspace) iff the following conditions are satisfied.

(i) For any three mutually distant points p, q, r of P' the unique chain through p, q, r is contained in P'.

(ii) Let $\boldsymbol{c}$ be a chain of Σ contained in P' and let p, q be distant points of P' with $p \in \boldsymbol{C}$. Then the unique chain through q naturally contacting $\boldsymbol{c}$ in p is contained in P'.

7.3.4. Lemma. *Let $\Sigma = \Sigma(K, R)$ be a chain geometry, where R is not necessarily commutative. For $i = 1, \ldots, 4$, let $\boldsymbol{c}_i, P_i, Q_i$ fulfil the assumptions of* (M). *If $P_1, \ldots, P_4$ are cocatenal but $P_1P_3Q_2Q_4$ are not cocatenal, then $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_4$ lie in a subspace Σ' of Σ isomorphic to $\Sigma(K, S)$ where S is a subalgebra of R with $\dim_K S = 2$, i.e. Σ' is a Benz plane.*

Proof. We may assume that $P_i = Q_i$ at most for $i = 2$. Using the transitivity properties of $\Gamma(R)$ we may choose P_1, P_2, P_3 freely. Choosing $P_1 = U = R(1, 0)$, the remaining points, with the possible exception of Q_3, belong to the residual space Σ_U and may be represented by elements of R, see 3.5.2. Here the following triples of points are collinear: $P_2Q_1Q_2, P_2P_3P_4, P_4Q_1Q_4$, and, for $P_2 = Q_2$, the line joining P_2 and Q_1 is tangent to the chain through $P_2P_3Q_3$. Choosing $P_3 = 0$, $P_2 = k \in K^*$ we obtain $P_4 = l \in K^*$. Since Q_3 is distant from P_3, we have $Q_3 = R(1, z)$ for suitable $z \in R$. Let $Q_1 = a$ for $a \in R \setminus K$.

1st case: $P_2 \neq Q_2$. We obtain

$$Q_2 = ax_1 + k(1 - x_1) = ax_1 + x_2,$$

$$Q_4 = ay_1 + l(1 - y_1) = ay_1 + y_2,$$

for $x_i, y_i \in K$, $x_1, y_1 \neq 0, 1$. Using a remark given after 3.6.1, we obtain

$$(1-s)Q_2^{-1} + sk^{-1} = z = (1-t)Q_4^{-1} + tl^{-1} \tag{1}$$

for suitable $s, t \in K$, $s, t \neq 0, 1$. Here necessarily $sk^{-1} \neq tl^{-1}$, since $sk^{-1} = tl^{-1}$ would imply $Q_2P_3Q_4$ collinear, whence $P_1P_3Q_2Q_4$ cocatenal.

2nd case: $P_2 = Q_2$. From 3.6.2 we obtain

$$k^{-1} + s(a-k) = y = tQ_4^{-1} + (1-t)l^{-1}, \quad s, t \in K^*. \tag{2}$$

From (1), (2), in each case we obtain an equation $ea^2 + fa + g = 0$ for $e, f, g \in K$, where necessarily $e \neq 0$.

Hence a is quadratic and the points P_i, Q_i are contained in a subspace isomorphic to $\Sigma(K, K[a])$. □

7.3.5. THEOREM. *Let R be a K-algebra, not necessarily commutative. Then $\Sigma(K, R)$ is Miquelian.* (For commutative K-algebras this theorem was proved in Benz [1973a], for skew fields in Ewald [1984].)

PROOF. Let P_i, Q_i fulfil the assumptions of (M) with $P_1, \ldots, P_4$ cocatenal. Then, if $P_1P_3Q_2Q_4$ are not cocatenal, (M) follows immediately from 7.3.2 and 7.3.4. Now assume $P_1P_3Q_2Q_4$ cocatenal, but $Q_1Q_2Q_3Q_4$ not cocatenal. Then P_i, Q_i necessarily are eight points. Using the permutation

$$\begin{pmatrix} P_1 & P_2 & P_3 & P_4 & Q_1 & Q_2 & Q_3 & Q_4 \\ P_4 & Q_3 & P_2 & Q_1 & Q_4 & P_3 & Q_2 & P_1 \end{pmatrix}$$

we are once more in the first situation. □

Now by Buekenhout [1966], 4.4, a set Q of points is a quadric iff every plane section of Q is a conic. From this together with 6.3.3 and 7.3.2 one may deduce the following proposition.

7.3.6. PROPOSITION. *Let Q be a nondegenerate quadratic set in a projective space of dimension at least 3. Then Q is a quadric iff $I(Q)$ is Miquelian.*

7.3.7. The second $(8_3, 6_4)$-configuration we want to present here is a part of the *bundle condition*. We shall formulate it in the following weak form.

(B) For $i = 1, \ldots, 4$, let p_i, q_i be eight pairwise distant points. If p_i, q_i, p_j, q_j are cocatenal for $1 \leqslant i < j \leqslant 4$ in 5 cases then they also are in the sixth.

7.3.8. LEMMA. *For any nondegenerate quadratic set Q the chain space $I(Q)$ fulfils the bundle condition.*

With respect to commutative algebras R we have the following results.

Call R *weakly quadratic* over K provided that any $a, b \in R^*$ such that a, b are quadratic over K and $ab \notin K$ necessarily imply that ab is quadratic over K.

7.3.9. THEOREM (Benz [1973c, 1988]). *If R is weakly quadratic over K, then in $\Sigma(K,R)$ the bundle condition is satisfied. Conversely, if $|K| > 11$ and in $\Sigma(K,R)$ the bundle condition holds, then R is weakly quadratic over K.*

Here, as in the Miquelian case, the assumptions of (B) need not be fulfilled in general. Namely, 3 chains $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ of $\Sigma(K,R)$ with

$$\big|(\boldsymbol{a} \cap \boldsymbol{b}) \cup (\boldsymbol{a} \cap \boldsymbol{c}) \cup (\boldsymbol{b} \cap \boldsymbol{c})\big| = 6$$

lie in a subspace Σ' of $\Sigma(K,R)$ isomorphic to $\Sigma(K,S)$ where S is a subalgebra of R with $\dim_K S = 2$, i.e. Σ' is a Benz plane. (This remark can be used to prove 6.3.4.)

Benz [1991] proves that for chain geometries over weakly quadratic algebras even stronger forms of the bundle condition are satisfied. Here in (B) one may admit $p_i = q_i$ provided the corresponding chains are contacting in p_i. Further, a geometric description of weakly quadratic algebras, the 'corner condition' (C), is given.

7.3.10. PROPOSITION. *For a commutative K-algebra R with $|K| > 3$ the following are equivalent:*

(a) *R is weakly quadratic;*

(b) *the corner condition* (C) *is satisfied in* $\Sigma(K,R)$:

(C) *Given any two distant points p, q, let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be chains with $a_i \cap a_j = \{p, q\}$. If two of the subspaces spanned by $\boldsymbol{a}_i, \boldsymbol{a}_j$ $(1 \leqslant i < j \leqslant 3)$ are Benz planes, then so is the third.*

7.3.11. REMARK. If R is a skew field, analogous statements are valid for $|K| > 5$ (Graf [1985]). The general case of noncommutative algebras seems not to have been treated yet.

8. Characterizations and direct products of chain geometries

We consider conditions for (weak) chain spaces to be chain geometries and related topics. Here we have to mention 7.3.2, which in the Möbius and Laguerre case is a rather old and famous result (Van der Waerden and Smid [1935]). And 7.3.2 is a good example of what is meant by 'characterization': if the configurational postulate (M) is valid, then we have a chain geometry, and conversely. For further characterizations of Miquelian Benz planes by their automorphism group, see Chapter 6.

8.1. *Möbius geometries*

We use the following

8.1.1. RESULT (Petkantschin [1940], Tits [1952]). *Let Γ be a sharply 3-transitive group of permutations of a set* M *with the following property.*

(I) *Any permutation of Γ interchanging two points is an involution.*

Then Γ is similar to the group of projectivities of a projective line over a commutative field.

The following two theorems give characterizations of Möbius geometries.

8.1.2. THEOREM. *Let $\Sigma = (P, C)$ be a circular space where the chains are the* 3*-sets of P. Then the following are equivalent*:

(a) *Σ possesses a group Γ of automorphisms acting sharply* 3*-transitively on P and having property* (I) *such that any involution has at most one fixed point*;

(b) *Σ is a Möbius geometry and Γ is its group of projectivities such that there is a commutative field L of characteristic* 2 *with $\Sigma \cong \Sigma(\mathrm{GF}(2), L)$ and $\Gamma \cong \Gamma(L)$* (*isomorphic as permutation groups*).

8.1.3. THEOREM. *Let $\Sigma = (P, C)$ be a circular space where some chain contains more than three points. Then the following is equivalent*:

(a) *Σ possesses a group Γ of automorphisms acting sharply* 3*-transitively on P and having property* (I).

(b) *Σ is a Möbius geometry and Γ its group of projectivities such that there is a commutative field L and a subfield K of L with $\Sigma \cong \Sigma(K, L)$ and $\Gamma \cong \Gamma(L)$* (*isomorphic as permutation groups*).

For a direct proof of 8.1.2 and 3, see Benz [1973a].

If Σ is a finite Möbius geometry, say $\Sigma = \Sigma(\mathrm{GF}(q), \mathrm{GF}(q^n))$, then Σ is a 3-$(q^n + 1, q + 1, 1)$-design (see Chapter 8).

For a more general definition of Möbius geometry, see Benz [1960]. In Benz [1973a], IV, the hypothesis $K \subseteq C(R)$ is omitted (as in the general definition of chain geometry in Bartolone [1989]).

The following theorem in Buekenhout [1971b] does not give a characterization of Möbius geometries in general but is closely connected to this problem.

Let Σ be a Möbius space. An *inversion* σ is an automorphism of Σ such that for any points x, y the points x, x^σ, y, y^σ are cocatenal. We call Σ *strongly Miquelian* if for any four cocatenal points a, a', b, b' there exists an inversion σ of Σ with $a^\sigma = a'$ and $b^\sigma = b'$.

8.1.4. THEOREM. *Let Σ be a Möbius space where Σ_p has finite dimension for some point p of Σ. Then Σ is strongly Miquelian iff Σ is isomorphic to $I(Q)$ for some ovoidal quadric Q.*

In a circular space Σ, any subspace of Σ generated by four noncocatenal points will be called a *circular plane*. Following Buekenhout [1971a] Σ is an *inversive space* provided every circular plane of Σ is an inversive plane. This condition is equivalent to the following

Σ is locally affine (i.e. a Möbius space); and if V_p is any affine subspace of $\Sigma_{p'}$ then $V_p \cup \{p\}$ is a subspace of Σ.

Any inversive space of dimension $\geqslant 3$ is isomorphic to $I(Q)$ for some ovoid Q. This follows from Mäurer [1968] (there a more general situation is considered, see also Heise [1970]). Further details are given in Chapter 6.

8.2. *Strong chain spaces*

The following results are formulated and proved in Herzer [1990, 1992b].

Let Γ be the full automorphism group of a chain space $\Sigma = (P, C)$. For points x, y the set $D_{x,y} = D_x \cap D_y$ is the set of all points distant from x and y, and $\Gamma_{x,y}$ is the stabilizer of x and y in Γ etc.

8.2.1. LEMMA. *For a K-algebra R let $\Sigma = \Sigma(K, R)$. There are three pairwise distant points o, e, w such that* (1), (2), (3) *hold.*

(1) *There exists a subgroup A of $\Gamma_{o,w}$ acting regularly on $D_{o,w}$.*

(2) *Let k be the unique chain containing o, e, w. There is $\iota \in \Gamma$ interchanging o and w and fixing e such that $(\iota\alpha)^2$ fixes k pointwise for every $\alpha \in A$.*

(3) *There exists $\sigma \in \Gamma_w$ with $o^\sigma = e$.*

PROOF. Choose $o = R(0, 1)$, $e = R(1, 1)$, $w = R(1, 0)$ and let A be the group of projectivities induced by all matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \in R^*.$$

Moreover, let ι be induced by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and σ by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then all assertions (1)–(3) are fulfilled. □

We will use the properties (1)–(3) for a characterization of a class of chain geometries. For this purpose we need the following

8.2.2. DEFINITION. The chain space $\Sigma = (P, C)$ is called *strong* iff the following conditions are fulfilled:

(a) Given any two points $x, y \in P$ there is a point $z \in P$ which is distant from x and y.

(b) For any chain C and point p let $C_p = C \cap D_p$ be the set of points of C distant from p. If $C_p \neq \varnothing$ then $|C \setminus C_p| + 1 < |C_p|$.

For finite C condition (b) simply means

$$|C_p| > \frac{1}{2}(|C| + 1)$$

and for infinite C condition (b) states that there is no injective mapping $C_p \to C \setminus C_p$. Using $p \in C$ one obtains $|C| \geqslant 4$.

One can show that if R is a K-algebra of finite dimension n and $|K| > 2n$ then $\Sigma(K, R)$ is a strong chain space.

8.2.3. THEOREM. *Let Σ be a strong chain space and Γ its group of automorphisms. Let o, e, w be three pairwise distant points of Σ such that the conditions* (1)–(3) *of* 8.2.1 *are fulfilled and that for Σ_w there is an underlying affine space of dimension $n \geqslant 3$. Then Σ is isomorphic to a chain geometry $\Sigma(K, R)$ where R is of dimension n.*[12]

To also give a characterization of strong chain spaces over commutative algebras the property (I) for an automorphism group Π of Σ now becomes

(I) Any permutation of Π interchanging two distant points is an involution.

We shall also formulate the property (T) 'sharply 3-$\|$-transitive':

(T) For any mutually distant points A, B, C and any mutually distant points A', B', C', there is exactly one $\gamma \in \Gamma$ with $A' = A^\gamma$, $B' = B^\gamma$ and $C' = C^\gamma$. ('Γ operates regularly on the set of triples of mutually distant points'.)

8.2.4. COROLLARY. *Let Σ be a strong chain space such that for any residual space of Σ there is an underlying affine space of dimension $n \geqslant 3$. If Σ has a group Π of automorphisms with properties* (I) *and* (T), *then Σ is isomorphic to $\Sigma(K, R)$ for a commutative K-algebra R, and Π is isomorphic to $\Gamma(R)$ as permutation group.*

Bartolone and Spera [1987] give a characterization of $\Gamma(R)$ as permutation group of $P(R)$ for a local ring R which in a certain extent is similar to that of 8.2.3. For a commutative local ring a characterization already was given by Melchior [1968]. He also gives a characterization of Laguerre geometries which is similar to 8.2.4 but needs no restrictions on the dimension of the residual space. Here a *Laguerre space* Σ is a nontrivial chain space where the parallel relation $\|$ is an equivalence relation on the point-set and every chain meets every parallel class of points.

Following Tits the automorphism group Γ is said to be of the *first kind* provided any involution of Γ with one fixed point still has another fixed point distant from the first one.

8.2.5. THEOREM. *Let Σ be a Laguerre space having a chain which contains more than three points. Then the following are equivalent:*

(a) *Σ has a group Π of automorphisms of the first kind with* (I) *and* (T);

(b) *Σ is a Laguerre geometry and Π its group of projectivities such that there is a commutative K-algebra R where* $\operatorname{char} K \neq 2$ *and R is a Laguerre algebra with $\Sigma \cong \Sigma(K, R)$ and $\Pi \cong \Pi(R)$* (*isomorphic as permutation groups*).

For $R = K \oplus N$ with $N^2 = \{0\}$, Mäurer [1965] characterizes the Laguerre geometry $\Sigma(K, R)$ using a group of automorphisms generated by 'reflections'. Mäurer [1967] gives a more general concept of Laguerre geometry, special examples of which are the chain spaces $I(Q)$, where Q is a cone over an ovoid (compare also Benz and Mäurer [1964] and Chapter 6).

[12] Here n is not necessarily finite, see Herzer [1992b] and the special result in Herzer [1991].

8.3. *Minkowski geometries*

We essentially follow Samaga [1979] (compare also Benz [1971, 1973a] and Halder [1979]). $M = (P, \mathcal{C}, (\rho_j)_{j\in J})$ will be called a *weak Minkowski chain space* (WMC *space*) provided $(P, \mathcal{C})$ is a nontrivial weak chain space and $(\rho_j)_{j\in J}$ is a family of equivalence relations on P such that the parallel relation on P is the union of the relations ρ_j, $j \in J$, i.e. $\forall p, q \in P$: $p \parallel q$ if and only if $\exists j \in J$: $p\rho_j q$, and the following conditions are satisfied:

(i) For each family $(a_j)_{j\in J}$ of points a_j there is exactly one point p such that $p\rho_j a_j$ for all $j \in J$.

(ii) For any chain c, any point p and any $j \in J$ there is (exactly one) $a_j \in c$ with $a_j \rho_j p$.

8.3.1. REMARKS. (a) Let $\boldsymbol{H}_j$ be the set of equivalence classes of ρ_j. Then (i), (ii) may be expressed in terms of the $\boldsymbol{H}_j$ as follows:

(i) $\forall j \in J \quad H_j \in \boldsymbol{H}_j \Rightarrow \left|\bigcap_{j\in J} H_j\right| = 1.$

(ii) $\forall c \in \mathcal{C}\ \forall H \in \bigcup_{j\in J} \boldsymbol{H}_j \quad |c \cap H| = 1.$

(b) Let C be a fixed chain and for $p \in C$ let $H_j(p)$ be the equivalence class of p with respect to ρ_j. Then the map $C \to \boldsymbol{H}_j$, $p \mapsto H_j(p)$, is bijective and we may identify C^J with P using the bijection

$$C^J \to P, \quad (p_j)_{j\in J} \mapsto x \text{ with } \bigcap_{j\in J} H_j(p_j) = \{x\}.$$

By abuse of notation on C^J the relation ρ_k is defined by

$$(x_j)_{j\in J}\rho_k(y_j)_{j\in J} \Leftrightarrow x_k = y_k. \qquad (*)$$

8.3.2. EXAMPLES. (1) Let J be an index set and $k \in J$ and $J' = J \setminus \{k\} \neq \varnothing$. For a given set C and $j \in J'$ let Γ_j be a sharply 3-transitive set of permutations of C. Let

$$M = \left(C^J, \prod_{j\in J'} \Gamma_j, \mathrm{I}\right)$$

be an incidence structure, where incidence is defined by

$$(x_j)_{j\in J}\,\mathrm{I}\,(\gamma_j)_{j\in J'} \Leftrightarrow \forall j \in J'\colon x_j = x_k^{\gamma_j}.$$

Then, together with the equivalence relations $(\rho_j)_{j\in J}$ defined by (*), M becomes a WMC space.

(2) Let C be a set and Γ be a sharply 3-transitive group of permutations on C. For an index set J define an equivalence relation $\sim$ on Γ^J by

$$(\alpha_j)_{j\in J} \sim (\beta_j)_{j\in J} \Leftrightarrow \exists \gamma \in \Gamma\ \forall j \in J\colon\ \beta_j = \gamma\alpha_j.$$

Let $\Gamma|\Gamma^J$ be the set of equivalence classes on Γ^J, where the class of (α_j) is denoted by $[\alpha_j]$.

Now $(C,\Gamma)^J = (C^J, \Gamma|\Gamma^J, \mathrm{I})$ is an incidence structure where the incidence I is defined by:

$$(x_j)\mathrm{I}[\alpha_j] \Leftrightarrow \text{there exists } x \in C \text{ such that } x_j = x^{\alpha_j} \text{ for all } j \in J.$$

Together with the family (ρ_j) of equivalence relations on C^J defined by (*), $(C,\Gamma)^J$ becomes a WMC space and will be called *Minkowski product*. Let Γ^J act on C^J by

$$(\alpha_j)\colon\ C^J \to C^J, \quad (x_j) \mapsto (x_j^{\alpha_j}).$$

Then Γ^J is a group of automorphisms of $(C,\Gamma)^J$ which is sharply transitive on the set of triples of mutually distant points and preserves ρ_j. (We have $[\alpha_j]^{(\beta_j)} = [\alpha_j\beta_j]$.)

8.3.3. DEFINITION. Let $M = (P, \mathcal{C}, (\rho_j)_{j\in J})$ be a WMC space. For $i,j \in J$, $i \neq j$, and $x_k \in P$ for $k = 1,\ldots,4$, call $x_1x_2x_3x_4$ an *(i,j)-rectangle* provided $x_1\rho_ix_2\rho_jx_3\rho_ix_4\rho_jx_1$ hold. We consider the following configurational postulates (G_{ij}) and (S_{ij}).

(G_{ij}) Let

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4' \end{matrix}$$

be (i,j)-rectangles where a_k, b_k, c_k, d_k, are cocatenal for each $k = 1,2,3$. Then there exists $d_4 \in P$ such that $d_1d_2d_3d_4$ is an (ij)-rectangle and a_4, b_4, c_4, d_4 are cocatenal.

(S_{ij}) Let $a_1a_2a_3a_4$ and $b_1b_2b_3b_4'$ be (i,j)-rectangles where a_1, b_1, a_3, b_3 are cocatenal. Then there exists $b_4 \in P$ such that $b_1b_2b_3b_4$ is an (i,j)-rectangle and $a_2b_2a_4b_4$ are cocatenal.

We say that (G) holds for M provided (G_{ij}) is valid for all $i,j \in J$, $i \neq j$. M is called *symmetric* provided (S_{ij}) holds in M for all $i,j \in J$, $i \neq j$.

The proofs of the following two theorems are analogous to proofs in Samaga [1979]. Direct proofs can be found in Neutzler [1989].

8.3.4. THEOREM. *The* WMC *space M is a Minkowski product iff* (G) *holds for M.*

8.3.5. THEOREM. *For a* WMC *space M the following statements are equivalent*:

(a) *M is a Minkowski geometry (i.e. the underlying incidence structure is isomorphic to $\Sigma(K, K^J)$ for a suitable field K and index set J)*;

(b) *M is isomorphic to the Minkowski product $(P_1(K), \mathrm{PGL}_2(K))^J$ where J, K are as in* (a);

(c) *M is a symmetric Minkowski product.*

8.3.6. REMARKS.

(a) The definition of Minkowski geometry in 8.3.5 extends the definition at the end of Section 2.5.

(b) A 'rectangle axiom' analogous to (G_{12}) is used also in Karzel [1968] for four *triples* of points in an affine plane (instead of quadruples here). It has something to do with associativity.

(c) The 'symmetry axiom' (S_{12}) expresses the property (I) for the sharply 3-transitive permutation group Γ.

(d) In Samaga [1979], a WMC space $M = \left(P, \mathcal{C}, \left(\rho_j\right)_{j=1,\ldots,n}\right)$ is called (B_n^*)-*geometry*. The 'weak Minkowski planes' (i.e. $n = 2$) are also called (B^*)-*geometries* (also known as Hyperbelstrukturen). They have some extraordinary properties, compare Chapter 6, Section 5.19. We should mention the following

8.3.7. RESULT (Artzy [1973a,b]). *Let M be a symmetric weak Minkowski plane. Then M is a Miquelian Minkowski plane (i.e. $M \cong \Sigma(K, K^{(2)})$ for a suitable field K).*

For the proof it is crucial that any symmetric weak Minkowski plane is a Minkowski plane. This part of the theorem was independently proved in Heise and Karzel [1973].

The concept of 'pseudocircles' in (weak) Minkowski planes allows interesting connections between (M), (B) and the symmetry postulate, compare Schröder [1974], Herzer [1979b] and Benz et al. [1981].

8.4. *Direct products of chain geometries*

The concept of Minkowski product introduced above can be used in a wider context.

8.4.1. Let C, P be sets and Γ a group of permutations on C and Δ a set of embeddings δ: $C \to P$. We call $\boldsymbol{X} = (C|P, \Gamma|\Delta)$ a *curve space* (over (C, Γ)) provided the following condition holds:

$$\Gamma\Delta = \Delta, \quad \text{i.e. for any } \gamma \in \Gamma \text{ and } \delta \in \Delta, \text{ also } \gamma\delta \in \Delta.$$

Then the elements of P are called *points* and the elements of Δ are called *curves*. (For more details and a better motivation of this concept, see Herzer [1987a].)

Using the equivalence relation on Δ given by Γ we write $[\delta]$ for the equivalence class of δ, i.e. for any $\alpha, \beta \in \Delta$ we have $[\alpha] = [\beta]$ provided $\beta = \gamma\alpha$ for some $\gamma \in \Gamma$. Let $\Gamma|\Delta$ be the set of equivalence classes of Δ. We define the following incidence structure associated to $\boldsymbol{X}$:

$$\mathrm{I}\boldsymbol{X} := \left(P, \Gamma|\Delta, \mathrm{I}\right)$$

with $y\,\mathrm{I}\,[\delta]$ if and only if $y \in \operatorname{im}\delta$ for any $y \in P$ and $\delta \in \Delta$. Hence the point set belonging to $[\delta]$ is just C^δ.

For $i = 1, 2$ let $\boldsymbol{X}_i = (C|P_i, \Gamma|\Delta_i)$ be curve spaces over (C, Γ). A *morphism* μ: $\boldsymbol{X}_1 \to \boldsymbol{X}_2$ is a pair (ζ, η), where ζ is a mapping from P_1 to P_2 and η is a mapping from Δ_1 to Δ_2 such that

$$\delta^\eta = \delta\zeta \quad \text{for any } \delta \in \Delta_1$$

(therefore $(\gamma\delta)^\eta = \gamma\delta^\eta$ for any $\gamma \in \Gamma$).

Let J be an index set and $\boldsymbol{X}_j = (C|P_j, \Gamma|\Delta_j)$ be curve spaces over (C, Γ) for $j \in J$. Then

$$\prod_{j\in J} \Delta_j$$

is a set of mappings

$$C \to \prod_{j\in J} P_j, \quad \text{i.e. for } (\delta_j) \in \prod_{j\in J} \Delta_j$$

defining

$$(\delta_j)\colon C \to \prod_{j\in J} P_j, \quad x \mapsto (x^{\delta_j}).$$

Obviously,

$$\prod_{j\in J} \boldsymbol{X}_j := \Bigl(C|\prod_{j\in J} P_j, \Gamma|\prod_{j\in J} \Delta_j\Bigr)$$

is a curve space over (C, Γ), which is in fact the direct product of the $\boldsymbol{X}_j$ in the category of curve spaces over (C, Γ).

8.4.2. Consider next the chain geometry $\Sigma(K, R)$ together with the projective line $P(K)$ with its group $\Gamma(K)$ of projectivities. Moreover let $\Delta(R)$ be the set of all mappings

$$\delta\colon P(K) \to P(R),\ K(s,t) \mapsto R(as + ct, bs + dt),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R).$$

Then

$$\boldsymbol{X}(K,R) := \bigl(P(K)|P(R), \Gamma(K)|\Delta(R)\bigr)$$

is a curve space such that $\mathrm{I}\boldsymbol{X}(K,R) = \Sigma(K,R)$.

Now it is easy to define direct products of chain geometries by means of curve spaces. Let $\Sigma(K, R_j)_{j\in J}$ be a family of chain geometries. Then

$$\prod_{j\in J} \Sigma(K, R_j) := \mathrm{I}\prod_{j\in J} \boldsymbol{X}(K, R_j).$$

We have the following

8.4.3. PROPOSITION.

$$\Sigma\Big(K, \prod_{j\in J} R_j\Big) = \prod_{j\in J} \Sigma(K, R_j)\,.$$

For example, if $\Sigma_j = \Sigma(K, K)$ (i.e. the trivial chain geometry over K) for every $j \in J$, we have the representation of a Minkowski geometry

$$\prod_{j\in J} \Sigma_j = \Sigma(K, R^J) = \mathrm{I}\big(P(K)|P(K), \Gamma(K)|\Gamma(K)\big)^J.$$

Hence the Minkowski product coincides with the direct product of projective lines over K.

In the same way (by suitable definitions of curve spaces), we obtain direct products of chain geometries $A(K, R_j)$ or $\Sigma_K(U_j, R_j)$ or $G_K(U_j, R_j)$ ($j \in J$), respectively (compare Herzer [1978, 1987a]). As a further application consider the following

8.4.4. THEOREM. *Suppose that the chain geometry $\Sigma(K, R_j)$ possesses a rational representation of degree r_j and dimension $d_j - 1$ for $j = 1, \ldots, s$. Let $R = R_1 \times \cdots \times R_s$. Then $\Sigma(K, R)$ possesses a rational representation of degree r and dimension $d - 1$, where $r = r_1 + \cdots + r_s$ and $d = d_1 \ldots d_s$, provided $|K| \geqslant r + 2$.*

PROOF. Let $(V_j, \widehat{C}_j)$ be such a rational representation of $\Sigma(K, R_j)$. Then the incidence structure belonging to the direct product of the corresponding curve spaces is a rational representation of $\Sigma(K, R)$ as desired. Its point set is $V_1 \times \cdots \times V_s$ and this can be considered as a Segre product of the V_j and so is itself a quasiprojective variety. From this the other assertions now follow. □

8.4.5. REMARKS.

(a) If in 8.4.4 the rational representations of $\Sigma(K, R_j)$ are smooth for $1 \leqslant j \leqslant s$, then certainly the rational representation of $\Sigma(K, R)$ constructed in the proof is smooth, too.

(b) The examples of embeddings of chain geometries given in Benz et al. [1981] are of this type.

(c) For $j = 1, \ldots, s$ let $R_j = K[a_j]$ and

$$R = \prod_{j=1}^{s} R_j$$

with $\dim_K(R) = n$. Then $G_K(R, R)$ is a (smooth) rational representation of $\Sigma(K, R)$ of dimension $2^n - 1$ and degree n.

PROOF. 'Degree n' is easily verified. For $\dim_K R_j = n_j$, by 5.3.3, the rational representation $G_K(R_j, R_j)$ of $\Sigma(K, R_j)$ has dimension $2^{n_j} - 1$. Clearly $n = n_1 + \cdots + n_s$. After having verified that the underlying point set of $G_K(R, R)$ is the Segre product of the point sets V_j of $G_K(R_j, R_j)$ one just has to apply 8.4.4 with its proof to get the desired assertion. □

9. Isomorphisms of chain geometries

9.1. *Jordan homomorphisms*

Let R, S be any rings (associative and with $1 \neq 0$). A *Jordan homomorphism* σ from R to S is an additive mapping σ: $R \to S$ with

(i) $(uvu)^\sigma = u^\sigma v^\sigma u^\sigma$ for all $u, v \in R$.

(ii) $1^\sigma = 1$.

Clearly, ring homomorphisms and antihomomorphisms are Jordan homomorphisms. Fundamental properties of a Jordan homomorphism σ are the following:

$$\sigma \text{ maps } R^* \text{ into } S^*, \qquad \forall a \in R^*\colon \left(a^{-1}\right)^\sigma = \left(a^\sigma\right)^{-1}.$$

If R and S are commutative and $1 + 1 \in R^*$ then σ is a ring homomorphism. If S is an integral domain, then σ is a ring homomorphism or antihomomorphism. (For more details, compare Herstein [1956] and Bartolone and Bartolozzi [1985]. Herzer [1987c] presents examples of Jordan isomorphisms which are neither ring isomorphisms nor anti-isomorphisms, even for commutative K-algebras where K is of characteristic 2.)

Now let R, S be K-algebras. The mapping σ: $R \to S$ is called *K-Jordan homomorphism* provided σ is a Jordan homomorphism which is K-semilinear. A K-Jordan isomorphism or automorphism, respectively, is defined as a K-Jordan homomorphism with the usual properties.

Clearly, a K-Jordan isomorphism $R \to S$ induces an isomorphism $A(K,R) \to A(K,S)$, since by 3.6.1 any affine chain is of the form

$$\{b\} \cup \left\{b + (a - dt)^{-1}\colon\ t \in K,\ (a - dt) \in R^*\right\}$$

for $a, b \in R$ and $d \in R^*$.

Meanwhile it has been proved that this mapping can be extended to an isomorphism $\varkappa$ of $\Sigma(K,R)$ onto $\Sigma(K,S)$ if R is of stable rank 2. (Recall that in this case any point $R(x,y)$ can be written in the form $R(a, 1 + ab)$ since (x,y) is unimodular). Namely we have the following

9.1.1. THEOREM (Bartolone [1989]). *Let R, S be K-algebras, R of stable rank* 2, *and σ: $R \to S$ be a K-Jordan homomorphism. Then the mapping*

$$\varkappa\colon\ P(R) \to P(S), \quad R(a, 1 + ab) \mapsto S\left(a^\sigma, 1 + a^\sigma b^\sigma\right),$$

is well defined and is a morphism from $\Sigma(K,R)$ to $\Sigma(K,S)$.

Here 'morphism' simply means that $\varkappa$ maps chains into chains.

9.2. *The structure theorem for chain geometries*[13]

An isomorphism $\varkappa$: $\Sigma(K,R) \to \Sigma(K,S)$ is called *fundamental* provided $R(1,0)^\varkappa = S(1,0)$, $R(1,1)^\varkappa = S(1,1)$ and $R(0,1)^\varkappa = S(0,1)$ hold. We see that $\varkappa$ in 9.1.1 is a fundamental isomorphism, provided $\varkappa$ is bijective.

[13] A more general theorem has been given in the meantime by Blunck [1994a]; see also the special case Blunck [1992].

Let R be a ring of stable rank 2 and $\varkappa$: $\Sigma(K,R) \to \Sigma(K,S)$ a fundamental isomorphism. We say that $\varkappa$ is *induced* by a K-Jordan isomorphism σ: $R \to S$ provided it is defined as in 9.1.1.

In Herzer [1987c] the following theorem was proved for finite dimensional K-algebras, but the proof can easily be extended to semiprimary K-algebras:

9.2.1. THEOREM. *Let R, S be K-algebras where R is semiprimary and $|K| > 3$ or R is a local ring and $|K| \geqslant 3$. Then any fundamental isomorphism $\Sigma(K,R) \to \Sigma(K,S)$ is induced by a K-Jordan isomorphism.*

Without use of 9.1.1 this theorem may be reformulated in the following way. If $\varkappa$: $\Sigma(K,R) \to \Sigma(K,S)$ is a fundamental isomorphism then σ: $R \to S$ is a K-Jordan isomorphism, where σ is defined by $R(x^\sigma, 1) = (R(x,1))^\varkappa$ for any $x \in R$. From 9.2.1 we obtain the following corollaries.

9.2.2. PROPOSITION. *Under the hypotheses of* 9.2.1 *there is a one-to-one correspondence between the set of all K-Jordan isomorphisms $R \to S$ and the set of all fundamental isomorphisms $\Sigma(K,R) \to \Sigma(K,S)$.*

Now consider the group $\operatorname{Aut}\Sigma(K,R)$ of all automorphisms of $\Sigma(K,R)$. Then the stabilizer of $R(1,0)$, $R(1,1)$, $R(0,1)$ in $\operatorname{Aut}\Sigma(K,R)$ is the group $F(K,R)$ of all fundamental automorphisms of $\Sigma(K,R)$ and we have

$$\operatorname{Aut}\Sigma(K,R) = F(K,R)\Gamma(R).$$

It should be observed that $\Gamma(R)$ need not be normal in $\operatorname{Aut}\Sigma(K,R)$.

9.2.3. PROPOSITION. *Under the hypotheses of* 9.2.1 *with $S = R$ the group $J(K,R)$ of all K-Jordan automorphisms of R is isomorphic to the group $F(K,R)$.*

It should be mentioned that this theorem also holds in case of a chain geometry over an alternative field, see Schaeffer [1981].

9.2.4. PROPOSITION. *Let $\Sigma(K,R)$ be a Möbius geometry (i.e. R a skew field with K in the centre of R). Representing $P(R)$ in the form $R \cup \{\infty\}$, any automorphism $\varkappa$ of $\Sigma(K,R)$ can be described in the following way. There is an automorphism or anti-automorphism α of R with $K^\alpha = K$ and there exists*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

such that $\forall x \in R$:

$$x^\varkappa = \infty \quad \textit{if } x^\alpha b + d = 0,$$

$$x^\varkappa = \left(x^\alpha b + d\right)^{-1}\left(x^\alpha a + c\right) \quad \textit{otherwise},$$

$$\infty^\varkappa = \infty \quad \textit{if } b = 0,$$

$$\infty^\varkappa = b^{-1}a \quad \textit{otherwise}.$$

For a direct proof, see Benz [1969], Mäurer, Metz and Nolte [1980].

Finally, let R be commutative and $\operatorname{char} K \neq 2$. (Note that the commutative semiprimary rings are just the *semilocal rings*, i.e. rings whith a finite number of maximal ideals.) We define $\mathrm{P\Gamma L}_K(2,R)$ as the group of all permutations of $P(R)$ induced by invertible R-semilinear mappings of R^2, where the companion ring automorphism α of R leaves K invariant. They can be described as the mappings

$$R(x,y) \to R(x^\alpha, y^\alpha)\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

and α is a ring automorphism of R with $K^\alpha = K$.

9.2.5. THEOREM. *Let the commutative K-algebra R be semilocal with $|K| > 3$ (or R local with $|K| \geqslant 3$) and* $\operatorname{char} K \neq 2$. *Then*

$$\operatorname{Aut} \Sigma(K,R) = \mathrm{P\Gamma L}_K(2,R).$$

Here we obtain other results from Limaye and Limaye [1977] and Mc Donald [1981].

9.2.6. PROPOSITION. *Let R be a commutative K-algebra with* $\operatorname{char} K \neq 2$. *We have* $\operatorname{Aut} \Sigma(K,R) = \mathrm{P\Gamma L}_K(2,R)$ *if either of the following conditions* (a), (b) *is fulfilled.*

(a) *If $P_0,\dots,P_5$ are any points of $P(R)$, there exists a point of $P(R)$ with $Q \not\parallel P_i$ for $i = 0,\dots,5$.*

(b) *R is primitive in the following sense: for every polynomial $f(x) \in R[x]$ whose entries generate R as an ideal (i.e. a primitive polynomial), there exists $a \in R$ such that $f(a)$ is a unit.*

To prove 9.2.6 one uses the *theorem of Von Staudt–Hua* (Fundamental theorem, see Chapter 2) in the following form.

If R is a commutative ring with $1+1 \in R^*$ fulfilling one of the conditions (a), (b) *and if γ is a permutation of $P(R)$ preserving harmonic quadruples (i.e. quadruples of points with cross ratio -1), then $\gamma \in \mathrm{P\Gamma L}(2,R)$, i.e. γ is induced by an R-semilinear mapping on R^2.*

For further details and extensions to wider classes of rings, see Bartolone and Bartolozzi [1985], and Benz et al. [1981].

10. n-chain geometries

In this section we shall introduce the concept of n-chain geometry. Here chains will be replaced by n-chains. These are projective geometries of dimension n over a field K, canonically embedded into a projective geometry of dimension n over a K-algebra R. The 1-chain geometries, in particular, coincide with the chain geometries of previous chapters.

10.1. *Short historical review*

The classical examples of n-chain geometries occur for the pair of fields $(\mathbb{R}, \mathbb{C})$ and deal with the problem of real embeddings or representations in complex geometries. The concept of chains for $n = 1$ and $n = 2$ already appears in Von Staudt [1856] who introduced it in a synthetic way. These considerations were continued by Juel [1885, 1890/91] and Segre [1890, 1891, 1892]. Here begins the description of 'anticollineations' and 'anticorrelations' of the complex projective n-space by means of n-chains. For example, for any 'anti-involution' of the complex plane the fixed structure is exactly a 2-chain, namely a real subplane. In modern terms this phenomenon is better known under the names *Baer involution* and *Baer subplane* of a projective plane, mainly in finite geometries. This concept was extensively developed in Burau [1961], Chapter III, compare also Juel [1934] (synthetic point of view) and Veblen and Young [1918], Chapter VI. Further considerations can be found in Young [1910] for $n = 2$ and Mac Gregor [1912] for $n = 3$.

10.2. *The general concept*

First, let R be a K-algebra as in the beginning of Section 2, so K is a subring of the centre of R with the same identity element. Let M be the canonical free left R-module of rank $n+1$. Then $P^n(R)$ consists of all cyclic submodules X such that there is a basis $v_0, v_1, \ldots, v_n$ of M with $X = Rv_0$. The elements of $P^n(R)$ are called *points*.

The points $X_0, \ldots, X_n$ form a *basis* of $P^n(R)$ provided there exists a basis $v_0, \ldots, v_n$ of M with $X_i = Rv_i$ for $i = 0, \ldots, n$. For $m \leqslant n$ the points $Y_0, \ldots, Y_m$ form a *standard set* provided there are points $Y_{m+1}, \ldots, Y_n$ such that $Y_0, \ldots, Y_n$ is a basis of $P^n(R)$. The points $Z_0, \ldots, Z_{m+1}$ form an *m-frame* provided any $m+1$ of these points form a standard set and its span contains the missing one.

For any basis $v_0, \ldots, v_n$ of M and $m \leqslant n$ we have a canonical embedding

$$P^m(K) \to P^n(R), \quad K(t_0, \ldots, t_m) \mapsto R\sum_{i=0}^{m} t_i v_i.$$

An *m-chain* is the image of $P^m(K)$ under such an embedding. Here m can be at most n. Any n-chain has the structure of the projective space $P^n(K)$ where the subspaces of dimension m are just the m-chains contained in it.

10.2.1. LEMMA. *Any given m-frame is contained in exactly one m-chain.*

We define $\Sigma^n(K, R) := (P^n(R), C)$ with C the set of all n-chains of $P^n(R)$ and call $\Sigma^n(K, R)$ the *n-chain geometry over R.*

The group $\Gamma_n(R) := \mathrm{PGL}(n+1, R)$ acts on $P^n(R)$ in a canonical way. ($\Gamma_n(R)$ is induced by the group $\mathrm{GL}(n+1, R)$ of invertible R-linear mappings on M.)

10.2.2. PROPOSITION. *$\Gamma_n(R)$ is a flag-transitive group of automorphisms of $\Sigma^n(K, R)$. Moreover for $1 \leqslant m \leqslant n$ the group $\Gamma_n(R)$ is transitive on the set of m-frames, hence transitive on the set of m-chains.*

A new problem for $n > 1$ is the classification of intersections of n-chains. (For $(K, R) = (\mathbb{R}, \mathbb{C})$ and a fixed basis of M this is the classical question of *types of reality* for n-chains). One may look for the maximal number of points of a standard set in the intersection and also for maximal m such that an m-frame (and hence an m-chain) is contained in the intersection, see Gibert [1983], Juel [1934], Burau [1961].

From now let K be a field. We consider R^n as an affine space over K and in M we use coordinates with respect to a fixed basis $e_0, \ldots, e_n$. We have the embedding

$$\iota\colon\ R^n \to P^n(R), \quad (x_1, \ldots, x_n) \mapsto R(1, x_1, \ldots, x_n).$$

We define $A^n(K, R) = (R^n, C)$, where C consists of all *affine n-chains* defined as the traces of the n-chains via ι. An affine n-chain is of the form

$$\Big\{ \big(x_1(t_0, \ldots, t_n), \ldots, x_n(t_0, \ldots, t_n)\big)\colon$$
$$K(t_0, \ldots, t_n) \in P^n(K), \text{ such that } \Sigma a_{j0} t_j \in R^* \Big\},$$

where

$$x_i(t_0, \ldots, t_n) := \Big(\sum_j a_{j0} t_j \Big)^{-1} \sum_k a_{ki} t_k$$

for a fixed $(a_{ij}) \in \mathrm{GL}(n+1, R)$.

Therefore the affine n-chains are rational n-surfaces.

To obtain the concept of residual space in $\Sigma^n(K, R)$ with respect to a *hyperplane H* of $P^n(R)$ (spanned by a standard set of n points) if suffices to consider the hyperplane H given by $x_0 = 0$. Then ι maps R^n onto the set of complements of H. And the trace of an n-chain c is an affine hyperplane in R^n iff c contains a standard set of n points lying in H. (We may set $t_0 = 1 = a_{00}$, and we have $a_{k0} = 0$ for $1 \leqslant k \leqslant n$.)

For $\dim_K(R) = r < \infty$ we have the representation of $\Sigma^n(K, R)$ on the Grassmannian $G_{nr+r-1, r-1}$, where n-chains become Veronese manifolds V_n^r. (For $n = 1$, compare Section 5.) The group $\Gamma_n(R)$ operates on this representation as group of collineations fixing the Grassmannian.

For commutative K-algebras R, characterizations of standard sets of points or sets of cocatenal points by means of determinants and generalizations of the cross ratio are given in Gibert [1983] and Werner [1987].

A fully developed theory of general n-chain geometries does not yet exist. But there does exist a good theory for the case where R is a quadratic extension field of K (where the dimension n may even be infinite).

10.3. *Burau geometries*

See Karzel, Kist and Kroll [1979]. Let $\Pi = (P, L)$ be a projective geometry of dimension at least 2, where P is the set of points and L is the set of lines. Lines are considered as point-sets. Let C be another collection of subsets of P, the elements of which are called *circles*. For $M \subseteq P$ the linear closure of M in Π is denoted by $\overline{M}$. We write $\overline{a, b}$ instead of $\overline{\{a, b\}}$. For $M \subseteq P$, let $C(M) = \{X \in C\colon\ X \subseteq M\}$.

10.3.1. Definition. Let $\boldsymbol{B} = (P, \boldsymbol{L}, \boldsymbol{C})$ where $\Pi = (P, \boldsymbol{L})$ and $\boldsymbol{C}$ are as described above. We call $\boldsymbol{B}$ a *Burau geometry* provided the following conditions hold.

(B1) The linear closure of a circle is a line.

(B2) $(M, \boldsymbol{C}(M))$ is an inversive plane, for any line $M \in \boldsymbol{L}$.

(B3) For any 5 different points $a, b, c, d, e \in P$ with $c = \overline{a,b} \cap \overline{d,e}$ the following holds: Any line through $f = \overline{a,e} \cap \overline{b,d}$ meeting the circle through a, b, c also meets the circle through c, d, e.

10.3.2. Proposition. *Let F be a quadratic extension of the field K and V an F-vector space of dimension at least 3. Let*

$$P(V,F) = \{Fv\colon\ 0 \neq v \in V\}, \quad V^* = V \setminus \{0\},$$

$$L_F(u,w) = \{F(xu + yw)\colon\ F(x,y) \in P^1(F)^{14}\},$$

$$L_K(u,w) = \{F(su + tw)\colon\ K(s,t) \in P^1(K)^{14}\},$$

$$\boldsymbol{L}_F(V) = \{L_F(u + w)\colon\ u, w \in V^*, Fu \neq Fw\},$$

$$\boldsymbol{L}_K(V) = \{L_K(u + w)\colon\ u, w \in V^*, Fu \neq Fw\}.$$

Then $B(V,K,F) = (P(V,F), \boldsymbol{L}_F(V), \boldsymbol{L}_K(V))$ is a Burau geometry.

10.3.3. Theorem. *Every Burau geometry $\boldsymbol{B} = (P, \boldsymbol{L}, \boldsymbol{C})$ is isomorphic to a geometry $B(V,K,F)$: There is a collineation $\varkappa$ of the projective geometry $\Pi(V,F) = (P(V,F), \boldsymbol{L}_F(V))$ onto the projective geometry $\Pi = (P, \boldsymbol{L})$ which maps circles onto circles.*

The automorphism problem for Burau geometries is solved in the following

10.3.4. Fundamental theorem. *Let $\boldsymbol{B} = B(V,K,F)$ be a Burau geometry. An invertible F-semilinear mapping from V onto V with companion field automorphism α induces an automorphism of $\boldsymbol{B}$ iff α leaves K invariant. Conversely, any automorphism of $\boldsymbol{B}$ is induced by an invertible semilinear mapping from V onto V.*

Let $\boldsymbol{B} = (P, \boldsymbol{G}, \boldsymbol{C})$ be a Burau geometry. A subset $M \subseteq P$ is called a *Von Staudt chain* provided $(M, \boldsymbol{C}(M))$ is a projective geometry. (In fact, M is an m-chain iff $(M, \boldsymbol{C}(M))$ is a projective geometry of dimension m.)

10.3.5. Theorem. *Let M be any maximal Von Staudt chain of $\boldsymbol{B}$. Then there exists exactly one collineation $\varkappa$ of (P, G) which fixes M pointwise. Moreover $\varkappa$ is an involutory automorphism of $\boldsymbol{B}$.*

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[14] Notation as in Section 4.

11. Bibliographical remarks

Chapter 14 is based on Benz' book (Benz [1973a]).

For the concept of chains as K-sublines of a projective line over a general K-algebra R, compare also Hubaut [1964], Thas [1969b], Hotje [1976] and Schröder [1979].

The concept of Sections 4 and 5 is presented in Herzer [1985a, 1984a] following ideas of Hubaut [1965], Werner [1982f], but was already used by Thas [1969a] in a special case.

The general concept of Section 10 mainly goes along the lines of Gibert [1983] and Werner [1987] who both handle the commutative case.

The following bibliography is restricted mainly to recent publications. An extensive bibliography containing also more older contributions can be found in Benz [1973a] and in the survey articles Benz [1960], Benz and Mäurer [1964], Benz, Leissner and Schaeffer [1972], Benz et al. [1981], and Bartolone and Bartolozzi [1985].

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CHAPTER 15

Discrete Non-Euclidean Geometry

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Contents

Introduction ... 845
1. Inner product spaces ... 846
2. Spherical geometry ... 850
   2.0. Summary ... 850
   2.1. Spherical harmonics ... 850
   2.2. Spherical codes ... 854
   2.3. Spherical designs ... 862
   2.4. The group case ... 868
   2.5. Other types of designs ... 874
3. Elliptic geometry ... 875
   3.0. Summary ... 875
   3.1. The elliptic plane ... 875
   3.2. Equidistant sets in elliptic $(d-1)$-space ... 879
   3.3. Regular two-graphs ... 884
   3.4. Few-distance sets in I^{d-1} ... 888
4. Hyperbolic geometry ... 890
   4.0. Summary ... 890
   4.1. Bolyai–Lobachevsky geometry ... 891
   4.2. Harmonics in $\mathbb{R}^{p,q}$ ... 898
   4.3. Graphs ... 904
   4.4. Integral lattices ... 908
   4.5. Distance matrices ... 913
References ... 916

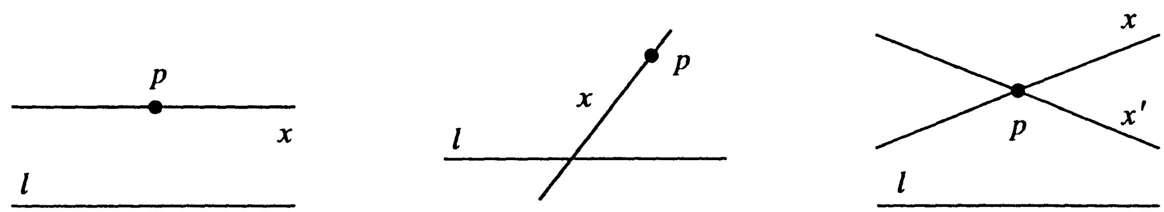
HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

Introduction

In the beginning there was Euclidean geometry, named after Euclid (about 300 B.C.). In thirteen books, called 'The Elements', an axiomatic treatment was given of the geometry of that time. The properties of points, lines, triangles, polygons, circles, pyramids, Platonic solids, etc. had been discovered earlier, mostly on an experimental basis. In 'The Elements' they were described in a scientific way.

Descartes (1596–1650) and Fermat (1601–1665) introduced algebraic tools in geometry, thus paving the way to the calculus of Newton (1642–1727) and Leibnitz (1646–1716). A century later showed two further developments of importance for geometry: projective geometry by Poncelet (1788–1867), and group theory by Galois (1811–1832). These subjects were brought together by Klein (1849–1925) in the Erlangen program, a classification of geometries according to the group of the transformations under which their theorems remain valid.

This program also unified the non-Euclidean geometries. Their earlier discovery by Gauss (1777–1855), Bolyai (1802–1860) and Lobachevsky (1792–1856) had solved the old problem of parallels. *Euclid's axiom* says that in a plane α, through a point P not on a line l there is 'exactly one' line x not intersecting l. It turned out that there are other consistent geometries: elliptic and hyperbolic geometry. They satisfy an axiom which differs from Euclid's axiom in replacing 'exactly one' by 'no' (elliptic) and by 'at least two' (hyperbolic), respectively:



Euclidean, elliptic, hyperbolic.

Spherical geometry also is 'non-Euclidean', but it fails the axiom that any two points determine exactly one line.

The present part of the Handbook deals with discrete figures of finite dimensional elliptic, spherical, hyperbolic and Euclidean type. We will start with the positive definite space $\mathbb{R}^d$ and the indefinite space $\mathbb{R}^{1,d}$ as a common framework. These spaces are selected in Section 1 on the basis of a simple criterion. In the spirit of Cayley (1821–1895), finite Euclidean and non-Euclidean sets have been characterized by Menger and Blumenthal in terms of matrices of inner products or distances. Such matrices and their generalizations also serve the description of certain notions of a combinatorial nature, such as distance spaces, codes, designs, graphs, matroids, root systems, integral lattices and finite groups. Reference to such notions will provide a major characteristic for our discussions.

On the other hand, the framework for our considerations is restricted to ordinary real (non-)Euclidean spaces. In fact the corresponding geometries, which traditionally have been a rich source for geometric models, recently witnessed a revival of interest (see

Milnor [1982]). Thus we shall not consider spaces over complex numbers, quaternions, etc., cf. Coxeter [1974], Hoggar [1982], Rosenfeld [1988]. Also, we shall not deal with Riemannian manifolds and their Laplacians.

The non-Euclidean geometries are briefly introduced in Section 1. They are investigated in much greater detail in the subsequent chapters. Each of these chapters starts with a short summary of its contents. Much attention has been given to the introduction of methods and tools. These come from linear algebra, group theory, harmonic analysis, topology, number theory and computation. They find their unification in the present geometric setting.

1. Inner product spaces

The present section gives a brief introduction to spherical space S^{p-1}, elliptic space I^{p-1}, hyperbolic space H^q, and Euclidean space E^{q-1}. The underlying spaces $\mathbb{R}^{p,0}$ and $\mathbb{R}^{1,q}$ are selected from arbitrary indefinite inner product spaces on the basis of the criterion of Theorem 1.2.

The inner product space $\mathbb{R}^{p,q}$ of signature p, q is most easily defined as the vector space of dimension $p+q$ over the reals provided withthe inner product

$$(x,y) := x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_{p+q}y_{p+q}.$$

In particular we will be interested in $\mathbb{R}^{p,0}$ and in $\mathbb{R}^{1,q}$ (sometimes $\mathbb{R}^{0,q}$ and $\mathbb{R}^{p,1}$ are more convenient). The reason for this special interest will soon become clear. We first give a coordinate-free definition of $\mathbb{R}^{p,q}$ in order to emphasize the role of the *Gram matrix* of the inner products of a set of vectors.

1.1. DEFINITION. $\mathbb{R}^{p,q}$ is the $\mathbb{R}$-vector space of dimension $p+q$ provided with a real bilinear form (the inner product $(\cdot,\cdot)$) of signature p, q. Vectors $x \in \mathbb{R}^{p,q}$ are called *positive*, *isotropic*, *negative*, according to whether their norm (x,x) is positive, zero, negative, respectively.

Remember (see Marcus and Minc [1964]) that the real symmetric matrices P and Q are said to be *congruent*, notation $P \sim Q$, whenever there exists a nonsingular square matrix T such that $Q = TPT^t$. For any symmetric matrix P we denote its rank by $\rho(P)$, the number of its positive (negative) eigenvalues by $\pi(P)$ ($\nu(P)$). Then $\rho(P) = \pi(P)+\nu(P)$. The pair $\pi(P), \nu(P)$ is called the *signature* of P. By Sylvester's law of inertia, $P \sim Q$ implies $\phi(P) = \phi(Q)$ for $\phi = \rho, \pi, \nu$. The signature of the vector space is the signature of the Gram matrix of any basis. Notice that by definition $\mathbb{R}^{p,q}$ is nondegenerate: only zero is orthogonal to all vectors. Also for a subspace of $\mathbb{R}^{p,q}$ the Gram matrices of any two bases are congruent. Hence in the subspace we may select an orthonormal basis whose Gram matrix is diagonal having entries 1, -1, 0. Notice that subspaces of $\mathbb{R}^{p,q}$ can be degenerate.

1.2. THEOREM. $\mathbb{R}^{p,0}$ *and* $\mathbb{R}^{1,q}$ *are characterized as those* $\mathbb{R}^{p,q}$ *which contain at least one positive vector and in which every subspace containing a positive vector is nondegenerate.*

PROOF. We first prove the sufficiency of the condition. Let Γ be a subspace of $\mathbb{R}^{p,q}$ with orthonormal basis $a_1, \ldots, a_n$ and Gram matrix A.

If Γ contains a positive vector then by assumption $\rho(A) = n$. We claim that either $\pi(A) = n$ or $\pi(A) = 1$. Indeed, at least one diagonal entry equals 1, say $a_{11} = 1$. Then $a_{22} = 1$, $a_{33} = -1$ would imply that $\langle a_1, a_2 + a_3 \rangle$ is degenerate, a contradiction.

If Γ does not contain a positive vector, then we claim that either $\rho(A) = \nu(A) = n$ or $\rho(A) = \nu(A) = n - 1$. Indeed, $\mathbb{R}^{p,q}$ contains $b \notin \Gamma$ with $(b, b) > 0$. Then $a_{11} = a_{22} = 0$ implies that $\langle a_1, a_2, b \rangle$ is degenerate, a contradiction.

Finally, $\langle a, b \rangle \subset \mathbb{R}^{p,q}$ with $\pi(\mathrm{Gram}(a, b)) = 2$ implies $(c, c) > 0$ for all $0 \neq c \in \mathbb{R}^{p,q}$. Indeed, $\dim\langle a, b, c \rangle = 3$ and $\pi(\mathrm{Gram}(a, b, c)) = 2$ contradict the above. Hence $\mathbb{R}^{p,q}$ equals $\mathbb{R}^{p,0}$ or $\mathbb{R}^{1,q}$.

The necessity of the condition is trivial for positive definite $\mathbb{R}^{p,0}$. For the case $\mathbb{R}^{1,q}$ we use the standard inner product

$$(x, y) = x_0 y_0 - x_1 y_1 - \cdots - x_q y_q.$$

There is a *cone* C of vectors of norm 0:

$$C := \{x \in \mathbb{R}^{1,q} \colon (x, x) = x_0^2 - x_1^2 - \cdots - x_q^2 = 0\}.$$

The vectors inside, on, outside C are positive, isotropic, negative, respectively. Independent vectors inside or on C have positive inner product. Subspaces of $\mathbb{R}^{1,q}$ are degenerate whenever they are tangent to C. Since no subspace containing an interior point can be tangent to C, also $\mathbb{R}^{1,q}$ must satisfy the condition of the theorem. This completes the proof. □

On the basis of $\mathbb{R}^{p,0}$ and $\mathbb{R}^{1,q}$, just characterized, we define the real non-Euclidean and Euclidean spaces S^{p-1}, I^{p-1}, H^q, E^{q-1} as follows. For each geometry we define the points in terms of the vectors of $\mathbb{R}^{p,0}$ or $\mathbb{R}^{1,q}$, and the distance matrix of a finite point set in terms of the Gram matrix of the corresponding vectors, see Menger [1928], Blumenthal [1953]. The lines, planes, 3-subspaces, etc. of the geometry are the point sets induced by the corresponding linear subspaces of the underlying vector space. The present definitions will be worked out in the subsequent Sections 2, 3 and 4.

1.3. *Spherical space* S^{p-1}. The points are the unit vectors in $\mathbb{R}^{p,0}$. The distance d between points is the angle between their vectors, so that $0 \leqslant d \leqslant \pi$. The distance matrix of $A = \{a_1, \ldots, a_n\} \subset S^{p-1}$ is the matrix

$$[\cos d(a_i, a_j)] = [(a_i, a_j)] = \mathrm{Gram}\, A.$$

In spherical space distance matrices are positive semidefinite.

1.4. *Elliptic space* I^{p-1}. The points are the 1-subspaces in $\mathbb{R}^{p,0}$. The distance d between points is the angle between the corresponding 1-subspaces, so that $0 \leqslant d \leqslant \pi/2$. The distance matrix of $\{\langle a_1 \rangle, \ldots, \langle a_n \rangle\} \subset I^{p-1}$ is the matrix

$$[\varepsilon_{ij} \cos d(a_i, a_j)] = \left[\frac{(a_i, a_j)}{\sqrt{(a_i, a_i)(a_j, a_j)}} \right] \approx [(a_i, a_j)],$$

where $\varepsilon_{ij} = \pm 1$ is defined by $(a_i, a_j) = \varepsilon_{ij} |(a_i, a_j)|$. In elliptic space distance matrices are positive semidefinite.

1.5. *Hyperbolic space* H^q. The points are the 1-subspaces inside the cone C in $\mathbb{R}^{1,q}$. All points are represented by vectors in one nappe of the cone, hence they have positive inner products. If we represent points by $\langle a\rangle$ and $\langle b\rangle$, then their distance is defined by

$$\cosh d(\langle a\rangle, \langle b\rangle) = \frac{(a,b)}{\sqrt{(a,a)(b,b)}}.$$

This is justified by the homogeneity of the right hand side, by $(a,b) > 0$ for vectors in one nappe of C, and by the anti-Cauchy–Schwarz inequality

$$(a,b)^2 \geqslant (a,a)(b,b).$$

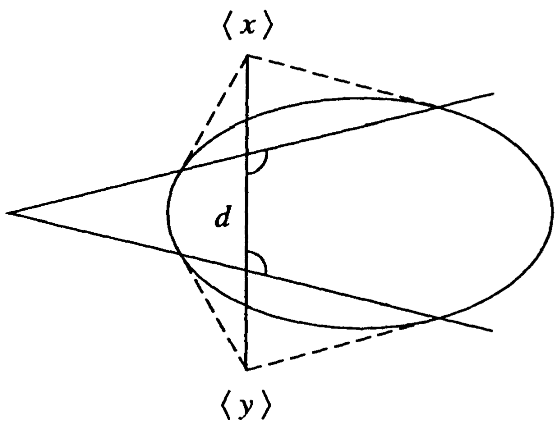
The distance matrix of $\{\langle a_1\rangle, \ldots, \langle a_n\rangle\} \subset H^q$

$$[\cosh d(\langle a_i\rangle, \langle a_j\rangle)] = \left[\frac{(a_i,a_j)}{\sqrt{(a_i,a_i)(a_j,a_j)}}\right] \approx [(a_i,a_j)]$$

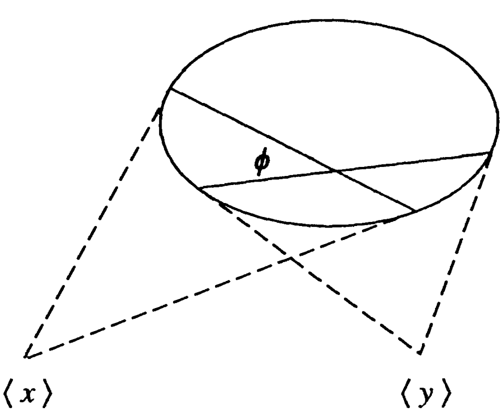
has the signature $\pi = 1,\ \nu = \rho - 1$.

Hyperbolic space H^q may be extended by its *ideal points*. These are the 1-subspaces on the cone C, again represented by vectors on the same nappe of C. The distance between ideal and ordinary points is not defined, but their mutual relations follow from the underlying Gram matrix $[(a_i,a_j)]$. This matrix is determined modulo pre- and post-multiplication by the same positive diagonal matrix, and again has signature $\pi = 1,\ \nu = \rho - 1$.

1.6. *Exterior hyperbolic* (*de Sitter*) *space*. The points are the 1-subspaces outside the cone C in $\mathbb{R}^{1,q}$. They maybe represented by vectors having norm -1.



$\cosh d = |(x,y)|$,



$\cos\phi = |(x,y)|$.

Exterior points $\langle x\rangle$ and $\langle y\rangle$ have a distance arcosh $|(x,y)|$ or an angle arccos $|(x,y)|$, depending on whether the plane $\langle x,y\rangle$ intersects the cone C or not. The distance (angle) between exterior points $\langle x\rangle$ and $\langle y\rangle$ equals the distance (angle) between the polar hyperplanes $\langle x\rangle^\perp$ and $\langle y\rangle^\perp$. For sets of points we better go back to the Gram matrix, which has signature $\pi = 1,\ \nu = \rho - 1$ or $\pi = 0,\ \nu = \rho$, depending on whether the subspace spanned by the points intersects the cone or not.

1.7. *Euclidean space* E^{q-1}. Our definition may seem somewhat unusual. It is phrased in terms of hyperbolic space H^q, and aims at the Cayley–Menger distance matrix, cf. Menger [1928]. Essentially, it goes back to F.L. Wachter (1792–1817), a student of Gauss, cf. Coxeter [1957].

The points, lines, ... of *Euclidean space* E^{q-1} are the lines, planes, ... of H^q which pass through a fixed ideal point $\langle a_0\rangle$ of H^q. The distance d_{ij} between the points $\langle a_0, a_i\rangle$ and $\langle a_0, a_j\rangle$ is defined, cf. Seidel [1955], by

$$d_{ij}^2 = \frac{\det(a_0, a_i, a_j)}{2\det(a_0, a_i)\det(a_0, a_j)} = \frac{(a_i, a_j)}{(a_0, a_i)(a_0, a_j)} - \frac{(a_i, a_i)}{2(a_0, a_i)^2} - \frac{(a_j, a_j)}{2(a_0, a_j)^2}.$$

It is convenient to represent the points of E^{q-1} by $\langle a_0, a_i\rangle$ where not only $\langle a_0\rangle$ but also $\langle a_i\rangle$ denotes an ideal point for H^q, so that $(a_i, a_i) = 0$. In addition, we make all $(a_0, a_i) = 1$ by normalization of the vectors a_i. Then the *Cayley–Menger distance matrix* of the Euclidean points $\langle a_0, a_1\rangle, \langle a_0, a_2\rangle, \dots, \langle a_0, a_n\rangle$ is the matrix

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 1 & & & & \\ \vdots & & d_{ij}^2 & & \\ 1 & & & & \end{bmatrix} \approx \begin{bmatrix} 0 & (a_0, a_i) \\ (a_0, a_j) & (a_i, a_j) \end{bmatrix}, \quad i, j = 1, \dots, n.$$

This matrix has size $n+1$ and signature $\pi = 1$, $\nu = \rho - 1$. Clearly $\rho = n+1$ if the points are independent.

Thus we have given the definitions of the geometries which will be the subjects of the subsequent chapters.

1.8. Remark. It is possible to generalize the present set-up by dropping the vector space structure, but keeping the inner products and concentrating on dependence. This leads to matroids. Below we give a brief sketch.

A *product set* is a set P in which each pair of elements a and b is provided with a real number, their product $ab = ba$ (generalizing the inner product). An element is *zero* if it has product 0 with all elements of P. Elements are identified if they have equal products with each element of P. For any finite $P = \{p_1, p_2, \dots, p_n\}$ the products $p_i p_j$ constitute the Gram matrix $G = G(p_1, \dots, p_n)$.

1.9. Definition. A *product space* is a product set P such that

$$0 < \operatorname{rk} G(p_1, \dots, p_m) < m$$

implies

$$\det G(p_1, \dots, p_m, x) = 0 \quad \text{for all } x \in P.$$

In a product space it is possible to introduce the notions of dependency and subspace. Only zero is dependent on $\varnothing$, and on sets consisting solely of elements 0. An element x is dependent on any finite set $\{p_1, \dots, p_m\}$ whenever $\operatorname{rk} G(x, p_1, \dots, p_m) = \operatorname{rk} G(p_1, \dots, p_m)$. The notion of dependency satisfies the usual properties, cf. Van der Waerden [1971], p. 105. Hence notions such as subspace may be introduced. This leads to the following theorem.

1.10. THEOREM. *The subspaces of a product space form a matroid lattice.*

This paves the way to a generalization of the (non-)Euclidean spaces defined above. We shall not pursue this here. For somewhat related work, in a more abstract direction, we refer to Dress and Havel [1986].

2. Spherical geometry

2.0. *Summary*

The geometry of the unit sphere S in the positive definite real space $\mathbb{R}^d$ is governed by spherical harmonics (Section 2.1). They are described by zonal spherical harmonics by use of Gegenbauer polynomials $Q_k(t)$ in one variable t. Finite subsets X of S will be viewed as codes, and as designs. For the codes (Section 2.2) the interest is in upper bounds for the cardinality $|X|$ of sets $X \subset S$ whose inner products are restricted to a prescribed subset of the interval $[-1, 1]$. We consider many examples, which relate for instance to semiregular polytopes, strongly regular graphs, and the integral lattices of Witt and Leech. Spherical designs (Section 2.3) are defined in terms of certain regularity properties, and draw the interest to lower bounds for $|X|$. Combinatorially most interesting are the cases where $|X|$ meets both the lower and the upper bound. The existence of spherical t-designs X in the general case only requires sufficiently large cardinality for X, but is much more restricted if X is the orbit of a finite group. In this group case (Section 2.4) invariants, representations and root systems are relevant. The present investigations also apply to the construction of cubature formulae for the sphere, where weights are introduced. In Section 2.5 we mention recent generalizations to the case of several spheres, also infinitely many, and to measures in Euclidean space $\mathbb{R}^d$. They form the counterpart of the binary combinatorial designs and codes in Boolean space $\mathbb{B}^d$.

2.1. *Spherical harmonics*

Let $\mathrm{Pol}_k(\mathbb{R}^d)$ denote the linear space of the polynomials of degree $\leqslant k$ in d variables. Let $\mathrm{Hom}_k(\mathbb{R}^d)$ denote the subspace of the homogeneous polynomials of degree k, and $\mathrm{Harm}_k(\mathbb{R}^d)$ the harmonic homogeneous polynomials h of degree k. *Harmonic polynomials* h satisfy Laplace's equation

$$\Delta h := \frac{\partial^2 h}{\partial x_1^2} + \cdots + \frac{\partial^2 h}{\partial x_d^2} = 0,$$

and Δ is the *Laplace operator*. Between these linear spaces the following direct sum relations exist:

$$\mathrm{Pol}_k(\mathbb{R}^d) = \sum_{i=0}^{k} \mathrm{Hom}_i(\mathbb{R}^d), \qquad \mathrm{Hom}_k(\mathbb{R}^d) = \mathrm{Harm}_k(\mathbb{R}^d) + q\mathrm{Hom}_{k-2}(\mathbb{R}^d)$$

where q: $\mathbb{R}^d \to \mathbb{R}$, $x \mapsto (x,x)$. For the dimensions of these spaces we have

$$\dim \mathrm{Hom}_k(\mathbb{R}^d) = \binom{d+k-1}{d-1}, \qquad \dim \mathrm{Pol}_k(\mathbb{R}^d) = \binom{d+k}{d},$$

$$\begin{aligned}\dim \mathrm{Harm}_k(\mathbb{R}^d) &= \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1}\\ &= \binom{d+k-2}{d-2} + \binom{d+k-3}{d-2}.\end{aligned}$$

Turning to the restrictions of these spaces to the unit sphere S in $\mathbb{R}^d$ we have the following isomorphisms:

$$\mathrm{Hom}_k(S) = \mathrm{Hom}_k(\mathbb{R}^d), \qquad \mathrm{Harm}_k(S) = \mathrm{Harm}_k(\mathbb{R}^d),$$

$$\mathrm{Pol}_k(\mathbb{R}^d) \neq \mathrm{Pol}_k(S) = \mathrm{Hom}_k(S) + \mathrm{Hom}_{k-1}(S),$$

since $(x,x)^i h_{k-2i}(x) \in \mathrm{Hom}_k(S)$. Introducing the positive definite *inner product*

$$\langle f, g\rangle := \int_S f(x)g(x)\,\mathrm{d}\sigma(x), \quad f,g \in \mathrm{Pol}_k(S),$$

where σ is the normalized Borel measure on the unit sphere S, we observe that the direct sums of the polynomial spaces over S are orthogonal sums. Iteration and substitution yields the following.

2.1.1. THEOREM.

$$\mathrm{Hom}_k(S) = \mathrm{Harm}_k(S) \perp \mathrm{Hom}_{k-2}(S),$$

$$\mathrm{Pol}_k(S) = \mathrm{Harm}_k(S) \perp \mathrm{Harm}_{k-1}(S) \perp \cdots \perp \mathrm{Harm}_1(S) \perp \mathrm{Harm}_0(S).$$

Thus each polynomial on S may be written as an orthogonal sum of so-called spherical harmonics.

REMARK. The general reference for the above is Stein and Weiss [1971]. With the right topology generalizations hold, such as

$$L_2(S) = \sum_{i=0}^{\infty}{}^{\perp} \mathrm{Harm}_i(S).$$

It is easily checked that, for any $y \in \mathbb{R}^d$, the following polynomials in $x \in \mathbb{R}^d$ are harmonic:

$$(x,y)^2 - (x,x)(y,y)/d, \qquad (x,y)^3 - 3(x,y)(x,x)(y,y)/(d+2).$$

Their restrictions to S are called *zonal* spherical harmonics, since for fixed y they are constant on the zones $\{x \in S\colon (x,y) = \text{const.}\}$. Here is how these *zonal spherical harmonics* are constructed in general. For any fixed $a \in S$ we consider the linear functional

$$l\colon \mathrm{Harm}_k(S) \to \mathbb{R}, \quad h \mapsto h(a).$$

Since $\langle\cdot,\cdot\rangle$ is a nondegenerate inner product on the linear space $\mathrm{Harm}_k(S)$, there exists a unique element, $\bar{a}$, say, in $\mathrm{Harm}_k(S)$ such that

$$l(h) = \langle \bar{a}, h\rangle = h(a) \quad \forall h \in \mathrm{Harm}_k(S).$$

For reasons to become clear soon write $\bar{a} =: Q_k(a,\cdot)$. Likewise, $b \in S$ corresponds to a unique $\bar{b} \in \mathrm{Harm}_k(S)$. Therefore, for $a, b \in S$ we have

$$\langle \bar{a}, \bar{b}\rangle = \bar{a}(b) = \bar{b}(a) = Q_k(a,b).$$

2.1.2. LEMMA. $Q_k(\gamma a, \gamma b) = Q_k(a,b)$, *for any element γ of the orthogonal group* $\mathrm{O}(d)$.

PROOF. For any $h \in \mathrm{Harm}_k(S)$, $\gamma \in O(d)$, $a \in S$, and $z := \gamma y$ we have:

$$\int_S h(y) Q_k(\gamma a, \gamma y)\,\mathrm{d}\sigma(y) = \int_S h\big(\gamma^{-1} z\big) Q_k(\gamma a, z)\,\mathrm{d}\sigma(z)$$
$$= h\big(\gamma^{-1}\gamma a\big) = h(a).$$

The result then follows from the uniqueness of $Q_k(a,\cdot)$ in $\langle h, Q_k(a,\cdot)\rangle = h(a)$. □

2.1.3. COROLLARY. *$Q_k(x,y)$ depends on the inner product (x,y) only, i.e.*

$$Q_k(x,y) = Q_k(t) \quad \textit{with } t = (x,y).$$

We shall identify $Q_k(t)$ as a Gegenbauer polynomial in one variable t, by use of the following *addition formula*.

2.1.4. THEOREM. *Let $f_{k,1}, \ldots, f_{k,\mu_k}$ denote an orthonormal basis of* $\mathrm{Harm}_k(S)$. *Then*

$$Q_k(x,y) = \langle \bar{x}, \bar{y}\rangle = \sum_{i=1}^{\mu_k} f_{k,i}(x) f_{k,i}(y), \quad \textit{for any } x, y \in S,$$

PROOF.

$$\langle \bar{x}, \bar{y}\rangle = \bar{x}(y) = \sum_{i=1}^{\mu_k} \langle f_{k,i}, \bar{x}\rangle f_{k,i}(y)$$
$$= \sum_{i=1}^{\mu_k} f_{k,i}(x) f_{k,i}(y).$$

□

2.1.5. COROLLARY. $\dim \mathrm{Harm}_k = Q_k(1)$.

PROOF.

$$Q_k(1) = \sum_{i=1}^{\mu_k} f_{k,i}^2(x) = \sum_{i=1}^{\mu_k} \int_S f_{k,i}^2(x)\,\mathrm{d}\sigma(x)$$

$$= \sum_{i=1}^{\mu_k} 1 = \mu_k.$$

□

The following formula is true for $k \neq l$ since $\mathrm{Harm}_k \perp \mathrm{Harm}_l$, and also for $k = l$ since $Q_k(x, y) = \langle \bar{x}, \bar{y} \rangle$:

$$\int_S Q_k(x, z) Q_l(z, y)\,\mathrm{d}\sigma(z) = \delta_{k,l} Q_k(x, y).$$

Putting $x = y$ and $(x, z) = t$ we obtain

$$|S_{d-1}| \int_{-1}^{1} Q_k(t) Q_l(t)(1 - t^2)^{1/2(d-3)}\,\mathrm{d}t = \delta_{k,l} |S_d| Q_k(1).$$

This means that $Q_k(t)$, $k = 0, 1, 2, \ldots,\ -1 \leqslant t \leqslant 1$, constitute a family of polynomials in one variable t which is orthogonal with respect to the weight function $(1 - t^2)^{(d-3)/2}$. The first few of these so called *Gegenbauer polynomials* are:

$$Q_0(t) = 1, \quad Q_1(t) = c_1 t, \quad 2Q_2(t) = c_2(t^2 - 1/d),$$

$$6Q_3(t) = c_3 \left(t^3 - \frac{3t}{d+2} \right),$$

$$24Q_4(t) = c_4 \left(t^4 - \frac{6t^2}{d+4} + \frac{3}{(d+2)(d+4)} \right),$$

$$120Q_5 = c_5 \left(t^5 - \frac{10t^3}{d+6} + \frac{15t}{(d+4)(d+6)} \right),$$

$$c_k = d(d+2)(d+4) \cdots (d+2k-2).$$

The Gegenbauer polynomial $Q_k(t)$ has degree k, and is even (odd) for k even (odd). The orthogonality of the family ensures that any polynomial $F(t)$ of degree μ has an expansion with unique Gegenbauer coefficients f_k:

$$F(t) = \sum_{k=0}^{\mu} f_k Q_k(t).$$

2.2. *Spherical codes*

Let A denote any subset of the interval $-1 \leqslant t < 1$. A *spherical A-code* in $\mathbb{R}^d$ is a finite subset X of the unit sphere $S \subset \mathbb{R}^d$ such that $(x,y) \in A$ for all $x, y \in X$, $x \neq y$. We will be interested in upper bounds for the cardinality $n := |X|$ of spherical A-codes with prescribed A (sometimes only $|A|$), and in the structure of *tight* codes, i.e. those which are extremal with respect to such bounds. For $|A| = 2$ this amounts to spherical *two-distance sets* (whose points have only two distances). For $A = [-1, \beta]$ this is the classical problem of nonoverlapping spherical caps of angular radius $(1/2)\arccos\beta$, cf. Gruber and Lekkerkerker [1987].

We will discuss two types of upper bounds: the absolute bound and the linear-programming bound. Spherical two-distance sets will be of special interest, because of their relations to the theory of graphs. The tight cases of our bounds will lead to graphs with a certain structure, and to equiangular lines, root systems and Newton numbers, the binary Golay code, the Steiner system 5-(24,8,1), and the lattices of Witt and Leech.

2.2.1. THEOREM. *Any spherical A-code X with $|A| = s$ satisfies*

$$n = |X| \leqslant \binom{d+s-1}{d-1} + \binom{d+s-2}{d-1}.$$

PROOF. For each $y \in X$ we define the polynomial F_y in d variables by

$$F_y(x) := \prod_{\alpha \in A} \frac{(y,x) - \alpha}{1 - \alpha}, \quad x \in S.$$

These are n polynomials of degree $\leqslant s$. Since $F_y(x) = \delta_{y,x}$ for $x, y \in X$, these polynomials are independent. Hence $n \leqslant \dim \mathrm{Pol}_s(S)$, and application of Section 2.1 yields the result since

$$\mathrm{Pol}_s(S) = \mathrm{Hom}_s(S) + \mathrm{Hom}_{s-1}(S), \qquad \dim \mathrm{Hom}_s(S) = \binom{d+s-1}{d-1}.$$

□

2.2.2. COROLLARY. *Spherical two-distance sets in $\mathbb{R}^d$ have the cardinality*

$$n \leqslant d(d+3)/2.$$

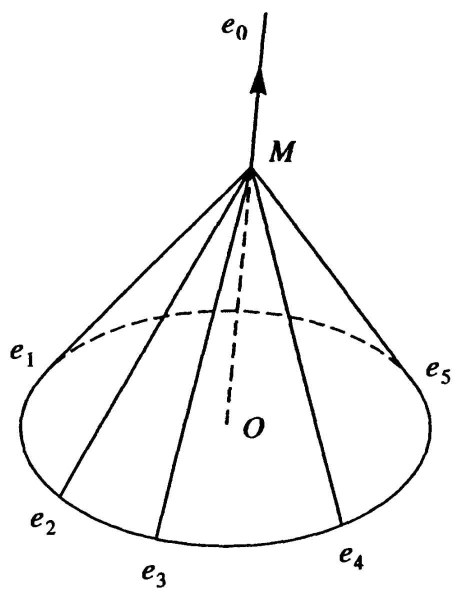
Only the following three tight cases are known to exist (the first is the regular pentagon):

$$n = 5,\ d = 2; \qquad n = 27,\ d = 6; \qquad n = 275,\ d = 22.$$

These cases can be constructed in the following way.

In $\mathbb{R}^3$ we consider the 6 diameters of the icosahedron (connecting the 6 pairs of antipodal vertices). From solid geometry we see that each pair of diameters has the same angle, in fact $\cos\phi = 1/\sqrt{5}$. Hence the diameters form a configuration of 6 equiangular lines in $\mathbb{R}^3$. Of course 6 unit vectors, one along each line, are not equiangular: they have inner products $+1/\sqrt{5}$ (acute angle), or $-1/\sqrt{5}$ (obtuse angle). Now we select a vector e_0 along any one of the lines, and take the other 5 unit vectors at obtuse angle with e_0. Their endpoints form a planar regular pentagon with angles

$$2\cos 72^\circ = \tau^{-1}, \qquad 2\cos 144^\circ = -\tau, \qquad \tau = (1+\sqrt{5})/2.$$



In $\mathbb{R}^7$ we construct 28 equiangular lines at $\cos\phi = 1/3$ as follows. Their spanning vectors are $(3,3,-1,-1,-1,-1,-1,-1)$ and the vectors with permuted coordinates. These vectors have mutual inner products 24, 8, -8, hence cosines $1/3$ and $-1/3$. They are situated in the 7-dimensional subspace of $\mathbb{R}^8$ with equation $x_1 + x_2 + \cdots + x_8 = 0$. Hence we have 28 equiangular lines in $\mathbb{R}^7$ at $\cos\phi = 1/3$. As above, we select a unit vector e_0 along any one of the lines, and the 27 unit vectors along the other lines at obtuse angle with e_0. The endpoints of these vectors form a configuration of 27 vectors in $\mathbb{R}^6$ with angles $\cos\alpha = 1/4$ and $\cos\beta = -1/2$, up to scaling a tight spherical two-distance set in $\mathbb{R}^6$.

This so-called *Schläfli configuration* occurs in many disguises: as 27 lines on a general cubic surface, as a semiregular Gosset polytope, as a rank 3 graph, in connection with the finite group $[3^{3,2,1}] \cong O_6^-(\mathbb{F}_2)$, with root systems, with cubic forms etc., cf. Coxeter [1973], Manin [1974].

The third case can be constructed similarly from a set of 276 equiangular lines in $\mathbb{R}^{23}$ at $\cos\phi = 1/5$. This configuration is unique and admits a doubly transitive action by Conway's group ·3. Later we will construct both the 276 lines and McLaughlin's graph on 275 vertices from vectors, by use of indefinite metric, cf. Example 4.2.7 and Theorem 3.3.9.

We continue our remarks about two-distance sets, partly to gain examples for future reference, partly because of the close relationship with the theory of *graphs*. In our terminology all graphs are undirected and have no loops and no multiple edges. Hence a graph on n vertices has a symmetric $n \times n$ *adjacency matrix* A with zero diagonal, which is defined by its entries $a_{ij} = 1$ if i and j are adjacent, $= 0$ otherwise. We let J denote the all-one matrix of any size. Clearly, a given two-distance set defines two complementary graphs (take any of the two distances as adjacency). Conversely we have:

2.2.3. THEOREM. *Let A denote the adjacency matrix of a regular graph on n vertices. Let g be the multiplicity of the smallest eigenvalue of A. Then the graph defines a two-distance set in $\mathbb{R}^d$ with $d = n - 1 - g$.*

PROOF. The graph is regular iff its adjacency matrix has constant row sums k, say, i.e. iff $AJ = kJ$. Hence A has largest eigenvalue k, with the all-one vector as its eigenvector. Let s be the smallest eigenvalue of A, of multiplicity $n - d - 1$, say. Then

$$G := A - sI - \frac{(k-s)}{n} J$$

has the smallest eigenvalue 0 of multiplicity $n - d$, hence G is symmetric, positive semidefinite of rank d. As a consequence, G is the Gram matrix of the inner products of n vectors in $\mathbb{R}^d$. The set of these vectors is spherical (since all $(x, x) = (-(n-1)s-k)/n)$ and has only two mutual distances (all (x, y) equal $(n - k + s)/n$ or $(-k + s)/n$). □

Our three examples are of the following type.

2.2.4. DEFINITION. A *graph* is *strongly regular* provided it is regular, and any pair of its vertices is adjacent to a constant number λ or μ of other vertices, depending only on whether the pair is adjacent (then λ) or nonadjancent (then μ):

$$AJ = kJ, \qquad A^2 = kI + \lambda A + \mu(J - I - A).$$

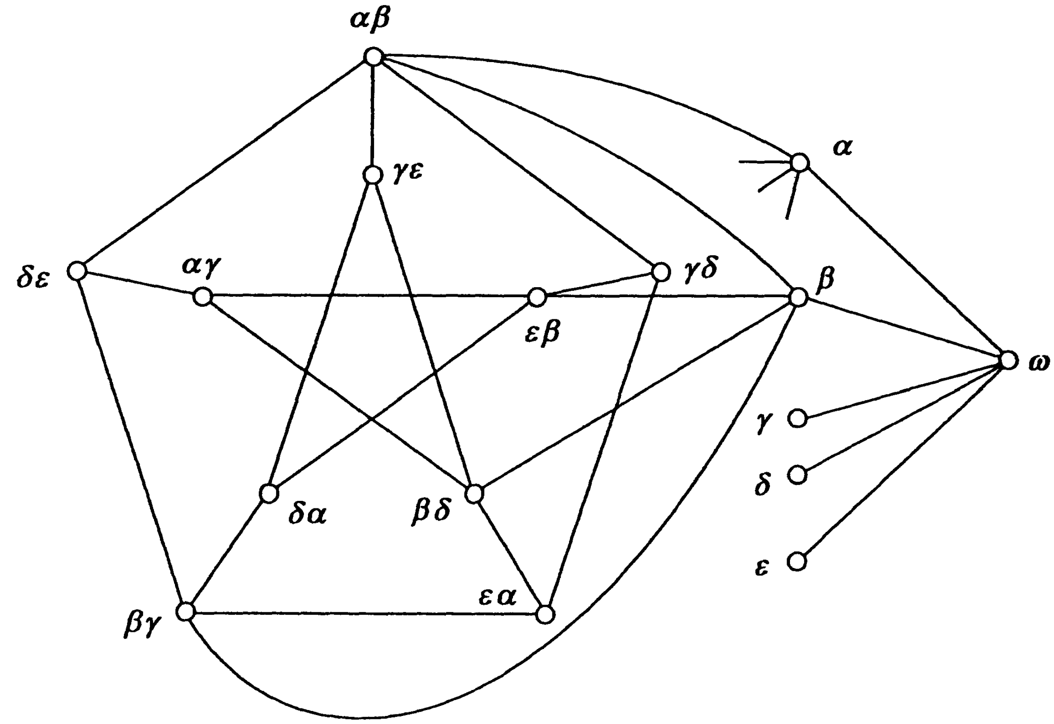
Strongly regular graphs are characterized by the property that they have three eigenvalues k, r, s. These satisfy

$$AJ = kJ, \qquad (A - rI)(A - sI) = (k + rs)J.$$

So there are easy relations between parameters n, k, λ, μ and eigenvalues k, r, s and their multiplicities $1, f, g$, e.g.,

$$n = 1 + f + g, \quad 0 = k + rf + sg, \quad (k - r)(k - s) = n(k + rs) = n\mu.$$

Now it is a matter of simple counting and calculation to determine the parameters and spectrum of the Petersen graph, the Clebsch graph, the Schläfli graph, and their complements (with $\overline{k}, \overline{\lambda}, \overline{\mu}, \overline{r}, \overline{s}, \overline{f}, \overline{g}$). To that end we draw the Petersen and Clebsch graphs, but we decided to delete the Schläfli graph since it has already been explained above as a two-distance set.



		Petersen	Clebsch	Schläfli
(n, k, λ, μ)	=	(10, 3, 0, 1),	(16, 5, 0, 2),	(27, 10, 1, 5).
$(n, \overline{k}, \overline{\lambda}, \overline{\mu})$	=	(10, 16, 3, 4),	(16, 10, 6, 6),	(27, 16, 10, 8).
(n, k, r, s, f, g)	=	(10, 3, 1, −2, 5, 4),	(16, 5, 1, −3, 10, 5),	(27, 10, 1, −5, 20, 6).
$(n, \overline{k}, \overline{r}, \overline{s}, \overline{f}, \overline{g})$	=	(10, 6, 1, −2, 4, 5),	(16, 10, 2, −2, 5, 10),	(27, 16, 4, −2, 6, 20).

The pentagon in $\mathbb{R}^2$, the icosahedron in $\mathbb{R}^3$, and applications of Theorem 2.2.3 with the three examples above provide spherical two-distance sets of the following sizes

$$5 \text{ in } \mathbb{R}^2, \quad 6 \text{ in } \mathbb{R}^3, \quad 10 \text{ in } \mathbb{R}^4, \quad 16 \text{ in } \mathbb{R}^5, \quad 27 \text{ in } \mathbb{R}^6.$$

Probably, all cardinalities are optimal for spherical two-distance sets, and possibly also for two-distance sets which are not restricted to the sphere in $\mathbb{R}^d$, cf. O. Kristensen (private communication via H. Tverberg), Seidel [1981]. It is about time that the upper bound of Corollary 2.2.2 be improved for large d. It is interesting to observe that $\mathbb{R}^d$ contains a two-distance set of size $d(d+1)/2$, for any d, from the vectors of type $(1^2 0^{d-1})$.

We return to general spherical A-codes in $\mathbb{R}^d$. Sometimes bounds better than the absolute bound of Theorem 2.2.1 may be achieved by use of the following *linear programming method*, due to Delsarte [1973]. We look for an appropriate polynomial $F(t)$ of degree μ, and Gegenbauer coefficients $f_0, f_1, \ldots, f_\mu$. We compare the following two expressions for a spherical A-code X:

$$|X|F(1) + \sum_{\alpha \in A} \operatorname{freq}(\alpha) F(\alpha) = \sum_{x,y \in X} F((x,y))$$

$$= f_0 |X|^2 + \sum_{k=1}^{\mu} f_k \sum_{x,y \in X} Q_k(x,y)$$

with

$$\operatorname{freq}(\alpha) := \#\{(x,y) \in X \times X\colon\ (x,y) = \alpha\} \geqslant 0,$$

$$\sum_{x,y\in X} Q(x,y) = \sum_{i\geqslant 0}\Big(\sum_{x\in X} f_{k,i}(x)\Big)^2 \geqslant 0,$$

by use of the Addition Theorem 2.1.4. The following theorem is a consequence of these considerations.

2.2.5. THEOREM. *Let $F(t)$ have non-negative Gegenbauer coefficients and $f_0 > 0$, and let $F(\alpha) \leqslant 0$ for all $\alpha \in A$. Then the cardinality of any A-code X satisfies $|X| \leqslant F(1)/f_0$.*

We illustrate this theorem and its generalizations by the following examples.

2.2.6. *Equiangular lines*

Suppose we have n equiangular lines at $\cos\phi = \alpha$ in $\mathbb{R}^d$. Let X be the set of n unit vectors in $\mathbb{R}^d$, one along each line. Take $A = \{\alpha, -\alpha\}$, and take

$$F(t) = \frac{t^2-\alpha^2}{1-\alpha^2} = \frac{1-d\alpha^2}{d(1-\alpha^2)} Q_0 + \frac{2}{d(d+2)(1-\alpha^2)} Q_2(t).$$

Application of Theorem 2.2.5 yields, for $\alpha^2 < 1/d$,

$$n = |X| \leqslant \frac{F(1)}{f_0} = \frac{d(1-\alpha^2)}{1-d\alpha^2}.$$

For $\alpha = 1/3$, and for $\alpha = 1/5$ these bounds read:

d	3	4	5	6	7
n	4	6	10	16	28

d	20	21	22	23
n	96	126	176	276

respectively. In each case, except for $d = 20$, $n = 96$, tight examples exist. Those in dimensions 21, 22, 23 are particularly interesting, since they involve the simple groups $\mathrm{U}(3,5^2)$, Higman–Sims HS, and Conway ·3, respectively, cf. Lemmens and Seidel [1973a], Taylor [1977].

2.2.7. *Root systems*

An ordinary *root system* in $\mathbb{R}^d$ is a set of lines at angles 60° or 90°. We are interested in bounds for the cardinality n of such systems. Let X denote the set of n unit vectors in $\mathbb{R}^d$, one along each line of a root system. Take $A = \{0, 1/2, -1/2\}$ and take

$$F(t) := t\Big(t^2 - \frac{1}{4}\Big) = \frac{6Q_3(t)}{d(d+2)(d+4)} + \frac{(10-d)Q_1(t)}{4d(d+2)}.$$

Application of Theorem 2.2.5 yields, for $d < 10$,

$$n = |X| \leqslant \frac{F(1)}{f_1} = \frac{3d(d+2)}{10-d}, \qquad \begin{array}{c|cccc} d & 5 & 6 & 7 & 8 \\ \hline n & 21 & 36 & 63 & 120 \end{array}.$$

The root systems $\mathsf{D}_5, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ are tight examples in dimensions 5, 6, 7, 8, cf. Section 2.2.10 and Bourbaki [1968].

2.2.8. *Tight* 3*-distance set from binary Golay code*
Given $-1 \leqslant \alpha \leqslant \beta \leqslant \gamma < 1$, the polynomial $F(t) := (t-\alpha)(t-\beta)(t-\gamma)$ is compatible with any subset A of $[-1,\alpha] \cup [\beta,\gamma]$. It has non-negative Gegenbauer coefficients and $f_0 > 0$, provided

$$\alpha+\beta+\gamma \leqslant 0, \quad \alpha+\beta+\gamma < -\alpha\beta\gamma d, \quad \alpha\beta+\beta\gamma+\alpha\gamma \geqslant -3/(d+2)$$

holds. If so, then Theorem 2.2.5 yields

$$n \leqslant -d(1-\alpha)(1-\beta)(1-\gamma)/(\alpha+\beta+\gamma+d\alpha\beta\gamma)$$

for the cardinality n of any A-code in $\mathbb{R}^d$. We will construct a tight $\{\alpha,\beta,\gamma\}$-code with

$$d = 24, \; n = 2048, \; \alpha = -9/23, \; \beta = -1/23, \; \gamma = 7/23.$$

Let I be the unit matrix and j be the all-one vector of size 11, let Q be the 11×11 circulant matrix circ(01011100010), and consider the 12×24 matrix

$$\begin{bmatrix} 0 & 0^t & 1 & j^t \\ j & I & j & Q \end{bmatrix}.$$

If $\{0,1\}$ is interpreted as the binary field $\mathbb{F}_2$, then the 2^{12} linear combinations with coefficients 0 and 1 of the 12 rows of the matrix are the 2^{12} code words of the linear extended binary *Golay code*. They have weight distribution as follows:

weight	0	8	12	16	24
number	1	759	2576	759	1

If the 2^{12} linear combinations are written with $\{+1,-1\}$ instead of $\{0,1\}$, then they are 2^{12} real vectors in $\mathbb{R}^{24}$, of the type

$$(+1)^{24}, \; (+1)^{16}(-1)^8, \; (+1)^{12}(-1)^{12}, \; (+1)^8(-1)^{16}, \; (-1)^{24},$$

with inner products $24, 8, 0, -8, -24$. These vectors are pairwise antipodal and form $1 + 759 + 1288 = 2^{11}$ lines in $\mathbb{R}^{24}$. We select along each line the vector whose 24-th coordinate equals $+1$, and project it into the hyperplane $x_{24} = 0$. Then we obtain 2^{11} vectors in $\mathbb{R}^{23}$ which have inner products $23, 7, -1, -9$, hence cosines $7/23, -1/23, -9/23$, as desired.

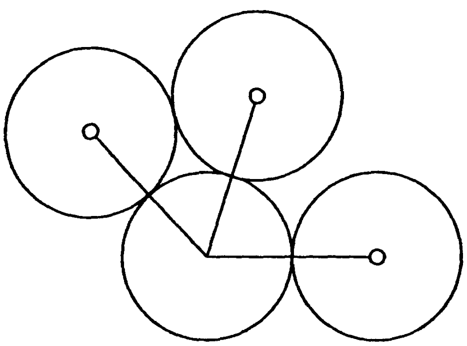
2.2.9. *The Steiner system* 5-(24,8,1)

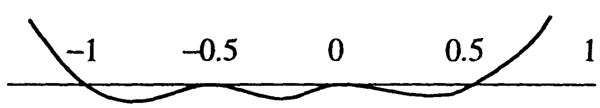
The 759 vectors of weight 8 in the extended Golay code form the *Steiner system* 5-(24,8,1). This block design consists of 759 blocks of size 8 in a 24-set, such that each 5-subset is in exactly one block. An easy count shows that it has block intersections 4,2,0, cf. Goethals and Seidel [1970]. We shall later encounter its derived designs 4-(23,7,1) and 3-(22,6,1), which have block intersections (3,1) and (2,0), respectively, cf. Section 3.3.9.

2.2.10. *Newton numbers*

Let τ_d denote the maximum number of nonoverlapping unit spheres that can touch a given unit sphere in $\mathbb{R}^d$, cf. Gruber and Lekkerkerker [1987]. Until recently only τ_1, τ_2, τ_3 were known. By use of the linear programming method Odlyzko and Sloane [1979] determined τ_8 and τ_{24}, and improved existing bounds for other τ_d. We illustrate this for the case $d = 8$. Take

$$A = \{\alpha \in \mathbb{R}: -1 \leqslant \alpha \leqslant \tfrac{1}{2}\}, \qquad F(t) = (t+1)\left(t+\tfrac{1}{2}\right)^2 t^2 \left(t - \tfrac{1}{2}\right).$$





The Gegenbauer coefficients $f_1, f_2, \ldots, f_6$ turn out to be non-negative, and $f_0 = 3/320$. Since $F(1) = 9/4$, application of Theorem 2.2.5 yields $\tau_8 \leqslant 240$.

Similarly, for the case $d = 24$ one can prove that $\tau_{24} \leqslant 196560$. In order to prove $\tau_8 = 240$ and $\tau_{24} = 196560$ we show that there exist sphere packings with these numbers of spheres in $\mathbb{R}^8$ and $\mathbb{R}^{24}$, respectively. In fact, it is known that these sphere-packings are unique, cf. Bannai and Sloane [1981].

2.2.11. *The lattices of Witt and Leech*

See McKay [1973]. Let $H = -I + S$ be a skew Hadamard matrix of size $4k$, i.e. H has entries ± 1 and

$$HH^t = 4kI, \qquad S + S^t = 0, \qquad \operatorname{diag}(H - I) = -2I.$$

We consider the following matrix B of size $8k$, and we calculate $B^t B$:

$$B_{8k} = \frac{1}{\sqrt{k+1}} \begin{bmatrix} (k+1)I_{4k} & H_{4k} - I_{4k} \\ O_{4k} & I_{4k} \end{bmatrix}, \quad B^t B = \begin{bmatrix} (k+1)I & H - I \\ H^t - I & 4I \end{bmatrix}.$$

The integral linear combinations of the columns constitute a *lattice* Λ_{8k} in $\mathbb{R}^{24}$, which has the following properties:

Λ_{8k} is *integral* (all lattice vectors have integral inner products), since B^tB is an integral matrix.

Λ_{8k} is *unimodular integral* (the parallelotope of basis vectors has volume one), since $\det B = 1$.

Λ_{8k} is *even integral* (the norms (x, x) of all lattice vectors are even), for odd k.

The case Λ_8 is the *Witt lattice* in $\mathbb{R}^8$, the case Λ_{24} is the *Leech lattice* in $\mathbb{R}^{24}$. Let us investigate these cases in some more detail.

For $k = 1$ we expose the matrix B, and some easy linear combinations of the 8 columns, which form $240 = 8 + 8 + 16(4 + 4 + 6)$ vectors of minimum norm $(x, x) = 1/2 \times 4 = 2$ in the lattice:

$$B_8 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & -1 & -2 & 1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 2 & -1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \pm 2 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & \pm 1 \\ 0 & 0 & \pm 1 & 0 & \pm 1 \\ 0 & \pm 2 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 & \pm 1 \\ 0 & 0 & 0 & \pm 1 & \pm 1 \end{matrix}$$

These 240 vectors in $\mathbb{R}^8$ all have norm $(r, r) = 2$ and inner products $(r, s) \in \{0, \pm 1, \pm 2\}$. They form the so-called *rootsystem* E_8. They form the centres of a packing of the unit sphere in $\mathbb{R}^8$. The pairs of opposite vectors are on 120 lines in $\mathbb{R}^8$ which have angles $60°, 90°$, so have cosines $\in \{1/2, 0\}$. The Witt lattice is a root lattice, cf. Section 4.4.11.

For $k = 3$, in dimension 24, we read from B^tB that the basis vectors all have norm 4. It is not difficult to show that all lattice vectors have norm at least 4, and $(x \pm y, x \pm y) \geqslant 4$ implies

$$(x, y) \in \{0, \pm 1, \pm 2, \pm 4\} \quad \text{for } X = \{x \in \Lambda\colon\ (x, x) = 4\}.$$

It turns out, cf. Conway and Sloane [1988], that the set X contains 196560 vectors of norm 4, which form the centres of a packing of the unit sphere in $\mathbb{R}^{24}$. They are pairwise antipodal and form 98280 lines at $\cos\phi \in \{0, 1/2, 1/4\}$. Finally, we observe that

$$240 = 2\binom{10}{3}, \qquad 196560 = 2\binom{28}{5}.$$

Indeed, the packings mentioned above correspond with the extreme cases for $d = 8$, $s = 1$ and $d = 24$, $s = 2$ in the following theorem, whose proof is similar to that of Theorem 2.2.1.

2.2.12. THEOREM. *There are at most $\binom{d+2s}{2s+1}$ lines in $\mathbb{R}^d$ having angles selected from at most $s + 1$ values, one of which being $\pi/2$.*

2.3. *Spherical designs*

Again we start with inequalities, and are interested in the case of equality. Let X be a finite subset of the unit sphere S in $\mathbb{R}^d$, of cardinality $|X| = n$, and let $k \in \mathbf{N}$.

2.3.1. THEOREM.

$$\sum_{x,y\in X} Q_k(x,y) \geqslant 0,$$

where $Q_k(t)$ is the k-th Gegenbauer polynomial.

PROOF. Use the addition formula of Theorem 2.1.4:

$$\sum_{x,y\in X} Q_k(x,y) = \sum_{i=1}^{Q_k(1)} \Big(\sum_{x\in X} f_{k,i}(x)\Big)^2 \geqslant 0. \qquad \square$$

2.3.2. THEOREM.

$$\frac{1}{n^2}\sum_{x,y\in X} (x,y)^k \geqslant \frac{1\cdot 3\cdot\ldots\cdot(k-1)}{d(d+2)\cdots(d+k-2)}$$

for even k, and $\geqslant 0$ for odd k.

PROOF. The proof uses symmetric k-tensors $\otimes^k x$ and their trace inner product

$$\langle \otimes^k x, \otimes^k y\rangle = (x,y)^k,$$

cf. Goethals and Seidel [1979]. Remember that if $x \in \mathbb{R}^d$ has coordinates $x_1,\ldots,x_d$ with respect to any orthonormal basis, then the components of $\otimes^k x$ are the monomials in $x_1,\ldots,x_d$ of degree k. The following norm is non-negative:

$$0 \leqslant \Big|\sum_{x\in X} \otimes^k x - n\int_S \otimes^k \xi \,\mathrm{d}\sigma(\xi)\Big|^2$$

$$= \sum_{x,y\in X}(x,y)^k - 2n\sum_{x\in X}\int_S (x,\xi)^k\,\mathrm{d}\sigma(\xi) + n^2\int_S\int_S (\xi,\eta)^k\,\mathrm{d}\sigma(\xi)\,\mathrm{d}\sigma(\eta).$$

The integral in the second term is constant with respect to x. The assertion follows from

$$\int_S\int_S (\xi,\eta)^k\,\mathrm{d}\sigma(\xi)\,\mathrm{d}\sigma(\eta) = \int_S (x,\xi)^k\,\mathrm{d}\sigma(\xi) = \int_S \xi_1^k\,\mathrm{d}\sigma(\xi),$$

which may be calculated by standard methods. $\square$

2.3.3. COROLLARY. *Equality in Theorem* 2.3.2 *holds iff*

$$\frac{1}{n}\sum_{x\in X}\otimes^k x = \int_S \otimes^k \xi \, \mathrm{d}\sigma(\xi).$$

REMARK. Theorems 2.3.1 and 2.3.2 are not equivalent, neither does

$$\sum Q_k(x,y) = 0$$

imply equality in Theorem 2.3.2. For instance, for the vertex set X of the regular pentagon in $\mathbb{R}^2$ equality in Theorem 2.3.2 holds for $k = 1,2,3,4,6,8$, but not for $k = 5,7,\ k \geqslant 9$. On the other hand,

$$\sum_{x,y\in X} Q_k(x,y) = \sum_{x,y\in X} 2\cos k\phi(x,y) = 0 \quad \text{for } k \not\equiv 0 \pmod 5.$$

There follow eight equivalent definitions for the notion of spherical t-design. We start with the original one, cf. Delsarte, Goethals and Seidel [1977].

2.3.4. DEFINITION. A finite subset $X \neq \varnothing$ of the unit sphere S in $\mathbb{R}^d$ is a *spherical design* of strength t whenever

$$\sum_{x\in X} p(x) = \sum_{x\in X} p(\gamma(x)), \quad \forall\gamma \in O(d),\ \forall p \in \mathrm{Pol}_t(S).$$

2.3.5. THEOREM. *For a finite nonempty subset X of $S \subset \mathbb{R}^d$ the following are equivalent:*

(1) *X is a spherical design of strength t;*

(2) $\displaystyle\frac{1}{n}\sum_{x\in X} p(x) = \int_S p(x)\,\mathrm{d}\sigma(x), \quad \forall p \in \mathrm{Pol}_t(S);$

(3) *the k-th moments of X equal the k-th moments of S, for $k = 1,2,\ldots,t$;*

(4) *equality in Theorem* 2.3.1, *for $k = 1,2,\ldots,t$;*

(5) *equality in Theorem* 2.3.2, *for $k = 1,2,\ldots,t$;*

(6) $\displaystyle\sum_{x\in X} h(x) = 0, \quad \forall h \in \mathrm{Harm}_k(S),\ k = 1,2,\ldots,t;$

(7) $\displaystyle\frac{1}{n}\sum_{x\in X} f(x)g(x) = \langle f,g\rangle \quad$ *for* $fg \in \mathrm{Pol}_t(S)$;

(8) Harm_k *is spanned by* $Q_k(x,\cdot)$, $x \in X$, $k = 1,\ldots,\lfloor t/2\rfloor$.

PROOF. (1) $\Leftrightarrow$ (2): integration over the orthogonal group $O(d)$.

(2) $\Leftrightarrow$ (3): Pol_t is spanned by the moments

$$\xi_1^{k_1}\cdots\xi_d^{k_d}, \quad k_1+\cdots+k_d=k,\ k=1,\dots,t.$$

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5): by Corollary 2.3.3, for $k=1,\dots,t$.

(2) $\Leftrightarrow$ (6): by spherical harmonics, and the vanishing of the integral over S for harmonics.

(2) $\Leftrightarrow$ (7): by specialization.

(7) $\Rightarrow$ (8): Let $f_{k,i}$, $i=1,\dots,Q_k(1)$, be an orthonormal basis of Harm_k, for $k\leqslant t/2$. Then

$$\delta_{ij}=\langle f_{k,i},f_{k,j}\rangle=\sum_{x\in X} f_{k,i}(x)f_{k,j}(x),$$

$$f_{k,i}(\xi)=\sum_{j=1}^{Q_k(1)}\sum_{x\in X} f_{k,i}(x)f_{k,j}(x)f_{k,j}(\xi)=\sum_{x\in X} f_{k,i}(x)Q_k(x,\xi).$$

The converse is easy. □

We give the characteristics and some first examples for spherical designs of small strength. For spherical 1-designs the condition reads $\sum_{x\in X} x=0$, saying that the vectors of X are *balanced*. For $k=2$ the condition says that the n vectors of X form a spherical eutactic star.

2.3.6. DEFINITION. A *eutactic star* of n vectors in $\mathbb{R}^d$ consists of the orthogonal projections of an orthogonal frame in an $\mathbb{R}^n$ which contains $\mathbb{R}^d$ as a subspace.

2.3.7. THEOREM. *Let $0<d<n$. A star of n vectors in $\mathbb{R}^d$ is proportional to a eutactic star iff its Gram matrix has two eigenvalues.*

PROOF. In $\mathbb{R}^n$ consider two orthonormal bases $c_1,\dots,c_n$ and $d_1,\dots,d_n$ with transition matrix T. Let $\mathbb{R}^d:=\langle d_1,\dots,d_d\rangle_{\mathbb{R}}$, and let P denote the orthogonal projection onto $\mathbb{R}^d$, then by definition $p_i:=Pc_i$ form a eutactic star in $\mathbb{R}^d$. The $n\times d$ matrix H having the entries

$$(p_i,d_j)=(Pc_i,d_j)=(c_i,d_j), \quad i=1,\dots,n,\ j=1,\dots,d,$$

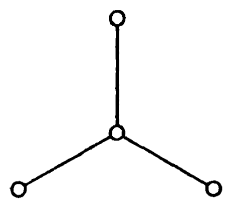
is part of the transition matrix T hence satisfies $H^tH=I_d$. It follows that

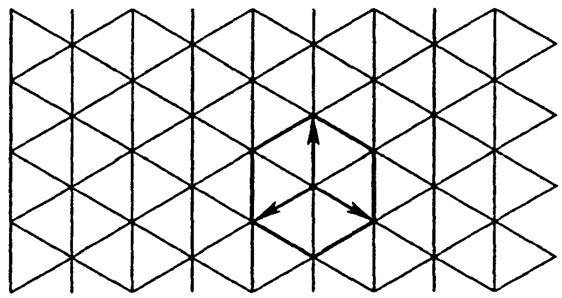
$$\mathrm{Gram}(p_1,\dots,p_n)=HH^t$$

has the spectrum $(1^d,0^{n-d})$. This proves one part of the theorem. The other part is left to the reader, cf. Seidel [1978]. □

In giving 5 examples of eutactic stars we cannot resist mentioning the following. The cubic lattice in $\mathbb{R}^n$ generated by the orthonormal frame projects into $\mathbb{R}^d$ as a lattice in two examples, but densely in three examples. However, if we restrict the lattice points to a strip of $\mathbb{R}^n$ about $\mathbb{R}^d$, then we obtain *aperiodic tilings* of $\mathbb{R}^d$, cf. Grünbaum and Shephard [1987], p. 582.

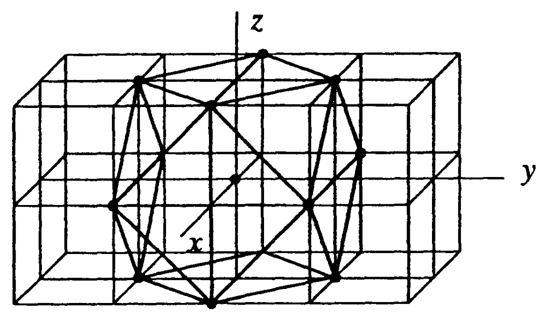
$n=3$, $d=2$, triangular lattice



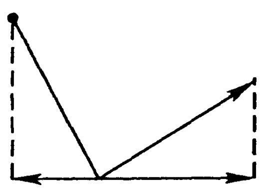


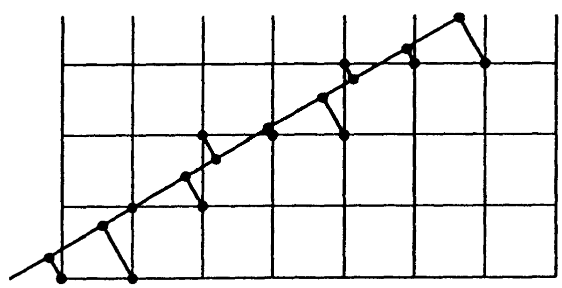
$n=6$, $d=3$, face-centered cubic lattice

$$\begin{bmatrix} 2 & 1 & 1 & 0 & - & 1 \\ 1 & 2 & 1 & 1 & 0 & - \\ 1 & 1 & 2 & - & 1 & 0 \\ 0 & 1 & - & 2 & - & - \\ - & 0 & 1 & - & 2 & - \\ 1 & - & 0 & - & - & 2 \end{bmatrix}$$

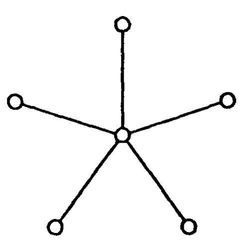


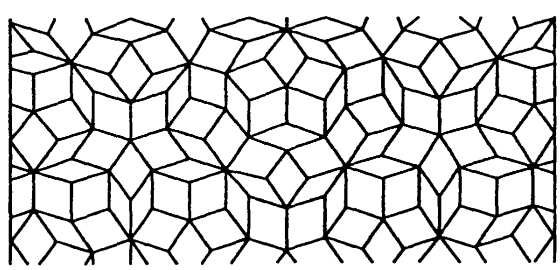
$n=2$, $d=1$, aperiodic sequence





$n=5$, $d=2$, Penrose-tiling $\mathbb{R}^2$



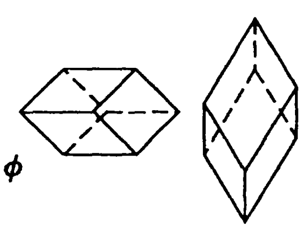


$n=6$, $d=3$, icosahedron tiling $\mathbb{R}^3$

$$\begin{bmatrix} \sqrt{5} & 1 & 1 & 1 & 1 & 1 \\ 1 & \sqrt{5} & 1 & - & - & 1 \\ 1 & 1 & \sqrt{5} & 1 & - & - \\ 1 & - & 1 & \sqrt{5} & 1 & - \\ 1 & - & - & 1 & \sqrt{5} & 1 \\ 1 & 1 & - & - & 1 & \sqrt{5} \end{bmatrix}$$

by rhomboedra

$$-\cos\psi = \frac{1}{\sqrt{5}} = \cos\phi$$



The matrices ($-$ means -1) are the Gram matrices of 6 vectors (one per diagonal) of the cuboctahedron and icosahedron, respectively.

We return to the significance of the condition for $k = 2$ of a spherical t-design. Let the n unit vectors of a set X in $\mathbb{R}^d$ satisfy equality for $k = 2$ in Theorem 2.3.2. Let the Gram matrix G of the inner products of the vectors of X have the nonzero eigenvalues $\Lambda_1, \Lambda_2, \ldots, \Lambda_d$, then

$$\operatorname{trace} G^2 = \sum_{x,y \in X} (x,y)^2 = \frac{n^2}{d} = \Lambda_1^2 + \cdots + \Lambda_d^2,$$

$$\operatorname{trace} G = n = \Lambda_1 + \cdots + \Lambda_d.$$

This implies $\Lambda_1 = \cdots = \Lambda_d$, since the sphere of the first equation is tangent to the flat of the second equation. Hence G has two eigenvalues 0 and n/d. Up to scaling, G is a symmetric idempotent matrix of size n and rank d, with a constant diagonal. Conversely, any such G is the Gram matrix of n spherical vectors which satisfy the condition for $k = 2$. Thus we have proved:

2.3.8. THEOREM. *Equality for $k = 2$ in Theorem 2.3.2 holds iff the vectors of the set X form a spherical eutactic star.*

If, in addition, the vectors are balanced then they form a spherical 2-design. However, our examples show that eutactic stars need not be balanced (and need not be spherical). As an application of this theorem we note that any strongly regular graph gives rise to a spherical 2-design. Indeed,

$$(A - rI)(A - sI) = \mu J, \qquad AJ = kJ,$$

implies that the Gram matrix

$$G := \frac{1}{r-s}\left(A - sI - \frac{k-s}{n} J\right)$$

is idempotent and has zero row sums. Some strongly regular graphs (the Smith graphs, cf. Cameron, Goethals and Seidel [1978]) yield spherical t-designs for $t > 2$. For instance, spherical 4-designs come from the unique strongly regular graphs on 27 and on 275 vertices, and spherical 3-designs from their subconstituents on 16 and on 112, 162 vertices, respectively. We will encounter further examples in the group case.

Whereas the cardinality of a spherical code is bounded from above, that of a spherical design is bounded from below.

2.3.9. THEOREM. *Let X be a spherical s-distance set and a $2e$-design. Then*

$$\dim \operatorname{Pol}_e(S) \leqslant |X| \leqslant \dim \operatorname{Pol}_s(S).$$

Equality on any one side implies equality on the other side.

We prove the first statement, and refer to Delsarte et al. [1977] for the more difficult proof of the second part. Consider the following sets of polynomials in $\xi \in S$:

$$\left\{\sum_{i=0}^{e} Q_i(x,\xi)\colon\ x \in X\right\},\quad \left\{a_x(\xi) := \prod_{\alpha\in A} \frac{(x,\xi)-a}{1-\alpha}\colon\ x \in X\right\}.$$

The first set spans $\mathrm{Pol}_e(S)$ by Theorem 2.3.1 and Theorem 2.3.5(8). The second set consists of independent polynomials in $\mathrm{Pol}_s(S)$, as in Theorem 2.2.1. This proves the inequality. □

Similarly we have:

2.3.10. THEOREM. *Let X be an antipodal spherical $(2e+1)$-design in $\mathbb{R}^d$, having s distances $\neq 0$ and $\neq \pi$. Then*

$$2\dim \mathrm{Hom}_e(S) \leqslant |X| \leqslant 2\dim \mathrm{Hom}_s(S).$$

Equality on any one side implies equality on the other side.

The spherical t-design is called *tight* whenever its cardinality attains the values mentioned in Theorems 2.3.9 and 2.3.10.

For $d = 2$ and any t, a tight t-design is a regular $(t+1)$-gon. For $d = 3$ the icosahedron is the only tight 5-design. For any d, the $d+1$ vertices of a regular simplex provide a tight 2-design, and the $2d$ vertices of the cross-polytope (the generalization of the octahedron) provide a tight 3-design. However, for $d \geqslant 3$ the 2^d vertices of the cube provide a 3-design (not a 4-design), but is not tight. All these statements will easily follow from the next paragraph.

2.3.11. THEOREM. *There exists no tight spherical $2e$-design in $\mathbb{R}^d$ for $d \geqslant 3$ and $e \geqslant 3$. There exists no tight antipodal* (see Brouwer, Cohen and Neumaier [1989]) *spherical $(2e+1)$-design in $\mathbb{R}^d$ for $d \geqslant 3$ and $e \geqslant 4$, except for $d = 24$, $t = 11$.*

This theorem was conjectured in Delsarte et al. [1977] and proved in Bannai and Damerell [1980]. The proof uses analytical and number-theoretical methods in relation with the rationality of the s zeros of the polynomial

$$\prod_{\alpha\in A}(t-\alpha).$$

We refer to the original papers for details.

Tight spherical t-designs exist for t = 2, 3, 4, 5, 7, 11. For each of these values we have encountered already a corresponding spherical code. The design property will follow from Section 3.4. For the relations between spherical designs and distance regular graphs we refer to Brouwer et al. [1989]. For generalizations and different approaches we mention also Godsil [1988] and Brouwer et al. [1989]. Finally we state the following existence theorem by Seymour and Zaslavsky [1984].

2.3.12. THEOREM. *For any positive integers d and t, and for all sufficiently large n, there exist spherical t-designs of size n in $\mathbb{R}^d$.*

This theorem is not constructive. It was proved in a general context and has many further consequences. Essentially, it is a deep generalization of the mean value theorem. Several new constructions will be published soon (B. Bojnak, N.J.A. Sloane).

2.4. *The group case*

Following Sobolev [1962], we let our spherical set X be an orbit under a finite subgroup Γ of the orthogonal group O(d) of $\mathbb{R}^d$. We investigate the relations between these groups and their orbits as spherical designs, cf. Goethals and Seidel [1979].

The *orbit* of $x_0 \in S$ under $\Gamma \subset \mathrm{O}(d)$ is the set $X_0 = x_0^\Gamma := \{\gamma(x_0)\colon\ \gamma \in \Gamma\}$.

On a function f, any $\gamma \in \Gamma$ acts as follows: $\gamma\colon\ f \mapsto f^\gamma$, where $f^\gamma(x) := f(\gamma(x))$.

A function $f\colon\ S \to \mathbb{R}$ is *Γ-invariant* whenever $f^\gamma = f$ for all $\gamma \in \Gamma$. For a given f its average over Γ

$$f^\Gamma = \operatorname*{ave}_\Gamma f := |\Gamma|^{-1} \sum_{\gamma\in\Gamma} f^\gamma$$

is a Γ-invariant function. Trivially, every Γ-invariant function is an average over Γ. The invariant polynomials of the linear space $\mathrm{Harm}_k(S)$ form a linear subspace, which we denote by $\mathrm{Harm}_k^\Gamma(S)$.

2.4.1. LEMMA. *The average of f over an orbit of Γ equals the average of f over Γ on the orbit.*

PROOF.

$$\frac{1}{|X_0|} \sum_{x\in X_0} f(x) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} f\big(\gamma(x_0)\big)$$

$$= \frac{1}{|\Gamma|} \Big(\sum_{\gamma\in\Gamma} f^\gamma \Big)(x_0) = \big(\operatorname*{ave}_\Gamma f\big)(x_0) =: f^\Gamma(x_0).$$

□

2.4.2. THEOREM. *The orbit x_0^Γ is a spherical t-design iff*

$$h(x_0) = 0 \quad \textit{for all } h \in \mathrm{Harm}_k^\Gamma(S),\ k = 1, 2, \ldots, t.$$

2.4.3. THEOREM. *Every Γ-orbit is a spherical t-design iff* $\mathrm{Harm}_k^\Gamma(S) = 0$ *for* $k = 1, 2, \ldots, t$.

PROOF. The theorems follow from criterion (6) of Theorem 2.3.5 for spherical t-designs, and the translation to the group case of Lemma 2.4.1. □

Theorem 2.4.3 invites to investigate groups Γ having no invariant harmonic polynomials of certain degrees, and justifies the following definition due to Bannai [1984].

2.4.4. Definition. A finite $\Gamma < \mathrm{O}(d)$ is *t-homogeneous* whenever each orbit is a spherical t-design.

However, Theorem 2.4.2 shows that this does not tell the whole story. For instance, if $\dim \mathrm{Harm}_k^{\Gamma}(S) = 1$ then there still are $x_0 \in \Gamma$ on whose orbit the harmonic polynomials of degree k vanish. Indeed,

2.4.5. Lemma. *Any harmonic polynomial has a zero on* S.

Proof. The integral over S of any harmonic polynomial h vanishes. Therefore, if h takes on S positive values, then h also takes on S negative values, hence the value 0. □

We will discuss an easy criterion for the application of Theorem 2.4.3. It is the harmonic version of the classical Molien–Poincaré series, cf. Conway and Sloane [1988], and it follows from that formula by application of Theorem 2.1.1, cf. Goethals and Seidel [1979].

2.4.6. Theorem.

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{1-\Lambda^2}{\det(I-\Lambda\gamma)}=\sum_{k\geqslant 0}h_k\Lambda^k$$

for finite $\Gamma < \mathrm{O}(d)$ *and* $h_k := \dim \mathrm{Harm}_k^{\Gamma}(S)$.

2.4.7. Example. The icosahedral group Γ in $\mathbb{R}^3$ has the Molien series

$$\frac{1}{(1-\Lambda^6)(1-\Lambda^{10})}=1+\Lambda^6+\Lambda^{10}+\cdots.$$

Using Theorem 2.4.6 we read off that the icosahedral group is 5-homogeneous, i.e. every orbit is a spherical 5-design. This applies, e.g., to the icosahedron, the dodecahedron, the icosidodecahedron, the football (truncated icosahedron); none of these is a spherical 6-design. We also read off that $\dim \mathrm{Harm}_6^{\Gamma}(S) = 1$. Let the harmonic polynomial h_6 generate this linear space, and let x_0 denote a zero of h_6. Then x_0^{Γ} is a spherical 9-design, which implies an improvement of the football, cf. Goethals and Seidel [1981b].

For the time being we will restrict to real *finite reflection groups*, referring to Conway and Sloane [1988], Flatto [1978], Goethals and Seidel [1981a]. Let Γ denote an irreducible finite group generated by reflections in $\mathbb{R}^d$. The ring R^{Γ} of the Γ-invariant polynomials on $\mathbb{R}^d$ has the following characteristic property. R^{Γ} has an algebraic basis consisting of d homogeneous polynomials, called *basic invariants*, of degrees

$1 + m_i$, $i = 1, \ldots, d$. The *exponents* $1 = m_1 \leqslant m_2 \leqslant \cdots \leqslant m_d$ are the logarithms in the base $\exp 2\pi i/h$, of the eigenvalues of the Coxeter–Killing transformations, and

$$m_i + m_{d+1-i} = h = \frac{2r}{d}, \qquad \sum_{i=1}^{d} m_i = r,$$

where h is the period of that transformation and r is the total number of reflections. Coxeter [1934] classified the finite reflection groups in terms of the root systems Φ with Weyl groups $W(\Phi)$, and calculated the exponents m_i as shown in the following table (we shall come back to the root systems in Sections 3.4 and 4.3).

Φ	$r = \lvert\Phi\rvert/2$	$\lvert W(\Phi)\rvert$	h	m_i
A_d $(d \geqslant 1)$	$d(d+1)/2$	$(d+1)!$	$d+1$	$1, 2, \ldots, d$
B_d $(d \geqslant 2)$	d^2	$2^d d!$	$2d$	$1, 3, \ldots, 2d-1$
D_d $(d \geqslant 4)$	$d(d-1)$	$2^{d-1} d!$	$2d-2$	$1, 3, \ldots, 2d-3, d-1$
I_2^p $(p \geqslant 5)$	p	$2p$	p	$1, p-1$
G_2	6	12	6	1,5
H_3	15	120	10	1,5,9
F_4	24	$2^7 \cdot 3^2$	12	1,5,7,11
H_4	60	120^2	30	1,11,19,29
E_6	36	$2^7 \cdot 3^4 \cdot 5$	12	1,4,5,7,8,11
E_7	63	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	18	1,5,7,9,11,13,17
E_8	120	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	30	1,7,11,13,17,19,23,29

The exponents serve to calculate $h_i = \dim \mathrm{Harm}_i^{\Gamma}(S)$ by use of the following theorem, which we already applied in Example 2.4.7.

2.4.8. THEOREM. *For a finite reflection group the harmonic Molien–Poincaré series reads*:

$$\sum_{i=0}^{\infty} h_i \Lambda^i = \prod_{i=2}^{d} \left(1 - \Lambda^{1+m_i}\right)^{-1}.$$

Once the dimensions and the degrees of Harm_i^{Γ} are known, it remains to determine the invariant harmonic polynomials themselves. To that end we mention the following theorem by Flatto [1978].

2.4.9. THEOREM. *For a finite reflection group $\Gamma \neq \mathsf{D}_{2d}$, let*

$$P_k(x, y) := |\Gamma|^{-1} \sum_{\gamma \in \Gamma} (x, \gamma(y))^k.$$

Then

$$\det\left[\frac{\partial(P_{1+m_1}, \ldots, P_{1+m_d})}{\partial(\xi_1, \ldots, \xi_d)}\right] = \prod_{i=1}^{d} J_i(y) \prod_{j=1}^{r} L_j(x),$$

where $L_j(x) = 0$ denote the reflecting hyperplanes, and $J_1(y), \ldots, J_d(y)$ are the unique (up to constants) basic invariants, satisfying

$$J_1(y) = (y, y), \qquad J_k(\partial/\partial y)J_l(y) = 0 \quad \text{for } 1 \leqslant k < l \leqslant d.$$

Clearly, the unique basic invariants $J_1, \ldots, J_d$ are harmonic. Theorem 2.4.9 implies that, provided $y \in S$ satisfies $\prod J_i(y) \neq 0$, a set of basic invariants is given by the polynomials

$$P_{1+m_1}(x, y), \ldots, P_{1+m_d}(x, y), \quad x \in \mathbb{R}^d.$$

We now give some examples of the above. The full symmetry groups of the tetrahedron, octahedron and icosahedron in $\mathbb{R}^3$ (the binary polyhedral groups) are the Weyl groups of the root systems A_3, B_3, H_3, of orders 24, 48, 120, respectively. Their harmonic Molien series are

$$\frac{1}{(1-\Lambda^3)(1-\Lambda^4)}, \qquad \frac{1}{(1-\Lambda^4)(1-\Lambda^6)}, \qquad \frac{1}{(1-\Lambda^6)(1-\Lambda^{10})}.$$

As a consequence, the tetrahedron, octahedron, icosahedron are spherical designs of strength 2, 3, 5, respectively.

The $120 = 96 + 8 + 16$ vertices of the 600-cell $\{3,3,5\}$ (a regular polytope in $\mathbb{R}^4$) are represented by the even permutations of

$$\frac{1}{2}(0, \pm 1, \pm\tau, \pm\tau^{-1}), \quad (\pm 1, 0, 0, 0), \quad \text{and} \quad \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1).$$

The Weyl group $W(\mathsf{H}_4)$ is isomorphic to $2 \cdot (A_5 \times A_5) \cdot 2$. The basic invariants have degrees 2, 12, 20, 30, hence the harmonic Molien series reads

$$1 + \Lambda^{12} + \Lambda^{20} + \Lambda^{24} + \Lambda^{30} + \cdots.$$

As a consequence, $(A_5 \times A_5) \cdot 2$ is 11-homogeneous, and the orbits of the zeros of the basic invariant of degree 12 provide spherical 19-designs in $\mathbb{R}^4$.

The Weyl groups $W(\mathsf{E}_6)$, $W(\mathsf{E}_7)$, $W(\mathsf{E}_8)$ are the groups of symmetries of Gosset's polytopes in $\mathbb{R}^6$, $\mathbb{R}^7$, $\mathbb{R}^8$. The first few terms in the harmonic Molien series for these groups are as follows

$$1 + \Lambda^5 + \Lambda^6 + \Lambda^8 + \cdots \quad \text{for } W(\mathsf{E}_6),$$
$$1 + \Lambda^6 + \Lambda^8 + \Lambda^{10} + \cdots \quad \text{for } W(\mathsf{E}_7),$$
$$1 + \Lambda^8 + \Lambda^{12} + \Lambda^{14} + \cdots \quad \text{for } W(\mathsf{E}_8).$$

Therefore, these groups are t-homogeneous for $t = 4, 5, 7$, respectively.

Our method to obtain spherical t-designs is not restricted to reflection groups. It works as soon as we know the Molien series. For instance, not of Coxeter-type is the Conway group $\cdot 0$, the group of all orthogonal transformation in $\mathbb{R}^{24}$ which preserve the Leech lattice, cf. Section 2.2. The first few terms of the harmonic Molien series are, Conway and Sloane [1988],

$$1 + \Lambda^{12} + \Lambda^{16} + \Lambda^{18} + \Lambda^{20} + \Lambda^{22} + 3\Lambda^{24} + \cdots.$$

Hence $\cdot 0$ acts 11-homogeneously, and produces spherical 15-designs.

We continue with some existence questions for spherical designs. Where Seymour and Zaslavski in Theorem 2.3.12 showed an abundance of spherical t-designs in $\mathbb{R}^d$, Bannai tells us that in the group case one should be more modest. In Bannai [1984] he proves the following theorems.

2.4.10. THEOREM. *If $\Gamma < \mathrm{O}(d)$ is t-homogeneous and $d \geqslant 3$, then $t \leqslant f(d)$, for a certain function $f(d)$ depending on d only.*

Bannai believes that t is bounded by an absolute constant, not depending on d. He also proves:

2.4.11. THEOREM. *If x_0^Γ is a spherical t-design for some $x_0 \in S$, then Γ is $\lceil t/2 \rceil$-homogeneous.*

The proof is performed by comparison of orbits.

2.4.12. THEOREM. *If x_1^Γ is a t_1-design, not a (t_1+1)-design, then x_2^Γ is a design of strength $\leqslant 2t_1+1$.*

Bannai gives two proofs. We reproduce the one which uses spherical harmonics. Since x_1^Γ is not a (t_1+1)-design, there exists $f \in \mathrm{Harm}_{t_1+1}^\Gamma$ with $f(x_1) \neq 0$. Suppose that, contrary to the claim, x_2^Γ is a $(2t+2)$-design, then

$$h(x_2) = 0, \quad \forall h \in \mathrm{Harm}_{2i}^\Gamma,\ 0 < 2i \leqslant 2t_1 + 2.$$

Now consider the square of f, a homogeneous polynomial of degree $2t_1+2$, and write by use of $h_{2i} \in \mathrm{Harm}_{2i}$

$$f^2 = h_{2t_1+2} + (x,x)h_{2t_1} + (x,x)^2 h_{2t_1-2} + \cdots + (x,x)^{t_1} h_2 + (x,x)^{t_1+1} h_0.$$

Since $f^2(x_1) \neq 0$ we have

$$h_0 = \int_S f^2(x)\,\mathrm{d}\sigma(x) > 0.$$

Since $f(x_2) = 0$, also $f^2(x_2) = 0$.

In the formula, take averages over Γ, then

$$f^2(x_2) = 0, \quad h_0 > 0, \quad \overline{h}_{2i}(x_2) = 0,\ 0 < 2i \leqslant 2t_1 + 2,$$

provide a contradiction. □

Finally, we discuss the relations between representations of Γ and the spherical designs generated by Γ.

A *representation* of a group Γ on a real vector space V is a homomorphism $\rho\colon \Gamma \to \mathrm{GL}(V)$. The representation is irreducible if V contains no proper Γ-invariant subspace. Thus, irreducibility means *real irreducibility*. Let Γ denote a finite subgroup of $\mathrm{O}(d)$, and let ρ_k denote its representation on Harm_k. With respect to any orthonormal basis $\{f_{k,i}\colon i = 1, \ldots, Q_k(1)\}$ of Harm_k, any $\gamma \in \Gamma$ is represented by an orthogonal matrix $A^k(\gamma)$ whose entries $A_{i,j}^k(\gamma)$ are given by

$$f_{k,j}(\gamma(x)) = \sum_{i=1}^{Q_k(1)} A_{ij}^k(\gamma) f_{k,i}(x),$$

for $x \in S$, $\gamma \in \Gamma$. Clearly, $A^k(\gamma^{-1})$ is the transposed of $A^k(\gamma)$. By (7) of Theorem 2.3.5 every Γ-orbit in S is a spherical t-design iff

$$\sum_{\gamma \in \Gamma} f_{k,j}(\gamma(x)) f_{l,\nu}(\gamma(x)) - |\Gamma| \delta_{k,l} \delta_{j,\nu}$$

vanishes for each $x \in S$, and relevant k, l, j, ν. By substitution, and by use of

$$\sum_{i=1}^{Q_k} \sum_{\mu=1}^{Q_k} f_{k,i}(x) f_{k,\mu}(x) \delta_{i,\mu} = \sum_{i=1}^{Q_k} f_{k,i}^2(x) = Q_k(1)$$

we arrive at the following lemma.

2.4.13. LEMMA. *Every Γ-orbit in S is a spherical t-design iff for each $\xi \in S$, for $k = l = \lfloor k/2 \rfloor$ and for $k = \lfloor t/2 \rfloor$, $l = \lfloor t/2 \rfloor - (-1)^t$, for $j = 1, \ldots, Q_k(1)$, for $\nu = 1, \ldots, Q_l(1)$ we have*

$$\sum_{i=1}^{Q_k(1)} \sum_{\mu=1}^{Q_l(1)} [k, l; j, \nu; i, \mu] f_{k,i}(x) f_{l,\mu}(x) = 0,$$

where

$$[k, l; j, \nu; i, \mu] := \sum_{\gamma \in \Gamma} A_{i,j}^k(\gamma) A_{\mu,\nu}^l(\gamma) - \frac{|\Gamma|}{Q_k(1)} \delta_{k,l} \delta_{j,\nu} \delta_{i,\mu}.$$

2.4.14. LEMMA.

$[k, l; j, \nu; i, \mu] = 0$ *if ρ_k and ρ_l are irreducible, $k \neq l$, $d > 2$;*

$[k, k; j, \nu; i, \mu] + [k, k; j, \nu; \mu, i] = 0$ *if ρ_k is irreducible;*

$[k, k; j, \nu; i, \mu] = 0$ *if ρ_k is absolutely irreducible.*

These formulae follow from well-known facts about real irreducible representations, cf. Goethals and Seidel [1981a].

2.4.15. THEOREM. *If the representations ρ_k on* Harm$_k$ *of Γ are irreducible, for $k \geqslant s$, then Γ is $2s$-homogeneous. If in addition no ρ_k has common constituents with ρ_{2s+1}, then Γ is $(2s+1)$-homogeneous.*

PROOF. If the ρ_k are irreducible, then Lemma 2.4.14 implies that the conditions of Lemma 2.4.13 are satisfied. Indeed, for $k \neq l$ the coefficients vanish, and for $k = l$ the form vanishes since the coefficients form a skew symmetric matrix in i and μ. Hence every Γ-orbit is a spherical $2s$-design. The second hypothesis implies that these conditions also hold for $k = e$, $l = e + 1$, since ρ_{e+1} is a direct sum of irreducible representations, each of which is distinct from ρ_e. Therefore, every Γ-orbit is a $(2e+1)$-design. □

2.4.16. REMARK. The converse of the first statement of Theorem 2.4.15 was claimed and proved in Theorem 6.7 of Goethals and Seidel [1981a]. However, Bannai [1984] convincingly demonstrated the falsity of both the statement and its proof, by counterexamples involving the unitary subgroup U(d) of O($2d$).

2.5. *Other types of designs*

Definition 2.3.5(2) of a spherical t-design amounts to the *approximation* of all of the unit sphere $S \subset \mathbb{R}^d$ by a finite subset $X \subset S$ such that for the averages we have

$$\operatorname*{ave}_X f = \operatorname*{ave}_{\text{all}} f \quad \text{for } f \in \mathrm{Pol}_t.$$

This definition also holds for a t-(v, k, Λ) design, cf. Seidel [1990]. Indeed, interpret the elements of a v-set V as the Boolean variables $x_1, \ldots, x_v$ which only take values belonging to $\mathbb{B} = \{0, 1\}$; the collection of all k-subsets of V as the flat

$$F := \{x \in \mathbb{B}^v\colon\ x_1 + \cdots + x_v = k\};$$

and Pol_t as the square-free polynomials in $x_1, \ldots, x_v$ of degree $\leqslant t$.

Analogous definitions hold for various generalizations, such as t-wise balanced designs t-(v, K, Λ) with several blocksizes $\in K$, Euclidean t-designs in several concentric spheres, lattices of strength t with infinitely many concentric spheres, and measures of strength in $\mathbb{R}^d$. We refer to Neumaier and Seidel [1988, 1992] and Delsarte and Seidel [1989] for these notions, for Fisher-type inequalities, and for relations to optimal experimental designs. For Euclidean space $\mathbb{R}^d$ we mention the most general definition of a measure of strength t, from which the others follow as special cases.

In $\mathbb{R}^d$, with the orthogonal group O(d), any union of concentric spheres is denoted by

$$RS := \bigcup_{r \in R} rS, \quad rS = \{x \in \mathbb{R}^d\colon\ (x, x) = r^2\}, \quad r \in R \subset \mathbb{R}.$$

This notation covers the unit sphere S, a union of p concentric spheres, the unit ball, the whole space $\mathbb{R}^d$, etc. We use σ for the standard Borel measure on the unit sphere S.

2.5.1. DEFINITION. A *measure* on RS has *strength* t whenever any one of the following equivalent conditions holds for all $f = \sum_{k=0}^{t} f_k \in \mathrm{Pol}_t(RS)$, $f_k \in \mathrm{Hom}_k$:

(i) $$\int_{RS} f\, \mathrm{d}\xi = \int_{RS} f\, \mathrm{d}\xi \circ \gamma \quad \forall \gamma \in \mathrm{O}(d);$$

(ii) $$\int_{RS} f\, \mathrm{d}\xi = \int_{RS} f\, \mathrm{d}\bar{\xi}, \quad \bar{\xi} = \int_{\mathrm{O}(d)} \xi \circ \gamma\, \mathrm{d}\gamma;$$

(iii) $$\int_{RS} f(y)\, \mathrm{d}\xi(y) = \sum_{k=0}^{t} \mu_k f_k(x)\, \mathrm{d}\sigma(x), \quad \mu_k = \int_{RS} \|y\|^k\, \mathrm{d}\xi(y).$$

The equivalence (i) $\Leftrightarrow$ (ii) is obvious, for the O(d)-invariant measure $\bar{\xi}$. The equivalence with (iii) follows from

$$\int_{RS} f\,\mathrm{d}\xi = \int_{\mathrm{O}(d)} \mathrm{d}\gamma \int_{RS} f\,\mathrm{d}\xi \circ \gamma = \int_{RS} \mathrm{d}\xi(y) \int_{\mathrm{O}(d)} f(\gamma^{-1}y)\,\mathrm{d}\gamma$$

$$= \sum_{k=0}^{t} \int_{RS} \|y\|^k \,\mathrm{d}\xi(y) \int_{\mathrm{O}(d)} f\left(\frac{\gamma^{-1}y}{\|y\|}\right) \mathrm{d}\gamma$$

$$= \sum_{k=0}^{t} \mu_k \int_S f_k(x)\,\mathrm{d}\sigma(x).$$

A *Euclidean t-design* is a measure of strength t having finite support. A *spherical t-design* has finite support on the unit sphere, with equal weights.

3. Elliptic geometry

3.0. *Summary*

Elliptic geometry is projective geometry provided with a metric. Like spherical geometry, elliptic geometry may be represented by use of the unit sphere. However, there are important distinctions between spherical and elliptic geometry. In spherical geometry two points need not define a unique line, and in elliptic geometry isometry of triangles need not imply congruence, properties which the other geometry does possess. In fact, elliptic space is a double cover of the corresponding spherical space. The main interest of elliptic geometry lies in its large equidistant pointsets, which for instance provide a setting for the 2-transitive representation of certain groups.

The metric specialties of the elliptic plane (Section 3.1) include equidistant 4-sets and 6-sets, represented by the diameters of the cube and the icosahedron, respectively. Thus we introduce the switching classes of graphs and the two-graphs of Section 3.2, with enumeration and cohomology. For the regular two-graphs of Section 3.3 a number of constructions based on finite geometries is explained. More generally, few-distance sets in elliptic geometry are dealt with in Section 3.4, in particular the root systems and sets related to the Leech lattice. A general reference is Seidel [1975].

3.1. *The elliptic plane*

In $\mathbb{R}^3$, with positive definite inner product, we define the *elliptic points* to be the 1-subspaces, the *elliptic lines* to be the 2-subspaces, and the *elliptic distance* $d(\langle a\rangle, \langle b\rangle)$ to be the angle between the corresponding 1-subspaces $\langle a\rangle$ and $\langle b\rangle$. Thus, cf. Section 1.4, the *elliptic plane* I^2 consists of the elliptic points and lines and the incidence relation of the underlying vector space. An easy theorem, demonstrating that parallelism does not exist in the elliptic plane, is the following.

3.1.1. THEOREM. *Two distinct elliptic points are on a unique elliptic line. Two distinct elliptic lines intersect in a unique elliptic point.*

3.1.2. THEOREM. *The elliptic plane is a metric space.*

PROOF. Since $d(\langle a\rangle, \langle a\rangle) = 0$,

$$d(\langle a\rangle, \langle b\rangle) = d(\langle b\rangle, \langle a\rangle) \geqslant 0$$

is trivial, we only have to prove the triangle inequality. Without loss of generality we put $(a, a) = (b, b) = (c, c) = 1$ and

$$d(\langle a\rangle, \langle b\rangle) = r, \qquad d(\langle a\rangle, \langle c\rangle) = q,$$

$$d(\langle b\rangle, \langle c\rangle) = p, \qquad 0 \leqslant r \leqslant q \leqslant p \leqslant \frac{1}{2}\pi.$$

Then the following holds:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & \pm\cos r & \pm\cos q \\ \pm\cos r & 1 & \pm\cos p \\ \pm\cos q & \pm\cos p & 1 \end{bmatrix},$$

$$D^+ := \begin{vmatrix} 1 & \cos r & \cos q \\ \cos r & 1 & \cos p \\ \cos q & \cos p & 1 \end{vmatrix} \geqslant 0,$$

or

$$D^- := \begin{vmatrix} 1 & -\cos r & -\cos q \\ -\cos r & 1 & -\cos p \\ -\cos q & -\cos p & 1 \end{vmatrix} \geqslant 0,$$

or both. Equality holds iff a, b, c are linearly dependent. Elementary calculation factors the first and the second determinant into

$$D^+ = 4\sin\frac{1}{2}(p+q+r)\sin\frac{1}{2}(-p+q+r)\sin\frac{1}{2}(p-q+r)\sin\frac{1}{2}(p+q-r),$$

$$D^- = -4\cos\frac{1}{2}(p+q+r)\cos\frac{1}{2}(-p+q+r)\cos\frac{1}{2}(p-q+r)\cos\frac{1}{2}(p+q-r).$$

This yields the following implications:

$$D^+ \geqslant 0 \Rightarrow -p+q+r \geqslant 0;$$

$$D^- \geqslant 0 \Rightarrow p+q+r \geqslant \pi;$$

$$D^- \geqslant 0 \Rightarrow D^+ \geqslant 0.$$

Hence the triangle inequality holds in any case. However, it can (and does) happen that both $D^+ \geqslant 0$ and $D^- < 0$ hold. □

3.1.3. THEOREM. *Isometry of triples in I^2 does not imply their congruence.*

PROOF. In the elliptic plane there exist noncongruent triples of points having the same distances. Such triples occur if $D^+ > 0$ and $D^- \geqslant 0$, with distances $\neq \pi/2$. Here is an example, made from the following two triples of vectors in $\mathbb{R}^3$ (together forming the root system D_3):

$$(0,1,1),\ (1,0,1),\ (1,1,0); \quad (0,1,-1),\ (-1,0,1),\ (1,-1,0);$$

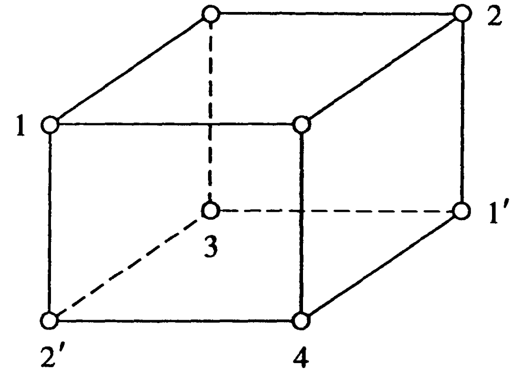
$$D^+ = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} > 0; \quad D^- = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0.$$

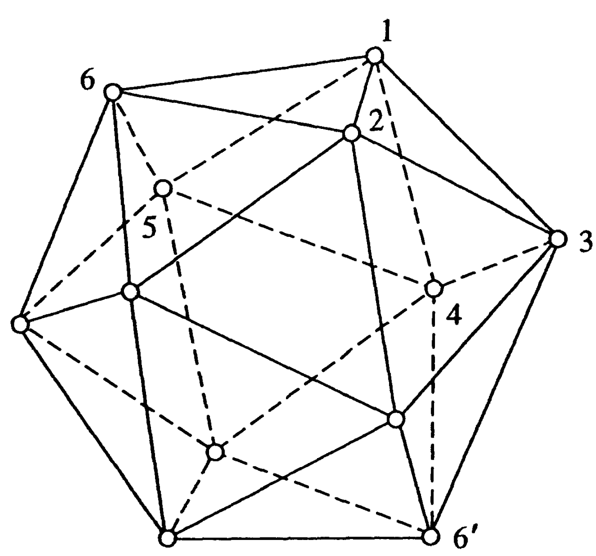
The triples are not congruent (one is collinear, the other is not), but both have three distances equal to $\pi/3$. More generally, two noncongruent elliptic triangles can be made from positive distances p, q, r, not $\pi/2$, with sum $\geqslant \pi$. However, if $p+q+r < \pi$ then all elliptic triangles having distances p, q, r are congruent. □

A further deviation from Euclidean geometry is provided by the following.

3.1.4. THEOREM. *I^2 contains equidistant 4-tuples, 5-tuples, 6-tuples; no 7-tuples.*

PROOF. The 4 diameters of the *cube* are equiangular with $\cos\phi = 1/3$. The 6 diameters of the *icosahedron* are equiangular with $\cos\psi = 1/\sqrt{5}$. It was proved by Haantjes that one cannot go beyond 6; this will also become clear from the rest of the present chapter. □





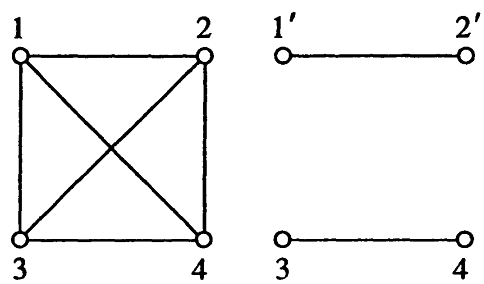
In the two above mentioned examples we select one vector along each diameter, of norm 3 for the cube, of norm $\sqrt{5}$ for the icosahedron. An appropriate choice yields the following Gram matrices ($-$ stands for -1, $+$ stands for 1):

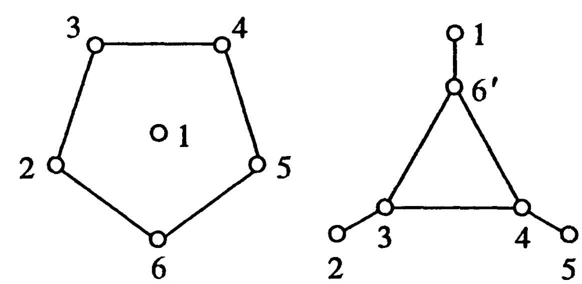
$$\begin{bmatrix} 3 & - & - & - \\ - & 3 & - & - \\ - & - & 3 & - \\ - & - & - & 3 \end{bmatrix}, \qquad \begin{bmatrix} \sqrt{5} & + & + & + & + & + \\ + & \sqrt{5} & + & - & - & + \\ + & + & \sqrt{5} & + & - & - \\ + & - & + & \sqrt{5} & + & - \\ + & - & - & + & \sqrt{5} & + \\ + & + & - & - & + & \sqrt{5} \end{bmatrix}.$$

By different choices we get different matrices, e.g., by switching the first and second vectors of the cube, and the sixth vector of the icosahedron, into their opposite vector, we obtain:

$$\begin{bmatrix} 3 & - & + & + \\ - & 3 & + & + \\ + & + & 3 & - \\ + & + & - & 3 \end{bmatrix}, \qquad \begin{bmatrix} \sqrt{5} & + & + & + & + & - \\ + & \sqrt{5} & + & - & - & - \\ + & + & \sqrt{5} & + & - & + \\ + & - & + & \sqrt{5} & + & + \\ + & - & - & + & \sqrt{5} & - \\ + & + & - & - & + & \sqrt{5} \end{bmatrix}.$$

Viewing these matrices as the $(-,+)$ adjacency matrices of graphs, we obtain two examples of *switching of graphs*:





The icosahedron provides a preliminary example for yet another notion that will interest us. Its 6 diameters have an automorphism group which is isomorphic to the alternating group on 5 symbols, of order 60. This group acts 2-transitively on the diameters, but not 3-transitively. Indeed, there are two types of triples of diameters (of elliptic points), namely those who carry a sharp, and those who carry a flat triple of vectors. The two types are represented by the Gram matrices of their spanning vectors as follows:

$$\frac{\text{sharp}}{\cos\phi = \frac{1}{\Lambda}} \begin{bmatrix} \Lambda & + & + \\ + & \Lambda & + \\ + & + & \Lambda \end{bmatrix}, \qquad \frac{\text{flat}}{\cos\psi = \frac{-1}{\Lambda}} \begin{bmatrix} \Lambda & - & - \\ - & \Lambda & - \\ - & - & \Lambda \end{bmatrix}.$$

For the cube all triples are flat. For the icosahedron there are 10 flat and 10 sharp triples. For each 4 elliptic points there is an even number of sharp triples, and an even number of flat triples. We refer the reader to the graphs and matrices above, and also to the quasi-crystals of Remark 2.3.8. The diameters of the icosahedron provide a first example for the notion of a *regular two-graph*, cf. Section 3.3.

Finally, the equidistant sextuple in I^2 given by the diameters of the icosahedron provides the ultimate counterexample in a question of metric embeddability.

Any metric space M is metrically embeddable in the Euclidean plane whenever each 5-subset of M is. This has been proved by Menger [1928], who called the number 5 (which cannot be reduced to 4) the *congruence order* of the Euclidean plane. Menger also proved that any M is metrically embeddable in Euclidean d-space whenever each $(d+3)$-subset of M is. Since $d+3$ is the smallest number with this property, it is called the congruence order of Euclidean d-space. Analogously, Blumenthal [1953] has determined the congruence order of hyperbolic d-space and spherical d-space to be $d+3$. However, for elliptic geometry the situation is completely different. For the elliptic plane the congruence order cannot be 5 but must be $\geqslant 7$. Indeed, let the metric space M consist of 100 points all of whose distances are $\arccos 1/\sqrt{5}$. Then each 6-subset of M is metrically embeddable in I^2, but M itself is not. It has been proved by Haantjes and Seidel [1947] that the congruence order of the elliptic plane equals 7. That of I^d for $d \geqslant 3$ is unknown, and turns the attention to equidistant sets of points in I^d.

3.2. *Equidistant sets in elliptic* $(d-1)$*-space*

In $\mathbb{R}^d$ with positive definite inner product an elliptic $(k-1)$-space I^{k-1} is defined to be a k-subspace, for $k = 1, 2, \ldots, d$. With the incidence of the underlying vector space this defines *elliptic space* I^{d-1} of dimension $(d-1)$, in which the elliptic distance between two points is defined as before. We shall be interested in few-distance sets in I^{d-1}, in particular in equidistant sets. Graph theory is useful to construct them.

For a *graph* G on n vertices, its $(+,-)$ *adjacency matrix* C is defined by its entries $c_{ii} = 0$, and $c_{ij} = -1$ if the vertices $i \neq j$ are adjacent, $c_{ij} = 1$ if they are nonadjacent. Thus C is a symmetric matrix with zero diagonal of size n. Let C have smallest eigenvalue $-\lambda$ of multiplicity $n-d$, say. Then $\lambda I + C$ has smallest eigenvalue 0 of multiplicity $n-d$, and rank d. Hence we can write $\lambda I + C = HH^t$, with $n \times d$ matrix H. The rows of H form n vectors in $\mathbb{R}^d$ whose Gram matrix equals $\lambda I + C$, hence which have norm λ and inner products ± 1. So we have n vectors in $\mathbb{R}^d$ at $\cos\phi = \pm 1/\lambda$. The n lines in $\mathbb{R}^d$ spanned by these vectors have $\cos\phi = 1/\lambda$, hence are *equiangular*, i.e. the angle between each pair of lines is the same. In other words, starting from our graph G on n vertices we have constructed n elliptic points in I^{d-1} having distances $\arccos 1/\lambda$.

Conversely, given n equiangular lines at $\cos\phi = 1/\lambda$ in $\mathbb{R}^d$, we construct a switching class of graphs on n vertices as follows. Select n vectors of norm λ, one along each line. The Gram matrix of such a selection has diagonal λ and off-diagonal entries ± 1. Deleting the diagonal we have the $(+,-)$ adjacency matrix C of a graph G. A different selection of vectors yields a different graph G', with adjacency matrix

$$C' = DCD, \quad D = \operatorname{diag}(\delta_1, \ldots, \delta_n),\ \delta_i = \pm 1.$$

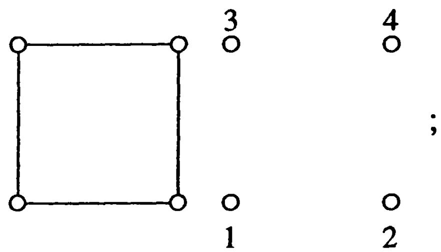
In terms of (1, 0) adjacency matrices this reads

$$A' = A + B_{p,q} \ (\text{mod } 2); \quad B_{p,q} := P^t \begin{bmatrix} O_p & J \\ J & O_q \end{bmatrix} P,$$

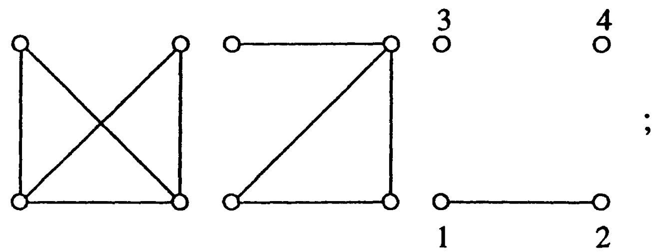
with $p + q = n$ and $n \times n$ permutation matrix P. So $B_{p,q}$ denotes a complete bipartite graph. We say that the graphs G and G' are *switching equivalent.* We refer to the two examples of Section 3.1, where switching of graphs is performed with respect to the vertices 1 and 2, and with respect to vertex 6, respectively. Clearly, switching of graphs is an equivalence relation. Thus we arrive at the following theorem by Van Lint and Seidel [1966].

3.2.1. THEOREM. *There is a one-to-one correspondence between equidistant point sets in elliptic space and switching classes of graphs.*

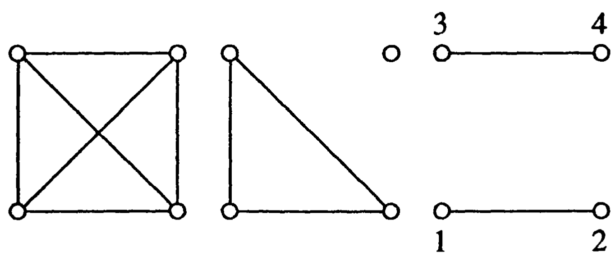
3.2.2. EXAMPLE. There are 3 switching classes of graphs on 4 vertices:



$\Delta = \varnothing$;



$\Delta = \{123, 124\}$;



$\Delta = \{123, 124, 134, 234\}$.

In each class we labelled one representative graph. The classes contain different sets Δ of *odd triples* of vertices (carrying an odd number of edges); these correspond to *flat triples* of elliptic points, cf. Section 3.1. We observe that for each switching class on 4 vertices the number of odd triples is even. Furthermore, the parity of the number of edges of any 3 vertices in a graph is invariant under switching.

There is a closed formula, due to Mallows and Sloane [1975], for the number of switching classes of unlabelled graphs. We present the essentials, in the setting of Cameron [1977].

3.2.3. THEOREM. *The number of switching classes of graphs on n vertices equals the number of Euler graphs on n vertices.*

PROOF. The labelled graphs on an n-set Ω are represented by their edges, hence are the elements of the vector space

$$V := V\left(\binom{n}{2}, \mathbb{F}_2\right) \quad \text{with basis } \binom{\Omega}{2}.$$

The number of common edges (mod 2) of two labelled graphs serves as a nondegenerate inner product for V. The labelled complete bipartite graphs on Ω constitute a linear subspace, called B, whose basis consist of the complete $(1, n-1)$ graphs (the stars). The quotient space V/B consists of the various switching classes v mod B of labelled graphs. A further linear subspace, called E, consists of the labelled *Euler graphs* (having even valency in each vertex). Now it is a matter of linear algebra to prove the following:

(i) $\dim E \cap B = 0$ for odd n, but $= n-2$ for even n;
(ii) the subspaces B and E are orthogonal;
(iii) $E^* \cong V/B$, where E^* is the dual of E.

The unlabelled graphs on Ω are the equivalence classes under the symmetric group Sym(n) on n symbols. Under the action of Sym(n) the number of orbits of E equals the number of orbits of its dual E^*, hence equals that of V/B.

This proves the theorem. □

It is interesting to observe that for $n = 4$ there are indeed 3 switching classes and 3 Euler graphs, cf. Example 3.2.2, but it is not true that each switching class contains one Euler graph.

3.2.4. REMARK. By the Burnside methods the closed formula is seen to be

$$\frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} 2^{v(\sigma)-b(\sigma)} : 2^{v(\sigma)} = \#(\mathrm{fix}\, V), \quad 2^{b(\sigma)} = \#(\mathrm{fix}\, B).$$

Thus, for the number N of noncongruent equidistant sets of n elliptic points we have:

n	3	4	5	6	7	8	9	10
N	2	3	7	16	54	243	2038	33120

Equidistant sets of elliptic points can be abstractly defined, in terms of two-graphs.

3.2.5. DEFINITION. A *two-graph* (Ω, Δ) is a set Ω and a collection of triples $\Delta \subset \binom{\Omega}{3}$, such that every 4-subset of Ω has an even number of triples in Δ.

A set of n equidistant elliptic points is a two-graph on n vertices, if Δ is taken to be the set of the flat triples of elliptic points. Equivalently, a switching class of graphs on a vertex set Ω is a two-graph on Ω, if Δ is taken to be the set of the odd triples of vertices. The condition that each 4-subset of Ω has an even number of triples in Δ was verified in Example 3.2.2.

Conversely, given a two-graph (Ω, Δ) one can construct a switching class of graphs on Ω whose odd triples agree with the triples of Δ. Indeed, partition $\Omega = \{w\} \cup \Omega_1 \cup \Omega_2$ in any way, and define a graph on Ω by the edges $w \sim w_1$, $w_i \sim w_i'$ iff $\{w_1, w_i, w_i'\} \in \Delta$, $w_1 \sim w_2$ iff $\{w, w_1, w_2\} \notin \Delta$, for any $w_1 \in \Omega_1$, $w_2 \in \Omega_2$, $i = 1, 2$. Thus, in view of Theorem 3.2.1, we arrive at the following:

3.2.6. THEOREM. *There is a one-to-one correspondence between switching classes, two-graphs and equidistant point sets in elliptic space.*

The *two* in two-graphs refers to 2-chains in topology. If fact, *two-graphs are 2-cocycles*, as we shall show below.

For an $(n+1)$-set Ω, let $\Omega^{(i)}$ denote the set of the i-subsets, and let $\subset$ denote inclusion of subsets. We introduce binary spaces of (co)chains and (co)boundary maps as follows:

$$\Gamma_k = \mathbb{F}_2\Omega^{(k+1)}, \qquad \Gamma^k = \langle \gamma\colon\ \Omega^{(k+1)} \to \mathbb{F}_2\rangle_{\mathbb{F}_2},$$

$$\partial_k\colon\ \Gamma_k \to \Gamma_{k-1}, \quad x \in \Omega^{(k+1)} \mapsto \sum \{y \in \Omega^{(k)}\colon\ y \subset x\},$$

$$\delta^k\colon\ \Gamma^k \to \Gamma^{k+1}, \quad x \in \Omega^{(k+1)} \mapsto \sum \{z \in \Omega^{(k+2)}\colon\ z \supset x\}.$$

Then mod 2 the maps, the (co)cycles $Z = \ker \Gamma$ and the (co)boundaries $B = \operatorname{im} \Gamma$ satisfy:

$$\partial_{k-1}\partial_k = 0,\ \delta^{k+1}\delta^k = 0; \qquad Z_k = B_k,\ Z^k = B^k;$$

$$\forall x \in \Gamma_k\ \forall f \in \Gamma^{k-1} \quad (f, \partial_k x) = (\delta^{k-1} f, x),$$

which is nothing but Stokes' law $f(\partial_k x) = (\delta^{k-1} f)(x)$.

Graphs are in Γ_1, Euler graphs in $Z_1 = B_1$. Graphs are in Γ^1, complete bipartite graphs in $B^1 = Z^1$. Switching classes of graphs are in Γ^1/B^1, and two-graphs are in $Z^2 = B^2 \subset \Gamma^2$. Indeed, a collection Δ of triples in Ω is indicated by the mod 2 sum $\gamma \in \Gamma^2$ of the characteristic functions of the triples. Furthermore,

$$(\delta\gamma)\{abcd\} = \gamma\{bcd\} + \gamma\{acd\} + \gamma\{abd\} + \gamma\{abc\}$$

vanishes, since by definition the set Δ contains an even number of triples from any 4-set $\{a, b, c, d\}$. Switching classes and two-graphs are equivalent objects, cf. Theorems 3.2.1 and 3.2.6, as a consequence of the isomorphism

$$\Gamma^1/B^1 \cong B^2 = Z^2.$$

Two-graphs have been proposed by G. Higman, as a setting for 2-transitive groups. We refer to Taylor [1977], Seidel [1976a], Seidel and Taylor [1981] for details. The next paragraph will contain several explicit constructions. We close the present paragraph by indicating the relations with cohomology of groups $H^1(\Gamma, B)$ and $H^2(\Gamma, \mathbb{Z}_2)$, which are due to Cameron [1977]. This deals with comparison of the groups of a two-graph

(n equiangular lines), a graph in its switching class (n vectors) and the corresponding 2-cover of K_n ($2n$ vectors).

As in Theorem 3.2.3, let the graphs on n vertices be denoted by the binary vector space V, and the complete bipartite graphs by B. Let $F = f + B \in V/B$ denote the switching class of a graph $f \in F$. The *automorphism groups* $\mathrm{Aut}\, f$ and $\mathrm{Aut}\, F$ are defined as follows in terms of linear transformations σ of V (which leave B invariant):

$$\sigma \in \mathrm{Aut}\, f \quad \text{iff} \quad (\sigma + 1)f = 0,$$

$$\sigma \in \mathrm{Aut}\, F \quad \text{iff} \quad (\sigma + 1)f \in B, \text{ for all } f \in F.$$

Clearly, $\mathrm{Aut}\, F$ is larger than the Aut of any graph in its switching class whenever

$$\forall f \in F\ \exists \sigma \in \mathrm{Aut}\, F \quad (0 \neq (\sigma + 1)f \in B).$$

Thus the more interesting two-graphs F can be phrased in terms of a nontrivial quotient of a *derivation* d:

$$d\colon \mathrm{Aut}\, F \to B, \quad \sigma \mapsto (\sigma + 1)(f + b),$$

and an *inner derivation* i:

$$i\colon \mathrm{Aut}\, F \to B, \quad \sigma \mapsto (\sigma + 1)b,$$

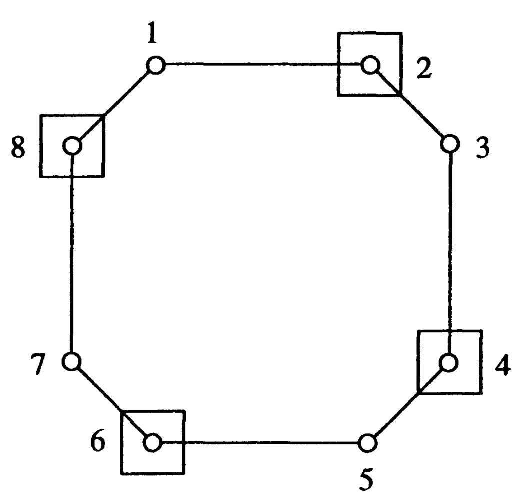
i.e. as a nontrivial element of $H^1(\mathrm{Aut}\, F, B)$.

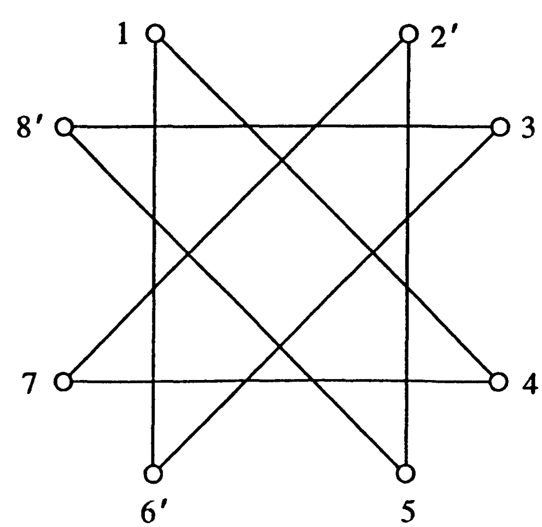
EXAMPLE. The diameters of the icosahedron form a two-graph T with 60 automorphisms. However, any graph in its switching class has only 10 or 6 automorphisms. The alternating group Alt(5) has a 2-transitive representation on T, but not on any graph.

We shall explain $H^2(\Gamma, \mathbb{F}_2)$ by examples. We distinguish between a two-graph (n equiangular lines) and its 2-cover ($2n$ unit vectors along these lines). The diameters of the icosahedron satisfy

$$\mathrm{Aut(2\text{-}cover)} = \mathbb{Z}_2 \times \mathrm{Aut(two\text{-}graph)} = \mathbb{Z}_2 \times \mathrm{Alt}(5),$$

of cardinality 120. However, there are examples of a different sort. Consider the octagon graph:





Switching with respect to 2,4,6,8 and the permutation $(2'6')(4'8')$ yields an automorphism of the corresponding two-graph. We introduce the following automorphisms:

$$\delta := (12345678)(1'2'3'4'5'6'7'8'); \qquad \rho := (26')(2'6)(48')(4'8);$$

$$\zeta := (11')(22')\dots(88') = \delta^3\rho\delta\rho = \delta^2(\delta\rho)^2.$$

Then δ and ρ generate an automorphism group $\widehat{\Gamma}$ of the 2-cover of order 32 (not the full group). Let Γ be the corresponding automorphism group of the two-graph; then Γ has order 16 and

$$\Gamma = \widehat{\Gamma}/\langle\zeta\rangle, \quad \text{but } \widehat{\Gamma} \neq \langle\zeta\rangle \times \Gamma,$$

i.e. $\widehat{\Gamma}$ does not split over $\langle\zeta\rangle$. Indeed, ζ is an element of the subgroup of Γ which is generated by the squares in $\widehat{\Gamma}$. Hence $\widehat{\Gamma}$ cannot contain a subgroup isomorphic to Γ which does not contain ζ. This is expressed by saying that $H^2(\Gamma, \mathbb{Z}_2) \neq 0$. A necessary condition is $n \equiv 0 \pmod 8$. We can complete our list of Remark 3.2.4 as follows, cf. Bussemaker, Mathon and Seidel [1981]:

n	4	5	6	7	8	9	10
$H^1 = H^2 = 0$	3	7	14	54	224	2038	32728
$H^1 \neq 0,\ H^2 = 0$	0	0	2	0	17	0	392
$H^1 \neq 0,\ H^2 \neq 0$	0	0	0	0	2	0	0

3.3. *Regular two-graphs*

The following list, Lemmens and Seidel [1973a], gives the state of affairs about the existence of sets of equidistant points in I^{d-1} (sets of equiangular lines in $\mathbb{R}^d$) of maximal cardinality n. In all cases these maximum numbers cannot be improved, except possibly for $d = 19$ and $d = 20$, when improvement to $n = 76$ and $n = 96$, respectively, is conceivable. We use $\sec\phi = 1/\cos\phi$.

d	2	3	4	5	6	7	...	14	15	16	17	18	19	20
$\max n$	3	6	6	10	16	28	...	28	36	40	48	48	72	90
$\sec\phi$	2	$\sqrt{5}$	3	3	3	3	...	3	5	5	5	5	5	5

d	21	22	23	...	42	43
$\max n$	126	176	276	...	276	344
Γ	$U(3,5^2)$	HS	·3			$U(3,7^2)$

The following general bounds hold, for d not too small,

$$d\sqrt{d} \leqslant \max n \leqslant d(d+1)/2,$$

but the asymptotic behaviour of $\max n$ is unknown.

Most of the extremal sets come from regular two-graphs, which we shall introduce now. Let the $(+,-)$ adjacency matrix C of a graph on n vertices have the eigenvalues $\lambda_1, \ldots, \lambda_d, (-\rho)^{n-d}$, then

$$\operatorname{trace} C = 0 = \lambda_1 + \cdots + \lambda_d - (n-d)\rho,$$

$$\operatorname{trace} C^2 = n(n-1) = \lambda_1^2 + \cdots + \lambda_d^2 + (n-d)\rho^2.$$

This implies $\rho^2(n-d) \leqslant d(n-1)$, with equality iff $\lambda_1 = \cdots = \lambda_d$ ($= \lambda$, say).

3.3.1. DEFINITION. A two-graph is *regular* if it has only two eigenvalues, λ and $(-\rho)$ say, i.e. if its adjacency matrix C satisfies $(C + \rho I)(C - \lambda I) = 0$.

3.3.2. DEFINITION. A two-graph (Ω, Δ) is *regular* if each pair of vertices of Ω is in a constant number of triples from Δ.

3.3.3. THEOREM. *The two definitions for regular two-graphs are equivalent, and the constant number is* $(\lambda + 1)(\rho - 1)/2$.

PROOF. The diagonal and the off-diagonal entries of the matrix product

$$(C + \rho I)(C - \lambda I)$$

read $-\lambda\rho + n - 1$, and $\pm(-\rho + \lambda + c - (n - 2 - c))$. Hence the eigenvalues λ and $-\rho$, and n and the constant c are related by $n = 1 + \lambda\rho$ and $2c = (\lambda + 1)(\rho - 1)$. □

For a regular two-graph, the trace equations yield

$$d\lambda = (n-d)\rho, \qquad d\lambda^2 + (n-d)\rho^2 = n(n-1),$$

whence $(n - 2d)\rho = d(\lambda - \rho)$. In the case $n \neq 2d$, the eigenvalues λ and ρ have product and quotient $\in \mathbb{Z}$, and are (odd) integers themselves. In the case $n = 2d$ we have

$$n = 2d, \qquad \lambda = \rho, \qquad n = 1 + \rho^2, \qquad C^2 = \rho^2 I,$$

which is the case of a *conference matrix*, cf. Goethals and Seidel [1967]. Thus we have proved:

3.3.4. THEOREM. *The eigenvalues of a regular two-graph are odd integers, unless C is a conference matrix.*

3.3.5. THEOREM. *Given one eigenvalue of a regular two-graph, there are only finitely many values for the other eigenvalue.*

PROOF. Elimination of n from the trace equations gives $d = \rho^2 + (\rho - \rho^3)/(\rho + \lambda)$. □

Our examples of a regular two-graph, which in particular will cover most of the extremal situations in our table, have three different settings. The first set of examples is based on alternating, symmetric and unitary geometry over finite fields with q elements, q a power of a prime $p \equiv 1 \pmod 4$. The second set comes from binary symplectic and orthogonal geometry. The third set is of sporadic type. We refer to Artin [1957], Shult [1972], Seidel [1973, 1976a], Taylor [1977], Seidel and Taylor [1981].

In the first setting we consider each time a vector space, a sesquilinear form B, and a set Ω of objects on which a 2-transitive group acts:

$V(2, \mathbb{F}_q)$ with alternating form D: $x_1y_2 - x_2y_1$;

$V(4, \mathbb{F}_q)$ with orthogonal E: $x_1y_1 + x_2y_2 + x_3y_3 - rx_4y_4$;

$V(3, \mathbb{F}_{q^2})$ with unitary H: $x_1\overline{y}_1 + x_2\overline{y}_2 + x_3\overline{y}_3$.

The sets Ω of objects are the following:

the $q+1$ lines through the origin,

the q^2+1 projective points of the ellipsoid $E(x,x) = 0$,

the q^3+1 projective points of the unital $H(x,x) = 0$.

These sets are subject to the 2-transitive action of the groups $\mathrm{SL}(2,q)$, $\mathrm{O}(4,q)$, $\mathrm{U}(3,q^2)$, respectively. In each case the set Δ of triples from Ω is defined in terms of the various sesquilinear forms B by the rule:

$$\{x,y,z\} \in \Delta \quad \text{iff } B(x,y)B(y,z)B(z,x) = \text{square}.$$

3.3.6. THEOREM. *The construction above yields the following regular two-graphs with a 2-transitive automorphism group:*

type Paley: $(n; \lambda, -\rho) = (q+1; \sqrt{q}, -\sqrt{q})$;

type Möbius: $(n; \lambda, -\rho) = (q^2+1; q, -q)$;

type Hermite: $(n; \lambda, -\rho) = (q^3+1; q^2, -q)$.

Indeed, each time (Ω, Δ) is a two-graph because of the definition of Δ, and a regular two-graph since a 2-transitive group acts on it. Only the calculation of parameters gives a little work on counting, for which we refer to the surveys.

In the second setting we consider $V(2m, \mathbb{F}_2)$ with the nondegenerate symplectic form S (which is symmetric) and the quadratic forms Q^ε, $\varepsilon = +, -$, defined by

$$Q(x+y) + Q(x) + Q(y) = S(x,y), \quad x,y \in V.$$

Without loss of generality these forms read

$$S(x,y) = x_1y_2 + x_2y_1 + \cdots + x_{2m-1}y_{2m} + x_{2m}y_{2m-1},$$

$$Q^+(x) = x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m},$$

$$Q^-(x) = x_1^2 + x_2^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m}.$$

The sets Ω of objects are the following:

the symplectic set $\Sigma(2m,2) \cong V$, with $S(x,x) = 0$, size 2^{2m},

the orthogonal sets $\Omega^\varepsilon(2m,2)$, with $Q^\varepsilon(x) = 0$, size $2^{m-1}(2^m+\varepsilon)$.

In each case, the set Δ of triples from Ω is defined in terms of the symplectic form S by the rule:

$$\{x,y,z\} \in \Delta \quad \text{iff } S(x,y) + S(y,z) + S(z,x) = 0.$$

Similar to Theorem 3.3.6 we obtain

3.3.7. THEOREM. *The construction above yields the following regular two-graphs with 2-transitive automorphism group*:
$\Sigma(2m,2)$, *symplectic two-graph*, $n = 2^{2m}$, spec $= \{1 \pm 2^m\}$,
$\Omega^\varepsilon(2m,2)$, *orthogonal two-graphs*, $n = 2^{m-1}(2^m + \varepsilon)$,
spec $= \{1 + \varepsilon 2^{m-1}, 1 - \varepsilon 2^m\}$.

These two-graphs on n vertices define strongly regular graphs on $n - 1$ vertices as follows. Take the vertices of $\Omega \setminus \{0\}$, and adjacency $u \sim v$ whenever $S(u,v) = 0$. Thus we obtain the following graphs:

$$S(2m,2)\text{: } n = 2^{2m} - 1, \text{ spec} = \{2, 1 + 2^m, 1 - 2^m\},$$

$$O^\varepsilon(2m,2)\text{: } n = 2^{m-1}(2^m + \varepsilon) - 1, \text{ spec} = \{2 - \varepsilon 2^{m-1}, 1 + \varepsilon 2^{m-1}, 1 - \varepsilon 2^m\}.$$

EXAMPLES. See Table.

n	6	10	16	28	36	64
two-graph	Ω^- (4,2)	Ω^+ (4,2)	Σ (4,2)	Ω^- (6.2)	Ω^+ (6,2)	Σ (6,2)
spec	5, −1	3, −3	5, −3	9, −3	5, −7	9, −7
SRG	O^- (4,2)	O^+ (4,2)	S (4,2)	O^- (6,2)	O^+ (6,2)	S (6,2)
$n - 1$	5	9	15	27	35	63
spec	4, −1	0, 3, −3	2, 5, −3	6, 9, −3	−2, 5, −7	2, 9, −7.

3.3.8. REMARK. The graphs $S(2m,2)$ and $O^\varepsilon(2m,2)$ satisfy the following *triangle property*:

$$\forall u \sim v \; \exists w \sim u, w \sim v \; \forall x \quad \big(x \sim \text{(one or all of } u, v, w)\big).$$

Indeed, take $w := u + v$ and use $S(u,x) + S(v,x) + S(u+v,x) = 0$. Conversely Shult [1972], cf. Seidel [1973], has proved that any graph (Ω, E) which satisfies the triangle property is not void and has no radical, actually being $S(2m,2)$ or $O^\varepsilon(2m,2)$. This characterization was the start for later developments on polar geometry, see Buekenhout and Shult [1974].

Finally, we recall the construction of the sporadic two-graphs on 176 and on 276 vertices by use of the Steiner system 5-(24, 8, 1), cf. Section 2.2.9. It is well known (Goethals and Seidel [1970]) that this system has 759 blocks in total, 253 blocks passing through any one point, and 176 blocks passing through any one and avoiding any other point. Both subsystems have intersection numbers 3 and 1, hence their blocks define a strongly regular graph. The graph on 176 vertices is in the switching class of a regular two-graph with eigenvalues 5 and −35. The graph on 253 vertices may be extended by a 23-clique to a regular two-graph on 276 vertices with eigenvalues 5 and −55. This settles the extremal sets of equiangular lines in $\mathbb{R}^{22}$ and in $\mathbb{R}^{23}$.

3.3.9. THEOREM. *The regular two-graph on* 276 *vertices is unique* (*up to taking complements*). *The Conway group* ·3 *is characterized as its full automorphism group.*

The proof is based on $276 = 3 \times 11 + 3^5$, and uses the ternary Golay code, cf. Goethals and Seidel [1975].

3.4. *Few-distance sets in* I^{d-1}

We will be interested in sets of elliptic points in I^{d-1} having few distances, i.e. in sets of lines in $\mathbb{R}^d$ having few angles. Let A denote the set of squares of cosines of the prescribed angles, and let X, of cardinality n, denote the set of the lines in $\mathbb{R}^d$ whose $\cos^2\phi \in A$. There are two types of bounds for $n = |X|$. The *absolute bound* only involves the dimension d and the cardinality of A. The *special bound* also involves the values of the prescribed cosines, but is of a limited applicability. The absolute bound can be obtained by the method of Theorem 2.2.1. We leave the proof to the reader. The special bound is an application of Theorem 2.2.5, and holds if $\alpha + \beta \leqslant K$ and if the denominators are positive. The following combined table contains both bounds for the first few cases.

A	absolute	special bound	K
$\{\alpha\}$	$\binom{d+1}{2}$	$\dfrac{d(1-\alpha)}{1-d\alpha}$	–
$\{0,\alpha\}$	$\binom{d+2}{3}$	$\dfrac{d(d+2)(1-\alpha)}{3-(d+2)\alpha}$	$\dfrac{6}{d+4}$
$\{\alpha,\beta\}$	$\binom{d+3}{4}$	$\dfrac{d(d+2)(1-\alpha)(1-\beta)}{3-(d+2)(\alpha+\beta)+d(d+2)\alpha\beta}$	$\dfrac{6}{d+4}$
$\{0,\alpha,\beta\}$	$\binom{d+4}{5}$	$\dfrac{d(d+2)(d+4)(1-\alpha)(1-\beta)}{15-3(d+4)(\alpha+\beta)+d(d+2)(d+4)\alpha\beta}$	$\dfrac{10}{d+6}$

For equidistant sets many examples were given in Sections 3.3 and 2.2 in which either bound is attained. We proceed with two-distance sets in elliptic space, in particular with the *root systems*. These are systems of vectors consisting of antipodal pairs along the following sets of lines in $\mathbb{R}^d$:

A_d, D_d, E_6, E_7, E_8, at angles 90°, 60°;
B_d, C_d, F_4, at angles 90°, 60°, 45°;
G_2 at angles 90°, 60°, 30°.

The sets of lines have the following cardinalities:

$$|\mathsf{A}_d| = \frac{1}{2}d(d+1); \ |\mathsf{D}_d| = d(d-1); \ |\mathsf{E}_8| = 120, \ |\mathsf{E}_7| = 63, \ |\mathsf{E}_6| = 36;$$

$$|\mathsf{B}_d| = |\mathsf{C}_d| = d^2; \ |\mathsf{F}_4| = 24; \ |\mathsf{G}_2| = 6.$$

The various systems of vectors have twice these cardinalities.

3.4.1. THEOREM. *The sets of lines* E_8 *and* G_2 *meet the absolute bound*; $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, F_4, G_2 *meet the special bound*; *of the infinite series, for* $d > 1$, *only* $\mathsf{A}_2, \mathsf{B}_2, \mathsf{C}_2, \mathsf{D}_4$, $\mathsf{B}_{12}, \mathsf{C}_{12}$ *meet the special bound.*

PROOF. A numerical verification of the bounds mentioned in our table. □

In order to define the sets of lines and to check our statements about their cardinalities and angles, we define the sets of vectors of the root systems, one antipodal pair per line. For an orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$, we use (cf. Coxeter [1973]):

the hyperoctahedron $\beta_d := \{\pm e_1, \ldots, \pm e_d\}$, of size $2d$;

the hypercube $\gamma_d := \{\pm e_1 \pm e_2 \pm \ldots \pm e_d\}$, of size 2^d;

the halved hypercube $h\gamma_d$, of size 2^{d-1}, consisting of the vectors with an even number of minuses in γ_d.

We define the antipodal sets of vectors as follows.

3.4.2. DEFINITION.

$$\overline{\mathsf{A}}_d = \{e_i - e_j\colon\ e_i \neq e_j \in \text{ basis of } \mathbb{R}^{d+1}\};$$

$$\overline{\mathsf{D}}_d = \{\pm e_i \pm e_j\colon\ i \neq j;\ e_i, e_j \in \beta_d\};$$

$$\overline{\mathsf{B}}_d = \beta_d \cup \overline{\mathsf{D}}_d;$$

$$\overline{\mathsf{C}}_d = 2\beta_d \cup \overline{\mathsf{D}}_d;$$

$$\overline{\mathsf{E}}_8 = 1/2(h\gamma_8) \cup \overline{\mathsf{D}}_8 = \overline{\mathsf{A}}_1 \perp \overline{\mathsf{E}}_7 = \overline{\mathsf{A}}_3 \perp \overline{\mathsf{E}}_6;$$

$$\overline{\mathsf{F}}_4 = (1/2)\gamma_4 \cup \beta_4 \cup \mathsf{D}_4;$$

$$\overline{\mathsf{G}}_2 = \mathsf{A}_3 \cup (\mathsf{A}_3 + \mathsf{A}_3).$$

Now our earlier statements about cardinalities and angles may be verified.

The vectors of $\overline{\mathsf{A}}_d, \overline{\mathsf{D}}_d, \overline{\mathsf{E}}_d$ all have norm 2. We represent the sets of lines $\mathsf{A}_d, \mathsf{D}_d, \mathsf{E}_d$ by selecting any one vector per line. Let $2I + C$ denote the Gram matrix of inner products of such a of vectors set. C has zero diagonal, and entries 0, 1, -1 elsewhere, and is determined by the root system up to switching.

3.4.3. THEOREM. *The root systems* A, D, E *are represented by eutactic stars*:

A_d: $(2I + C)(C - (d-1)I) = 0$,	E_8: $(2I + C)(C - 28I) = 0$,
D_d: $(2I + C)(C - 2(d-2)I) = 0$,	E_7: $(2I + C)(C - 16I) = 0$,
	E_6: $(2I + C)(C - 10I) = 0$.

PROOF. Since Theorem 3.7 of Cameron, Goethals, Seidel and Shult [1976] contains a misprint in the equation for E_8, we prove that case; the other cases are proved similarly. E_8 is represented by 120 vectors in $\mathbb{R}^8$, hence its matrix C has smallest eigenvalue -2 of multiplicity 112. Any vector of E_8 has nonzero inner product with $119 - 63 = 56$ others. Denoting the remaining eigenvalues of C by $\lambda_1, \lambda_2, \ldots, \lambda_8$, we have

$$\operatorname{trace} C = 0 = \lambda_1 + \cdots + \lambda_8 - 2 \times 112,$$

$$\operatorname{trace} C^2 = 120 \times 56 = \lambda_1^2 + \cdots + \lambda_8^2 + 4 \times 112.$$

These equations only admit $\lambda_1 = \cdots = \lambda_8 = 28$. □

3.4.4. THEOREM. *The root systems* B, C, F, G *are represented by eutactic stars*:

B_d, C_d: $(2I + C)(C - 2(d-1)I) = 0$,

F_4: $(2I + C)(C - 10I) = 0$, G_2: $(2I + C)(C - 4I) = 0$.

We again represent the lines by vectors of norm 2. Now the off-diagonal entries of their Gram matrix C read: twice $0, \pm 1, \pm\sqrt{2}$; $0, \pm 1, \pm\sqrt{3}$, respectively. But one still proves that C has two eigenvalues (we leave out the details). By Theorem 2.3.7 this means that the vectors form a eutactic star. □

We mention some further cases of few-angle sets achieving the absolute bound. Most famous is the absolute bound n for the antipodal pairs of vectors of minimum norm 4 in the Leech lattice in $\mathbb{R}^{24}$. The parameters are:

$$n = 98280 = \binom{28}{5}, \quad \cos\phi \in \{0, \tfrac{1}{2}, \tfrac{1}{4}\}, \qquad (4I + C)(C - 16380I) = 0.$$

The absolute bound also is achieved for two sets of lines in the equidistant locus (of dimension 23) of any two vectors in the Leech lattice at mutual squared distance equal to 4, and to 6, respectively. Here the parameters are:

$$n = 2300 = \binom{25}{3}, \qquad \cos\phi \in \{0, \tfrac{1}{3}\}, \quad \text{in } \mathbb{R}^{23};$$

$$n = 276 = \binom{24}{2}, \qquad \cos\phi = \tfrac{1}{5}, \quad \text{in } \mathbb{R}^{23}.$$

For our final example, we return to $\mathbb{R}^3$, to the 10 diameters of the dodecahedron. They form a 2-angle set of lines meeting the special bound with

$$(\cos\phi)^2 = \alpha = \tfrac{5}{9}, \qquad (\cos\psi)^2 = \beta = \tfrac{1}{9}.$$

The relation $\cos\phi = \sqrt{5}/3$ defines the Petersen graph. The Gram matrix $3I + C$ defines a eutactic star since

$$(3I + C)(C - 7I) = 0.$$

4. Hyperbolic geometry

4.0. *Summary*

Hyperbolic geometry is the geometry of indefinite metric. The term is also used as a short name for Bolyai-Lobachevsky geometry H^q, the geometry inside and on the light cone in $\mathbb{R}^{1,q}$. In Section 4.1 the underlying Gram matrices of $\mathbb{R}^{1,q}$ lead to properties of parallels, orthogonality and volume, first in hyperbolic plane H^2, then in hyperbolic space H^q. In Section 4.2 harmonic methods are introduced for general indefinite $\mathbb{R}^{p,q}$. They produce bounds for sets with few inner products. Graphs and root systems are dealt with in Section 4.3, in the setting of $\mathbb{R}^{p,1}$ and its subspaces. Certain aspects of the theory of integral lattices in definite and indefinite spaces are surveyed in Section 4.4. In the final Section 4.5 a new application of hyperbolic methods is given to distance matrices, in particular to those of negative and hypermetric type.

4.1. *Bolyai–Lobachevsky geometry*

We first introduce the hyperbolic plane H^2 and later hyperbolic space H^q of dimension q. The general references are Coxeter [1957, 1969].

$\mathbb{R}^3$ is provided with the *indefinite* inner product

$$(a,b) := -a_1b_1 - a_2b_2 + a_3b_3 \quad \text{for } a=(a_1,a_2,a_3),\ b=(b_1,b_2,b_3).$$

Then nonzero vectors may have zero norm. The *cone* C is the set

$$C := \{x \in \mathbb{R}^3\colon (x,x)=0\} = \{x \in \mathbb{R}^3\colon x_3^2 = x_1^2 + x_2^2\}.$$

Inside the cone we have $(x,x) > 0$, and outside $(x,x) < 0$. For any two vectors a and b the quadratic form

$$(\lambda a + \mu b, \lambda a + \mu b) = \lambda^2(a,a) + 2\lambda\mu(a,b) + \mu^2(b,b)$$

takes positive, zero, and negative values according to whether the vector $\lambda a+\mu b$ is inside, on, and outside the cone, respectively. Hence the discriminant of the form determines whether the 2-subspace $\lambda a + \mu b$ is intersecting, tangent or exterior with respect to C. For instance, the following anti-Cauchy–Schwarz property holds.

4.1.1. LEMMA. $(a,b)^2 \geqslant (a,a)(b,b)$, *for* a, b *inside* C.

If both a and b are taken inside one nappe of C then $(a,b) \geqslant \sqrt{(a,a)(b,b)}$.

The hyperbolic plane is defined as follows. The *hyperbolic points* are the 1-subspaces inside the cone C. The *hyperbolic lines* are the 2-subspaces restricted to the inside of C. The *hyperbolic plane* H^2 consists of the hyperbolic points and lines, with the incidence relation of the underlying vector space. The *hyperbolic distance* between hyperbolic points is

$$d(\langle a\rangle, \langle b\rangle) := \operatorname{arcosh} \frac{|(a,b)|}{\sqrt{(a,a)(b,b)}}.$$

The modulus sign may be deleted if the vectors a and b are taken in the same nappe of C. The definition of the hyperbolic distance $d = d(\langle a\rangle, \langle b\rangle)$ is justified by the following:

(i) invariance with respect to $a \mapsto \lambda a$;

(ii) $\cosh d \geqslant 1$, and $= 1$ iff $\langle a\rangle = \langle b\rangle$;

(iii) the distance d is a metric.

Indeed, the following theorem is true.

4.1.2. THEOREM. *The hyperbolic plane is a metric space.*

PROOF. In order to prove the triangle inequality we take without loss of generality:

$$(a,a)=(b,b)=(c,c)=1;\quad (a,b)>0,\ (a,c)>0,\ (b,c)>0;$$

$$d(\langle a\rangle,\langle b\rangle)=r,\ d(\langle a\rangle,\langle c\rangle)=q,\ d(\langle b\rangle,\langle c\rangle)=p,\ 0\leqslant r\leqslant q\leqslant p.$$

Then $p\leqslant q+r$ is a consequence of

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} -a_1 & -b_1 & -c_1 \\ -a_2 & -b_2 & -c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}=\begin{bmatrix} 1 & \cosh r & \cosh q \\ \cosh r & 1 & \cosh p \\ \cosh q & \cosh p & 1 \end{bmatrix},$$

$$0\leqslant 4\sinh\tfrac{1}{2}(p+q+r)\sinh\tfrac{1}{2}(-p+q+r)\sinh\tfrac{1}{2}(p-q+r)\sinh\tfrac{1}{2}(p+q-r).$$

□

The *ideal* hyperbolic points are the 1-subspaces on the cone, and the *exterior* hyperbolic points are the 1-subspaces outside the cone. Hence the 2-subspaces of two hyperbolic lines have in common an ordinary, an ideal, or an exterior hyperbolic point, i.e. the hyperbolic lines are intersecting, *parallel* or nonintersecting. Thus we arrive at the parallel axiom for the hyperbolic plane mentioned in the Introduction. The mutual position of the hyperbolic lines

$$u_1x_1+u_2x_2+u_3x_3=0,\quad v_1x_1+v_2x_2+v_3x_3=0$$

is determined by the (negative) inner product of the exterior points $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$. Indeed, normalizing to $(u,u)=(v,v)=-1$ we have:

$$-(u,v)=\cos\phi,\qquad -(u,v)=1,\qquad -(u,v)=\cosh d,$$

according to whether the hyperbolic lines intersect at angle ϕ, are parallel, or are nonintersecting at distance d. Thus parallel lines are said to have zero angle, and nonintersecting lines have a common perpendicular. Two hyperbolic lines are orthogonal whenever

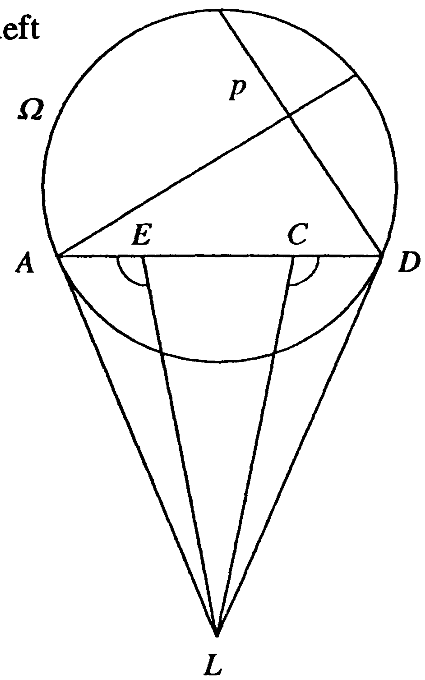
$$(u,v)=-u_1v_1-u_2v_2+u_3v_3=0.$$

The hyperbolic plane has a simple model in the plane $x_3=1$, where points $(x_1,x_2,1)$ are denoted as (x,y). In this so-called *Klein model* the hyperbolic points are the points inside the circle Ω: $x^2+y^2=1$, and the hyperbolic lines are the chords of this circle. The following theorem can be proved by elementary analytic geometry.

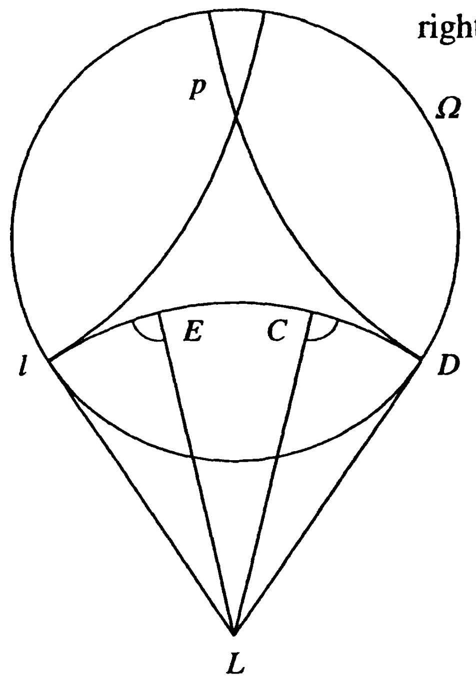
4.1.3. THEOREM. *In the Klein model any two lines are orthogonal whenever they are conjugate with respect to* $x^2+y^2=1$.

As a consequence, the Klein model does not preserve angles and distances. The *Poincaré model*, which uses circles $\perp \Omega$ instead of chords of Ω, does preserve angles, but not distances. This is illustrated below: left is a picture in the model of F. Klein (1849–1925), right is the same picture in the model of H. Poincaré (1854–1912). The middle picture, taken from Coxeter [1969], indicates how one model is obtained from the other by stereographic projection.

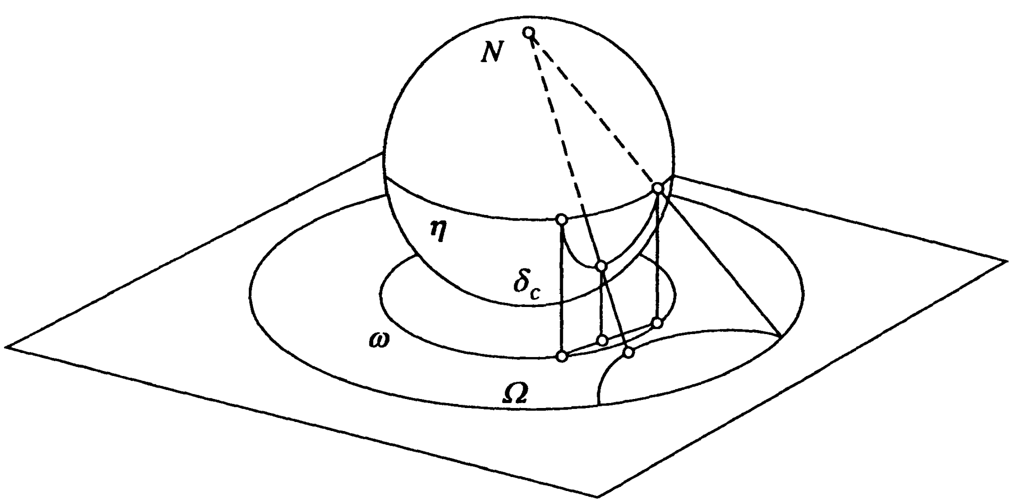
left



right



middle



In the hyperbolic plane extended by its ideal points, lines have infinite length. However, triangles have finite area, also if they have ideal vertices.

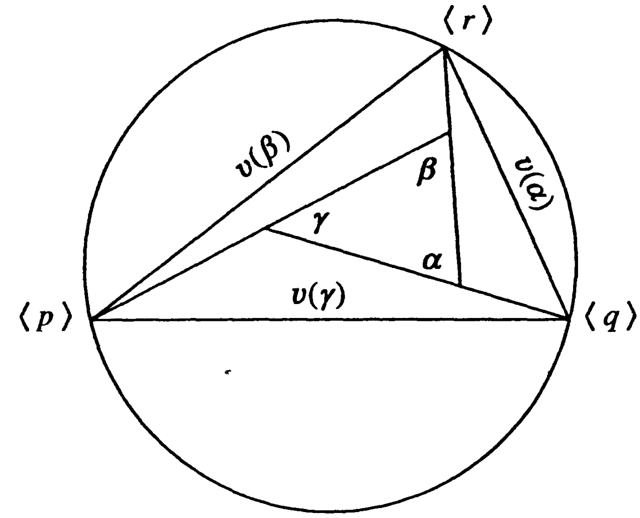
4.1.4. THEOREM. *In the extended hyperbolic plane any triangle with angles α, β, γ has area $\pi - \alpha - \beta - \gamma$.*

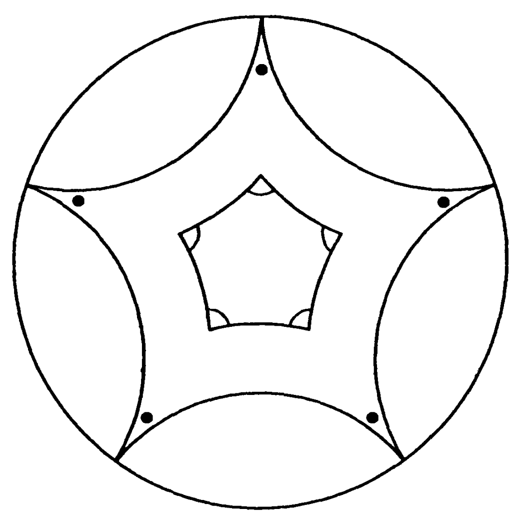
This theorem goes back to Gauss. For the proof, cf. Coxeter, [1969], pp. 295–299.

We illustrate the theorem for an ideal triangle $\langle p\rangle, \langle q\rangle, \langle r\rangle$. For the triples of vectors p, q, r and $p(q,r), q(p,r), r(p,q) / (p,q)(q,r)(r,p)$ the Gram matrices read

$$\begin{bmatrix} 0 & (p,q) & (p,r) \\ (p,q) & 0 & (q,r) \\ (p,r) & (q,r) & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

respectively. Hence for any ideal hyperbolic triangle the vectors may be chosen such that they have the standard Gram matrix $J_3 - I_3$. As a consequence, any ideal triangle can be mapped onto any other ideal triangle by an orthogonal transformation of the underlying $\mathbb{R}^3$. Hence all ideal triangles have the same area, say π.





4.1.5. Example. We illustrate the use of Gram matrices in hyperbolic geometry by considering regular pentagons in H^2. Let $P = \mathrm{circ}(01000)$ denote the cyclic permutation matrix of size 5. The symmetric matrices $A = P + P^{-1}$ and $B = P^2 + P^{-2}$ are simultaneously diagonalizable with eigenvalues

$$2, \tau^{-1}, \tau^{-1}, -\tau, -\tau \quad \text{and} \quad 2, -\tau, -\tau, \tau^{-1}, \tau^{-1},$$

where $\tau = (1+\sqrt{5})/2$ denotes the golden ratio. For which positive α and β is $\alpha A + \beta B$ the Gram matrix of an ideal regular pentagon? Then $\alpha A + \beta B$ must have one positive, two negative and two zero eigenvalues, hence $\tau A + \tau^{-1} B$ applies. For which positive α and β is $I + \alpha A + \beta B$ the Gram matrix of a regular pentagon which is not ideal? Then we must have

$$1 - \tau\alpha + \tau^{-1}\beta = 0, \qquad 1 + \alpha\tau^{-1} - \beta\tau < 0, \quad \alpha > 0,\ \beta > 0,$$

hence, e.g., $\alpha = \tau$, $\beta = \tau^2$. This corresponds to a rectangular regular pentagon, with sides τ and diagonals τ^2. By reflecting this pentagon in its sides we obtain the tessellation $\{5, 4\}$ of the hyperbolic plane, cf. Coxeter [1956].

We now turn to Bolyai–Lobachevsky geometry H^q of *dimension* q. In $\mathbb{R}^{1,q}$ we denote the coordinates of x by $(x_0; x_1, \ldots, x_q)$ and the inner product of x and y by

$$(x, y) = x_0 y_0 - x_1 y_1 - \cdots - x_q y_q.$$

The hyperbolic points are the rays inside and on the positive nappe of the cone C with equation $x_0^2 = x_1^2 + \cdots + x_q^2$. Those on the cone are the ideal hyperbolic points. We define hyperbolic distance as before, and the hyperbolic k-subspaces as the restrictions to the cone of the $(k+1)$-subspaces of $\mathbb{R}^{1,q}$. In the flat F with equation $x_0 = 1$ the cone C appears as the 'ideal' sphere $C \cap F$. The geometry is represented by the *Klein model* consisting of points, lines, planes etc. inside and on the sphere $C \cap F$, and by the *Poincaré model*, consisting of the hemispheres of various dimensions orthogonal to $C \cap F$. We refer to Coxeter [1956], Wilker [1981] for further details.

A hyperbolic simplex in H^q consists of $q+1$ independent hyperbolic points, possibly ideal. Its Gram matrix, of signature $\pi = 1$, $v = q$, has positive entries, except possibly for zeros on the diagonal. The hyperbolic points are determined by vectors along their halflines. Each vector is determined up to a positive real factor. Hence the Gram matrix A is determined modulo pre- and post-multiplication by a positive diagonal matrix D. What are the possibilities for DAD?

4.1.6. THEOREM. *Any hyperbolic simplex in H^q is uniquely represented by a set of $q+1$ spanning vectors whose Gram matrix has all row sums and column sums equal to* 1 (*is doubly stochastic*).

This follows from a theorem by Sinkhorn [1964], stating that a non-negative symmetric A with positive row sums admits a unique diagonal matrix D such that DAD is doubly stochastic.

In particular, an ideal hyperbolic simplex is uniquely represented by a doubly stochastic Gram matrix with zero diagonal and positive entries elsewhere. There are other interesting representations of an ideal simplex as well. For any $1 \leqslant k \leqslant q+1$ the class DAD contains a unique matrix K where k-th row and column have all entries equal to 1, apart from the (k,k)-entry 0. This K is the *Cayley–Menger distance matrix* of q independent points in Euclidean $(q-1)$-space, which are obtained from the lines in H^q connecting the k-th with the remaining ideal points, cf. Section 1.7. Passing from one ideal point k with distance matrix K to any other ideal point l with distance matrix L amounts to *inversion*. Indeed, the $q+1$ ideal points in H^q correspond to $q+1$ points of Möbius $(q-1)$-space (inversive $(q-1)$-space), Wilker [1981].

4.1.7. EXAMPLE. In $\mathbb{R}^{1,3}$ the Gram matrix of the vectors p, q, r, s of an ideal hyperbolic tetrahedron in H^3 may be taken as follows:

$$A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{bmatrix},$$

$$\det A = -(a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

Hence the positive a, b, c satisfy the triangle inequality. In fact, by stereographic projection from $\langle p \rangle$ it is seen that planes $\langle p, q \rangle$, $\langle p, r \rangle$, $\langle p, s \rangle$ constitute the vertices Q, R, S of a Euclidean triangle with sides a, b, c. As a consequence, any ideal hyperbolic 3-simplex has equal opposite dihedral angles α, β, γ with $\alpha + \beta + \gamma = \pi$. Interestingly, the volume of an ideal tetrahedron in H^3 can be expressed in α, β, γ by use of the function Л of Lobachevsky:

$$\mathrm{vol}(\text{3-simplex}) = \text{Л}(\alpha) + \text{Л}(\beta) + \text{Л}(\gamma), \qquad \text{Л}(\alpha) = -\int_0^{\alpha} \log|2 \sin t| \, dt.$$

Recently, hyperbolic geometry started a second phase in its existence, Milnor [1982]. One reason for the revived interest was Thurston's work on hyperbolic 3-manifolds, cf. Munkholm [1980]. In this connection Gromov gave a proof of Mostow's rigidity theorem on the basis of the following theorem, conjectured by Thurston and proved by Haagerup and Munkholm [1981].

4.1.8. THEOREM. *In H^q the volume of a hyperbolic simplex is maximal iff the simplex is ideal and regular.*

A hyperbolic simplex is ideal and regular iff its Gram matrix reads $J_{q+1} - I_{q+1}$. The theorem makes one think of possible connections with the following, cf. Seidel [1986].

4.1.9. THEOREM. *The permanent of a doubly stochastic $n \times n$ matrix attains a unique minimum for the matrix J_n/n.*

This conjecture by Van der Waerden (1928) was proved by Egoritshev and by Falikman, cf. Van Lint [1982]. The key inequality in their proof is the anti-Cauchy–Schwarz inequality of Lemma 4.1.1 applied to the indefinite inner product resulting from the following theorem.

4.1.10. THEOREM. *In real n-space V, let $a_1, a_2, \ldots, a_{n-2}$ denote vectors having positive coordinates. Then the permanent $\mathrm{per}(a_1, a_2, \ldots, a_{n-2}, x, y)$ acts on V as a nondegenerate inner product of signature $1, (n-1)$.*

There are extensive theories about the notion of volume of polyhedra in hyperbolic space, going back to Lobachevsky, cf. Coxeter [1956], Böhm and Hertel [1981], Vinberg [1985], Im Hof [1985]. The building blocks are the *orthoschemes*, simplices whose $q+1$ hyperplanes have normals with a tridiagonal Gram matrix. A Coxeter orthoscheme has 'natural' angles π/p, $p \in \mathbb{N}$, $p \geqslant 2$, between its hyperplanes. It is associated to a diagram with

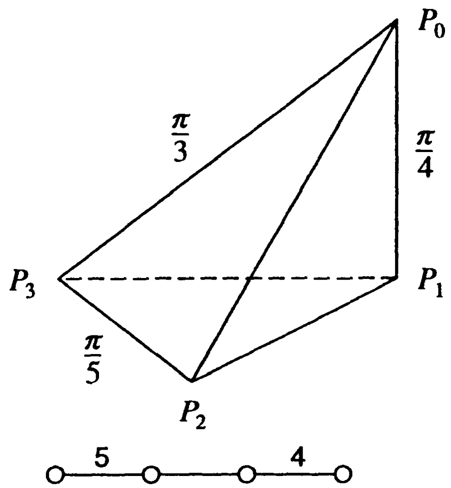
for hyperplanes at angle π/p, parallel, nonintersecting (having a common perpendicular), respectively. Im Hof generalized to *complete orthoschemes*, consisting of $q+3$ hyperplanes. They are:

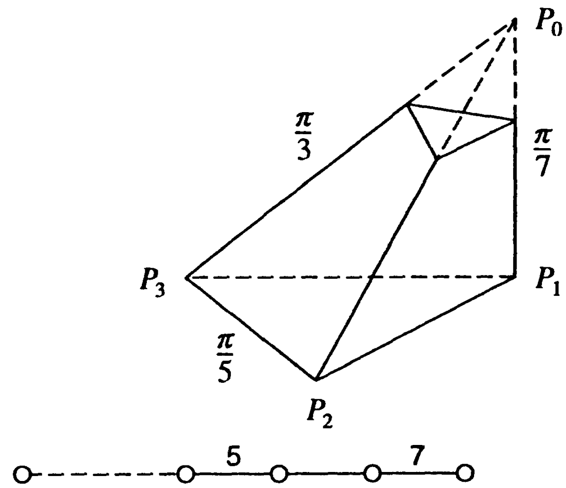
ordinary on $q+1$ vertices, if two adjacent hyperplanes are outside C;
simply truncated on $q+2$ vertices, if one hyperplane and one vertex $\langle x\rangle$ are outside C;
doubly truncated on $q+3$ vertices, if two vertices $\langle x\rangle$ and $\langle y\rangle$ are outside C.

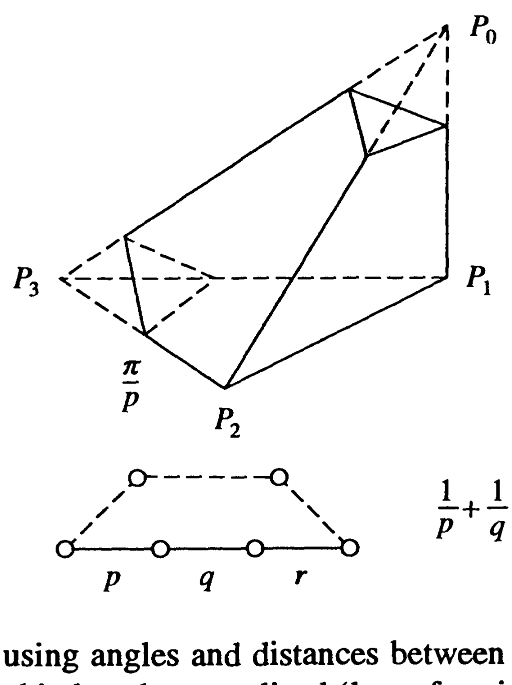
The truncation is performed by the hyperplanes $x^{\perp}$ and $y^{\perp}$. Such orthoschemes of Coxeter type and of finite volume have been enumerated in Coxeter [1956] and in Im Hof [1985].

4.1.11. THEOREM. *Ordinary orthoschemes exist iff $q \leqslant 5$, complete orthoschemes iff $q \leqslant 9$.*

Here are three examples in H^3, and their diagrams:







$$\frac{1}{p}+\frac{1}{q}<\frac{1}{2},\quad \frac{1}{q}+\frac{1}{r}<\frac{1}{2}$$

We have been freely using angles and distances between hyperplanes in H^q. Here is a sketch of the theory behind such generalized 'law of cosines'. A similar theory holds in elliptic space and also, but more complicated, in Euclidean space, cf. Seidel [1955]. For elliptic space the notion of Clifford parallelism amounts to *isoclinic* subspaces in $\mathbb{R}^p$

(all roots $\lambda_1, \dots, \lambda_n$ in the corresponding Theorem 4.1.12 are equal, cf. Lemmens and Seidel [1973b]). Let

$$\Gamma = \langle a_1, \dots, a_m \rangle, \qquad \Delta = \langle b_1, \dots, b_n \rangle, \quad m \geqslant n, \begin{bmatrix} A & C \\ C^t & \lambda B \end{bmatrix},$$

denote two subspaces of $\mathbb{R}^{1,q}$ whose bases have nondegenerate Gram matrices A and B, and mutual inner products C. Any function of the inner products, which is independent of the choice of the bases, is an *invariant* of Γ and Δ with respect to orthogonal transformations.

4.1.12. THEOREM. *For the subspaces Γ and Δ a complete set of invariants is provided by the n roots of*

$$\det \begin{bmatrix} A & C \\ C^t & \lambda B \end{bmatrix} = 0 = \det \left(\lambda B - C^t A^{-1} C \right).$$

The equation has $n-1$ roots $0 \leqslant \lambda_i \leqslant 1$, and the remaining root λ_n can be > 1, $= 1$, or < 1. Interpreting the subspaces in hyperbolic space H^q, and putting $\lambda_i = \cos^2 \phi_i$ and $\lambda_n = \cosh^2 d_n$, or $\cos^2 \phi_n$, we arrive at the following theorem, cf. Seidel [1955].

4.1.13. THEOREM. *The angles ϕ_i and the distance d_n between two subspaces $\langle a_1, \dots, a_m \rangle$ and $\langle b_1, \dots, b_n \rangle$ of hyperbolic space H^q are related as above to the roots of the equation*

$$\det \begin{bmatrix} \cosh d(\langle a_i \rangle, \langle a_j \rangle) & \cosh d(\langle a_i \rangle, \langle b_v \rangle) \\ \cosh d(\langle b_\mu \rangle, \langle a_j \rangle) & \lambda \cosh d(\langle b_\mu \rangle, \langle b_v \rangle) \end{bmatrix} = 0$$

$$(i, j = 1, \dots, m; \ \mu, \nu = 1, \dots, n).$$

We close our survey of certain aspects of discrete Bolyai–Lobachevsky geometry by mentioning the following theorem, proved by Blokhuis [1984], in the spirit of earlier results in Sections 2 and 3.

4.1.14. THEOREM. *An s-distance set in H^q has cardinality $\leqslant \binom{q+s}{s}$.*

4.2. *Harmonics in $\mathbb{R}^{p,q}$*

Jumping to the other extreme we consider the $\mathbb{R}$-vector space $\mathbb{R}^{p,q}$ of dimension $p+q$, provided with the inner product

$$(x, y) := x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q},$$

for $x = (x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q})$. The interest in $\mathbb{R}^{p,q}$ is justified by the following theorem.

4.2.1. THEOREM. *Any symmetric matrix M of size n over the reals is the Gram matrix of n vectors in $\mathbb{R}^{p,q}$, where p, q is the signature of M.*

PROOF. From linear algebra we have

$$M = S\Lambda S^t, \qquad \Lambda = \Lambda_p^+ \oplus \Lambda_q^- \oplus O_{n-p-q},$$

for an orthogonal S and diagonal Λ in which we put together the p positive eigenvalues in Λ_p^+, the q negative eigenvalues in Λ_q^-, and the zero eigenvalues in O_{n-p-q}. In S we delete the $n-p-q$ last columns and adjust the others such that Λ_p^+ becomes I_p and Λ_q^- becomes $-I_q$. Then the rows of the adjusted $n \times (p+q)$ matrix T apply, since

$$M = T \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} T^t.$$

□

Let $\mathrm{Pol}(\mathbb{R}^d)$ denote the linear space of the polynomials in d variables over $\mathbb{R}$. Associated to any polynomial $f(x)$ is the differential operator $f(\partial)$, with

$$\partial = (\partial_1, \ldots, \partial_p, -\partial_{p+1}, \ldots, -\partial_{p+q}) \quad \text{where } \partial_i = \frac{\partial}{\partial x_i}.$$

We define the following inner product for $f, g \in \mathrm{Pol}(\mathbb{R}^{p+q})$:

$$\langle f, g \rangle := \big(f(\partial)g\big)(0).$$

This is a symmetric, nondegenerate inner product which is invariant with respect to the orthogonal group of $\mathbb{R}^{p,q}$, cf. Bannai, Blokhuis, Delsarte and Seidel [1984], Blokhuis [1984], Blokhuis and Seidel [1986]. It induces an orthogonal decomposition

$$\mathrm{Pol}\big(\mathbb{R}^{p+q}\big) = \sum\nolimits^{\perp} \mathrm{Hom}_k\big(\mathbb{R}^{p+q}\big),$$

into components $\mathrm{Hom}_k(\mathbb{R}^d)$ which consist of the homogeneous polynomials in d variables of degree k. The inertia of the inner product on $\mathrm{Hom}_k(\mathbb{R}^{p+q})$ may be determined as follows. The monomial $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{p+q}^{\alpha_{p+q}}$ has

$$\langle x^\alpha, x^\alpha \rangle = (-1)^{\alpha_{p+1} \cdots \alpha_{p+q}} \prod_{i=1}^{p+q} \alpha_i!\,.$$

Hence Hom_k admits the orthogonal decomposition

$$\mathrm{Hom}_k\big(\mathbb{R}^{p+q}\big) = \mathrm{Hom}_k^+\big(\mathbb{R}^{p+q}\big) \perp \mathrm{Hom}_k^-\big(\mathbb{R}^{p+q}\big)$$

into ε-definite subspaces $\mathrm{Hom}_k^\varepsilon$ spanned by the monomials x^α of degree k having sign $\langle x^\alpha, x^\alpha \rangle = \varepsilon$, for $\varepsilon = 1$ and for $\varepsilon = -1$. The dimensions of $\mathrm{Hom}_k^\varepsilon$ can be calculated. Associated to $b(x) = (x, x)$ is the Laplace operator

$$\Delta = \Delta_{p,q} = b(\partial) = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}.$$

It maps Hom_k onto Hom_{k-2}. The kernel of this map is called $\mathrm{Harm}_k = \mathrm{Harm}_k(\mathbb{R}^d)$. The mapping

$$\Pi\colon \mathrm{Hom}_k \to \mathrm{Harm}_k$$

is the orthogonal projection with respect to $\mathrm{Hom}_k = \mathrm{Harm}_k \perp b(x)\mathrm{Hom}_{k-2}$. It is not difficult to show, that Π commutes with the decomposition of Hom_k into its ε-definite parts. Therefore, Harm_k decomposes correspondingly:

$$\mathrm{Harm}_k = \mathrm{Harm}_k^+ \perp \mathrm{Harm}_k^-, \qquad \mathrm{Harm}_k^\varepsilon = \Pi \mathrm{Hom}_k^\varepsilon.$$

From this, one calculates the dimensions $\mu_k := \dim \mathrm{Harm}_k^+$, $\nu_k := \dim \mathrm{Harm}_k^-$. In particular, for $\mathbb{R}^{p,1}$ it follows that (see Blokhuis [1984]):

$$\dim \mathrm{Harm}_k^+ = \binom{p+k-1}{p-1}, \qquad \dim \mathrm{Harm}_k^- = \binom{p+k-2}{p-1},$$

in accordance with the formula for $\dim \mathrm{Harm}_k$:

$$\binom{p+k}{p} - \binom{p+k-2}{p} = \binom{p+k-1}{p-1} + \binom{p+k-2}{p-1}.$$

For any $a \in \mathbb{R}^{p+q}$, the homogeneous polynomial $g(x) := (a,x)^k/k!$ is the unique element of Hom_k which has the reproducing property

$$\langle g, f\rangle = f(a) \quad \text{for all } f \in \mathrm{Hom}_k.$$

Indeed, the property follows by Taylor, and the uniqueness by the nondegeneracy of the inner product in connection with the linear functional $f \mapsto f(a)$. For any $a \in \mathbb{R}^{p,q}$ the harmonic polynomial $\bar{a}(x) := \Pi(a,x)^k/k!$ is the unique element of Harm_k which satisfies

$$\langle \bar{a}, f\rangle = f(a) \quad \text{for all } f \in \mathrm{Harm}_k.$$

Hence for any $a, b \in \mathbb{R}^{p,q}$ we have

$$\langle \bar{a}, \bar{b}\rangle = \bar{b}(a) = \bar{a}(b) = \Pi(a,b)^k/k!\,.$$

This expression depends on the inner product (a,b) only, rather than on the $2(p+q)$ coordinates of a and b. In fact, cf. Blokhuis [1984], it equals the normalized Gegenbauer polynomial $Q_k(t)$ of degree k in one variable t, with $t = (a,b)$ and $d = p+q$. Here normalized Gegenbauer polynomials are precisely the Gegenbauer polynomials introduced in Section 3.1, but with the c_k deleted (taken $= 1$).

By use of the orthonormal bases

$$\{f_{k,i}\colon i = 1,\ldots,\mu_k\} \ \text{ for } \mathrm{Harm}_k^+, \qquad \{g_{k,j}\colon j = 1,\ldots,\nu_k\} \ \text{ for } \mathrm{Harm}_k^-,$$

we write any $h \in \mathrm{Harm}_k$ as

$$h = \sum_{i=1}^{\mu_k} \langle h, f_{k,i}\rangle f_{k,i} - \sum_{j=1}^{\nu_k} \langle h, g_{k,j}\rangle g_{k,j}.$$

Applying this to the harmonics $\bar{a}(x)$ and $\bar{b}(x)$ introduced above, we arrive at the *addition formula*, Bannai et al. [1984]:

4.2.2. THEOREM.

$$\sum_{i=1}^{\mu_k} f_{k,i}(a) f_{k,i}(b) - \sum_{j=1}^{\nu_k} g_{k,j}(a) g_{k,j}(b) = Q_k\big((a,b)\big),$$

for any $a, b \in \mathbb{R}^{p,q}$.

This formula furnishes strong bounds for the cardinality of few-distance sets in $\mathbb{R}^{p,q}$. Let X denote a finite subset of the 'unit sphere' in $\mathbb{R}^{p,q}$:

$$X \subset S^{p,q} := \{x \in \mathbb{R}^{p,q}\colon (x,x) = 1\}, \quad |X| = n.$$

Thus $S^{p,0}$ is the ordinary unit sphere in $\mathbb{R}^p$. In Lorentz space our considerations include both the unit sphere $S^{1,q}$ inside the light cone and the unit sphere $S^{p,1}$ outside C. Furthermore, we define for the inner products in X:

$$A := \{(x,y)\colon x \neq y \in X\}, \quad 1 \notin A, \ |A| =: S.$$

The annihilator polynomial $\phi(t)$ of A is expanded in terms of the Gegenbauer polynomials by

$$\phi(t) := \prod_{\alpha \in A} \frac{t-\alpha}{1-\alpha} = \sum_{k=0}^{s} \phi_k Q_k(t).$$

We translate the addition formula by use of the matrices

$$F_k := [f_{k,i}(x)], \qquad G_k := [g_{k,j}(x)], \qquad Q_k := [Q_k(x,y)]$$

for $x \in X,\ y \in X,\ i = 1,2,\ldots,\mu_k,\ j = 1,2,\ldots,\nu_k$,

$$H := [F_0; G_0; \ldots; F_s; G_s], \qquad \Delta := \bigoplus_{k=0}^{s} \phi_k \operatorname{diag}\big(1^{\mu_k}, (-1)^{\nu_k}\big).$$

Then the addition formula reads $F_k F_k^t - G_k G_k^t = Q_k$, and the expansion of the annihilator yields

$$H \Delta H^t = \sum_{k=0}^{s} \phi_k \big(F_k F_k^t - G_k G_k^t\big) = \sum_{k=0}^{s} \phi_k Q_k = I_n.$$

This implies that n cannot exceed the number of positive diagonal entries in Δ, which proves:

4.2.3. THEOREM. *A finite subset X of the unit sphere in $\mathbb{R}^{p,q}$, whose annihilator has Gegenbauer coefficients $\phi_0, \phi_1, \ldots, \phi_s$, satisfies*

$$n = |X| \leqslant {\sum_{k=0}^{S}}' \dim \operatorname{Harm}_k^{\varepsilon_k},$$

where $\varepsilon_k = \operatorname{sign} \phi_k$, *and the summation is over the indices k with $\phi_k \neq 0$.*

4.2.4. COROLLARY. *$n \leqslant \binom{d+s-1}{s}$ for $\mathbb{R}^{1,d-1}$ and for $\mathbb{R}^{d-1,1}$.*

PROOF. $\dim \mathrm{Harm}_k^\varepsilon \leqslant \dim \mathrm{Harm}_k^+ = \binom{d+k-2}{k}$. □

Our examples mainly deal with $\mathbb{R}^{p,1}$, with the inner product

$$(x,y) = -x_0y_0 + x_1y_1 + \cdots + x_py_p.$$

The vectors $x = (x_0; x_1, \ldots, x_p)$ have positive norm iff they lie outside the cone $x_1^2 + \cdots + x_p^2 = x_0^2$. If the plane spanned by the vectors x and y does not intersect the cone, then the angle $\arccos(x,y)$ between the vectors makes sense. Likewise the angle $\arccos|(x,y)|$ is defined between the lines spanned by x and by y. We will be first interested in sets of equiangular lines in $\mathbb{R}^{p,1}$, i.e. sets of lines such that for each pair the angle is defined and equal, say $\arccos\alpha$.

4.2.5. THEOREM. *Any set of equiangular lines in $\mathbb{R}^{p,1}$ has cardinality $n \leqslant p(p+1)/2$.*

PROOF. For each line we select one unit vector in either direction. The n vectors thus obtained have mutual inner products $\pm\alpha$. The annihilator polynomial is expressed in the Gegenbauer polynomials as follows:

$$t^2 - \alpha^2 = t^2 - \frac{1}{p+1} + \frac{1}{p+1} - \alpha^2 = 2Q_2(t) + \left(\frac{1}{p+1} - \alpha^2\right) Q_0(t).$$

Application of Theorem 4.2.3. yields

$$n \leqslant \tfrac{1}{2}p(p+1) \quad \text{for } (p+1)\alpha^2 \geqslant 1,$$

$$n \leqslant \tfrac{1}{2}p(p+1) + 1 \quad \text{for } (p+1)\alpha^2 < 1.$$

In the first case the bound is sharp, as will be seen from the examples. In the second case improvement by 1 is possible. This follows from the next theorem and the impossibility of $1/(p+2) < \alpha^2 < 1/(p+1)$, cf. Blokhuis [1984].

4.2.6. THEOREM. *A set of equiangular lines in $\mathbb{R}^{p,1}$ at angle* $\arccos\alpha$ *with $p\alpha^2 < 1$ has cardinality*

$$n \leqslant \frac{p(1-\alpha^2)}{1-p\alpha^2}.$$

PROOF. The Gram matrix G of the n unit vectors in $\mathbb{R}^{p,1}$ of the proof of Theorem 4.2.5. has at most p positive eigenvalues. Hence $C = (G-I)/\alpha$ has zero diagonal, entries ±1 elsewhere, and $n-p$ eigenvalues $\leqslant -1/\alpha$. For the remaining eigenvalues $\lambda_1, \ldots, \lambda_p$ we have

$$0 = \operatorname{trace} C \leqslant \lambda_1 + \cdots + \lambda_p - (n-p)/\alpha,$$

$$n(n-1) = \operatorname{trace} C^2 \geqslant \lambda_1^2 + \cdots + \lambda_p^2 + (n-p)/\alpha^2.$$

Hence

$$(n-p)^2/\alpha^2 \leqslant (\lambda_1 + \cdots + \lambda_p)^2 \leqslant p(\lambda_1^2 + \cdots + \lambda_p^2) \leqslant p\big(n(n-1) - (n-p)/\alpha^2\big).$$

In case $p < 1/\alpha^2$ this is equivalent to our claim. Equality holds iff C has two eigenvalues $\lambda_1 = \cdots = \lambda_p$ and $-1/\alpha$. Then the lines span a positive definite subspace of dimension p. □

4.2.6. EXAMPLE. The lines spanned by the $p(p+1)/2$ vectors $(\sqrt{2}/2; 1^2 0^{p-1})$ are equiangular at arccos 1/3. All vectors are perpendicular to the vector $w = (2\sqrt{2}; 1^{p+1})$, which has norm $(w, w) = p - 7$. Hence for $p \geqslant 8$ we have equiangular lines an $\mathbb{R}^{p,1}$, whose cardinality meets the bound of Theorem 4.2.5. For $p \leqslant 6$ we have a Euclidean set in $\mathbb{R}^{p+1}$, whose cardinality meets the bound of Theorem 4.2.6 for $p = 3$ and $p = 4$, and almost for $p = 5$ (only $(3\sqrt{2}/2; 1^6)$ is missing). For $p = 7$ the vector w has norm 0, and our set is in the tangent hyperplane. If we factor out the radical $\langle w \rangle$ then we obtain 28 equiangular lines in Euclidean $\mathbb{R}^7 = w^\perp/\langle w \rangle$, as in Section 3.3.

4.2.7. EXAMPLE. The 276 lines at angle arccos 1/5 in $\mathbb{R}^{23}$, cf. Theorem 3.3.9, may be represented in $\mathbb{R}^{23,1}$ by the 23 vectors $(3\sqrt{2}; (-1)^1, 1^{22})$ and the 253 vectors $(\sqrt{2}; 1^7, 0^{16})$ according to the blocks of the Steiner system 4-(23,7,1). All vectors are perpendicular to $(7\sqrt{2}; 2^{23})$, hence lie in a positive definite subspace of dimension 23.

Our theorems may be applied to two-angle sets of lines as well, for instance to sets of n lines having angles $\pi/2$ and $\arccos \alpha$ in d-dimensional space $\mathbb{R}^{d-1,1}$. The n unit vectors, one along each line, have the annihilator

$$t(t^2 - \alpha^2) = tQ_2(t) + \big(3 - (d+2)\alpha^2\big)Q_1(t).$$

Application of Theorem 4.2.3 yields, e.g., $n \leqslant \binom{d+1}{3}$ for $(d+2)\alpha^2 = 3$.

4.2.8. EXAMPLE. Equality is realized by the following 165 lines at 90° and 60° in $\mathbb{R}^{9,1}$. They are spanned by the $120 + 45$ vectors $(1; 1^3, 0^7)$ and $(0; 1, -1, 0^8)$, and they are in $\mathbb{R}^{9,1}$ since they are perpendicular to $(3; 1^{10})$.

4.2.9. EXAMPLE. A further example of application of Theorem 4.2.3 is obtained by considering the normalized Gegenbauer polynomial

$$120Q_5(t) = t^5 - \frac{10t^3}{d+6} + \frac{15t}{(d+4)(d+6)}.$$

For $d = 26$ this reads

$$120Q_5(t) = t\Big(t^2 - \frac{1}{4}\Big)\Big(t^2 - \frac{1}{16}\Big).$$

In this case Theorem 4.2.3 says that in $\mathbb{R}^{25,1}$ there are at most $\binom{29}{5}$ lines at $\cos\phi \in \{0, 1/2, 1/4\}$. It would be interesting to know whether this bound can be attained, all the more by the existence of $\binom{28}{5}$ such lines in the Leech lattice in $\mathbb{R}^{24}$, cf. Sections 2.2 and 3.4.

4.3. *Graphs*

The use of indefinite space $\mathbb{R}^{p,1}$ for the representation of graphs is demonstrated. Sometimes such representations generate interesting integral lattices. These are considered more systematically in Section 4.4.

In Theorem 4.2.1 any symmetric matrix was seen to be the Gram matrix of vectors in $\mathbb{R}^{p,q}$. This applies in particular to the matrix $\lambda I - A$, where A is the (1, 0)-adjacency matrix of a graph on n vertices. If $\lambda = \lambda_1$, the largest eigenvalue of A, then the graph is represented by n vectors in positive definite space. If $\lambda = \lambda_2$, the second largest eigenvalue of A, then the graph is represented by n vectors outside of the light cone in $\mathbb{R}^{p,1}$. We consider the cases $I - A$ and $2I - A$.

Let $\mathbb{R}^{p,1}$ have the orthonormal basis $e_0, e_1, \dots, e_p$, with

$$-(e_0, e_0) = 1 = (e_1, e_1) = \cdots = (e_p, e_p).$$

The only graphs having positive semidefinite $I - A$ are the one-factors, which are represented in $\mathbb{R}^p$ by the cross-polytope $B^{(p)} := \{\pm e_1, \dots, \pm e_p\}$. The truncated cross-polytope

$$D^{(p)} := \{e_0 \pm e_i \pm e_j\colon\ 1 \leqslant i \neq j \leqslant p\} \subset \mathbb{R}^{p,1}$$

consists of $2p(p-1)$ vectors of norm 1 having mutual inner products in $\{0, -1, -2, -3\}$. Any subset of $D^{(p)}$ avoiding the inner products -2 and -3 has a Gram matrix $I - A$, hence yields a graph represented in $\mathbb{R}^{p,1}$. For instance, the subset

$$\{e_0 - e_i - e_j\colon\ i,\ j \in V,\ \{i,j\} \in E\}$$

yields the complement of the line graph of the graph (V, E). If we hang on one-factors of the type $e_0 - e_i \pm e_{i,a}$ with $a = 1, \dots, a_i$, then we obtain the complement of Hoffman's *generalized line graph*

$$L\big((V, E); a_1, \dots, a_p\big)$$

cf. Cameron et al. [1976]. Other interesting sets of unit vectors in $\mathbb{Z}^{p,1}$ are, cf. Du Val [1937], Neumaier and Seidel [1983].

$$H^{(p)} := \big\{x \in \mathbb{Z}^{p,1}\colon\ (x,x) = 1 = \big(x, w^{(p)}\big)\big\}, \quad w^{(p)} = -3e_0 + e_1 + \cdots + e_p.$$

For $p \geqslant 9$ this set is infinite, but for smaller p the set is finite, and given by the vertices of the uniform polytope $(p-4)_{21}$:

p	=	3	4	5	6	7	8	>8
$\lvert H^{(p)}\rvert$	=	6	10	16	27	56	240	∞

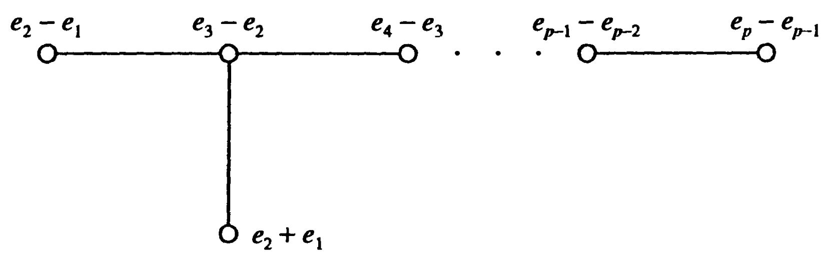
For $p = 4$ it consists of 10 vectors which produce the Petersen graph:

$$(e_0 - e_1 - e_4,\ e_4,\qquad e_0 - e_3 - e_4,\ e_0 - e_1 - e_2,\ e_1)$$
$$(e_0 - e_2 - e_3,\ e_0 - e_2 - e_4,\ e_3,\qquad e_2,\qquad e_0 - e_1 - e_3)$$

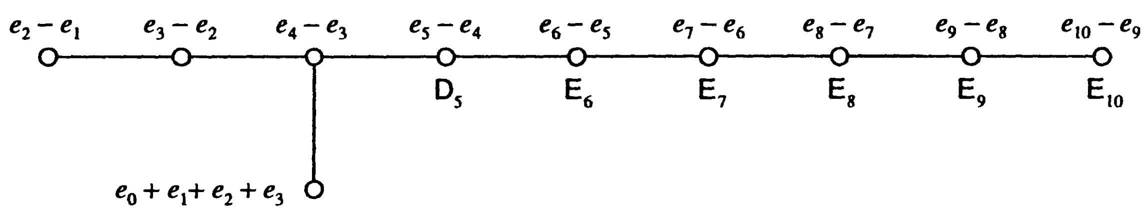
Likewise, for $p = 3$ we obtain the hexagon, for $p = 5$ the Clebsch graph on 16, and for $p = 6$ the Schläfli graph on 27 vertices. For $p = 7$ inner products $0, -1, -2$ occur, producing the double cover corresponding to the regular two-graph on 28 vertices, cf. Section 3.3. For $p = 8$ inner products $0, \pm 1, \pm 2$ occur, producing the 240 norm 2 vectors of the root system E_8, cf. Section 2.2.

We now turn to $2I - A$. The vertices of certain graphs with adjacency matrix A, sometimes called Dynkin diagrams, are given in terms of vectors, first of $\mathbb{R}^p$, then of $\mathbb{R}^{p,1}$.

4.3.1. DEFINITION. The following diagram



defines the graphs D_p. The graphs A_{p-1} are obtained by suppressing the vertex $e_1 + e_2$. The graphs E_p are defined as follows:

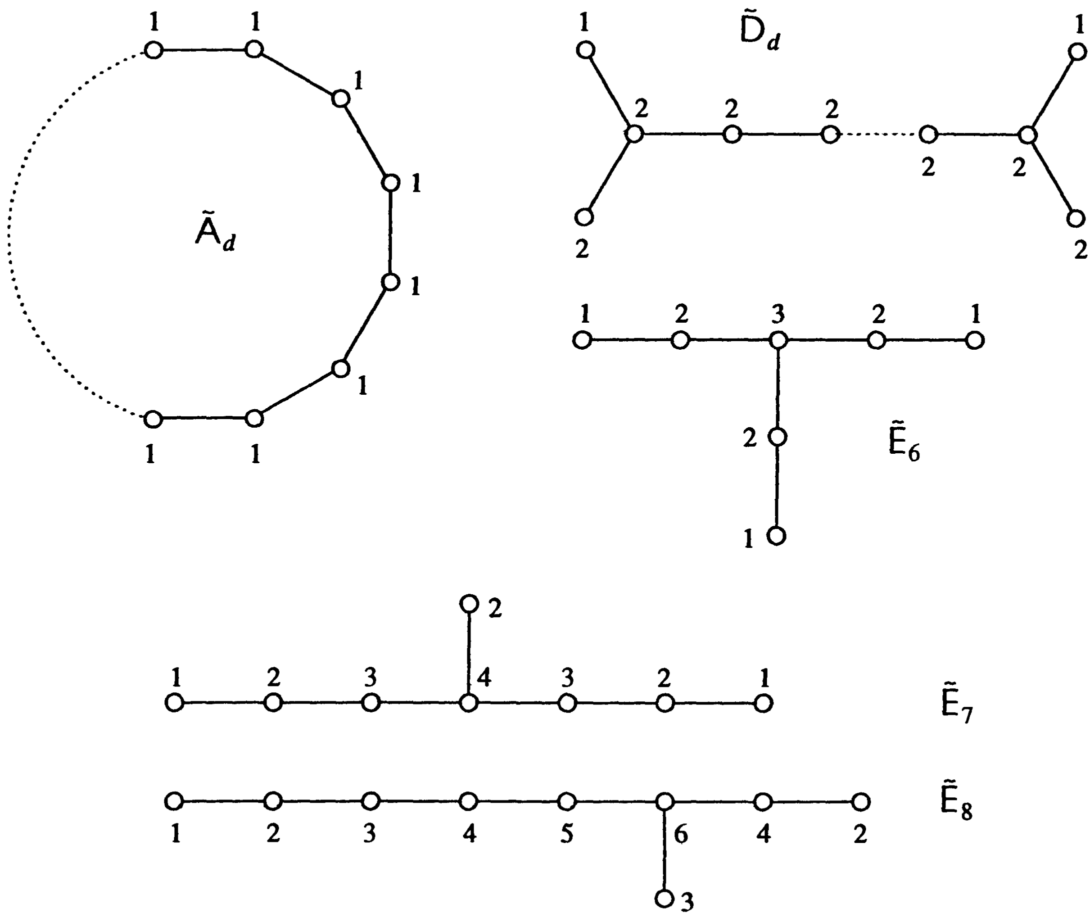


4.3.2. THEOREM. $\mathsf{A}_{p-1}, \mathsf{D}_p, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{E}_9, \mathsf{E}_{10}$ *have* $\det(2I - A) = p, 4, 3, 2, 1, 0, -1$, *respectively. The matrix* $2I - A$ *is positive definite, except for* E_9 *and* E_{10}, *when* A *has maximum eigenvalue* $\lambda_1 = 2$ *and* $2.0066\ldots$, *respectively.*

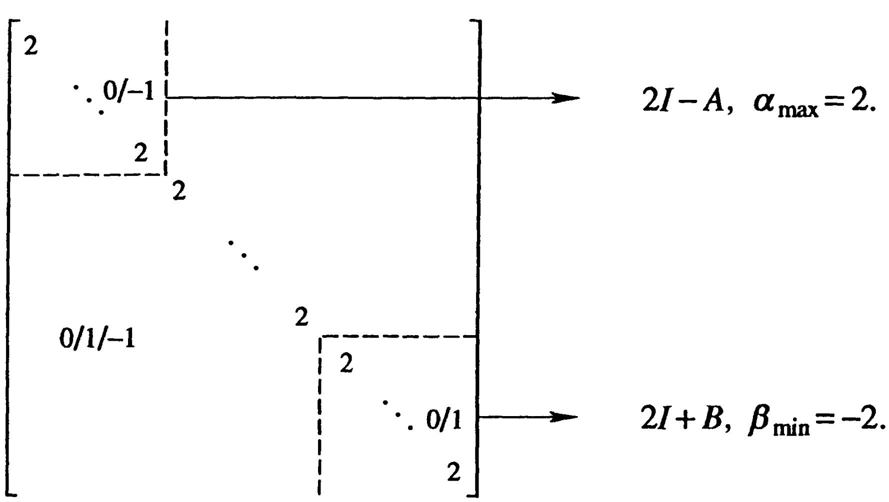
The proof is by evaluating determinants, cf. Milnor and Husemoller [1973], Bourbaki [1968].

The graphs $\mathsf{A}_{p-1}, \mathsf{D}_p, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ are the only connected graphs having $\lambda_1 < 2$. They are proper subgraphs of graphs having $\lambda_1 = 2$, which are called *Coxeter graphs*, or extended Dynkin diagrams. The following enumeration is not difficult to achieve (see Coxeter [1973], Lemmens and Seidel [1973a]). For each graph we have indicated the eigenvector corresponding to the maximum eigenvalue $\lambda = 2$.

4.3.3. THEOREM. *The only Coxeter graphs* (*connected graphs with* $\lambda_1 = 2$) *are the following* $\bar{\mathsf{A}}_d, \bar{\mathsf{D}}_p, \bar{\mathsf{E}}_{6,7,8}$:



The graphs of Theorem 4.3.3 are related to the root systems A, D, E, i.e. the sets of lines at 60°, 90° which we encountered in Definition 3.4.2 and Theorem 3.4.3. Indeed, let $2I + C$ denote the Gram matrix of a set of vectors of norm 2, one along each line. Upon rearrangement and switching the positive semidefinite matrix $2I + C$ can be written as follows.



First, we select as many roots as possible having inner products 0 and -1 only. For their Gram matrix $2I - A$, the (0, 1) matrix A indicates a graph having maximum eigenvalue 2. Such graphs have been determined in Theorem 4.3.3.

Secondly, we select as many roots as possible having inner products 0 and 1 only. For their Gram matrix $2I + B$, the (0, 1) matrix B indicates a graph having minimum eigenvalue -2.

Examples of such graphs are the line graphs (encountered earlier in this section), but also other graphs such as the strongly regular graphs on 10, 16, 27, 28 vertices, cf. Section 3.2, the complement of a one-factor

$$\begin{bmatrix} J-I & J-I \\ J-I & J-I \end{bmatrix},$$

and the generalized line graphs. How to determine *all* graphs having smallest eigenvalue -2? Let B denote the adjacency matrix of such a graph. Then make up the positive semidefinite matrix $2I + B$, the corresponding set of vectors in Euclidean space, and the set of lines at 60° and 90° spanned by these vectors. We then *close* this set by adding to each pair of lines at 60° the third line of their planar star. Finally, we apply the following theorem, cf. Cameron et al. [1976].

4.3.4. THEOREM. *The irreducible sets of lines at* 60°, 90° *which are star-closed, are precisely the root systems* $\mathsf{A}_d, \mathsf{D}_d, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$.

As a consequence, all graphs with minimum eigenvalue -2 are obtained from D_d (the generalized line graphs) and from E_8 (the exceptional graphs).

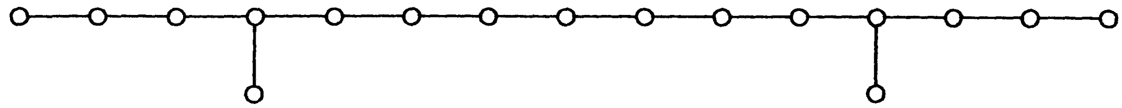
We return to graphs whose adjacency matrix has $\lambda_2 = 2$. These graphs are called *reflexive graphs*. They are the Lorentzian counterparts of the spherical and the Euclidean graphs which occur in the theory of reflection groups, see Neumaier and Seidel [1983]. As an example, we construct 25 vectors in $\mathbb{Z}^{19,1}$ having Gram matrix $2I - A$. Let N denote the 10×15 vertex-edge incidence matrix of the Petersen graph. Then $NN^t = 3I + A$ (Petersen), has spectrum $6^1, 4^5, 1^4$. This implies that

$$\begin{bmatrix} 2I & -N \\ -N^t & 2I \end{bmatrix}$$

has spectrum $1 \times (2 - \sqrt{6})$, 5×0, $1 \times (2 + \sqrt{6})$, $18 \times$ positive. Hence this matrix is the Gram matrix of 25 vectors in $\mathbb{R}^{19,1}$ with norm 2 and inner products 0, -1. It is possible to give these vectors integral coordinates, hence the set is in $\mathbb{Z}^{19,1}$. Its graph

$$\begin{bmatrix} 0 & N \\ N^t & 0 \end{bmatrix}$$

is the incidence graph on the $10 + 15$ vertices and edges of the Petersen graph. This diagram is due to Vinberg [1975], see also Vinberg and Kaplinskaja [1978], who also describe sets of vectors and diagrams in lower dimensions by deleting suitable vectors: 22 in $\mathbb{R}^{18,1}$, and 17 in $\mathbb{R}^{15,1}$ with graph



and various extended Dynkin diagrams in $\mathbb{R}^{16}$: $\overline{E}_8 \oplus \overline{E}_8$, $\overline{D}_8 \oplus \overline{D}_8$, $\overline{D}_{16}$.

A further example of reflexive graphs is obtained from the Steiner system 3-(22, 6, 1). It consists of 77 blocks of size 6 from 22 symbols such that each triple of symbols is in precisely one block. It turns out that the blocks intersect in either 2 or 0 symbols, cf. Section 2.2.9. The corresponding 77 vectors $(\sqrt{2}; 1^6 0^{16})$ all are perpendicular to $(3\sqrt{2}; 1^{22})$, hence are in $\mathbb{R}^{21,1}$ and have Gram matrix $2(2I - A)$. It turns out that the graph A is strongly regular with smallest eigenvalue -3.

We shall return to reflexive graphs in Section 4.4. In fact, the vectors at angles 90° and 60° in $\mathbb{R}^{p,1}$ which correspond to a reflexive graph will be seen to generate a reflexive lattice.

4.4. *Integral lattices*

We give a brief introduction, following Serre [1970] with a touch of Vinberg [1972] and Conway and Sloane [1988]. An *integral lattice* L is a free Abelian group provided with a nondegenerate symmetric bilinear form $(x, y) \in \mathbb{Z}$. The lattice L is *unimodular* whenever the map

$$L \longrightarrow \mathrm{Hom}(L, \mathbb{Z}), \quad y \mapsto (x, y),$$

is an isomorphism, i.e. whenever the Gram matrix $G = [(e_i, e_j)]$ of a basis $e_1, \ldots, e_d$ has determinant ± 1. We borrow the notions *rank* $d(L)$ and *signature* $\tau(L)$ from the Gram matrix in $L \otimes \mathbb{R}$, and distinguish between *definite* (only positive or only negative norms) and *indefinite* (otherwise) lattices. We say that the lattice L has *type* II if

$$\forall x \in L \quad \big((x, x) \equiv 0 \ (\mathrm{mod}\ 2)\big);$$

type I otherwise. These types are distinguished by the following invariant. $\overline{L} := L/2L$ is a vector space over $\mathbb{F}_2$, and $(\overline{x}, \overline{x})$ is linear, since $(\overline{x}+\overline{y}, \overline{x}+\overline{y}) = (\overline{x}, \overline{x}) + (\overline{y}, \overline{y})$. Then there is a unique $\overline{u} \in \overline{L}$ such that $(\overline{u}, \overline{x}) = (\overline{x}, \overline{x})$ for all $\overline{x} \in \overline{L}$. Hence there exists $u \in L$ with $(u, x) \equiv (x, x)$ mod 2 for all $x \in L$. From

$$(u + 2x, u + 2x) = (u, u) + 4\big((u, x) + (x, x)\big) \equiv (u, u) \bmod 8$$

it follows that

$$\sigma(L) := (u, u) \in \mathbb{Z}/8\mathbb{Z}$$

is an *invariant*. For type II lattices we have $\sigma(L) = 0$ since $u = 0$ applies. This yields Corollary 4.4.3 below. Finally,

$$\phi(L_1 \oplus L_2) = \phi(L_1) + \phi(L_2) \quad \text{for } \phi = d, \tau, \pi.$$

We now give a number of examples and general theorems.

4.4.1. EXAMPLES. The lattice I_+ is $\mathbb{Z}$ with the form xy. It has $d = 1$, $\sigma(I_+) \equiv 1 \pmod 8$ since $x^2 \equiv x \cdot 1 \pmod 2$.

The lattice I_- is $\mathbb{Z}$ with $-xy$. It has $d = 1$ and $\sigma(I_-) = -1$.

The lattice $pI_+ \oplus qI_-$ has $\sigma \equiv p - q \pmod 8$.

The lattice

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is $\mathbb{Z}^2$ with $(x, y) = x_1y_2 + x_2y_1$. It has $\sigma = \tau = 0$, $d = 2$, $\det = -1$.

The standard lattice $\mathbb{Z}^d$ with $(x, y) = \sum x_iy_i$ has $\det = 1$.

The lattice $\mathsf{D}_d := \langle e_i + e_j \rangle_{\mathbb{Z}} = \{x \in \mathbb{Z}^d\colon \sum x_i \text{ even}\}$ has index 2 in $\mathbb{Z}^d$.

The lattice $\mathsf{E}_d := \langle \mathsf{D}_d, e = (e_1 + \cdots + e_d)/2 \rangle_{\mathbb{Z}}$ contains D_d of index 2, since $2e \in \mathsf{D}_d$. As a consequence, each E_{4k} is integral unimodular, and E_{8m} is of type II.

For the following theorems about integral unimodular lattices L we refer to Serre [1970].

4.4.2. THEOREM. $\sigma(L) \equiv \tau(L) \bmod 8$.

4.4.3. COROLLARY. *For L of type* II, *the signature* $\tau \equiv 0 \pmod 8$.

4.4.4. THEOREM. *Indefinite L of type* I *must be $pI_+ \oplus qI_-$.*

4.4.5. THEOREM. *Indefinite L of type* II *must be $pU \oplus q\mathsf{E}_8$.*

Theorems 4.4.4 and 4.4.5 yield a complete classification of indefinite integral unimodular lattices. The starting point in the proof is the selection of an indivisible (the coordinates have gcd $= 1$) vector e having $(e, e) = 0$, and of a vector f with $(e, f) = 1$, and $(f, f) = 1$ or 0 depending on whether L is of type I or type II. By taking suitable linear combinations one can achieve

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \oplus L' \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus L',$$

with integral unimodular L', thus splitting off a sublattice $I_+ \oplus I_-$ or U, respectively.

In the space $\mathbb{R}^{p,1}$ with $(x, y) = -x_0y_0 + x_1y_1 + \cdots + x_py_p$ the only even unimodular integral lattice (only for $p \equiv 1 \bmod 8$) is

$$II_{p,1} = U \oplus (\mathsf{E}_8)^{(p-1)/8}.$$

Splitting off $I_+ \oplus I_-$ or U yields integral unimodular lattices in Euclidean $\mathbb{R}^{p-1}$, which by no means are unique. Indeed, for $p \equiv 1 \pmod 8$ Euclidean $\mathbb{R}^{p-1}$ contains the following nonisomorphic type II lattices.

Only E_8 for $p = 9$. The nonisomorphic E_{16} and $\mathsf{E}_8 \oplus \mathsf{E}_8$ for $p = 17$.

The Leech lattice and the 23 Niemeier lattices for $p = 25$ (minimum norm 4 and 2, respectively) (see Venkov [1980]).

Several millions for $p = 33$, all generated by vectors norm 2 and 4, with 15 exceptions, cf. Koch and Venkov [1989].

On the other hand, any integral lattice in Euclidean $\mathbb{R}^{p-1}$ may be lifted into an integral lattice in $\mathbb{R}^{p,1}$ by taking the integral linear combinations with suitable vectors. In terms of the Gram matrices of bases this reads (take $\varepsilon = 1$ ($\varepsilon = 0$) in the odd (even) case):

$$[\text{indefinite}_{p,1}] \cong \begin{bmatrix} 0 & 1 \\ 1 & \varepsilon \end{bmatrix} \oplus [\text{Euclid}_{p-1}] .$$

4.4.6. EXAMPLE. The relation between $II_{9,1}$ and E_8. In $II_{9,1}$ the following $2 \times 165 = 2 \times (120 + 45)$ vectors are nearest to the origin:

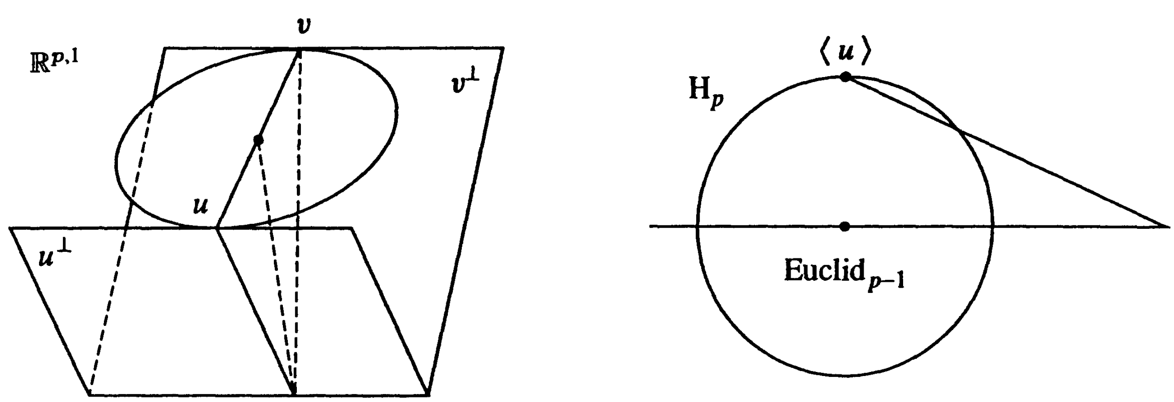
$$\pm(1; 1^3 0^7), \quad \pm(0; 1(-1)0^8), \quad \text{all} \perp (3; 1^{10}).$$

Working modulo the norm 0 vectors

$$u = (3; 1^8 10) \quad \text{and} \quad -v = (3; 1^8 01), \quad \text{Gram}(u, v) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we find 2×120 vectors in E_8, in $u^\perp \cap v^\perp$ with the equations

$$x_9 = x_{10} = 0, \qquad 3x_0 = x_1 + \cdots + x_8.$$



$\mathbb{R}^{p-1} = u^\perp \cap v^\perp = u^\perp / \langle u \rangle$ Euclid^{p-1}

Projectively, the relation between the indefinite and the Euclidean lattice amounts to stereographic projection. However, it depends on the choice of the isotropic vector u, which Euclidean lattice is obtained. For instance, Conway and Sloane [1988]:

$II_{9,1}$ with $u = (3; 1^9)$ yields the unique lattice E_8;

$II_{17,1}$ with $u = (3; 1^9 0^8)$ yields $\mathsf{E}_8 \otimes \mathsf{E}_8$;

$II_{25,1}$ with $u = (5; 1^{25})$ yields a Niemeier lattice;

with $u = (70; 0, 1, 2, \ldots, 24)$ yields the Leech lattice.

For indefinite integral lattices and their Euclidean projections several properties agree:

indefinite$_{p,1}$	unique	type	unimodular	reflexive
Euclid$_{p-1}$	all	type	unimodular	reflexive

4.4.7. DEFINITION. A lattice is *reflexive* if (i) its automorphism group contains a subgroup of finite index which is generated by reflections, and (ii) the roots span the space.

For reflexive lattices the determination of the automorphism group and its fundamental domain becomes tractable.

In Section 4.3 we have constructed a reflexive graph corresponding to 25 vectors with Gram matrix $2I - A$. These vectors span the lattice $I_{19,1}$, which therefore is reflexive. Furthermore, Euclidean space $\mathbb{R}^{19}$ (and hence any $\mathbb{R}^p$ for $p \geqslant 19$) contains a unimodular lattice which is not spanned by norm 2 vectors. Indeed, to the lattice $\mathsf{E}_6 \oplus \mathsf{E}_6 \oplus \mathsf{E}_6 \oplus \mathbb{Z}\sqrt{3}$, which has det Gram (basis) = 3^4, we add the integral linear combinations with the two glue vectors $(0, a, -a, 1/\sqrt{3})$ and $(-a, 0, a, 1/\sqrt{3})$, where $a = (1^4(-2)^2; 0^2)/3$. Since $3a \in \mathsf{E}_6$ and $3/\sqrt{3} = \sqrt{3}$, we obtain an integral unimodular Euclidean lattice in $\mathbb{R}^{19}$, which is not spanned by norm 2 vectors. From these facts one concludes the following theorem, due to Vinberg [1972].

4.4.8. THEOREM. *All unimodular integral Euclidean lattices of dimension* $\leqslant 18$ *are reflexive. Unimodular integral lattices in* $\mathbb{R}^{p,1}$ *are reflexive iff* $p \leqslant 19$.

4.4.9. REMARK. Nikulin proved that for (even) integral $(p, 1)$-lattices, $p \geqslant 4$, only finitely many are reflexive. In contrast to this, Kneser shows that a very large class of internal lattices of signature p, q, with $p \geqslant 2$, $q \geqslant 2$, is reflexive, cf. Nikulin [1981], Kneser [1981].

In the final part of this section we drop the unimodularity, but specialize into other directions. Let L denote an integral lattice.

4.4.10. DEFINITION. The *discriminant* $\delta(L)$ is the determinant of the Gram matrix of any integral basis of L. The *dual lattice* L^* is the lattice defined by

$$L^* := \{x \in \mathbb{R} \otimes L\colon\ (x, y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

Clearly $L^* = L$ iff L is unimodular, i.e. $\delta(L) = 1$. Otherwise $L \subset L^* \subset L/\delta(L)$, and L^*/L is an Abelian group of order $\delta(L)$. This is illustrated for root lattices.

4.4.11. DEFINITION. A *root lattice* is an integral lattice which is generated by vectors of norm 2.

Euclidean space contains the following root lattices of types A, D, E. Some of these have been defined earlier, in Sections 4.4.1 and in 2.2.11.

$$\mathsf{A}_d := \{x \in \mathbb{R}^{d+1}\colon\ x_i \in \mathbb{Z},\ x_0 + x_1 + \cdots + x_d = 0\},$$

$$\mathsf{D}_d := \{x \in \mathbb{R}^d\colon\ x_i \in \mathbb{Z},\ x_1 + x_2 + \cdots + x_d \in 2\mathbb{Z}\},$$
$$\mathsf{E}_8 := \langle \mathsf{D}_8, (e_1 + \cdots + e_8)/2 \rangle \mathbb{Z},$$
$$\mathsf{E}_7 := \{x \in \mathsf{E}_8\colon\ x_1 + \cdots + x_8 = 0\},$$
$$\mathsf{E}_6 := \{x \in \mathsf{E}_8\colon\ x_1 + \cdots + x_6 = x_7 + x_8 = 0\}.$$

Each of these root lattices is generated by the root system of Definition 3.4.2, and by the Coxeter graph of Definition 4.3.1, which carries the same name.

It is not difficult to show that the discriminants of these lattices have the following values, cf. Section 4.3.2:

$$\delta(\mathsf{A}_d) = d + 1, \quad \delta(\mathsf{D}_d) = 4, \quad \delta(\mathsf{E}_6) = 3, \quad \delta(\mathsf{E}_7) = 2, \quad \delta(\mathsf{E}_8) = 1.$$

We refer to Bourbaki [1968] and to Brouwer, Cohen and Neumaier [1988] for further details about these lattices, and their duals:

$$\mathsf{A}_d^* = \langle \mathsf{A}_d, (e_1 + \cdots + e_d - de_{d+1})/(d+1) \rangle_{\mathbb{Z}};$$
$$\mathsf{D}_d^* = \langle \mathsf{D}_d, e_1, (e_1 + e_2 + \cdots + e_d)/2 \rangle_{\mathbb{Z}};$$
$$\mathsf{E}_7^* = \langle \mathsf{E}_7, (e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3e_7 - 3e_8)/4 \rangle_{\mathbb{Z}};$$
$$\mathsf{E}_6^* = \langle \mathsf{E}_6, (e_1 + e_2 + e_3 + e_4 - 2e_5 - 2e_6)/2 \rangle_{\mathbb{Z}}.$$

4.4.12. THEOREM. *In Euclidean space every root lattice is the direct sum of lattices of type* A, D, E.

This theorem, which goes back to Witt, is the companion to Theorem 4.3.4 on root systems. It can be proved in a similar way, cf. Brouwer et al. [1989], 3.10.4. The theorem is useful in the classification of certain graphs.

The characterization theorems for lattices and root systems of types A, D, E also apply to the Leech lattice. This lattice has no roots at all, since every vector has norm $\geqslant 4$. However, the Leech lattice contains *deep holes* of radius $\sqrt{2}$, i.e. balls whose boundary does, but whose interior does not contain lattice points. Let h denote the centre of a deep hole, and let the boundary contain the distinct lattice points $0, z_2, z_3, \ldots, z_p$. For $h_i := z_i - h$ we have

$$4 \leqslant (z_i - z_j, z_i - z_j) = (h_i - h_j, h_i - h_j) = 2 + 2 - 2(h_i, h_j).$$

Hence (h_i, h_j) takes the values 0 and -1, and

$$\mathrm{Gram}(h_1, h_2, \ldots, h_p) = 2I - A \quad \text{with } (1,0)\text{-matrix } A.$$

Therefore, the boundary points of a deep hole correspond to a Coxeter graph, which by Theorem 4.3.3 is a disjoint union of Coxeter graphs of type A, D, E. Conway and Sloane found that there are 23 types of deep holes and that the Leech lattice can be reconstructed from each deep hole, cf. Conway and Sloane [1988].

4.5. *Distance matrices*

A final application to distance matrices and embeddability problems brings us back to indefinite metric, and to root lattices. For further development concerning distance spaces in relation to combinatorics, we refer to Neumaier [1980] and to Godsil [1988].

4.5.1. DEFINITION. A *distance matrix* is a real symmetric matrix which has zero diagonal and positive entries elsewhere.

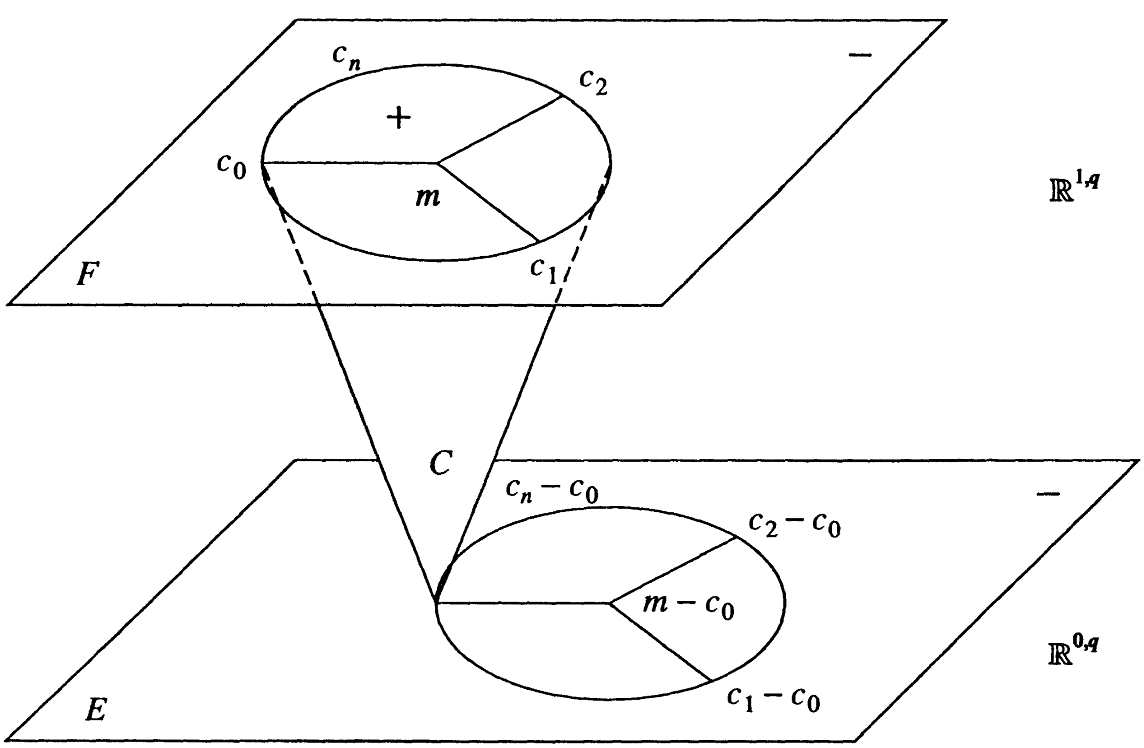
Let the distance matrix D have size $n+1$, say. By Theorem 4.2.1, D may be viewed as the Gram matrix of the inner products of $n+1$ vectors $c_0, c_1, \dots, c_n$ in $\mathbb{R}^{p,q}$, where p, q denote the signature of D, so that $\operatorname{rk} D = p + q \leqslant n+1$.

4.5.2. DEFINITION. A distance matrix D has *negative type* whenever $(x, x) < 0$ for every nonzero vector

$$x = \sum_{i=0}^{n} \xi_i c_i, \quad \xi_i \in \mathbb{R}, \ \sum_{0}^{n} \xi_i = 0.$$

Since these vectors x constitute a Euclidean subspace E of dimension $p + q - 1$, we must have $p = 1$, and the indefinite space is $\mathbb{R}^{1,q}$.

4.5.3. THEOREM. *A distance matrix of negative type, of size $n+1$ and rank $q+1$, is the Gram matrix of $n+1$ vectors in $\mathbb{R}^{1,q}$. The endpoints of these vectors are in the intersection sphere of the light cone C and a flat F of dimension q.*



PROOF. We already observed that

$$D = \mathrm{Gram}(c_0, c_1, \ldots, c_n), \quad c_0, \ldots, c_n \subset C \subset \mathbb{R}^{1,q}.$$

The Euclidean subspace $E = \mathbb{R}^{0,q}$ contains the vectors $c_1 - c_0, \ldots, c_n - c_0$. Let F denote the flat parallel to E containing the endpoint of c_0, then

$$c_0, c_1, \ldots, c_n \in F \cap C, \quad F \parallel E, \quad E \cap C = 0.$$

The unique vector $m \in F$, $m \perp E$, satisfies

$$(m, c_i) = \text{const}, \qquad (m - c_i, m - c_i) = \text{const},$$

$$(m - c_i, c_i - c_0) = (c_0, c_i) \quad \text{for } i = 0, 1, \ldots, n.$$

Hence m is the centre of the sphere $F \cap C$. □

Conversely we have:

4.5.4. THEOREM. *Any negative definite space E with basis $\{e_1, \ldots, e_q\}$ can be embedded in an $\mathbb{R}^{1,q}$ with isotropic basis $\{c_0, c_1, \ldots, c_q\}$ such that $e_i = c_i - c_0$ and*

$$\mathrm{Gram}(c_0, c_1, \ldots, c_q) = -\frac{1}{2}\begin{bmatrix} 0 & (e_i, e_i) \\ (e_j, e_j) & (e_i - e_j, e_i - e_j) \end{bmatrix}.$$

PROOF. Given the negative definite $E = \langle e_1, \ldots, e_q \rangle_{\mathbb{R}}$ we define $e_0 \in \mathbb{R}^{1,q}$, $(e_0, e_0) = 1$, $(e_0, e_i) = 0$, and ask for $c_0, c_1, \ldots, c_q \in \mathbb{R}^{1,q}$ such that

$$c_i - c_0 = e_i, \qquad (c_0, c_0) = 0 = (c_i, c_i), \quad i = 1, \ldots, q.$$

These requirements determine $\gamma_0 > 0$, $\gamma_1, \ldots, \gamma_q$ in

$$c_0 = \gamma_0 e_0 + \sum_{j=1}^{q} \gamma_j e_j,$$

whence also $c_i = e_i + c_0$. □

REMARK. If all $(e_i, e_i) = -2$ then $\mathrm{Gram}(c_0, c_1, \ldots, c_q)$ is the *Cayley–Menger* matrix of the vectors $\{e_1, \ldots, e_q\}/\sqrt{2}$.

Distance matrices of negative type play a role in problems of metric embeddability, cf. Schoenberg [1938], Assouad and Deza [1980]. In this connection Deza [1960] and Kelly [1970] introduced the following notion.

4.5.5. DEFINITION. A distance matrix $D(c_0, c_1, \ldots, c_n)$ is *hypermetric* whenever

$$(y, y) < 0 \quad \text{for every } y = \sum_{i=0}^{n} \eta_i c_i, \ \eta_i \in \mathbb{Z}, \ \sum_{0}^{n} \eta_i = 1.$$

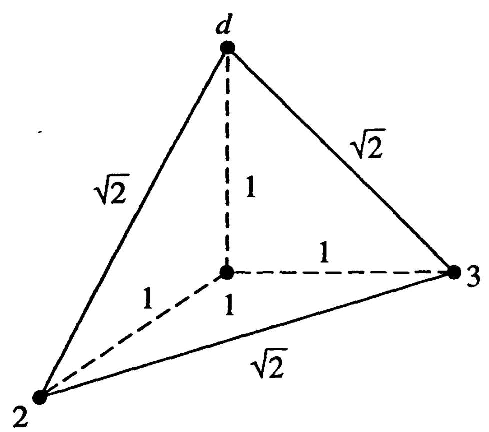
4.5.6. THEOREM. *Any hypermetric distance matrix is of negative type.*

This is easy to prove. As a consequence, we observe from the pictures of Theorem 4.5.3:

4.5.7. THEOREM. *The distance matrix $D(c_0, c_1, \ldots, c_n)$ is hypermetric whenever the sphere $F \cap C$ constitutes a hole in the lattice induced in F by $\langle c_0, c_1, \ldots, c_n \rangle_{\mathbb{Z}}$.*

However, one must be careful, as the following example from Winkler [1985] shows:

4.5.8. EXAMPLE. Let D denote the Cayley–Menger distance matrix of the 'corner-simplex' in Euclidean $(d-1)$-space:



$$, \quad D = \begin{bmatrix} 0 & 1 & u' \\ 1 & 0 & u' \\ u & u & 2J - 2I \end{bmatrix} \begin{matrix} 1 \\ 1 \\ d-1 \end{matrix}$$

This D has size $d + 1$ and signature $1, d$, hence D can be viewed as the Gram matrix of $d + 1$ independent vectors $c_0, c_1, \ldots, c_d \in \mathbb{R}^{1,d}$. We investigate whether D has negative type, by checking the intersection of the light cone C and the subspace $E = \langle c_1 - c_0, \ldots, c_d - c_0 \rangle_{\mathbb{R}}$. This can be done by the sign of

$$\mu := (m, m) \quad \text{for } m = E^{\perp}, \ (m, c_0) = \cdots = (m, c_d) = 1,$$

say. The Gram matrix of $m, c_0, c_1, \ldots, c_d$ reads

$$\begin{matrix} 1 \\ 1 \\ 1 \\ d-1 \end{matrix} \begin{bmatrix} \mu & 1 & 1 & u' \\ 1 & 0 & 1 & u' \\ 1 & 1 & 0 & u' \\ \mu & \mu & \mu & 2J - 2I \end{bmatrix} \approx \begin{bmatrix} \mu & 1 & 1 & 0' \\ 1 & 0 & 1 & 0' \\ 1 & 1 & 0 & u' \\ 0 & 0 & u & -2I \end{bmatrix}$$

$$\approx \begin{bmatrix} \mu & 1 & 1 & 0' \\ 1 & 0 & 1 & 0' \\ 1 & 1 & \frac{1}{2}(d-1) & 0' \\ 0 & 0 & 0 & -2I \end{bmatrix};$$

where u denotes the all-one column of size $d-1$. Since the vectors are dependent, we have

$$0 = \det = (-2)^{d-1}\left(2 - \mu - \frac{d-1}{2}\right) = (-2)^{d-2}(2\mu + d - 5).$$

If $d \geqslant 6$ then $\mu < 0$, hence $E \cap C \neq 0$ and D is not of negative type. If $d = 4$ then $\mu = 1/2$ and D has negative type; however, D is not hypermetric since

$$x = -c_0 - c_1 + c_2 + c_3 + c_4$$

has $(x, x) > 0$.

Finally, we turn to distance matrices $D(c_0, c_1, \ldots, c_n)$ of negative type which have *integral* entries (c_i, c_j). In this case the lattices

$$\langle c_0, c_1, \ldots, c_n \rangle_{\mathbb{Z}} \subset \mathbb{R}^{1,q} \quad \text{and} \quad L := \langle c_1 - c_0, \ldots, c_n - c_0 \rangle_{\mathbb{Z}} \subset E$$

are *even* integral lattices since $(c_i, c_i) = 0$ and $(c_i + c_j, c_i + c_j) = 2(c_i, c_j)$. Referring to the proof of Theorem 4.5.3 we also observe that in E the vectors $m - c_0, m - c_1, \ldots, m - c_n$ belong to the coset $L + m - c_0$ of L in the dual lattice L^*. They are vectors of *minimum norm* in L^* whenever $F \cap C$ constitutes a hole in the lattice induced in F, i.e. whenever the integral distance matrix D is hypermetric.

These observations produce the results of Terwilliger and Deza [1987]. Indeed, suppose that our distance matrix D of negative type has enough entries $(c_i, c_j) = 1$. Then L is a Euclidean root lattice, hence a disjoint union of lattices of types $\mathsf{A}, \mathsf{D}, \mathsf{E}$. The vectors $m - c_0, m - c_1, \ldots, m - c_n$ belong to the dual lattice L^*. If D is hypermetric, then these vectors have minimum norm in L^*, hence can be determined. This characterizes finite integral hypermetric distance matrices.

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CHAPTER 16

Distance Preserving Transformations

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Contents

Introduction 923
1. Alexandrov-type theorems 924
2. Beckman–Quarles-type theorems 930
3. Zeeman-type theorems 935
References 938

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Introduction

Like many concepts in mathematics, geometrical notions of 'distance' can often be many and varied. Some conservatively remain close to the canonical Euclidean distance, with its triangle inequality, additivity along lines, etc. Others wander far afield and break most of the Euclidean rules: distance need no longer be positive (or even real), symmetry disappears, and about all that remains is a space with scalars somehow assigned to pairs of its points (Benz [1981a]).

Once we have settled on a suitable definition of distance, however, the next question is immediate: which transformations of the space will preserve this distance? Depending on just what is assumed about the transformations (should they be somehow regular? should they be bijective? need they preserve *all* distances? etc.), the answers to this question, even for quite elementary spaces, can give rise to a wide range of interesting geometrical problems of greatly varying difficulty.

Consider, for example, the problem of determining the distance preserving transformations of ordinary real Euclidean n-space $\mathbb{E}_n$. We take $\mathbb{E}_n$ to be $\mathbb{R}^n$ with the standard dot product

$$\boldsymbol{r} \cdot \boldsymbol{s} := r_1 s_1 + r_2 s_2 + \cdots + r_n s_n$$

for all $\boldsymbol{r} := (r_1, r_2, \ldots, r_n)$, $\boldsymbol{s} := (s_1, s_2, \ldots, s_n) \in \mathbb{E}_n$; the distance between two arbitrary points $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathbb{E}_n$ is then $\{(\boldsymbol{r}-\boldsymbol{s})\cdot(\boldsymbol{r}-\boldsymbol{s})\}^{1/2}$. The translations $\boldsymbol{r} \mapsto \boldsymbol{r}+\boldsymbol{a}$ for all $\boldsymbol{a}$ in $\mathbb{E}_n$ clearly preserve this distance, as do those linear transformations $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ of $\mathbb{E}_n$ with $\bar{\boldsymbol{r}}\cdot\bar{\boldsymbol{s}} = \boldsymbol{r}\cdot\boldsymbol{s}$ for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathbb{E}_n$ (the orthogonal transformations of $\mathbb{E}_n$). We would like to know whether there are any others: can a distance preserving transformation of $\mathbb{E}_n$ be anything besides an affine transformation with orthogonal linear part?

The answer is trivial if we assume that the mapping $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is already affine. Without loss of generality, $\mathbf{0} \mapsto \mathbf{0}$; then, from the *polarization identity*

$$\boldsymbol{r}\cdot\boldsymbol{s} = \frac{1}{2}\big\{(\boldsymbol{r}-\mathbf{0})\cdot(\boldsymbol{r}-\mathbf{0}) + (\boldsymbol{s}-\mathbf{0})\cdot(\boldsymbol{s}-\mathbf{0}) - (\boldsymbol{r}-\boldsymbol{s})\cdot(\boldsymbol{r}-\boldsymbol{s})\big\},$$

so since squares of distances are preserved, dot products are preserved. The situation is only slightly less trivial if we again assume that the mapping $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is affine but that it need only preserve pairs of points some fixed nonzero distance apart. Affinity immediately ensures that *all* distances are preserved, and the previous case applies.

Life becomes a little more interesting if we drop the assumption of affinity. We are now in the realm of characterization problems: we must *derive* the regularity of the mapping (in this case, affinity) from the geometry it preserves (distances between pairs of points). If all distances are preserved this is still not very difficult, provided the mapping is assumed surjective. As above, we may assume $\mathbf{0} \mapsto \mathbf{0}$; then dot products are preserved as before, and we must prove the mapping to be linear. For any $\boldsymbol{r}, \boldsymbol{s}$ in $\mathbb{E}_n$ and any scalars α and β, set $\boldsymbol{t} := \alpha\boldsymbol{r} + \beta\boldsymbol{s}$; then for all $\boldsymbol{p} \in \mathbb{E}_n$,

$$\begin{aligned}
\big(\bar{\boldsymbol{t}} - \alpha\bar{\boldsymbol{r}} - \beta\bar{\boldsymbol{s}}\big)\cdot\bar{\boldsymbol{p}} &= \bar{\boldsymbol{t}}\cdot\bar{\boldsymbol{p}} - \alpha\bar{\boldsymbol{r}}\cdot\bar{\boldsymbol{p}} - \beta\bar{\boldsymbol{s}}\cdot\bar{\boldsymbol{p}} \\
&= \boldsymbol{t}\cdot\boldsymbol{p} - \alpha\boldsymbol{r}\cdot\boldsymbol{p} - \beta\boldsymbol{s}\cdot\boldsymbol{p} \\
&= (\boldsymbol{t} - \alpha\boldsymbol{r} - \beta\boldsymbol{s})\cdot\boldsymbol{p} = 0.
\end{aligned}$$

Since the mapping is surjective, $\bar{t} - \alpha\bar{r} - \beta\bar{s}$ is orthogonal to all of $\mathbb{E}_n$, so $\bar{t} = \alpha\bar{r} + \beta\bar{s}$ and the mapping is linear. (See Rätz [1970a,b, 1971] for similar arguments on more abstract spaces.)

If we now go a step further and assume that our mapping need not preserve all distances, we arrive at a considerably more difficult characterization problem. Affinity is now harder to obtain, and even more so if neither injectivity nor surjectivity is assumed. We shall not provide further proofs here; the ultimate generalization for $\mathbb{E}_n$, the Beckman–Quarles theorem, will be discussed together with its many variations in Section 3. As we shall see there and in the other sections following, not only can different assumptions affect the effort necessary to find distance preserving transformations, they can in some cases affect *which* transformations result.

In the following sections, we will consider distance preserving transformations on many different spaces and under many different assumptions. The theorems fall broadly into groups: Alexandrov-type theorems (for mappings preserving distance zero between distinct points), Beckman–Quarles-type theorems (for nonzero distances), and Zeeman-type theorems (dealing with distance-related ordered structures). Rather than bore the reader with a detailed catalogue of these theorems, we instead concentrate on their roots and on some of the ideas and difficulties encountered in their proofs. Readers wishing to delve further into the details will find as extensive bibliography at the end, and a more elaborate mathematical discussion of many of the topics in Benz [1992].

1. Alexandrov-type theorems

Alexandrov's theorem concerns distance preserving transformations of Minkowski space-time, the geometrical space of special relativity theory. Minkowski space-time is an amalgam of ordinary three-dimensional Euclidean space with time; specifically, if Euclidean space is coordinatized as usual by rectangular coordinates $(x, y, z) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$ is measured by some clock, then the *events* of Minkowski space-time are described by coordinates $(t, x, y, z) \in \mathbb{R}^4$. Structure for the set of events comes from the basic axiom of relativity theory: unreflected light signals travel in straight lines with the same speed in all directions. If units of measurement are chosen to make this speed numerically 1, then a light signal can travel between two events with coordinates $\boldsymbol{r}_1 := (t_1, x_1, y_1, z_1)$ and $\boldsymbol{r}_2 := (t_2, x_2, y_2, z_2)$ whenever

$$1 = \frac{\text{distance travelled}}{\text{time taken}} = \frac{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}^{1/2}}{|t_2 - t_1|}.$$

By squaring and rearranging, we obtain the equivalent relation

$$-(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = 0.$$

Note the resemblance to the Euclidean dot product 'square-distance'. If we define

$$(\boldsymbol{r}_1, \boldsymbol{r}_2) := -t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$

for all events $\boldsymbol{r}_1 := (t_1, x_1, y_1, z_1)$ and $\boldsymbol{r}_2 := (t_2, x_2, y_2, z_2) \in \mathbb{R}^4$, then an unreflected light signal can travel between two events $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ whenever

$$(\boldsymbol{r}_2 - \boldsymbol{r}_1, \boldsymbol{r}_2 - \boldsymbol{r}_1) = 0. \qquad (*)$$

Mathematically, *Minkowski space-time* $\mathbb{M}_4$ is defined to be $\mathbb{R}^4$ together with the *metric* given by $(\cdot\,,\cdot)$, and *Lorentz transformations* are defined to be linear bijections of $\mathbb{M}_4$ which preserve this metric.

It is relatively simple to show that *affine* transformations of $\mathbb{M}_4$ which preserve the relation $(*)$ must be essentially Lorentz transformations. This presumption of affinity was in fact used by Einstein in his 1905 derivation of Lorentz transformations: 'Zunächst ist klar, daß die Gleichungen linear sein müssen, wegen der Homogenitätseigenschaften, welche wir Raum und Zeit beilegen'[1] (Einstein [1905]). The significance of Alexandrov's theorem (Alexandrov [1950]) is that affinity (or for that matter, *any* regularity assumption, other than bijectivity) is totally unnecessary: constancy of light-speed suffices. We state the theorem for Minkowski space-time $\mathbb{M}_n$, with $n-1$ spatial dimensions.

ALEXANDROV'S THEOREM. *For $n \geqslant 3$, let $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ be a bijection from $\mathbb{M}_n$ onto itself such that for all $\boldsymbol{r}$, $\boldsymbol{s} \in \mathbb{M}_n$,*

$$(\boldsymbol{r} - \boldsymbol{s},\ \boldsymbol{r} - \boldsymbol{s}) = 0 \quad \textit{if and only if} \quad (\bar{\boldsymbol{r}} - \bar{\boldsymbol{s}},\ \bar{\boldsymbol{r}} - \bar{\boldsymbol{s}}) = 0.$$

Then $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is essentially a Lorentz transformation, i.e. it has the form $\boldsymbol{r} \mapsto \alpha \boldsymbol{L}\boldsymbol{r} + \boldsymbol{a}$ for some nonzero $\alpha \in \mathbb{R}$, $\boldsymbol{a} \in \mathbb{M}_n$, and a linear bijection $\boldsymbol{L}$: $\mathbb{M}_n \mapsto \mathbb{M}_n$ satisfying $(\boldsymbol{L}\boldsymbol{r}, \boldsymbol{L}\boldsymbol{s}) = (\boldsymbol{r}, \boldsymbol{s})$ for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathbb{M}_n$.

To convey some of the geometric flavour of Alexandrov's theorem, we outline via diagrams the basic ideas of its proof for $\mathbb{M}_3$. (Only minor modifications are necessary for a proof for $\mathbb{M}_n$ for any $n > 3$.) By convention, we draw the t-axis vertically in all diagrams; we then call lines or planes in $\mathbb{M}_3$ *timelike, null*, or *spacelike* whenever the angle they make with the horizontal x-y-plane is greater than, equal to, or less than 45°, respectively. The steps below follow partly Alexandrov and partly Borchers and Hegerfeldt, who rediscovered this theorem in 1972 (Borchers and Hegerfeldt [1972a]; see also Hua [1981], § 9.3, for a proof via Hermitian matrices). The crux of the proof is to show that the mapping is affine; a standard way to do this is to show that it maps all lines into lines and then apply the following well-known characterization theorem.

FUNDAMENTAL THEOREM OF AFFINE GEOMETRY. *For $n \geqslant 2$, a bijection from $\mathbb{R}^n$ onto itself which maps all lines in $\mathbb{R}^n$ onto lines must be affine.*

[1] 'First it is clear that the equations must be linear, because of the homogeneity properties which we assume for space and time'.

It then follows easily that the mapping $r \mapsto \bar{r}$ is a Lorentz transformation up to a translation and a dilatation.

PROOF. 1. For any fixed $a \in \mathbb{M}_3$, the *null cone* with vertex a is defined to be the set

$$\mathcal{C}(a) := \{r \in \mathbb{M}_3\colon\ (r - a, r - a) = 0\}.$$

Clearly, for all $a \in \mathbb{M}_3$,

$$\mathcal{C}(a) \leftrightarrow \mathcal{C}(\bar{a}),$$

i.e. $r \mapsto \bar{r}$ preserves null cones and their vertices. Furthermore, since any null line is the intersection of two tangent null cones (Figure 2.1(a): two null cones are tangent whenever the vertex of one lies on the other), null lines are also preserved.

2. A null cone with vertex on some null plane is tangent to that plane along a null line (Figure 2.1(b)). Points of $\mathbb{M}_3$ lie on this null plane if and only if they lie either on the null line or on no null cone with vertex on this line. Since null cones and null lines are preserved, null planes are also preserved.

3. Since any spacelike line is the intersection of two null planes (Figure 2.1(c)), spacelike lines are preserved.

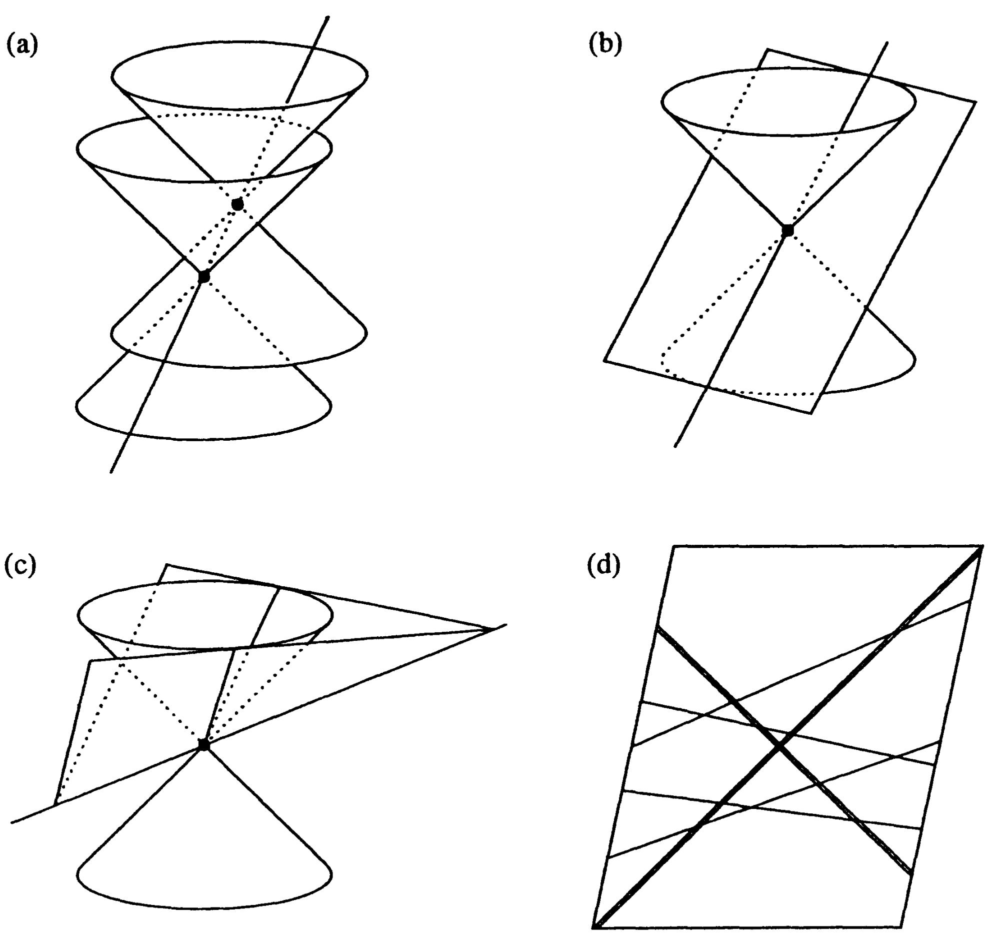

Figure 2.1. Proof of Alexandrov's theorem.

4. The points in a timelike plane are covered by infinitely many intersecting null and spacelike lines in that plane (Figure 2.1(d)). It follows that their images are also coplanar, so timelike planes map into planes. Furthermore, since any timelike line is the intersection of two timelike planes, timelike lines map into lines.

We thus have that the mapping $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ maps all three types of lines into lines; the same is true for its inverse. By the fundamental theorem of affine geometry, then, the mapping is affine, and is thus a Lorentz transformation. □

Note that Alexandrov's theorem is restricted to real spaces of dimension at least 3: it fails when $n = 2$. To see why, rewrite the metric

$$(\boldsymbol{r}, \boldsymbol{r}) = r_1^2 - r_2^2$$

on $\mathbb{M}_2$ by the change of variables $\tilde{r}_1 := r_1 + r_2$, $\tilde{r}_2 := r_1 - r_2$ as

$$(\boldsymbol{r}, \boldsymbol{r}) = \tilde{r}_1 \tilde{r}_2.$$

The relation $(\boldsymbol{r}-\boldsymbol{s}, \boldsymbol{r}-\boldsymbol{s}) = 0$ then holds between the points $\boldsymbol{r} := (\tilde{r}_1, \tilde{r}_2)$ and $\boldsymbol{s} := (\tilde{s}_1, \tilde{s}_2)$ whenever

$$(\tilde{r}_1 - \tilde{s}_1)(\tilde{r}_2 - \tilde{s}_2) = 0, \tag{$*$}$$

which holds if and only if $\tilde{r}_1 = \tilde{s}_1$ or $\tilde{r}_2 = \tilde{s}_2$. Clearly, for arbitrary bijective functions g, h: $\mathbb{R} \mapsto \mathbb{R}$, the transformation

$$(\tilde{r}_1, \tilde{r}_2) \mapsto \big(g(\tilde{r}_1), h(\tilde{r}_2)\big)$$

will preserve the relation $(*)$, but need not be affine or even continuous. Additional assumptions, such as those in Benz [1967], Rätz [1983] and Stettler [1981] are necessary.

Before discussing Alexandrov-type theorems for more general spaces, we introduce some useful notation and terminology. Euclidean n-space and Minkowski space are both examples of an important class of 'distance spaces' called metric vector spaces. A succinct discussion of these spaces appears in Artin [1957] and a more leisurely approach in Snapper and Troyer [1971]; we outline here only a few of the more basic ideas.

A *metric vector space* $\mathcal{V}$ is a finite dimensional vector space $\mathcal{V}$ over a field K not of characteristic 2 upon which is defined a *metric*, or symmetric bilinear form $(\cdot, \cdot)$. The *separation* between two points $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathcal{V}$ is defined by

$$|\boldsymbol{r}, \boldsymbol{s}| := (\boldsymbol{r} - \boldsymbol{s}, \boldsymbol{r} - \boldsymbol{s}).$$

Two vectors $\boldsymbol{v}, \boldsymbol{w}$ in $\mathcal{V}$ are said to be *orthogonal* whenever $(\boldsymbol{v}, \boldsymbol{w}) = 0$; a *null vector* is then a self-orthogonal one. $\mathcal{V}$ is said to be *nonsingular* if no nonzero vector is orthogonal to all of $\mathcal{V}$. (Both $\mathbb{E}_n$ and $\mathbb{M}_n$ are nonsingular.) If $\mathcal{V}$ contains no nonzero null vectors, it is said to be *anisotropic* (like $\mathbb{E}_n$). The *Witt index* of $\mathcal{V}$ is the largest possible dimension of a subspace of $\mathcal{V}$ consisting only of null vectors. The Witt index of $\mathbb{M}_n$ is 1.

An *isometry* of $\mathcal{V}$ is a linear bijection $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ on $\mathcal{V}$ such that, for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathcal{V}$, $(\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}) = (\boldsymbol{r}, \boldsymbol{s})$. A *motion* of $\mathcal{V}$ is the composition of an isometry with translations, and a *similarity* of $\mathcal{V}$ is the composition of an isometry with translations and dilatations. (Thus, in this terminology, Alexandrov's theorem states that bijections of $\mathbb{M}_n$ which preserve separation zero in both directions must be similarities).

If $\mathcal{V}$ is real (i.e. if $K = \mathbb{R}$), we have the *law of inertia*: for appropriate coordinate systems on a nonsingular real metric vector space $\mathcal{V}$, the metric takes the form

$$(\boldsymbol{r}, \boldsymbol{s}) = \sum_{i=1}^{p} r_i s_i - \sum_{i=p+1}^{n} r_i s_i \tag{$**$}$$

for all $\boldsymbol{r} := (r_1, r_2, \ldots, r_n)$ and $\boldsymbol{s} := (s_1, s_2, \ldots, s_n)$ in $\mathcal{V}$. The number p is independent of the coordinate system, and the pair $(p, n-p)$ is called the *signature* of $\mathcal{V}$.

We note that it is also possible to define metric vector spaces of infinite dimension or over fields of characteristic 2 (via quadratic forms instead of symmetric bilinear forms); for simplicity we shall not do so. However, the reader should be aware that some of the results discussed below may be valid for such spaces (although perhaps with different proofs) even when this is not explicitly stated. Such results (e.g., Vroegindewey [1984]) are included in the bibliography.

We now return to our examination of Alexandrov-type theorems.

Instead of transformations preserving the relation $|\boldsymbol{r}, \boldsymbol{s}| = 0$, it is also possible to examine transformations preserving the relation $|\boldsymbol{r}, \boldsymbol{s}| < 0$ or $|\boldsymbol{r}, \boldsymbol{s}| \leqslant 0$ for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathbb{M}_n$. Preservation of these relations is equivalent to preservation of the 'solid' cones

$$\mathcal{P}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathbb{M}_n\colon\ |\boldsymbol{r}, \boldsymbol{a}| < 0\}$$

and

$$\mathcal{Q}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathbb{M}_n\colon\ |\boldsymbol{r}, \boldsymbol{a}| \leqslant 0\}$$

in $\mathbb{M}_n$. Bijections preserving the first type of cones were shown by Borchers and Hegerfeldt [1972a] to be essentially Lorentz transformations; the same result was proven in Alexandrov [1969] for the second type. (See also Alexandrov [1975] and Guts [1982].) Alexandrov also considered other types of cones, consisting of lines through a vertex, and showed that, under appropriate conditions, mappings preserving these cones must be affine. Since a detailed survey of these and related theorems appears in Alexandrov [1975], we will not elaborate further on them here.

On more general metric vector spaces, the above proof of Alexandrov's theorem breaks down at the first step: tangent null cones need no longer intersect in a line. Nevertheless, the theorem does hold with a few modifications.

For real spaces, a minor change is necessary for spaces of signature (p, p) (Artinian spaces, in the terminology of Snapper and Troyer [1971]). In this case there exist *pseudo-isometries*, or linear bijections $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ of the space with $(\bar{\boldsymbol{r}}, \bar{\boldsymbol{r}}) = -(\boldsymbol{r}, \boldsymbol{r})$ (e.g., for a metric in the form $(**)$, the mapping

$$(r_1, r_2, \ldots, r_p, r_{p+1}, \ldots, r_n) \mapsto (r_{p+1}, \ldots, r_n, r_1, r_2, \ldots, r_p)$$

is a pseudo-isometry). This modification is sufficient: separation zero preserving bijections of nonanisotropic nonsingular real metric vector spaces must be either similarities, or possibly (for signature (p,p)) 'pseudo-similarities' (Borisov [1960]; see also Huckenbeck [1987], where the assumption of bijectivity is removed, and Astrakov [1990]).

For spaces over arbitrary fields K, a further modification is necessary: we need *semilinear* transformations of $\mathcal{V}$. These consist of a bijection $\boldsymbol{L}\colon \mathcal{V} \mapsto \mathcal{V}$ and an automorphism τ of K such that for all $\boldsymbol{r}, \boldsymbol{s} \in \mathcal{V}$ and $\lambda \in K$, $\boldsymbol{L}(\boldsymbol{r}+\boldsymbol{s}) = \boldsymbol{L}\boldsymbol{r} + \boldsymbol{L}\boldsymbol{s}$ and $\boldsymbol{L}(\lambda \boldsymbol{r}) = \lambda^{\tau} \boldsymbol{L}\boldsymbol{r}$. Then a separation zero preserving bijection of a nonsingular, nonanisotropic metric vector space $\mathcal{V}$ over a field K must be *semi-affine*, i.e. up to a translation, it takes the form $\boldsymbol{r} \mapsto \boldsymbol{L}\boldsymbol{r} + \boldsymbol{a}$ for some $\boldsymbol{a} \in \mathcal{V}$ and some semilinear bijection $(\boldsymbol{L}, \tau)$ of $\mathcal{V}$. Such a semilinear bijection must satisfy

$$(\boldsymbol{L}\boldsymbol{r}, \boldsymbol{L}\boldsymbol{s}) = \lambda(\boldsymbol{r}, \boldsymbol{s})^{\tau}$$

for some nonzero $\lambda \in K$ and all $\boldsymbol{r}, \boldsymbol{s} \in \mathcal{V}$ (Lester [1977]).

If, instead of transformations defined on the *whole* of $\mathbb{M}_n$, we consider transformations defined only on a subset, the resulting separation zero preserving transformations need not be affine. As examples, we mention the mappings

$$\boldsymbol{r} \mapsto \frac{\boldsymbol{r}}{(\boldsymbol{r}, \boldsymbol{r})} \quad \big(\text{undefined on the null cone } (\boldsymbol{r}, \boldsymbol{r}) = 0\big),$$

and, for a nonzero null vector $\boldsymbol{n} \in \mathbb{M}_n$,

$$\boldsymbol{r} \mapsto \frac{\boldsymbol{r} + (\boldsymbol{r}, \boldsymbol{r})\boldsymbol{n}}{1 + 2(\boldsymbol{r}, \boldsymbol{n})} \quad \big(\text{undefined on the null hyperplane } (\boldsymbol{r}, \boldsymbol{n}) = -\tfrac{1}{2}\big),$$

both of which preserve separation zero where defined. Compositions of mappings of these forms with similarities of $\mathbb{M}_n$ are called *conformal transformations*. The following theorem appears in Alexandrov [1976] and Guts [1982]; it was rediscovered at least twice, in Lester [1983a], and Popovici and Rădulescu [1981].

THEOREM. *Let $\mathcal{D}$ be an open connected subset of $\mathbb{M}_n$, $n \geqslant 3$, and let $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ be an injective mapping from $\mathcal{D}$ into $\mathbb{M}_n$ such that for all $\boldsymbol{r}$, $\boldsymbol{s} \in \mathcal{D}$,*

$$|\boldsymbol{r}, \boldsymbol{s}| = 0 \quad \textit{if and only if} \quad |\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}| = 0.$$

Then $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ must be a restriction to $\mathcal{D}$ of a conformal transformation of $\mathbb{M}_4$.

(This theorem holds as well for injective mappings $\mathcal{D} \mapsto \mathbb{M}_n$ preserving the cones $\mathcal{P}(\boldsymbol{a})$ and $\mathcal{Q}(\boldsymbol{a})$ described earlier (Alexandrov [1976], Guts [1982]).) A partial generalization to real metric vector spaces of other signatures appears in Lester [1986a].

Alexandrov's theorem can also be investigated on other relativistic space-times. By taking 'space' to be the surface of sphere in Euclidean 4-space and adjoining a time coordinate, e.g., we obtain *Einstein's cylinder universe*

$$\mathcal{C}_4 := \big\{(\rho, \boldsymbol{r})\colon\ \rho \in \mathbb{R},\ \boldsymbol{r} \in \mathbb{E}_4,\ (\boldsymbol{r}, \boldsymbol{r}) = 1\big\}.$$

Since the distance between points r and s on the sphere is given by $\cos^{-1}(r, s)$, a light signal can travel between events (ρ, r) and (σ, s) in C_4 whenever

$$1 = \frac{\text{distance travelled}}{\text{time taken}} = |\rho - \sigma|^{-1} \cos^{-1}(r, s),$$

i.e. whenever $(r, s) = \cos(\rho - \sigma)$.

The transformations of C_4 preserving this relation are derived in Lester [1982a]; unlike those for Minkowski space-time, these can be somewhat pathological. Similar derivations exist for De Sitter space-time (Lester [1983b]) and for Robertson–Walker space-time (Lester [1982b]).

2. Beckman–Quarles-type theorems

As indicated in Section 1, the original Beckman–Quarles theorem (Beckman and Quarles [1953]) concerns distance preserving transformations of real Euclidean n-space $\mathbb{E}_4$.

BECKMAN–QUARLES THEOREM. *Let $\rho > 0$ be some fixed scalar, and let f: $r \mapsto \bar{r}$ be a mapping from $\mathbb{E}_n$ $(n \geqslant 2)$ into itself such that for all $r, s \in \mathbb{E}_n$,*

$$|r, s| = \rho \quad \textit{implies} \quad |\bar{r}, \bar{s}| = \rho.$$

Then f must be a motion of $\mathbb{E}_n$.

Elementary proofs of this theorem can be found in Benz [1987], Schröder [1990] and Lenz [1991]. Note the generality of the theorem: it is not assumed that the inverse mapping also preserves separation ρ: the inverse mapping need not even exist. In the original theorem as stated in Beckman and Quarles [1953], the mapping was not even assumed to be single-valued (although this turns out to be a minor inconvenience).

To give some feeling for the type of geometrical ideas involved, we give a proof of this theorem for the special case when the inverse mapping exists and also preserves separation ρ. We will use several times the following form of the triangle inequality on $\mathbb{E}_n$.

For a given positive scalar α and for all points a and b in $\mathbb{E}_n$, $|a, b| \leqslant \alpha$ if and only if there exists a c in $\mathbb{E}_n$ with $|a, c| = |b, c| = \alpha/4$. If $|a, b| = \alpha$, then c is unique, and is in fact the midpoint of ab.

PROOF. 1. Two points a and b are a separation 4ρ apart if and only if there exists a unique point c (their midpoint) a distance ρ from each (Figure 3.1a). Then $r \mapsto \bar{r}$ preserves this relation: pairs of points a distance 4ρ apart and their midpoints map into like pairs and *their* midpoints.

2. Any points a and b with $|a, b| = \rho/4$ can be made part of a configuration like that in Figure 3.1b, with p, a, q collinear, p, b, r collinear, and

$$|p, b| = |b, q| = |q, r| = |r, a| = |a, p| = \rho.$$

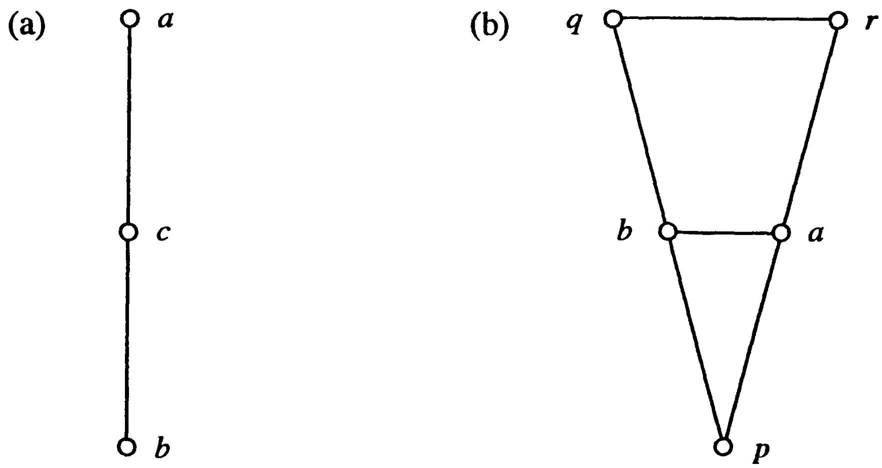

Figure 3.1. The points of steps 1 and 2.

Since such configurations are preserved, separation $\rho/4$ is preserved.

3. By considering overlapping triples of collinear points like those in step 1, it follows that *all* separations of the form $k\rho$ for $k = 1, 2, \ldots,$ are preserved. Then by reiterating step 2, we have that all separations of the form $4^{-n}k\rho$ for integers $k > 0$ and n are preserved.

4. Assume that for some points $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{E}_n$, $|\bar{a}, \bar{b}| > |\boldsymbol{a}, \boldsymbol{b}|$. Since the numbers of the form $4^{-n}k\rho$ for integers $k > 0$ and n are dense in $\mathbb{R}$, there exist integers k and n with $|\bar{a}, \bar{b}| > 4^{-n}k\rho > |\boldsymbol{a}, \boldsymbol{b}|$. From the triangle inequality as above, there exists a point $\boldsymbol{c} \in \mathbb{E}_n$ with

$$|\boldsymbol{a}, \boldsymbol{c}| = |\boldsymbol{b}, \boldsymbol{c}| = \left(4^{-n}k\rho\right)/4 = 4^{-(n+1)}k\rho.$$

Since distances $4^{-(n+1)}k\rho$ are preserved, $|\bar{a}, \bar{c}| = |\bar{b}, \bar{c}| = 4^{-(n+1)}k\rho$, so from the triangle inequality again, $|\bar{a}, \bar{b}| \leqslant 4^{-n}k\rho$, a contradiction.

Similarly, if $|\bar{a}, \bar{b}| < |\boldsymbol{a}, \boldsymbol{b}|$ for some $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{E}_n$, then the same argument applied to the inverse mapping gives another contradiction. It follows that $|\bar{a}, \bar{b}| = |\boldsymbol{a}, \boldsymbol{b}|$, so since this holds for all $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{E}_n$, we have from the discussion in Section 1 that the mapping is a motion. □

Because of the Euclidean nature of its proof (as above and also in Beckman and Quarles [1953], Benz [1987], and Modenov and Parkhomenko [1965]), the Beckman–Quarles theorem generalizes most readily to spaces with similar, 'Euclidean-like' distances or everywhere positive distances, where arguments like that above involving the triangle inequality or other familiar geometrical ideas can be used. Generalizations exist for finite Euclidean planes (Farrahi [1975b]), rational Euclidean planes (Farrahi [1976a]), constructible Euclidean planes (Farrahi [1976b]), Euclidean planes over Galois fields (Benz [1983b], Farrahi [1978b], Radó [1985]) Lobachevsky space (Kuz'minykh [1979b]), rational Euclidean space (Farrahi [1979], Lenz [1987]), pre-Hilbert spaces (Schröder [1979b]) and normed spaces (Benz [1985a]). (Many of these also require additional assumptions; e.g., we may have to assume the mapping to be injective or to preserve more than one separation.) The Beckman–Quarles theorem holds as well for mappings defined on an open ball of radius $r < \rho$ in $\mathbb{R}^n$ (Kuz'minykh [1979b]). There also exist variations such as the following, which appear as corollaries of a more complicated theorem in Kuz'minykh [1979a].

1. Let $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ be an injection from $\mathbb{E}_n$ ($n \geqslant 3$) into itself such that, for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathbb{E}_n$, if $|\boldsymbol{r}, \boldsymbol{s}|^{1/2}$ is rational, then $|\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}|^{1/2}$ is algebraic. Then $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is a similarity of $\mathbb{E}_n$.

2. Let α and β be distinct primes, and let $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ be an injection from $\mathbb{E}_n$ ($n \geqslant 3$) into itself such that, for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathbb{E}_n$, if $|\boldsymbol{r}, \boldsymbol{s}|^{1/2} = \alpha$ or $|\boldsymbol{r}, \boldsymbol{s}|^{1/2} = \beta$, then $|\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}|^{1/2}$ is prime. Then $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is a motion of $\mathbb{E}_n$.

Similar results exist for Lobachevsky spaces (Kuz'minykh [1979a]).

When distances become non-Euclidean or nonpositive, the Beckman–Quarles theorem becomes more difficult to generalize. Not only do we lose the framework of Euclidean or other familiar geometries (for example, we lose the ubiquitous equilateral triangles of Beckman and Quarles [1953], Benz [1987], and Modenov and Parkhomenko [1965]) but we must often construct entirely different proofs for different values of ρ, for spaces of different Witt indices, or for spaces of different dimensions.

We consider first generalizations to two-dimensional spaces. (Note that, unlike Alexandrov's theorem, the Beckman–Quarles theorem does hold in the two-dimensional case.)

The only nonsingular two-dimensional real metric vector space other than $\mathbb{E}_2$ is $\mathbb{M}_2$, with metric

$$(\boldsymbol{r}, \boldsymbol{s}) := r_1 s_1 - r_2 s_2$$

for all $\boldsymbol{r} := (r_1, r_2)$ and $\boldsymbol{s} := (s_1, s_2)$ in $\mathbb{M}_2$. Given a mapping $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ on $\mathbb{M}_2$ which preserves a separation $\rho \neq 0$, it is useful to simplify matters by a coordinate transformation (Benz [1983a]):

$$\tilde{r}_1 := \tfrac{1}{4}\rho r_1 + r_2, \quad \tilde{r}_2 := \tfrac{1}{4}\rho r_1 - r_2.$$

Then it can easily be verified that, for any $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{M}_2$, $|\boldsymbol{r}, \boldsymbol{s}| = \rho$ if and only if

$$(\tilde{r}_1 - \tilde{s}_1)(\tilde{r}_2 - \tilde{s}_2) = 1.$$

Thus, since the coordinate transformation is linear, we may without loss of generality assume that $|\boldsymbol{r}, \boldsymbol{s}| = (r_1 - s_1)(r_2 - s_2)$ and that $|\boldsymbol{r}, \boldsymbol{s}| = 1$ implies $|\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}| = 1$ for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathbb{M}_2$. In this form, the Beckman–Quarles theorem for $\mathbb{M}_2$ was proven in Benz [1977c] under the additional assumption that the inverse mapping exists and also preserves separation 1, and in Benz [1987] without this assumption.

Over an arbitrary field K (with $\operatorname{char} K \neq 2$), there is only one two-dimensional nonsingular nonanisotropic metric vector space: the Artinian plane, spanned by two nonorthogonal null vectors. As with $\mathbb{M}_2$, for a mapping $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ preserving a separation $\rho \neq 0$, we may assume that, for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathcal{V}$,

$$|\boldsymbol{r}, \boldsymbol{s}| = (r_1 - s_1)(r_2 - s_2)$$

and

$$|\boldsymbol{r}, \boldsymbol{s}| = 1 \quad \text{implies} \quad |\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}| = 1.$$

For the Artinian plane coordinatized this way, $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ has been shown to be semilinear up to a translation under the following conditions.

(a) Radó [1970]: $\operatorname{char} K \neq 2$ or 3 and the inverse mapping exists and also preserves separation 1.

(b) Benz [1981b]: $\operatorname{char} K \neq 2,\ 3,\ 5$ or 7, and -3 and -11 are squares in K.

(c) Benz [1981b]: $\operatorname{char} K \neq 2,\ 3$ or 5; -3 is not a square in K, and K has the form

$$K = \left\{x + \frac{4x}{(x-1)(1-y^2)}\colon\ x, y \in K,\ x \neq 1 \neq y^2\right\}.$$

(Many fields are of this form, e.g., Pythagorean fields and the Galois field GF(p) for $p \geqslant 11$ (Tallini [1981]).)

(d) Benz [1982a]: $\operatorname{char} K \neq 2$ or 3, K is quadratically closed, the inverse mapping exists and also preserves separation 1.

(e) Benz [1982a]: $\operatorname{char} K = 2$ or 3, and GF(5^2) or GF(7^2) is a subfield of K.

(f) Benz [1982b]: $K = \mathrm{GF}(p^n)$ and any one of the following conditions holds:

(i) $p \neq 2,\ 3,\ 5$ or 7.

(ii) $p = 5$ or 7 and n is even.

(iii) $p = 7$ and n is odd and divisible by 3.

(g) Benz [1983a]: $\operatorname{char} K \neq 2,\ 3,\ 5$ or 7, and $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is injective.

(h) Radó [1983], Schaeffer [1986]: $\operatorname{char} K \neq 2$, 3 or 5, or $K \neq \mathrm{GF}(5^m)$ and $m > 1$.

With the aid of a computer, results for some exceptional cases ($K = \mathrm{GF}(q)$ for $q = 5,\ 7,\ 11,\ 32,\ 64,\ 81,\ 128$) have also been obtained (Samaga [1982, 1984a]). There also exist similar theorems for Hjelmslev's plane (Benz [1985b]) and for planes over other commutative rings (Schaeffer [1984]).

Since two-dimensional spaces can be easily coordinatized (as above or otherwise), Beckman–Quarles theorems for these spaces can often be handled by elementary algebra, functional equations or other 'one-variable' methods. For higher-dimensional spaces, more sophisticated linear algebraic tactics are often necessary. For nonsingular real metric vector spaces $\mathcal{V}$ of dimension at least 3, the Beckman–Quarles theorem generalizes exactly: a mapping $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ from $\mathcal{V}$ into itself such that $|\boldsymbol{r}, \boldsymbol{s}| = \rho$ implies $|\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}| = \rho$ for all $\boldsymbol{r}, \boldsymbol{s}$ in $\mathcal{V}$ must be a motion. The earliest proofs of this result (with and without additional assumptions) were for $\mathbb{M}_n$, $n \geqslant 3$: in Schröder [1979a], Benz [1980a] for $\rho < 0$, and in Benz [1977b], Lester [1981] for $\rho > 0$. To see why different proofs are necessary for ρ's of different signs, look at the affine quadrics

$$\mathcal{R}(\boldsymbol{a}) := \left\{\boldsymbol{r} \in \mathbb{M}_n\colon\ |\boldsymbol{r}, \boldsymbol{a}| = \rho\right\}$$

for $\boldsymbol{a}$ in $\mathbb{M}_n$. In view of the proof of Alexandrov's theorem in Section 2, an obvious tactic for proving the Beckman–Quarles theorem on $\mathbb{M}_n$ is to use the fact that such quadrics are preserved to try to obtain linearity. Now, if $\rho > 0$, $\mathcal{R}(\boldsymbol{a})$ is a hyperboloid of one sheet. Any two such hyperboloids always intersect; moreover, their intersection lies in a hyperplane, a useful fact on the way to a proof that the mapping is affine (Benz [1977b]). If $\rho < 0$, however, $\mathcal{R}(\boldsymbol{a})$ is a hyperboloid of two sheets. This time two such hyperboloids $\mathcal{R}(\boldsymbol{a})$ and $\mathcal{R}(\boldsymbol{b})$ need not intersect at all, and intersect in a single point (the midpoint of $\boldsymbol{ab}$) if and only if $|\boldsymbol{a}, \boldsymbol{b}| = 4\rho$. From here, arguments like that in the above proof can be used to start a proof that the mapping preserves all separations (Schröder [1979a]). The proof for real metric vector spaces of arbitrary signature is considerably difficult (Lester [1979a], Huckenbeck [1986]).

For a metric vector space $\mathcal{V}$ over a more general field K, the Beckman–Quarles theorem is known to hold in certain special cases. As with real spaces, different cases arise for different values of ρ. The following definition generalizes terminology applying to $\mathbb{M}_n$ to a general metric vector space $\mathcal{V}$ over a field K not of characteristic 2.

DEFINITION. Any nonzero $\rho \in K$ is said to be *spacelike* whenever there exist vectors $\boldsymbol{n} \neq \boldsymbol{0}$, $\boldsymbol{u} \in \mathcal{V}$ with $(\boldsymbol{u}, \boldsymbol{u}) = \rho$, $\boldsymbol{n}$ null and orthogonal to $\boldsymbol{u}$; and *timelike* otherwise.

(If $\mathcal{V}$ has dimension at least 3 and Witt index 1, K contains both spacelike and timelike ρ's. For other nonsingular, nonanisotropic metric vector spaces, all ρ's are spacelike.)

The following results appear in Schröder [1980]. Let $\mathcal{V}$ be a nonsingular metric vector space of dimension at least 3 over a field K, and for $\rho \neq 0$ in K, let $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ be a bijection from $\mathcal{V}$ to itself such that for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathcal{V}$,

$$|\boldsymbol{r}, \boldsymbol{s}| = \rho \quad \text{if and only if} \quad |\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}| = \rho.$$

Then

(i) if K has at least 4 elements and ρ is spacelike, then $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is semi-affine,

(ii) if $\operatorname{char} K \neq 2$, 3 or 5, $\mathcal{V}$ has Witt index 1 and ρ is timelike, then $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is semi-affine.

Another result (Radó [1986]): Let $\mathcal{V}$ be a nonsingular metric vector space of dimension at least 3 over GF(p^m), $p > 2$, $m \geqslant 3$ and for $\rho \neq 0$ in $\mathcal{V}$, $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ be a bijection from $\mathcal{V}$ to itself such that for all $\boldsymbol{r}$ and $\boldsymbol{s}$ in $\mathcal{V}$,

$$|\boldsymbol{r}, \boldsymbol{s}| = \rho \quad \text{implies} \quad |\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}}| = \rho.$$

Then

(i) if $n \not\equiv 0, -1, -2 \pmod{p}$, $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is semi-affine,

(ii) if the discriminant of $\mathcal{V}$ satisfies a certain condition (see Radó [1986] for details), $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ is semi-affine.

To conclude, we note that Beckman–Quarles type theorems also exist for distances other than metric vector space distances. We mention, for example, Alpers [1990] and Alpers and Schröder [1991] (angles), Lester [1985] (distances between lines in three-space), Lester [1986b, 1987a] (angles between spheres), Lester [1983b] (separation in De Sitter space-time). There exist also results on transformations between spaces of different dimensions (Dekster [1985], Radó, Andreescu and Valcan [1986], Kuz'minykh [1988]).

3. Zeeman-type theorems

Like Alexandrov's theorem, Zeeman's theorem deals with Minkowski space-time $\mathbb{M}_4$: the relation to be preserved is now *causality* between pairs of events.

Not every event in $\mathbb{M}_4$ which precedes another (as defined by their t-coordinates) can cause the other event: there must be sufficient time for a causal influence (e.g., a material particle of some sort) to propagate from the first event to the second. A consequence of special relativity theory is that material particles travel at speeds less than the speed of light; thus an event $\boldsymbol{r}_1 := (t_1, x_1, y_1, z_1)$ can cause another $\boldsymbol{r}_2 := (t_2, x_2, y_2, z_2)$ in this sense whenever

$$\frac{\text{distance travelled}}{\text{time taken}} < 1 \quad \text{and} \quad t_2 > t_1,$$

i.e. whenever

$$\frac{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}^{1/2}}{|t_2 - t_1|} < 1 \quad \text{and} \quad t_2 > t_1.$$

By squaring and rearranging, we obtain the relation

$$|\boldsymbol{r}_2, \boldsymbol{r}_1| < 0 \quad \text{and} \quad t_2 > t_1, \qquad (*)$$

which we symbolize by $\boldsymbol{r}_1 \prec \boldsymbol{r}_2$.

The translations, dilatations and Lorentz transformations of $\mathbb{M}_4$ all preserve the relation $|\boldsymbol{r}_2, \boldsymbol{r}_1| < 0$; however not all of these preserve the temporal order of events, as defined by the inequality '$t_2 > t_1$'. Those Lorentz transformations which do indeed preserve temporal order are called *orthochronous* orthochronous Lorentz transformations, and form a subgroup of the full Lorentz group. Orthochronous Lorentz transformations, translations and dilatations by a *positive* scalar all preserve the causal relation $\boldsymbol{r}_1 \prec \boldsymbol{r}_2$.

The *causal automorphisms* of $\mathbb{M}_4$ are those bijections of $\mathbb{M}_4$ which preserve the relation $\boldsymbol{r}_1 \prec \boldsymbol{r}_2$. As with the light-speed preserving transformations of Alexandrov's theorem, we would like to know whether there are other causal automorphisms of $\mathbb{M}_4$ besides those described above. If we assume they are affine, the answer is easily shown to be negative. Zeeman's theorem (Zeeman [1964]) shows that the assumption of affinity is superfluous. We state the theorem for $\mathbb{M}_n$; the relation $\prec$ is defined on $\mathbb{M}_n$ exactly as in (*).

ZEEMAN'S THEOREM. *Let $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ be a bijection from $\mathbb{M}_n$ ($n \geqslant 3$) onto itself such that for all $\boldsymbol{r}$ and $\boldsymbol{s} \in \mathbb{M}_n$,*

$$\boldsymbol{r} \prec \boldsymbol{s} \quad \textit{if and only if} \quad \bar{\boldsymbol{r}} \prec \bar{\boldsymbol{s}}.$$

Then, up to translations and dilatations by a positive scalar, $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ must be an orthochronous Lorentz transformation.

Before proceeding, it should be mentioned that although the theorem as stated above was technically first proven by Zeeman, it is in fact an easy corollary of an earlier theorem on causality by Alexandrov. Alexandrov and Ovchinnikova [1953] examined a more complete form of causality which includes causality by photons (light particles). Photons travel with speed 1, so an event $\boldsymbol{r}_1 := (t_1, x_1, y_1, z_1)$ can cause another $\boldsymbol{r}_2 := (t_2, x_2, y_2, z_2)$ by *any* influence whenever

$$|\boldsymbol{r}_2, \boldsymbol{r}_1| \leqslant 0 \quad \text{and} \quad t_2 > t_1.$$

We symbolize this relation by $\boldsymbol{r}_1 \preccurlyeq \boldsymbol{r}_2$; clearly, it can be defined similarly on $\mathbb{M}_n$ for any $n \geqslant 2$.

As with the light-speed theorem, Alexandrov expressed the preservation of causality in terms of the preservation of cones. For any $\boldsymbol{a} \in \mathbb{M}_n$, define the 'future-cones'

$$\mathcal{P}^{\uparrow}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathbb{M}_n\colon\ \boldsymbol{a} \prec \boldsymbol{r}\}$$

and

$$\mathcal{Q}^{\uparrow}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathbb{M}_n\colon\ \boldsymbol{a} \preccurlyeq \boldsymbol{r}\} \cup \{\boldsymbol{a}\}$$

(i.e. these are the 'top halves' of the corresponding cones of Section 2). Mappings $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ preserve Zeeman's causality whenever $\mathcal{P}^{\uparrow}(\boldsymbol{a}) \mapsto \mathcal{P}^{\uparrow}(\bar{\boldsymbol{a}})$ for all events $\boldsymbol{a} \in \mathbb{M}_n$, and Alexandrov's causality whenever $\mathcal{Q}^{\uparrow}(\boldsymbol{a}) \mapsto \mathcal{Q}^{\uparrow}(\bar{\boldsymbol{a}})$ for all events $\boldsymbol{a} \in \mathbb{M}_n$. As shown in Alexandrov and Ovchinnikova [1953], the latter are essentially orthochronous Lorentz transformations. But, as pointed out in Alexandrov [1975], Alexandrov's cones can easily be expressed in terms of Zeeman's cones:

$$\mathcal{Q}^{\uparrow}(\boldsymbol{a}) = \bigcap_{\boldsymbol{a} \in \mathcal{P}^{\uparrow}(\boldsymbol{c})} \mathcal{P}^{\uparrow}(\boldsymbol{c}),$$

so Zeeman's theorem immediately reduces to the theorem of Alexandrov and Ovchinnikova. Nevertheless, to avoid confusion with the theorem of Section 2 and to be consistent with other authors (to whom Zeeman's paper appears better known) we shall observe the technicality and continue to refer to the theorem as Zeeman's theorem. (The theorem is also claimed by Hua [1981], p. 97.)

Zeeman begins by showing that causal automorphisms also preserve the 'upper light-cones'

$$\mathcal{C}^{\uparrow}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathbb{M}_n\colon\ |\boldsymbol{r}, \boldsymbol{a}| = 0,\ a_1 < t\} \cup \{\boldsymbol{a}\}$$

for all $\boldsymbol{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{M}_n$. (At this point the theorem can also be reduced to the theorem of Section 2). He then shows that causal automorphisms are affine using projective quadrics, a Cauchy functional equation and other methods. Other authors (Briginshaw [1980], Flato and Sternheimer [1966]) take a different approach: if we define the 'past cones'

$$\mathcal{P}^{\downarrow}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathbb{M}_n\colon\ |\boldsymbol{r}, \boldsymbol{a}| < 0,\ a_1 > t\}$$

for all $a := (a_1, a_2, \ldots, a_n) \in \mathbb{M}_n$, then sets of the form $\mathcal{P}^{\uparrow}(a) \cap \mathcal{P}^{\downarrow}(b)$ for $a \prec b$ (called *causal intervals* or *Alexandrov neighbourhoods*) form a base for the usual topology on $\mathbb{M}_n$. The causal automorphisms of $\mathbb{M}_n$ preserve these sets, so causal automorphisms are continuous; they can then easily be shown to be affine.

If we consider causal automorphisms defined only on a subset of $\mathbb{M}_n$, we need not obtain Lorentz transformations: as in Section 2 for Alexandrov's theorem, we obtain restrictions of conformal transformations (Alexandrov [1976]). Some caution must be exercised here, however, since, as pointed out in Gamba and Luzatto [1964], some conformal transformations (like the inversion $r \mapsto (r, r)^{-1} r$ mentioned in Section 2) reverse causality. To negate this effect, conformal transformations preserving causality must contain more than one such transformation in their decompositions: the precise forms necessary are given in Alexandrov [1976].

As a theorem in relativity theory, Zeeman's theorem can also be generalized to other space-times. For example, consider De Sitter space-time, which can be modeled by a hyperboloid of one sheet in $\mathbb{M}_5$,

$$\mathcal{S}_4 := \{ r \in \mathbb{M}_5 \colon (r, r) = 1 \}.$$

Separation between the events r and s in $\mathcal{S}_4$ is given here by $\{\cos^{-1}(r, s)\}^2$, and causality is defined by

$$r \prec s \quad \text{whenever} \quad (r, s) > 1 \quad \text{and} \quad r_1 < s_1$$

for events $r := (r_1, r_2, r_3, r_4, r_5)$ and $s := (s_1, s_2, s_3, s_4, s_5) \in \mathcal{S}_4$. Clearly, the Lorentz transformations of $\mathbb{M}_5$ map $\mathcal{S}_4$ onto itself, preserve separation, and, if they are orthochronous, preserve causality on $\mathcal{S}_4$. Zeeman's theorem for $\mathcal{S}_4$ states that these are its only causal automorphisms (Lester [1984]).

The causal automorphisms of Einstein's cylinder universe and Robertson–Walker space-times have also been determined (Lester [1984, 1987b]). Unlike those for $\mathcal{S}_4$, these can be somewhat pathological.

Unlike the light-speed relation of Section 2, causality is defined not only in terms of the metric of $\mathbb{M}_4$, but also in terms of scalar inequalities. This makes it somewhat more difficult to generalize to other metric vector spaces than Alexandrov's theorem: we need an appropriate order structure on the field of scalars and a space with a 'causality' invariant under some suitable subgroup of isometries. For real spaces, Minkowski space (with Witt index 1) is essentially the only possibility. For more general spaces, we have the following theorem (Vroegindewey, Kreinovic and Kosheleva [1979]).

THEOREM. *Let K be a commutative field with a nontrivial partial order $\leqslant$, and assume that all nonzero squares in K are positive. Let $\mathcal{V} = K^n$ ($n \geqslant 3$), and define the metric $(\cdot, \cdot)$ on $\mathcal{V}$ by*

$$(r, s) := -r_1 s_1 + r_2 s_2 + r_3 s_3 + \cdots + r_n s_n$$

for all $\boldsymbol{r} := (r_1, r_2, \ldots, r_n)$ *and* $\boldsymbol{s} := (s_1, s_2, \ldots, s_n) \in \mathcal{V}$. *Also define the cones*

$$\mathcal{C}^{\uparrow}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathcal{V}\colon\ |\boldsymbol{r}, \boldsymbol{a}| = 0,\ a_1 \leqslant r_1\},$$

$$\mathcal{P}^{\uparrow}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathcal{V}\colon\ |\boldsymbol{r}, \boldsymbol{a}| < 0,\ a_1 < r_1\},$$

$$\mathcal{Q}^{\uparrow}(\boldsymbol{a}) := \{\boldsymbol{r} \in \mathcal{V}\colon\ |\boldsymbol{r}, \boldsymbol{a}| \leqslant 0,\ a_1 \leqslant r_1\}$$

for all $\boldsymbol{a} := (a_1, a_2, \ldots, a_n) \in \mathcal{V}$. *Then* $\mathcal{V}$ *has Witt index* 1, *and any bijection from* $\mathcal{V}$ *onto itself which preserves cones of any one of these types is semilinear up to a translation.*

We can also examine mappings preserving more general cones on real spaces. Alexandrov considered *ray-cones*, formed by sets of rays originating from a vertex in affine space. Several theorems on ray-cone preserving mappings, including a more general version of the following, appear in Alexandrov [1972].

THEOREM. *Let* $C(\mathbf{0})$ *denote a strictly convex ray-cone* (as defined in Alexandrov [1972]) *with vertex* $\mathbf{0}$ *in an affine space* $\boldsymbol{A}$, *and for all* $\boldsymbol{a}$ *in* $\boldsymbol{A}$, *let* $C(\boldsymbol{a})$ *denote the translation of* $C(\mathbf{0})$ *with vertex* $\boldsymbol{a}$. *Let* $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ *be a bijection from* $\boldsymbol{A}$ *onto itself such that for all* $\boldsymbol{a} \in \boldsymbol{A}$, $C(\boldsymbol{a})$ *maps onto* $C(\tilde{\boldsymbol{a}})$ *for some* $\tilde{\boldsymbol{a}} \in \boldsymbol{A}$. *Then* $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ *maps lines into lines, and is thus affine when ever* $\boldsymbol{A}$ *is finite-dimensional.*

Another result for convex cones in normed linear spaces appears in Rothaus [1966].

On spaces other than metric vector spaces, causal structures can be defined by an appropriate order structure on the space. The following theorem of Gheorghe and Mihul [1969] is an example.

THEOREM. *Let a causal relation be given on* $\mathbb{R}^n$ $(n \geqslant 3)$ *by a partial order* $\preccurlyeq$ *such that*

(i) $\preccurlyeq$ *is compatible with linearity* (*i.e. it is preserved by translations and dilatations by positive scalars*),

(ii) $\mathbb{R}^n$ *is directed with respect to* $\preccurlyeq$ (*i.e. for all* $\boldsymbol{r}$ *and* $\boldsymbol{s}$ *in* $\mathbb{R}^n$ *there exists a* $\mathbf{t}$ *in* $\mathbb{R}^n$ *such that* $\boldsymbol{r} \preccurlyeq \mathbf{t}$ *and* $\boldsymbol{s} \preccurlyeq \mathbf{t}$),

(iii) *the mapping* $(r_1, r_2, \ldots, r_n) \mapsto (-r_1, r_2, \ldots, r_n)$ *on* $\mathbb{R}^n$ *reverses the order* $\preccurlyeq$,

(iv) *the cone* $C := \{\boldsymbol{r} \in \mathbb{R}^n\colon\ \mathbf{0} \preccurlyeq \boldsymbol{r}\}$ *is closed in the usual* $\mathbb{R}^n$ *topology.*

Let $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ *be a bijection from* $\mathbb{R}^n$ *onto itself such that for all* $\boldsymbol{r}$ *and* $\boldsymbol{s}$ *in* $\mathbb{R}^n$,

$$\boldsymbol{r} \preccurlyeq \boldsymbol{s} \quad \text{if and only if} \quad \bar{\boldsymbol{r}} \preccurlyeq \bar{\boldsymbol{s}}.$$

Then $\boldsymbol{r} \mapsto \bar{\boldsymbol{r}}$ *must be affine.*

Other abstract variations of Zeeman's theorem may be found in Noll and Schäffer [1978], and Schäffer [1979].

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CHAPTER 17

Metric Geometry

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Contents

Introduction	947
1. Quadratic Forms	947
1.1. Definition of quadratic forms	947
1.2. Orthogonality	949
1.3. The kernel of a quadratic form	951
1.4. Orthogonal decompositions of quadratic forms	953
1.5. Isomorphisms	955
1.6. Isometries	956
1.7. Geometric equivalence of quadratic forms	958
1.8. Metric vector spaces of dimension 2	959
1.9. Metric vector spaces of dimension 3 and 4	961
2. Quadrics	963
2.1. Quadratic sets	964
2.2. Geometric characterizations of quadrics	966
2.3. Symmetries	967
2.4. Geometric characterizations of oval quadrics	969
3. Affine metric geometry	970
3.1. Concepts of affine metric geometry	971
3.2. Isomorphisms of affine metric geometry	974
3.3. Motions and similarities	977
3.4. Affine metric planes	980
3.5. Geometric characterizations of affine metric spaces	987
4. Projective metric geometry	990
4.1. Isomorphisms of projective metric geometry	990
4.2. Quadric circle geometry	991
4.3. The stereographic projection	992
4.4. Affine metric geometry and quadric circle geometry	995
4.5. Subgeometries of projective metric spaces	1000
4.6. Geometric characterizations of subgeometries of projective metric spaces	1002
References	1005

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Introduction

In this chapter, we will consider metric concepts such as distance, congruence, orthogonality, angle, circle, quadric, reflection, motion, similarity, and we want to state corresponding theorems.

These subjects, which are well-known in classical Euclidean geometry, remain meaningful in a much more general context which can be described algebraically by quadratic forms on vector spaces.

In view of the growing importance of quadratic forms in various branches of mathematics as well as in other parts of geometry as, e.g., in the theory of polar spaces (cf. Chapters 2 and 12), it seems worthwhile to consider metric geometry in a frame which is based on the theory of quadratic forms.

There is also a foundational aspect of metric geometry. Over the ages, geometers have tried to recognize the geometric kernel of geometric structures and to deduce all known properties from it, as Euclid did.

The reason for doing this is not only aesthetical but also methodological. Indeed, form and content of theorems, methods and proofs are strongly influenced by the chosen foundation.

Therefore, in this chapter we develop both algebraic and foundational views on metric geometry.

Because algebraic definitions in general are more explicit, while geometric definitions often are more intrinsic, we prefer first to look at the algebraic part and then to develop corresponding geometric concepts.

So, we will start with algebraic aspects of the geometric theory of quadratic forms and then consider quadrics, affine metric geometry, projective metric geometry and subgeometries of projective metric geometry.

For proofs of theorems stated in this chapter and for detailed comments we refer to Schröder [1992] and to the papers specified in the given context.

1. Quadratic Forms

1.1. *Definition of quadratic forms*

In this section, well-known concepts of Linear Algebra are generalized to vector spaces which may have infinite dimension or characteristic 2.

As basic notions we will define orthogonality, kernel of a quadratic form, equimetry, isometry, symmetry, and metric product. In particular, we will introduce theorems which yield remarkable consequences for affine metric and projective metric geometry.

Let V be a vector space over a commutative field $\mathbb{F}$.

A mapping $Q\colon V \to \mathbb{F}$ is called a *quadratic form* on V, provided

(1) $Q(\lambda x) = \lambda^2 \cdot Q(x),\ \forall \lambda \in \mathbb{F},\ \forall x \in V$;

and the mapping

(2) $f\colon V \times V \to \mathbb{F},\ (x, y) \mapsto Q(x+y) - Q(x) - Q(y)$

is bilinear.

We call f the *bilinear form associated with* Q.

Let Q: $V \to \mathbb{F}$ be a quadratic form. Then, we have

(3) $f(x,y) = f(y,x)$, $\forall x, y \in V$,

(4) $Q(x+y) = Q(x) + Q(y) + f(x,y)$, $\forall x, y \in V$,

(5) $f(x,x) = 2 \cdot Q(x)$, $\forall x \in V$;

and by induction we obtain

$$(6)\ Q\left(\sum_{i=0}^{r} \lambda_i x_i\right) = \sum_{i=0}^{r} \lambda_i^2 \cdot Q(x_i) + \sum_{0 \leqslant i < j \leqslant r} \lambda_i \lambda_j \cdot f(x_i, x_j)$$

for $i, j, r \in \mathrm{N} := \{0, 1, 2, \ldots\}$, $\lambda_i \in \mathbb{F}$ and $x_i \in V$.

Let $|M|$ denote the cardinality of the set M. If B is a basis of V (cf. Baer [1952]), the elements of V are representable in the form

$$(7)\ \sum_{b \in B} \lambda_b \cdot b, \text{ where } \lambda_b \in \mathbb{F} \text{ and } |\{b \in B\colon\ \lambda_b \neq 0\}| \in \mathrm{N}.$$

For $B_2 := \{\{x, y\}\colon\ x, y \in B \wedge x \neq y\}$, by using (7), we get

$$(8)\ Q\left(\sum_{b \in B} \lambda_b \cdot b\right) = \sum_{b \in B} \lambda_b^2 \cdot Q(b) + \sum_{\{c,d\} \in B_2} \lambda_c \cdot \lambda_d \cdot f(c,d)$$

and

$$(9)\ f\left(\sum_{c \in B} \lambda_c \cdot c, \sum_{d \in B} \mu_d \cdot d\right) = 2 \cdot \sum_{b \in B} \lambda_b \mu_b Q(b) + \sum_{\{c,d\} \in B_2} (\lambda_c \mu_d + \lambda_d \mu_c) \cdot f(c,d),$$

which means that Q is completely determined by the elements $Q(b)$ and $f(c,d)$ of $\mathbb{F}$ and that Q is a 'quadratic' function of the 'coordinates' λ_b $(b, c, d \in B)$.

If α: $B \to \mathbb{F}$ and β: $B_2 \to \mathbb{F}$ are arbitrary mappings, the mapping

$$(10)\ Q'\colon\ V \to \mathbb{F},\ \sum_{b \in B} \lambda_b b \mapsto \sum_{b \in B} \lambda_b^2 \cdot \alpha(b) + \sum_{\{c,d\} \in B_2} \lambda_c \cdot \lambda_d \cdot \beta(\{c,d\}),$$

turns out to be a quadratic form on V. Obviously, by varying α and β in (10), we obtain all quadratic forms of V into $\mathbb{F}$ (cf. Bourbaki [1959]).

Let $\operatorname{char} \mathbb{F}$ be the *characteristic* of $\mathbb{F}$. Then, from (5) we get

(11) $\operatorname{char} \mathbb{F} \neq 2 \Rightarrow Q(x) = f(x,x)/2$, $\forall x \in V$,

which means that Q is determined by f whenever $2 \neq 0$ in $\mathbb{F}$.

On the other hand, (5) leads to

(12) $\operatorname{char} \mathbb{F} = 2 \Rightarrow f(x,x) = 0$, $\forall x \in V$,

and considering (8) and (9) we see that Q is *not* determined by f if $2 = 0$.

The triple $(V, \mathbb{F}, Q)$ is called a *metric vector space*.

Well-known examples of metric vector spaces are the *Euclidean* space $(\mathbb{R}^n, \mathbb{R}, Q_e)$, where Q_e is given by

(13) Q_e: $\mathbb{R}^n \to \mathbb{R}$, $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1^2 + \cdots + \lambda_n^2$ $(n \in \dot{\mathrm{N}} := \mathrm{N} \setminus \{0\})$,

and the *Minkowskian* space $(\mathbb{R}^n, \mathbb{R}, Q_m)$, where Q_m is given by

(14) Q_m: $\mathbb{R}^n \to \mathbb{R}$, $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1^2 + \cdots + \lambda_{n-1}^2 - \lambda_n^2$ $(n \in \mathbb{N} \setminus \{0,1\})$.

From the *projective* point of view, as we will see, Q_e describes an *elliptic* space, and Q_m describes a *hyperbolic* space. Therefore, Q_e is called *Euclidean or elliptic*, and Q_m is called *Minkowskian or hyperbolic*.

The concept of metric vector spaces generalizes the examples under consideration to Euclidean and non-Euclidean metric structures, where the metric from the algebraic point of view is given by the quadratic form Q.

In this context, $Q(x)$ is called the *square length of the vector* x with respect to the generalized metric, given by Q. Formula (4) is a generalized 'Law of Cosines'.

1.2. *Orthogonality*

Let $(V, \mathbb{F}, Q)$ be a metric vector space and let f be the bilinear form associated with Q.

1.2.1. In view of 1.1 (13), (14), vectors $x, y \in V$ are called *orthogonal* with respect to Q, written $x \perp y$, iff (i.e., if and only if) $f(x, y) = 0$. From 1.1 (3) we obtain

(1) $x \perp y \Leftrightarrow y \perp x$, $\forall x, y \in V$.

For $a \in V$, the mapping $V \to V$, $x \mapsto f(x, a)$, is a linear form, and thus we obtain

(2) For $a \in V$, the set $a^\perp := \{x \in V\colon\ x \perp a\}$ is a hyperplane of V or equal to V.

We will use the following abbreviations:

'$U \leqslant W$' indicates that U and W are (vector) subspaces of V such that $U \subseteq W$.

For $M \subseteq V \vee M \in V$, $\operatorname{span} M$ denotes the vector subspace spanned by M, i.e. the smallest vector subspace of V which contains M.

For $r \in \dot{\mathbb{N}} := \{1, 2, 3, \dots\}$ and $U_1, \dots, U_r \leqslant V$, we write

$$U_1 + \cdots + U_r \quad \text{or} \quad U_1 \oplus \cdots \oplus U_r$$

instead of $\operatorname{span}(U_1 \cup \cdots \cup U_r)$, where the notation $U_1 \oplus \cdots \oplus U_r$ indicates, that the additional property

$$\bigl(x_i, y_i \in U_i,\ \forall i \in \{1, \dots, r\}\bigr) \wedge \sum_{i=1}^{r} x_i = \sum_{i=1}^{r} y_i \Rightarrow \bigl(x_i = y_i,\ \forall i \in \{1, \dots, r\}\bigr)$$

is fulfilled.

For $M \subseteq V$, M nonempty, we define

$$M^\perp := \{x \in V\colon\ x \perp a\ \ \forall a \in M\}.$$

Furthermore, we put $\varnothing^\perp := V$. Then, $a^\perp = \{a\}^\perp$ for $a \in V$.

By using (2), we obtain

(3) $M \subseteq V \wedge M \neq \varnothing \Rightarrow O \in M^\perp = \bigcap_{a \in M} a^\perp \leqslant V$,

(4) $O \in V^\perp = \bigcap_{a \in V} a^\perp \leqslant V$,

(5) $O^\perp = V$,

where O is the zero vector of V.

1.2.2. The vector subspace $V^\perp$ of V is called the *radical* of Q, of f, of $\perp$ and of $(V, \mathbb{F}, Q)$. Moreover, $Q, f, \perp$ and $(V, \mathbb{F}, Q)$ are called *regular* in case of $V^\perp = \{O\}$, and *irregular* otherwise.

The quadratic form Q is called *singular* whenever $Q = 0$, while f and $\perp$ are called *isotropic* iff $V = V^\perp$. Obviously, $Q = 0$ implies $V = V^\perp$, but the converse is not true in case of $\operatorname{char} \mathbb{F} = 2$.

EXAMPLE. Consider

$$Q_r\colon\ \mathbb{F}^n \to \mathbb{F},\ (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1^2 + \cdots + \lambda_r^2 \quad (r, n \in \dot{\mathbb{N}} \wedge r \leqslant n).$$

Then $\operatorname{char} \mathbb{F} \neq 2$ implies $\dim V^\perp = n - r$, but $\operatorname{char} \mathbb{F} = 2$ implies $V^\perp = V$, $\forall r \leqslant n$ (cf. 1.1 (9)).

1.2.3. If A and B are subsets of V, we easily obtain

(1) $A \supseteq B \Rightarrow A^\perp \leqslant B^\perp$,

(2) $A \subseteq A^{\perp\perp} := (A^\perp)^\perp$,

(3) $A^\perp = (\operatorname{span} A)^\perp = ((A^\perp)^\perp)^\perp$.

The sets A, B are called *orthogonal*, written $A \perp B$, whenever $A \subseteq B^\perp$. Obviously, 1.2.1 (1) leads to $A \perp B \Leftrightarrow B \perp A$.

For any family of subsets $(M_i)_{i \in I}$ of V, by using (1)–(3) we obtain

(4) $\displaystyle\bigcup_{i \in I} M_i^\perp \subseteq \left(\bigcap_{i \in I} M_i\right)^\perp$,

(5) $\displaystyle\bigcap_{i \in I} M_i^\perp = \left(\bigcup_{i \in I} M_i\right)^\perp$,

(6) $\displaystyle M \subseteq V \wedge M_i \perp M,\ \forall i \in I \Rightarrow \operatorname{span}\left(\bigcup_{i \in I} M_i\right) \perp M \cup V^\perp$,

and for $a_1, \ldots, a_n \in V$ and $U_1, \ldots, U_n \leqslant V$ $(n \in \dot{\mathbb{N}})$ we obtain

(7) $a_1^\perp \cap \cdots \cap a_n^\perp = (\operatorname{span}\{a_1, \cdots, a_n\})^\perp = (\mathbb{F}a_1 + \cdots + \mathbb{F}a_n)^\perp$,

(8) $U_1^\perp \cap \ldots \cap U_n^\perp = (U_1 + \ldots + U_n)^\perp$,

(9) $U_1^\perp = (U_1 + V^\perp)^\perp$,

where $\mathbb{F}a_i := \{\lambda a_i\colon\ \lambda \in \mathbb{F}\} = \operatorname{span} a_i$ for $i = 1, \ldots, n$.

1.2.4. If U is a vector subspace of V, by the axiom of choice, there exists a basis B of U, and its cardinal number $|B|$ is the dimension of U, noted $\dim U$ (cf. Baer [1952]). Moreover, there exists a vector subspace W of V such that $U \oplus W = V$ (cf. Baer [1952]). It turns out that W is isomorphic to the quotient space V/U, which implies $\dim W = \dim V/U$. The subspaces U and W are called *complementary in* V, and $\dim V/U$ is called the *codimension of* U *in* V.

By definition, the vector subspace U is a *line* (resp., a *plane*, a *hyperplane*) of V provided $\dim U = 1$ (resp., $\dim U = 2$, $\dim V/U = 1$).

For $n \in \mathbb{N}$, we define $\mathcal{U}_n := \{U\colon\ U \leqslant V \wedge \dim U = n\}$.

Beside the well-known formulas concerning the dimension of vector subspaces S, T of V (cf. Baer [1952] and Bourbaki [1974]) we need the following formulas, which are easily proved:

(1) $\dim(S \cap T) + \dim V/S = \dim T + \dim V/(S+T)$,

(2) $S \not\subseteq T \wedge \dim V/T = 1 \Rightarrow \dim V/(S \cap T) = 1 + \dim V/S$.

1.2.5. Assume $U \leqslant V$ and $\dim U = r \in \mathbb{N}$. Then, by using 1.2.3 (2) and 1.2.4 (1), (2), we obtain

(1) $U^\perp = (U + V^\perp)^\perp \wedge \dim V/U^\perp = \dim U/(U \cap V^\perp) = \dim(U + V^\perp)/V^\perp \leqslant r$,

(2) $U \cap U^\perp = \{O\} \Rightarrow U \oplus U^\perp = V \wedge \dim V/U^\perp = r$,

(3) $V^\perp = \{O\} \Rightarrow U = U^{\perp\perp} \wedge \dim V/U^\perp = r$ (cf. Bourbaki [1959]).

Obviously, for $W \leqslant V$, $Q' := Q|_W$ (i.e. Q restricted to W) is a quadratic form on W, and $f' := f|_{W \times W}$ is the bilinear form associated with Q'. The corresponding orthogonality is given by $\perp' := \perp|_{W \times W}$, and $(W, \mathbb{F}, Q')$ is a metric vector space. Therefore W is called *regular* (resp., *irregular*, *singular*), provided Q' is regular (resp., irregular, singular), and *isotropic*, provided f' is isotropic. We obtain

(4) W is regular iff $W \cap W^\perp = \{O\}$.

Assume now that W and $V^\perp$ are complementary in V. Then, by using the above notations, we get

(5) $W \oplus V^\perp = V \wedge W \cap W^\perp = \{O\} \wedge W^\perp = V^\perp$;

and the mapping

(6) $\varphi\colon\ V = W \oplus V^\perp \to W,\ w + v \mapsto w\ (w \in W,\ v \in V^\perp)$

turns out to be an epimorphism with the properties

(7) $f(x,y) = f'(\varphi(x), \varphi(y)),\ \forall x, y \in V$,

(8) $\varphi(x) = \varphi(y) \Leftrightarrow x + V^\perp = y + V^\perp,\ \forall x, y \in V$.

Because of (5) and (7), f' is regular, and f is completely determined by f' and $V^\perp$. From (3)–(8) we obtain

(9) $U \leqslant V \wedge \dim U \in \mathbb{N} \Rightarrow U^{\perp\perp} = (U + V^\perp)^{\perp\perp} = U + V^\perp$.

Occasionally, we state theorems based on

1.2.6. EMBEDDING THEOREM. *Given the metric vector space $(V, \mathbb{F}, Q)$, there exists a regular metric vector space $(V_r, \mathbb{F}, Q_r)$ such that*

$$V \leqslant V_r \wedge \dim V_r/V = \dim V^\perp \wedge Q_r|_V = Q.$$

For a proof, cf. Frank [1984b], p. 159.

1.3. *The kernel of a quadratic form*

Let $(V, \mathbb{F}, Q)$ be a metric vector space and let f be the bilinear form associated with Q.

1.3.1. The kernel of the quadratic form Q is defined to be the set

$\ker Q := Q^{-1}(\{0\}) = \{x \in V\colon\ Q(x) = 0\}$.

The elements of $\ker Q$ are called *singular*, while the elements of $V \setminus \ker Q$ are called *Euclidean*. By 1.1 (1)–(5), we get

(1) $O \in \ker Q \wedge (\mathbb{F}x \subseteq \ker Q,\ \forall x \in \ker Q) \wedge (\operatorname{char}\mathbb{F} \neq 2 \Rightarrow V^{\perp} \subseteq \ker Q)$.

Obviously, Q is *singular* iff $\ker Q = V$.

Generalizing the notations of 1.1, Q is called *Euclidean* or *elliptic*, iff $\ker Q = \{O\}$, and Q is called *Minkowskian* or *hyperbolic*, iff $\ker Q$ contains at least two distinct lines of V, but no plane of V.

Moreover, Q is called *Galilean*, iff $\ker Q$ is a line of V, and Q is called *2-singular*, iff $\ker Q$ is the union of two distinct hyperplanes of V.

For $U \leqslant V$ (cf. 1.2.1), the notations introduced for $Q|_U$ are used for U as well.

A singular subspace U of V is called *maximal* provided $U \leqslant W \leqslant V \wedge Q|_W = 0$ implies $U = W$.

From (1) we deduce

(2) $\dim V = 0 \Rightarrow Q = 0 \wedge \ker Q = V$.

(3) $\dim V = 1 \Rightarrow \ker Q = \{0\} \dot{\vee} \ker Q = V$,

where '$\dot{\vee}$' means 'either-or'.

Furthermore, the following statements are easily proved:

1.3.2. Suppose $\dim V = 2$. Then Q is either singular or Galilean or Euclidean or 2-singular. Moreover, Q is 2-singular iff it is Minkowskian.

1.3.3. Assume $\ker Q \leqslant V$. Then, the following statements are valid:

(1) $\operatorname{char}\mathbb{F} \neq 2$ implies $\ker Q = V^{\perp}$.

(2) $\operatorname{char}\mathbb{F} = 2$ implies $\ker Q \leqslant V^{\perp}$.

(3) If W and $\ker Q$ are complementary subspaces of V, then W is Euclidean, and we obtain $Q(a + x) = Q(a),\ \forall (a, x) \in W \times \ker Q$, i.e. Q is completely determined by $\ker Q$ and $Q|_W$.

1.3.4. The set $V_Q^{\perp} := \ker Q \cap V^{\perp}$ always is a subspace of V and $V^{\perp}$. In particular, $\ker Q \subseteq V^{\perp}$ implies $\ker Q \leqslant V$. Moreover, $\operatorname{char}\mathbb{F} \neq 2$ implies $V_Q^{\perp} = V^{\perp}$, and $\operatorname{char}\mathbb{F} = 2$ implies $\dim V^{\perp}/V_Q^{\perp} \leqslant |\dot{\mathbb{F}}/\{\lambda^2\colon \lambda \in \dot{\mathbb{F}}\}|$, where $\dot{\mathbb{F}} := \{x \in \mathbb{F}\colon x \neq 0\}$ (cf. Dieudonné [1955], p. 33, and Havlicek [1985]).

For example, in case of $V = \mathbb{F}^2$ and $\operatorname{char}\mathbb{F} = 2$ and $Q((x, y)) = x^2$ for $x, y \in \mathbb{F}$, we obtain $V^{\perp} = V$ and $V_Q^{\perp} = \mathbb{F}(0, 1)$.

1.3.5. Suppose $\ker Q$ is the union of two distinct hyperplanes H_1, H_2 and define $D := H_1 \cap H_2$. Then there exist elements $a \in H_1 \setminus D$ and $b \in H_2 \setminus D$ such that

(1) $Q((\lambda a + \mu b + d) = \lambda\mu,\ \forall (\lambda, \mu, d) \in \mathbb{F} \times \mathbb{F} \times D$

holds, and we obtain

(2) $V^{\perp} = D$.

1.3.6. Assume $\ker Q \not\leqslant V$. Then we get

(1) span($\ker Q$) is equal to V.

(2) If $\ker Q$ contains a hyperplane of V, then Q is 2-singular.

(3) There exists an element $a \in \ker Q \setminus V^{\perp}$. If b is an arbitrary element of $V \setminus a^{\perp}$, then $\varepsilon := \mathbb{F}a + \mathbb{F}b$ is a Minkowskian plane with the properties $\varepsilon \oplus \varepsilon^{\perp} = V$ and $\dim V/\varepsilon^{\perp} = 2$.

1.3.7. REMARKS.

(1) Suppose $\dim V = 1$ and $a \in V \setminus \{O\}$. Then,

$$Q_1\colon\ V \to \mathbb{F}, \quad \lambda a \mapsto \lambda^2,$$

is a Euclidean quadratic form and Q_1 is regular iff $\operatorname{char}\mathbb{F} \neq 2$.

(2) Assume $\mathbb{F}$ is an ordered field. Then, according to notation 1.1 (7), with respect to a basis B of V, for any mapping $\alpha\colon\ B \to \dot{\mathbb{F}}_+ := \{x \in \mathbb{F}\colon\ x > 0\}$,

$$Q_2\colon\ V \to \mathbb{F}, \quad \sum_{b\in B} \lambda_b \cdot b \mapsto \sum_{b\in B} \lambda_b^2 \alpha(b),$$

is a Euclidean quadratic form on V. Modifying this example (cf. 1.2.5), it turns out that every subspace of V is the kernel of a quadratic form, if $\mathbb{F}$ is ordered.

(3) Suppose, Q is Euclidean, $\operatorname{char}\mathbb{F} \neq 2$ and $\mathbb{F}^{(2)} := \{x^2\colon\ x \in \mathbb{F}\} = \mathbb{F}$. Then, an easy computation leads to $\dim V \leqslant 1$.

(4) Assume Q is Euclidean and $\mathbb{F}$ is finite. Then one obtains $\dim V \leqslant 2$, and $\dim V = 2$ implies $Q(V) = \mathbb{F}$ (cf. Bachmann [1973], p. 123, Jacobson [1974], p. 342, Scharlau [1985], p. 39).

(5) If M is a hyperplane or a union of two distinct hyperplanes, then, considering (1), 1.3.3 and 1.3.5, it is easy to find a quadratic form Q'' such that $\ker Q'' = M$.

(6) If $(V, \mathbb{F}, Q)$ is a Minkowskian space of dimension 2, i.e. a Minkowskian plane, then, because of 1.3.5, there exists a basis $\{a, b\}$ of V such that

$$Q(\lambda a + \mu b) = \lambda\mu, \quad \forall \lambda, \mu \in \mathbb{F},$$

and Q is regular. In case of $\operatorname{char}\mathbb{F} \neq 2$, the basis $\{a, b\}$ may be replaced by $\{c, d\}$, where $c := a + b$ and $d := a - b$, and then we obtain

$$Q(\lambda c + \mu d) = \lambda^2 - \mu^2, \quad \forall \lambda, \mu \in \mathbb{F}$$

(cf. 1.1 (14)).

(7) Assume $x \in \ker Q \setminus \{O\}$ and choose $y \in V \setminus \mathbb{F}x$. Then, because of $x \perp x$, 1.3.2, 1.3.3, and 1.3.5, for the plane $\varepsilon := \mathbb{F}x + \mathbb{F}y$ we get

(i) ε is Minkowskian iff $x \not\perp y$ holds;

(ii) ε is singular or Galilean iff $x \perp y$ holds.

(8) Suppose $|\mathbb{F}| \geqslant 3$ and let $T, H \leqslant V$ such that $T \not\subseteq H$. Then, because of (7), $Q|_{T\setminus H} = 0$ implies $Q|_T = 0$.

1.4. *Orthogonal decompositions of quadratic forms*

Again, let $(V, \mathbb{F}, Q)$ be a metric vector space and let f be the bilinear form associated with Q.

1.4.1. Assume $U_1, \ldots, U_r$ ($r \in \mathbb{N} \setminus \{0, 1\}$) are subspaces of V such that

(1) $(U_1 \oplus \cdots \oplus U_r = V) \wedge (U_i \perp U_j$ for $i \neq j$ $(i, j \in \{1, \ldots, r\}))$

is valid. Then, instead of (1), we write

(2) $U_1 \perp \cdots \perp U_r = V$

as well and call V the *orthogonal sum* of $U_1, \ldots, U_r$. Furthermore, $(U_1, \ldots, U_r)$ is called an *orthogonal decomposition* of Q and V, if (2) holds.

If $V = \{O\}$ or if V is the orthogonal sum of r Minkowskian planes ($r \in \dot{\mathbb{N}}$), i.e. by 1.3.7 (6), there exists a basis $\{a_1, \ldots, a_r, b_1, \ldots, b_r\}$ of V such that

$$Q\left(\sum_{i=1}^{r}(\lambda_i a_i + \mu_i b_i)\right) = \sum_{i=1}^{r} \lambda_i \mu_i, \quad \forall \lambda_i, \mu_i \in \mathbb{F},$$

then $(V, \mathbb{F}, Q)$ and Q are called *Artinian*. According to 1.3.1, a vector subspace U of V is called *Artinian*, whenever $Q|_U$ is Artinian. It turns out that each Artinian subspace of V is regular.

The following theorems are well known (cf. 1.2.5, Artin [1957] and Bourbaki [1959]):

1.4.2. Assume $\dim V/V^\perp \in \mathbb{N}$ and $\operatorname{char} \mathbb{F} \neq 2$. Then there exists a basis B of V such that $B \setminus \{b\} \subseteq b^\perp \ \forall b \in B$ holds.

We call B an *orthogonal basis* of V.

1.4.3. Let U be a regular and finite-dimensional vector subspace of V and suppose $\operatorname{char} \mathbb{F} = 2$. Then, U is an orthogonal sum of regular planes, each of which is Euclidean or Minkowskian.

In particular, $\operatorname{char} \mathbb{F} = 2 \wedge \dim V/V^\perp \in \mathbb{N}$ implies $\dim V/V^\perp \in 2\mathbb{N}$.

1.4.4. Suppose, V contains a maximal singular subspace of finite dimension r. Then, every maximal singular subspace of V has dimension r, and r is called the *Witt index* $\operatorname{ind} Q$ of Q and of $(V, \mathbb{F}, Q)$.

1.4.5. Assume $V = A \perp E$, where A is Artinian of dimension $2r$ and E is Euclidean. Then, $E = A^\perp$ and $r = \operatorname{ind} Q|_A = \operatorname{ind} Q$.

1.4.6. Suppose $V_Q^\perp := \ker Q \cap V^\perp = \{O\}$. Then, for every finite-dimensional singular subspace U of V, there exists a singular subspace U' of V such that $U \cap U' = \{O\}$ and that $U \oplus U'$ is Artinian of dimension $2 \cdot \dim U$.

1.4.7. WITT'S DECOMPOSITION THEOREM. *Assume $V_Q^\perp$ and W are complementary in V and $Q|_W$ has finite Witt index r. Then, there exists an Artinian subspace A of dimension $2r$ and an Euclidean subspace E such that $W = A \perp E$ and $V = A \perp E \perp V_Q^\perp$. Moreover, the maximal singular subspaces of V are the sets $U + V_Q^\perp$, where U is an arbitrary maximal singular subspace of W.*

1.5. *Isomorphisms*

Let $(V, \mathbb{F}, Q)$ and $(V', \mathbb{F}', Q')$ be metric vector spaces. The notations, introduced for $(V, \mathbb{F}, Q)$, will be used for $(V', \mathbb{F}', Q')$ as well, but with a prime.

1.5.1. For $\lambda \in \dot{\mathbb{F}} := \mathbb{F} \setminus \{0\}$, λQ: $V \to \mathbb{F}$, $x \mapsto \lambda \cdot Q(x)$, is a quadratic form on V, and λf: $V \times V \to \mathbb{F}$, $(x, y) \mapsto \lambda \cdot f(x, y)$, is the bilinear form associated with λQ.

From the geometric point of view, as we will see, there is no essential difference between Q and λQ.

Therefore, two quadratic forms Q_1: $V \to \mathbb{F}$ and Q_2: $V \to \mathbb{F}$ are called *geometrically equivalent*, written $Q_1 \sim Q_2$, provided there exists a nonzero element $\lambda \in \dot{\mathbb{F}}$ such that $Q_2 = \lambda Q_1$. Obviously, $\sim$ is an equivalence relation on the set of all quadratic forms of V into $\mathbb{F}$.

Beside this definition, we need the concept of semilinearity. If ρ: $\mathbb{F}(+, \cdot) \to \mathbb{F}'(+, \cdot)$ is an isomorphism, a bijection σ: $V \to V'$ is called a *ρ-linear* (*or semilinear*) *bijection* of $(V, \mathbb{F})$ onto $(V', \mathbb{F}')$, iff the conditions

(SL1) $\sigma(x + y) = \sigma(x) + \sigma(y)$, $\forall x, y \in V$,

(SL2) $\sigma(\lambda x) = \rho(\lambda)\sigma(x)$, $\forall \lambda \in \mathbb{F}$, $\forall x \in V$,

are fulfilled. σ is called *linear* iff ρ is the identity mapping (cf. Baer [1952]).

By direct verification, we obtain

1.5.2. Suppose, σ is a ρ-linear bijection of $(V, \mathbb{F})$ onto $(V', \mathbb{F}')$. Then,

$$Q_0 := \rho \circ Q \circ \sigma^{-1}: V' \to \mathbb{F}'$$

is a quadratic form with the associated bilinear form

$$f_0: V' \times V' \to \mathbb{F}', \quad (x, y) \mapsto \rho \circ f(\sigma^{-1}(x), \sigma^{-1}(y)),$$

and $\sigma(\ker Q) = \ker Q_0$.

Furthermore, $x \perp y \Leftrightarrow \sigma(x) \perp_0 \sigma(y)$, $\forall x, y \in V$, where $\perp_0$ is the orthogonality relation corresponding to f_0.

1.5.3. A ρ-linear bijection σ of $(V, \mathbb{F})$ onto $(V', \mathbb{F}')$ is called an *isomorphism* of $(V, \mathbb{F}, Q)$ onto $(V', \mathbb{F}', Q')$ and in case of $(V, \mathbb{F}, Q) = (V', \mathbb{F}', Q')$ an *automorphism* of $(V, \mathbb{F}, Q)$, provided $\rho \circ Q \circ \sigma^{-1} \sim Q'$ holds. A *linear* isomorphism σ of $(V, \mathbb{F}, Q)$ onto $(V', \mathbb{F}, Q')$ is called an *equimetry*, and in case of $Q \circ \sigma^{-1} = Q'$ an *isometry*.

The metric vector spaces $(V, \mathbb{F}, Q)$ and $(V', \mathbb{F}', Q')$ as well as the associated quadratic forms Q and Q' are called *isomorphic* (*equimorphic*, *isometric*) provided there exists an isomorphism (an equimetry, an isometry) of $(V, \mathbb{F}, Q)$ onto $(V', \mathbb{F}', Q')$.

Obviously, the *automorphisms* of $(V, \mathbb{F}, Q)$ form a group Aut$(V, \mathbb{F}, Q)(\circ)$, and the set of *isometries* of $(V, \mathbb{F}, Q)$, which is denoted by $I(Q)$, is a subgroup of Aut$(V, \mathbb{F}, Q)$.

It is easily proved that the metric notions and concepts introduced so far are invariant under isomorphisms. Moreover, we get (cf. Bourbaki [1959]):

1.5.4. Suppose, $(V, \mathbb{F}, Q)$ and $(V', \mathbb{F}', Q')$ are Artinian and of equal (finite) dimension. If ρ: $\mathbb{F}(+,\cdot) \to \mathbb{F}'(+,\cdot)$ is an isomorphism, then for each $\alpha \in \mathbb{F}'$ there exists an isomorphism σ of $(V, \mathbb{F}, Q)$ onto $(V', \mathbb{F}', Q')$ such that $\alpha \cdot \rho \circ Q = Q' \circ \sigma$. In particular, if $\mathbb{F} = \mathbb{F}'$, $(V, \mathbb{F}, Q)$ and $(V', \mathbb{F}, Q')$ are isometric.

1.5.5. Suppose $V = A \perp E \perp V_Q^{\perp}$ and $V' = A' \perp E' \perp V'^{\perp'}_{Q'}$ are Witt decompositions of V, resp., V', where A, A' are Artinian and E, E' are Euclidean. Then, in case of $\mathbb{F} = \mathbb{F}'$, we obtain:

(1) If $V_Q^{\perp}$ and $V'^{\perp'}_{Q'}$ are isomorphic and if $\dim A = \dim A'$, then every existing isometry (equimetry, isomorphism) φ: $E \to E'$ can be extended to an isometry (equimetry, isomorphism) $\overline{\varphi}$: $V \to V'$.

(2) If $V = V'$ and if $Q = Q'$, then the Euclidean spaces E and E' are isometric.

As corollaries, we get

1.5.6. Assume $V = \mathbb{F}^3$ and $\ker Q \not\leq V$.

Then, up to equimetry, we obtain $Q \in \{Q_{2s}, Q_m\}$, where

$$Q_{2s}((\lambda, \mu, \nu)) = \lambda\mu \quad \text{and} \quad Q_m((\lambda, \mu, \nu)) = \lambda\mu - \nu^2, \ \forall \lambda, \mu, \nu \in \mathbb{F}.$$

Obviously $\operatorname{ind} Q_{2s} = 2$ and $\operatorname{ind} Q_m = 1$, where Q_{2s} is 2-singular and Q_m is Minkowskian.

1.5.7. Assume $V = \mathbb{F}^4$ and $\ker Q \not\leq V$.

Then, up to equimetry, we obtain $Q \in \{Q_{2s}, Q_c, Q_a, Q_m\}$, where

$$Q_{2s}((\kappa, \lambda, \mu, \nu)) = \kappa\lambda, \qquad Q_c((\kappa, \lambda, \mu, \nu)) = \kappa\lambda - \mu^2,$$
$$Q_a((\kappa, \lambda, \mu, \nu)) = \kappa\lambda + \mu\nu, \qquad Q_m((\kappa, \lambda, \mu, \nu)) = \kappa\lambda + \mu^2 + q\mu\nu - p\nu^2$$

for $\kappa, \lambda, \mu, \nu \in \mathbb{F}$. Here, q, p are fixed elements of $\mathbb{F}$ such that $x^2 + qx \neq p$, $\forall x \in \mathbb{F}$. Obviously, $\operatorname{ind} Q_{2s} = 3$, $\operatorname{ind} Q_c = \operatorname{ind} Q_a = 2$ and $\operatorname{ind} Q_m = 1$, where Q_{2s}, Q_a, Q_m is 2-singular, Artinian, Minkowskian respectively, and where $\ker Q_c$ is called a *cone*. Observe, that Q_m does not exist for every field $\mathbb{F}$, because $Q_m|_\varepsilon$ for $\varepsilon := \mathbb{F}(0,0,1,0) + \mathbb{F}(0,0,0,1)$ is Euclidean (cf. 1.3.7 (3)).

1.6. *Isometries*

Let $(V, \mathbb{F}, Q)$ be a metric vector space and let f be the bilinear form associated with Q.

1.6.1. If φ: $V \to V$ is a linear bijection, $\operatorname{fix} \varphi := \{x \in V\colon \varphi(x) = x\}$ is the set of vectors fixed by φ, and $p(\varphi) := \{\varphi(y) - y\colon y \in V\}$ called the *path* of φ (cf. Ellers [1977]).

It turns out that $\operatorname{fix} \varphi$ and $p(\varphi)$ are vector subspaces of V with the properties

(1) $p(\varphi) \cong V/\operatorname{fix} \varphi \wedge \dim p(\varphi) = \dim V/\operatorname{fix} \varphi$,

(2) $\operatorname{fix} \varphi = \operatorname{fix} \varphi^{-1} \wedge p(\varphi) = p(\varphi^{-1})$.

In particular, if φ is an isometry of $(V, \mathbb{F}, Q)$, i.e. if

(3) $Q(\varphi(x)) = Q(x) \wedge f(\varphi(x), \varphi(y)) = f(x, y),\ \forall x, y \in V$
holds (cf. 1.5.3), then we get
(4) $p(\varphi) \perp \operatorname{fix}\varphi$,
(5) $\dim p(\varphi) \in \mathbb{N} \wedge p(\varphi) \cap p(\varphi)^\perp = \{O\} \Rightarrow p(\varphi)^\perp = \operatorname{fix}\varphi \wedge p(\varphi) \oplus \operatorname{fix}\varphi = V$,
(6) $\dim p(\varphi) \in \mathbb{N} \wedge V^\perp = \{O\} \Rightarrow p(\varphi)^\perp = \operatorname{fix}\varphi \wedge (\operatorname{fix}\varphi)^\perp = p(\varphi)$.
Moreover, if σ: $(V, \mathbb{F}, Q) \to (V', \mathbb{F}', Q')$ is an isomorphism, we obtain
(7) $\psi := \sigma \circ \varphi \circ \sigma^{-1}$ is an isometry of $(V', \mathbb{F}', Q')$,
(8) $\operatorname{fix}\psi = \sigma\ (\operatorname{fix}\varphi) \wedge p(\psi) = \sigma(p(\varphi))$
(cf. Ellers [1977]).

1.6.2. Suppose $a \in V \setminus \ker Q$, i.e. a is Euclidean. Then the mapping

(1) $\widetilde{a}$: $V \to V,\ x \mapsto x - \dfrac{f(x,a)}{Q(a)} \cdot a$

turns out to be an isometry with the properties
(2) $\widetilde{a} = \widetilde{a}^{-1}$,
(3) $\operatorname{fix}\widetilde{a} = a^\perp \wedge \widetilde{a}(a) = -a$,
(4) $\operatorname{char}\mathbb{F} \neq 2 \vee a \notin V^\perp \Rightarrow p(\widetilde{a}) = \mathbb{F}a$,
(5) $\widetilde{a} = \widetilde{b} \Leftrightarrow \mathbb{F}a = \mathbb{F}b,\ \forall a, b \in V \setminus (V^\perp \cup \ker Q)$.
We call $\widetilde{a}$ the *Q-symmetry with respect to a* and denote the set of all such Q-symmetries by $S_0(Q)$. (Here, 'S_0' indicates that the zero vector O is fixed.) If $\widetilde{a}$ is not the identity mapping, then $\widetilde{a}$ is a reflection at the hyperplane $a^\perp$.

1.6.3. We are interested in products of Q-symmetries and define

$$S_0^r(Q) := \{\sigma_1 \circ \cdots \circ \sigma_r\colon\ \sigma_1, \ldots, \sigma_r \in S_0(Q)\} \quad \text{for } r \in \dot{\mathbb{N}}.$$

In Chevalley [1954], Dieudonné [1955], Ellers [1977], Götzky [1968], Kneser [1970], the following results are proved:
(1) For $r \in \dot{\mathbb{N}}$ and $\varphi_1, \ldots, \varphi_r \in I(Q)$, we have

$$p(\varphi_1 \circ \cdots \circ \varphi_r) \subseteq p(\varphi_1) + \cdots + p(\varphi_r).$$

Thus $\varphi \in S_0^r(Q)$ implies $\dim p(\varphi) \leqslant r$.
(2) Suppose $\varphi \in I(Q)$ and $\sigma \in S_0(Q)$ such that $\operatorname{fix}\varphi \nsubseteq \operatorname{fix}\sigma$. Then, $\operatorname{fix}\sigma \circ \varphi = \operatorname{fix}\varphi \circ \sigma = \operatorname{fix}\sigma \cap \operatorname{fix}\varphi$ is a hyperplane of $\operatorname{fix}\varphi$.
(3) Assume $\varphi \in I(Q)$ and $\sigma \in S_0(Q)$ such that $\{O\} \neq p(\sigma) \subseteq p(\varphi)$. Then, $\operatorname{fix}\varphi$ is a hyperplane of $\operatorname{fix}\sigma \circ \varphi$ and of $\operatorname{fix}\varphi \circ \sigma$.
(4) Suppose, φ is an isometry such that $\dim p(\varphi) = r \in \dot{\mathbb{N}}$. Then,
(i) $V^\perp = \{O\} \wedge |\mathbb{F}| \geqslant 3$ implies $\varphi \in S_0^{r+2}(Q)$,
(ii) $V^\perp = \{O\} \wedge |\mathbb{F}| \geqslant 3 \wedge \dim V = n \in \dot{\mathbb{N}}$ implies $\varphi \in S_0^{n-1}(Q) \cup S_0^n(Q)$,
(iii) Q Euclidean or Minkowskian and dim $V^\perp \leqslant 1$ imply $\varphi \in S_0^r(Q)$,
(iv) $|\mathbb{F}| \geqslant 3 \wedge Q \neq 0 \wedge \varphi|_{V^\perp} = \mathrm{id}_{V^\perp}$ implies $\varphi \in S_0^t(Q)$ for some $t \in \dot{\mathbb{N}}$.
For further results, cf. Ellers [1977].

1.6.4. From the foundational point of view, it is of general interest to look at the situations in which a product of three Q-symmetries is again a Q-symmetry (cf. Bachmann [1937, 1973], Ewald [1974], Karzel [1955a,b, 1971], Lingenberg [1966, 1979b], Nolte [1980a], Quaisser [1970], Schröder [1982, 1984a,b], Sperner [1954]). Indeed, considerations of this type lead to remarkable characterizations of metric geometries (cf. 4.6.2–4).

In this context, consider three linearly dependent vectors a, b, c of $V \setminus \ker Q$ and call

$$d := \begin{cases} \alpha f(a,b) \cdot a - \alpha Q(a) \cdot b & \text{if } c = \alpha a, \\ \alpha Q(a) \cdot c + \gamma Q(c) \cdot a & \text{if } b = \alpha a + \gamma c \quad (\alpha, \gamma \in \mathbb{F}), \end{cases}$$

the *4-th Q-vector of* (a, b, c). In terms of the Clifford algebra $C(Q)(+, \circ)$ corresponding to $(V, \mathbb{F}, Q)$ (cf. Bourbaki [1959] and Schröder [1987b]), we simply have $d = a \circ b \circ c = c \circ b \circ a$, and by using this algebra, or by direct computation, we obtain

(1) $d \in \operatorname{span}\{a, b, c\}$,

(2) $Q(d) = Q(a) \cdot Q(b) \cdot Q(c) \neq 0$,

(3) $2d = f(b,c) \cdot a - f(c,a) \cdot b + f(a,b) \cdot c$,

(4) $\mathbb{F}a = \mathbb{F}c \Rightarrow \mathbb{F}d = \mathbb{F}\,\widetilde{a}(b) = \widetilde{a}(\mathbb{F}b)$,

(5) $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = \widetilde{d} = \widetilde{c} \circ \widetilde{b} \circ \widetilde{a}$ (*3-reflection lemma*).

There exists a converse of (5) if $\dim V^{\perp} \leqslant 1$; otherwise, there are counterexamples (observe that $a, b, c \in V \setminus \ker Q \wedge b, c \in a + \widetilde{V_Q^{\perp}} \Rightarrow \widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = (a - b + c)^{\sim}$, and see 1.6.2 (3)).

(6) 3-REFLECTION THEOREM. *Suppose $a, b, c \in V \setminus \ker Q$ and* $\dim V^{\perp} \leqslant 1$. *Then, a, b, c are linearly dependent iff $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c}$ is an element of $S_0(Q)$.*

The proof of (6) can be given by using (5) and 1.6.3 (1)–(3) (cf. Bachmann [1973] and Lingenberg [1979b] for the case $\dim V = 3$).

1.7. *Geometric equivalence of quadratic forms*

Let $(V, \mathbb{F}, Q_i)$ be a metric vector space of dimension $\geqslant 1$, let f_i be the bilinear form associated with Q_i, and denote the corresponding orthogonality relation by $\perp_i$ $(i = 1, 2)$.

We want to state geometric conditions under which a quadratic form is determined up to geometric equivalence (cf. Schröder [1986a]). We obtain

1.7.1. Assume $\operatorname{char} \mathbb{F} \neq 2$. Then, $\perp_1 = \perp_2 \Leftrightarrow Q_1 \sim Q_2$.

1.7.2. Suppose $\ker Q_1 \not\leqslant V$. Then, $\ker Q_1 = \ker Q_2 \Leftrightarrow Q_1 \sim Q_2$.

1.7.3. If $\ker Q_1$ is a hyperplane of V, we have $\ker Q_1 = \ker Q_2 \Leftrightarrow Q_1 \sim Q_2$.

1.7.4. Assume $V^{\perp_1} \neq V$. Then, $S_0(Q_1) = S_0(Q_2) \Leftrightarrow Q_1 \sim Q_2$.

1.7.5. The set of *Euclidean* lines $\{U \in \mathcal{U}_1\colon\ Q_{|U} \neq 0\}$ will be denoted by $\boldsymbol{R}_Q$. Moreover, we define

$$^3\boldsymbol{R}_Q := \{(g,h,k) \in \boldsymbol{R}_Q^3\colon\ \dim(g+h+k) \leqslant 2\}$$

and consider

$$\mu_Q\colon\ ^3\boldsymbol{R}_Q \to \boldsymbol{R}_Q, \quad (\mathbb{F}a, \mathbb{F}b, \mathbb{F}c) \mapsto \mathbb{F}d \quad (a,b,c \in V \setminus \ker Q),$$

where d is the 4-th Q-vector of (a,b,c). It turns out that μ_Q is well defined. We call μ_Q the *metric product associated with* Q and obtain (cf. Schröder [1986a])

(*) $\mu_{Q_1} = \mu_{Q_2} \Leftrightarrow Q_1 \sim Q_2$.

This means that the metric product in any case determines the given quadratic form up to geometric equivalence, while other notions as $\perp$, $\ker Q$, $S_0(Q)$ possess this property only under additional assumptions.

The following theorem is essential for several proofs of theorems stated later on (cf. Schröder [1987a]):

1.7.6. Extension theorem. *Let V be a vector space over a (commutative) field $\mathbb{F}$ and assume $4 \leqslant \dim V \leqslant \infty$. Furthermore let L, H be some fixed vector subspaces of V such that* $\dim V/L \geqslant 4$ *and* $H \neq V$.

If $\boldsymbol{L}_3$ is defined by

$$\boldsymbol{L}_3 := \{U\colon\ L \leqslant U \leqslant V \wedge \dim U/L = 3 \wedge U \nleq H\}$$

and if, for each $U \in \boldsymbol{L}_3$, a quadratic form $Q_U\colon\ U \to \mathbb{F}$ is given such that the condition

(*) $U \cap W \nleq H \Rightarrow Q_U|_{U\cap W} \sim Q_W|_{U\cap W}\ \forall U, W \in \boldsymbol{L}_3$

holds, then there exists a quadratic form $Q\colon\ V \to \mathbb{F}$ such that

$Q|_U \sim Q_U\ \forall U \in \boldsymbol{L}_3$.

Moreover, again we get a true statement, if the symbols '$\sim$' are replaced by '$=$'.

1.8. *Metric vector spaces of dimension 2*

In this section we want to point out the close relation between metric vector spaces and quadratic algebras in case of dimension 2.

Suppose, $(V, \mathbb{F}, Q)$ is a metric vector space of dimension 2 such that $Q \neq 0$. Because we are interested in Q up to geometrical equivalence only, let us assume that there exists an element $e \in V$ such that $Q(e) = 1$.

1.8.1. Let i be an element of $V \setminus \mathbb{F}e$ and define $p := -Q(i)$ and $q := f(e,i)$, where f is the bilinear form associated with Q.

By using the multiplication

(1) $(xe + yi) \cdot (ue + vi) := (xu + pyv)e + (xv + yu + qyv)i$ for $x, y, u, v \in \mathbb{F}$,

we obtain an associative and commutative $\mathbb{F}$-algebra $V(+,\cdot)$ with unit element e, and by using the identification

(2) $x = xe,\ \forall x \in \mathbb{F}$,

$\mathbb{F}(+,\cdot)$ turns out to be a subfield of the ring $V(+,\cdot)$ such that $1 = e$. In terms of Bourbaki [1974], p. 439, $V(+,\cdot)$ is a *quadratic $\mathbb{F}$-algebra of type* (p,q).

The mapping

(3) $\kappa_Q\colon\ V \to V,\ x + iy \mapsto \overline{x+iy} := x + qy - iy\ (x, y \in \mathbb{F})$,

is called the *conjugation* of $V(+,\cdot)$ and is an automorphism of the algebra $V(+,\cdot)$ with the following properties (cf. Schröder [1974a]):

$$
\begin{array}{lrcll}
(4) & Q(z) &=& Q(\overline{z}) = z\overline{z} \in \mathbb{F}, & \forall z \in V,\\
(5) & Q(z\cdot w) &=& Q(z)\cdot Q(w), & \forall z, w \in V,\\
(6) & f(z,w) &=& z\overline{w} + \overline{z}w, & \forall z, w \in V,\\
(7) & \overline{(\overline{z})} &=& z, & \forall z \in V,\\
(8) & z &=& \overline{z}, & \forall z \in \mathbb{F},\\
(9) & z + \overline{z} &\in& \mathbb{F}, & \forall z \in V,\\
(10) & \multicolumn{4}{l}{\kappa_Q = \mathrm{id}_V \Leftrightarrow q = 0 \wedge \operatorname{char}\mathbb{F} = 2.}
\end{array}
$$

Because $z^2 = f(z,e)\cdot z - Q(z)$, $\forall z \in V$, the multiplication of the algebra $V(+,\cdot)$ depends on Q and e only. Another choice of e yields an isomorphic algebra (cf. Schröder [1980]), and therefore, $V(+,\cdot)$ is called 'the' *algebra associated with* Q.

This algebra is called *separable* if $\kappa_Q \neq \mathrm{id}_V$ and *inseparable* otherwise.

In case of $\operatorname{char}\mathbb{F} \neq 2$, the element i may be chosen in $\mathbb{F}^{\perp}$ which means $q = 0$.

In case of $\operatorname{char}\mathbb{F} = 2 \wedge V^{\perp} = \{0\}$, the element i may be chosen such that $q = 1$, and in case of $\operatorname{char}\mathbb{F} = 2 \wedge V^{\perp} \neq \{0\}$ we have $V^{\perp} = V$, i.e. $q = 0$.

1.8.2. If $\mathbb{L}(+,\cdot)$ is an arbitrary quadratic $\mathbb{F}$-algebra – we always assume $\mathbb{F} \subset \mathbb{L}$ according to 1.8.1 (2) – and if $\kappa_{\mathbb{L}}$ is the corresponding conjugation (cf. Bourbaki [1974], p. 439 ff.), $Q_{\mathbb{L}}\colon\ \mathbb{L} \to \mathbb{F},\ z \mapsto z\cdot\kappa_{\mathbb{L}}(z)$, is a quadratic form on the 2-dimensional $\mathbb{F}$-vector space $\mathbb{L}$.

If $\mathbb{L}'(+,\cdot)$ is a quadratic $\mathbb{F}'$-algebra, $\mathbb{L}(+,\cdot)$ and $\mathbb{L}'(+,\cdot)$ are called *isomorphic*, provided there exists a ring isomorphism $\sigma\colon\ \mathbb{L}(+,\cdot) \to \mathbb{L}'(+,\cdot)$ such that $\sigma(\mathbb{F}) = \mathbb{F}'$. It turns out that the algebras $\mathbb{L}(+,\cdot), \mathbb{L}'(+,\cdot)$ are isomorphic iff the corresponding metric vector spaces $(\mathbb{L}, \mathbb{F}, Q_{\mathbb{L}})$, $(\mathbb{L}', \mathbb{F}', Q_{\mathbb{L}'})$ are isomorphic in the sense of 1.5.3 (cf. Schröder [1980, 1986a]).

1.8.3. Assume $z \in V$. Then, $Q(z) \neq 0$ yields $z\cdot(\overline{z}\cdot Q(z)^{-1}) = Q(z)\cdot Q(z)^{-1} = 1$, and $z\cdot w = 1$ implies $Q(z)\cdot Q(w) = 1 \neq 0$. Therefore, $\dot{V} := \{z \in V\colon\ Q(z) \neq 0\}$ is the group of the invertible elements of the algebra $V(+,\cdot)$, and we obtain

(*) $V(+,\cdot)$ is a field iff Q is Euclidean.

In particular, $V(+,\cdot)$ is isomorphic to $\mathbb{C}(+,\cdot)$, if $\mathbb{F} = \mathbb{R}$ and if Q is Euclidean.

If Q is Galilean, then we always may choose i such that $\mathbb{F}i = \ker Q$. This implies $p = q = 0$, and then $V(+,\cdot)$ is a local ring with the maximal ideal $\mathbb{F}i$, called the *ring of dual numbers over* $\mathbb{F}$ (cf. Bourbaki [1974]).

If Q is Minkowskian, we may chose $i \in \ker Q \setminus \{0\}$ such that $f(i,e) = 1$, which means $p = 0$ and $q = 1$. Here we get $\ker Q = \mathbb{F}i \cup \mathbb{F}\overline{i}$, where $\mathbb{F}i$ and $\mathbb{F}\overline{i}$ are the maximal ideals of the ring $V(+,\cdot)$, and then $V(+,\cdot)$ is called the *ring of double numbers over* $\mathbb{F}$. Taking the $\mathbb{F}$-basis $\{i, \overline{i}\}$ instead of $\{e, i\}$ it is easily proved that $V(+,\cdot)$ in this case is isomorphic to the direct ring product $\mathbb{F}(+,\cdot)\times\mathbb{F}(+,\cdot)$ with component-wise addition *and* multiplication.

1.8.4. By using the multiplication and the conjugation introduced in 1.8.1, we obtain very simple representations of the isomorphisms, the equimetries and the isometries in the case of dimension 2.

Assume, $\mathbb{L}(+,\cdot)$ is a quadratic $\mathbb{F}$-algebra and $\mathbb{L}'(+,\cdot)$ is a quadratic $\mathbb{F}'$-algebra isomorphic to $\mathbb{L}(+,\cdot)$. Then, the isomorphisms from $(\mathbb{L},\mathbb{F},Q_{\mathbb{L}})$ onto $(\mathbb{L}',\mathbb{F}',Q_{\mathbb{L}'})$ are the mappings of the type

(*) $\varphi\colon\ \mathbb{L}\to\mathbb{L}',\ z\mapsto a\cdot z^{\sigma},$

where a is an invertible element of $\mathbb{L}'(+,\cdot)$ and σ is a ring isomorphism from $\mathbb{L}(+,\cdot)$ onto $\mathbb{L}'(+,\cdot)$ such that $\sigma(\mathbb{F})=\mathbb{F}'$ (cf. Schröder [1980, 1986a]).

In particular, in case of $(\mathbb{L}(+,\cdot),\mathbb{F})=(\mathbb{L}'(+,\cdot),\mathbb{F}')$, φ is an equimetry of $(\mathbb{L},\mathbb{F},Q_{\mathbb{L}})$ iff $\sigma|_{\mathbb{F}}=\mathrm{id}_{\mathbb{F}}$, and φ is an isometry of $(\mathbb{L},\mathbb{F},Q_{\mathbb{L}})$ iff $\sigma|_{\mathbb{F}}=\mathrm{id}_{\mathbb{F}}\wedge a\bar{a}=1$.

Moreover, we have the relation $(\sigma|_{\mathbb{F}}=\mathrm{id}_{\mathbb{F}}\Leftrightarrow\sigma\in\{\mathrm{id}_{\mathbb{L}},\kappa_{\mathbb{L}}\})$ if $Q_{\mathbb{L}}$ is not Galilean (cf. Schröder [1979a]).

1.8.5. For $a\in V\setminus\ker Q$ we obtain

(1) $\widetilde{a}\colon\ V\to V,\ z\mapsto -a\cdot\bar{z}\cdot\bar{a}^{-1}$

(cf. 1.6.2 (1)), which is a very handy representation of the Q-symmetries.

For $a,b,c\in V\setminus\ker Q$, the 4-th Q-vector d of (a,b,c) is given by

(2) $d=a\bar{b}c,$

and this representation yields a very simple proof of the 3-reflection lemma 1.6.4 (5) for the case of dimension 2. Thus, the multiplication and the conjugation introduced in 1.8.1 lead to remarkable simplifications for computations with quadratic forms. The following section shows analogous results for certain cases of dimension 3 and 4.

1.9. *Metric vector spaces of dimension 3 and 4*

1.9.1. For the following assume $\mathbb{L}(+,\cdot)$ is a quadratic $\mathbb{F}$-algebra or $\mathbb{L}(+,\cdot)=\mathbb{F}(+,\cdot)$. Again we presume $\mathbb{F}\subseteq\mathbb{L}$ such that the unit elements of $\mathbb{F}$ and $\mathbb{L}$ coincide (cf. 1.8.1 (2)). Moreover, if $\mathbb{L}=\mathbb{F}$ we put $\bar{x}:=x,\ \forall x\in\mathbb{F}$.

Let V be the vector space $\mathbb{L}\times\mathbb{F}\times\mathbb{F}$ over $\mathbb{F}$, consisting of the elements $(z;\xi,\eta)$ such that $z\in\mathbb{L}\wedge\xi,\eta\in\mathbb{F}$, and consider the quadratic form

(1) $Q\colon\ V\to\mathbb{F},\ (z;\xi,\eta)\mapsto z\bar{z}-\xi\eta.$

Then, Q is Minkowskian of dimension 3 or 4 iff $\mathbb{L}(+,\cdot)$ is a field.

Moreover, Q is Artinian of dimension 4 iff $\mathbb{L}(+,\cdot)$ is isomorphic to the ring of double numbers over $\mathbb{F}$, and $\ker Q$ is a cone of dimension 4 iff $\mathbb{L}(+,\cdot)$ is isomorphic to the ring of dual numbers over $\mathbb{F}$ (cf. 1.5.7). The cases under consideration will later on lead to the plane quadric circle geometries (see 4.2 and Benz [1973]).

Assume now that $\mathbb{F}',\mathbb{L}',V',Q'$ are defined analogously to $\mathbb{F},\mathbb{L},V,Q$ and that the metric vector spaces $(V,\mathbb{F},Q)$ and $(V',\mathbb{F}',Q')$ are isomorphic. Then, by using the identifications

(2) $(z;\xi,\eta)=\begin{pmatrix} z & -\xi \\ \eta & -\overline{z}\end{pmatrix}$ and $(z';\xi',\eta')=\begin{pmatrix} z' & -\xi' \\ \eta' & -\overline{z'}\end{pmatrix}$

for $z\in\mathbb{L}\wedge\xi,\eta\in\mathbb{F}\wedge z'\in\mathbb{L}'\wedge\xi',\eta'\in\mathbb{F}'$, we get the following representation of isomorphisms by 2×2 matrices:

1.9.2. *The isomorphisms from $(V,\mathbb{F},Q)$ onto $(V',\mathbb{F}',Q')$ are the mappings of the type*

$$(1)\quad \varphi\colon\begin{cases} V \to V' \\ \begin{pmatrix} z & -\xi \\ \eta & -\overline{z}\end{pmatrix} \to \lambda\cdot\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot\begin{pmatrix} z^\sigma & -\xi^\sigma \\ \eta^\sigma & -\overline{z}^\sigma\end{pmatrix}\cdot\begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a}\end{pmatrix},\end{cases}$$

where λ is an element of $\mathbb{F}'\setminus\{0'\}$, a,b,c,d are elements of $\mathbb{L}'$ such that $ad-bc$ is invertible in $\mathbb{L}'(+,\cdot)$, and $\sigma\colon\ \mathbb{L}(+,\cdot)\to\mathbb{L}'(+,\cdot)$ is a ring isomorphism such that $\sigma(\mathbb{F})=\mathbb{F}'$.

By using the abbreviations

$$(2)\quad \begin{pmatrix} a & b \\ c & d\end{pmatrix}^\diamond := \begin{pmatrix} d & -b \\ -c & a\end{pmatrix},\quad \overline{\begin{pmatrix} a & b \\ c & d\end{pmatrix}} := \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d}\end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}^\sigma := \begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma\end{pmatrix}$$

for $a,b,c,d\in\mathbb{L}'$ respectively $\mathbb{L}$, because $A\cdot A^\diamond=A^\diamond\cdot A=\det A\ \forall A\in M_{2,2}(\mathbb{L}')$, the representation (1) may be rewritten as

(3) $\varphi\colon\ V\to V',\ X\mapsto\lambda\cdot A\cdot X^\sigma\cdot\overline{A}^\diamond=\lambda\cdot\det\overline{A}\cdot A\cdot X^\sigma\cdot\overline{A}^{-1}$.

Observe that $^\diamond$ is an involutory anti-automorphism of $M_{2,2}(\mathbb{L}')$ and that $^{-}$ is an involutory or identical automorphism of $M_{2,2}(\mathbb{L}')$ such that

(4) $\overline{X}=-X^\diamond;\ Q'(X)=X\cdot\overline{X}=-\det X;$
$f'_Q(X,Y)=X\cdot\overline{Y}+Y\cdot\overline{X},\ \forall X,Y\in V',$

holds, where $^{-}$ and $^\diamond$ commute (cf. Schröder [1987b]).

Each vector subspace of dimension 1 of $\ker Q\setminus V^\perp$ can be represented as $\mathbb{F}(z\overline{w};z\overline{z},w\overline{w})$, where z,w are elements of $\mathbb{L}$ such that $z\mathbb{L}+w\mathbb{L}=\mathbb{L}$. Moreover, it turns out that we may use the identification

(5) $\mathbb{F}(z\overline{w};z\overline{z},w\overline{w})=\mathbb{L}(z,w)$.

Let $\dot{F}_Q$ be the set $\{\mathbb{L}(z,w)\colon\ z,w\in\mathbb{L}\wedge z\mathbb{L}+w\mathbb{L}=\mathbb{L}\}$ and let $\dot{F}_{Q'}$ be defined analogously. Then it is easily verified that the mapping φ defined in (1) induces a mapping

(6) $\overline{\varphi}\colon\ \dot{F}_Q\to\dot{F}_{Q'},\ \mathbb{L}(z,w)\mapsto\mathbb{L}'(az^\sigma+bw^\sigma,cz^\sigma+dw^\sigma)$,

and this is an isomorphism between quadric circle geometries (cf. 4.2, Benz [1973], and Schröder [1986a]).

Thus, the representation (1) is a consequence of a corresponding result of circle geometry; a direct proof of (1) can be obtained by using transitivity properties of $\dot{F}_{Q'}$ (cf. Benz [1973], p. 88) and the fundamental theorem 1.8.2.

1.9.3. According to the above, the Q-symmetries of $(V, \mathbb{F}, Q)$ can be described by

(1) $\widetilde{B}$: $V \to V$, $X \mapsto -B \cdot \overline{X} \cdot \overline{B}^{-1}$, $\forall B \in V \setminus \ker Q$,

and for linear dependent elements $A, B, C \in V \setminus \ker Q$, the 4-th Q-vector D of (A, B, C) is given by

(2) $D = A \cdot \overline{B} \cdot C$.

Again, this representation yields a very simple proof of the 3-reflection lemma for the cases under consideration.

If $Q_{\mathbb{L}}$ is not Galilean or if $\operatorname{char} \mathbb{F}$ is unequal to 2, 1.9.2 (3) delivers a representation of all isometries of $(V, \mathbb{F}, Q)$ onto $(V', \mathbb{F}', Q')$, if A, λ, σ are restricted such that $\sigma|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}}$ and $\lambda \cdot \det \overline{A} \in \{1, -1\}$. Moreover, if $Q_{\mathbb{L}}$ is not Galilean and if $\mathbb{L}$ is unequal to $\mathbb{F}$, we obtain the representation

(3) $X \to AX\overline{A}^{-1}$ or $X \to -A\overline{X}\overline{A}^{-1}$

for every isometry of $(V, \mathbb{F}, Q)$, where $\det A$ is an element of $\mathbb{F} \setminus \{0\}$.

1.9.4. Beside the description of isomorphisms given in 1.9.2, which includes the Minkowskian spaces of dimension 3 and 4, the results of 1.9.2 may be used to describe equimetries and isometries of some other spaces of dimension 3.

For this purpose, let $\mathbb{F}, \mathbb{L}, V, Q$ be given according to 1.9.1 and such that $\mathbb{F} \neq \mathbb{L}$. For $r \in \mathbb{F}$ consider the hyperplane $V_r := \{(z; \xi, \eta) \in V\colon\ \eta = r\xi\}$.

The quadratic form Q, restricted to V_r, is given by

(1) Q_r: $V_r \to \mathbb{F}$, $(z; \xi, r\xi) \mapsto z\overline{z} - r\xi^2$.

Now let σ be a ring automorphism of $\mathbb{L}(+, \cdot)$ with the property $\sigma|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}}$ and assume

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a regular element of $M_{2,2}(\mathbb{L})$ such that $c = r\overline{b}$ and $(r \neq 0 \Rightarrow d = \overline{a})$. Then, for $\lambda \in \mathbb{F} \setminus \{0\}$, the mapping

(2) φ: $V_r \to V_r$, $X \mapsto \lambda \cdot A \cdot X^{\sigma} \cdot \overline{A}^{\diamond}$,

defined according to 1.9.1 (2) and 1.9.2 (2), is an equimetry of $(V_r, \mathbb{F}, Q_r)$.

If $Q_{\mathbb{L}}$ is Galilean, in general there are equimetries of $(V_r, \mathbb{F}, Q_r)$ which are not represented by (2).

On the other hand, if $Q_{\mathbb{L}}$ is not Galilean and if $\mathbb{L}$ is separable, every equimetry of $(V_r, \mathbb{F}, Q_r)$ is represented by (2). In case of $r = 0$ this is easily verified by using 1.8.4, and in case of $r \neq 0$ this is a consequence of 1.9.2, of 1.6.3 (4) (iii), of Schröder [1979], p. 155 f., of Snapper and Troyer [1971], p. 361, and of $(\operatorname{char} \mathbb{F} = 2 \Rightarrow V_r^{\perp} = \mathbb{F}(0; 1))$.

2. Quadrics

Quadrics are surfaces in projective spaces which can be described by quadratic forms and have remarkable geometric properties.

A consideration of symmetries of quadrics leads to projective metric geometry. On the other hand, by stereographic projection, to each nondegenerate quadric corresponds an affine metric geometry. Thus, quadrics form a link between different types of metric geometries.

For example, let us look at the following classical situation.

By stereographic projection, the real Euclidean plane turns out to be a subgeometry of the real Möbius plane, which is the geometry of plane sections of the sphere $s := \{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+z^2=1\}$.

But s is a quadric of the projective closure $\Pi(\mathbb{R}^4,\mathbb{R})$ of $\mathbb{R}^3$, and the Möbius geometry of s may be extended to the usual three-dimensional hyperbolic geometry in the sense of F. Klein as well as to the corresponding projective-metric hyperbolic geometry which is the projective version of the four-dimensional Lorentz–Minkowski geometry.

Later on, we will develop these connections in full generality (cf. 4.2–4). In this paragraph, we restrict our attention to properties and geometric characterizations of quadrics.

2.1. *Quadratic sets*

2.1.1. Assume V is a left vector space over a division ring (= skew field) $\mathbb{F}$ and let $\mathcal{U}_n$ denote the set of n-dimensional vector subspaces of V ($n \in \mathbb{N}$). For $M \subseteq V$, define $M^\pi := \{X \in \mathcal{U}_1\colon X \subseteq M\}$.

Then, for $\boldsymbol{P} := \mathcal{U}_1$ and $\boldsymbol{L} := \{U^\pi\colon U \in \mathcal{U}_2\}$, $\Pi := \Pi(V,\mathbb{F}) := (\boldsymbol{P},\boldsymbol{L})$ is the *projective space which corresponds to the vector space* V (see Chapters 2, 4 and 7).

The set $\mathcal{U}^\pi := \{U^\pi\colon U \leqslant V\}$ (cf. 1.2.1) is the set of *projective subspaces* of Π. For $U \leqslant V$, we call $\operatorname{Dim} U^\pi := \dim U - 1$ the *projective dimension of* U^π and $\dim U$ the *rank* of U^π. Thus, $\mathcal{U}_n^\pi := \{U^\pi\colon U \in \mathcal{U}_n\}$ ($n \in \mathbb{N}$) is the set of projective subspaces of Π of rank n and of projective dimension $n-1$. In particular, $\operatorname{Dim}\Pi := \dim V - 1$ is the *dimension* of Π. The elements of $\boldsymbol{P} = \mathcal{U}_1$ (resp., $\boldsymbol{L} = \mathcal{U}_2^\pi$, $\boldsymbol{E}^\pi := \mathcal{U}_3^\pi$, $\boldsymbol{H}^\pi := \{U^\pi\colon U \leqslant V \wedge \dim V/U = 1\}$) are called *points* (resp., *lines*, *planes*, *hyperplanes*) of Π (cf. Chapters 1 and 2).

For $\boldsymbol{M} \subseteq \boldsymbol{P}$, the projective subspace $\langle \boldsymbol{M}\rangle := \bigcap\{U^\pi \in \mathcal{U}^\pi\colon U^\pi \supseteq \boldsymbol{M}\}$ is called the *projective hull* of $\boldsymbol{M}$. The point set $\boldsymbol{M}$ is called *independent*, provided $\langle \boldsymbol{M} \setminus \{X\}\rangle \neq \langle \boldsymbol{M}\rangle$, $\forall X \in \boldsymbol{M}$, holds (cf. Karzel, Sörensen and Windelberg [1973] and Chapters 1–7). For $A_1,\dots,A_n \in \boldsymbol{P}$ ($n \in \dot{\mathbb{N}}$), we also write $\langle A_1,\dots,A_n\rangle$ instead of $\langle\{A_1,\dots,A_n\}\rangle$.

Π is called *Desarguesian* provided $\dim V \geqslant 3$, and *Pappian* provided $\mathbb{F}$ is commutative and $\dim V \geqslant 3$. If $\dim V = 3$, Π is called a *Desarguesian* (resp., *Pappian*) *projective plane*.

Beside the projective spaces defined above there are non-Desarguesian projective planes $(\boldsymbol{P},\boldsymbol{L})$ which by definition are projective spaces of dimension 2 as well, and whose hyperplanes are the lines (cf. Chapters 1 and 4).

2.1.2. Let $(V,\mathbb{F},Q)$ be a metric vector space. In the projective space $\Pi := \Pi(V,\mathbb{F}) = (\boldsymbol{P},\boldsymbol{L})$ the point set

(1) $\boldsymbol{F}_Q := \{\mathbb{F}x\colon x \in V \setminus \{O\} \wedge Q(x) = 0\} = (\ker Q)^\pi$

is called the *quadric associated with* Q.

Obviously, Q is Euclidean iff $\boldsymbol{F}_Q$ is empty, and Q is singular iff $\boldsymbol{F}_Q = \boldsymbol{P}$. Moreover, $\boldsymbol{F}_Q$ is a *subspace* of Π iff $\ker Q$ is a vector subspace of V, and $\boldsymbol{F}_Q$ is the union of two hyperplanes of Π, called *cross of hyperplanes*, iff Q is 2-*singular*. By 1.3.2, we obtain

(2) $\operatorname{Dim} \Pi = 1 \Rightarrow |\boldsymbol{F}_Q| \in \{0,1,2\} \vee \boldsymbol{F}_Q = \boldsymbol{P}$,
and because of 1.2.5, we get

(3) $U \leqslant V \wedge Q' := Q|_U \Rightarrow \boldsymbol{F}_{Q'} = \boldsymbol{F}_Q \cap U^{\pi}$.

From (2) and (3) we derive

(4) $|l \cap \boldsymbol{F}_Q| \leqslant 2 \ \vee \ l \subseteq \boldsymbol{F}_Q, \ \forall l \in \boldsymbol{L}$.

If σ is a semilinear induced collineation (cf. Brauner [1976]) which maps Π onto a projective space Π', then, because of 1.5.2, $\sigma(\boldsymbol{F}_Q)$ is a quadric of Π'. Thus the fundamental theorem of projective geometry (cf. Brauner [1976]) shows that quadrics are objects of projective geometry. For a synthetical definition of quadrics see 2.2.1–3, 2.4.3, 2.4.5 and Havlicek [1985].

2.1.3. Assume $\Pi = (\boldsymbol{P}, \boldsymbol{L})$ is an arbitrary (possibly non-Desarguesian) projective space. A subset $\boldsymbol{F}$ of $\boldsymbol{P}$ is called a *2-set* of Π, provided the condition

(1) $|l \cap \boldsymbol{F}| \leqslant 2 \ \vee \ l \subseteq \boldsymbol{F}, \ \forall l \in \boldsymbol{L}$,

holds.

Let $\boldsymbol{F}$ be a 2-set of Π. A line $l \in \boldsymbol{L}$ is called a *tangent* of $\boldsymbol{F}$ if $|l \cap \boldsymbol{F}| = 1 \ \vee \ l \subseteq \boldsymbol{F}$, a *secant* of $\boldsymbol{F}$ if $|l \cap \boldsymbol{F}| = 2$ and a *passing line* of $\boldsymbol{F}$ if $|l \cap \boldsymbol{F}| = 0$.

$\boldsymbol{F}$ is called *proper*, provided a secant of $\boldsymbol{F}$ exists, i.e. $\boldsymbol{F}$ is not a subspace of Π (cf. Karzel et al. [1973]).

A point $A \in \boldsymbol{F}$ is called a *simple point* (resp., a *double point*) of $\boldsymbol{F}$, provided the union τ_A of all tangents of $\boldsymbol{F}$ passing through A together with $\{A\}$ is a hyperplane of Π (resp., equals $\boldsymbol{P}$).

If $\boldsymbol{F}$ is a 2-set which consists of simple points and double points *only*, then $\boldsymbol{F}$ is called a *quadratic set* (cf. Buekenhout [1969]).

A proper 2-set is called a *calotte* (cf. Barlotti [1955] and Halder and Heise [1976]), if it does not contain a line. A calotte, which is a quadratic set, is called an *ovoid*, and if $\operatorname{Dim} \Pi = 2$ an *oval* as well.

A quadratic set $\boldsymbol{F}$ is called *ruled*, if there exists a line l contained in $\boldsymbol{F}$ and consisting of simple points *only*.

2.1.4. If Π is a Pappian space, every quadric of Π is an example of a quadratic set (cf. Buekenhout [1966a, 1969]).

In particular, the quadric $\boldsymbol{F}_Q$ is an ovoid iff Q is Minkowskian.

If $\operatorname{Dim} \Pi = 2$, every proper quadric is a cross of lines or an oval quadric (cf. 1.5.6 and Figure 2.1 a), b)).

In case of $\operatorname{Dim} \Pi = 3$, every proper quadric is a cross of planes or a (so-called) cone or a ruled quadric or an ovoidal quadric (cf. 1.5.7 and Figure 2.1 c)–f)).

2.1.5. Suppose $\Pi = (\boldsymbol{P}, \boldsymbol{L})$ is an arbitrary projective space of dimension $\geqslant 2$ and $\boldsymbol{F}$ is a quadratic set of Π. Then, according to Buekenhout [1969], we obtain

(1) If ε is a plane of Π, then $\boldsymbol{F} \cap \varepsilon$ is a subspace or a cross of lines or an oval of ε.

(2) If $\boldsymbol{F}'$ is an arbitrary subset of $\boldsymbol{P}$ and if $\boldsymbol{F}' \cap \varepsilon$ is a quadratic set in ε for every plane ε, then $\boldsymbol{F}'$ is a quadratic set in Π.

(3) If $\boldsymbol{F}$ is proper, then $\langle \boldsymbol{F} \rangle = \boldsymbol{P}$.

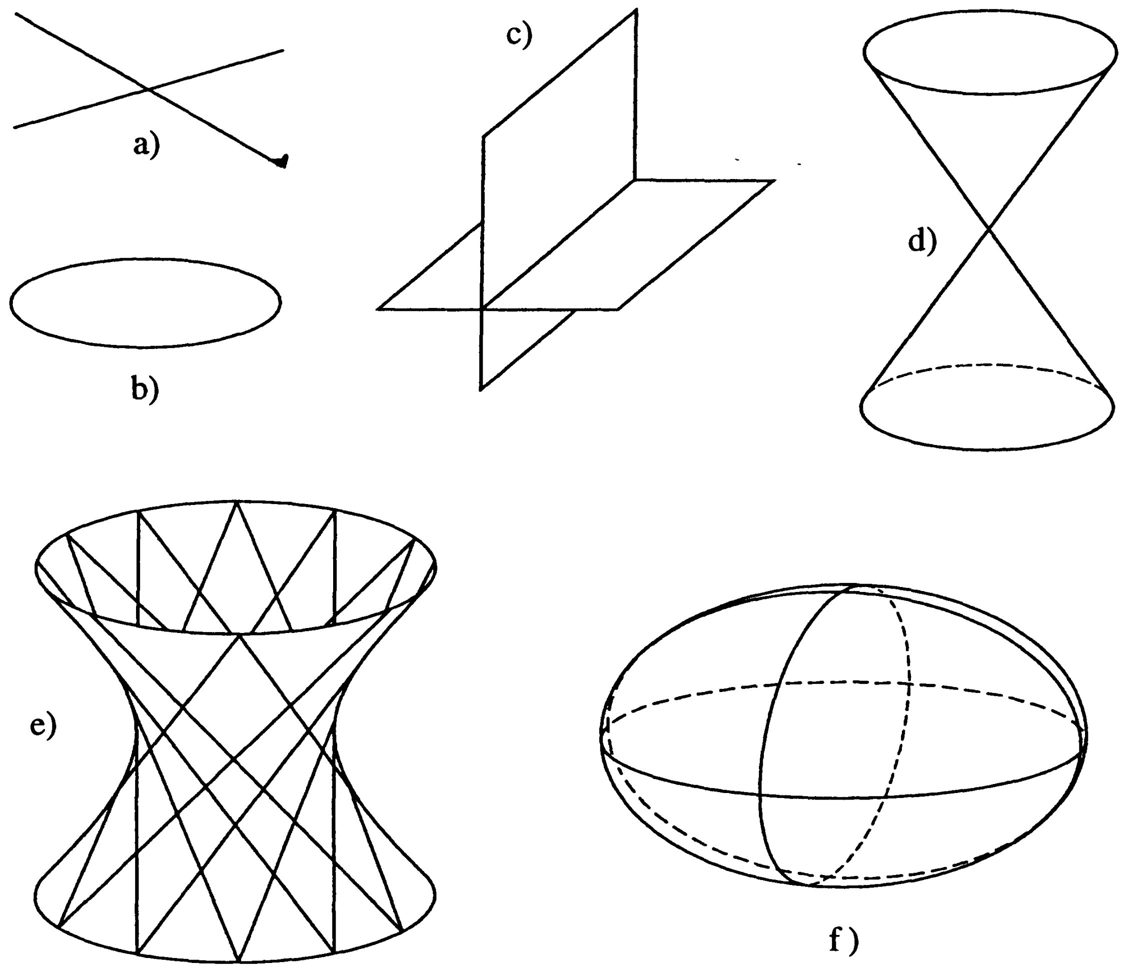

Figure 2.1. Proper quadrics.

(4) If $\boldsymbol{F}$ contains a hyperplane of Π, then $\boldsymbol{F}$ is a hyperplane or a cross of hyperplanes of Π, or $\boldsymbol{F} = \boldsymbol{P}$.

(5) If $\boldsymbol{F}$ is an ovoid, then $\boldsymbol{F}$ consists of simple points only.

(6) The set $\boldsymbol{D}$ of all double points of $\boldsymbol{F}$ is a subspace of Π. In case of $\varnothing \neq \boldsymbol{D} \neq \boldsymbol{F}$, $\boldsymbol{F}$ is proper, and there exists a subspace $\boldsymbol{T}$ of Π such that

$$\boldsymbol{D} \cap \boldsymbol{T} = \varnothing \wedge \langle \boldsymbol{D} \cup \boldsymbol{T} \rangle = \boldsymbol{P} \wedge \boldsymbol{F} = \bigcup \{\langle X, Y \rangle \colon \ X \in \boldsymbol{D} \wedge Y \in \boldsymbol{T} \cap \boldsymbol{F}\},$$

where $\boldsymbol{T} \cap \boldsymbol{F}$ is a proper quadratic set of $\boldsymbol{T}$ which consists of simple points with respect to $\boldsymbol{T}$.

Moreover, if Π is Pappian, $\boldsymbol{F}$ is a quadric in Π iff $\boldsymbol{T} \cap \boldsymbol{F}$ is a quadric in $\boldsymbol{T}$.

2.2. *Geometric characterizations of quadrics*

We now state a main result of Buekenhout [1969] which he proved for finite dimensions and which Schröder [1987a] proved for arbitrary dimensions:

2.2.1. THEOREM OF BUEKENHOUT. *Let Π be an arbitrary projective space and assume $3 \leqslant \operatorname{Dim} \Pi \leqslant \infty$. If $\boldsymbol{F}$ is a ruled quadratic set of Π, then Π is Pappian and $\boldsymbol{F}$ is a quadric of Π.*

This fails to be true, if $\boldsymbol{F}$ is not ruled.

As another important result, proved by Tits [1962b] for ovoids of finite dimensions and by Schröder [1987a] for arbitrary dimensions, we state:

2.2.2. THEOREM OF TITS. *Let Π be a Pappian projective space and assume $3 \leqslant \operatorname{Dim} \Pi \leqslant \infty$. Furthermore, let $\boldsymbol{F}$ be a quadratic set of Π which contains a simple point A. If $\varepsilon \cap \boldsymbol{F}$ is a quadric of ε for each plane ε which contains A and which is not contained in τ_A, then $\boldsymbol{F}$ is a quadric of Π.*

Schröder [1988] shows that the set of planes considered in this theorem may be replaced by a proper subset which consists of certain bundles of planes. Moreover, counterexamples are considered there.

Because of 2.1.5 (6) and the above, the question, which quadratic sets are quadrics, amounts to the question, which (plane) ovals are quadrics.

For spaces of finite order, a famous result of Segre [1954] together with the theorem of Tits and a theorem of Wedderburn (cf. Jacobson [1974]) leads to the following statement, which for finite dimensions is a corollary of theorems of Barlotti [1955] and Tallini [1956a,b]:

2.2.3. *Let Π be a Desarguesian projective space and assume*

$$|\mathbb{F}| \in 2\dot{\mathbb{N}} + 1 \ \wedge\ 2 \leqslant \operatorname{Dim} \Pi \leqslant \infty.$$

Then, Π is Pappian, and every proper quadratic set of Π is a quadric.

In case of $|\mathbb{F}| \in 2\mathbb{N}$, this fails to be true (cf. Tits [1962b] and Halder and Heise [1976], p. 218 ff.). For further characterizations of oval quadrics cf. Section 2.4.

2.3. *Symmetries*

Let $(V, \mathbb{F}, Q)$ be a metric vector space and let $\Pi := \Pi(V, \mathbb{F}) = (\boldsymbol{P}, \boldsymbol{L})$ be the corresponding projective space.

2.3.1. Obviously, some results obtained in Section 1 may be used to prove properties of quadrics. For example, $(V_Q^{\perp})^{\pi}$ is the set of double points of $\boldsymbol{F}_Q$, because 1.3.7 (7) yields

$$(A^{\perp})^{\pi} = \tau_A, \quad \forall A \in \boldsymbol{F}_Q.$$

Moreover, if Q has finite Witt-index r, for every $\boldsymbol{M} \in \mathcal{U}^{\pi}$ which is contained in $\boldsymbol{F}_Q$ there exists a subspace $\boldsymbol{T} \in \mathcal{U}^{\pi}$ such that $\boldsymbol{M} \subseteq \boldsymbol{T} \subseteq \boldsymbol{F}_Q$ and $\operatorname{Dim} \boldsymbol{T} = r - 1$. This means that $\boldsymbol{F}_Q$ is the union of projective subspaces of dimension $r - 1$ and that $\boldsymbol{F}_Q$ does not contain a projective subspace of dimension r.

Further properties of subspaces of $\boldsymbol{F}_Q$ are proved in Buekenhout [1969]. If $\boldsymbol{F}_Q$ is ruled and if $\operatorname{Dim} \Pi = 3$, it is well known that there exist two disjoint sets $\boldsymbol{L}_1, \boldsymbol{L}_2$ consisting of pairwise disjoint lines such that $\boldsymbol{F}_Q = \bigcup \boldsymbol{L}_1 = \bigcup \boldsymbol{L}_2$ (cf. Figure 2.1 e)).

2.3.2. Suppose $a \in V \setminus \ker Q$ and $A := \mathbb{F}a$. The Q-symmetry $\widetilde{a}$ (cf. 1.6.2) is a linear permutation of V and thus induces a projectivity

(1) $\widetilde{A}\colon \boldsymbol{P} \to \boldsymbol{P},\ \mathbb{F}x \mapsto \mathbb{F}\Big(x - \dfrac{f(x,a)}{Q(a)}a\Big)$

of Π (cf. Buekenhout [1969]) with the properties:

(2) $\widetilde{A} \circ \widetilde{A} = \mathrm{id}_{\boldsymbol{P}}$,
(3) $\widetilde{A} = \mathrm{id}_{\boldsymbol{P}} \Leftrightarrow A^{\perp} = V \Leftrightarrow A \subseteq V^{\perp} \wedge \operatorname{char}\mathbb{F} = 2$,
(4) $\widetilde{A}(\boldsymbol{T}) = \boldsymbol{T} \Leftrightarrow \boldsymbol{T} \ni A \vee \boldsymbol{T} \subseteq (A^{\perp})^{\pi},\ \forall \boldsymbol{T} \in \mathcal{U}^{\pi}$,
(5) $(\operatorname{char}\mathbb{F} \neq 2 \Rightarrow A \notin (A^{\perp})^{\pi}) \wedge (\operatorname{char}\mathbb{F} = 2 \Rightarrow A \in (A^{\perp})^{\pi})$,
(6) $\widetilde{A}(\boldsymbol{F}_Q) = \boldsymbol{F}_Q$,
(7) $X \in \boldsymbol{F}_Q \cap (A^{\perp})^{\pi} \Rightarrow \widetilde{A}(X) = X \wedge \langle A, X\rangle \cap \boldsymbol{F}_Q = \{X\}$,
(8) $X \in \boldsymbol{F}_Q \setminus (A^{\perp})^{\pi} \Rightarrow \widetilde{A}(X) \neq X \wedge \langle A, X\rangle \cap \boldsymbol{F}_Q = \{X, \widetilde{A}(X)\}$.

We call $\widetilde{A}$ the *Q-symmetry with respect to A*.

Because of (4), $\{A\} \cup (A^{\perp})^{\pi}$ is the set of fixed points of $\widetilde{A}$. Therefore A and $(A^{\perp})^{\pi}$ and called the *centre* and the *axis of* $\widetilde{A}$.

In case of $\widetilde{A} \neq \mathrm{id}_{\boldsymbol{P}}$, the axis of $\widetilde{A}$ is a hyperplane of Π (cf. 1.2.1 (2)). If $\widetilde{A} = \mathrm{id}_{\boldsymbol{P}}$ (cf. (3)), A is called a *Q-knot*; here, because of (7), every line passing through A is a tangent or a passing line of $\boldsymbol{F}_Q$.

The relation $\perp$, restricted to the elements of $\boldsymbol{P}$ and then written $\widetilde{\perp}$, is called the *polarity induced by Q*. Obviously, we have

(9) $X \in (Y^{\perp})^{\pi} \Leftrightarrow X \widetilde{\perp} Y \Leftrightarrow Y \widetilde{\perp} X \Leftrightarrow Y \in (X^{\perp})^{\pi},\ \forall X, Y \in \boldsymbol{P}$.

We call X a *pole* of $(X^{\perp})^{\pi}$ and $(X^{\perp})^{\pi}$ the *polar space* of X.

The points of $\boldsymbol{R}_Q := \boldsymbol{P} \setminus \boldsymbol{F}_Q$ (cf. 1.7.5) are called *Q-regular* or *Euclidean*, while the points of $\boldsymbol{F}_Q$ are called *Q-singular*. The set of Q-symmetries with respect to Q-regular points is denoted by $\widetilde{\boldsymbol{R}}_Q$.

The set ${}^3\boldsymbol{R}_Q$, defined in 1.7.5, consists of the triples of collinear Q-regular points, and the *metric product* μ_Q: ${}^3\boldsymbol{R}_Q \to \boldsymbol{R}_Q$ is a mapping such that $A, B, C, \mu_Q(A,B,C)$ are collinear points for all $(A,B,C) \in {}^3\boldsymbol{R}_Q$ (cf. 1.6.4). We call $ABC := \mu_Q(A,B,C)$ the *4-th Q-point* of (A,B,C), and by 1.6.4, we obtain

(10) PROJECTIVE 3-REFLECTION LEMMA. *Assume* $(A,B,C) \in {}^3\boldsymbol{R}_Q$. *Then,*

$$\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} = (ABC)^{\sim} = \widetilde{C} \circ \widetilde{B} \circ \widetilde{A} \quad \textit{and} \quad \widetilde{A} \circ \widetilde{B} \circ \widetilde{A} = (\widetilde{A}(B))^{\sim}.$$

When looking for a converse of (10), one has to observe that $\widetilde{\mathbb{F}a} \circ \widetilde{\mathbb{F}b} \circ \widetilde{\mathbb{F}c} = \widetilde{\mathbb{F}d}$ implies $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = \widetilde{d}$ or $\widetilde{a} \circ \widetilde{b} \circ \widetilde{c} = -\widetilde{d}$ for $a, b, c, d \in V \setminus \ker Q$. Thus, by using 1.6.1–4, one obtains

2.3.3. PROJECTIVE 3-REFLECTION THEOREM. *Assume that* $A, B, C \in \boldsymbol{R}_Q$ *and* $\dim V^{\perp} \leqslant 1$.

Except in the case where

$(*)$ $\dim \Pi = 3 \wedge \operatorname{char}\mathbb{F} \neq 2 \wedge V^{\perp} = \{O\} \wedge A \perp B \perp C \perp A$,

the points A, B, C *are collinear iff* $\widetilde{A} \circ \widetilde{B} \circ \widetilde{C}$ *is an element of* $\widetilde{\boldsymbol{R}}_Q$.

2.4. *Geometric characterizations of oval quadrics*

Let $(V, \mathbb{F}, Q)$ be a metric vector space of dimension 3 and let $\Pi := \Pi(V, \mathbb{F}) = (\boldsymbol{P}, \boldsymbol{L})$ be the corresponding *projective plane*.

2.4.1. Assume $\boldsymbol{F}$ is a proper quadratic set of an arbitrary projective plane $\Pi' = (\boldsymbol{P}', \boldsymbol{L}')$ (cf. 2.1.5 (1)). For simple points A, B of $\boldsymbol{F}$, let AB denote the line l which is determined by $l = \langle A, B \rangle \vee l = \tau_A = \tau_B$. A sequence $(A_i)_{i \in \mathbb{Z}_6}$ of simple points of $\boldsymbol{F}$ is called an *inscribed hexagon* of $\boldsymbol{F}$, provided

(1) $A_i A_{i+1} \not\subseteq \boldsymbol{F} \wedge A_i \notin \{A_{i+2}, A_{i+3}A_{i+4}\}$, $\forall i \in \mathbb{Z}_6$,

holds. An inscribed hexagon $(A_i)_{i \in \mathbb{Z}_6}$ of $\boldsymbol{F}$ is called *Pascalian* provided there exists a line $l \in \boldsymbol{L}$ such that the condition

(2) $A_i A_{i+1} \cap A_{i+3} A_{i+4} \cap l \neq 0$, $\forall i \in \mathbb{Z}_6$,

is fulfilled.

According to Buekenhout [1969], by using 2.3.2 (10) one obtains a simple proof of

2.4.2. Theorem of Pappus and Pascal. *Assume* $\dim \Pi = 2$ *and* $\boldsymbol{F}_Q$ *is a proper quadric. Then, every inscribed hexagon of* $\boldsymbol{F}_Q$ *is Pascalian* (cf. Figure 2.2 a), b), c)).

By Pickert [1959] and Buekenhout [1966b] the following remarkable converse of 2.4.2 is proved (for other proofs cf. Artzy [1968], Burn [1968], Karzel and Sörensen [1971], Mäurer [1981], Schröder [1968]).

2.4.3. Theorem of Pickert and Buekenhout. *Assume* Π' *is an arbitrary projective plane and* $\boldsymbol{F}$ *is a proper quadratic set of* Π'. *If each inscribed hexagon of* $\boldsymbol{F}$ *is Pascalian, then* Π' *is a Pappian projective plane and* $\boldsymbol{F}$ *is a quadric of* Π'.

2.4.4. Assume $\operatorname{char} \mathbb{F} \neq 2$. If $\boldsymbol{F}_Q$ is an oval, Q is regular, and the mapping $\mathcal{U}_1 \cup \mathcal{U}_2 \to \mathcal{U}_1 \cup \mathcal{U}_2$, $X \mapsto X^{\perp}$, induces an incidence preserving permutation of $\boldsymbol{P} \cup \boldsymbol{L}$ which interchanges the points with the lines, and the singular points with the corresponding tangents of $\boldsymbol{F}_Q$. By using this duality, from 2.4.2 one obtains a 'dual' theorem, the theorem of Brianchon (cf. Figure 2.2 d), e)).

2.4.5. Suppose $\Pi' = (\boldsymbol{P}', \boldsymbol{L}')$ is an arbitrary projective plane and $\boldsymbol{F}$ is an oval of Π'. We call $\boldsymbol{F}$ *symmetric* with respect to a point $A \in \boldsymbol{P}' \setminus \boldsymbol{F}$, if there exists a nonidentical collineation σ of Π which fixes $\boldsymbol{F}$ and every line passing through A.

By Buekenhout [1969] and Mäurer [1981], the following geometric characterizations of oval quadrics are given:

Theorem of Buekenhout and Mäurer. *Suppose there exists a line* l *such that* $\boldsymbol{F}$ *is symmetric with respect to every point of* $l \setminus \boldsymbol{F}$. *Then,* Π' *is Pappian and* $\boldsymbol{F}$ *is a quadric, if*

(i) l *is a secant or a tangent of* $\boldsymbol{F}$ *and* Π *is a Moufang plane; or,*

(ii) l *is a passing line of* $\boldsymbol{F}$ *and* π *is a Desarguesian plane of characteristic* $\neq 2$.

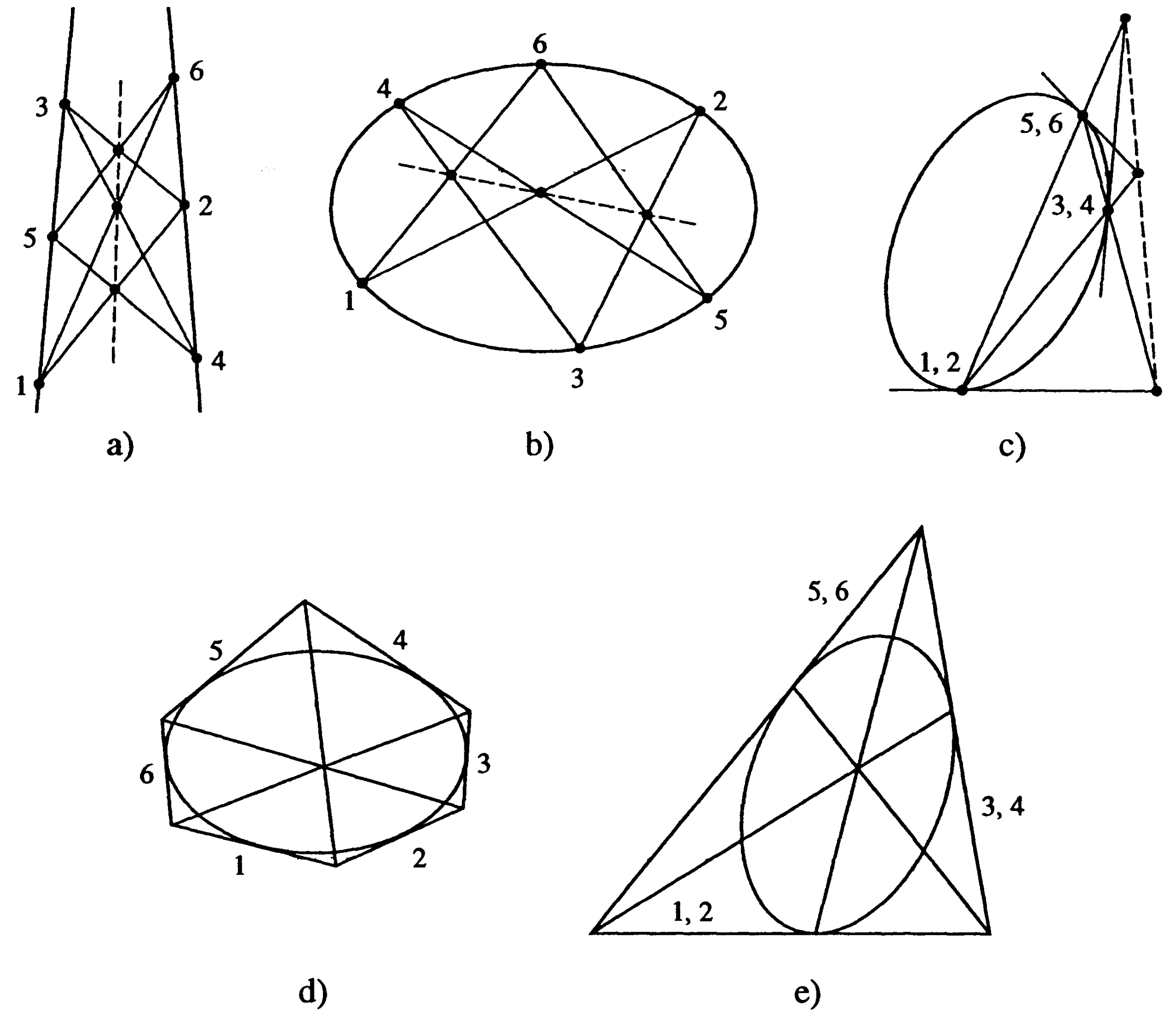

Figure 2.2. Theorems of Pappus, Pascal, and Brianchon.

Observe, that every Desarguesian plane is a Moufang plane, but that the converse is not true (cf. Chapter 4).

3. Affine metric geometry

Affine metric geometry on one hand is closely related with the theory of metric vector spaces. On the other hand, there are important differences.

Apart from the fact that the set of points of an affine space is 'homogeneous', while in a vector space the zero vector is distinguished in many respects from all other vectors, there are affine and affine metric concepts, as parallelity, translations, motions, and circles which are not primarily concepts of vector geometry. But such concepts are well known from Euclidean and Minkowskian geometry, and therefore they shall be developed here in full generality.

3.1. *Concepts of affine metric geometry*

3.1.1. Let V be a left vector space over a division ring $\mathbb{F}$ and define $\mathcal{U} := \{U\colon\ U \leqslant V\}$ and $\mathcal{U}_n := \{U \in \mathcal{U}\colon\ \dim U = n\}$ for $n \in \mathbb{N}$.

For the elements of $\mathcal{L} := \{a+B\colon\ a \in V \wedge B \in \mathcal{U}_1\}$, a parallelity relation $\parallel$ is defined by

$$a+B \parallel c+D \ :\Leftrightarrow\ B = D,\ \forall a,c \in V,\ \forall B,D \in \mathcal{U}_1,$$

and $\boldsymbol{A} := \boldsymbol{A}(V,\mathbb{F}) := (V,\mathcal{L},\parallel)$ is called the *affine space corresponding to the vector space* V (cf. Chapter 2).

The set $\mathcal{T} := \{a+U\colon\ a \in V \wedge U \leqslant V\} \cup \{\varnothing\}$ is the set of *affine subspaces* of $\boldsymbol{A}$. For $a \in V$ and $U \in \mathcal{U}$, we call $\dim(a+U) := \dim U$ the (*affine*) *dimension* of $a+U$. Moreover, we define $\dim\varnothing := -1$ and call $\dim V$ the *dimension of* A. The elements of $\mathcal{P} := V$, $\mathcal{L}$, $\mathcal{E} := \{a+U\colon\ a \in V \wedge U \in \mathcal{U}_2\}$, $\mathcal{G} := \{a+U\colon\ a \in V \wedge U \in \mathcal{U} \wedge \dim V/U = 2\}$ and $\mathcal{H} := \{a+U\colon\ a \in V \wedge U \in \mathcal{U} \wedge \dim V/U = 1\}$ are called *points*, *lines*, *planes*, (affine) *hyperlines* and (affine) *hyperplanes*, respectively, of $\boldsymbol{A}$.

For $(a,l) \in V \times \mathcal{L}$, there is exactly one line m such that $m \ni a \wedge m \parallel l$. We call m the *parallel line of* l *through* a and denote m by $(a \parallel l)$. Moreover, we use the abbreviation $l\| := \{m \in \mathcal{L}\colon\ m \parallel l\}$ and call $l\|$ the *parallel class* of direction l.

For $a,a' \in V$ and $U,U' \leqslant V$, the affine subspaces $a+U, a'+U'$ are called *parallel*, written $a+U \parallel a'+U'$, provided $U \subseteq U' \vee U \supseteq U'$ holds.

For $M \subseteq V$, the affine subspace $\overline{M} := \bigcap\{T \in \mathcal{T}\colon\ T \supseteq M\}$ is called the *affine hull* of M.

A subset M of V is called *affinely independent*, if $\overline{M \setminus \{x\}}$ is a proper subset of $\overline{M}$ for every $x \in M$; otherwise, M is called *affinely dependent*.

For $a_0,\ldots,a_n \in V$ ($n \in \dot{\mathbb{N}}$), we also write $\overline{a_0,\ldots,a_n}$ instead of $\overline{\{a_0,\ldots,a_n\}}$. We obtain $n \geqslant \dim\overline{a_0,\ldots,a_n}$ and

$$\overline{a_0,\ldots,a_n} = a_0 + \sum_{i=1}^{n} \mathbb{F}(a_i - a_0) = \left\{ \sum_{i=0}^{n} \lambda_i a_i\colon\ \lambda_i \in \mathbb{F} \wedge \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

Moreover, $\{a_0,\ldots,a_n\}$ is affinely independent, iff $n = \dim\overline{a_0,\ldots,a_n}$, and this equation holds, iff the vectors $a_1 - a_0,\ldots,a_n - a_0$ are linearly independent.

$\boldsymbol{A}$ is called *Desarguesian* if $\dim V \geqslant 2$, and *Pappian* if $\mathbb{F}$ is commutative and $\dim V \geqslant 2$. For $\dim V = 2$, $\boldsymbol{A}$ is called a Desarguesian (resp., Pappian) *affine plane*.

Beside the affine spaces defined above, there are non-Desarguesian affine planes which by definition are affine spaces of dimension 2 as well and whose hyperplanes are the lines (cf. Chapter 2).

3.1.2. If $\pi := (\boldsymbol{P},\boldsymbol{L})$ is an arbitrary projective space of dimension $\geqslant 2$ and if $\boldsymbol{H}$ is a hyperplane of Π, then, by '*deleting*' $\boldsymbol{H}$, one obtains an affine space $\boldsymbol{A} := \boldsymbol{A}(\Pi,\boldsymbol{H}) := (\mathcal{P},\mathcal{L},\parallel)$, where $\mathcal{P} := \boldsymbol{P} \setminus \boldsymbol{H}$, $\mathcal{L} := \{l \setminus \boldsymbol{H}\colon\ l \in \boldsymbol{L} \wedge l \not\subseteq \boldsymbol{H}\}$ and

$$l \setminus \boldsymbol{H} \parallel m \setminus \boldsymbol{H} \ :\Leftrightarrow\ l \cap m \cap \boldsymbol{H} \neq \varnothing$$

for $l, m \in \boldsymbol{L}$ such that $l, m \nsubseteq \boldsymbol{H}$.

Every affine space of dimension $\geqslant 2$ can be obtained in this way and, up to isomorphism, Π is determined uniquely by $\boldsymbol{A} = \boldsymbol{A}(\Pi, \boldsymbol{H})$ (cf. Chapter 2). Therefore, Π is called the *projective closure* of $\boldsymbol{A}$ and $\boldsymbol{H}$ is called the *hyperplane at infinity* of $\boldsymbol{A}$. If T is a nonempty affine subspace of $\boldsymbol{A}$, there exists a uniquely determined subset $[T]$ of $\boldsymbol{H}$ such that $T \cup [T]$ is a projective subspace of Π; we call $[T]$ the *hyperplane at infinity* of T. If g, h are lines of $\boldsymbol{A}$, then $[g]$ (and $[h]$ as well) consists of one point of $\boldsymbol{H}$, the *point at infinity* of g, and we have $g \parallel h \Leftrightarrow [g] = [h]$.

Assume now $\boldsymbol{A} = \boldsymbol{A}(V, \mathbb{F})$, where V is a left vector space over a division ring $\mathbb{F}$. Then, $V \times \mathbb{F}$ is a left vector space over $\mathbb{F}$ of dimension $\dim V + 1$, and we may identify Π and $\Pi(V \times \mathbb{F}, \mathbb{F})$ such that

$$\big(a = \mathbb{F}(a, 1),\ \forall a \in V = \mathcal{P}\big) \ \wedge\ \big([\mathbb{F}a] = \{\mathbb{F}(a, 0)\},\ \forall a \in V \setminus \{O\}\big).$$

Thus, $\Pi(V \times \{0\}, \mathbb{F})$ describes the hyperplane $\boldsymbol{H}$ at infinity of $\boldsymbol{A}$, and, by using the canonical linear injection $V \to V \times \mathbb{F},\ a \mapsto (a, 0)$, we get $\Pi(V, \mathbb{F}) \cong \boldsymbol{H}$.

In this sense we will interpret $\Pi(V, \mathbb{F})$ as the hyperplane at infinity of $\boldsymbol{A}(V, \mathbb{F})$.

3.1.3. Let $(V, \mathbb{F}, Q)$ be a metric vector space and let $\boldsymbol{A} := \boldsymbol{A}(V, \mathbb{F}) := (V, \mathcal{L}, \parallel)$ be the affine space corresponding to $(V, \mathbb{F})$.

(a) For $a, b \in V$, $d_Q(a, b) := Q(a - b)$ is called the *Q-distance of a and b*.

(b) The relation $\equiv_Q$, defined on $V \times V$ by

(1) $(a, b) \equiv_Q (c, d) :\Leftrightarrow Q(a - b) = Q(c - d), \quad \forall a, b, c, d \in V$,

is called the *congruence relation with respect to Q*.

(c) The relation $\perp$, defined on $\mathcal{L}$ by

(2) $a + \mathbb{F}b \perp c + \mathbb{F}d :\Leftrightarrow f(b, d) = 0,\ \forall a, b, c, d \in V$ such that $b, d \neq O$

(cf. (1.1) (2)), is called the *affine orthogonality relation with respect to Q*.

(d) Notations, introduced in Section 1 for a vector subspace U of V will be used for all '*parallel*' affine spaces $x + U$ $(x \in V)$ as well.

Thus, $\mathcal{L}_Q^r := \{x + l\colon\ x \in V \wedge l \in \boldsymbol{R}_Q\}$ (cf. 1.7.5) is the set of *Euclidean* lines of $\boldsymbol{A}$ which will be called *Q-regular lines* as well, and $\mathcal{L}_Q^s := \{x + l\colon\ x \in V \wedge l \in \boldsymbol{F}_Q\}$ (cf. 1.7.5 and 2.1.1) is the set of *singular* lines of $\boldsymbol{A}$.

(e) According to 3.1.2, the quadric

$$\boldsymbol{F}_Q^\infty := \{\mathbb{F}(a, 0)\colon\ a \in V \setminus \{O\} \wedge Q(a) = 0\}$$

of $\Pi(V \times \{0\}, \mathbb{F})$ consists of the points at infinity of the singular lines. Therefore, $\boldsymbol{F}_Q^\infty$ is called the *quadric at infinity with respect to Q*. Obviously, $\boldsymbol{F}_Q^\infty$ is empty iff Q is Euclidean, $\boldsymbol{F}_Q^\infty$ is an ovoid iff Q is Minkowskian and $\boldsymbol{F}_Q^\infty$ equals $[V]$ iff Q is singular.

(f) For $a \in V$ and $\rho \in \mathbb{F}$, the point set

(3) $\mathcal{O}_\rho(a) := \{x \in V\colon\ Q(x - a) = \rho\}$

is called the *affine quadric with centre a and Q-radius ρ*. By using the linear form $\psi_a\colon\ V \to \mathbb{F},\ x \mapsto -f(x, a)$, for $\alpha := Q(a) - \rho$ we obtain

(4) $\mathcal{O}_\rho(a) = \{x \in V\colon\ Q(x) + \psi_a(x) + \alpha = 0\}$.

(g) If $\varphi\colon\ V \to \mathbb{F}$ is an *arbitrary* linear form, for $\beta \in \mathbb{F}$, the point set

(5) $\mathcal{O} := \{x \in V\colon\ Q(x) + \varphi(x) + \beta = 0\}$
is called an *affine quadric* of $\boldsymbol{A}$ with respect to Q.

The set $\mathcal{O}$ is not necessarily of type (3). By considering the quadratic form
$Q'\colon\ V \times \mathbb{F} \to \mathbb{F},\ (x, \lambda) \mapsto Q(x) + \lambda \cdot \varphi(x) + \lambda^2 \cdot \beta,$
it turns out that $\mathcal{O} \cup \boldsymbol{F}_Q^\infty$ is the quadric $\boldsymbol{F}_{Q'}$ of the projective closure $\Pi(\boldsymbol{A})$ of $\boldsymbol{A}$. Thus, every affine quadric together with the quadric at infinity is a (projective) quadric.

Because of 2.1.2 this means that every affine collineation maps affine quadrics onto affine quadrics (cf. Chapter 2).

Assume l is a line of $\boldsymbol{A}$ and $\mathcal{O}$ is defined according to (5). Then, l is called a *tangent* of $\mathcal{O}$ provided $l \cup [l]$ is a tangent of $\mathcal{O} \cup \boldsymbol{F}_Q^\infty$, i.e. the condition

$$(l \in \mathcal{L}_Q^T \wedge |l \cap \mathcal{O}| = 1) \vee (l \in \mathcal{L}_Q^s \wedge (l \subseteq \mathcal{O} \vee l \cap \mathcal{O} = \varnothing))$$

is fulfilled. A tangent of $\mathcal{O}$ which does not intersect $\mathcal{O}$ is also called an *asymptote* of $\mathcal{O}$.

(h) If $\mathcal{O}$ is an affine quadric with respect to Q and if T is an affine subspace of $\boldsymbol{A}$, the intersection $\mathcal{O} \cap T$ is called a *Q-sphere of rank* $\dim \overline{\mathcal{O} \cap T}$.

The Q-spheres of rank 2 are called *Q-circles* of $\boldsymbol{A}$, and the set of all Q-circles is denoted by $\mathcal{C}_Q$.

A Q-sphere $\mathcal{O} \cap T$ is called *proper* provided $\mathcal{O} \cap T \neq \overline{\mathcal{O} \cap T}$.

Assume B is a nonempty affinely independent subset of V. Then there exists an affine quadric $\mathcal{O}'$ with respect to Q which contains B. If $\mathcal{O}$ is an arbitrary affine quadric with respect to Q which contains B, then we obtain $\mathcal{O} \cap \overline{B} = s(B)$, where

$$s(B) := \Big\{ \sum_{b \in B} \lambda_b b \in V\colon\ \lambda_b \in \mathbb{F} \wedge |\{b \in B\colon\ \lambda_b \neq 0\}| \in \mathbb{N} \wedge \sum_{b \in B} \lambda_b = 1 \wedge Q\Big(\sum_{b \in B} \lambda_b b\Big) = \sum_{b \in B} \lambda_b Q(b) \Big\}$$

(cf. Schröder [1981b]). This means that $s(B)$ is the only Q-sphere of $\overline{B}$ which contains B.

In particular, if $B = \{a_0, \ldots, a_n\}$ such that $|B| = n + 1 \geqslant 2$, the Q-sphere $s(a_0, \ldots, a_n) := s(B)$ may be represented by

$$s(a_0, \ldots, a_n) = \Big\{ a_0 + \sum_{i=1}^{n} \lambda_i (a_i - a_0)\colon\ \lambda_i \in \mathbb{F} \wedge Q\Big(\sum_{i=1}^{n} \lambda_i (a_i - a_0)\Big) = \sum_{i=1}^{n} \lambda_i Q(a_i - a_0) \Big\}.$$

Moreover, for $|\mathbb{F}| \geqslant 3$ one obtains $s(B) = \overline{B}$ iff all lines $\overline{a_i, a_j}$ for $i \neq j$ are singular (cf. Schröder [1981b]).

(i) For regular lines $l = a + \mathbb{F}b$ and $m = c + \mathbb{F}d$ $(a, c \in V \wedge b, d \in V \setminus \ker Q)$, the *set* $\{l, m\}$ is called a *(nonoriented) angle* with *Q-measure*

(6) $<_Q\{l, m\} := (f(b,d))^2 \cdot (Q(b))^{-1} \cdot (Q(d))^{-1}$.

For $g, h, l, m \in \mathcal{L}^r_Q$, we write $\{g, h\} \simeq_Q \{l, m\}$ provided $<_Q\{g, h\} = <_Q\{l, m\}$. We call $\simeq_Q$ the *angle Q-relation.*

(j) If g, h are regular lines of $\boldsymbol{A}$, the *(ordered) pair* (g, h) is called an *oriented angle* of $\boldsymbol{A}$.

Oriented angles $(g, h), (l, m)$ are called *Q-conform*, written $(g, h) \mathrel{\widehat{-}_Q} (l, m)$, provided

(7) $\mu_Q((O \parallel g), (O \parallel h), (O \parallel m)) = (O \parallel l)$ (in this order!)

holds. It turns out that $\widehat{-}_Q$ is an equivalence relation on $\mathcal{L}^r_Q \times \mathcal{L}^r_Q$ with the property

(8) $(g, h) \mathrel{\widehat{-}_Q} (l, m) \Rightarrow \{g, h\} \simeq_Q \{l, m\}, \ \forall g, h, l, m \in \mathcal{L}^r_Q$.

Moreover, if $Q'\colon V \to \mathbb{F}$ is a quadratic form, we obtain

(9) $\widehat{-}_Q = \widehat{-}_{Q'} \Leftrightarrow \mu_Q = \mu_{Q'} \Leftrightarrow Q \sim Q'$ (cf. Schröder [1986a]).

From (9) we deduce that the metric structure, which is induced on the affine space $\boldsymbol{A}$ by the quadratic form Q, is determined by the Q-conformity of oriented angles up to geometric equivalence.

Therefore, the pair $(\boldsymbol{A}, \widehat{-}_Q)$ is called the *affine metric space* which corresponds to the metric vector space $(V, \mathbb{F}, Q)$. For examples, cf. Section 3.4.

3.2. *Isomorphisms of affine metric geometry*

Let $(\boldsymbol{A}, \widehat{-}_Q)$, $(\boldsymbol{A}', \widehat{-}_{Q'})$ be affine metric spaces which correspond to the metric vector spaces $(V, \mathbb{F}, Q)$, $(V', \mathbb{F}', Q')$, where $\boldsymbol{A} = (V, \mathcal{L}, \parallel)$ and $\boldsymbol{A}' = (V', \mathcal{L}', \parallel')$.

3.2.1. A collineation φ of $\boldsymbol{A}$ onto $\boldsymbol{A}'$ (cf. Chapter 2) is called an *isomorphism* of $(\boldsymbol{A}, \widehat{-}_Q)$ onto $(\boldsymbol{A}', \widehat{-}_{Q'})$, and in case of $(V, \mathbb{F}, Q) = (V', \mathbb{F}', Q')$ an *automorphism* of $(\boldsymbol{A}, \widehat{-}_Q)$, provided

(1) $(g, h) \mathrel{\widehat{-}_Q} (k, l) \Leftrightarrow (\varphi(g), \varphi(h)) \mathrel{\widehat{-}_{Q'}} (\varphi(k), \varphi(l)), \ \forall (g, h), (k, l) \in \mathcal{L} \times \mathcal{L}$,

holds. Because $\widehat{-}_Q$, resp., $\widehat{-}_{Q'}$ is defined only for elements of $\mathcal{L}^r_Q \times \mathcal{L}^r_Q$, resp., $\mathcal{L}^r_{Q'} \times \mathcal{L}^r_{Q'}$ by using statement (1) for $g = h = k = l \in \mathcal{L}^r_Q$, we obtain

(2) $\varphi(\mathcal{L}^r_Q) = (\mathcal{L}^r_{Q'}) \wedge \varphi(\mathcal{L}^s_Q) = (\mathcal{L}^s_{Q'})$

for every isomorphism φ of $(\boldsymbol{A}, \widehat{-}_Q)$ onto $(\boldsymbol{A}', \widehat{-}_{Q'})$.

From the fundamental theorem of affine geometry (cf. Brauner [1976]) and from Schröder [1986a] we deduce

3.2.2. FIRST FUNDAMENTAL THEOREM OF AFFINE METRIC GEOMETRY. *Assume* $\dim V \geqslant 2$. *Then, a bijection* $\varphi\colon V \to V'$ *is an isomorphism of* $(\boldsymbol{A}, \widehat{-}_Q)$ *onto* $(\boldsymbol{A}', \widehat{-}_{Q'})$ *iff*

$$\overline{\varphi}\colon V \to V, \quad x \mapsto \varphi(x) - \varphi(O),$$

is an isomorphism of $(V, \mathbb{F}, Q)$ *onto* $(V', \mathbb{F}', Q')$ (cf. 1.5.3).

3.2.3. Suppose $\dim V \geqslant 2$. If φ is an isomorphism of $(\boldsymbol{A}, \widehat{-}_Q)$ onto $(\boldsymbol{A}', \widehat{-}_{Q'})$, for $c := \varphi(O)$ and $d := \overline{\varphi}^{-1}(c)$, the semilinearity of $\overline{\varphi}$ (cf. 3.2.2) leads to the representation

(1) $\varphi(x) = \overline{\varphi}(x) + c = \overline{\varphi}(x + d), \ \forall x \in V$.

This means that φ is *semilinear up to a translation* (cf. Brauner [1976]). Moreover, according to 1.5.3, 3.1.3, there exist an isomorphism ρ: $\mathbb{F}(+,\cdot) \to \mathbb{F}'(+,\cdot)$ and an element $\alpha \in \dot{\mathbb{F}}'$ such that the statements

(2) $\alpha \cdot \rho \circ Q = Q' \circ \overline{\varphi}$,

(3) $\alpha \cdot \rho(d_Q(x,y)) = d_{Q'}(\varphi(x), \varphi(y))$, $\forall x, y \in V$,

(4) $(a,b) \equiv_Q (c,d) \Leftrightarrow (\varphi(a), \varphi(b)) \equiv_{Q'} (\varphi(c), \varphi(d))$, $\forall a,b,c,d \in V$,

(5) $g \perp h \Leftrightarrow \varphi(g) \perp' \varphi(h)$, $\forall g, h \in \mathcal{L}$,

(6) $\varphi(\mathcal{C}_Q) = \mathcal{C}_{Q'}$,

(7) $\rho(<_Q \{l,m\}) = <_{Q'} \{\varphi(l), \varphi(m)\}$, $\forall l, m \in \mathcal{L}^r_Q$,

(8) $\{g,h\} \simeq_Q \{l,m\} \Leftrightarrow \{\varphi(g), \varphi(h)\} \simeq_{Q'} \{\varphi(l), \varphi(m)\}$, $\forall g,h,l,m \in \mathcal{L}$,

hold (cf. Schröder [1986a]).

The isomorphism φ is called a *similarity* iff $\overline{\varphi}$ is linear, i.e. ρ is the identity mapping, and a *motion* iff $\alpha = 1$ and $\overline{\varphi}$ is linear (cf. 1.5.3).

Thus, according to (1) and 1.5.3, a similarity is a product of an equimetry and a translation, and a motion is a product of an isometry and a translation.

If φ is a similarity, we call $\overline{\varphi}$ the *linear part* of φ and $\varphi(O)$ the *translation vector* of φ.

3.2.4. The similarities of $(\boldsymbol{A}, \widehat{-}_Q)$ form a subgroup Σ_Q of the group $\mathrm{Aut}(\boldsymbol{A}, \widehat{-}_Q)(\circ)$ of all automorphisms of $(\boldsymbol{A}, \widehat{-}_Q)$, and the set $\boldsymbol{M}_Q$ of all motions of $(V, \mathbb{F}, Q)$ is a subgroup of Σ_Q. If $\boldsymbol{T_A}$ denotes the group of translations of $\boldsymbol{A}$, we have

$$\boldsymbol{M}_Q = \boldsymbol{T_A} \circ I(Q) = I(Q) \circ \boldsymbol{T_A}$$

(cf. 1.5.3 and 3.2.3 (1)).

For $\varphi \in \Sigma_Q$, $\mathrm{fix}\,\varphi := \{x \in V\colon \varphi(x) = x\}$ is called the *axis* of φ and $p(\varphi) := \{\varphi(y) - y\colon y \in V\}$ is called the *path* of φ (cf. 1.6.1).

It turns out that fix φ and $p(\varphi)$ for $\varphi \in \Sigma_Q$ are affine subspaces of $\boldsymbol{A}$. More precisely, we obtain

(1) $p(\varphi) = \varphi(O) + p(\overline{\varphi}) \neq \varnothing \wedge \dim p(\varphi) = \dim V/\,\mathrm{fix}\,\overline{\varphi}$,

(2) $\varphi \in \boldsymbol{T_A} \Leftrightarrow \mathrm{fix}\,\overline{\varphi} = V \Leftrightarrow \dim p(\varphi) = 0$,

(3) $\mathrm{fix}\,\varphi = \varnothing \Leftrightarrow O \notin p(\varphi) \Leftrightarrow p(\varphi) \cap p(\overline{\varphi}) = \varnothing$.

Let $\varphi \in \Sigma_Q$ and assume $\mathrm{fix}\,\varphi \neq \varnothing$. Then, for $a \in \mathrm{fix}\,\varphi$ and $\tau_a\colon V \to V,\ x \mapsto x + a$, we get

(4) $\varphi = \tau_a \circ \overline{\varphi} \circ \tau_a^{-1} \wedge \varphi(x) = \overline{\varphi}(x-a) + a$, $\forall x \in V$,

(5) $\mathrm{fix}\,\varphi = a + \mathrm{fix}\,\overline{\varphi}$.

Moreover, by 1.6.1 we obtain

(6) $\varphi \in \boldsymbol{M}_Q \wedge \mathrm{fix}\,\varphi \neq \varnothing \Rightarrow p(\varphi) \perp \mathrm{fix}\,\overline{\varphi}$.

3.2.5. Assume $a \in V, b \in V \setminus \ker Q$ and $\tau_a\colon V \to V,\ x \mapsto x + a$. Then,

$$(1)\ \sigma := \sigma_{a,b} := \tau_a \circ \widetilde{b} \circ \tau_a^{-1}\colon V \to V,\ x \mapsto x - \frac{f(x-a,b)}{Q(b)} \cdot b$$

(cf. 1.6.2) is a motion with the properties

(2) $\sigma = \sigma^{-1}$,

(3) $(\sigma \neq \mathrm{id}_V \Rightarrow p(\sigma) = \mathbb{F}b) \wedge \mathrm{fix}\, \sigma = a + b^{\perp}$,
(4) $\sigma(l) = l \Leftrightarrow l \parallel \mathbb{F}b \vee l \subseteq a + b^{\perp}$, $\forall l \in \mathcal{L}$,

called the *affine Q-symmetry* with *direction* $\mathbb{F}b\|$ and *axis* $a + b^{\perp}$ (cf. 3.2.4). Let $S(Q)$ be the set of all these affine Q-symmetries (cf. 1.6.2).

By direct computation, we get

(5) $\alpha \in S(Q) \Leftrightarrow \overline{\alpha} \in S_o(Q)$, $\forall \alpha \in \mathrm{Aut}(\boldsymbol{A}, \widehat{-}_Q)$.

Moreover, if φ is an isomorphism of $(\boldsymbol{A}, \widehat{-}_Q)$ onto $(\boldsymbol{A}', \widehat{-}_{Q'})$, in case of $\dim V \geqslant 2$ we obtain

(6) $\varphi \circ \sigma_{a,b} \circ \varphi^{-1} = \sigma_{\varphi(a), \overline{\varphi}(b)}$, $\forall (a,b) \in V \times (V \setminus \ker Q)$,
(7) $\varphi \circ S(Q) \circ \varphi^{-1} = S(Q') \wedge \varphi \circ \boldsymbol{T_A} \circ \varphi^{-1} = \boldsymbol{T_{A'}}$,
(8) $\varphi \circ \boldsymbol{M}_Q \circ \varphi^{-1} = \boldsymbol{M}_{Q'} \wedge \varphi \circ \Sigma_Q \circ \varphi^{-1} = \Sigma_{Q'}$.

The following statement shows, under which additional assumptions the metric structure of $(\boldsymbol{A}, \widehat{-}_Q)$ is fixed by one of the sets $\mathcal{L}_Q^r$, $\mathcal{C}_Q$, $S(Q)$ or by one of the relations $\equiv_Q$, $\simeq_Q$, $\perp$. In each such case the structure $(\boldsymbol{A}, \widehat{-}_Q)$ may as well be given in the form $(\boldsymbol{A}, \mathcal{L}_Q^r)$, $(\boldsymbol{A}, \mathcal{C}_Q)$, $(\boldsymbol{A}, S(Q))$, $(\boldsymbol{A}, \equiv_Q)$, $(\boldsymbol{A}, \simeq_Q)$ or $(\boldsymbol{A}, \perp)$ (cf. Schröder [1986a]):

3.2.6. SECOND FUNDAMENTAL THEOREM OF AFFINE METRIC GEOMETRY. *Assume* $\dim V \geqslant 2$ *and let* φ *be a collineation of* $\boldsymbol{A}$ *onto* $\boldsymbol{A}'$. *Then,* φ *is an isomorphism of* $(\boldsymbol{A}, \widehat{-}_Q)$ *onto* $(\boldsymbol{A}', \widehat{-}_{Q'})$ *if (at least) one of the following conditions is fulfilled*:

(1) $\ker Q \notin \mathcal{U} \wedge \varphi(\mathcal{L}_Q^r) = (\mathcal{L}_{Q'}^r)$,
(2) $\ker Q \in \mathcal{U} \wedge \dim V/\ker Q = 1 \wedge \varphi(\mathcal{L}_Q^r) = \mathcal{L}_{Q'}^r$,
(3) $|\mathbb{F}| = 2 \wedge \varphi(\mathcal{L}_Q^r) = \mathcal{L}_{Q'}^r$,
(4) $|\mathbb{F}| \geqslant 3 \wedge \varphi(\mathcal{C}_Q) = \mathcal{C}_{Q'}$,
(5) $V^{\perp} \neq V \wedge \varphi \circ S(Q) \circ \varphi^{-1} = S(Q')$,
(6) $V^{\perp} \neq V \wedge \big((a,b) \equiv_Q (c,d) \Leftrightarrow (\varphi(a), \varphi(b)) \equiv_{Q'} (\varphi(c), \varphi(d)),\ \forall a,b,c,d \in V\big)$,
(7) $V^{\perp} \neq V$
$\wedge \big(\{g,h\} \simeq_Q \{l,m\} \Leftrightarrow \{\varphi(g), \varphi(h)\} \simeq_{Q'} \{\varphi(l), \varphi(m)\},\ \forall g,h,l,m \in \mathcal{L}\big)$,
(8) $\mathrm{char}\, \mathbb{F} \neq 2 \wedge \big(g \perp h \Leftrightarrow \varphi(g) \perp' \varphi(h),\ \forall g,h \in \mathcal{L}\big)$.

A point set B of $\boldsymbol{A}$ is an *affine basis of* A, provided $\overline{B} = V$ and B is affinely independent. Using this notation, by 3.2.2 and Bourbaki [1959], we obtain

3.2.7. TRANSITIVITY THEOREM OF AFFINE METRIC GEOMETRY. *Assume* ρ: $\mathbb{F}(+,\cdot) \to \mathbb{F}'(+,\cdot)$ *is an isomorphism,* B *is an affine basis of* $\boldsymbol{A}$ *and* κ *is an injection of* B *into* V' *such that* $\kappa(B)$ *is an affine basis of* $\boldsymbol{A}'$.

Then κ *can be extended uniquely to a collineation* φ *of* $\boldsymbol{A}$ *onto* $\boldsymbol{A}'$ *such that* $\overline{\varphi}$: $V \to V$, $x \mapsto \varphi(x) - \varphi(O)$ *is* ρ*-linear. Moreover,* φ *is an isomorphism of* $(\boldsymbol{A}, \widehat{-}_Q)$ *onto* $(\boldsymbol{A}', \widehat{-}_{Q'})$ *iff there exists an element* $\alpha \in \dot{\mathbb{F}}'$ *such that*

(∗) $d_{Q'}(\kappa(a), \kappa(b)) = \alpha \cdot \rho(d_Q(a,b))$, $\forall a, b \in B$,

holds. In particular, φ *is a motion iff* $\overline{\varphi}$ *is linear and* (∗) *holds for* $\alpha = 1$.

While these two theorems establish geometric characterizations of isomorphisms, Alpers [1989a] gives the following geometric characterizations of similarities:

(1) *Assume* $Q \neq 0$ *and* φ *is a collineation of* $\boldsymbol{A}$ *such that* $d_Q(\varphi(X), \varphi(Y)) = \lambda \cdot d_Q(X, Y)$, $\forall X, Y \in V$, *where* λ *is a fixed element of* $\dot{\mathbb{F}}$. *Then,* φ *is a similarity of* $(\boldsymbol{A}, \widehat{-}_Q)$.

(2) *Assume* $\dim V/V^{\perp} \geqslant 2$ *and* φ *is a collineation of* $\boldsymbol{A}$ *such that*

$$<_Q\{\varphi(g), \varphi(h)\} = <_Q\{g, h\}, \ \forall g, h \in \mathcal{L}_Q^r.$$

Then, except for the case $|\mathbb{F}| = 4$ *and* Q *is 2-singular,* φ *is a similarity of* $(\boldsymbol{A}, \widehat{-}_Q)$.

For other characterizations of isomorphisms of affine metric geometry, see Chapter 16.

3.3. *Motions and similarities*

Let $(\boldsymbol{A}, \widehat{-}_Q)$ be an affine metric space which corresponds to the metric vector space $(V, \mathbb{F}, Q)$ and assume $\boldsymbol{A} = (V, \mathcal{L}, \|)$.

3.3.1. We are now interested in products of affine Q-symmetries and define

$$S^r(Q) := \{\sigma_1 \circ \cdots \circ \sigma_r \colon \sigma_r, \ldots, \sigma_r \in S(Q)\} \quad \text{for } r \in \dot{\mathbb{N}}.$$

For $\alpha, \beta \in S(Q)$, we obtain

(1) $\alpha \circ \beta \in \boldsymbol{T_A} \Leftrightarrow p(\alpha) = p(\beta)$.

To get an answer to the question, which translations are products of Q-symmetries, we need

(2) TRANSITIVITY OF Q-SYMMETRIES. *Assume* $a, b \in V$ *such that*

$$a - b \in V \setminus (V^{\perp} \cup \ker Q).$$

Then, there exists one and only one Q*-symmetry* $\widetilde{\chi}_{a,b}$ *which interchanges* a *and* b. *The axis* $\chi_{a,b}$ *of* $\widetilde{\chi}_{a,b}$ *is an affine hyperplane, called the* Q*-symmetry axis or the perpendicular bisector of* (a, b). *Moreover,* $\chi_{a,b} = \{x \in V \colon (a, x) \equiv_Q (x, b)\}$.

Observe that $\widetilde{\chi}_{a,b} = \sigma_{d,a-b}$ for $d := a + Q(a-b) \cdot (f(a-b, c))^{-1} \cdot c$, where c is an arbitrary element of $V \setminus (a-b)^{\perp}$.

Because of (1) and (2), for $e \in V \setminus (V^{\perp} \cup \ker Q)$, the translation $\tau_e \colon V \to V$, $x \mapsto x + e$, is the product $\widetilde{\chi}_{O,e} \circ \sigma_{O,e}$. In case of $V^{\perp} \neq V \wedge |\mathbb{F}| \geqslant 3$, each element of $\ker Q \cup V^{\perp}$ is the sum of two elements of $V \setminus (\ker Q \cup V^{\perp})$, and thus we get

(3) $V^{\perp} \neq V \wedge |\mathbb{F}| \geqslant 3 \Rightarrow \boldsymbol{T_A} \subseteq S^4(Q)$.

From (3), 1.6.3, and 3.2.4 we conclude that in many cases a motion is a product of affine Q-symmetries.

To be more precise, for $a \in V$ let us consider the set

$$M_Q(a) := \{\varphi \in M_Q \colon \varphi(a) = a\}.$$

By using the translation τ_a we get an isomorphism

$$I(Q)(o) \to \boldsymbol{M}_Q(a)(o),\ \varphi \mapsto \tau_a \circ \varphi \circ \tau_a^{-1},$$

which maps $S_o(Q)$ onto the set $S_a(Q) := \{\sigma \in S(Q)\colon\ \sigma(a) = a\}$. Thus, for $r \in \dot{\mathbb{N}} \wedge a \in V \wedge \varphi \in I(Q)$, we obtain

(4) $\varphi \in S_o^r(Q) \Leftrightarrow \tau_a \circ \varphi \circ \tau_a^{-1} \in S^r(Q) \cap \boldsymbol{M}_Q(a)$,

and this means that the results of 1.6.3 concerning $I(Q)$ hold similarly for motions which have a fixed point.

Let us consider now a motion ψ without fixed points and assume

$$p(\psi) \not\subseteq V^\perp \cup \ker Q.$$

Then, there exists an element $b \in V$ such that $\psi(b) - b \notin V^\perp \cup \ker Q$, and because of (2), for $\sigma := \widetilde{\chi}_{b,\psi(b)}$ we obtain $\sigma \circ \psi(b) = b$. Thus, $\sigma \circ \psi$ has a fixed point, and in case of $\sigma \circ \psi \in S^r(Q)$ for some $r \in \dot{\mathbb{N}}$, we conclude $\psi \in S^{r+1}(Q)$.

If ζ is a motion without fixed points and such that $p(\zeta) \subseteq V^\perp \cup \ker Q$, we may use the representation $\zeta = \overline{\zeta} \circ \tau$ for some $\tau \in \boldsymbol{T_A}$ (cf. 3.2.3 (1)) to look for a decomposition of ζ into a product of Q-symmetries.

If $V^\perp = \{O\} \wedge |\mathbb{F}| \geqslant 3 \wedge \dim V = n \in \dot{\mathbb{N}}$, because of 1.6.3 (4), our considerations lead to

(5) $\boldsymbol{M}_Q = S^{n+1}(Q) \cup S^{n+2}(Q)$,

and moreover we obtain

(6) $\boldsymbol{M}_Q = S^n(Q) \cup S^{n+1}(Q)$,

if Q is Euclidean, or if $n \geqslant 3$ and Q is Minkowskian.

For elements $H_1, \ldots, H_r$ of $\mathcal{H} \cup \{\mathcal{P}\}$ ($r \in \dot{\mathbb{N}}$) we say that $H_1, \ldots, H_r$ *belong to a pencil*, if $H_1 \cap \ldots \cap H_r$ is an affine hyperline of $\boldsymbol{A}$ or if $H_1, \ldots, H_r$ are pairwise parallel (cf. 3.1.1). We then obtain from 1.2.5 (3), 1.6.4 (5), (6) and 3.2.4:

3.3.2. AFFINE 3-REFLECTION LEMMA. *Assume $a, b, c \in V \setminus \ker Q$ and $d, e, f \in V$ such that $\mathbb{F}a = \mathbb{F}b = \mathbb{F}c \vee (d = e = f \wedge \dim(\mathbb{F}a + \mathbb{F}b + \mathbb{F}c) = 2)$.*

Then, for $\alpha := \sigma_{d,a}$, $\beta := \sigma_{e,b}$, $\gamma := \sigma_{f,c}$ we get

(1) $\delta := \alpha \circ \beta \circ \gamma \in S(Q)$,

(2) $p(\delta) \subseteq p(\alpha) + p(\beta) + p(\gamma)$,

(3) $\operatorname{fix}\alpha, \operatorname{fix}\beta, \operatorname{fix}\gamma, \operatorname{fix}\delta$ *belong to a pencil.*

3.3.3. AFFINE 3-REFLECTION THEOREM. *Assume, Q is regular and α, β, γ are elements of $S(Q)$. Then, $\alpha \circ \beta \circ \gamma$ is an element of $S(Q)$ iff* $\operatorname{fix}\alpha, \operatorname{fix}\beta, \operatorname{fix}\gamma$ *belong to a pencil.*

By using standard arguments of Bachmann [1973], from these two theorems we deduce

3.3.4. *Let Q be regular and assume $a \in V$, $U \in \mathcal{U}_2$ such that $U \not\subseteq \ker Q$, and $D := a + U^\perp$. Then,*

$$\boldsymbol{M}_Q(D) := \{\varphi \in \boldsymbol{M}_Q\colon\ \operatorname{fix}\varphi \supseteq D\}$$

is a subgroup of M_Q, *and for*

$$S_D := \{\sigma \in S(Q)\colon \operatorname{fix}\sigma \supseteq D\} \quad \textit{and} \quad S_D^2 := \{\sigma \circ \tau\colon \sigma, \tau \in S_D\},$$

we obtain $M_Q(D) = S_D \mathbin{\dot\cup} S_D^2$. *Moreover,* S_D^2 *is an Abelian normal subgroup of* $M_Q(D)$ *of index* 2, *and*

$$S_D^2 \setminus \{\mathrm{id}_V\} = \{\varphi \in M_Q\colon \operatorname{fix}\varphi = D\}.$$

We call S_D^2 the group of *rotations with axis D.*

The following statement can be proved similar as in the classical Euclidean case:

3.3.5. CANONICAL DECOMPOSITION OF MOTIONS. *If* φ *is a motion such that* $p(\varphi)$ *is regular and of finite dimension, then there exists exactly one motion* $\alpha \in M_Q$ *and exactly one translation* $\tau \in T_A$ *such that* $\varphi = \alpha \circ \tau = \tau \circ \alpha$ *and* $\operatorname{fix}\alpha \neq \varnothing$. *Moreover, we obtain*

$$\tau(O) \in \operatorname{fix}\overline{\varphi} = p(\overline{\varphi})^{\perp}; \quad \alpha(O) \in p(\alpha) = p(\overline{\varphi}); \quad \tau(O) + \alpha(O) = \varphi(O);$$

$$\overline{\alpha} = \overline{\varphi}; \quad \operatorname{fix}\varphi \neq \varnothing \Leftrightarrow \alpha = \varphi; \quad \tau(\operatorname{fix}\alpha) = \operatorname{fix}\alpha.$$

3.3.6. The group Δ_A of dilatations of A (cf. Chapter 2) is a subgroup of Σ_Q, and we have $M_Q \leqslant \Delta_A \circ I(Q) \leqslant \Sigma_Q$.

Assume $Q \neq 0$. If a similarity $\varphi \in \Sigma_Q$ is represented according to 3.2.3 (1), (2), a simple computation (cf. Snapper and Troyer [1971]) leads to

(1) $\varphi \in \Delta_A \circ I(Q) \Leftrightarrow \alpha \in \dot{\mathbb{F}}^{(2)} := \{x^2\colon x \in \dot{\mathbb{F}}\}$,

where the *square ratio* α of φ is determined by

(2) $\alpha \cdot Q(x) = Q(\varphi(x) - \varphi(O))$, $\forall x \in V$.

For a detailed discussion of the case

$$\Delta_A \circ I(Q) = \Sigma_Q,$$

cf. Snapper and Troyer [1971]. Because of 1.4.3 and according to Snapper and Troyer [1971], p. 361, we get

(3) If $(A, \widehat{-}_Q)$ is regular and of finite odd dimension, then $\Delta_A \circ I(Q) = \Sigma_Q$.

(4) If Q is Euclidean and if $\mathbb{F} = \mathbb{R}$, then $\Delta_A \circ I(Q) = \Sigma_Q$.

On the contrary, from 1.3.7 (4), 1.4.7, 1.9.9 (1) and Schröder [1985], p. 86, we deduce

(5) For $\dim A \in 2\dot{\mathrm{N}}$ and $|\mathbb{F}| \in 2\mathrm{N} + 1$ we have $\Delta_A \circ I(Q) \neq \Sigma_Q$.

Observe that the mapping $\delta\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (1 + x, -y)$, is a similarity of the Minkowskian plane belonging to $Q\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto xy$, where the square ratio of δ is $-1 \notin \mathbb{R}^{(2)}$. Moreover, δ has no fixed point. Contrary to this, by considering the kernel of the linear mapping $\overline{\varphi} - \mathrm{id}_V$, one obtains

(6) If $(A, \widehat{-}_Q)$ is Euclidean and of finite dimension, every element φ *of* $\Sigma_Q \setminus M_Q$ has exactly one fixed point.

3.4. *Affine metric planes*

The theory of affine metric planes contains the classical plane Euclidean geometry and thus a lot of fascinating theorems. It is a surprising experience that most of the classical theorems mutatis mutandis hold in arbitrary affine metric planes (cf. Bachmann [1973], Baer [1944], Baptist and Schröder [1988], Benz [1973], Blanck and Schröder [1978], Ewald [1971], Schröder [1974a,b, 1985]).

Lack of space forces us to restrict our attention here to a few of these theorems. For this purpose let us consider a metric vector space $(\mathbb{L}, \mathbb{F}, Q)$, where $\mathbb{L}$ is a quadratic $\mathbb{F}$-algebra and $Q := Q_{\mathbb{L}}$ (cf. 1.8.1–2). Let $\boldsymbol{A} := \boldsymbol{A}(\mathbb{L}, \mathbb{F})$ and $\widehat{-} := \widehat{-}_Q$. We will use the notations introduced in 1.8.1–3, 3.1.3 and 3.2.1–5 to develop properties of the affine metric plane $(\boldsymbol{A}, \widehat{-})$.

3.4.1. Because of $Q \neq 0$ and according to 2.1.2 (2) and 3.1.3 (e), the quadric at infinity $\boldsymbol{F}^{\infty}$ of $(\boldsymbol{A}, \widehat{-})$ contains at most two points. Thus, the set $\mathcal{L}^s$ of singular lines of $(\boldsymbol{A}, \widehat{-})$ consists of one pencil respectively two pencils of parallel lines if Q is Galilean respectively Minkowskian, and $\mathcal{L}^s$ is empty iff Q is Euclidean.

Assume $\mathcal{L}$ is the set of lines and $\mathcal{L}^r = \mathcal{L} \setminus \mathcal{L}^s$ is the set of Euclidean lines. Let us extend the notations introduced in 3.1.3 (j) and define the oriented angles to be the elements of $\mathcal{L} \times \mathcal{L}$.

Suppose $(g, h) \in \mathcal{L} \times \mathcal{L}$ such that

$$g = a + \mathbb{F}b, \qquad h = c + \mathbb{F}d \quad (a, c \in \mathbb{L} \wedge b,\ d \in \mathbb{L} \setminus \{0\}).$$

Then,

(1) $\angle(g, h) := \mathbb{F}\bar{b}d$ is called the *measure* of (g, h).

For $g, h, k, l \in \mathcal{L}$ and $m \in \mathcal{L}^s$ we obtain

(2) $\angle(g, h) = \mathbb{F} \Leftrightarrow g \| h \wedge g, h \in \mathcal{L}^r$,

(3) $\angle(g, h) = \{0\} \Leftrightarrow g \| h \wedge g, h \in \mathcal{L}^s$,

(4) $\{0\} \neq \angle(g, h) \parallel m \Leftrightarrow g \nparallel h \wedge (g \| \kappa_{\mathbb{L}}(m) \vee h \| m)$,

(5) $g \| h \wedge k \| l \Rightarrow \angle(g, h) = \angle(k, l) \wedge \angle(g, k) = \angle(h, l)$,

(6) $\angle(g, h) = \angle(k, l) \wedge g \| k \wedge k \in \mathcal{L}^r \Rightarrow h \| l$,

(7) $\angle(g, h) = \angle(k, l) \Rightarrow \angle(h, g) = \angle(l, k)$,

(8) $\angle(g, h) = \angle(k, l) \wedge h, k \in \mathcal{L}^r \Rightarrow \angle(g, k) = \angle(h, l)$,

and for $g, h, k, l \in \mathcal{L}^r$ we obtain

(9) $\angle(g, h) = \angle(k, l) \Leftrightarrow (g, h) \widehat{-} (k, l)$ (cf. Schröder [1974b]).

With respect to the addition

(10) $\mathbb{F}z(+)\mathbb{F}w := \mathbb{F}z \cdot w$ for $z, w \in \mathbb{L} \setminus \{0\}$,

the measures form an Abelian semigroup, and the measure function is *additive* in the sense of

(11) $\angle(g, h)(+)\angle(h, k) = \angle(g, k)$ for $g, k \in \mathcal{L} \wedge h \in \mathcal{L}^r$.

Moreover, we get

(12) Assume $g \in \mathcal{L}^r$, $z \in \mathbb{L}$ and $\omega \in \mathcal{L} \times \mathcal{L}$ such that $\angle\omega \neq \{0\}$. Then there exists exactly one element $h \in \mathcal{L}$ such that $h \ni z$ and $\angle(g, h) = \angle\omega$.

If $\mathbb{L}$ is inseparable, we obtain $\angle(g, h) = \angle(h, g)$ for $g, h \in \mathcal{L}$. On the other hand, if $\mathbb{L}$ is separable, (1) leads to

(13) $\angle(g, h) = \angle(h, g) \Leftrightarrow g \| h \vee g \perp h$.

3.4.2. Let C be the set of circles of $(\boldsymbol{A}, \widehat{-})$ and let a, b, c be noncollinear points of A. According to 3.1.3 (h), there exists exactly one circle $k = s(a, b, c)$ which passes through a, b, c. If at least one of the lines $\overline{a, b}$, $\overline{b, c}$, $\overline{c, a}$ is singular, then k is the union of two distinct singular lines which are parallel if Q is Galilean and which intersect if Q is Minkowskian (cf. 2.1.4, and 3.1.3 (i)). On the other hand, if the lines $\overline{a, b}$, $\overline{b, c}$, $\overline{c, a}$ are regular, $k \cup \boldsymbol{F}^{\infty}$ is an oval quadric of $\Pi(\boldsymbol{A})$. This means that for $z, w \in k$ there exists exactly one regular line g such that $g \cap k = \{z, w\}$; if $z = w$, g is the tangent of k at z.

A circle is called *singular* if it is the union of two singular lines, and it is called *regular* otherwise. Let C^s be the set of singular circles and $C^r = C \setminus C^s$ the set of regular circles (Figure 3.1). We obtain $C^r \neq \varnothing \Leftrightarrow |\mathbb{F}| \geqslant 2 + |\boldsymbol{F}^{\infty}|$.

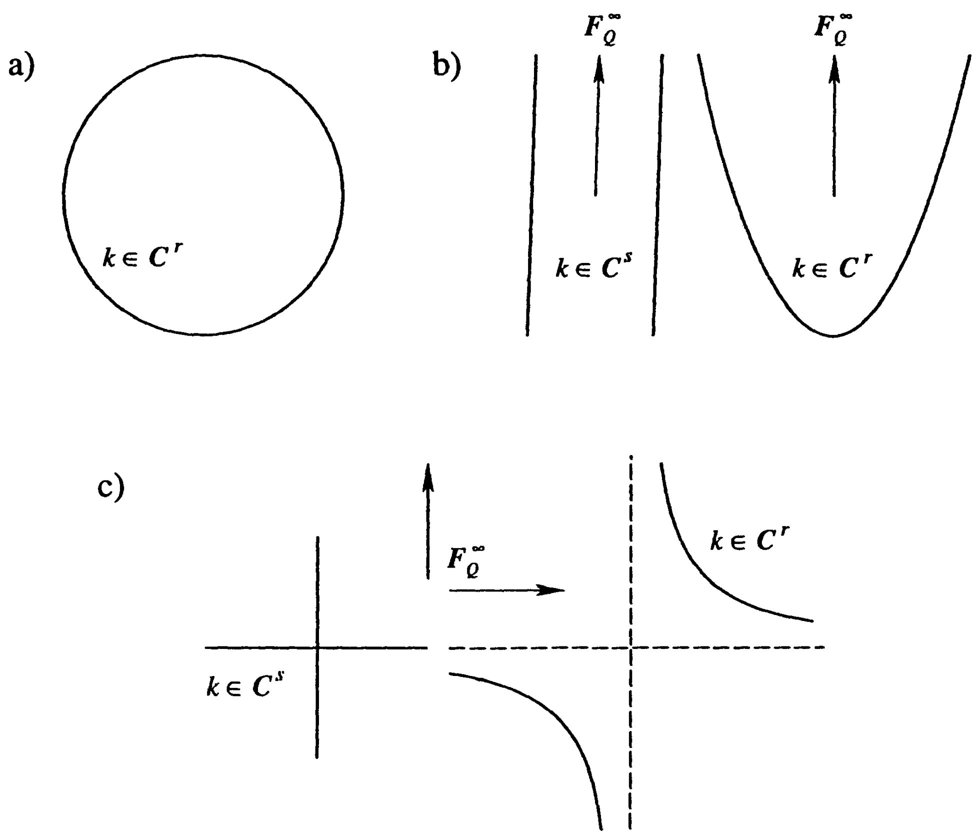

Figure 3.1. Circles: a) Q Euclidean, b) Q Galilean, c) Q Minkowskian.

3.4.3. One of the most important theorems of plane affine metric geometry is the following theorem which describes the connection between circles and angles (cf. Benz [1973] and Schröder [1974b, 1980, 1985]):

THEOREM OF INSCRIBED ANGLES. *Assume a, b, c, z are distinct points such that a, b, c are noncollinear. Then we obtain (Figure 3.2)*

(1) $z \in s(a, b, c) \Leftrightarrow \angle(\overline{a, z}, \overline{z, c}) \subseteq \angle(\overline{a, b}, \overline{b, c})$.

Moreover, if t_a, t_c are tangents of $s(a, b, c)$ at a, resp., c, we obtain

(2) $\angle(t_a, \overline{a, c}) \subseteq \angle(\overline{a, b}, \overline{b, c}) \supseteq \angle(\overline{a, c}, t_c)$.

Figure 3.2 d)–g) shows that the equality of measures of angles in case of $\mathcal{L}^s \neq \varnothing$ is determined by the incidence geometry of regular and singular lines.

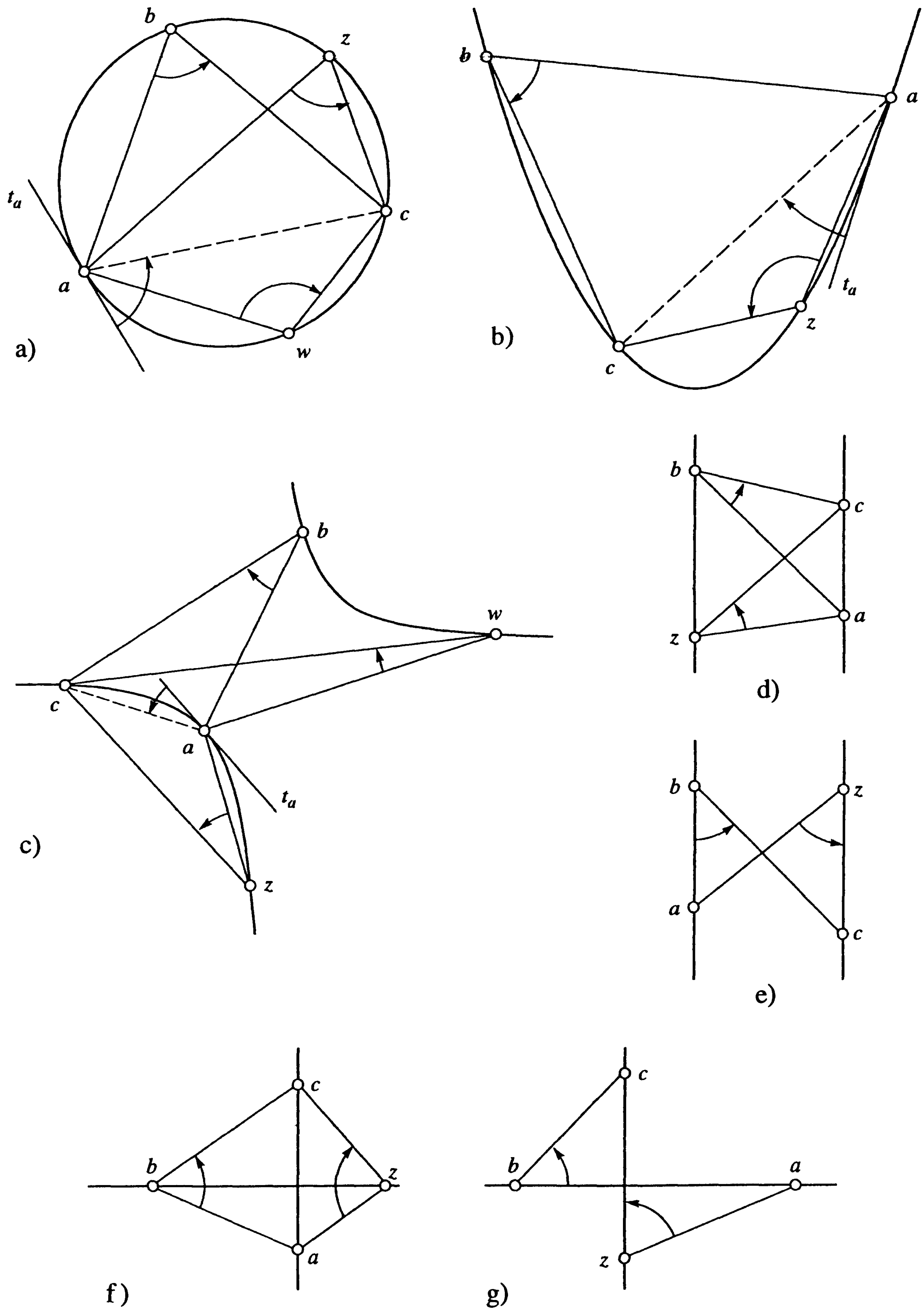

Figure 3.2. a) Q Euclidean; b), d), e) Q Galilean; c), f), g) Q Minkowskian.

3.4.4. As a corollary we get the following

TANGENT THEOREM. *Assume* $a, b \in \mathbb{L}$ *and* $t \in \mathcal{L}$ *such that* $a \in t \wedge b \notin t \wedge \overline{a, b} \in \mathcal{L}^r$. *Then, in case of* $\mathbf{C}^r \neq \varnothing$, *there exists exactly one circle* k *which contains* a, b *such that* t *is the tangent of* k *at* a (cf. Benz [1973] and Schröder [1974b]).

3.4.5. For $g_\nu \in \mathcal{L}$ ($\nu \in \mathbb{Z}_4$) let us write $q(g_1, \ldots, g_4)$ provided

$$g_1 \cap \ldots \cap g_4 = \varnothing, \quad g_\nu \nparallel g_{\nu+1} \text{ and } (g_\nu = g_{\nu+2} \Rightarrow g_\nu \in \mathcal{L}^r) \quad \text{for } \nu \in \mathbb{Z}_4.$$

Assume $q(g_1, \ldots, g_4)$. Then we define $\{a_\nu\} := g_{\nu-1} \cap g_\nu$ for $\nu \in \mathbb{Z}_4$ and call $a_1, \ldots, a_4$ the *vertices* of the *quadrangle* $(g_1, \ldots, g_4)$. We write $\mathrm{C}q(g_1, \ldots, g_4)$ provided there exists a circle k which contains $a_1, \ldots, a_4$ such that g_ν is a tangent of k in case of $a_\nu = a_{\nu+1}$. The circle k is uniquely determined, if it exists (cf. Figure 3.3 a)–c)). Obviously we have

(*) $\mathrm{C}q(g_1, \ldots, g_4) \Leftrightarrow \mathrm{C}q(g_4, \ldots, g_1) \Leftrightarrow \mathrm{C}q(g_4, g_1, g_2, g_3)$

3.4.6. By using the notations of 3.4.5, the theorem of inscribed angles 3.4.3 may be reformulated as follows:

Assume $q(g_1, g_2, g_3, g_4)$. *Then we obtain*

$$\mathrm{C}q(g_1, g_2, g_3, g_4) \Leftrightarrow \angle(g_1, g_4) = \angle(g_2, g_3)$$

(cf. Figure 3.3 a)–c)).

From this we easily derive

3.4.7. *Assume* $\mathrm{C}q(g_1, \ldots, g_4)$ *and* $q(h_1, \ldots, h_4)$ *such that* $g_\nu \| h_\nu$ *for* $\nu = 1, \ldots, 4$. *Then we obtain* $\mathrm{C}q(h_1, \ldots, h_4)$ (Figure 3.3 d)).

3.4.8. FIRST THEOREM OF MIQUEL. *Assume* $\mathrm{C}q(g, h, k, l)$ *and* $q(g', h, k, l)$ *such that* $g \| g'$. *Then we obtain* $\mathrm{C}q(g', h, k, l)$ (Figure 3.4 a)–f)).

For the case that $g' \cap h$ and $k \cap l$ are points of a common singular line, this theorem, interpreted with respect to $\Pi(A)$, encloses all versions of the theorem of Pascal (cf. 2.4.2 and Figure 3.4 b)–f)).

3.4.9. SECOND THEOREM OF MIQUEL. *Assume*

$$\mathrm{C}q(g, g', h', h) \wedge \mathrm{C}q(h, h', k', k) \wedge q(g, g', k', k) \wedge g' \cap h' \cap k' \neq \varnothing.$$

Then we obtain $\mathrm{C}q(g, g', k', k)$ (Figure 3.4 g)).

By repeated applications of this theorem, one obtains the *theorem of Clifford* concerning n lines and n circles ($n \geqslant 4$; cf. Yaglom [1968]).

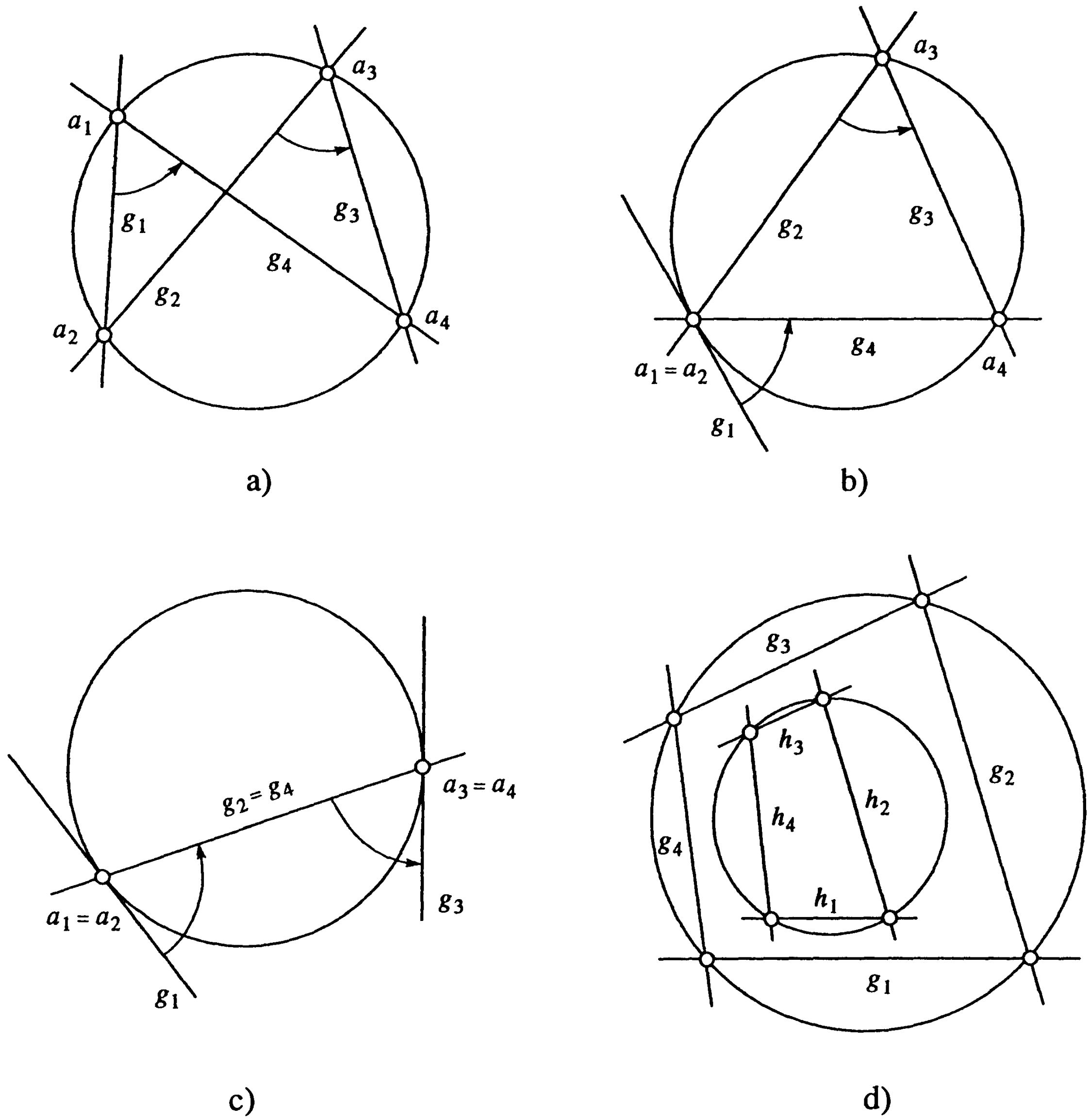

Figure 3.3.

3.4.10. Third theorem of Miquel. *For* $\nu \in \mathbb{Z}_4$ *assume*

$$\mathrm{C}q(f_\nu, g_\nu, h_\nu, g_{\nu-1});\ \mathrm{C}q(f_1, \ldots, f_4);\ q(h_1, \ldots, h_4);$$
$$f_{\nu+1} \cap f_\nu \cap g_\nu \neq \varnothing;\ h_{\nu+1} \cap h_\nu \cap g_\nu \neq \varnothing;\ g_1, \ldots, g_4 \in \mathcal{L}^r.$$

Then we obtain $\mathrm{C}q(h_1, \ldots, h_4)$ (Figure 3.4 h), i)).

As a further important statement we obtain

3.4.11. Secant segments theorem. *Assume*

$$q(g, h, k, l) \wedge g \nparallel k \wedge g \cap k \not\subseteq h \cup l.$$

Then, for $a \in l \cap g,\ b \in g \cap h,\ c \in h \cap k,\ d \in k \cap l,\ z \in g \cap k$, *we obtain* (Figure 3.5)

$(*)$ $\mathrm{C}q(g, h, k, l) \Leftrightarrow (a - z)\overline{(b - z)} = (c - z)\overline{(d - z)}$.

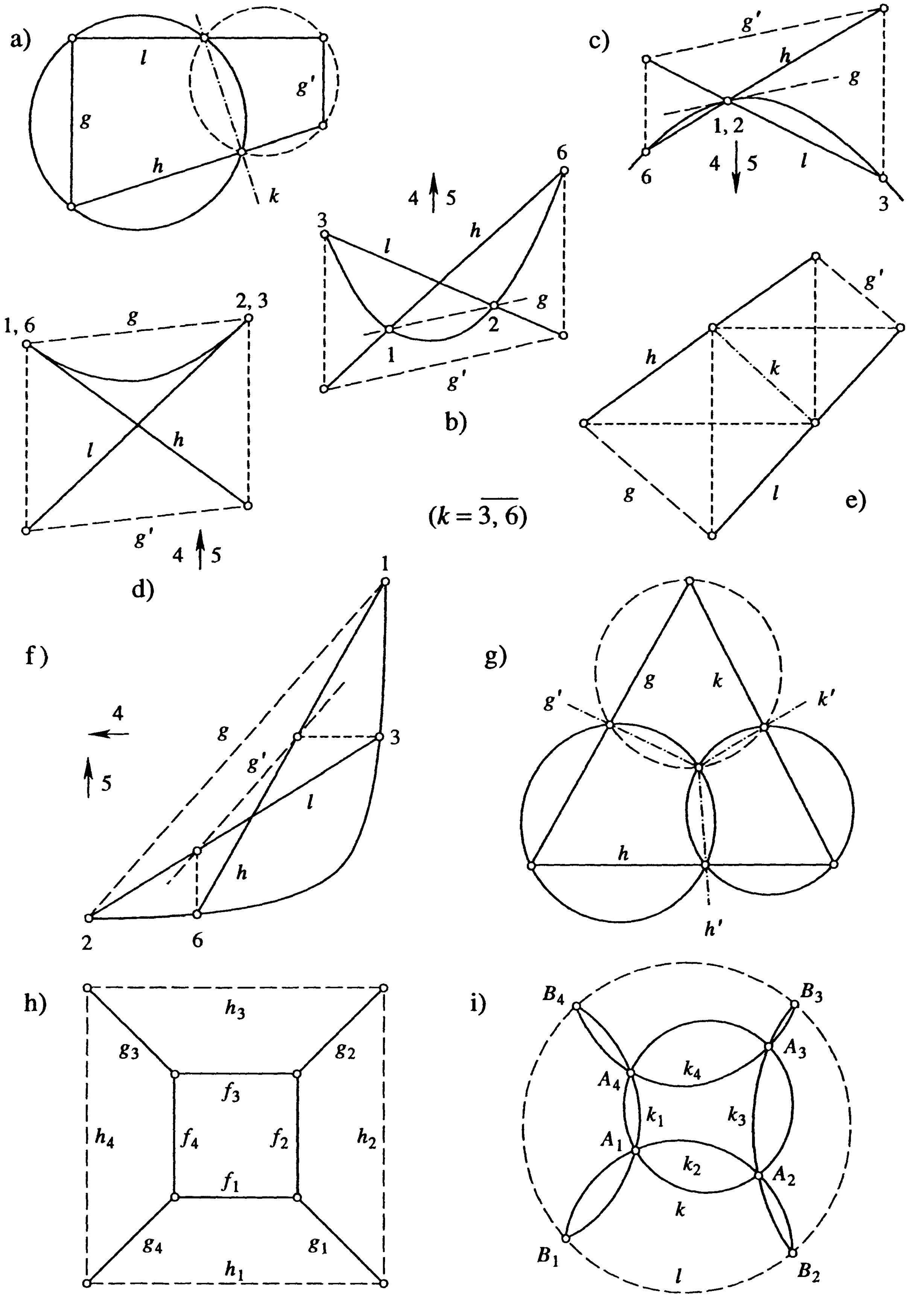

Figure 3.4. Theorems of Miquel.

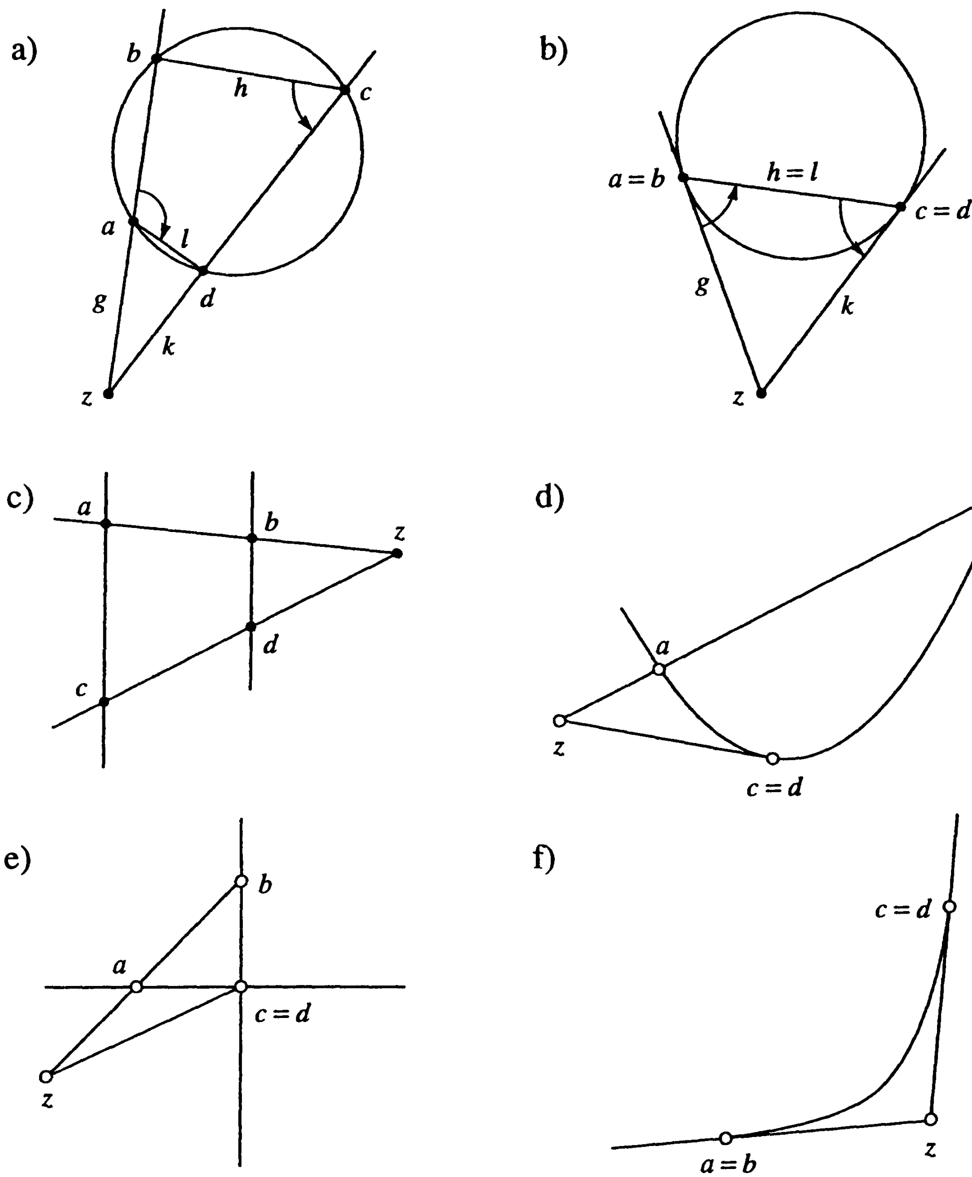

Figure 3.5.

3.4.12. As a corollary we get the

ISOSCELES TRIANGLE THEOREM. *Suppose, a, c, z are noncollinear points such that* $\overline{a,c} \in \mathcal{L}^r$. *Then we obtain*

$$d_Q(z,a) = d_Q(z,c) \Leftrightarrow \angle(\overline{a,c}, \overline{c,z}) = \angle(\overline{z,a}, \overline{a,c})$$

(cf. 3.1.3(a) and Figure 3.5).

3.4.13. By using 3.4.11, one obtains elementary proofs of the bundle theorems (cf. 4.4.7, Benz [1973] and Schröder [1974b, 1985]). Moreover, the basic properties of inversions can be proved.

3.4.14. According to 1.8.4 and 3.2.3, the automorphisms of $(\boldsymbol{A}, \widehat{-})$ are the mappings of the type

(1) $\xi\colon\ \mathbb{L} \to \mathbb{L},\ z \mapsto a \cdot z^{\sigma} + b \quad (a \in \dot{\mathbb{L}},\ b \in \mathbb{L}),$

where σ is a ring automorphism of $\mathbb{L}(+,\cdot)$ such that $\sigma(\mathbb{F}) = \mathbb{F}$. The mapping ξ is a similarity iff $\sigma|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}}$ and a motion iff $\sigma|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}}$ and $a\bar{a} = 1$.

The angle preserving collineations of $\boldsymbol{A}$ are the similarities of type

(2) $\varphi\colon\ \mathbb{L} \to \mathbb{L},\ z \mapsto az + b \quad (a \in \dot{\mathbb{L}},\ b \in \mathbb{L}),$

i.e. such a collineation φ is characterized by

(3) $\angle(\varphi(g), \varphi(h)) = \angle(g, h),\ \forall g, h \in \mathcal{L}.$

The angle reverting collineations of $\boldsymbol{A}$ are the similarities of type

(4) $\psi\colon\ \mathbb{L} \to \mathbb{L},\ z \mapsto a\bar{z} + b \quad (a \in \dot{\mathbb{L}},\ b \in \mathbb{L}),$

i.e. such a collineation ψ is characterized by

(5) $\angle(\psi(g), \psi(h)) = \angle(h, g),\ \forall g, h \in \mathcal{L}.$

Let $\boldsymbol{E}^+$ be the set of angle preserving collineations of $\boldsymbol{A}$, and $\boldsymbol{E}^-$ the set of its angle reverting collineations. Then $\boldsymbol{E}^+(\circ)$ and $(\boldsymbol{E}^+ \cup \boldsymbol{E}^-)(\circ)$ are groups, and we obtain the following

TRANSITIVITY THEOREM. *Suppose $a, b, c, d \in \mathbb{L}$ such that $a \neq b \wedge c \neq d \wedge \overline{a, b}, \overline{c, d} \in \mathcal{L}^r$. Then, there exists exactly one element $\varphi \in \boldsymbol{E}^+$ such that $\varphi(a) = c \wedge \varphi(b) = d$ and exactly one element $\psi \in \boldsymbol{E}^-$ such that $\psi(a) = c \wedge \psi(b) = d$.*

3.5. *Geometric characterizations of affine metric spaces*

3.5.1. The beauty of Euclid's approach to geometry has fascinated geometers through the ages. Therefore, till this day they try to develop intrinsic geometrical characterizations of geometric structures and objects and to find axiomatic characterizations which are as natural as possible.

Since Euclid's work contains gaps, as Pasch [1882] pointed out, mathematicians looked for improvements of Euclid's approach.

In his famous book 'Grundlagen der Geometrie', Hilbert [1899] proposed a system of 20 axioms for the Euclidean $\mathbb{R}^3$, which is consistent with Euclid's outlook but correct by present-day standards. Moreover, Hilbert's approach delivers a common axiomatic basis for Euclidean and hyperbolic geometry and thus shows, how different geometric structures may be characterized by a common system of axioms.

There have been attempts to improve Hilbert's system of axioms, which are based on a description of incidence, betweenness and congruence. Among others, Schwabhäuser and Szmielew and Tarski [1983] establish two systems of 11 axioms, which describe the Euclidean $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, by betweenness and equidistance, and Schröder [1985] introduces two systems of 7 axioms for the Euclidean $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, which are based on a description of equidistance.

The systems of Schröder [1985], which use ideas of Baer [1944] and Benz [1980], contain subsystems of 5 axioms, which characterize all Euclidean spaces of dimension 2 and 3, respectively, of characteristic $\neq 2$, including the finite cases.

A geometric characterization of all separable Euclidean planes by 6 independent axioms based on equidistance was introduced by Schnabel [1981], and Schaeffer [1979] characterized the class of all Euclidean planes by means of circle geometry, including the inseparable cases. In [1979b], Schröder described all regular Euclidean spaces of dimension $\geqslant 2$ within the frame of affine spaces by imposing a symmetry condition on an arbitrary set of affine reflections at hyperplanes.

3.5.2. There exist a lot of characterizations of certain classes of metric geometries. Let us mention here the approach by using reflection geometry in the sense of F. Bachmann (cf. Bachmann [1937, 1973], Ewald [1974], Karzel [1955a,b, 1958a,b, 1971], Karzel and Kist [1979], Lingenberg [1966, 1967, 1979b], Nolte [1966, 1974, 1979], Ott [1971], Quaisser [1985], Schröder [1984a,b], Sörensen [1986], Sperner [1954], Wolff [1967] and the references in Bachmann [1973]), the approach by using circle geometry in the sense of W. Benz (cf. Benz [1973], Dienst [1977], Leissner [1974], Mäurer [1965, 1968], Quaisser [1973], Samaga [1991a,b], Schaeffer [1974b,c], Schröder [1980, 1981b] and the references in Benz [1973]), the approach by characterizing a *congruence relation* in the sense of D. Hilbert (cf. Hilbert [1899], Baer [1944], Karzel [1975, 1985], Karzel and König [1981], Karzel, Sörensen and Windelberg [1973], Karzel and Stanik [1979], Kroll and Sörensen [1976], Sörensen [1975, 1984], Schröder [1981a, 1985, 1986b], Schnabel [1981], Wähling [1977]), and the approach by characterizing an *orthogonality relation* in the sense of H. Lenz (cf. Baer [1944], Lenz [1954a,b, 1962b], Rautenberg and Quaisser [1969], Quaisser [1970], Smith [1973]).

Obviously, each of these approaches has its special advantages and seems to be appropriate for a characterization of certain classes of metric geometries. But up until now, no common natural characterization for all metric geometries has been developed.

For example, the reflection geometry developed by Bachmann [1973] yields a common characterization of Euclidean, elliptic and hyperbolic planes of characteristic $\neq 2$. This approach and its far-reaching generalizations will be discussed in more detail in Section 4.6. Here, we only mention that Karzel [1955a,b] showed how geometries of characteristic 2 can be characterized by means of reflection geometry, and that Lingenberg [1959a] and Ott [1971] showed, respectively, how to include the Minkowskian planes and the separable Galilean planes. A special difficulty in such considerations is the proof of Desargues' theorem.

3.5.3. The approach via an orthogonality relation in the sense of H. Lenz admits characterizations of affine and projective metric structures of arbitrary dimension, as far as the characteristic 2 is excluded (cf. Lenz [1954a,b, 1962b]).

An approach which is appropriate for affine metric spaces of characteristic $\neq 2$ is introduced by Schröder [1981a]:

Assume $\boldsymbol{A} := (\mathcal{P}, \mathcal{L}, \|)$ is an affine space which contains a parallelogram whose diagonals intersect, and in case of $\dim \boldsymbol{A} = 2$ suppose that $\boldsymbol{A}$ is a translation plane. Furthermore, let $\equiv$ be an equivalence relation on $\mathcal{P} \times \mathcal{P}$ such that $(a, a) \equiv (b, b)$ holds

$\forall a, b \in \mathcal{P}$, and call $\chi_{a,b} := \{x \in \mathcal{P}\colon (a,x) \equiv (x,b)\}$ for $a, b \in \mathcal{P}$ the *perpendicular bisector* of (a,b) with respect to the *congruence relation* $\equiv$.

Then, $(\boldsymbol{A}, \equiv)$ is an affine metric space which contains a regular plane (cf. 3.1.3 (b) and 3.2.5), iff the following three conditions are fulfilled:

(PG) If (a,b,c,d) is a parallelogram, then $(a,b) \equiv (c,d)$.

(H) For $a, b \in \mathcal{P}, \chi_{a,b}$ is a hyperplane, or $\chi_{a,b} = \mathcal{P}$.

(R) There are points a, b, c such that $\chi_{a,b} \nparallel \chi_{b,c}$.

3.5.4. There exist geometric characterizations of the whole class of affine metric spaces. In order to show this, the following theorem, which reduces the problem from arbitrary dimensions to a problem of dimension two, is crucial (cf. 1.7.8, 3.2.6 and Schröder [1981b]):

THEOREM. *Assume $\boldsymbol{A} = (\mathcal{P}, \mathcal{L}, \|)$ is a Pappian affine space of dimension $\geqslant 3$. Let $\mathcal{L}^r$ be a subset of $\mathcal{L}$ and let $\boldsymbol{C}$ consist of subsets of $\mathcal{P}$ such that $\overline{k}$ is a plane for each element k of $\boldsymbol{C}$. If for every plane ε of $\boldsymbol{A}$ a metric is defined such that $\{k \in \boldsymbol{C}\colon k \subseteq \varepsilon\}$ and $\{l \in \mathcal{L}^r\colon l \subseteq \varepsilon\}$ are the corresponding sets of circles and regular lines, then there exists a metric on $\boldsymbol{A}$ such that $\boldsymbol{C}$ and $\mathcal{L}^r$ are the corresponding sets of circles and regular lines, iff the metric structures of the planes are compatible according to the following condition*:

(B) *Suppose $k, k' \in \boldsymbol{C}$ and $g, h, h' \in \mathcal{L}^r$ such that*

$$\dim(\overline{g \cup h \cup h'}) = 3 \wedge g \cap k = g \cap k' \neq \varnothing$$
$$\wedge (g \cap h \cap h' \neq \varnothing \vee g\|h\|h') \wedge |h \cap k| = |h' \cap k'| = 2.$$

Then, there exists an element in $\boldsymbol{C}$ which contains $(h \cap k) \cup (h' \cap k')$.

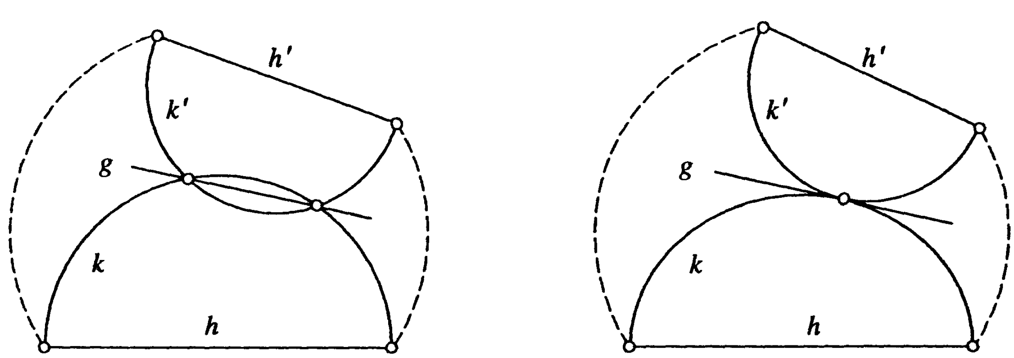

Figure 3.6. Axiom (B); cf. Figure 4.1.

By using this statement and results obtained by Schaeffer [1979] and Schröder [1980], Schröder [1981b] characterizes the affine metric spaces by conditions on an equivalence relation $\widehat{-}$, which is defined between oriented angles (cf. 3.1.3).

4. Projective metric geometry

4.1. *Isomorphisms of projective metric geometry*

4.1.1. If $(V, \mathbb{F}, Q)$ is a metric vector space, then the projective space $\Pi = \Pi(V, \mathbb{F}) = (\boldsymbol{P}, \boldsymbol{L})$ is furnished with a *polarity* $\widetilde{\perp}$, with a set of *symmetries* $\widetilde{\boldsymbol{R}}_Q$, with a metric product μ_Q and with a *quadric* $\boldsymbol{F}_Q$ induced by Q (cf. 2.1.1–2 and 2.3.2).

Moreover, according to 3.1.3(i), we may introduce a *Q-distance* (*Cayley-distance*)

(1) $\delta_Q(\mathbb{F}a, \mathbb{F}b) := (f(a,b))^2 \cdot (Q(a))^{-1} \cdot (Q(b))^{-1}$

for regular points $\mathbb{F}a, \mathbb{F}b \in \boldsymbol{R}_Q$ $(a, b \in V \backslash \ker Q)$, where f is the bilinear form associated with Q, and we can define a *Cayley-relation* $\sim_Q$ on $\boldsymbol{R}_Q \times \boldsymbol{R}_Q$ by

(2) $(A,B) \sim_Q (C,D) :\Leftrightarrow \delta_Q(A,B) = \delta_Q(C,D)$ for $A,B,C,D \in \boldsymbol{R}_Q$.

Thus, Π has several metric concepts which depend on Q. These concepts do not change if Q is replaced by λQ, where $\lambda \in \dot{\mathbb{F}}$.

On the other hand, from 1.7.5 we deduce that Q is fixed by μ_Q up to geometric equivalence.

Therefore, the pair (Π, μ_Q) is called a *projective metric space.*

We call (Π, μ_Q) *regular, irregular, singular,* provided Q is of the corresponding type (cf. 1.2.2), and we call char $\mathbb{F}$ the *characteristic* of Π. For historical reasons, (Π, μ_Q) is defined to be *elliptic* iff Q is Euclidean (i.e., iff $\boldsymbol{F}_Q$ is empty), and *hyperbolic* iff Q is Minkowskian (i.e., iff $\boldsymbol{F}_Q$ is an ovoid).

For the following, let (Π, μ_Q) and $(\Pi', \mu_{Q'})$ be projective metric spaces which correspond to the metric vector spaces $(V, \mathbb{F}, Q)$ and $(V', \mathbb{F}', Q')$, where $\Pi = (\boldsymbol{P}, \boldsymbol{L})$ and $\Pi' = (\boldsymbol{P}', \boldsymbol{L}')$.

4.1.2. A collineation φ of Π onto Π' is called an *isomorphism* of (Π, μ_Q) onto $(\Pi', \mu_{Q'})$ and in the case $(V, \mathbb{F}, Q) = (V', \mathbb{F}', Q')$ an *automorphism* of (Π, μ_Q), iff the conditions $\varphi(\boldsymbol{R}_Q) = \boldsymbol{R}_{Q'}$ and

(∗) $\varphi(\mu_Q(A,B,C)) = \mu_{Q'}(\varphi(A), \varphi(B), \varphi(C)), \ \forall (A,B,C) \in {}^3\boldsymbol{R}_Q$

are fulfilled (cf. 2.3.2).

From the fundamental theorem of projective geometry (cf. Brauner [1976]) and from Schröder [1986a] we deduce

4.1.3. First fundamental theorem of projective metric geometry. *Assume* $\dim V \geqslant 3$. *Then, a bijection* φ: $\boldsymbol{P} \to \boldsymbol{P}'$ *is an isomorphism of* (Π, μ_Q) *onto* $(\Pi', \mu_{Q'})$, *iff there exists a* ρ*-linear isomorphism* σ *of* $(V, \mathbb{F}, Q)$ *onto* $(V', \mathbb{F}', Q')$ *such that* $\varphi(\mathbb{F}x) = \mathbb{F}'\sigma(x)$, $\forall x \in V \setminus \{O\}$ (cf. 1.5.3).

4.1.4. Suppose $\dim V \geqslant 3$. If φ is an isomorphism of (Π, μ_Q) onto $(\Pi', \mu_{Q'})$ which is represented according to 4.1.3, the statements 3.2.3–4, 4.1.2–3 lead to

(1) $\varphi(\boldsymbol{F}_Q) = \boldsymbol{F}_{Q'}$,

(2) $A \,\widetilde{\perp}\, B \Leftrightarrow \varphi(A) \,\widetilde{\perp}'\, \varphi(B), \ \forall A, B \in \boldsymbol{P}$,

(3) $\varphi \circ \widetilde{A} \circ \varphi^{-1} = (\widetilde{\varphi}(A))^{\sim}, \ \forall A \in \boldsymbol{R}_Q$,

(4) $\varphi \circ \widetilde{\boldsymbol{R}}_Q \circ \varphi^{-1} = \widetilde{\boldsymbol{R}}_{Q'}$,

(5) $\rho(\delta_Q(A,B)) = \delta_{Q'}(\varphi(A),\varphi(B))$, $\forall A,B \in \boldsymbol{R}_Q$,
(6) $(A,B)\sim_Q(C,D) \Leftrightarrow (\varphi(A),\varphi(B))\sim_{Q'}(\varphi(C),\varphi(D))$, $\forall A,B,C,D \in \boldsymbol{P}$.

The isomorphism φ is called a *Q-projectivity*, provided σ is an equimetry, and a *Q-motion* provided σ is an isometry.

Observe that the results of the preceding chapters on equimetries, isometries, Q-symmetries, similarities, and motions apply to Q-projectivities and Q-motions as far as these mappings leave the zero vector O of V fixed.

The following statement shows, under which additional assumptions the metric structure of (Π,μ_Q) is fixed by one of the sets $\boldsymbol{R}_Q, \widetilde{\boldsymbol{R}}_Q$ or by one of the relations $\widetilde{\perp}$, $\sim_Q$. In each such case, the structure (Π,μ_Q) may as well be given in the form $(\Pi,\boldsymbol{R}_Q)$, $(\Pi,\widetilde{\boldsymbol{R}}_Q)$, $(\Pi,\widetilde{\perp})$, $(\Pi,\sim_Q)$ (cf. Schröder [1986a]):

4.1.5. SECOND FUNDAMENTAL THEOREM OF PROJECTIVE METRIC GEOMETRY. *Assume* $\dim V \geqslant 3$ *and let* φ *be a collineation of* Π *onto* Π'. *Then,* φ *is an isomorphism of* (Π,μ_Q) *onto* $(\Pi',\mu_{Q'})$ *if (at least) one of the following conditions is fulfilled*:

(1) $\boldsymbol{F}_Q$ *is a proper quadric or a hyperplane, and* $\varphi(\boldsymbol{R}_Q) = \boldsymbol{R}_{Q'}$.
(2) $\boldsymbol{F}_Q$ *is a proper quadric or a hyperplane, and* $\varphi(\boldsymbol{F}_Q) = \boldsymbol{F}_{Q'}$.
(3) $\operatorname{char}\mathbb{F} \neq 2$ *and* $(A \mathbin{\widetilde{\perp}} B \Leftrightarrow \varphi(A) \mathbin{\widetilde{\perp}'} \varphi(B)$, $\forall A,B \in \boldsymbol{P})$.
(4) $V^{\perp} \neq V \wedge \varphi \circ \widetilde{\boldsymbol{R}}_Q \circ \varphi^{-1} = \widetilde{\boldsymbol{R}}_{Q'}$.
(5) $V^{\perp} \neq V \wedge$
$((A,B)\sim_Q(C,D) \Leftrightarrow (\varphi(A),\varphi(B))\sim_{Q'}(\varphi(C),\varphi(D))$, $\forall A,B,C,D \in \boldsymbol{P})$.

4.2. *Quadric circle geometry*

While Section 2 deals with quadrics considered as part of a projective space, here we want to look at the 'inner' geometry of a quadric.

For this purpose let $(V,\mathbb{F},Q)$, $(V',\mathbb{F}',Q')$ be metric vector spaces of dimension $\geqslant 4$ such that the quadrics $\boldsymbol{F}_Q, \boldsymbol{F}_{Q'}$ are proper and not the union of hyperplanes. Let $\boldsymbol{P},\boldsymbol{P}'$; $\boldsymbol{L},\boldsymbol{L}'$; $\boldsymbol{E},\boldsymbol{E}'$ be the sets of points, lines and planes respectively of the projective spaces $\Pi := \Pi(V,\mathbb{F})$, $\Pi' := (V',\mathbb{F}')$.

4.2.1. We shall consider the subsets
(1) $\dot{\boldsymbol{F}}_Q := \boldsymbol{F}_Q \setminus \boldsymbol{D}_Q$, $\dot{\boldsymbol{F}}_{Q'} = \boldsymbol{F}_{Q'} \setminus \boldsymbol{D}_{Q'}$
of $\boldsymbol{P}$, $\boldsymbol{P}'$ respectively, where $\boldsymbol{D}_Q := (V^{\perp} \cap \ker Q)^{\pi}$, $\boldsymbol{D}_{Q'} := (V'^{\perp'} \cap \ker Q')^{\pi}$ is the set of double points of the quadric $\boldsymbol{F}_Q$, $\boldsymbol{F}_{Q'}$ respectively (cf. 1.3.4, 2.1.3, 2.3.1).

The point set $\boldsymbol{D}_Q$ is a projective subspace $\neq \boldsymbol{P}$ of Π, because $\boldsymbol{F}_Q$ is a proper quadric (cf. 2.1.5), and we obtain $\boldsymbol{D}_Q \subseteq \tau_A$, $\forall A \in \dot{\boldsymbol{F}}_Q$, where $\tau_A = (A^{\perp})^{\pi}$ is the tangent hyperplane of $\boldsymbol{F}_Q$ at A (cf. 2.3.1). Moreover, for $A \in \dot{\boldsymbol{F}}_Q$, every line l of Π which passes through A and is not contained in τ_A, is a secant of $\boldsymbol{F}_Q$ such that $l \cap \boldsymbol{F}_Q = l \cap \dot{\boldsymbol{F}}_Q$ (cf. 2.1.3).

For any plane ε of Π we call $\varepsilon \cap \dot{\boldsymbol{F}}_Q$ a *proper circle* of $\dot{\boldsymbol{F}}_Q$, provided

$$\langle \varepsilon \cap \dot{\boldsymbol{F}}_Q \rangle = \varepsilon \wedge \varepsilon \nsubseteq \boldsymbol{F}_Q$$

holds. Let C^Q be the set of all proper circles of $\dot{F}_Q$.

The proper circle $\varepsilon \cap \dot{F}_Q$ is called *singular* provided $\varepsilon \cap F_Q$ is the union of two lines. Otherwise, $\varepsilon \cap F_Q = \varepsilon \cap \dot{F}_Q$ is an oval quadric in ε, and then $\varepsilon \cap F_Q$ is called a *regular* circle (cf. 2.1.5).

Suppose δ, ε are planes of Π such that $k := \delta \cap \dot{F}_Q$ and $l := \varepsilon \cap \dot{F}_Q$ are proper circles which contain a common point $A \in \dot{F}_Q$. Then we say that *k and l touch at A*, written $k, l|_Q A$, provided $k = l$ or $\{A\} \neq \delta \cap \varepsilon \subseteq \tau_A$. The relation $|_Q$ thus defined on $C^Q \times C^Q \times \dot{F}_Q$ is called the *touch relation* corresponding to Q.

Observe that a proper circle k is singular iff there exist a proper circle l and a point $A \in \dot{F}_Q$ such that $k, l|_Q A$, $k \neq l$ and $|k \cap l| \geqslant 2$.

We call $M(Q) := (\dot{F}_Q, C^Q, |_Q)$ the *quadric circle geometry* associated with Q.

Clearly, the results and definitions concerning $\dot{F}_Q$ hold for $\dot{F}_{Q'}$ analogously.

4.2.2. Assume $M(Q)$ and $M(Q')$ are quadric circle geometries and φ: $\dot{F}_Q \to \dot{F}_{Q'}$ is a bijection. Then, φ is called an *isomorphism* of $M(Q)$ onto $M(Q')$ and in case of $(V, \mathbb{F}, Q) = (V', \mathbb{F}', Q')$ an *automorphism* of $M(Q)$, provided the conditions

(1) $\varphi(C^Q) = C^{Q'}$,

(2) $k, l|_Q A \Leftrightarrow \varphi(k), \varphi(l)|_{Q'} \varphi(A)$, $\forall (k, l, A) \in C^Q \times C^Q \times \dot{F}_Q$,

are fulfilled.

It turns out that $|_Q$ is determined by C^Q if $|\mathbb{F}| \geqslant 3$ and that ((1) $\Rightarrow$ (2)) holds if $|\mathbb{F}| \geqslant 3$. Therefore, the relations $|_Q$, $|_{Q'}$ are introduced in this context in order to include the case $|\mathbb{F}| = |\mathbb{F}'| = 2$.

Because singular circles differ from regular circles with respect to the touch relation, proper circles are always mapped onto proper circles of the same type by isomorphisms.

The close relation between a quadric circle geometry and the 'surrounding' projective metric geometry becomes obvious by the following statement (cf. Schröder [1986a]):

4.2.3. FUNDAMENTAL THEOREM OF QUADRIC CIRCLE GEOMETRY. *Assume* Iso_π *is the set of isomorphisms of* (Π, μ_Q) *onto* $(\Pi', \mu_{Q'})$ *and* Iso_M *is the set of isomorphisms of* $M(Q)$ *onto* $M(Q')$. *Then, in case of* $\mathrm{Iso}_\pi \neq \varnothing$ *or* $\mathrm{Iso}_M \neq \varnothing$, *we obtain*

(1) *The mapping* χ: $\mathrm{Iso}_\pi \to \mathrm{Iso}_M$, $\xi \mapsto \xi|_{\dot{F}_Q}$, *is a bijection.*

(2) Iso_M *consists of the mappings of the type* $\dot{F}_Q \to \dot{F}_{Q'}$, $\mathbb{F}x \mapsto \mathbb{F}'\sigma(x)$, *where* σ *is an isomorphism of* $(V, \mathbb{F}, Q)$ *onto* $(V', \mathbb{F}', Q')$.

As a corollary we obtain

4.2.4. *Let* $\mathrm{Aut}_\pi(\circ)$, $\mathrm{Aut}_M(\circ)$ *be the automorphism groups of* (Π, μ_Q), $M(Q)$ *respectively. Then,* χ: $\mathrm{Aut}_\pi \to \mathrm{Aut}_M$, $\xi \mapsto \xi|_{\dot{F}_Q}$, *is a group isomorphism.*

4.3. *The stereographic projection*

As was pointed out in 3.1.2 and 3.1.3 (e)–(g), the projective metric geometry which belongs to a metric vector space $(V, \mathbb{F}, Q)$ may be interpreted as the 'geometry at infinity' of the corresponding affine metric space. But a lot of metric 'information' is lost if one changes from the affine geometry to its geometry 'at infinity'.

Therefore it seems worthwhile to consider a change from affine metric geometry to projective metric geometry, where no 'information' is lost. This can be done by stereographic projection, which also yields a close relation between affine metric geometry and quadric circle geometry and thus, according to 4.2.3, between affine metric geometry and projective metric geometry.

In the following, let $(V, \mathbb{F}, Q)$ be a nonsingular metric vector space of dimension $\geqslant 2$ and let $(\boldsymbol{A}, \widehat{-}_Q)$ be the corresponding affine metric space according to 3.1.3.

We will use the notations of Section 3. In particular, $\mathcal{C}_Q$ is the set of Q-circles of $\boldsymbol{A}$.

4.3.1. Beside the vector space V, consider the vector space $\overline{V} := V \times \mathbb{F} \times \mathbb{F}$ over $\mathbb{F}$, consisting of the elements $(z; \xi, \eta)$ such that $z \in V$ and $\xi, \eta \in \mathbb{F}$, and consider the quadratic form

(1) $\overline{Q}\colon \overline{V} \to \mathbb{F},\ (z; \xi, \eta) \mapsto Q(z) - \xi\eta,$

which is associated with the bilinear form

(2) $\overline{f}\colon \overline{V} \times \overline{V} \to \mathbb{F},\ ((z; \xi, \eta), (a; \alpha, \beta)) \mapsto f(z, a) - \xi\beta - \eta\alpha$

(cf. 1.9.1). In the projective metric space $\overline{\Pi} := \Pi(\overline{V}, \mathbb{F}) = (\overline{\boldsymbol{P}}, \overline{\boldsymbol{L}})$, the quadric

(3) $\boldsymbol{F} := \boldsymbol{F}_{\overline{Q}} = \{\mathbb{F}(z; \xi, \eta)\colon\ Q(z) = \xi\eta\}$

is proper and not the union of hyperplanes. Thus, the quadric circle geometry $\boldsymbol{M}(\overline{Q}) = (\dot{\boldsymbol{F}}, \boldsymbol{C}^{\overline{Q}}, |_{\overline{Q}})$, is defined according to 4.2.1.

Let us consider now the mapping

(4) $^*\colon V \to \dot{\boldsymbol{F}},\ z \mapsto z^* := \mathbb{F}(z; Q(z), 1),$

which obviously is injective such that $V^* := \{z^*\colon\ z \in V\} = \boldsymbol{F} \setminus \tau_N$ for $N := \mathbb{F}(O; 1, 0)$ and $\tau_N := (N^\perp)^\pi$. The inverse mapping

(5) $_*\colon V^* \to V,\ z^* \mapsto z \quad (z \in V),$

is called the *stereographic projection* of $\dot{\boldsymbol{F}}$ onto V with respect to the *centre* N.

If we use the identification $z = \mathbb{F}(z; 0, 1)$ for $z \in V$, then $\boldsymbol{A}$ is contained in the hyperplane τ_S for $S := \mathbb{F}(O; 0, 1)$, such that $V = \tau_S \setminus \tau_N$, and then the points z, z^*, N are collinear in $\overline{\Pi}$ for all $z \in V$ (Figure 4.1; this explains the name 'stereographic projection').

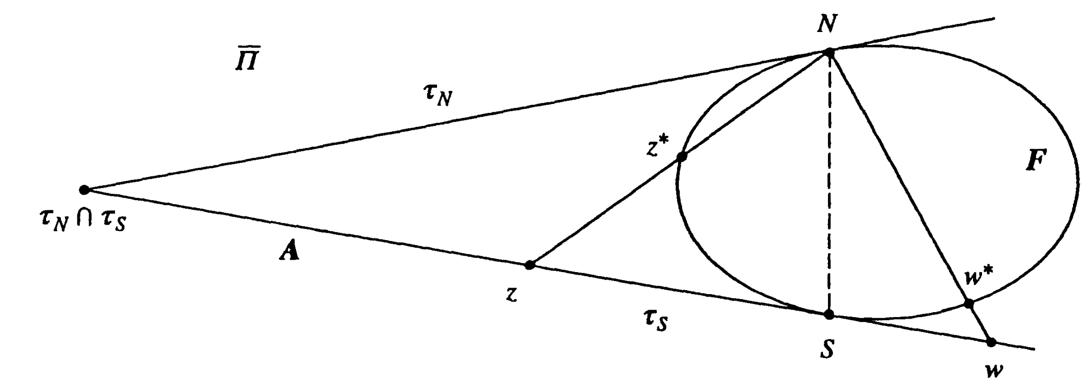

Figure 4.1.

4.3.2. Suppose M is an affinely independent subset of V consisting of at least two points (cf. 3.1.1). Then, if $\langle\ \rangle$ is the hull operator in $\overline{\Pi}$, we obtain

(1) $M^* := \{z^*\colon\ z \in M\}$ is an independent point set in $\overline{\Pi}$, and $N \notin \langle M^*\rangle$.

(2) The Q-sphere $s(M)$ (cf. 3.1.3 (h)) is mapped by * onto a '*section*' of V^*:

$$(s(M))^* = V^* \cap \langle M^*\rangle = (\boldsymbol{F} \cap \langle M^*\rangle) \setminus \tau_N.$$

(3) Assume the affine hull $\overline{M}$ of M is a singular affine subspace of $(\boldsymbol{A}, \widehat{-}_Q)$. Then we have

$$\langle M^*\rangle \subseteq \boldsymbol{F} \quad \text{and} \quad \overline{M}^* = \langle M^*\rangle \setminus \tau_N = \langle M^* \cup \{N\}\rangle \setminus \tau_N.$$

(4) Assume $\overline{M}$ is a nonsingular affine subspace of $(\boldsymbol{A}, \widehat{-}_Q)$. Then we have

$$\langle M^*\rangle \not\subseteq \boldsymbol{F} \quad \text{and} \quad \overline{M}^* = V^* \cap \langle M^* \cup \{N\}\rangle.$$

4.3.3. Suppose M' is an independent subset of V^* (with respect to $\overline{\Pi}$) such that $|M'| \geqslant 2$ and $\langle M'\rangle \not\ni N$. Then we obtain

(1) $M := {}_*(M')$, as defined in 4.3.1 (5), is an affinely independent subset of V.

(2) Assume $\langle M'\rangle \subseteq \boldsymbol{F}$. Then, $\overline{M}$ is a singular affine subspace of $(\boldsymbol{A}, \widehat{-}_Q)$. Moreover, $\langle M'\rangle \setminus \tau_N = \langle M' \cup \{N\}\rangle \setminus \tau_N = \overline{M}^*$.

(3) Assume $\langle M'\rangle \not\subseteq F$. Then, $\overline{M}$ is a nonsingular affine subspace of $(A, \widehat{-}_Q)$. Moreover, $\langle M'\rangle \cap V^* = (s(M))^*$ and $\langle M' \cup \{N\}\rangle \cap V^* = \overline{M}^*$.

4.3.4. Consider now the quadric circle geometry $\boldsymbol{M}(\overline{Q}) = (\dot{F}, \boldsymbol{C}^{\overline{Q}}, |_{\overline{Q}})$. Because of 4.3.1–3, the mapping * may be interpreted as an embedding of the affine metric space $(\boldsymbol{A}, \widehat{-}_Q)$ into the quadric circle geometry $\boldsymbol{M}(\overline{Q})$ such that the following properties hold (cf. Benz [1973] and Schröder [1986a]):

(1) Assume l is a line of $\boldsymbol{A}$. Then there exists one and only one proper circle $\widehat{l} \in \boldsymbol{C}^{\overline{Q}}$ such that

$$N \in \widehat{l} \wedge \tau_N \not\supseteq \widehat{l} \wedge \widehat{l} \cap V^* = l^*.$$

The circle $\widehat{l}$ is singular (regular) iff l is singular (regular).

(2) If r is a proper circle of $\boldsymbol{M}(\overline{Q})$ such that $N \in r \wedge \tau_N \not\supseteq r$, then there exists one and only one line l of A such that $\widehat{l} = r$.

(3) For arbitrary lines l, m of $\boldsymbol{A}$ we have

$$l \parallel m \Leftrightarrow \widehat{l}, \widehat{m} |_{\overline{Q}} N.$$

(4) Assume k is a proper circle of $(\boldsymbol{A}, \widehat{-}_Q)$. Then there exists one and only one proper circle $\widehat{k} \in \boldsymbol{C}^{\overline{Q}}$ such that

$$N \notin \langle\widehat{k}\rangle \wedge \tau_N \not\supseteq \widehat{k} \wedge \widehat{k} \cap V^* = k^*.$$

The circle $\widehat{k}$ is singular (regular) iff k is singular (regular).

(5) If s is a proper circle of $\boldsymbol{M}(\overline{Q})$ such that $N \notin \langle s\rangle \wedge \tau_N \not\supseteq s$, then there exists one and only one proper circle k of $\boldsymbol{A}$ such that $\widehat{k} = s$.

The following statements show that the embedding is complete in the sense that every isomorphism between affine metric structures may be extended to an isomorphism between the corresponding projective metric spaces and thus to an isomorphism between the corresponding quadric circle geometries (cf. 4.2.3):

4.3.5. For $a \in V$, consider the translation

(1) $\varphi\colon V \to V,\ z \mapsto z + a.$

Then it is easily verified that

(2) $\Phi\colon \overline{V} \to \overline{V},\ (z;\xi,\eta) \mapsto (z + \eta a; \xi + f(z,a) + \eta Q(a), \eta),$

is an isometry of $(\overline{V}, \mathbb{F}, \overline{Q})$ which induces a $\overline{Q}$-motion

(3) $\widehat{\varphi}\colon \overline{\boldsymbol{P}} \to \overline{\boldsymbol{P}},\ \mathbb{F}X \mapsto \mathbb{F}\Phi(X),$

of $(\overline{\Pi}, \mu_{\overline{Q}})$ such that

(4) $\widehat{\varphi}(\boldsymbol{F}) = \boldsymbol{F} \wedge \widehat{\varphi}(N) = N \wedge \widehat{\varphi}(z^*) = (\varphi(z))^*,\ \forall z \in V.$

Beside $(V, \mathbb{F}, Q)$, consider now a nonsingular metric vector space $(V', \mathbb{F}', Q')$ of dimension $\geqslant 2$. The notations introduced in 4.3.1 hold *mutatis mutandis* for $(V', \mathbb{F}', Q')$ and will be used with a dash for $(V', \mathbb{F}', Q')$:

4.3.6. Assume ψ is a ρ-linear isomorphism of $(V, \mathbb{F}, Q)$ onto $(V', \mathbb{F}', Q')$. Then, there exists an element $\lambda \in \mathbb{F}' \setminus \{0\}$ such that $\lambda \cdot \rho \circ Q = Q' \circ \psi$, and the mapping

(1) $\Psi\colon \overline{V} \to \overline{V}',\ (z;\xi,\eta) \mapsto (\psi(z); \lambda \cdot \rho(\xi), \rho(\eta)),$

is a ρ-linear isomorphism of $(\overline{V}, \mathbb{F}, \overline{Q})$ onto $(\overline{V'}, \mathbb{F}', \overline{Q'})$ which induces an isomorphism

(2) $\widehat{\psi}\colon \overline{\boldsymbol{P}} \to \overline{\boldsymbol{P}},\ \mathbb{F}x \mapsto \mathbb{F}\Psi(x),$

of $(\overline{\Pi}, \mu_{\overline{Q}})$ onto $(\overline{\Pi}', \mu_{\overline{Q'}})$ such that

(3) $\widehat{\psi}(\boldsymbol{F}) = \boldsymbol{F}' \wedge \widehat{\psi}(N) = N' \wedge \widehat{\psi}(z^*) = (\psi(z))^*,\ \forall z \in V.$

By using 4.3.5–6, one obtains

4.3.7. FUNDAMENTAL THEOREM ON STEREOGRAPHIC PROJECTION. *If φ is an isomorphism of $(\boldsymbol{A}, \widehat{-}_Q)$ onto $(\boldsymbol{A}', \widehat{-}_{Q'})$, then there exists one and only one isomorphism $\widehat{\varphi}$ of $(\overline{\Pi}, \mu_{\overline{Q}})$ onto $(\overline{\Pi}', \mu_{\overline{Q}'})$ such that*

$$\widehat{\varphi}(\boldsymbol{F}) = \boldsymbol{F}' \wedge \widehat{\varphi}(N) = N' \wedge \widehat{\varphi}(z^*) = (\varphi(z))^*,\ \forall z \in V.$$

The mapping $\widehat{\varphi}$ is a $\overline{Q}$-projectivity (a $\overline{Q}$-motion) iff φ is a similarity (a motion).

If ζ is an isomorphism of $(\overline{\Pi}, \mu_{\overline{Q}})$ onto $(\overline{\Pi}', \mu_{\overline{Q}'})$ such that $\zeta(N) = N'$, then there exists an isomorphism ψ of $(\boldsymbol{A}, \widehat{-}_Q)$ onto $(\boldsymbol{A}', \widehat{-}_{Q'})$ such that $\zeta = \widehat{\psi}$.

4.4. *Affine metric geometry and quadric circle geometry*

Let $(V, \mathbb{F}, Q)$ be a nonsingular metric vector space of dimension $\geqslant 2$ and let $(\boldsymbol{A}, \widehat{-})$, $(\overline{\Pi}, \mu_{\overline{Q}})$, $\boldsymbol{M}(\overline{Q})$ be defined as in Section 4.3. We will use the identification $z = z^*$, $\forall z \in V$, (cf. 4.3.1) and thus will consider $(\boldsymbol{A}, \widehat{-})$ as a subgeometry of $\boldsymbol{M}(\overline{Q})$ and $(\overline{\Pi}, \mu_{\overline{Q}})$.

4.4.1. First of all, let us consider the $\overline{Q}$-symmetry $\widetilde{A}$, where A is a point of $\overline{\Pi}$ which does not belong to $\boldsymbol{F}$ (cf. 2.3.2). According to 4.2.3, by restriction to $\dot{\boldsymbol{F}}$, $\widetilde{A}$ induces an automorphism $\check{A}$ of $\boldsymbol{M}(\overline{Q})$.

Because of 2.3.2 (2), we obtain $\check{A} = \check{A}^{-1}$, and because of 2.3.2 (4),

$$\operatorname{fix} \check{A} := \dot{\boldsymbol{F}} \cap (A^{\perp})^{\pi}$$

is the set of fixed points of $\check{A}$. We call $\check{A}$ an *inversion* of $M(\overline{Q})$. If fix $\check{A}$ is empty, $\check{A}$ is called *elliptic*. If $\langle \text{fix}\, \check{A}\rangle$ is the hyperplane $(A^{\perp})^{\pi}$, $\check{A}$ is called a *hyperbolic inversion* at fix $\check{A}$.

From 2.1.5 one easily derives that every inversion is either elliptic or hyperbolic, if $\text{char}\,\mathbb{F} \neq 2$ (for $\text{char}\,\mathbb{F} = 2$, there may exist inversions of a third type; e.g., consider the case $\mathbb{F} \neq \mathbb{L}$ with $Q := Q_{\mathbb{L}}$, $\kappa_{\mathbb{L}} = \text{id}_{\mathbb{L}}$ and $B := (0; 1, 1)$ in 1.9.3 (1)).

The affine Q-symmetry $\sigma_{a,b}$ of $(\boldsymbol{A}, \widehat{-}_Q)$ (cf. 3.2.5 (1)) with direction $\mathbb{F}b\|$ and axis $a + b^{\perp}$ $(a \in V,\ b \in V \setminus \ker Q)$ extends to the inversion $\check{A}$, where A is the point $\mathbb{F}(b; f(a, b), 0)$ of $\tau_N \setminus \boldsymbol{F}$, and for every point B of $\tau_N \setminus \boldsymbol{F}$, $\check{B}|_{V^*}$ is a Q-symmetry of $(\boldsymbol{A}, \widehat{-}_Q)$.

From elementary geometry, a lot of fascinating properties of inversions are well known (cf. Coxeter [1957, 1969], Morley and Morley [1954], Pedoe [1957], Schwerdtfeger [1979], Yaglom [1968]).

We want to mention here the following transitivity properties, which help to transform problems of $(\overline{\Pi}, \mu_Q)$ and $M(\overline{Q})$ into problems of $(\boldsymbol{A}, \widehat{-}_Q)$:

(1) Assume, X and Y are distinct elements of $\dot{\boldsymbol{F}}$ such that $\langle X, Y\rangle$ is a secant of $\boldsymbol{F}$. We call such points *distant* and write $X \not\approx Y$. From 2.3.2 (8) we deduce that there is an inversion which interchanges X and Y.

(2) Assume, X and Y are distinct elements of $\dot{\boldsymbol{F}}$ such that $\langle X, Y\rangle$ is a subset of $\boldsymbol{F}$. We call such points *near* and write $X \approx Y$. From (1) and 2.1.5 we deduce that there is a product of two inversions which maps X onto Y.

(3) Let $\{X, Y, Z\}$, $\{X', Y', Z'\}$ be subsets of distinct elements of a regular circle of $M(\overline{Q})$. Then there is a product φ of at most two inversions such that $\varphi(X) = X'$; $\varphi(Y) = Y'$; $\varphi(Z) = Z'$ (cf. Buekenhout [1966b]).

(4) Assume $|\mathbb{F}| \geqslant 3$ and $X, Y, X', Y' \in \dot{\boldsymbol{F}}$ such that $\langle X, Y\rangle, \langle X', Y'\rangle$ are intersecting secants of $\boldsymbol{F}$. Then there is a product φ of at most three inversions such that $\varphi(X) = X' \wedge \varphi(Y) = Y'$.

(5) Assume $|\mathbb{F}| \geqslant 3$ and $X, Y, X', Y' \in \dot{\boldsymbol{F}}$ such that $X \not\approx Y \wedge X' \not\approx Y'$. Then, there is a product φ of at most four inversions such that $\varphi(X) = X' \wedge \varphi(Y) = Y'$.

(6) Assume $X, Y, X', Y' \in \dot{\boldsymbol{F}}$ such that $X \not\approx Y \wedge X' \not\approx Y'$. Then, according to Alpers [1989b], (2.22), there exists a $\overline{Q}$-motion φ such that $\varphi(X) = X' \wedge \varphi(Y) = Y'$.

From (1), (2), and 4.3.7 we deduce

4.4.2. *Let ψ be an automorphism of $(\overline{\Pi}, \mu_{\overline{Q}})$. Then there exists an automorphism α of $(\boldsymbol{A}, \widehat{-})$ and a product φ of at most two $\overline{Q}$-symmetries such that $\psi = \varphi \circ \widehat{\alpha}$, where $\widehat{\alpha} := \varphi^{-1} \circ \psi$ is the unique extension of α according to* 4.2.3, 4.3.7.

4.4.3. From the above we deduce that every automorphism of $M(\overline{Q})$ may be represented as a product, consisting of an extended automorphism of $(\boldsymbol{A}, \widehat{-}_Q)$ and at most two inversions.

This means that questions concerning transitivity properties of $(\overline{\Pi}, \mu_{\overline{Q}})$ and $M(\overline{Q})$ may be transposed to questions which can be formulated in $(A, \widehat{-}_Q)$. For example, we may apply 3.2.7.

General transitivity theorems, formulated in the language of $(\overline{\Pi}, \mu_{\overline{Q}})$, are scarcely known. In fact, the transitivity mainly depends on the inner structure of $\mathbb{F}$ and $\overline{Q}$ and leads to difficult problems of algebra (cf. Alpers [1989b]). Therefore, we will only formulate here the following theorem, which is a corollary of 3.4.14 and 4.4.2, and which is well known from circle geometry (cf. Benz [1973]):

4.4.4. *Assume* $\dim V = 2$ *and let* $\{X, Y, Z\}$, $\{X', Y', Z'\}$ *be triples of pairwise distant points of* $M(\overline{Q})$. *Then, there exists a linearly induced automorphism* ψ *of* $M(\overline{Q})$ *such that* $\psi(X) = X'$, $\psi(Y) = Y'$ *and* $\psi(Z) = Z'$.

4.4.5. Suppose Q is Euclidean. Then, according to 4.3.2, $\boldsymbol{F}$ is an ovoid of $\overline{\Pi}$, and we obtain

$$\boldsymbol{F} = \dot{\boldsymbol{F}} \wedge \tau_N \cap \boldsymbol{F} = \{N\} \wedge V^* \cup \{N\} = V \cup \{N\} = \boldsymbol{F}.$$

Therefore, we may 'complete' $(\boldsymbol{A}, \widehat{-})$ to $M(\overline{Q})$ by 'adding' the only point N, which, as seen from $(\boldsymbol{A}, \widehat{-}_Q)$, has now to be considered as 'the point at infinity'. In this sense, $M(\overline{Q})$ is the *Möbius-closure* of $(\boldsymbol{A}, \widehat{-}_Q)$, which must be well distinguished from the projective closure of $\boldsymbol{A}$ (cf. 3.1.2).

We obtain $\widehat{l} = l \cup \{N\}$ for every line of $\boldsymbol{A}$ and $\widehat{k} = k$ for every circle of $(\boldsymbol{A}, \widehat{-}_Q)$, and thus the circles of $M(\overline{Q})$ are the circles of $(\boldsymbol{A}, \widehat{-}_Q)$ together with the 'extended' lines of $\boldsymbol{A}$. Two lines l, m of $\boldsymbol{A}$ are parallel in $\boldsymbol{A}$ iff the extended lines $\widehat{l}, \widehat{m}$ touch at N.

Clearly, statements on lines and circles of $(\boldsymbol{A}, \widehat{-}_Q)$ may be transposed to statements on circles of $M(\overline{Q})$, and *vice versa*.

If Q is not Euclidean, we have $\tau_N \cap \boldsymbol{F} \neq \{N\}$. For example, in case of $\dim V = 2$, $\tau_N \cap \boldsymbol{F}$ is a line iff Q is Galilean, and a cross of lines iff Q is Minkowskian.

In any case, the points of $\tau_N \cap \dot{\boldsymbol{F}}$ must be added to the points of $\boldsymbol{A}$ in order to obtain the *circle geometric closure* $M(\overline{Q})$ of $(\boldsymbol{A}, \widehat{-}_Q)$.

4.4.6. To give an idea of the interplay between circle geometry and affine metric geometry, let us consider the 'bundle theorem':

Assume $\dim V = 2$. For distinct elements $k, l \in \boldsymbol{C}^{\overline{Q}}$, we call

$$B(k,l) := \{m \in \boldsymbol{C}^{\overline{Q}}\colon \langle m \rangle \supseteq \langle k \rangle \cap \langle l \rangle\}$$

the *bundle* determined by k and l. We say that circles *belong to a bundle* provided they form a subset of a bundle.

By observing 2.1.1–5 and 4.2.1, for distinct elements $k, l \in C_{\overline{Q}}$, we obtain

(1) Every point of $\dot{\boldsymbol{F}} \setminus (k \cap l)$ coincides with exactly one element of $B(k, l)$.

(2) Assume $m, n \in B(k, l)$ and $m \neq n$. Then, $B(m, n) = B(k, l)$.

(3) Assume $|k \cap l| \geqslant 2$. Then, $B(k, l) = \{m \in \boldsymbol{C}^{\overline{Q}}\colon m \supseteq k \cap l\}$.

(4) Assume $A \in k \cap l$ and $k, l|_{\overline{Q}}A$. Then,

$$B(k,l) = \{m \in \boldsymbol{C}^{\overline{Q}}\colon (m, k|_{\overline{Q}}A) \wedge (m, l|_{\overline{Q}}A)\}.$$

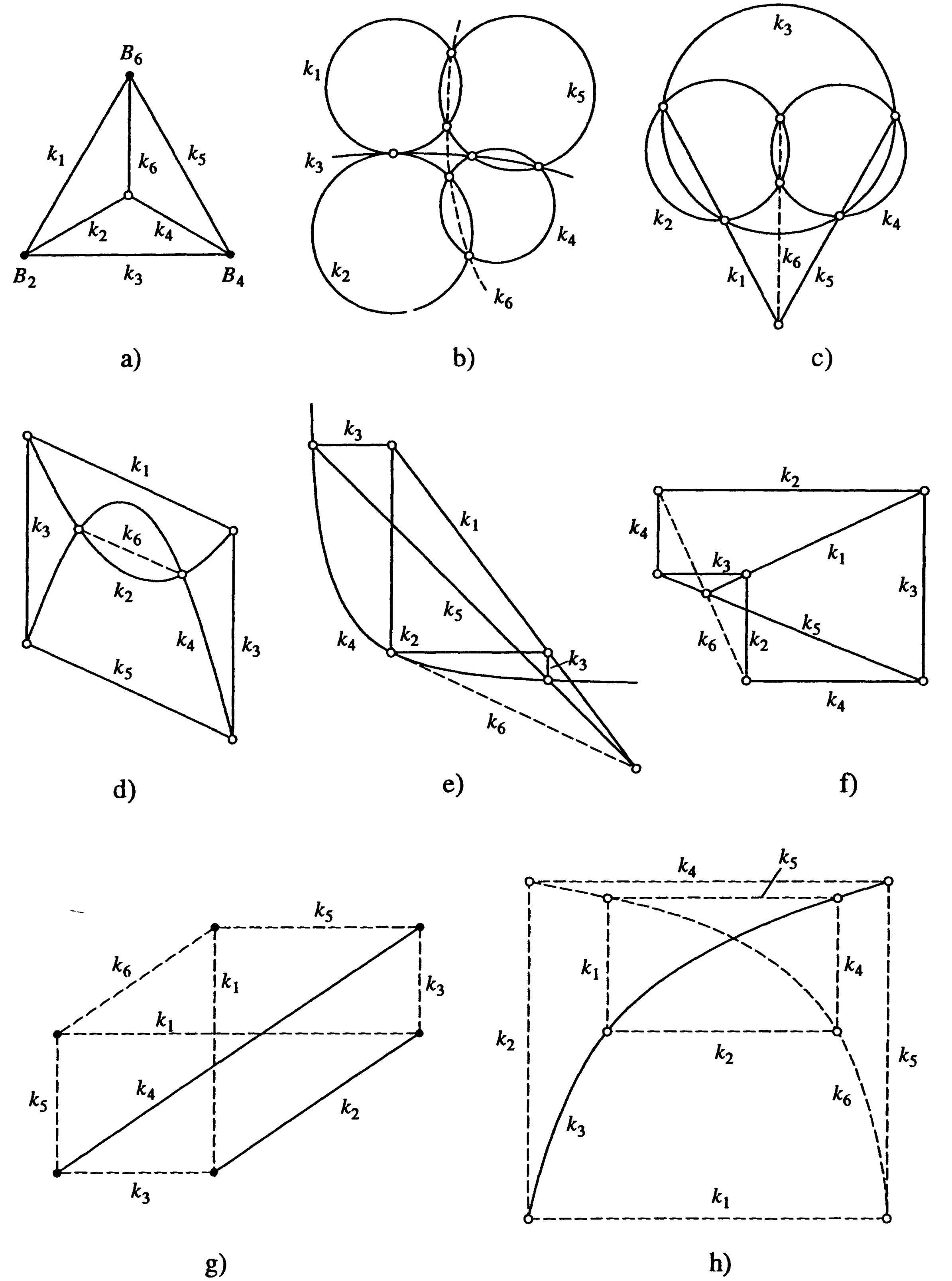

Figure 4.2. Affine metric versions of 4.4.7 for N = point at infinity: a) scheme; b), c) Q Euclidean; d) Q Galilean; e)–h) Q Minkowskian; c)–f) $N \in k_2 \cap k_4 \cap k_6$; g) $N \in k_6$.

By using simple properties of the incidence geometry of the three-dimensional projective space $\overline{\Pi}$ (cf. Figure 4.2 a)), we obtain

4.4.7. BUNDLE THEOREM. *Assume* $\dim V = 2$ *and let* $k_1, \dots, k_6$ *be pairwise distinct elements of* $C^{\overline{Q}}$ *such that* $k_1 \cap k_3 \cap k_5 = \varnothing$ *and* $k_2 \cap k_4 \cap k_6 \neq \varnothing$. *If each of the triples* (k_1, k_2, k_3), (k_3, k_4, k_5), (k_5, k_6, k_1) *belongs to a bundle, then,* k_2, k_4, k_6 *belong to a bundle* (cf. Figure 4.2 b)–h)).

Observe that this theorem may be used to construct the elements of a bundle $B(k,l)$ for the case $k \cap l = \varnothing$.

The circle geometric version of the bundle theorem yields different affine versions of this theorem (cf. Figure 4.2), and it turns out that analogously the three theorems of Miquel 3.4.8–10 are special cases of the following circle geometric version (cf. Samaga [1991a,b] and Chapter 6, 5.10):

4.4.8. THEOREM OF MIQUEL. *Assume* $\dim V = 2$ *and* $k, k_\nu \in C^{\overline{Q}}$ *and* $A_\nu, B_\nu \in \dot{F}$ $(\nu \in \mathbb{Z}_4)$ *such that*

(1) $|\{A_1, B_2, A_3, B_4\}| = 4$,

(2) $k \cap k_{\nu+1} = \{A_\nu, A_{\nu+1}\} \wedge k_\nu \cap k_{\nu+1} = \{A_\nu, B_\nu\}$, $\forall \nu \in \mathbb{Z}_4$.

Then, except in the case $|\mathbb{F}| = 2$ *and* $\dot{F} \neq F$, *there exists exactly one circle* l *such that*

(*) $l \cap k_{\nu+1} = \{B_\nu, B_{\nu+1}\}$, $\forall \nu \in \mathbb{Z}_4$

(cf. Figure 3.4).

REMARK. From (1) and (2) one deduces

(3) $(A_\nu \not\approx A_{\nu+1} \vee A_\nu = A_{\nu+1}) \wedge (A_\nu \not\approx B_\nu \vee A_\nu = B_\nu)$, $\forall \nu \in \mathbb{Z}_4$,

iff $|\mathbb{F}| \geqslant 3 \vee \dot{F} = F$, and it turns out that the theorem remains true in case of $|\mathbb{F}| = 2 \wedge \dot{F} \neq F$, if (3) is added to the assumptions (1), (2).

4.4.9. The geometry of angles of $(A, \widehat{-}_Q)$ (cf. 3.1.3 (i), (j)) may be extended to $M(\overline{Q})$ by using the following procedure (cf. Benz [1973]):

If k, l are regular circles of $M(\overline{Q})$ such that $k \cap l \neq 0$, then the set $\{k, l\}$ is called a *(nonoriented) angle*, and the ordered pair (k, l) is called an *oriented angle*.

Assume now that k, l, m, n are regular circles of $M(\overline{Q})$ which pass through a common point $P \in \dot{F}$. Then, there exists a $\overline{Q}$-motion φ such that $\varphi(P) = N$ (cf. 4.4.1 (1), (2)), and we define

(1) $<_Q^M \{k, l\} := <_Q \{\varphi(k)', \varphi(l)'\}$,

(2) $(k, l) \mathrel{\widehat{-}_Q^M} (m, n) :\Leftrightarrow (\varphi(k)', \varphi(l)') \mathrel{\widehat{-}_Q} (\varphi(m)', \varphi(n)')$,

where $X' := X \setminus \tau_N$ for $X \subseteq \dot{F}$. Here, $<_Q^M \{k, l\}$ is called the *Q-measure* of $\{k, l\}$, while $\widehat{-}_Q^M$ is called the *Q-conformity*. It turns out that the measure $<_Q^M$ and the conformity $\widehat{-}_Q^M$ are well defined (cf. 3.2.1 (1), 3.2.3 (7), 4.3.4, 4.3.7). Moreover, every automorphism of $M(\overline{Q})$ preserves the relation $\widehat{-}_Q^M$ and maps angles of equal Q-measure onto angles of equal Q-measure.

4.5. *Subgeometries of projective metric spaces*

In this section, we again use the notations of Section 4.3.

4.5.1. According to 4.2.4, there is no difference between the automorphism groups of $(\overline{\Pi}, \mu_{\overline{Q}})$ and $\boldsymbol{M}(\overline{Q})$, and according to 4.3.7, we find the automorphisms of $(\boldsymbol{A}, \widehat{-}_Q)$ simply by taking those automorphisms of $(\overline{\Pi}, \mu_{\overline{Q}})$ which fix the point $N \in \dot{\boldsymbol{F}}$. In this sense, $\boldsymbol{M}(\overline{Q})$ and $(\boldsymbol{A}, \widehat{-}_Q)$ may be considered as subgeometries of $(\overline{\Pi}, \mu_{\overline{Q}})$.

4.5.2. There are other criteria which may be taken for obtaining subgeometries of a projective metric space.

For example, the well-known Klein model of the real hyperbolic plane gives rise to a general definition of *Klein spaces*.

Beside the projective metric space $(\overline{\Pi}, \mu_{\overline{Q}})$, defined in 4.3.1, we will consider now the projective metric space (Π, μ_Q), where $\Pi := \Pi(V, \mathbb{F}) = (\boldsymbol{P}, \boldsymbol{L})$ (cf. 4.1.1):

4.5.3. Let $\boldsymbol{I}_Q^{\pi}(\circ)$ be the group of Q-motions of (Π, μ_Q). Clearly, $\boldsymbol{I}_Q^{\pi}(\circ)$ operates on $\boldsymbol{P}$, and thus $\boldsymbol{P}$ is the disjoint union of the $\boldsymbol{I}_Q^{\pi}$-orbits

(1) $[X] := \{\varphi(X)\colon\ \varphi \in \boldsymbol{I}_Q^{\pi}\}\ (X \in \boldsymbol{P})$.

Because $(V^{\perp})^{\pi}$ and $\boldsymbol{F}_Q$ are fixed by every element of $\boldsymbol{I}_Q^{\pi}$, we obtain

(2) $[X] \subseteq (V^{\perp})^{\pi} \vee [X] \cap (V^{\perp})^{\pi} = \varnothing,\ \forall X \in \boldsymbol{P}$,

(3) $[X] \subseteq \boldsymbol{F}_Q \vee [X] \cap \boldsymbol{F}_Q = \varnothing,\ \forall X \in \boldsymbol{P}$.

Define $\dot{\mathbb{F}}^{(2)} := \{\lambda^2\colon\ \lambda \in \dot{\mathbb{F}}\}$. Then, $\dot{\mathbb{F}}/\dot{\mathbb{F}}^{(2)}(\cdot)$ and $\mathbb{F}/\dot{\mathbb{F}}^{(2)}(\cdot)$ are the group and semigroup *of square classes* of $\mathbb{F}(+,\cdot)$. For $x \in V \setminus \{O\}$ and $X = \mathbb{F}x$, we define

(4) $\dot{Q}(X) := \{Q(\lambda x)\colon\ \lambda \in \dot{\mathbb{F}}\} = Q(x) \cdot \dot{\mathbb{F}}^{(2)}$.

Then, from $\dot{Q}(X) = \dot{Q}(\varphi(X)),\ \forall \varphi \in \boldsymbol{I}_Q^{\pi}$, we deduce

(5) $[X] = [Y] \Rightarrow \dot{Q}(X) = \dot{Q}(Y),\ \forall X, Y \in \boldsymbol{P}$.

According to Alpers [1989b], 3.3, we obtain

(6) $[X] = [Y] \Leftrightarrow \dot{Q}(X) = \dot{Q}(Y),\ \forall X, Y \in \boldsymbol{P} \setminus ((V^{\perp})^{\pi} \cup \boldsymbol{F}_Q)$,

(7) $[X] = [Y] \Leftrightarrow \dot{Q}(X) = \dot{Q}(Y),\ \forall X, Y \in ((V^{\perp})^{\pi} \setminus \boldsymbol{F}_Q)$,

which shows that there exists a direct correspondence between the square classes of $\mathbb{F}$ and the $\boldsymbol{I}_Q^{\pi}$-orbits.

Assume now $\alpha \in \dot{\mathbb{F}}/\dot{\mathbb{F}}^{(2)}$ such that $\langle \boldsymbol{N}_\alpha \rangle = \boldsymbol{P}$ for $\boldsymbol{N}_\alpha := \{X \in \boldsymbol{P}\colon\ \dot{Q}(X) = \alpha\}$. Then we call $\boldsymbol{N}_\alpha$ a *nucleus*, and from 1.6.4 (2) we deduce

(8) $\mu_Q(X, Y, Z) \in \boldsymbol{N}_\alpha,\quad \forall (X, Y, Z) \in {}^3\boldsymbol{N}_\alpha := \boldsymbol{N}_\alpha^3 \cap {}^3\boldsymbol{R}_Q$.

If $\boldsymbol{K}$ is a subset of $\boldsymbol{P}$, let

(9) $\boldsymbol{L}_{\boldsymbol{K}} := \{l \cap \boldsymbol{K}\colon\ l \in \boldsymbol{L} \wedge |l \cap \boldsymbol{K}| \geqslant 2\}$

be the set of *$\boldsymbol{K}$-lines*.

If $\boldsymbol{K}$ is a *union of nuclei* such that

(10) $\mu_Q(X, Y, Z) \in \boldsymbol{K},\quad \forall (X, Y, Z) \in {}^3\boldsymbol{K} := \boldsymbol{K}^3 \cap {}^3\boldsymbol{R}_Q$,

then we call $\boldsymbol{K}$ the set of *points*, $\boldsymbol{L}_{\boldsymbol{K}}$ the set of *lines* and $\mu_Q|_{{}^3\boldsymbol{K}}$ the metric product of the *Klein space* $((\boldsymbol{K}, \boldsymbol{L}_{\boldsymbol{K}}), \mu_Q|_{{}^3\boldsymbol{K}})$.

For example, condition (10) is fulfilled if every threefold product of elements of $\dot{Q}(\boldsymbol{K}) := \{\dot{Q}(X)\colon X \in \boldsymbol{K}\}$ is an element of $\dot{Q}(\boldsymbol{K})$. Because of $\alpha = \alpha^{-1}$, $\forall \alpha \in \dot{\mathbb{F}}/\dot{\mathbb{F}}^{(2)}$, the latter condition means that $\dot{Q}(\boldsymbol{K})$ is a coset of a subgroup of $\dot{\mathbb{F}}/\dot{\mathbb{F}}^{(2)}(\cdot)$.

According to (8), every nucleus is the point set of a Klein space, and obviously, $\boldsymbol{P} \setminus \boldsymbol{F}_Q$ is the point set of a Klein space, iff $\langle \boldsymbol{P} \setminus \boldsymbol{F}_Q \rangle = \boldsymbol{P}$.

If $\dot{\mathbb{F}} = \dot{\mathbb{F}}^{(2)}$ (examples: (1) $\mathbb{F} = \mathbb{C}$; (2) $|\mathbb{F}| \in \mathbb{N}$ and $\operatorname{char} \mathbb{F} = 2$), there is at most one possibility to get a Klein space. If $|\dot{\mathbb{F}}/\dot{\mathbb{F}}^{(2)}| = 2$ (examples: (1) $\mathbb{F} = \mathbb{R}$; (2) $\mathbb{F}$ is a Euclidean field; (3) $|\mathbb{F}| \in \mathbb{N}$ and $\operatorname{char} \mathbb{F} \neq 2$) there are at most three possibilities to get a Klein space.

4.5.4. We want to present here a general example of a *nucleus*. For this purpose, let $(V, \mathbb{F}, Q)$, $\overline{V}, \overline{Q}, \overline{\Pi}, \overline{\boldsymbol{P}}, \boldsymbol{F}, \dot{\boldsymbol{F}}, N, S$ be given according to 4.3.1 and assume $\operatorname{char} \mathbb{F} \neq 2$ and $|\mathbb{F}| \geqslant 5$. Define $V' := V \times \mathbb{F}$ and consider the quadratic form

(1) $Q'_r\colon V' \to \mathbb{F},\ (z;\xi) \mapsto Q(z) - r\xi^2$,

where r is a fixed element of $\dot{\mathbb{F}}$.

Obviously, by using the identification

(2) $(z;\xi) = (z;\xi, r\xi),\ \forall (z;\xi) \in V'$,

the vector space V' may be considered to be the hyperplane $\{(z;\xi,\eta) \in \overline{V}\colon\ \eta = r\xi\}$ of $\overline{V}$ (cf. 4.3.1), and then the corresponding projective space $\Pi_r := \Pi(V', \mathbb{F}) =: (\boldsymbol{P}_r, \boldsymbol{L}_r)$ is a hyperplane of $\overline{\Pi}$.

We state that

(3) $\boldsymbol{K}_r := \{\mathbb{F}(2rz; r + Q(z))\colon\ z \in V \wedge r \neq Q(z)\}$

is a parameter representation for a nucleus of the projective metric space $(\Pi_r, \mu_{Q'_r})$, which is related in a natural way with the geometry of $(\overline{\Pi}, \mu_{\overline{Q}})$.

Indeed: We have $Q'_r = \overline{Q}|_{V'}$, and for $Z_r := \mathbb{F}(O; -1, r)$ we obtain

(4) $Z_r \notin \boldsymbol{P}_r \wedge Z_r \in \langle N, S \rangle \wedge Z_r \notin \boldsymbol{F} \wedge (Z_r^{\perp})^{\pi} = \boldsymbol{P}_r$,

(5) $X \in \boldsymbol{K}_r \Leftrightarrow |\langle X, Z_r \rangle \cap \dot{\boldsymbol{F}}| = 2,\ \forall X \in \boldsymbol{P}_r$,

(6) $\boldsymbol{K}_r = \{X \in \boldsymbol{P}_r\colon\ \dot{Q}'_r(X) = \dot{Q}'_r(\mathbb{F}(O;1,r)) = -r \cdot \dot{\mathbb{F}}^{(2)}\}$.

Because of (4), (5), and $|\mathbb{F}| \geqslant 5$, for $Y \in \boldsymbol{F} \setminus \{N, S\}$ there exists an $X \in \boldsymbol{K}_r$ such that $Y \in \langle X, N, S \rangle$. This yields $\overline{\boldsymbol{P}} = \langle \boldsymbol{F} \rangle \subseteq \langle \boldsymbol{K}_r \cup \{Z_r\} \rangle$, and thus $\langle \boldsymbol{K}_r \rangle$ is the hyperplane $\boldsymbol{P}_r$. For this reason and because of (6), $\boldsymbol{K}_r$ is a nucleus of $(\Pi_r, \mu_{Q'_r})$.

The automorphisms of the Klein space $((\boldsymbol{K}_r, L_{\boldsymbol{K}_r}), \mu_{Q'_r}|_{{}^3\boldsymbol{K}_r})$ can be defined analogously to 4.1.2, and obviously, every automorphism of $(\overline{\Pi}, \mu_{\overline{Q}})$ which fixes Z_r induces an automorphism of this Klein space. The converse question, whether there are other automorphisms, is not solved in general.

Now assume that Q is Euclidean. If we use the usual identification $z = \mathbb{F}(z;1)$, $\forall z \in V$, in case of $-r \notin Q(V)$, we get the representation

(7) $\boldsymbol{K}_r = \{(r + Q(z))^{-1} \cdot 2rz \in V\colon\ z \in V \wedge r \neq Q(z)\}$,

and then, for $r := 1$ and $\mathbb{F} := \mathbb{R}$, $\boldsymbol{K}_r$ is the point set of Klein's model of a real hyperbolic space.

4.5.5. Looking at the classical situation, the considerations of 4.5.4 give rise to the following general definition of Poincaré spaces:

In the projective metric space $(\overline{\Pi}, \mu_{\overline{Q}})$ let Z be a point of $\overline{\boldsymbol{P}} \setminus (\overline{V}^{\perp})^{\pi}$, i.e. a point which coincides with a secant of $\boldsymbol{F}$. According to 4.4.1 (6), without loss of generality we may assume $Z = Z_r = \mathbb{F}(O; -1, r) \in \langle N, S\rangle \setminus \{S\}$, where r is a fixed element of $\mathbb{F}$.

Suppose $|\mathbb{F}| \geqslant 4$. Then, we call

(1) $\mathcal{P}_r := \{\{X,Y\}: X,Y \in \dot{F} \wedge X \neq Y \wedge Z_r \in \langle X,Y\rangle\}$

the set of *r-points*,

(2) $\mathcal{L}_r := \{k \in \boldsymbol{C}^{\overline{Q}}\colon \exists\{X,Y\} \in \mathcal{P}_r$ such that $k \ni X, Y\}$

the set of *r-lines*,

(3) $\mathcal{C}_r := \{k \cup \widehat{Z}_r(k)\colon k \in \boldsymbol{C}^{\overline{Q}} \setminus \mathcal{L}_r\}$

the set of *r-circles*, and we call $\mathcal{P}(r) := ((\mathcal{P}_r, \mathcal{L}_r), \mathcal{C}_r)$ a *Poincaré space*.

Because $|\mathbb{F}| \geqslant 4$ and $\dim V \geqslant 2$, every r-line contains at least two different r-points, and every r-point coincides with at least five different r-lines. The automorphisms of $\mathcal{P}(r)$ can be defined analogously to 4.2.2, and then, obviously, every automorphism of $(\overline{\Pi}, \mu_{\overline{Q}})$ which fixes Z_r, induces an automorphism of $\mathcal{P}(r)$.

For r-points, we obtain the parameter representation

(4) $\{\mathbb{F}(z; Q(z), 1), \mathbb{F}(rz; 1, Q(rz))\}$,

where z is an element of V such that $rQ(z) \neq 1$.

By using the identification $z = z^*$ (cf. 4.3.1 (4)), we obtain the parameter representation

$$(5)\ \left\{z, \frac{z}{rQ(z)}\right\} \quad (z \in V \wedge Q(rz) \notin \{0, r\})$$

in case of $Q(rz) \neq 0$ and

(6) $\{z, \mathbb{F}(rz; 1, 0)\} \quad (z \in V \wedge Q(rz) = 0)$

otherwise, and thus, $\mathcal{P}(r)$ may essentially be regarded as a subgeometry of $(V, \widehat{-}_Q)$, in as much as the r-points, the r-lines, and the r-circles are 'extended' objects of $(V, \widehat{-}_Q)$. In particular, $\mathcal{P}(0)$ may be identified with $(V, \widehat{-}_Q)$.

The subjects of the quadric circle geometry $\boldsymbol{M}(Q)$ as well as the subjects of the affine metric geometry $(V, \widehat{-}_Q)$ may be transferred in a modified manner to $\mathcal{P}(r)$, and thus $\mathcal{P}(r)$ may be considered as a 'rich' structure.

For example, there exists a general trigonometry (cf. Schröder [1974a]).

We want to point out that every element of $\mathcal{P}_r$ can be mapped by an inversion onto an arbitrary element of $\mathcal{P}_r$ and that r-points and the r-lines in case of $2r \neq 0$ correspond to the nucleus considered in 4.5.4 according to

(7) $\mathcal{P}_r = \{\langle X, Z_r\rangle \cap \dot{F}\colon X \in \boldsymbol{K}_r\}$ and $\mathcal{L}_r = \{\langle l \cup \{Z_r\}\rangle \cap \dot{F}\colon l \in \boldsymbol{L}_{\boldsymbol{K}_r}\}$.

Finally we want to mention that the structure $((\dot{F}, \mathcal{L}_r), \mathcal{C}_r)$ represents a generalized *spherical space*.

4.6. *Geometric characterizations of subgeometries of projective metric spaces*

As was pointed out in 3.5.2, reflection geometry yields one of the most important methods for obtaining intrinsic characterizations of metric structures.

In particular, the approach of Bachmann [1973] shows the basic ideas for characterizations by using reflection geometry:

4.6.1. Assume, $G(\cdot)$ is a group and J is the set of all involutions of G. We use the abbreviation $\alpha_1, \alpha_2, \ldots \mid \beta_1, \beta_2, \ldots$ to indicate that each of the products $\alpha_\nu \cdot \beta_\mu$ is an element of J ($\alpha_\nu, \beta_\mu \in G$; $\nu, \mu \in \{1, 2, \ldots\}$). If S is a subset of J which generates G and which is invariant in G, the pair (G, S) is called a *Bachmann plane*, if the following axioms are fulfilled:

(B1) Given $a, b \in D := S^2 \cap J$, there exists $A \in S$ such that $a|A \wedge b|A$.

(B2) For $a, b \in D$ and $A, B \in S$, $(a, b|A, B)$ implies $a = b \vee A = B$.

(B3) For $x \in D \cup S$ and $A, B, C \in S$, $(x|A, B, C)$ implies $ABC \in S$.

(B4) There are $X, Y, U \in S$ such that $X|Y \wedge XYU, XU, YU \notin J$.

According to Bachmann [1973], p. 120, each Bachmann plane is a subgeometry of a projective metric plane $((P, L), \mu_Q)$ of characteristic $\neq 2$ such that the following properties hold for $A, B, C \in S$ and $a \in D$:

(1) F_Q is empty or a point or an oval.

(2) S is a subset of $R_Q = P \setminus F_Q$.

(3) $a^* := \{X \in S\colon\ aX \in J\}$ is a line of (P, L).

(4) $(A, B, C) \in {}^3R_Q \Leftrightarrow A \cdot B \cdot C = \mu_Q(A, B, C) \in S$ (cf. 1.7.5).

(5) $A|B \Leftrightarrow A \perp B$.

(6) $G(\cdot)$ is isomorphic to a subgroup of the group of motions of $((P, L), \mu_Q)$.

By using a dual interpretation, taking S as the set of lines and D as the set of points, it turns out that all Euclidean planes and all elliptic planes of characteristic $\neq 2$ are Bachmann planes, and that the real Klein plane and every direct generalization to an ordered coordinate field is a Bachmann plane. The complete determination of all Bachmann planes still is an open problem (cf. Klopsch [1976]).

Among the characterizations, inspired by Bachmann's work, we want to mention the following characterization of Nolte [1979] and Schröder [1982], which describes the whole class of nontrivial projective metric spaces.

4.6.2. *Suppose $\Pi = (P, L)$ is a (possibly non-Desarguesian) projective space of dimension $\geqslant 2$ and F is a 2-set of Π* (cf. 2.1.3) *such that $|\varepsilon| = 7$ implies $|\varepsilon \cap F| \notin \{4, 6\}$ for every plane ε.*

Assume now, $S := P \setminus F$ is nonempty and generates a group $G(\cdot)$ such that points $A, B, C \in S$ are collinear iff $A \cdot B \cdot C \in S$ holds.

Then, if $A \cdot A = 1$, $\forall A \in S$, there exists a vector space V over a field $\mathbb{F}$ and a quadratic form $Q\colon V \to \mathbb{F}$ such that

$$\Pi = \Pi(V, \mathbb{F}) \ \wedge\ F = F_Q \ \wedge\ \mu_Q(A, B, C) = A \cdot B \cdot C, \quad \forall (A, B, C) \in {}^3S.$$

4.6.3. There have been diverse attempts to generalize Bachmann's approach such that a large class of subgeometries of projective metric spaces is characterized (cf. 3.5.1). The farthest reaching concept seems to be the following (cf. Schröder [1984b]):

Let (P, L) be a linear space (cf. Chapters 3, 4, 6) and assume S is a nonempty subset of P. We define

$${}^3S := \{(A, B, C) \in S \times S \times S\colon\ A, B, C \text{ are collinear}\}$$

and call a mapping μ: ${}^3\boldsymbol{S} \to \boldsymbol{S}$, $(A,B,C) \mapsto \mu(A,B,C) =: ABC$ a *metric product* on $\boldsymbol{S}$, if the conditions

(MP1) $AAB = B \wedge ABB = A, \ \forall A,B \in \boldsymbol{S}$,

(MP2) $(A,B,C),(C,D,E),(E,F,G),(B,D,F),(ABC,CDE,EFG) \in {}^3\boldsymbol{S}$
$\Rightarrow (A,BDF,G) \in {}^3\boldsymbol{S} \wedge (ABC)(CDE)(EFG) = A(BDF)G$

are fulfilled.

For example, if $\boldsymbol{S}$ generates a group $G(\cdot)$ such that

$$((A,B,C) \in {}^3\boldsymbol{S} \Leftrightarrow A \cdot B \cdot C \in \boldsymbol{S}) \quad \text{and} \quad A \cdot A = 1 \text{ for } A,B,C \in \boldsymbol{S},$$

then μ_G: ${}^3\boldsymbol{S} \to \boldsymbol{S}$, $(A,B,C) \mapsto A \cdot B \cdot C$, obviously is a metric product.

One reason for introducing metric products was the fact that the group $G(\cdot)$ in 4.6.2 is not always canonically determined by the geometry under consideration (cf. Frank [1984b]) and that the assumptions concerning $\boldsymbol{S}$ and $G(\cdot)$ in 4.6.2 may be replaced by the condition that there exists a metric product μ on $\boldsymbol{S}$ (cf. Schröder [1984b]). It turns out that the condition (MP2) is something like a projective version of the theorem of Miquel.

If ε is a plane of $(\boldsymbol{P},\boldsymbol{L})$, let $\boldsymbol{L}(\varepsilon) := \{l \in \boldsymbol{L}\colon\ l \subset \varepsilon\}$ be the set of *lines* of ε and let

$$\boldsymbol{L}^\pi(\varepsilon) := \{l \in \boldsymbol{L}(\varepsilon)\colon\ |l| \geqslant 3 \wedge l \cap m \neq \varnothing,\ \forall m \in \boldsymbol{L}(\varepsilon)\}$$

be the set of *projective lines* of ε. A plane ε of $(\boldsymbol{P},\boldsymbol{L})$ is called a *Sperner plane*, provided

$$\boldsymbol{L}^\pi(\varepsilon) \neq \varnothing \ \wedge\ |\{l \in \boldsymbol{L}^\pi(\varepsilon)\colon\ l \ni A\}| \notin \{1,2\}, \quad \forall A \in \varepsilon.$$

By using these definitions, the result of F. Bachmann stated in 4.6.1 can be generalized as follows (cf. Schröder [1984b]):

Suppose $(\boldsymbol{P},\boldsymbol{L})$ is a linear space of dimension $\geqslant 2$ such that every element of the set $\boldsymbol{E}$ of planes of $(\boldsymbol{P},\boldsymbol{L})$ is a Sperner plane. Assume further that the conditions

(L1) $\delta,\varepsilon,\eta \in \boldsymbol{E} \wedge \delta \cap \eta, \varepsilon \cap \eta \in \boldsymbol{L} \Rightarrow |(\delta \cap \varepsilon) \setminus \eta| \neq 1$;

(L2) $\dim(\boldsymbol{P},\boldsymbol{L}) = 3 \wedge (g,h,k,l \in \boldsymbol{L}) \wedge |\{g,h,k,l\}| = 4 \wedge (k,l \not\subseteq \overline{g \cup h})$
$\wedge(\overline{g \cup h}, \overline{g \cup k}, \overline{g \cup l}, \overline{h \cup k}, \overline{h \cup l} \in \boldsymbol{E}) \Rightarrow \overline{k \cup l} \in \boldsymbol{E}$

are fulfilled, where $\overline{}$ *denotes the hull operator in $(\boldsymbol{P},\boldsymbol{L})$.*

Then, the existence of a metric product μ: ${}^3\boldsymbol{P} \to \boldsymbol{P}$ implies that there exists a vector space $\boldsymbol{V}$ over a field $\mathbb{F}$ and a quadratic form Q: $V \to \mathbb{F}$ such that $((\boldsymbol{P},\boldsymbol{L}),\ \mu)$ is a substructure of the projective metric space (Π,μ_Q) for $\Pi := \Pi(V,\mathbb{F}) =: (\boldsymbol{P}^\pi,\boldsymbol{L}^\pi)$, fulfilling the conditions

(1) $\boldsymbol{P} \subseteq \boldsymbol{P}^\pi \setminus \boldsymbol{F}_Q \wedge \operatorname{ind} Q \leqslant 1$,

(2) $\mu(A,B,C) = \mu_Q(A,B,C),\ \forall (A,B,C) \in {}^3\boldsymbol{P}$,

(3) $\boldsymbol{L} = \{l \cap \boldsymbol{P}\colon\ l \in \boldsymbol{L}^\pi\}$.

Observe that condition (L2) is superfluous, if $(\boldsymbol{P},\boldsymbol{L})$ has dimension $\geqslant 4$, and that both conditions (L1), (L2) are superfluous if $\dim(\boldsymbol{P},\boldsymbol{L}) = 2$.

4.6.4. For intrinsic characterizations of certain classes of circle geometries, we refer to Benz [1958a, 1966, 1968, 1973], Chen [1970], Dienst [1977], Ewald [1956, 1957], Herzer [1979], Hotje [1976, 1978], Mäurer [1965, 1968], Schaeffer [1974b,c], Van der Waerden and Smid [1935] and in particular to Chapter 14.

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CHAPTER 18

Pointless Geometries

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Contents

Introduction . . . 1017
1. Historical remarks . . . 1017
   1.1. The first attempts . . . 1017
   1.2. More recent proposals . . . 1018
2. Pointless topologies . . . 1018
   2.1. Frames . . . 1018
   2.2. Points of a frame . . . 1019
3. Connection structure . . . 1020
   3.1. Whitehead's axioms . . . 1020
   3.2. Grzegorczyk's axioms . . . 1021
   3.3. The system of B.L. Clarke . . . 1022
4. The metrical approach . . . 1023
   4.1. Pointless metrical spaces . . . 1023
   4.2. Definition of diameters and distances . . . 1023
   4.3. The points . . . 1024
5. Physical geometry . . . 1025
   5.1. The first axioms . . . 1025
   5.2. Points and Cauchy filters . . . 1026
   5.3. Topological and metric structures . . . 1027
   5.4. The basic theorems of physical geometry . . . 1028
References . . . 1029

HANDBOOK OF INCIDENCE GEOMETRY
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Introduction

Since the times of Euclid, the concept of point has been assumed as the main primitive term for an axiomatic foundation of geometry. Is this choice a necessary one? The answer is 'no', and in this chapter I shall expose some attempts that have been made to build up geometry by assuming as primitive the notions of region or solid, and thereafter defining the points in a suitable way.

In pointless geometry, regions are considered as individuals, i.e. in the vocabulary of logic, first order objects, while points are represented by classes (or sequences), i.e. second order objects. Obviously, expressions like 'pointless geometry' or 'geometry without points', as used in the literature and in the title of this chapter, should be understood as contractions of 'geometry without the point as a primitive concept'. As a matter of fact, the authors of the papers that will be examined just aim at giving a good definition of 'point'.

The choice of considering the concept of 'region' as primitive makes me exclude the most famous pointless geometry: the Von Neumann Continuous Geometry, which is based on the concept of closed subspace. The same reason makes me discard category theory, although this theory can be viewed as a no-point approach of the whole field of mathematics.

In Section 1, I shall sketch some historical information about pointless geometry. In Section 2, I shall examine pointless topology, and in Section 3 connection structure. Section 4 is devoted to pointless metric spaces and Section 5 to the physical geometry of H.J. Schmidt.

1. Historical remarks

1.1. *The first attempts*

The literature on pointless geometry is not too large and each author usually ignores the previous attempts at the subject.

As soon as 1835, Lobachevsky gives an example of pointless geometry, assuming as primitive the notions of solid and contact between two solids. Several types of contact, superficial, linear and pointwise, define the surfaces, lines and points, respectively. Unfortunately, Lobachevsky's definitions are a little obscure and far from a rigorous treatment. No subsequent paper on pointless geometries quotes Lobachevsky's geometry.

Several years later, Whitehead [1919] and [1920] analyses how the objects (e.g., volumes) and the relations (e.g., inclusion of volumes) supplied by nature can be combined in order to obtain an abstract notion of point. In Nicod [1930] such philosophical analysis is assumed as a starting point for a new approach to pure geometry. Later on, Whitehead [1929], following an idea of De Laguna [1922], introduces the connectedness relation between regions to replace the inclusion relation. In Grzegorczyk [1960] and Clarke [1981] precise axiom system for the connection relation are proposed (see Section 3).

In 1927, Tarski, in the framework of Lesniewski's mereology, sketches a solution to the problem of a geometry of solids based on the notions of sphere and inclusion between spheres. The concentricity relation is defined and enables us to define the points as complete classes of concentric spheres.

1.2. *More recent proposals*

The works of Ehresmann [1957/58], Benabou [1957/58], Papert and Papert [1957/58] give rise to pointless topology, i.e. an abstract treatment of a class of lattices (the frames) extending the class whose members are the lattices of open subsets of the topological spaces. Such lattices are interesting in the frameworks of topology, computability and intuitionistic logic (see Section 2).

Schmidt [1979] gives a complete treatment of pointless geometry in which, in addition to regions and inclusion between regions, the translations and rotations are assumed as primitive concepts (see Section 5). Weihrauch and Schreiber [1981] proposes a suitable system of axioms for the *partial orders with weight and distance.* Although such structures are examined in the framework of abstract computability theory, they turn out to be a promising starting point for a metrical approach of pointless geometry (see Section 4).

Finally, one may notice that a large number of papers related to time logics can perhaps be viewed as a chapter on pointless one-dimensional geometry. The *periods of time* (the regions) are assumed as primitive together with the *inclusion relation* and the *temporal order,* while the *instants* (the points) are defined in a suitable way (see, e.g., Hamblin [1971]). I do not consider these works in this chapter.

2. Pointless topologies

2.1. *Frames*

The class $\mathcal{T}$ of open subsets of a topological space constitutes a complete lattice such that

$$x \wedge \left(\bigvee x_i\right) = \bigvee (x \wedge x_i)$$

holds for every $x \in \mathcal{T}$ and every family $(x_i)_{i \in I}$ of elements of $\mathcal{T}$. In literature the complete lattices satisfying such distributive laws are called *frames* or *locals* and extensively examined. In the following, if $\mathcal{R} = (R, \leqslant)$ is a frame, we call *regions* the elements of R and *inclusion* relation the relation $\leqslant$.

The frames may be organized into a category FR by suitably defining morphisms. The definition is given in order to obtain a category extending that of topological spaces. Now, if f is a continuous map from a topological space $(X, \mathcal{T})$ into another $(X', \mathcal{T}')$, it determines, via f^{-1}, a map from $\mathcal{T}'$ to $\mathcal{T}$ preserving infinite joins and finite meets. This suggests calling *frame-morphism* from a frame $\mathcal{R}$ into a frame $\mathcal{R}'$, any lattice morphism from $\mathcal{R}'$ into $\mathcal{R}$ which preserves infinite joins.

2.2. *Points of a frame*

In the famous Birkhoff–Stone representation theorem for distributive lattices (see Johnstone [1982]), the points are identified with the prime filters. This is unsatisfactory from a geometrical point of view, because the built-up spaces always are totally disconnected, compact spaces, so that the usual geometrical spaces are excluded. As a matter of fact, by identifying the points with the prime filters, we obtain too many points, and in pointless geometry, a good definition of 'point' leads to consider some other types of filters.

In the case of frames, we define a *point* as any *completely prime* filter, i.e. a filter P such that, for every family $(x_i)_{i\in I}$, $\bigvee x_i \in P$ implies $x_i \in P$ for a suitable x_i. The points of a frame $\mathcal{R}$ are just the elements of $\mathcal{R}$ in the category FR. We denote by $\mathbb{P}$ the set of points and say that a point *belongs* to a region r, briefly $P \in r$, provided $r \in P$.

The following proposition shows that it is possible to identify $\mathbb{P}$ with the set of the $\wedge$-irreducible elements of $\mathcal{R}$. Remember that an element u of a distributive lattice L is $\wedge$-*irreducible* provided that, for every $x, y \in L$, $u \geqslant x \wedge y$ implies that either $u \geqslant x$ or $u \geqslant y$. In the lattice of the open subsets of a Euclidean space E, u is $\wedge$-irreducible if and only if it is the complement of a point of E.

PROPOSITION 1. *A class P of elements of $\mathcal{R}$ is a point if and only if there is a $\wedge$-irreducible region $u_{\mathcal{P}}$ such that P is the complement of the ideal generated by $u_{\mathcal{P}}$, i.e.*

$$P = \{x \in R\colon\ x \nleqslant u_{\mathcal{P}}\}.$$

The map $\pi\colon\ R \to \mathfrak{P}(\mathbb{P})$ defined by putting

$$\pi(r) = \{P \in \mathbb{P}\colon\ P \in r\} = \{P \in \mathbb{P}\colon\ r \in P\}$$

preserves infinite joins and finite meets and $\pi(\mathcal{R})$ is a topology on $\mathbb{P}$. If there are enough points in $\mathcal{R}$, then π is a lattice isomorphism between $\mathcal{R}$ and the topology $\pi(\mathcal{R})$.

PROPOSITION 2. *For every frame $\mathcal{R}$, the following propositions are equivalent*:

(i) *$\mathcal{R}$ is spatial, i.e. it is isomorphic to the lattice structure of a suitable topology*;

(ii) *every element x of $\mathcal{R}$ is a meet of $\wedge$-irreducible elements.*

Proposition 1 shows that the choice of open sets as primitive terms is somewhat unsatisfactory from the point of view of pointless geometry. Indeed, in a sense, the points are present directly in a spatial lattice under the form of (complements of) $\wedge$-irreducible elements, while it should be desirable to define them by an *abstraction* process as suggested by Whitehead. This leads to consider as 'privileged model' of the concept of region some particular type of open sets; e.g., regular open sets (a subset x is called *regular open*, or *regular* if x is the interior of its own closure).

3. Connection structure

3.1. *Whitehead's axioms*

In the following we call *connection structure* any pair $(\mathcal{R}, C)$ where $\mathcal{R}$ is a set and C a binary relation on $\mathcal{R}$. The elements of $\mathcal{R}$ are called *regions*, C is a *connection relation* and if xCy, then we say that x is *connected* with y. For any $z \in \mathcal{R}$, the set of all x's such that zCx is denoted by $C(z)$. Several binary relations can be defined in a connection structure. Namely, by setting

$$x \leqslant y \Leftrightarrow C(x) \subseteq C(y)$$

we obtain a preorder relation $\leqslant$ that we call *inclusion*. The *overlapping relation* $\bigcirc$ is defined by setting

$$x \bigcirc y \Leftrightarrow z \text{ exists such that } z \leqslant x \text{ and } z \leqslant y.$$

Finally, the *nontangential inclusion* $\ll$ is defined by

$$x \ll y \Leftrightarrow C(x) \subseteq \bigcirc(y),$$

where, for any $z \in \mathcal{R}$, $\bigcirc(z)$ is defined as $C(z)$.

Connection structures were first considered in De Laguna [1922] and successively in Whitehead [1929] where a very large sequence of properties that the connection structures had to verify is exposed (in Chapter 2, Whitehead exposed 31 assumptions!). The aim was to analyze the abstraction process leading to the notions of point, line and surface. No attempt was made by Whitehead to frame his analysis into a mathematical theory. In particular, no attempt was made to reduce his system of assumptions and definitions to a logical minimum. Also, it is not clear whether such a system is able to define the Euclidean geometry or not. The following set of axioms is equivalent to the first 12 assumptions (see Gerla and Tortora [1992]).

(A1) C is symmetric.
(A2) There is no maximum for $\subseteq$.
(A3) For every x and y there exists z connected with both x and y.
(A4) C is reflexive.
(A5) $C(x) = C(y) \Rightarrow x = y$.
(A6) Any region z contains two regions x and y that are not connected.

The points are defined by the basic notion of *abstractive set*. An abstractive set is a set α of regions such that

– α is totally ordered by the nontangential inclusion,
– there is no region included in every element of α.

Intuitively, an abstractive set can 'converge' either to a point, a line or an area. We say that an abstractive set α *covers* an abstractive set β if every region in α contains a region in β. A corresponding equivalence relation is defined by setting $\alpha \equiv \beta$ provided that α covers β and β covers α. Any complete class of equivalence modulo $\equiv$ is called a *geometrical element*. The covering relation induces an order relation in the class of geometrical elements: any minimal geometrical element is called a *point*.

Whitehead, in order to define the concept of straight segment, assumed that a class of regions exists whose elements are called *ovals*. The idea was that the ovals are the convex regions of the Euclidean space. Obviously, suitable properties were assumed for the class of ovals. The *straight segment* between two points P and Q is defined as the minimal geometrical element defined by an abstractive set covering P and Q whose elements are ovals.

3.2. *Grzegorczyk's axioms*

Grzegorczyk [1960] added to the primitive $\leqslant$, the relation of *being separated*. In order to emphasize the similarity with Whitehead's ideas, I assume as primitive the negation of this relation, namely the relation C of *being connected*.

Then, Grzegorczyk's axioms become:

(G_0) $(\mathcal{R}, \leqslant)$ is a mereological field, where a *mereological field* is the structure obtained by deleting the zero element in a complete Boolean algebra.
(G_1) C is reflexive.
(G_2) C is symmetric.
(G_3) If $x \leqslant y$ then $C(x)$ is included in $C(y)$.

We say that a set p of regions is *representative* of a point if:
(i) p is without minimum and totally ordered with respect to $\ll$;
(ii) given any two regions u and v, $u \bigcirc x$ and $v \bigcirc x$ for every $x \in p$ implies uCv.

We denote by S the class of representatives of points and we call a *point* the filter $\mathcal{P}$ generated by an element p of S. Notice that two elements p and p' of S define the same point provided that, for every $x \in p$ there exists $y \in p'$ such that $x \geqslant y$, and for every $y \in p'$ there exists a $x \in p$ such that $y \geqslant x$.

Moreover, we say that a point $\mathcal{P}$ *belongs* (is *adherent*) to a region r provided that r is an element of $\mathcal{P}$ (r overlaps with every element of $\mathcal{P}$). We denote by $\mathbb{P}$ the set of points and by $\mathcal{P}(r)$ the set of points belonging to r. The following two axioms deal with the existence of points.

(G_4) Every region has at least one point.
(G_5) If xCy, then there is a point $\mathcal{P}$ such that $\mathcal{P}$ is adherent to both x and y.

Grzegorczyk proves the following two basic theorems.

THEOREM 1. *Let $\mathcal{T}$ be a Hausdorff topology, $\mathcal{R}$ the class of nonempty regular elements of $\mathcal{T}$ and put, for every $x, y \in \mathcal{R}$, xCy if $\bar{x} \cap \bar{y} \neq \varnothing$. Then $(\mathcal{R}, \subseteq, C)$ satisfies* (G_0)–(G_3). *Moreover, if every point is the intersection of a decreasing (with respect to $\ll$) family of open sets, then $(\mathcal{R}, \subseteq, C)$ also satisfies* (G_4)–(G_5).

THEOREM 2. *Assume that $(\mathcal{R}, \subseteq, C)$ satisfies* (G_0)–(G_5), *and let $\mathfrak{T}$ be the topology on $\mathbb{P}$ generated by $\{\pi(x)\colon x \in \mathcal{R}\}$, then:*
(i) *$\{\pi(x)\colon x \in \mathcal{R}\}$ is the class of the nonempty regular elements of $\mathfrak{T}$;*

(ii) $x \leqslant y \Leftrightarrow \pi(x) \subseteq \pi(y)$;

(iii) $x \ll y \Leftrightarrow \overline{\pi(x)} \subseteq \pi(y)$;

(iv) $xCy \Leftrightarrow \overline{\pi(x)} \cap \overline{\pi(y)} \neq \varnothing$;

(v) $\mathcal{P}$ *is adherent to* $x \Leftrightarrow \mathcal{P} \in \overline{\pi(x)}$.

Let $\mathcal{T}$ be a topology and $(\mathcal{R}, \leqslant, C)$ the connection structure associated with it by Theorem 1. It is an open question as to whether the topological space obtained in Theorem 2 coincides with $\mathcal{T}$ (at least for the most usual topological spaces).

3.3. *The system of B.L. Clarke*

A more direct reference to Whitehead [1929] can be found in Clarke [1981, 1985]. Clarke considers structures of type $(\mathcal{R}, C)$ for which the following axioms hold.

(A_1) C is reflexive.
(A_2) C is symmetric.
(A_3) If $C(x) = C(y)$ then $x = y$.
(A_4) If $X \subseteq \mathcal{R}$ and X is nonempty, then X admits fusion, where x is the *fusion* of X provided that $C(x) = \bigcup \langle C(z)\colon\ z \in X \rangle$.

A *point* is defined as a nonempty set $\mathcal{P}$ of regions such that:

(i) if $x \in \mathcal{P}$ and $y \in \mathcal{P}$ then xCy;

(ii) if $x \in \mathcal{P}$, $y \in \mathcal{P}$ and $x \bigcirc y$, then $x \wedge y \in \mathcal{P}$;

(iii) if $x \in \mathcal{P}$ and $y \geqslant x$ then $y \in \mathcal{P}$;

(iv) if $x \vee y \in \mathcal{P}$ then $x \in \mathcal{P}$ or $y \in \mathcal{P}$.

As usual, we say that a point $\mathcal{P}$ *belongs* to a region x, and write $\mathcal{P} \in x$, provided that $x \in \mathcal{P}$. Clarke suggests the following existence axioms:

(A_5) if xCy then there exists a point $\mathcal{P}$ such that $\mathcal{P} \in x$ and $\mathcal{P} \in y$

which is the reciprocal of (i).

The following proposition (see Biacino and Gerla [1991]) shows that, in a sense, the system (A_1)–(A_5) characterizes the mereological fields, i.e. the complete Boolean algebras.

THEOREM 3. *If* $(\mathcal{R}, C)$ *satisfies* (A_1)–(A_5), *then* $(\mathcal{R}, \leqslant)$ *is a mereological field, and* C *is the overlapping relation. Conversely, if* $(\mathcal{R}, \leqslant)$ *is a mereological field and* C *is the overlapping relation, then* $(\mathcal{R}, C)$ *satisfies* (A_1)–(A_5).

The fact that the connection relation coincides with the overlapping relation seems far from Whitehead's purposes, but it is in accordance with Leonard and Goodman [1940].

4. The metrical approach

4.1. *Pointless metrical spaces*

In the previous sections we were still at a topological level; to justify the word 'geometry' we have to consider richer structures. In this section this is achieved by considering metrical concepts. We call a *pointless pseudometric space*, briefly *ppm-space*, any structure $\mathcal{R} = (R, \leqslant, \delta, ||)$ where $(R, \leqslant)$ is a partial order and $||$: $R \to [0, \infty]$, δ: $R \times R \to [0, \infty)$ are functions satisfying, for every $x, y, z \in R$, the following axioms.

(A_1) If $x \geqslant y$ then $|x| \geqslant |y|$.
(A_2) If $x \geqslant y$ then $\delta(y, z) \geqslant \delta(z, x)$.
(A_3) $\delta(x, x) = 0$.
(A_4) $\delta(x, y) \leqslant \delta(x, z) + \delta(z, y) + |z|$.

A similar set of axioms was first defined by Weihrauch and Schreiber [1981] in the framework of computability theory (see also Pultr [1984a,b,c, 1989]). We call the number $\delta(x, y)$ the *distance* between x and y, and $|x|$ the *diameter* of x, and we say that x is *bounded* if $|x|$ is finite. If there exists in $(R, \leqslant)$ a minimum region, say O, we call O the *empty region*. Notice that, as with the pseudometric spaces, every (nonempty) subset of a ppm-space defines a ppm-space. From (A_1)–(A_4) it follows that $\delta(x, y) = \delta(y, x)$ and that if x and y overlap, then $\delta(x, y) = 0$.

If R is equal to a class of nonempty subsets of a pseudometric space (M, d), taking $\leqslant$ as the inclusion relation and δ and $||$ as the usual distance and diameter functions defined by

$$\delta(X, Y) = \inf\{d(x, y)\colon\ x \in X,\ y \in Y\}$$

and

$$|X| = \sup\{d(x, y)\colon\ x \in X,\ y \in X\},$$

we then obtain a ppm-space. We call *canonical* any space obtained by these means.

We call *pointless metric space*, briefly *pm-space*, any ppm-space satisfying

(A_5) if $|x| \geqslant |y|$ and $\delta(z, x) \leqslant \delta(y, z)$ for every $z \in R$, then $x \geqslant y$.

Notice that ppm-spaces (pm-spaces) generalize pseudometric (metric) spaces; the pseudometric (metric) spaces being the ppm-spaces (pm-spaces) in which every region is an atom whose diameter is zero.

4.2. *Definition of diameters and distances*

Let $(R, \leqslant)$ be an ordered set and $||$: $R \to [0, \infty]$ be a function. For $x, y \in R$, a path from x to y is any finite sequence $b_1, \ldots, b_n$, of nonempty bounded regions such that b_{i+1} overlaps b_i for each $i = 1$ to $n - 1$, x overlaps b_1 and y overlaps b_n. We say that

$(R, \leqslant, ||)$ is *connected* if for every pair of nonempty elements of R there is a path from x to y. We define the function $\delta_{||}$: $R \times R \to [0, \infty]$ by $\delta_{||}(x, y) = 0$ if $x \bigcirc y$, and

$$\delta_{||}(x, y) = \inf \left\{ |b_1| + \cdots + |b_n|: \ b_1 \ldots b_n \text{ is a path from } x \text{ to } y \right\}$$

otherwise. This definition was given in Weihrauch and Schreiber [1981]. Moreover, for every function δ: $R \times R \to [0, \infty)$ we put:

$$|z|_\delta = \sup \left\{ \delta(x, y) - \delta(x, z) - \delta(z, y): \ x, y \in R \right\}.$$

PROPOSITION 1. *Let* $(R, \leqslant)$ *be an ordered set together with a function* $||$: $R \to [0, \infty)$ *satisfying* (A_1) *and assume that* $(R, \leqslant, ||)$ *is connected. Then* $(R, \leqslant, \delta_{||}, ||)$ *is a ppm-space. If* δ: $R \times R \to [0, \infty)$ *is a function satisfying* (A_2) *and* (A_3), *then* $(R, \leqslant, \delta, ||_\delta)$ *is a ppm-space.*

PROPOSITION 2. *If* $(R, \leqslant, \delta, ||)$ *is a ppm-space, then* $(R, \leqslant, \delta, ||_\delta)$ *is a ppm-space such that* $||_\delta \leqslant ||$, *that is,* $||_\delta$ *is the smallest diameter compatible with* δ. *If* $(R, \leqslant, ||)$ *is connected, then* $(R, \leqslant, \delta_{||}, ||)$ *is a ppm-space such that* $\delta_{||} \geqslant \delta$, *that is,* $\delta_{||}$ *is the largest distance compatible with* $||$.

Notice that if $\mathcal{R}$ is a ppm-space, then, in the general case, δ and $\delta_{||}$ are not the same, neither are $||$ and $||_\delta$.

4.3. *The points*

In Weihrauch and Schreiber [1981], points are defined as in Section 3.2, except that $\ll$ is defined by stating that $x \ll y$ if there exists a positive λ such that $\delta(x, z) + |z| < \lambda \Rightarrow z \leqslant y$. In Gerla [1990], I define the points by a procedure similar to the completion of a metric space using Cauchy sequences. We call a *Cauchy sequence* every sequence $\langle p_n \rangle$ of bounded regions such that

(i) $\lim |p_n| = 0$ and (ii) $\forall \varepsilon > 0\ \exists \nu$: $\delta(p_h, p_k) < \varepsilon \ \ \forall h \geqslant \nu, \ \forall k \geqslant \nu$.

Decreasing sequences with vanishing diameters are examples of Cauchy sequences. There is no need to have any Cauchy sequence in a ppm-space.

PROPOSITION 3. *Assuming that the class* S *of Cauchy sequences of* $\mathcal{R}$ *is nonempty and defining* d: $S \times S \to [0, \infty)$ *by* $d(\langle p_n \rangle, \langle q_n \rangle) = \lim \delta(p_n, q_n)$ *for every* $\langle p_n \rangle \in S$ *and* $\langle q_n \rangle \in S$, (S, d) *is a pseudometric space.*

We denote by $(\mathbb{P}, d)$ the metric space obtained as a quotient of (S, d) by the relation $\equiv$ defined by $\langle p_n \rangle \equiv \langle q_n \rangle$ if $d(\langle p_n \rangle, \langle q_n \rangle) = 0$. Moreover, we call a point every element of $\mathbb{P}$. As a consequence, a point P is a class

$$[\langle p_n \rangle] = \left\{ \langle q_n \rangle \in S: \ \langle q_n \rangle \equiv \langle p_n \rangle \right\}$$

and d: $\mathbb{P} \times \mathbb{P} \to [0, \infty)$ is defined by putting, for every P, Q in $\mathbb{P}$:

$$d(P,Q) = d(\langle p_n \rangle, \langle q_n \rangle) = \lim \delta(p_n, q_n)$$

where $\langle p_n \rangle \in S$ and $\langle q_n \rangle \in S$ represent P and Q, respectively.

If the pm-space $\mathcal{R}$ is a metric space, then the associated metric space $(\mathbb{P}, d)$ obviously is its completion. If $\mathcal{R}$ is the canonical pm-space of open balls of a Euclidean space E, then $(\mathbb{P}, d)$ coincides with E.

If $P \in \mathbb{P}$ and $r \in \mathcal{R}$, we say that P *belongs* to r, briefly $P \in r$, provided that there is a sequence $\langle p_n \rangle$ representing P with $p_n \leqslant r$ for every $n \in N$. We denote by $\pi(r)$ the set of all points belonging to r.

Axioms (A_1)–(A_5) do not guarantee the existence of points in a ppm-space. In order to get this, we have to add some new axiom. For example, we may assume that every region contains arbitrarily small regions.

(A_6) $\forall \varepsilon > 0 \ \ \forall r \in \mathcal{R} \ \ \exists r' \leqslant r$ such that $|r'| < \varepsilon$.

Obviously, (A_6) is equivalent to saying that every region has points. In this case, the class $\mathcal{R}' = \pi(\mathcal{R}) = \{\pi(r)\colon\ r \in \mathcal{R}\}$ of subsets of $\mathbb{P}$ defines a canonical ppm-space $\mathcal{R}'$, the *canonical ppm-space associated with* $\mathcal{R}$.

THEOREM 4. *Assume that $\mathcal{R}$ is a ppm-space satisfying* (A_6). *Then*:

(i) $(\mathbb{P}, d)$ *is a complete metric space*;

(ii) *for every $r \in R$, $\pi(r)$ is a closed subset*;

(iii) *if $r \leqslant s$, then $\pi(r) \subseteq \pi(s)$*;

(iv) $\delta(r,s) \leqslant \delta(\pi(r), \pi(s)) \leqslant \delta_{||}(r,s)$ *and* $|\pi(r)| = \sup\{\delta(u,v)\colon\ u \leqslant r,\ v \leqslant r\}$.

Moreover, if $\delta = \delta_{||}$ and $|| = ||_\delta$, then the function π: $\mathcal{R} \to \mathcal{R}'$ is an isomorphism between $\mathcal{R}$ and its associated canonical space $\mathcal{R}'$.

We conclude this section by noting that, since many classical geometries may be defined in terms of axioms about metric spaces, pointless metric geometry leads to complete axiomatizations of these geometries without the primitive notion of 'point'.

5. Physical geometry

5.1. *The first axioms*

Schmidt [1979] proposed perhaps the most complete treatment of Euclidean pointless geometry. The term 'physical geometry' means that this geometry is understood as a theory of the physical space. As a consequence, axioms are thought as empirical laws governing the behaviour of rigid bodies, rather than as a mathematical device to produce the desired theorems. The goal is to construct a set $\mathbb{P}$ of points, a topology N^{top} on $\mathbb{P}$ and a group $\overline{T}$ of transformations of $\mathbb{P}$ such that $(\mathbb{P}, N^{\text{top}}, \overline{T})$ is isomorphic to the Euclidean space E equipped with the usual topology and the group generated by the translations and the rotations.

The following are the first two axioms of physical geometry:

(R_1) $(R, \leqslant)$ is a *weakly distributive lattice* (i.e. $r \wedge z = 0$ and $r' \wedge z = 0$ implies $(r \vee r') \wedge z = 0$) with an empty region 0, and $R \neq \{0\}$.
(R_2) T is a group of automorphisms of $(R, \leqslant)$.

Let r and r' be two regions. If r contains each overlapping displacement of r', i.e. if $\tau r' \leqslant r$ for every $\tau \in T$ such that $r' \wedge \tau r' \neq 0$, then we say that r' is a *kernel* of r.

(R_3) Every nonempty region has a kernel.

As usual, the group T determines an equivalence relation in R; we call *shapes* the related equivalence classes.

(R_4) Each region can be covered by finitely many regions of any given shape (with possible overlappings).

5.2. *Points and Cauchy filters*

To define the points, Schmidt uses a construction similar to the completion of a uniform space by the use of minimal Cauchy filters. The points are defined as suitable filters of $(R, \leqslant)$ as is usual in lattice theory. Now, it is very natural to require that a point may be represented by regions as small as we wish and therefore to only consider filters F such that, for every nonempty region r, there exists $s \in F$ such that $s \leqslant \tau r$ for a suitable $\tau \in T$. Since F is a filter, this is equivalent to require that $\tau r \in F$. We thus have the following definition:

DEFINITION 1. A *Cauchy filter* (briefly, C-filter) is any filter of R containing regions of every shape.

Now, the C-filters are not suitable candidates for a definition of 'point'. In fact, it is possible that, in a sense, two different C-filters F and F' be *infinitely close*, i.e. for every $r \neq 0$ there is $\tau \in T$ such that $\tau r \in F$ and $\tau r \in F'$. A similar situation arises when we complete a metric space: it is possible that two different Cauchy sequences represent the same point. This leads us to identify two infinitely close filters F and F' and to call 'point' any complete equivalence class of the relation 'is infinitely close to'. Equivalently, it is possible to identify the points with suitable filters representative of such classes; such filters are the minimal C-filters. Indeed, two minimal, infinitely close C-filters coincide, and one may prove that every C-filter contains an infinitely close minimal filter. Thus we get the following definition.

DEFINITION 2. A *point* P of $(R, \leqslant, T)$ is any C-filter, minimal in the class of the C-filters.

We denote by $\mathbb{P}$ the set of points, and put $\pi(r) = \{P \in \mathbb{P}\colon r \in P\}$.
The following proposition shows that $\pi\colon R \to P(\mathbb{P})$ is a lattice representation in a weak sense.

PROPOSITION 1. *For every* $r, s \in R$:

(i) *if* $r \leqslant s$ *then* $\pi(r) \leqslant \pi(s)$;

(ii) $\pi(r) = 0$ *if and only if* $r = 0$;

(iii) $\pi(r \wedge s) = \pi(r) \cap \pi(s)$;

(iv) $\pi(r \vee s) \supseteq \pi(r) \cup \pi(s)$.

Notice that, since a C-filter is not always prime, in general in (iv) the equality does not hold. In other words, it is possible for P to be a point of the union $r \vee s$ of two regions r and s while being neither a point of r nor a point of s.

5.3. *Topological and metric structures*

The next step is to define a suitable topology on $\mathbb{P}$. Schmidt proceeds as follows: remember that a *uniform structure*, or *uniformity*, on a set X is a filter U of $X \times X$ such that:

(U_1) every element of U contains the diagonal of $X \times X$;
(U_2) if $V \in U$ then $V^{-1} \in U$;
(U_3) for each $V \in U$ there exists a $W \in U$ such that $W \circ W \subseteq V$.

Each element V of U is then called an *entourage*. If $x, y \in X$ and $(x, y) \in V$, we say that x and y are V-close. A *fundamental system of entourages* for U is any set B of entourages such that every entourage contains an element of B. Every uniformity induces a topology, in which the filter of neighbourhoods of a point x is defined by the sets $V(x) = \{y \in x\colon (x, y) \in V\}$ with $V \in U$.

Schmidt defines an uniformity on $\mathbb{P}$ with a fundamental system of entourages

$$N = (N_r)_{r \in R},$$

where

$$N_r = \{(P, Q) \in \mathbb{P} \times \mathbb{P}\colon P, Q \in \tau r \text{ for a suitable } \tau \in T\};$$

N^{top} is defined as the topology induced by this uniformity. One may prove that the elements of $\pi(R)$ are open and relatively compact with respect to this topology.

On account of the very rich structure of a physical space, it is also possible to define a metric in $\mathbb{P}$ by utilizing chains of regions. A *chain of shape* $[v]$ is a sequence $s_1, \ldots, s_n$ of regions of the same shape $[v]$ such that each s_i overlaps s_{i+1}. If P is a point of s_1 and Q a point of s_n, we say that $s_1, \ldots, s_n$ is a *chain between P and Q*. Given a shape $[v]$, we denote by $\lambda(P, Q, v)$ the minimal length of the chains of shape $[v]$ between P and Q, if such exists. If we choose as unity of length a pair of points, say u and u', and set $\delta(P, Q, v) = \lambda(P, Q, v)/\lambda(u, u', v)$, then the *distance* $d(P, Q)$ is defined by

$$d(P, Q) = \lim_{r \to 0} \delta(P, Q, r),$$

where this equality means that for any $\varepsilon > 0$ there exists a $r \neq 0$ such that for all $v \leqslant r$ we have $|\delta(P, Q, v) - d(P, Q)| < \varepsilon$.

Obviously the above definition of distance is meaningful only if, given any shape $[v]$, the following two statements hold:

– for every pair of points p and q, there exists a chain of shape $[v]$ between p and q;
– for every pair of points p and q, the above mentioned limit exists.

This forces us to retain two additional axioms. The first one specifies that every transport can be made of arbitrarily small transport, and so enables us to prove that two points always are connected by a suitable chain:

(R_5) For every $\sigma \in T$ and every $s_1, \ldots, s_n \in R \setminus \{0\}$, there exist $\tau_1, \ldots, \tau_m \in T$ such that $s_i, \tau_1 s_i, \tau_2 s_i, \ldots, \tau_m s_i, \sigma s_i$ is a chain of shape $[s_i]$ for $i = 1$ to n.

The formulation of axiom (R_6) requires a previous definition of the group $\overline{T}$ of congruent mappings of $\mathbb{P}$. Now, every element τ of T induces a map

$$\tilde{\tau}\colon \mathbb{P} \to \mathbb{P} \quad \text{defined by} \quad \tilde{\tau}(P) = \{\tau x\colon\ x \in P\}.$$

The map $\sim$ is a homomorphism from T into the permutation group on $\mathbb{P}$. Unfortunately, the group $\widetilde{T} = \{\tilde{\tau}\colon\ \tau \in T\}$ is not an adequate candidate to represent the whole congruence group of $\mathbb{P}$. Indeed, $\widetilde{T}$ operates only *almost transitively*, and not transitively, in general. This means that, for every point P, the set $\{\tilde{\tau}P\colon\ \tilde{\tau} \in \widetilde{T}\}$ is dense in $\mathbb{P}$ but different from $\mathbb{P}$, in general. Moreover, $\widetilde{T}$ may be incomplete. Let us define the topology t on T, whose open subbase is constituted by the sets

$$T(r) = \{\tau \in T\colon\ \tau r \wedge r \neq 0\}, \quad r \neq 0,$$

and their images by translations. T is a topological group with respect to t, and t determines a topology $\tilde{t}$ on $\widetilde{T}$ by the way of the homomorphism $\sim$. One may prove that $\tilde{t}$ coincides with the topology of compact convergence c; however, $\widetilde{T}$ is not complete with respect to this very natural topology.

Consequently, Schmidt builds up a completion $\overline{T}$ of $\widetilde{T}$. Let us define by c^s and c^d the left and right uniformities determined by c, and set $c^z = c^s \vee c^d$. Then the group $\overline{T}$ is defined as the c^z-completion of $\widetilde{T}$.

It is possible to give the following further axiom ensuring the existence of $\lim_{v \to 0} \delta(P, Q, v)$. Remember that, given two points P and Q, the *orbit* $J_P Q$ is the subset $\{\tau Q\colon\ \tau \in \overline{T},\ \tau P = P\}$.

(R_6) There exist two points $P, Q \in \mathbb{P}$ such that the orbit $J_P Q$ dissects the space $\mathbb{P}$, i.e. $\mathbb{P} \setminus J_P Q$ is not connected.

Axioms (R_1)–(R_6) enable us to prove that the pair $(\mathbb{P}, d)$ is a complete, locally compact metric space and that, if another unity of length is chosen, the corresponding metric coincides with d, up to a constant factor.

5.4. *The basic theorems of physical geometry*

By (R_1)–(R_6) we are able to prove the following basic theorem.

THEOREM 2. *$(\mathbb{P}, N^{\text{top}})$ is a connected, locally compact, uniform Hausdorff space. Moreover, $\overline{T}$ is a complete group of uniform homeomorphisms acting transitively on $\mathbb{P}$ and at least one of its orbits dissects the space $(\mathbb{P}, N^{\text{top}})$.*

In accordance with the results of Freudenthal [1955/56], such properties guarantee that $(\mathbb{P}, N^{\text{top}}, \overline{T})$ belongs to a very small class of possible geometries. Euclidean geometry could be obtained by adding two axioms asserting that the curvature vanishes and that the space has dimension 3.

We first have to define the dimension of a physical space. Given a nonempty region r (the radius), we say that a set V of regions is *r-approximately overlapping* if there is a nonempty region c (the centre) such that every element v of V overlaps with a suitable τr containing c. The dimension of R is the smallest number N such that each region can be covered by a finite number of arbitrarily small regions such that at most $N+1$ are r-approximately overlapping. The following is a precise definition.

DEFINITION 3. The number $\dim R$ is the smallest number N such that, given two nonempty regions s and v, a covering $v_1, \ldots, v_n$ of s and a nonempty region r exist such that:

(i) for every i, $v_i \leqslant \tau v$ for a suitable $\tau \in T$;

(ii) at most $N+1$ elements of $v_1, \ldots, v_n$ are r-approximately overlapping.

(R_7) $\dim R = 3$.

The vanishing of the curvature is guaranteed by the following.

(R_8) There exists a neighbourhood $U \in N^{\text{top}}$ such that if u, v, x, y, z are points of U then $d(x,u) = d(z,u)$, $d(x,v) = d(y,v)$, $d(x,z) = 2d(u,z)$ and $d(x,y) = 2d(v,y)$ imply $d(z,y) = 2d(u,v)$.

Axioms (R_1)–(R_8) enable us to prove the main theorem of Schmidt [1979].

THEOREM 3 (Schmidt [1979]). *The structure $(\mathbb{P}, N^{\text{top}}, \overline{T})$ is isomorphic to the Euclidean structure $(E, \mathcal{T}, I_S)$, where E is the Euclidean three-dimensional space, $\mathcal{T}$ its natural topology and I_S the group generated by the translations and rotations of E.*

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CHAPTER 19

Geometry over Rings

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Contents

Introduction 1035
1. Free modules and their subspaces 1036
2. Rings of stable rank 2 1038
3. Algebraic description of $P_n(R)$ 1042
4. Axiomatic approach: V-spaces 1044
5. Collineations and affine collineations. Coordinatization of V-spaces 1048
6. Homomorphisms of V-spaces 1053
7. Projective spaces over full matrix rings 1056
8. Klingenberg spaces 1058
9. Hjelmslev geometries 1063
10. Transvection V-planes 1065
11. Faulkner planes 1069
12. Affine ring planes and Barbilian domains 1074
13. Leißner planes 1079
14. Open problems 1080
References 1081

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Introduction

In classical geometry coordinates are elements of a field or division ring (skew field). Is it possible to extend this to rings in general? The first explicit step in this direction was made, to my knowledge, by C. Segre [1911], who studied three-dimensional projective geometry over the ring of dual numbers, $\mathbb{R}[\varepsilon]$ with $\varepsilon^2 = 0$, and some related extensions of the reals. Dual numbers were not uncommon in geometry around the turn of the century; cf. Grünwald [1906], Petersen [1898], Study [1903], and also Klein [1926]. Their appearance in projective geometry, therefore, is not amazing and was already implicitly made in mechanics and line geometry, for instance. In Hjelmslev [1929–1949] they are used for what J. Hjelmslev (= J. Petersen) considered a more 'natural' approach to congruence geometry, i.e. more in accordance with physical reality than the traditional approach using real numbers. A systematic study of projective planes over large classes of associative rings was initiated by D. Barbilian. His very general approach in Barbilian [1940, 1941] remained rather unsatisfactory, however; his axioms were partly of a geometric nature, partly algebraic as pertaining to the ring of coordinates, and there were a number of difficulties which Barbilian could not overcome.

W. Klingenberg took up the line of Segre and Hjelmslev and gave an axiomatic treatment of projective and affine planes over Hjelmslev rings, which are local rings satisfying some extra conditions (see Klingenberg [1954a,b, 1955]), and of projective geometry over local rings in general (see Klingenberg [1956]). F. Bingen [1966] made an algebraic study of projective geometry over semiprimary rings; he defines a projective space as the structure consisting of all free submodules having a complement in a free module of finite rank.

Carrying further the line of Barbilian and Bingen, the present author has given an axiomatic description of projective planes and higher-dimensional geometries in Veldkamp [1981, 1984, 1985a, 1987]; a most satisfactory situation is reached by using rings of stable rank 2, a somewhat wider class than semiprimary rings. Partly in collaboration with J.C. Ferrar he also developed a theory of homomorphisms of such geometries, which are, roughly speaking, noninjective collineations (see Ferrar and Veldkamp [1985, 1987], Veldkamp [1985a,b,c, 1987]).

There are approaches to projective planes over much larger classes of associative rings, e.g., one by F. Knüppel [1987]. The axiomatic description for these planes is not yet complete, however. More satisfactory is the situation in the affine case. Affine planes over *two-sided units rings* (i.e. rings in which $\alpha\beta = 1$ implies $\beta\alpha = 1$) have been treated by W. Leißner [1975]. F. Radó [1980] generalized this to planes over arbitrary associative rings with unity.

The planes we have been speaking about so far might be called Desarguesian in analogy with the classical theory of projective planes. The theory has been extended in the non-Desarguesian direction. A first example of such a plane was studied by T.A. Springer and the present author [1968]. J.R. Faulkner [1983a,b] studied the ring analogs of Moufang planes. Faulkner [1989] presents a very general theory encompassing all projective planes whose coordinate rings are alternative, which is somewhat weaker than associative, and having two-sided units.

Some words about terminology. The names *Hjelmslev planes* (*spaces*) and *Klingenberg planes* (*spaces*) will be used here as is standard nowadays. Terms as *Barbilian plane* and *Barbilian space* in the existing literature are a source of confusion since almost each author chooses his favourite assumptions about their precise meaning. Following Faulkner [1989] we therefore call *Barbilian space* (*plane*) any structure consisting of points and hyperplanes (or lines) provided with an incidence and a neighbour relation. Depending on the specific axioms imposed on a Barbilian plane or space, we call it a *Faulkner plane*, or a *Veldkamp plane* or *space*; we think that clarity is preferable to modesty, and other authors have already used the latter terminology. The axiomatic set-up for an affine plane over a two-sided units ring will be named *Leißner plane*.

In this chapter we will first present the theory of projective planes and spaces over associative rings of stable rank 2, both from an algebraic and an axiomatic point of view. Then we deal with homomorphisms of these. Projective spaces over full matrix rings are treated as an example; they can be interpreted in terms of ordinary projective spaces over division rings. We next go over to spaces over local rings (Klingenberg spaces) and over Hjelmslev rings (Hjelmslev spaces), presented here as special instances of the general theory developed before, together with a few other special cases. Then comes a section on transvection planes, after which we pay attention to Faulkner planes. We complete this chapter with sections on affine planes over two-sided units rings and their axiomatic characterization as Desarguesian Leißner planes, and a short list of open problems.

More lattice-theoretic approaches to ring geometry will not come into focus here. For a survey of that field, the reader may consult Chapter 21 of this Handbook by U. Brehm, M. Greferath and S.E. Schmidt.

1. Free modules and their subspaces

Until further notice we consider only associative rings with unit element 1. For a ring R, we denote by R^* the group of its units (invertible elements). R° is the *oppositive ring*, i.e. R with the same addition but with reversed multiplication: $\alpha \circ \beta = \beta\alpha$.

We denote the free right (left) module on n generators over a ring R by R^n (${}^nR$, respectively). Elements of R^n are column vectors

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = (\xi_1, \ldots, \xi_n)^T$$

(T denoting transposition of a matrix), elements of ${}^nR$ are row vectors $(\lambda_1, \ldots, \lambda_n)$. The dual pairing

$$\langle \cdot \mid \cdot \rangle \colon {}^nR \times R^n \to R$$

is defined by

$$\langle \ell \mid x \rangle = \ell x = \langle \lambda_1, \ldots, \lambda_n \mid \xi_1, \ldots, \xi_n \rangle = \sum_{i=1}^{n} \lambda_i \xi_i$$

for $\ell = (\lambda_1, \dots, \lambda_n) \in {}^nR$ and $x = (\xi_1, \dots, \xi_n)^T \in R^n$. Thus, we consider ${}^nR$ and R^n as each other's dual. The mapping

$$(\alpha_1, \dots, \alpha_n) \mapsto (\alpha_1, \dots, \alpha_n)^T$$

identifies the left module ${}^nR$ with the right module $(R^\circ)^n$, which enables us to carry over definitions, and often results too, from right modules to left modules, by simply interchanging the order of the factors in all products.

By a *subspace* of R^n we understand a free direct summand, i.e. a submodule $L \cong R^p$ for some p such that $R^n = L \oplus M$ for another submodule M. If L is a subspace of R^n and N a submodule of L, then N is a subspace of R^n if and only if it is a subspace of L. For if $R^n = N \oplus M$, then $L = N \oplus L \cap M$, and the converse is also easy.

An element a of R^n generates a subspace aR if and only if $\ell \in {}^nR$ exists such that $\langle \ell \mid a \rangle = 1$. For if aR is a subspace, then it is free and $R^n = aR \oplus M$ for a submodule M. So each $x \in R^n$ can be written in a unique way as $x = a\lambda + y$ with $\lambda \in R$ and $y \in M$, and ℓ: $x \mapsto \lambda$ is a linear form as required. Conversely, if $\langle \ell \mid a \rangle = 1$, then

$$R^n = aR \oplus \ker(\ell)$$

since we can write any $x \in R^n$ as

$$x = a\langle \ell \mid x \rangle + (x - a\langle \ell \mid x \rangle) \quad \text{with } x - a\langle \ell \mid x \rangle \in \ker(\ell),$$

and aR is free since $a\lambda = 0$ implies $\lambda = \langle \ell \mid a \rangle \lambda = \langle \ell \mid a\lambda \rangle = 0$. We call $a \in R^n$ such that $\langle \ell \mid a \rangle = 1$ for some $\ell \in {}^nR$ *unimodular.* $a = (\alpha_1, \dots, \alpha_n)^T$ is unimodular if and only if $R = R\alpha_1 + \cdots + R\alpha_n$ (sum of left ideals).

Some comments on our definition of subspace are in order here. If we assume the ring R to have stable rank 2 (see Section 2), then $R^n = L \oplus M$ with L free implies that M is free, i.e. any complement of a subspace is a subspace (see 2.8(ii)). For a general R this need not be true any more, so then it may be desirable to modify the definition, e.g., as follows. A submodule L of R^n is a *subspace* if $L \cong R^p$ and $R^n = L \oplus M$ with $M \cong R^{n-p}$, for some p, $0 \leqslant p \leqslant n$. If R has invariant basis number (cf. 2.6), it suffices to require that L and M are free, for then their ranks must be complementary. With this definition, however, aR need not be a subspace if a is unimodular. (The converse remains true.) Anyway, for rings of stable rank 2 it all amounts to the same, as we shall see.

The group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices, called the *general linear group* over R, acts on R^n on the left, and on ${}^nR$ on the right, and

$$\langle \ell \mid Ax \rangle = \langle \ell A \mid x \rangle \quad \text{for } A \in \mathrm{GL}_n(R),\ \ell \in {}^nR,\ x \in R^n.$$

So the action of A on R^n induces the contragradient action $\ell \mapsto \ell A^{-1}$ on ${}^nR$. It is clear that $\mathrm{GL}_n(R)$ transforms subspaces of R^n into subspaces.

E_{ij} is the $n \times n$ matrix with 1 as i, j-entry and zeros elsewhere. The invertible matrices

$$E_{ij}(\alpha) = I_n + \alpha E_{ij} \quad \text{with } i \neq j \text{ and } \alpha \in R$$

(I_n denoting the $n \times n$ indentity matrix) are called *elementary matrices*. The *elementary group* $\mathrm{E}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by the elementary matrices.

It is sometimes convenient to reduce geometric questions to the case of rings with zero radical. We recall that the *Jacobson radical* of a ring R, denoted by $\operatorname{rad} R$, is the intersection of all maximal left ideals in R, and also the intersection of all maximal right ideals. It is a two-sided ideal, and $R/\operatorname{rad} R$ has zero radical. An element α of R belongs to $\operatorname{rad} R$ if and only if $1 + \alpha\xi$ has a right inverse for all $\xi \in R$, or equivalently, if $1 + \xi\alpha$ has a left inverse for all ξ. For more information, the reader should consult textbooks like Adamson [1971], Cohn [1971, 1977], Divinsky [1965], Jacobson [1956, 1980].

1.1. PROPOSITION. *For $\xi \in R$, let $\bar{\xi}$ denote ξ mod $\operatorname{rad} R$. A vector $x = (\xi_1, \ldots, \xi_n)^T$ is unimodular in R^n if and only if $\bar{x} = (\bar{\xi}_1, \ldots, \bar{\xi}_n)^T$ is unimodular in $(R/\operatorname{rad} R)^n$.*

PROOF. If $\bar{x}$ is unimodular, there are $\lambda_1, \ldots, \lambda_n \in R$ such that

$$\sum_{i=1}^{n} \bar{\lambda}_i \bar{\xi}_i = \bar{1}.$$

This means that

$$\sum_{i=1}^{n} \lambda_i \xi_i = 1 + \alpha \quad \text{for some } \alpha \in \operatorname{rad} R.$$

Since $1 + \alpha$ is invertible, we get

$$\sum_{i=1}^{n} (1 + \alpha)^{-1} \lambda_i \xi_i = 1,$$

which shows that x is unimodular. The proof in the opposite direction is immediate. □

The above result can be extended to general subspaces: a submodule of $(R/\operatorname{rad} R)^n$ is a subspace if and only if it is the image of a subspace of R^n under the natural projection $R^n \to (R/\operatorname{rad} R)^n$; see Veldkamp [1988].

We shall define the *projective space of dimension n over a ring R*, denoted by $P_n(R)$, as the structure consisting of all subspaces of the free right module R^{n+1}, for $n \geqslant 0$. Now, in its generality this is a rather wild structure. The geometry will behave very decently if we assume that the ring R has stable rank 2 (see Section 2). The notion of stable rank of a ring comes from algebraic K-theory where it plays an important role in results on modules and the groups $\mathrm{GL}_n(R)$ and $\mathrm{E}_n(R)$, so it is not so amazing that it turns up here; cf. Bass [1968, 1974]. In the next section we therefore make the reader acquainted with stable rank.

2. Rings of stable rank 2

In the previous section we introduced unimodular vectors. The following condition allows to reduce unimodular vectors in R^n to similar vectors in R^{n-1}.

CONDITION SR_n ($n \geqslant 2$). *For each unimodular* $(\xi_1, \ldots, \xi_n)^T \in R^n$ *there exist* $\alpha_1, \ldots, \alpha_{n-1} \in R$ *such that* $(\xi_1 + \alpha_1\xi_n, \ldots, \xi_{n-1} + \alpha_{n-1}\xi_n)^T$ *is unimodular in* R^{n-1}.

For $n = 2$, in particular, this means: for each unimodular $(\xi_1, \xi_2)^T \in R^2$ there exists $\alpha \in R$ such that $\xi_1 + \alpha\xi_2$ has a left inverse. A ring may or may not satisfy the condition SR_n, but once it satisfies SR_n, it satisfies SR_m for all $m > n$.

2.1. PROPOSITION. SR_n *implies* SR_{n+1} *for all* $n \geqslant 2$.

For a proof, see Krusemeyer [1973], or Veldkamp [1985a], 2.1.

2.2. DEFINITION. The *stable rank* of a ring R, denoted by $\mathrm{sr}(R)$, is the least n such that SR_n is valid for R, and ∞ if no such n exists.

WARNING. What we call SR_n here is often called SR_{n-1} in the literature, so then the stable rank of a ring is 1 lower than the one considered here.

For a large part of this chapter we focus attention on projective geometry over rings of stable rank 2. A complete account of results on stable rank and rings of stable rank 2 as needed in this context is given in Veldkamp [1985a], §2, where we have collected all relevant results and their proofs, which come from a large number of sources (Bass [1964, 1968, 1974], Van der Kallen [1976], Krusemeyer [1973], Vasershtein [1969, 1971], Veldkamp [1981]). Here we confine to quoting the most important results and examples, referring the reader to Veldkamp [1985a] or the original sources for most of the proofs.

To begin with, some examples. Division rings of course have stable rank 2 since every nonzero element has an inverse. A full matrix ring $M_n(D)$ over a skew field has stable rank 2. More generally, a semiprimary ring, i.e. a ring R such that $R/\operatorname{rad} R$ is a semisimpleArtin ring, has stable rank 2.

The ring of integers $\mathbb{Z}$ does not have stable rank 2. For $(5,7)^T$ is a unimodular vector in $\mathbb{Z}^2$, but no element of the form $5 + 7n$ is invertible in $\mathbb{Z}$. In fact, $\mathrm{sr}(\mathbb{Z}) = 3$. A similar argument shows that a polynomial ring $K[X_1, \ldots, X_n]$ over a field K must have stable rank $\geqslant 3$.

2.3. PROPOSITION. *The opposite ring* R° *has the same stable rank as R itself.*

This result has a far-reaching consequence. For identifying a left module ${}^nR$ with the right module $(R^\circ)^n$ over the opposite ring as explained in the beginning of Section 1 we can carry over the definition of unimodular vectors and the condition SR_n to left modules: a vector $\ell = (\lambda_1, \ldots, \lambda_n)$ is unimodular in ${}^nR$ if there exists $x \in R^n$ such that $\langle \ell \mid x \rangle = 1$, i.e. if $\lambda_1 R + \cdots + \lambda_n R = 1$; the condition SR_n for left modules then says:

SR'_n ($n \geqslant 2$): *For each unimodular* $(\lambda_1, \ldots, \lambda_n) \in {}^nR$ *there exist* $\alpha_1, \ldots, \alpha_{n-1} \in R$ *such that* $(\lambda_1 + \lambda_n\alpha_1, \ldots, \lambda_{n-1} + \lambda_n\alpha_{n-1})$ *is unimodular in* ${}^{n-1}R$.

Now 2.3 says that, if SR_n holds for R, then it holds for the opposite ring R° too, and therefore SR'_n holds for R. In other words, the condition SR'_n is equivalent to SR_n.

2.4. PROPOSITION.

(i) $\mathrm{sr}(R) = \mathrm{sr}(R/\mathrm{rad}\, R)$.

(ii) *If* φ: $R \to S$ *is a surjective homomorphism, then* $\mathrm{sr}\,(S) \leqslant \mathrm{sr}\,(R)$. *In particular, if* R *has stable rank* 2, *then* S *has stable rank* 2.

(iii) *The stable rank of the direct product of (any finite or infinite number of) rings* R_α *is the maximum of all* $\mathrm{sr}\,(R_\alpha)$.

The proof of (i) is an immediate consequence of Proposition 1.1. In the more general situation of (ii), equality need not hold; e.g., $\mathbb{Z}$ has stable rank 3, but $\mathbb{Z}/(p)$, with p prime, has stable rank 2 since it is a field. The proof of (iii) is immediate from the definition: if SR_n holds for all R_α, then it also holds for their product.

In a matrix ring over a division ring an element has a left inverse if and only if it has a right inverse; hence the same is true in semiprimary rings, since $1 + \alpha$ is invertible in R if $\alpha \in \mathrm{rad}\, R$. A nice result of Kaplansky and Lenstra says that all rings of stable rank 2 have this property (cf. Van der Kallen [1976], p. 122, Lemma 5, or Veldkamp [1985a], 2.10).

2.5. PROPOSITION. *If* $\mathrm{sr}(R) = 2$, *then* $\xi\eta = 1$ *implies* $\eta\xi = 1$ *in* R.

It is well known that commutative rings have invariant basis number, i.e. $R^s \cong R^t$ implies $s = t$; see, e.g., Jacobson [1974]. For noncommutative rings this is not true in general; cf. Cohn [1966a], McDonald [1984]. It is a pleasant circumstance for projective geometry in the sense of this chapter that rings of stable rank 2 behave decently in this respect.

2.6. PROPOSITION. *A ring* R *of finite stable rank has invariant basis number, i.e.* $R^s \cong R^t$ *implies* $s = t$.

The following result is of crucial importance. To demonstrate the role played by the condition SR_n, we shall include a complete proof.

2.7. PROPOSITION. $\mathrm{E}_n(R)$ *is transitive on unimodular vectors in* R^n, *provided* $n \geqslant \mathrm{sr}(R)$.

PROOF. Let $x = (\xi_1, \ldots, \xi_n)^T$ be unimodular in R^n. By SR_n, which holds since $n \geqslant \mathrm{sr}\,(R)$, we can find $\alpha_1, \ldots, \alpha_{n-1}$ in R such that $(\xi_1 + \alpha_1\xi_n, \ldots, \xi_{n-1} + \alpha_{n-1}\xi_n)^T$ is unimodular, say

$$\sum_{j=1}^{n-1} \mu_j(\xi_j + \alpha_j\xi_n) = 1.$$

The transformation

$$\prod_{i=1}^{n-1} E_{ni}\big((1 - \xi_n)\mu_i\big) \cdot \prod_{i=1}^{n-1} E_{in}(\alpha_i)$$

transforms x into $(\xi_1 + \alpha_1\xi_n, \ldots, \xi_{n-1} + \alpha_{n-1}\xi_n, 1)^T$, which is mapped upon $(0, \ldots, 0, 1)^T$ by

$$\prod_{i=1}^{n-1} E_{in}(-\xi_i - \alpha_i\xi_n).$$

Thus, $\mathrm{E}_n(R)$ can transform any unimodular vector into $(0, \ldots, 0, 1)^T$, which proves transitivity on unimodular vectors. □

A number of consequences of this are derived in the following corollary.

2.8. COROLLARY.

(i) *Every unimodular vector in R^n is part of a basis, if $n \geqslant \mathrm{sr}(R)$.*

(ii) *If $R^n = L \oplus M$ and L is free, then M is free, provided $\mathrm{sr}(R) = 2$; in other words, any complement of a subspace is a subspace.*

(iii) *If $\mathrm{sr}(R) = 2$, then for any two bases $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of R^n, with $n \geqslant 1$, there exists $A \in \mathrm{E}_n(R)$ such that $Aa_i = b_i$ for $i < n$ and $Aa_n = b_n\gamma$ for some $\gamma \in R^*$.*

PROOF. Part (i) is clear from 2.7.

(ii) Let $L \cong R^p$. We proceed by induction on p. In case $p = 1$, pick a unimodular vector a spanning L. By (i), R^n has a basis $e_1, \ldots, e_n$ with $e_1 = a$. Each e_j for $j > 1$ can be written as $e_j = e_1\alpha_j + f_j$ with $\alpha_j \in R$ and $f_j \in M$. The elements $f_2, \ldots, f_n$ are easily seen to form a basis of M. In case $p > 1$, write $L = L_1 \oplus L_2$ with $L_1 \cong R$ and $L_2 \cong R^{p-1}$. Then $R^n = L_1 \oplus L_2 \oplus M$. From the case $p = 1$ we infer that $L_2 \oplus M \cong R^{n-1}$. By induction hypothesis we get that M is free.

(iii) Here we use induction on n. If $n = 1$, we have nothing to prove, so assume now $n > 1$. It suffices to consider the case with

$$b_1 = (1, 0, \ldots, 0)^T, \; \ldots, \; b_n = (0, \ldots, 0, 1)^T.$$

We first map a_1 on b_1 using an element of $\mathrm{E}_n(R)$, which is possible by 2.7. Let $a_2, \ldots, a_n$ be moved to $a'_2, \ldots, a'_n$ under this transformation. Since $b_1, a'_2, \ldots, a'_n$ form a basis, we can find $\ell \in {}^nR$ with $\langle \ell \mid b_1 \rangle = 1$ and $\langle \ell \mid a'_j \rangle = 0$ for $j > 1$, so $\ell = (1, \lambda_2, \ldots, \lambda_n)$ for certain λ_i. The matrix $A = E_{12}(\lambda_2) \ldots E_{1n}(\lambda_n) \in E_n(R)$ leaves $b_1 = (1, 0, \ldots, 0)^T$ fixed, and its inverse $A^{-1} = E_{12}(-\lambda_2) \cdots E_{1n}(-\lambda_n)$ moves ℓ in ${}^nR$ to $(1, 0, \ldots, 0)$, so A must map $a'_2, \ldots, a'_n$ on vectors a''_i in R^n having first coordinate 0. Now by induction assumption, $a''_2, \ldots, a''_n$ can be mapped upon $b_2, \ldots, b_n\gamma$ by a product of elementary matrices. □

In view of 2.6 we can define for rings R with $\mathrm{sr}(R) = 2$ the *rank* of a free module R^n as $\mathrm{rk}(R^n) = n$. For a subspace L of R^n we have $\mathrm{rk}(L) \leqslant n$, with equality only for $L = R^n$. For any complement M of L in R^n, $\mathrm{rk}(M) = n - \mathrm{rk}(L)$. 2.8(ii) says that a submodule L is a subspace if and only if it has a basis which is part of a basis of R^n.

We now have a closer look at the relations between a right module R^n and the corresponding left module ${}^nR$, its dual. For any basis $e_1, \ldots, e_n$ of R^n we can define the *dual basis* $e_1^*, \ldots, e_n^*$ of ${}^nR$ by $\langle e_i^* \mid e_j \rangle = \delta_{ij}$, where δ_{ij} denotes the Kronecker symbol: $\delta_{ij} = 1$ if $i = j$, and 0 otherwise.

2.9. DEFINITION. For any subset $S \subseteq {}^nR$, we put:

$$S^\perp = \{x \in R^n\colon\ \langle \ell \mid x\rangle = 0 \text{ for all } \ell \in S\}.$$

Similarly we define $T^\perp$ in ${}^nR$ for subsets T of R^n.

In general, the correspondence $S \mapsto S^\perp$ does not have the nice properties we know from vector spaces. But if we restrict to rings of stable rank 2 and to subspaces, things go very smoothly again.

2.10. PROPOSITION. *If R has stable rank 2, the map $S \mapsto S^\perp$ is bijective from the set of rank d subspaces of ${}^nR$ onto the set of rank $n-d$ subspaces of R^n; its inverse is given by $T \mapsto T^\perp$. These maps reverse inclusion*: $S_1 \subseteq S_2 \Leftrightarrow S_1^\perp \supseteq S_2^\perp$.

PROOF. Let S be a subspace of ${}^nR$ of rank d. Then ${}^nR$ has a basis $a_1^*, \ldots, a_n^*$ such that the first d vectors $a_1^*, \ldots, a_d^*$ form a basis of S. Take the corresponding dual basis $a_1, \ldots, a_n$ in R^n. It is clear that $S^\perp$ is the subspace of R^n spanned by $a_{d+1}, \ldots, a_n$, and that $S^{\perp\perp} = S$. The rest of the proof is clear. □

3. Algebraic description of $P_n(R)$

From now on we consider, until further notice, only associative rings with unity of stable rank 2, unless we explicitly state otherwise.

Part of the theory of projective spaces over rings can be developed for larger classes of rings, but since we eventually aim at the stable rank 2 case we shall, for clarity of exposition, restrict to that right from the beginning.

At the end of Section 1 we have already given a tentative definition of the projective space $P_n(R)$. We repeat that here in a more extended form.

3.1. DEFINITION. $P_n(R)$, the *n-dimensional projective space over the ring R*, is the structure consisting of all subspaces of the free right module R^{n+1}, for $n \geqslant 0$. A *subspace of (projective) dimension d*, or *d-subspace*, of $P_n(R)$ is a subspace of rank $d+1$ of R^{n+1}, for $-1 \leqslant d \leqslant n$. A subspace of dimension 0 is called a *point*, a subspace of dimension $n-1$ a *hyperplane*.

Subspaces L and M of $P_n(R)$ are said to be *incident*, which is written as $L \mid M$, if $L \supseteq M$ or $L \subseteq M$.

The structure of all subspaces of the free left module ${}^{n+1}R$ is denoted by $P_n(R)^*$ and called the *dual space* of $P_n(R)$.

In Section 1 we saw that ${}^{n+1}R$ can be identified with the right module $(R^\circ)^{n+1}$ over the opposite ring by means of the mapping

$$(\alpha_0, \alpha_1, \ldots, \alpha_n) \mapsto (\alpha_0, \alpha_1, \ldots, \alpha_n)^T.$$

Under this map left subspaces in ${}^{n+1}R$ correspond to right subspaces in $(R^\circ)^{n+1}$, and vice versa. So $P_n(R)^*$ is in fact the ordinary projective space $P_n(R^\circ)$ over the dual ring.

But there are good reasons to maintain the interpretation of $P_n(R)^*$ as consisting of the left subspaces of ${}^{n+1}R$, as will soon become clear.

A point in $P_n(R)$ is a subspace of rank 1 in R^{n+1}, so of the form aR with a unimodular in R^{n+1} as we saw in Section 1; we denote this by $\ulcorner a \urcorner$, or $\ulcorner \alpha_0, \alpha_1, \ldots, \alpha_n \urcorner$ if $a = (\alpha_0, \alpha_1, \ldots, \alpha_n)^T$. Notice that $\ulcorner a \urcorner = \ulcorner b \urcorner$ if and only if $b = a\lambda$ for some $\lambda \in R^*$.

A hyperplane L in $P_n(R)$ is a rank n subspace in R^{n+1}, so by 2.10 we can write L as

$$L = (R\ell)^\perp = \{x \in R^{n+1}\colon\ \langle \ell \mid x \rangle = 0\}$$

for a rank 1 subspace $R\ell$ of ${}^{n+1}R$, which means that ℓ must be unimodular. We use the notation

$$L = \llcorner \ell \lrcorner = \llcorner \lambda_0, \lambda_1, \ldots, \lambda_n \lrcorner$$

for $\ell = (\lambda_0, \lambda_1, \ldots, \lambda_n)$. Here $\llcorner \ell \lrcorner = \llcorner m \lrcorner$ if and only $m = \lambda \ell$ for some $\lambda \in R^*$. We can also view the hyperplane $\llcorner \ell \lrcorner$ in $P_n(R)$ as a point in the dual space $P_n(R)^*$. More generally, the mapping

$$L \mapsto L^\perp$$

gives a bijective correspondence between d-dimensional subspaces in $P_n(R)^*$ and $(n - d - 1)$-dimensional subspaces in $P_n(R)$, by 2.10. From the above we see that incidence between points and hyperplanes is described by

$$\ulcorner a \urcorner \mid \llcorner \ell \lrcorner \Leftrightarrow \langle \ell \mid a \rangle = 0.$$

If L and M are subspaces, the submodules $L+M$ and $L \cap M$ need not be subspaces. We therefore introduce the relation of transversality between subspaces, and some relations derived from that:

3.2. Definition. Let L and M be subspaces of $P_n(R)$, i.e. subspaces of R^{n+1}. We call L and M *transversal*, denoted by $L\tau M$, if and only if $L + M$ and $L \cap M$ are subspaces as well. In particular,

$$L \not\approx M,\ L \text{ is } \textit{distant} \text{ from } M \Leftrightarrow L\tau M \text{ and } L \cap M = 0,$$

$$L \overset{*}{\not\approx} M,\ L \text{ is } \textit{dually distant} \text{ from } M \Leftrightarrow L\tau M \text{ and } L + M = R^{n+1}.$$

The negations of $\not\approx$ and $\overset{*}{\not\approx}$ are denoted by $\approx$ and $\overset{*}{\approx}$, respectively; if $L \approx M$, we call L a *neighbour* of M, and if $L \overset{*}{\approx} M$, we say L is *dually a neighbour* of M.

3.3. Proposition.

(i) *If L and M are subspaces of $P_n(R)$, i.e. subspaces of R^{n+1}, then $L\tau M$ if and only if R^{n+1} has a basis which contains bases for L, M and $L \cap M$ (and then also for $L + M$).*

(ii) *For subspaces L and M in $P_n(R)^*$ we have $L\tau M$ in $P_n(R)^*$ if and only if $L^\perp \tau M^\perp$ in $P_n(R)$, and $L \not\approx M$ ($L \overset{*}{\not\approx} M$) in $P_n(R)^*$ if and only if $L^\perp \overset{*}{\not\approx} M^\perp$ ($L^\perp \not\approx M^\perp$, respectively) in $P_n(R)$.*

(iii) $\ulcorner a \urcorner \not\approx \llcorner \ell \lrcorner \Leftrightarrow \langle \ell \mid a \rangle \in R^* \Leftrightarrow \ulcorner a \urcorner \overset{*}{\not\approx} \llcorner \ell \lrcorner.$

PROOF. Part (i) is easily seen. (ii) If $L\tau M$ in $P_n(R)^*$, take a basis $a_0^*, \ldots, a_n^*$ of ${}^{n+1}R$ which contains bases for L, M and $L \cap M$. Then the dual basis $a_0, \ldots, a_n$ in R^{n+1} contains bases for $L^\perp$, $M^\perp$ and $L^\perp \cap M^\perp$; hence we have $L^\perp \tau M^\perp$. The converse goes the same way, and the rest is clear.

(iii) $\ulcorner a \urcorner \not\approx \llcorner \ell \lrcorner$ means that R^{n+1} has a basis $e_0 = a, e_1, \ldots, e_n$ such that $\langle \ell \mid e_i \rangle = 0$ for $i > 0$. Since ℓ is unimodular, $\langle \ell \mid x \rangle = 1$ for some $x \in R^{n+1}$. This can only happen if $\langle \ell \mid e_0 \rangle \in R^*$. If, conversely, $\langle \ell \mid a \rangle = 1$, then $R^{n+1} = \ulcorner a \urcorner \oplus \llcorner \ell \lrcorner$ as we saw in Section 1, so $\ulcorner a \urcorner \not\approx \llcorner \ell \lrcorner$ and $\ulcorner a \urcorner \overset{*}{\not\approx} \llcorner \ell \lrcorner$. □

Using the correspondence $L \mapsto L^\perp$ as in 2.10, and part (ii) of the above proposition we see that the Principle of Duality holds for projective spaces $P_n(R)$.

3.4. PRINCIPLE OF DUALITY. *If a statement S about projective subspaces using the incidence relation $|$, the transversality relation τ and $\approx$, $\overset{*}{\approx}$ holds for all spaces $P_n(R)$ over rings R of stable rank 2 with given dimension $n \geqslant 0$, then the dual statement S^* is also true for all spaces $P_n(R)$. More specifically, if S is true in $P_n(R)$, then S^* is true in $P_n(R)^* = P_n(R^\circ)$.*

Here the dual statement S^* is obtained from S by replacing 'd-subspace' by '$(n-d-1)$-subspace', so in particular 'point' by 'hyperplane' and vice versa, and interchanging $\approx$ and $\overset{*}{\approx}$.

We finally define independence of points in $P_n(R)$; this transfers the notion of basis of a subspace in R^{n+1} to the projective space.

3.5. DEFINITION. Points $\ulcorner a_1 \urcorner, \ldots, \ulcorner a_k \urcorner$ in $P_n(R)$, with $1 \leqslant k \leqslant n+1$, are *independent* provided there exists hyperplanes $\llcorner h_2 \lrcorner, \ldots, \llcorner h_k \lrcorner$ such that

$$\ulcorner a_{i-1} \urcorner \not\approx \llcorner h_i \lrcorner \quad \text{and} \quad \llcorner h_i \lrcorner \mid \ulcorner a_i \urcorner, \ldots, \ulcorner a_k \urcorner \quad \text{for } i = 2, \ldots, k.$$

Independent hyperplanes are defined in the dual way.

3.6. PROPOSITION. *Unimodular elements $a_1, \ldots, a_k$ form a basis of a subspace of R^{n+1} if and only if the points $\ulcorner a_1 \urcorner, \ldots, \ulcorner a_k \urcorner$ are independent in $P_n(R)$.*

The proof is left to the reader.

From these considerations it is clear that the distant relation between points and hyperplanes determines the transversality relation between subspaces in general. We use this in the axiomatic set-up that we describe in the next section.

4. Axiomatic approach: V-spaces

For an axiomatic characterization of projective spaces over rings, we introduce the notion of Barbilian space; this is a set of points and a set of hyperplanes together with an incidence and a neighbour relation. A Barbilian space which satisfies the axioms formulated in 4.2 is called a Veldkamp space or V-space; these will turn out to be the projective spaces coordinatized by rings of stable rank 2, if we make additional assumptions about the existence of transvections, dilatations and similar transformations in the plane case. The approach with points and hyperplanes as basic objects is a generalization of a self-dual set-up for spaces over skew fields given in Esser [1951]; see also Chapter 2, Section 4. For a more detailed treatment, we refer to Veldkamp [1981, 1985a, 1987]; in those papers the names *Barbilian space* and *Barbilian plane* are used instead of V*-space* and V*-plane.*

4.1. DEFINITION. A (*projective*) *Barbilian space*

$$\mathcal{P} = (P, H, |, \approx)$$

consists of a nonempty set P of *points* and a nonempty set H of *hyperplanes*, together with two relations between P and H, denoted by $|$ and $\approx$ and called *incidence* and the *neighbour relation*, respectively.

Characters $a, b, c, x, y, z, \ldots$, possibly with subscripts and/or accents, denote elements of P, and similarly $h, k, \ell, m, n, \ldots$ elements of H.

For $x \mid h$ or $x \approx h$ we shall also write $h \mid x$ or $h \approx x$, respectively. If $x \not\approx h$, we call x and h *distant.*

Dualizing a statement means interchanging the words 'point' and 'hyperplane', i.e. replacing P by H and vice versa.

Points $a_1, \ldots, a_k$ are called *independent* provided there exist hyperplanes $h_2, h_3, \ldots, h_k$ such that

$$a_{i-1} \not\approx h_i \quad \text{and} \quad h_i \mid a_i, \ldots, a_k \quad \text{for } i = 2, \ldots, k.$$

Independent hyperplanes are defined in the dual way. Points a_1, a_2 are *distant*, $a_1 \not\approx a_2$, if they are independent, and independent hyperplanes h_1, h_2 are called *dually distant*: $h_1 \overset{*}{\not\approx} h_2$.

A Barbilian space is called a V*-space* of dimension n or V*-*n*-space*, $n \geqslant 1$, provided it satisfies the axioms below.

4.2. AXIOMS for a V-n-space.

V1. *If* $x \mid h$, *then* $x \approx h$.

V2. *If* $a_1, \ldots, a_k$ *are independent points* $(1 \leqslant k \leqslant n)$, *then there exists a hyperplane* $h \mid a_1, \ldots, a_k$. *This* h *is unique if* $k = n$, *and is then denoted by* $a_1 \vee \cdots \vee a_n$, *called the join of these points.*

V2′. *If* $h_1, \ldots, h_k$ *are independent hyperplanes* $(1 \leqslant k \leqslant n)$, *then there exists a point* $a \mid h_1, \ldots, h_k$. *This* a *is unique if* $k = n$, *and is then denoted by* $h_1 \wedge \cdots \wedge h_n$, *called the meet of these hyperplanes.*

V3. *If* a *is a point and* $h_1, \ldots, h_n$ *are independent hyperplanes with* $a \mid h_1, \ldots, h_{n-1}$ *and* $h_1 \wedge \cdots \wedge h_n \not\approx a$, *then* $a \not\approx h_n$.

V4. *For any two points* x, y *there exists a hyperplane* h *with* $h \not\approx x, y$.

V5. *If* $a_1, \ldots, a_{n-1}$ *are independent points and* h_1, h_2 *independent hyperplanes such that* $a_1, \ldots, a_{n-1} \mid h_1, h_2$, *then* $x \mid h_1, h_2$ *and* $h \mid a_1, \ldots, a_{n-1}$ *imply* $x \mid h$.

REMARKS.

1. In case $n = 2$ axiom V5 is superfluous, as it then follows from axiom V2′ that the only point x lying on h_1, h_2 is a_1. If $n = 1$, then V5 is a void statement.

2. In general, it follows from axioms V1 up to V4 that any x and h satisfying the assumptions in axiom V5 are neighbours: $x \approx h$. But in $P_n(R)$ the stronger statement of axiom V5 is valid, so we cannot do without this as long as no proof for axiom V5 from the preceding axioms is available. See Veldkamp [1987], p. 227, for details.

3. If the neighbour relation is taken to be the incidence relation, i.e. $x \approx h$ if and only if $x \mid h$, then the above axioms reduce to axioms for an ordinary projective space. In that case axiom V5 is superfluous as follows from the preceding remark.

From Veldkamp [1987], §1, we quote without proofs a number of consequences of the axioms.

4.3. PROPOSITION. *The principle of duality holds for V-n-spaces, i.e. the duals of axioms* V1 *to* V5 *are also valid and therefore any consequence of these axioms can be dualized.*

4.4. PROPOSITION.

(i) *If $a_1, \ldots, a_k$ are independent points, then $a_{\pi(1)}, \ldots, a_{\pi(k)}$ are independent for any permutation π.*

(ii) *Any subset of a set of independent points is a set of independent points.*

(iii) *Any row of independent points has at most $n + 1$ members and can be extended to a row of $n + 1$ independent points in a V-n-space.*

(iv) *If $a_1, \ldots, a_{n+1}$ are independent points in a V-n-space and*

$$h_i = a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_{n+1} \quad \text{for } 1 \leqslant i \leqslant n+1,$$

then $h_1, \ldots, h_{n+1}$ are independent hyperplanes.

4.5. PROPOSITION. *The points and hyperplanes of a projective space $P_n(R)$, $n \geqslant 1$, over a ring R of stable rank* 2 *satisfy the axioms for a V-n-space, if we take incidence and the neighbour relation as defined in Section* 3.

PROOF. For axioms V1, V2, V2′ and V5 this is readily verified using Section 3. Axiom V3 can be translated as follows: if ℓ is a line (intersection of $n - 1$ independent hyperplanes!) and h a hyperplane in $P_n(R)$ with $\ell \tau h$, $\ell \not\subseteq h$, then $a \mid \ell$, $a \not\approx \ell \cap h$ imply $a \not\approx h$. The proof of the latter statement is immediate.

To prove axiom V4 is satisfied we essentially need the assumption that R has stable rank 2. Consider two points x and y in $P_n(R)$. We may choose a basis in R^{n+1} in such a way that

$$x = \ulcorner 1, 0, \ldots, 0 \urcorner \quad \text{and} \quad y = \ulcorner \eta_0, \eta_1, \ldots, \eta_n \urcorner$$

with unimodular $(\eta_0, \eta_1, \ldots, \eta_n)^T$. By SR_{n+1}, which is a consequence of SR_2 by 2.1, we can find $\alpha_i \in R$ such that

$$(\eta_0 + \alpha_0 \eta_n, \ldots, \eta_{n-1} + \alpha_{n-1} \eta_n)^T$$

is unimodular in R^n. Applying SR_n to this vector we find a unimodular vector

$$(\eta_0+\alpha_0\eta_n+\beta_0(\eta_{n-1}+\alpha_{n-1}\eta_n),\ldots,\eta_{n-2}+\alpha_{n-2}\eta_n+\beta_{n-2}(\eta_{n-1}+\alpha_{n-1}\eta_n))$$

in R^{n-1}. Continuing this way we eventually arrive at a relation

$$\gamma(\eta_0+\delta_1\eta_1+\cdots+\delta_n\eta_n)=1 \quad \text{for certain } \gamma,\delta_i\in R.$$

This says that the hyperplane $h=\llcorner\gamma,\gamma\delta_1,\ldots,\gamma\delta_n\lrcorner$ is distant from y. It is also distant from x since

$$\langle\gamma,\gamma\delta_1,\ldots,\gamma\delta_n \mid 1,0,\ldots,0\rangle=\gamma,$$

which is invertible (cf. 2.5). This proves axiom V4 for $P_n(R)$. □

From the points and hyperplanes in $P_n(R)$ together with incidence and the neighbour relation we can reconstruct all subspaces of $P_n(R)$. We do the same *in abstracto* for V-spaces.

4.6. DEFINITION. Let $a_0,a_1,\ldots,a_d$ be independent points in a V-n-space $\mathcal{P}=(P,H,\mid,\approx)$. The *flat spanned by* $a_0,a_1,\ldots,a_d$ is the set

$$\begin{aligned}F&=F(a_0,a_1,\ldots,a_d)\\&=\{x\in P\colon\ x\mid h \text{ for all } h\in H \text{ with } h\mid a_0,a_1,\ldots,a_d\}.\end{aligned}$$

$a_0,a_1,\ldots,a_d$ are said to form a *basis* of F. Further, $F(\varnothing)=\varnothing$, the *empty flat*.

Making use of axiom V5, in particular, one can prove (see Veldkamp [1987], Proposition 2.2):

4.7. PROPOSITION. *For any set of independent points $a_0,a_1,\ldots,a_d$ in a V-n-space there exist $n-d$ independent hyperplanes $h_{d+1},\ldots,h_n$ (which is meant to be the empty set if $d=n$) such that*

$$F=F(a_0,\ldots,a_d)=\{x\in P\colon\ x\mid h_{d+1},\ldots,h_n\}.$$

$h_{d+1},\ldots,h_n$ are called a dual basis of F, and F is also denoted by

$$F=F^*(h_{d+1},\ldots,h_n).$$

Dualizing the definition of a flat yields a *dual flat* as a set of hyperplanes. But the above proposition and its dual say that a dual flat consists of all the hyperplanes containing a flat, and vice versa, so we can identify the two notions, which we shall always do.

A point a will from now on be identified with the flat $F(a)=\{a\}$, and a hyperplane h with the flat $F^*(h)=\{x\in P\colon\ x\mid h\}$.

In Veldkamp [1987], 2.3, it is shown that any two bases of a flat F have the same number of elements, say $d+1$ with $-1\leqslant d\leqslant n$; we call d the *dimension* of F and F a *d-flat*.

Transversality of flats is defined by transferring 3.3(i) to the present situation.

4.8. DEFINITION. Two flats A and B are *transversal*, $A\tau B$, if there exists a set of independent points such that each of A and B is spanned by a subset thereof.

If $A\tau B$, say $A = F(a_0, \ldots, a_d, a_{d+1}, \ldots, a_k)$ and $B = F(a_0, \ldots, a_d, a_{k+1}, \ldots, a_\ell)$ with $a_0, \ldots, a_\ell$ independent, then $A \cap B$ is the flat $F(a_0, \ldots, a_d)$ and there is a minimal flat $A + B$, called the *sum* of A and B, which contains both A and B, viz., $F(a_0, \ldots, a_k, \ldots, a_\ell)$. See Veldkamp [1987], Proposition 2.6, for a proof of these statements and of the dimension rule:

$$\dim(A + B) + \dim(A \cap B) = \dim A + \dim B \quad \text{if } A\tau B.$$

Again we call flats A and B *distant*, $A \not\approx B$, (*dually distant*, $A \overset{*}{\not\approx} B$) if $A\tau B$ and $A \cap B = 0$ ($A + B = P$, respectively). For a point a and a hyperplane h the notions $F(a) \not\approx F^*(h)$ and $F(a) \overset{*}{\not\approx} F^*(h)$ coincide with the axiomatically defined relation $a \not\approx h$ in 4.1.

In Veldkamp [1987], §2, the reader will find a more detailed treatment of flats, transversality, etc. The modular law (see Chapter 2, Proposition 2.2(iv)), which is well known from lattice theory and projective geometry (cf. Baer [1952] and Birkhoff [1948]), is also valid in the present situation with obvious adaptations.

4.9. MODULAR LAW for V-spaces. *If A, B and C are flats satisfying $A\tau B$, $(A + B)\tau C$ and $A \subseteq C$, then $B\tau C$, $A\tau B \cap C$ and*

$$(A + B) \cap C = A + B \cap C.$$

In case $n \geqslant 3$ the axioms given in 4.2 are sufficient to prove that every V-n-space is isomorphic to a projective space $P_n(R)$ over a ring R of stable rank 2; see 5.12. As the reader knows from Chapter 2 one has to add Desargues' Theorem as an axiom for projective planes to obtain a skew field for the coordinatization. Equivalent to Desargues' Theorem is the existence of 'all' transvections and dilatations. It is in this way that we shall proceed for V-planes, the case $n = 2$. We therefore turn to collineations and related notions in the following section.

5. Collineations and affine collineations. Coordinatization of V-spaces

The notions of collineation, transvection and dilatation are carried over from ordinary projective geometry in a straightforward manner. A typical aspect of Barbilian spaces is the occurrence of affine collineations: a kind of collineations in affine space which need not be restrictions of collineations in projective space. This is the geometric translation of the fact that a ring may contain nonzero elements which are not invertible.

5.1. DEFINITION. Let $\mathcal{P} = (P, H, |, \approx)$ and $\mathcal{P}' = (P', H', |, \approx)$ denote Barbilian spaces. A *collineation* ψ from $\mathcal{P}$ to $\mathcal{P}'$ is a mapping ψ: $P \cup H \to P' \cup H'$ which induces bijective mappings $P \to P'$ and $H \to H'$ such that for all $x \in P$ and $h \in H$

$$x \mid h \Leftrightarrow \psi x \mid \psi h, \qquad x \approx h \Leftrightarrow \psi x \approx \psi h.$$

If a collineation from $\mathcal{P}$ to itself leaves all points on a hyperplane h and all hyperplanes through a point c fixed, it is called a *(c, h)-transvection* (or *central transvection*) if $c \mid h$, and a *(c, h)-dilatation* (or just *dilatation*) in case $c \not\approx h$. In either case, c is called a *centre* and h an *axis* of the transvection or dilatation.

Of course, one may also consider (c, h)-collineations with $c \approx h$, $c \nmid h$. But we do not need these in the further theory.

The existence of an axis does not imply the existence of a centre, i.e. a collineation may leave all points of some hyperplane fixed without having a centre. For an example, see Veldkamp [1981], 4.7, [1985], 5.7 or [1988], Remark 3 after 4.1.

A (c, h)-transvection or dilatation may have more than one centre and axis. Consider, e.g., the plane $P_2(R)$ over the ring of dual numbers $R = \mathbb{R}[\varepsilon]$, where $\varepsilon^2 = 0$, $\varepsilon \neq 0$. The diagonal matrix $\mathrm{diag}(1+\varepsilon, 1, 1)$ acting on R^3 induces in $P_2(R)$ a $(\ulcorner 1,0,0 \urcorner, \llcorner 1,0,0 \lrcorner)$-dilatation, which also has $\ulcorner 1,\varepsilon,0 \urcorner$ as a centre since this point is fixed and distant from the axis $\llcorner 1,0,0 \lrcorner$, and $\llcorner 1,\varepsilon,0 \lrcorner$ is another axis.

It is immediate that a collineation between V-spaces carries independent points to independent points and preserves dimension. It maps d-flats on d-flats, preserves transversality, and maps sum and intersection of transversal flats on sum and intersection, respectively, of the images.

Every bijective semilinear transformation A in R^{n+1}, i.e. an automorphism σ of R acting coordinate-wise followed by a bijective linear transformation B, induces a collineation in $P_n(R)$, which we denote by $[A]$, or by $[B, \sigma]$. Its action on hyperplanes is given by

$$[A]\llcorner h \lrcorner = \llcorner \sigma(h) B^{-1} \lrcorner.$$

Since $\langle \sigma(h)B^{-1} \mid B\sigma(x) \rangle = \sigma(\langle h \mid x \rangle)$, it is clear that $[A]$ indeed preserves incidence and the neighbour relation. It is part of the coordinatization theorem, 5.12, that all collineations in $P_n(R)$ are of this kind.

In $P_n(R)$, the $(\ulcorner 0,1,0,\ldots,0 \urcorner, \llcorner 1,0,\ldots,0 \lrcorner)$-transvection mapping $\ulcorner 1,0,\ldots,0 \urcorner$ on $\ulcorner 1,\alpha,0,\ldots,0 \urcorner$ is induced by the linear transformation $E_{21}(\alpha)$. The $(\ulcorner 1,0,\ldots,0 \urcorner, \llcorner 1,0,\ldots,0 \lrcorner)$-dilatation mapping $\ulcorner 1,1,0,\ldots,0 \urcorner$ on $\ulcorner 1,\alpha,0,\ldots,0 \urcorner$ with invertible α is induced by $\mathrm{diag}(\alpha^{-1},1,1,\ldots,1)$; notice that the condition 'α invertible' means that

$$\ulcorner 1,\alpha,0,\ldots,0 \urcorner \not\approx \ulcorner 1,0,\ldots,0 \urcorner,$$

which is a necessary consequence of $\ulcorner 1,1,0,\ldots,0 \urcorner \not\approx \ulcorner 1,0,0,\ldots,0 \urcorner$. The action of this dilatation on the points $\not\approx h = \llcorner 1,0,\ldots,0 \lrcorner$, its axis, is given by

$$\ulcorner 1,\xi_1,\ldots,\xi_n \urcorner \mapsto \ulcorner \alpha^{-1},\xi_1,\ldots,\xi_n \urcorner = \ulcorner 1,\xi_1\alpha,\ldots,\xi_n\alpha \urcorner.$$

Now this still makes sense if α is not invertible: we then get an incidence-preserving mapping defined on the points $\not\approx h$ and the points on h, and also on the hyperplanes $\overset{*}{\not\approx} h$ and h itself: $h \mapsto h$ and

$$k = \llcorner \lambda_0,\lambda_1,\ldots,\lambda_n \lrcorner \mapsto \llcorner \lambda_0\alpha,\lambda_1,\ldots,\lambda_n \lrcorner$$

if $(\lambda_1,\ldots,\lambda_n)$ is unimodular in R^n, which means that $k \overset{*}{\not\approx} h$. Thus we find something like an 'affine dilatation', viz., multiplication of the 'affine space' by a not necessarily invertible α. Let us formalize this in definitions.

5.2. DEFINITION. For a hyperplane h in a projective Barbilian space $\mathcal{P} = \{P, H, |, \approx\}$ the *affine Barbilian space* $\mathcal{P}^h$ is defined by

$$\mathcal{P}^h = \{x \in P\colon\ x \not\approx h \text{ or } x \mid h\} \cup \{m \in H\colon\ m \overset{*}{\not\approx} h \text{ or } m = h\}$$

provided with the incidence and neighbour relation of $\mathcal{P}$. If $\mathcal{P}$ is a V-space, we call $\mathcal{P}^h$ an *affine V-space.*

An *h-affine dilatation* with *centre c*, where $c \not\approx h$, is a mapping $\psi\colon\ \mathcal{P}^h \to \mathcal{P}^h$ carrying points to points and hyperplanes to hyperplanes, which has the properties

$$x \mid h \Rightarrow \psi x = x, \qquad x \not\approx h \Rightarrow \psi x \not\approx h, \qquad \psi c = c,$$
$$x \mid m \Rightarrow \psi x \mid \psi m, \qquad x \approx m \Rightarrow \psi x \approx \psi m.$$

Here x denotes any point and m any hyperplane of $\mathcal{P}^h$.

The duals of the above definitions yield the *dual affine Barbilian space* or *V-space* $\mathcal{P}^a$ with respect to a point a and *dual a-affine dilatation* (with some *axis $h \not\approx a$*).

More generally, one can define *h-affine collineations* (not necessarily dilatations) in $\mathcal{P}^h$; see Veldkamp [1985a], 5.8, and [1987], §3.

In the definition of h-affine dilatation as given above the condition '$x \approx m \Rightarrow \psi x \approx \psi m$' can in fact be omitted, since it is a consequence of the other conditions. For a proof of this in the plane case, see Veldkamp [1985a], 5.9; the proof in higher dimensions is analogous though more complicated, and is written out in Veldkamp [1988], 4.3.

Since a d-flat in $\mathcal{P}^h$, i.e. the intersection with $\mathcal{P}^h$ of a d-flat in $\mathcal{P}$ transversal to h, is the sum of $d+1$ independent points $a_0, \ldots, a_d$ with either all $a_i \mid h$ or $a_0 \not\approx h$ and $a_i \mid h$ for $i > 0$, an h-affine dilatation defines a mapping from d-flats to d-flats in $\mathcal{P}^h$.

In classical projective geometry a central collineation is completely determined by its centre, its axis and the image of one other point. A similar result holds in V-spaces.

5.3. PROPOSITION. *Let h be a hyperplane and c, a and b points in a V-space of dimension $\geqslant 2$ such that $a, b \not\approx h$, $a \not\approx c$ and $b \mid a + c$.*

If $c \mid h$ (or $c \not\approx h$ and $b \not\approx c$), there exists at most one (c, h)-transvection ((c, h)-dilatation, respectively) which maps a on b. We denote this by $T_{c,h;a,b}$.

If $c \not\approx h$ but no further condition on b is imposed (i.e. $b \approx c$ is admitted), there exists at most one h-affine dilatation with centre c mapping a on b. Notation: $\widehat{T}_{c,h;a,b}$. If $c \not\approx h$ and $b \not\approx c$, and $T_{c,h;a,b}$ exists, then $\widehat{T}_{c,h;a,b}$ exists, too, and is just the restriction of $T_{c,h;a,b}$ to the affine space $\mathcal{P}^h$.

A proof is found in Veldkamp [1981], (4.4) and (4.14), or in Veldkamp [1985a], 5.4 and 5.13, both for the plane case; adaptation to higher dimensional spaces is easy, and for the case of a dilatation written out in Veldkamp [1988], 4.4.

5.4. DEFINITION. Let $\mathcal{P} = (P, H, |, \approx)$ be a V-n-space with $n \geqslant 2$, and $c \in P$, $h \in H$. The space $\mathcal{P}$ is called *(c, h)-transitive* in either of the following cases:

$c \mid h$, and all transvections $T_{c,h;a,b}$ exist for points a, b satisfying the conditions in 5.3;

$c \not\approx h$, and all dilatations $T_{c,h;a,b}$ as well as all h-affine dilatations $\widehat{T}_{c,h;a,b}$ exist for points a, b satisfying the conditions in the respective cases.

Further, if $c \not\approx h$, the space $\mathcal{P}$ is called *dually (c, h)-transitive* if all dilatations $T_{c,h;a,b}$ and all dual c-affine dilatations with axis h exist.

The situation with V-n-spaces is the same as with ordinary projective spaces. If $n \geqslant 3$, the transitivity properties defined above are consequences of the axioms given in 4.2. For $n = 2$ this is not true – think of non-Desarguesian planes! – and we have to add axioms requiring (c, h)-transitivity and dually to characterize planes over rings of stable rank 2.

5.5. THEOREM. *Every V-n-space with $n \geqslant 3$ is (c, h)-transitive for all $c \mid h$ and all $c \not\approx h$, and also dually (c, h)-transitive for all $c \not\approx h$.*

A proof of this result can be found in Veldkamp [1987], §3. It is based on the same simple idea as in the classical case of projective spaces, but the presence of a neighbour relation causes considerable complications and makes the proof long and tedious. A simpler proof would be most welcome here.

The axioms we add in the plane case are separated in the existence of transvections, which suffices to prove certain results analogous to what is known from classical transvection planes, and the existence of (ordinary, affine and dual affine) dilatations.

5.6. EXTRA AXIOMS for V-planes. Let $\mathcal{P} = (P, H, |, \approx)$ be a projective V-2-space.

V6. *$\mathcal{P}$ is (c, h)-transitive for all $c \in P$ and $h \in H$ with $c \mid h$.*

V7. *$\mathcal{P}$ is (c, h)-transitive and dually (c, h)-transitive for all $c \in P$ and $h \in H$ with $c \not\approx h$.*

A V-plane is called a *transvection V-plane* if it satisfies axiom V6, and *Desarguesian* if both V6 and V7 hold.

In the classical case of axiomatically defined projective planes the above definitions of transvection planes and Desarguesian planes are equivalent to the usual definitions; cf. Hughes and Piper [1973].

After Definition 5.1 we already remarked that a collineation leaving all points of some hyperplane h fixed need not have a centre. In fact, a product of two transvections T_1 and T_2 with the same axis h need not be a central transvection; this may occur, notably, if $T_1T_2x \approx x$ for $x \not\approx h$, so T_1T_2x and x need not have a unique line connecting them. We therefore define transvections as products of central transvections, more precisely:

5.7. DEFINITION. For a hyperplane h in a Barbilian n-space, $n \geqslant 2$, the group generated by all (c, h)-transvections with arbitrary centres $c \mid h$ is called the *h-transvection group*, denoted by G_h. Its elements are called *transvections* with *axis* h, or *h-transvections*. Dually we have for a point a the a-transvection group G_a generated by all (a, m)-transvections with hyperplanes $m \mid a$.

5.8. PROPOSITION. *Let $\mathcal{P}$ be a V-n-space with $n \geqslant 3$ or a transvection V-plane. For any hyperplane h in $\mathcal{P}$ the h-transvection group G_h is commutative and its action on the points $\not\approx h$ is sharply transitive.*

In Veldkamp [1985a], §6 (or [1981], (4.19)–(4.24), but that contains a mistake in (4.21) which has been repaired in [1985a]), the reader finds a proof for the plane case, which is easily adapted to higher dimensions. Basically, it follows the lines of classical proofs for projective planes such as found, e.g., in Artin [1957] or Hughes and Piper [1973], but one has to proceed more cautiously since neighbouring points need not be connected by a line.

Before formulating other transitivity properties which can be proved with the aid of 5.8, we have to give a couple of definitions.

5.9. DEFINITION. By a *frame* in a Barbilian n-space we understand an ordered set of $n+2$ points such that each subset of $n+1$ points thereof is independent.

In a ring space $P_n(R)$ frames are obtained by taking a basis $e_0, e_1, \ldots, e_n$ of R^{n+1}: the points $\ulcorner e_0 \urcorner, \ulcorner e_1 \urcorner, \ldots, \ulcorner e_n \urcorner, \ulcorner e_0 + e_1 + \cdots + e_n \urcorner$ then form a frame. In a V-n-space, every set of $n+1$ independent points is contained in a frame; see Veldkamp [1987], 2.13.

5.10. DEFINITION. In a Barbilian space the *little projective group* LPG is the group generated by all transvections. The *full projective group* PG is the group generated by all transvections and dilatations.

5.11. PROPOSITION.

(i) *In V-n-spaces with $n \geqslant 3$ and in transvection V-planes the little projective group* LPG *is transitive on ordered sets of* $n+1$ *independent points.*

(ii) *In V-n-spaces with $n \geqslant 3$ and in Desarguesian V-planes the full projective group* PG *is transitive on frames.*

Proofs for the plane case are given in Veldkamp [1985a], 6.10 and 7.4; similar proofs work in higher dimension.

For the coordinatization of a V-n-space one has to start from a given frame. Part (ii) of the above proposition ensures that the choice of this frame does not make an essential difference: the coordinate ring is always the same up to isomorphism. The introduction of coordinates in a V-n-space can be done as is usual in ordinary projective space: one chooses a hyperplane 'at infinity' h_∞ and a frame in $\mathcal{P}$ with $n-1$ points on h_∞, then first coordinatizes the affine space $\mathcal{P}^{h_\infty}$ using transvections and dilatations to define addition and multiplication in the coordinate ring, and then extends the coordinates to the whole projective V-space. The last step is much more complicated then in the classical case of projective spaces, for one has to take care not only of the points on h_∞, which are not the real troublemakers, but also of the points $\approx h_\infty$. An important by-product of the coordinatization is the description of all collineations (often referred to as the Fundamental Theorem of Projective Geometry, see Chapter 2, 3.1), and of the groups LPG and PG in particular.

5.12. THEOREM. *Every projective V-space of dimension $n \geqslant 3$ or Desarguesian V-plane ($n = 2$) is isomorphic to a projective space $P_n(R)$ over a ring R of stable rank 2. R is unique up to isomorphism. Collineations are induced by bijective semilinear transformations. A collineation belongs to the full or to the little projective group if and only if it is induced by an element of* $\mathrm{GL}_{n+1}(R)$ *or* $\mathrm{E}_{n+1}(R)$, *respectively.*

A proof for the plane case is written out in Veldkamp [1981], §5. A fairly elaborate sketch of the proof is given in Veldkamp [1985a], §8, which also contains some corrections to the proof in [1981]. In higher dimensions almost the same proof works.

6. Homomorphisms of V-spaces

A homomorphism between projective spaces is, loosely speaking, a noninjective incidence-preserving mapping. A basic example is ψ: $P_n(\mathbb{Q}) \to P_n(\mathrm{GF}(p))$ defined on points by

$$\psi \ulcorner \alpha_0, \alpha_1, \ldots, \alpha_n \urcorner = \ulcorner \bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_n \urcorner,$$

where we have taken coordinates $\alpha_i \in \mathbb{Z}$ with $\gcd(\alpha_0, \alpha_1, \ldots, \alpha_n) = 1$ and $\bar{\alpha}$ stands for α mod p, and similarly on hyperplanes. A systematic study of homomorphisms between Desarguesian projective planes and spaces was made by W. Klingenberg, who in his [1956] paper showed that homomorphisms $P_n(K) \to P_n(L)$, with K and L fields (but his arguments work for division rings as well) can be described in the following way. Recoordinatize $P_n(K)$ by a valuation ring A in K and consider any ring homomorphism φ: $A \to L$; then

$$\ulcorner \alpha_0, \alpha_1, \ldots, \alpha_n \urcorner \mapsto \ulcorner \varphi(\alpha_0), \varphi(\alpha_1), \ldots, \varphi(\alpha_n) \urcorner$$

is a homomorphism of projective spaces, and thus one gets them all. In the above example ψ: $P_n(\mathbb{Q}) \to P_n(\mathrm{GF}(p))$ the appropriate valuation ring is the ring $\mathbb{Z}_p$ of p-adic integers, i.e. rationals $\frac{m}{n}$ with $p \nmid n$. We shall come back to this in Section 8.

Valuation rings are local rings and therefore have stable rank 2. An extension of the notion of homomorphism to projective spaces over rings of stable rank 2 therefore seems natural. This was done in Ferrar and Veldkamp [1985, 1987] and Veldkamp [1985b,c, 1987]. In Veldkamp [1985a], §§10–14, a coherent presentation is given for the plane case.

6.1. DEFINITION. Let $\mathcal{P} = (P, H, |, \approx)$ and $\mathcal{P}' = (P', H', |, \approx)$ be V-spaces of the same dimension $n \geqslant 2$. A *full homomorphism* ψ: $\mathcal{P} \to \mathcal{P}'$ is a mapping

$$\psi : P \cup H \to P' \cup H' \quad \text{with } \psi P \subseteq P',\ \psi H \subseteq H'$$

satisfying the conditions

(a) $x \mid h \Rightarrow \psi x \mid \psi h$,

(b) for all $x, y \in P$ there exists $h \in H$ such that $h \not\approx x, y$ and $\psi h \not\approx \psi x, \psi y$.

ψ is called *neighbour-preserving* (*n-p* for short) if

$$x \approx h \Rightarrow \psi x \approx \psi h$$

and *distant-preserving* (*d-p* for short) if

$$x \not\approx h \Rightarrow \psi x \not\approx \psi h.$$

A *eumorphism* is a surjective full homomorphism which is both neighbour- and distant-preserving.

Instead of *full homomorphism* we shall usually say '*homomorphism*'.

In Veldkamp [1987], §5, homomorphisms are defined from V-n-spaces to V-n'-spaces with $n, n' \geqslant 2$. In that case some extra conditions are needed to avoid derailments. The algebraic description is then the same as in the case of equal dimension we treat here, with some obvious modifications. In particular, one finds $n' \geqslant n$.

For the rest of this section we deal with homomorphisms between *V-n-spaces of equal dimension* only.

For *d-p* homomorphisms condition (b) is superfluous in view of axiom V4 for V-spaces: given x, y take any $h \in H$ such that $h \not\approx x, y$, then automatically $\psi h \not\approx \psi x, \psi y$ if ψ is *d-p*.

For *n-p* homomorphisms, (b) can be weakened to:

for all $x, y \in P$ there exists $h \in H$ such that $\psi h \not\approx \psi x, \psi y$.

This condition is automatically fulfilled if the *n-p* homomorphism is surjective for points.

For ordinary projective spaces, i.e. if $x \approx h$ if and only if $x \mid h$, condition (b) is equivalent to the condition: ψP contains a frame in $\mathcal{P}'$, used in Carter and Vogt [1980]; in Ferrar and Veldkamp [1985], 1.3, this is proved for the plane case, and a similar argument works in higher dimensions.

The notion of full homomorphism is self-dual since from (a) and (b) one can derive the dual of b); see Veldkamp [1987], 5.2. In 5.3 of the same article it is shown that a homomorphism ψ from $\mathcal{P}$ to $\mathcal{P}'$ defines a mapping from the flats of $\mathcal{P}$ to those of $\mathcal{P}'$ which preserves inclusion and dimensions. If A is a flat of $\mathcal{P}$, the image flat ψA is the flat spanned by the images of the points of A.

We saw in the introduction of this section that in the classical case of homomorphisms ψ: $P_n(K) \to P_n(L)$, with K and L fields, one has to recoordinatize $P_n(K)$ by a valuation ring A in K, and then ψ is described by a homomorphism φ: $A \to L$. For projective spaces over rings of stable rank 2 we follow a similar way. For this purpose we need a special type of subrings, called *admissible subrings*. To define these we first have to introduce the notion of principal denominator set in a ring.

6.2. DEFINITION. Let S be an associative ring with 1. A *principal denominator set* (abbreviated PDS) in S is a subset T satisfying the following conditions.

(1) T is a left and right denominator set in S, i.e. a multiplicative subsemigroup not containing zero divisors and satisfying the left and right Ore conditions:

$$T\sigma \cap S\tau \neq \varnothing, \text{ resp., } \sigma T \cap \tau S \neq \varnothing \quad \text{for all } \sigma \in S,\ \tau \in T.$$

(2) $T \supseteq S^*$, the set of units in S.

(3) If $(S\sigma_1 + S\sigma_2) \cap T \neq \varnothing$, then $S\sigma_1 + S\sigma_2 = S\tau$ for some $\tau \in T$, and similarly for right ideals.

This definition is based on Ferrar and Veldkamp [1987], where the reader finds a detailed treatment. In Ferrar and Veldkamp [1985] one finds a less simple version of the definition, using the name 'planar denominator set' for a PDS.

6.3. DEFINITION. Let R be a ring of stable rank 2. A subring S of R which has the same identity element 1 is called an *admissible subring* if S has stable rank 2 and contains a PDS T such that all elements of T are invertible in R and

$$R = ST^{-1} = \{\sigma\tau^{-1}\colon\ \sigma \in S,\ \tau \in T\}.$$

From the Ore conditions it follows that also

$$R = T^{-1}S = \{\tau^{-1}\sigma\colon\ \sigma \in S,\ \tau \in T\}.$$

Further, $T = R^* \cap S$. For proofs of these and following results on admissible subrings, the reader is referred to Ferrar and Veldkamp [1985, 1987] and Veldkamp [1987].

If S is an admissible subring of R, each unimodular vector $x \in R^{n+1}$ can be written as $x = y\tau^{-1}$ with y unimodular in S^{n+1} and $\tau \in T$, and this y is unique up to right multiplication by an element of S^*. Thus we find a bijective correspondence between points of $P_n(R)$ and points of $P_n(S)$. This can be extended to higher dimensional spaces. Thus, $P_n(R)$ and $P_n(S)$ have the same subspaces and the same incidence relation. The neighbour relation $\sim$ in $P_n(S)$ need not be the same as the neighbour relation $\approx$ in $P_n(R)$, however. For, if ℓ and x are unimodular in ${}^{n+1}S$ and S^{n+1}, respectively, then

$$\llcorner\ell\lrcorner \not\sim \ulcorner x\urcorner \Leftrightarrow \langle \ell \mid x\rangle \in S^*$$
$$\Rightarrow \langle \ell \mid x\rangle \in R^* \Leftrightarrow \llcorner\ell\lrcorner \not\approx \ulcorner x\urcorner,$$

but, conversely, $\langle \ell \mid x\rangle \in R^*$ does not imply $\langle \ell \mid x\rangle \in S^*$. So the neighbour relation $\sim$ in $P_n(S)$ is *coarser* then the neighbour relation $\approx$ in $P_n(R)$: $a \approx h$ implies $a \sim h$, but not conversely.

Before we can give the algebraic description of homomorphisms, one more definition is needed.

6.4. DEFINITION. Let R and R' be rings, S a subring of R and $\varphi\colon\ S \to R'$ a ring homomorphism. We call S *maximal for* φ if φ cannot be extended to a homomorphism from a larger subring of R to R'.

Now the ring analog of Klingenberg's theorem on homomorphisms between projective spaces can be stated.

6.5. THEOREM. *Let* $\psi\colon\ P_n(R) \to P_n(R')$, $n \geqslant 2$, *be a full homomorphism,* R *and* R' *having stable rank* 2. *Then after choosing appropriate frames in the two spaces we can find a unique admissible subring* S *of* R *and a homomorphism* $\varphi\colon\ S \to R'$ *carrying* 1 *to* 1 *with* S *being maximal for* φ *such that* ψ *is the homomorphism* $\widetilde{\varphi}$ *induced by* φ:

$$\widetilde{\varphi}\ulcorner\xi_0,\ldots,\xi_n\urcorner = \ulcorner\varphi(\xi_0),\ldots,\varphi(\xi_n)\urcorner,$$

$$\widetilde{\varphi}\llcorner\lambda_0,\ldots,\lambda_n\lrcorner = \llcorner\varphi(\lambda_0),\ldots,\varphi(\lambda_n)\lrcorner$$

for $\xi_i, \lambda_i \in S$. *Conversely, for any admissible subring* S *of* R *and any homomorphism* $\varphi\colon S \to R'$ *with* $\varphi(1) = 1$, *the induced mapping* $\widetilde{\varphi}$ *is a full homomorphism. Further,*

$\widetilde{\varphi}$ *is n-p if and only if* $\ker(\varphi) \subseteq \operatorname{rad} S$ *and* $\varphi(S)^* = R'^* \cap \varphi(S)$;

$\widetilde{\varphi}$ *is d-p if and only if* $S = R$;

$\widetilde{\varphi}$ *is injective if and only if* φ *is injective*;

$\widetilde{\varphi}$ *is surjective if and only if* $\varphi(S)$ *is an admissible subring of* R', *and an n-p homomorphism* $\widetilde{\varphi}$ *is surjective if and only if* φ *is surjective.*

As a consequence, $\widetilde{\varphi}$ *is a eumorphism if and only if* $S = R$, $\varphi(R) = R'$ *and* $\ker(\varphi) \subseteq \operatorname{rad} R$.

Examples of admissible subrings are valuation rings in division rings (see 8.6–8). Finite rings have no proper subrings which are admissible:

6.6. PROPOSITION. *If* S *is an admissible subring of a finite ring* R, *then* $S = R$.

PROOF. Let T be the PDS in S such that $R = ST^{-1}$. By definition, $T \supseteq S^*$; on the other hand, every nonunit in a finite ring is a zero divisor, so it cannot belong to T. It follows that $T = S^*$ and, therefore, $R = S$. □

For more examples and special cases, see Ferrar and Veldkamp [1985].

7. Projective spaces over full matrix rings

In this section we consider projective spaces over a full matrix ring $M_d(D)$ with D a division ring. We recall that these rings have stable rank 2. For the projective space $P_n(M_d(D))$ a simple interpretation in terms of the ordinary projective space $P_{(n+1)d-1}(D)$ exists, which has been developed by J.A. Thas [1971].

An element of the free right module $M_d(D)^{n+1}$ is a column vector

$$A = (A_0, A_1, \ldots, A_n)^T = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{bmatrix}$$

with all $A_i \in M_d(D)$. So A is an $(n+1)d \times d$ matrix over D, which we shall consider as a linear transformation

$$A\colon D^d \to D^{(n+1)d}$$

of right vector spaces over D. Similarly, an element of the left module ${}^{n+1}M_d(D)$ is a $d \times (n+1)d$ matrix

$$L = (L_0, L_1, \ldots, L_n),$$

which we shall consider as a linear transformation

$$L\colon\ D^{(n+1)d} \to D^d.$$

A is unimodular if $LA = I_d$ (the $d \times d$ unit matrix) for some L as above. This is the case precisely if the subspace $\mathrm{im}(A) = AD^d$ of $D^{(n+1)d}$ has rank d over D. Similarly, unimodularity of L means that the subspace $\ker(L)$ of $D^{(n+1)d}$ has rank nd.

For unimodular $d \times (n+1)d$ matrices A and B,

$$\begin{aligned} \ulcorner A \urcorner = \ulcorner B \urcorner &\Leftrightarrow B = AR \text{ for some } R \in \mathrm{GL}_d(D) \\ &\Leftrightarrow \mathrm{im}(A) = \mathrm{im}(B). \end{aligned}$$

Thus we get a bijective correspondence $\ulcorner A \urcorner \leftrightarrow \mathrm{im}(A)$ between points of $P_n(M_d(D))$ and $(d-1)$-subspaces of $P_{(n+1)d-1}(D)$. In the same way we have a bijective correspondence between hyperplanes of $P_n(M_d(D))$ and $(nd-1)$-subspaces of $P_{(n+1)d-1}(D)$, viz., $\llcorner L \lrcorner \leftrightarrow \ker(L)$ since

$$\begin{aligned} \llcorner L \lrcorner = \llcorner M \lrcorner &\Leftrightarrow M = RL \text{ for some } R \in \mathrm{GL}_d(D) \\ &\Leftrightarrow \ker(L) = \ker(M). \end{aligned}$$

Incidence $\ulcorner A \urcorner \mid \llcorner M \lrcorner$ is defined by $LA = 0$, which means that $\mathrm{im}(A) \subseteq \ker(L)$. So incidence in $P_n(M_d(D))$ corresponds to inclusion in $P_{(n+1)d-1}(D)$. For the distant relation we find

$$\begin{aligned} \ulcorner A \urcorner \not\approx \llcorner L \lrcorner &\Leftrightarrow LA \in \mathrm{GL}_d(D) \\ &\Leftrightarrow \mathrm{im}(A) \cap \ker(L) = 0, \end{aligned}$$

so a point and a hyperplane in $P_n(M_d(D))$ are distant if and only if the corresponding subspaces in $P_{(n+1)d-1}(D)$ have empty intersection.

From this we derive that points $\ulcorner A_1 \urcorner, \ldots, \ulcorner A_\ell \urcorner$ in $P_n(M_d(D))$ are independent if and only if the corresponding subspaces $\mathrm{im}(A_1), \ldots, \mathrm{im}(A_\ell)$ in $P_{(n+1)d-1}(D)$ are in general position, i.e. if each of the subspaces $\mathrm{im}(A_i)$ in $k^{(n+1)d}$ has a basis which is part of one and the same basis of $k^{(n+1)d}$. It is clear that the flat $F(\ulcorner A_1 \urcorner, \ldots, \ulcorner A_\ell \urcorner)$ is uniquely determined by the subspace $\mathrm{im}(A_1) + \cdots + \mathrm{im}(A_\ell)$ of dimension $\ell d - 1$ in $P_{(n+1)d-1}(D)$. Thus we get a bijection between $(\ell - 1)$-flats in $P_n(M_d(D))$ and $(\ell d - 1)$-subspaces in $P_{(n+1)d-1}(D)$. Two flats in $P_n(M_d(D))$ are transversal if and only if the corresponding subspaces in $P_{(n+1)d-1}(D)$ have an intersection of dimension $vd - 1$ for some v.

Now consider a collineation ψ in $P_n(M_d(D))$. This is induced by an invertible semilinear transformation in the free module $M_d(D)^{n+1}$. A ring automorphism of $M_d(D)$ is of the form

$$Y \mapsto A\sigma(Y)A^{-1},$$

where σ is an automorphism of D acting on the entries of the matrix Y, and $A \in \mathrm{GL}_d(D)$ (see, e.g., Artin, Nesbitt and Thrall [1944], Jacobson [1943, 1956, 1980]). So we get for ψ the form

$$\psi\colon \ulcorner X \urcorner \mapsto \ulcorner B\Big(\overset{n+1}{\oplus} A\Big)\sigma(X)A^{-1}\urcorner = \ulcorner B\Big(\overset{n+1}{\oplus} A\Big)\sigma(X)\urcorner.$$

Here X is a unimodular element of $M_d(D)^{n+1}$, i.e. an $(n+1)d \times d$ matrix of right column rank d, B is an invertible $(n+1)\times(n+1)$ matrix over $M_d(D)$, which is nothing but an invertible $(n+1)d \times (n+1)d$ matrix over D. Further, $A \in \mathrm{GL}_d(D)$ and $\overset{n+1}{\oplus} A$ denotes the $(n+1)d \times (n+1)d$ matrix having $n+1$ copies of A along the main diagonal and zeros elsewhere. Finally, σ is an automorphism of D acting entry-wise on the matrix X.

The effect of ψ on the $(d-1)$-dimensional subspace $\mathrm{im}(X)$ of $P_{(n+1)d-1}(D)$ is:

$$\begin{aligned}\mathrm{im}(X) &\mapsto \mathrm{im}\,\Big(B\Big(\overset{n+1}{\oplus} A\Big)\sigma(X)\Big)\\ &= B\Big(\overset{n+1}{\oplus} A\Big)\sigma(\mathrm{im}(X)).\end{aligned}$$

Here we let σ act coordinate-wise on the vectors in $D^{(n+1)d}$. This means that ψ corresponds in $P_{(n+1)d-1}(D)$ to the collineation $\ulcorner B\big(\overset{n+1}{\oplus} A\big), \sigma\urcorner$, i.e. σ acting on the coordinates followed by the linear transformation $B\big(\overset{n+1}{\oplus} A\big)$. This means that collineations in $P_n(M_d(D))$ are induced by collineations in $P_{(n+1)d-1}(D)$. Since every point of $P_{(n+1)d-1}(D)$ is the intersection of $(d-1)$-subspaces, a collineation in $P_{(n+1)d-1}(D)$ is completely determined by its action on $(d-1)$-subspaces, i.e. by the corresponding collineation in $P_n(M_d(D))$. Thus the correspondence between collineations in $P_n(M_d(D))$ and those in $P_{(n+1)d-1}(D)$ is bijective. Under this correspondence, the full projective group carries over to the full projective group, and similarly for the little projective group.

8. Klingenberg spaces

W. Klingenberg [1956] has given an axiomatic description of projective planes and 3-spaces over local rings. Remember that a ring R is local provided that $R/\mathrm{rad}\,R$ is a division ring, i.e. R has a unique maximal left ideal, which is also the unique maximal right ideal. In the context of V-spaces we can characterize these by the additional assumption that the neighbour relation for points is transitive.

The natural homomorphism π of a local ring R onto the division ring $D = R/\mathrm{rad}\,R$ induces a eumorphism $\tilde{\pi}$ of $P_n(R)$ onto an ordinary projective space $P_n(D)$. This leads to a characterization of projective spaces over local rings as *spaces with homomorphism*, or *Klingenberg spaces*.

8.1. PROPOSITION. *For $P_n(R)$, with $n \geqslant 2$ and R of stable rank 2, the following are equivalent.*

(a) *The neighbour relation $\approx$ for points is transitive.*

(b) *The dual neighbour relation $\overset{*}{\not\approx}$ for hyperplanes is transitive.*

(c) *R is a local ring.*

PROOF. In Veldkamp [1981], (2.10), a proof for the plane case was given. Here we prove the equivalence of (a) and (c) in arbitrary dimension $n \geqslant 2$.

Assume, there exist points x, y and z with $x \approx y \approx z$ but $x \not\approx z$. Take hyperplanes h_1, h_2 with $h_1 \mid x, z$ and $h_2 \mid y$, and a point $u \not\approx h_1, h_2$. Then $u \not\approx y$, so we can choose a hyperplane $h \mid u, y$. Since $u \not\approx h_1$, we have $h \overset{*}{\not\approx} x + z$. Let $y' = h \cap (x + z)$. From $x \approx y$ it follows that $x \approx h$, whence $x \approx y'$ (cf. axiom V3 and 4.5). Similarly, $z \approx y'$.

So we may assume x, y and z to be collinear. For $x \not\approx z$, the existence of a point $y \mid x + z$ with $x \approx y \approx z$ is equivalent to

($*$) There exists a unimodular $(\eta_0, \eta_1) \in R^2$ with neither η_0 nor η_1 invertible.

Indeed, we may take

$$x = \ulcorner 1, 0, \dots, 0 \urcorner, \;\; z = \ulcorner 0, 1, 0, \dots, 0 \urcorner \text{ and } y = \ulcorner \eta_0, \eta_1, 0, \dots, 0 \urcorner.$$

Then $y \approx z$ is equivalent to $y \approx \llcorner 1, 0, \dots, 0 \lrcorner$, which means that η_0 is not a unit, and similarly with x and η_1.

Now by SR_2, (η_0, η_1) is unimodular if and only if $\eta_0 + \alpha\eta_1$ is a unit for some $\alpha \in R$, so we may assume $\eta_0 + \alpha\eta_1 = 1$. Hence ($*$) is equivalent to the existence of $\eta_1 \in R$ such that neither η_1 nor $1 - \alpha\eta_1$ is a unit for some α, i.e. a noninvertible $\eta_1 \notin \operatorname{rad} R$. This amounts to saying that $R/\operatorname{rad} R$ is not a division ring. This proves the equivalence of (a) and (c). For (b) $\Leftrightarrow$ (c), just dualize and notice that the opposite ring R° is local if and only if R is local. □

Now let R be any local ring. $D = R/\operatorname{rad} R$ is a division ring, so it has stable rank 2. By 2.4(i), R itself then also has stable rank 2. The homomorphism

$$\pi\colon\; R \to D, \quad \xi \mapsto \xi + \operatorname{rad} R,$$

is surjective and $\ker(\pi) = \operatorname{rad} R$, so by 6.5 the induced homomorphism

$$\widetilde{\pi}\colon\; P_n(R) \to P_n(D)$$

is a eumorphism. Hence the neighbour relation in $P_n(R)$ is determined by the incidence relation in $P_n(D)$:

$$x \approx h \text{ in } P_n(R) \;\Leftrightarrow\; \widetilde{\pi}x \mid \widetilde{\pi}h \text{ in } P_n(D).$$

This leads to the definition of Klingenberg spaces.

8.2. DEFINITION. A *projective Klingenberg space*, or *K-space*, of *dimension n*, where $n \geqslant 2$, $\mathcal{P} = (P, H, |)$ consists of a nonempty set P of *points* and a nonempty set H of *hyperplanes* with an incidence relation $|$ between P and H, together with a *homomorphism* π of $\mathcal{P}$ onto an n-dimensional projective space $\mathcal{P}_0 = (P_0, H_0, |)$, i.e. a surjective mapping π: $P \to P_0$ and $H \to H_0$ such that $x \mid h$ implies $\pi x \mid \pi h$. The following axioms have to be satisfied.

K1. *If $a_1, \ldots, a_k$ are points in P such that $\pi a_1, \ldots, \pi a_k$ are independent in $\mathcal{P}_0$ and $1 \leqslant k \leqslant n$, then there exists a hyperplane $h \mid a_1, \ldots, a_k$. This h is unique if $k = n$.*
K1$'$ = *the dual of* K1.
K2. *If $a_1, \ldots, a_{n-1} \in P$ and $h_1, h_2 \in H$ are such that $\pi a_1, \ldots, \pi a_{n-1}$ are independent, $\pi h_1, \pi h_2$ are independent and $a_1, \ldots, a_{n-1} \mid h_1, h_2$, then $x \mid h_1, h_2$ and $h \mid a_1, \ldots, a_{n-1}$ imply $x \mid h$.*

In Klingenberg [1956] one finds a different formulation, with points, lines and planes, for the cases $n = 2$ and $n = 3$ only; this is equivalent to the formulation given above. Lück [1970] deals with Klingenberg spaces of arbitrary finite dimension (the name 'Hjelmslev-räume' in the title is misleading, since in Hjelmslev planes and spaces one requires additional properties, see Section 9). In his axioms the basic objects are points and lines. Instead of *K-space* one also uses the name *geometry with homomorphism* in the literature.

From the remark preceding Definition 8.2 it is clear that V-spaces over local rings are K-spaces. Conversely, in a K-space one can introduce a neighbour relation between P and H by

$$x \approx h \Leftrightarrow \pi x \mid \pi h.$$

The above axioms K1, K1$'$ and K2 are then reformulations of V2, V2$'$ and V5 in 4.2. It is an easy exercise to prove the remaining axioms V1, V3 and V4 for a K-space, using the homomorphism π. Thus, any K-space is a V-space. It follows that for $n \geqslant 3$ one can coordinatize a K-space with a ring R of stable rank 2, which must be a local ring by 8.1. The same result holds for Desarguesian K-planes, i.e. K-planes in which axioms V6 and V7 of 5.6. hold.

In this context it is interesting to give a proof of W. Klingenberg's theorem about homomorphisms between ordinary projective spaces, which we mentioned in the introduction of Section 6. We start with some preparations on rings.

8.3. DEFINITION. A ring R is called a *left* (*right*) *chain ring* if for any $\alpha, \beta \in R$ either $\alpha \in R\beta$ or $\beta \in R\alpha$ ($\alpha \in \beta R$ or $\beta \in \alpha R$, respectively).

8.4. LEMMA. *Let R be a left or a right chain ring. Then the left or right ideals, respectively, are linearly ordered by inclusion, and R is a local ring.*

PROOF. Consider a left chain ring R. Let I and J be left ideals with $I \not\subseteq J$. Pick $\alpha \in I \setminus J$. For any $\xi \in J$, it is excluded that $\alpha \in R\xi$ since $\alpha \notin J$, whence $\xi \in R\alpha$ and so $\xi \in I$. Thus, $J \subseteq I$. It follows that the union of all left ideals is the unique maximal left ideal, i.e. rad R. Therefore R is a local ring. □

The geometric meaning of being a chain ring is given in the following proposition.

8.5. PROPOSITION. *Let R be a local ring. For every $n \geqslant 2$ the following are equivalent.*
(a) *R is a left chain ring.*
(b) *In $P_n(R)$ any two points have at least one line in common.*
Similarly, for right chain rings and the property that any two hyperplanes have at least one $(n-2)$-flat in common.

PROOF. (a) $\Rightarrow$ (b). Consider any two points a, b. We may coordinatize in such a way that $a = \ulcorner 1, 0, \dots, 0 \urcorner$ and $b = \ulcorner \beta_0, \beta_1, \dots, \beta_n \urcorner$. Since R is a left chain ring, we may assume that, e.g., $\beta_i = \gamma_i \beta_1$ for each $i > 1$. The point $c = \ulcorner 0, 1, \gamma_2, \dots, \gamma_n \urcorner$ is distant from a, for $a \not\approx \llcorner 1, 0, \dots, 0 \lrcorner \mid c$. The line connecting a and c clearly contains b.
(b) $\Rightarrow$ (a). Consider $\alpha, \beta \in R$. The points $\ulcorner 1, 0, \dots, 0 \urcorner$ and $\ulcorner 1, \alpha, \beta, 0, \dots, 0 \urcorner$ have a line in common. This implies that the points $\ulcorner 1, 0, 0 \urcorner$ and $\ulcorner 1, \alpha, \beta \urcorner$ have a line in common in $P_2(R)$, say $\llcorner 0, \lambda_1, \lambda_2 \lrcorner$ with $\lambda_1\alpha + \lambda_2\beta = 0$. Since (λ_1, λ_2) must be unimodular, at least one of λ_1, λ_2 is not contained in $\operatorname{rad} R$, say $\lambda_1 \notin \operatorname{rad} R$. Since R is local, this implies that λ_1 is invertible and, therefore, $\alpha = -\lambda_1^{-1}\lambda_2\beta$. This proves that R is a left chain ring. □

The above proposition can be generalized to arbitrary rings R of stable rank 2. Condition (a) must then be replaced by either of the following equivalent conditions.
(a′) *For any $\eta_1, \eta_2 \in R$ there exists $\alpha \in R$ such that*

$$R\eta_1 + R\eta_2 = R(\eta_2 + \alpha\eta_1).$$

(a″) *Each $(\eta_1, \eta_2) \in R^2$ is right proportional to a unimodular $(\gamma_1, \gamma_2) \in R^2$.*
For a proof of this, see Veldkamp [1984], 3. Condition (b) can be strengthened to
(b′) *Any k points have at least one $(k-1)$-flat in common in $P_n(R)$, for all k with $1 \leqslant k \leqslant n$.*
See Veldkamp [1988], 6.2, for a proof.
Finally, we have to introduce valuation rings; we just extend the standard definition in the commutative case to the noncommutative situation (cf. Jacobson [1980], Krull [1935]; in Schilling [1950] the condition is added that A be invariant under inner automorphisms of D, but we do not need that).

8.6. DEFINITION. A *valuation ring* in a division ring D is a subring A with the property that x or $x^{-1} \in A$ for each nonzero $x \in D$.

8.7. LEMMA. *A ring is a valuation ring if and only if it is a left and right chain domain.*

PROOF. If A is a valuation ring, then for nonzero σ and τ in A either $\sigma\tau^{-1} \in A$ or $\tau\sigma^{-1} = (\sigma\tau^{-1})^{-1} \in A$, which means $\sigma \in A\tau$ or $\tau \in A\sigma$. Similarly one sees that A is a right chain ring.
Conversely, let A be a chain ring without zero divisors. Then $R\backslash\{0\}$ is a multiplicative subsemigroup not containing zero divisors, and the Ore conditions follow from R being a chain ring (cf. 6.2, (1)), whence the quotient division ring D exists. Any nonzero $x \in D$ can be written as $x = \sigma\tau^{-1}$ with nonzero $\sigma, \tau \in A$, so $x^{-1} = \tau\sigma^{-1}$. Then $x \in A$ or $x^{-1} \in A$ according to whether $\sigma \in A\tau$ or $\tau \in A\sigma$. □

8.8. THEOREM. *Let D and D' be division rings. If ψ: $P_n(D) \to P_n(D')$, with $n \geqslant 2$, is a full homomorphism, then there exists a valuation ring A in D and a homomorphism φ: $A \to D'$ with* $\ker(\varphi) = \operatorname{rad} A$ *such that $\psi = \widetilde{\varphi}$, the homomorphism induced by φ:*

$$\widetilde{\varphi}\ulcorner \alpha_0, \alpha_1, \ldots, \alpha_n \urcorner = \ulcorner \varphi(\alpha_0), \varphi(\alpha_1), \ldots, \varphi(\alpha_n) \urcorner$$

if all $\alpha_i \in A$, with not all $\alpha_i \in \operatorname{rad} A$.

Conversely, any valuation ring A in D defines a homomorphism

$$\widetilde{\pi}\colon\ P_n(D) \to P_n(A/\operatorname{rad} A)$$

induced by the projection π: $A \to A/\operatorname{rad} A$.

PROOF. Let ψ: $P_n(D) \to P_n(D')$ be a full homomorphism. By 6.5 we can recoordinatize $P_n(D)$ by an admissible subring A of D such that ψ is induced by a homomorphism φ: $A \to D'$.

Since the neighbour relation in $P_n(D)$ coincides with incidence, ψ is n-p; so by 6.5

$$\varphi(A)^* = D'^* \cap \varphi(A) = \varphi(A) \setminus \{0\}.$$

This means that $\varphi(A)$ is a division ring. Consequently, A is a local ring and $\ker(\varphi) = \operatorname{rad} A$. In $P_n(A)$ any two points have a line in common, for $P_n(A)$ is the same as $P_n(D)$ if we forget the neighbour relation. By 8.5 this means that A is a chain ring, and therefore it is a valuation ring. Its quotient division ring is D since A is admissible in D.

If, conversely, A is a valuation ring in D, then A is an admissible subring in D with $A \setminus \{0\}$ as PDS. The Ore conditions and condition (3) in 6.2 are easy consequences of A being a chain ring.

It should be noticed that if A is a valuation ring in D and φ: $A \to D'$ a homomorphism with $\ker(\varphi) = \operatorname{rad} A$, then A is automatically maximal for φ. For if $x \in D \setminus A$, then $x^{-1} \in \operatorname{rad} A$ (for $x^{-1} \in A \setminus \operatorname{rad} A$ would imply that $x \in A$), whence $\varphi(x^{-1}) = 0$. It follows that φ cannot be extended to any larger subring of D. □

The reader should not think that valuation rings are the only admissible subrings in division rings. More generally, one can have Bezout domains of stable rank 2. A *Bezout domain* is a ring without zero divisors in which every finitely generated left or right ideal is principal. A Bezout domain always has a division ring as ring of fractions, so the projective spaces they coordinatize are just the ordinary projective spaces over division rings, but provided with a neighbour relation. See Veldkamp [1984] and Ferrar and Veldkamp [1985] for more information.

For a survey of, not necessarily Desarguesian, finite Klingenberg planes and their combinatorial aspects, the reader is referred to Drake and Jungnickel [1985].

Marshall Hall's coordinatization of arbitrary projective planes with planar ternary rings (see Hall [1943] or Hughes and Piper [1973]) has been generalized to the class of all K-planes in Keppens [1987], or see Keppens [1988]. For this purpose, the planar ternary rings have been generalized to *planar sexternary rings*, which consist of a set R, a distinguished subset R_0, and six ternary operations defined on $R \times R \times R$, $R_0 \times R \times R$, $R \times R_0 \times R$, $R_0 \times R \times R_0$, $R \times R_0 \times R_0$ and $R_0 \times R_0 \times R_0$, respectively, which have to fulfil certain conditions. R_0 plays the role of the maximal ideal in a local ring here. If R coordinatizes a Klingenberg plane $\mathcal{P}$, the corresponding projective plane $\mathcal{P}_0$, which is the canonical epimorphic image of $\mathcal{P}$, is coordinatized by the – properly defined – quotient planar ternary ring R/R_0.

9. Hjelmslev geometries

W. Klingenberg [1954a,b, 1955] made a systematic study of planes over Hjelmslev rings (cf. Definition 9.2). These are V-planes with three extra geometric assumptions:

– the neighbour relation $\approx$ for points, and also $\overset{*}{\approx}$ for lines, is transitive; in case of a Desarguesian V-plane this means that the coordinate ring R is local as we saw in 8.1;

– two points always have at least one line in common, and dually; for a local ring R this amounts to the validity of the chain conditions (see 8.5);

– if two points are neighbours, they have more than one line in common, and dually.

The algebraic meaning of the latter condition is, that all nonunits in R are zero divisors. To be more precise, call $\alpha \in R$ a *left* (*right*) *zero divisor* if there exists $\lambda \in R$, $\lambda \neq 0$, such that $\alpha\lambda = 0$ ($\lambda\alpha = 0$, respectively).

9.1. PROPOSITION. *Let R be any ring of stable rank 2. The following conditions are equivalent.*

(a) *In $P_n(R)$, with $n \geqslant 2$, two points are neighbours if and only if they have either no line or at least two lines in common.*

(b) *Each element of R is either a unit or a right zero divisor.*

Dually for hyperplanes and left zero divisors.

PROOF. (a) $\Rightarrow$ (b). Consider a nonunit $\alpha \in R$. Then

$$x = \ulcorner 1, 0, \ldots, 0 \urcorner \approx \ulcorner 1, \alpha, 0, \ldots, 0 \urcorner = y.$$

These points are contained in the line

$$\ell = \ulcorner 1, 0, \ldots, 0 \urcorner + \ulcorner 0, 1, 0, \ldots, 0 \urcorner,$$

whence they must lie on a second line. Take a flat F of minimal dimension $k \geqslant 2$ containing ℓ such that x and y lie on two lines contained in F; we replace $P_n(R)$ by $P_k(R)$.

Assume $k > 2$. Let ℓ and ℓ' be two lines through x and y. Take hyperplanes h and h' which contain ℓ and ℓ', respectively, and pick a point $u \not\approx h, h'$. The plane $p = u + \ell'$ cannot contain ℓ because k was chosen to be minimal. Since $u \not\approx h$ we have $p \overset{*}{\not\approx} h$, so $p \cap h$ exists and is a line ℓ'' containing x and y. Since $\ell'' \neq \ell$, we have found a flat h of dimension $k - 1$ containing two lines through x and y, which contradicts the minimality of k.

We must conclude that $k = 2$. The points $x = \ulcorner 1,0,0 \urcorner$ and $y = \ulcorner 1,\alpha,0 \urcorner$ are connected by $\ell = \llcorner 0,0,1 \lrcorner$ and by a second line $\ell' = \llcorner 0,\lambda,\mu \lrcorner$ with $\lambda \neq 0$. Since ℓ' passes through y, we have $\lambda\alpha = 0$.

(b) $\Rightarrow$ (a). Let x and y be neighbours. Assume x and y have a line in common. We may assume that $n = 2$ and that $x = \ulcorner 1,0,0 \urcorner$, $y = \ulcorner \eta_0,\eta_1,0 \urcorner$, so they have the line $\llcorner 0,0,1 \lrcorner$ in common. $x \approx y$ implies that η_1 is not invertible, for otherwise $y \not\approx \llcorner 0,1,0 \lrcorner \mid x$ and hence $y \not\approx x$. So there exists $\lambda \neq 0$ with $\lambda\eta_1 = 0$ by (b). The line $\llcorner 0,\lambda,1 \lrcorner$ is distinct from $\llcorner 0,0,1 \lrcorner$ and passes through x and y. □

RingsArtinian which are left and right Artinian, i.e. which satisfy the minimum conditions for left and right ideals, have the property that every noninvertible element is a left and right zero divisor; see Baer [1942] or Veldkamp [1981], (2.7).

If condition (a) holds in a V-space, the neighbour relation is determined by the incidence relation. For, observe that the neighbour relation between points and hyperplanes is already determined by the neighbour relation between points: $x \not\approx h$ if and only if $x \not\approx y$ for all points $y \mid h$; 'only if' follows from the definition of distant points in 4.1, and 'if' is easily seen with the aid of axiom V3 in 4.2.

The above considerations lead to the definition of an interesting class of rings.

9.2. DEFINITION. A ring R is called a *Hjelmslev ring* or *H-ring* if it is a left and right chain ring and every nonunit is a left and right zero divisor in R.

Notice that an H-ring is a local ring by 8.4, and therefore it has stable rank 2. The V-spaces coordinatized by H-rings are discerned by some simple geometric conditions, as we have seen above.

9.3. PROPOSITION. *A V-space of dimension $n \geqslant 3$ or Desarguesian V-plane ($n = 2$) is coordinatized by an H-ring if and only if the following conditions hold.*

(i) *The neighbour relation $\approx$ for points (or, equivalently, the dual neighbour relation $\overset{*}{\approx}$ for hyperplanes) is transitive.*

(ii) *Two points are neighbours if and only if they have more than one line in common.*

(iii) *Two hyperplanes are dual neighbours if and only if they have more than one $(n-2)$-flat in common.*

Examples of H-rings are the rings $K[t]/(t^n)$, where K is any field and t transcendent over K and, more generally, certain homomorphic images of valuation rings; see Törner [1987]. For a deeper study of H-rings and the planes they coordinatize, see Törner [1974].

The axioms in Klingenberg [1954a] for planes over H-rings, called Hjelmslev planes or H-planes, are rather different from the axioms for V-planes as given in 4.2. We shall not write them down here, but refer the interested reader to the original paper. A. Kreuzer [1987, 1988] has extended Klingenberg's axiomatic approach to higher dimensional Hjelmslev spaces.

B. Artmann [1969] has introduced refined neighbour relations in Hjelmslev planes by distinguishing various degrees of neighbourness. In the formulation of this notion, epimorphisms (i.e. surjective full homomorphisms) play a central role.

There is a vast literature on Hjelmslev planes and related topics: papers by Artmann, Bacon, Baker et al., Benz, Clark, Cronheim, Drake, Jungnickel, Klingenberg, Lorimer, Machala, Mathiak, Seier, and Törner. For a survey of the finite case, with special attention to combinatorial aspects, one may consult Drake and Jungnickel [1985]. Several papers of the above authors are devoted to *affine Hjelmslev and Klingenberg planes*; this is then defined as a somewhat wider class than the affine planes one obtains from a projective plane by deleting a line at infinity and all neighbouring points and lines.

10. Transvection V-planes

A first example of a transvection V-plane not being an ordinary projective plane was studied by T.A. Springer and the present author [1968], then called *Hjelmslev–Moufang plane*; this study continued a series of papers by the authors and others on octonion planes, a class of planes which are transvection planes but not Desarguesian. Motivating background for these studies was the fact that the full projective group and the unitary groups in those planes are simple algebraic groups of type E_6 and F_4, respectively. J.R. Faulkner [1983a,b] created a general theory of planes over alternative rings of stable rank 2; these are transvection V-planes satisfying the quadrangle section condition. For a survey, see Faulkner and Ferrar [1985].

10.1. *Split octonions.* Besides quaternion division rings there also exist quaternion algebras containing zero divisors, called *split quaternion algebras*; any such algebra is isomorphic to a full matrix algebra $M_2(K)$ over a field K. In the same way there are alternative but not associative algebras of dimension 8 over a groundfield K called *octonion algebras*. These can be obtained as follows. Take any quaternion algebra H with involution (anti-automorphism of order 2) $x \mapsto \bar{x}$. Consider the vector space

$$C = H \oplus H\ell,$$

where ℓ is a new symbol. Define multiplication in C by

$$(a + b\ell)(c + d\ell) = (ac + \mu\bar{d}b) + (da + b\bar{c})\ell$$

for some fixed $\mu \neq 0$ in K. This makes C an algebra over K which is not associative but satisfies the *alternative laws*

$$x(xy) = (xx)y \quad \text{and} \quad (xy)y = x(yy).$$

The mapping

$$x = a + b\ell \mapsto \bar{x} = \bar{a} - b\ell$$

is an involution in C, i.e. $\overline{xy} = \bar{y}\bar{x}$ and $\bar{\bar{x}} = x$. The norm N on C defined by

$$N(x) = x\bar{x},$$

is quadratic, i.e. $N(\lambda x) = \lambda^2 N(x)$ for $\lambda \in K$,

$$(x, y) = N(x + y) - N(x) - N(y)$$

is a nondegenerate bilinear form on C, and N permits composition:

$$N(xy) = N(x)N(y).$$

Now two cases may occur. Either N is anisotropic, i.e. $N(x) \neq 0$ for all $x \neq 0$; in this case every $x \neq 0$ has an inverse in C, viz., $x^{-1} = N(x)^{-1}\bar{x}$, and C is called an *octonion division algebra*. Or N is isotropic, i.e. there exist $x \neq 0$ with $N(x) = 0$; in the latter case C contains zero divisors, viz., the elements x with $N(x) = 0$. We then call C a *split octonion algebra*. If C is split, we may take $\mu = 1$. Up to isomorphism, there is exactly one split octonion algebra over any field K, irrespective of the quaternion algebra H we use for its construction. For more details, the reader is referred to Jacobson [1958] or Van der Blij and Springer [1959].

10.2. *Projective planes over split octonions.* Given an octonion algebra C, one can construct a V-plane $P_2(C)$. If C is an octonion division algebra, $P_2(C)$ is an ordinary projective plane which is not Desarguesian but is a transvection plane. If C is split, one gets a V-plane in which two points may have more than one line in common, and dually. For the construction of $P_2(C)$ one cannot make use of a 'free module' C^3 since C is not associative. For one thing, if $x \in C^3$ and $a, b \in C$, then $(xa)b$ need not be a right multiple of x; thus, there is no good notion of rank 1 submodules in C^3.

A good substitute for free modules of rank 3 are the *Jordan algebras* $J = H_3(C)$ of 3×3 Hermitian matrices x over C:

$$x = \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} \quad \text{with } \xi_i \in K \text{ and } x_i \in C.$$

One provides J with the Jordan product

$$x \circ y = \frac{1}{2}(xy + yx).$$

For technical reasons one has to assume that the ground field K has characteristic $\neq 2, 3$.

To catch the idea of how to use $H_3(C)$ for the construction of the plane $P_2(C)$, replace C by the ordinary field of complex numbers, $\mathbb{C}$, for a moment. A point in $P_2(\mathbb{C})$ is a subspace $x\mathbb{C}$ in $\mathbb{C}^3$. We can represent $x\mathbb{C}$ by the projection X of $\mathbb{C}^3$ on $x\mathbb{C}$ with respect to the standard unitary inner product. X is a primitive idempotent Hermitian 3×3 matrix, i.e. an element of $H_3(\mathbb{C})$ satisfying $X \circ X = X$ which is not the sum of two other idempotents. So points in $P_2(\mathbb{C})$ correspond to primitive idempotents in the

Jordan algebra $H_3(\mathbb{C})$. Now, returning to the split octonion algebra C again, define a *point* of $P_2(C)$ as a primitive idempotent or a nilpotent element in $H_3(C)$ up to scalar multiplication by elements of K. The latter type, the nilpotent ones, are absent in the case of the Jordan algebra $H_3(\mathbb{C})$ over the complexes, but have to be included to let things run. *Lines* of $P_2(C)$ are of the form x^* with x a point – the reader will notice that duality is built in right from the beginning. For these points and lines one can define incidence and the neighbour relation. For example, $x \approx y^*$ is defined by $(x, y) = 0$, where the bilinear form $(\ , \ )$ on $H_3(C)$ is defined by

$$(x,y) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + (x_1, y_1) + (x_2, y_2) + (x_3, y_3)$$

for

$$x = \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{bmatrix}.$$

Here (x_i, y_i) means the bilinear form on C corresponding to the norm N. We shall not enter into further details, which involve a lot of long and technical computations in the Jordan algebra $J = H_3(C)$ and the exceptional simple Lie algebra of type E_6 related to J. The reader is referred to Springer and Veldkamp [1968] for the basic theory of $P_2(C)$, and to Veldkamp [1968, 1969] for the full and little projective group and unitary groups in $P_2(C)$. In Faulkner [1970] the theory is extended to groundfields K of arbitrary characteristic, so including 2 and 3, by using the theory of quadratic Jordan algebras.

10.3. *Planes over alternative rings of stable rank* 2. A not-necessarily associative ring A is said to be *alternative* if $\alpha(\alpha\beta) = (\alpha\alpha)\beta$ and $(\alpha\beta)\beta = \alpha(\beta\beta)$ for $\alpha, \beta \in A$. We shall always assume that A has an identity element 1. We call A of *stable rank* 2 if it satisfies

SR_2^*. *$\alpha\xi + \eta$ is left invertible implies $\xi + \beta\eta$ is left invertible for some $\beta \in A$.*

For associative A the above condition is equivalent to the usual SR_2. For, let $\alpha_1\xi_1 + \alpha_2\xi_2 = 1$, then applying SR_2^* to $\xi = \xi_1$, $\eta = \alpha_2\xi_2$ we get β such that $\xi_1 + \beta\alpha_2\xi_2$ is left invertible, which proves SR_2; the converse implication is obvious.

Just as in the previous section, 10.2, for alternative A there is no such thing as a free module, so to construct a projective plane over A one has to follow a different approach. If A is not an octonion algebra, Jordan algebras cannot help either. A way out is given by the Jordan pairs (see Loos [1975]). Using a certain Jordan pair which can be attached to any alternative ring A, J.R. Faulkner [1983a] has constructed groups $\mathrm{E}_2(A)$ and $\mathrm{E}_3(A)$ which in the associative case are the usual elementary groups as in Section 1. From these groups a plane $P_2(A)$ can be constructed having $\mathrm{E}_3(A)$ as its little projective group which is the usual $P_2(A)$ in the associative case. To give the reader an idea of this construction, we shall describe it in the associative case.

So let A be associative. An element g of $G = \mathrm{E}_3(A)$ acts on the points of $P_2(A)$ in the usual way: $x \mapsto gx$, and on the lines in the contragradient way: $\ell \mapsto \ell g^{-1}$. Fix the point $a_0 = \ulcorner 1, 0, 0 \urcorner$ and the line $\ell_0 = \llcorner 1, 0, 0 \lrcorner$. The set of points of $P_2(A)$ is in bijective correspondence with the coset space G/B^- with B^- the stabilizer of a_0 in G:

$$x \leftrightarrow gB^- \quad \text{with } ga_0 = x.$$

The subgroup B^- consists of all matrices

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} \\ 0 & \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{21} & \alpha_{22} \end{bmatrix}$$

which belong to G. Similarly we take the stabilizer B^+ of ℓ_0 in G, so the subgroup of all

$$\begin{bmatrix} \alpha_{00} & 0 & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_{12} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} \end{bmatrix} \in G,$$

and get a bijective correspondence between the lines of $P_2(A)$ and the coset space G/B^+:

$$\ell \leftrightarrow gB^+ \quad \text{with } \ell_0 g^{-1} = \ell.$$

Since G acts transitively on point-line pairs x, ℓ with $x \not\sim \ell$, we see

$$\begin{aligned} x \not\sim \ell &\Leftrightarrow \exists g \in G \text{ with } ga_0 = x,\ \ell_0 g^{-1} = \ell \\ &\Leftrightarrow \exists g \in G \text{ such that } x \leftrightarrow gB^+ \text{ and } \ell \leftrightarrow gB^+. \end{aligned}$$

To find a similar description for incidence, we introduce the element

$$w = E_{21}(1)E_{12}(-1)E_{21}(1) \in G,$$

so

$$w = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $wa_0 = \ulcorner 0, 1, 0 \urcorner \mid \ell_0 = \llcorner 1, 0, 0 \lrcorner$. Using the transitivity of G on incident point-line pairs, we find

$$\begin{aligned} x \mid \ell &\Leftrightarrow \exists g \in G \text{ with } gwa_0 = x,\ \ell_0 g^{-1} = \ell \\ &\Leftrightarrow \exists g \in G \text{ such that } x \leftrightarrow gwB^- \text{ and } \ell \leftrightarrow gB^+. \end{aligned}$$

Thus, the geometry of $P_2(A)$ can be described in terms of the group G, its subgroups B^- and B^+ and the particular element w.

Now, conversely, in any group $G = E_3(A)$ for alternative A one can define the subgroups B^- and B^+ and the element w in the appropriate way, take G/B^- as set of points and G/B^+ as set of lines and then define incidence and the distant relation by

$$\begin{aligned} x \mid \ell &\Leftrightarrow x = gwB^- \text{ and } \ell = gB^+ \text{ for some } g \in G, \\ x \not\sim \ell &\Leftrightarrow x = gB^- \text{ and } \ell = gB^+ \text{ for some } g \in G. \end{aligned}$$

This turns out to be a transvection V-plane whose little projective group is isomorphic to G. We refer the reader to Faulkner [1983a,b] for the details.

To discern the planes thus obtained within the class of transvection V-planes, an extra condition is needed, the quadrangle section condition.

10.4. *The quadrangle section condition.* Fix independent points a, b, c in a V-plane. Consider points $x \mid a \vee b$, $x \not\approx b$ and $y \mid b \vee c$, $y \not\approx c$. Then $x \not\approx b \vee c$ and $y \not\approx a \vee c$, so we can define $z = (x \vee y) \wedge (a \vee c)$ and find $z \not\approx c$. We use the notation

$$z = x * y.$$

The quadrangle section condition is:

(QS) *If a, b, c are independent points, $y \mid b \vee c$, $y \not\approx c$, and $z \mid c \vee b$, $z \not\approx b$, then there is a $p \mid b \vee c$, $p \not\approx c$, such that*

$$x * p = ((x * y) * z) * y \quad \textit{for all } x \mid a \vee b,\ x \not\approx b.$$

See Figure 1, where $r = x * y$, $s = (x * y) * z$, and $t = ((x * y) * z) * y = x * p$.

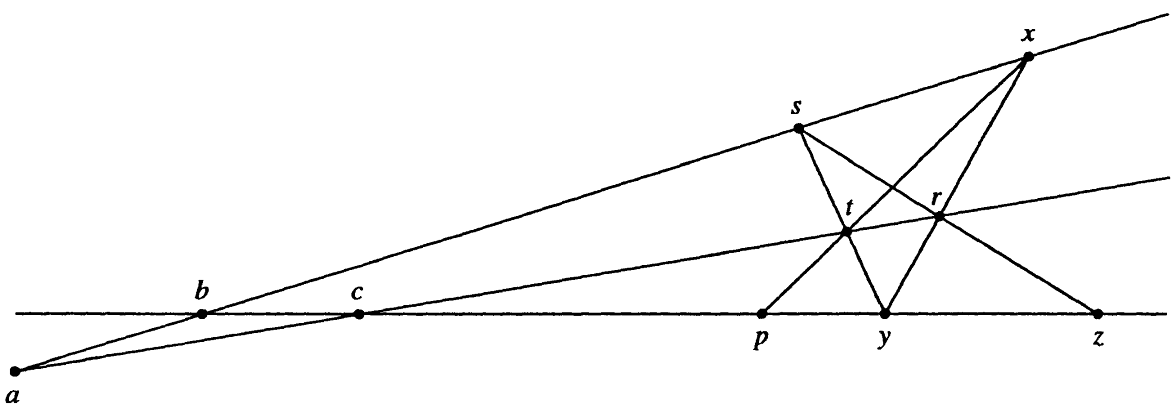


Figure 1.

From the (c, ℓ)-transitivity for all points c and lines ℓ with either $c \mid \ell$ or $c \not\approx \ell$ it is not hard to prove that (QS) holds in Desarguesian V-planes. Whether it holds in all transvection V-planes is an open question. In Faulkner [1983b] it is shown that (QS) and its dual hold in all planes $P_2(A)$ with alternative A of stable rank 2 as in 10.3. Conversely, it is shown in that paper that any transvection V-plane $\mathcal{P}$ satisfying (QS) and its dual is isomorphic to a plane $P_2(A)$ for an alternative ring A of stable rank 2, which is unique up to isomorphism. The ring A is constructed as the set of points x on a fixed line $a \vee b$ with $x \not\approx b$, where addition is defined by transvections, and multiplication with the aid of the operation $*$ defined above. (QS) and its dual are needed, e.g., to prove the alternative laws.

11. Faulkner planes

J.R. Faulkner [1989] has extended the theory of projective planes over rings so as to include all alternative rings for which $\alpha\beta = 1$ implies $\beta\alpha = 1$. These include the rings of stable rank 2 (see 2.5 for a proof in the associative case) and appear to be the most general class of rings that give a geometric structure of the kind we have been considering in this chapter; see Section 11.1 below. A high price has to be paid for this great generality. Even in the associative case one cannot use the free module R^3 for the description of the plane $P_2(R)$ any more. Some analog of the group $\mathrm{E}_3(R)$ has to be constructed and then certain coset spaces of that are the set of points and that of lines for the plane; cf. the procedure described in Section 10.3. A more principal difference with the classical situation is that for each plane defined by a certain set of axioms there is still one coordinate ring attached to it, but that conversely a given ring may be attached to several distinct planes. This leads to a notion of coverings of planes, somewhat analogous to coverings of manifolds, and to a notion of homotopy in Faulkner planes, although no topology is present here. To reassure the reader who feels upset by this loss of uniqueness of the plane coordinatized by a given ring, in the case of rings of stable rank 2 the plane is uniquely determined by the ring in this set-up; on the other hand, there are examples of even commutative associative rings with more than one plane corresponding to it.

There can be no question of paying due attention to Faulkner's long and technically complicated paper here. We shall only explain some fundamental ideas.

11.1. *Why two-sided units rings?* Consider any associative ring R which contains an element η that has a left inverse but no right inverse. So there exists $\lambda \in R$ with $\lambda\eta = 1$ but also a nonzero $\mu \in R$ with $\mu\eta = 0$. In the projective plane $P_2(R)$ the points $\ulcorner 1, 0, 0 \urcorner$ and $\ulcorner 1, \eta, 0 \urcorner$ are distant, since $\ulcorner 1, 0, 0 \urcorner \not\approx \llcorner 1, -\lambda, 0 \lrcorner \mid \ulcorner 1, \eta, 0 \urcorner$. However, these points have two distinct lines in common, viz., $\llcorner 0, 0, 1 \lrcorner$ and $\llcorner 0, \mu, 1 \lrcorner$.

This is so drastically different from the kind of geometry we have been dealing with in this chapter that it would require an altogether different approach. For this reason, Faulkner has only considered rings where $\alpha\beta = 1$ implies $\beta\alpha = 1$.

11.2. Definition. A *Faulkner plane* or *F-plane* is a Barbilian space which satisfies axioms V1, V2, V2′ and V3 with $n = 2$ (see 4.2) and, moreover,

F4. *For any point x there is a line ℓ with $x \not\approx \ell$.*
F5. For any points x, y on a line ℓ there is a sequence of points $x_0 = x, x_1, \dots, x_k = y$ with $x_i \mid \ell$ and $x_{i-1} \not\approx x_i$ for $i = 1, \dots, k$. If $x \not\approx y$, there is such a sequence with $k = 2$.

Every V-plane is an F-plane; in that case one can always take $k = 2$ in axiom F5. The principle of duality holds for F-planes as Faulkner has shown.

An F-plane is called *connected* if for any two points x, y there exists a sequence of points and lines $x_0 = x$, $\ell_1, x_1, \ell_2, x_2, \dots, \ell_r, x_r = y$ such that $x_{i-1} \mid \ell_i \mid x_i$ for $i = 1, \dots, r$. In general, an F-plane is the disjoint union of connected F-planes. V-planes are connected, since for any two points x, y there is a line $\ell \not\approx x, y$; if we pick a point $x_1 \mid \ell$, then $x \not\approx x_1 \not\approx y$ and we can take $\ell_1 = x \vee x_1$ and $\ell_2 = x_1 \vee y$ to get the sequence $x, \ell_1, x_1, \ell_2, y$ connecting x and y.

Transvection F-planes are defined as for V-planes provided the F-planes are connected. A *d-p* homomorphism is a mapping from points to points and lines to lines preserving incidence and the distant relation.

Now we introduce coverings of planes.

11.3. DEFINITION. For a line ℓ in a Barbilian plane, let $I_\ell = \{x \in P\colon\ x \mid \ell\}$ and $D_\ell = \{x \in P\colon\ x \not\sim \ell\}$.

A surjective *d-p* homomorphism $\rho\colon \mathcal{P}_1 \to \mathcal{P}_2$ of Barbilian planes is called a *covering* if the restrictions $\rho\colon I_\ell \to I_{\rho\ell}$ and $\rho\colon D_\ell \to D_{\rho\ell}$ are bijections for all lines ℓ, and dually for all points; then $\mathcal{P}_1$ is called a *covering* of $\mathcal{P}_2$.

If $\rho\colon \mathcal{P}_1 \to \mathcal{P}_2$ is a covering of V-planes, then ρ is bijective since any two points in $\mathcal{P}_1$ belong to some D_ℓ, and one easily derives that ρ is n-p also, whence it is an isomorphism. But with F-planes proper coverings exist. It is shown in Faulkner [1989] that every F-plane $\mathcal{P}$ has a unique *universal covering*, i.e. a covering $\mathcal{U}(\mathcal{P})$ which covers all other coverings of $\mathcal{P}$, and that $\mathcal{U}(\mathcal{P})$ is a transvection F-plane if $\mathcal{P}$ is so. $\mathcal{U}(\mathcal{P}) = \mathcal{P}$ if $\mathcal{P}$ is a V-plane.

11.4. *The tangent bundle plane.* The existence of transvections seems not to suffice for the introduction of coordinates. In the case of V-planes we added the existence of dilatations as well as affine and dual affine dilatations in the Desarguesian case, while for transvection planes the quadrangle section condition (QS) and its dual were added in Section 10. In Faulkner [1989] a different approach is followed using tangent bundle planes. We shall explain this notion for the case of a plane $P_2(R)$ over an associative ring R of stable rank 2. The reader who is acquainted with the theory of algebraic groups will notice the analogy with the way tangent vectors and the Lie algebra of an algebraic group are introduced in, e.g., Borel [1969] and Demazure and Gabriel [1970].

Given R we introduce the ring of dual numbers $R[\varepsilon] = R[T]/(T^2)$ over R, so $\varepsilon^2 = 0$. A vector $(\xi_0 + \varepsilon\eta_0, \xi_1 + \varepsilon\eta_1, \xi_2 + \varepsilon\eta_2)^T$, with $\xi_i, \eta_i \in R$, is easily seen to be unimodular in $R[\varepsilon]^3$ if and only if $(\xi_0, \xi_1, \xi_2)^T$ is unimodular in R^3, and similarly in the free left modules. Since $\alpha + \varepsilon\beta$, with $\alpha, \beta \in R$, has a left inverse in $R[\varepsilon]$ if and only if α is a unit in R (and then $\alpha^{-1} - \varepsilon\alpha^{-1}\beta\alpha^{-1}$ is the inverse of $\alpha + \varepsilon\beta$), one easily sees that $R[\varepsilon]$ has stable rank 2 if R itself has. Thus we have the plane $P_2(R[\varepsilon])$ and the projection $\pi\colon P_2(R[\varepsilon]) \to P_2(R)$ defined by

$$\pi \ulcorner x + \varepsilon y \urcorner = \ulcorner x \urcorner \quad \text{for } x, y \in R^3 \text{ with } x \text{ unimodular,}$$

and similarly for lines. Notice that π is a eumorphism. For a point $\ulcorner x \urcorner \in P_2(R)$, the inverse image $\pi^{-1}\ulcorner x \urcorner$ consists of all points $\ulcorner x + \varepsilon y \urcorner$ with $y \in R^3$. Let $x = (\xi_0, \xi_1, \xi_2)^T$ and fix $\lambda_i \in R$ such that $\sum \lambda_i \xi_i = 1$. If $y = (\eta_0, \eta_1, \eta_2)^T$, we multiply $x + \varepsilon y$ on the right by $1 - \varepsilon \sum \lambda_i \eta_i$, which is invertible. This reduces $\ulcorner x + \varepsilon y \urcorner$ to a form $\ulcorner x + \varepsilon z \urcorner$ with $z \in \ell^\perp$, where $\ell = (\lambda_0, \lambda_1, \lambda_2) \in {}^3R$. It is easily verified that the map

$$\ell^\perp \to \pi^{-1}\ulcorner x \urcorner, \quad z \mapsto \ulcorner x + \varepsilon z \urcorner,$$

is injective and so it is a bijection. Since ℓ is unimodular in ${}^3R$, $\ell^\perp$ is a free submodule of rank 2 of R^3. We consider the vectors $z \in \ell^\perp$ as *tangent vectors* and $\ell^\perp$ as *tangent space to $P_2(R)$ at $\ulcorner x \urcorner$*.

We will now prepare a coordinate-free definition of tangent space. First notice that $z \mapsto \ulcorner x + z \urcorner$ is a bijection between $\ell^\perp$ and $D_{\llcorner \ell \lrcorner}$, the set of points distant from $\llcorner \ell \lrcorner$

in $P_2(R)$. Since the group G_ℓ of transvections with axis $\llcorner \ell \lrcorner$ is sharply transitive on $D_{\llcorner \ell \lrcorner}$ (cf. 5.8), we find a bijection between $\ell^\perp$ and G_ℓ, viz., $z \mapsto T_z$ where T_z is the transvection with axis $\llcorner \ell \lrcorner$ which carries $\ulcorner x \urcorner$ to $\ulcorner x + z \urcorner$. Now

$$T_z = \ulcorner I + z\ell \urcorner$$

where I denotes the 3×3 identity matrix and $z\ell$ the matrix product of the column vector z and the row vector ℓ. If z' is another vector from $\ell^\perp$,

$$(I + z\ell)(I + z'\ell) = I + (z + z')\ell,$$

since $\ell z' = 0$. Hence

$$T_z T_{z'} = T_{z+z'}.$$

Thus we have an isomorphism between G_ℓ and the additive group of $\ell^\perp$. Call $z = \bar{\sigma} x$ if $\sigma = T_z$ with $z \in \ell^\perp$. Combining this with the bijection found above between $\pi^{-1}\ulcorner x \urcorner$ and $\ell^\perp$ we get a bijection

$$\varphi_\ell \colon \pi^{-1}\ulcorner x \urcorner \to G_\ell, \quad \ulcorner x + \varepsilon z \urcorner \mapsto T_z \quad \text{for } z \in \ell^\perp,$$

with inverse $\sigma \mapsto \ulcorner x + \varepsilon \bar{\sigma} x \urcorner$.

All this still depends on the choice of an $\ell \in {}^3R$ with $\ell x = 1$, i.e. of a line $\llcorner \ell \lrcorner \not\approx \ulcorner x \urcorner$. Take another $\ell' \in {}^3R$ such that $\ell' x = 1$. Dualizing the above considerations we see that $\ell' = \ell + m$ for a unique $m \in {}^3R$ with $mx = 0$. The unique transvection θ with centre $\ulcorner x \urcorner$ which carries $\llcorner \ell \lrcorner$ to $\llcorner \ell' \lrcorner$ (i.e. an element of the $\ulcorner x \urcorner$-transvection group; cf. 5.7 and the dual of 5.8) acts on lines as $\llcorner n \lrcorner \mapsto \llcorner n(I + xm) \lrcorner$, so its action on points is induced by

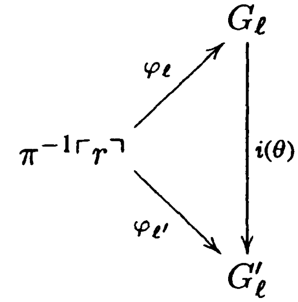
$$(I + xm)^{-1} = I - xm$$

(for $(I + xm)(I - xm) = I$ since $mx = 0$).

$i\,(\theta)$: $\sigma \mapsto \theta\sigma\theta^{-1}$ is an isomorphism of G_ℓ onto $G_{\ell'}$. Corresponding to this we have an R-isomorphism $\ell^\perp \to \ell'^\perp$, $z \mapsto (I - xm)z$. Now observe that

$$x + \varepsilon(I - xm)z = x + \varepsilon(z - xmz) = (x + \varepsilon z)(1 - \varepsilon mz)$$

and that $1 - \varepsilon mz$ is a unit in $R[\varepsilon]$. Thus, we see that $\ulcorner x + \varepsilon(I - xm)z \urcorner = \ulcorner x + \varepsilon z \urcorner$. This means that the diagram



is commutative, i.e. $\varphi_{\ell'} = i(\theta) \circ \varphi_\ell$.

Now, for a point a of $P_2(R)$, consider the set of all pairs (ℓ, σ) where ℓ is a line, $\ell \not\approx a$ and $\sigma \in G_\ell$. Define an equivalence relation by $(\ell_1, \sigma_1) \equiv (\ell_2, \sigma_2)$ if $\sigma_2 = \theta\sigma_1\theta^{-1}$ for the unique transvection $\theta \in G_a$ with $\theta\ell_1 = \ell_2$. Call the equivalence class $[\ell, \sigma]$ of (ℓ, σ) a *tangent vector* at a, and the set $\mathcal{T}_a$ of all tangent vectors at a the *tangent space* at a. Define addition in $\mathcal{T}_a$ by

$$[\ell, \sigma_1] + [\ell, \sigma_2] = [\ell, \sigma_1\sigma_2].$$

This makes $\mathcal{T}_a$ a commutative group isomorphic to the transvection group G_ℓ.

For $a = \ulcorner x \urcorner$ as above, the bijection φ_ℓ: $\pi^{-1}a \to G_\ell$ defines a bijection between $\pi^{-1}a$ and $\mathcal{T}_a$. The map $z \mapsto [\ell, T_z]$ from the free R-module $\ell^\perp$ of rank 2 to $\mathcal{T}_a$ is an isomorphism for the additive group structure of $\ell^\perp$. We write $a + \varepsilon v$ for $\ulcorner x + \varepsilon z \urcorner$ if $v = [\ell, T_z]$.

The abstract definition of tangent vector and tangent space we have arrived at can be given in arbitrary transvection F-planes. For any such plane $\mathcal{P}$ a tangent bundle plane $\mathcal{T}(\mathcal{P})$ can then be defined with points $a + \varepsilon v$, a a point of $\mathcal{P}$ and $v \in \mathcal{T}_a$, and lines defined in a similar way, and appropriately defined incidence and neighbour relation. It can then be proved that if $\mathcal{P}$ is a transvection F-plane, then $\mathcal{T}(\mathcal{P})$ is a connected F-plane. It is not known whether $\mathcal{T}(\mathcal{P})$ must be a transvection plane. Therefore an extra assumption is made.

11.5. Definition. A *Lie transvection F-plane* is a transvection F-plane (and therefore, by definition, connected) $\mathcal{P}$ such that $\mathcal{T}(\mathcal{P})$ is also a transvection F-plane.

If $\mathcal{P}$ is a Lie transvection F-plane, then its universal covering $\mathcal{U}(\mathcal{P})$ is also a Lie transvection F-plane, and $\mathcal{U}(\mathcal{T}(\mathcal{P})) \cong \mathcal{T}(\mathcal{U}(\mathcal{P}))$.

11.6. *The coordinate ring of a Lie transvection F-plane.* Now a coordinate ring R can be attached to every Lie transvection F-plane; as an additive group R is isomorphic to the group of (c, ℓ)-transvections for any fixed line ℓ and fixed point $c \mid \ell$. This ring R can be shown to be an alternative two-sided units ring and is unique up to isomorphism. Conversely, starting from an alternative two-sided units ring R one can construct a Lie transvection F-plane $\mathcal{P}$ having R as its coordinate ring. This plane is not uniquely determined by R. In fact, $\mathcal{P}$ and its universal covering $\mathcal{U}(\mathcal{P})$ have the same coordinate ring R, and all planes with coordinate ring R have the same universal covering. Conversely, from $\mathcal{U}(\mathcal{P})$ one can form certain 'quotients' which are Lie transvection F-planes covered by $\mathcal{U}(\mathcal{P})$ and have R as coordinate ring. In case R has stable rank 2, all these planes coincide, but examples of commutative associative rings do exist where this is not the case. The construction of planes from rings is by group-theoretical methods analogous to the procedure described in Section 10.3. If $\mathcal{P}$ has coordinate ring R, its tangent bundle plane $\mathcal{T}(\mathcal{P})$ has as coordinate ring $R[\varepsilon]$ with $\varepsilon^2 = 0$.

For a more precise description of the results and for the proofs, the reader is referred to Faulkner [1989].

12. Affine ring planes and Barbilian domains

We now turn to affine planes. We begin with an algebraic description of affine planes over arbitrary associative rings R with 1. The set of points will be the free right module R^2. For the definition of lines and of the distant relation $\not\approx$ between points we introduce the notion of Barbilian domain B in R^2. Lines are then sets of the form $a + bR$ with $b \in B$, and $x \not\approx y$ is defined by $x - y \in B$. In this generality, distant points may have more than one line in common. If one wants to avoid this, R must be a two-sided units ring, i.e. a ring where $\alpha\beta = 1$ implies $\beta\alpha = 1$; cf. 11.1 and 12.8.

The axiomatic point of view will be presented in the following section. There we mainly confine ourselves to planes where distant points have a unique line in common, i.e. to planes over two-sided units rings.

Affine planes over special classes of rings (H-rings, local rings, commutative rings, ...) have been studied by several authors; here again, W. Klingenberg [1954a, 1956] has done pioneer work. W. Leißner [1975] made a general study of affine planes over two-sided units rings, introducing among other things Barbilian domains. His approach was generalized to arbitrary associative rings with unit by F. Radó [1980]. Our presentation is a variation on Leißner and Radó with occasionally a theme of our own such as, e.g., the characterization of Barbilian domains in 12.3.

12.1. Before taking the main route we make a small digression on $\mathrm{E}_n(R)$ and related subgroups of $\mathrm{GL}_n(R)$. We begin with defining certain elements. First, for $i \neq j$ and $\lambda \in R^*$, let

$$C_{ij}(\lambda) = E_{ij}(\lambda)E_{ji}\big(-\lambda^{-1}\big)E_{ij}(\lambda).$$

For example, for $n = 2$

$$C_{12}(\lambda) = \begin{bmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{bmatrix}.$$

Further,

$$D_{ij}(\lambda) = C_{ij}(\lambda)C_{ij}(-1)$$

is the diagonal matrix with λ in the i-th position, λ^{-1} in the j-th position and 1 elsewhere.

Since $C_{ij}(1)^{-1} = C_{ij}(-1) = C_{ji}(1)$, and

$$C_{ij}(1)E_{ij}(\lambda)C_{ji}(1) = E_{ji}(-\lambda),$$

we see that $\mathrm{E}_n(R)$ is generated by the matrices $E_{ij}(\lambda)$ and $C_{ij}(1)$ with $i > j$ and $\lambda \in R$.

$\mathrm{GE}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ generated by $\mathrm{E}_n(R)$ and all invertible diagonal matrices. Using $D_{ij}(\lambda) \in \mathrm{E}_n(R)$ one sees that $\mathrm{GE}_n(R)$ is already generated by $\mathrm{E}_n(R)$ and the matrices $\mathrm{diag}(\lambda, 1, \ldots, 1)$ for $\lambda \in R^*$.

NOTATION. We shall denote the $n \times n$ matrix whose column vectors are $a_1, a_2, \ldots, a_n \in R^n$ by $M(a_1, a_2, \ldots, a_n)$. The *unit vectors* in R^n will be denoted $e_1 = (1, 0, \ldots, 0)^T$, $e_2 = (0, 1, 0, \ldots, 0)^T$, etc.

12.2. DEFINITION. For any associative ring R with 1, a subset B of R^2 is called a *Barbilian domain* provided it satisfies the following three conditions.

B1. $e_1 \in B$.
B2. *If $a \in B$, then there exists $b \in B$ such that $M(a, b) \in \mathrm{GL}_2(R)$.*
B3. *If $a, b \in B$ and $M(a, b) \in \mathrm{GL}_2(R)$, then $a + b\lambda \in B$ for all $\lambda \in R$.*

If B is a Barbilian domain, we denote by $M(B)$ the set of all matrices $M(a, b) \in \mathrm{GL}_2(R)$ with $a, b \in B$.

From B2 we see that $a \in B$ if and only if $M(a, b) \in M(B)$ for some vector $b \in R^2$. We shall say that a subset S of $\mathrm{GL}_2(R)$ *determines the Barbilian domain B* if $S = M(B)$; then $B = Se_1$.

12.3. LEMMA. *A subset S of $\mathrm{GL}_2(R)$ determines a Barbilian domain if and only if S is a union of left cosets $A\,\mathrm{GE}_2(R)$ including $\mathrm{GE}_2(R)$ itself.*

PROOF. Consider a union S of left cosets of $\mathrm{GE}_2(R)$ including that subgroup itself. We show that $B = Se_1$ is a Barbilian domain and that $M(B) = S$. Clearly, B1 holds for B.

If $M(a, b) \in S$, then $M(b, -a) = M(a, b)C_{12}(1) \in S$, whence $b \in B$. This shows B2 for B, and also that $S \subseteq M(B)$.

Now consider $a, b \in B$ such that $M(a, b) \in \mathrm{GL}_2(R)$. There exists a' such that $M(a, a') \in S$. Since $M(a, a') \in \mathrm{GL}_2(R)$, it defines a bijective linear transformation of R^2. So there exist $\lambda, \lambda' \in R$ such that $b = a\lambda + a'\lambda'$. From

$$M(a, a\lambda + a'\lambda') \in \mathrm{GL}_2(R)$$

it follows that $M(a, a'\lambda') \in \mathrm{GL}_2(R)$, whence λ' is a unit. This implies that

$$M(a, b) = M(a, a') \begin{bmatrix} 1 & 0 \\ 0 & \lambda' \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \in S.$$

From this we see that $M(B) \subseteq S$ and further that

$$M(a + b\lambda, b) = M(a, b)E_{21}(\lambda) \in S,$$

which proves B3.

Conversely, let $S = M(B)$ for a Barbilian domain B. If $A = M(a, b) \in S$, it follows by B3 that

$$AE_{21}(\lambda) = M(a + b\lambda, b) \in S.$$

Further, $a - b \in B$, so $M(a - b, a) \in S$. Using B3 again we find $M(-b, a) \in S$, i.e. $AC_{21}(1) \in S$. Since the matrices $E_{21}(\lambda)$ and $C_{21}(1)$ generate $\mathrm{E}_2(R)$, we find that $A\mathrm{E}_2(R) \subseteq S$. Using $D_{12}(\lambda) \in \mathrm{E}_2(R)$ we see that $AD_{12}(\lambda) \in S$, whence $a\lambda \in B$ for $\lambda \in R^*$. It follows that $A\,\mathrm{diag}(\lambda, 1) \in S$, so $A\,\mathrm{GE}_2(R) \subseteq S$.

$e_1 \in B$ implies that $M(e_1, b) \in S$ for some $b \in B$. Since $M(e_1, b) \in \mathrm{GE}_2(R)$, we find that $\mathrm{GE}_2(R) \subseteq S$. This completes the proof. □

We now show that the collection of Barbilian domains in R^2 has a smallest as well as a largest element, and some more properties of Barbilian domains.

12.4. PROPOSITION.
(i) *The sets*

$$B_0 = B_0(R) = \mathrm{E}_2(R)e_1 = \mathrm{GE}_2(R)e_1 \quad \textit{and} \quad B_1 = B_1(R) = \mathrm{GL}_2(R)e_1$$

are Barbilian domains in R^2 *with* $M(B_0) = \mathrm{GE}_2(R)$ *and* $M(B_1) = \mathrm{GL}_2(R)$.
(ii) *If* B *is any Barbilian domain in* R^2, *then* $B_0 \subseteq B \subseteq B_1$. *Consequently, all elements of* B *are unimodular. Further,* $e_2 \in B$, *and* $a\lambda \in B$ *if* $a \in B$ *and* $\lambda \in R^*$.

PROOF. (i) From $D_{12}(\lambda) \in \mathrm{E}_2(R)$ for $\lambda \in R^*$ it follows that $\mathrm{E}_2(R)e_1 = \mathrm{GE}_2(R)e_1$, so it is a Barbilian domain by the lemma. Similarly for B_1.
(ii) If $S = M(B)$, then $\mathrm{GE}_2(R) \subseteq S \subseteq \mathrm{GL}_2(R)$, whence $B_0 \subseteq B \subseteq B_1$.
If $a \in B_1$, then $A = M(a,b) \in \mathrm{GL}_2(R)$ for some b. From $A^{-1}M(a,b) = I$ it follows that a in unimodular.
$C_{21}(1) \in \mathrm{E}_2(R)$ implies $e_2 \in B_0$. If $M(a,b) \in M(B)$, then $M(a,b)D_{12}(\lambda) \in M(B)$ for $\lambda \in R^*$, so $a \in B$ implies $a\lambda \in B$. □

12.5. COROLLARY. *There is a unique Barbilian domain in* R^2 *if and only if* R *is a* GE_2 *ring, i.e. if* $\mathrm{GE}_2(R) = \mathrm{GL}_2(R)$.

For GE_2 rings, see Cohn [1966b], or Hahn and O'Meara [1989], 1.2. Examples of GE_2 rings are Euclidean rings, and rings of stable rank 2. In the latter case, the Barbilian domain consists of all unimodular vectors as follows from 2.8. Examples of rings that are not GE_2 rings are the ring of integers in $\mathbb{Q}(\sqrt{-d})$ where d is a squarefree positive integer $\neq 1, 2, 3, 7, 11$ (see Cohn [1966b], Theorems (6.1) and (6.2), where more examples are given), and the polynomial ring $R = k[x,y]$ over a field k. In the latter case,

$$\begin{bmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{bmatrix}$$

belongs to $\mathrm{GL}_2(R)$, but not to $\mathrm{GE}_2(R)$; see Silvester [1981], pp. 118–121.

12.6. DEFINITION. For an associative ring R with 1 and a Barbilian domain $B \subset R^2$ the *affine plane* $A_2(R,B)$ consists of *points* and *lines*, together with a *distant relation* $\not\approx$ between points, and a *parallel relation* $\|$ between lines:
– *points* are all elements of R^2;
– *lines* are the subsets $a + bR$ of R^2 with $a \in R^2$ and $b \in B$;
– points a and b are *distant*, $a \not\approx b$, if $b - a \in B$;
– lines $\ell = a + bR$ and $\ell' = a' + b'R$ are *parallel*, $\ell \parallel \ell'$, if $bR = b'R'$.
Such a plane is called an *affine ring plane*.
Further, lines ℓ and m are called *cross lines*, or ℓ is said to *cross* m, if every $\ell' \parallel \ell$ has exactly one point of intersection with every $m' \parallel m$. Notation: $\ell \dagger m$.
If R^2 has a unique Barbilian domain B, we may simply write $A_2(R)$ for $A_2(R,B)$.

Notice that in this set-up lines are sets of points, so the incidence relation $x \mid \ell$ is just $x \in \ell$.

12.7. LEMMA.

(i) *$a + bR \parallel a' + b'R$ if and only if $b' = b\lambda$ for some $\lambda \in R^*$.*

(ii) *$a + bR \dagger a' + b'R$ if and only if $M(b, b') \in \mathrm{GL}_2(R)$. We denote this by $b \dagger b'$.*

PROOF. (i) Since b and b' are unimodular, $bR = b'R$ if and only if $b' = b\lambda$ for some $\lambda \in R^*$.

(ii) $a + bR \dagger a' + b'R$ means that for every $y \in R^2$ the equation $M(b, b')x = y$ has exactly one solution x, i.e. $M(b, b')$ is bijective. □

The affine plane $A_2(R)$ over a ring R of stable rank 2 is nothing else than the affine plane $P_2(R)^{\ell_\infty}$ for $\ell_\infty = \llcorner 1, 0, 0 \lrcorner$ with its affine lines, $\ell \parallel m$ meaning that $\ell \wedge \ell_\infty = m \wedge \ell_\infty$. Here $\ell \dagger m$ is equivalent to: $\ell \tau m$ and $\ell \wedge m$ is a point $\not\approx \ell_\infty$.

12.8. PROPOSITION. *If $a \not\approx b$ in $A_2(R, B)$, the line $a + (b - a)R$ joins a and b, and is contained in any other line doing so. The following are equivalent:*

(i) *Any two distant points have a unique line in common in $A_2(R, B)$.*

(ii) *R is a two-sided units ring.*

PROOF. If $b - a \in B$, the line $a + (b - a)R$ passes through a and b. If $b \in a + cR$, then $b - a \in cR$, so $(b - a)R \subseteq cR$. Since $b - a$ is unimodular, $b - a = c\alpha$ implies that α has a left inverse. So if R is a two-sided units ring, α is a unit, whence $a + (b - a)R = a + cR$.

Conversely, assume (i) holds. For $\alpha, \beta \in R$ with $\beta\alpha = 1$,

$$\begin{bmatrix} \alpha & \alpha\beta - 1 \\ 0 & \beta \end{bmatrix} = E_{12}(-1)E_{21}(1)E_{12}(-1)E_{12}(\beta)E_{21}(-\alpha)E_{12}(\beta),$$

which belongs to $\mathrm{E}_2(R)$. Hence $a = (\alpha, 0)^T \in B$. Now $0 \not\approx a$ and 0 and a lie on the lines e_1R and aR, whence $e_1R = aR$ by (i). This implies that $\alpha \in R^*$. □

We now introduce a geometric condition which ensures that the Barbilian domain B is minimal in R^2, i.e. $B = B_0$.

12.9. DEFINITION. Two lines ℓ, m in an affine ring plane are said to be *coherent* provided there exists a series of lines $\ell_0 = \ell, \ell_1, \ldots, \ell_s = m$ such that $\ell_{i-1} \dagger \ell_i$ for $i = 1, \ldots, s$. For $\ell = a_0 + b_0R$ and $m = a_s + b_sR$ we may also say that b_0 and b_s are *coherent*.

An affine ring plane is called *coherent* if any two lines in it are coherent.

Notice that the property of being coherent is in fact the dual of axiom F5 for Faulkner planes (see 11.2).

12.10. PROPOSITION. *$A_2(R, B)$ is coherent if and only if $B = B_0$.*

PROOF. We first show: If $a, b, c \in R^2$ with $a \dagger b \dagger c$, then there exists $A \in \mathrm{GE}_2(R)$ such that $M(a,b)A = M(b,c)$.

$a \dagger b$ means that a and b form a basis of R^2, whence $c = a\alpha + b\beta$ for certain α, β. Now $b \dagger c$ if and only if $\alpha \in R^*$. One easily verifies that

$$M(a,b)\begin{bmatrix} -\alpha & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\beta & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = M(b,c).$$

Now assume that $A_2(R,B)$ is coherent. Pick any $a \in B$. Since e_1 and a are coherent, and also $e_2 \dagger e_1$, there exist $b_1, \dots, b_\ell \in B$ such that

$$e_1 \dagger e_2 \dagger b_1 \dagger \cdots \dagger b_\ell \dagger a \dagger b_\ell.$$

By what we proved above we find $A \in \mathrm{GE}_2(R)$ such that $M(e_1, e_2)A = M(a, b_\ell)$. It follows that $a = Ae_1$, i.e. $a \in B_0$.

Conversely, if $a \in B_0$, we can write $a = Ae_1$ for some $A \in \mathrm{E}_2(R)$. Then

$$M(e_1, e_2)A = M(a, b)$$

for some $b \in B_0$. A is a product of matrices $E_{21}(\lambda)$ and $C_{21}(1)$. Now notice that $u, v \in B_0$ with $u \dagger v$ means $M(u,v) \in \mathrm{GL}_2(R)$, from which it follows that $M(u,v)E_{21}(\lambda)$ as well as $M(u,v)C_{21}(1)$ belongs to $\mathrm{GL}_2(R)$ and has both columns in B_0, so these columns are cross in B_0. It follows that $e_1 \dagger e_2 \dagger \cdots \dagger a$, i.e. e_1 and a are coherent. □

12.11. DEFINITION. A bijective mapping φ between affine ring planes is called an *affine collineation*, or *isomorphism*, provided both φ and φ^{-1} preserve the distant relation, map lines onto lines and parallel lines onto parallel lines. The affine collineations of $A_2(R,B)$ onto itself form its *automorphism group* $\mathrm{Aut}(A_2(R,B))$, those leaving 0 fixed forming a subgroup $\mathrm{Aut}_0(A_2(R,B))$.

The *translations* $x \mapsto x + a$ of course form a normal subgroup of $\mathrm{Aut}(A_2(R,B))$, which we denote by $T(R^2)$.

The ring automorphisms of R induce bijective transformations of R^2 by acting coordinatewise on the vectors $(\xi_1, \xi_2)^T$. The group generated by $\mathrm{GL}_2(R)$ and the ring automorphisms is denoted by $\Gamma\mathrm{L}_2(R)$ (bijective semilinear transformations), the subgroup generated by $\mathrm{GE}_2(R)$ and the ring automorphisms by $\Gamma\mathrm{E}_2(R)$.

The following can be said about the structure of $\mathrm{Aut}(A_2(R,B))$.

12.12. PROPOSITION. *Let B be a Barbilian domain in R^2.*

(i) $\mathrm{Aut}(A_2(R,B)) = T(R^2) \rtimes \mathrm{Aut}_0(A_2(R,B))$.[1]

If R is a two-sided units ring, then we further have:

(ii) $\mathrm{Aut}_0(A_2(R,B))$ *is a subgroup of* $\Gamma\mathrm{L}_2(R)$.

(iii) $\mathrm{Aut}_0(A_2(R,B_0)) = \Gamma\mathrm{E}_2(R)$.

(iv) $\mathrm{Aut}_0(A_2(R,B) = \Gamma\mathrm{L}_2(R)$ *if and only if* $B = B_1$.

---

[1] Editorial note: the symbol $\rtimes$ stands for a semidirect product of groups.

PROOF. (i) is immediate. For (ii), see Leißner [1975], II, Satz 5. (iii) is an easy consequence of the fact that $M(B_0) = \mathrm{GE}_2(R)$ (see 12.4) and the remark that $A \in \Gamma\mathrm{L}_2(R)$ induces a collineation of a plane $A_2(R, B)$ if and only if $AB = B$. Similarly, (iv) follows from the fact that $M(B_1) = \mathrm{GL}_2(R)$. □

13. Leißner planes

In this section we give an axiomatic description of affine planes $A_2(R, B)$ for two-sided units rings R. We follow Leißner [1975], with some modifications as in Radó [1980].

13.1. DEFINITION. We consider a structure $\mathcal{A} = (P, L, \not\approx, \|)$ consisting of a nonempty set of *points* P, a nonempty family L of nonempty subsets of P, called *lines*, provided with an antireflexive symmetric relation $\not\approx$ between points, called *distant relation*, and an equivalence relation $\|$ between lines, the *parallel relation*. Two lines ℓ and m are said to be *cross lines*, $\ell \dagger m$, if $| \ell' \cap m' | = 1$ for all $\ell' \| \ell$ and $m' \| m$. We call this an *affine Leißner plane*, or *L-plane*, if it satisfies the following axioms.

L1. *Any two distant points a, b have exactly one line in common, denoted by $a \vee b$.*
L2. *For any line ℓ and any point $a \in \ell$ there is a point $b \in \ell$ with $a \not\approx b$.*
L3. *For each $a \in P$ and $\ell \in L$ there exists exactly one $m \in L$ such that $a \in m$ and $m \| \ell$. Notation: $m = (a \| \ell)$.*
L4. *For every line ℓ there exists a line m such that $m \dagger \ell$.*
L5. *If ℓ and m are cross lines and $a \in \ell$ with $a \not\approx \ell \cap m$, then $a \not\approx x$ and $a \vee x \dagger m$ for all $x \in m$.*

Affine planes $A_2(R, B)$ for two-sided unit rings R are L-planes.

If one defines $a \not\approx b$ by $a \neq b$ and requires that $\ell \nparallel m$ implies $\ell \dagger m$ for lines, then the above axioms define ordinary affine planes. So to characterize affine planes over associative rings we have to add Desargues-like conditions again.

13.2. *Collineations* of L-planes are defined as in 12.11. An *ℓ-translation* of

$$\mathcal{A} = (P, L, \not\approx, \|),$$

for given $\ell \in L$, is a collineation φ such that $\varphi\ell' = \ell'$ for all $\ell' \| \ell$, and $a, b \in m$ implies $\varphi b \in (\varphi a \| m)$. $\mathcal{A}$ is *ℓ-transitive* if for any two points $a, b \in \ell$ there exists an ℓ-translation mapping a on b.

An *affine c-dilatation* of $\mathcal{A}$, for $c \in P$, is a mapping δ: $P \to P$ such that $\delta\ell \subseteq \ell$ for all lines ℓ through c, and $a, b \in m$ implies $\varphi b \in (\varphi a \| m)$. If δ is bijective, then it is a collineation and δ^{-1} is also an affine dilation; see Radó [1980]. $\mathcal{A}$ is said to be *c-transitive* if for every line ℓ through c and any two points $a, b \in \ell$ with $a \not\approx c$ there exists an affine c-dilation carrying a to b.

ℓ-translations and affine c-dilatations are, of course, similar to central transvections and affine dilatations as in Section 5. ℓ-transitivity and c-transitivity can be reformulated in terms of configuration conditions for the plane $\mathcal{A}$, the (*little*) *parallelodromy condition* and the *Desargues condition with respect to c*, respectively. Refer to Leißner [1975, 1976] and Radó [1980] for details.

13.3. DEFINITION. A Leißner plane $\mathcal{A}$ is called a *translation L-plane* if it satisfies

L6. $\mathcal{A}$ is ℓ-transitive for every line ℓ.

A *Desarguesian L-plane* is a translation L-plane which satisfies

L7. $\mathcal{A}$ is c-transitive for all points c.

The axioms L1 up to L7 suffice to characterize affine planes over two-sided units rings as has been shown in Leißner [1975].

13.4. THEOREM. *An L-plane is Desarguesian if and only if it is isomorphic to an affine plane $A_2(R, B)$ over a two-sided units ring R with a Barbilian domain $B \subset R^2$. The ring R is determined up to isomorphism.*

13.5. The above result has been generalized to arbitrary associative rings with 1 by F. Radó [1980]. By 12.8 one then has to drop the assumption of uniqueness of the line connecting two distant points.

A generalization to higher-dimensional spaces over rings can be found in Leißner, Severin and Wolf [1985]; see also Leißner [1987]. A different approach to higher-dimensional affine ring geometry is given in Veldkamp [1993].

For a description of a translation L-plane by an Abelian group with a certain family of subgroups, called a B-congruence, see Leißner [1976] and, more generally, Radó [1980]. These are generalizations of the André representation of affine translation planes by congruence partitions of Abelian groups. See also Chapter 5 of this Handbook by M. Kallaher.

14. Open problems

We end up with some open problems.

PROBLEM 1. To build a theory of projective spaces of infinite dimension over rings of stable rank 2, or a wider class of rings – cf. Problem 2 below. This should comprise an algebraic approach as in Section 3 as well as an axiomatic approach. Already in the case of spaces over division rings the principle of duality gets lost since there are far more hyperplanes than points (cf. Baer [1952]), so a selfdual approach with points and hyperplanes is not appropriate.

PROBLEM 2. To extend the algebraic description of projective spaces over rings using free modules to wider classes of rings. If one wants to maintain the property that distant points have a unique line connecting them, the rings must have two-sided units only (cf. 11.1 and 12.8). For an axiomatic approach one might think of Faulkner's axioms (see 11.2) generalized to higher dimensions, combined with the existence of transvections and (ordinary, affine, and dual affine) dilations for the plane case. See also Knüppel [1987] for the plane case.

PROBLEM 3. In many rings each element is a sum of two invertible elements (cf. Raphael [1974]). Is it possible to axiomatize planes over such rings without using affine and dual affine dilatations?

PROBLEM 4. Find a proof for 5.5 simpler than the one given in Veldkamp [1987].

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CHAPTER 20

Applications of Buildings

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Contents

1. The spherical building of a reductive group 1087
2. Topological properties, the Steinberg representation 1091
3. Affine buildings 1095
4. The Borel–Serre compactification 1101
5. An application to cohomology of S-arithmetic groups 1106
6. Covolumes of arithmetic groups 1110
7. Applications of buildings in differential geometry 1111
References 1112

HANDBOOK OF INCIDENCE GEOMETRY
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In this chapter we shall discuss some applications of the theory of buildings, exposed in Chapters 11 and 12. In those chapters buildings were viewed as combinatorial objects. In our applications we shall use geometrical realizations of these as topological spaces. We shall discuss their constructions. The buildings of the present chapter will almost always be such topological spaces.

Our review of applications is far from complete. In particular, we have not gone into the applications in the representation theory of p-adic Lie groups. We refer to Cartier [1979].

We shall have to use basic notions from the theory of linear algebraic groups. The reader who is not familiar with them will find a brief review in Borel [1966]. A more extensive reference is Borel [1991].

1. The spherical building of a reductive group

We denote by G a connected reductive linear algebraic group over the field k. We shall define here the building $\mathcal{B}$ ($= \mathcal{B}(G) = \mathcal{B}(G, k)$), a topological space. For more details, see Curtis, Lehrer and Tits [1980].

1.1.

First consider the case where $G = S$ is a split torus over k. Denote by $X^*(S)$, resp., $X_*(S)$, its group of characters, resp., cocharacters (homomorphisms of algebraic groups $\mathbb{G}_m \to S$). These are free Abelian groups of finite rank which are in duality. The pairing

$$\langle\, , \rangle\colon\ X^*(S) \times X_*(S) \to \mathbb{Z}$$

is defined by

$$\langle \chi, \lambda \rangle = \chi \circ \lambda \in \operatorname{Hom}(G_m, G_m) \xrightarrow{\ \sim\ } \mathbb{Z},$$

where $n \in \mathbb{Z}$ is identified with the homomorphism $\xi \mapsto \xi^n$ of G_m. Define $\mathcal{B}(S)$ to be the sphere whose points are the rays in the real vector space $X_*(S) \otimes \mathbb{R}$, with the obvious topology.

Now assume S to be a k-subtorus of our group G. To any $b \in \mathcal{B}(S)$ one associates the parabolic k-subgroup $P(b)$ of G characterized by the property that its Lie algebra is the subspace of the Lie algebra of G spanned by the weight vectors for S whose weights have non-negative values on the point b.

1.2. *The building*

We can now define the building $\mathcal{B}(G)$. First let $\mathcal{B}_1$ be the disjoint union of the spheres $\mathcal{B}(S)$, where S runs over all maximal k-split tori of G. If $g \in G(k)$ let $\operatorname{ad}(g)$ denote the bijection of $\mathcal{B}_1$ induced by the inner automorphism $x \mapsto {}^g x = gxg^{-1}$ of $G(k)$.

We define an equivalence relation $b \sim b'$ on $\mathcal{B}_1$ by $b' = \operatorname{ad}(g)(b)$ for some $g \in P(b)(k)$ (that this is an equivalence relation follows readily from the observation that $P(b') = {}^g P(b) = P(b)$).

The spherical building $\mathcal{B} = \mathcal{B}(G)$ is defined to be the quotient of $\mathcal{B}_1$ by this equivalence relation. We provide it with the quotient topology. Notice that if k is finite, the space $\mathcal{B}(G)$ is compact.

It follows from the definitions that if $b \in \mathcal{B}$ the parabolic k-subgroup $P(b)$ is well defined. Also, $G(k)$ operates on $\mathcal{B}$ (more generally, the group $\mathrm{Aut}_k(G)$ of k-automorphisms operates on $\mathcal{B}$).

1.2.1. LEMMA.

(i) *The stabilizer of* $b \in \mathcal{B}$ *in* $G(k)$ *is* $P(b)(k)$.

(ii) *If* S *is a maximal* k*-split torus of* G*, the canonical map* $\mathcal{B}_1 \to \mathcal{B}$ *restricts to an injection of* $\mathcal{B}(S)$ *into* B.

1.3. *Apartments*

If S is as in 1.2.1(ii) we may and shall identify $\mathcal{B}(S)$ with a subset of $\mathcal{B}$, the *apartment* defined by S. If $b \in \mathcal{B}$, we have $b \in \mathcal{B}(S)$ if and only if $S \subset P(b)$.

We have the following properties.

1.3.1. *Any two points of* $\mathcal{B}$ *lie in an apartment.*

1.3.2. *If* A *and* A' *are two apartments, there exists* $g \in G(k)$ *mapping* A *onto* A' *and fixing all points of* $A \cap A'$.

1.3.1 is a consequence of the existence of a Tits system on $G(k)$ (see Chapter 11, Section 4.3) and 1.3.2 comes from a result about conjugacy of maximal k-split tori in algebraic groups.

1.4. *Functoriality*

Let S' be any k-split subtorus of G and let S be a maximal k-split torus in G containing S'. We have an injection $\mathcal{B}(S') \to \mathcal{B}(S)$. Composing it with the injection $\mathcal{B}(S) \to \mathcal{B}(G)$ we obtain an injection $\mathcal{B}(S') \to \mathcal{B}(G)$, which is independent of the choice of S. Now let G' be another connected reductive k-group and let $f\colon G' \to G$ be a k-monomorphism. If $\mathcal{B}_1'$ is the disjoint union of the $\mathcal{B}(S')$, for S' a maximal k-split subtorus of G' then f induces a map $\mathcal{B}_1' \to \mathcal{B}(G)$, which in fact factors through an injective map $\mathcal{B}(f)\colon \mathcal{B}(G') \to \mathcal{B}(G)$. Thus $\mathcal{B}$ becomes a functor between the appropriate categories.

If G' is a subgroup of G we may and shall identify $\mathcal{B}(G')$ with a closed subset of $\mathcal{B}(G)$.

1.5. *Fixed points*

If $g \in G(k)$ we denote by $Z_G(g)^\circ$ its connected centralizer. If g is semisimple this is a connected reductive k-subgroup of G.

1.5.1. PROPOSITION. *If* g *is a semisimple element of* $G(k)$ *then* $\mathcal{B}(Z_G(g)^\circ)$ *is the fixed point set* $\mathcal{B}(G)^g$ *of* g *acting on* $\mathcal{B}(G)$.

The proof is straightforward.

Now let l be a Galois extension of k (finite or infinite), with Galois group Γ. Then Γ acts on the building $\mathcal{B}(G, l)$. By a discussion similar to that of Section 1.4, one has an injection $\mathcal{B}(G, k) \to \mathcal{B}(G, l)$.

1.5.2. PROPOSITION. *$\mathcal{B}(G, k)$ is the fixed point set $\mathcal{B}(G, l)^{\Gamma}$.*

If $b \in \mathcal{B}(G, l)$ is fixed under Γ then the l-subgroup $P(b)$ is Γ-stable, hence is defined over k and contains a maximal k-split torus of G. The proposition follows from these observations.

1.6. *Combinatorial structures*

Let G and $\mathcal{B}$ be as before. If P is a minimal parabolic k-subgroup of G the (closed) chamber defined by it is the set of $b \in \mathcal{B}$ with $P(b) \supset P$. Denote by $\Delta = \Delta(G; k)$ the combinatorial building, i.e. the simplicial complex whose simplices are the proper parabolic k-subgroups of G, ordered by reverted inclusion. The vertices of Δ are the maximal parabolic k-subgroups and $(P_0, \ldots, P_r)$ is a simplex if and only if $P_0 \cap P_1 \cap \cdots \cap P_r$ is parabolic.

If S is a maximal k-split torus, the parabolic k-subgroups of G containing S define a subcomplex of Δ, which is the apartment of Δ defined by S. Let Σ be the set of apartments. Then (Δ, Σ) is a building in the sense of Chapter 11, 3.2.1. We have the map P: $b \mapsto P(b)$ of $\mathcal{B}$ to Δ. Let

$$\sigma^d = \left\{ (x_0, \ldots, x_d) \in \mathbb{R}_+^d \colon \sum_{i=0}^{d} x_i = 1 \right\}$$

be the standard simplex in $\mathbb{R}^d$.

1.6.1. LEMMA. *Assume G to be semisimple. Let Q be a proper parabolic k-subgroup of G, defining a d-dimensional simplex of Δ. Then*

$$\{ b \in \mathcal{B} \colon P(b) \supset Q \}$$

is a closed subset of $\mathcal{B}$, homeomorphic to σ^d.

Let S be a maximal k-split torus contained in Q. The set in question then lies in the apartment $\mathcal{B}(S)$. The lemma follows by using the relative root system of (G, S).

It is not hard to see that $\mathcal{B}$ is a geometric realization of Δ, see Curtis et al. [1980].

1.7. *Metric structure on $\mathcal{B}$*

Let S be maximal k-split torus of G and denote by N and Z the normalizer and centralizer of S in G. Then $W = N/Z$ is the relative Weyl group. Choose a Euclidean metric on $X_*(S) \otimes \mathbb{R}$ which is W-invariant. By transport via inner automorphisms we obtain a Euclidean metric on $X_*(S') \otimes \mathbb{R}$ for any maximal k-split torus S' and also an angular metric on any apartment, with values in $[0, \pi]$.

Now let $b, b' \in \mathcal{B}$ and let A be an apartment containing these points. The angular distance $d(b, b')$ of b and b' in A does not depend on the choice of A, as a consequence of 1.3.2. The function d is $G(k)$-invariant.

1.7.1. PROPOSITION.

(i) *d is a metric on $\mathcal{B}$.*

(ii) *The metric space $(\mathcal{B}, d)$ is complete.*

To prove the triangular inequality for d one uses a 'retraction' of $\mathcal{B}$ onto an apartment, given by the following lemma.

1.7.2. LEMMA. *Let A be an apartment. There exists a map ρ: $\mathcal{B} \to A$ with the following properties*:

(a) *the restriction of ρ to A is the identity*;

(b) $d(\rho(b), \rho(b)) \leqslant d(b, b')$ *for all* $b, b' \in \mathcal{B}$.

To prove (ii) one uses the following lemma.

1.7.3. LEMMA. *Let $b \in \mathcal{B}$. There exists an open ball D with centre b such that if* $\mathrm{ad}(g)(b) \in D$ *for some $g \in G(k)$ we have* $\mathrm{ad}(g)(b) = b$.

One chooses D such that $P(b') \subset P(b)$ for all $b' \in D$.

The metric d is continuous for the product topology on $\mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ has the topology of Section 1.2. Hence the latter topology is stronger than the metric topology defined by d. If k is finite they coincide.

1.8. *Opposite points*

The points $b, b' \in \mathcal{B}$ are *opposite* if $d(b, b') = \pi$. This means that for any apartment A containing them, b and b' are antipodes in the sphere A.

If b and b' are opposite, the parabolic subgroups $P(b)$ and $P(b')$ are opposite, i.e. their intersection L is a Levi subgroup of both, which means $P(b)$ is the semidirect product of L and its unipotent radical $R_u(P(b))$ (a normal subgroup). From this one sees the following.

1.8.1. *Let b lie in the chamber defined by the minimal parabolic k-subgroup P. Then the group $R_uP(b)(k)$ acts simply transitively on the set of points of $\mathcal{B}$ opposite to b.*

Let b and b' be nonopposite. We denote by $[b, b']$ the set of $a \in \mathcal{B}$ with

$$d(b, b') = d(b, a) + d(a, b').$$

1.8.2. LEMMA.

(i) *$[b, b']$ lies in all apartments containing b and b'.*

(ii) *If $g \in G(k)$ fixes b and b' it fixes all points of $[b, b']$.*

(iii) *$[b, b']$ is homeomorphic to the segment* $[0, 1]$.

We call $[b, b']$ the *geodesic* joining b and b'.

2. Topological properties, the Steinberg representation

We keep the previous notation. As an application of the theory of spherical buildings, we shall deduce the main properties of the Steinberg representation of the finite group $G(k)$, where k is a finite field.

2.1. *The topological space $\mathcal{B}(G)$*

For the moment, k is arbitrary. We denote by l the k-rank of G. Let P be a minimal parabolic k-subgroup.

2.1.1. THEOREM. *The metric space $\mathcal{B}(G)$ has the homotopy type of a bouquet of $(l-1)$-spheres, indexed by the elements of $R_u(P)(k)$.*

The proof runs as follows. Fix $b \in \mathcal{B}$ with $P(b) = P$ and denote by $\mathcal{O}$ the set of points of $\mathcal{B}$ opposite to b. For each $a \in \mathcal{O}$, choose an open ball D_a such that $P(b') = P(a)$ for all $b' \in D_a$. The D_a are mutually disjoint. Let

$$C = \mathcal{B} - \bigcup_{a \in \mathcal{O}} D_a.$$

The main point of the proof is the contractibility of C, which is a consequence of 1.8.2. The last assertion follows from 1.8.1.

As a consequence we can determine the homology and cohomology groups of $\mathcal{B}(G)$ with constant coefficients. We only formulate the result for cohomology.

2.1.2. COROLLARY. *Let $l > 1$. For any Abelian group M we have*

$$H^0\big(\mathcal{B}(G), M\big) = M, \quad H^i\big(\mathcal{B}(G), M\big) = 0 \quad \textit{for } 0 < i < l-1$$

and $H^{l-1}(\mathcal{B}(G), M) = M^R$, where $R = R_u(P)(k)$.

(For $l = 1$ one has to make a slight modification.)

We used the metric topology of $\mathcal{B}$.

If $\mathcal{B}$ is compact, the corollary can also be proved more directly, by using the exact sequence

$$\cdots \to H_c^i\left(\bigcup_a D_a, M\right) \to H^i(\mathcal{B}, M) \to H^i(C, M) \to \cdots,$$

observing that $H^i(C, M) = 0$ for $i \neq 0$ because of the contractibility of C, and also that $H_c^i(D_a, M) = 0$ if $i < l-1$ and is equal to M if $i = l-1$. As usual, H_c denotes cohomology with compact support.

As another application of 1.8.2 we have the following result.

2.1.3. PROPOSITION. *Assume that k is perfect. Let $g \in G(k)$ be a non semisimple element. Then the fixed point set $\mathcal{B}^g$ is contractible.*

Let $g = g_s g_u$ be the Jordan decomposition of g. Then $g_s, g_u \in G(k)$ (here one uses that k is perfect). Using 1.5.1 one sees that it is sufficient to deal with the case where g is unipotent. In that case one easily sees that $\mathcal{B}^g$ cannot contain two opposite points if $g \neq 1$, and the contractibility follows by using 1.8.2.

2.2. *The Steinberg representation*
Now assume k be to finite. Then $\mathcal{B}$ is a compact metric space, on which the finite group $G(k)$ acts. The *Steinberg representation* St_G of $G(k)$ is the representation of $G(k)$ on the reduced cohomology groups $V = \widetilde{H}^{l-1}(\mathcal{B}, \mathbb{Q})$. (Remember that $\widetilde{H}^i(X, M)$ is the kernel of the canonical map $H^i(X, M) \to H^i(pt, M)$, so in particular $\widetilde{H}^i = H^i$ if $i \neq 0$. The use of $\widetilde{H}^{l-1}$ allows us to include the case $l = 1$).

By using 2.1.2 one sees that $\dim V = N$, where $N = |R_u(P)(k)|$ as in 2.1.2. This is also the order of a p-Sylow subgroup of $G(k)$, where $p = \mathrm{char}\, k$.

We shall also write St_G or St for the character of the Steinberg representation. Its values are determined in Curtis et al. [1980], via a use of Hopf's trace formula. We shall sketch here a somewhat different version of this argument. We assume $l > 1$, the case $l = 1$ being easy. We also assume, for the sake of simplicity, that G is semisimple.

2.3.
If P is any proper parabolic k-subgroup we denote by σ_P the set of $b \in \mathcal{B}$ with $P(b) = P$. It is isomorphic to the interior of the simplex $\sigma^{l-s(P)-1}$, where $s(P)$ denotes the semisimple k-rank. For $i \in [0, l-1]$ put

$$\mathcal{B}_i = \bigcup_{s(P) \geqslant l-i-1} \sigma_P.$$

Then $(\mathcal{B}_i)_{0 \leqslant i \leqslant l-1}$ is an ascending filtration of $\mathcal{B}$ by closed subsets and $\mathcal{B}_i - \mathcal{B}_{i-1}$ is a disjoint union of copies of σ^i. The group $G(k)$ operates on all $\mathcal{B}_i$. If $g \in G(k)$ fixes a point of some σ_P then it fixes all of σ_P. It follows that the fixed point set $\mathcal{B}^g$ is filtered by the fixed point sets $\mathcal{B}_i^g$ and that $\mathcal{B}_i^g - \mathcal{B}_{i-1}^g$ is the disjoint union of the σ_P with $s(P) = l - i - 1$ which are fixed pointwise by g. If X is a topological space with reasonable properties (which $\mathcal{B}$ has) and f a continuous map of X into itself we write

$$\chi(f, X) = \sum_p (-1)^p \operatorname{tr}\left(f, H_c^p(X, \mathbb{Q})\right),$$

where H_c^p denotes cohomology with compact support. One knows that f induces endomorphisms of all $H_c^p(X, \mathbb{Q})$ with trace $\operatorname{tr}(f, H_c^p(X, \mathbb{Q}))$.

We write $\chi(X) = \chi(\mathrm{id}, X)$ for the Euler characteristic of X.

2.3.1. LEMMA. *If $g \in G(k)$ then $\chi(g, \mathcal{B}) = \chi(\mathcal{B}^g)$.*

We have exact sequences

$$\cdots \to H_c^p(\mathcal{B}_i - \mathcal{B}_{i-1}, \mathbb{Q}) \to H_c^p(\mathcal{B}_i, \mathbb{Q}) \to H_c^p(\mathcal{B}_{i-1}, \mathbb{Q})$$
$$\to H_c^{p+1}(\mathcal{B}_i - \mathcal{B}_{i-1}, \mathbb{Q}) \to \cdots,$$

whence

$$\chi(g, \mathcal{B}_i) = \chi(g, \mathcal{B}_{i-1}) + \chi(g, \mathcal{B}_i - \mathcal{B}_{i-1})$$

and

$$\chi(g, \mathcal{B}) = \sum_{i=0}^{l-1} \chi(g, \mathcal{B}_i - \mathcal{B}_{i-1}).$$

By the remarks made before

$$\chi(g, \mathcal{B}_i - \mathcal{B}_{i-1}) = \chi\big((\mathcal{B}_i - \mathcal{B}_{i-1})^g\big).$$

By a similar argument we have

$$\chi(\mathcal{B}^g) = \sum_{i=0}^{l-1} \chi\big((\mathcal{B}_i - \mathcal{B}_{i-1})^g\big),$$

and the lemma follows.

Denote by Ind the induction functor for characters of finite groups. Fix a Borel subgroup B of G which is defined over k.

2.3.2. LEMMA.

$$\mathrm{St}_G = \sum_{P \supset B} (-1)^{s(P)} \mathrm{Ind}_{P(k)}^{G(k)}(1)$$

where P runs through the parabolic k-subgroups containing B.

We have

$$H_c^p(\mathcal{B}_i - \mathcal{B}_{i-1}, \mathcal{Q}) = 0 \quad \text{if } p \neq i,$$

since $\mathcal{B}_i - \mathcal{B}_{i-1}$ is a disjoint union of open simplices of dimension i. It also follows that the representation of $G(k)$ on $H_c^i(\mathcal{B}_i - \mathcal{B}_{i-1}, \mathcal{Q})$ is the direct sum of the permutation representations by left translations in the sets $G(k)/P(k)$, where P runs through the proper parabolic k-subgroups containing B which have semisimple k-rank $l - i - 1$.

Hence

$$\chi(g, \mathcal{B}) = \sum_{i} \chi(g, \mathcal{B}_i - \mathcal{B}_{i-1}) = \sum_{\substack{P \supset B \\ P \neq G}} (-1)^{l - s(P) - 1} \mathrm{Ind}_{P(k)}^{G(k)}(1)(g).$$

On the other hand, by 2.1.2 we have

$$\chi(g, \mathcal{B}) = 1 + (-1)^{l+1} \mathrm{St}_G(g).$$

The lemma follows.

2.3.3. PROPOSITION.
(i) *$g \in G(k)$ is not semisimple then* $\mathrm{St}_G(g) = 0$.
(ii) *If $g \in G(k)$ is semisimple then* $\mathrm{St}_G(g) = (-1)^{s(G)+s(Z)}\,\mathrm{St}_Z(g)$, *where* $Z = Z_G(g)^\circ$.

If g is not semisimple then $\mathcal{B}^g$ is contractible (2.1.3), whence $\chi(\mathcal{B}^g) = 1$. By 2.3.1 $\chi(g, \mathcal{B}) = 1$, and (2) shows that $\mathrm{St}_G(g) = 0$. If g is semisimple then apply 2.3.1, 1.5.1 and 2.1.2.

The proposition holds for an arbitrary connected reductive k-group G.

We can also prove by topological means the following result.

2.3.4. PROPOSITION. *The Steinberg representation is absolutely irreducible.*

Since the character of St is real (see 2.3.2) this amounts to showing that the fixed point set of $G(k)$, acting diagonally in $H^{l-1}(\mathcal{B}, \mathcal{C}) \otimes H^{l-1}(\mathcal{B}, \mathcal{C})$ is one-dimensional, or that the fixed point set $H^{2l-2}(\mathcal{B} \times \mathcal{B}, \mathcal{C})^{G(k)}$ is one-dimensional. Now

$$H^*(\mathcal{B} \times \mathcal{B}, \mathcal{C})^{G(k)} \simeq H^*\big(\mathcal{B} \times \mathcal{B}/G(k), \mathcal{C}\big).$$

Let A be an apartment in $\mathcal{B}$, defined by the maximal k-split torus S and let $W = N(S)/Z(S)$. We then have a homeomorphism

$$\mathcal{B} \times \mathcal{B}/G(k) \simeq (A \times A)/W, \qquad (*)$$

as a consequence of 1.3.1. Since

$$H^{2l-2}\big((A \times A)/W, \mathcal{C}\big) \simeq \big(H^{l-1}(A, \mathcal{C}) \otimes H^{l-1}(A, \mathcal{C})\big)^W$$

the desired result follows, because $H^{l-1}(A, \mathcal{C})$ is one-dimensional (A being a sphere).

REMARK. $(*)$ holds for arbitrary k.

2.4. *Locally compact fields*
The building $\mathcal{B}$, with its metric topology, is compact if and only if k is finite (as follows, e.g., from 1.8.1). But with a different topology it can become compact.

Let k be a non-Archimedean local field and let P be a minimal parabolic k-subgroup. Denote by C the fixed point set of $P(k)$ in $\mathcal{B}$ (a chamber, see 1.6). It is not hard to see that $\mathcal{B} = G(k) \cdot C$, whence there is a map (of sets) $\lambda\colon\ G(k)/P(k) \times C \to \mathcal{B}$. The homogeneous space $G(k)/P(k)$ of the locally compact group $G(k)$ is compact and so is C. We give $G(k)/P(k) \times C$ the product topology and denote by $\mathcal{B}_t$ the set $\mathcal{B}$ provided with the weakest topology for which λ is continuous. Then $\mathcal{B}_t$ is a compact Hausdorff space.

2.4.1. PROPOSITION. *$\widetilde{H}^i(\mathcal{B}_t, \mathbb{Z}) = 0$ if $i \neq l-1$, and $\widetilde{H}^{l-1}(\mathcal{B}_t, \mathbb{Z})$ is a free Abelian group.*

For more details, see Borel and Serre [1976], §1.

2.4.2. Now let $k = \mathbb{R}$. One can also define in this case a space like $\mathcal{B}_t$. Let P and C be as before and let K be a maximal compact subgroup of $G(\mathbb{R})$ such that $G(\mathbb{R}) = KP(\mathbb{R})$. Then $\mathcal{B}_t = K \cdot C$ and one can proceed in a similar manner. One can identify the resulting space with the unit sphere of tangent space to $G(\mathbb{R})/K$ at the origin. See Curtis et al. [1980], 1.4.

3. Affine buildings

In this section k denotes a field with a discrete valuation ω: $k \to \mathbb{Z} \cup \{\infty\}$, which is complete. We denote by $\mathfrak{o}$ the ring of integers of k and by π a uniformizing element. The residue field $\tilde{k} = \mathfrak{o}/\pi\mathfrak{o}$ is assumed to be perfect.

As before, G is a connected reductive k-group. Let S be a maximal k-split subtorus of G. The groups $X^*(S)$ and $X_*(S)$ of characters and cocharacters of S are in duality via the pairing $\langle\, , \rangle$ of Section 1.1. We denote by V the real vector space $X_*(S) \otimes \mathbb{R}$, which we identify with the dual of $X^*(S) \otimes \mathbb{R}$.

3.1. *Apartments*

The valuation ω extends uniquely to a valuation of the algebraic closure $\bar{k}$ to $\mathbb{Q} \cup \{\infty\}$, also denoted by ω. We then define a homomorphism

$$\mu\colon\ S = S(\bar{k}) \to V = \operatorname{Hom}(X^*(S), \mathbb{R})$$

by

$$\mu(s)(\chi) = -\omega\big(\chi(s)\big), \quad \text{where } s \in S,\ \chi \in X^*(S).$$

Let $Z = Z_G(S)$ be the centralizer of S. Since the character group of Z is a subgroup of finite index of $X^*(S)$, the homomorphism μ extends to a homomorphism μ: $Z \to V$. Let N be the normalizer of S and denote by ${}^vW = N/Z$ the Weyl group relative to k (v stands for 'vectorial'). We have a homomorphism ν of N to the group of affine bijections of V such that

(a) if $n \in N$ and $w = nZ$ then the linear part of $\nu(n)v$ is $w \cdot v$ ($v \in V$),

(b) if $s \in Z, v \in V$ then $\nu(s)v = v + \mu(s)$.

We denote by $A = A(S)$ the affine space defined by V. It is the *apartment* defined by S.

3.2. *Affine roots*

Denote by ${}^vR \subset X^*(S)$ the (relative) root system of (G, S). Any $\alpha \in {}^vR$ defines a reflection $s_\alpha \in {}^vW$ and a root subgroup X_α, which is a connected unipotent k-subgroup of G (of dimension one if G is split over k, but not in general).

If $x \in X_\alpha(k) - \{1\}$ there is a unique element $m_\alpha(x)$ in $X_{-\alpha}xX_{-\alpha} \cap N(k)$ whose image in ${}^vW$ is s_α. Then $s_x = \nu(m_\alpha(x))$ is an affine reflection in V, with

$$s_x(v) = v - \big(\langle \alpha, v\rangle + \varphi_\alpha(x)\big)\alpha^\vee,$$

where $\varphi_\alpha(x) \in \mathbb{R}$ ($\alpha^\vee$ denotes the coroot defined by α). If $n \in N$ we have

$$m_{w\alpha}\big(nxn^{-1}\big) = nm_\alpha(x)n^{-1}$$

where $w = nZ$ and $\varphi_{w\alpha}(nxn^{-1}) - \varphi_\alpha(x)$ is independent of x. We put $\varphi_\alpha(1) = \infty$.

3.2.1. EXAMPLE. $G = \mathrm{GL}_n$. Take S to be the diagonal subtorus. We identify $X^*(S)$ and $X_*(S)$ with $\mathbb{Z}^n$, such that the pairing $\langle\, ,\rangle$ is the canonical one. The root system ${}^vR$ consists of the elements $\varepsilon_i - \varepsilon_j$ of $\mathbb{Z}^n$ $(i, j \in [1, n],\ i \neq j)$, $\{\varepsilon_i\}$ being the canonical basis. If $\alpha = \varepsilon_i - \varepsilon_j$ then X_α consists of the matrices $n = (1 + te_{ij})$, where $t \in \bar{k}$ and $\{e_{ij}\}$ is the standard matrix basis. We then have $\varphi_\alpha(x) = \omega(t)$.

We put $X_{\alpha,r} = \{x \in X_\alpha(k)\colon\ \varphi_\alpha(x) \geqslant r\}$.

3.2.2. PROPOSITION.

(i) *$X_{\alpha,r}$ is a subgroup of $X_\alpha(k)$.*

(ii) *If $\alpha, \beta \in {}^vR$ and $\alpha \notin -\mathbb{R}^+\beta$ then the commutator $(X_{\alpha,r}, X_{\beta,s})$ is contained in the subgroup of G generated by the $X_{p\alpha+q\beta,pr+qs}$ where p, q are integers $\geqslant 1$.*

The proof is difficult, see Bruhat and Tits [1984].

If $\alpha \in {}^vR$, $2\alpha \notin {}^vR$ put $X_{2\alpha} = \{1\}$. The groups $X_{\alpha,r}$ define a filtration of the quotient $X_a(k)/X_{2\alpha}(k)$. Let J_α be the set of $r \in \mathbb{R}$ for which this filtration jumps, i.e.

$$X_{\alpha,r} \not\subset \bigcup_{s>r} X_{\alpha,s}X_{2\alpha}(k).$$

An *affine root* of (G, S) is a pair $a = (\alpha, r)$ where $\alpha \in {}^vR$ and $r \in J_\alpha$. The affine root $a = (\alpha, r)$ is viewed as an affine function on A, with

$$a(v) = \langle \alpha, v\rangle + r.$$

It defines an *affine reflection* s_a, given by

$$s_a(v) = v - a(v)\alpha^\vee.$$

These generate a group of affine transformations W of A, the *affine Weyl group* of (G, S). If G is semisimple, then it is the affine Weyl group of a reduced root system (which is not necessary proportional to ${}^vR$).

3.3. *The affine building*

One now constructs a building by gluing together the various apartments $A(S)$. The gluing procedure is as follows (notation being as in Sections 3.1 and 3.2).

Let $A = A(S)$. If $b \in A$ denote by $Q(b)$ the subgroup of $G(k)$ generated by $Z(k)$ and the X_a where a runs through the affine roots with $a(b) \geqslant 0$.

Let $\mathcal{B}_1$ be the disjoint union of the $A(S)$, S running through the maximal k-split subtori of G. We define an equivalence relation similar to the one of Section 1.2: $b \in A(S)$ is equivalent to $b' \in A(S)$ if $b' = \mathrm{ad}(g)(b)$ for some $g \in Q(b)$. The quotient $\mathcal{B}_1/\{\sim\}$ is the *affine building* $\mathcal{B}_a(G, k)$ (or simply $\mathcal{B}_a$). The group $G(k)$ acts on it.

Apartments of $\mathcal{B}_a$ are introduced as in Section 1.3. We have the analogs of 1.3.1 and 1.3.2. Notice that now the apartments are affine spaces.

Let A be as before. Starting with a W-invariant distance on the affine space A one constructs a distance d on $\mathcal{B}_a$, in the same way as in Section 1.4. We have the properties of 1.7.1 for the metric space $(\mathcal{B}_a, d)$.

In the present situation one does not have opposite points. One concludes that for any pair of points $b, b' \in \mathcal{B}_a$ there exists a segment $[b, b']$ with the properties of 1.8.2. As a consequence one sees that there exists a unique midpoint $m \in [b, b']$ such that $d(b, m) = d(m, b')$.

We have the following new properties.

3.3.1. PROPOSITION.

(i) *The metric space $(\mathcal{B}_a, d)$ is contractible.*

(ii) *Let $a, b, b' \in \mathcal{B}_a$ and let m be the midpoint of $[b, b']$. Then*

$$d(b,a)^2 + d(b',a)^2 \geqslant 2d(m,a)^2 + 2^{-1}d(b,b')^2.$$

(i) follows easily from the existence of segments. The proof of (ii) is along the lines of the proof of 1.7.1. The inequality of (ii) can be interpreted as meaning that $\mathcal{B}_a$ has 'negative curvature'. An important consequence is the following result.

3.3.3. PROPOSITION. *Any bounded subgroup of $G(k)$ fixes a point of $\mathcal{B}_a$.*

Recall that a subgroup Γ of $G(k)$ is bounded if, viewing G as a k-subgroup of some GL_n, the matrix coordinate functions are bounded on $G(k)$, for a norm corresponding to ω. For these results, see Bruhat and Tits [1972], §3.

3.3.4. EXAMPLE. $G = \mathrm{GL}_n$. Its building $\mathcal{B}_a$ can be described as follows (see Bruhat and Tits [1972], pp. 238–239).

Let M be the vector space k^n. An *additive norm* on M is a function φ: $M \to \mathbb{R} \cup \{\infty\}$ such that for $x, y \in M$, $\xi \in k$

$$\varphi(x+y) \geqslant \min(\varphi(x), \varphi(y)),$$

$$\varphi(\xi x) = \omega(\xi) + \varphi(x),$$

$$\varphi(x) = \infty \quad \text{if and only if} \quad x = 0.$$

If $|\xi| = c^{-\omega(\xi)}$ $(c > 1)$ is an absolute value on k associated to ω then $|x| = c^{-\varphi(x)}$ defines a non-Archimedean norm on M.

We can identify $\mathcal{B}_a$ with the set of additive norms, the $G(k)$-action being the obvious one.

Let $e = (e_1, \ldots, e_n)$ be an unordered basis of M. Denote by A_e the set of $\varphi \in \mathcal{B}_a$ with

$$\varphi\left(\sum_{i=1}^{n} \xi_i e_i\right) = \inf_{1 \leqslant i \leqslant n} \left(\omega(\xi_i) - \alpha_i\right),$$

where $\alpha = (\alpha_1, \dots, a_n) \in \mathbb{R}^n$. Then A_e is an apartment, and any apartment is of this form, with a unique unordered basis e. If $\varphi' \in A_e$ is given by the previous formula with $\alpha' \in \mathbb{R}^n$, then the distance $d(\varphi, \varphi')$ can be taken to be the Euclidean distance of α and α'. This defines the metric of $\mathcal{B}_a$ (cf. 1.3.1).

A *lattice* Λ in M is an $\mathfrak{o}$-module of the form

$$\Lambda = \mathfrak{o}e_1 + \cdots + \mathfrak{o}e_n,$$

where $(e_1, \dots, e_n)$ is a basis. The result of 3.3.3 in this case implies (and is equivalent to) the following result: any bounded subgroup of $G(k)$ stabilizes a lattice in M.

It is also if interest to observe that now 1.3.1 implies the 'elementary divisor theorem' for lattices: If Λ and Λ' are two lattices in M there exists a basis $(e_1, \dots, e_n)$ of M and a set of integers $(h_1, \dots, h_n)$ such that

$$\Lambda = \mathfrak{o}e_1 + \cdots + \mathfrak{o}e_n, \qquad \Lambda' = \pi^{h_1}\mathfrak{o}e_1 + \cdots + \pi^{h_n}\mathfrak{o}e_n.$$

The group $\mathbb{R}$ acts on the set of additive norms by translations, whence an action of $\mathbb{R}$ on $\mathcal{B}_a$ by isometries. It can be shown that $\mathcal{B}_a/\mathbb{R}$ is the building of SL_n.

3.3.5. *Combinatorial structure.* We continue with the general situation. Let A be an apartment of $\mathcal{B}_a$. For each affine root a relative to A (see Section 3.2) let H_a be its zero set. The closure of a connected component of the complement $A - \bigcup_a H_a$ is a (closed) *chamber* in A. We then also have the notion of chamber in $\mathcal{B}_a$. The set of chambers can be made into a chamber system. We have analogies of the results of Section 1.6.

3.3.6. EXAMPLE. $G = \mathrm{SL}_2$. Put $M = k^2$. Using the facts stated in 3.3.1 one now checks that the chambers correspond to the lattices $\Lambda \subset M$, up to homothety. Two chambers Λ and Λ' are adjacent if these lattices are homothetic to a pair where $\Lambda' \subset \Lambda$ and Λ/Λ' is a vector space of dimension 1 over the residue field. Also, the building $\mathcal{B}_a$ is now a tree, whose edges correspond to the chambers. The edges coming together in a vertex of the graph corresponds to the points of the projective line over the residue field $\tilde{k}$. See Serre [1980] for more details.

3.4. *A Tits system on $G(k)$*

We keep the previous notation. Fix an apartment $A = A(S)$ of $\mathcal{B}_a$. Fix a chamber C in A and let $I \subset G(k)$ be the pointwise stabilizer of C. This is an *Iwahori subgroup* of $G(k)$. As before, let N be the normalizer of S.

3.4.1. THEOREM. *Let G be semisimple and simply connected. Then $(G(k), I, N(k))$ is a Tits system.*

The corresponding Weyl group is the affine Weyl group (Section 3.2). For the proof, see Bruhat and Tits [1972], §6.5. For the notion of Tits system, see Chapter 11, Section 4.3.

3.4.2. EXAMPLE. $G = \mathrm{SL}_n$. We continue with the example of 3.3.4. Let $A = A_e$ be as before. Then $\varphi \in A$ we described by a vector $\alpha \in \mathbb{R}^n$, up to a translation by an element of $\mathbb{R}$. One shows that a chamber C is given by the $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfying

$$\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n$$

(a translation invariant condition). One concludes that the corresponding Iwahori subgroup I consists of the $g = (g_{ij}) \in \mathrm{SL}_n(k)$ with $g_{ij} \in \mathfrak{o}$ for $i < j$ and $g_{ij} \in \pi\mathfrak{o}$ for $i > j$.

It is clear that $N(k)$ is the subgroup of $\mathrm{SL}_n(k)$ whose elements have in each row and column only one nonzero entry.

3.5. *Decompositions*

We use the following notation. Let K be the stabilizer in $G(k)$ of

$$0 \in V = \mathrm{Hom}\left(X^*(S), \mathbb{R}\right),$$

and take a chamber $C \subset A$ with vertex 0. Denote by $\widetilde{C}$ the convex cone in V, with vertex 0, generated by C. This is a Weyl chamber in V for the finite Weyl group ${}^vW$, defining a system of positive roots ${}^vR^+$ in the root system ${}^vR$. Let U be the subgroup of G generated by the root groups X_α with $\alpha \in {}^vR^+$. Put $Y = \nu(Z(k)) \subset V$ and $Y_+ = Y \cap \widetilde{C}$, $Z(k)_+ = \nu^{-1}(Y_+)$.

3.5.1. *Iwasawa decomposition.* $G(k) = KZ(k)U(k)$ and the map

$$KzU(k) \mapsto \nu(z) \quad \left(z \in Z(k)_+\right)$$

defines a bijection

$$K\backslash G(k)/U(k) \xrightarrow{\sim} Y_+.$$

3.5.2. *Cartan decomposition.* $G(k) = KZ(k)K$ and the map

$$KzK \mapsto \nu(z) \quad \left(z \in Z(k)_+\right)$$

defines a bijection $K\backslash G(k)/K \xrightarrow{\sim} Y_+$.

For the proofs, see Bruhat and Tits [1972], §4. The results are to be compared with the analogous results for real Lie groups.

3.6. *Groups over local fields*

Assume k to be a local field, i.e. assume that the residue field $\tilde{k}$ is finite. Denote by k_{et} a maximal *étale* (i.e. algebraic unramified) extension of k. Then k_{et} is complete for the extension of ω and the residue field $\tilde{k}_{\mathrm{et}}$ is an algebraic closure of $\tilde{k}$. The Galois extension k_{et}/k has a Galois group Γ isomorphic to the Galois group $(\tilde{k}_{\mathrm{et}}/\tilde{k})$ (isomorphic to $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$). By a result similar to 1.5.2, we have that $\mathcal{B}_a(G,k)$ is the fixed point set of Γ in $\mathcal{B}_a(G, k_{\mathrm{et}})$.

3.6.1. LEMMA. *G has a Borel subgroup which is defined over k_{et}.*

See Bruhat and Tits [1987], §4.

3.6.2. PROPOSITION. *Assume that G is quasisimple, simply connected and anisotropic over k. Then G is isomorphic over k_{et} to some SL_n.*

Quasisimplicity means that the root system of G is irreducible and anisotropy that G contains no nontrivial k-split tori.

Since G is anisotropic the building $\mathcal{B}_a(G,k)$ is empty, which means that the Galois groups Γ acts on $\mathcal{B}_a(G,k_{\mathrm{et}})$ without fixed points (see 1.5.2).

Let B be a Borel subgroup of G defined over k_{et} (see the previous lemma). The subgroup B determines a chamber $C \subset \mathcal{B}_a(G,k_{\mathrm{et}})$ and Γ acts on C. By considering the possible affine root systems one sees that Γ can only act on C without fixed points if G is of type A_{n-1}, whence the result.

The proposition was proved originally by Kneser [1965a,b]. The argument sketched here comes from Bruhat and Tits [1987].

In the present case, k a local field, the building $\mathcal{B}_a = \mathcal{B}_a(G,k)$, with its metric topology, is locally compact. Let $\mathcal{B}_t$ the topological spherical building of Section 2.4.

3.6.3. PROPOSITION (compactification theorem). *There exists a topology $\mathcal{T}$ on the disjoint union $\widetilde{\mathcal{B}}_a = \mathcal{B}_a \coprod \mathcal{B}_t$ with the following properties.*

(a) *$\widetilde{\mathcal{B}}_a$ is compact and contractible.*

(b) *$\mathcal{T}$ is $G(k)$-stable and the action of $G(k)$ is continuous.*

(c) *$\mathcal{B}_a$ (resp., $\mathcal{B}_t$) is an open (resp., closed) subset of $\widetilde{\mathcal{B}}_a$.*

This is due to Borel and Serre (see Borel and Serre [1976], §5). For a similar 'combinatorial' result, see Ronan [1989], 9.3.

3.6.4. COROLLARY. *$\widetilde{H}^i_c(\mathcal{B}_a,\mathbb{Z}) = 0$ if $i \neq l$ and $\widetilde{H}^l_c(\mathcal{B}_a,\mathbb{Z})$ is a free Abelian group. Here l is again the k-rank of G.*

This is an easy consequence of 3.6.3 and 2.4.1. One has a representation of $G(k)$ on $\widetilde{H}^l_c(\mathcal{B}_a,\mathbb{C})$, which is analogous to the Steinberg representation of Section 2. See Ronan [1989], 5.10.

3.7. *Group schemes*

In the work of Bruhat and Tits 1984] certain group schemes are used. We give here a few basic facts about these. Here we use some notions from algebraic geometry.

3.7.1. We assume for simplicity that G is a connected, simply connected and semisimple algebraic group over a non-Archimedean local field k. We choose an apartment A in $\mathcal{B}_a(G,k)$ and a chamber C in A. Let X be a proper subset of the set Δ of vertices of C. The stabilizer in $G(k)$ of the facet of C given by $\Delta - X$ is denoted by P_X and is called a *parahoric* subgroup. If $X = \varnothing$ then $P_\varnothing = I$ is an Iwahori subgroup, see Section 3.4. All subgroups of $G(k)$ which are conjugate to some P_X are called *parahoric* subgroups of type X. Parahoric subgroups are selfnormalizing. If P_X has a type X such that $\Delta - X$ consists of one point then P_X is a maximal open and compact subgroup of $G(k)$. In particular there are $l+1$ $G(k)$-conjugacy classes of such subgroups, where l is the k-rank of G.

3.7.2. Given P_X as above, Bruhat and Tits construct a group scheme $\mathcal{P}_X$ over the ring $\mathfrak{o}$ of integers in k with the following properties.

(i) $\mathcal{P}_X$ is smooth of finite type with generic fiber G.

(ii) $\mathcal{P}_X(\mathfrak{o}) = P_X$.

(iii) Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. Then the reduction homomorphism

$$\mathcal{P}_X(\mathfrak{o}) \to \mathcal{P}_X(\mathfrak{o}/\mathfrak{p})$$

is surjective and its kernel is connected and unipotent.

(iv) The special fiber of $\mathcal{P}_X$ is connected (G is simply connected by assumption). The relative Dynkin diagram of the reductive quotient of the special fiber can be determined from the extended relative Dynkin diagram of G and from X.

3.7.3. Using 3.7.2, Bruhat and Tits obtain a new proof of the triviality of the Galois cohomology set $H^1(k, G)$. This was proved originally by Kneser [1965a]. His proof depends on a case by case analysis.

3.7.4. The Iwahori subgroup $I = P_\varnothing \subset P_X$ is the preimage of a Borel subgroup in $\mathcal{P}_X(\mathfrak{o}/\mathfrak{p})$. Hence from 3.7.2(iv) one can derive explicit formulas for $[P_X : I]$ and for the Euler–Poincaré volume of P_X, see 5.2.3.

4. The Borel–Serre compactification

Let G be a connected semisimple linear algebraic group defined over $\mathbb{Q}$ such that $G(\mathbb{R})$ is noncompact. We describe the Borel–Serre construction of a manifold with corners $\overline{X}$ which contains as its interior the space X of maximal compact subgroups of $G(\mathbb{R})$. The space $\overline{X}$ can be viewed as an analog at the infinite place of

$$\widetilde{\mathcal{B}}_a = \mathcal{B}_a \coprod \mathcal{B}_t,$$

see 3.6.3. For full details, see Borel and Serre [1973].

4.1. *The geodesic action*

4.1.1. It is known that all maximal compact subgroups of $G(\mathbb{R})$ are $G(\mathbb{R})$-conjugate. If K is such a subgroup then K is selfnormalizing in $G(\mathbb{R})$ and meets all connected components of $G(\mathbb{R})$. Hence if we fix K we can identify the space X of maximal compact subgroups of $G(\mathbb{R})$ with $K\backslash G(\mathbb{R})$. Let θ be the Cartan involution corresponding to K and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the real Lie algebras of $G(\mathbb{R})$ and K. Then one has a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into eigenspaces of θ. The exponential map exp induces a diffeomorphism $\mathfrak{p} \xrightarrow{\sim} X$. In particular X is diffeomorphic to a Euclidean space.

4.1.2. Let P be a proper parabolic subgroup of the real algebraic group $G \times_{\mathbb{Q}} \mathbb{R}$. We write in Sections 4.1 to 4.5 simply P instead of $P(\mathbb{R})$. If N is the unipotent radical of P then P is the semidirect product $P = L \times N$ of N and a maximal reductive subgroup L of P, called a Levi subgroup. Let $x \in X$ and denote by K_x the corresponding maximal compact subgroup, with Cartan involution θ_x. It is known that P contains exactly one θ_x-stable Levi subgroup, which will be denoted by L_x. The connected component of the maximal central $\mathbb{R}$-split subtorus of L_x is denoted by S_x. Let π: $P \to P/N$ be the canonical projection and denote the connected component of the maximal $\mathbb{R}$-split central subtorus of P/N by S_P. Then π induces an isomorphism $S_x \xrightarrow{\sim} S_P$.

4.1.3. We fix $x \in X$ and identify $X \xrightarrow{\sim} K_x\backslash G(\mathbb{R})$. Since $G(\mathbb{R}) = K_x \cdot P$ (Iwasawa decomposition, P proper parabolic) we have $X \xrightarrow{\sim} K_x \cap P\backslash P$ and $K_x \cap P$ is maximal compact in P. For $a \in S_P$ and $y = xp$, $p \in P$, we define the *geodesic action* of S_P on X, denoted by $(y, a) \mapsto y \circ a$, as follows. Let $y = (K_x \cap P)p$ and let $a_x \in S_x$ correspond to $u \in S_P$ under the isomorphism $S_x \xrightarrow{\sim} S_P$ of 4.1.2. Then $y \circ a = (K_x \cap P)a_x p$. Now it is not difficult to show:

4.1.4. PROPOSITION. *The geodesic action is a well-defined right action of S_P on X. The action is independent of x and commutes with the right action of P on X.*

4.1.5. EXAMPLE. Consider

$$G = \mathrm{SL}_2 \quad \text{and} \quad K = \mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R}).$$

We identify $\mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})$ with the upper half-plane $H = \{z \in C\colon \operatorname{Im} z > 0\}$ by the map

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{di - b}{-ci + a}, \quad i = \sqrt{-1}.$$

Then $g \in \mathrm{SL}_2(\mathbb{R})$ acts on H by

$$zg = \frac{dz - b}{-cz + a}.$$

Let P be the group of upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$. We identify $S_P = \mathbb{R}_+^*$ with

$$S_i := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_+^* \right\}.$$

Let α be the root of S_i on P, i.e.

$$\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^2.$$

Then $a \in S_P$ acts on $z \in H$ by $z \circ a = (x + iy) \circ a = x + i\alpha(a)^{-1}y$, $x, y \in \mathbb{R}$. Hence $z \circ S_i = z \circ S_P$ is the vertical line through x. If a tends to zero then $z \circ a$ tends to infinity.

4.1.6. In Section 4.2 we shall need the geodesic action in a situation which is closely related to the $\mathcal{Q}$-structure of G. We introduce the relevant notation.

(i) Let P be a proper parabolic subgroup of G, which is assumed to be defined over $\mathcal{Q}$. If $U = R_u P$ is the unipotent radical of P we denote the connected component of the set of real points of the maximal $\mathcal{Q}$-split central subtorus of P/U by A_P. Then $A_P \subset S_P$, see 4.1.2, and the geodesic action of A_P on X is defined.

(ii) Next we introduce natural coordinates in A_P as follows. We fix a minimal parabolic subgroup P_0 defined over $\mathcal{Q}$ in G and choose a maximal $\mathcal{Q}$-split torus $T \subset P_0$. Let Δ be the set of simple $\mathcal{Q}$-roots of G with respect to T and P_0. If $J \subset \Delta$ is a subset we put

$$T_J := \left(\bigcap_{\alpha \in J} \ker \alpha \right)^0.$$

Let $Z(T_J)$ be the centralizer of T_J in G and put $P_J := Z(T_J) \cdot U$, where U is the unipotent radical of P_0. Then P_J is a parabolic subgroup which contains $P_0 = P_\varnothing$. If $J = \Delta$ then $P_J = G$. It is known that a given $\mathcal{Q}$-parabolic P is $G(\mathcal{Q})$-conjugate to exactly one P_J. For all this, see Borel and Tits [1965], §5. Let $A_J := A_{P_J}$, as defined above. Then we get a canonical isomorphism $A_P \xrightarrow{\sim} A_J$. But the roots in $\Delta - J$ define a basis of $X^*(P_J) \otimes \mathcal{Q}$. Hence there is an isomorphism

$$A_J \xrightarrow{\sim} \mathbb{R}_+^{*\Delta - J} \quad \text{with } a \mapsto (\ldots, \alpha(a), \ldots),\ \alpha \in \Delta - J.$$

(iii) Let $x \in X$ and assume that P and A_P are as above. We write $P(\mathbb{R}) = L_x \cdot N$, $N = U(\mathbb{R})$, see 4.1.2, and we denote by A_x the preimage of A_P under the isomorphism π: $S_x \xrightarrow{\sim} S_P$. Then we have the decomposition $P(\mathbb{R}) = M_x A_x N$ where M_x (essentially) is the subgroup of L_x on which all squares of characters of L_x become trivial. We point out that in general M_x and A_x are not defined over $\mathcal{Q}$.

4.2. *The manifold* $\overline{X}$

4.2.1. Let P be a $\mathcal{Q}$-parabolic subgroup of G. We have a canonical isomorphism

$$A_P \xrightarrow{\sim} \mathbb{R}_+^{*\Delta - J},$$

see 4.1.6(ii), and A_P acts componentwise on $\mathbb{R}^{A-J}$. Hence we can form a completion $\bar{A}_P$ of A_P as fibre product $\bar{A}_P = A_P \times_{A_P} \mathbb{R}_+^{\Delta - J}$. We define *the corner* $X(P)$ *of* X *associated to* P as the fibre product

$$X(P) = X \times_{A_P} \bar{A}_P,$$

where A_P acts on X by the geodesic action. We observe that $X(G) = X$. We define the *face* $e(P)$ *of* X *associated to* P by $e(P) = X/A_P$. We observe that $e(G) = X$. It is clear that $X(P)$ has the structure of a manifold with corners, see Borel and Serre [1973], Appendice. If $P \subset Q$ are $\mathcal{Q}$-parabolic subgroups we have a natural inclusion $X(Q) \subset X(P)$, and $X(Q)$ has the structure of an open submanifold of $X(P)$. Therefore on

$$\overline{X} = \varinjlim_{P \in \mathcal{P}} X(P)$$

there is exactly one structure of a manifold with corners such that the $X(P)'s$ are open submanifolds. Here $\mathcal{P}$ denotes the set of all $\mathcal{Q}$-parabolic subgroups of G.
The following result is not difficult to prove.

4.2.2. PROPOSITION. *The manifold with corners $\overline{X}$ is Hausdorff and countable at infinity.*

4.2.3. We collect further properties of $\overline{X}$.

(i) One has a disjoint union

$$X(P) = \bigcup_{Q} e(Q)$$

where the sum runs over all $\mathbb{Q}$-parabolics Q of G such that $P \cap Q$ is parabolic. This follows from the fact that $(x, a) \in X \times_{A_P} \bar{A}_P$ can be represented by (x', a'), where for a' only coordinates 0 or 1 occur, and the fact that subsets of Δ correspond to parabolics of G. In particular here $e(P)$ occurs and is identified with $X \times 0_P/A_P$, where 0_P is the zero element in $\mathbb{R}^{\Delta - J}$. Hence we also get a disjoint union

$$\partial \overline{X} = \bigcup_{P} e(P)$$

of the boundary $\partial \overline{X}$ of $\overline{X}$ where P runs over all proper $\mathbb{Q}$-parabolics of G.

(ii) For $x \in X$ let $P(\mathbb{R}) = M_x A_x N$ be the corresponding decomposition, see 4.1.6(iii). We can identify $e(P)$ with $(M_x \cap K \backslash M_x) \times N$ and get an isomorphism $e(P) \times A_P \xrightarrow{\sim} X$ which extends to an isomorphism $e(P) \times \bar{A}_P \xrightarrow{\sim} X(P)$. These isomorphisms depend on the choice of x.

(iii) Let $\overline{e(P)}$ be the closure of $e(P)$ in $\overline{X}$. Using 4.2.3(i) we get

$$\overline{e(P)} = \bigcup_{Q \subset P} e(Q).$$

In particular $\overline{e(P)} \cap \overline{e(Q)}$ is equal to $\overline{e(P \cap Q)}$ if $P \cap Q$ is parabolic, and empty otherwise.

(iv) The $\overline{e(P)}$'s are contractible. To see this, one observes that

$$e(P) \simeq (M_x \cap K \backslash M_x) \times N$$

is contractible. This holds since $M_x \cap K \backslash M_x$ is a symmetric space and since N is isomorphic to some $\mathbb{R}^n$. The assertion then follows since $\overline{e(P)}$ is homotopic to $e(P)$.

(v) The $\overline{e(P)}$'s for proper P form a locally finite cover of $\partial \overline{X}$. This follows from (iii) and the fact that a given parabolic subgroup is contained in only finitely many parabolic subgroups.

4.2.4. Let k be a number field and choose a finite set S of finite places of k. Denote by $O_S \subset k$ the ring of S-integers. It consists of all the elements of k which are integral outside S. Let G be an algebraic group defined over k and assume that ρ: $G \to \mathrm{GL}_n$ is a k-rational faithful representation. A subgroup Γ_S of $G(k)$ is called S-arithmetic if it is commensurable to $\rho^{-1}(\mathrm{GL}_n(O_S))$. This definition does not depend on the choice of ρ, see Borel [1969]. If $S = \varnothing$ then $\Gamma_S = \Gamma$ is called arithmetic. We can view G via Weil restriction as a group $H = \mathrm{res}_{\mathbb{Q}}\, G$ which is defined over $\mathbb{Q}$. The arithmetic subgroups of H and G coincide. In Sections 4 and 5 we only work with groups defined over $\mathbb{Q}$.

We observe that $G(\mathbb{Q})$ acts on $\overline{X}$ by transport of structure. Hence an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ also acts. The following result is a main motivation for the construction of $\overline{X}$.

4.2.5. PROPOSITION. *An arithmetic group $\Gamma \subset G(\mathcal{Q})$ acts properly and discontinuously on $\overline{X}$. The quotient $\overline{X}/\Gamma$ is compact.*

REMARK.

(i) The proof of 4.2.5 requires the main results of reduction theory (see Borel [1969]) and is not at all formal. In particular, if $\mathcal{Q}$-rank $G = 0$, then $\partial\overline{X} = \varnothing$ and X/Γ is compact. This is the compactness criterion of reduction theory.

(ii) If $\mathcal{Q}$-rank $G = 1$ then the connected components of $\partial\overline{X}/\Gamma$ are parametrized by the Γ-conjugacy classes of proper parabolic subgroups of G. If $\mathcal{Q}$-rank $G > 1$ then $\partial\overline{X}/\Gamma$ is connected.

(iii) For $G = \mathrm{SL}_n$ there is an equivalent construction of a compactification of X/Γ due to Siegel [1966]. If $\mathcal{Q}$-rank $G = 1$ then Raghunathan [1968] has constructed an equivalent compactification. For arbitrary $\mathcal{Q}$-rank of G Harder [1971] has constructed a manifold with boundary $M \subset X/\Gamma$ which can be viewed as a retract of $\overline{X}/\Gamma$.

(iv) We mention that if X/Γ has the structure of a complex algebraic variety then there exist other compactifications of X/Γ which are algebraic varieties, see Baily and Borel [1966], Kempf, Knudsen, Mumford and Saint-Donat [1973].

For $v \in S$ let $X_v = \mathcal{B}_a(G, \mathcal{Q}_v)$ be the affine building, see Section 3.3. Then

$$G_S = G(\mathbb{R}) \times \prod_{v \in S} G(\mathcal{Q}_v)$$

acts on

$$\overline{X}_S = \overline{X}_\infty \times \prod_{v \in S} X_v$$

componentwise. Here X_∞ denotes the space of maximal compact subgroups of $G(\mathbb{R})$ and $\overline{X}_\infty$ is as in 4.2.1. We embed Γ_S diagonally in G_S. Hence Γ_S acts on $\overline{X}_S$.

4.2.6. COROLLARY. *An S-arithmetic group Γ_S acts properly and discontinuously on $\overline{X}_S$. The quotient $\overline{X}_S/\Gamma_S$ is compact.*

This is proved by induction on $|S|$. If $S = \varnothing$ we use 4.2.5. For the induction step we use that $\Gamma_S \backslash G(\mathcal{Q}_p)/U_p$ is finite where U_p is an open and compact subgroup of $G(\mathcal{Q}_p)$, see Borel and Serre [1976].

4.2.7. We know that the $\overline{e(P)}$ for proper parabolic P's form a locally finite cover of $\partial\overline{X}$, see 4.2.3(v). If $P_0, \ldots, P_d$ are proper parabolic $\mathcal{Q}$-subgroups such that

$$\overline{e(P_0)} \cap \cdots \cap \overline{e(P_d)} \neq \varnothing$$

we call $(P_0, \ldots, P_d)$ a d-simplex. By 4.2.3(iii) we have

$$\overline{e(P_0)} \cap \cdots \cap \overline{e(P_d)} \neq \varnothing$$

if and only if $P_0 \cap \cdots \cap P_d$ is a parabolic subgroup of G. Hence the geometrical realization of the nerve of the covering $g\{\overline{e(P)}\}$ of $\partial\overline{X}$ is exactly the spherical building $\mathcal{B}(G; \mathbb{Q})$, see Section 1.6. The $\overline{e(P)}$ are locally and globally contractible, see 4.2.3(iv). This essentially implies that $\partial\overline{X}$ and $\mathcal{B}(G, \mathbb{Q})$ endowed with the topology of Section 1.2 have the same homotopy type, see Borel and Serre [1973], Theorem 8.2.1. Now 2.1.1 applies and we arrive at:

4.2.8. PROPOSITION. *Let l be the $\mathbb{Q}$-rank of G. Then $\partial\overline{X}$ has the homotopy type of a bouquet of $(l-1)$-spheres indexed by the elements of $R_u(P)(\mathbb{Q})$ where P is a minimal parabolic subgroup defined over $\mathbb{Q}$.*

5. An application to cohomology of S-arithmetic groups

In this chapter we sketch results due to Borel and Serre on the cohomological dimension of an S-arithmetic group Γ_S and describe work of Harder and Serre on the Euler–Poincaré characteristic of such groups.

5.1. *The cohomological dimension of S-arithmetic groups*

5.1.1. We use the notation which has been introduced in Sections 3 and 4. For a finite place v of $\mathbb{Q}$ we put $X_v = \mathcal{B}_a(G, \mathbb{Q}_v)$, see Section 3.3. We remark that

$$\dim X_v = \mathbb{Q}_v\text{-rank}(G \times \mathbb{Q}_v)$$

and write

$$d_v = \dim X_v, \qquad d_\infty = \dim X_\infty, \qquad l = \mathbb{Q}\text{-rank}\, G.$$

We abbreviate

$$d = d_\infty + \sum_{v \in S} d_v.$$

We recall that the *cohomological dimension* $\operatorname{cd}(\Gamma)$ of a group Γ is the smallest integer q such that $H^i(\Gamma, M) = 0$ for all Γ-modules M and all $i > q$. The group Γ is said to be of *type* (FL) if the trivial Γ-module $\mathbb{Z}$ admits a finite projective resolution consisting of free finite $\mathbb{Z}[\Gamma]$-modules.

5.1.2. PROPOSITION. *Let $\Gamma_S \subset G(\mathbb{Q})$ be an S-arithmetic torsion-free subgroup. Then the following hold.*

(i) *Γ_S is a finitely presentable group.*

(ii) *Γ_S is of type* (FL).

(iii) *The cohomological dimension of Γ_S is $d - l$.*

(iv) *$H_c^{d-l}(\overline{X_S}, \mathbb{Z}) \xrightarrow{\sim} H^{d-l}(\Gamma_S, \mathbb{Z}[\Gamma_S]) =: I_S$ and I_S is a dualizing module for Γ_S, which means there is a distinguished class $e \in H^{d-l}(\Gamma_S, I_S)$ such that for all Γ_S-modules M and all $q \in \mathbb{N}$ the cap product with e induces an isomorphism*

$$H^q(\Gamma_S, M) \xrightarrow{\sim} H_{d-l-q}(\Gamma_S, I_S \otimes M).$$

(v) *If $l = 0$ and $\sum_{v \in S} d_v = 0$ then $I_S \simeq \mathbb{Z}$. Otherwise I_S is a free $\mathbb{Z}$-module of infinite rank.*

We sketch the argument. For full details, see Borel and Serre [1976].

The space $\overline{X}_S$ is locally and globally contractible and $\overline{X}_S/\Gamma_S$ is compact. Hence (i) follows from Behr [1962]. Since Γ_S is torsion-free the space $\overline{X}_S/\Gamma_S$ is triangulizable. Therefore the simplicial complex made out of a pullback of a triangulation of $\overline{X}_S/\Gamma_S$ to $\overline{X}_S$ gives the desired resolution and (ii) holds. Clearly, (iii) is a consequence of (iv). To prove (iv) we use that $\overline{X}_S$ is contractible and get for the reduced cohomology group $\widetilde{H}^i(\overline{X}_S, \mathbb{Z}) = 0$ for all i. By Poincaré duality

$$H_c^i(\overline{X}_\infty, \mathbb{Z}) = H_{d_\infty - i}\big(\overline{X}_\infty, \partial\overline{X}_\infty, \mathbb{Z}\big)$$

and the boundary operator in homology gives

$$\widetilde{H}_{d_\infty - i}(\overline{X}_\infty, \partial\overline{X}_\infty, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}_{d_\infty - i - 1}\big(\partial(\overline{X}_\infty), \mathbb{Z}\big).$$

But $\partial(\overline{X}_\infty)$ is a bouquet of $(l-1)$-spheres, see 4.2.8. Hence $\widetilde{H}_{d_\infty - i - 1}(\partial(\overline{X}_\infty), \mathbb{Z})$ is zero unless $d_\infty - i - 1 = l - 1$, see also 2.4.1. Now we use 3.6.4. and the Künneth formula and get

$$H_c^i(\overline{X}_S, \mathbb{Z}) = H_c^{d_\infty - l}(X_\infty, \mathbb{Z}) \otimes \bigotimes_{v \in S} H_c^{d_v}(X_v, \mathbb{Z}) \quad \text{if } i = d - l,$$

and zero otherwise. In particular, (v) follows. For the fact that then I_S is dualizing, see Brown [1982].

To prove that

$$H_c^{d-l}(\overline{X}_S, \mathbb{Z}) = H^{d-l}\big(\Gamma_S, \mathbb{Z}[\Gamma_S]\big)$$

we use the simplicial complex C_* coming from a triangulation of $\overline{X}_S/\Gamma_S$. Then

$$H^*(\Gamma_S, \mathbb{Z}[\Gamma_S])$$

is computed from

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma_S]}\big(C_*, \mathbb{Z}[\Gamma_S]\big) = \mathrm{Hom}_{\mathbb{Z}}^f(C_*, \mathbb{Z})$$

where the upper index f indicates that we only take homomorphisms with support on finitely many cells. But the last complex gives $H_c^*(\overline{X}_S, \mathbb{Z})$, and (iv) is proved.

REMARKS.

(i) We did not deal with S-arithmetic groups over function fields. To some extent the situation is simpler here, see Harder [1977]. On the other hand, there are complications. For example, it is not true that in general such a Γ_S contains a torsion free subgroup of finite index. Moreover 5.1.2 (i) and (ii) are false in general, see Behr [1987].

(ii) The results of this chapter hold for G reductive. The restriction to semisimple G simplifies some technical aspects of the construction of $\overline{X}_\infty$.

(iii) For the structure of $H^{d_\infty - l}(\overline{X}_S, C)$ as a G_S module, see Borel [1976]. There is a connection with modular symbols, see Ash and Rudolph [1970].

5.2. *Euler–Poincaré characteristics of S-arithmetic groups*

5.2.1. Let Γ_S be a torsion-free S-arithmetic group and M a finite-dimensional $\mathbb{Q}$-vector space with Γ_S-action. The *Euler–Poincaré characteristic* is defined as

$$\chi(\Gamma_S, M) = \sum_{i=0}^{\infty}(-1)^i \dim_{\mathbb{Q}} H^i(\Gamma_S, M).$$

Since Γ_S is of type (FL) we easily see that $\chi(\Gamma_S, M) = \chi(\Gamma_S, \mathbb{Q}) \dim_{\mathbb{Q}}(M)$. We explain how Harder and Serre compute $\chi(\Gamma_S) = \chi(\Gamma_S, \mathbb{Q})$. They show that $\chi(\Gamma_S)$, if not zero, essentially is a covolume of Γ_S in

$$G_S = G(\mathbb{R}) \times \prod_{v\in S} G(\mathbb{Q}_v).$$

5.2.2. Let H be locally compact unimodular group and let μ be an invariant measure on H. If $\Gamma \subset H$ is a discrete subgroup then μ induces a left invariant measure on H/Γ which is again denoted by μ. The measure μ is said to be an *Euler–Poincaré measure* on H if

$$\chi(\Gamma) = \int_{H/\Gamma} \mu$$

for all discrete subgroups $\Gamma \subset H$ of type (FL) such that H/Γ is compact. Of course, on H there is at most one Euler–Poincaré measure.

5.2.3. We explain the construction of the Euler–Poincaré measure μ on

$$G_S = G(\mathbb{R}) \times \prod_{v\in S} G(\mathbb{Q}_v).$$

For full details, see Serre [1971]. The measure μ is constructed as a product

$$\mu = \mu_\infty \otimes \bigotimes_{v\in S} \mu_v$$

of Euler–Poincaré measures. The existence of μ_∞ follows immediately from the classical Gauss–Bonnet theorem and from the fact that there exist discrete torsion-free cocompact subgroups $\Gamma_\infty \subset G(\mathbb{R})$. The measure μ_∞ can be explicitly expressed in curvature terms only depending on X_∞. The construction of μ_v is due to Serre [1971] and is as follows. Let R be a (finite) set of cells of X_v representing all $G(\mathbb{Q}_v)$-orbits of cells of X_v. We choose some positive invariant measure μ on $G(\mathbb{Q}_p)$ and put

$$\chi(\mu) := \sum_{\sigma\in R}(-1)^{\dim\sigma}\mu(G_\sigma)^{-1}$$

where $\mu(G_\sigma)$ is the volume with respect to μ of the open and compact subgroup G_σ of $G(\mathbb{Q}_v)$ consisting of the elements which stabilize σ. Then $\mu_v = \chi(\mu)\mu$ is independent of all choices.

5.2.4. PROPOSITION.

(i) $\mu_v \neq 0$, $\operatorname{sign} \mu_v = (-1)^{d_v}$.

(ii) μ_v *is the Euler–Poincaré measure on* $G(\mathcal{Q}_v)$.

(iii) $\mu_\infty \neq 0$ *if and only if* $G(\mathbb{R})$ *contains a compact Cartan subgroup. If* $\mu_\infty \neq 0$ *then* $\operatorname{sign}(\mu_\infty) = (-1)^{d_\infty/2}$.

The second claim is easy. To prove (i) Serre actually computes the μ_v-volume of an Iwahori subgroup of $G(\mathcal{Q}_v)$. The third claim follows from a result of Hopf and Samelson and from Hirzebruch's proportionality principle, see Serre [1971].

REMARK. Euler–Poincaré measures play an important role in Kottwitz [1988] proof of Weil's conjecture on Tamagawa numbers.

5.2.5. PROPOSITION. *Let*

$$\mu = \mu_\infty \otimes \bigotimes_{v \in S} \mu_v$$

be the Euler–Poincaré measure and let $\Gamma_S \subset G_S$ *be a* S*-arithmetic subgroup. Then*

$$\chi(\Gamma_S) = \int_{G_S/\Gamma_S} \mu.$$

The proposition is proved by induction on $|S|$. The difficult part is the proof for $S = \varnothing$. Harder [1971] argues roughly as follows. There is a family of compact manifolds with boundary $M_t \subset X_\infty/\Gamma$ such that

$$\bigcup_{t=1}^{\infty} M_t = X_\infty/\Gamma.$$

For M_t the Gauss–Bonnet theorem for manifolds with boundary gives

$$\chi(M_t) = \int_{M_t} \omega + \int_{\partial M_t} \pi$$

where ω and π are certain differential forms not depending on Γ. Harder shows that

$$\lim_{t \to \infty} \int_{\partial M_t} \pi = 0.$$

REMARKS.

(i) $\chi(\Gamma_S)$ can be computed explicitly provided Γ_S is given by sufficiently concrete local data, see Section 6.

(ii) The explicit formulas for $\chi(\Gamma_S)$ have implications for denominators of Bernoulli numbers, see Serre [1971], Brown [1974].

(iii) There is a vast literature dealing with the connection between cohomology of arithmetic groups, automorphic forms and representation theory. For this the interested reader should consult the book of Borel and Wallach [1980] and the survey articles of Borel [1974, 1984] and Schwermer [1990].

6. Covolumes of arithmetic groups

6.1. Let G be a connected semisimple and (for simplicity) simply connected algebraic group over a number field k. We denote by $\mathbb{A}$ the adele ring of k and we choose a k-rational differential form ω of highest degree on G. Then ω determines the Tamagawa measure, again denoted by ω, on $G(\mathbb{A})$, and local positive invariant measures ω_v on $G(k_v)$ for all places v of k, see Tamagawa [1966]. Here k_v is the completion of k with respect to v.

Let $\Gamma \subset G(k)$ be an arithmetic group (see 4.2.4). We assume that

$$G_\infty = G(k \times_{\mathbf{Q}} \mathbb{R})$$

is noncompact (otherwise Γ is finite). Then the closures Γ_v of Γ in $G(k_v)$, v non-Archimedean, are open and compact subgroups and almost all of them are *hyperspecial*, see Tits [1979], 2.4. We call Γ *parahoric* if all these Γ_v are parahoric.

6.2. We write ω_∞ for the measure induced by ω on G_∞. Then one has

$$\mathrm{vol}_{\omega_\infty}(G_\infty/\Gamma) = D_k^{\dim G/2} \prod_{v \text{ finite}} \mathrm{vol}_{\omega_v}(\Gamma_v)^{-1}.$$

Here $D_k \in \mathbf{N}$ is the absolute value of the discriminant of k over $\mathcal{Q}$ and in the product v runs in the set of finite places of k. This follows from strong approximation (see Platonov [1969, 1970]) and Weil's conjecture on Tamagawa numbers, see Kottwitz [1988].

To obtain a useful formula for the covolume of Γ one has to replace ω_∞ by a natural measure ω_e, which for example can be constructed from the Killing form of G. Almost all $\mathrm{vol}_{\omega_v}(\Gamma_v)$ are known. If we assume Γ_v to be parahoric, $\mathrm{vol}_{\mu_v}(\Gamma_v)$ is known, see 5.2.4 and 3.7.4, where μ_v is the Euler–Poincaré measure. By comparing ω_∞ with ω_e and ω_v with μ_v an explicit formula for $\mathrm{vol}_{\omega_e}(G_\infty/\Gamma)$ can be derived, see Prasad [1989] and Harder [1971]. Here we assume that Γ is parahoric. The formula for the covolume of Γ is essentially a product of special values of L- and ζ-functions and a factor which is determined by the combinatorial data describing the type of the parahoric group Γ. Using this and various finiteness results Borel and Prasad [1989], 7.8, prove:

6.3. PROPOSITION. *Let $c > 0$ be given. Assume that k runs through the number fields. Then there are only finitely many choices of k up to isomorphism, of k-isomorphism classes of absolutely almost simple groups G defined over k and of $G(k)$-conjugacy classes of arithmetic groups $\Gamma \subset G$ such that* $\mathrm{vol}_{\omega_e}(G_\infty/\Gamma) < c$.

Similarly for S-arithmetic groups. There is a related result for groups with a prescribed bound of class numbers, see Borel and Prasad [1989], 7.2.

6.4. For simplicity we now assume that G is adjoint and that G is defined over a number field k. Let G_∞ be noncompact. Then it is possible to classify up to $G(k)$-conjugacy the maximal arithmetic subgroups. It is shown that the maximal arithmetic groups are normalizers of certain (globally) parahoric arithmetic groups P of $\widetilde{G}(k)$, $\widetilde{G}$ the simply connected cover of G, see Rohlfs [1979], Margulis and Rohlfs [1986]. Using this, it is shown in Margulis and Rohlfs [1986] (see also Borel [1976]), that the covolumes with respect to ω_e of the arithmetic subgroups which lie in a fixed commensurability class are all integral multiples of a fixed (in general unknown) positive real number.

7. Applications of buildings in differential geometry

Let $k = \mathbb{R}$. A topologized version of the combinatorial building Δ of Section 1.6 was introduced in Burns and Spatzier [1987a]. See also Chapter 23, Section 6. It has nice applications in differential geometry. We discuss them briefly.

7.1. Let Δ_1 be the set of vertices of the combinatorial building Δ. The group $G(\mathbb{R})$ acts on Δ_1 with finitely many orbits. Each orbit is, as a set, a homogeneous space $G(\mathbb{R})/P(\mathbb{R})$, where P is a parabolic $\mathbb{R}$-subgroup of G. The Lie group topology of $G(\mathbb{R})$ defines a topology on Δ_1, making it a compact Hausdorff space. The set Δ_r of r-simplices of Δ can be viewed as a subset of $(\Delta_1)^{r+1}$ which is closed in the product topology. We provide Δ_r with the induced topology. We thus make Δ a compact topological space. This is an example of the notion of topological buildings studied in Burns and Spatzier [1987a]. In this paper an analog of Tits' characterization of the spherical buildings associated to groups of rational points of reductive groups is proved.

7.2. *The applications.* Burns and Spatzier [1987b] use their result [1987a] to prove a differential geometric characterization of locally symmetric Riemannian manifolds.

Let M be a complete Riemannian manifold with bounded nonpositive sectional curvature. Assume that the universal cover of M is irreducible and that the rank of M is at least two. The latter condition means that along each geodesic there are at least two independent parallel Jacobi vector fields.

7.2.1. PROPOSITION. *If M is as above, then M is a locally symmetric space.*

This is proved in Burns and Spatzier [1987b].

7.2.2. *Isoparametric maps.* A C^∞-map

$$f = (f_1, \ldots, f_r)\colon\ \mathbb{R}^{n+r} \to \mathbb{R}^r$$

is *isoparametric* if

(a) f has a regular value,

(b) the Euclidean scalar product $(\operatorname{grad} f_i, \operatorname{grad} f_j)$ and the Laplacian Δf_i of f_i is constant along all fibers of f $(i, j \in [1, r])$,

(c) the Lie bracket

$$[\operatorname{grad} f_i, \operatorname{grad} f_j] \quad (i, j \in [1, r])$$

is a linear combination of the grad f_k, whose coefficients are constant along the fibers of f.

An *isoparametric subvariety* for f is the fibre of a regular value of f. The foliation of $\mathbb{R}^{n+r}$ defined by the fibers of f is an *isoparametric foliation.*

If X is a Riemannian symmetric space and if K is the isotropy group of a point $x \in X$ then the orbits of K in the tangent space T_xX define an isoparametric foliation of that space.

7.2.3. PROPOSITION. *Let f be as in 7.2.2. Assume*

(a) *the isoparametric subvarieties are compact, irreducible (not the product of two lower dimensional isoparametric submanifolds) and full (not contained in a proper affine subspace),*

(b) $r \geqslant 3$.

Then the isoparametric foliation of f is isomorphic to the one obtained as above from a Riemannian symmetric space.

This is proved by Thorbergsson [1991], using the theory of topological buildings.

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CHAPTER 21

Projective Geometry on Modular Lattices

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Contents

1. Introduction 1117
2. Preliminaries 1119
3. Algebraic representation of modular lattices 1121
   3.1. Representation of complemented modular lattices 1121
   3.2. Representation of primary lattices 1122
   3.3. Partial algebraic representation of modular lattices with a homogeneous basis 1122
4. Faltings' generalization of projective geometry 1124
5. A lattice-geometric characterization of torsion free modules over left Ore domains 1125
6. Projective lattice geometry 1128
   6.1. Concept 1128
   6.2. Desargues' postulate 1130
   6.3. Z-geometries 1130
   6.4. The U-property 1131
   6.5. A fundamental algebraization theorem 1131
   6.6. Point-uniform geometries 1132
   6.7. Reformulations of previous results 1133
   6.8. Point-irreducible geometries 1134
   6.9. Barbilian spaces in projective lattice geometries 1135
7. Generalizations of the fundamental theorem 1136
   7.1. Representation of join and disjointness preserving mappings 1137
   7.2. Representation of lattice homomorphisms and lattice isomorphisms 1138

References 1140

HANDBOOK OF INCIDENCE GEOMETRY
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1. Introduction

Incidence and *Order* are basic concepts for a foundation of modern synthetic geometry. These concepts describe the relative location or containment of geometric objects and have led to different lines of geometry, an *incidence-geometric* and a *lattice-theoretic* one.

An incidence-geometric axiomatization of classical projective geometry goes back to Veblen and Young [1917] at the beginning of this century.[1] Using today's terminology they defined a projective space as a linear space where every line is incident with at least three points and where the Veblen–Young[2] axiom is satisfied. A lattice-theoretic approach to projective geometry was developed by K. Menger and G. Birkhoff in the early thirties and based on the fact that every projective space is associated with a *projective geometry* defined as the partially ordered set of its linear subspaces; indeed, the theory of lattices as founded by G. Birkhoff allows a characterization of projective geometries as algebraic modular lattices which are atomistic and irreducible (cf., e.g., Maeda and Maeda [1970]).

Modularity is one of the fundamental properties of classical projective geometry. It makes projections into join-preserving mappings and yields perspectivities to be (interval) isomorphisms. It is therefore natural that order-theoretic generalizations of projective geometry are based on modular lattices and even more, the theory of modular lattices may be considered as a most general concept of projective geometry. In particular, the partially ordered set of all submodules of a module forms a (complete) modular lattice; even more general, any sublattice of the lattice of all normal subgroups of a group is a modular lattice. A complete characterization, however, of those modular lattices which are isomorphic to submodule lattices seems to be unobtainable, though for several classes of lattices such a characterization has been achieved. The origin of all these results is the lattice-theoretic version of the well-known representation theorem of classical projective geometry which states that an algebraic modular lattice of rank $\geqslant 4$ is induced by a vector space if and only if the lattice is atomistic and irreducible. Within the purely lattice-theoretic frame this result could be extended in essentially two ways: the first is due to J. von Neumann who gave a representation of complemented modular lattices of *order* $\geqslant 4$ by Von Neumann regular rings (cf. Von Neumann [1960]). A representation of another class of modular lattices was given by Baer [1942] when he developed a unified theory of projective spaces and finite Abelian groups. Baer clarified the connection between primary lattices of *geometric dimension* $\geqslant 6$ (in the sense of Jónsson and Monk [1969]) and modules over completely primary uniserial rings; a stronger result was achieved by Inaba [1948] who extended Baer's result to geometric dimension $\geqslant 4$.

The establishment of a lattice-theoretic version of Desargues' postulate by Jónsson [1954] (cf. also Schützenberger [1945]) allowed improvements of the results mentioned before even in the planar case. Subsequently, complemented *Arguesian* lattices of order $\geqslant 3$ were algebraically characterized by Jónsson [1961] and, in a joint work of Jónsson and Monk [1969], a representation of primary Arguesian lattices of 'geometric dimension' $\geqslant 3$ was obtained. For more details concerning the work of Von Neumann, Jónsson, Baer, Inaba and Monk, see Sections 2 and 3.

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[1] See also Veblen and Bussey [1906].

[2] See Chapter 2.

Further systematic research on the algebraic representation of modular lattices containing a homogeneous basis by free modules was done by Artmann [1968] and by Day and Pickering [1983]. Artmann showed that each modular lattice of order $\geqslant 4$ is associated with a (coordinate) ring; Day and Pickering improved this result for Arguesian lattices including the planar case, i.e. the case of order 3. All in all they achieved what could be called a partial representation of a large class of modular (or Arguesian) lattices (see also Section 3).

To obtain even stronger results in the algebraic representation of modular lattices, a conceptual expansion has proved to be valuable and is based on the following fact: the set of all 1-generated submodules of a module contains a lot of important information about the structure of the given module and furthermore has some nice geometric properties in the underlying submodule lattice. In general, however, there is no way to recover the set of all 1-generated submodules merely from the structure of the submodule lattice of the given module. To keep this additional information available, it is therefore natural to consider a modular lattice together with a subset of *points* satisfying a list of axioms which synthetically characterize general properties of 1-generated submodules of a module. K. Faltings was the first exponent of such a lattice-geometric approach to projective geometry. Faltings [1975] considered what are called 'Modulare Verbände mit Punktsystem' (*modular lattices with a point-system*) and utilized R. Baer's algebraization technique in order to obtain an algebraic representation of a fairly large class of modular lattices with point system. Unfortunately, however, his approach did not allow the formulation of algebraic counterparts of all the geometrical properties he postulated, and therefore the class of modules he used in his characterization is difficult to describe. A comparable approach came from Brehm [1983, 1984a], who gave a complete lattice-geometric characterization of submodule lattices of torsion free left modules of uniform rank $\geqslant 3$ over left Ore domains: he also considered modular lattices together with a point system. For more details, see Sections 4 and 5.

We already indicated that every classical projective space is canonically associated with a projective geometry, i.e. the lattice of its linearly closed sets. Here the set of points of the underlying space can be recognized as the set of atoms in the corresponding lattice. This yields a canonical 1–1 correspondence between projective spaces and their geometries.

Faigle and Herrmann [1981] showed that this correspondence is preserved in a more general situation: they marked the set of all join-irreducibles of a modular lattice of finite length as a point set of a projective space and showed that the lattice of all linear sets of this projective space is isomorphic to the original lattice. Some more comprehensive investigations concerning the relation between modular lattices and (generalized concepts of) projective spaces are due to Schmidt [1987], who considered point systems in algebraic modular lattices on an axiomatic basis which is similar to that of Faltings [1975]. All in all it turned out that even in a very general situation the relationship between projective spaces as incidence structures and their associated projective geometries as complete order structures is similar to what is known in the case of classical projective geometry.

In the search for lattice-geometric concepts which reflect algebraic properties of rings and modules even more efficiently, the choice of yet another subset in a complete lattice has recently proved to be valuable: in *projective lattice geometries* as first introduced by S.E. Schmidt in 1988 one distinguishes a set of points, and also a set of free points, within a complete lattice and postulates a list of axioms to be satisfied. This approach has several advantages in that there are natural notions of *hyperplane*, *basis*, *dimension* etc. Furthermore every module produces an example of a projective (lattice) geometry defining the 1-generated submodules to be the points and the freely 1-generated submodules to be the free points of that geometry. The reader will find more details in Section 6.

We finally emphasize that lattice-geometric approaches as discussed here consider complete geometrical structures whose geometrical objects form complete (modular) lattices. This marks an essential conceptual difference to geometrical approaches considering point-hyperplane structures. For a detailed survey of these, the reader should consult Chapter 19 of this Handbook by F.D. Veldkamp. The interdependencies between these two lines have first been investigated by Artmann [1972] and Dugas [1977], who illustrated the connection between primary lattices and Hjelmslev planes. At the end of Section 6 we will briefly report some more general investigations by Greferath and Schmidt [1992b] and show how *Barbilian spaces* occur as substructures of *projective lattice geometries*.

Besides the algebraic representation of projective geometries by unitary modules a further subject is of eminent interest. This is the generalization of the fundamental theorem of projective geometry[3] and hence, the representation of mappings between submodule lattices. Descriptions of certain isomorphisms between submodule lattices of various classes of modules were given by Von Neumann [1960], Skornjakov [1960], Faltings [1975] and Brehm [1983] and led to semilinear isomorphisms between the underlying modules. It is clear that the latter isomorphisms map (freely) 1-generated submodules onto (freely) 1-generated submodules; the interest in further generalizations has resulted in representation theorems which work even without this premise. A survey of these results will be given in Section 7: under certain weak assumptions on the modules involved, each isomorphism between submodule lattices is representable by a Morita equivalence. Even more generally we will report a result of Brehm [1987] giving a representation of lattice homomorphisms which preserve arbitrary joins.

2. Preliminaries

We first give a short introduction to lattice- and ring-theoretic notions and notations which are necessary for the reader to understand what follows. For more details on the essentials of rings, modules and lattices, we refer to Anderson and Fuller [1974], Birkhoff [1948], Crawley and Dilworth [1973] and Maeda and Maeda [1970].

CONVENTION. All rings occurring are associative and have a unit, and all modules are unitary left modules unless stated otherwise.

[3] See Chapter 4, 2.2.1.

The lattice of all submodules of a module ${}_RM$ will be denoted by $L({}_RM)$, and by $\operatorname{ann}(x)$ we will mean the annihilator ideal of an element x of M in ${}_RR$.

Let $(L, \leqslant)$ be a complete lattice. Then $\sum X$ and $\prod X$ denote the *join* (i.e. the least upper bound) and the *meet* (i.e. the greatest lower bound) of any subset X of L; further let 1 denote the top element of L and 0 the bottom element of L, i.e. $1 = \sum L = \prod \varnothing$ and $0 = \prod L = \sum \varnothing$. For $x, y, z \in L$ define

$$x + y := \sum\{x, y\}, \qquad xy := \prod\{x, y\}, \quad x + yz := x + (yz)$$

and

$$x \oplus y := x + y \quad \text{if } x \text{ and } y \text{ are } \textit{disjoint}, \text{ i.e. } xy = 0;$$

furthermore x and y are *perspective* if there exists $z \in L$ such that $x \oplus z = y \oplus z$. The *interval of x over y* is given by

$$x/y := \{t \in L\colon\ y \leqslant t \leqslant x\}.$$

L is *complemented* if each element x of L has a *complement*, i.e. an element y in L such that $x \oplus y = 1$.

L is said to be *modular* if

$$(x + y)z = x + yz \quad (\textit{modular identity})$$

holds for all $x, y, z \in L$ with $x \leqslant z$.

L is called *Arguesian* if

$$(x_0 + y_0)(x_1 + y_1)(x_2 + y_2) \leqslant x_0(x_1 + p) + y_0(y_1 + p)$$

holds for all $x_0, x_1, x_2, y_0, y_1, y_2 \in L$ where

$$p := p_2(p_0 + p_1) \quad \text{and} \quad p_i := (x_j + x_k)(y_j + y_k)$$

for all i, j, k with $\{i, j, k\} = \{0, 1, 2\}$. Arguesian lattices are modular and define a variety in the sense of universal algebra (cf. Jónsson [1954]).

A subset X of L is *join-dense* in L if $t = \sum\{x \in X\colon\ x \leqslant t\}$ holds for all $t \in L$. An element t is *compact* in L if for $X \subseteq L$ with $t \leqslant \sum X$ there always exists a finite subset X_0 of X with $t \leqslant \sum X_0$. An *algebraic* lattice is defined as a complete lattice where the set of compact elements is join-dense. An *upper continuous* lattice is a lattice such that

$$t\sum X = \sum\left\{t\sum X_0\colon\ X_0 \subseteq X,\ X_0 \text{ finite}\right\}$$

holds for arbitrary $t \in L$ and $X \subseteq L$. Clearly every algebraic lattice is upper continuous.

A family $(x_i)_{i \in I}$ of elements of L is called a *spanning family of L* if

$$\sum\{x_i\colon\ i \in I\} = 1;$$

it is called *independent in* L if each x_i is nonzero and disjoint from

$$\sum \{x_j\colon\ j \in I \setminus \{i\}\}.$$

A *homogeneous basis of order* n of L is defined as an n-element independent spanning family of L any two elements of which are perspective (cf. Von Neumann [1960], p. 93); L is of *order* n if it possesses a homogeneous basis of order n.

If in L the cardinalities of all *chains* (totally ordered subsets) are bounded by a finite number, then L is of *finite length.* The *rank* of an element x of L is defined as the number of nonzero elements in a chain of maximal length in $x/0$. The elements of rank 1 are called *atoms*; L is *atomistic* if its set of atoms is join-dense in L.

A nonzero element of L is called *join-irreducible* if it is not the join of any two smaller elements. If L is of finite length, then its set of join-irreducibles is join-dense in L, and the *Kurosh–Ore dimension* of L is defined as the least number of spanning join-irreducibles in L.

A nonzero element x of L is *uniform* if the meet of any two nonzero elements of $x/0$ is again nonzero. The *uniform rank* of L is the maximal number of independent uniform elements of L; it will be denoted by $\varrho(L)$. For $a \in L$ define $\varrho(a) := \varrho(a/0)$.

The uniform rank $\varrho(M)$ of a module ${}_RM$ is given by $\varrho(L({}_RM))$.

A *cycle* is an element x of L such that $x/0$ is a chain. L is called *cyclic* if the set of its cycles is join-dense in L.

A lattice L is *irreducible* if $L \cong L_1 \times L_2$ always implies $|L_1| = 1$ or $|L_2| = 1$ (cf. Maeda [1958], S. 19); it will be called *relatively irreducible* if each of its intervals is irreducible.

3. Algebraic representation of modular lattices

3.1. *Representation of complemented modular lattices*

The classical coordinatization theorem for projective geometry was vastly extended by J. von Neumann in 1935–1937 when he gave lectures on *Continuous Geometry.* His previous work on rings of operators in Hilbert spaces and a synthetic approach to quantum mechanics had led to the investigation of what are called (*Von Neumann*) *regular* rings on the one hand, and continuous *dimension functions* on the other hand (cf. Von Neumann [1960]). The existence of such functions was considered as a striking phenomenon at the time that Von Neumann's work became known. It made 'continuous geometry' a generalization of classical projective geometry with its discrete dimension function.

Introducing the notion of *homogeneous basis* and *order* for a complemented modular lattice (see the preliminaries), Von Neumann established the coordinatization of complemented modular lattices of order $\geqslant 4$ by regular rings (cf. Von Neumann [1960], Theorem 9.2). A ring R is called (*Von Neumann*) *regular* if for each $x \in R$ there exists an element y of R with $xyx = x$, and it can be shown that the set of all principal left ideals of a regular ring forms a complemented sublattice of the lattice of all left ideals.

THEOREM 3.1. *Every complemented modular lattice of order $\geqslant 4$ is isomorphic to the lattice of principal left ideals of a regular ring.*

This result is usually also referred to in the following version (cf. Maeda [1958], S. 226, Satz 3.3).

THEOREM 3.2. *Every complemented modular lattice of order $n \geqslant 4$ is isomorphic to the lattice of all finitely generated submodules of a free module of rank n over a regular ring. The latter ring is uniquely determined up to isomorphy.*

In 1954, B. Jónsson introduced the Arguesian law (cf. Jónsson [1954]) as a lattice-theoretic reflection of Desargues' postulate. This law is stronger than the modular identity, and by using it, Jónsson [1961, 1962] proved that Von Neumann's theorem also holds for the planar case.

THEOREM 3.3. *Every complemented Arguesian lattice of order $\geqslant 3$ is isomorphic to the lattice of principal left ideals of a regular ring.*

3.2. *Representation of primary lattices*

The algebraization of primary lattices has been achieved by Baer [1942], Inaba [1948] and Jónsson and Monk [1969]. It was incited by Baer's interest in developing a unified theory of projective spaces and finite Abelian p-groups.[4]

A cyclic lattice of finite length, the dual of which is also cyclic, is called *semiprimary*; it is called *primary* if furthermore every interval which is not a chain has at least three atoms. The algebraic counterpart of the latter is given by the following class of rings: A ring R is said to be *completely primary and uniserial* if each of its left or right ideals (and in particular the zero ideal) is a power of the Jacobson radical of R. Indeed, the lattice of all submodules of any finitely generated module over a completely primary and uniserial ring is primary (cf. Jónsson and Monk [1969], Theorem 6.7).

The main representation result of Inaba [1948] (Baer [1942]) states that every primary lattice which contains at least 4 (6) independent cycles of maximal rank is induced by a module over a primary and uniserial ring. For Arguesian lattices this result was extended to the planar case by Jónsson and Monk [1969].

THEOREM 3.4. *Every Arguesian primary lattice which contains at least* 3 *independent cycles of maximal rank is induced by a module over a completely primary and uniserial ring.*

3.3. *Partial algebraic representation of modular lattices with a homogeneous basis*

Some very general considerations concerning the partial representation of modular lattices are due to Artmann [1968] and Day and Pickering [1983]. Artmann proved a result

[4] The notion of primary lattice goes back to Inaba [1948].

clarifying the relationship between modular lattices with a homogeneous basis of order $\geqslant 4$ and (coordinate) rings. His work was continued by Day and Pickering who proved a comprehensive extension of Artmann's result for Arguesian lattices in the case of order 3. Furthermore they worked on the algebraic representation of what they called hyperplanes in modular lattices with a homogeneous basis. Their result was far-reaching, although they succeeded in giving a full algebraic representation only under constraining additional conditions on the lattice under consideration (cf. Day and Pickering [1983], Corollary 5.10).

All the investigations which we mentioned use appropriate generalizations of the concept of (projective) coordinate systems of classical projective geometry. Indeed, the concept of a *homogeneous basis* as used in Von Neumann [1960] and Artmann [1968] and that of a *spanning diamond* as used in Day and Pickering [1983] are equivalent in that homogeneous bases of *bounded* modular lattices (i.e. a lattice with 0 and 1) can be constructed with spanning diamonds of these lattices, and vice versa. To understand what Artmann, Day and Pickering showed we will briefly recall how the 'arithmetic on lines' in classical projective geometry generalizes to bounded modular lattices which contain a homogeneous basis. For a detailed survey, see Stevenson [1972], p. 193 ff, Artmann [1968], p. 632 ff, and Day and Pickering [1983], p. 510 ff.

Let L be a bounded modular lattice with a homogeneous basis $(f_i)_{i=0,\ldots,n-1}$ of order $n \geqslant 3$ and let $(f_i')_{i=1,\ldots,n-1}$ be an independent family of elements of L, such that f_0 is perspective to f_i via f_i' for all $i = 1, \ldots, n-1$. Without loss of generality we assume $f_0 \oplus f_i = f_0 \oplus f_i' = f_i \oplus f_i'$ and on the set $R := \{x \in L\colon\ x \oplus f_1 = f_0 \oplus f_1\}$ we define two binary operations $\boxplus$ and $\circ$ by

$$a \boxplus b := ((a + f_2')(f_1 + f_2) + (b + f_2)(f_1 + f_2'))(f_0 + f_1),$$

$$a \circ b := ((a + f_2')(f_1 + f_2) + (b + f_*)(f_0 + f_2))(f_0 + f_1),$$

where $f_* := (f_1' + f_2')(f_1 + f_2)$. We refer to $(R, \boxplus, \circ, f_0, f_1')$ as the (universal) algebra associated with $(f_0, (f_i, f_i')_{i=1,\ldots,n-1})$ in L.

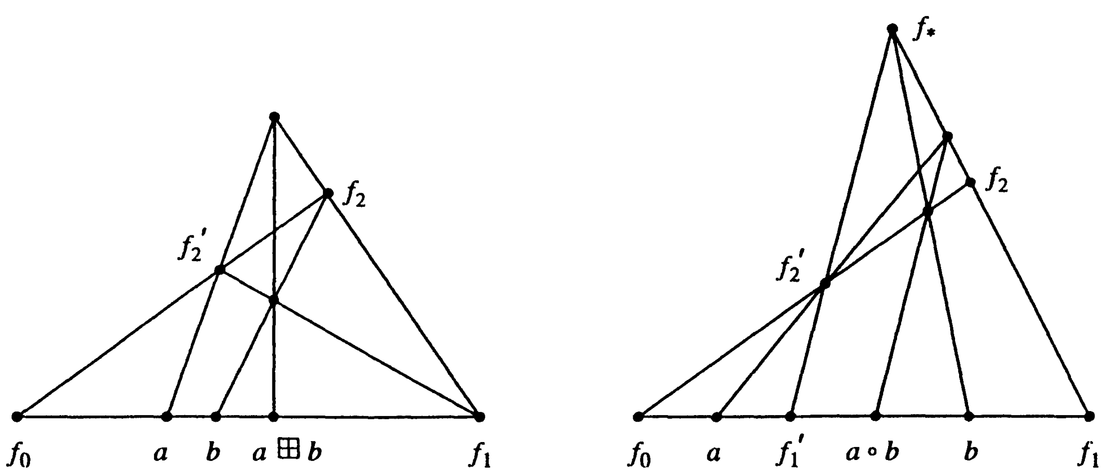

Figure 1. Definition of operations $\boxplus$ and $\circ$.

Now the results of Artmann, Day and Pickering concerning (ternary) coordinate rings can be summarized as follows.

THEOREM 3.5. *The algebra belonging to a homogeneous basis of order n of a bounded modular lattice L is a ring, provided $n \geqslant 4$, or $n \geqslant 3$ and L is Arguesian.*

4. Faltings' generalization of projective geometry

Faltings [1975] derives a representation of certain modular lattices with point system by modules.

He first defines a *point system* (*Punktsystem*) of a complete modular lattice L as a subset E of L such that the following axioms are satisfied.

(P1) $t \oplus a = b + a$ always implies $t \in E$, for all $a, b \in E$ and $t \in L$.
(P2) E is join-dense in L.
(P3) For all $x, y \in L$ and $e \in E$ with $e \leqslant x + y$ there exist $c, d \in E$ with $c \leqslant x$ and $d \leqslant y$ such that $c + d = d + e = e + c$.
(P4) If T is any nonempty subset of E such that $e \leqslant c + d$ with $e \in E$ and $c, d \in T$ implies $e \in T$, then $T = \{e \in E\colon\ e \leqslant \sum T\}$.

Faltings defines a *hyperplane* of a modular lattice L with point system E as an element h of L satisfying:

(H1) There exists a nonzero $p \in E$ with $h \oplus p = 1$.
(H2) For $q \in E$ with $h + q = 1$ there always exists $p \in E$ with $p \leqslant q$ such that $h \oplus p = 1$.
(H3) For all $e \in E$ with $e \leqslant h$ and $p \in E$ with $h \oplus p = 1$ there exists $q \in E$ such that $h \oplus q = 1$ and $e + p = p + q = q + e$.

The lattice of all submodules of a module $_RM$, together with the set of all 1-generated submodules of $_RM$, is a fundamental example of such a modular lattice with point system. We will call a modular lattice L with point system E *module-induced*, if there exists a module $_RM$ and a lattice isomorphism between L and $L(_RM)$ which maps E onto the set of all 1-generated submodules of $_RM$. If H and P are submodules of $_RM$ such that $H \oplus P = M$ and $_RP \cong {_RR}$, then H is a hyperplane of the modular lattice with point system induced by $_RM$.

For his main result concerning the representation of modular lattices with point system, Faltings needs the existence of hyperplanes having some additional strong and somehow technical properties. A hyperplane of a modular lattice with point system (L, E) is called *regular* if it satisfies the conditions (R0) to (R4) of Faltings [1975], p. 110 ff. The condition (R0) states for a hyperplane h of (L, E) that every 3-*generated element* of (L, E) (i.e. every element which is the join of three elements of E) is disjoint from some appropriate complement of h in L. A sufficient condition for a hyperplane h to be regular is that every 5-generated element of (L, E) is disjoint from a suitable complement of h. Another sufficient condition is that every element of E is uniform in L and that (R0) is satisfied (cf. Faltings [1975], Theorem 5.3). A modular lattice with point system which is induced by a free module of infinite rank over an arbitrary ring always contains regular hyperplanes.

We now state Faltings' main representation result (cf. Faltings [1975], Theorem 4.6) which generalizes the classical representation theorem for projective geometries.

THEOREM 4.1. *Every modular lattice with point system possessing a regular hyperplane is induced by a module.*

5. A lattice-geometric characterization of torsion free modules over left Ore domains

In the present section we will characterize submodule lattices of torsion free left modules of uniform rank at least 3 over left Ore domains. More precisely we regard lattices together with a distinguished subset of *points* which shall correspond to the set of nontrivial 1-generated submodules. The characterizing theoretic properties (axioms) are simple, natural, independent and of a geometric flavour. All what follows in this section is based on Brehm [1983, 1984a].

A nonzero ring R is called a *left Ore domain* if it has no zero divisors and if $Ra \cap Rb \neq \{0\}$ for arbitrary nonzero $a, b \in R$. A module $_RM$ over a left Ore domain is called *torsion free* if $rx = 0$ implies $r = 0$ or $x = 0$ for all $r \in R$ and $x \in M$. It is well known that each left Ore domain R possesses a *skew field of left quotients* $K \supseteq R$ with

$$K = \{a^{-1}b\colon\ a, b \in R,\ a \neq 0\}.$$

If $_RM$ is a torsion free module over a left Ore domain R and if K is the skew field of left quotients of R, then the canonical mapping $i\colon\ M \to K \otimes_R M$ is injective and thus M can be regarded as an R-submodule of a K-vector space. This fact makes the torsion free left modules over left Ore domains an interesting class of modules. Indeed, it gives the crucial clue for the proof of the representation Theorem 5.1.

Using some definitions and notions from the preliminaries, such as ρ for the uniform rank, we now formulate the axioms.

THE AXIOMS. Let L be a complete modular lattice and a subset P of L.

(A1) P is join-dense in L.
(A2) All $p \in P$ are compact in L.
(A3) For all $p_1, p_2 \in P$ with $p_1p_2 = 0$ there exists $p_3 \in P$ with

$$p_1 + p_2 = p_1 + p_3 = p_2 + p_3.$$

(A4) For all $p, q \in P,\ a \in L \setminus \{0\}$ with $p \leqslant q + a$ there exists $q' \in P$ with $q' \leqslant a$ and $p \leqslant q + q'$.
(A5) Either $\rho(L) = 3$ and L is Arguesian, or $\rho(L) \geqslant 4$.
(A6) All $p \in P$ are uniform.
(A7) For all $a, b, c \in L$ with $ac = 0$ and $(a + b)c \neq 0$ there exists $c' \in L$ with $ac' = 0$ and $bc' \neq 0$.

We now formulate the coordinatization theorem.

THEOREM 5.1. *Let L be a complete modular lattice and P a subset of L. Then (L, P) satisfies the axioms* (A1)–(A7) *if and only if there exists a left Ore domain R, a torsion free R-module M of uniform rank $\geqslant 3$ and a lattice isomorphism $f\colon L \to L({}_RM)$ with $f(P) = \{Rx\colon x \in M \setminus \{0\}\}$.*

We give here a short outline of the necessary lemmas and the general idea of the proof. For more detail, cf. Brehm [1983, 1984a]. The next lemma points out the meaning of the individual axioms for the lattice of submodules. We recall that a module ${}_RM$ is *nonsingular* if for each nonzero $x \in M$ there exists a nonzero $\lambda \in R$ such that $\operatorname{ann}(x) \cap R\lambda = \{0\}$.

LEMMA 5.2. *Let ${}_RM$ be a module and set $L := L({}_RM)$ and*

$$P := \big\{Rx\colon x \in M \setminus \{0\}\big\}.$$

Then:

(a) *For (L, P) axioms* (A1)–(A4) *hold.*
(b) *L is Arguesian.*
(c) *For (L, P) axiom* (A5) *holds if and only if $\rho(M) \geqslant 3$.*
(d) *For (L, P) axiom* (A6) *holds if and only if every nonzero* 1*-generated submodule of ${}_RM$ is uniform.*
(e) *If ${}_RM$ is a nonsingular module, then* (A7) *holds for (L, P).*
(f) *If for each $x \in M \setminus \{0\}$ there exists $r \in R$ and $y \in M$ such that $rx \neq 0$, $Rrx \cap Ry \neq 0$ and $\operatorname{ann}(y) \subseteq \operatorname{ann}(r)$, then axiom* (A7) *for $L({}_RM)$ implies that ${}_RM$ is nonsingular.*

REMARK 5.3. Let ${}_RM$ be a torsion free module over a left Ore domain R with $\rho(M) \geqslant 3$. Then ${}_RM$ is nonsingular, and every nonzero 1-generated submodule of ${}_RM$ is uniform. Thus axioms (A1)–(A7) hold for (L, P) with $L := L({}_RM)$ and $P := \{Rx\colon x \in M \setminus \{0\}\}$.

LEMMA 5.4. *Let L be a lattice with* 0*. Define an equivalence relation $\sim$ on L by*

$$a \sim b \Leftrightarrow \forall c \in L\ (ac = 0 \Leftrightarrow bc = 0).$$

Then the following hold:

(a) *The relation $\sim$ is a congruence relation if and only if* (A7) *holds in L.*
(b) *If L is upper continuous and satisfies* (A7), *then $L/{\sim}$ is a complemented upper continuous lattice. Furthermore, the natural homomorphism* $\operatorname{nat}\colon L \to L/{\sim}$ *preserves arbitrary joins and the uniform rank and satisfies* $\operatorname{nat}^{-1}[\{0\}] = \{0\}$.

Using the above (b) and a lattice-theoretic version of the classical coordinatization theorem of projective geometry (see, e.g., Crawley and Dilworth [1973]), we easily obtain the following

LEMMA 5.5. *Let L be a complete modular lattice and P a subset of L such that the axioms* (A1)–(A7) *hold. Then there exists a vector space ${}_KV$ and a lattice isomorphism i: $L/{\sim} \to L({}_KV)$ with $i \circ \mathrm{nat}(P) = \{Kx\colon\ x \in V \setminus \{0\}\}$.*

Having obtained the vector space ${}_KV$ one finally has to construct a subring R of the skew field K and a submodule M of ${}_RV$ together with a lattice isomorphism f: $L \to L({}_RM)$ such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\ f\ } & L({}_RM) \\ {\scriptstyle \mathrm{nat}}\downarrow & & \downarrow{\scriptstyle K\otimes_R -} \\ L/{\sim} & \xrightarrow{\ i\ } & L({}_KV) \end{array}$$

Applying 5.2 (d) and (f), we finally obtain that R is a left Ore domain and ${}_RM$ is a torsion free R-module.

COMMENT 5.6. The approach to use the classical theorem in order to get first ${}_KV$, then construct a subring R of K and finally (in the main part of the proof) construct ${}_RM$ as a submodule of ${}_RV$, has the advantage that one need not prove any associative or distributive laws and that one can already use the constructed vector space ${}_KV$ in an early stage of the proof.

REMARK 5.7.

(a) Let L be a complete modular lattice and P a subset of L such that (A1), (A2) and (A4) hold. Then the following are equivalent:

(i) (A6) and (A7) hold for (L, P);

(ii) $\rho(p + a) + \rho(pa) = 1 + \rho(a)$ for all $p \in P$ and $a \in L$.

Thus the axioms (A6) and (A7) can be replaced by the simpler condition (ii).

(b) Each of the axioms (A1)–(A7) is independent of the combination of the other axioms. Simple examples showing this are given in Brehm [1984a].

(c) Instead of considering a lattice L together with a subset P one can start with L and a given compact element p_0 of L and then construct $P \subseteq L$ as the set of all elements of L which are projective to p_0. For more details, see Brehm [1984a].

(d) The torsion free modules ${}_RM$ over left Ore domains for which the set of all uniform compact elements of $L({}_RM)$ coincides with the set of all nonzero 1-generated submodules, are characterized by the property that $Rx + Ry$ is free for all $x, y \in M$. Thus, in this case the set P of points can be distinguished by purely lattice-theoretic properties.

Lattice-theoretic characterizations of some other additional ring properties for left Ore domains such as commutativity or the *right* Ore property can be found in Brehm [1983].

6. Projective lattice geometry

As we have remarked in the introduction, the interest in pursuing projective geometry on modules has led to some generalizations of the classical concept of projective geometry. In the following we shall give a survey on the concept and several results of projective lattice geometry as introduced in Greferath and Schmidt [1992a]. This concept is a general framework that allows a reformulation of earlier approaches and furthermore the achievement of more comprehensive representation theorems. At the end of our discourse we will show how point-hyperplane structures occur as substructures of projective (lattice) geometries and hence indicate a link to Chapter 19 of this Handbook. The results referred to in the following are all based on Greferath and Schmidt [1992a,b, 1994a,b] (combined with Greferath [1991] and Schmidt [1987, 1991]). Before going on, we will give some additional technical notions.

Let $(L, \leqslant)$ be a complete lattice. For $X \subseteq L$ and $t \in L$ we abbreviate

$$X(t) := \{x \in X\colon\ x \leqslant t\}, \qquad X/t := \{x + t\colon\ x \in X\}$$

and

$$X//t := \{x \oplus t\colon\ x \in X \text{ with } xt = 0\};$$

for any natural number n we define $X^{(n)}$ (or $X^{[n]}$) as the set of all $\sum_{i=1}^{n} x_i$ where $(x_i)_{i=1,\dots,n}$ is an (independent, respectively) family of elements contained in X. A triple (x, y, z) of elements of L is *balanced* if $x + y = y + z = z + x$ holds; it is *directly balanced* if additionally $xy = yz = zx = 0$ is satisfied. Finally, let $K(L)$ denote the set of all compact elements of L.

6.1. *Concept*

A *projective* (*lattice*) *geometry* is a triple $G = (L, E, F)$ where L is a complete lattice and $F \subseteq E \subseteq K(L)$, satisfying the following axioms.

(E1) $0 \in E$ and E is join-dense in L.
(E2) For $x, y \in L$ and $e \in E(x + y)$ there always exist $c \in E(x)$ and $d \in E(y)$ such that c, d, e are balanced.
(E3) For $c, d \in E$ there always exists $e \in E$ such that c, d, e are balanced.
(F1) For $a, e \in E$ and $f \in F$ with $f \oplus a = e + a$ there always exists $p \in F(e)$ with $f \oplus a = p \oplus a$.
(F2) For $e \in E$ and $f \in F$ with $ef = 0$ there always exists $g \in F$ with $eg = 0$ such that e, f, g are balanced.
(F3) For $f, g \in F$ with $fg = 0$ there always exists $p \in F$ such that f, g, p are directly balanced.

By (E1) and (E2) it is easily seen that L is algebraic and modular. For $x, y \in L$ we say that x *is contained in* y if $x \leqslant y$. The elements of E and F are called *points* and *free points*, respectively. A point f is *unimodular* if it is free and has a *complement in*

G, i.e. $f \oplus x = 1$ for some $x \in L$; any complement of a unimodular point is called a *hyperplane*.

An element of L is said to be *n-generated* in G if it is the join of n points, i.e. if it is an element of $E^{(n)}$; G is said to be n-generated, if 1 is n-generated; and G is *finitely generated* if it is n-generated for some natural number n, i.e. if $1 \in K(L)$.

A *basis* of G is an independent spanning family of free points of G; if G possesses a basis of $n + 1$ elements, then it is said to be of *dimension* n, but we emphasize that a projective geometry may have various different dimensions or even no dimension at all.

Every basis of a projective geometry is *homogeneous* in the sense of Von Neumann [1960].

EXAMPLES 6.1.

(a) For a unitary left R-module M let $E(_RM)$ be the set of all 1-generated submodules of $_RM$ and let $F(_RM)$ consist of those submodules of $_RM$ which are isomorphic to $_RR$. Then $G(_RM) := (L(_RM), E(_RM), F(_RM))$ is a projective geometry. $G(_RM)$ is finitely generated iff $_RM$ is so and it has a basis iff $_RM$ is a free module. A point p in $G(_RM)$ is unimodular in $G(_RM)$ iff $p = Rv$ holds for some *unimodular* element v of $_RM$ (i.e. v is mapped to 1 by some R-linear mapping from M to R).

A projective geometry G is said to be *module-induced* if there exists a module $_RM$ such that $G \cong G(_RM)$ (i.e. there exists an order isomorphism between the underlying lattices which induces a bijection between the sets of points and also between the sets of free points). We emphasize that the underlying ring of a module-induced projective geometry G is determined up to isomorphism, whenever the geometry contains at least 3 independent free points. Furthermore, if G is of dimension at least 2, then the underlying module is determined up to isomorphism.

(b) For an algebraic modular lattice L, a very general example of a projective geometry is given by $(L, K(L), \varnothing)$. For lattice-theoretic constructions of even smaller point sets, see Schmidt [1991], Theorem 2.8. If L is atomistic and irreducible and A denotes the set of all atoms of L, then $(L, A \cup \{0\}, A)$ defines a *classical projective geometry*.

(c) Let $G = (L, E, F)$ and $G' = (L', E', F')$ be projective geometries. Then for all $t, x \in L$ with $t \leqslant x$ and every nonzero natural number n one can derive the following projective geometries:

$G(x) := (L(x), E(x), F(x))$, the *subgeometry of x in G*;

$G/x := (L/x, E/x, F//x)$, the *quotient geometry of G over x*;

$G(x/t) := G(x)/t$, the *minor geometry of x/t in G*;

$G^{(n)} := (L, E^{(n)}, F^{[n]})$, the *($n$-)cluster geometry of G*;

$G_{\mathrm{red}} := (L, E, \varnothing)$, the *reduced geometry of G*;

$G \times G' := (L \times L', E \times E', F \times F')$, the *direct product of G and G'*.

From Greferath and Schmidt [1992a], 2.5(a), we cite that if G and G' are module-induced, then so are all the geometries defined above.

REMARK 6.2. Each projective lattice geometry (L, E, F) is naturally associated with a *projective space* $(E^{(2)}, E, F)$. This latter structure allows an intrinsic axiomatization which makes use of a generalized version of the well-known *Veblen–Young axiom*. In regard to this the category of all projective lattice geometries is equivalent to that of all projective spaces in the above sense (cf. Schmidt [1987], 4.1.6). This clearly generalizes the correspondence between classical projective spaces and classical projective geometries.

6.2. *Desargues' postulate*

From Schmidt [1991] we cite a form of Desargues' postulate in order to illustrate how an axiom of classical projective geometry has a quite natural extension to projective lattice geometry.

(D) If $a^0, a^1, a^2, b^0, b^1, b^2, e \in E$ such that $e \leqslant a^i + b^i$ for all $i \in \{0,1,2\}$, then there exist $a_i, b_i, c_i \in E$ with $a_i \leqslant a^i$, $b_i \leqslant b^i$ for all $i \in \{0,1,2\}$ such that (e, a_i, b_i), (c_i, a_j, a_k), (c_i, b_j, b_k) and (c_0, c_1, c_2) are balanced triples for all $\{i,j,k\} = \{0,1,2\}$.

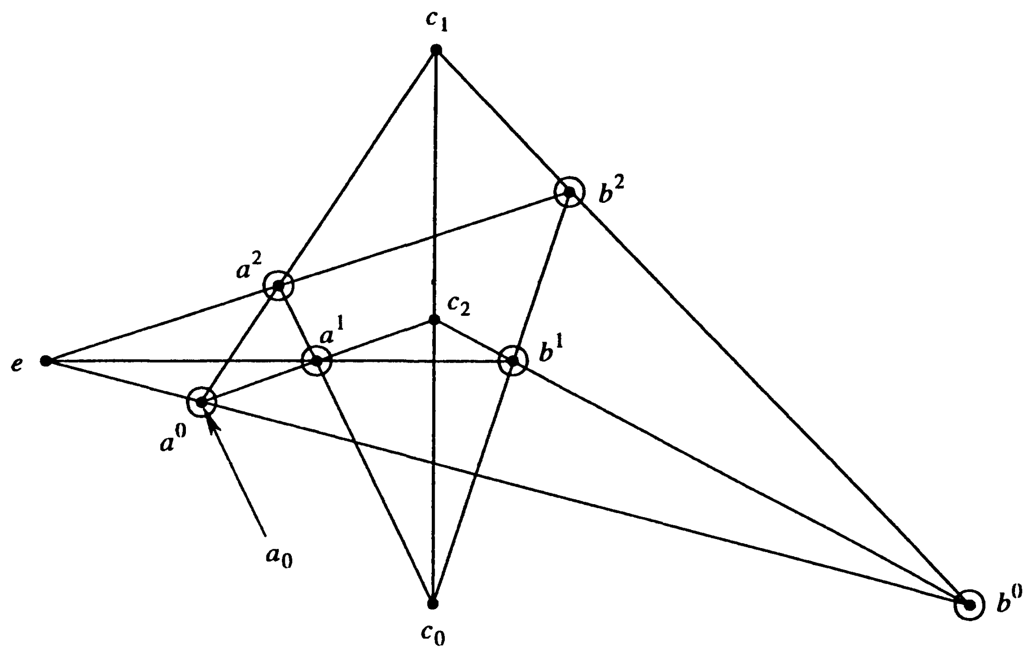

Figure 2. Desargues' postulate.

A projective geometry where Desargues' postulate holds will be called *Desarguesian*. The underlying lattice of a Desarguesian projective geometry is always Arguesian (see Schmidt [1991]).

6.3. *Z-geometries*

In Greferath and Schmidt [1992a] a *Z-geometry* has been defined as a projective geometry in which the subgeometry of a point never properly contains a unimodular point. This notion was introduced to give a geometric counterpart of the property of a ring to be a two-sided units ring, i.e. a ring where $\lambda\mu = 1$ always implies $\mu\lambda = 1$ for arbitrary ring elements λ, μ. We will now briefly report some important facts about Z-geometries.

PROPOSITION 6.3.

(a) *For a module $_RM$ with $F(_RM) \neq \varnothing$ the following are equivalent:*

(i) *$G(_RM)$ is a Z-geometry;*

(ii) *R is a two-sided units ring.*

(b) *In a Z-geometry $G = (L, E, F)$, the axioms* (F1) –(F3) *as a whole are equivalent to the following axiom:*

(F1*) *Whenever $e + a = f \oplus a$ holds for $a, e \in E$ and $f \in F$, then $ea = 0$ and $e \in F$.*

(c) CONSTRUCTION LEMMA: *Let $(L, E, \varnothing)$ be a projective geometry where every point is of finite rank at most n (in L), for a fixed natural number n. If F is defined as the set of all points of rank n, then (L, E, F) is a Z-geometry.*

6.4. *The U-property*

For most of what follows a special property has turned out to be crucial. A projective geometry is said to have the *U-property* if each of its points is contained in a unimodular point.

The algebraic counterpart of the U-property is the following property of a ring R. For all $\alpha, \beta \in R$ there exist $\gamma, \delta, \varepsilon \in R$ such that $\gamma\delta = \alpha$, $\gamma\varepsilon = \beta$ and $\delta R + \varepsilon R = R$, in other words, every pair (α, β) is a left multiple of a unimodular pair (δ, ε). We call rings with this property *proper right Bézout rings*[5]; for domains and local rings the properties to be right Bézout and proper right Bézout are equivalent (cf. also Cohn [1985], p. 86). The following proposition illustrates an important connection; its proof is straightforward.

PROPOSITION 6.4. *Let R be a nonzero ring.*

(a) *R is a proper right Bézout ring iff $G(_RR^2)$ has the U-property.*

(b) *If R is a proper right Bézout ring, then the projective geometry of every free module over R has the U-property.*

6.5. *A fundamental algebraization theorem*

For a projective geometry G and a natural number n, we first introduce a condition measuring some degree of freedom within the geometry:

(C_n) In G there exists a hyperplane h such that every n-generated element of G is disjoint from some complement of h.

Projective geometries which satisfy (C_n) always contain subgeometries of dimension n and if a projective geometry has an infinite basis, then (C_n) is satisfied for every natural number n. We now state a representation theorem which is based on Theorem 4.1 and which will turn out to be basic for most of what follows. For a proof, see Greferath and Schmidt [1992a], 3.2.

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[5] A right *Bézout ring* is a ring where every finitely generated right ideal is principal.

THEOREM 6.5. *A projective geometry G is module-induced whenever it has one of the following properties.*

(a) *G contains a complemented element whose subgeometry satisfies* (C_5).

(b) *Every point of G is uniform and G contains a complemented element whose subgeometry satisfies* (C_3).

COROLLARY 6.6. *Every projective geometry with infinite basis is module-induced.*

A general representation of finite-dimensional projective geometries seems to be out of reach. An application of Day and Pickering [1983], however, allows at least a representation of hyperplanes in Desarguesian projective geometries with finite basis. For a proof of the following theorem, see Greferath [1991].

THEOREM 6.7. *In a Desarguesian projective geometry, the subgeometry of every at least 1-dimensional hyperplane is module-induced.*

6.6. *Point-uniform geometries*

We will continue with an interesting application of Theorem 6.5. A *point-uniform geometry* is a projective geometry in which every point is uniform; obviously, point-uniform geometries are Z-geometries. A special class of point-uniform geometries allows an algebraic representation.

THEOREM 6.8. *A point-uniform projective geometry which is not 3-generated*[6] *and satisfies the U-property is induced by a module, provided each of its free points is unimodular.*

A special class of point-uniform geometries are the (weak) Hjelmslev geometries. A *weak Hjelmslev geometry* is a projective geometry where every point is a cycle, every free point is unimodular and where the U-property is satisfied. If furthermore every nonfree point is contained in at least two unimodular points, then the geometry is called a *Hjelmslev geometry*.

In order to see the algebraic counterparts of these defining properties, we recall that a ring in which the set of all left (right) ideals forms a chain is called a *left* (*right*, respectively) *chain ring*. A (*left-weak*) *Hjelmslev ring* is a left and right chain ring where each element is a unit or a (right) zero divisor.[7] The next lemma gives a reflection of some algebraic properties of rings and modules in their projective geometries – and in particular illustrates the algebraic counterpart of the difference between weak Hjelmslev geometries and Hjelmslev geometries; its proof is immediate.

PROPOSITION 6.9. *Let R be a ring and $G := G({}_RR^n)$ where $n \geqslant 2$.*

(a) *R is a left chain ring iff in G every point is a cycle.*

[6] Keeping in mind that every n-generated projective geometry is also k-generated for $k \geqslant n$, it is clear that a projective geometry which is not 3-generated, is neither 2- nor 1-generated.

[7] An element λ of a ring R is a *right* zero divisor, if there exists a nonzero element μ of R such that $\mu\lambda = 0$.

(b) *If in G the set of free points is trivially ordered, then in R every element is a unit or a right zero divisor.*

(c) *R is a right chain ring iff in G each point is contained in a join-irreducible unimodular point.*

(d) *If R is a right chain ring, then all right zero divisors are left zero divisors iff in G each nonfree point is contained in at least two unimodular points.*

(e) *R is a (left-weak) Hjelmslev ring iff G is a (weak) Hjelmslev geometry.*

By applying Greferath and Schmidt [1992a], 3.21 and 3.22, one obtains the following representation theorem for (weak) Hjelmslev geometries.

THEOREM 6.10. *For a finitely but not* 3*-generated projective geometry G the following are equivalent.*

(a) *G is a (weak) Hjelmslev geometry.*

(b) *G is induced by a free left module over a (left-weak) Hjelmslev ring.*

It is not difficult to prove that the left Artinian left-weak Hjelmslev rings coincide with the completely primary uniserial rings in the sense of Jónsson and Monk [1969]. An application of Kreuzer [1992] shows even more: the left Noetherian left-weak Hjelmslev rings coincide with the completely primary uniserial rings. This together with Greferath and Schmidt [1992a], 3.24, results in the following representation theorem for *point-Noetherian* weak Hjelmslev geometries, i.e. weak Hjelmslev geometries where every element contained in a point is compact.

THEOREM 6.11. *For a projective geometry G which is not* 3*-generated the following are equivalent:*

(a) *G is a point-Noetherian weak Hjelmslev geometry.*

(b) *G is induced by a free left module over a completely primary uniserial ring.*

6.7. *Reformulations of previous results*

In the following we will reprove and reformulate some previous results from the literature. As the first example, we mention the main result of Inaba [1948]: A primary lattice L can easily be completed to a Z-geometry (L, E, F), which is even more a Hjelmslev geometry provided it has a basis (cf. Greferath and Schmidt [1992a], 3.25). Together with 3.16 of the latter, this yields Inaba's result concerning the representation of primary lattices (cf. also 2.1 in this chapter).

COROLLARY 6.12. *Every primary lattice which contains at least* 4 *independent cycles of maximal rank is induced by a module over a completely primary and uniserial ring.*

Another class of point-uniform geometries is that of all Ore geometries. An *Ore geometry* is a point-uniform projective geometry in which each nonzero point is free. Now Theorem 5.1 can be reformulated in terms of projective lattice-geometry as follows.

THEOREM 6.13. *For a projective geometry G which contains a subgeometry of dimension at least 3, or at least 2 if G is Desarguesian, the following are equivalent.*

(a) *G is an Ore geometry.*

(b) *G is induced by a torsion free left module over a left Ore domain.*

Also Von Neumann's coordinatization of complemented modular lattices (of order $\geqslant 4$) may be reflected within projective lattice geometry. A *Von Neumann geometry* is defined as a projective geometry where every point has a complement and where the meet of any two compact elements is again compact. It is easily verified that every projective geometry which is induced by a submodule of a free left module over a Von Neuman regular ring is a Von Neumann geometry.

The following theorem shows how the coordinatization Theorems 3.1 and 3.3 may be applied to Von Neumann geometries.

THEOREM 6.14. *For a Von Neumann geometry G with n-basis where $n \geqslant 4$ (or $n = 3$ and G is Desarguesian) there always exists a Von Neumann regular ring R such that the reduced geometries of $G^{(n)}$ and $G({}_RR)$ are isomorphic.*

6.8. *Point-irreducible geometries*

Faigle and Herrmann [1981] investigate modular lattices (of finite length) and their connection with projective geometry on partially ordered sets. The geometric aspect of their approach is conceptually based on the choice of all join-irreducibles of a lattice as *points* of an associated projective space.

This motivated the introduction of the following special class of projective lattice geometries in Greferath and Schmidt [1994a]. A *PI-geometry* (point-irreducible geometry) is a projective geometry where each nonzero point is join-irreducible. Any module over a local ring induces a PI-geometry and every weak Hjelmslev geometry as defined earlier is a PI-geometry.

A large class of PI-geometries having the U-property allows an algebraic representation. This is due to the fact that the algebraic counterpart of the U-property in PI-geometries can easily be formulated, as we saw in Proposition 6.9(c). For projective geometries, which are not 5-generated, the following extension of Theorem 6.10 is proved in Greferath and Schmidt [1994a].

THEOREM 6.15. *For a finitely but not 5-generated projective geometry G the following are equivalent.*

(a) *G is a PI-geometry having the U-property.*

(b) *G is induced by a free left module over a right chain ring.*

Similar to the situation where primary lattices and Hjelmslev geometries are involved, there is a way to give a purely lattice-theoretic application of what we have just seen. This results from the fact that every relatively irreducible modular lattice of finite length can easily be completed to a PI-geometry. Together with Theorem 6.15 this yields the following characterization of free left modules over Artinian right chain rings.

THEOREM 6.16. *For a poset L and a natural number $n \geqslant 6$ the following are equivalent.*

(a) *L is a relatively irreducible modular lattice of finite length, the Kurosh–Ore dimension of L is n and in L all maximal join-irreducibles are complemented and of equal rank.*

(b) *$L \cong L({}_R R^n)$ for some Artinian right chain ring R.*

6.9. *Barbilian spaces in projective lattice geometries*

A *Barbilian space* is a quadruple $B = (P, H, |, \not\approx)$, consisting of a set P of *points* and a set H of *hyperplanes*, together with two relations $|$ and $\not\approx$ between P and H which are called *incidence* and *distant relation*, respectively, such that every point is distant from some hyperplane and that every hyperplane is distant from some point of B. The negation of $\not\approx$ is called *neighbour relation* and will be denoted by $\approx$. If P (and hence H) is empty, then B is called *empty*.

From a projective geometry G we naturally obtain a Barbilian space in the following way. Let P be the set of all unimodular points of G and H the set of all hyperplanes of G. Between P and H we introduce an *incidence relation* $|$ and a *distant relation* $\not\approx$. For all $(p, h) \in P \times H$ let $p|h$ iff $p \leqslant h$ and let $p \not\approx h$ iff p is a complement of h in G. Then $(P, H, |, \not\approx)$ is called the *Barbilian space of* G, denoted by $\mathrm{Barb}(G)$.

A large class of Barbilian spaces, called Veldkamp spaces, allows an algebraic representation as given in Chapter 19 of this Handbook. Projective geometries whose Barbilian space is a Veldkamp space can be characterized by a simple property. In order to illustrate this we recall: A Barbilian space $B = (P, H, |, \not\approx)$ is called a *Veldkamp space of dimension* n (where n is a nonzero natural number) if it is nonempty and the axioms (V1)–(V5) of Chapter 19, Section 4.2, are satisfied. A projective geometry is *stable* if any two of its unimodular points have a common complement. Note that the underlying ring of an at least 1-dimensional projective geometry is of stable rank 2 (see Chapter 19, Section 2.2) iff the geometry is stable.

As a result we report the following characterization from Greferath and Schmidt [1992b], Theorem 1.

THEOREM 6.17. *For a projective geometry G and a nonzero natural number n the following are equivalent.*

(a) *G is a stable geometry of dimension n.*

(b) $\mathrm{Barb}(G)$ *is a Veldkamp space of dimension n.*

It is clear that this combined with Veldkamp's representation of Veldkamp spaces, yields a partial representation of stable geometries of (finite) dimension $\geqslant 3$. This partial result is far away from being extended to a complete representation of stable geometries. However, as a recent result which in particular extends Theorem 6.15, we finally state a complete representation for a subclass of the class of all stable geometries. For a proof, see Greferath and Schmidt [1994b].

THEOREM 6.18. *For a projective geometry G of dimension $\geqslant 5$ the following are equivalent.*

(a) *G is stable and has the U-property.*

(b) *G is induced by a free left module over a proper right Bézout ring of stable rank 2.*

7. Generalizations of the fundamental theorem

Representation of mappings between submodule lattices. The fundamental theorem of projective geometry gives a representation of collineations between projective spaces by semilinear isomorphisms of the associated vector spaces and has been generalized in various ways. Our goal here is to give an algebraic description of a large class of mappings and, in particular, of isomorphisms between certain submodule lattices. For results concerning mappings in a point-hyperplane set-up, see Veldkamp's Chapter 19, Section 6, of this Handbook.

It is well known and not difficult to show, that tensoring all submodules of a module ${}_RM$ with a bimodule ${}_SB_R$, which is flat[8] as a right R-module, induces a lattice homomorphism from $L({}_RM)$ to $L({}_SB \otimes {}_RM)$ (cf. Bourbaki [1972], I, §2.6). Thus for any S-module monomorphism

$$h\colon\ {}_SB \otimes {}_RM \to {}_SN$$

the mapping $L({}_RM) \to L({}_SN)$ with $U \mapsto h(B \otimes {}_RU)$ is a lattice homomorphism which preserves arbitrary joins. Under certain conditions on the module ${}_RM$ the converse is also true, namely, each lattice homomorphism which preserves arbitrary joins has an essentially unique representation of this kind. This result has been shown by Brehm [1984b] in a manuscript, an outline of which appeared in Brehm [1987]. It generalizes the fundamental theorem in that the modules and also the mappings considered are more general. A further representation theorem can be proved for even more general mappings between submodule lattices of a large class of modules. These mappings are, geometrically spoken, those which map subspaces to subspaces and preserve arbitrary joins and disjointness of points and lines. (We do not assume these mappings to preserve finite meets!)

In the classical fundamental theorem of projective geometry one assumes the underlying vector space to be of dimension at least 3; in case of modules it turns out that it is not sufficient to assume that the underlying module contains a free submodule of rank at least 3. The results reported here can be shown assuming either of two different sets of rather technical but sufficiently weak conditions which aim at different classes of modules.

The first set of conditions is called the *triangle property* and is a slight modification of a set of conditions introduced by Stephenson [1967]. It holds, e.g., for modules over left Ore domains or, more generally, for modules over left orders in strongly regular left self-injective rings, provided these modules contain free submodules of rank 3.

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[8] That is, the functor $B\otimes_R$ preserves monomorphisms.

The second set is called the *splitting property* and holds, e.g., for free modules of rank $\geqslant 3$ over rings, in which left annihilators of elements are generated by idempotents. This class of rings clearly covers that of all Von Neumann regular rings and that of all Baer rings.

Considering lattice isomorphisms between submodule lattices of modules which have either the triangle property or the splitting property we can represent both the isomorphism and its inverse and obtain the underlying rings to be Morita equivalent.[9] It then follows that the lattice isomorphism in question is induced by the corresponding equivalence of categories combined with a module isomorphism.

For more details about all the theorems and propositions which follow, see Brehm [1984b, 1987].

7.1. *Representation of join and disjointness preserving mappings*

We first describe the two different sets of conditions which are requested in the proof of our representation theorems.

The triangle property. A module ${}_RM$ has the *triangle property* (t-property) provided the following holds for arbitrary elements x_1, x_2, x_3, x_4 of M.

(i) There exists an element y of M with $\operatorname{ann}(y) = 0$ and $Ry \cap (Rx_1 + Rx_2) = 0$.

(ii) If $\operatorname{ann}(x_i) = 0$ and $Rx_i \cap Rx_{i+1} = 0$ for all $i = 1, 2, 3$, then there exists an element y of M with $\operatorname{ann}(y) = 0$ such that $Ry \cap Rx_i = 0$ for all $i = 1, 2, 3, 4$.

(iii) If $\operatorname{ann}(x_1) = 0 = \operatorname{ann}(x_2)$ and $Rx_i \cap Rx_j = 0$ for all $i \neq j \in \{1, 2, 3\}$ but $Rx_3 \cap (Rx_1 + Rx_2) \neq 0$, then there exists an element y of M with $\operatorname{ann}(y) = 0$ such that $Ry \cap (Rx_i + Rx_j) = 0$ for all $i, j = 1, 2, 3$.

For a large class of modules, even the following simpler but stronger (*quadrangle*) property holds which clearly implies the t-property.

(iv) For arbitrary elements x_1, x_2, x_3, x_4 of M there exists an element y of M with $\operatorname{ann}(y) = 0$ such that $Ry \cap (Rx_i + Rx_{i+1}) = 0$ for each $i = 1, \ldots, 4$, where $x_5 := x_1$.

The splitting property. A module ${}_RM$ has the *splitting property* (s-property) if there exist submodules H_i of ${}_RM$ and elements x_i of H_i ($i = 0, 1, 2$) such that the following conditions hold, where $W := (H_0 + H_1) \cup (H_1 + H_2) \cup (H_2 + H_0)$.

(v) $M = H_0 + H_1 + H_2$, $H_j \cap H_k = 0$, $\operatorname{ann}(x_i) = 0$ and $Rx_i \cap (H_j \oplus H_k) = 0$, for all $\{i, j, k\} = \{0, 1, 2\}$.

(vi) $Rx = \sum\{Rrx$: there exist $y, z \in W$ with $rx - y \in W$ and $rx - z \in W$ such that $Ry \cap (Rrx + Rz) = 0\}$ for all $x \in M$.

Now we state a representation theorem for a large class of mappings between submodule lattices of a large class of modules. For a detailed proof, the reader should see Brehm [1984b, 1987].

[9] Two rings R and S are called *Morita equivalent* if there exist functors F: ${}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ and G: ${}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$ between the categories of R-modules and S-modules and functor equivalences $FG \cong 1_{{}_S\mathrm{Mod}}$, $GF \cong 1_{{}_R\mathrm{Mod}}$. Morita equivalences induce lattice isomorphisms between the respective submodule lattices.

THEOREM 7.1. *Let $_RM$ be a unitary module which has the triangle property or the splitting property and let $f\colon L(_RM) \to L(_SN)$ be a mapping that preserves (arbitrary) joins such that $Rx \cap (Ry + Rz) = 0$ implies*

$$f(Rx) \cap \big(f(Ry) + f(Rz)\big) = 0 \quad \textit{for all } x, y, z \in M.$$

Then there exists a bimodule $_SB_R$ and an R-balanced mapping $h\colon {}_SB \times M \to {}_SN$ such that

$$f(U) = \big\langle h[B \times U]\big\rangle \quad \textit{for all } U \in L(_RM)$$

and there exists an $x_0 \in M$ such that $\operatorname{ann}(x_0) = 0$ and $h(\_, x_0)$ is injective. The latter mapping and the bimodule are uniquely determined in the following sense: for k and $_SC_R$ in place of h and $_SB_R$, there exists a unique bimodule isomorphism $\varphi : {}_SC_R \to {}_SB_R$ such that $k(c, x) = h(\varphi(c), x)$ for all $c \in C$ and $x \in M$.

In the following propositions we give some simpler but also more restrictive conditions which are sufficient for a module to have the splitting or the triangle property, respectively.

PROPOSITION 7.2. *Let $_RM$ be a module such that the following conditions hold.*

(vii) *There exist submodules $H_i \subseteq {}_RM$ and $x_i \in H_i$ with $\operatorname{ann}(x_i) = 0$ such that $M = H_0 \oplus H_1 \oplus H_2$.*

(viii) *For all $a_0 \in H_0$, $a_1 \in H_1$ there exists an $r \in R$ such that $ra_0 = 0$ or $ra_1 = 0$, and $\operatorname{ann}(1-r)a_0 \subseteq \operatorname{ann}(1-r)a_1$ or $\operatorname{ann}(1-r)a_1 \subseteq \operatorname{ann}(1-r)a_0$.*

Then $_RM$ has the s-property.

It is clear that if for each $a_0 \in H_0$ there exists an $r \in R$ with $r^2 = r$ and $\operatorname{ann}(a_0) = Rr$, then (viii) is satisfied, because $ra_0 = 0$ and $\operatorname{ann}(1-r)a_0 = Rr \subseteq \operatorname{ann}(1-r)a_1$ for each $a_1 \in H_1$. This implies in particular that a free R-module of rank at least 3 has the s-property if R is (Von Neumann) regular (see Section 1) or if R is a Baer ring.

If the set of annihilators of elements of $H_0 \cup H_1$ is linearly ordered by inclusion then (viii) is clearly satisfied.

PROPOSITION 7.3. *If S is a left self-injective strongly regular ring, and R a left order (i.e. S is a left quotient ring of its subring R), then $_RM$ has the quadrangle property if and only if $_RM$ contains a free R-submodule of rank* 3.

A large class of rings satisfying the assumption of the last proposition is that of all left Ore domains.

7.2. *Representation of lattice homomorphisms and lattice isomorphisms*

THEOREM 7.4. *If $_RM$ has the triangle property or the splitting property and if*

$$f\colon L(_RM) \to L(_SN)$$

is a mapping, then the following are equivalent.

(a) *f is a lattice homomorphism preserving arbitrary joins.*

(b) *There exists a bimodule ${}_SB_R$ and an S-module monomorphism*

$$h\colon\ {}_SB \otimes {}_RM \to {}_SN$$

such that B_R is flat and $f(U) = h(B \otimes U)$ for all $U \in L({}_RM)$.

Furthermore, if (b) *also holds with a mapping k and a bimodule ${}_SC_R$ in place of h and ${}_SB_R$, then there exists a unique bimodule isomorphism $\varphi\colon\ {}_SC_R \to {}_SB_R$ such that $k = h \circ (\varphi \otimes_R \mathrm{id}_M)$.*

REMARK 7.5. (a) If the lattice homomorphisms $f_1\colon\ L({}_RM) \to L({}_SN)$ and $f_2\colon\ L({}_SN) \to L({}_TP)$ are represented as in the above theorem by $({}_SB_R, h_1)$ and $({}_TC_S, h_2)$, respectively, then, using the same theorem, $f_2 \circ f_1$ is represented by $({}_TC \otimes {}_SB_R, k)$, where

$$k\colon\ ({}_TC \otimes {}_SB) \otimes {}_RM \to {}_TP$$

is defined by

$$k\big((c \otimes b) \otimes x\big) := h_2\big(c \otimes h_1(b \otimes x)\big)$$

for all $b \in B, c \in C$ and $x \in M$.

(b) There are examples showing that the assumption on ${}_RM$ in the Theorems 7.1 and 7.4 cannot be replaced by the condition that ${}_RM$ contains a free submodule of rank 3. (Choose ${}_RM := {}_RR$ where R is a local ring having a left ideal isomorphic to ${}_RR^3$).

If ${}_RM$ and ${}_SN$ have the t-property or the s-property and $f\colon\ L({}_RM) \to L({}_SN)$ is a lattice isomorphism, then we can apply our last theorem to represent f and f^{-1}. It turns out that f is induced by a Morita equivalence together with a module isomorphism. Using Morita's characterization of equivalent rings (cf., e.g., Anderson and Fuller [1974]) we obtain:

THEOREM 7.6. *If ${}_RM$ and ${}_SN$ have the triangle property or the splitting property, then for any mapping $f\colon\ L({}_RM) \to L({}_SN)$ the following are equivalent.*

(a) *f is a lattice isomorphism.*

(b) *There exists a bimodule ${}_SB_R$ such that the rings S and* $\mathrm{End}(B_R)$ *are canonically isomorphic, B_R is a progenerator (i.e. a finitely generated projective generator) and there exists an S-module isomorphism $h\colon\ {}_SB \otimes_R M \to {}_SN$ such that $f(U) = h(B \otimes U)$ for all $U \in L({}_RM)$.*

Moreover ${}_SB_R$ and h of (b) *are unique in the sense described in the two theorems before.*

Note that if in addition f maps freely 1-generated submodules to freely 1-generated submodules then ${}_SB_R \cong {}_SS_R$ with $sr = s\alpha(r)$, for all $s \in S,\ r \in R$, where $\alpha\colon\ R \to S$ is an isomorphism; thus f is induced by a semilinear mapping with respect to α.

Finally we shall mention some earlier related results on the representation of lattice isomorphisms.

Stephenson [1967] has shown in his thesis that a lattice isomorphism

$$f\colon\ L({}_RM) \longrightarrow L({}_SN)$$

which maps some freely 1-generated submodule of ${}_RM$ to one of ${}_SN$ is induced by a semilinear mapping, assuming for ${}_RM$ either a condition which is only slightly different from the t-property, or condition (vii). Stephenson's result generalizes a previous similar result of Skornjakov [1960], who assumed another variant of the t-property.

Stephenson also shows that a lattice isomorphism $f\colon\ L({}_RM) \to L({}_SN)$ is induced by a Morita equivalence if ${}_RM$ is free of infinite rank and ${}_SN$ contains a free element.

Camillo [1984] shows that lattice isomorphisms between free modules of finite rank $\geqslant 3$, satisfying in addition a condition similar to the s-property are induced by a Morita equivalence together with a semilinear isomorphism. His condition is that ${}_RR^n$ has a basis $(x_i)_{i=1,\dots,n}$ such that every submodule of ${}_RR^n$ is the sum of submodules A, each of which has either of the following properties:

(i) A has zero projection to Rx_i for some i;

(ii) There exist distinct indices i, j such that $A \cap (Rx_i + Rx_j) = 0$.

The condition is satisfied, e.g., if R is a serial ring or a semihereditary ring or a domain or if R satisfies the bounded annihilator condition with bound $N \leqslant n - 2$.

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CHAPTER 22

Finite Diagram Geometries Extending Buildings

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Contents

1. Extending buildings with restrictions 1145
   1.0. Introduction 1145
   1.1. Sporadic groups: a source and a goal 1145
   1.2. Other sources 1146
   1.3. A setting in terms of diagram, geometry, group and characteristic 1147
   1.4. A selection of allowed rank 2 residues 1148
   1.5. Classical cases 1150
   1.6. A stock of possible restrictions 1151
   1.7. Flag-transitive generalized polygons 1153
   1.8. Flag-transitive rank 2 geometries of small deficiency 1154
   1.9. The flag-transitive linear spaces 1155
   1.10. Finite or infinite universal covers? 1157
2. Geometries over Coxeter diagrams of spherical type 1159
   2.1. Spherical diagrams and finiteness conditions 1159
   2.2. Irreducible spherical diagrams 1161
   2.3. The locally classical case 1166
   2.4. Finite thick C_3 geometries 1167
   2.5. Finite thick C_n and F_4 geometries 1169
3. Geometries over Coxeter diagrams of nonspherical type 1171
   3.1. Hyperbolic and other diagrams 1171
   3.2. Flag-transitive locally classical finite geometries belonging to Coxeter diagrams 1178
   3.3. Some generalizations 1182
   3.4. The tradition of thin geometries 1187
   3.5. A characterization of buildings 1189

HANDBOOK OF INCIDENCE GEOMETRY
Edited by F. Buekenhout

4. Extensions of geometries over Coxeter diagrams by linear spaces 1189
4.1. Introduction 1189
4.2. The diagram L_n 1191
4.3. The diagram $\mathsf{L}_{n-1} \cdot \mathsf{C}_2$ 1195
4.4. The diagram $\mathsf{L} \cdot \mathsf{C}_3^*$ 1209
4.5. The diagram $\mathsf{L} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}^*$ 1214
4.6. The diagrams $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$ and $\mathsf{D}_{n-1}^* \cdot \mathsf{L}$ 1220
4.7. The diagrams $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$ and $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$ 1225
5. Extensions of geometries over Coxeter diagrams by the Petersen graph geometry 1230
5.1. Extensions of projective geometries by the Petersen geometry 1230
5.2. Extensions of affine geometries 1234
5.3. Other extensions 1234
5.4. Tilde geometries 1238
5.5. The relationship between tilde and Petersen geometries 1240
5.6. More examples 1242
6. Other extensions 1243
6.1. Back to sporadic groups 1243
6.2. Extensions by partial geometries 1244
6.3. All geometries for a given group 1244
References 1245

1. Extending buildings with restrictions

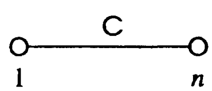
1.0. *Introduction*

In this chapter all geometries are residually connected and firm. In this first section, all geometries are finite. A geometry Γ is an *extension* of a geometry Γ' if Γ admits an element or a flag whose residue is isomorphic to Γ'. Our purpose is to report on an expansion of the theory of buildings based on diagrams. This started in 1975 and resulted in a strong body of activity and results.

1.1. *Sporadic groups: a source and a goal*

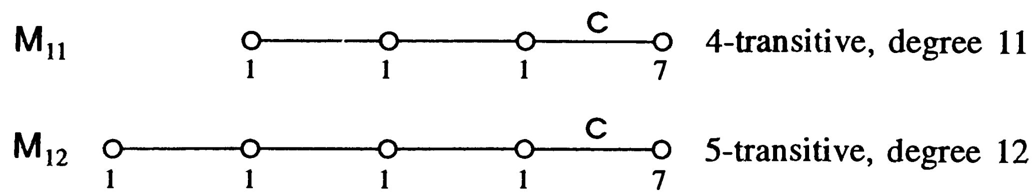
Buildings of spherical type (in particular finite) are the natural geometric counterpart of simple groups of Lie–Chevalley type (in particular finite). In this context, sporadic groups appear as orphans and call for an extension of the theory of buildings. This situation reproduces to some extent (not entirely) the conditions under which Tits started research on geometries that would explain the five exceptional simple complex Lie groups E_6, E_7, E_8, F_4 and G_2. Among the sporadic groups, the five Mathieu groups were definitely loaded with geometric meaning (via Steiner systems) after Witt [1938]. Some inclusions or 'towers' of simple groups (Tits [1969]) went in the same direction. The three Fischer groups Fi_{22}, Fi_{23}, Fi_{24} extend geometrically the Mathieu groups M_{22}, M_{23}, M_{24}. The latter appear as three consecutive 'extensions' of the projective plane of order 4 and the former as extensions of the rank 3 polar space corresponding to U(6, 2), whose planes are precisely projective planes of order 4. It also became clear that it was possible to work with such structure in some generality (Buekenhout and Hubaut [1977]).

These observations together with some other ones were leading to a unifying concept: adding a new rank 2 diagram to the list of Coxeter diagrams it became possible to explain various sporadic geometries in a uniform way.

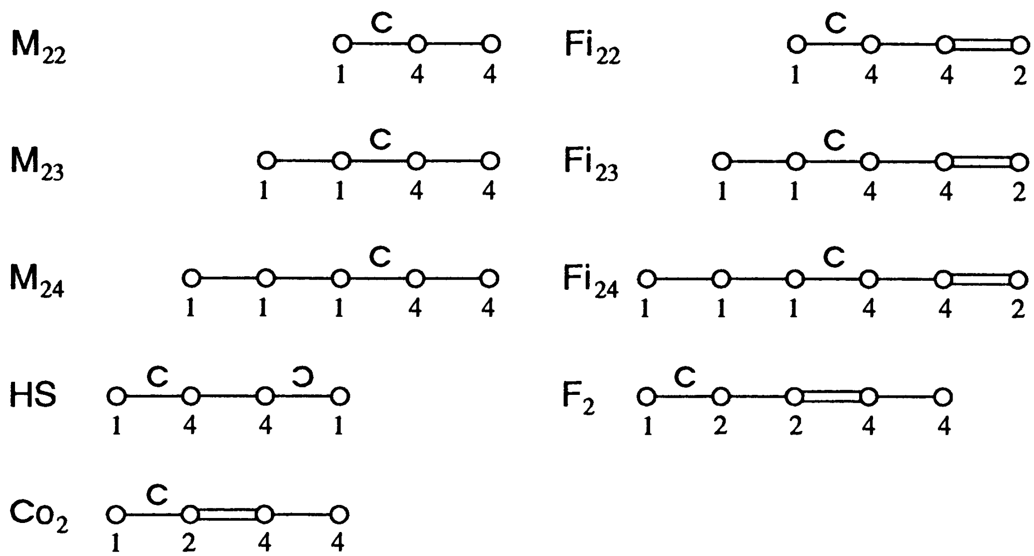


This diagram represents a 'circle' or 'complete graph' of $n+2$ points in which every line is incident to 2 points and every point is incident with $n+1$ lines, which means that each pair of points is identified with a line. In the first printed exposition of these topics (Buekenhout [1979]) 17 of the 26 sporadic finite simple groups were related to a diagram geometry allowing circles and generalized polygons as residues of rank 2. We give this table with some improvements that came later. The rest of the chapter will display a lot more. At present, 21 sporadic groups constitute the preceding list. The exceptions are J_4, O'N, $\mathsf{F}_3 = \mathsf{Th}$, $\mathsf{F}_5 = \mathsf{HN}$ and Ly (for this notation, see Conway et al. [1985]).

The small Mathieu groups

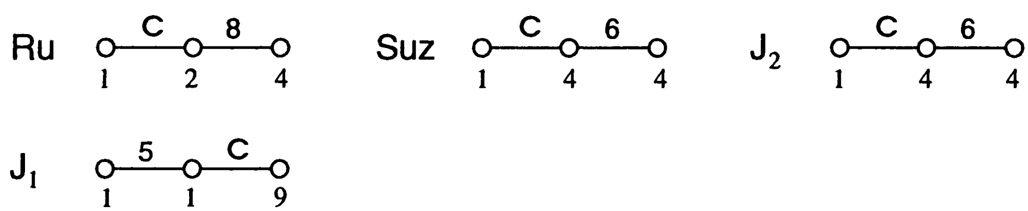


Extensions of the projective plane of order four



Extensions of the classical generalized quadrangle of order (3,9)

Others



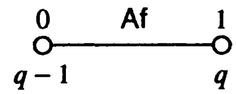
An observation made on this empirical evidence, was that sporadic geometries arise more easily in connection with small fields.

All developments in this chapter are partially motivated by these early observations and by attempts to include them in theories explaining all of the finite simple groups from a geometric viewpoint.

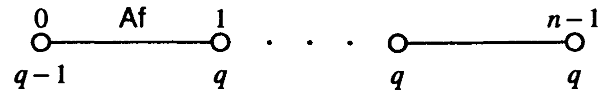
1.2. *Other sources*

The early developments explained in 1.1 went along with some classical geometrical facts that were not covered by the theory of buildings. For instance, affine spaces!

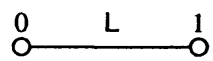
If the class of affine planes of order q is represented by the diagram



then an affine space of dimension n and order q is represented by

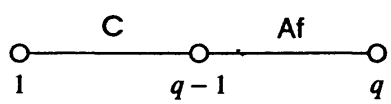


Since a convenient common generalization of projective planes and affine planes is the class of linear spaces, the latter are represented by a new diagram

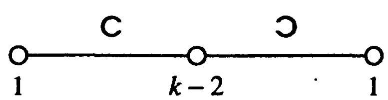


and 'the *linear spaces with dimension n*' (see Chapter 6) 'or *geometric lattices*' are then represented by

The inversive plane of order q, classically seen as a rank 2 geometry consisting of points and circles, has to be considered as a rank 3 geometry (with trivial lines of 2 points) over the diagram



As early as 1976, D. Hughes (unpublished lectures) was observing that *biplanes*, namely symmetric 2-designs with $\lambda = 2$ gave rise to a geometry over



and he started studying all geometries with the intersection property (IP) (see Chapter 3) over this diagram under the name of *semibiplanes*.

The oldest source of all is the tradition of regular solids and polytopes (see 3.4).

1.3. *A setting in terms of diagram, geometry, group and characteristic*

1.3.1. We shall now formalize our subject at four levels of complexity constantly used afterwards. The objects we consider are one of:

– a diagram Δ (whose set of nodes is I);

– a pair (Δ, Γ), where Δ is a diagram and Γ is a geometry belonging to Δ;

– a triple (Δ, Γ, G), where (Δ, Γ) are as above and G is a subgroup of Aut(Γ), the group of automorphisms of Γ;

– a 4-tuple (Δ, Γ, G, p), where (Δ, Γ, G) is as above and p is a prime number called the '*characteristic of* Γ *and* G'.

The last level may come as a surprise but it is needed. When we deal with a simple group G like U(4, 2) $\simeq$ Sp(4, 3), it acts on two different rank 2 buildings corresponding to $p = 2$ and $p = 3$.

Most of the time we shall deal with one of the first three levels.

1.3.2. *The Δ-level.* We do always assume that Δ is of finite rank, that it is connected (as a graph on the set of types) and that it is '*special*' in the sense of Chapter 3, 3.4, namely, for each pair of types $i \neq j$ in Δ, there are parameters g_{ij}, d_{ij}, d_{ji} whose geometric role is to restrict the corresponding rank 2 residues to (g_{ij}, d_{ij}, d_{ji})-gons. We take some liberty with this rule when we consider diagrams like

As a matter of fact, nonconnected subdiagrams are allowed in Δ.

An *endnode* of Δ is a node (type) 0 such that $\Delta \setminus \{0\}$ is connected.

In Δ, a *Coxeter stroke* is a pair of nodes $\{i, j\}$ such that $g_{ij} = d_{ij} = d_{ji}$. A *singular stroke* of Δ is a pair of nodes $\{i, j\}$ which is not a Coxeter stroke. We want to relate Δ to some other diagrams:

(1) Min Δ, *the minimal circuit diagram.* This is a Coxeter diagram and for all nodes $i \neq j$, $(\text{Min}\,\Delta)_{ij} = g_{ij}$, the gonality corresponding to Δ;

(2) Max Δ, the *maximal diagram.* This is a Coxeter diagram in which $(\text{Max}\,\Delta)_{ij}$ is the maximum of $\{d_{ij}, d_{ji}\}$.

(3) Cox Δ, the *Coxeter part of* Δ, obtained from Δ by deleting all singular strokes and all nodes belonging to no Coxeter stroke.

Convention: if Cox Δ is empty we do actually decide to replace it by the rank 1 Coxeter diagram ○.

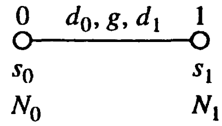
1.3.3. *The (Δ, Γ)-level.* Now Δ is as in the preceding section and Γ is a geometry belonging to Δ.

The geometry Γ is always firm, pure and residually connected. Moreover, we assume that for each $i \in \Delta$, Γ has an i-order s_i. Sometimes we consider that the i-orders are part of the Δ-level. We make use of the concepts introduced in Chapter 3.

1.3.4. *The (Δ, Γ, G)-level.* Here (Δ, Γ) is as in the preceding section and G is a group of automorphisms of Γ. We always assume (with explicitly stated exceptions) that G is flag-transitive on Γ. We use Aut(Γ) to denote the full group of automorphisms of Γ.

1.4. *A selection of allowed rank 2 residues*

1.4.1. *The Δ-level.* So far, the diagram Δ as restricted in 1.3.2 and with rank $\Delta = 2$, has the following shape



Here N_0 (resp., N_1) denotes the number of elements of type 0 (resp., 1). It is either of *Coxeter type* ($g = d_0 = d_1$) or of *singular type* (one of d_0, d_1 is not equal to g). In the latter case, the Coxeter rank is 1. This is far too general for our purposes. Let us introduce more concepts in order to restrict the situation or restrict and open it gradually.

The 0-*deficiency* δ_0 of Δ (resp., 1-*deficiency* δ_1) is $\delta_0 = d_0 - g$ (resp., $\delta_1 = d_1 - g$). The *deficiency* of Δ is the pair $\delta(\Delta) = (\delta_0, \delta_1)$. Thus, if $\delta(\Delta) = (0,0)$, Δ is of Coxeter type. The restrictions to be considered in this chapter are inspired by the idea that δ_0 and δ_1 are small. It is not clear yet whether this can be forced in some natural way, e.g., with assumptions on one of the levels (Δ, Γ) or (Δ, Γ, G).

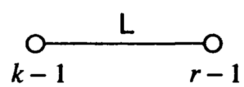
In view of our lack of knowledge here, it is presently indicated to leave the definition of 'small' open. Practically, however, the results dealt with are restricted to $\delta_i \leqslant 1$ $(i = 0, 1)$.

1.4.2. *The* (Δ, Γ)*-level and* $\delta_i \leqslant 1$ $(i = 0, 1)$.

(1) If $\delta = (0,0)$, the Feit–Higman theorem (see Chapter 9) restricts g to $\{3, 4, 6, 8, 12\}$ unless Γ is thin. This phenomenon keeps working also in the next smallest case.

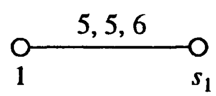
(2) If $\delta = (0,1)$, then $g = 3$ or 5 (see Chapter 10, 6.2 and 7.2).

For $g = 3$, Γ is a linear space (actually a Steiner system) belonging to



where k, r are the 'usual' parameters for linear spaces.

For $g = 5$, Γ belongs to



where $s_1 \in \{2, 6, 56\}$ (see Chapter 10, 6.1). If $s_1 = 2$, Γ is the Petersen graph. If $s_1 = 6$, Γ is the Hoffman–Singleton graph, and if $s_1 = 56$ the existence of Γ remains open.

(3) It is a striking fact to observe that the case $\delta = (0,1)$ is more restricted than the case of generalized polygons.

PROBLEM 1. Is there a 'generalized Feit–Higman' theorem for $\delta = (1,1)$ and more generally, for any δ?

(4) Here are some properties that are used sometimes as additional axioms.

(LL) Any two points (resp., lines) are incident with at most one line (resp., point).

(MF) (*multiplicity freeness*) No two points and no two lines have the same residue.

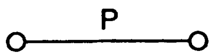
Observe that (LL) implies (MF). Also, if one of the orders s_0, s_1 is equal to one, then (MF) and (LL) are equivalent. Observe also, that in the rank 2 case, the intersection property (IP) means either that (LL) holds or that Γ is a generalized digon.

(Th) (*thickness*) Each rank one residue has at least three elements.

1.4.3. *The* (Δ, Γ, G)*-level.*

(1) For $\delta_i \leqslant 1$ and G '*geodesic transitive*' (see Chapter 10), there is a classification and we get $g \leqslant 8$ (Buekenhout and Van Maldeghem [1994], see Chapter 10, Section 7).

(2) The case $\delta_0 = g = 5$ and $\delta_1 = 6$ was almost complete at the (Δ, Γ)-level and so the data of G leaves only the Petersen and Hofman–Singleton graphs.



(3) The case $\delta_0 = g = 3$, $\delta_1 = 4$ is (almost) completely classified (see 1.9).

(4) Therefore Problem 1 deserves a companion:

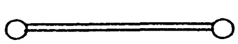
PROBLEM 2. Classify all flag-transitive (g, d_0, d_1)-gons with $g < d_1$.

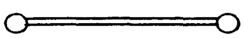
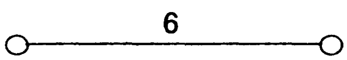
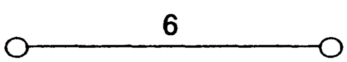
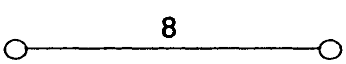
1.5. *Classical cases*

As one might expect, there is some possible confusion about the use of the term 'classical'. It could mean that G is a classical group, that Γ is an underlying building (generalized n-gon) and that Δ is of Coxeter type.

Quite a number of variations are conceivable. In particular, it may be reasonable to allow for a group G of Lie–Chevalley type.

Here is our choice (it is in agreement with Buekenhout and Van Maldeghem [1994]). Compare with Chapter 9, Section 4.

The classical (Δ, Γ, G)	
 q, q; $\frac{q^3-1}{q-1}$, $\frac{q^3-1}{q-1}$	$\mathrm{PSL}(3,q) \leqslant G \leqslant \mathrm{P\Gamma L}(3,q)$ q any prime power Selfdual.
 q, q; $\frac{q^4-1}{q-1}$, $\frac{q^4-1}{q-1}$	(1) $\mathrm{PSp}(4,q) \leqslant G \leqslant \mathrm{P\Gamma Sp}(4,q)$ q any prime power (2) $\mathrm{P\Omega}(5,q) \leqslant G \leqslant \mathrm{P\Gamma O}(5,q)$
Classes (1) and (2) are dual of each other, and for q even, they are selfdual.	

 q $\quad$ q^2 $(1+q)(1+q^3)$ $\quad$ $(1+q^2)(1+q^3)$	(1) $\mathrm{P}\Omega^-(6,q) \leqslant G \leqslant \mathrm{Aut}\,\mathrm{P}\Omega^-(6,q)$ (2) $\mathrm{PSU}(4,q) \leqslant G \leqslant \mathrm{Aut}\,\mathrm{PSU}(4,q)$
Classes (1) and (2) are dual of each other, in particular $\mathrm{P}\Omega^-(6,q)$ and $\mathrm{PSU}(4,q)$ are isomorphic.	
 q $\quad$ 1 $(q+1)^2$ $\quad$ $2(q+1)$	$\mathrm{PSL}(2,q) \times \mathrm{PSL}(2,q) = \mathrm{P}\Omega^-(3,q) \leqslant G \leqslant$ $\mathrm{Aut}\,\mathrm{P}\Omega^+(3,q)$
Caution: some of these G are not flag-transitive. The largest G has 2 blocks of imprimitivity on the lines. This case is rejected in 2.3 and in Sections 3 and 4.	
 q^2 $\quad$ q^3 $(1+q^2)(1+q^5)$ $\quad$ $(1+q^3)(1+q^5)$	$\mathrm{PSU}(5,q) \leqslant G \leqslant \mathrm{Aut}\,\mathrm{PSU}(5,q)$
Caution: it may happen that the dual situation is not considered as classical.	
 q $\quad$ q $(1+q)(1+q^2+q^4)$ $\quad$ $(1+q)(1+q^2+q^4)$	$\mathrm{G}_2(q) \leqslant G \leqslant \mathrm{Aut}\,\mathrm{G}_2(q)$ Selfdual if q is a power of 3. Otherwise, beware of duality.
 q^3 $\quad$ q $(1+q^3)(1+q^4+q^8)$ $\quad$ $(1+q)(1+q^4+q^8)$	${}^3\mathrm{D}_4(q) \leqslant G \leqslant \mathrm{Aut}\,{}^3\mathrm{D}_4(q)$
Caution: the dual situation is not necessarily considered as classical.	
 q $\quad$ q^2 $(1+q)(1+q^3+q^6)$ $\quad$ $(1+q^2)(1+q^3+q^6)$	${}^2\mathrm{F}_4(q) \leqslant G \leqslant \mathrm{Aut}\,{}^2\mathrm{F}_4(q)$ $q = 2^{2m+1}$
Think of duality as earlier. Also, the Tits group ${}^2\mathrm{F}_4(2)'$ of index 2 in ${}^2\mathrm{F}_4(2)$ may be considered as classical. Earlier, this could also apply to $\mathrm{Sp}(4,2)'$ and $\mathrm{G}_2(2)'$.	

1.6. *A stock of possible restrictions*

We give a choice of properties that are sometimes used as axioms at one or the other level and we complete then with some other suggestions. The rank 2 case has been dealt with in 1.5, hence we assume $\mathrm{rk}\,\Delta \geqslant 3$.

1.6.1. *The Δ-level.* Here are various natural conditions.

(R2) All rank 2 residues (*subdiagrams*) are restricted as in 1.5, namely they have small deficiency (open version) or all deficiencies are $\leqslant 1$ (closed version).
(Deg) The Coxeter rank is small (open version) or it does not exceed some bound (closed version).

For instance, the Coxeter degree is $\leqslant 1$ (simple extension).

(Sph) Min Δ is of spherical type (weak version) or Max Δ is of spherical type (strong version).
(Sph Cox) Cox Δ is of spherical type.

Observe that (Sph) implies (Sph Cox). Also, Min Δ = Max Δ implies Min Δ = Cox Δ = Max Δ.
Afterwards, we will often assume (R2), (Deg $\leqslant 1$) and (Sph Cox).

1.6.2. *The (Δ, Γ)-level.* We may impose on Γ some of the global properties discussed in Chapter 3 or some other.

(1) *Variations on the intersection property* (IP) (see Chapter 3 and Pasini [1994d]).

(IP) as usual.
(MF) (*multiplicity freeness*) Every rank 2 residue is a generalized digon or it is (MF) as in 1.4.2.
(TMF) (*truncational multiplicity freeness*) = (MF) on Γ and on all truncations of Γ.
$(\mathrm{IP})_s$ All residues of rank $\leqslant s$ satisfy (IP).
(O) For all $i \in \Delta$, if 2 elements of the same type j have the same i-shadow then they are equal (and inductively for connected subdiagrams). Δ itself is supposed to be connected.

Observe the following implication pattern.

$$\begin{array}{ccccccc} (\mathrm{IP}) & \Rightarrow & (\mathrm{IP})_3 & \Rightarrow & (\mathrm{IP})_2 & \Rightarrow & (\mathrm{MF}) \\ & & & & & \nearrow & \\ & & & & (\mathrm{TMF}) & & \end{array}$$

(2) (NT) No truncation. It assumes that (Δ, Γ) is not a proper truncation of some (Δ', Γ').
(3) (UC) (Δ, Γ) is simply 2-connected, namely equal to its universal 2-cover.
(4) Consider $(\Delta, \Gamma, \mathcal{A})$ where $\mathcal{A}$ is a family of apartments (see Chapter 3) and submit these data to various properties.

1.6.3. *The (Δ, Γ, G)-level.* We can ask the following for all nodes, all endnodes, some nodes, etc.
(1) G acts primitively on the elements of each type i (property (Pri)).
(2) G acts faithfully on the set of elements of each given type (property (Fth)).
(3) G is inductively (Pri), namely (Pri) holds in every flag residue.
(4) Some kind of Moufang condition (see Chapter 9).

(5) Concerning the Borel subgroup, namely the stabilizer of a chamber, we may ask for properties like:

– B is solvable;

– $N_G(B) = B$;

– if U is the maximal normal nilpotent subgroup of B, then $B = N_G(U)$.

1.7. *Flag-transitive generalized polygons*

The classification of all pairs (Γ, G) where Γ is a generalized g-gon and G is a flag-transitive automorphism group is an open problem.

1.7.1. *The case* $g = 2$. The question is equivalent to the classical *factorization of groups* namely the search for a triple (G, A, B) where G is a group, A and B are subgroups of G and $G = AB$. For recent work on this matter, see Liebeck, Praeger and Saxl [1990].

1.7.2. *The case* $g = 3$. Much has been published on this. Let us mention some striking facts.

(1) If the projective plane is 'classical', namely Desarguesian, and if G is not classical, namely G does not contain PSL$(3, q)$, then one of the following holds (see Higman [1962])

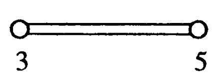
– $q = 2$ and G is a Frobenius group of order $7 \cdot 3$;

– $q = 8$ and G is a Frobenius group of order $73 \cdot 9$.

(2) If Γ is non-Desarguesian of order n, then G is a Frobenius group of odd order $(n^2 + n + 1)(n + 1)$ and $n^2 + n + 1$ is prime (Kantor [1987a]). No example is known.

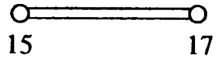
Actually we get somewhat more (Feit [1990]). The order n is not a power of 2, it is divisible by 8 and for every divisor d of n, $d^{n+1} \equiv 1 \pmod{p}$. Finally, $n > 14400008$.

1.7.3. *The case* $g \in \{4, 6, 8\}$. For $g = 4$, two sporadic and nonclassical examples are known (Kantor [1986]), namely:

(1) 

$G = 2^6 : \text{Alt}(6)$

There are 64 points.

(2) 

$G = 16^3 : (18 \cdot \varepsilon)$,

where ε is some small integer. There are 16^3 points.

Actually, there are some sporadic cases with a classical quadrangle (Theorem 3.2.3).

For $g = 6$ or 8, there is a recent attack on small cases (Buekenhout and Van Maldeghem [1993]). Here it is not necessary to assume flag-transitivity for G. The result runs as follows. Assume G is almost simple, namely $S \trianglelefteq G \leqslant \text{Aut}\, S$ where S is a non-Abelian simple group. Assume also that S is an ATLAS-group namely one of the small simple groups displayed in the main Section of the ATLAS (Conway, Curtis, Norton, Parker and Wilson [1985]). Finally, assume that G acts point-transitively on Γ. Then (Γ, G) is classical. There are six such pairs, up to duality and isomorphism.

1.8. *Flag-transitive rank 2 geometries of small deficiency*

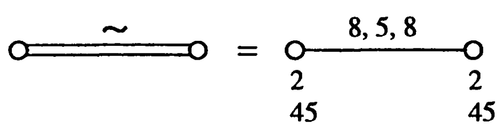
We do no longer assume that Γ is a generalized g-gon. Hence the deficiency $(\delta_0, \delta_1) = (d_0 - g, d_1 - g)$ is no longer $(0, 0)$. We consider small cases for (δ_0, δ_1).

1.8.1. $(\delta_0, \delta_1) = (0, 1)$. Then $g = 3$ or $g = 5$ (see Chapter 10, 6.2 and 7.2). For $g = 3$, Γ is a linear space. This situation is under (almost) full control. See 1.9.

For $g = 5$, Γ is a Moore graph and so it is either the Petersen graph, the Hoffman–Singleton graph or a hypothetical graph $\mathcal{G}_{57}$ of valency 57. We refer to Chapter 10, 6.1. In the first case, G is Alt(5) or Sym(5). In the second case, G is PSU(3, 5) or an overgroup of it. In the third case G has not been classified; however, by Aschbacher [1971] it is known that G does not act as a rank 3 group on the points.

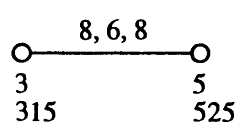
1.8.2. $(\delta_0, \delta_1) = (1, 1)$. There is a classification under much stronger transitivity assumptions for G (see Chapter 10, Section 7). The list that arises provides interesting candidates for extensions to rank $\geqslant 3$ situations.

1.8.3. *Other cases.* No classification seems to be available. There are a number of isolated examples that are interesting for various reasons. We make no claim as to completeness of the following list.

(1) 

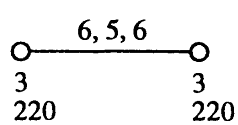
$G = 3 \cdot \mathrm{Sym}(6)$

The geometry is a 3-fold cover of the generalized quadrangle $W(2)$. The group G acts transitively on maximal ordered geodesic paths of length 9 whose first element is a point (resp., line).

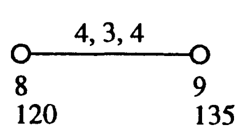
(2) 

$G = \mathrm{J}_2$ or $\mathrm{J}_2 \cdot 2$

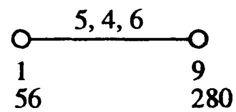
The automorphism group is transitive on the maximal geodesics beginning with a point (length 9). Maximal geodesics beginning with a line have length either 7 or 9. The group is transitive on both kinds separately.

(3) 

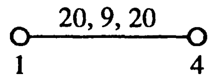
$G = \mathrm{M}_{12}$

(4) 

$G = \mathrm{Alt}(9)$

(5) 

$G = \mathrm{PSL}(3, 4)$

(6) 

$G = \mathrm{J}_3$ (Weiss)

There are 17442 points.

1.9. *The flag-transitive linear spaces*

This section is based entirely on Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [1990]. We give a brief description of the pairs (S, G) such that S is a linear space whose lines have $\geqslant 3$ points while no line contains all points and G is a flag-transitive group of automorphisms of S. One of the following occurs.

(a) G is almost simple: i.e. G has a non-Abelian simple normal subgroup N such that $N \trianglelefteq G \leqslant \mathrm{Aut}(N)$;

(b) G is affine: i.e. the set of points of S carries the structure of an affine space $\mathrm{AG}(n, p)$ of dimension n over a field of prime order p, invariant under G, and G contains the group T of all translations of $\mathrm{AG}(n, p)$ (so $T \trianglelefteq G \leqslant \mathrm{AGL}(n, p)$).

1.9.1. *Examples with G almost simple.* In this case there are four infinite families of pairs (S, G) and one sporadic example.

(1) *Desarguesian projective spaces.* Here $S = \mathrm{PG}(d, q)$ is a projective space of dimension $d \geqslant 2$ over $\mathbb{F}_q$. Any group G with $\mathrm{PSL}(d+1, q) \leqslant G \leqslant \mathrm{P\Gamma L}(d+1, q)$ acts flag-transitively on S. There is one additional sporadic example with $S = \mathrm{PG}(3, 2)$ and $G = \mathrm{Alt}(7)$ in its 2-transitive action on the points.

(2) *Hermitian unitals.* Let V be a 3-dimensional vector space over $\mathbb{F}_{q^2}$ with a non-degenerate Hermitian form. The Hermitian unital $U_H(q)$ has as points the $q^3 + 1$ totally isotropic 1-spaces in V, and as lines the sets of $q + 1$ points lying in a nondegenerate 2-space. Any group G with $\mathrm{PSU}(3, q) \leqslant G \leqslant \mathrm{P\Gamma U}(3, q)$ acts flag-transitively on $U_H(q)$. Note that when $q = 2$ this example belongs to the affine case as $\mathrm{PSU}(3, 2)$ is soluble.

(3) *Ree unitals.* For any integer $e \geqslant 0$ and $q = 3^{2e+1}$, there is a Ree group ${}^2\mathrm{G}_2(q)$ with a 2-transitive action on $q^3 + 1$ points. Any pair of points is fixed by a unique involution, and the fixed point set of such an involution is a set of $q + 1$ points which we call a line. The resulting linear space, denoted by $U_R(q)$, is called a *Ree unital.* Any group G with ${}^2\mathrm{G}_2(q) \leqslant G \leqslant \mathrm{Aut}({}^2\mathrm{G}_2(q))$ acts flag-transitively on $U_R(q)$.

(4) *Witt–Bose–Shrikhande spaces.* Let $q = 2^n$, $n \geqslant 3$. The Witt–Bose–Shrikhande space $W(q)$[1] is defined from the group $\mathrm{PSL}(2, q)$ as follows: the points are the subgroups of $\mathrm{PSL}(2, q)$ isomorphic to the dihedral group of order $2(q + 1)$, the lines are the involutions of $\mathrm{PSL}(2, q)$, a point being incident with a line if and only if the subgroup contains the involution. Any group G with $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{P\Gamma L}(2, q)$ acts flag-transitively on $W(q)$. Note that $W(8)$ is isomorphic to $U_R(3)$.

[1] This $W(q)$ is different from the quadric $W(q)$ used in Chapter 9 and in 1.8.3 (Editor's note).

1.9.2. *Examples with G affine.* In this case G is a subgroup of the affine group $\mathrm{AGL}(n,p)$ containing the translation group T. If G_0 is the stabilizer of the point $0 \in S$, then $G = TG_0$ and G_0 is an irreducible subgroup of $\mathrm{GL}(n,p)$.

(1) *Desarguesian affine spaces.* Here S is the affine space $\mathrm{AG}(d,q)$ with $d \geqslant 2$ and $q^d = p^n$, and $G_0 \leqslant \Gamma\mathrm{L}(d,q)$. If Z is the centre of $\mathrm{GL}(d,q)$ then by the flag-transitivity of G, the group ZG_0 is transitive on the nonzero vectors of S, and so is one of the transitive linear groups determined by Hering. Thus if G is not 1-dimensional, one of the following holds:

(a) G is 2-transitive (hence known).

(b) $d = 2$, q is 11 or 23 and G is one of three soluble flag-transitive groups.

(c) $d = 2$, $q = 9$, 11, 19, 29 or 59, $G_0^{(\infty)} = 2.\mathrm{Alt}(5)$ (where $G_0^{(\infty)}$ is the last term in the derived series of G_0).

(d) $d = 4$, $q = 3$ and $G_0^{(\infty)} = 2.\mathrm{Alt}(5)$.

(2) *Non-Desarguesian translation affine planes.* The examples here are:

(a) The Lüneburg planes. These are affine planes of order q^2, where $q = 2^{2e+1}$ $(e \geqslant 1)$, and ${}^2\mathrm{B}_2(q) \leqslant G_0 \leqslant \mathrm{Aut}({}^2\mathrm{B}_2(q))$.

(b) The Hering plane of order 27 (see Chapter 5). Here $G_0 = \mathrm{SL}(2,13)$ and G is 2-transitive on the points of S.

(c) The nearfield plane of order 9. Here there are seven possibilities for G_0.

(3) *Hering spaces.* These are two flag-transitive linear spaces on 3^6 points with lines of size 3^2. In both cases $G_0 = \mathrm{SL}(2,13)$ and G is 2-transitive on the points.

1.9.3. *The 1-dimensional affine case.* So far, the case where $G \leqslant \mathrm{A}\Gamma\mathrm{L}(1,q)$ remains open. We shall see below that there are many examples in this case.

(1) *Translation affine planes.* All flag-transitive affine planes which do not belong to 1.9.2 (2) have a 1-dimensional affine group.

(2) *Generalized Netto systems.* Let $q = p^n$ with p an odd prime, n an integer, and let $k \geqslant 3$ be an odd integer such that $k(k-1)$ divides $q-1$. Let K be the set of all k-th roots of unity in $\mathbb{F}_q$, and define a point-line incidence structure $N(k,q)$ as follows. The set of points is $\mathbb{F}_q$; the lines are the images of K under the group $\mathrm{AG}^{k-1}\mathrm{L}(1,q)$ consisting of all mappings $x \mapsto a^{k-1}x + b$, where $a, b \in \mathbb{F}_q$, $a \neq 0$. That group and its overgroups in $\mathrm{A}\Gamma^{k-1}\mathrm{L}(1,q)$ act flag-transitively on $N(k,q)$. However, $N(k,q)$ need not be a linear space; here is a necessary and sufficient condition for $N(k,q)$ to be a linear space:

$(*)$ For any primitive k-th root of unity ε in $\mathbb{F}_q$, the elements $\varepsilon - 1, \varepsilon^2 - 1, \ldots, \varepsilon^{k-1} - 1$ are in distinct cosets of the multiplicative group consisting of all $(k-1)$-th powers of nonzero elements of $\mathbb{F}_q$.

In particular, if $k = 3$ then $(*)$ holds provided $q \equiv 3 \pmod 4$. Therefore $N(3,q)$ is a linear space for every prime power $q \equiv 7 \pmod{12}$. These spaces are usually called *Netto triple systems*. When $k > 3$ we call the resulting linear spaces generalized Netto systems. If $k = 5$ then $(*)$ is equivalent to the condition that $5^{(q-1)/4} \neq 1$ in $\mathbb{F}_q$, with $q \equiv 21 \pmod{40}$. For $q < 800$ this gives four linear spaces with $q = 61$, 421, 661, 701. Only two Desarguesian projective planes arise as generalized Netto systems: $N(3,7) \cong \mathrm{PG}(2,2)$ and $N(9,73) \cong \mathrm{PG}(2,8)$.

(3) *Kantor's inflation trick.* Suppose G is a flag-transitive automorphism group of a nontrivial linear space S with point-set P and let L be a line of S. If the stabilizer G_L acts flag-transitively on some linear space S' with point-set L, then all images of any line L' of S' under G are the lines of a flag-transitive linear space S'' on the point-set P.

Start, e.g., with $P = \mathrm{GF}(q^n)$, $L = \mathrm{GF}(q)$ and $G = \mathrm{AG}^m\mathrm{L}(1, q^n)$: they define a flag-transitive linear space S if and only if $(m, (q^n - 1)/(q - 1)) = 1$, and then $G_L = \mathrm{AG}^m\mathrm{L}(1, q)$. If there is a generalized Netto system $S'' = N(m + 1, q)$, then we just get more generalized Netto systems $S'' = N(m + 1, q^n)$ for those values of n satisfying the above greater common divisor condition. However, we get new examples if there is a translation affine plane S' of order $\sqrt{q}$ admitting $\mathrm{AG}^m\mathrm{L}(1, q)$ as a flag-transitive automorphism group; such translation planes do exist in great profusion if $\sqrt{q} = 2^e$, where $e \geqslant 5$ is odd, and also if $\sqrt{q}$ is an odd prime power whose exponent is not a power of 2.

1.9.4. THEOREM. *Suppose G is a flag-transitive group of automorphisms of the linear space S. Then one of the following occurs:*

(I) *(S, G) is one of the (known) examples described in Sections* 1.9.1, 1.9.2;

(II) *S has $q = p^a$ points (p prime) and G is a subgroup of the group* $\mathrm{A\Gamma L}(1, q)$ *of 1-dimensional semilinear affine transformations*;

(III) *all lines have exactly two points and G is any 2-transitive group acting on the points of S*;

1.10. *Finite or infinite universal covers?*

1.10.1. *A fundamental question.* Assume that a diagram Δ is given and that we want all finite geometries Γ or all finite pairs (Γ, G) belonging to it. A basic method is to look for the universal objects Γ or (Γ, G), if there are any, and to find the other objects as quotients of those.

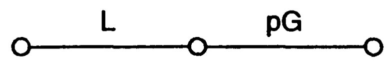
1.10.2. *Diagrams of finite type of infinite type and of almost finite type.* Given a geometry Γ of rank n over a set of types I, let $C = C(\Gamma)$ be the chamber system of Γ, as defined in Chapter 3, 3.9. The chamber system C can be viewed as a graph, whose vertices are the chambers of Γ, two chambers being *adjacent* if they are i-adjacent for some type $i \in I$. We take the diameter of this graph as the *diameter* of Γ, denoting it by d_Γ.

We say that Γ is *locally finite* provided every flag of Γ of corank 1 is contained in finitely many chambers. Trivially, Γ is finite if and only if it is locally finite and $d_\Gamma < \infty$.

Furthermore, if Γ is finite and $s + 1$ is an upper bound for the size of residues of flags of Γ of corank 1, then Γ has at most the following number of chambers:

$$(1 + ns)\bigl(1 + ns + (ns)^2 + \cdots + (ns)^{d_\Gamma - 1}\bigr).$$

Given a diagram Δ, we say that Δ is of *finite type* if there is a positive integer d such that every geometry belonging to Δ has diameter $d_\Gamma \leqslant d$. We say that Δ is of *infinite type* if at least one of the geometries belonging to Δ has infinite diameter. Note that there are diagrams which are neither of finite type nor of infinite type. We call them *almost finite*. For instance, pG denotes the class of partial geometries (see Chapter 10 and remember that all partial geometries are finite, by definition). Then all geometries belonging to the following diagram are finite; however, it is not difficult to construct such geometries with arbitrarily large diameters (Del Fra and Ghinelli [1991]).



Similarly, all locally finite geometries belonging to the diagram pG · L are finite (Hughes [1991]); however there are examples with arbitrarily large diameter (Hughes [1991], Pasechnik [1991]).

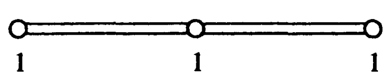
We shall see in Section 2, that Coxeter diagrams provide explicit examples of finite type or infinite type but not of almost finite type.

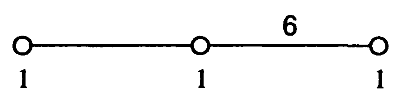
1.10.3. *About hopeless problems.* Here we want to fight a fairly standard pessimistic argumentation which is a source of confusion.

Assume that Δ is given.

Assume furthermore that the universal 2-cover of (Γ, G) is known and infinite. Finally, assume that infinitely many finite quotients (Γ, G) are known for that universal 2-cover.

Under such circumstances the pessimistic view is that a classification of all (Γ, G) belonging to Δ is hopeless. Let us mention two counter-examples, namely,





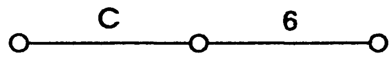
The preceding arguments apply. Nevertheless, a complete classification of all (Γ, G) belonging to these diagrams is available from Coxeter and Moser [1964]. See 3.4 for more details. As a matter of fact, this situation does not imply that optimism must be the rule either.

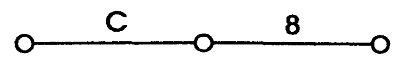
1.10.4. *Weetman's theorems for graph extensions.* Proving that a classification problem at the (Δ, Γ) level or at the (Δ, Γ, G) level, is of finite (or almost finite, or infinite) type is sometimes related to a question about extensions of graphs. Let $\mathcal{G}$ be a graph. Call $\mathcal{G}$ *regular* if all vertices of $\mathcal{G}$ belong to the same number of edges. If x is a vertex of $\mathcal{G}$, the *neighbourhood* $x^\perp$ of x is the subgraph of $\mathcal{G}$, induced by $\mathcal{G}$, on the set of vertices y such that $\{x, y\}$ is an edge of $\mathcal{G}$.

The *girth* of $\mathcal{G}$ is the smallest possible length of a circuit in $\mathcal{G}$, if there are any circuits. Let a graph $\mathcal{H}$ be called *locally* $\mathcal{G}$ if for any vertex $x \in \mathcal{H}$, $x^\perp$ is isomorphic to $\mathcal{G}$.

THEOREM A (Weetman [1994a]). *If $\mathcal{G}$ is a regular graph of girth $\geqslant 6$, then there exists a graph $\mathcal{H}$ of infinite diameter which is locally $\mathcal{G}$.*

This result applies to many situations like the search for geometries over diagrams as follow.





Another contribution of Weetman takes the other direction.

Consider a connected finite graph $\mathcal{G}$ with v vertices and valency k. Assume that there are integers λ and μ such that for any distinct vertices x, y we have $|x^\perp \cap y^\perp| = \lambda$ (resp., μ) provided that $y \in x^\perp$ (resp., $y \notin x^\perp$). Then $\mathcal{G}$ is called *strongly regular.*

THEOREM B (Weetman [1994b]). *Let $\mathcal{G}$ be a strongly regular graph with parameters (v, k, λ, μ). Then every locally $\mathcal{G}$ graph has a diameter bounded by $k + 1$ provided any one of the following conditions is realized*:

(i) $v \leqslant 2k + 1$;

(ii) $\mu > \lambda$;

(iii) *$\mathcal{G}$ is the collinearity graph of a partial geometry.*

Current research by A. Brouwer is improving on this result.

We come to another striking result. If $\mathcal{G}$ is a graph, the *line graph* $L(\mathcal{G})$ has the edges of $\mathcal{G}$ as *vertices*, and as edges the pairs of edges E, E' of $\mathcal{G}$ such that E and E' have a common vertex in $\mathcal{G}$.

THEOREM C (Ivanov, Shpectorov, Pasechnik: private communication, 1993). *Let $\mathcal{G}$ be a trivalent graph. Assume that any two disjoint edges of $\mathcal{G}$ are contained in disjoint circuits of $\mathcal{G}$. Then there is an infinite connected graph which is locally $L(\mathcal{G})$.*

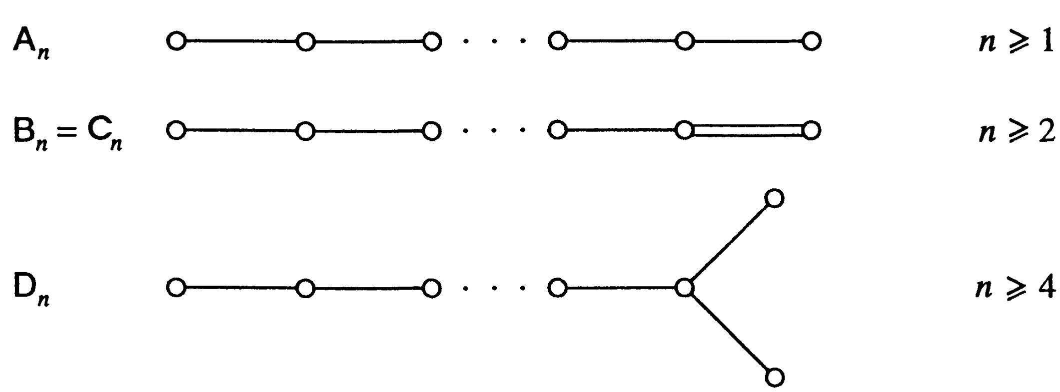
This applies, e.g., to the case where $\mathcal{G}$ is the Petersen graph.

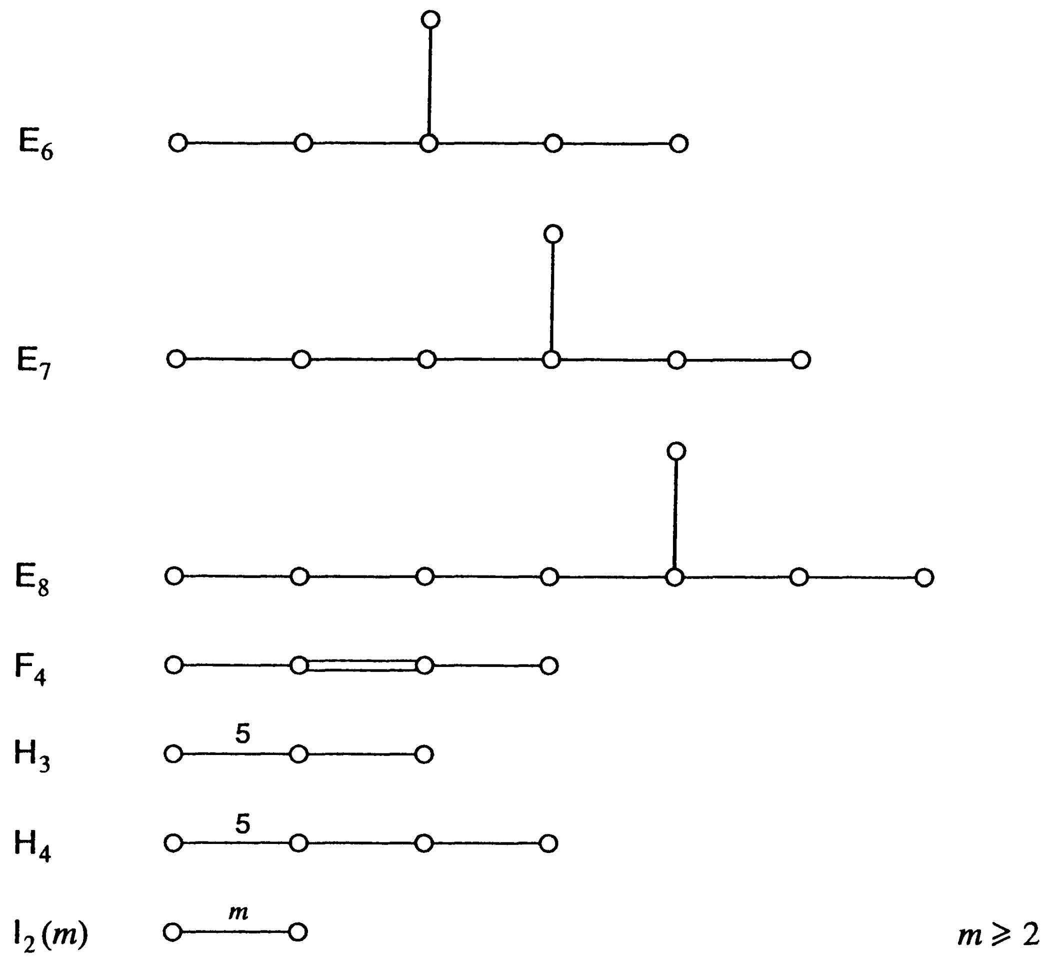
2. Geometries over Coxeter diagrams of spherical type

2.1. *Spherical diagrams and finiteness conditions*

We start at the Δ-level. Here, Γ is not necessarily finite.

The following irreducible (i.e. connected) Coxeter diagrams are said to be *spherical* (see Chapter 11, Section 2).





A reducible Coxeter diagram is *spherical* if each of its connected components is spherical. Coxeter diagrams of spherical type peculiarly embody a finiteness information. To explain this it is convenient to use the language of chamber systems and to proceed to the (Δ, Γ)-level.

2.1.1. THEOREM. *All spherical Coxeter diagrams are of finite type. All nonspherical Coxeter diagrams are of infinite type.*

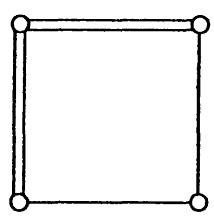
The first statement of the theorem is a consequence of a more general result of Tsaranov [1990a] on monoids generated by idempotents satisfying certain 'Coxeter type' relations. The second statement becomes evident if we remark the following: for every Coxeter diagram there is a (unique) Coxeter complex belonging to it, and a Coxeter complex is finite if and only if its Coxeter diagram is spherical (see Chapter 11, Section 2).

2.1.2. *Reduction to connected components.* Let Γ be a geometry over a set of types I and with disconnected basic diagram. We know from 3.1 that, if $\{J_1, J_2, \ldots, J_m\}$ is the partition of I in the connected components of the basic diagram of Γ, then

$$\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_m,$$

with Γ_i isomorphic to the residue of an (arbitrary) flag of Γ of cotype J_i, for $i = 1, 2, \ldots m$. We call this the *Direct Sum Theorem.* Thus, to know the structure of Γ, we only need its summands $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$. Therefore, if we aim to classify geometries belonging to a given family of diagrams, we can restrict to the connected diagrams of that family. In particular, dealing with spherical Coxeter diagrams, we can restrict ourselves to A_n, C_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4 and $\mathsf{I}_2(m)$.

2.1.3. *Again on infiniteness and finiteness.* Every chamber system of rank $n \geqslant 3$ admits a universal 2-cover (Chapter 11, Proposition 6.1.6), and diameters can be defined for arbitrary chamber systems just as for (chamber systems of) geometries. Thus, we can state the following definition: a diagram Δ of rank $n \geqslant 3$ is of *strongly infinite type* if, for every geometry Γ belonging to it, the universal 2-cover of $C(\Gamma)$ has infinite diameter. Spherical Coxeter diagrams are of finite type (Theorem 2.1.1). Therefore, all spherical Coxeter diagrams are of strongly finite type. On the other hand, if Δ is a nonspherical Coxeter diagram of rank 3, then the universal 2-cover of a geometry belonging to Δ is a building (Chapter 11, Theorem 6.3.1), hence it has infinite diameter since a building has finite diameter if and only if its Coxeter diagram is spherical (Chapter 11, Sections 2.1 and 5.2). Therefore Δ is of strongly infinite type. The restriction to rank 3 cases is essential for the above: a nonspherical Coxeter diagram of rank $n \geqslant 4$ might not be of strongly infinite type (even if it is of infinite type, by Theorem 2.1.1). For instance, let Δ be the following diagram.



There is a finite $\tilde{2}$-simply connected geometry belonging to Δ (Stroth [1989]). Needless to say, this geometry is not a building.

For a discussion of these questions in the case of diagrams that are not of Coxeter type, we refer to 4.1.2.

For questions implying the finiteness of geometries, see Cameron [1992].

2.2. *Irreducible spherical diagrams*

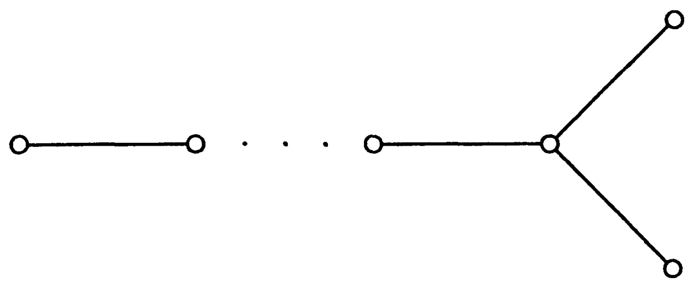
The list of irreducible spherical Coxeter diagrams includes infinitely many rank 2 cases, namely A_2, C_2, $\mathsf{I}_2(5)$, $\mathsf{I}_2(6), \ldots, \mathsf{I}_2(m), \ldots$. These diagrams represent generalized m-gons, which have been examined in Chapter 9. We will not insist on them here.

A great deal of information on buildings of type A_n, C_n, D_n, E_6, E_7, E_8 and F_4 is given in Chapter 11, Section 4, and in Chapter 12. We recall some facts.

Buildings of type A_n are n-dimensional (possibly reducible) projective geometries (a projective geometry is said to be *reducible* if some of its lines have just two joints). In particular, a thick building of type A_n, $n \geqslant 3$, is a PG(n, K), for some division ring K.

Buildings of type C_n are nondegenerate polar spaces of rank n (see Chapter 12, Section 4). Reading a linear diagram from left to right we can speak of a 'first' node, a 'last' node, etc. We say that a polar space is *thick-lined* if it is thick at the first node of the C_n diagram, i.e. if all of its lines are *thick* (they have at least 3 points). Thick-lined polar spaces of rank $n \geqslant 3$ have been classified by Tits [1974] (see also Chapter 12). In particular, the finite ones are obtained from a nondegenerate alternating or Hermitian form f or a nonsingular quadratic form g in a finite vector space V, taking as elements the nontrivial subspaces of V that are totally isotropic for f (totally singular for g, respectively). They are the rank n analogues of the classical finite generalized quadrangles (Chapter 9, Section 4) and the hyperbolic quadric $Q^+_{2n-1}(q)$, defined by a nonsingular quadratic form of Witt index n in Vect$(2n, q)$.

Buildings of type D_n are related to polar spaces of rank n that are thin at the last node of the C_n diagram. Indeed, if Γ is a building of type D_n, and if we form the O-shadow space of Γ (Chapter 3, 3.6), with O the 'initial' node of the D_n diagram



then we obtain a polar space of rank n i.e. thin at the last node of the C_n diagram (Chapter 12, Section 5). Conversely, if Γ is a polar space of rank $n \geqslant 4$ thin at the last node of the C_n diagram, then there is a (uniquely determined) building B of type D_n such that Γ is the O-shadow space of B, with O as above (Chapter 12, 5.12). We call B the *unfolding* of Γ.

The hyperbolic quadric $Q^+_{2n-1}(K)$, defined by a nonsingular quadratic form of Witt index n in Vect$(2n, K)$ with K a commutative field, is a polar space of rank n, thin at the last node of the diagram. Tits [1974] has proved that every thick building of type D_n is the unfolding of a hyperbolic quadric $Q^+_{2n-2}(K)$, for some commutative field K.

Thick buildings of type E_6, E_7, E_8 and F_4 have been classified by Tits [1974] (see also Chapter 12, Section 7). There are no thick buildings of type H_3 or H_4 (Tits [1977]; see also Chapter 11, Theorem 5.3.8).

Thus, all thick buildings of irreducible spherical type and rank $n \geqslant 3$ are known.

Thin buildings are precisely Coxeter complexes (Chapter 11, Section 2), which are uniquely determined by their Coxeter diagrams.

Scharlau [1987] has proved that every building Γ that is neither thick nor thin can be obtained by 'assembling' a number of copies of the chamber system of a thick building $\overline{\Gamma}$, uniquely determined by Γ but not belonging to the same diagram as Γ, possibly of rank less than that of Γ. We call $\overline{\Gamma}$ the *thick support* of Γ. It can be constructed as follows. Given two distinct chambers C, D of Γ, if they are the only chambers containing the flag $C \cap D$, then we say that they are *thin-adjacent*. Let $\approx$ denote the transitive closure of the relation 'being thin-adjacent'. We take the equivalence classes of $\approx$ as chambers of the building $\overline{\Gamma}$ we want to define. We say that two types i, j of Γ are *thin-equivalent* if there are chambers C_1, C_2, D_1, D_2 of Γ such that $C_1 \approx C_2$, $D_1 \approx D_2$, $C_1 \tilde{i} D_1$ and $C_2 \tilde{j} D_2$. The relation defined in this way is an equivalence relation on the set of types of Γ (see Scharlau [1987]). Its equivalence classes are taken as types for $\overline{\Gamma}$. Given a type $\bar{i}$ of $\overline{\Gamma}$ and two chambers $\overline{C}$, $\overline{D}$ of $\overline{\Gamma}$, we say that $\overline{C}$ and $\overline{D}$ are *$\bar{i}$-adjacent* if $C\tilde{i}D$ for some $i \in \bar{i}$, $C \in \overline{C}$ and $D \in \overline{D}$. Thus, we have defined a chamber system $\overline{\Gamma}$, which turns out to be a building (Scharlau [1987]).

For instance, if Γ is a thick-lined polar space thin at the last node of the C_n diagram, then its thick support is just its unfolding. If Γ is a building of type F_4 thick at the first two nodes of the F_4 diagram and thin at the last two nodes, then the thick support $\overline{\Gamma}$ of Γ is a thick building of type D_4 and Γ is the shadow space of $\overline{\Gamma}$ with respect to the central node of the D_4 diagram.

Every non-thick-lined polar space Γ can be decomposed as a 'product' of thick polar spaces and/or sets of pairwise noncollinear points (Buekenhout and Sprague [1982]). If Γ is not thin, then this decomposition is an instance of the construction by Scharlau. The thick support of Γ belongs to a reducible Coxeter diagram with irreducible components of type C_m and/or A_1. Every degenerate projective geometry Γ can be decomposed as a product of nondegenerate projective geometries and/or thick lines and/or points. If Γ is not thin, then we have a special case of Scharlau's construction; the thick support of Γ belongs to a reducible Coxeter diagram with irreducible components of type A_m.

Assume now that we want to classify a class of geometries belonging to an irreducible spherical Coxeter diagram Δ of rank $n \geqslant 3$. We can try the following strategy:

Step 1. Prove that all geometries of the class we are considering, are 2-quotients of buildings.

Step 2. Study the quotients of the buildings of type Δ and check which of them belong to that class.

The next theorem is the main tool for the first step of the above program.

2.2.1. THEOREM (Tits [1981a]). *Let Δ be a Coxeter diagram of rank $n \geqslant 3$, let Γ be a geometry belonging to Δ and let $\widetilde{C}$ be the universal* 2*-cover of* (*the chamber system $C(\Gamma)$ of*) *Γ.*

If Δ does not contain any induced subdiagram of type C_3 or H_3, then $\widetilde{C}$ is (*the chamber system of*) *a building. If Δ contains C_3 or H_3 as induced subdiagrams, then $\widetilde{C}$ is* (*the chamber system of*) *a building if and only if every rank* 3 *residue of Γ of type C_3 or H_3 is* 2*-covered by a building.*

See Chapter 11, Theorem 6.3.1. The next corollary is a trivial consequence of the above.

2.2.2. COROLLARY. *All geometries of type A_n ($n \geqslant 3$), D_n, E_6, E_7 and E_8 are* 2*-quotients of buildings. A geometry of type C_n ($n \geqslant 4$) or F_4 is a* 2*-quotient of a building if and only if all residues of Γ of type C_3 are* 2*-quotients of buildings.*

The reader is referred to 3.7 of Chapter 3 for the statement of the Intersection Property (IP). We recall that (IP) holds in every building (see 6.4.3 of Chapter 11).

2.2.3. COROLLARY.

(i) *All geometries of type A_n are* (*possibly reducible*) *projective geometries.*

(ii) *All geometries of type* C_n *satisfying* (IP) *are polar spaces.*
(iii) *All geometries of type* D_n *are buildings.*
(iv) *All geometries of type* E_6 *are buildings.*
(v) *All geometries of type* E_7 *and* E_8 *that satisfy* (IP) *are buildings.*
(vi) *All geometries of type* F_4 *satisfying* (IP) *are buildings.*

The above have been obtained by Tits [1981a], Propositions 6 and 9, exploiting covers (Corollary 2.2.2) and elementary properties of buildings of spherical type. However, Tits assumed some weak version of (IP) in (iii) and (iv) of Corollary 2.2.3. An elementary proof that (IP) holds in every D_n geometry has later been found (Timmesfeld [1983], Lemma 3.2; see also Pasini [1994d], Chapter 7; Timmesfeld attributed this result to Meixner). Thus, (iii) holds. Exploiting (iii) it is easy to obtain (iv) using the same argument as in Tits [1981a]; indeed Tits needed some kind of intersection property for E_6 only to make it sure that D_5 residues were buildings.

The statement (i) can also be obtained with no use of coverings, as a corollary of the characterization of n-dimensional matroids as geometries belonging to the diagram (L_n) (Chapter 6, 6.1.5; Section 5.2 of the present chapter).

It is also possible to prove (ii) in a straightforward way, with no use of coverings, generalizing to higher rank cases the argument used by Tits [1981a] for C_3 (see Chapter 7 of Pasini [1993], Section 7.4).

Once (ii) has been stated, (iii) follows easily from it. Indeed, let B be a D_n geometry. As we have previously remarked, (IP) holds in B (we only need (i) to prove this and (i) can be obtained without using coverings, as we have remarked earlier). Therefore, if Γ is the shadow space of B with respect to the initial node of the D_n diagram, (IP) holds in Γ. Hence Γ is a polar space by (ii) and B is the unfolding of Γ, whence a building.

Another elementary (but not easy) proof of (iii) which does not even exploit (ii) has been found by Huybrechts [1992].

By (ii) and using the axioms given in 10.13 of Tits [1974] for buildings of type F_4 it is also possible to prove (vi) straightforwardly, with no use of coverings.

Thick buildings of type E_6, E_7 and E_8 can be characterized in a number of ways as point-line systems satisfying certain axioms (Cohen and Cooperstein [1983]; see also Chapter 12). Exploiting (i) and (iii) it is not difficult to prove that every thick E_6 geometry, and every thick geometry of type E_7 or E_8 with (IP), satisfies the respective sets of axioms. However, this would not yet be a 'covering-free' proof of (iv) and (v) in the thick case. Indeed Cohen and Cooperstein [1983] exploit the statements of (iv) and (v) (proved via coverings) to show that the geometries characterized by their axioms are indeed buildings of type E_6, E_7 and E_8. Moreover, they need to work with thick lines, thus leaving nonthick cases (in particular, Coxeter complexes) out of their theory. Because of this, they cannot provide a straightforward way to recover apartments. Hence they need (iv) and (v) to finish. Thus, (iv) and (v) are the only two statements for which an elementary proof is not known, namely a proof avoiding coverings.

2.2.4. THEOREM (Brouwer and Cohen [1983]).
(i) *All finite geometries of type* E_7 *and* E_8 *are buildings.*

(ii) *Let Γ be a finite C_n geometry thick at the first node of the C_n diagram or a finite thick F_4 geometry. Then either Γ is a building or the universal 2-cover of Γ is not a building.*

This theorem has been obtained by Brouwer and Cohen as a consequence of a lemma on orbits of automorphisms of finite graphs satisfying certain regularity conditions, which generalizes a classical result on polarities of finite projective planes (Dembowski [1968], 3.3.1).

The finiteness and thickness hypothesis are essential in the above. Indeed, it is not difficult to construct proper 2-coverings $f\colon \widetilde{\Gamma} \to \Gamma$ with $\widetilde{\Gamma}$ a nonthick or infinite geometry of type E_7, E_8 or F_4 or a non-thick-lined or infinite polar space. The reader can find examples of this kind in Tits [1981a] (1.4, 1.6), Brouwer and Cohen [1983] (Section 4), Rees [1985] (2.2i), Pica [1991] (Section 5); also Pasini [1994d] (Chapter 8 and Section 11.1 of Chapter 11).

2.2.5. THEOREM (Rees [1987]). *Let Γ be a C_n geometry thin at the last node of the C_n diagram. Then the universal 2-cover of Γ is a polar space.*

Rees proved this theorem in the case of $n = 3$, showing that a C_3 geometry with the above property is 2-simply connected only if it can be unfolded to an A_3 geometry; i.e. only if it is the shadow space of a 3-dimensional projective geometry with respect to the central node of the A_3 diagram (by Corollary 2.2.3(i)). The statement in the general case follows from the statement in the rank 3 case and from Theorem 2.2.1.

By Theorems 2.2.5 and 2.2.4 we obtain the following result.

2.2.6. COROLLARY. *Let Γ be a finite C_n geometry thick at the first node and thin at the last node of the C_n diagram. Then Γ is a polar space.*

By Theorem 2.2.5, all thin C_3 geometries are 2-quotients of the C_3 Coxeter complex, namely of the octahedron. Actually, there is no need of Theorem 2.2.5 for this. Indeed, it is very easy to prove that there is just one thin C_3 geometry besides the octahedron, obtained identifying antipodal elements of the octahedron, called hemioctahedron. Similarly, there are just two thin geometries of type H_3: the dodecahedron and its 2-quotient obtained identifying antipodal elements, called hemidodecahedron (Pasini [1994d], Lemma 13.23). Therefore by Theorem 2.2.1 we have the following.

2.2.7. THEOREM. *Every thin geometry belonging to a Coxeter diagram of rank $n \geqslant 3$ is a 2-quotient of a Coxeter complex.*

2.2.8. *Résumé.* Let us summarize what we have seen till now: all geometries of type A_n, D_n and E_6 are buildings (Corollary 2.2.3).

All geometries of type E_7 and E_8 are 2-quotients of buildings (Corollary 2.2.2). They are buildings if and only if they satisfy (IP) (Corollary 2.2.3). All finite thick geometries of type E_7 and E_8 are buildings (Theorem 2.2.4).

A C_n geometry is a building (in this case, a nondegenerate polar space of rank n) if and only if it satisfies (IP) (Corollary 2.2.3). All C_n geometries thin at the last node of the C_n diagram are 2-quotients of polar spaces (Theorem 2.2.5). All finite C_n geometries that are thick at the first node and thin at the last node of the C_n diagram are polar spaces (Corollary 2.2.6).

A geometry of type F_4 is a building if and only if (IP) holds in it (Corollary 2.2.3). All thin F_4 geometries are 2-quotients of the F_4 Coxeter complex (Theorem 2.2.7).

All thin geometries of type H_3 and H_4 are 2-quotients of the Coxeter complexes of those types (Theorem 2.2.7). The universal 2-cover of a thick geometry of type H_3 or H_4 is never a building, because there are no thick buildings of those types (Chapter 11, Theorem 5.3.8). Anyhow, all geometries of type H_3 or H_4 admitting finite orders are thin, as the ordinary pentagon is the only finite generalized 5-gon admitting orders (see Chapter 9, 2.2).

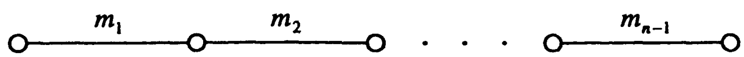
2.2.9. *The Main Gap:* C_3 *and* H_3. If all C_3 and H_3 geometries were 2-quotients of buildings, then every geometry of Coxeter type would be a 2-quotient of a building (Theorem 2.2.8). Unfortunately, things do not go so easily: there are 2-simply connected geometries of type C_3 and H_3 that are not buildings. Some examples of this kind can be obtained by free constructions (Tits [1981a], 1.6), hence they are infinite. Other examples are not thick (see Pasini [1986b] for some C_3 examples). A thick finite C_3 example also exists, namely the Alt(7)-*geometry*, described in 3.1.9 of Chapter 11. Let us recall the construction and some properties of that geometry. It is well known that there are precisely 30 ways of drawing a model of PG(2, 2) on a given set S of 7 points. The alternating group Alt(7), viewed as a group of permutations of S, has two orbits of size 15 on that set of 30 copies of PG(2, 2). We take one of those orbits as set of 'planes', S as set of 'points' and the 3-subsets of S as 'lines', stating that all 'points' are incident with all 'planes', a 'line' x and a 'plane' u are incident if x is one of the 7 lines of the model u of PG(2, 2), and a point a and a line x are incident if $a \in x$. The geometry defined in this way is flag-transitive with Alt(7) as automorphism group (it is called the Alt(7)-geometry because of this reason). It is 2-simply connected (see 6.4 of Chapter 12 of Pasini [1994d] for a straightforward proof of this claim). Residues of 'planes' are isomorphic to PG(2, 2) and residues of 'points' are isomorphic to the classical generalized quadrangle $W(2)$ (notation as in Chapter 9).

2.3. *The locally classical case*

We say that a thick locally finite geometry Γ belonging to a Coxeter diagram is *locally classical* if all rank 2 residues of Γ other than generalized digons are (finite thick) classical generalized polygons (Chapter 9, Section 3), with the convention that the dual of a classical generalized polygon also is classical (this is not the case in Chapter 9). Note that, on the other hand, $Q_3^+(q)$ is not allowed as a rank 2 residue in a locally classical geometry, since it is not thick. Compare also with 1.5.

2.3.1. THEOREM (Aschbacher [1984]). *The* Alt(7)-*geometry is the only flag-transitive locally classical* C_3 *geometry that is not a polar space.*

2.3.2. THEOREM (Aschbacher [1984]). *Let Γ be a flag-transitive locally classical geometry belonging to a Coxeter diagram of string shape and rank $n \geqslant 4$:*



($m_i = 3, 4, 6$ or 8 for $i = 1, 2, \ldots, n-1$). Then no rank 3 residue of Γ is isomorphic to the Alt(7)*-geometry.*

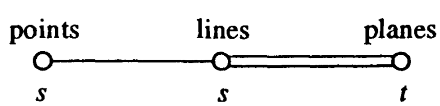
Assembling the previous two theorems with Corollaries 2.2.2 and 2.2.3 and with Theorem 2.2.4 we obtain the following.

2.3.3. COROLLARY. *Let Γ be a flag-transitive locally classical geometry belonging to a diagram of irreducible spherical type and rank $n \geqslant 3$. Then Γ is either a building or the* Alt(7)*-geometry.*

Thus, the existence of a finite thick 2-simply connected C_3 geometry that is not a building is not at all an obstacle to a classification of geometries of irreducible spherical type and rank $n \geqslant 3$, if we restrict our interest to the flag-transitive locally classical case. The restriction to locally classical geometries may be questioned. Of course, it is unlikely that unknown finite C_3 geometries give new finite projective planes or new finite generalized quadrangles. Anyway, this remains an open problem. Aschbacher needed to restrict himself to the locally classical case in Theorems 2.3.1 and 2.3.2 in order to exploit a result in Seitz [1973] (see 3.2.3) on flag-transitive automorphism groups of finite classical generalized polygons. However, some generalizations of Theorem 2.3.1 and Corollary 2.3.3 can be obtained avoiding the result of Seitz, as we will see in the next two sections.

2.4. *Finite thick C_3 geometries*

In this subsection Γ is a finite thick C_3 geometry. All thick generalized polygons admit orders (see, e.g., Chapter 3 of Pasini [1994d]). Therefore, Γ admits the diagram and orders s, t with $1 < s,\ t < \infty$. We call the elements of Γ points, lines and planes, as follows:



We say that Γ is *flat* if all points of Γ are incident with all planes (note that the Alt(7)-geometry is flat). If Γ is neither a polar space nor flat, then we say that it is *anomalous*. The reader can guess the motivation of the latter definition: we conjecture that every finite thick C_3 geometry is either a polar space or flat (and that the Alt(7)-geometry is the only flat finite thick C_3 geometry); thus, the word 'anomalous' suggests nonexistence.

The next theorem assembles results by Pasini [1993] and Lunardon and Pasini [1989].

2.4.1. THEOREM. *Let Γ be flag-transitive. Then Γ is one of the following*:

(i) *a polar space*;
(ii) *the* Alt(7)-*geometry*;
(iii) *an anomalous* C_3 *geometry with the following properties*:
(iii.a) *residues of planes of* Γ *are (unknown) non-Desarguesian flag-transitive finite projective planes of even order* s, *with* $s+1 \equiv 0 \pmod 3$ *and* s^2+s+1 *prime*; *the stabilizer in* $\mathrm{Aut}(\Gamma)$ *of a plane* u *of* Γ *acts in the residue of* u *as a Frobenius group of order* $(s+1)(s^2+s+1)$;
(iii.b) $d^2 < s < t$ *and* $(s-1)d^2 + d \leqslant t \leqslant s^2 - s$, *with* $d = (s^2, t)$; *hence the residues of the points of* Γ *cannot be isomorphic to any of the known finite generalized quadrangles.*

In the anomalous case (iii), the orders s, t must actually satisfy more conditions besides those stated in (iii.b). Those additional conditions involve a third number, called the Ott–Liebler number of Γ (see Pasini [1993]; also Lunardon and Pasini [1990] or Chapter 14 of Pasini [1994d]). More restrictions can be obtained in the anomalous case (iii) assuming the primitivity of the action of $\mathrm{Aut}(\Gamma)$ on the set of points of Γ (see Pasini [1993] and Lunardon and Pasini [1990]). Theorem 2.3.1 has been obtained by Aschbacher exploiting the theorem of Seitz, Theorem 2.4.1 depends on a theorem by Kantor [1987a] on flag-transitive finite projective planes: see 1.7.2, 2. So much for the flag-transitive case. Henceforth we do not assume flag-transitivity but we make assumptions on the orders s and t.

2.4.2. *Orders of known type.* We say that the orders s, t of Γ are of *known type* if they satisfy one of the relations holding between orders of known finite thick generalized quadrangles (see Chapter 9); i.e. if one of the following holds:
(1) $s = t$,
(2) $t = s^2$,
(3) $s = t^2$,
(4) $t^2 = s^3$,
(5) $s^2 = t^3$,
(6) $t = s + 2$ (and $s > 2$),
(7) $s = t + 2$ (and $t > 2$).

2.4.3. THEOREM (Pasini [1986b]). *Let* Γ *admit orders of known type. Then* Γ *is one of the following*:
(i) *a polar space (orders as in* (1), (2), (3) *or* (4));
(ii) *a flat geometry with orders as in* (1) *or* (4);
(iii) *an anomalous geometry with orders as in* (3) *and with* $(t^4-t^3+t^2-t+1)(t^2+t+1)$ *points.*

This theorem depends on the theory of irreducible representations of Hecke algebras of finite geometries (Ott [1981, 1985], Liebler [1985]; see also Lunardon and Pasini [1990]). An 'ancestor' of Theorem 2.4.3 is an earlier result by Ott [1985]; he considered case (1), proving that Γ is either a polar space or flat in that case.

We remark that the Hermitian variety $H_4(q^2)$ is the only known finite generalized quadrangle with orders as in (4) (see Chapter 9). However, if Γ is flat with orders as in (4), then the residues of the points of Γ cannot be isomorphic to $H_4(q^2)$. Indeed, if Γ is flat, then, given two distinct points a, b of Γ, there are $st + 1$ lines incident with both a and b and they form an ovoid in the residue of a. However, no ovoid exists in $H_4(q^2)$ (Chapter 9, Section 6).

Note that cases (5), (6) and (7) do never occur.

2.5. *Finite thick C_n and F_4 geometries*

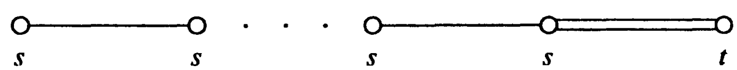
From 2.5.1 to 2.5.5, Γ is a finite thick geometry of type C_n, of rank $n \geqslant 4$.

2.5.1. LEMMA (Pasini [1987]). *If none of the C_3 residues of Γ is anomalous, then all C_3 residues of Γ are 2-quotients of polar spaces.*

By this lemma and by Corollary 2.2.2 and Theorem 2.2.4(ii) we obtain the following.

2.5.2. LEMMA. *If none of the C_3 residues of Γ is anomalous, then Γ is a polar space.*

2.5.3. *Orders.* As in the rank 3 case, Γ admits orders s, t, with $1 < s, t < \infty$:



The elements of Γ of the last type will be called *maximal subspaces* or *max* for short. If u is a max of Γ, the residue Γ_u of u is an $(n-1)$-dimensional projective geometry of order s (Corollary 2.2.3(i)). As $n \geqslant 4$, we have $s = p^m$ for some prime p and some positive integer m and $\Gamma_u = \mathrm{PG}(n-1, p^m)$. In particular, residues of Γ of type A_2 are Desarguesian projective planes.

We call p the *characteristic* of Γ and we say that the orders s, t of Γ are *feasible* if t also is a power of p. Note that, by Theorem 2.4.3, the orders s, t of Γ are feasible if they are of known type.

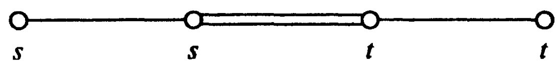
2.5.4. THEOREM (Lunardon and Pasini [1990]). *Let s, t be feasible. Then Γ is a polar space.*

Let us recall the main steps of the proof. First, we prove that, if s, t are feasible, then they are of known type. This has been done in Section 4 of Lunardon and Pasini [1990]. In an earlier paper Pasini [1990a] it has been proved that the anomalous case (iii) of Theorem 2.4.3 cannot occur as a C_3 residue inside a higher rank C_n geometry. Therefore every C_3 residue of Γ is either a polar space or flat, by Theorem 2.4.3. The conclusion follows from Lemma 2.5.2.

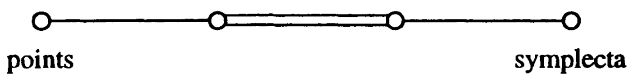
2.5.5. THEOREM (Pasini [1988]). *Let Γ be flag-transitive. Then Γ is a polar space.*

This theorem can be proved in a few lines. Let Γ be flag-transitive. By a well-known theorem of Higman [1962], the stabilizer G_u in Aut(Γ) of a max u of Γ acts as a classical group in $\Gamma_u = \mathrm{PG}(n-1, p^m)$, except possibly when $n = 3$ and $s = 2$; in this case, G_u may act as Alt(7) in Γ_u. In any case, the stabilizer in Aut(Γ) of a residue of Γ of type A_2 acts as a classical group in that residue. Therefore the anomalous case (iii) of Theorem 2.4.1 never occurs for C_3 residues of Γ. Hence Γ is a polar space, by Lemma 2.5.2.

2.5.6. F_4 *geometries.* Henceforth Γ is a finite thick geometry of type F_4. As in the C_n case, Γ admits orders s, t, with $1 < s, t < \infty$:



We assume that the diagram is oriented in such a way that $s \leqslant t$. The elements of Γ corresponding to the first and the last node of the diagram will be called *points* and *symplecta* (or *symps*), respectively:



We begin with a lemma on the case of uniform order.

2.5.7. LEMMA (Pasini [1986c]). *Let $s = t$. Then there are some points and some symplecta whose residues are polar spaces.*

We can now prove the following.

2.5.8. THEOREM (Pasini [1988]). *Let Γ be flag-transitive. Then Γ is a building.*

The proof is quite easy. If some C_3 residues are anomalous as in (iii) of Theorem 2.4.1, then $s < t$ (as we have assumed $s \leqslant t$) and anomalous residues can occur only as residues of symps. Therefore, residues of points are either polar spaces or isomorphic to the Alt(7)-geometry (Theorem 2.4.1). Hence s and t are powers of the same prime and (iii.b) of Theorem 2.4.1 cannot hold. Therefore, either all C_3 residues are polar spaces, hence Γ is a building by Corollary 2.2.2 and by Theorem 2.2.4(ii), or we have $s = t = 2$ and residues of points or of symps are isomorphic to the Alt(7)-geometry. However, the latter case is impossible by Lemma 2.5.7 and because Γ is flag-transitive. Therefore Γ is a building. Assembling Theorems 2.5.5, 2.5.8, 2.4.1, 2.2.4 and Corollary 2.2.3, we obtain the following final result.

2.5.9. COROLLARY. *Let Γ be a flag-transitive finite thick geometry belonging to an irreducible spherical Coxeter diagram of rank $\geqslant 3$. Then Γ is one of the following*:

(i) *a building*;

(ii) *the* Alt(7)-*geometry*;

(iii) *an (unknown) anomalous* C_3 *geometry as in* (iii) *of Theorem* 2.4.1.

3. Geometries over Coxeter diagrams of nonspherical type

3.1. *Hyperbolic and other diagrams*

3.1.1. *Why hyperbolic diagrams?* All thin geometries of rank $\geqslant 3$ belonging to a Coxeter diagram are 2-quotients of Coxeter complexes (Theorem 2.2.7). On the other hand, all buildings (hence all Coxeter complexes) of rank $\geqslant 3$ are 2-simply connected (Chapter 11, Proposition 3.3.11); i.e. they are their own universal 2-covers. Therefore, Coxeter complexes of rank $\geqslant 3$ are precisely 2-simply connected thin geometries belonging to Coxeter diagrams. Furthermore, residues of nonmaximal flags of buildings are still buildings (Chapter 11, Proposition 3.2.9(b)). Hence buildings (in particular, Coxeter complexes) are residually simply connected in the meaning of 3.13 in Chapter 3. On the other hand, the chamber system of a thin geometry of rank n can be realized as an n-dimensional numbered simplicial complex (Chapter 11, Sections 1 and 2.2), and an m-covering $f\colon \Gamma \to \Gamma'$ of thin geometries of rank $m+1$ (7.8 of Chapter 3 or 6.1 of Chapter 11) is a covering of the corresponding simplicial complexes in the usual topological meaning: the restrictions of f to simplices and to stars of vertices are isomorphisms. As a consequence of the above, the simplicial realization of the chamber system of a Coxeter complex is simply connected in the usual topological sense.

A Coxeter complex is finite if and only if its Coxeter diagram is of spherical type (Chapter 11, Section 2; see also 2.1.2 of this chapter). The simplicial realization of the chamber system of a Coxeter complex belonging to a spherical diagram of rank n is the $(n-1)$-dimensional sphere in $\mathbb{R}^n$ (Chapter 11, Section 2), which is the only compact, simply connected $(n-1)$-dimensional topological manifold, all of whose points are simple, and with empty border (note that a topological translation of 'finite' is 'compact'). Thus, Theorem 2.2.7 could be rephrased as follows: the chamber system of a thin geometry belonging to a Coxeter diagram of rank n and spherical type is a tessellation of a homomorphic image of the real $(n-1)$-dimensional sphere. The above also explains why the word 'spherical' has been chosen for these Coxeter diagrams.

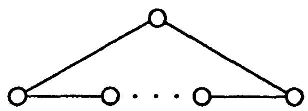
REMARK. It should be clear now that the antique classification of Platonic polyhedra is nothing but a special case of Theorem 2.1.2.

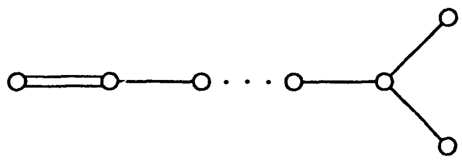
According to the above topological point of view, the Coxeter diagrams to examine just after the spherical ones should be those with a chamber system for the corresponding Coxeter complex, i.e. realizable as tessellation of a real noncompact simply connected topological manifold S such that every point of S has a neighbourhood in S isomorphic to $\mathbb{R}^{n-1}$, where n is the rank of the diagram (note that the m-dimensional real sphere is locally $\mathbb{R}^m$ and that a tessellation of the m-dimensional real sphere can also be viewed as a partition of $\mathbb{R}^{m+1}$ by simplicial cones). Hence, we should consider diagrams such that for every node i, the diagram induced on the set of nodes other than i is spherical. Coxeter diagrams of this kind are called *hyperbolic diagrams*.

By 8.2 of Chapter 3, we can focus on connected hyperbolic diagrams.

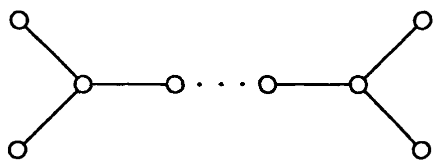
3.1.2. *Affine diagrams.* An important family of connected hyperbolic diagrams is the family of affine diagrams. An *affine diagram* of rank n is a connected hyperbolic diagram such that the chamber system of the corresponding Coxeter complex is a tessellation of the $(n-1)$-dimensional real Euclidean space $\mathbb{R}^{n-1}$. Here is the list of affine diagrams, with their names (Bourbaki [1968], Chapter 6, 4.3, Theorem 4):

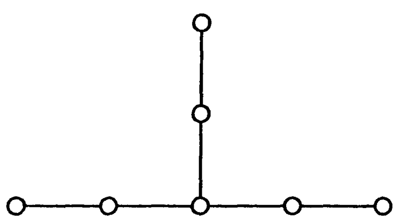
$\tilde{A}_1$ 

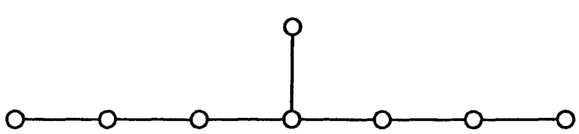
$\tilde{A}_m$  (rank $n=m+1 \geqslant 3$)

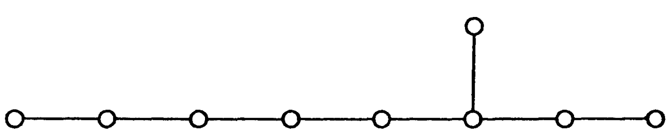
$\tilde{B}_m$  (rank $n=m+1 \geqslant 4$)

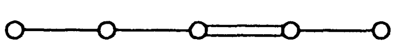
$\tilde{C}_m$  (rank $n=m+1 \geqslant 3$)

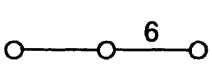
$\tilde{D}_m$  (rank $n=m+1 \geqslant 5$)

$\tilde{E}_6$ 

$\tilde{E}_7$ 

$\tilde{E}_8$ 

$\tilde{F}_4$ 

$\tilde{G}_2$ 

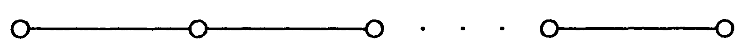
Another reason to focus on affine diagrams is the following: thick buildings belonging to affine diagrams of rank $n \geqslant 4$ have been classified (by Tits [1986]; see also

Ronan [1989]); all of them arise from suitable BN-pairs in groups defined over local fields. An essential observation and tool is that a building of affine type gives rise to a 'building at infinity' which is of spherical type.

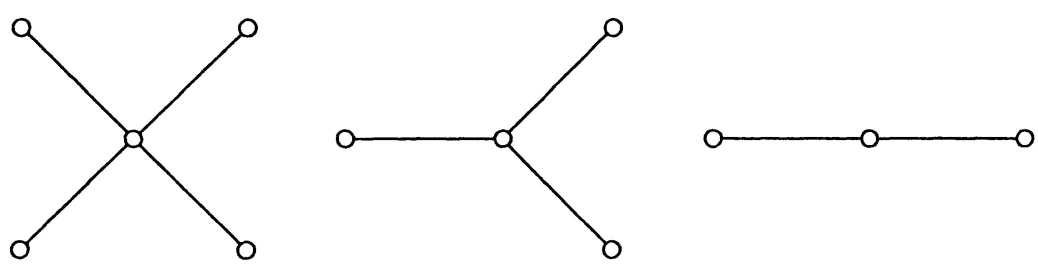
3.1.3. *Nice finite geometries over a Coxeter diagram.* One might expect that most nice finite geometries belonging to nonspherical Coxeter diagrams actually belong to hyperbolic (in particular, affine) diagrams. However, the real situation is quite different. Indeed, assume that we understand 'nice' as 'flag-transitive and locally classical'. Then nonspherical Coxeter diagrams of rank $\geqslant 3$ for which nice finite geometries can exist have shapes that we will now describe before stating the results in the next section.

3.1.4. *Timmesfeld diagrams.* This section is entirely at the Δ-level.

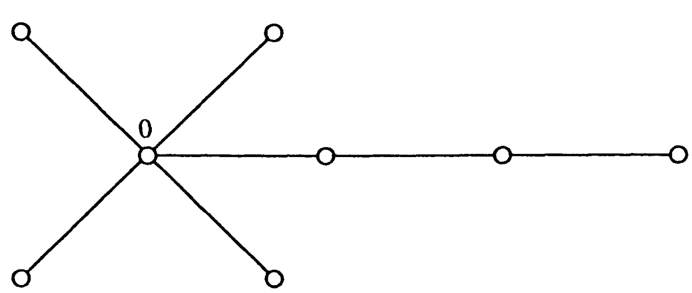
As the Coxeter diagrams we are dealing with should represent locally classical locally finite geometries, all their edges are labelled by 3, 4, 6 or 8 (see Chapter 9). Dropping the labels of the edges, of a Coxeter diagram we just obtain the underlying basic diagram (Chapter 3, 3.3), which is indeed a simple graph. Thus, we can use for Coxeter diagrams the terminology used for graphs, speaking of bipartite graphs, complete bipartite graphs, complete graphs, cycles etc. It is also useful to state the following conventions. By a *string* of *length* ℓ we mean a graph as follows, with $\ell + 1$ vertices:



A complete bipartite graph with at least 3 vertices where one class of the bipartition consists of only one vertex, say 0, will be called a *star* of *centre* 0. The following are examples of stars with 5, 4 and 3 vertices, respectively:



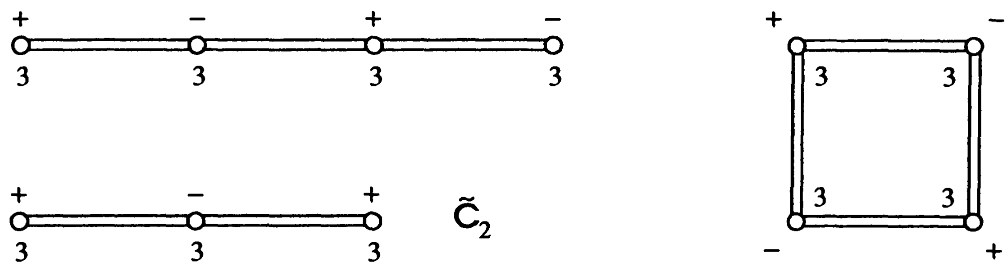
By a *comet* we mean a graph obtained by attaching a string of length $\geqslant 1$ to the centre of a star. The vertices of the star different from the centre are said to form the *hair* of the comet, whereas the string attached to the centre will be called the *tail* of the comet. Here is an example with 4 vertices in the hair and a tail of length 3 (0 denotes the centre of the star).



Note that a star with $n \geqslant 4$ vertices can be viewed as a comet in $n-1$ different ways, selecting anyone of its rays as its tail. If a diagram Δ is a star, we prefer to call it a comet when all of its edges but one have the same label, say m, the tail being the unique edge with label $\neq m$.

We can now describe the diagrams we will meet in the next subsection. We call them *Timmesfeld diagrams.*

(1) Connected bipartite graphs with at least 3 vertices, all edges labelled by 4, and order 3 at every node. Here are three examples.

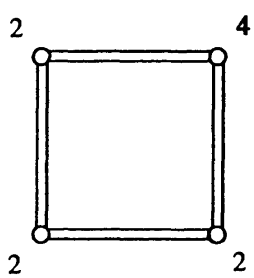


The signs + and − are used to denote the two classes of the bipartition. This family of diagrams contains just one diagram of hyperbolic type, namely the affine diagram $\tilde{C}_2$.

(2) Complete bipartite graphs all of whose edges have label 4, with at least two nodes in each of the two classes X, Y of the bipartition, and orders as in one of the following two cases:

(a) all nodes in X are labelled by the same order, say s, all nodes in Y are labelled by the same order, say t and, assuming $s \leqslant t$, we have $(s,t) = (2,2)$, (2,4) or (4,8);

(b) all nodes in X are labelled by 2 and all nodes in Y are labelled by 2 and 4. Each of the orders 2 and 4 occurs at least once in Y. The following is an example of this situation.



Note that none of the diagrams of this family is hyperbolic.

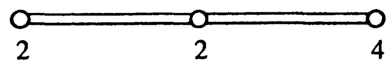
(3) Complete bipartite graphs all of whose edges have label 6, with at least two nodes in each of the two classes X, Y of the bipartition, all nodes in X being labelled by the same order, say s, all nodes in Y being labelled by the same order, say t, and, assuming $s \leqslant t$, we have $(s,t) = (2,2)$, (2,8) or (3,3). None of these diagrams is hyperbolic.

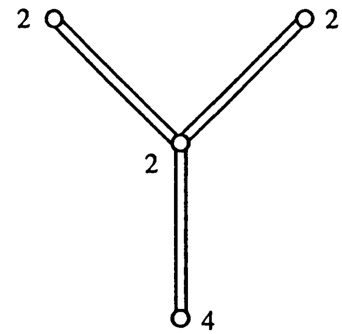
(4) Complete bipartite graphs where all edges have label 8, there are at least two nodes in each of the two classes X, Y of the bipartition, all nodes in X are labelled by 2 and all nodes in Y are labelled by 4.

(5) Stars with all edges labelled by 4, and orders as in one of the following two cases

(a) the centre of the star is labelled by an order t and all other nodes are labelled by the same order s, with $(s,t) = (2,2)$, (2,4), (4,2), (4,8), (8,4), (3,3), (3,9), (9,3) or (7,7);

(b) the star has order 2 at the centre and orders 2 and 4 at the remaining nodes. Each order 2 and 4 occurs at least once on the remaining nodes. Here are two examples.





Note that the affine diagram $\tilde{C}_2$ is the only hyperbolic diagram in this family.

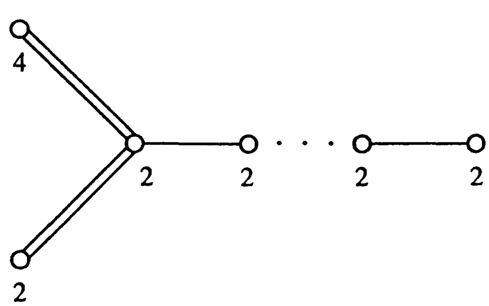
(6) Stars with all edges labelled by 6, the centre labelled by an order t and all remaining nodes labelled by the same order s, with $(s,t) = (2,2)$, (2,8), (3,3) or (3,27). None of these diagrams is hyperbolic.

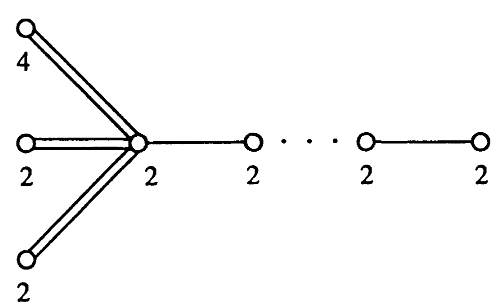
(7) Stars with all edges labelled by 8, the centre labelled by an order t and all remaining nodes labelled by the same order s, with $\{s,t\} = \{2,4\}$. None of these diagrams is hyperbolic.

(8) Comets, with all edges in the tail labelled by 3 and all remaining edges labelled by 4, and orders as in one of the following two cases:

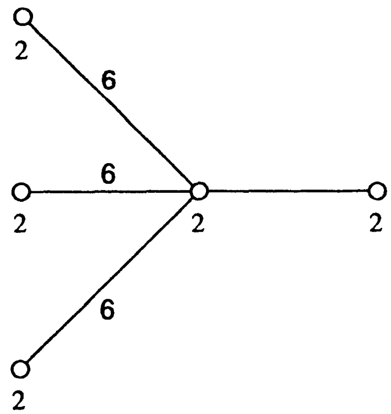
(a) all nodes in the hair are labelled by the same order, say s, and all other nodes are labelled by the same order, say t, with $(s,t) = (2,2)$, (2,4), (4,2), (8,4), (3,3), (3,9), (9,3) or (7,7);

(b) the nodes in the hair are labelled by 2 or 4 and each of these values occurs at least once, as in the following examples:

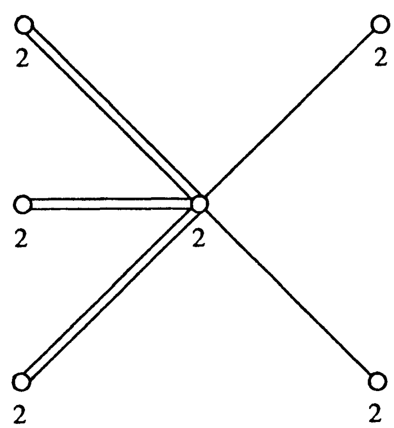


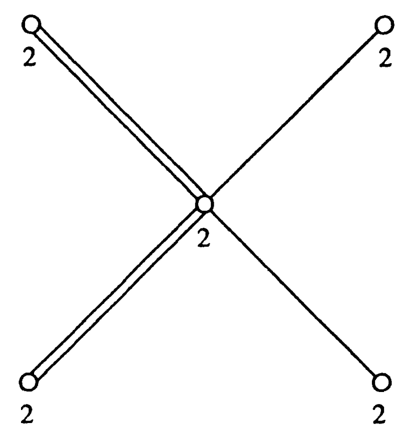


(9) Comets, with tail of length 1, the edge of the tail labelled by 3, all other edges labelled by 6, order 2 at every node. None of these diagrams is hyperbolic. The following is an example of rank 5.



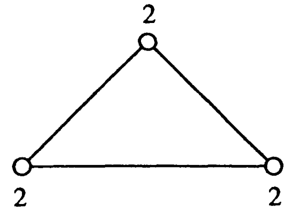
(10) Diagrams obtained attaching two edges labelled by 3 to the centre of a star with all edges labelled by 4. Order 2 at every node. Here are two examples, of rank 6 and 5 respectively.

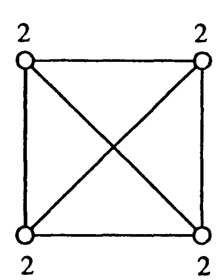




None of these diagrams is hyperbolic.

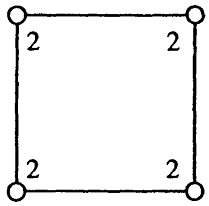
(11) Complete graphs with at least 3 vertices, all edges labelled by 3 and all nodes labelled by the same order $s = 2$ or 8. Here are two examples of rank 3 and 4, respectively, with $s = 2$.



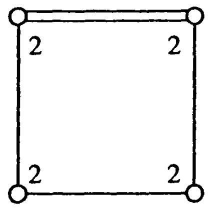


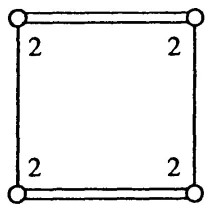
The affine diagram $\widetilde{A}_2$ is the only hyperbolic diagram in this family.

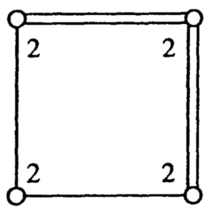
(12) The affine diagram $\widetilde{A}_3$, with order 2 at every node.

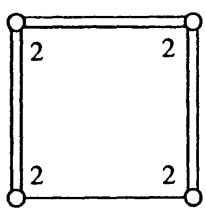


(13) The following cycles of length 4, with order 2 at every node.

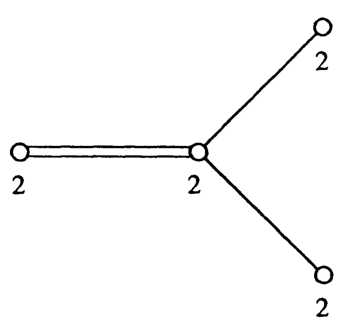
(13.a)  (hyperbolic but not affine)

(13.b)  (hyperbolic but not affine)

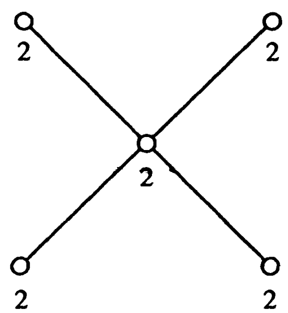
(13.c)  (non-hyperbolic)

(13.d)  (non-hyperbolic)

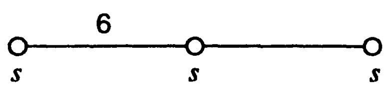
(14) The affine diagram $\widetilde{B}_3$ with order 2 at every node.



(15) The affine diagram $\widetilde{D}_3$ with order 2 at every node.



(16) The affine diagram $\tilde{G}_2$



with $s = 2$ or 5.

3.2. *Flag-transitive locally classical finite geometries belonging to Coxeter diagrams*

In this section Γ is a flag-transitive locally classical finite geometry belonging to a Coxeter diagram (locally classical geometries have been defined in 2.3). We are interested in a classification of these geometries. By 2.1.3 we can assume that the diagram of Γ is connected. If Γ has rank 2, than it is a classical generalized m-gon ($m = 3,\ 4,\ 6$ or 8), because Γ is locally classical. All finite classical generalized m-gons are known (see Chapter 9). Thus, we assume that Γ has rank $n \geqslant 3$.

In the next theorem we collect contributions by Timmesfeld [1983, 1984a,b, 1985, 1986, 1987, 1989], Stroth [1986, 1987, 1988a,b,c, 1990], Heiss [1989, 1990] and Meixner [1986a, 1990a,b, 1994e].

For a thorough survey of this matter, in particular, a list of examples, we refer to Meixner [1990b].

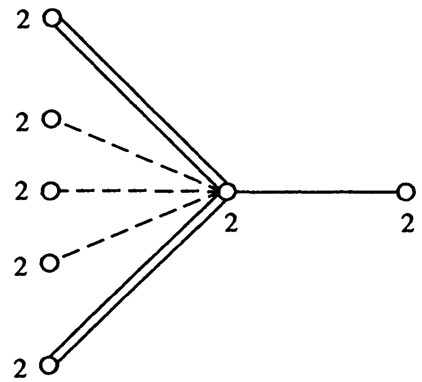
3.2.1. THEOREM. *Let Γ be as above. Then Γ is one of the following*:

(i) *a finite building*;

(ii) *the* Alt(7)-*geometry*;

(iii) *a geometry with diagram and orders as in one of* (1)–(16) *of the previous section* (*a geometry over a Timmesfeld diagram*).

More information can be given on case (iii) of the above theorem. Let Γ be as in that case. By Theorems 2.3.1, 2.3.2 and 2.2.1, the universal 2-cover of Γ is a (necessarily infinite) building, except possibly in cases (8), (10), (13) and (14) of 3.1.4, with a tail of length 1 and order 2 at every type in case (8):



Indeed these are the only cases where the Alt(7)-geometry can occur as a C_3 residue. Stroth has examined the possibility of C_3 residues isomorphic to the Alt(7)-geometry in Stroth [1986, 1987, 1988b], obtaining the following.

3.2.2. THEOREM. *In each of the diagrams* (13.a)–(13.d) *of case* (13) *all* C_3 *residues are isomorphic to the* Alt(7)*-geometry* (*hence the universal* 2 *cover of* Γ *is not a building*).

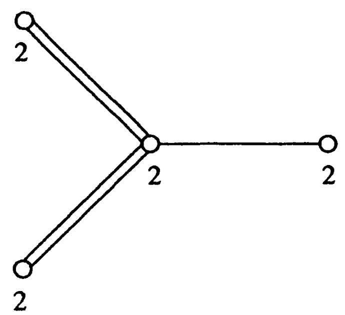
In case (8), *at most one of the* C_3 *residues may be isomorphic to the* Alt(7)*-geometry.*

In cases (10) *and* (14) *all* C_3 *residues are polar spaces* (*hence* Γ *is a* 2*-quotient of a building*).

In cases (13.a), (13.b) and (13.c) we can say more. If Γ belongs to (13.a) or (13.b), then it is uniquely determined and Aut(Γ) = $U_3(5)$ (Stroth [1987], Heiss [1990]). In particular Γ, being unique, is 2-simply connected. Constructions of this geometry have been provided by Neumaier [1984] and Kantor [1985]. Turning to (13.c), we have to distinguish two cases. Given an element x of Γ with $\Gamma_x \cong$ PG(3, 2), let G_x be the stabilizer of x in $G =$ Aut(Γ) and let $\overline{G}_x$ be the action of G_x on Γ_x. By a theorem of Higman [1962], either $\overline{G}_x =$ PSL(4, 2) or $\overline{G}_x =$ Alt(7). In the first case Γ is uniquely determined and G is the sporadic group McL (Stroth [1989]); this geometry has been constructed in Ronan and Stroth [1984]; however, the uniqueness proof of Stroth [1989] also entails an existence proof, by constructing Γ as a system of cosets of suitable subgroups of McL, and a proof of the 2-simple connectedness of Γ.

Infinitely many examples are known for the latter case, with $U_4(q)$ or $O_6^+(q)$ as their automorphism groups for every prime power $q > 2$ (Kantor [1986], p. 133). They are 2-quotients of one infinite nonbuilding geometry constructed by Kantor [1987], which has been proved to be 2-simply connected by Li [1985]. One flag-transitive example is known for case (13.d) (Wester [1985]).

Let us consider the rank 4 case of (8) with one C_3 residue isomorphic to the Alt(7)-geometry.



Infinitely many examples are known (Kantor [1986], p. 132) with $O_7(q)$ as automorphism group, for every prime power $q > 2$ (see also Aschbacher and Smith [1983] for the case $q = 3$). All of them are 2-quotients of an infinite nonbuilding geometry constructed by Kantor [1985], which has been proved to be 2-simply connected by Li [1985].

We have not found sufficient information in the literature on the case of (8) with rank $\geqslant 5$ and one C_3 residue isomorphic to the Alt(7) geometry. However, it is likely that none of the finite examples that might arise for that case will be 2-simply connected. If so, (13.a), (13.b) and (13.c) with PSL(4, 2) induced on A_3 residues would be the only 'absolutely finite' situations in the context of (iii) of Theorem 2.4.1. So much for the exceptional case where the Alt(7)-geometry occurs as a rank 3 residue. Thus, from now on, Γ is as in (iii) of Theorem 2.4.1, but with no C_3 residue isomorphic to the Alt(7) geometry (hence Γ is a quotient of an infinite building).

Even if all rank 3 residues of Γ are buildings, Γ might be exceptional in another sense, which we will explain in a few lines. We first recall some known facts on flag-transitive automorphism groups of finite classical generalized m-gons (m = 3, 4, 6 or 8).

Let Γ be a finite classical generalized m-gon (m = 3, 4, 6 or 8). We recall that Γ is the building naturally associated to a simple or essentially simple group G_0 of Lie type and rank 2. We say 'essentially simple' to allow $G_0 = \mathrm{Sp}(4,2)$, $G_2(2)$ or ${}^2F_4(2)$, which are not simple but are not far from being simple: in these cases, the derived subgroup G_0' of G_0 is simple and has index 2 in G_0. By a *classical automorphism group* of Γ we mean a subgroup G of Aut(Γ) such that $G \geqslant G_0$ if G_0 is simple and $G \geqslant G_0'$ if $G_0 = \mathrm{Sp}(4,2)$, $G_2(2)$ or ${}^2F_4(2)$.

3.2.3. THEOREM (Seitz [1973], Higman and McLaughlin [1961]). *Let G be a flag-transitive subgroup of* Aut(Γ), *with Γ a finite classical generalized m-gon. Then either G is a classical automorphism group of Γ or Γ and G are as in one of the following cases*:

(i) $\Gamma = \mathrm{PG}(2,2)$ *and* G = Frob(7.3);

(ii) $\Gamma = \mathrm{PG}(2,8)$ *and* G = Frob(73.9);

(iii) $\Gamma = Q(4,3)$ (*dually*, $W(3)$; notation as in Chapter 9) *and* $G = 2^4$: Alt(5), 2^4 : Sym(5) *or* 2^4 : Frob(5.4);

(iv) $\Gamma = Q^-(5,3)$ (*dually*, $H(3,9)$) *and* $G = \mathrm{PSL}(3,4)\cdot 2_2$, $\mathrm{PSL}(3,4)\cdot 2_3$ *or* $\mathrm{PSL}(3,4)\cdot 2^2$ (notation as in Conway et al. [1985]).

The case of $\Gamma = \mathrm{PG}(2,q)$ has been settled by Higman and McLaughlin [1961]. We mentioned this in 1.7.3. The cases where Γ is a generalized m-gon with m = 4, 6, 8 have been examined by Seitz [1973]. We warn that some of the cases mentioned in (iii) and (iv) above were missing in Seitz [1973]; Seitz soon wrote a correction, but he never published it. The reader should consult Meixner [1986b] (Lemma 4.1) and Meixner [1990a] (Section 2) for a complete proof of (iii) and (iv).

If Γ and G are as in one of (i)-(iv) above, then we say that G is *exceptional.*

3.2.4. *Locally classical and locally exceptional groups.* Let Γ be as in (iii) of Theorem 3.2.1, with no C_3 residue isomorphic to the Alt(7)-geometry. Let $G \leqslant \mathrm{Aut}(\Gamma)$ be flag-transitive in Γ. Given two types i, j joined in the diagram of Γ and a flag F of Γ of cotype $\{i, j\}$, let G_F be the stabilizer of F in G, K_F the elementwise stabilizer of Γ_F in G_F and $\overline{G}_F = G_F/K_F$ (action of G_F in Γ_F). We write X_{ij}, Q_{ij}, $\overline{X}_{ij}$ and Γ_{ij} for G_F, K_F, $\overline{G}_F$ and Γ_F, respectively (this is an abuse, but it is quite harmless, as G is flag-transitive). We say that G is *locally classical* if, for every choice of distinct types i, j as above, $\overline{X}_{ij}$ is a classical automorphism group of Γ_{ij}. On the contrary, if Γ_{ij} and $\overline{X}_{ij}$ are as in one of (i)–(iv) of Theorem 3.2.3 for every pair of types i, j, then we say that G is *locally exceptional.*

The locally classical and locally exceptional cases look as the two extreme cases and one might believe that intermediate situations are possible. Actually, it is not so. We first remark that, if one of (i) or (ii) of Theorem 3.2.3 occurs for some pair of types, then the same case occurs for every edge of the diagram (this can be easily seen considering actions induced on rank 1 residues and exploiting the connectedness of the diagram of Γ). Meixner [1990a] has proved that, if (iii) (respectively, (iv)) of Theorem 3.2.3 sometimes occur, then (iii) (respectively, (iv)) occurs at every edge of the diagram. Hence we get the following result.

3.2.5. LEMMA. *The group G is either locally classical or locally exceptional.*

In the next theorem we assemble results by Timmesfeld [1983] and Meixner [1990a].

3.2.6. THEOREM. *Let G and Γ be as above. In case* (11) *of Section* 3.1, *the group G is locally exceptional.*

In case (1) *of Section* 3.1, *if the diagram is not a complete bipartite graph, then G is locally exceptional*; *otherwise G is either locally classical or locally exceptional.*

In case (5) *of Section* 3.1 *with $s = t = 3$ or $(s, t) = (3, 9)$, both possibilities can occur: G is either locally classical or locally exceptional.*

In all other cases G is locally classical.

Thus, the existence of finite flag-transitive examples for (11) and (1) depends on the exceptional phenomena mentioned in (i), (ii) and (iii) of Theorem 3.2.3. Another exceptional phenomenon is involved in (10), (12) and (14) of Section 3.1: in those cases, the action induced on A_3 residues is the flag-transitive action of Alt(7) in PG(3, 2) (Stroth [1988a,b]), which is the only example of a flag-transitive subgroup of $\mathrm{P\Gamma L}(n, q)$ ($n \geqslant 3$) not containing $\mathrm{PSL}(n + 1, q)$ (see Higman [1962]).

Infinitely many examples are known for the rank 3 case $\widetilde{\mathsf{A}}_2$ of (11), both with $s = 2$ and $s = 8$ (Ronan [1984], Koehler, Meixner and Wester [1984a,b, 1985], Meixner [1984, 1986b], Kantor [1986]) and for (12) (see Kantor [1987b]). On the other hand, we do not know of any finite flag-transitive examples for the cases of (11) of rank > 3. Infinitely many finite flag-transitive chamber systems have been constructed by Kantor [1987b] for (11), of arbitrarily large ranks, but they are not geometries.

Infinitely many finite examples with G locally exceptional have been constructed in Kantor, Meixner and Wester [1990] for the particular case of $\widetilde{\mathrm{C}}_2$ with order 3 at every node. We do not know of any other finite examples. Flag-transitive locally exceptional automorphism groups of 2-simply connected chamber systems with finite chamber stabilizers are constructed as amalgamated products in Meixner [1990a], for every rank n. However, that construction does not give us any information on the existence of finite examples, even if the existence of groups of that kind is a necessary condition for the existence of finite examples with G locally exceptional.

The reader is referred to Kantor [1986] (in particular to the list given on pages 131–133) for information on examples with Γ as in (iii) of Theorem 3.2.1, no C_3 residues isomorphic to the Alt(7)-geometry and G locally classical. Only two sporadic simple groups arise in this context, namely the Suzuki group Suz, acting on a geometry with diagram $\widetilde{\mathrm{C}}_2$ and orders 2,2,4 (discovered by Ronan and Smith [1980]; see also Yoshiara [1988]), and the Lyons group Ly, acting on a geometry belonging to $\widetilde{\mathrm{G}}_2$, with order 5 at every type (discovered by Kantor [1981]).

3.2.7. *Taking quotients of buildings.* As we have previously remarked, the finite geometries we considered in Theorem 3.2.6 are quotients of buildings (of nonspherical type). The existence of a finite flag-transitive 2-quotient of a building $\widetilde{\Gamma}$ of nonspherical type depends on the existence of a flag-transitive subgroup G of Aut($\widetilde{\Gamma}$) admitting a normal subgroup N acting semiregularly on the set of flags of $\widetilde{\Gamma}$ of corank 2 and such that G/N is finite (Chapter 11, Proposition 6.1.8). Furthermore, we want to get geometries, thus N must be such that the quotient $\widetilde{\Gamma}/N$ is a geometry, not only a chamber system. Then $\widetilde{\Gamma}/N$ would be a finite geometry with the same rank 2 residues as $\widetilde{\Gamma}$ and flag-transitive automorphism group containing a copy of G/N. Groups with the above properties are quite rare, as Theorem 3.2.1 also says. For instance, there are only a finite number of pairs $(\widetilde{\Gamma}, G)$ with $\widetilde{\Gamma}$ a locally classical building of affine type and rank $\geqslant 3$ and $G \leqslant$ Aut($\widetilde{\Gamma}$) flag-transitive and admitting a normal subgroup N as above (see Kantor, Liebler and Tits [1987]; also Kantor [1988], Section 3 and Chapter 11, 6.5.6). However, given one of those pairs $(\widetilde{\Gamma}, G)$, there are always infinitely many ways to choose $N \trianglelefteq G$ of finite index in G and acting semiregularly on the set of chambers of $\widetilde{\Gamma}$, even if we want N to be maximal with respect to the above properties (see Koehler, Meixner and Wester [1984a,b, 1985], Kantor [1985, 1986] (Sections C.3, C.4, C.5), Kantor [1987b, 1988, 1990], Kantor et al. [1990]).

3.3. *Some generalizations*

3.3.1. *Remarks on the proof of Theorems 3.2.1, 3.2.2 and 3.2.6.* The method of proof used by Timmesfeld, Stroth and Meixner is based on the analysis of stabilizers of elements of Γ in a flag-transitive subgroup G of Aut(Γ), starting with the rank 3, then exploiting induction for cases of rank $n > 3$. We focus on the rank 3 case. The stabilizer of an element of type k is X_{ij} (notation as in 3.2.4; i, j, k are the types of Γ). Theorem 3.2.3 allows to get control over the possibilities for $\overline{X}_{ij}$ when i and j are joined in the diagram of Γ. When i and j are not joined in that diagram, then $X_{ij} = X_i X_j = X_j X_i$, where X_i and X_j are the stabilizers of a flag of cotype i and j, respectively. We can compare $\overline{X}_{ij}$ and $\overline{X}_{jk}$ for pairs of joined types $\{i, j\}$, $\{j, k\}$: we have $X_j = X_{ij} \cap X_{jk}$ and the structure inherited by X_j/Q_{ij} from the (supposed) structure of $\overline{X}_{ij}$ must fit with the structure inherited by X_j/Q_{jk} from $\overline{X}_{jk}$. Many pairings of supposed structures for $\overline{X}_{ij}$ and $\overline{X}_{jk}$ turn out to be inconsistent (they do not fit on X_j). However this kind of analysis does not give us sufficient information on X_{ij}, X_{jk} and X_j. It might be that a number of pairings of supposed structures for $\overline{X}_{ij}$ and $\overline{X}_{jk}$ still survive through the previous analysis, while they actually cannot occur. In order to go on, it is necessary to examine further Q_{ij}, Q_{jk} and/or the stabilizer $B = X_i \cap X_j = X_j \cap X_k$ of a chamber of Γ (note that Q_{ij}, $Q_{jk} \leqslant B$). Only sections of B are visible in $\overline{X}_{ij}$ and $\overline{X}_{jk}$. To obtain (and then exploit) some information on the hidden part of B, we can consider a Sylow p-subgroup S of B, where p is the common characteristic of the ground fields of the residues of Γ of type $\{i, j\}$ and $\{j, k\}$ (note that $O_p(Q_{ij}) \leqslant O_p(B) \leqslant S$). Let $V = \Omega_1(Z(S))$ be the maximal elementary Abelian subgroup of the centre $Z(S)$ of S and $V_{ij} = \langle V^{X_{ij}} \rangle$, $V_{jk} = \langle V^{X_{jk}} \rangle$. Trivially, $V_{ij} \trianglelefteq X_{ij}$. Furthermore, V_{ij} is an elementary Abelian p-group. Hence it is a GF(p)-module for X_{ij}. Note that $V_{ij} \trianglelefteq O_p(X_{ij})$. Furthermore, G can be chosen in such a way that $O_{\hat{p}}(X_{ij}) = O_p(Q_{ij})$ (see, e.g., Timmesfeld [1983], §4, [1985], §3, [1987], §3); hence $V_{ij} \trianglelefteq O_p(Q_{ij})$ and $\overline{X}_{ij}$ is involved in X_{ij}/V_{ij}. The investigation goes on by determining feasible actions of X_{ij} and X_{jk} on V_{ij} and V_{jk}, respectively, recalling that $X_j = X_{ij} \cap X_{jk}$ acts on both V_{ij} and V_{jk} and that $V_j \leqslant V_{ij} \cap V_{jk}$, where $V_j = \langle V^{X_j} \rangle$. This analysis can be done by using a number of results on modules for Chevalley groups and, possibly, considering conjugates of X_{ij} and X_{jk}, and focusing on certain pairs of conjugates, called 'critical pairs' in Stroth [1988a,b] and Timmesfeld [1989]. If Γ is not as in Theorem 3.2.1 (resp., 3.2.2, 3.2.6), then a contradiction is eventually derived.

For the rank $n \geqslant 4$, a similar strategy can be applied, using stabilizers of elements of suitable types i, j and induction on the rank that has been initiated by the rank 3 case.

We call the above method '*comparison of parabolics*', the stabilizers in G of the elements of Γ being called *maximal parabolic subgroups* of G. This is also basically the technique used in Aschbacher [1984] to prove Theorems 2.3.1 and 2.3.2. Parabolics can also be defined for any rank. If m is a positive integer less than the rank n of Γ, then stabilizers in G of flags of Γ of corank m are called *parabolics of rank* m (compare 3.2.3 of Chapter 3). In particular, parabolics of rank 1 are called *minimal parabolics*.

The stabilizer in G of a chamber of Γ is called the *Borel* subgroup of G (Chapter 3, 3.2.3); we will always denote it by B. We are abusively referring to 'the' Borel subgroup of G as if it were unique, whereas it depends on the choice of a chamber of Γ, in general; however, this abuse is harmless in the context of this section. The different Borel subgroups are conjugate in G.

3.3.2. *The case with finite Borel subgroup.* It should be evident from 3.3.1 that there is no need to assume Γ finite in order to obtain the statements of Theorems 3.2.1, 3.2.2 and 3.2.6.

Indeed, the main step in the comparison of parabolics is the investigation of the action of the stabilizer G_a in G of an element a of Γ on the elementwise stabilizer Q_a of Γ_a or on the module $V_a = \langle V^{G_a} \rangle$, where $V = \Omega_1(Z(S))$ and S is a Sylow p-subgroup of the Borel subgroup B of G. If B is finite, then Q_a is finite, as $Q_a \leqslant B$, hence S and $V = \Omega_1(Z(S))$ can be defined. Therefore, we can replace the hypothesis that Γ is finite in Theorems 3.2.1, 3.2.2 and 3.2.6 by the weaker hypothesis that Γ admits some flag-transitive automorphism group with finite Borel subgroup, obtaining the same conclusions (actually, this is the hypothesis assumed in Timmesfeld [1987, 1989], Stroth [1988b,c, 1990], Meixner [1986a, 1990a,b]).

We now recall some facts on the lifting of automorphisms to universal 2-covers. Let $G \leqslant \mathrm{Aut}(\Gamma)$ be flag-transitive in Γ and let f: $\widetilde{\Gamma} \rightarrow \Gamma$ be the universal 2-covering of Γ. The group G lifts to a flag-transitive subgroup $\widetilde{G}$ of $\mathrm{Aut}(\widetilde{\Gamma})$ (Chapter 11, Proposition 6.1.8). A rank 2 parabolic X_{ij} of G lifts to a subgroup Y_{ij} of $\widetilde{G}$ containing the corresponding rank 2 parabolic $\widetilde{X}_{ij}$ of $\widetilde{G}$ and the group N of deck transformations of f. Since f is a 2-covering, we have $N \cap \widetilde{X}_{ij} = 1$. Hence $Y_{ij} = N : \widetilde{X}_{ij}$ and, since $Y_{ij}/N \cong X_{ij}$, we obtain $\widetilde{X}_{ij} \cong X_{ij}$. The rank 1 parabolic $\widetilde{X}_i$ of $\widetilde{G}$ is mapped onto the corresponding rank 1 parabolic X_i of G by the above isomorphism and the Borel subgroup $\widetilde{B}$ of $\widetilde{G}$ is mapped onto the Borel subgroup B of G. Thus, there is no way to distinguish G from $\widetilde{G}$ as far as we only look at rank 2 parabolics and their intersections.

Thus, we could safely restrict our interest to 2-simply connected geometries in Theorems 3.2.1, 3.2.2 and 3.2.6. If furthermore we restrict our interest to cases of nonspherical type where no C_3 residue is the Alt(7)-geometry, then Theorems 3.2.1 and 3.2.6 give us a classification of locally classical locally finite buildings of irreducible nonspherical type admitting some flag-transitive automorphism group with finite chamber stabilizer. There is no evident a priori reason for such a building to admit finite flag-transitive quotients, even if this is what happens in many cases.

3.3.3. *More remarks on parabolics.* Let G be a flag-transitive subgroup of $\mathrm{Aut}(\Gamma)$. Let us keep the notation of 3.2.4 and 3.3.1 and let I be the set of types of Γ. The following is trivial

(1) $X_i \cap X_j = B$ $(i, j \in I,\ i \neq j)$.

The next property follows from the flag-transitivity of G and from the assumption that Γ is residually connected (Kantor [1986], Buekenhout and Cohen [1994], Pasini [1994d, Chapter 10]:

(2) *all parabolic subgroups and the group G are generated by minimal parabolics.*

As flags are sets of mutually incident elements, the 'dual' of (2) holds also:

(3) *all parabolic subgroups and the Borel subgroup are intersections of maximal parabolics.*

3.3.4. *Direct sums and automorphism groups.* Let the basic diagram of Γ be disconnected, with connected components $I_1, I_2, \ldots, I_m$. By the Direct Sum Theorem (see 2.1.2), we have $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_m$ with Γ_h isomorphic to a residue of Γ of type I_h $(h = 1, 2, \ldots, m)$. Then we get immediately:

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2) \times \cdots \times \mathrm{Aut}(\Gamma_m). \tag{1}$$

However, if G is a flag-transitive subgroup of Aut(Γ), it is in general not true that $G = G_1 \times G_2 \times \cdots \times G_m$ with G_h a subgroup of Aut(Γ_h) ($h = 1, 2, \ldots, m$). The only thing we can say in general is that

$$X_i X_j = X_j X_i \quad \text{for } i \in I_h,\ j \in I_k,\ h \neq k. \tag{2}$$

3.3.5. *Chamber systems.* Let $\mathcal{C} = (C, (\Phi_i)_{i \in I})$ be a chamber system over a finite set of types I (Chapter 3.13; Chapter 11, Section 1.2; here we write Φ_i for $\overset{i}{\sim}$). We assume that $\mathcal{C}$ is connected, i.e. the (lattice-theoretic) join of all Φ_i is the trivial equivalence relation on C (namely, the equivalence relation having C as its unique equivalence class). As we are mainly interested in chamber systems admitting diagrams (Chapter 11, Section 6.2), we also assume that $\Phi_i \cap \Phi_j$ is the identity relation on C, for every choice of distinct types $i, j \in I$.

J-stars (Chapter 11, Section 1.2) now take the place of residues of type J. The cardinal number $|J|$ is called the *rank* of a J-star. Stars of rank 2 are chamber systems of rank 2 geometries, by the previous assumption on the intersections $\Phi_i \cap \Phi_j$ ($i, j \in I$). Thus, diagrams and basic diagrams (Chapter 3, Section 8.2) can be defined for chamber systems just as for geometries. We say that $\mathcal{C}$ belongs to a Coxeter diagram $M = (m_{ij})_{i,j \in I}$ if all $\{i, j\}$-stars of $\mathcal{C}$ are generalized m_{ij}-gons ($i, j \in I$, $i \neq j$). We say that $\mathcal{C}$ is *locally finite* if all rank 1 stars of $\mathcal{C}$ are finite. It $\mathcal{C}$ is locally finite and belongs to a Coxeter diagram, then $\mathcal{C}$ is said to be *locally classical* if every rank 2 star of $\mathcal{C}$ is either a generalized digon or a finite, thick classical generalized polygon. Automorphisms of chamber systems have been defined in Chapter 11 (Section 1.2). We denote the automorphism group of a chamber system $\mathcal{C}$ by Aut($\mathcal{C}$). A subgroup G of Aut($\mathcal{C}$) is said to be *transitive* on $\mathcal{C}$ if it is transitive on the set of chambers of $\mathcal{C}$. We say that $\mathcal{C}$ is *transitive* if Aut($\mathcal{C}$) is transitive. *Parabolic* subgroups and *Borel* subgroups can be defined in a transitive subgroup G of Aut($\mathcal{C}$) just as in the case of geometries, substituting J-cells for flags of cotype J. We use for them the same notation as for geometries.

Since we have assumed that $\Phi_i \cap \Phi_j$ is the identity relation for $i \neq j$, property (1) of 3.3.3 remains true for chamber systems. As $\mathcal{C}$ is connected by assumption, the transitivity of G implies (2) of 3.3.3 also in the case of chamber systems. On the other hand, (3) of 3.3.3 is false in general for chamber systems (see Pasini [1994b] for some counterexamples).

The following construction is an analogue of the direct sum of geometries for chamber systems. Given chamber systems

$$\mathcal{A} = \big(A, (\Psi_j)_{j \in J}\big) \quad \text{and} \quad \mathcal{B} = \big(B, (\Xi_k)_{k \in K}\big)$$

over mutually disjoint sets of types J and K, set $I = J \cup K$, $C = A \times B$ and define Φ_i ($i \in I$) as follows: let $(a, b), (c, d) \in C$; given $j \in J$, we set $(a, b) \equiv (c, d)(\Phi_j)$ if $a \equiv c(\Psi_j)$ and $b = d$; given $k \in K$, we set $(a, b) \equiv (c, d)(\Phi_k)$ if $a = c$ and $b \equiv d(\Xi_k)$. Then $\mathcal{C} = (C, (\Phi_i)_{i \in I})$ is a chamber system over the set of types I. We call $\mathcal{C}$ the *direct product* of $\mathcal{A}$ and $\mathcal{B}$ and we denote it by $\mathcal{A} \times \mathcal{B}$.

Note that, if $\mathcal{A}$ and $\mathcal{B}$ are chamber systems of geometries Γ_1, Γ_2, respectively, then $\mathcal{A} \times \mathcal{B}$ is (isomorphic to) the chamber system of $\Gamma_1 \oplus \Gamma_2$. Note also that, given $j \in J$

and $k \in K$, then all $\{i, j\}$-stars of $\mathcal{A} \times \mathcal{B}$ are generalized digons; i.e. j and k are not joined in the basic diagram of $\mathcal{A} \times \mathcal{B}$.

However, the converse of the above does not hold. Namely, there is no general analogue of the Direct Sum Theorem for chamber systems. That is, if the basic diagram of a chamber system $\mathcal{C}$ is disconnected with connected components $I_1, I_2, \ldots, I_m$, this does not imply that $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_m$ for suitable chamber system $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ over the sets of types $I_1, I_2, \ldots, I_m$. The reader can find some counterexamples in Pasini [1994b].

REMARK. If we want to state Theorem 2.2.1 of Section 2 for chamber systems instead of geometries, then assuming that all rank 3 stars of type C_3 or H_3 are 2-covered by buildings would not be sufficient to obtain the conclusion. We should assume the same hypothesis for stars of type A_3 and also for rank 3 stars with disconnected diagram (Tits [1981], 5.3). The previous remarks on chamber systems with disconnected basic diagrams should make it clear why that hypothesis is necessary for rank 3 stars with disconnected diagram. As for the A_3 case, we warn that there are chamber systems of type A_3 that are not chamber systems of projective geometries (Tits [1981], 6.1.6(b)), hence they are not chamber systems of geometries, in view of Corollary 2.2.3(i) of Section 2. A locally classical (but not transitive) example of this kind is known, due to Timmesfeld (private communication).

3.3.6. *A generalization of Theorems 3.2.1 and 3.2.6 to chamber systems.* Properties (1) and (2) of 3.3.3 are the only two general properties of parabolic subgroups used by Timmesfeld, Stroth and Meixner when they deal with the comparison of parabolics in the nonspherical case. Those two properties hold for chamber systems as well, as we saw in 3.3.5. Furthermore, if we work with some subgroup of the automorphism group of a reducible geometry, then we do not know whether we can split the group as a direct product of subgroups corresponding to the direct summands of the geometry as in (1) of 3.3.4. Thus, we are left with relations as (2) of 3.3.4, which only describe the shape of the basic diagram, and remain valid with that meaning in the case of chamber systems. Hence, in the analysis of (iii) of Theorem 3.2.1 and in the proof of Theorem 3.2.6 there is no need to assume that the structure we are dealing with is a geometry rather than a chamber system.

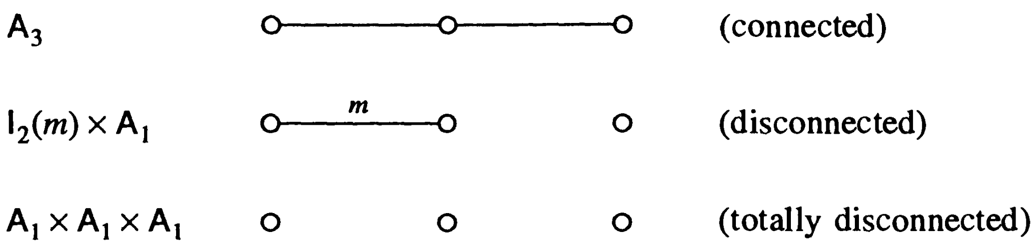
As for the spherical case, Timmesfeld [1984b], 3.1) has proved the following.

3.3.7. LEMMA. *Every locally classical, locally finite transitive chamber system belonging to a connected Coxeter diagram of spherical type is a geometry (whence it is either a building or the* Alt(7)*-geometry, by Theorem* 2.3.1).

Therefore:

3.3.8. THEOREM. *The statements of Theorems* 3.2.1 *and* 3.2.6 *also hold for locally classical locally finite chamber systems with connected Coxeter diagram admitting some transitive automorphism group with finite Borel subgroup.*

3.3.9. *Universal covers.* As we have remarked in 3.3.5, no analogue of the Direct Sum Theorem holds for chamber systems in general. This difficulty has no relevance for an information of 'local' nature as that given by (iii) of Theorem 3.2.1 and by Theorem 3.2.6. However, it becomes relevant if we want to say more. For instance, we do not know whether a chamber system $\mathcal{C}$ as in Theorem 3.3.8, with nonspherical diagram and no C_3 star isomorphic to (the chamber system of) the Alt(7)-geometry, is always 2-covered by a building. We should ask whether all stars $\mathcal{C}$ of rank 3 with one of the following diagrams are 2-quotients of buildings (see the Remark of 3.3.5):



The answer is always affirmative for A_3, in view of Lemma 3.3.7. On the contrary, things are not so easy for the other two (disconnected) types and these questions remain open. Coming back to a chamber system $\mathcal{C}$ as in Theorem 3.3.8, things are not so bad as they look. The universal 2-cover of $\mathcal{C}$ is a building in all rank 3 cases of (1), (5), (6) and (7) of Section 3.1, in the rank 4 cases of (2) and (3), in all cases of (11) and in cases (12) and (16). Indeed all induced rank 3 subdiagrams are connected in these cases and none of them is of type C_3. We can settle more cases using the following result.

3.3.10. LEMMA (Pasini [1994b], Lemma 23). *Let $\mathcal{C}$ be a locally classical, locally finite transitive chamber system belonging to a (possibly disconnected) Coxeter diagram Δ such that, for every set of types J of size* 3 *such that the diagram induced by Δ on J is disconnected, there is a set of types K of size at least* 3, *such that $|J \cap K| \geqslant 2$ and Δ induces a connected Coxeter diagram of spherical type on K. Assume furthermore that the action induced on a star of $\mathcal{C}$ of rank* 3 *by its stabilizer in* Aut($\mathcal{C}$) *never is the alternating group* Alt(7).

Then the universal 2-*cover of $\mathcal{C}$ is a building.*

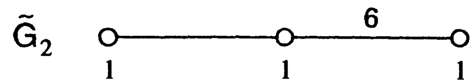
By this lemma, the universal 2-cover of $\mathcal{C}$ is a building if the diagram of $\mathcal{C}$ is as in the rank 4 case of (9) or as in (15) of 3.14.

Let $\mathcal{C}$ still be as in Theorem 3.3.8, with a connected nonspherical diagram and no C_3 star isomorphic to the Alt(7)-geometry. Let $G \leqslant \mathrm{Aut}(\mathcal{C})$ be transitive with finite Borel subgroup. In many cases with G locally classical, the minimal parabolics of G form a weak parabolic system in the sense of Timmesfeld [1983]. Corollary 4.9 of Timmesfeld [1983] is a partial group-theoretic substitute of the Direct Sum Theorem.

3.4. *The tradition of thin geometries*

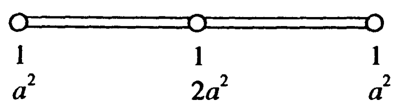
3.4.1. This tradition stems from the antique classification of the five regular convex polyhedra in 3-space, extended to the regular convex polytopes and a series of objects of increasing complexity and generality (see, e.g., Coxeter [1973], Grünbaum [1977a], Danzer and Schulte [1982]). There have been spectacular developments over the last ten years (see McMullen and Schulte [1994b]). Here we shall briefly describe some results that appear as good indicators for the main stream of the present chapter. All of this occurs at the (Δ, Γ, G)-level. We will suppose Γ finite.

3.4.2. THEOREM (Coxeter and Moser [1964]). *Let (Γ, G) belong to one of the diagrams*



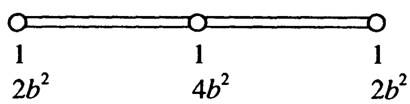
Then, either (Γ, G) is the corresponding Coxeter geometry-group pair or one of the following holds.

(1)



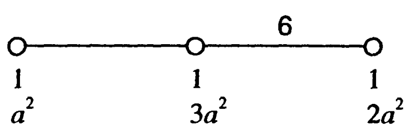
$a \geqslant 2$, $G = (\mathbb{Z}_a \times \mathbb{Z}_a) \cdot D_8$, $|G| = 8a^2$. *Notation:* $\{4,4\}_{(a,0)}$.

(2)



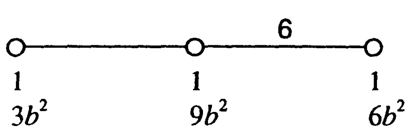
$b \geqslant 2$, $G = (\mathbb{Z}_{2b} * \mathbb{Z}_{2b}) \cdot D_8$, $|G| = 16b^2$. *Here, the symbol $*$ stands for an amalgamated product fusing the involutions of the factors. Notation:* $\{4,4\}_{(b,b)}$.

(3)



$a \geqslant 2$, $G = (\mathbb{Z}_a \times \mathbb{Z}_a) \cdot D_{12}$, $|G| = 12a^2$. *Notation:* $\{3,6\}_{(a,0)}$.

(4)



$b \geqslant 1$, $G = (\mathbb{Z}_{36} * \mathbb{Z}_{36}) \cdot D_{12}$, $|G| = 36b^2$. *Here, the amalgamated product fuses the subgroups of order* 3 *of the factors* $\mathbb{Z}_{36}$. *Notation:* $\{3,6\}_{(b,b)}$.

Moreover, there is a unique pair (Γ, G) in each of these families, for each given value of the parameters a, b.

This provides an interesting situation in which there are infinitely many nonisomorphic quotients of a pair (Γ, G) and where control is nevertheless achieved over them.

3.4.3. The finite geometries of types $\{4,4\}_{(a,0)}$, $\{4,4\}_{(b,b)}$, $\{3,6\}_{(a,0)}$, $\{3,6\}_{(b,b)}$ together with the finite Coxeter geometries of rank 3, namely $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, can be used as rank 3 residues of thin regular geometries of rank 4, or more. This theme has given rise to many results, in various directions such as universal covers and constructions of infinitely many quotients.

For more details on these developments, we refer to McMullen and Schulte [1989, 1990a,b,c, 1994a,b], Monson and Weiss [1990], A.I. Weiss [1986].

3.5. *A characterization of buildings*

We end this section with a remarkable result due to Heiss [1992], in the direction developed by Timmesfeld.

We start at the (Δ, Γ, G)-level and we need the concept of an apartment of Γ, discussed already in Chapters 3 and 11. Here an *apartment* of Γ is a thin subgeometry of Γ admitting the diagram Δ.

THEOREM. *Let Δ be a connected Coxeter diagram and let (Γ, G) be a finite locally classical flag-transitive geometry over Δ such that every rank 3 residue belonging to a connected subdiagram of Δ is a building.*

Assume that there is an apartment A of Γ such that the stabilizer of A in G acts flag-transitively on A. Then Γ is a building.

4. Extensions of geometries over Coxeter diagrams by linear spaces

4.1. *Introduction*

4.1.1. *Our next purpose.* As we observed in 1.1, buildings are not general enough in order to explain all finite simple groups in terms of incidence geometry. After Section 3, we can see that geometries over Coxeter diagrams are not improving very much on that situation, since the only sporadic groups arising in that more general context, are McL, Suz and Ly. Hence, it seems necessary to introduce slightly more general diagrams.

One of the best motivated choices, as we see in Section 1 (1.1, 1.4) is to allow rank 2 residues that are $(g, g, g+1)$-gons. Then $g = 3$ or 5 (1.4.2). Here we shall consider only the case $g = 3$, hence we deal with diagrams whose strokes are of Coxeter type or L_2 (linear space) type. The case of $g = 5$, is studied in Section 6.

Let us mention briefly a different approach due to Stroth and Wong [1992]. These authors consider a transitive chamber system with finite Borel subgroup, orders equal to 2 or 4, Sym(3) or Sym(5) as induced actions on residues of rank 1, direct products or $(Z_3 \times Z_3) \cdot 2$, $(\mathrm{Alt}(5) \times \mathrm{Alt}(5)) \cdot 2$ or $(\mathrm{Alt}(5) \times \mathrm{Sym}(3)) \cdot 2$ as actions induced on those rank 2 residues that are generalized digons; they also assume that the residues of rank 2 other than generalized digons are isomorphic with some of the known 2-local geometries, compatible with the above hypothesis on rank 1 residues (see Ronan and Stroth [1984], Section 2, for the definition of a 2-local geometry). A number of sporadic simple groups are known to act flag-transitively on such geometries (Ronan and Stroth [1984], Ronan and Smith [1980]), and this is an excellent reason to study chamber systems of that kind. Under the above assumptions, Stroth and Wong obtain a classification of the feasible diagrams, in the style of Theorem 3.2.1.

Quite another feature of the coming subject has to be pointed out. Assume that Δ is some diagram whose strokes are either of Coxeter type or of linear space type and that we can classify all geometries belonging to Δ. Most often the classification gives rise to one or no or more infinite families that can be described in a uniform way and to a finite number of isolated examples that require *ad hoc* description. It is rather natural to think of the first case as being 'generic' and to think of the second as being 'exceptional' or 'sporadic'.

4.1.2. *Some remarks on non-Coxeter diagrams.* Theorem 2.1.2 gives us a description of Coxeter diagrams of finite type. What about diagrams that are not of Coxeter type? The following is the first thing one might think of: a diagram is of finite type, or at least not of infinite type, if it 'resembles' a spherical Coxeter diagram. However, it is not clear what the word 'resembles' means precisely. For instance, one might say that the following diagrams of rank $n \geqslant 3$ resemble the Coxeter diagram C_n:

$L \cdot C_{n-1}$  $n \geqslant 3$

$L_{n-1} \cdot C_2$  $n \geqslant 3$

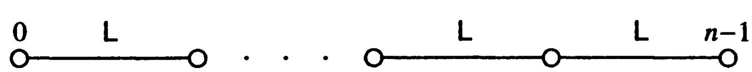
If Γ is a geometry belonging to $L \cdot C_{n-1}$, admitting finite orders at each type and if the Intersection Property (IP) holds in Γ, then $d_\Gamma < \infty$ and it is bounded by a number depending on the orders of Γ at the first two nodes of the diagram Pasini [1991a]. However, this does not prove that $L \cdot C_{n-1}$ is not of infinite type. In some special cases we can say a bit more.

The Coxeter diagram C_n is included in $L \cdot C_{n-1}$ and it is of finite type, as we know. The following also is a special case of $L \cdot C_{n-1}$.

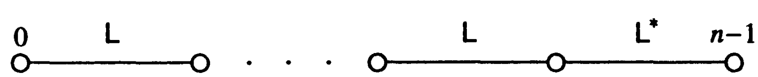
$Af \cdot C_{n-1}$ 

It is proved in Pasini [1990b] that all geometries belonging to $Af \cdot C_{n-1}$ and satisfying the Intersection Property (IP) have diameter $\leqslant 3$. Perhaps the same conclusion can be obtained without assuming (IP). If so, then $Af \cdot C_{n-1}$ would be of finite type. On the other hand $L_{n-1} \cdot C_2$ is of infinite type when $n \geqslant 3$ (see Meixner and Pasini [1993]; also Pasini and Yoshiara [1992]).

The following diagram of rank $n \geqslant 2$ resembles the Coxeter diagram A_n. It characterizes n-dimensional linear spaces (see 4.2.1).

L_n 

It is not difficult to prove that every n-dimensional linear space has diameter $n(n+1)/2$. Therefore L_n is of finite type. The following diagram of rank n resembles L_n:

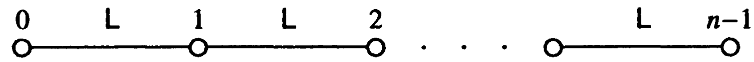
$L_{n-1} \cdot L^*$ 

However, if $n \geqslant 4$, then $\mathsf{L}_{n-1} \cdot \mathsf{L}^*$ is of infinite type (see 4.5.10).

Of course, a case-by-case analysis is not the best way to understand which non-Coxeter diagrams are of infinite or finite type. Some attempts have been made in order to cope with this problem in more systematic ways, focusing on gonalities of rank 2 residues (as in Ronan [1981a] and Pasini [1986a]) or on diameters of rank 2 residues (as in Tsaranov [1990a]) or on both (as in Pasini and Tsaranov [1993]). Some important results are available (in Ronan [1981a] and Tsaranov [1990a], for instance). However, a complete solution of this problem does not seem to be in reach for the time being.

4.2. *The diagram* L_n

All n-dimensional linear spaces (Chapter 6) belong to the following diagram of rank n, which we called L_n in 1.2:

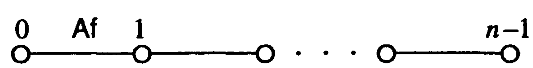
L_n 

Furthermore, (see 2.15, 2.2.3 and Chapter 6, 6.1.5), the diagram L_n characterizes n-dimensional linear spaces; namely:

4.2.1. THEOREM (Buekenhout [1979], Theorem 7). *All geometries belonging to* L_n *are* n*-dimensional linear spaces.*

Buekenhout is assuming (IP) but this is not needed (see Pasini [1994b], Lemma 7.4). By Theorem 4.2.1, further results or problems concerning special cases of L_n are questions on n-dimensional linear spaces.

4.2.2. *Projective and affine geometries.* The Coxeter diagram A_n is a special case of L_n. By Theorem 4.2.1 and by Jónsson [1959] and Veblen and Young [1917] it characterizes n-dimensional (possibly reducible) projective geometries (compare Corollary 2.2.3 of Section 2; see also Chapter 2, Section 3). The following diagrams are also specializations of L_n:

$\mathsf{Af} \cdot \mathsf{A}_{n-1}$ 

(rank $n \geqslant 2$)

$\mathsf{A}_{n-1} \cdot \mathsf{L}$ 

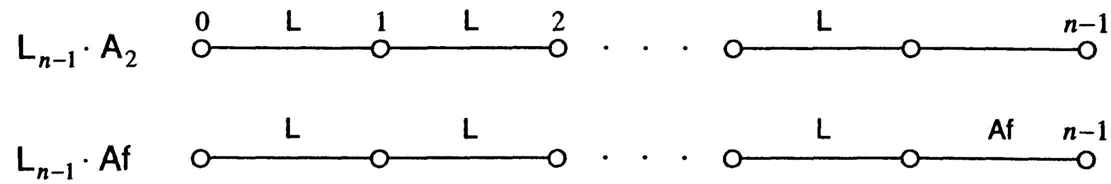
(rank $n \geqslant 3$)

$\mathsf{Af} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}$ 

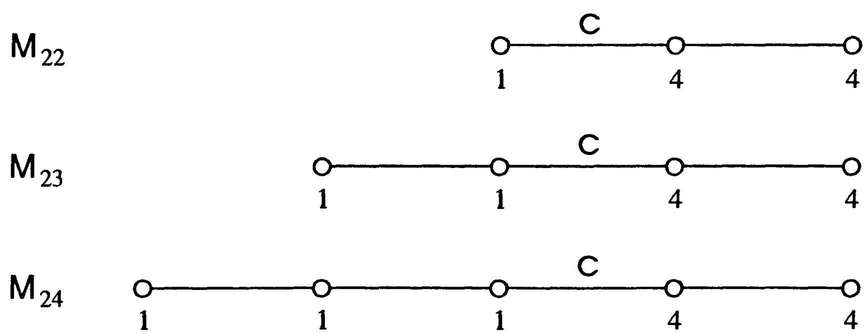
(rank $n \geqslant 3$)

The diagram $\mathsf{Af} \cdot \mathsf{A}_{n-1}$ characterizes n-dimensional affine geometries (Jónsson [1959]). The diagram $\mathsf{A}_{n-1} \cdot \mathsf{L}$ characterizes n-truncations of (possibly reducible or infinite dimensional) projective geometries (Chapter 6, 2.1). When $n \geqslant 4$ the diagram $\mathsf{Af} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}$ characterizes n-truncations of (possibly infinite dimensional) affine geometries (Chapter 6, 2.1). All 3-dimensional linear spaces belonging to $\mathsf{Af} \cdot \mathsf{L}$ and with at least 4 points on every line are 3-truncations of affine geometries (Buekenhout [1969]; see also Chapter 6, 4.2). On the other hand, there are 3-dimensional linear spaces belonging to Af · L with $q = 2$ or 3 points on every line which are not truncations of affine geometries. They are called *Hall triple systems* when $q = 3$ (Chapter 6, 4.12) and *Steiner quadruple systems* when $q = 2$. Note that all inversive planes with circles of size 4 are Steiner quadruple systems (Chapter 2, 3.7).

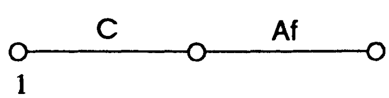
4.2.3. *Designs for the Mathieu groups and inversive planes.* We now consider the following two diagrams of rank $n \geqslant 3$.



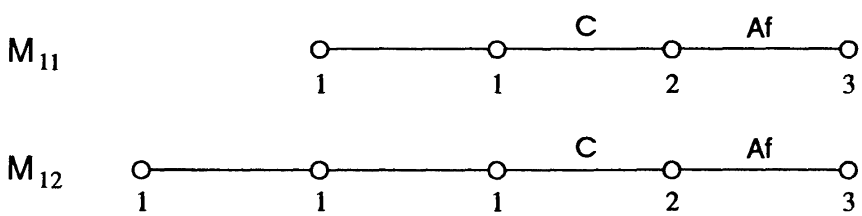
Note that the Coxeter diagram A_n and the diagram $Af \cdot A_{n-1}$ are special cases of $L_{n-1} \cdot A_2$. Hence n-dimensional projective and affine geometries are examples for $L_{n-1} \cdot A_2$. The Witt designs for the Mathieu groups M_{22}, M_{23} and M_{24} have diagram and orders as follows (Chapter 6, 2.3 and 1.1):



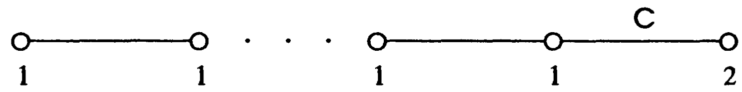
Hence they are examples for $L_{n-1} \cdot A_2$. Inversive planes (Chapter 6, 5.4) are characterized by the following diagram:



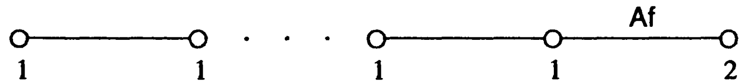
Hence they are examples for $L_2 \cdot Af$. The Witt designs for the Mathieu groups M_{11} and M_{12} have diagrams and orders as follows (Chapter 6, 2.3):



Hence they are examples for $L_{n-1} \cdot Af$. The n-truncation of a thin $(n+1)$-dimensional projective geometry (i.e. of a completely trivial $(n+1)$-dimensional linear space, according to 2.2 of Chapter 6) has diagram and orders as follows.



This diagram with these orders can also be represented as a special case of $\mathsf{L}_{n-1} \cdot \mathsf{Af}$:



4.2.4. THEOREM. *Let Γ be a geometry belonging to $\mathsf{L}_{n-1} \cdot \mathsf{A}_2$ admitting finite order at every type. Then Γ is one of the following:*

(i) *the thin n-dimensional projective geometry;*

(ii) *the projective geometry* $\mathrm{PG}(n, q)$, *for some prime power q;*

(iii) *the affine geometry* $\mathrm{AG}(n, q)$, *for some prime power q;*

(iv) *the Witt design for one of the Mathieu groups* M_{22}, M_{23} *or* M_{24};

(v) *an (unknown)* 3*-dimensional linear space with orders* s, t, t, $s > 1$ *and* $t = (s+1)^2$;

(vi) *an (unknown)* 3*-dimensional linear space with orders* s, t, t *and* $t = (s+1)^3 + (s+1)$;

(vii) *an (unknown)* 4*-dimensional linear space with orders* $1, 3, 68, 68$.

The above statement is obtained from a theorem of Doyen and Hubaut [1971] (see also Chapter 6, 4.8), a result by Hughes [1963], and a lot of additional work to reduce some cases (Pasini [1994d], 7.2.3). Note that $\mathrm{Aut}(\Gamma)$ is flag-transitive in each of the cases (i)–(iv). In each of those, Γ admits a smallest flag-transitive automorphism group G_0 and we have

$$G_0 = \mathrm{Aut}(\Gamma) = \mathrm{Sym}(n) \quad \text{in case (i)},$$

$$G_0 = \mathrm{PSL}_{n+1}(q) \leqslant \mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(n+1, q) \quad \text{in case (ii)},$$

$$G_0 = \mathrm{ASL}(n, q) \leqslant \mathrm{Aut}(\Gamma) = \mathrm{A\Gamma L}(n, q) \quad \text{in case (iii)},$$

$$G_0 = \mathsf{M}_{22} < \mathrm{Aut}(\Gamma) = \mathsf{M}_{22} \cdot 2,\ G_0 = \mathrm{Aut}(\Gamma) = \mathsf{M}_{23} \quad \text{and}$$

$$G_0 = \mathrm{Aut}(\Gamma) = \mathsf{M}_{24} \quad \text{in case (iv)}.$$

By easy computations on orders of locally finite n-dimensional linear spaces (Pasini [1994d], 7.2.1) together with a theorem of Dembowski [1968], 6.2.14, on orders of finite inversive planes and a result of Kantor [1974], Section 5, Remark 2, on the nonexistence of extensions of certain inversive planes, the following can be obtained (Pasini [1994d], 7.2.4).

4.2.5. THEOREM. *Let Γ be a geometry belonging to $\mathsf{L}_{n-1}\cdot\mathsf{Af}$ and admitting finite orders. Then Γ is one of the following:*

(i) *the n-truncation of the thin $(n+1)$-dimensional projective geometry;*

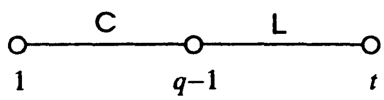
(ii) *an inversive plane;*

(iii) *the Witt design for one of the Mathieu groups M_{11} or M_{12};*

(iv) *an (unknown) 4-dimensional linear space with orders $1, 1, 12, 13$.*

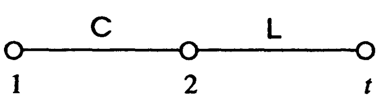
Aut(Γ) is flag-transitive in cases (i) and (iii), with Aut(Γ) = Sym($n+1$), M_{11}, M_{12}, respectively; however Alt($n+1$) also acts flag-transitively in (i). If Γ is a classical inversive plane in (ii) (Chapter 6, 5.8(a)), then Aut(Γ) = PΓO$^-(4,q)$ for some prime power q and it is flag-transitive on Γ. In this case, the smallest flag-transitive automorphism group of Γ is $\mathrm{O}^-(4,q) \cong \mathrm{PSL}(2,q^2)$. We now describe two more families of flag-transitive 3-dimensional linear spaces.

Let X be a copy of PG$(1,q)$ inside PG$(1,q^d)$ ($q \geqslant 3$, $d \geqslant 2$). We take PG$(1,q^d)$ as set of points, the images of X under the action of PΓL$(2,q^d)$ as planes and all pairs of points of PG$(1,q^d)$ as lines. Thus, we obtain a 3-dimensional linear space with diagram and orders as follows.



$t = q(q^{d-1} - 1)/(q-1)$

and with PΓL$(2,q^d)$ as (flag-transitive) automorphism group. Moreover, the residual linear space is the affine space AG(d,q). We call it the *classical WLA-circular space* of *type* (q,d), stressing the terminology of 5.28 of Chapter 6. Note that classical WLA-circular spaces of type $(q,2)$ are just classical inversive planes. The other family of flag-transitive 3-dimensional linear spaces arises as follows. We take PG$(1,q) = \{\infty\} \cup$GF(q) as set of points, with $q \equiv 7 \pmod{12}$. If K is the set of solutions of the equation $X^3 = 1$ in GF(q) and PΣL$(2,q)$ is the extension of PSL$(2,q)$ by Aut(GF(q)), then we take the images of K under the action of PΣL$(2,q)$ as planes. As lines we take the pairs of points of PG$(1,q)$. The geometry obtained in this way has diagram and orders as follows



$t = (q-3)/2$

and PΣL$(2,q)$ as (flag-transitive) automorphism group. We call it a *classical locally Netto system* (compare Chapter 6, 7.5). Using the classification of finite flag-transitive linear spaces (Buekenhout, Delandtsheer and Doyen [1988]), Delandtsheer [1991] has proved the following (see also Chapter 6, 7.5).

4.2.6. THEOREM. *Let Γ be a flag-transitive locally finite n-dimensional linear space with $n \geqslant 3$. Then Γ is one of the following:*

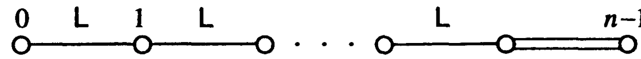
(i) *the thin n-dimensional projective geometry or the n-truncation of a thin m-dimensional projective geometry with $m > n$;*

(ii) *a projective geometry* PG(n,q) *or the n-truncation of* PG(m,q) *for some $m > n$;*

(iii) *an affine geometry* AG(n, q) *or the* m*-truncation of* AG(m, q) *for some* $m > n$;
(iv) *a classical* WLA*-circular space*;
(v) *a classical locally Netto system*;
(vi) *the Witt design of one of the five Mathieu groups.*

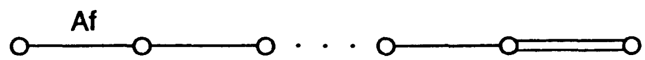
4.3. *The diagram* $\mathsf{L}_{n-1} \cdot \mathsf{C}_2$

The diagram L_n can be viewed as a generalization of A_n. The next one includes C_n as a special case:

$\mathsf{L}_{n-1} \cdot \mathsf{C}_2$ 

(rank $n \geqslant 3$)

The following are special cases of the above:

$\mathsf{Af} \cdot \mathsf{C}_{n-1}$ 

($n \geqslant 3$)

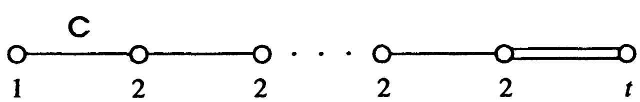
$\mathsf{c}^{n-2} \cdot \mathsf{C}_2$ 

($n \geqslant 3$)

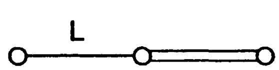
$\mathsf{c}^{n-3} \cdot \mathsf{C}_3$ 

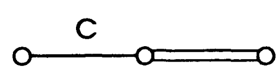
($n \geqslant 4$)

Note that $\mathsf{Af} \cdot \mathsf{C}_{n-1}$ with orders $1, 2, 2, \ldots 2, t$ coincides with

$\mathsf{c}^1 \cdot \mathsf{C}_{n-1}$ 

We write $\mathsf{L} \cdot \mathsf{C}_{n-1}$ and $\mathsf{C} \cdot \mathsf{C}_{n-1}$ for short, instead of $\mathsf{L}_2 \cdot \mathsf{C}_{n-1}$ and $\mathsf{c}^1 \cdot \mathsf{C}_{n-1}$. In particular, $\mathsf{L} \cdot \mathsf{C}_2$ and $\mathsf{c} \cdot \mathsf{C}_2$ denote the following diagrams:





We now describe some examples for $\mathsf{L}_{n-1} \cdot \mathsf{C}_2$, not of type C_n.

4.3.1. *Affine polar spaces and their quotients.* Affine polar spaces are defined in Chapter 12, 8.5. Every affine polar space is obtained by removing a hyperplane H (Chapter 12, Section 3) from a thick-lined polar space $\mathcal{P}$ of rank $\geqslant 3$. The affine polar space $\Gamma = \mathcal{P} - H$ inherits the Intersection Property (IP) from $\mathcal{P}$ and belongs to $\mathsf{Af} \cdot \mathsf{C}_{n-1}$

$\mathsf{Af} \cdot \mathsf{C}_{n-1}$ 

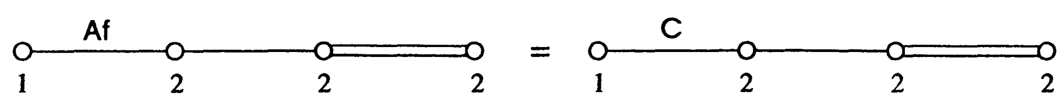
where n is the rank of $\mathcal{P}$. Furthermore, $\Gamma = \mathcal{P} - H$ determines uniquely the polar space $\mathcal{P}$ and its hyperplane H, up to isomorphisms (Cohen and Shult [1990]). Hence Aut(Γ) can be identified with the setwise stabilizer Aut$(\mathcal{P})_H$ of H in Aut($\mathcal{P}$). Γ is also 2-simply connected (Pasini [1990b], Corollary of Theorem 5). Therefore every 2-quotient of Γ

arises by factorizing over a subgroup A of $\mathrm{Aut}(\mathcal{P})_H$, and Γ/A is flag-transitive if and only if the normalizer of A in $\mathrm{Aut}(\mathcal{P})_H$ is flag-transitive in Γ by a general result on 2-covers of chamber systems (see Ronan [1980]). The Intersection Property (IP) holds in Γ/A if and only if A is contained in the elementwise stabilizer of H in $\mathrm{Aut}(\mathcal{P})$ (see Pasini [1990c]). If this is the case, then we say that the quotient Γ/A is *standard.* Of course, Γ can also be viewed as a standard quotient of itself, the *improper* one (with $A = 1$).

4.3.2. THEOREM (Cuypers and Pasini [1992]). *All geometries belonging to* $\mathsf{Af} \cdot \mathsf{C}_{n-1}$ *with* $n \geqslant 4$ *and satisfying* (IP) *are (possibly improper) standard quotients of affine polar spaces.*

4.3.3. THEOREM (Cuypers [1992]). *All locally finite geometries belonging to* $\mathsf{Af} \cdot \mathsf{C}_2$ *and satisfying* (IP) *are (possibly improper) standard quotients of affine polar spaces.*

4.3.4. *A geometry for the alternating group* Alt(8). See Neumaier [1984]. There are 30 ways to build a model of AG(3, 2) on a set S of 8 objects. The symmetric group Sym(8) acts on that set of 30 models of AG(3, 2) as a transitive group of permutations, whereas the alternating group Alt(8) has two orbits of size 15 on that set of models of AG(3, 2). Let V be one of those orbits. Take S as set of points, all unordered pairs of elements of S as lines, all unordered quadruples of elements of S as planes and V as set of 3-spaces. Define the incidence relation between points, lines and planes in the natural way, by symmetrized inclusion, state that all points and all lines are incident with all 3-spaces and that a plane u and an element $v \in V$ are incident iff u is one of the planes of the model v of AG(3, 2). Thus, we obtain a geometry Γ with diagram $\mathsf{Af} \cdot \mathsf{C}_3$ and orders 1, 2, 2, 2,



admitting Alt(8) as flag-transitive automorphism group. Note that residues of points of Γ are isomorphic to the Alt(7)-geometry. We call Γ the *Neumaier geometry.*

4.3.5. *Four families for the diagram* $\mathsf{c}^{n-2} \cdot \mathsf{C}_2$

(a) It is well known that a model of the generalized quadrangle $W(2)$ of order 2 can be constructed taking as points the unordered pairs of objects of a set S of size 6 and as lines the partitions of S in 3 pairs (Chapter 9, 5.1.1). The exceptional isomorphism Sym(6) $\cong$ Sp(4, 2) is evident from that model of $W(2)$. We can generalize that construction starting with a set S of size $2n+2$ ($n \geqslant 2$), taking as elements of type $i = 0, 1, \ldots, n-2$ all sets of $i+1$ mutually disjoint unordered pairs of elements of S and as elements of type $n-1$ the partitions of S in $n+1$ pairs. Define the incidence relation as symmetrized inclusion. Then we obtain a geometry $\Gamma_n^{(1)}$ of rank n belonging to $\mathsf{c}^{n-2} \cdot \mathsf{C}_2$ (the symbol $\mathsf{c}^0 \cdot \mathsf{C}_2$ should be read as C_2) and with orders $1, 1, \ldots, 1, 2, 2$. We have $\mathrm{Aut}(\Gamma_n^{(1)}) = \mathrm{Sym}(2n+2)$, flag-transitive on Γ. However, $\mathrm{Alt}(2n+2)$ also acts flag-transitively on Γ_n.

Needless to say, $\Gamma_2^{(1)} = W(2)$. Note also that $\Gamma_n^{(1)}$ is 2-simply connected (Meixner [1991]) and satisfies (IP) for every $n \geqslant 3$. Therefore, $\Gamma_3^{(1)}$ is an affine polar space, by Theorem 4.3.3. Actually, it is the affine polar space obtained from the quadric $Q_6(2)$ by deleting a hyperplane H of PG(6, 2) such that $H \cap Q_6(2) = Q_5^+(2)$.

(b) Let f be a nonsingular quadratic form in PG(n+1, 3) ($n \geqslant 3$) with discriminant -1. We define a graph $\mathcal{G}$, taking as vertices the points of PG($n+1$, 3) of norm 1 with respect to f, and stating that two of those points are adjacent in $\mathcal{G}$ precisely when the line joining them in PG($n+1$, 3) is tangent to the quadric defined by f. Take the vertices, the edges and the maximal cliques of $\mathcal{G}$ as elements of type 0,1 and $n-1$, respectively, and the k-cliques of $\mathcal{G}$ ($k = 3, 4, \ldots, n-1$) as elements of type $k-1$. Define the incidence relation as symmetrized inclusion. Then we obtain a geometry $\Gamma_n^{(2)}$ for $\mathrm{c}^{n-2} \cdot \mathrm{C}_2$ with orders $1, 1, \ldots, 1, 2, 2$. The following is a (minimal) flag-transitive automorphism group of $\Gamma_n^{(2)}$, for $n = 3, 4, 5, 6, 7, 8, \ldots$:

$$\mathrm{O}(5,3), \mathrm{O}^+(6,3), \mathrm{O}(7,3), \mathrm{O}^-(8,3), \mathrm{O}(9,3), \mathrm{O}^+(10,3), \ldots$$

$\Gamma_3^{(2)}$ and $\Gamma_4^{(2)}$ are 2-simply connected and $\Gamma_n^{(2)}$ is $(n-1)$-simply connected for every $n \geqslant 6$ (see Meixner [1991]). On the other hand, $\Gamma_5^{(2)}$ admits a 2-simply connected 3-fold 4-cover where the central nonsplit extension $3 \cdot \mathrm{O}(7,3)$ acts flag-transitively (Meixner [1991]). We denote that 4-cover of $\Gamma_5^{(2)}$ by $3\Gamma_5^{(2)}$. $\Gamma_n^{(2)}$ satisfies (IP) for all $n \geqslant 3$. Therefore $\Gamma_3^{(2)}$ is an affine polar space, by Theorem 4.3.3. Actually it is the affine polar space obtained from $Q_6(2)$ by deleting a hyperplane H of PG(6, 2) such that $H \cap Q_6(2) = Q_5^-(2)$. The minimal flag-transitive automorphism group of this affine polar space is the stabilizer $\mathrm{O}^-(6,2)$ ($\cong \mathrm{U}(4,2)$) of H in $\mathrm{O}(7,2)$. On the other hand, $\mathrm{O}(5,3)$ is the minimal flag-transitive automorphism group of $\Gamma_3^{(2)}$. Therefore, $\mathrm{O}(5,3) \cong \mathrm{O}^-(6,2)$ (i.e. $\mathrm{Sp}(4,3) \cong \mathrm{U}(4,2)$), as is otherwise well known.

(c) We can repeat the previous construction starting from a nonsingular quadratic form of discriminant 1 in PG($n+2$, 3) ($n \geqslant 2$), now obtaining a geometry $\Gamma_n^{(3)}$ belonging to $\mathrm{c}^{n-2} \cdot \mathrm{C}_2$ and with orders $1, 1, \ldots, 1, 4, 2$. The sequence of minimal flag-transitive automorphism groups is similar to the preceding one:

$$\mathrm{O}(5,3), \mathrm{O}^-(6,3), \mathrm{O}(7,3), \mathrm{O}^+(8,3), \mathrm{O}(9,3), \mathrm{O}^-(10,3), \ldots$$

for $n = 2, 3, 4, 5, 6, 7, \ldots$.

The geometry $\Gamma_n^{(3)}$ is $(n-1)$-simply connected for every $n \geqslant 4$ (see Meixner [1991]). On the other hand, $\Gamma_3^{(3)}$ admits a 2-simply connected 3-fold 2-cover where a central nonsplit extension $3 \cdot \mathrm{O}^-(6,3)$ of $\mathrm{O}^-(6,3)$ acts flag-transitively (Meixner [1991]; see also Del Fra, Ghinelli, Meixner and Pasini [1990]). We denote that cover of $\Gamma_3^{(3)}$ by $3\Gamma_3^{(3)}$. As $\Gamma_3^{(3)}$ is not simply connected, $\Gamma_n^{(3)}$ need not be 2-simply connected when $n \geqslant 4$, even if it is $(n-1)$-simply connected. Actually, a 3^7-fold 2-cover is known for $\Gamma_5^{(3)}$, where the subgroup $3^7 \cdot \mathrm{O}(7,3)$ of Fi_{24}' acts flag-transitively (Meixner [1991]). Note that $\Gamma_2^{(3)}$ is nothing but the generalized quadrangle $H_3(2^2)$ of order (4,2) (compare Chapter 9, 5.1.1). We are again facing the exceptional isomorphism $\mathrm{O}(5,3) \cong \mathrm{U}(4,2)$ (i.e. $\mathrm{Sp}(4,3) \cong \mathrm{U}(4,2)$).

(d) We now start from the Hermitian variety $H_{n+1}(2^2)$ in $\mathrm{PG}(n+1,4)$ ($n \geqslant 2$) and define a graph $\mathcal{G}$ taking the nonisotropic points of $\mathrm{PG}(n+1,4)$ as vertices and orthogonal pairs as edges. As in (b), we take the vertices, the edges and the maximal cliques of $\mathcal{G}$ as elements of type 0,1 and $n-1$, respectively, and the k-cliques of $\mathcal{G}$ as elements of type $k-1$, for $k = 2, 3, \ldots, n-1$. We obtain a geometry $\Gamma_n^{(4)}$ belonging to $\mathrm{c}^{n-2} \cdot \mathrm{C}_2$ (to C_2 when $n = 2$) and with orders $1, 1, \ldots, 1, 3, 3$. The minimal flag-transitive automorphism group of $\Gamma_n^{(4)}$ is $\mathrm{U}_{n+2}(2)$.

$\Gamma_3^{(4)}$ is 2-simply connected and $\Gamma_n^{(4)}$ is $(n-1)$-simply connected for all $n \geqslant 5$ (see Meixner [1991]). On the other hand, $\Gamma_4^{(4)}$ admits a 2-simply connected 4-fold 2-cover, which we denote by $2^2\Gamma_4^{(4)}$, and where $2^2 \cdot \mathrm{U}(6,2)$ acts flag-transitively (Meixner [1991]). Thus, $\Gamma_n^{(4)}$ might not be 2-simply connected when $n \geqslant 5$, even if it is $(n-1)$-simply connected. $2^2\Gamma_4^{(4)}$ also admits a flag-transitive 2-quotient, where $2 \cdot \mathrm{U}(6,2)$ acts and which is a 2-fold 2-cover of $\Gamma_4^{(4)}$ (see Meixner [1991]). We denote it by $2\Gamma_4^{(4)}$.

Note that $\Gamma_2^{(4)}$ is the generalized quadrangle $W(3)$ of order 3 and symplectic type (Chapter 9, 5.1.2). Thus, we are facing the exceptional isomorphism $\mathrm{U}(4,2) \cong \mathrm{Sp}(4,3)$, once more.

4.3.6. *Ten sporadic examples of type* $\mathrm{c}^{n-2} \cdot \mathrm{C}_2$ *and* $\mathrm{c}^{n-3} \cdot \mathrm{C}_3$. We only give few informations on the next examples: diagrams, orders and (flag-transitive) automorphism groups; however, we suggest references for more information. The examples we are going to mention arise in connection with the sporadic simple groups McL, Co_3, Suz, Co_1, HS, Fi_{22}, Fi_{23} and Fi'_{24}.

It will be useful to have stated the following conventions: elements corresponding to the first and last node of the diagram will be called *points* and *blocks*, respectively:

(a) Diagram $\mathrm{c} \cdot \mathrm{C}_2$; orders 1,3,9 (hence residues of points are isomorphic to $Q_5^-(3)$; see Chapter 9, 5.1.2). Flag-transitive automorphism groups: McL and Aut(McL) (= McL·2). See Buekenhout and Hubaut [1977], Buekenhout [1985], (22), Del Fra et al. [1990], Yoshiara [1991], Pasini [1990d], Goethals and Seidel [1975].

(b) Diagram $\mathrm{c}^2 \cdot \mathrm{C}_2$; orders 1,1,3,9. Residues of points are isomorphic to the above geometry for McL. Unique flag-transitive automorphism group: Co_3. See Buekenhout [1985], (23), Meixner [1994c].

(c) Diagram $\mathrm{c}^2 \cdot \mathrm{C}_2$; orders 1,1,3,9. Group: $2 \times \mathrm{Co}_3$. This geometry is the (2-fold) universal 2-cover of the previous geometry for Co_3.

(d) Diagram $\mathrm{c} \cdot \mathrm{C}_2$; orders 1,9,3 (hence residues of points are isomorphic to $H_3(3^2)$; see Chapter 9, 5.2). Flag-transitive automorphism groups: Suz and Aut(Suz)(= Suz · 2). See Buekenhout [1985], (6), Weiss and Yoshiara [1990], Del Fra et al. [1990], Yoshiara [1991], Pasini [1990d], Patterson and Wong [1976].

(e) Diagram $\mathrm{c}^2 \cdot \mathrm{C}_2$; orders 1,1,9,3. Residues of points are isomorphic to the above geometry for Suz. Unique flag-transitive automorphism group: Co_1. See Buekenhout [1985], (7), Meixner [1994c]. This geometry admits a 2^{24}-fold cover, where a non-split extension $2^{24} \cdot \mathsf{Co}_1$ acts. It is not known for the time being if that cover is 2-simply connected.

(f) Diagram $\mathsf{c} \cdot \mathsf{C}_2$; orders 1,9,3 (residues of points isomorphic to $H_3(3^2)$). Unique flag-transitive automorphism group: Aut(HS) (= HS · 2). See Yoshiara [1990], Weiss Yoshiara [1990], Pasini [1991b], Del Fra et al. [1990].

(g) Diagram $\mathsf{c} \cdot \mathsf{C}_3$; orders 1,4,4,2. Residues of points are isomorphic to the Hermitian variety $H_5(2^2)$ and residues of blocks are isomorphic to the Witt design for M_{22} (see 4.2.3). Flag-transitive automorphism group: Fi_{22} and Aut(Fi_{22}) (= $\mathsf{Fi}_{22} \cdot 2$). See Buekenhout and Hubaut [1977], Buekenhout [1985], (46), Meixner [1991], Van Bon and Weiss [1994], Pasechnik [1994b], Fischer [1971].

(h) Diagram $\mathsf{c}^2 \cdot \mathsf{C}_3$; orders 1,1,4,4,2. Residues of points are isomorphic to the above geometry for Fi_{22} and residues of blocks are isomorphic to the Witt design for M_{23} (see 4.2.3). Unique flag-transitive automorphism groups: Fi_{23}. See Buekenhout [1985], Meixner [1991], Van Bon and Weiss [1993], Pasechnik [1994b], Fischer [1971].

(i) Diagram $\mathsf{c}^3 \cdot \mathsf{C}_3$; orders 1,1,1,4,4,2. Residues of points are isomorphic to the above geometry for Fi_{23} and residues of blocks are isomorphic to the Witt design for M_{24} (see 4.2.3). Flag-transitive automorphism groups: Fi'_{24} and Aut(Fi'_{24}) (= Fi_{24}). See references given in (h).

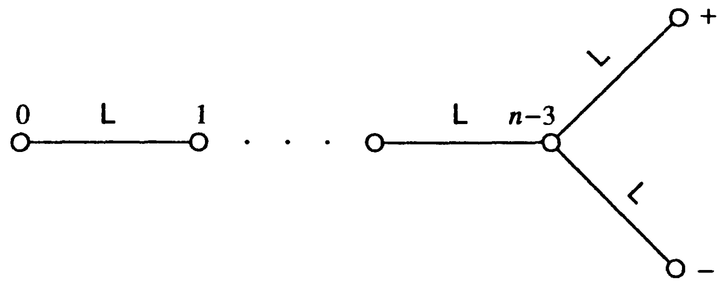
(j) Diagram $\mathsf{c}^3 \cdot \mathsf{C}_3$; orders 1,1,1,4,4,2. Groups: the central nonsplit extension $3 \cdot \mathsf{Fi}'_{24}$ of Fi'_{24} and $3 \cdot$ Aut(Fi'_{24}) = Aut($3 \cdot \mathsf{Fi}'_{24}$). This geometry is the (3-fold) universal 2-cover of the previous geometry for Fi'_{24} (see Ronan [1981]).

4.3.7. *Extensions of nonclassical generalized quadrangles*
There are just three flag-transitive simply connected geometries of type $\mathsf{c} \cdot \mathsf{C}_2$ satisfying the Intersection Property and with residues of points isomorphic to the (nonclassical) generalized quadrangle $T_2^*(O)$ of order (3,5) or to its dual (Yoshiara [1993]; see Chapter 9, 9.5.3(a), for the definition of $T_2^*(O)$). The example with point-residues of type $T_2^*(O)$ has automorphism group $2^{1+8} \cdot (\mathrm{Alt}(5) \times \mathrm{Alt}(5))2$. The automorphism groups of the other two examples are $2^{1+12} : 3\,\mathrm{Sym}(7)$ and $2^{6+6} : \mathrm{PSL}_3(2)$. Note that only four flag-transitive nonclassical finite thick generalized quadrangles are presently known: $T_2^*(O)$ with O a conic of PG(2, 4) plus its nucleus, and $T_2^*(O)$ with O the Lunelli–Sce hyperoval in PG(2, 16) (see Chapter 7, 1.3(d)), and their duals. If we discard the flag-transitivity, then we find more classes of geometries of type $\mathsf{c} \cdot \mathsf{C}_2$ with residues of points isomorphic to nonclassical finite thick generalized quadrangles. One infinite family with point-residues of type AS(q) (see Chapter 9, 4.3.2) has been constructed and characterized by Thas [1985]; proper 2-covers of the examples by Thas have been constructed by Cameron [1991]. There is also an infinite family with point residues of type $T_2^*(O)$ (see Pasini [1994c]), which does not seem to have anything to do with the 3 flag-transitive examples by Yoshiara mentioned above.

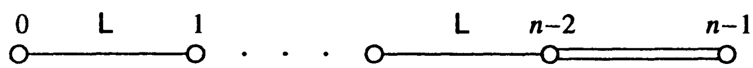
4.3.8. *Folding, unfolding and others.* The examples mentioned till now are thick at the last node of the diagram. However, many (even flag-transitive) geometries of type $\mathsf{L}_{n-1} \cdot \mathsf{C}_2$ exist that are thin at the last node of the diagram. The reader should consult Cameron, Hughes and Pasini [1990], Cameron [1991], Del Fra et al. [1990], Meixner and Pasini [1993, 1994b], Pasini and Yoshiara [1992], 6.2, 6.3, for examples of this kind.

Most of them arise from geometries belonging to the following diagrams, by a folding construction similar to that by which a polar space thin at the last node of C_n is obtained from a building of type D_n by folding.

$L(D)_n$



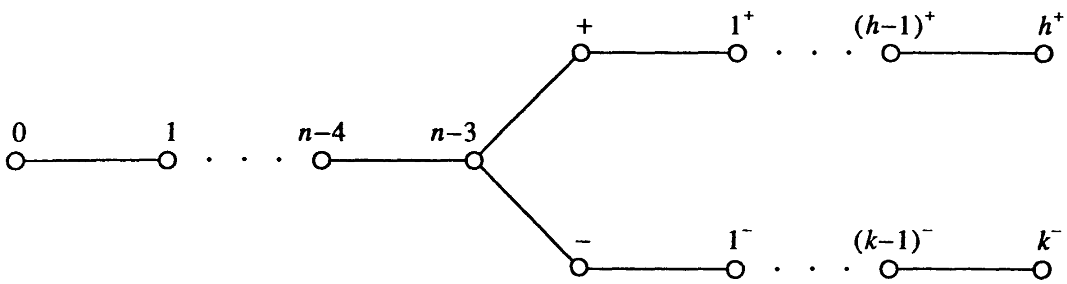
Indeed, given a geometry B belonging to $L(D)_n$ ($n \geqslant 3$), we can form a new geometry $\Gamma = \mathrm{Fl}(\Delta)$, which we call the *folding* of B, taking as elements of type $0, 1, \ldots, n-3$ the elements of B of that type, as elements of type $n-2$ the flags of B of type $\{+, -\}$ and as elements of type $n-1$ the elements of B of type $+$ and $-$, with the incidence relation inherited from B. The geometry Γ belongs to $L_{n-1} \cdot C_2$



and it is thin at the type $n-1$. We call Δ the *unfolding* of Γ and we denote it by $\mathrm{Unf}(\Gamma)$.

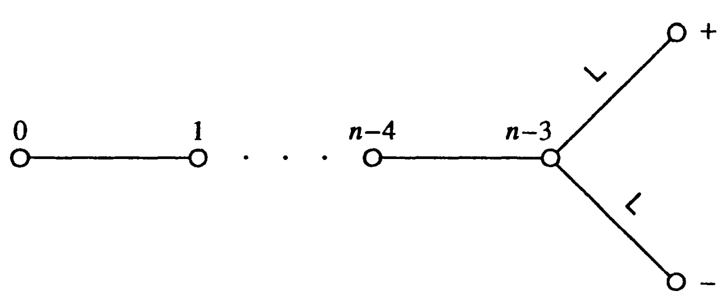
We now describe some examples for $L(D)_n$.

(a) Let B^* be a building belonging to the following Coxeter diagram of rank $n+h+k$



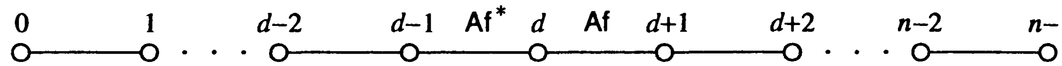
($0, 1, \ldots, n-3, +, -, 1^+, \ldots, h^+, 1^-, \ldots, k^-$ are types). By deleting the elements of B^* of type $1^+, \ldots, h^+, 1^-, \ldots, k^-$ we obtain a geometry B belonging to the following special case of $L(D)_n$:

$L_{n-1} \cdot L(D)_2$



where the rank 2 residues corresponding to the two horns of the diagram are point-line systems of projective geometries. We call B the $\{1^+,\ldots,h^+,1^-,\ldots,k^-\}$-*truncation* of B^*.

(b) Let $S_d(n,K)$ be the d-shadow space of PG(n,K), $0 \leqslant d \leqslant n-1$, $n \geqslant 3$ (see Chapter 12, Section 5; we take $0,1,\ldots,n-1$ as types for PG(n,k), as usual). The d-dimensional subspaces of PG(n,K) are the points of $S_d(n,K)$. The other elements of PG(n,K) can be identified with some subspaces of $S_d(n,K)$. Let H be a hyperplane of $S_d(n,K)$ (Chapter 12, Section 3) and let $G_{H,d}(n,K)$ be the geometry obtained removing from PG(n,K) those elements that, viewed as points or subspaces of $S_d(n,K)$, are contained in H. If $d=0$ or $n-1$, then $G_{H,d}(n,K)$ is an affine or dual affine geometry. Let $1 \leqslant d \leqslant n-2$. Then $G_{H,d}(n,K)$ belongs to the following diagram of rank n, which we call $\mathsf{A}_d \cdot \mathsf{Af}^* - \mathsf{AF} \cdot \mathsf{A}_{n-d-1}$:

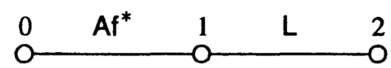


We call $G_{H,d}(n,K)$ an *affine d-Grassmann geometry*. If we truncate $G_{H,d}(n,K)$ deleting the elements of type $> d+1$, then we obtain a geometry for the following diagram:

$\mathsf{A}_d \cdot \mathsf{Af}^* \cdot \mathsf{L}$

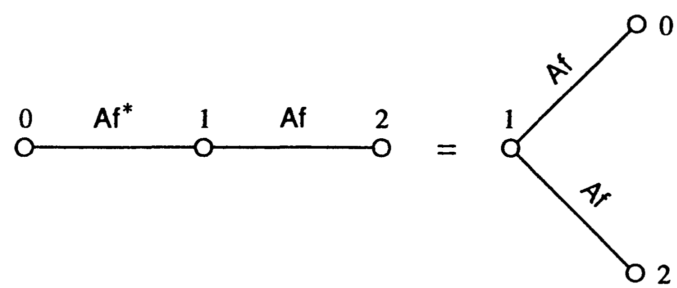
In particular, if $d=1$, then we get the following special case of $\mathsf{L(D)}_3$:

$\mathsf{Af}^* \cdot \mathsf{L}$



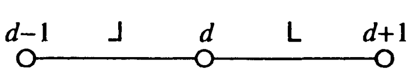
In particular, if $d=1$ and $n=3$, then $G_{H,d}(n,K)$ belongs to the diagram

$\mathsf{Af}^* \cdot \mathsf{Af}$



and its folding (with respect to the type 1) is an affine polar space (indeed $S_1(3,K)$ is a polar space).

More generally, if we truncate $G_{H,d}(n,K)$ removing all elements of type $\leqslant d-2$ or $\geqslant d+2$, then the geometry obtained in this way belongs to $\mathsf{L(D)}_3$:



A classification of affine d-Grassmann geometries amounts to a classification of the hyperplanes of $S_d(n,K)$. This has been done in Shult [1992], Hall and Shult [1993]. If K is commutative, then all hyperplanes of $S_d(n,K)$ can be obtained from alternating $(d+1)$-linear forms in Vect$(n+1,K)$, taking the totally isotropic $(d+1)$-dimensional linear subspaces of Vect$(n+1, K)$. This includes the following as a special case: fix a subspace U of PG(n, K) of dimension $n-d-1$ and take all d-dimensional subspaces X such that $U \cap X \neq \varnothing$. When K is noncommutative, the latter is the only way to obtain a hyperplane of $S_d(n, K)$.

(c) Let $Q = Q_3^+(q)$ as embedded in PG$(3, q)$, $q = 2^h$, $h \geqslant 2$. Take as elements of type $+$ (of type $-$) the points of PG$(3, q)$ not in Q (the planes of PG$(3, q)$ secant for Q) and as elements of type 0 the lines of PG$(3, q)$ external to Q. Keep the incidence relation inherited from PG$(3, q)$, except for requiring the additional condition $x^\perp \neq X$ for two elements x, X of type $+$ and $-$, respectively, to be incident. Thus, we obtain a geometry of type $\mathsf{L(D)}_3$ where residues of type $\{0, +\}$ and $\{0, -\}$ are Witt–Bose–Shrikhande spaces (see 1.9).

We will now state some classification theorems.

4.3.9. THEOREM. *Let Γ be a locally finite flag-transitive geometry belonging to* $\mathsf{L}_{n-2} \cdot \mathsf{C}_3$ (*see below*) *with $n \geqslant 4$. Then Γ is one of the following*:

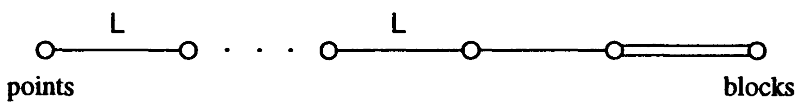
(i) *a geometry of type C_n thin at all types, except possibly at the type corresponding to the last node of the diagram*;

(ii) *a polar space of rank n*;

(iii) *a (possibly improper) standard quotient of an affine polar space of rank n*;

(iv) *the Neumaier geometry* (4.3.4);

(v) *one of the geometries for* Fi_{22}, Fi_{23}, Fi'_{24} *and* $3 \cdot \mathsf{Fi}'_{24}$ *mentioned in* 4.3.6 (g)–(j).

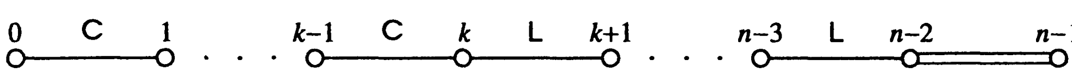
$\mathsf{L}_{n-2} \cdot \mathsf{C}_3$ 

$(n \geqslant 4)$

OUTLINE OF THE PROOF. We can get control over residues of blocks of Γ by Theorem 4.2.6. By that theorem we see that the diagram of Γ is actually C_n, $\mathsf{Af} \cdot \mathsf{C}_{n-1}$ or $\mathsf{c}^m \cdot \mathsf{C}_3$, $m = 1, 2$ or 3. In the thick-lined C_n case we have (ii) by Theorem 2.4.1. The thin-lined C_n case is (i). Unfortunately, not so much is known on this case in general. When the diagram of Γ is $\mathsf{Af} \cdot \mathsf{C}_{n-1}$, it is not difficult to prove that either Γ is as in (iv) or (IP) holds in Γ (Pasini and Yoshiara [1992], Proposition 3.3). In the latter case, (iii) follows from Theorem 4.3.2.

Thus, we are left with the case of $\mathsf{c}^m \cdot \mathsf{C}_3$, $m = 1, 2$ or 3, which is the hard one. The reader should consult Buekenhout and Hubaut [1977] and Meixner [1991] (also Van Bon and Weiss [1994] or Pasechnik [1994b] for the proof that Γ is as in (v).

We shall need the following convention. Given a geometry Γ_0 of type $\mathsf{L}_{m-1} \cdot \mathsf{C}_2$ and a geometry Γ_1 belonging to the following special case of $\mathsf{L}_{n-1} \cdot \mathsf{C}_2$, with $n = m + k$, $k > 0$:



$(0, 1, \ldots, k-1, k, k+1, \ldots, n-1$ are the types), if the residues of all flags of Γ_1 of type $\{0, 1, \ldots, k-1\}$ are isomorphic to Γ_0, then we say that Γ_1 is a *circular extension* of Γ_0.

In the next theorem Γ is a flag-transitive locally finite geometry of type $\mathsf{L}_{n-1} \cdot \mathsf{C}_2$, with $n \geqslant 3$. We assume that the residues of Γ of type C_2 are classical generalized quadrangles (according to Chapter 9, grids and dual grids are considered as nonclassical generalized quadrangles, even if they are of type $Q_3^+(q)$). When $n > 3$, we also assume that Γ is not of type $\mathsf{L}_{n-2} \cdot \mathsf{C}_3$, to avoid any overlapping with the previous Theorem 4.3.9.

4.3.10. THEOREM. *Let Γ be as above. Then Γ is one of the following*:

(i) *a polar space of rank $n = 3$*;

(ii) *the* Alt(7)-*geometry*;

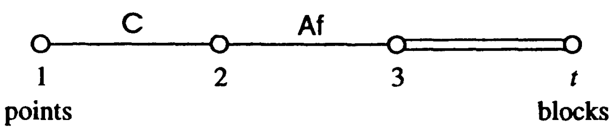
(iii) *a (possibly improper) standard quotient of an affine polar space of rank $n = 3$*;

(iv) *one of the geometries for* McL, Co_3, $2 \times \mathsf{Co}_3$, Suz, Co_1 *and* Aut(HS) *mentioned in* 4.3.6 (a)–(f) *or a cover of the geometry for* Co_1;

(v) *a member $\Gamma_n^{(i)}$ ($i = 1, 2, 3$ or 4, $n \geqslant 3$) of one of the four families described in* 4.3.5;

(vi) *one of the geometries $3\Gamma_5^{(2)}$, $3\Gamma_3^{(3)}$, $2^2\Gamma_4^{(4)}$ or $2\Gamma_4^{(4)}$ mentioned in* 4.3.5 ((b), (c) *and* (d), *respectively*) *or a circular extension of one of them*;

(vii) *an (unknown) geometry with diagram and orders as follows*:



$t = 3$ or $t = 9$

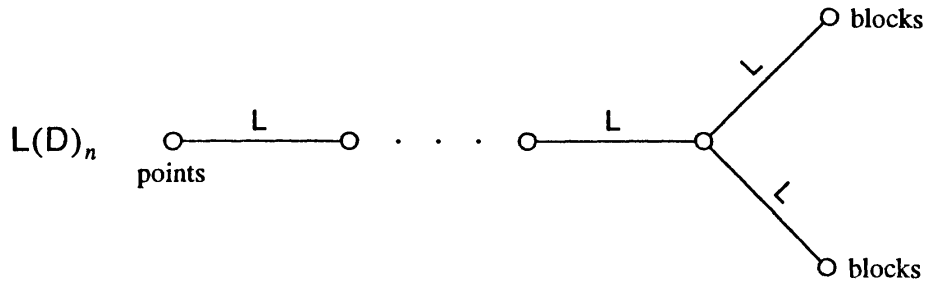
REMARK. By (iii) of this theorem (and Theorem 4.2.6) residues of points (resp., of blocks) in (vii) are possibly improper standard quotients of an affine polar space (resp., the classical inversive plane arising from $Q_3^-(3)$). The 3^7-fold cover of $\Gamma_4^{(3)}$ mentioned in 4.3.5(c) is the only example known for (vi) besides $3\Gamma_5^{(2)}$, $3\Gamma_3^{(3)}$, $2^2\Gamma_4^{(4)}$ and $2\Gamma_4^{(4)}$.

OUTLINE OF THE PROOF. Combining the classification of finite flag-transitive linear spaces (see 1.9) with Seitz's Theorem 3.2.3 ((iii), (iv)) it is not difficult to prove that, if $n = 3$, then Γ belongs to C_3, $\mathsf{Af} \cdot \mathsf{C}_2$ or $\mathsf{c} \cdot \mathsf{C}_2$ (Ghinelli [1993]). If Γ belongs to C_3, then we have (i) or (ii) by Aschbacher's Theorem 2.3.1. If Γ belongs to $\mathsf{Af} \cdot \mathsf{C}_2$, then, again using Theorem 3.2.3 ((iii), (iv)), it is possible to prove that Γ satisfies (IP) (Del Fra [1994]), hence we have (iii) by Theorem 4.3.3.

Let Γ belong to $\mathsf{c} \cdot \mathsf{C}_2$, with orders $1, s, t$ and $s > 2$ (when $s = 2$ the diagram is $\mathsf{Af} \cdot \mathsf{C}_2$, whence we are in (iii)). Combining Seitz's Theorem 3.2.3 ((iii), (iv)) with a theorem of Suzuki [1966] and Tits [1971] on transitive 1-point extensions of PSL$(2, q)$, it is possible to prove that $(s, t) = (3, 3), (3.9), (4.2)$ or $(9,3)$ (see Del Fra et al. [1990], Buekenhout and Hubaut [1977]). If $(s, t) = (3, 3)$ or $(4,2)$, then $\Gamma = \Gamma_3^{(4)}$, $\Gamma_3^{(3)}$ or $3\Gamma_3^{(3)}$ (Buekenhout and Hubaut [1977], Del Fra et al. [1990]; see also Yoshiara [1991], Pasini [1990d]). If $(s, t) = (3, 9)$, then Γ is the geometry for McL of 4.3.6(a) (Buekenhout and Hubaut [1977]; see also Goethals and Seidel [1975]; and Yoshiara [1991], Pasini [1990d]). If $(s, t) = (9, 3)$, then Γ is the geometry for Suz of 4.3.6(d) or the geometry for Aut(HS) of 4.3.6(f) (Weiss and Yoshiara [1990]; see also Yoshiara [1991], Pasini [1994a, 1990d]; Meixner [1994b]).

Assume now that Γ has rank $n \geqslant 4$. Comparing the classification obtained in the rank 3 case and Theorem 4.2.6 it is not difficult to prove that either we have (vii) or that Γ belongs to $c^{n-2} \cdot C_2$ (see Pasini [1994d], Proposition 15.8). In the latter case, Γ is the geometry for Co_3, $3 \times Co_3$ or Co_1 as in (iv) or it is as in (v) or (vi) (Meixner [1994c, 1991]).

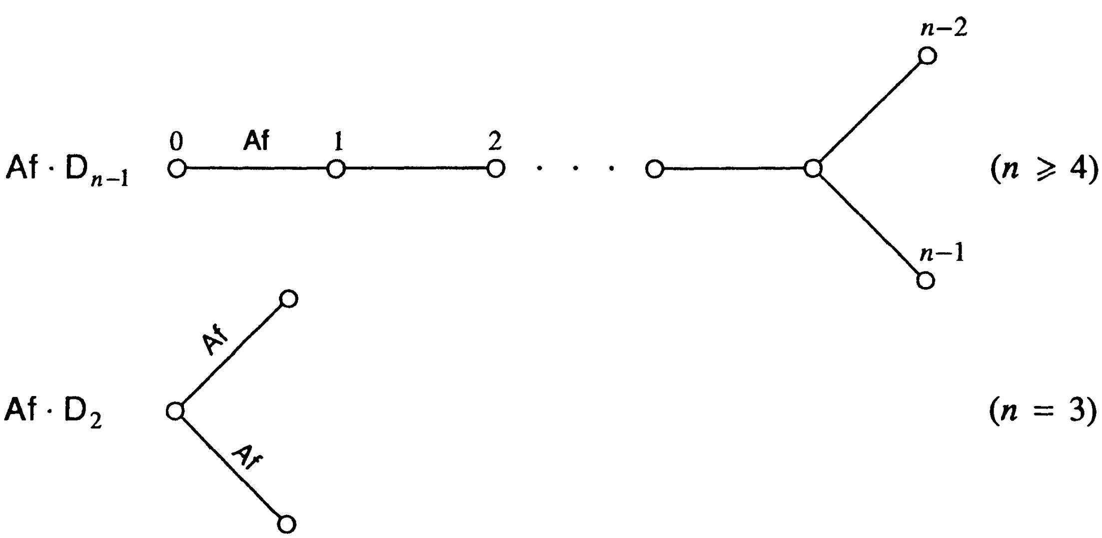
As we have remarked in 4.3.8, many geometries of type $L_{n-1} \cdot C_2$ thin at the last node of the $L_{n-1} \cdot C_2$ diagram are obtained by folding a geometry belonging to $L(D)_n$:



The large number of examples of type $L_{n-1} \cdot C_2$ related to geometries of type $L(D)_n$ is not an entirely coincidental occurrence. Indeed:

4.3.11. LEMMA (Rinauro [1990]). *Every 2-simply connected geometry belonging to* $L_{n-1} \cdot C_2$, *thin at the last node of the diagram and satisfying* (IP), *is obtained by folding a 2-simply connected geometry belonging to* $L(D)_n$.

4.3.12. *Affine sections of* D_n *buildings.* Thus, we turn to $L(D)_n$. First we mention one more family of examples for $L(D)_n$ besides those described in 4.3.8. Let $\mathcal{B}$ be a building of type D_n ($n \geqslant 3$; observe that $D_3 = A_3$). Let $\mathcal{P} = \mathrm{Fl}(\mathcal{B})$ be the polar space obtained folding $\mathcal{B}$ and let H be a hyperplane of $\mathcal{P}$. If we delete H from $\mathcal{P}$ and if we unfold the resulting affine polar space, then we obtain a geometry Γ belonging to the following special case of $L(D)_n$.



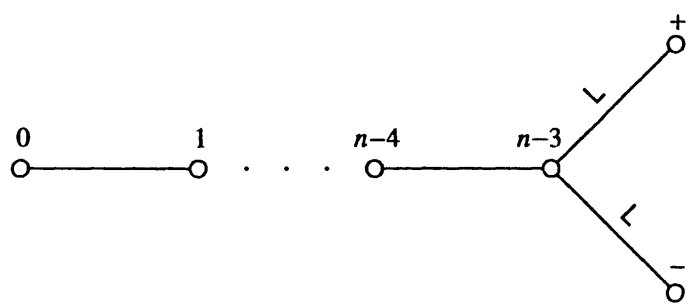
We call Γ an *affine section* of $\mathcal{B}$. The rank 3 case of this section is already discussed in 4.3.8(b): affine sections of D_3 buildings are just affine 1-Grassmann geometries of rank 3.

4.3.13. THEOREM (Pasini and Yoshiara [1992], Proposition 4.3). *All geometries belonging to* $\mathsf{Af}\cdot\mathsf{D}_{n-1}$ ($n \geqslant 3$) *are affine sections of buildings of type* D_n.

4.3.14. *Again on* $\mathsf{L(D)}_n$

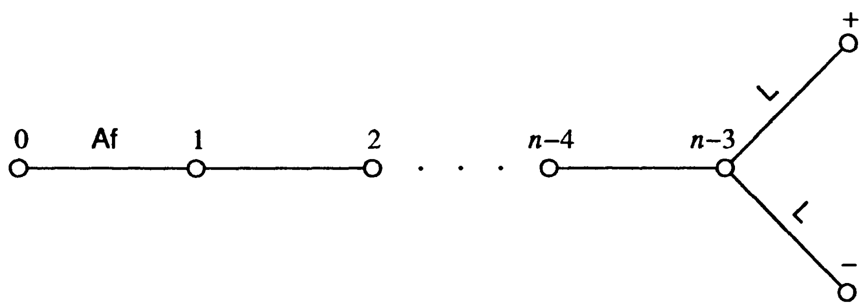
By Theorem 4.3.12 and by Lemma 4.3.11, every (possibly infinite) geometry of type $\mathsf{Af}\cdot\mathsf{C}_2$, thin at the third node of the $\mathsf{Af}\cdot\mathsf{C}_2$ diagram and satisfying (IP) is a (possibly improper) standard quotient of an affine polar space (compare Theorem 4.3.3). Assembling Theorem 4.2.6, Corollary 2.2.3(iii), Theorems 4.3.10 and 4.3.13, it is easy to see that, if Γ is a locally finite flag-transitive geometry belonging to $\mathsf{L(D)}_n$ with $n \geqslant 4$ and if Γ is neither a building of type D_n nor an affine section of a building of type D_n, then Γ belongs to one of the following special cases of $\mathsf{L(D)}_n$.

(1)



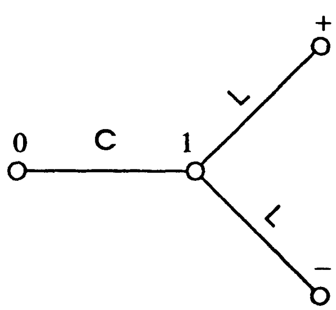
($n \geqslant 4$)

(2)

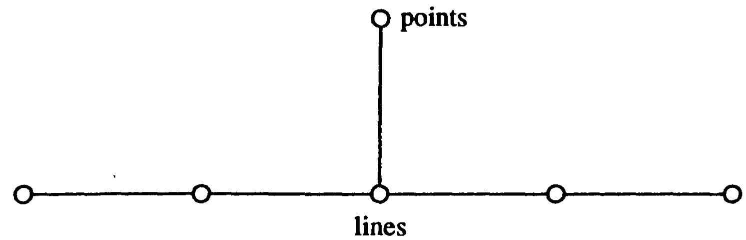


($n \geqslant 4$)

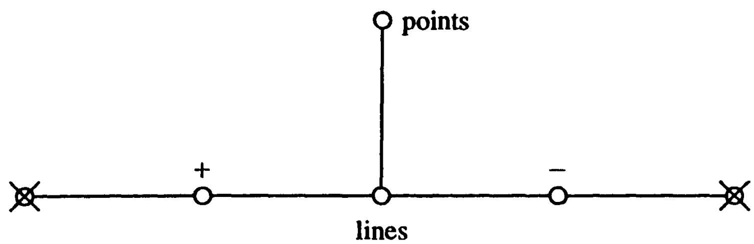
(3)



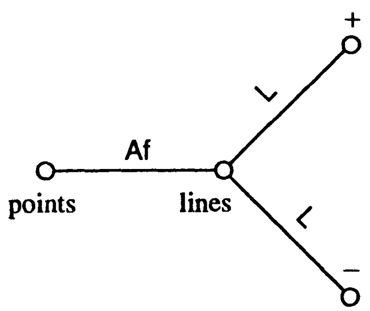
Here residues of type $\{n-3, +\}$ and $\{n-3, -\}$ are point-line systems of projective geometries in (1) and (2), whereas in (3) residues of elements of type $+$ and $-$ are classical WLA-circular spaces or classical locally Netto systems (Theorem 4.2.6, (iv) and (v)). No examples are known for case (3). On the other hand, examples for (2) are easy to construct. For instance, let Γ be a thick building of type E_6 and let H be a hyperplane of the following point-line system of Γ:



Deleting (cf. 4.3.12) the elements of the types crossed out in the next picture



we obtain a geometry with the following diagram:

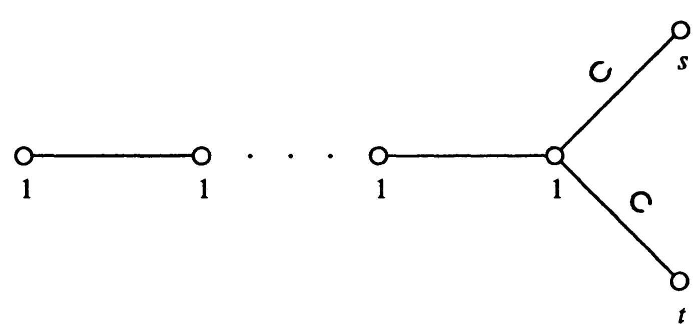


where the residues of the elements of type + and − are truncated affine geometries.

A family of examples for (1) has been described in 4.3.8(a). Needless to say, more examples can be obtained from those of 4.3.8(a) by taking 2-quotients. For instance, Sp(6, 2), which is involved in the Coxeter group of affine type $\tilde{E}_7$, acts flag-transitively on a 2-quotient of the geometry obtained by truncating the Coxeter complex of type $\tilde{E}_7$ (Meixner and Pasini [1993], 2.2, Remark).

Furthermore, there are examples for the following special case of (1) which have nothing to do with those considered in 4.3.8(a):

$\mathrm{c(D)}_n$



$(1, 1, \ldots, 1, s, t$ are orders). For instance, there is a family of geometries belonging to the above diagram $\mathrm{c(D)}_n$ with $s = t = 2$, of arbitrary rank n and with $\mathrm{Sym}(n{+}1)\times\mathrm{Sym}(n{+}1)$ as flag-transitive automorphism group. There is also a family of $\mathrm{c(D)}_n$ geometries with $s = t = 3$, of arbitrary rank n, with $2^{(n+1)n/2}\cdot\mathrm{Sym}(n{+}2)$ as flag-transitive automorphism group (Meixner and Pasini [1994a]). These geometries admit 2-fold quotients when n is odd. The first member of this family appears as a residue of type $\mathrm{c(D)}_3$ in an example of rank 5 admitting the non-split extension $2\cdot\mathrm{M}_{22}$ as flag-transitive automorphism group with point stabilizers isomorphic to $2\cdot\mathrm{PGL}_3(4)$. In all of these examples, the stabilizer of a block acts as $\mathrm{Sym}(n+s-1)$ or $\mathrm{Alt}(n+s-1)$ on the $n+s-1$ points of that block. According to Meixner and Pasini [1994a], if B is a flag-transitive geometry belonging to $\mathrm{c(D)}_n$, with $\mathrm{Sym}(n+s-1)$ or $\mathrm{Alt}(n+s-1)$ induced on residues of blocks, then B is a (possibly improper) 2-quotient, either of a truncation of a Coxeter complex as in 4.3.8(2) or of some $\mathrm{c(D)}_n$ geometry mentioned previously. A similar statement holds for geometries of type $\mathrm{c}^{n-2}\cdot\mathrm{C}_2$ with orders $1, 1, \ldots, 1, s, 1$ and block stabilizer acting as $\mathrm{Alt}(n+s-1)$ or $\mathrm{Sym}(n+s-1)$ on the block residue: the universal 2-cover of such a geometry in the unfolding of some $\mathrm{c(D)}_n$ geometry described earlier (Meixner and Pasini [1994a]). Observe that for $n \geqslant 7$, the action induced on a block residue of a flag-transitive $\mathrm{c(D)}_n$ or $\mathrm{c}^{n-2}\cdot\mathrm{C}_2$ geometry is necessarily an alternating or symmetric group. Thus, flag-transitive $\mathrm{c(D)}_n$ or $\mathrm{c}^{n-2}\cdot\mathrm{C}_2$ geometries with $n \geqslant 7$ are classified.

The following theorem by Sprague [1985] also gives us some information on $\mathrm{L(D)}_3$.

4.3.15. THEOREM. *Let Γ belong to the following diagram*

$\mathrm{L}^*\cdot\mathrm{A}_{n-2}\cdot\mathrm{L}$ 

$(n \geqslant 3)$

and assume the following:

(i) *the dual of every residue of type* $\{0, 1\}$ *is spanned as a linear space by a finite number of points*;

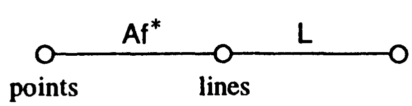
(ii) *the Intersection Property* (IP) *holds in* Γ.

Then Γ is obtained by deleting the elements of dimension $< d$ and $\geqslant n + d$ of a projective geometry of dimension $m \geqslant n$ (possibly $m = \infty$), for some non-negative integer $d \leqslant m - n$.

Actually, Sprague assumes that $n = 3$ and that the rank 1 residues of Γ are finite; however, the above more general statement is implicit in the proof. The diagram $\mathrm{L(D)}_3$ is just the rank 3 case of the above diagram $\mathrm{L}^*\cdot\mathrm{A}_{n-2}\cdot\mathrm{L}$.

However, there are many (even finite) geometries belonging to $\mathrm{L(D)}_3$ that are not truncations of projective geometries as in Theorem 4.3.17 (hence (IP) fails to hold in them); see 4.3.8(b) and 4.3.8(c), for instance.

In particular, no geometry belonging to the following diagram is a truncated projective geometry:

$\mathrm{Af}^*\cdot\mathrm{L}$ 

except, possibly, when lines have just 3 points. Truncated 1-Grassmann geometries are examples for Af* · L (4.3.8(b)). The following is proved in Cuypers [1994c].

4.3.16. THEOREM. *All geometries belonging to* Af* · L *with at least* 4 *points on a line are (possibly truncated) affine* 1*-Grassmann geometries.*

Af* · L geometries with lines of size 3 are classified in Hall [1989a,b]. Actually, more than the above is known.

4.3.17. THEOREM (Cuypers [1994d]). *All geometries belonging to* A_d · Af* · L

(rank $d+2$)

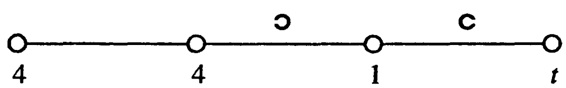
with at least 5 *points on a line are (possibly truncated) affine d-Grassmann geometries.*

If we now assemble the above with Theorem 4.2.6, we come close to a classification of flag-transitive geometries belonging to the following diagram of rank $d \geqslant 4$

A_d · L* · L

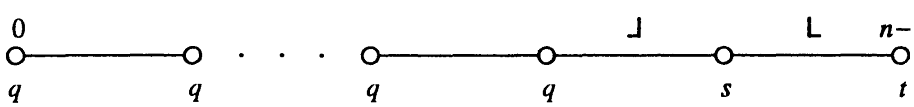
($1 \leqslant q, s, t < \infty$, $q \geqslant 4$). Besides possibly truncated projective geometries and affine d-Grassmann geometries, we can only have geometries of rank 4 with orders $4, 4, 1, t$:

A_2 · c* · c



By Theorem 4.2.6, the dual of the Witt design for M_{23} is the only example for A_2 · c* · c with $t = 1$. If we truncate the Witt design for M_{24}, deleting the points, and dualize, then we obtain an example for A_2 · c* · c with $t = 2$. It is likely that no more flag-transitive examples exist.

4.3.18. THEOREM (Cuypers and Pasini [1994]). *Let Γ be a flag-transitive geometry over the diagram*



$n \geqslant 4$

and assume $q \geqslant 4$. Then Γ is one of:

(i) *a possibly truncated projective geometry,*

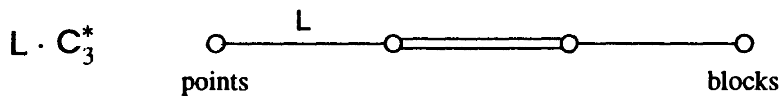
(ii) *a possibly truncated $(n-2)$-affine Grassmann geometry,*

(iii) *the Witt design for* M_{23} *(order* $4, 4, 1, 1$*),*

(iv) *a truncation of the Witt design for* M_{24}.

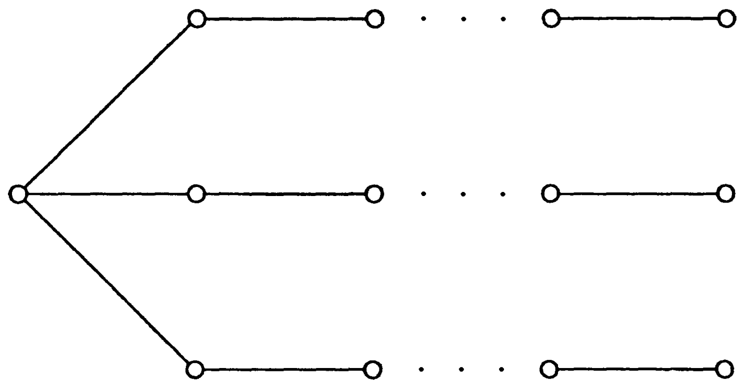
4.4. *The diagram* $\mathsf{L}\cdot\mathsf{C}_3^*$

We now consider the following diagram:

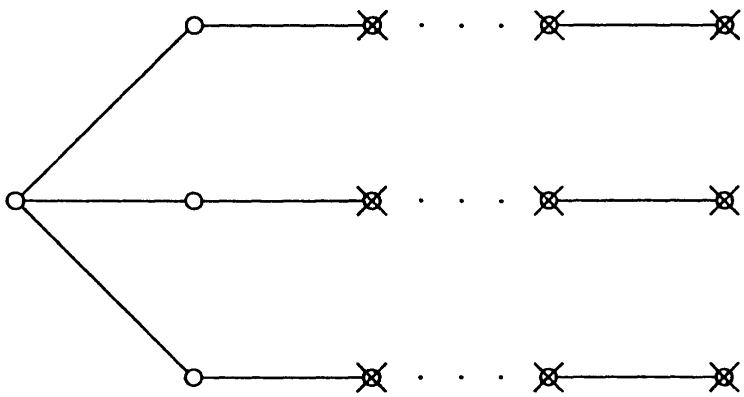


The Coxeter diagram F_4 is a special case. Hence buildings of type F_4 are examples. We describe other examples in the next paragraphs.

4.4.1. *Examples by truncations.* Let B be a building belonging to a Coxeter diagram of the following form:

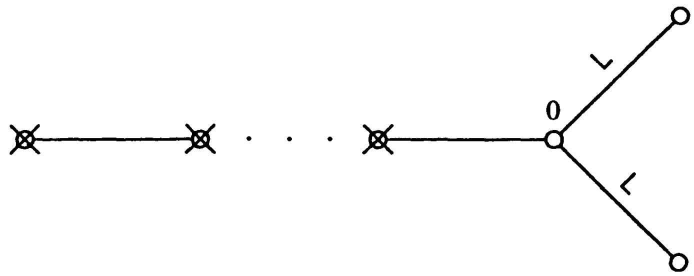


We can truncate B, deleting the elements of the type crossed out on the next picture:



Then we take the shadow space of what is left, with respect to the type 0 (see Chapter 3, Section 3.6; or compare the construction of metasymplectic spaces (Chapter 12) from buildings of type D_4). Thus, we obtain a geometry Γ belonging to $\mathsf{L}\cdot\mathsf{C}_3^*$, thin at the third and fourth node of the diagram $\mathsf{L}\cdot\mathsf{C}_3^*$. Needless to say, 2-quotients of Γ can also be considered. For instance, an example with orders 1,2,1,1 and $Sp_4(3)$ as flag-transitive automorphism group is mentioned in Neumaier [1983] (see also Buekenhout [1985], (78)). Actually, it is a 2-quotient of a geometry obtained in the above way from the Coxeter complex of affine type $\widetilde{\mathsf{E}}_6$ (Pasechnik, private communication).

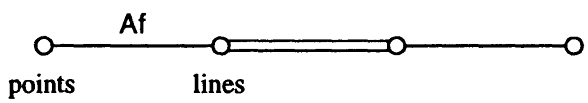
Comparing 4.3.8(a) and 4.3.17, we see that the previous construction can be generalized, starting from any geometry B belonging to the diagram (1) of 4.3.17, truncating the elements of the types crossed out in the next picture



and taking the shadow space with respect to the type 0. We can also modify the previous construction deleting a hyperplane of the 0-shadow space of B, if B is thick and if hyperplanes exist in that space.

4.4.2. *Affine sections of* F_4 *buildings.* The last remark suggests another way to construct examples for $\mathsf{L} \cdot \mathsf{C}_3^*$. Let B be a building of type F_4 with thick lines and let H be a hyperplane of the metasymplectic space of B (Chapter 12, 6.5). Deleting (all elements of B contained in) H we obtain a geometry Γ belonging to the following special case of $\mathsf{L} \cdot \mathsf{C}_3^*$:

$\mathsf{Af} \cdot \mathsf{C}_3^*$



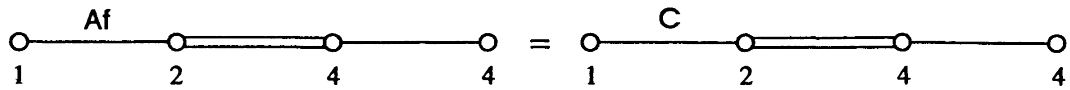
We call Γ an *affine section* of B. Trivially, Γ inherits (IP) from B. Affine sections of F_4 buildings look like natural candidates to the title of 'classical' examples of $\mathsf{L} \cdot \mathsf{C}_3^*$. Unfortunately, they are not necessarily 2-simply connected. For instance, two affine sections of the building for the Chevalley group $\mathsf{F}_4(2)$ are considered in Yoshiara [1994a], but only one of them is 2-simply connected; the other one admits a (2-simply connected) double cover (example (a) of the list of the next paragraph). The building for the twisted group ${}^2\mathsf{E}_6(2)$ admits affine sections with orders 1,2,4,4. Their universal 2-covers are isomorphic to the geometry (c) in the list of 4.4.3.

The building of type F_4 with orders 2,2,1,1 (for $\mathrm{O}^+(8,2) \cdot 3$) admits an affine section Γ where $(3 \times \mathrm{U}(4,2)) : 2$ acts flag-transitively. The orders of Γ are 1,2,1,1, just as in the geometry obtained from the Coxeter complex of affine type $\widetilde{\mathsf{E}}_6$ by truncating and taking 0-shadows as in 4.4.1. The Coxeter group of type $\widetilde{\mathsf{E}}_6$ is $\mathbb{Z}^6 : (\mathrm{U}(4,2) : 2)$. Thus, it would not be surprising if $\widetilde{\Gamma}$ were the universal 2-cover of Γ.

4.4.3. *Some sporadic examples.*

(a) There are two affine sections of the building for $\mathsf{F}_4(2)$, admitting $\mathrm{Sp}(8,2)$ and$(2^{1+8} \times 2^6)\,\mathrm{Sp}(6,2)$, respectively, as (flag-transitive) automorphism groups (Yoshiara [1994a]). The geometry for $\mathrm{Sp}(8,2)$ is 2-simply connected, whereas the other one admits a 2-simply connected double cover. This double cover is the first exceptional example we are going to mention.

(b) The second Conway group Co_2 is known to act flag-transitively on a geometry of type $\mathsf{Af}\cdot\mathsf{C}_3^*$ with orders 1,2,4,4:



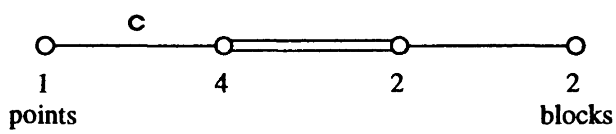
(Buekenhout [1985], (73)). This geometry is not 2-simply connected, but it admits a 2-simply connected double cover with $2 \times \mathsf{Co}_2$ as automorphism group (Ronan [1981b]).

(c) Yoshiara [1994a] proved that there are just two flag-transitive 2-simply connected geometries of type $\mathsf{Af}\cdot\mathsf{C}_3^*$ with orders 1,2,4,4. One of them is the geometry for $2 \times \mathsf{Co}_2$ mentioned before. The automorphism group of the other one is described by means of generators and relations. Unfortunately, Yoshiara could not compute the precise order of that group. On the other hand, the building for ${}^2\mathsf{E}_6(2)$ admits flag-transitive affine sections with orders 1,2,4,4. These cannot be 2-quotients of the geometry for $2 \times \mathsf{Co}_2$. Hence they are 2-quotients of the latter (mysterious) geometry.

(d) There is a geometry of type $\mathsf{Af}\cdot\mathsf{C}_3^*$ with orders 1,2,2,2 admitting the Fischer group Fi_{22} as flag-transitive automorphism group. It is a subgeometry of the next example. Stabilizers of points and blocks are isomorphic to $2^6 : \mathrm{Sp}(6,2)$ and $(2\times 2^{1+5} : \mathrm{U}(4,2)) : 2$, respectively (see Yoshiara [1994a]).

(e) Yoshiara [1994a] mentions a geometry for the Fischer group Fi_{24}' with orders 1,4,2,2 belonging to the following special case of $\mathsf{L}\cdot\mathsf{C}_3^*$:

$\mathsf{c}\cdot\mathsf{C}_3^*$



Residues of blocks are isomorphic with the geometry $3\Gamma_3^{(3)}$ of 4.3.5(c), stabilizers of points are isomorphic with $2^8 : \mathrm{O}^-(8,2)$ (standard module for $\mathrm{O}^-(8,2)$) and stabilizers of blocks act as $3\cdot\mathrm{O}^-(6,3)\cdot 2$ on residues of blocks (Yoshiara, private communication, 1992).

(f) Another geometry for Fi_{24}' belongs to $\mathsf{c}\cdot\mathsf{C}_3^*$ with orders 1,3,3,3. Buekenhout [1985], (24), and Ronan and Stroth [1984].

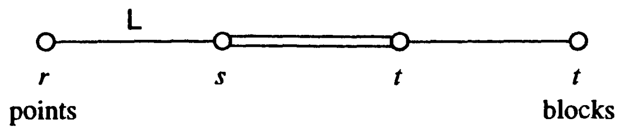
(g) A geometry of type $\mathsf{c}\cdot\mathsf{C}_3^*$ with orders 1,9,3,3 is also known for the Fischer and Griess Monster $\mathsf{M} = \mathsf{F}_1$ (Buekenhout [1985], and Buekenhout and Fischer, unpublished, 1982).

(h) Finally, there is a geometry of type $\mathsf{Af}\cdot\mathsf{C}_3^*$ with orders 1,2,2,2 where residues of points are isomorphic to the Alt(7)-geometry and flag-transitive automorphism group 2^4 : Alt(7) (see Pasini and Yoshiara [1994b]). Note that (IP) does not hold in this geometry, whereas it holds in the previous examples.

4.4.4. *Some classification work.* A classification of locally finite flag-transitive geometries of type $\mathsf{L}\cdot\mathsf{C}_3^*$ does not seem to be in reach for the moment. We do not know any analogue of Theorem 4.3.3 for affine sections of F_4 buildings (furthermore, hyperplanes of polar spaces have been classified, by Cohen and Shult [1990], whereas not so much is known on hyperplanes of buildings of type F_4, at the moment). Note that a geometry of type

$\mathsf{Af}\cdot\mathsf{C}_2^*$ satisfying (IP) and thick at all types except possibly at the first one, need not be an affine section of an F_4 building: four counterexamples have been given in 4.4.3 ((a), (b), (c) and (d)). However, those counterexamples are thin at the first node of the diagram; we do not know what happens in the thick case.

However, some partial results have been obtained. Let Γ be a flag-transitive geometry of type $\mathsf{L}\cdot\mathsf{C}_3^*$ with finite orders r, s, t, t.



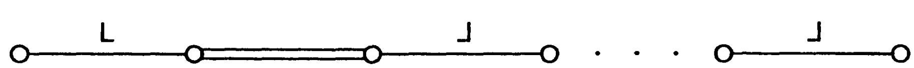
As the F_4 case is settled (Theorem 2.5.8), we assume $r < s$. We also assume $t > 1$, as the case $t = 1$ is likely to be very hard to deal with (compare the constructions of 7.4.1).

For $r = 1$ and Γ flag-transitive, the C_3 residues are either polar spaces or the Alt(7)-geometry (Pasini and Yoshiara [1994a]). If the latter case occurs, then Γ is the geometry mentioned in 4.4.3(h) (Pasini and Yoshiara [1994b]). Thus, we can assume that residues of points of Γ are polar spaces. All possibilities for residues of blocks can now be determined, by Theorem 4.3.10: they are (possibly improper) standard quotients of affine polar spaces of rank 3, or one of the geometries $\Gamma_3^{(3)}$, $3\Gamma_3^{(3)}$, $\Gamma_3^{(4)}$ of 4.3.5 ((c), (d)) or one of the geometries for McL, Suz and Aut(HS) mentioned in 4.3.6 ((a), (d), (f)). However, the geometries for McL and Aut(HS) cannot occur here (Yoshiara [1994a]). On the other hand, $3\Gamma_3^{(3)}$, $\Gamma_3^{(4)}$ and the geometry for Suz occur as block-residues in (e), (f), (g) of 4.4.3, respectively. Yoshiara [1994a] has proved that, if Γ has orders 1,2,4,4 and is 2-simply connected, then it is as in (b) or (c) of 4.4.3, and he has obtained a partial result on the case with orders 1,2,2,2. However, we are still far from a satisfactory classification theorem.

4.4.5. *A graphical characterization of* Co_2. This is about the geometry Γ described in 4.4.3(b). The point-graph (collinearity graph) $\mathcal{G}$ of Γ, has 2300 vertices. For each vertex p of $\mathcal{G}$, the subgraph induced on the neighbours of p is the distance 1- or -2 graph of the dual polar graph related to PSU(6, 2). Denote by $\mathcal{H}$ this dual polar graph, and by $\mathcal{H}_{1,2}$ its distance 1- or -2 graph.

THEOREM (Cuypers [1994b]). *Let C be a connected graph, such that for each vertex p of C, the induced subgraph on the neighbours of p is isomorphic to $\mathcal{H}_{1,2}$. Then C is isomorphic to $\mathcal{G}$.*

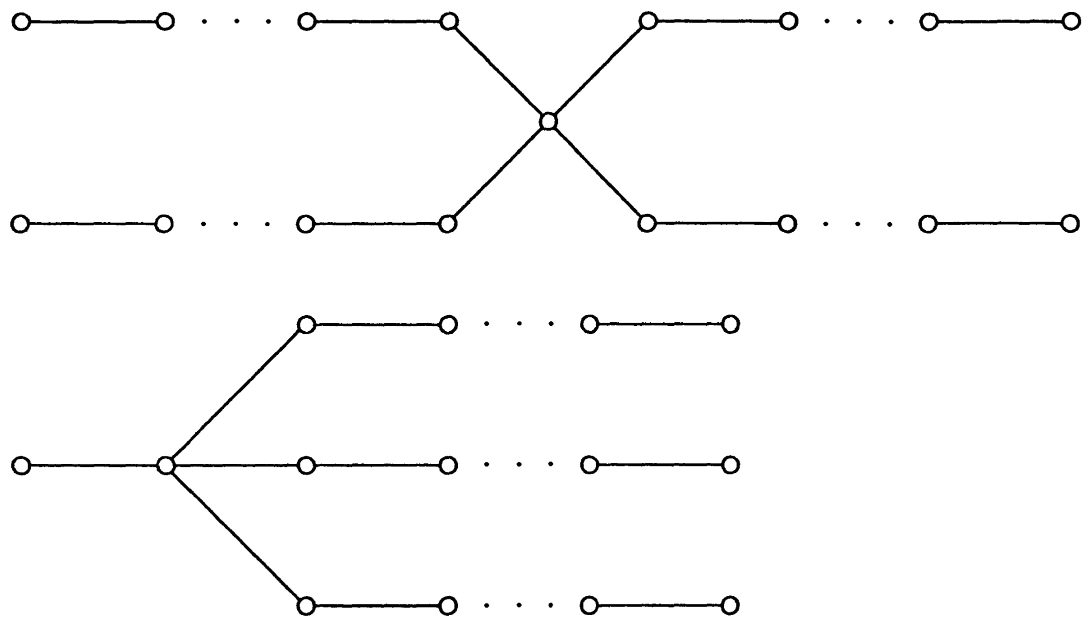
4.4.6. *Dreaming of generalizations.* The following is a natural generalization of $\mathsf{L}\cdot\mathsf{C}_3^*$.

(1) 

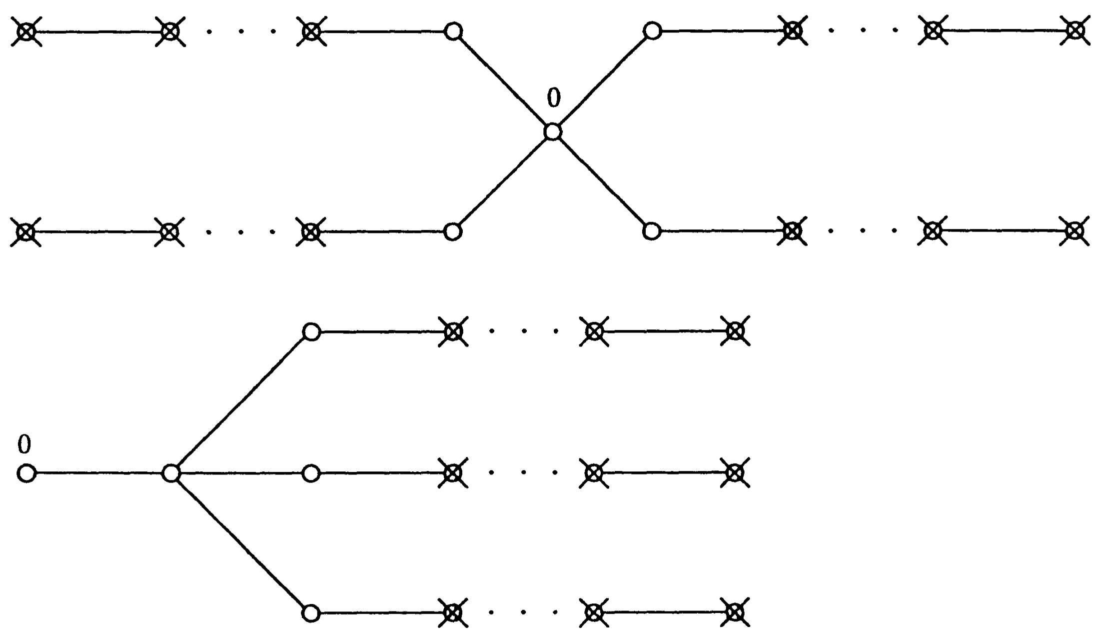
A particular case of (1) is as follows.

(2) 

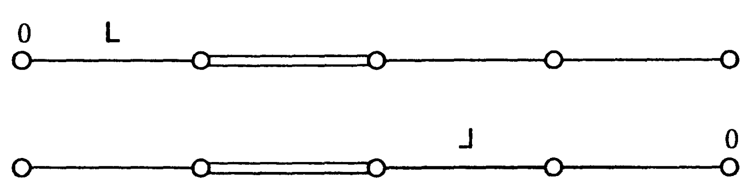
Examples for (1) not belonging to (2) are easy to obtain by some modification of the constructions of 4.4.1. For instance, if we start from buildings belonging to one of the following diagrams



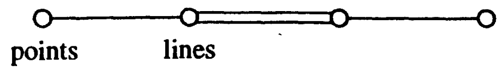
and truncate them, deleting the elements crossed out in the next pictures, and take shadows

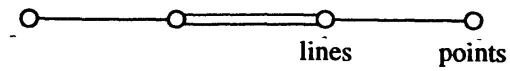


with respect to the type 0, then we obtain geometries belonging to the following diagrams:

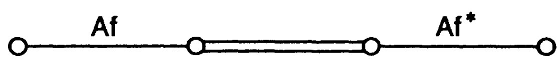


Examples for the rank 4 case of (1) not belonging to $\mathsf{L}\cdot\mathsf{C}_3^*$ can be constructed as follows. A thick building Γ of type F_4 can be viewed as a metasymplectic space in two ways (Chapter 12, 6.5):

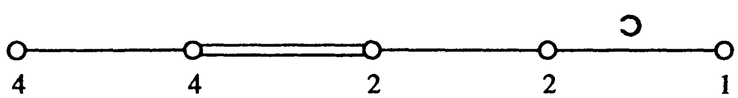


or


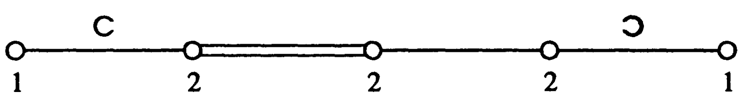
Let $\mathcal{M}(\Gamma)$ and $\mathcal{M}^*(\Gamma)$ be these two metasymplectic spaces and let H and H^* be hyperplanes of $\mathcal{M}(\Gamma)$ and $\mathcal{M}^*(\Gamma)$, respectively. Deleting H and H^* from Γ we obtain a geometry Γ belonging to the following special case of (1):

(3)


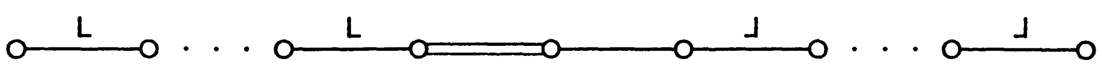
We call Γ an *affine-dual-affine section* of Γ. Examples different from those mentioned above seem to be quite rare. However, some of them exist. For instance, there is a geometry with diagram and orders as follows



with the Baby Monster $\mathsf{BM} = \mathsf{F}_2$ as flag-transitive automorphism group (Buekenhout [1985], (74)). There is also an example with orders 1,2,2,2,1:



with flag-transitive automorphism group 2^4 : Alt(8) and C_3 residues isomorphic with the Alt(7)-geometry (Pasini and Yoshiara [1994]). Actually, this is the only geometry of rank $n \geqslant 5$ belonging to a diagram of the following form



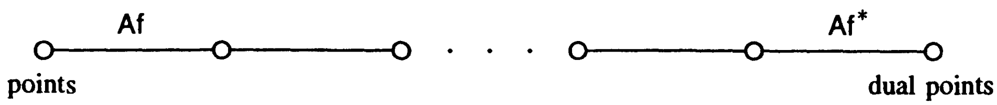
and with C_3 residues isomorphic to the Alt(7)-geometry (Pasini and Yoshiara [1994b] and Theorem 2.3.2). There is also an example for (3) with orders 1,2,2,1 and admitting M_{22} as flag-transitive automorphism group. The Mathieu group M_{22} is not involved in the Chevalley group $\mathsf{F}_4(2)$. Therefore this example has nothing to do with affine-dual-affine sections of buildings of type F_4.

4.5. *The diagram* $\mathsf{L}\cdot\mathsf{A}_{n-2}\cdot\mathsf{L}^*$

We denote the following diagram of rank $n \geqslant 3$ by $\mathsf{L}\cdot\mathsf{A}_{n-2}\cdot\mathsf{L}^*$:

The diagram $\mathsf{L} \cdot \mathsf{A}_{n-1}$ studied in 4.2 is a special case. Thus, if Γ is a geometry belonging to $\mathsf{L} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}^*$, we always assume that none of the residues of Γ of type $\{0, 1\}$ or $\{n-2, n-1\}$ is a projective plane.

4.5.1. *Affine-dual-affine geometries.* Let p and H be a point and a hyperplane of $\mathrm{PG}(n, K)$, $n \geqslant 3$. By deleting H with all its elements and p with its star, we obtain a geometry Γ belonging to the following special case of $\mathsf{L} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}^*$ which we call $\mathsf{Af} \cdot \mathsf{A}_{n-2} \cdot \mathsf{Af}^*$:



We call Γ an *affine-dual-affine* geometry (*defined over* K). $\mathrm{Aut}(\Gamma)$ is the stabilizer in $\mathrm{P\Gamma L}(n+1, K)$ of p and H and it is flag-transitive on Γ. Furthermore, Γ satisfies (IP).

4.5.2. THEOREM (Lefèvre-Percsy and Van Nypelseer [1990], Lefèvre-Percsy [1990], Van Nypelseer [1991]). *Every geometry belonging to* $\mathsf{Af} \cdot \mathsf{A}_{n-2} \cdot \mathsf{Af}^*$ *and satisfying* (IP) *is an affine-dual-affine geometry.*

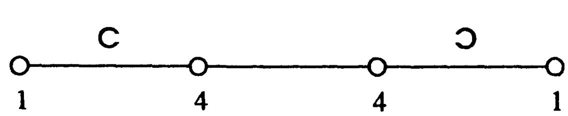
Actually, the authors replace (IP) by the weaker assumption (LL) on the point-line system (see 1.4.2(4)). However, it is not difficult to prove, with the help of 4.2.1, that this property and (IP) are equivalent in $\mathsf{L} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}^*$ geometries. It easily follows from this theorem that affine-dual-affine geometries are 2-simply connected. Hence, if Γ is an affine-dual-affine geometry, every 2-quotient of Γ can be obtained by factorizing over a subgroup A of $\mathrm{Aut}(\Gamma)$ (see Ronan [1980]) and it is not difficult to prove that A must fix the star of p and the hyperplane H elementwise, where p and H are the point and the hyperplane of $\mathrm{PG}(n, K)$ that were removed in order to get Γ.

Note that none of the proper 2-quotients of Γ satisfies (IP). $\mathrm{Aut}(\Gamma)$ is the stabilizer in $\mathrm{P\Gamma L}(n, K)$ of p and H and it is flag-transitive on Γ. When K is commutative and $p \notin H$, then $\mathrm{Aut}(\Gamma) = \Gamma\mathrm{L}(n, K)$ and the flag-transitive quotients of Γ are obtained factorizing by a subgroup of the centre Z of $\Gamma\mathrm{L}(n, K)$. Clearly, Γ/Z is the minimal quotient of Γ, and the system of points and dual points of Γ/Z is isomorphic to the system of points and hyperplanes of $\mathrm{PG}(n-1, K)$ with $\notin$ as incidence relation. In the finite case, the latter is a well known 2-$(q^n - 1, q^{n-1}, q^{n-2})$-design. Note that $Z = 1$ when $K = \mathrm{GF}(2)$.

Let now $p \in H$. In this case Γ admits just one flag-transitive proper quotient Γ/A, which is *flat* (all points incident with all dual points). If $K = \mathrm{GF}(2)$, then $A = Z_2$. If furthermore $n = 3$, then Γ is isomorphic to the geometry obtained from the Coxeter complex of type D_4 by truncation of one of the three noncentral nodes of the diagram (see 4.5.5).

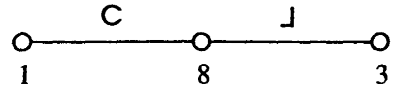
4.5.3. *Sporadic examples.*

(a) The Higman–Sims group HS acts on a geometry with diagram and orders as follows (Buekenhout [1985], (104); also Hughes [1983]).

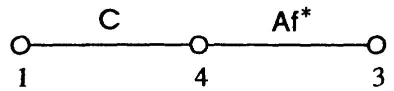


Property (IP) holds in this geometry and residues of elements of the last (resp., first) type are isomorphic with the Witt design for the Mathieu group M_{22}.

(b) A geometry for the Janko group J_2 belongs to the following rank 3 case of $L \cdot A_{n-2} \cdot L^*$ (Buekenhout [1985], (104)) with orders 1,8,3.

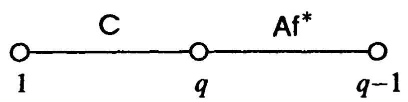


(c) If we delete a point and its star from the Witt design for M_{22}, then we obtain a geometry for the following diagram



admitting PSL(3, 4) as flag-transitive automorphism group.

(d) Given a plane π of PG$(3, 2^h)$, a hyperoval O of π and a point $p \notin \pi$, let C be the cone projecting O from p. Take the points on C other than p as points, the pairs of points on distinct lines of C as lines, and the planes of PG$(3, 2^h)$ not on p as blocks. Thus, we obtain a geometry Γ belonging to the following diagram:



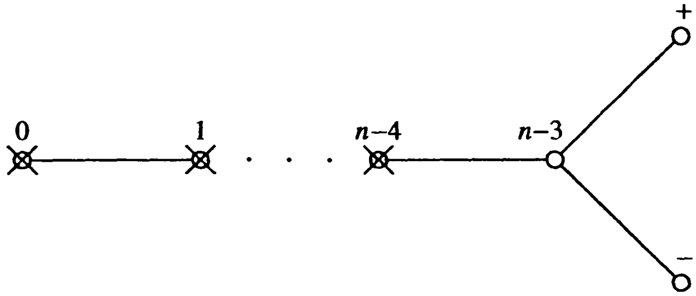
$(q = 2^h)$

If $q = 2$, then Γ is the affine-dual-affine geometry defined over GF(2) with 8 points. If $q = 4$ then Γ admits the following flag-transitive automorphism groups: 2^6 : Alt(5), 2^6 : Sym(5), 2^6 : Alt(6), 2^6 : Sym(6).

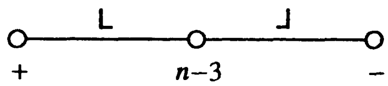
4.5.4. THEOREM (Hughes [1983]). *The geometry for* HS *mentioned in* 4.5.3(a) *is the only geometry belonging to* $L \cdot A_2 \cdot L^*$ *with orders* 1,4,4,1 *satisfying* (IP).

The use of (IP) in the (LL) form is implicit in the given reference.

4.5.5. *Truncations of buildings of type* D_n. Let B be a building of type D_n. If we delete the elements corresponding to the types crossed out in the picture



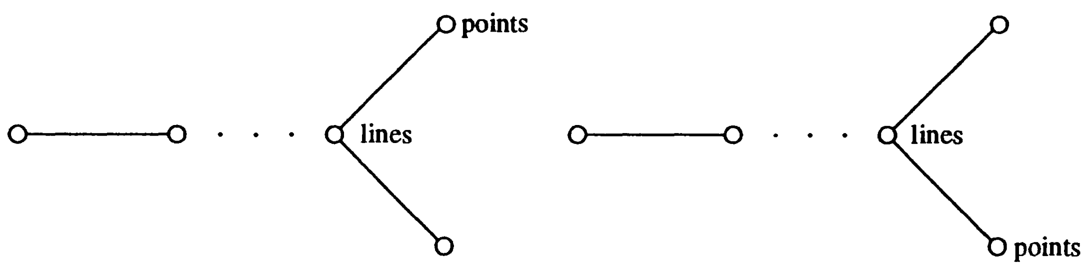
then we obtain a geometry Γ belonging to the rank 3 case of $L \cdot A_{n-2} \cdot L^*$:

L · L* 

We can also take proper 2-quotients of Γ, even if B does not admit any proper 2-quotients (by Corollary 2.2.3(iii)). For instance, if B is the Coxeter complex of type D_4 and we truncate it as above, then we obtain the affine-dual-affine geometry of rank 3 defined over GF(2) with 8 points, which admits a proper 2-quotient.

REMARK. The above makes it clear that, although every truncation of a 2-quotient is a quotient (possibly a 2-quotient) of a corresponding truncation, the inverse is false in general: not every 2-quotient of a truncation of a geometry B is a truncation of a 2-quotient of B.

If B is thick, we can slightly modify the previous construction, by deleting a hyperplane in one of the two following point-line systems of B (see Chapter 12):



Then we truncate as above. Note that doing this with PG$(3, K)$ in each of the above point-line systems is just constructing an affine-dual-affine geometry of rank 3.

4.5.6. *Semibiplanes.* These are the finite geometries with (IP) on a diagram $\mathsf{c} \cdot \mathsf{c}^*$. They were studied by Hughes as early as 1976, at the (Δ, Γ)-level (see Hughes [1981]).

Truncations of Coxeter complexes of type D_n as in 4.5.5 are examples of semibiplanes. However, many other examples exist, including many biplanes (the case in which any two points are collinear). We mention two of them, for the groups PSL(2, 11) and M_{12}, respectively (see Buekenhout [1985], (26) and (32); also Pasini and Yoshiara [1992], 8.4). The semibiplane for PSL(2, 11) is actually a biplane and it arises from the exceptional 2-transitive action of PSL(2, 11) on 11 objects. For more examples of flag-transitive semibiplanes, see Pasini and Yoshiara [1992], Grams and Meixner [1994], Baumeister [1992].

The latter author assumes that the stabilizer G_0 of a point is almost simple but that it is not a rank one group of Lie type. Then the full automorphism group G satisfies one of the following:

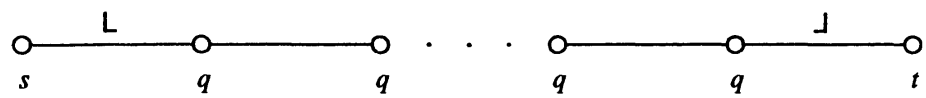
(1) $G = 2^{n-1} : G_0$ for some 2-transitive group G_0;

(2) $G = 2 \cdot M_{22}$, $G_0 \cong \mathrm{Alt}(7)$.

A classification (up to quotients) is obtained when the order at the middle node is at most 19, under the assumption that G_0 is almost simple.

Combining Theorems 4.5.2 and 4.5.4 with Theorem 4.2.6 we easily obtain the following.

4.5.7. THEOREM. *Let Γ be a flag-transitive geometry, belonging to* $\mathsf{L} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}^*$, *of rank* $n \geqslant 4$ *and with finite orders* $s, q, \ldots, q, t$



with $t \leqslant s < q$. *Assume furthermore that* (IP) *holds in* Γ. *Then* Γ *is one of the following*:

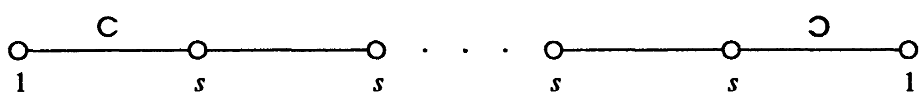
(i) *an affine-dual-affine geometry*;

(ii) *the geometry for the group* HS, *mentioned in* 4.5.3;

(iii) *an (unknown) geometry with diagram and orders as follows*:

It is conceivable to obtain the same conclusion without assuming (IP) and ruling out (iii) too, provided we add quotients of affine-dual-affine geometries to the list (i), (ii), (iii). Here is a step in that direction.

4.5.8. THEOREM (Pasini and Yoshiara [1992]). *Let Γ be a flag-transitive geometry of rank $n \geqslant 4$ belonging to the following special case of* $\mathsf{L} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}^*$:



$1 < s < \infty$

Then Γ is one of the following:

(i) *one of the two affine-dual-affine geometries of rank n defined over* GF(2);

(ii) *the minimal quotient of an affine-dual-affine geometry with 2^n points defined over* GF(2);

(iii) *the geometry for the group* HS *mentioned in* 4.5.3(a).

REMARK. Lefèvre-Percsy (private communication) has proved that (iii) of Theorem 4.5.7 is actually impossible.

LAST MINUTE RESULT (Del Fra and Pasini). *If $n \geqslant 4$ and Γ is flag-transitive of type* $\mathsf{L} \cdot \mathsf{A}_{n-2} \cdot \mathsf{L}^*$, *then Γ is one of*:

(i) PG(n, q);

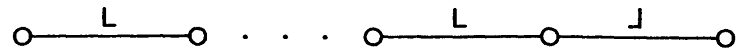
(ii) AG(n, q);

(iii) *a possibly improper quotient of an affine-dual-affine geometry*; *or*

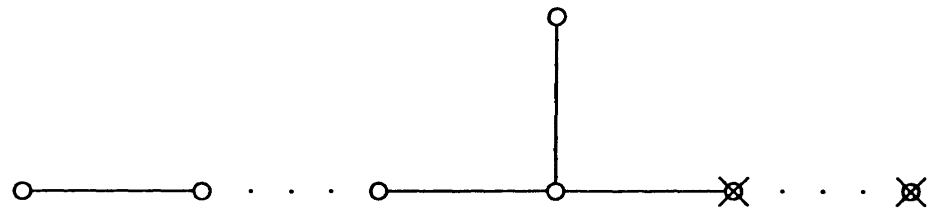
(iv) *the geometry for* HS.

4.5.9. *Some generalizations.* The following is a generalization of $\mathsf{L}\cdot\mathsf{A}_{n-2}\cdot\mathsf{L}^*$:

$\mathsf{L}_{n-1}\cdot\mathsf{L}^*$

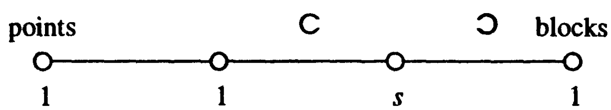


Examples are obtained by truncation of buildings, as follows and, possibly, taking 2-quotients.

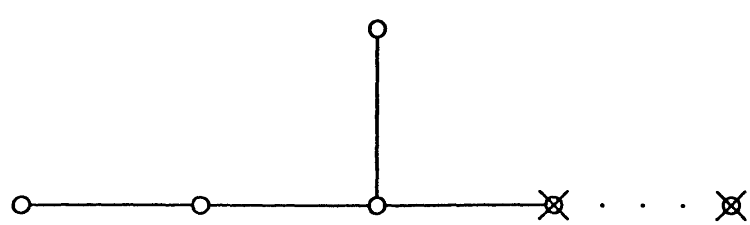


There are also examples, as follows, which are not obtained by truncating Coxeter complexes:

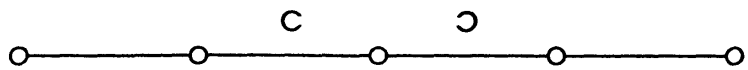
$\mathsf{c}^2\cdot\mathsf{c}^*$



One of these geometries has orders 1,1,3,1 and admits the Mathieu group M_{11} as flag-transitive automorphism group (Buekenhout [1985], (27); Meixner [1994a]; see also Pasini and Yoshiara [1992], 8.4). We denote this geometry by Γ_1. It has 12 points (actually, it arises from the exceptional action of M_{11} on 12 points) and residues of points are isomorphic to the biplane for PSL(2, 11) mentioned in 4.5.6. The Mathieu group M_{11} also acts flag-transitively on another geometry of type $\mathsf{c}^2\cdot\mathsf{c}^*$, with orders 1,1,7,1 and 11 points (Meixner [1994a]; also Pasini and Yoshiara [1992], 8.4), arising from the 4-transitive action of M_{11} on 11 points. We call it Γ_2. The geometry Γ_2 admits a 2-simply connected double cover $2\Gamma_2$, on which $2\times\mathsf{M}_{11}$ acts flag-transitively (Meixner [1994a]). This author has proved that (possibly improper) 2-quotients of truncated Coxeter complexes as follows

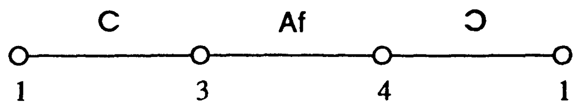


and the geometry Γ_1 for M_{11} are the only locally finite flag-transitive examples for the diagram $\mathsf{c}^2\cdot\mathsf{c}^*$ on which the stabilizer of a block u in the automorphism group of the geometry does not act as a subgroup of $\mathrm{A\Gamma L}(2, s+3)$ on the $s+3$ points of u (in short: that stabilizer is big enough); actually, Meixner does not make this assumption, but it is needed to get rid of a gap in his proof. Each of Γ_1 and $2\Gamma_2$ occurs as a rank 3 residue in a geometry belonging to the diagram



with orders 1, 1, 3, 1, 1 and 1, 1, 7, 1, 1, respectively, and $2M_{12}$ (central nonsplit extension) as flag-transitive automorphism group (Meixner [1994a]).

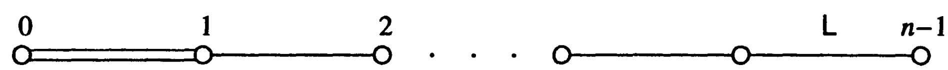
Still another example (Buekenhout, unpublished) is for the simple group Sp(4, 4) with 85 points, 120 hyperplanes and the diagram:



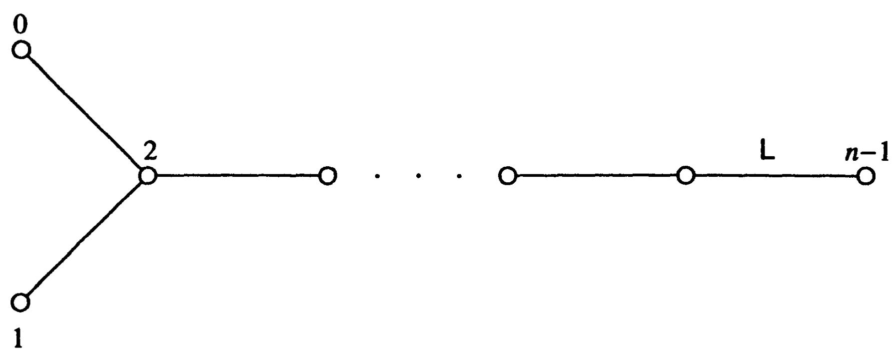
4.6. *The diagrams* $C^*_{n-1} \cdot L$ *and* $D^*_{n-1} \cdot L$

Henceforth, we label a stroke of a diagram by $L^{(k)}$ to denote all linear spaces spanned by at most $k+1$ points ($2 \leqslant k < \infty$). We use the label $L^{(\omega)}$ for the class of linear spaces admitting a finite spanning set of points. We consider the following diagrams of rank $n \geqslant 3$.

$C^*_{n-1} \cdot L$

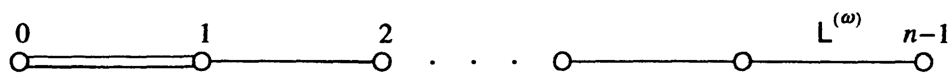


$D^*_{n-1} \cdot L$

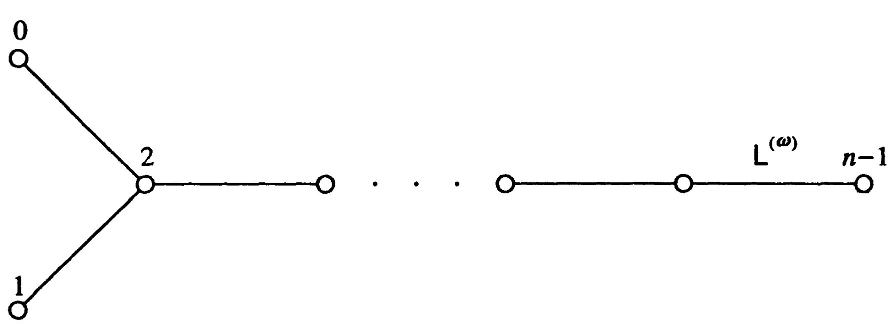


The following are special cases.

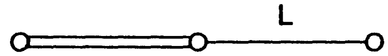
$C^*_{n-1} \cdot L^{(\omega)}$



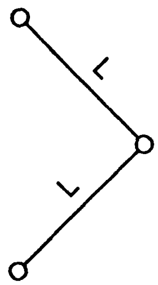
$D^*_{n-1} \cdot L^{(\omega)}$



$\mathsf{C}_2^* \cdot \mathsf{L}$

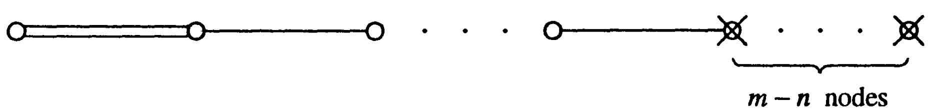


$\mathsf{D}_2^* \cdot \mathsf{L}$

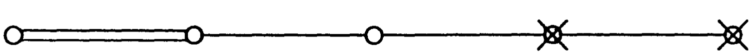


The diagrams $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$ and $\mathsf{D}_{n-1}^* \cdot \mathsf{L}$ are mutually related, as follows: a geometry belonging to $\mathsf{D}_{n-1}^* \cdot \mathsf{L}$ can be folded to a geometry of type $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$ thin at the first node of the $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$ diagram, just as buildings of type D_n can be folded to polar spaces of rank n. Conversely, if Γ is a 2-simply connected geometry belonging to $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$, thin at the first node of the $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$ diagram and satisfying (IP), then Γ can be 'unfolded' to a geometry of type $\mathsf{D}_{n-1} \cdot \mathsf{L}$ (Rinauro [1990]). We now describe some examples for $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$ and $\mathsf{D}_{n-1}^* \cdot \mathsf{L}$.

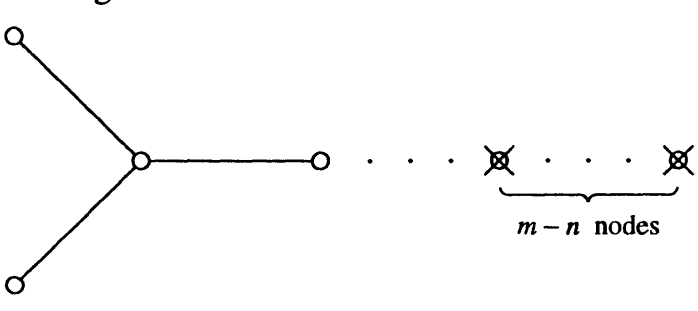
4.6.1. *Truncations.* Let $\mathcal{P}$ be a polar space of rank $m \geqslant n \geqslant 3$. If we truncate $\mathcal{P}$ by deleting the elements of the types crossed out in the next picture



then we obtain a geometry Γ belonging to $\mathsf{C}_{n-1}^* \cdot \mathsf{L}$. We call Γ a *lower truncation* of $\mathcal{P}$. Note that Γ satisfies (IP). We can also take 2-quotients of Γ. Note that if we truncate a 2-quotient of $\mathcal{P}$ by deleting the last $m - n$ types as in the above picture, then we obtain a 2-quotient of Γ. On the other hand, Γ may admit 2-quotients that are not truncations of 2-quotients of $\mathcal{P}$ (see 4.5.5, Remark). Note that (IP) may hold in a truncation of a proper 2-quotient $\mathcal{P}/A$ of $\mathcal{P}$ even if, by Corollary 2.2.3(ii), (IP) does not hold in $\mathcal{P}/A$. For instance, let $\mathcal{P}$ be the Coxeter complex of type C_5 and let σ be the centre of the corresponding Coxeter group. It is straightforward to check that (IP) holds in the following truncation of $\mathcal{P}/\langle\sigma\rangle$:



Similarly, if B is a building of type D_m with $m \geqslant n \geqslant 3$, then we can truncate B as follows and we get



a geometry Γ belonging to $\mathsf{D}^*_{n-1} \cdot \mathsf{L}$. We call it a *lower truncation* of B. The truncation Γ might admit proper 2-quotients even if no proper 2-quotients of B exist (see 4.5.5).

4.6.2. THEOREM (Ronan [1986]). *Every geometry of type $\mathsf{D}^*_{n-1} \cdot \mathsf{L}^{(\omega)}$ of rank $n \geqslant 4$ is a (possibly improper) 2-quotient of a lower truncation of a building of type D_m, for some $m \geqslant n$. Every geometry of type $\mathsf{C}^*_{n-1} \cdot \mathsf{L}^{(\omega)}$ with $n \geqslant 4$ and C_3 residues 2-covered by polar spaces is a (possibly improper) 2-quotient of a lower truncation of a polar space of rank m, for some $m \geqslant n$.*

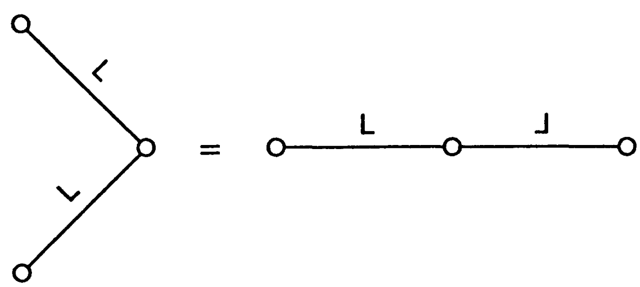
4.6.3. COROLLARY. *Let Γ be a locally finite flag-transitive geometry of type $\mathsf{C}^*_{n-1} \cdot \mathsf{L}$ with $n \geqslant 4$. Assume furthermore that Γ is thick at all types except possibly at the first node of the $\mathsf{C}^*_{n-1} \cdot \mathsf{L}$ diagram. Then Γ is either:*

(i) *a (possibly improper) 2-quotient of a lower truncation of a polar space; or*

(ii) *an (unknown) geometry of type $\mathsf{C}^*_3 \cdot \mathsf{L}$ ($n = 4$) with C_3 residues isomorphic to the* Alt(7)*-geometry.*

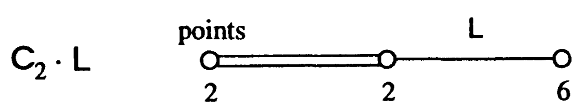
PROOF. The residues of Γ of type $\mathsf{A}_{n-2} \cdot \mathsf{L}$ are truncations of projective geometries (4.2.2). Hence the C_3 residues of Γ are either polar spaces or isomorphic to the Alt(7)-geometry. The latter case occurs only if $n = 4$ (Theorem 2.4.1). In that case Γ is as in (ii). In the first case, we have (i) by Theorem 4.6.2. □

4.6.4. *Examples of rank* 3. Many examples exist for the rank 3 case $\mathsf{C}_2 \cdot \mathsf{L}$ of $\mathsf{C}^*_{n-1} \cdot \mathsf{L}$ (see Pasechnik [1991], Hughes [1990a]). Some of them are foldings of geometries belonging to the rank 3 case $\mathsf{D}_2 \cdot \mathsf{L}$ ($= \mathsf{L} \cdot \mathsf{L}^*$) of $\mathsf{D}_{n-1} \cdot \mathsf{L}$ (see 4.5):



We only mention two examples that neither arise from $\mathsf{D}_2 \cdot \mathsf{L}$ nor are 2-quotients of lower truncations of polar spaces.

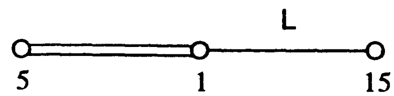
(a) There is a geometry with diagram and orders as follows



admitting M_{24} as flag-transitive automorphism group and with residues of points isomorphic to the point-line system of PG(3, 2) (see Ronan and Smith [1980]). This geometry satisfies (IP) and it is not a 2-quotient of a truncated polar space (note that M_{24} is not involved in Sp(8, 2).

(b) There is also a geometry of type $\mathsf{C}_2 \cdot \mathsf{L}$ for the third Janko group J_3, with orders as follows (Tits [1981b]).

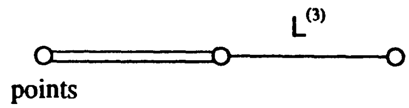
$C_2 \cdot L$



The only polar space that gives rise to lower truncations with the above orders is the thin lined polar space of rank 17 with order 5 at the last node of the C_{17} diagram. The automorphism group of this polar space is the wreath product Sym(6) wr Sym(17), which does not involve J_3. Hence the above mentioned geometry for J_3 is not a 2-quotient of a truncated polar space.

4.6.5. THEOREM (Ronan [1982]). *Let Γ be a geometry belonging to the following special case of* $C_{n-1} \cdot L$:

$C_2 \cdot L^{(3)}$



Assume that (IP) *holds in Γ and that the Pasch Axiom* (*Chapter* 2) *holds in residues of points of Γ. Then Γ is one of the following*:

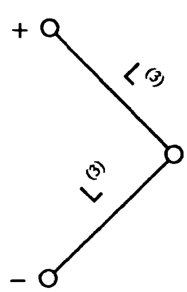
(i) *a lower truncation of a polar space of rank* 4;

(ii) *the folding of the affine-dual-affine geometry defined over* GF(2) *with* 7 *points and* 7 *planes* (*see* 4.5.2, *remarks after the Theorem*);

(iii) *the geometry for* M_{24} *mentioned in* 4.6.4(a).

4.6.6. COROLLARY (Ronan [1986]). *Let Γ be a geometry belonging to the following special case of* $D_{n-1} \cdot L$:

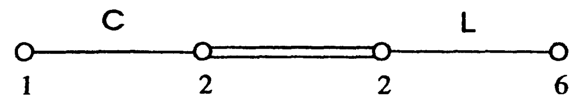
$D_2 \cdot L^{(3)}$



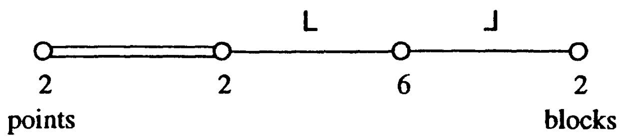
Assume that (IP) *holds in Γ and that the Pasch Axiom holds in all residues of elements of Γ of type + and −. Then Γ is either a lower truncation of a building of type* D_4 *or the affine-dual-affine geometry defined over* GF(2) *with* 7 *points and* 7 *planes.*

4.6.7. *Extensions of the geometry of* 4.6.4(a) *for* M_{24}. The geometry for M_{24} mentioned in 4.6.4(a) appears as a residue in a number of interesting flag-transitive geometries, which we now mention in a sketchy way, giving their diagrams and orders, minimal flag-transitive automorphism groups and some references. It will be clear from the diagram which are the residues isomorphic to the geometry of 4.6.4(a).

(a) Group: $2^{11} \cdot M_{24}$. Diagram and orders (Neumaier [1983], Buekenhout [1985], (83)):

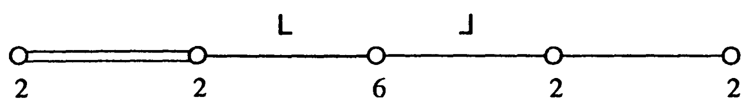


(b) Group: the Conway group Co_1. Diagram and orders as follows:



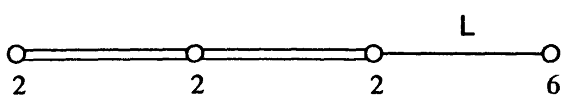
(see Ronan and Smith [1980], Buekenhout [1985], (80)). Segev [1988] has proved that this geometry for Co_1 is the unique flag-transitive geometry belonging to the above diagram where residues of points are truncations of the D_4 building for $O^+(8,2)$ and residues of blocks are isomorphic with the geometry for M_{24} of 4.6.4(a).

(c) Group: the Monster $M = F_1$. Diagram and orders as follows (Ronan and Smith [1980], Buekenhout [1985], (82)):

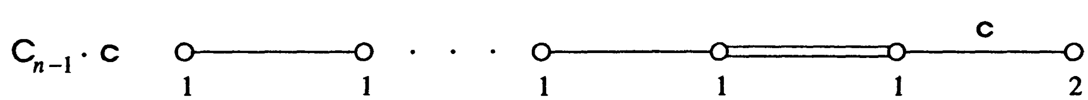


The previous geometry (for Co_1) is a residue of this one.

(d) Group: Fi'_{24}. Diagram and orders as follows (Ronan and Smith [1980], Buekenhout [1985], (81)).



4.6.8. *The Suzuki chain.* Let Γ_0 be the folding of the affine-dual-affine geometry defined over GF(2) with 7 points and 7 planes (Theorem 4.6.5(ii)) and let Γ be a geometry of rank $n \geqslant 4$ with diagram and orders as follows:

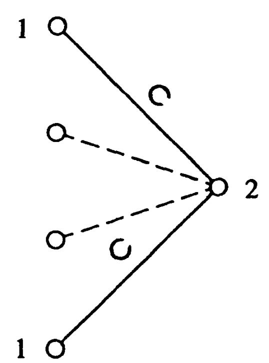


If all residues of Γ of type $C_2 \cdot c$ are isomorphic to Γ_0, then we say that Γ is an $(n-3)$-*times linear extension* of Γ_0 (compare 4.3.11). Similarly, if Γ and Γ' are a s-times and a t-times linear extension of Γ, respectively, with $s < t$, and if all residues of Γ' of type $C_{s+2} \cdot c$ are isomorphic with Γ, then we say that Γ' is a $(t-s)$-*times linear extension* of Γ.

For every $k = 1, 2, \ldots, 8$, Γ_0 admits a flag-transitive k-times linear extension Γ_k satisfying (IP) (Neumaier [1984] and Soicher [1992]). The automorphism groups are the following.

k	group
1	$G_2(2) = \mathrm{Aut}(U(3,3))$
2	$J_2 \cdot 2$
3	$G_2(4) \cdot 2$
4	$3 \cdot (\mathrm{Suz} \cdot 2)$ (central nonsplit extension)
5	$2 \times Co_1$
6	$2 \cdot (Co_1 \text{ wr } 2)$ (where wr means wreath product)
7	$2^m \cdot (Co_1 \text{ wr } 2)$ for some (unknown) $m \geqslant 2$.

For every $k = 2,3,\ldots,8$, Γ_k is a linear extension of Γ_{k-1}. Furthermore, Γ_k is $(k+2)$-simply connected, for all $k = 2,3,\ldots,6$ (Soicher [1992]; see also Pasechnik [1993]). Γ_5 admits a 2-quotient on which $\mathrm{Suz} \cdot 2$ acts flag-transitively (Neumaier [1982], Soicher [1992], Pasechnik [1993]). We call it $\Gamma_4/3$. Both Γ_5 and Γ_6 admit 2-quotients, with flag-transitive automorphism group Co_1 and Co_2 wr 2, respectively (Soicher [1992]). We call them $\Gamma_5/2$ and $\Gamma_6/2$. Furthermore, Γ_2, Γ_3, Γ_4, $\Gamma_4/3$, Γ_5, $\Gamma_5/2$, Γ_6, $\Gamma_6/2$ are the only flag-transitive linear extensions of Γ_1 of rank $\leqslant 9$ satisfying (IP) (Soicher [1992]; see also Pasechnik [1993]). However, we are still far from a complete classification of all flag-transitive linear extensions of Γ_0, because we do not know much on the case of $k = 7$ and we do not know if any k-times linear extensions exist for $k > 7$, and because none of the geometries $\Gamma_1, \Gamma_2, \ldots, \Gamma_7$ is 2-simply connected. Indeed the universal 2-cover $\widetilde{\Gamma}_k$ of Γ_k is the folding of a geometry B_k whose diagram is a star with $(k+2)$ rays, as follows

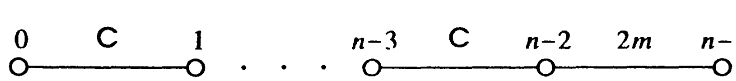


and $\mathrm{Aut}(\widetilde{\Gamma}_k) = \mathrm{Aut}(B_k) \cdot H$, with H permuting transitively the $(k+2)$ rays of that star (Pasini [1994a]). On the other hand, $\mathrm{Aut}(\Gamma_k)$ has not such a structure, for every $k = 1,2,\ldots,8$.

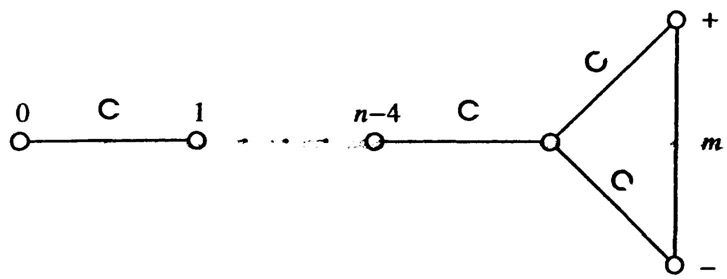
Brouwer, Fon-Der-Flaas and Shpectorov [1993] prove that the universal cover $\widetilde{\Gamma}_1$ of Γ_1 is 3-fold and that its flag-transitive automorphism group is $(3 \times U(3,3)) : 2$. They prove that there are only three extensions of Γ_0 satisfying (IP), namely Γ_1, $\widetilde{\Gamma}_1$ and a geometry of 48 points whose group is not flag-transitive. Their results are graph-theoretical and make no use of group theory. For recent work in this section, see Cuypers [1994e].

4.7. *The diagrams* $c^{n-2} \cdot I_2(2m)$ *and* $c^{n-1} \mid I_2(m)$

By $c^{n-2} \cdot I_2(2m)$ we denote the following diagram of rank $n \geqslant 3$:



This diagram is related with the following that we call $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$:



Indeed, let B be a geometry belonging to $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$. We can define a geometry $\Gamma = \mathrm{Fl}(B)$ (the *folding* of B) belonging to $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$ and thin at the last node, in the same way as we have defined the folding of a geometry of type $\mathsf{L(D)}_n$ in 4.3.8. Note that the residues of Γ of type $\{n-2, n-1\}$ are the flag-geometries of the residues of B of type $\{+, -\}$ (the *flag-geometry* of a geometry $\mathcal{G}$ of rank 2 is the J-shadow space of $\mathcal{G}$, with J the set of types of $\mathcal{G}$, see Chapter 11, 6.4.1; see also Chapter 12, 5).

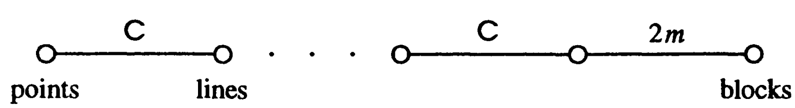
The geometry B is uniquely determined (up to isomorphism) by its folding $\Gamma = \mathrm{Fl}(B)$. Thus, we call B the *unfolding* of Γ, and we write $B = \mathrm{Unf}(\Gamma)$. An analogue of Lemma 4.3.14 also holds (Pasini [1994a]).

Note that $\Gamma = \mathrm{Fl}(B)$ is flag-transitive if and only if B is flag-transitive and admits a duality interchanging types $+$ and $-$. Thus, every theorem on (flag-transitive) geometries of type $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$ thin at the last node, entails a theorem on (flag-transitive) geometries of type $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$.

The geometries of type $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$ we are interested in are locally finite and thick at the types $+$ and $-$. Thus, we assume $m = 2, 3, 4, 6$ or 8 (Chapter 9, 2.1). We also assume $m \neq 2$, as we have already discussed the case of $m = 2$ in 4.3.16.

4.7.1. *A few definitions.* By a *pseudo-classical* generalized $2m$-gon we mean the flag-geometry of a classical generalized m-gon. We say that a geometry B belonging to $\mathsf{c}^{n-2} \mid \mathsf{I}_2(m)$ is *locally-classical* if its residues of type $\{+, -\}$ are classical generalized m-gons. We say that a geometry Γ belonging to $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$ is *locally classical* (*locally pseudo-classical*) if its residues of type $\{n-2, n-1\}$ are classical (pseudo-classical) generalized $2m$-gons.

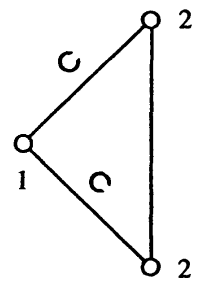
Given a geometry Γ of type $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$, we call the elements of Γ of the first, second and last type, *points*, *lines* and *blocks*, respectively.



If B belongs to $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$, then its *points, lines* and *blocks* are those of $\mathrm{Fl}(B)$.

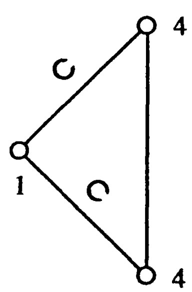
4.7.2. EXAMPLES. The diagrams $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$ and $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$ ($m = 3, 4$ or 6) are of strongly infinite type (2.1.4 and Ronan [1981a]). Nevertheless a number of finite examples are known for them. Thin finite examples are not difficult to obtain, by taking suitable quotients of Coxeter complexes. We will mention some (finite) flag-transitive locally classical or locally pseudoclassical examples. We only give very sketchy information on these examples. In most cases it will be clear from the orders which are the classical (or pseudoclassical) generalized m-gons (or $2m$-gons) occurring as rank 2-residues and otherwise we shall explain them. The folding of each of the next $\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$ examples is flag-transitive.

(1) Group: $\mathsf{U}(3,3) = \mathsf{G}(2,2)'$.



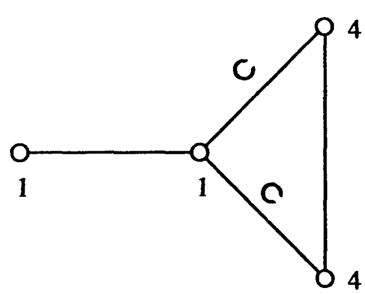
Group for the folding: $\mathsf{G}_2(2)$. (Weiss [1990].)

(2) Group: $\mathsf{U}(4,3)$ (also $\mathsf{U}(4,3)\cdot 2$).



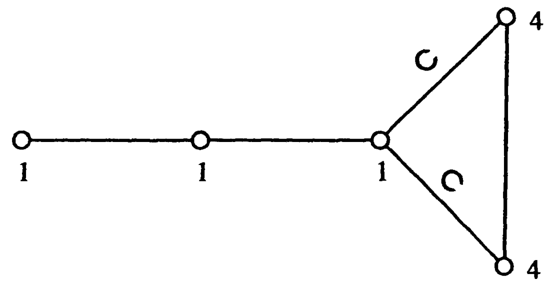
Group for the folding: $\mathsf{U}(4,3)\cdot 2$ (also $\mathsf{U}(4,3)\cdot 2^2$). (Buekenhout [1985], (51), Weiss [1990].)

(3) Group: the McLaughlin group McL. Group for the folding: McL $\cdot$ 2. (Buekenhout [1985], (52), Weiss [1990].)



Residues of points are isomorphic with the geometry (2) for $\mathsf{U}(4,3)$.

(4) Group: the Conway group Co_3. Group for the folding: $\mathsf{Co}_3 \times 2$. (Weiss [1990].)



Residues of points are isomorphic with the geometry (3) for McL.

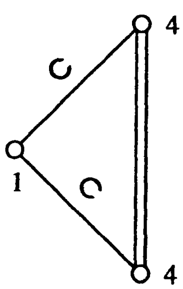
(5) Group: the Held group He. Group for the folding: He · 2. (Buekenhout [1985], (53).) Diagram and orders as in (2).

(6) Group 3 × PSL(2, 7) (Van Bon [1994]). Diagram and orders as in (1). Group for the folding: 3 · PGL(2, 7). There is a 3-fold quotient with automorphism group PSL(2, 7) and PGL(2, 7) on the folding. (Van Bon [1994].) We call this (6)bis.

(7) Group 2^6 : Frob(21), diagram and orders as in (1) (Van Bon [1994]). The group for the folding is $2^6 : (\mathbb{Z}_7 : \mathbb{Z}_6)$.

(8) Group 2^2 : Frob(21). Diagram and orders as in (1). Group for the folding: action of $2^2 : (\mathbb{Z}_7 : \mathbb{Z}_6)$. (Van Bon [1994]).

(9) Group: the Held group He. Group for the folding: He · 2. (Buekenhout [1985], (50), Weiss [1991a].)

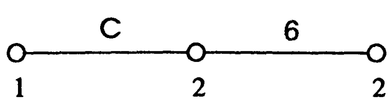


(10) Group: PSL(3, 4) · 2. Group for the folding: PSL(3, 4) · 2^2. (Weiss [1991a].) Diagram and orders as in (9).

(11) Group: 2 · PSL(3, 4) · 2. Diagram and orders as in (10). Group for the folding: 2 · PSL(3, 4) · 2^2. (Weiss [1991a].)

Remark: this is a double cover of the geometry (10) for PSL(3, 4) · 2.

(12) Group: the second Janko group J_2 (also $J_2 \cdot 2$).



Residues of points are isomorphic to $H(2)$ (notation as in Chapter 9, 3.3). (Buekenhout [1985], (94), Weiss [1991b].)

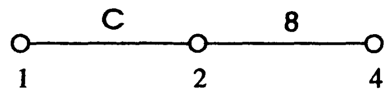
(13) Group: the Suzuki group Suz (also Suz · 2). Diagram and orders as in (12). Residues of points are isomorphic to $H(2)$. (Weiss [1991b].)

(14) Group: $2^6 : G_2(2)'$ (also $2^6 : G_2(2)$). Diagram and orders as in (12). Residues of points are isomorphic to the dual of $H(2)$. (Weiss [1991b].)

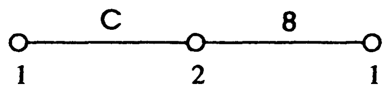
(15) Group: $2^7 : \mathsf{G}_2(2)'$ (also $2^7 : \mathsf{G}_2(2)$). Diagram and orders as in (12). Residues of points are isomorphic with the dual of $H(2)$. (Weiss [1991b].) This geometry is a double cover of the geometry (13) for $2^6 : \mathsf{G}_2(2)'$.

(16) Group: $\mathrm{Sp}(6, 2)$. Diagram and orders as in (15) (Van Bon, private communication). There are 120 points and the point stabilizer is $\mathsf{G}_2(2)$.

(17) Group: the Rudvalis group Ru. (Buekenhout [1985], (98), Weiss [1991a], Meixner [1994d].)



(18) Group: the Mathieu group M_{12}. (Weiss [1991a].)



(19) Group: $2 \cdot \mathsf{M}_{12}$ (central nonsplit extension), Weiss [1991a]. Diagram and orders as in (18). This is a double cover of the example (18) for M_{12}. None of these two examples admits an unfolding.

All the examples satisfy (IP) except (6)bis and (8). All examples of rank 3, except (5) and (8) satisfy the following property which is the negation of (SN) in Buekenhout [1985] (compare 5.1.4, 'triangle axiom').

(Non-SN) There are triples of pairwise collinear points which are not incident with any common block.

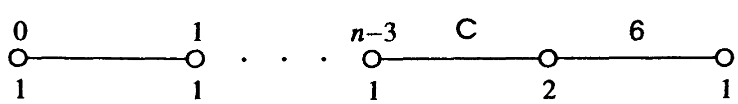
4.7.3. THEOREM (Weiss [1990, 1991a,b]). *Let Γ be a flag-transitive locally finite and locally classical geometry of type* $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$ ($m = 3$ *or* 4). *Assume furthermore that property* (Non-SN) *holds in residues of Γ of type* $\mathsf{c} \cdot \mathsf{I}_2(2m)$ (*when* $n = 3$, (Non-SN) *holds in* Γ). *Then Γ is one of the geometries mentioned in* (10)–(14) *of* 4.7.2.

4.7.4. THEOREM (Weiss [1990, 1991a,b], Van Bon [1994]). *Let Γ be a flag-transitive locally finite and locally pseudoclassical geometry of type* $\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$. *Assume furthermore that* (Non-SN) *holds in residues of Γ of type* $\mathsf{c} \cdot \mathsf{I}_2(2m)$. *Then Γ is one of the following*:

(i) *the folding of one of the examples* (1), (2), (3), (4), (6)bis, (7), (9), (10), (11) *or* (12) *of* (4.7.2);

(ii) *one of the two examples* (18) *or* (19) *of* 4.7.2;

(iii) *an* (*unknown*) *geometry of rank* $n \geqslant 4$ *with diagram and orders as follows*



where the residues Γ_F *of a flag* F *of type* $\{0, 1, \ldots, n-4\}$ *is of type* (6) *or* (7) *in* 4.7.2.

4.7.5. COROLLARY. *Let Γ be a locally finite locally classical flag-transitive geometry belonging to* $\mathsf{C}^{n-1} \mid \mathsf{I}_2(m)$ *and let Γ admit a duality interchanging the types + and −. Assume furthermore that* (Non-SN) *holds in residues of Γ of type* $\mathsf{C}^2 \mid \mathsf{I}_2(m)$ *(when* $n = 3$ (Non-SN) *holds in Γ). Then Γ is either one of the examples mentioned in* (i) *of Theorem* 4.7.4 *or the unfolding of an* (*unknown*) *geometry as in* (iii) *of Theorem* 4.7.4.

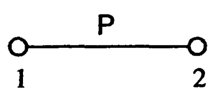
REMARK. The flag-geometry of a classical generalized m-gon $\mathcal{G}$ is flag-transitive if and only if $\mathcal{G}$ is flag-transitive and self-dual. The only self-dual classical generalized m-gons are: PG$(2, q)$, the generalized quadrangles $W(2^h)$ and the generalized hexagons $H(3^h)$ (Chapter 9). This restricts severely the cases to examine in the proof of Theorem 4.7.5. For recent work in this section, see Cuypers [1994b,e].

5. Extensions of geometries over Coxeter diagrams by the Petersen graph geometry

In this section, all geometries are finite.

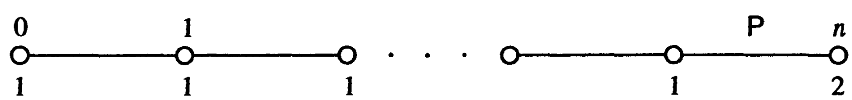
5.1. *Extensions of projective geometries by the Petersen geometry*

5.1.1. Here a *projective geometry* is a member of the class of geometries belonging to a spherical Coxeter diagram A_n, $n \geqslant 2$, and we assume that it has the same order $s \geqslant 1$ at each node. Other cases (nonuniform order) correspond to open problems in the sequel. In this Chapter P stands for the Petersen geometry with 10 points, each incident with 3 lines, and 15 lines, each incident with 2 points.



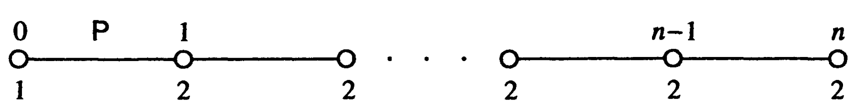
There are two major classes of diagrams to consider first.

$\mathsf{A}_n\mathsf{P}$



$n \geqslant 2$

PA_n



$n \geqslant 2$

5.1.2. *Geometries of type* $\mathsf{A}_2\mathsf{P}$. There are four known examples that we describe briefly.

(1) Aut Γ = Sym(7). Let Ω denote a set of cardinality 7 on which Sym(7) acts. The 0-elements of Γ are the 21 pairs of elements of Ω. The 1-elements of Γ are the 105 pairs $\{\{a, b\}, \{a, c\}\}$ with $a, b, c \in \Omega$ and a, b, c distinct. The 2-elements are 105 trios $\{\{a, b\}, \{b, c\}, \{a, c\}\}$ with a, b, c as above.

The flag-transitive groups of Γ are Sym(7) and Alt(7).

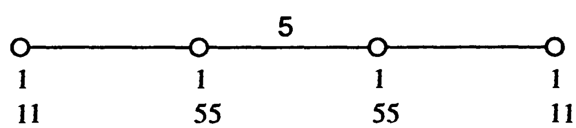
(2) Γ is a 3-fold 2-covering of the preceding. It has 63 0-elements, 315 1-elements, 315 2-elements and Aut $\Gamma = 3 \cdot$ Sym(7), the nontrivial 3-cover of Sym(7). The flag-

transitive groups of Γ are $3 \cdot \mathrm{Sym}(7)$ and $3 \cdot \mathrm{Alt}(7)$. Examples (1) and (2) did occur in Hall [1980] as 'locally Petersen' graphs.

(3) Consider PG(3, 5) and an ovoidal quadric Q in it. This is invariant by the orthogonal group $G = \mathrm{PO}^{-}(4, 5)$. There are 130 points not on Q. The transitive action of G on these points has two blocks of imprimitivity of size 65. Let B be one of these. The orthogonality relation induced by the quadratic form related to Q, on the set B gives a graph of valency 10 which is 'locally Petersen'. The geometry Γ consists of the 65 points in B and the complete subgraphs (of size 2 and 3) in B. The flag-transitive groups acting on Γ are PSL(2, 25) and PΣL(2, 25), a subgroup of index 2 in PΓL(2, 25), namely the extension of PSL(2, 25) by the field automorphism. This graph (geometry), was first observed and studied by S. Doro (see Hall [1980]).

(4) Consider the sporadic 2-transitive action of $\mathrm{PSL}(2, 11) = G$ on a set of 11 points, say Ω. The stabilizer of a point is Alt(5) acting as a rank 3 group on the 10 remaining points.

Hence G has two orbits on the trios of Ω, of respective sizes 55 and 110. The elements of Γ are the points of Ω, the pairs of points of Ω and the 55 trios of the smallest G-orbit on trios. A flag-transitive group acting on Γ is necessarily G.



This geometry was observed in Buekenhout [1985]. It is a truncation of a thin geometry due to Grünbaum [1977b]. The group was found by Coxeter [1984], who gets a representation of this as a regular skew-polytope in Euclidean 10-space.

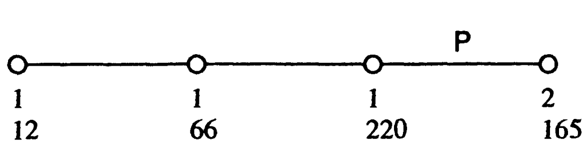
5.1.3. THEOREM (Shpectorov [1989]). *Let Γ be a geometry of type* $\mathsf{A}_2\mathsf{P}$ *and let G be a flag-transitive automorphism group of Γ. Then Γ and G are as in the descriptions* (1) *to* (4) *of* 5.1.2.

5.1.4. Can we hope to derive the same result with no presence of a flag-transitive group? This is likely.

Assume that Γ satisfies the *triangle axiom*: any set of 3 points a, b, c of Γ and 3 lines ℓ, m, n such that $a * \ell * b * m * c * n * a$ (where $*$ means 'incident') is contained in the residue of some plane.

Then it is readily seen that Γ amounts to a 'locally Petersen' graph and from Hall [1980], Γ is one of (1), (2), (3). At the other extreme, if all points are pairwise collinear, there are exactly 11 points and we get rather easily the uniqueness of Γ, hence (4). It is reasonable to expect a complete solution.

5.1.5. PROBLEM. How about $\mathsf{A}_n\mathsf{P}$ with $n \geqslant 3$? The following examples are known.

(5) 

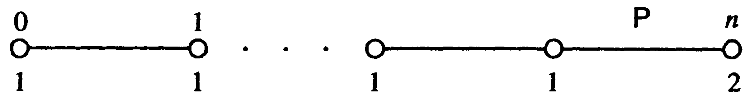
Aut $\Gamma = \mathsf{M}_{11}$ (Buekenhout [1985]), 3-transitive action on 12 points;

(6) 

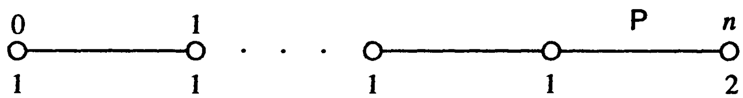
Aut $\Gamma =$ He.

This is due to Ivanov and Shpectorov [1988]. The stabilizer of a point is $3 \cdot \mathrm{Sym}(7)$, a maximal subgroup of He. Other stabilizers of elements are not maximal.

(7) Diagram as in (6). This is obtained from $\mathrm{Sp}_4(5)$ with point stabilizer PSL(2, 25) : 2 (Ivanov and Shpectorov [1988]). There are 300 points.

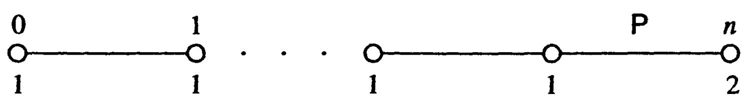
(8) 

Aut $\Gamma = \mathrm{Sym}(2n+3)$, $n \geqslant 3$ (Buekenhout [1985]).
The points are the $\binom{2n+3}{2}$ duos; two of these are collinear if they have a common member. Elements of Γ are the sets of pairwise collinear duos. The series starts with $n = 2$ which is (1).

(9) 

Aut Γ is some elementary Abelian 3-group T extended by $\mathrm{Sym}(2n+1)$. The order of T is not known. The construction is due to Shpectorov and Pasechnik, personal communication. See 5.4.2.

We give another description of these geometries. Let $\widehat{\Gamma}$ be one of the $\mathsf{c}^{n-2} \cdot \mathsf{C}_2$ geometries described in 4.3.5(a). Choose a point p of $\widehat{\Gamma}$. Remove p from $\widehat{\Gamma}$, as well as all elements of $\widehat{\Gamma}$ containing p. The resulting geometry Γ is (9) and its group is $\mathrm{Sym}(2n+1)$.

(10) 

(Brouwer, private communication, 1988).

Aut Γ is the series of groups

$$(1/2)\,\mathrm{P\Gamma O}^-(4,5)\ (n=2),\ \ \mathrm{P\Gamma O}(5,5)\ (n=3),\ \ (1/2)\,\mathrm{P\Gamma O}^-(6,5)\ (n=4), \ldots .$$

It generalizes (3) in a fairly obvious way. The first member of this series also appears in Ivanov and Shpectorov [1988]. Points are the nonsingular points (of some imprimitivity block for n even) of the corresponding quadratic form and the elements are all sets of pairwise orthogonal points.

It seems reasonable to expect a full classification with or without flag-transitive group of automorphisms.

5.1.6. *Geometries of type* PA_n (*P-geometries*), $n \geqslant 2$. Examples were first observed in Buekenhout [1985] for M_{22} and, on this basis, by Brouwer (private communication, 1983) for M_{23}. Much more involved examples and a complete classification are due to deep work of Ivanov and Shpectorov [1988, 1989, 1990, preprint], Shpectorov [1985, 1992], Ivanov [1992a,b,c].

THEOREM (Ivanov and Shpectorov, in preparation). *Let* Γ *be a geometry of type* PA_n, $n \geqslant 2$, *and* G *a flag-transitive automorphism group of* Γ. *Then* (Γ, G) *is in a list of eight pairs with the following characteristic properties*:

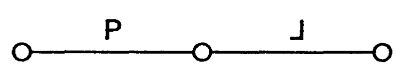
(1) $n = 2$, $G = \mathsf{M}_{22}$;
(2) $n = 2$, $G = 3 \cdot \mathsf{M}_{22}$;
(3) $n = 3$, $G = \mathsf{M}_{23}$;
(4) $n = 3$, $G = \mathsf{Co}_2$;
(5) $n = 3$, $G = 3^{23} \cdot \mathsf{Co}_2$;
(6) $n = 3$, $G = \mathsf{J}_4$;
(7) $n = 4$, $G = \mathsf{F}_2$;
(8) $n = 4$, $G = 3^{4371} \cdot \mathsf{F}_2$.

An early step of the proof was to show that for $n = 3$, the universal 2-cover of Γ is necessarily the universal 2-cover of one of the above geometries, corresponding to M_{23}, Co_2 or J_4 (Shpectorov [1989, thesis]).

PROBLEM. Classify for $n = 2$, without any group-theoretical assumption. Thanks to a lemma of Shpectorov [1985] the diameter is bounded.

5.1.7. Here is a little extension of the preceding theorem.

PROPOSITION (Buekenhout and King [1993]). *Let* (Γ, G) *belong to the diagram,* G *flag-transitive*

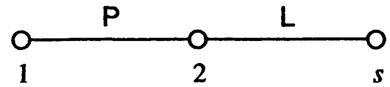


and assume that G *acts primitively on the* 2*-elements. Then* Γ *belongs to* PA_2.

PROBLEM. Classify without the primitivity assumption. There are six known examples with $6, 18, 16, 32, 32$ and 64 elements of type 2. If there are some repeated lines, the first is the only possible example. If there are no repeated lines but there are triples of collinear 2-elements not in any 0-element residue, then the examples with $18, 16, 32, 32$ 2-elements are the only to occur. The first and third case are the smallest members of two families coming as quotients from two distinct infinite simply connected examples. How to characterize the example on 64 elements of type 2? And are there any other finite examples? (King and Pasini [1994].)

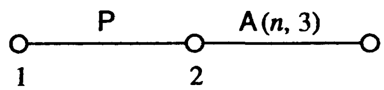
5.2. *Extensions of affine geometries*

5.2.1. If we think of a generalization of 7.1.6, a natural problem is to ask for all



with a flag-transitive automorphism group, and linear space residues that are no longer projective spaces. Then the most natural case is that of an affine space of some dimension n over the field of order 3. This situation is studied in Pasechnik [1992].

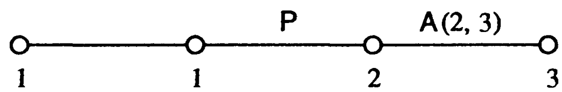
PAf_n



An example with $n = 2$ was given by Ivanov [1987b]. It is shown that there are examples for all n. For $n = 2$, a complete classification is given. There are exactly two pairs (Γ, G) in that case.

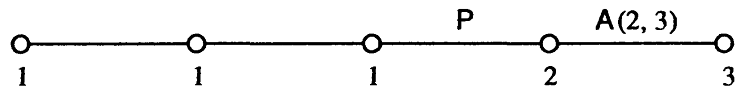
5.2.2. Moreover, Pasechnik (personal communication) produces two spectacular extensions as follows.

M_{11}



(11 points)

M_{12}

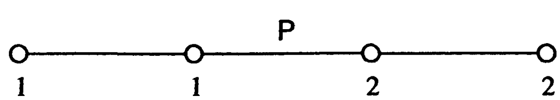


(12 points)

Quite remarkably, the stabilizer of each element of these geometries, is a maximal subgroup.

5.3. *Other extensions*

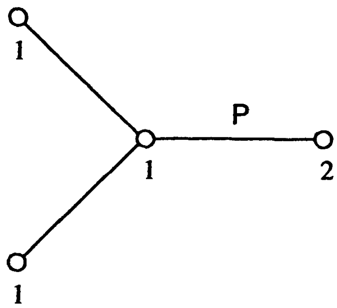
5.3.1. There is another remarkable geometry produced in Ivanov and Shpectorov [1986]. Here $G = \mathsf{O'N}$ and the diagram is as follows:

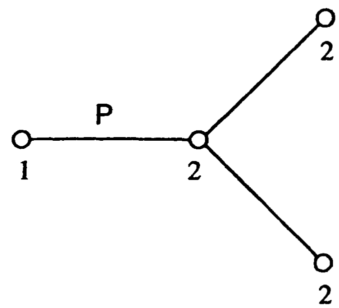


The same diagram occurs for McL (Ivanov and Shpectorov [1988]). The same authors announce a geometry for HS on a diagram $\mathsf{P} \cdot \mathsf{C}_2$.

PROBLEM. Classify all pairs (Γ, G) over this diagram. The point-stabilizer must be finite and 'small' because the kernel of the action is $\leqslant 2$.

5.3.2. PROBLEMS. Classify the flag-transitive geometries over the following diagrams.





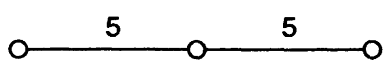
5.3.3. PROBLEM. Classify the following:

An example is known with $G = \mathsf{M}_{24}$ (Buekenhout [1985]).

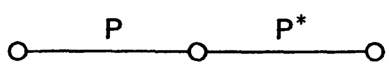
5.3.4. PROBLEM. Classify the following:

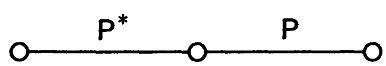
By 7.1.6 the residues of PA_2 type can be of two kinds: either M_{22} or $3 \cdot \mathsf{M}_{22}$. From earlier sections, $s = 2$ or 4.

5.3.5. PROBLEM. Classify the following. This may be difficult because the minimal circuit diagram is of infinite type.

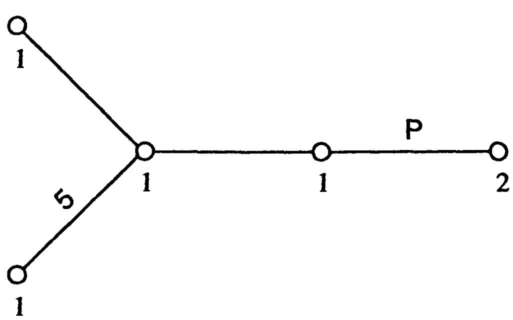


In particular, therefore the universal cover is infinite:





5.3.6. Here is another striking example (Ivanov and Shpectorov [1986]).
$G = \mathsf{O'N}$ (also, $G = 3 \cdot \mathsf{O'N}$)



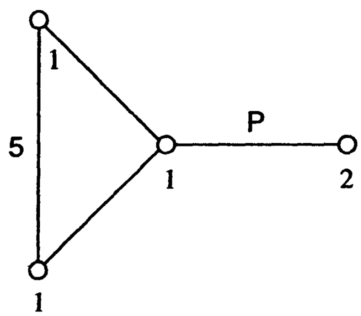
It involves residues:

M_{11}

and

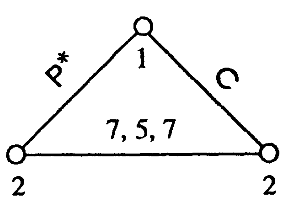
J_1

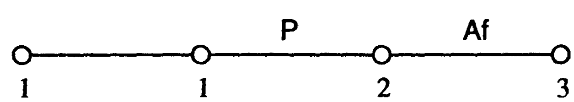
Another interesting example is due to Tsaranov [1990b]. Its existence has also been checked on CAYLEY, by Michel Dehon. It admits J_1 as a flag-transitive group and admits the following diagram.

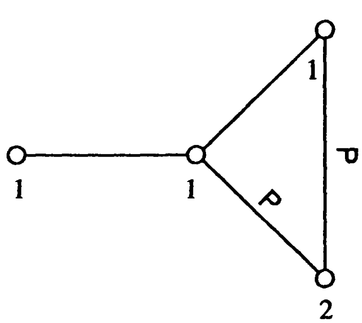


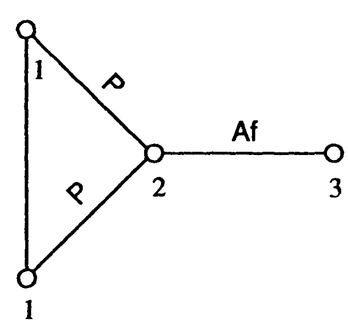
5.3.7. *A computer search for* M_{11}. M. Dehon (personal communication, 1992) has determined all geometries Γ of rank $\geqslant 3$ with $G = \mathsf{M}_{11}$ under the additional assumption that each element stabilizer is maximal in G. This was realized with a CAYLEY programme.

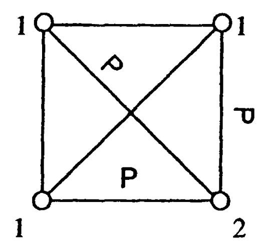
There are six such geometries involving P. Their diagrams are:

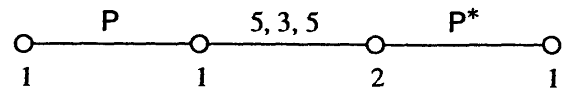








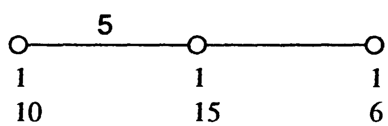




An earlier search on rank 4 chamber systems for J_1 and U(4, 2) was completed in Komissartschik and Tsaranov [1990]. The rank 4 geometries are necessarily in this list but they have not been detected so far.

5.3.8. *A link with regular polytopes.* Extensions of the rank 2 Petersen geometry P are related to questions about regular polytopes, more generally, regular thin geometries.

Indeed, P is the $\{0, 1\}$-truncation of the hemidodecahedron HD, a rank 3 geometry over the diagram H_3.

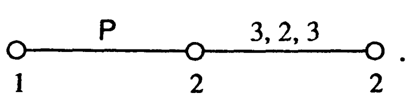
HD


This observation leads to the following questions.

(1) Given a geometry Γ extending HD, does it lead to an extension of P by (the obvious) truncation?

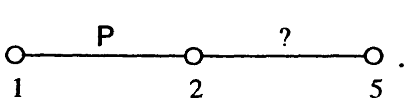
(2) Given an extension of P, does it occur as a truncation of some HD-extension? In 5.1.2, case (4), we obtain an example. We give some examples of diagrams in order to illustrate these ideas further. The square node is the node that determines the truncation to be made.

leads to

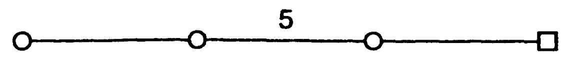


Examples are known (McMullen and Schulte [1990]) in particular with $G =$ Sym(7) and Sym(7) $\times$ Alt(5), for

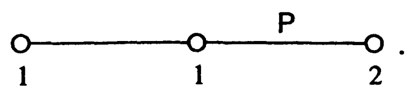
gives



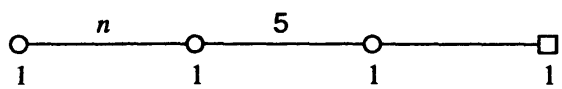
The diagram



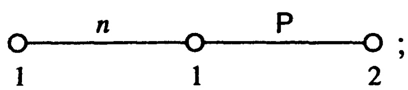
leads to

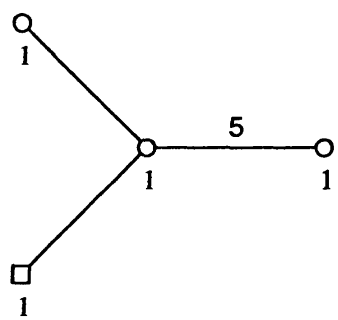


For work on the former, see Coxeter and Weiss [1984], also Van den Cruyce [1985]. Similarly,

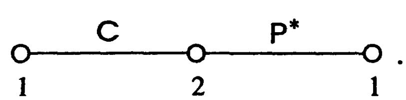


gives





gives



The answer to question (2) may be negative. This is the case with Sym(7) and (7) of 5.1.5.

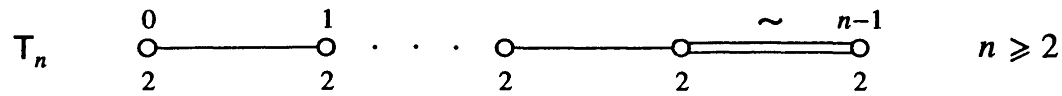
5.3.9. PROBLEM. How about extensions of the $(5,5,6)$-gon corresponding to the Hoffman–Singleton graph?

5.4. *Tilde geometries*

5.4.1. We now discuss a remarkable class of geometries that happens to be related structurally to PA_n geometries although it does not involve the Petersen graph in its diagram.

We work exclusively at the (Δ, Γ, G)-level.

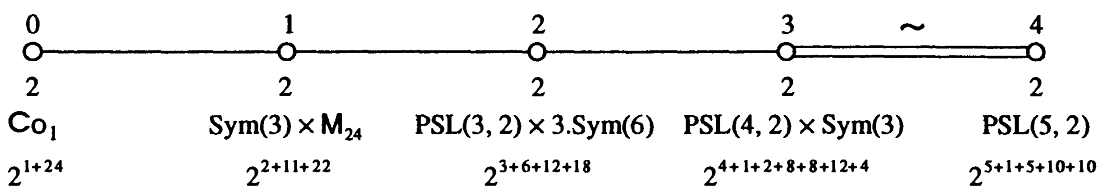
A *tilde geometry* is a geometry belonging to the following diagram.



The case $n = 2$, was described in 1.8.3 (Example 1). This diagram does clearly escape the philosophy of small deficiency advocated so far in this chapter with our choices of linear spaces and Petersen graphs in order to extend Coxeter diagrams. On the other hand, T_n is too densely fed with sporadic groups to be ignored.

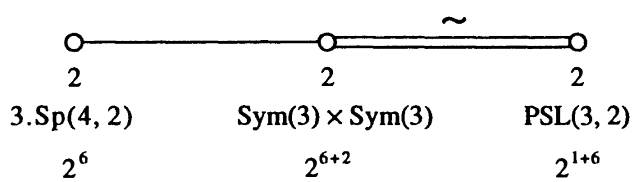
5.4.2. EXAMPLES.

(1) The first examples were observed in Ronan and Stroth [1984] for the sporadic groups M_{24} ($n = 3$), He ($n = 3$), Co_1 ($n = 4$), $\mathsf{M} = \mathsf{F}_1$ ($n = 5$). The examples for M_{24}, Co_1 and M are residually related as can be seen on the next picture for M (Ivanov [1992a]):



The upper group is the group induced on the corresponding residue and the lower 2-group is the kernel of the action on the residue. For instance, $2^{1+24} \cdot \mathsf{Co}_1$ is the stabilizer of a 0-element.

For the Held group we have:



(2) As it turns out, there is also a generic family of examples and, unlike most other discoveries of diagram geometries, this family was found later than the sporadic examples. It is due to Ivanov and Shpectorov [1993] and independently to Meierfrankenfeld (unpublished). It goes as follows.

Let $\overline{n}$ be the number of 2-dimensional subspaces of an n-dimensional vector space over the field of order 2. The authors produce a nonsplit extension $3^{\overline{n}} \cdot \mathrm{Sp}(2n, 2)$. For $n = 2$, this is precisely the group $3 \cdot \mathrm{Sym}(6)$ of the geometry T_2. Coming back to the general case, it is shown that there is a T_n-geometry whose automorphism group is $3^{\overline{n}} \cdot \mathrm{Sp}(2n, 2)$, a point residue giving the (T_{n-1})-geometry corresponding to $3^{\overline{n-1}} \cdot \mathrm{Sp}(2(n-1), 2)$. This T_n-geometry is called of *symplectic type.*

5.4.3. THEOREM (Rowley [1989], Heiss [1991], Shpectorov and Stroth [1994], Ivanov [1992a,b], Shpectorov, in preparation). *If (Γ, G) provides a tilde geometry than (Γ, G) is either of symplectic type or it is one of the four examples related to* M_{24}, He, Co_1 *and* M.

Here are some guidelines about the proof. For $n = 3$, Rowley [1989] shows that the Borel subgroup must have order 2^9 or 2^{10}, the second alternative being realized by M_{24} and He. The former alternative inspired the discovery of a geometry of symplectic type and of the generic family. Observe that 2^9 is the order of the Borel subgroup of the group Sp(6, 2) related to a geometry (polar space, building) of type C_3. The T_2-geometry is obtained as a 1-cover of the generalized quadrangle for Sp(4, 2) and the T_n-geometry of symplectic type is likewise obtained as a 1-cover of the C_n building geometry corresponding to $Sp(2n, 2)$.

The case $n = 3$, was finished in Heiss [1991], (see also Heiss [1992]). The general case for n, with rank 3 residues of symplectic type $3^7 \cdot Sp(6, 2)$ is settled in Shpectorov and Stroth [1994]. The T_3-geometry for the group He has no extension of type T_4 according to Shpectorov (in preparation).

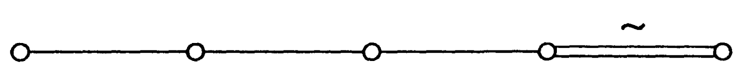
The extensions of the T_3-geometry for the group M_{24} are settled for $n = 4$ by Ivanov [1992b], for $n = 5$ by Ivanov [1992a], and for $n = 6$ by Shpectorov (in preparation). A key unpublished observation used for $n = 4$ on, made by Ivanov and Shpectorov in 1989, was that the methods used in Shpectorov [1989] in order to find the amalgams for PA_n geometries with $n = 3$, work as well for amalgams of tilde geometries of any type known at that time.

5.5. *The relationship between tilde and Petersen geometries*

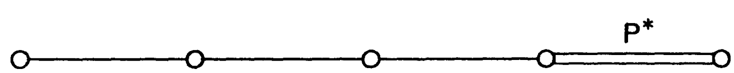
All of this is quoted from a conversation with Shpectorov. The geometries of types PA_n and T_n are intertwined in two different ways and related to geometries of type C_n as well.

This appears already for $n = 2$. Indeed, PA_1 (the incidence graph of the line-graph of the Petersen graph) is a subgeometry of the generalized quadrangle whose 3-fold 1-cover is T_2. It has the 15 points that are required and one more line of 3 points is necessary on each point p. It consists of the two points at maximal distance from p.
The general situation will be explained now. There are two kinds of inclusions.

To illustrate the first kind we start with the case of M and T_5:

M 

This geometry has a subgeometry as follows.

BM 

In order to get this, one takes the involution a in the centre of a subgroup $2 \cdot BM$ of M. The subgeometry of fixed elements of a, contains (properly) the BM geometry of type PA_4.

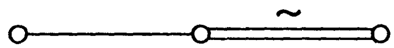
The same method applies to the following.

Co_1 

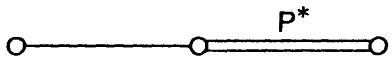
contains

Co_2 

;

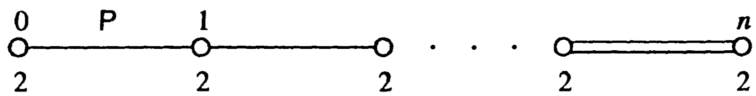
M_{24} 

contains

M_{22} 

.

Let us turn to the second kind of inclusion. Start with (Γ, G) of type PA_n.



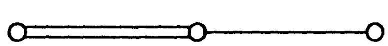
$n \geqslant 2$

Consider the 0-truncation (Γ^0, G) which means that all 0-elements are removed. In Γ^0, the Borel subgroup is twice as big as it was in Γ and the minimal parabolic subgroups of type $1, \ldots, n$ have the form $2^{x-2} \cdot \mathrm{Sym}(5)$, $2^x \cdot \mathrm{Sym}(3), \ldots, 2^x \cdot \mathrm{Sym}(3)$ for some x. Take the subgroup $2^{x-2} \cdot \mathrm{Sym}(4)$ of the first member in this series. Together with the other members of the series, it generates a subgroup giving either a polar space of type $\mathrm{Sp}(2(n-1), 2)$ or a tilde geometry.

5.5.1. *Link with apartments.* In this way, the following derivations occur.

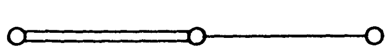
M_{23} 

leads to

Alt(7)  ;

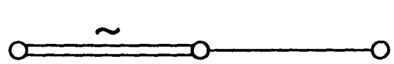
Co_2 

leads to

Sp(6, 2)  ;

$3^{23}Co_2$ 

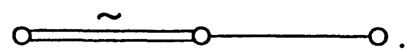
leads to

$3^7 \cdot \mathrm{Sp}(6,2)$ 

;

J_4 

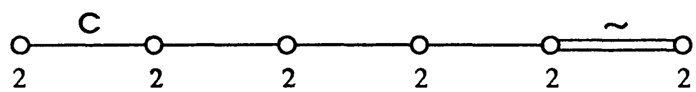
leads to

M_{24} 

The geometry Γ for $G = 3^7 \cdot \mathrm{Sp}(6,2)$ is close to buildings in a striking way, observed by Heiss [1992]. There are apartments in Γ (see Chapter 3), and if N is the stabilizer of an apartment A, then N acts flag-transitively on A and $G = BNB$ where B is the stabilizer of a chamber. (The analog for buildings is proved in Chapter 11, 4.3.) This means that any two chambers of Γ are contained together in some apartment. This situation is characterized by Heiss [1992].

5.6. *More examples*

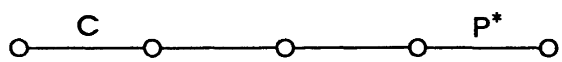
5.6.1. Sphectorov (private communication) knows pairs (Γ, G) with Γ and G as follows.



$G = (M \times M) : 2$

$G = (BM \times BM) : 2$

$G = (2 \cdot BM * 2 \cdot BM) : 2$

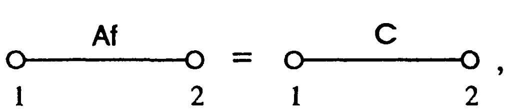


$G = (J_4 \times J_4) : 2$

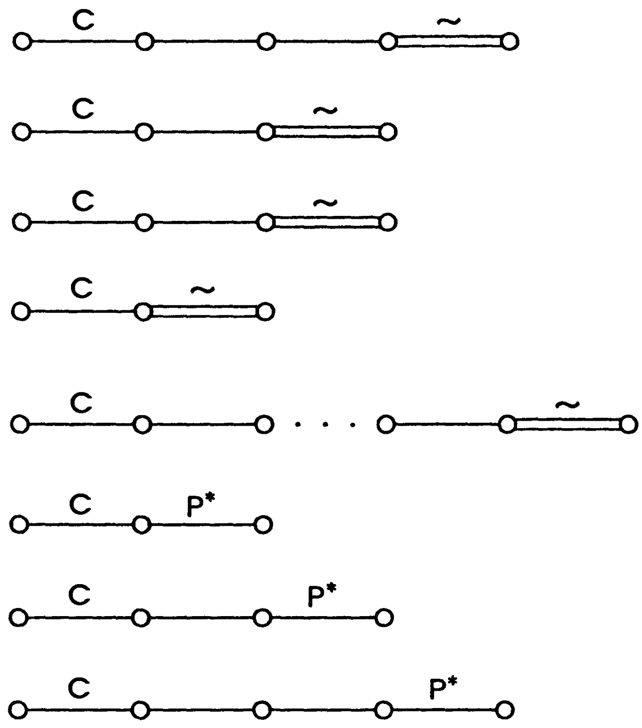
Here $*$ denotes a central product.

The proof relies on some Cayley graph of the groups M, BM, 2BM, J_4 with a particular class of involutions as generators. The method is the same in all cases. We explain it for M. Let $\mathcal{A}$ be the class of central involutions of M and $a \in \mathcal{A}$. Then $C_M(a) = 2^{1+24} \cdot \mathrm{Co}_1$ which is a point stabilizer in the tilde geometry for M, while a line of the latter has 3 points corresponding to 3 involutions of a 2^2-group. In order to get Γ for $(M \times M) : 2$ we take as points the elements of M, and consider the Cayley graph on M with $\mathcal{A}$ as set of generators. Then the neighborhood of $a \in \mathcal{A}$ is the set $\mathcal{A}$ and everything comes out as expected, except that the 2 on top of $M \times M$ is not obvious.

5.6.2. The embeddings of Petersen and tilde geometries in projective spaces over the field of order 2, together with the method of affine extensions, remembering that



gives:



$2^{24} \cdot Co_1$

$2^{11} \cdot M_{24}$

$2^{52} \cdot He$

$2^6 \cdot 3 \cdot Sp(4, 2)$

$2^n(2^n - 1)3^{\bar{n}} \cdot Sp(2n, 2)$

$2^6 \cdot Sym(5)$

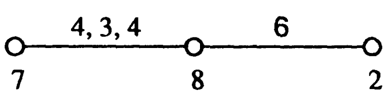
$2^{11} \cdot M_{22} \cdot 2,\ 2^{12} \cdot 3 \cdot M_{22} \cdot 2$

$2^{23} \cdot Co_2$

6. Other extensions

6.1. *Back to sporadic groups*

As mentioned in 1.1, one of the goals of research on extensions of buildings is to contribute to a uniform geometric interpretation of all finite simple groups. It should not be forgotten that a sporadic group may have to be understood as a rank 2 group or even as a rank one group. Here we leave these low rank cases aside. We observe that most sporadic groups have been met so far, in this chapter, with at least one flag-transitive action on a geometry whose rank ranges from 3 to 6. The exceptions are the Thompson group F_3 and the Harada–Norton group F_5. The former group occurs as a flag-transitive group acting on a geometry belonging to the diagram and orders as follows.

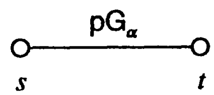


The generalized hexagon corresponds to ${}^3D_4(2)$ while the other residues are isomorphic to the partial geometry corresponding to Alt(9) (see Chapter 10, Section 1). As to the group F_5, it belongs to the following (see Buekenhout [1985]):

This provides motivation for further extensions, in particular on the basis of rank 2 strokes of the diagram with deficiency (1,1) (see 1.4.1). Then partial geometries (other than generalized quadrangles and linear spaces) come in the first place. This is our next topic.

6.2. *Extensions by partial geometries*

In this section we consider the following diagram of rank 2:

pG_α 

$1 \leqslant \alpha \leqslant \min(s+1, t+1)$

Members of that diagram are rank 2 geometries of orders s, t such that for every non-incident point-line pair (p, ℓ) there are exactly α points on ℓ that are collinear with p. These geometries are called (proper) *partial geometries* provided $1 \neq \alpha < \min(s,t)$

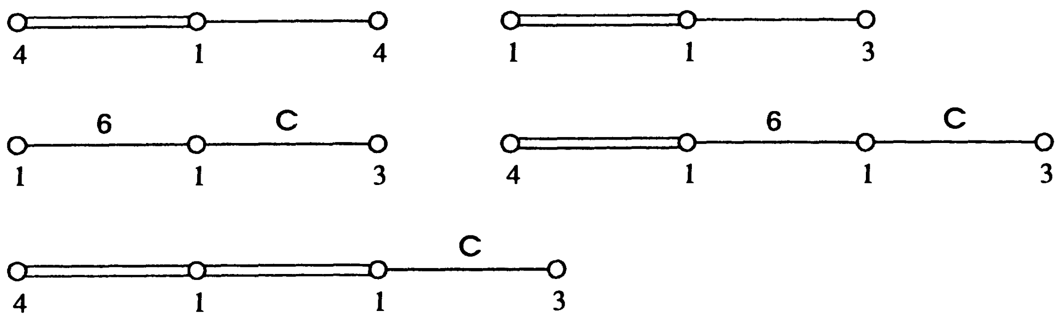
This setting allows for many developments at the (Δ, Γ) level and there are many constructions of examples. We refer to Hughes [1990b, 1991, 1992], Hobart and Hughes [1990, 1992], Del Fra, Ghinelli and Pasini [1992], Del Fra and Ghinelli [1992], Del Fra, Ghinelli and Hughes [1992]. See also Chapter 10.

6.3. *All geometries for a given group*

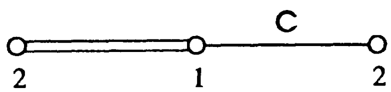
In this chapter our main concern has been the classification of all pairs (Γ, G) belonging to a given diagram Δ of some specified type. In other words, we look for the triples (Δ, Γ, G) with Δ fixed. As a matter of fact, we can also fix G. Given some group G, we may ask for all triples (Δ, Γ, G) submitted to some set of restrictions to be decided. This view was expressed and worked out on small groups G like Sym(4) and Alt(5) in Buekenhout [1986]. This question and the related algorithms are discussed in Chapter 3, 3.2.3. The same line of thought is developed in Komissartschik and Tsaranov [1990] where the pairs (Γ, G) are somewhat more general as here. They determine for instance all pairs $(\Gamma, \mathrm{PSU}(4,2))$ with $\mathrm{rank}\,\Gamma \geqslant 3$, under the condition (Pri) that stabilizers of elements of Γ be maximal subgroups of G_0. They get 71 'geometries' of rank 3, 22 of rank 4, and 3 of rank 5. On the basis of a CAYLEY programme, Dehon [1992] handles the case $(\Gamma, \mathrm{Aut\,PSU}(4,2))$ and he finds 27 (resp., 77,87,20,4) geometries of rank 2 (resp., 3,4,5,6). In Tsaranov [1992], the case of the Janko group J_1 is dealt with under the condition (Pri) and $\mathrm{rk}\,\Gamma \geqslant 4$. The result provides 18 'geometries', all of rank 4. A systematic development of CAYLEY programmes devoted to the search of 'all' Γ, given G has been built in Brussels and tested on various groups, by a team consisting of M. Hermand, F. Buekenhout, V. Gobbe, M. Dehon, I. De Schutter and D. Leemans. It appears that a fairly small group like PGL(2,7) may act flag-transitively on hundreds of rank 3 geometries, and this is but an example. Restricting to connected diagrams is not a very strong condition. At another extreme, assuming property (Pri) leaves a fairly small number of examples and confirms the earlier results quoted at the beginning of the present section.

Let us mention some interesting examples arising from this experimental work (Dehon, De Schutter, Buekenhout):

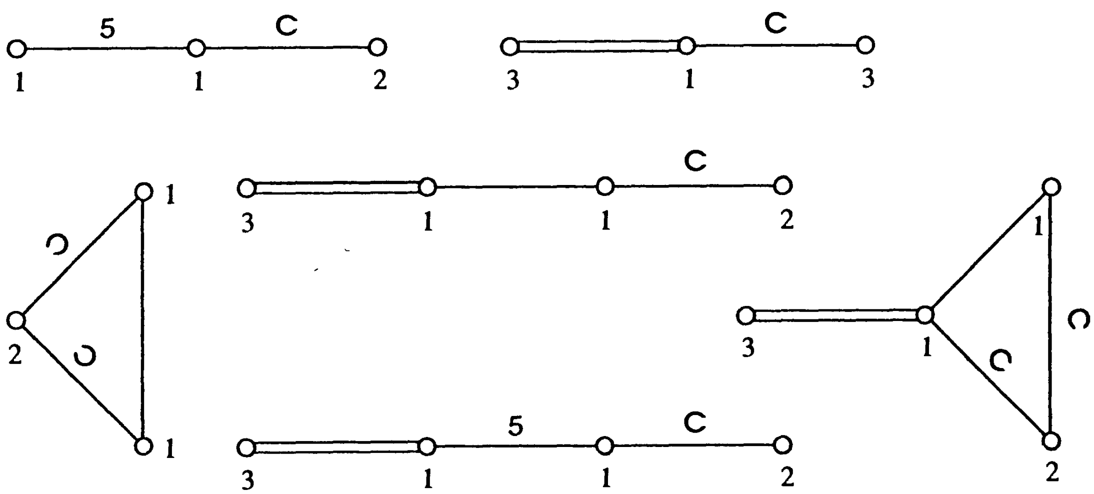
(1) $5^2 : GL(2,5)$ admits



(2) $3^2 : GL(2,3)$ admits



(3) $2^4 : GL(2,4)$ admits



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CHAPTER 23

Linear Topological Geometries

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Contents

Introduction . . . 1257
1. Topological projective spaces . . . 1258
2. Stable spaces . . . 1264
3. Geometry and topology of stable planes . . . 1270
4. Topological projective planes . . . 1283
5. Homogeneous stable planes . . . 1293
   Results on planes of arbitrary dimension . . . 1293
   2-dimensional planes . . . 1299
   4-dimensional planes . . . 1301
   8-dimensional planes . . . 1302
   Symmetric planes . . . 1304
6. Generalized polygons and buildings . . . 1306
7. Some related topics . . . 1310
References . . . 1312

HANDBOOK OF INCIDENCE GEOMETRY
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Introduction

Many branches of modern geometry can be traced back to Hilbert [1899]; topological geometry is no exception. We refer to Karzel and Kroll [1988] for the historical development since Hilbert, see also Mainzer [1980], Klingenberg [1971]. Here we offer some observations concerning the role of topology in the foundations of geometry. Continuity certainly is an essential part of the geometric imagination. In fact, topological elements are indispensable for characterizations of the classical geometries. This was clear at the turn of the century, despite the fact that the abstract concepts and notions of topology had not yet emerged. Thus, Hilbert expressed the necessary continuity assumptions in terms of ordering properties. The control (and, to some extent, the elimination) of the influence of ordering axioms constituted one direction of research in the first decades of this century. The concept of a topological incidence structure, in the sense of this chapter, appears for the first time in Kolmogoroff [1932], as far as we know. The notion of a topological projective plane was introduced and studied systematically by Skornjakov [1954], Salzmann [1955, 1957] and Freudenthal [1957a]. Unlike Hilbert's system of axioms, this topological approach includes complex and quaternion projective spaces, as well as the octonion plane; compare the lucid review in Freudenthal [1957b]. There is a related approach, which eliminates all incidence axioms in favour of assumptions on topological transformation groups characterizing the groups of 'motions', compare Hilbert [1903]; this line of development culminates in papers of Tits and Freudenthal, see 2.13.

Affine and projective spaces are linear in the sense that any two points have at most one joining line; in fact, in Sections 1–5 we consider only geometries where any two points are joined by a unique line. There are many interesting geometries which are not linear, see Chapters 14, 19, 21, and 24 on circle geometries.

In this chapter, we emphasize the interplay between incidence and continuity. A complete axiomatic characterization of the classical geometries requires metric components as well, see Chapter 17, and 5.2, 5.12, 5.31 in the topological context. Ordering properties, and the related convexity theory, will receive only cursory attention. Older completeness assumptions (like Hilbert's maximality axiom) have been replaced by compactness, which is a topological finiteness condition. Together with connectedness assumptions, it leads to results which have many formal analogies to results on finite planes; often the topological results are more incisive, cf. 4.13.

Topological geometry, in the sense of this chapter, connects with many other areas of mathematics, in particular with differential geometry (see 5.27, 6.7, 7.2) and with variants of synthetic differential geometry in the sense of Busemann (Section 5.18) or in the sense of Haupt, see Haupt and Künneth [1967].

Section 1 deals with topological projective spaces (of finite dimension). Desarguesian topological projective spaces are understood very well; they are equivalent to vector spaces of finite dimension over topological division rings, see the representation theorem 1.5. In Section 2 we present Groh's concept of a stable n-space. This is a semimodular lattice with a topology, with axioms which express geometric properties of open subsets of topological projective n-spaces. In fact, by an important result of Groh [1986a,b], every rich stable n-space with $n \geqslant 3$ is embeddable as an open subset into a topological projective space, see 2.9.

For *stable planes* (rich stable 2-spaces), the corresponding statement is false. This is no surprise, because stable planes are generalizations of topological projective planes, and this class of planes contains many non-Desarguesian examples. The *hyperbolic plane* (the interior of the unit circle in $\mathbb{R}^2$) is a prime example of a stable plane. In Section 3 we begin a systematic investigation of the topological and geometrical structure of stable planes which are locally compact and not totally disconnected. The geometry imposes severe restrictions on the topology of these planes. In particular, the topological dimension of lines and pencils is 1, 2, 4, 8, or infinite, see 3.29.

In Section 4 we specialize to topological projective planes; we consider mainly (locally) compact connected projective planes. Drawing on results from Section 3, we give a rather condensed account of the main achievements of Salzmann and his school; see Salzmann, Betten, Grundhöfer, Hähl, Löwen and Stroppel [1994] for a detailed exposition. A programmatic idea of Salzmann is to classify compact connected projective planes by the topological dimensions of their automorphism groups (see 4.19, 4.20). This has the advantage that various types of interesting non-Desarguesian planes appear in a systematic fashion, ordered by the 'size' of their automorphism groups.

In Section 5, we return to stable planes and report on classifications under various homogeneity conditions. Naturally, there is some overlap between Sections 4 and 5. Compared to projective planes, there is a much larger variety of stable planes which deserve to be called classical. Stable planes are often useful in the study of topological projective planes, because the concept of stable planes allows us to pass to any open subgeometry (like an open orbit of some group). The notion of a symmetric plane links topological geometry with differential geometry; a symmetric plane is a stable plane carrying a compatible structure of a symmetric space.

Topological generalized polygons and topological buildings form a fairly new direction in topological geometry; some recent results in this area are mentioned in Section 6. We conclude in Section 7 with some remarks on topics related to the general area of topological geometry.

Our friend and teacher H. Reiner Salzmann has accompanied the writing of this survey from the very beginning by his encouragement and active help. We wish to express our gratitude to him for his detailed advice and for sharing his insight through many hours of conversation.

1. Topological projective spaces

We want to establish an axiom system describing the combination of topological and geometrical properties encountered in finite-dimensional projective spaces over topological division rings (skew fields). Later, we mean to generalize our axioms so as to include, e.g., hyperbolic spaces; see Section 2. It will turn out that projective spaces over topological division rings are the only Desarguesian models (see 1.5). Their structure is by now well understood.

The rough idea is to describe the geometry as a lattice and to require continuity of the lattice operations $\vee$ (*join*) and $\wedge$ (*intersection*). However, we have to be careful about the details, because in the models just mentioned, the operations $\vee$ and $\wedge$ are *not* globally continuous; e.g., a sequence of lines l_n not contained in a hyperplane h may converge to a line l contained in h, and then neither $l_n \vee h$ nor $l_n \wedge h$ converge as they should if $\vee$ and $\wedge$ were continuous. In particular, the notion of *topological lattice*, defined as a lattice with continuous operations, is inappropriate for our purposes; this observation is made more precise in Choe and Groh [1989]. In order to obtain a continuous join operation $x \vee y = z$, we have to restrict the general join $\vee$ in such a way that the dimensions of x, y, and z are all fixed.

1.1. DEFINITION. *Topological projective spaces.* Let P be a projective space (not necessarily Desarguesian). Consider P as a lattice, with $\min P = 0$ and $\max P = 1$, and let $\dim x$ denote the projective dimension of $x \in P$, so that points have dimension 0. Assume that $\dim P := \dim 1 = n$, where $2 \leqslant n < \infty$. Decompose P into the sets $P_k := \{x \in P\colon \dim x = k\}$ for $k \leqslant n$ and define

$$^{k+1}(P_k \times P_0) := \{(x,y) \in P_k \times P_0\colon \dim(x \vee y) = k+1\},$$

$$_{k-1}(P_k \times P_{n-1}) := \{(x,y) \in P_k \times P_{n-1}\colon \dim(x \wedge y) = k-1\}.$$

Suppose that the sets P_k, $0 \leqslant k \leqslant n-1$, are endowed with topologies, not the coarsest, such that $\vee$ is continuous on $^{k+1}(P_k \times P_0)$ and $\wedge$ is continuous on $_{k-1}(P_k \times P_{n-1})$. Then P together with these topologies is called a *topological projective n-space.*

Sometimes we think of the given topologies as one topology by considering P as the topological sum, which contains each P_k as an open subspace.

This definition was given by Misfeld [1968]. Other definitions were proposed by Sörensen [1969], Heise and Sörensen [1970], Doignon [1971], Szambien [1986a], Zanella [1990b]. All these notions are equivalent in some sense. A strictly weaker notion of topological projective spaces was introduced previously by Lenz [1965]; compare Misfeld [1968] and Hartmann [1988]. Lenz assumes continuity of central projections. Yet another definition requires the graph of the incidence relation to be a closed subset of $P \times P$. The latter two definitions are strong enough only in the compact case, see, e.g., 4.4 (compare also 6.1 and 6.4). For example, the graph remains closed if the topology of P is replaced by a finer one. In contrast to Misfeld's definition, the weaker ones do not lead to a representation theorem like 1.5 below, except in the compact case.

Topological projective n-spaces are topological buildings of type A_n, see Section 6.

1.2. PROBLEM. *Local approach.* It is an open question whether or not topological projective n-spaces can be characterized by the condition that all rank 2 residues of P (with connected diagrams, i.e. all 2-dimensional intervals of the lattice P) are topological projective planes. An example of Zanella [1990a] shows that it is not enough to require that all hyperplanes are topological projective spaces.

1.3. NOTE. From the minimized continuity requirements in the definition of topological projective spaces, far stronger continuity properties can be deduced. We shall state the known results in the general framework of topological n-spaces, see 2.4. We mention in particular that every element x of a topological projective space P gives rise to two derived topological projective spaces, namely the intervals $[0, x]$ and $[x, 1]$ of the lattice P. Those facts are needed to prove some of the results presented in this section.

1.4. *Projective spaces over topological division rings.* We describe the approach given in Kühne and Löwen [1992] leading to these standard examples of topological projective spaces. It is essentially analytic and, therefore, rather simple. For proofs and details not given here, we refer to that paper. A synthetic approach has been given by Misfeld [1968].

Let F be a *topological division ring*, i.e. a division ring such that, with respect to a given topology on F different from the coarsest topology, the operations of addition, multiplication and inversion are continuous on their natural domains of definition. Examples of topological division rings abound; in fact, every field of infinite cardinality a admits 2^{2^a} pairwise nonisomorphic field topologies, see Kiltinen [1973]. Compare also the remarks before 4.2.

Let $n \geqslant 2$ and consider the $(n+1)$-dimensional (left) vector space $V = F^{n+1}$ over F. Endow V with the product topology, which incidentally makes F^{n+1} a topological vector space. The n-dimensional projective space $P = \mathrm{P}_n F$ consists of the *Grassmannians*

$$P_k = G_{n+1,k+1} = \{x\colon\ x \leqslant V,\ \operatorname{rank} x = k+1\}.$$

Here, as in Chapter 2, $\operatorname{rank} x$ denotes vector space dimension, as opposed to projective dimension:

$$\dim x = \operatorname{rank} x - 1.$$

Using Gaussian elimination, it is easy to see that the set $E_{k+1} \subseteq V^{k+1}$ of all independent $(k+1)$-tuples $(v_1, \ldots, v_{k+1})$ in V is open. We endow P_k with the quotient topology with respect to the surjective map

$$\pi_k\colon\ E_{k+1} \to P_k$$

defined by

$$(v_1, \ldots, v_{k+1}) \mapsto \operatorname{span}(v_1, \ldots, v_{k+1}).$$

In other words, a set $U \subseteq P_k$ is open if and only if the set of all bases of all elements $x \in U$ is open in V^{k+1}. It turns out that π_k becomes an open map and P becomes a topological projective n-space in this way (see 1.5).

The following result is due to Misfeld [1968]; remember that projective spaces of dimension $\geqslant 3$ are Desarguesian and hence isomorphic to some $\mathrm{P}_n F$, see Chapter 2, 2.3.

1.5. REPRESENTATION THEOREM. *Up to isomorphism, the Desarguesian topological projective n-spaces are precisely the projective spaces* $\mathrm{P}_n F$ *over topological division rings as constructed in* 1.4.

Two Desarguesian topological projective spaces $\mathrm{P}_n F_1$ *and* $\mathrm{P}_m F_2$ *are isomorphic if and only if* (i) $n = m$ *and* (ii) *the topological division rings* F_1 *and* F_2 *are isomorphic.*

For a simple proof, see Kühne and Löwen [1992]. The difficult part is the verification that $\mathrm{P}_n F$ is indeed a topological projective space. The proof uses the action of the general linear group GL(V); the topology defined in 1.4 makes P_k a homogeneous space of this group. The other half of the first assertion, as well as the second assertion, follows easily because as in Chapter 2, 2.1, the topological division ring F can be reconstructed on the point row of an arbitrary affine line, endowed with the topology inherited from P_0; conversely, this restriction determines the entire topology of P, cf. 2.6 below. Invariance of dimension follows from Chapter 2, 1.2; see also Chapter 2, 2.4.

By the representation theorem, all 'nonstandard' examples of topological projective spaces are non-Desarguesian topological projective planes ($n = 2$). The interest of the planar case lies in the abundance of non-Desarguesian examples; see Section 4.

Next, we give an alternative description of the topology of $\mathrm{P}_n F$ by introducing *Grassmann coordinates*. If F is locally compact and connected, then F is homeomorphic to $\mathbb{R}^d$ by Pontrjagin's theorem, see 1.10 below; in this case, 1.6 will imply that each layer P_k is a topological manifold (locally homeomorphic to $\mathbb{R}^{du}$, where $u = (k+1)(n-k)$). These *Grassmann manifolds* are standard objects of topology, see Milnor and Stasheff [1974].

1.6. THEOREM. *Let* $y \in P_{n-k-1}$ *be an element of the topological projective n-space* $\mathrm{P}_n F$. *Then the set* $U \subseteq P_k$ *of all complements of* y *is open and homeomorphic to* $F^{(k+1)(n-k)}$.

In particular, the affine space obtained by deleting a hyperplane $y \in P_{n-1}$ *has an open point set homeomorphic to* F^n.

PROOF. The set $U \subseteq P_k$ is open by general results about stable n-spaces, see 2.5. The transitive action of GL(V) allows us to assume that y is spanned by the standard basis vectors $e_{k+2}, \ldots, e_{n+1}$, where $e_i = (0, \ldots, 1, \ldots, 0)$ has 1 in the i-th place. Elements of U have bases of the form $b_i = e_i + a_i,\ i \leqslant k+1,\ a_i \in y$. Clearly, the map f sending the matrix A with rows $a_1, \ldots, a_{k+1}$ to span$(b_1, \ldots, b_{k+1})$ is a continuous bijection of the space $F^{(k+1)\times(n-k)}$ of matrices onto U. Now we have an open quotient map from some open set $W \subseteq E_{k+1}$ onto U, cf. 1.4. Therefore, continuity of f^{-1} can be proved by showing that for sufficiently small open subsets $W_1 \subseteq W$ we can associate to every basis $w \in W_1$ of $x \in U$ a basis of x of the form $b_1, \ldots, b_{k+1}$ in a continuous manner. This is indeed possible via Gaussian elimination. □

By Chapter 2, 3.1, the automorphisms of an abstract projective space $\mathrm{P}_n F$ are precisely the maps induced by all F-semilinear automorphisms of the vector space F^{n+1}. Such a semilinear automorphism is given by φ: $x \mapsto x^\sigma A$, where σ is an automorphism of the division ring F, which is applied to the coordinates of x, and $A \in F^{(n+1)\times(n+1)}$ is a regular matrix.

Now let F be a topological division ring. Clearly, φ is a homeomorphism of F^{n+1} if and only if σ is a homeomorphism of F. In this case, φ induces a homeomorphism of $\mathrm{P}_n F$ by definition of the topology. Conversely, for $A = \mathbf{1}$ (unit matrix), it is easy to see that continuity of the induced map ψ on P_0 is equivalent to continuity of σ. Indeed, ψ fixes some affine subspace, whose point set can be identified with F^n by 1.6, and ψ induces the semilinear map $v \mapsto v^\sigma$ on F^n. In view of the general fact (see 2.6) that the topology of P is determined by that of P_0, this implies

1.7. THEOREM. *The automorphisms of the Desarguesian topological projective n-space $\mathrm{P}_n F$ are precisely the maps induced by all semilinear automorphisms $x \mapsto x^\sigma A$ of F^{n+1} such that σ is an automorphism of the topological division ring F.*

Let us call a topological property *hereditary for topological projective spaces* if for every given Desarguesian topological projective n-space $\mathrm{P}_n F$, $n \geqslant 2$, either all or none of the spaces $F, P_0, P_1, \ldots, P_{n-1}$ possess this property.

1.8. THEOREM. *A topological property is hereditary for topological projective spaces if it is preserved by the formation of finite products and open subsets, and by continuous open surjections. In particular, discreteness, first and second countability, and local compactness are hereditary. Moreover, connectedness and total disconnectedness are also hereditary.*

The proof is given in Kühne and Löwen [1992] and Zanella [1989a].

Note that F admits the doubly transitive group of all affine maps $t \mapsto at + b$; therefore, F is either connected or totally disconnected, and hence the same is true for every P_k. However, P is never connected since all P_k are open.

1.9. THEOREM. *If F is a locally compact topological division ring, then $\mathrm{P}_n F$ is compact (i.e. every P_k is compact) and has a countable basis for its topology.*

For the proof, see Löwen [1989], Zanella [1989a], Kühne and Löwen [1992].

1.10. *The Cayley–Dickson process.* Using induction, we define a sequence of $\mathbb{R}$-algebras $\mathbb{F}_k$, $k \geqslant 0$, endowed with involutory anti-automorphisms $x \mapsto \overline{x}$, called *conjugations.* The induction starts with $\mathbb{F}_0 = \mathbb{R}$ and $\overline{x} = x$ for every $x \in \mathbb{R}$. The inductive step is given by $\mathbb{F}_{k+1} = \mathbb{F}_k \oplus \mathbb{F}_k$ with component-wise addition and with multiplication and conjugation defined by

$$(a, b) \cdot (c, d) = (ac - \overline{d}b, da + b\overline{c}),$$

$$\overline{(a, b)} = (\overline{a}, -b).$$

Then $\mathbb{F}_1 = \mathbb{C}$ is the topological field of complex numbers and $\mathbb{F}_2 = \mathbb{H}$ is the topological division ring of *quaternions*. The topological semifield (division algebra) $\mathbb{F}_3 = \mathbb{O}$ of *octonions* is not associative, but every two elements of $\mathbb{O}$ generate an associative subalgebra, hence $\mathbb{O}$ is said to be *biassociative*. For $k \geqslant 4$, the algebra $\mathbb{F}_k$ has zero divisors and does not even define a projective plane.

We have described this construction because, by a theorem of Pontrjagin [1932], the only locally compact connected topological division rings are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Moreover, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only locally compact connected biassociative semifields; see, e.g., Ebbinghaus et al. [1990] and note that such a semifield is an algebra over the closed subfield $\mathbb{R}$ generated by 1. For more information, see again Ebbinghaus et al. [1990] and Salzmann et al. [1994], Chapter 1.

Combining Pontrjagin's theorem with the representation theorem 1.5 and the permanence results 1.8 and 1.9, we obtain the following theorem of Kolmogoroff [1932], which is one of the earliest results of topological geometry; compare also Misfeld [1969].

1.11. THEOREM. *The only Desarguesian topological projective n-spaces, $n \geqslant 2$, with a (locally) compact connected point space P_0 are the real, complex, and quaternion projective spaces* $\mathrm{P}_n\mathbb{R}$, $\mathrm{P}_n\mathbb{C}$, $\mathrm{P}_n\mathbb{H}$.

The locally compact (totally) disconnected topological division rings can be described as central extensions of the p-adic number fields $\mathbb{Q}_p$ or of the fields of Laurent series over finite prime fields, see, e.g., Grundhöfer and Salzmann [1990], XI.7.7. Only the finite commutative (totally ramified) extensions of $\mathbb{Q}_p$ have not been completely classified. With this restriction, also the compact disconnected Desarguesian projective n-spaces are known.

1.12. COROLLARY. *The lines of a Desarguesian topological projective n-space with a locally compact connected point space are homeomorphic to the sphere $\mathbb{S}_l$ of dimension $l = 1, 2,$ or 4.*

For the proof, one only has to remark that a line is compact by 1.9 and can be written as $L = A \cup \{\infty\}$, where A is an affine line; A can be identified with the coordinate division ring $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ (see Chapter 2, 2.1), which is homeomorphic to $\mathbb{R}^l$. Therefore, L is the one-point compactification $\mathbb{S}_l$ of $\mathbb{R}^l$.

1.13. *Projective spaces of infinite dimension.* If one wants to associate a 'topological projective space' to an infinite-dimensional topological vector space, then the first obstacle is that one has to modify the notion of the projective space belonging to a vector space. The lattice of all vector subspaces is not suited for this purpose because most subspaces fail to be closed and are topologically ill-behaved. A good starting point is to consider a dual pair of topological vector spaces (V, W) over F with a continuous canonical pairing $V \times W \to F$, and to consider the lattice generated by all one-dimensional subspaces of V and all kernels of the continuous linear forms in W. This was done by Pfrommer [1973]. He proved that in this situation, a topological projective space in a sense similar to Sörensen [1969] or Doignon [1971] is obtained, provided that each of the spaces V,

W separates the elements of the other one. He characterizes the spaces obtained from reflexive Banach spaces over connected fields and those obtained from locally convex vector spaces over nondiscrete ordered fields.

A similar problem concerns the characterization of the lattice of all closed subspaces of some Hilbert space. This is known in Mathematical Physics under the caption 'Logic of Quantum Mechanics'. See Kalmbach [1983, 1986] or Pták and Pulmannová [1991] for an introduction, and the bibliography Pavicić [1992]. A contribution to this problem from the geometric point of view is made by Szambien [1986b].

2. Stable spaces

In order to include open subgeometries of topological projective spaces, the geometric axioms considered in the last section will be relaxed. In exchange, the topological axioms have to be sharpened, thus making up for some of the lost strength of the geometry. For dimension $n \geqslant 3$, the resulting axiom system characterizes open subgeometries of topological projective spaces, see 2.9. The subject is well understood, but there might be room for further study in connection with homogeneous stable spaces, see 2.11 ff.

2.1. *Geometric lattices.* In what follows, let $(G, \vee, \wedge)$ (or G for short) be an *n-dimensional geometric lattice*, i.e. a lattice with $\min G = 0$ and $\max G = 1$ such that

(i) there is a maximal chain $0 < x_1 < \cdots < x_n < 1$ of length $n+2$,
(ii) G is *atomistic*, i.e. every element is a join of upper neighbours of 0,
(iii) G is *semimodular*, i.e. if $x, y \in G$ have a common lower neighbour, then they also have a common upper neighbour.

Semimodularity is often defined in a different way, because the finite length condition (i) is usually not required. Chapter 6 or Grätzer [1978], Stern [1991] can serve as a general reference.

For $0 < x \in G$, there is an integer $k =: \dim x \in \{0, \ldots, n\}$ such that the interval $[0, x]$ is a k-dimensional geometric lattice. Clearly, $\dim 1 = n$. If $x \neq 1$, then $[x, 1]$ is an $(n-k-1)$-dimensional geometric lattice. We complete the dimension function by setting $\dim 0 = -1$. Then we have the *dimension inequality*

$$\dim(x \vee y) + \dim(x \wedge y) \leqslant \dim x + \dim y.$$

We decompose G as the disjoint union of the sets

$$G_k := \{x \in G\colon\ \dim x = k\}, \quad -1 \leqslant k \leqslant n.$$

The elements of G_k will be called *k-flats*. For k-flats with $k = 0, 1, n-1$ we also use the names *points, lines, hyperplanes*, respectively. We define

$${}^k(G_i \times G_j) := \{(x, y) \in G_i \times G_j\colon\ \dim(x \vee y) = k\},$$

$${}_k(G_i \times G_j) := \{(x, y) \in G_i \times G_j\colon\ \dim(x \wedge y) = k\},$$

and we denote by ${}^k\vee$ the restriction of $\vee$ to the set of all pairs (x, y) with $\dim(x \vee y) = k$. Dually, we define ${}_k\wedge$.

An n-dimensional geometric lattice is just a dimensional linear space (an n-DLS) in the sense of Chapter 6, 1.2. However, we prefer to use the language of lattices because it simplifies the formulation of continuity axioms, see 2.2.

After a first treatment of 3-dimensional stable spaces by Betten [1981, 1983b, 1985, 1987], the following notion was coined by Groh [1986a]; he used the name 'topological n-space'.

2.2. DEFINITION. *Stable n-spaces.* A stable n-space consists of an n-dimensional geometric lattice $(G, \vee, \wedge)$ together with a topology on G such that properties (S i) to (S iv) below hold.

(S i) $\dim\colon G \to \{-1, \ldots, n\}$ is continuous, i.e. G carries the sum topology determined by the open sets G_k.

For a pair $(x, y) \in G_j \times G_{n-1}$, the dimension of $x \wedge y$ may be less than $j - 1$. We require

(S ii) *Stability*: For all $j \geqslant 1$, the set ${}_{j-1}(G_j \times G_{n-1})$ is open in $G_j \times G_{n-1}$,

i.e. each pair (x, y) consisting of a j-flat x and a hyperplane y with $(j-1)$-dimensional intersection has a neighbourhood consisting of pairs with the same property. The analogous statement for $j = 0$ can be proved, see 2.4(d) below; it asserts that the incidence relation between points and hyperplanes is closed in $G_0 \times G_{n-1}$.

(S iii) *Continuity of intersection*: For all $j \geqslant 1$, the operation ${}_{j-1}\wedge$ is continuous on ${}_{j-1}(G_j \times G_{n-1})$.

(S iv) *Continuity of join*: For all $j \geqslant 0$, the operation ${}^{j+1}\vee$ is continuous on the set ${}^{j+1}(G_j \times G_0)$, which consists of the nonincident pairs in $G_j \times G_0$.

Stronger stability and continuity properties can be deduced from these axioms, see 2.4 below. A stable n-space is said to be *rich* if $n \geqslant 2$ and G is nondiscrete, and every line contains at least 3 points. This merely excludes trivial examples.

Note that truncation of a stable n-space in the sense of Chapter 6, 1.4 (i.e. omission of the levels $G_k, \ldots, G_{n-1}$) does not produce a stable k-space because the stability axiom is violated. For example, the points and lines of a stable 3-space do not form a stable 2-space. So even at this point it is clear that stable spaces are dimensional linear spaces of a rather special kind.

2.3. *Special cases.* The concept of stable n-space unifies several notions that have been studied separately before. Let $n \geqslant 2$.

(a) A *topological projective n-space* in the sense of Section 1 is the same as a stable n-space P that happens to be a projective space. Indeed, Groh [1986a] shows that topological projective spaces satisfy the stability axiom. On the other hand, the topology of a stable n-space is nontrivial (as required for topological projective spaces) by 2.4(a) below.

(b) A *topological affine n-space* can be defined as a stable n-space A which is an affine space. From every topological projective n-space P, we can obtain a topological affine n-space by deleting an arbitrary hyperplane x together with the interval $(0,x]$ (which is a closed subset of P, cf. 2.4(e) below), and by taking the induced topology on the remainder of P. Compare also 2.8 for open subgeometries in general. Conversely, let A be a rich topological affine n-space with $n \geqslant 3$. Then it follows from the general embedding theorem of Groh, see 2.9 below, that A can be obtained by the above construction from some topological projective n-space. This is not true for $n = 2$, see Eisele [1990, 1991a,b, 1993a,b]. For $n \geqslant 3$, our remarks imply that our definition is equivalent with various other definitions of topological affine spaces given by Fick [1977], Sörensen [1970, 1977], Zanella [1989b].

(c) A *stable plane* is a rich stable 2-space $S = S_0 \cup S_1 \cup \{0,1\}$. The geometric axioms just require that any two distinct points $x, y \in S_0$ are joined by a unique line $x \vee y \in S_1$. The stability axiom says that the set $D = {}_0(S_1 \times S_1)$ of pairs of distinct intersecting lines is open in $S_1 \times S_1$. Moreover, ${}_0\wedge\colon D \to S_0$ and ${}^1\vee$ are continuous maps. The usual definition of a stable plane, see Löwen [1976b], also requires that S is a Hausdorff space, but this can be proved; see 2.4(a) below. Stable planes were introduced by Skornjakov [1957] and Salzmann [1967a,b]. Stable planes and, in particular, topological affine and projective planes, will occupy us in Sections 3 to 5.

(d) A *topological affine* or *projective plane* is a stable 2-space S which is affine or projective, respectively. If S is nondiscrete, then S is a stable plane. In the projective case, the stability axiom merely requires that the set of pairs of distinct lines is open in $S_1 \times S_1$, i.e. that S_1 is a Hausdorff space. This can be proved if the topologies of S_0 and S_1 are not the coarsest, see 2.4(a) below.

In the affine case, stability means that the set of parallel line pairs is closed in $S_1 \times S_1$. Continuity of the operation of drawing parallels is a strictly stronger condition, as was shown by Eisele [1991b]. Some authors require this continuity property in their definitions of topological affine planes, see the articles mentioned in (b) and Grundhöfer [1987a]. By Eisele's result, the projective completion of a topological affine plane in our sense need not be a topological projective plane. There are counterexamples with continuous parallelism (Eisele [1991a]) or of Lenz type V (Eisele [1992b]). A topological projective completion does exist, however, if S_0 is locally compact and connected; see 3.13.

2.4. THEOREM. *The following assertions are true in every rich stable n-space G.*

(a) *G is a Hausdorff space* (Groh [1986a], 4.13).

(b) *For every $k \geqslant 1$, the map ${}^k\vee$ is continuous and open* (Groh [1986a], 4.14 *and* 4.16).

(c) $\dim(x \vee y)$ *is a lower semicontinuous function of* $(x,y) \in G \times G$ (Groh [1986a], 4.12). *In other words, the set* $\{(x,y)\colon \dim(x \vee y) \geqslant k\} \subseteq G \times G$ *is open for every* k.

(d) *The incidence graph* $I := \{(x,y)\colon x \leqslant y\} \subseteq G \times G$ *is closed.*

(e) *The intervals* $[0,x]$ *and* $[x,1]$ *are closed in G for every* $x \in G$.

(f) *For* $0 < k < n$ *and* $x \in G_k$, *the interval* $[0,x]$ *is a stable k-space, and* $[x,1]$ *is a stable* $(n-k-1)$*-space* (Groh [1986a], 4.20).

PROOF. (d) follows from (c) because $G_i \times G_j \setminus I = \{(x,y)\colon \dim(x \vee y) > \max(i,j)\}$. Assertion (e) follows from (d). □

The definition of stable n-spaces is not self-dual. However, the dual of a topological projective n-space is a topological projective n-space with the same topology. Therefore, the duals of statements (b) and (c) are true in this case: ${}_k\wedge$ is open and continuous, and $\dim(x \wedge y)$ is upper semicontinuous. By the general embedding theorem 2.9, the dual of assertion (b) carries over to all rich stable n-spaces with $n \geqslant 3$. It is easy to prove that the map ${}_0\wedge$ is open in stable planes. The dual of (c) is false in general.

2.5. NOTE. Call a pair $(x,y) \in G \times G$ *skew* if $x \wedge y = 0$, and *complementary* if moreover $x \vee y = 1$. It follows from 2.4(c) that the set of skew pairs and the set of complementary pairs are open in $G \times G$; for example, (x,y) is complementary if and only if

$$\dim(x \vee y) = n = \dim x + \dim y + 1.$$

Groh [1986a], 4.18, proves the following result, using continuity and openness of ${}^k\vee$ and a weak form of the dual assertions.

2.6. THEOREM. *If the underlying lattice is given, then the topology of a stable n-space G is completely determined by its restriction to any layer G_k, $0 \leqslant k \leqslant n-1$.*

This implies that a lattice automorphism φ of $(G, \vee, \wedge)$ is a homeomorphism of G if and only if it induces a homeomorphism of the given layer G_k. In this case, we call φ an automorphism *of the stable n-space G.*

Next, we aim at Groh's general embedding theorem [1986b], which characterizes rich stable n-spaces, $n \geqslant 3$, as open subspaces of nondiscrete topological projective n-spaces. First we introduce open subspaces of stable n-spaces in general.

2.7. *Open subspaces.* Let G be a rich stable n-space, and let $U \subseteq G_0$ be an open set of points. Define $G(U) \subseteq G$ as the set of all $x \in G$ containing some point of U. Define lattice operations in $G(U)$ to be the same as in G, except for elements $x, y \in G(U)$ such that $x \wedge y \notin G(U)$. In this case, put $x \wedge y := 0$ in $G(U)$.

Clearly, $G(U)$ is open in G. Groh [1986a], 5.6a, proves that each $x \in G(U)$ is spanned by the points of x contained in U. This improves his result 4.20c:

2.8. THEOREM. *If U is a nonvoid open set of points of a rich stable n-space, then $G(U)$ is a rich stable n-space; it will be called an open subspace of G.*

Groh [1986b] shows that a rich stable n-space with $n \geqslant 3$ satisfies the Desargues condition and is weakly modular; in fact, for each point x, the interval $[x, 1]$ is a Desarguesian topological projective $(n-1)$-space. He uses this to prove the following conclusive theorem.

2.9. EMBEDDING THEOREM. *Up to isomorphism, the rich stable n-spaces, $n \geqslant 3$, are precisely all open subspaces (in the sense of 2.7) of topological projective n-spaces $\mathrm{P}_n F$ over nondiscrete topological division rings F. Every automorphism of an open subspace extends to a unique automorphism of $\mathrm{P}_n F$.*

PROOF. The second assertion is obtained as follows: Groh proves that the inclusion of an open subspace of $\mathrm{P}_n F$ is a strong embedding in the sense of Kantor [1974], and our assertion follows by Kantor's uniqueness lemma (*loc. cit.*). □

Before Groh's paper, special cases of 2.9 had been obtained. Using ordering assumptions (which usually enforce that $F = \mathbb{R}$), such results were given by Klein [1873], Pasch [1882], Schur [1891], Moufang [1931]. Other previously treated cases are sliced spaces (Sörensen [1970]) and real 3-spaces with lines homeomorphic to $\mathbb{R}$ (Betten and Horstmann [1983]). The extensibility of automorphisms (strong embedding) was proved by Lenz [1958] for $F = \mathbb{R}$, by Löwen [1982b] for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (including the planar case), and by Frank [1992] in general. Lenz and Frank even consider noninjective self-maps.

A similar theorem does not hold for stable planes in general. See 3.19 and Stroppel [1993a], Section 3, for examples of stable planes that cannot be embedded as open subplanes in any topological projective plane (Desarguesian or not), because their disjoint line pairs behave too wildly. Other nonembeddable examples are provided by certain planes associated with partial spreads (see 3.14), and by the modified real cylinder plane, see 5.3 and Löwen [1981c]. It is not even clear whether or not the Desargues condition implies embeddability; see 5.18 for a partial result and for further comments.

The division ring F is connected or totally disconnected according as whether the point set of $\mathrm{P}_n F$ contains a nontrivial connected subset or not, see Theorem 1.8 and the comment following it. Combining the embedding theorem 2.9 with the classification of locally compact fields (1.11 and the subsequent remark), we thus obtain the following.

2.10. COROLLARY. *For $n \geqslant 3$, a rich stable n-space G with a locally compact point set G_0 is isomorphic to an open subspace of $\mathrm{P}_n \mathbb{F}$, where $\mathbb{F}$ is a nondiscrete locally compact division ring* (cf. 1.11). *If, in addition, G_0 contains a nontrivial connected set, then $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.*

Especially nice examples of open subspaces of $\mathrm{P}_n \mathbb{F}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, are obtained using the complements of orthogonal or Hermitian quadrics or connected components of such complements; we call them *Hermitian n-spaces*. In particular, *hyperbolic spaces* are of this type; their point sets are open balls in Euclidean spaces. See also 5.2. By a theorem of Witt, connected Hermitian spaces are homogeneous in the sense of the following definition.

2.11. *Homogeneity.* We call a stable space *homogeneous* (or *point homogeneous*, to be precise) if the automorphism group is transitive on the point set. According to 2.9, homogeneous rich stable n-spaces for $n \geqslant 3$ correspond to open orbits of subgroups of the projective automorphism group $\mathrm{P\Gamma L}_{n+1}(F)$ in the point set of $\mathrm{P}_n F$, where F is a nondiscrete topological division ring. The situation is more complicated for $n = 2$, see Section 5.

The following is an interesting special case. A *sliced stable space* is an open subspace S of a topological projective space obtained by deleting the points of some k-flat. If S carries a sharply point-transitive group Φ of automorphisms, then S is called a *topological sliced incidence group*. An algebraic characterization has been obtained, in this topological context, by Sörensen [1971a]. A characterization of the topological incidence group $\mathrm{P}_3\mathbb{R}$ with $\Phi = \mathrm{SO}_3\mathbb{R}$ is given by Misfeld [1971b].

Betten [1985] determined the open subspaces of $\mathrm{P}_3\mathbb{R}$ having lines homeomorphic to $\mathbb{R}$ and admitting a sufficiently large group. Klein [1993] investigates and characterizes stable n-spaces, $n \geqslant 3$, that are symmetric spaces in the sense of differential geometry. This generalizes the notion of symmetric planes considered in 5.27 to spatial geometries.

2.12. *Homogeneous open cones.* 'Homogeneous convex' open cones in real and complex affine spaces have been classified, see Vinberg [1963], Vinberg, Gindikin and Piateckij-Shapiro [1963], Koszul [1962], Vinberg and Kats [1967], and further references given in these papers. These cones always admit a sharply transitive action of a group of triangular matrices. Via the Lie algebra of this group, an algebra structure can be defined on the cone, and is then used to classify.

However, this is based on a notion of homogeneity different from ours; only automorphisms induced by affine maps are allowed, so that even hyperbolic spaces are not considered as homogeneous. Nevertheless, we can apply this theory in our context, if we consider a cone in $\mathbb{R}^{n+1}$ as a set of points in $\mathrm{P}_n\mathbb{R}$ by passing from vectors to one-dimensional subspaces. This yields a classification of homogeneous (in our sense) convex open subspaces of real affine spaces.

2.13. *The Helmholtz–Lie space problem.* This is a programme aiming at a related class of homogeneous geometries. Freudenthal [1965] gives an excellent survey. The ideas of Helmholtz led to a characterization of Euclidean and hyperbolic spaces (and their motion groups) as homogeneous spaces of Lie groups such that every point stabilizer is sharply transitive on the set of maximal flags of the tangent space. The first step of the proof is to show that a linear action with the latter property is an orthogonal action with respect to some scalar product (Baer [1950]; see also Hausknecht and Löwen [1985] and references given there).

Tits and Freudenthal generalized this approach and classified transitive transformation groups having some weak topological 'stiffness' property and satisfying the condition that every point stabilizer has an orbit which dissects the point space. As in the original Helmholtz–Lie problem, the existence of a system of lines invariant under the given group is not presupposed. For the special case of 2-manifolds, these ideas first appeared in Hilbert [1903]; compare Anhang IV in Hilbert [1930]. This theory was further developed by Kolmogoroff [1930], among others, and culminates in the work of Freudenthal and Tits. See, again, Freudenthal [1965].

2.14. *Ordered space geometries.* A different approach to convex open subgeometries of real affine spaces has been given by Cantwell [1974, 1978] and Cantwell and Kay [1978].

Their theory develops the ideas of Pasch [1882]. The axioms require that there is a unique line joining any two given points, carrying an order relation isomorphic to the real line, and that the following version of Pasch's axiom (see Chapter 2, 2.3) holds: given three points a, b, c and $x \in (a, c)$ (open interval) and $y \in (x, b)$, there exists $z \in (a, b)$ so that $y \in (c, z)$. On this axiomatic basis, Cantwell develops a theory of convexity including Helly and Radon type theorems, and constructs a topology homeomorphic to $\mathbb{R}^n$ on the point set. Cantwell and Kay finally prove an embedding theorem very much like 2.9. Related work was done by Doignon [1976] and Lenz [1992]. For the planar case, compare 3.31 and 7.4.

Another theory of ordered space geometries is developed in Haupt and Künneth [1967], Section 7. Their axiom system presupposes a compact metric space, and describes separation properties of the set of hyperplanes in $\mathrm{P}_n\mathbb{R}$ and of its universal covering, the system of maximal great spheres in the sphere $\mathbb{S}_n$. They do not intend to give an axiomatic characterization of these models.

3. Geometry and topology of stable planes

Recall first that nondiscrete topological projective planes are special stable planes (2.3; see also 3.11 below). Without the assumption of local compactness, little can be said about stable planes; so this assumption will prevail in the remainder of this chapter. Most of the time, we shall also assume that the point sets of the planes under consideration are connected, or at least not totally disconnected. The latter condition will turn out to be equivalent to the requirement that the topological dimension is different from zero.

Although there are many stable planes that cannot be embedded as open subplanes in any topological projective plane (cf. 2.9 and the subsequent remarks; see also 3.14 and 3.19), all existing results indicate that there is a strong similarity between the topology of an arbitrary locally compact connected stable plane and the topology of an open subplane of one of the four *classical projective planes* $\mathrm{P}_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. This phenomenon will be referred to as *domination by classical planes*; it has a decisive influence on the appearance of the whole theory.

In addition to this type of topological results, the present section will deal with geometric questions such as duality and projective embeddings, and with the structure of the automorphism group. Most notably, the latter is a locally compact transformation group of the spaces of points and lines and has a topology with a countable basis. The theory of stable planes is well developed, but there are still major open problems, see 3.22, 3.24, 3.38, 5.11, 5.28. Many of them are open even in the projective case.

To a large extent, the results about stable planes presented in this section originate from similar theorems about projective planes, mostly due to Salzmann. See the survey articles Salzmann [1967b, 1981b]; compare also Section 4.

3.1. *Notation and examples.* Instead of using the lattice-theoretic notation of the previous sections, it will be more convenient to write a stable plane as a pair $S = (P, \mathcal{L})$ consisting of two topological spaces P (point space) and $\mathcal{L}$ (line space). We shall identify lines with their point rows, and we write $\mathcal{L}_x = \{L \in \mathcal{L}\colon\ x \in L\}$ for the *pencil* of all lines containing $x \in P$. Sometimes we write pq instead of $p \vee q$ for the line joining two points.

The axioms are given in 2.3(c). Note that P is assumed to be nondiscrete. Remember that lines are closed in the point set and pencils are closed in the line set (see 2.4(e)). When we say that a stable plane S has a particular topological property, we shall always mean that the point space has that property. By a compact projective plane, we mean a topological projective plane with a compact point set, etc. A stable plane which is neither a projective plane nor an affine plane will be called a *proper stable plane*.

Examples of stable planes are provided by all open subplanes of topological projective planes (see 2.7). For other construction methods and concrete examples, see 3.14, 3.19, 5.2, 5.3, 5.6, 5.10, 5.30, 5.33, and Groh [1981]. The verification of the axioms is sometimes facilitated by results of Skornjakov [1957] (compare Salzmann [1967b], Section 2, and Salzmann et al. [1994], Section 31) and Löwen [1986c].

3.2. *Projectivities.* In a topological projective plane, every line L is homeomorphic to every line pencil $\mathcal{L}_x$, $x \notin L$, via the map $\pi\colon\ y \mapsto y \vee x$. The maps π and π^{-1} are called *perspectivities.* By composition of perspectivities we obtain *projectivities*, and we see that any two lines or pencils are homeomorphic. The group of all projectivities of a line L onto itself is a triply transitive group of homeomorphisms of L. For a systematic study of the interrelations between properties of this group and geometric properties of the plane, see the survey Löwen [1981e]; see also 4.23, 4.24.

In a stable plane, however, a projectivity from a line L_1 to a line L_2 may not be defined on all of L_1, but only on some open subset, which is mapped homeomorphically onto an open subset of L_2. Via perspectivities, a line is homeomorphic to an open subset of any line pencil not containing it. Two lines need not be homeomorphic. It is not even known whether or not the corresponding statement for pencils is true. There is no group of projectivities of a line onto itself, and triple transitivity cannot be proved. Yet, locally, some weak form of 2-transitivity remains true, see Löwen [1976b], 1.10. This is sufficient for applications such as 3.6.

3.3. *Product structure.* In a projective plane $(P, \mathcal{L})$, consider two points a_1, a_2 and a point $p \notin L = a_1 \vee a_2$. The affine plane $A = P \setminus L$ is homeomorphic to the Cartesian product of the two affine lines $p \vee a_i \setminus \{a_i\}$, $i \in \{1, 2\}$; a homeomorphism is given by $x \mapsto (xa_2 \wedge pa_1, xa_1 \wedge pa_2)$.

In a stable plane, however, this homeomorphism is not in general defined on all of A. All that we get by the above considerations is that the point set is *locally* homeomorphic to the product of two lines. The same is true for the line set. Together with local homogeneity of the lines, granted by the projectivities, this implies that any two elements of $P \cup \mathcal{L}$ have homeomorphic open neighbourhoods.

3.4. *Ternary fields.* For any topological projective plane and any frame of reference, the corresponding ternary field K (see references in Chapter 19, 8.8) is a topological ternary field with respect to the topology obtained by identifying K with a punctured line, see 4.5. The plane is completely determined by the ternary field. In a stable plane, a ternary field exists only 'locally'; the precise meaning of this statement and its proof are given in Löwen [1976b], 1.21. The local ternary field does not determine the entire plane, but it is useful because some arguments based on coordinates in projective planes can be generalized. In particular, the proof of the dimension theorem 3.29 for stable planes relies on the existence of this local ternary field.

The (local) ternary field and the transitivity properties of projectivities (3.4 and 3.2) make it possible to imitate standard arguments from the theory of topological fields in order to investigate the topological properties of the ternary field and, hence, of the lines. Via projectivities and the product structure (see 3.3), the results usually carry over to line pencils and to the point space and the line space. A theorem of Menger, Moore and Mazurkiewicz can be used to show that compact, locally connected pencils are arcwise connected. This line of development finally led to the following two theorems, taken from Löwen [1976b], 1.9, 1.11, 1.12 and 1.14.

3.5. THEOREM. *In a locally compact (nondiscrete) stable plane $S = (P, \mathcal{L})$, the point space and the line space are metrizable, and their topologies have countable bases.*

3.6. THEOREM. *In every locally compact stable plane $S = (P, \mathcal{L})$, one of the following assertions is true.*

(i) *Every pencil $\mathcal{L}_x$ is totally disconnected, and $l = \dim \mathcal{L}_x = 0$, or*

(ii) *every pencil is connected and even locally and globally arcwise connected, and has positive dimension $l > 0$.*

Here, $\dim X$ denotes the topological dimension of a space X. For a discussion of this notion, see 3.21 below. In case (i), also every line L and the point and line spaces are totally disconnected and 0-dimensional; cf. Löwen [1976b], 1.11. Incidentally, this implies that P, L, $\mathcal{L}$ and $\mathcal{L}_x$ are homeomorphic to Cantor's triadic set or to the Cantor set minus a point. In case (ii), the line space $\mathcal{L}$ is connected, since any two pencils have an element in common. Moreover, P, L and $\mathcal{L}$ are locally arcwise connected by 3.2 and 3.3. Easy examples (open subplanes of projective planes) show that P may be disconnected. The dimension of P and $\mathcal{L}$ is $2l$, see 3.25 below. In what follows, we shall usually assume that S is a locally compact, locally connected stable plane; in other words, we consider case (ii).

Consider the ternary field K coordinatizing a locally compact connected projective plane, and let e_t, $0 \leqslant t \leqslant 1$, be a path in K joining the elements $0, 1 \in K$. Then the maps $x \mapsto e_t x$, $0 \leqslant t \leqslant 1$, form a *contraction* of K, i.e. a homotopy between the identity map of K and a constant map. Freudenthal [1957a] proved this fact and recognized its significance. The same proof also shows that K is locally contractible, a property which is equally important for the proofs of 3.25 and 3.28. In stable planes, only local contractibility can be proved. We obtain the following result, see Löwen [1976b], 1.12, and Löwen [1983b].

3.7. THEOREM. *In a locally compact, locally connected stable plane $S = (P, \mathcal{L})$, the following statements hold.*

(a) *The spaces $P, \mathcal{L}, L \in \mathcal{L}$ and $\mathcal{L}_x$ are locally contractible.*

(b) *For $x \in L$, every compact subset of $\mathcal{K} := \mathcal{L}_x \setminus \{L\}$ can be contracted in $\mathcal{K}$; in particular, all homology and homotopy groups of $\mathcal{K}$ are zero.*

(c) *If S is projective, then the affine point set $P \setminus L$ and the affine lines are contractible.*

In a locally compact, locally connected stable plane, every line containing the point x meets the boundary of a suitably chosen compact neighbourhood of x; in this way, Löwen [1976b], 1.17, proves the following result.

3.8. THEOREM. *Let x be a point of a locally compact, locally connected stable plane $S = (P, \mathcal{L})$. Then we have:*

(a) *The pencil $\mathcal{L}_x$ is compact.*

(b) *More generally, a subset $\mathcal{A} \subseteq \mathcal{L}$ has compact closure in $\mathcal{L}$ if and only if there is some compact set $A \subseteq P$ meeting every line $L \in \mathcal{A}$.*

3.9. DEFINITION. A line L in a stable plane $S = (P, \mathcal{L})$ is said to be *projective* if L meets every other line.

If S is an open subplane of some projective plane R, then a projective line of S coincides with some line of R.

The open image set of a perspectivity $L \to \mathcal{L}_x$ consists of all lines in $\mathcal{L}_x$ meeting L. Since $\mathcal{L}_x$ is compact and connected (3.8 and 3.6), we obtain the following results 3.10–12, cf. Löwen [1976b], 1.15 ff. They exhibit special properties that distinguish locally compact, locally connected stable planes among general 2-dimensional linear spaces.

3.10. THEOREM. *For every line L of a locally compact, locally connected stable plane, the following assertions are equivalent.*

(i) *The line L is projective.*

(ii) *The line L is compact.*

(iii) *There is a point $x \notin L$ such that L intersects all lines $K \in \mathcal{L}_x$.*

3.11. THEOREM. *For every locally compact, locally connected stable plane $S = (P, \mathcal{L})$, the following assertions are equivalent.*

(i) *The plane S is projective.*

(ii) *The point set P is compact.*

(iii) *There is a pencil $\mathcal{L}_x$ consisting of projective lines.*

3.12. THEOREM. *In every locally compact, locally connected stable plane $(P, \mathcal{L})$, the set $\mathcal{P} \subseteq \mathcal{L}$ of all compact (hence projective) lines is open. Clearly, $\mathcal{P}$ may be empty.*

The proofs of the following results about projective completions are unexpectedly involved; this becomes explicable in view of the nonembeddable examples mentioned in 2.3(d) and 4.5.

3.13. THEOREM.

(a) *The projective completion of a locally compact, locally connected affine plane is a topological projective plane in a natural way.*

(b) *Every locally compact connected topological ternary field coordinatizes a compact connected projective plane* (cf. 4.6b).

Part (a) is proved in Löwen [1981a], 2.2, in a more general form. In fact, the topological projective completion exists for all almost projective locally compact, locally connected stable planes as defined in 3.16 below.

3.14. *Examples: shear planes.* We describe a class of planes constructed by Löwe [1994], containing very homogeneous planes. Some of them, nevertheless, admit no projective embedding. Let $\mathcal{P}$ be a *partial spread* in $\mathbb{R}^{2l}$, i.e. a set of mutually complementary l-dimensional subspaces. Assume that $\mathcal{P}$ is an l-manifold with the topology induced by the topological projective space $\mathrm{P}_{2l}\mathbb{R}$. Form a partial plane with point set $\mathbb{R}^{2l}$ by taking as lines all translates of all elements of $\mathcal{P}$. Add a point at infinity to every line, and then dualize. This gives a stable plane $S(\mathcal{P})$. If $\mathcal{P}$ can be embedded in a spread $\mathcal{S}$, then $S(\mathcal{P})$ is an open subplane of the dual translation plane defined by $\mathcal{S}$, see 4.10. The cylinder planes (see 5.2) are of this type. If the topological closure of $\mathcal{P}$ is not a partial spread, then $S(\mathcal{P})$ cannot be embedded as an open subplane into any topological projective plane.

The group $\mathbb{R}^{2l}$ acts on the shear planes like a dual translation group; in particular, it fixes all elements of some l-dimensional set of lines and acts transitively on the remaining lines. This property characterizes shear planes, see Löwe [1994].

3.15. *Duality.* The *dual* of a topological projective plane $S = (P, \mathcal{L})$ is the plane $S^* = (\mathcal{L}, P)$ with the same topologies and with the roles of $\vee$ and $\wedge$ exchanged. The obvious fact that S^* is a topological projective plane implies that every general result about topological projective planes remains true after exchanging points and lines.

This is not so for stable planes in general. In fact, a stable plane is projective if and only if its dual in the above sense is a stable plane. Instead of the dual plane, we can form the *opposite plane* S^* of a stable plane $S = (P, \mathcal{L})$, provided that the set $\mathcal{P}$ of projective (hence compact) lines is nonempty. Indeed, if we define $S^* = (\mathcal{P}, P')$, where P' is the set of all points lying on at least one projective line, then any two 'points' in $\mathcal{P}$ can be joined by a 'line' in P', and the stability axiom holds because the set $\mathcal{P}$ is open in $\mathcal{L}$ (3.12). We had to replace P by P' because we want every line to be determined by its points. The characterization 3.11 of projective planes implies that a 'bi-opposite' plane S^{**} exists only if S is projective.

3.16. *Affine and coaffine elements.* Two lines are said to be *parallel* if they are disjoint or equal. The *parallel axiom* requires that for any point p and line L there is exactly one line parallel to L and containing p. If the parallel axiom holds in a stable plane S, then S is an affine plane (see 2.3(d)). If the parallel axiom holds for a fixed point p and an arbitrary line L, then p is called an *affine point*. An *affine line* is defined similarly.

A projective line $K \in \mathcal{P}$ or a point $p \in P'$ lying on some projective line are said to be *coaffine* if they become affine elements when considered as elements of the opposite plane S^*. Group-theoretic conditions securing the existence of affine or coaffine elements will be given below, see 3.41 and 3.42. Also, if reflections exist and behave badly with respect to uniqueness or convergence, then there are coaffine elements, see Löwen [1981a], Section 3.

A stable plane S is called an *almost projective plane* if S itself or the complement of some line is an affine plane. The plane S is affine if, and only if, all its points are affine. By analogy, we define a *coaffine plane* as a plane S in which all points are coaffine. This just means that the opposite plane S^* is an affine plane. It is not difficult to see that S then is a topological projective plane with one point deleted, also called a *punctured projective plane*. The following result is taken from Löwen [1981a], 1.6, 1.9. The first half of statement (a) is a direct consequence of the definition, but the second half depends on 3.10 and, thus, on the topology.

3.17. THEOREM. *In every locally compact, locally connected stable plane $S = (P, \mathcal{L})$, the following statements hold.*

(a) *A point x is coaffine if, and only if, x lies on exactly one nonprojective line A. Then all points of A are coaffine.*

(b) *The coaffine points form a closed set $C \subseteq P$, and the lines consisting of coaffine points form a closed set $\mathcal{C} \subseteq \mathcal{L}$.*

3.18. COROLLARY. *If S contains a coaffine line, then S is a coaffine plane.*

PROOF. By assumption, $P \setminus L \subseteq C$, and 3.17(b) implies that $C = P$. □

3.19. *Examples: Planes with (co-)affine elements.* Several planes showing pathological behaviour of the sets of affine or coaffine elements are exhibited in Löwen [1981a], Section 5. The open subplanes of $\mathrm{P}_2\mathbb{R}$ or $\mathrm{P}_2\mathbb{C}$ obtained by removing some closed set of points lying on a conic show that a plane may have some affine lines without being affine. Nonaffine planes containing affine points exist, but they are more difficult to describe (*loc. cit.*).

Suppose we want to construct an open subplane S of a topological projective plane such that S contains coaffine points. We can delete one point, to obtain a coaffine plane. The only other possibility is to produce an almost projective plane by deleting several, but not all, points of one line L. The remaining points of L will precisely be the coaffine points of the subplane. Deleting those points, we obtain an affine plane.

We shall describe another plane that also contains precisely one line K consisting of coaffine points. However, the complement of that line will not be affine; consequently, the plane is not an open subplane of a topological projective plane. In the real affine plane, replace all lines of negative slope by all translates of the curve $y = x^{-1}$, $x > 0$. The resulting stable plane S is an example of an arc plane, cf. 5.10. Its affine lines are precisely the ordinary lines of positive slope s, $0 < s < \infty$. By the results of Löwen [1981a], these lines can be made projective by adding a point at infinity to each of them. Together, the points at infinity form a line K with the properties announced above.

This extended plane is called the *modified real dual cylinder plane*, and is denoted by MDC($\mathbb{R}$). It is interesting also because of its homogeneity, cf. 5.3, 5.15, 5.22.

More complicated constructions yield examples of planes with precisely two lines or countably many lines consisting of coaffine points (*loc. cit.*). Clearly, such planes are not open subplanes of projective planes.

These examples show that the complement of a line $K \in \mathcal{C}$ (as defined in 3.17(b)) need not be an affine plane, and that neither does the opposite plane of a plane having $\operatorname{card} \mathcal{C} \geqslant 2$. Nevertheless, the topology of such complements and opposite planes resembles that of an affine plane; this statement is taken from Löwen [1981a], 1.8, where it is misprinted (but the proof is correct):

3.20. THEOREM. *In a locally compact, locally connected stable plane $S = (P, \mathcal{L})$, consider the set $\mathcal{C}$ of all lines consisting of coaffine points. The following statements hold.*

(a) *For $K \in \mathcal{C}$, the subplane $P \setminus K$ has mutually homeomorphic lines, and $P \setminus K$ is homeomorphic to the product of any two lines.*

(b) *If* $\operatorname{card} \mathcal{C} \geqslant 2$, *then the lines of the opposite plane S^* are mutually homeomorphic, and the point set $\mathcal{P}$ of S^* is homeomorphic to the product of two lines.*

3.21. *Topological dimension.* One way to define the *dimension* of a topological space X is by induction: One puts $\dim \varnothing = -1$, and $\dim X \leqslant n$ provided every point of X has arbitrarily small neighbourhoods whose boundaries have dimension $< n$. We put $\dim X = \infty$ if there is no integer n such that $\dim X \leqslant n$ holds. For separable metric spaces X (such as the spaces associated with locally compact stable planes), this *small inductive dimension* agrees with most other dimension functions commonly considered in topology; in particular, $\dim \mathbb{R}^n = n$, see Engelking [1978]. Since dim is defined by local properties, the dimension of a topological n-manifold (a Hausdorff space locally homeomorphic to $\mathbb{R}^n$) is also equal to n.

Let now $S = (P, \mathcal{L})$ be a stable plane. Since S is not a topological space, $\dim S$ can only mean the geometric dimension of S defined in 2.1, which is 2. On the other hand, since $P, \mathcal{L}$ and $L \in \mathcal{L}$ are not geometries, dim applied to any of these refers to the topological dimension just defined. Throughout Sections 3 to 5, we shall use the symbol $l = \dim L$ to denote the topological dimension of a line in the plane under consideration. Remember that $l > 0$ if S is locally compact and locally connected (3.6 and subsequent remarks). In the classical planes $\mathrm{P}_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, the value of l is 1, 2, 4, or 8, respectively, compare 1.12. We shall see (3.29) that in an arbitrary locally compact, locally connected stable plane, l can take no other values, except for one unpleasant possibility:

3.22. PROBLEM. *Finiteness of dimension.* One of the major unsolved problems about stable planes is to decide whether or not $l = \dim L$ can be infinite for a locally compact stable plane. This seems to be rather unlikely. For the time being, we have to make $l < \infty$ a hypothesis whenever we want to apply topological theorems on finite-dimensional spaces. This does happen quite often.

3.23. THEOREM. *If the lines of a locally compact stable plane are l-manifolds, $l < \infty$, then the line pencils are homeomorphic to the l-sphere $\mathbb{S}_l$.*

Note that lines and line pencils are mutually homeomorphic in the projective case.

PROOF. The proof uses the contractibility result 3.7(b) and a theorem of M. Brown, which characterizes $\mathbb{R}^l$ as the union of an ascending chain of open sets each of which is homeomorphic to an open ball; see Löwen [1976b], 1.19, for details. □

3.24. PROBLEM. *Manifold lines.* The last result makes it desirable to prove that lines of a locally compact, locally connected stable plane are manifolds. This is another one of the major unsolved problems. No counterexamples are known; see also 3.27 and 3.28. If the lines are manifolds, then the same is true for the point and line spaces by the product structure 3.3, but the reverse conclusion cannot be drawn, because some manifolds can be decomposed as Cartesian products of nonmanifolds.

It is known, however, that lines have the local homological properties of l-manifolds if l is finite, see Löwen [1983b]. The proof uses the contractibility properties of pencils (see 3.7(b)) and a sheaf-theoretic characterization of generalized manifolds due to Bredon. This result suffices to obtain most of the benefit that can be drawn from manifolds. In particular, Löwen (*loc. cit.*) proves the following.

3.25. THEOREM. *Let $S = (P, \mathcal{L})$ be a locally compact stable plane with l-dimensional lines, $0 < l < \infty$. Then the following assertions hold.*

(a) $\dim P = \dim \mathcal{L} = 2l$.

(b) *Domain invariance: A subset of P homeomorphic to an open subset of P is itself open in P. Similar statements hold for subsets of $\mathcal{L}$ or of $L \in \mathcal{L}$. In fact,*

(c) *a closed subset of L (of P, $\mathcal{L}$) contains interior points if and only if it has the same topological dimension as L or P, $\mathcal{L}$, respectively.*

(d) *A pencil $\mathcal{L}_x$ and the line space $\mathcal{L}$ are not dissected by any closed subset of dimension less than $l - 1$ or $2l - 1$, respectively.*

3.26. *Topological pigeonhole principle.* The domain invariance property 3.25(b) allows us to use a substitute for counting arguments, as follows. Suppose we have a continuous injection from one line pencil into another one of the same plane. Then domain invariance implies that this map is open. Since pencils are compact and connected, this means that injectivity implies surjectivity, as with finite sets of equal cardinalities.

By a theorem of Bing and Borsuk, the l-manifolds with $l \leqslant 2$ can be characterized as l-dimensional locally contractible, locally homogeneous metric spaces. According to 3.2, 3.5 and 3.7, the lines of locally compact stable planes satisfy these conditions, so we have the following partial answer to the manifold problem 3.24, see Löwen [1976b], 1.13.

3.27. THEOREM. *Let S be a locally compact stable plane with lines of dimension $l = 1$ or $l = 2$. Then the lines are l-manifolds, and the line pencils are homeomorphic to the sphere $\mathbb{S}_l$.*

By comparing the local and global homology groups of line pencils, Löwen [1983b] computes both kinds of homology groups. See also Salzmann et al. [1994], Sections 52 and 54. This leads to the following theorem.

3.28. THEOREM. *In a locally compact stable plane with l-dimensional lines,* $0 < l < \infty$, *the line pencils* $\mathcal{L}_x$ *are homotopy equivalent to the sphere* $\mathbb{S}_l$.

In fact, the same proof also shows that the multiplicative loop of an associated local ternary field (see 3.4; in the projective case, this is a line or a pencil minus two elements) is homotopy equivalent to $\mathbb{S}_{l-1}$. Now there is a famous result of Adams [1960] about the nonexistence of continuous multiplications with (homotopy) units on spheres other than $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{S}_3$ and $\mathbb{S}_7$. Hence, the only possible finite values of $\dim P$ are the dimensions $2l = 2$, 4, 8 and 16 of the classical planes $\mathrm{P}_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$; see Löwen [1983b]. This is a strong indication of classical domination (see the introduction). Moreover, this deep result is the basis of all closer investigations of stable planes and, in particular, of compact connected projective planes.

3.29. THEOREM ON DIMENSIONS. *The lines of a locally compact stable plane have topological dimension* $l \in \{0, 1, 2, 4, 8\}$ *or, possibly,* $l = \infty$.

3.30. THEOREM. *If a locally compact stable plane* $S = (P, \mathcal{L})$ *of finite positive dimension* $2l$ *contains a projective line* L, *then the point space* P *is not contractible*; *in fact, the homotopy group* $\pi_l P$ *is nonzero.*

PROOF. (Compare Löwen [1976b], 1.23.) Since L is projective, there is a homotopy equivalence $f\colon \mathbb{S}_l \to L$ as in 3.28. We show that f represents a nonzero element of $\pi_l P$. So assume that F is a homotopy contracting f in P. If F is surjective onto P, then S is a projective plane by 3.11, and then our claim follows from Löwen [1983b], cf. 4.8. If $p \in P$ does not belong to the image set of F, then we can map all images $F(x, t)$ back into L by the central projection with centre p, since L is projective. The resulting homotopy contracts f in L, which is absurd because $\mathbb{S}_l$ is not contractible. □

It seems reasonable to conjecture that the following 'dual' assertion is true: In any locally compact stable plane of finite positive dimension, the line space $\mathcal{L}$ is not contractible; more precisely, every line pencil $\mathcal{L}_x$ should represent a nonzero element of the homotopy group $\pi_l\mathcal{L}$. If this could be proved, then one could hope to use it in order to classify stable planes with a line-transitive group of automorphisms, employing the same techniques as for 4.14. As it stands, such a group might be comparatively small. See also 5.11.

3.31. *Order and convexity in 2-dimensional stable planes.* Let $S = (P, \mathcal{L})$ be a 2-dimensional locally compact stable plane. By 3.25(a), we have $l = 1$, and the lines are 1-manifolds by 3.27. The local product structure of P, see 3.3, implies that, locally, P is dissected by every line.

If P is homeomorphic to $\mathbb{R}^2$ and all lines are connected, then the lines are homeomorphic to the real line $\mathbb{R}$ by 3.10 and 3.30. The Jordan curve theorem says that every line dissects the point space into two components. These *half-planes* constitute an ordering of the plane satisfying Pasch's axiom; compare also 7.4. Every point has a neighbourhood basis consisting of the interiors of triangles, so the topology can be recovered from the ordering. The ordering can be described using only the geometry. In particular, all collineations are continuous. Details are given in Salzmann [1967b], Section 2, and in Salzmann et al. [1994], Section 31. Compare also Cantwell's work described in 2.14 (Cantwell [1974, 1978]), which develops the rudimentary theory of convex sets inherent in the preceding remarks. Cantwell in fact characterizes planes with lines homeomorphic to $\mathbb{R}$ by their order properties. A similar characterization is given by Polley [1973], allowing also lines homeomorphic to the circle $\mathbb{S}_1$; in that case, Pasch's axiom does not hold. A related result of Prieß-Crampe [1967], [1983], p. 231, states that a projective plane admits an Archimedean ordering if and only if it can be embedded into a 2-dimensional compact projective plane.

For 2-dimensional locally compact stable planes in general, results related to order and convexity can be obtained locally. If projective lines exist, then the results can also be applied to the opposite plane. This is used by Salzmann [1969a] to determine the possible point and line spaces of 2-dimensional stable planes with connected lines. Salzmann assumes that the point space has *finite connectivity* (i.e. a finitely generated fundamental group). This hypothesis was eliminated by Löwen [1972], using the Freudenthal compactification of the point set, so that we have the following result.

3.32. THEOREM. *Let $S = (P, \mathcal{L})$ be a locally compact 2-dimensional stable plane with connected lines. Then both P and $\mathcal{L}$ are homeomorphic to their counterparts in one of the following prototypes*:

(i) *The real affine plane*; *here, $\mathcal{L}$ is a Möbius strip, and there are no projective lines.*

(ii) *The real projective plane*; *then, S is a projective plane.*

(iii) *The exterior real hyperbolic plane* (5.2), *i.e. the subplane of* $\mathrm{P}_2\mathbb{R}$ *induced on the complement of a closed circular disc*; *here, P is a Möbius strip and $\mathcal{L}$ is homeomorphic to the point set of* $\mathrm{P}_2\mathbb{R}$; *there are both projective and nonprojective lines.*

For planes of higher dimension, no similar theorem has been proved. Certainly, one would have to make strong assumptions in order to restrict the immense number of possibilities existing even for open subplanes of classical projective planes. For example, one could assume that every line is homeomorphic to $\mathbb{R}^l$ or to $\mathbb{S}_l$. For projective planes, however, we have strong results; see 4.8.

3.33. *Automorphisms.* By definition, an *automorphism* of a stable plane $S = (P, \mathcal{L})$ is a homeomorphism γ: $P \to P$ that sends lines onto lines. It is easily seen that then γ induces a homeomorphism of $\mathcal{L}$. *Isomorphisms* φ: $S_1 \to S_2$ are defined in a similar manner. If S is locally compact and locally connected, then every continuous collineation of S is an automorphism of S, i.e. continuity of the inverse γ^{-1} follows from the other conditions, see Löwen [1976b], 2.1; the easy proof uses the fact that line pencils are compact (see 3.8).

Several weaker notions have proved to be interesting. If we require only that γ sends lines *into* lines, then we obtain the notion of *lineations*, studied in Grundhöfer and Stroppel [1992]. In the theory of symmetric planes (see 5.27), it is necessary to work with the condition that every connected component of a line is mapped into a line, see Löwen [1979a], 4.8. If γ is required to be injective rather than bijective, then in some cases the number of possibilities is greatly increased; consider, e.g., shifts of a half-plane in the real affine plane. These maps form a semigroup rather than a group; see Stroppel [1992d].

In general, automorphisms of an open subplane S of a topological projective plane R do not extend to automorphisms of R; see, however, the remarks following 2.9 on subplanes of the classical planes. In fact, there are rigid projective planes (without any nontrivial automorphisms) with the property that by deleting two point rows we obtain the disjoint union of two subplanes isomorphic to the real half-plane (Steinke [1985]); the half-plane itself has many automorphisms. However, if S is a dense open subplane of R and R is locally compact and connected, then all automorphisms of S extend to R, see Löwen [1981c].

3.34. *Topology of the automorphism group.* Consider the group $\Sigma = \operatorname{Aut} S$ of all automorphisms of a locally compact, locally connected stable plane S. We endow Σ with the *compact-open topology*, i.e. the topology generated by the subbasis consisting of all sets $[C, U] = \{\sigma \in \Sigma\colon\ C^\sigma \subseteq U\}$, where $C \subseteq P$ is compact and $U \subseteq P$ is open. Since P is a metric space (see 3.5), this topology can be described as the topology of uniform convergence on compact subsets of P, or, if S happens to be a compact projective plane, as the topology of uniform convergence on P, see Dugundji [1966]. If we replace the point space P by the line space $\mathcal{L}$ in this definition, then we obtain the same topology on Σ, see Grundhöfer and Stroppel [1992].

The main feature of the compact-open topology is that it makes Σ a *topological transformation group* of P. This means that Σ is a topological group (multiplication and inversion are continuous) and that the evaluation map $\Sigma \times P \to P$ sending (σ, x) to x^σ also is continuous. Continuity of inversion depends on the properties of P (local connectedness) and is by no means trivial; see references given in Löwen [1976b] for this result of Arens.

3.35. THEOREM. *The automorphism group $\Sigma = \operatorname{Aut} S$ of a locally compact, locally connected stable plane $S = (P, \mathcal{L})$ is a locally compact, second countable topological transformation group of both P and $\mathcal{L}$.*

PROOF. The proof is given in Löwen [1976b], Section 2. Local compactness is obtained using the theorem of Arzela and Ascoli, which characterizes sets of mappings that are compact with respect to the compact-open topology. A quite similar proof has been given for compact buildings with closed incidence by Burns and Spatzier [1987a], see 6.5. For compact projective planes, the proof can be simplified (Salzmann [1975a]), and local connectedness is not needed in this case (Grundhöfer [1986]). □

We mention an important technical tool that is used in the proof of the last theorem and elsewhere, namely conditions ensuring that the limit of a pointwise convergent sequence of automorphisms is an automorphism. For results in this direction, see Löwen [1976b], 2.8 and 3.7, Löwen [1981a], Grundhöfer and Stroppel [1992], (12).

3.36. *Degree of homogeneity.* As a consequence of 3.35, the automorphism group Σ of a locally compact, locally connected stable plane S is a separable metric space, hence the dimension $d = \dim \Sigma$ can be defined in many equivalent ways, cf. 3.21. For example, d also is the maximum of all integers k such that Σ contains a subset homeomorphic to the k-cube $[0, 1]^k$, compare Salzmann et al. [1994], 93.6. The integer d measures the size of the automorphism group, so we call it the *degree of homogeneity* of S.

Consider the classical projective planes $\mathrm{P}_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. For $\mathbb{F} \neq \mathbb{O}$, we can use 1.7 in order to determine Σ. All automorphisms of the topological division ring $\mathbb{F}$ are inner automorphisms, except for complex conjugation. We obtain that $\Sigma = \mathrm{PGL}_3\,\mathbb{F} \rtimes \Phi$, where $\Phi = \mathbb{Z}_2$ in the complex case and $\Phi = \mathbb{1}$ otherwise. The automorphism group of $\mathrm{P}_2\mathbb{O}$ is the exceptional Lie group $\mathsf{E}_6(-26)$, see, e.g., Salzmann et al. [1994], Section 18.

This implies that the degree of homogeneity of $\mathrm{P}_2\mathbb{F}$ is $d_l = 8$, 16, 35, or 78, respectively, for $l = \dim \mathbb{F} = 1$, 2, 4 or 8. The general experience is that the value of d for any other plane S with lines of dimension l is much smaller than d_l. In fact, for projective planes, the maximal value of d is known; it is close to $d_l/2$. We shall come back to this point in the next two sections.

If a finite group Δ acts on a set X, then one has Lagrange's formula, which relates the orders of Δ and of the *stabilizer* $\Delta_x = \{\delta \in \Delta\colon x^\delta = x\}$ of $x \in X$ to the index $\mathrm{card}(\Delta/\Delta_x)$, which equals the cardinality of the *orbit* $x^\Delta = \{x^\delta\colon \delta \in \Delta\}$ of x. In order to work with the degree of homogeneity, one needs some substitute for this formula. This is provided by the *dimension formula*

$$\dim \Delta = \dim x^\Delta + \dim \Delta_x,$$

which is valid for all closed (hence locally compact) subgroups Δ of Σ and all $x \in P \cup \mathcal{L}$. Note that a locally compact orbit x^Δ is homeomorphic to the coset space Δ/Δ_x, endowed with the quotient topology with respect to the canonical surjection $\Delta \to \Delta/\Delta_x$. The dimension formula is in fact valid for arbitrary orbits, see Salzmann et al. [1994], 96.10.

Moreover, in order to exploit information about d, one needs *stiffness* properties of Σ, i.e. upper bounds for the dimensions of subgroups with large fixed point sets. These bounds depend on the dimension of the plane, and will be treated in the appropriate subsections of 5; see also 4.18.

3.37. *Lie groups.* For all known examples of locally compact, locally connected stable planes S, the automorphism group $\Sigma = \mathrm{Aut}\,S$ is a *Lie group*. This means that Σ is a differentiable manifold, and the group operations are differentiable. In this case, $d = \dim \Sigma$ may be interpreted as the dimension of this manifold or as the vector space dimension of its tangent space at $1 \in \Sigma$, which supports the associated Lie algebra. Some remarks on the structure theory of Lie groups will be made in 5.19.

As an example, consider the automorphism group of a classical projective plane $P_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. By the description given in 3.36, this group essentially is the quotient of the Lie group $GL_3\mathbb{F}$ modulo some closed subgroup isomorphic to the centre of the multiplicative group $\mathbb{F}^\times$; this implies that it is a Lie group.

3.38. PROBLEM. Is $\Sigma = \mathrm{Aut}\,\mathcal{S}$ a Lie group for every locally compact, locally connected stable plane?

Note that it is an extremely difficult open problem of topology whether every locally compact transformation group acting effectively on a manifold is a Lie group (Hilbert–Smith conjecture). Our problem above might be less difficult due to the impact of the geometry which is left invariant by the group.

One wants to classify planes using their automorphism groups. In order to do this, a positive answer to the above problem is highly desirable, but also partial results are welcome. In the case of projective planes, the known partial results suffice for all practical purposes, see 4.12. For general stable planes, the answer is positive if $\dim P = 2$; in fact, the general Hilbert–Smith problem has an affirmative solution for 2-manifolds. If $\dim P = 4$, then the identity component Σ^1 is a Lie group if all lines are homeomorphic to $\mathbb{R}^2$ (Löwen [1976b]), or if $d = \dim \Sigma \geqslant 5$ (Löwen [1978]). If Σ is transitive on P or on $\mathcal{L}$, then Σ is a Lie group by Szenthe's solution of Hilbert's 5th problem (Szenthe [1974]). Note that Szenthe's proof works only for first countable groups, but 3.35 says that we are dealing with such groups. Therefore, we can even drop Szenthe's hypothesis that Σ/Σ^1 be compact, compare the proof of Löwen [1983a], 3.10. If S contains an isotropic point (see 5.1), then again Σ is a Lie group (see 5.5a).

3.39. *Special automorphisms.* Consider a group $\Delta \leqslant \mathrm{Aut}\,S = \Sigma$ of automorphisms of some stable plane. We write $\Delta_{[c,A]}$ for the subgroup of all *centro-axial collineations* with *axis* A and *centre* c; this group consists of all $\delta \in \Delta$ that fix every element incident with c or A. The meaning of $\Delta_{[c]}$ and $\Delta_{[A]}$ is now obvious; elements of these groups are called *central* or *axial* collineations, respectively. It is easy to see that every central or axial collineation γ is continuous, hence an automorphism; see Löwen [1976b], 3.2. By a *reflection* at $x = c$ or $x = A$ we mean an involution $\sigma \in \Sigma_{[x]}$.

Of course, an involution $\sigma \in \Sigma$ of a stable plane need not have any fixed points; just consider an arbitrary involution and pass to a subplane by deleting the fixed point set. In this case, σ will be called a *free* involution. By a *Baer involution*, we mean an involution $\sigma \in \Sigma$ whose fixed point set is a subplane of dimension $l = \dim L$, where L is a line of S. Simple domain invariance arguments show that this coincides with the usual notion of Baer involutions in the projective case. The following result generalizes a well-known theorem of Baer.

3.40. THEOREM. *Let σ be an involutory automorphism of a locally compact, locally connected stable plane. Then σ is free or central or axial, or σ is a Baer involution.*

PROOF. Let F be the fixed point set of σ. If F consists of just one point c, then $\sigma \in \Sigma_{[c]}$ by Löwen [1979a], 1.5. Let L be a nonfixed line. Then the map $f\colon p \mapsto p \vee p^\sigma$

is a local homeomorphism between the set $L \setminus F$ and the set of fixed lines of σ. If F contains a triangle, then the set of fixed lines containing at least two fixed points is open in the set of all fixed lines and hence is locally homeomorphic to L. It follows that σ is a Baer involution. Finally, we treat the case that F is contained in a line A and contains more than one point. We have to show that σ has axis A. By a fixed point theorem of Smith (see the proof of Löwen [1979a], 1.5), every $x \in F$ lies on at least one fixed line other than A. Using the map f near a hypothetical nonfixed point $a \in A$ we are now able to construct fixed points outside A, a contradiction. □

For more results about reflections, concerning uniqueness, commuting reflections, convergence of sequences of reflections, etc., see Löwen [1981a], Stroppel [1993c]. The following two results are taken from Löwen [1981a], 3.12, 3.15, 3.16.

3.41. THEOREM. *Let c be a point of a locally compact, locally connected stable plane $S = (P, \mathcal{L})$ such that the group $\Sigma_{[c]}$ of central collineations is not compact. Then c is affine, or S is projective or contains a line consisting of coaffine points.*

3.42. THEOREM. *If S satisfies the hypothesis of* 3.41 *for every point c, then S is an almost projective plane as defined in* 3.16.

4. Topological projective planes

Topological projective planes are a special type of stable planes, hence the results from the previous sections apply. We shall concentrate on (locally) compact planes; for deeper results we also have to assume connectedness (cf. 4.3). In fact, Section 3 contains incisive restrictions for the topology of a compact connected projective plane, most of which were first proved for projective planes and then extended to stable planes.

The theory of compact connected projective planes has developed vigorously. As a consequence, one can now classify the most homogeneous planes of this type. See Salzmann et al. [1994] for a detailed treatment.

The following definition is due to Skornjakov [1954] and Salzmann [1955].

4.1. DEFINITION. A *topological projective plane* is a projective plane $\mathcal{P} = (P, \mathcal{L})$ with topologies on P and $\mathcal{L}$ such that the two geometric operations of joining distinct points and intersecting distinct lines are continuous; we also exclude the coarsest topologies on P and on $\mathcal{L}$.

This definition agrees with 1.1, specialized to projective planes, compare 2.3(d). Freudenthal [1957a] stipulates that the diagonal point $ac \wedge bd$ of a quadrangle a, b, c, d should depend continuously on the quadrangle (a, b, c, d); this requires a topology only on the point set P. That definition is equivalent to 4.1, see Arumugam [1985].

The point space P and the line space $\mathcal{L}$ of a topological projective plane $(P, \mathcal{L})$ are regular topological spaces; furthermore P and $\mathcal{L}$ are either both connected or both totally disconnected, see Skornjakov [1954], Salzmann [1955], Salzmann et al. [1994], 41.4, 42.1; for the locally compact case, see also 3.6.

Some projective planes cannot be made into topological projective planes, see Szambien [1989]. On the other hand, for every topological division ring F, the Desarguesian projective plane P_2F over F is a topological projective plane with respect to the topologies inherited from F^3, see 1.4, 1.5 or Pickert [1975], p. 265. According to Kiltinen [1973], every infinite field F of cardinality a carries 2^{2^a} distinct field topologies; thus P_2F is a topological projective plane in too many ways. (It seems to be an open problem whether every division ring of infinite dimension over its centre admits a nontrivial topology rendering it a topological division ring). In order to obtain substantial results, it seems necessary to admit only decent topologies. By the classical geometries (or the classical fields), we are led to consider only locally compact connected topologies.

4.2. PROPOSITION. *Let $(P, \mathcal{L})$ be a topological projective plane such that P is locally compact and connected. Then P, $\mathcal{L}$, each line and each pencil are compact and (locally and globally arcwise) connected.*

For proofs, see Skornjakov [1954], Salzmann [1957], §5, compare also 3.6, 3.3, 3.11. It is an open problem whether or not every locally compact (totally disconnected) projective plane is compact. As yet, we do not have a coherent theory on compact totally disconnected projective planes.

4.3. *Compact totally disconnected projective planes.* Only few results are available without connectedness assumptions. Apart from the theory of topological fields (see, e.g., Warner [1989]), we mention the construction of topologies for projective planes (Grundhöfer [1987a]), local compactness of the automorphism group (Grundhöfer [1986]), construction and classification of nearfields (Rink [1986], Grundhöfer [1989], Hanke and Wähling [1990]) and nearfield planes (Rink [1985]), finite epimorphic images (Grundhöfer [1988], Grundhöfer and Van Maldeghem [1990]), and continuously differentiable ovals (Tillmann [1991]); see also 4.7.

In what follows, we consider primarily compact connected projective planes. For compact topologies, Definition 4.1 can be rephrased as follows (see Grundhöfer [1987a], 2.1).

4.4. LEMMA. *A projective plane $(P, \mathcal{L})$ with compact topologies on P and $\mathcal{L}$ is a topological projective plane if and only if the incidence relation I (the set of all flags) is closed in the product $P \times \mathcal{L}$.*

According to Hartmann [1989b], the condition that I be closed may be replaced by continuity of all central projections together with the dual condition.

4.5. *Ternary fields.* A ternary field (K, τ), with the ternary operation τ: $K^3 \to K$, is the algebraic equivalent of an affine (or projective) plane with a fixed frame of reference, see Chapter 19, 8.8, and Pickert [1975], 1.5.

For any topological projective plane (or any topological affine plane with continuous parallelism) and any frame of reference, the corresponding ternary field is a topological ternary field with respect to the topology obtained by identifying K with a punctured line; by definition, this means that τ and its inverse operations are continuous, see Salzmann [1957], §2, [1967b], 7.5, or Grundhöfer and Salzmann [1990], XI.2.2, for details. This assertion is a direct consequence of the geometric interpretation of τ and its inverse operations.

The converse of this relationship is not true in general: Eisele [1992b] constructs topological semifields which do not belong to topological affine planes as defined in 2.3d (the trouble is caused by the lines of infinite slope). Note that every topological alternative field does define a topological projective plane, see Salzmann [1957], § 14. Furthermore, Eisele [1990, 1991a,b, 1993a,b] has examples of topological affine planes which cannot be extended to topological projective planes. Fortunately, these pathologies cannot occur in the locally compact connected world.

4.6. THEOREM.

(a) *A ternary field* (K, τ) *with* K *homeomorphic to* $\mathbb{R}^n$ *is a topological ternary field if and only if the ternary operation* $\tau\colon K^3 \to K$ *is continuous.*

(b) *Each locally compact connected ternary field coordinatizes a locally compact connected affine plane* $\mathcal{A}$, *and* $\mathcal{A}$ *extends to a compact connected projective plane.*

Part (a), due to Knarr and Weigand [1986], is an application of domain invariance. For proofs of (b) see Skornjakov [1954], Theorem 9, Salzmann [1967b], 7.14, 7.15, Grundhöfer [1987a], 4.2. It is not known whether a version of this theorem holds for locally compact totally disconnected topologies, cf. Grundhöfer [1987a] for some partial results.

All compact Moufang planes can be determined:

4.7. THEOREM. *Let* $\mathcal{P}$ *be a compact projective Moufang plane. If* $\mathcal{P}$ *is connected, then* $\mathcal{P}$ *is isomorphic to the plane* $\mathrm{P}_2\mathbb{F}$ *over one of the four classical semifields* $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. *If* $\mathcal{P}$ *is not connected, then* $\mathcal{P}$ *is the Desarguesian plane* P_2F *over a locally compact totally disconnected division ring* F *(and these division rings can be described explicitly).*

This can be proved by classifying all locally compact (nondiscrete) alternative fields, see the survey Grundhöfer and Salzmann [1990], XI.7, and Grundhöfer [1987b]; see also 1.11 for the connection with Pontrjagin [1932]. Details about the structure of the division rings F mentioned above can be found in Grundhöfer and Salzmann [1990], XI.7.7.

The projective plane $\mathrm{P}_2\mathbb{O}$ over Cayley's octonions $\mathbb{O}$ cannot be described by homogeneous coordinates in the usual way, since $\mathbb{O}$ is not associative. See Salzmann et al. [1994], Section 17, for a homogeneous description of the octonion plane and its topology. The descriptions in Salzmann [1957], §14, Aslaksen [1991] are less homogeneous.

From Section 3 we recall the following crucial result of Löwen on the topology of compact connected projective planes $(P, \mathcal{L})$. Let $l = \dim L$ denote the topological dimension (3.21) of a line $L \in \mathcal{L}$. If $l < \infty$, then

$$l \in \{1, 2, 4, 8\};$$

furthermore $\dim P = \dim \mathcal{L} = 2l$, and each line and each pencil is homotopy equivalent to the sphere $\mathbb{S}_l$.

This is just a special case of 3.24(a), 3.27, 3.28; note that $l > 0$ by 3.6, 3.3, and that lines and pencils are homeomorphic (via projectivities). A proof can also be found in Salzmann et al. [1994], Section 54. This result is a geometrical generalization of the well-known fact that real division algebras (of finite dimension) exist only in dimensions 1, 2, 4, 8.

It is an open problem whether the assumption $l < \infty$ is necessary. We conjecture that compact projective planes of infinite topological dimension do not exist (cf. 3.22). This is true for planes over ternary fields with associative addition or multiplication, see Salzmann [1967b], 7.21, 7.22.

The following characterization of Baer subplanes is a fairly direct consequence of Löwen's result. A closed subplane $\mathcal{P}_0 = (P_0, \mathcal{L}_0)$ of a compact connected plane $\mathcal{P} = (P, \mathcal{L})$ of finite dimension is a Baer subplane if and only if $\dim P = 2 \dim P_0$; see Löwen [1983b], Salzmann et al. [1994], 55.5.

4.8. *Classical domination.* As a consequence of Löwen's result and 4.7, each compact connected projective plane $\mathcal{P}$ of finite topological dimension is associated (via its topological dimension) with one of the four classical planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Thus $\mathcal{P}$ may be studied as a distorted version of one of the four classical planes; this heuristic idea of 'classical domination' turns out to be very fruitful, as is seen below.

If $\mathcal{P} = (P, \mathcal{L})$ has dimension $\dim P = 2l \leqslant 4$, then the point rows in P and the line pencils in $\mathcal{L}$ are even homeomorphic to the sphere $\mathbb{S}_l$, see 3.26, 3.22. This agrees with a conjecture of Freudenthal [1957a]. As a consequence, P, $\mathcal{L}$ and the flag space I are in fact homeomorphic to their classical counterparts in the real or complex projective plane, see 3.32, Breitsprecher [1971, 1972], Buchanan [1979b], Salzmann et al. [1994], Section 53. It is an open problem whether or not this is also true for $l \in \{4, 8\}$, cf. 3.22 and Buchanan [1979a]. According to Löwen [1983b], Salzmann et al. [1994], Chapter 5, the (co-)homology of P, $\mathcal{L}$ and I depends only on l, i.e. it is isomorphic to the (co-)homology of the classical counterpart, provided that $l < \infty$.

Now we describe some construction principles, in order to exhibit many examples of nonclassical compact projective planes. Some of these examples appear again in the classification results below. For further constructions, see 4.21.

4.9. *Constructions of* 2*-dimensional compact projective planes.* These planes have the special feature that their topology can be described by an ordering (the usual ordering on punctured lines, which are homeomorphic to $\mathbb{R}$), see 3.31, 7.4. In fact, the ordering allows us to deform the line segments within some convex region in $\mathbb{R}^2$ in a fairly arbitrary way. According to Skornjakov [1957], Salzmann [1967b], 2.8, 2.12, every affine plane $(\mathbb{R}^2, \mathcal{L})$ whose lines are closed in $\mathbb{R}^2$ and homeomorphic to $\mathbb{R}$ gives rise to a compact connected projective plane.

The plane in §23 of the first five editions of Hilbert [1899] is probably the earliest published example of a non-Desarguesian projective plane; it is obtained by replacing the line segments in the interior of an ellipse by segments of circles that pass through a suitably fixed exterior point. This plane admits only two collineations, see Anisov [1992].

Further examples are the Moulton planes: starting with the real affine plane on $\mathbb{R}^2$, we fix a real number $k > 1$ and replace each line of negative slope by its image under the bijection

$$(x,y) \mapsto \begin{cases} (x,y) & \text{for } x \leqslant 0, \\ (x,ky) & \text{for } x \geqslant 0. \end{cases}$$

It is easy to see that these planes are not Desarguesian, see Hilbert [1930], §23. Variations of this construction lead to planes with trivial collineation groups, see Steinke [1985].

Now we describe the parabola model of the real affine plane. Apart from the vertical lines, it consists of the parabola $\{(x,y) \in \mathbb{R}^2\colon\ y = x^2\}$ and its images under the 'shifts' $(x,y) \mapsto (x+a, y+b)$ with $a, b \in \mathbb{R}$; the map $(x,y) \mapsto (x, y-x^2)$ is an isomorphism of the parabola model onto the usual model. We say that an affine plane on $\mathbb{R}^2$ is a *shift plane* if it is generated in a similar way by a suitable curve $\{(x,y) \in \mathbb{R}^2\colon\ y = f(x)\}$ instead of the parabola.

The following criterion yields many examples: Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable such that the derivative $f'\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Then the curve

$$\{(x,y) \in \mathbb{R}^2\colon\ y = f(x)\}$$

generates a shift plane as defined above, see Salzmann [1965], p. 258. In fact, the differentiability of f is not necessary. According to Groh [1976], a function $f\colon \mathbb{R} \to \mathbb{R}$ generates a shift plane if and only if f is strictly convex and

$$\lim_{x \to \pm\infty} \big(f(x) - ax\big) = +\infty \quad \text{for every } a \in \mathbb{R},$$

or if $-f$ satisfies these conditions.

More examples can be found in 5.3, 5.6, 5.10, 5.14, Salzmann et al. [1994], Section 31.

Joussen [1981] proved that every finitely generated free plane (see Chapter 13, 2.2) admits an Archimedean ordering; hence by Prieß-Crampe [1967], [1983], p. 231, the completions of these free planes are 2-dimensional compact projective planes. This shows that no configurational proposition holds in the class of all 2-dimensional compact projective planes.

4.10. *Constructions of translation planes.* Many (topological) translation planes have been constructed by describing a coordinatizing (topological) quasifield, as in Chapter 5 and Grundhöfer and Salzmann [1990]. For example, the *mutations* of $\mathbb{F} \in \{\mathbb{H}, \mathbb{O}\}$ have the usual addition of $\mathbb{F}$ and the new multiplication

$$x * y = txy + (1-t)yx \quad \text{for } x, y \in \mathbb{F},$$

where $t \in \mathbb{R} \setminus \{1/2\}$ is a fixed real parameter. If $t \neq 0, 1$, then $(\mathbb{F}, +, *)$ is a proper semifield. For more examples of semifields, see Hähl [1975b, 1976, 1986a], Plaumann and Strambach [1975], Benkart, Britten and Osborn [1982] and the references given there.

Kalscheuer [1940] constructed nearfields $(\mathbb{H}, +, *)$ by defining

$$x * y = x\varphi(x)^{-1}y\varphi(x) \quad \text{for } x, y \in \mathbb{H}^{\times},$$

where $\varphi(x) = \exp(it \log|x|)$ with a fixed real parameter t. In fact, Kalscheuer proved that these nearfields, together with $\mathbb{R}$ and $\mathbb{C}$, are the only locally compact connected nearfields, cf. 4.16.

The class of all proper (locally compact connected) quasifields is extremely large. For example, let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, let ϱ: $[0, \infty) \to [0, \infty)$ be any homeomorphism with $1^{\varrho} = 1$, and define

$$x * y = x\big(y_0 + |x|^{\varrho}|x|^{-1}(y - y_0)\big) \quad \text{for } x, y \in \mathbb{F},$$

where y_0 is the real part of y. Then $(\mathbb{F}, +, *)$ is a topological quasifield, see Hähl [1980b, 1987a]. For more examples, see Betten [1977] and references given there, Plaumann and Strambach [1970], Hähl [1980a, 1984, 1986a], Buchanan and Hähl [1977].

Compact projective translation planes may also be described in terms of spreads (Chapter 5, Section 1) which are compact subsets of the appropriate Grassmann manifold, see Löwen [1989]. Various notions and constructions for (abstract or topological) spreads are treated in a unifying geometric fashion by Knarr [1991]. As a consequence of his results, every strict (or bounded) contraction ψ: $\mathbb{F} \to \mathbb{F}$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ leads to a compact connected projective translation plane; in fact, the corresponding spread in $\mathbb{F}^2$ consists of the subspaces $\{0\} \times \mathbb{F}$ and $\{(x, ax + a^{\psi}\bar{x}) \colon x \in \mathbb{F}\}$ with $a \in \mathbb{F}$. We remark that (noncompact) spreads exist in all vector spaces $\mathbb{R}^{2l}$ with $l > 1$, see Bernardi [1973]. Incidentally, the resulting (nontopological) translation planes show that the analogue of Skornjakov's result mentioned in 4.9 is false for $l > 1$.

See also Buchanan and Hähl [1978], Riesinger [1991].

4.11. *Automorphism groups.* The automorphism group of a compact (connected) projective plane $\mathcal{P} = (P, \mathcal{L})$ is the group Σ of all continuous collineations of $\mathcal{P}$. It is possible to show that Σ, endowed with the compact-open topology, is a locally compact group with countable basis, acting on P and on $\mathcal{L}$ as a topological transformation group, see 3.34, 3.35.

The following result is quite useful for the investigation of automorphism groups: if $\mathcal{P}$ has finite positive dimension, then every automorphism of $\mathcal{P}$ fixes some point and some line of $\mathcal{P}$. For the proof, one applies the Lefschetz fixed point theorem, which is possible because the homology groups of P and $\mathcal{L}$ are known, see Löwen [1983b], Breitsprecher [1971], Salzmann et al. [1994], 55.19.

Concerning discontinuous collineations, we just mention that the complex projective plane $P_2\mathbb{C}$ is the only compact projective plane which is known to admit discontinuous collineations (see Salzmann et al. [1994], 55.22, for more information).

4.12. *Homogeneity.* According to Kegel and Schleiermacher, see Chapter 13, Section 5, every projective plane can be embedded into some projective plane which satisfies the strong homogeneity condition that the collineation group is transitive on triangles. For compact connected projective planes $\mathcal{P}$ however, complete classifications are possible under much weaker homogeneity assumptions, see below. These classifications make essential use of the theory of Lie groups; this is possible because the customary homogeneity assumptions imply that the automorphism group Σ of $\mathcal{P}$ is a Lie group, see 3.38, Löwen and Salzmann [1982]. For example, if Σ has an open orbit in the point space of $\mathcal{P}$, then Σ is a Lie group by Szenthe [1974], see the remarks after 3.38.

In fact, one might conjecture that the automorphism group Σ of any compact connected projective plane is a Lie group (see 3.38); this is true for planes of dimension 2 or 4, see Salzmann [1962a], 4.1, [1970], 3.9, and for all translation planes.

4.13. THEOREM. *Let $\mathcal{P} = (P, \mathcal{L})$ be a compact connected projective plane. If the automorphism group Σ of $\mathcal{P}$ acts transitively on the point set P, then $\mathcal{P} \cong \mathrm{P}_2\mathbb{F}$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, and Σ contains a conjugate of the elliptic motion group.*

This result of Löwen [1981d] is one of the highlights of the theory; the proof uses the classification by Borel and De Siebenthal of all homogeneous spaces with Euler characteristic 3; see also Salzmann [1975a], Salzmann et al. [1994], Section 63.

The classical Moufang planes $\mathrm{P}_2\mathbb{F}$ as above are the only compact connected projective planes which admit an automorphism group isomorphic to a classical motion group; see 5.12, Löwen [1986b] or Salzmann et al. [1994], Section 62, for a more detailed and more general statement.

Point homogeneous affine planes exist in abundance (translation planes, shift planes 4.9). The following result on line homogeneous affine planes is proved in Salzmann [1975b], Salzmann et al. [1994], 63.1, compare also 5.11.

4.14. THEOREM. *Let L be a line of a compact connected projective plane $\mathcal{P} = (P, \mathcal{L})$. If $\mathcal{P}$ admits an automorphism group fixing L and acting transitively on $\mathcal{L} \setminus \{L\}$, then $\mathcal{P} \cong \mathrm{P}_2\mathbb{F}$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

4.15. THEOREM. *Let $\mathcal{P}$ be a compact connected projective plane with automorphism group Σ. Then the following statements are equivalent:*

(a) *The stabilizer Σ_L of some line L of $\mathcal{P}$ acts doubly transitively on L.*

(b) *$\mathcal{P}$ has Lenz type at least* III.

(c) *$\mathcal{P}$ is isomorphic to $\mathrm{P}_2\mathbb{F}$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ or to a Moulton plane* (4.9).

For the proof, see Salzmann [1964, 1972a, 1974] or Salzmann et al. [1994], 63.1, 63.5, 64.18.

Theorem 4.7 determines all compact connected planes of Lenz type VII. In view of 4.10, there is no hope to classify all planes of Lenz–Barlotti type IV a1 (i.e. all translation planes); the same is probably true for planes of Lenz type V (i.e. planes over semifields). In contrast, the compact connected planes of Lenz–Barlotti type IV a2, i.e. the planes over locally compact connected nearfields, are classified by the following result of Kalscheuer

[1940], see also Tits [1952, 1956], Salzmann [1967b], 7.26, Salzmann et al. [1994], 64.22.

4.16. THEOREM. *The compact connected nearfield planes are precisely the Desarguesian planes* $\mathrm{P}_2\mathbb{R}$, $\mathrm{P}_2\mathbb{C}$, $\mathrm{P}_2\mathbb{H}$ *and the planes over the Kalscheuer nearfields* (4.10).

As another transitivity condition for compact connected projective planes $\mathcal{P} = (P, \mathcal{L})$, we mention the existence of an open orbit in the flag space $I \subset P \times \mathcal{L}$; planes satisfying this condition have been called *flexible.* If $\dim P = 2$, then $\mathcal{P}$ is flexible if and only if the automorphism group of $\mathcal{P}$ has dimension at least 3, and these planes are known explicitly, see 4.21 and Salzmann [1967b], Section 4. See Betten [1990, 1991] for recent results towards a classification of flexible planes $\mathcal{P}$ with $\dim P = 4$. For $\dim P > 4$ it seems preferable to redefine flexibility as finiteness of the number of flag orbits, in order to avoid the multitude of planes of Lenz type V.

4.17. *Dimensions of automorphism groups.* Let $\mathcal{P} = (P, \mathcal{L})$ be a compact connected projective plane with automorphism group Σ. We denote by $\dim \Sigma$ the dimension of the topological space Σ, compare 3.21, 4.11. Clearly $\dim \Sigma$ is a natural measure for the size of Σ, similar to the order of a finite group, see 3.36, 3.37.

Let us look at the classical Moufang planes: the projective planes $\mathrm{P}_2\mathbb{F}$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ have automorphism groups $\Sigma = \mathrm{PGL}_3\,\mathbb{R}$, $\mathrm{PGL}_3\,\mathbb{C} \rtimes \mathbb{Z}_2$, $\mathrm{PGL}_3\,\mathbb{H}$, $\mathsf{E}_6(-26)$, respectively, with dimensions 8, 16, 35, 78, cf. 3.36.

All planes $\mathcal{P}$ where $\dim \Sigma$ is sufficiently large (about half of the value in the dominating classical case) have been classified, see 4.19.

4.18. *Stabilizers of quadrangles.* Let $\mathcal{P} = (P, \mathcal{L})$ be a compact connected projective plane of finite dimension $\dim P = 2l$. In order to exploit an assumption saying that $\dim \Sigma$ is large, one needs some information on the stabilizer $\Sigma_{o,u,v,e}$ of a quadrangle o, u, v, e in $\mathcal{P}$. This stabilizer is isomorphic to the automorphism group of the ternary field coordinatizing $\mathcal{P}$ with respect to o, u, v, e, cf. Pickert [1975], 1.24, p. 37. In the classical planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, these stabilizers are the compact groups 1, $\mathbb{Z}_2$, $\mathrm{SO}_3\,\mathbb{R}$, G_2, respectively, of dimensions 0, 0, 3, 14.

We conjecture that the stabilizer $\Sigma_{o,u,v,e}$ is always compact, and that its dimension is never larger than the corresponding classical value. These conjectures are closely connected with the question whether $\mathcal{P}$ can have totally disconnected closed subplanes; subplanes of this type have been ruled out only for $l \leqslant 2$, see Salzmann [1959a, 1969b]. A proof of these conjectures would simplify many arguments in connection with 4.19.

At present, we only have the following general information. If $l = 1$, then $\Sigma_{o,u,v,e}$ is trivial (Salzmann [1962a], 2.3). If $l = 2$, then $\Sigma_{o,u,v,e}$ contains at most 2 elements, see Salzmann [1969b]. If $l = 4$, then $\dim \Sigma_{o,u,v,e} \leqslant 4$, see Bödi [1992, 1993], Salzmann [1979a]. If $l = 8$, then $\dim \Sigma_{o,u,v,e} \leqslant 14$ by Salzmann [1979b]; in fact, by recent work of Bödi, either $\dim \Sigma_{o,u,v,e} \leqslant 11$, or the connected component $(\Sigma_{o,u,v,e})^1$ is isomorphic to G_2.

4.19. THEOREM. *Let $\mathcal{P} = (P, \mathcal{L})$ be a compact connected projective plane with automorphism group Σ.*

(a) *Let* $\dim P = 2$. *If* $\dim \Sigma > 4$, *then* $\mathcal{P} \cong \mathrm{P}_2\mathbb{R}$. *If* $\dim \Sigma = 4$, *then* $\mathcal{P}$ *is isomorphic to a Moulton plane. All planes* $\mathcal{P}$ *with* $\dim \Sigma \geqslant 2$ *are known.*

(b) *Let* $\dim P = 4$. *If* $\dim \Sigma > 8$, *then* $\mathcal{P} \cong \mathrm{P}_2\mathbb{C}$. *All planes* $\mathcal{P}$ *with* $\dim \Sigma \geqslant 7$ *are known*; *in fact, if* $\dim \Sigma \geqslant 8$, *then* $\mathcal{P}$ *is a translation plane.*

(c) *Let* $\dim P = 8$. *If* $\dim \Sigma > 18$, *then* $\mathcal{P} \cong \mathrm{P}_2\mathbb{H}$. *The planes* $\mathcal{P}$ *with* $\dim \Sigma = 18$ *form three families of planes over special semifields. All planes* $\mathcal{P}$ *with* $\dim \Sigma \geqslant 17$ *are known.*

(d) *Let* $\dim P = 16$. *If* $\dim \Sigma > 40$, *then* $\mathcal{P} \cong \mathrm{P}_2\mathbb{O}$. *If* $\dim \Sigma = 40$, *then* $\mathcal{P}$ *is the plane over a mutation of* $\mathbb{O}$, see 4.10.

This remarkable theorem summarizes the results of many papers. Part (a) is due to Salzmann, Strambach, Groh, Schellhammer, Lippert, and Pohl, see Salzmann [1967b] for a survey on planes with $\dim \Sigma \geqslant 3$, and references in Pohl [1990]. For part (b) see Salzmann [1973b,c], Betten [1973c, 1990], Knarr [1983, 1986], Löwen [1990], and for part (c) see Salzmann [1990] and the references given there, in particular Hähl [1978, 1984, 1986a]. Finally, for part (d) compare Salzmann [1987], Hubig [1990], Lüneburg [1992] and Hähl [1978, 1988, 1990]. See also 4.20 below. Many (but not all) assertions of 4.19 are proved in Salzmann et al. [1994], Chapters 3, 7, 8. Here we sketch the general strategy of the proof, and in 4.21 we describe some of the nonclassical planes appearing in 4.19, 4.20.

The assumptions imply that Σ is a Lie group (see 4.12). If the connected component Σ^1 is semisimple, hence an almost direct product (see 5.19) of quasisimple Lie groups, then one analyzes the involutions in these factors and their centralizers; usually Σ^1 turns out to be quasisimple, and the plane $\mathcal{P}$ can be reconstructed from the reflections in Σ^1. If Σ^1 is not semisimple, then Σ^1 has a minimal normal subgroup Θ which is a central torus group $\mathbb{R}/\mathbb{Z}$ or a vector group $\mathbb{R}^n$. In the second case, one studies the irreducible linear representation of Σ^1 on $\Theta \cong \mathbb{R}^n$; typically, Θ consists of translations, and n is rather large (because Σ^1 is large); this situation leads to translation planes. In fact, 4.19 relies on the following classification theorem for translation planes.

4.20. THEOREM. *Let $\mathcal{P} = (P, \mathcal{L})$ be a compact connected projective translation plane with automorphism group Σ.*

(a) *If* $\dim P = 2$, *then* $\mathcal{P}$ *is isomorphic to the real projective plane* $\mathrm{P}_2\mathbb{R}$.

(b) *If* $\dim P = 4$ *and* $\dim \Sigma \geqslant 7$, *then* $\mathcal{P}$ *is known* (*ten families of planes depending on up to five real parameters*).

(c) *If* $\dim P = 8$ *and* $\dim \Sigma \geqslant 17$, *then the possibilities for* $\mathcal{P}$ *are exhausted by the planes over Kalscheuer's nearfields* (4.10), *five one-parameter families of semifield planes* (*including the planes over mutations of* $\mathbb{H}$), *and a two-parameter family of proper translation planes.*

(d) *Let* $\dim P = 16$. *If* $\dim \Sigma \geqslant 38$, *then* $\mathcal{P}$ *is known. If* $\dim \Sigma \geqslant 40$, *then* $\mathcal{P}$ *is the plane over a mutation of* $\mathbb{O}$, cf. 4.10.

For part (a) see Salzmann [1957], §12. Part (b) is due to Betten [1972b, 1973a,b, 1975, 1976, 1977]. Note that there exist finite analogues of the two 4-dimensional translation planes with $\dim \Sigma = 8$ which appear in Betten [1973a], Theorem 3, Betten [1973b], see Chapter 5, 2.X. Statements (c), (d) summarize a large body of results of Hähl, see in particular Hähl [1978, 1986a, 1990]. Hähl's classification proceeds by considering the possibilities for the maximal compact subgroups of Σ, see Hähl [1978], 1.6, 1.7.

We remark that every compact projective translation plane of dimension 16 has kernel $\mathbb{R}$, see Buchanan and Hähl [1977].

4.21. *Nonclassical planes with large groups.* According to Salzmann [1981a], the Hughes planes constructed as in Chapter 4, 1.3, 2,3, from Kalscheuer's nearfields (see 4.10) are compact projective planes. Apart from special (dual) translation planes, these Hughes planes are the only compact projective planes of dimension 8 admitting a collineation group of dimension (at least) 17, see Salzmann [1990].

These planes have octonion analogues: Salzmann [1982a] constructs a one-parameter family of 16-dimensional compact projective planes $\mathcal{P}_\alpha$, $\alpha > 0$, which contain a subplane $\mathrm{P}_2\mathbb{H}$ such that all collineations of $\mathrm{P}_2\mathbb{H}$ extend to collineations of $\mathcal{P}_\alpha$. As shown by Hähl [1986b], these properties characterize the planes $\mathcal{P}_\alpha$. The collineation groups of $\mathcal{P}_\alpha$ have dimension 36.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be such that the point set $\mathbb{R}^{2n}$ with lines

$$\{(x, f(x-a)+b)\colon x \in \mathbb{R}^n\} \quad \text{and} \quad \{a\} \times \mathbb{R}^n,$$

where $a, b \in \mathbb{R}^n$, is an affine plane. Then this affine plane (and its projective completion) is called the *shift plane* generated by f, compare 4.9, Betten [1979a], Knarr [1988b]. Shift planes of this type generated by continuous functions f exist only for $n \leqslant 2$, see Weigand [1987], 5.4. By construction, every shift plane admits $\mathbb{R}^{2n}$ as a collineation group.

For $n = 1$, only the skew parabolas $f\colon \mathbb{R} \to \mathbb{R}$ with

$$f(x) = x^d \quad \text{for} \quad x \geqslant 0 \quad \text{and} \quad f(x) = c|x|^d \quad \text{for } x \leqslant 0,$$

generate shift planes with a group of larger dimension. For $c = 1, d = 2$ we obtain the parabola model of the plane over $\mathbb{R}$. In all other cases, these shift planes have collineation groups of dimension 3, see Salzmann [1965].

Knarr [1983] has found a remarkable shift plane of dimension 4, which is generated by the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (xy - x^3/3, y^2/2 - x^4/12).$$

Apart from special (dual) translation planes, this plane is the only compact projective plane of dimension 4 admitting a collineation group of dimension (at least) 7, see the references for 4.19(b). See also Betten [1979a], Knarr [1983, 1986], Betten and Knarr [1987], Salzmann et al. [1994], Section 74, for more results on shift planes.

Radial distortions of $\mathbb{C}, \mathbb{H}, \mathbb{O}$ lead to double groups and planes of Lenz–Barlotti type II.2, see Plaumann and Strambach [1974]; see also Knarr [1987b].

For further examples, see also 5.3, 5.10 and Betten [1984], Weigand [1987], Sperner [1990].

4.22. *Rigid planes.* One expects that an 'arbitrarily chosen' plane has a rather small automorphism group. Indeed, it is not difficult to construct many 2-dimensional compact projective planes that are rigid (i.e. whose automorphism group is trivial), see 4.9. In marked contrast, no rigid compact projective plane of higher dimension is known at present.

4.23. *Groups of projectivities.* Let $\mathcal{P}$ be a projective plane, and denote by Π the group of projectivities of a line onto itself. It is well known that classical planes $\mathcal{P}$ have rather small groups Π, see Chapter 2, 9.2. This is also true in the topological context:

4.24. THEOREM. *If $\mathcal{P}$ is a compact connected projective plane, then Π is locally compact with respect to the compact-open topology if and only if $\mathcal{P} \cong \mathrm{P}_2\mathbb{F}$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

This was proved by Strambach [1977]; see also Löwen [1977, 1981e].

For results concerning groups of affine projectivities (products of parallel projections) in affine planes, see Strambach [1977, 1986], Grundhöfer and Strambach [1986]. See also Betten [1979b, 1983a], Betten and Wagner [1982], Betten and Weigand [1985] for concrete descriptions of groups of projectivities in various planes.

5. Homogeneous stable planes

Homogeneous stable planes are certainly more difficult to classify than homogeneous compact projective planes (cf. 4.13, 4.14, 4.19). For one thing, there are more planes to be considered and more homogeneous ones to be classified. In addition, there are extra technical difficulties; e.g., ternary fields are not available as a tool for proving stiffness results like 4.18. On the other hand, in a proper stable plane there are some helpful extra constraints on the automorphism group; indeed, the automorphisms often behave as though there were a surrounding projective plane with an invariant set of ideal points. For example, the collineation group of the real hyperbolic plane (see 5.2) coincides with the hyperbolic motion group of the real projective plane.

The additional difficulties account for the fact that homogeneous stable planes in general are less thoroughly understood than the projective ones. In particular, there will be no subsection on 16-dimensional planes, for lack of material; some results on stiffness, however, will be stated together with those for 8-dimensional planes.

There is a particularly beautiful subject, naturally belonging to the realm of stable planes, namely, symmetric planes. It brings incidence geometry into close contact with differential geometry. After all, these two geometric theories have a common origin in the study of Euclidean and non-Euclidean geometries. In some way, symmetric planes are for stable planes what translation planes are for projective planes.

Results on planes of arbitrary dimension

5.1. *Transitivity properties.* We say that a stable plane $S = (P, \mathcal{L})$ is *flag-homogeneous* if the automorphism group $\Sigma = \mathrm{Aut}\, S$ is transitive on the *flag space* I, i.e. on the graph of the incidence relation. Similarly, the plane is said to be *point-homogeneous* or *line-homogeneous* if Σ is transitive on P or on $\mathcal{L}$, respectively. A point p is called *isotropic* if the stabilizer Σ_p is transitive on the line pencil $\mathcal{L}_p$. Isotropic lines are defined dually.

We illustrate these notions by describing some particularly homogeneous examples that are contained as open subplanes (see 2.7) in the classical projective planes $P_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. They can be obtained using the orbits of unitary groups acting on $P_2\mathbb{F}$, and they will provide the main examples of symmetric planes as defined in 5.27.

5.2. *Hermitian planes.* First consider a division ring $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let f be a sesquilinear form on $\mathbb{F}^3$ which is either symmetric ($f(y,x) = f(x,y)$; then $\mathbb{F} \neq \mathbb{H}$) or Hermitian or skew-Hermitian, ($f(y,x) = \pm\overline{f(x,y)}$). We do not assume that f is nondegenerate. We define an open set M_f of points of $P_2\mathbb{F}$ by taking all one-dimensional subspaces $\langle x\rangle \leqslant \mathbb{F}^3$ such that $f(x,x) > 0$ (for Hermitian or real symmetric forms f) or such that $f(x,x) \neq 0$ (for complex symmetric forms and for quaternion skew-Hermitian forms). By 2.7, the set M_f defines a stable plane $S_f = (M_f, \mathcal{L}_f)$. This plane is point-homogeneous, since, by a theorem of Witt, the unitary group $\mathrm{PU}(f)$ operates transitively on the set M_f. All these planes will be called *Hermitian planes*, and this denomination is also extended to the planes with point set $M_f \cup M_{-f}$, which may or may not be point-homogeneous. Note that we distinguish between symmetric and (skew-)Hermitian forms f; nevertheless, we use the terms *Hermitian plane* and *unitary group* in all cases.

By well-known classification theorems for sesquilinear forms, it suffices to consider symmetric forms

$$f(x,y) = x_1a_1y_1 + x_2a_2y_2 + x_3a_3y_3$$

and Hermitian forms

$$f(x,y) = x_1a_1\overline{y_1} + x_2a_2\overline{y_2} + x_3a_3\overline{y_3}$$

on $\mathbb{F}^3$, with $a_j \in \{0, 1, -1\}$, as well as skew-Hermitian forms

$$f(x,y) = x_1a_1\overline{y_1} + x_2a_2\overline{y_2} + x_3a_3\overline{y_3}$$

on $\mathbb{H}^3$, with $a_j \in \{0, \mathrm{i} = \sqrt{-1}\}$. A detailed discussion of the various cases can be found in Löwen [1979a], Section 2, Löwen [1983a], 1.1, Löwen [1982a], 2.1, Stroppel [1993f], Section 4. Here, we merely give a brief description.

The inequality $x_1\overline{x_1} + x_2\overline{x_2} + x_3\overline{x_3} > 0$ defines the entire plane $S_f = P_2\mathbb{F}$. The corresponding group $\mathrm{PU}(f)$ is the *elliptic motion group* $\mathrm{PU}_3\mathbb{F}$; it is a maximal compact connected subgroup of the full automorphism group of $P_2\mathbb{F}$ and acts flag-transitively. Viewed in this context, $P_2\mathbb{F}$ is called the *elliptic plane*.

Using the inequality $f(x) = x_1\overline{x_1} - x_2\overline{x_2} - x_3\overline{x_3} > 0$, we obtain the *interior hyperbolic plane* $S_f = \mathrm{IH}(\mathbb{F})$. It contains no projective lines in the sense of 3.9. The converse inequality $f(x) < 0$ defines the *exterior hyperbolic plane* $S_{-f} = \mathrm{EH}(\mathbb{F})$. It contains projective lines, coming from the lines of the projective plane that miss the hyperbolic quadric $\{x\colon f(x) = 0\}$. The corresponding unitary group $\mathrm{PU}(f) = \mathrm{PU}(-f)$ is the *hyperbolic motion group* $\mathrm{PU}_3(\mathbb{F}, 1)$. The complement of the quadric is the *united* *hyperbolic plane* $S_f \cup S_{-f} = \mathrm{UH}(\mathbb{F})$. The interior hyperbolic plane is flag-homogeneous, but the exterior one is not even line-homogeneous, since it contains both projective and nonprojective lines.

The flag-homogeneous affine plane $\mathrm{A}_2\mathbb{F}$ (we also call it the *Euclidean plane*) can be defined by $x_1\overline{x_1} \neq 0$. Note that the unitary group of this form is the full automorphism group of the affine plane, except for the complex case, where it is the group of linear affinities. By contrast, the *Euclidean motion group* is defined as usual to be the group $\mathbb{F}^2 \rtimes \mathrm{SU}_2\mathbb{F}$ with its natural action on the affine plane $\mathbb{F}^2$.

The *complex oval plane* is the subplane $\mathrm{O}(\mathbb{C})$ of $\mathrm{P}_2\mathbb{C}$ defined by $x_1^2 + x_2^2 + x_3^2 \neq 0$, with group $\mathrm{PO}_3\mathbb{C}$, and the quaternion *skew-hyperbolic plane* $\mathrm{SH}(\mathbb{H}) \leqslant \mathrm{P}_2\mathbb{H}$ has the defining inequality

$$x_1 i\overline{x_1} + x_2 i\overline{x_2} + x_3 i\overline{x_3} \neq 0;$$

its group $\mathrm{PU}(f)$ is usually called an *anti-unitary group*. The *cylinder planes* $\mathrm{C}(\mathbb{F})$ are defined by $x_1\overline{x_1} - x_2\overline{x_2} > 0$. Note that $\mathrm{C}(\mathbb{R})$ is isomorphic to a real half-plane, obtained by cutting $\mathbb{R}^2$ along a line. There also is a *skew cylinder plane* $\mathrm{SC}(\mathbb{H})$, and there are united cylinder planes $\mathrm{UC}(\mathbb{F})$. The complement of a point (the *dual Euclidean plane* $\mathrm{DA}_2\,\mathbb{F}$) and the complement of two lines (the *Minkowski plane* $\mathrm{M}(\mathbb{F})$, $\mathbb{F} \neq \mathbb{H}$) also are Hermitian planes.

In the octonion plane, sesquilinear forms are not available to define Hermitian planes. However, the motion groups described so far have analogues in the automorphism group of the octonion plane. Their open orbits can be used to define Hermitian planes. The information needed on subgroups of $\mathrm{Aut}\,\mathrm{P}_2\mathbb{O}$ can be found in Tits [1953, 1954] or in Salzmann et al. [1994], Sections 15, 17, 18. In the 'nondegenerate' cases, motion groups can be obtained simply as centralizers of polarities, as for the Desarguesian planes. A satisfactory treatment of the 'degenerate' cases has not yet been given.

We consider some examples. The octonion hyperbolic motion group $\mathsf{F}_4(-20)$ has open orbits $\mathrm{IH}(\mathbb{O})$ and $\mathrm{EH}(\mathbb{O})$ with properties similar to the other hyperbolic planes. The automorphism group of the affine plane $\mathrm{A}_2\mathbb{O}$ contains a subgroup $\Phi_8 \cong \mathrm{Spin}_9\mathbb{R}$ fixing some affine point o and acting transitively on the line pencil $\mathcal{L}_o$, cf. 5.4 below. This group is well-defined up to conjugation and can be used to define the *Euclidean motion group* of the octonion plane as the group $\mathbb{O}^2 \rtimes \Phi_8$ with its natural action.

5.3. *Variations.* If $S_f = (M_f, \mathcal{L}_f)$ is a Hermitian plane, then we can construct another plane, also admitting the unitary group $\mathrm{PU}(f)$, by passing to the dual projective plane $\mathrm{P}_2{}^*\mathbb{F}$ and taking the subplane with point set $\mathcal{L}_f$. The resulting stable plane S_f' will be called a *dual Hermitian plane*, cf. Löwen [1982a], Stroppel [1993f]. Apart from the projective, dual Euclidean and exterior hyperbolic planes, which are both Hermitian and dual Hermitian planes, we obtain the dual (skew, united) cylinder planes and dual Minkowski planes, all of which are almost projective planes in the sense of 3.16. Note that the opposite plane (see 3.15) of S_f' is S_f.

There are several possibilities to apply local distortions to certain (2-dimensional) Hermitian planes, in a way compatible with the action of (some large part of) the unitary group. The following planes obtained in this way will be called *modified Hermitian planes*.

Salzmann [1962b] constructs modifications of the real projective plane compatible with the connected component of the hyperbolic motion group. These *modified real projective planes* $H_t(\mathbb{R})$ are obtained by replacing the exterior parts of the lines meeting the interior hyperbolic plane with suitable parts of conics; the construction depends on a real parameter $t > 0$. By passing to the subplanes induced on the point sets of the hyperbolic planes, we obtain the *modified real hyperbolic planes* $EH_t(\mathbb{R})$ and $UH_t(\mathbb{R})$, cf. Löwen [1979c].

The *modified real dual cylinder plane* $MDC(\mathbb{R})$ has been constructed in 3.19 by adding a line at infinity to the arc plane S defined by the curve $y = x^{-1}$, $x > 0$. By forming the disjoint union of S with $C(\mathbb{R})$ and identifying lines in a suitable way we obtain the *modified real cylinder plane* $MC(\mathbb{R})$, see Löwen [1981c]. This plane cannot be embedded as an open subplane into any larger stable plane.

5.4. *Isotropic points.* For the definition, see 5.1. As an example, consider the classical $2l$-dimensional affine plane $A_2\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. In its automorphism group (in fact, in the Euclidean motion group defined in 5.2), we can find a subgroup $\Phi_l \cong \mathrm{Spin}_{l+1}\mathbb{R}$ acting transitively on some line pencil. Here, $\mathrm{Spin}_{l+1}\mathbb{R}$ denotes the universal covering group of $SO_{l+1}\mathbb{R}$ for $l > 1$, and $\mathrm{Spin}_2\mathbb{R}$ is defined as $SO_2\mathbb{R}$ itself. For $\mathbb{F} \neq \mathbb{O}$, the group Φ_l can be obtained as $SU_2\mathbb{F}$ with its natural action on the affine plane. The existence of Φ_8 follows, e.g., from flag-transitivity of $\mathrm{Aut}\,P_2\mathbb{O}$ together with the following theorem, which is proved, partly in Löwen [1983a], and for the remaining part, in Salzmann et al. [1994], Section 62.

5.5. THEOREM. *Let $S = (P, \mathcal{L})$ be a locally compact, locally connected stable plane containing an isotropic point p. Then the following assertions hold.*

(a) *The lines of S are l-manifolds, $l < \infty$, and $\Sigma^1 = (\mathrm{Aut}\,S)^1$ is a Lie group.*

(b) *If $l > 1$, then Σ_p contains a subgroup $\Phi \cong \mathrm{Spin}_{l+1}\mathbb{R}$.*

(c) *The action of Φ on S closely resembles the action of its classical counterpart Φ_l described above. In particular, Φ is transitive on $\mathcal{L}_p$ and the centre of Φ is generated by a reflection $\sigma \in \Sigma_{[p]}$. If $l > 2$, then Σ contains a reflection at every line in $\mathcal{L}_p$. If S is projective, then the action of Φ on P is equivalent to the action of Φ_l on the point set of the classical projective plane of dimension $2l$.*

Conversely, if Σ_p contains a subgroup Φ locally isomorphic to $SO_{l+1}\mathbb{R}$, then p is isotropic, and $\Phi \cong \mathrm{Spin}_{l+1}\mathbb{R}$ acts as described above.

A general statement like assertion (b) does not hold for $l = 1$. On the one hand, the Moulton planes contain an isotropic point (the universal fixed point of the automorphism group) whose stabilizer does not contain $SO_2\mathbb{R}$. On the other hand, every point of the interior real hyperbolic plane is isotropic, and the point stabilizers are isomorphic to $O_2\mathbb{R}$.

5.6. *Isotropic points: Examples.* The Moulton planes (see 4.9) are examples of planes containing exactly one isotropic point. Further examples can be obtained by generalizing the description of the Moulton planes given by Betten [1972a], see Schellhammer [1981];

compare Salzmann et al. [1994], Section 34. Examples with $l = 2$ are given by Sperner [1990]. All these examples are projective planes.

A proper stable plane with precisely one isotropic point has been constructed by Strambach [1968]. One starts from the real affine plane and replaces all lines that do not pass through the origin with all images of the hyperbola $\{(x, x^{-1})\colon\ 0 < x \in \mathbb{R}\}$ under the group $\mathrm{SL}_2\mathbb{R}$. This plane will be referred to as *Strambach's* $\mathrm{SL}_2\mathbb{R}$*-plane*. It does not admit a *strong* (i.e., group-preserving) open embedding into any compact projective plane. However, Stroppel recently constructed a (weak) open embedding into some compact projective plane. Strambach's plane can also be embedded as a Baer subplane into a 4-dimensional plane with similar properties, admitting the group $\mathrm{SL}_2\mathbb{C}$ (Löwen [1986d], Stroppel [1993b]). There is, however, no 8-dimensional analogue of Strambach's plane (Stroppel [1990]).

In contrast, existence of two isotropic points in a stable plane has strong consequences.

5.7. THEOREM. *Let $S = (P, \mathcal{L})$ be a locally compact, locally connected stable plane containing at least two isotropic points. Then S is isomorphic to one of the following planes:*

(a) *the Hermitian planes* $\mathrm{A}_2\mathbb{F}$, $\mathrm{P}_2\mathbb{F}$, $\mathrm{IH}(\mathbb{F})$ *or* $\mathrm{UH}(\mathbb{F})$, *where* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, see 5.2, *or*

(b) *the modified real projective or modified united hyperbolic planes*, $\mathrm{H}_t(\mathbb{R})$ *or* $\mathrm{UH}_t(\mathbb{R})$, see 5.3.

The proof of this result (Löwen [1983a]) uses 5.5 in order to construct a flag-homogeneous open subplane, and to show that this subplane is a symmetric plane in the sense of 5.27. The underlying symmetric space is in fact *isotropic*, i.e. the point stabilizer Σ_p is transitive on the set of geodesics passing through p, not only on the line pencil $\mathcal{L}_p$. Then, classification theorems for isotropic symmetric spaces are applied.

5.8. COROLLARY. *Let $S = (P, \mathcal{L})$ be a locally compact, locally connected stable plane. If S is flag-homogeneous, then S is a projective, affine, or interior hyperbolic plane over* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

Isotropic lines are less useful than isotropic points unless they are projective lines, in which case they can be treated as isotropic points of the opposite plane, cf. 3.15. In this way, all locally compact, locally connected planes containing at least two projective isotropic lines can be determined; they are Hermitian planes or modifications as in 5.3, see Löwen [1983a].

5.9. *Point-homogeneous planes.* In contrast to the projective case (see 4.13), the class of point-homogeneous stable planes is definitely too large to be completely understood; cf. the remarks on the affine case after 4.13 and the examples 5.10 (arc planes).

According to Stroppel [1993f], each quaternion Hermitian plane S_f is characterized by the properties that it is point-homogeneous and admits an effective action of the unitary group $\mathrm{PU}(f)$. Compare also 5.12.

5.10. *Arc planes.* A rich class of 2-dimensional point-homogeneous planes, generalizing the affine shift planes 4.9, has been studied systematically by Groh; see Groh [1979, 1982b]. His planes are obtained from a suitable family of generating arcs (convex curves) in $A_2\mathbb{R}$. For each generating arc a, one replaces all lines parallel to some secant of a by all translates of a. Concrete examples are given in 5.14. A similar procedure works in the cylinder plane $C(\mathbb{R})$.

5.11. PROBLEM. *Line homogeneous planes.* It is conceivable that line-homogeneity is a much stronger condition than point-homogeneity. This belief is supported by the classification of line-homogeneous affine planes (see 4.14). The proof of this result rests on the existence of a compact group Φ as considered in connection with isotropic points (see 5.5). The example of translation groups shows that a point-transitive group can be homotopically trivial, and hence need not contain any nontrivial compact subgroups. If the conjecture about the homotopy of $\mathcal{L}$ formulated after 3.30 could be proved, then it would follow that a line-transitive group cannot be homotopically trivial. For the 2-dimensional case, see Salzmann [1967b], 4.15 and 3.32.

By 3.29, the following result concerns all stable planes of finite positive dimension. The proof is given in Löwen [1986b], using 5.5 and 5.8. Note that the connected component $PSO_3(\mathbb{R}, 1)^1$ of the real hyperbolic motion group can also act on the modified hyperbolic planes, since it does not contain the reflections at interior lines.

5.12. THEOREM. *Let* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ *and* $l = \dim \mathbb{F}$. *If S is a locally compact, $2l$-dimensional stable plane, and if* $\operatorname{Aut} S$ *contains a closed subgroup Δ isomorphic to the elliptic, hyperbolic, or Euclidean motion group of* $P_2\mathbb{F}$ (see 5.2), *then S is isomorphic to one of the Hermitian planes* (5.2) *invariant under this group, i.e. to* $P_2\mathbb{F}$, $A_2\mathbb{F}$, $DA_2(\mathbb{F})$, $IH(\mathbb{F})$, $EH(\mathbb{F})$ *or* $UH(\mathbb{F})$. *Up to duality, the action of Δ is the standard one.*

For compact groups, the hypothesis of the last theorem can be replaced by a rather weak dimension estimate; in fact, Stroppel [1993d] obtains the following result, essentially by reduction to the classification of point-homogeneous projective planes (see 4.13), using 3.11 and 3.25(c) among other things. For $l = 2$, the examples of Sperner [1990] show that the bound is sharp.

5.13. THEOREM. *Let S, $\mathbb{F}$ and l be as in the last theorem, and let* $\Delta \leqslant \operatorname{Aut} S$ *be a compact group such that* $\dim \Delta$ *is larger than the dimension of the point stabilizer of the elliptic motion group of* $P_2\mathbb{F}$, *i.e. larger than* $1, 4, 13, 36$ *for* $l = 1, 2, 4$ *or* 8, *respectively. Then Δ is isomorphic to the elliptic motion group, and hence S is isomorphic to* $P_2\mathbb{F}$.

To conclude this subsection on theorems valid for all dimensions, we quote the results of Stroppel [1993f, 1992c] that an Abelian group of automorphisms has dimension not exceeding $2l = \dim P$ (with additional information if the bound is attained), and that the dimension of a solvable group is bounded by $5l$. Moreover, we mention Stroppel's results on planar or quasiperspective groups (Stroppel [1992a] and 5.24, 5.25), and we refer to the characterization of symmetric planes by a transitivity condition (see 5.29) and to the characterization of almost projective planes (see 3.42).

2-dimensional planes

If S is a locally compact 2-dimensional stable plane, then $\operatorname{Aut} S$ is a Lie group, see the remarks after 3.38. It has strong stiffness properties, because the fixed points of an automorphism $\sigma \neq \mathbb{1}$ cannot form a subplane of positive dimension by 3.25(c) and 3.29. See Salzmann [1967b], 3.6, for a more specific result; compare also 4.18. Stiffness combined with the dimension formula (see 3.36) is the basic tool which is used in order to prove the following classification results. Many of these results were found by Strambach, and later reproved in a more general or more complete form.

By 3.32, the point space of a 2-dimensional proper stable plane with connected lines is homeomorphic to $\mathbb{R}^2$ or a Möbius strip. In both cases, the planes with a sufficiently high degree of homogeneity are known. For affine and projective planes, refer to 4.19.

5.14. THEOREM. *Let $S = (\mathbb{R}^2, \mathcal{L})$ be a stable plane with connected lines and with point set $\mathbb{R}^2$. Assume that S is not an affine plane. Then the following assertions hold.*

(a) *If* $\operatorname{Aut} S$ *is at least* 4*-dimensional, then S is isomorphic to the cylinder plane* $\mathrm{C}(\mathbb{R})$, *and* $\dim \operatorname{Aut} S = 4$.

(b) *The planes with* $\dim \operatorname{Aut} S = 3$ *are the following*:
  (1) *the interior hyperbolic plane* $\mathrm{IH}(\mathbb{R})$,
  (2) *Strambach's* $\mathrm{SL}_2\mathbb{R}$*-plane*, see 5.6,
  (3) *a* 3*-parameter family and a* 4*-parameter family of planes whose groups fix precisely one line, and finally*
  (4) *four arc planes* (5.10), *generated by the graphs of the following functions*:
    (i) x^s *for* $x > 0$, *where* $s \leqslant -1$, *or*
    (ii) x^s *and* rx^s *for* $x > 0$, *where* $r \leqslant -1$ *and* $s < 0$, *or*
    (iii) e^x *for* $x \in \mathbb{R}$, *or*
    (iv) e^x *and* $-\operatorname{sgn}(s)e^{sx}$ *for* $x \in \mathbb{R}$, *where* $s \leqslant -1$ *or* $s > 1$.

PROOF. For (a) see Salzmann [1967b], Section 4. Part (b) was proved by Salzmann, Strambach, Ostmann, Betten, Groh, Lippert, and Pohl. See Strambach [1970b], Groh [1982a], Groh, Lippert and Pohl [1983] and references given in these papers. □

5.15. THEOREM. *Let $S = (P, \mathcal{L})$ be a stable plane with connected lines such that P is a Möbius strip. Then the following assertions hold.*

(a) *If* $\operatorname{Aut} S$ *is at least* 4*-dimensional, then S is an almost projective plane isomorphic to the dual cylinder plane* $\mathrm{DC}(\mathbb{R})$ *or to the dual Euclidean plane* $\mathrm{DA}_2\,\mathbb{R}$ (see 5.3 and 5.2), *or to a projective Moulton plane with its distinguished point deleted.*

(b) *If* $\operatorname{Aut} S$ *is* 3*-dimensional, then S is an almost projective plane in the sense of* 3.16, *or S is isomorphic to* $\mathrm{EH}_t(\mathbb{R})$ *or to one of the arc planes* (4i) *or* (4iii) *of* 5.14 *with points at infinity added. This includes the plane* $\mathrm{MDC}(\mathbb{R})$ *of* 5.3; cf. also 3.19.

PROOF. Betten [1968]. The planes (4i) have been overlooked in case 1 of the proof of his result 5.8. □

Of course, the almost projective planes that occur in part (b) of the last theorem are all known. Two-dimensional locally compact stable planes with a group of dimension $\geqslant 3$ were treated by Hubig [1987] without assuming that lines are connected; however, he did not consider groups locally isomorphic to $\mathbb{R} \times L_2$, where L_2 denotes the non-Abelian connected 2-dimensional Lie group. In view of the following result of Löwen [1983c], he only needed to consider solvable groups.

5.16. THEOREM. *Let S be a 2-dimensional locally compact stable plane and let $\Delta \neq 1$ be a semisimple* (*see* 5.19) *connected closed subgroup of* Aut S. *Then S is a Δ-invariant open subplane of one of the following planes, with the natural action of the specified group Δ.*

(i) *The real projective plane with* $\Delta \in \{\mathrm{PGL}_3\mathbb{R},\ \mathrm{PSO}_3\mathbb{R},\ \mathrm{PSO}_3(\mathbb{R},1)^1,\ \mathrm{SL}_2\mathbb{R}\}$.

(ii) *A modified real projective plane* $\mathrm{H}_t(\mathbb{R})$ (5.3) *with* $\Delta = \mathrm{PSO}_3(\mathbb{R},1)^1$.

(iii) *A projective Moulton plane with Δ isomorphic to the universal covering group of* $\mathrm{SL}_2\mathbb{R}$.

(iv) *Strambach's* $\mathrm{SL}_2\mathbb{R}$*-plane* (see 5.6) *with* $\Delta = \mathrm{SL}_2\mathbb{R}$.

5.17. THEOREM. *Let S be a 2-dimensional stable plane such that every point is the centre of a nontrivial central collineation. Then S is an almost projective plane obtained from* $\mathrm{P}_2\mathbb{R}$ *by deleting a closed subset of some line, or S is one of the hyperbolic or cylinder planes* $\mathrm{C}(\mathbb{R})$, $\mathrm{UC}(\mathbb{R})$, $\mathrm{IH}(\mathbb{R})$, $\mathrm{EH}(\mathbb{R})$, $\mathrm{UH}(\mathbb{R})$.

PROOF. Löwen [1984b]. □

Planes admitting many axial reflections have been classified, cf. 5.22.

5.18. *Locally Desarguesian planes.* According to Polley [1968, 1972a], a 2-dimensional locally compact stable plane is isomorphic to an open subplane of $\mathrm{P}_2\mathbb{R}$ if it has connected lines and satisfies the Desargues condition locally. In fact, the triply degenerate Desargues condition suffices, see Polley [1972b]. A similar result is proved in Bröcker [1971] for geometries which locally are stable planes but may have multiple joins for distant points.

Polley's first step is to apply Busemann's theory of G-spaces, interpreting lines as geodesics. He obtains that a locally Desarguesian plane is locally isomorphic to the real hyperbolic plane. This method does not work for planes of higher dimensions, but the analogous result might well be true. Next, Polley uses the fact that the hyperbolic plane is strongly embedded in the real projective plane (i.e. all automorphisms extend), in order to construct a global embedding of the given plane via a monodromy argument. The strong embedding property is valid for every $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, see Löwen [1982b]. The monodromy argument needs the hypotheses that lines are connected and that the point set is simply connected (or some reduction to this case). If disconnected lines are allowed, then it is easy to construct planes which are locally but not globally Desarguesian; e.g., take the complement of the distinguished line in the usual affine description of a Moulton plane, or see Löwen [1979a], 4.10, Stroppel [1993a], 3.3, for examples with connected point sets. With the precautions mentioned, it might be possible to formulate and prove a general theorem.

4-dimensional planes

5.19. *Structure of the automorphism group.* Let S be a 4-dimensional locally compact stable plane. If $\Sigma = \operatorname{Aut} S$ has dimension $d \geqslant 5$, then the identity component Σ^1 is a Lie group, see the remarks after 3.38. The action of Σ is stiff in the sense that every nontrivial automorphism fixing every point of some nondegenerate and nondiscrete subplane is a Baer involution and cannot belong to Σ^1; moreover, every connected set of fixed points of a nontrivial automorphism $\sigma \in \Sigma^1$ is contained in some line, see Löwen [1978]. Compare also 4.18.

In general, a Lie group Δ can be decomposed as an almost semidirect product $\Delta = \Xi\Lambda$, where Ξ is the largest solvable normal subgroup and Λ is semisimple. *Almost semidirect* means that $\Xi\Lambda$ is isomorphic to a factor group of $\Xi \rtimes \Lambda$ modulo some discrete normal subgroup. The *solvable radical* Ξ is a closed subgroup. The *Levi complement* Λ, which is unique up to conjugacy, may not be closed. It is an almost direct product of a finite number of quasisimple Lie groups, where a Lie group Φ is said to be *quasisimple* if $\Phi \neq \mathbb{1}$ and every proper normal subgroup is discrete. All quasisimple Lie groups are known explicitly.

Let $d \geqslant 5$, so that Σ^1 is a Lie group. The solvable radical of Σ^1 is at most 10-dimensional (Stroppel [1992c]). A nontrivial Levi complement $\Phi \leqslant \Sigma^1$ is even quasisimple, and is in fact isomorphic to one of the quasisimple groups that act on $\mathrm{P}_2\mathbb{C}$, or possibly to a more than 2-sheeted covering group of $\mathrm{PSL}_2\mathbb{R}$ (Löwen [1978]).

For $A \in \mathcal{L}$ and $c \in P \setminus A$, the group $\Sigma_{[c,A]}$ (see 3.39) is isomorphic to some closed subgroup of the multiplicative group of complex numbers,see Seidel [1988].

Unlike the projective case, where the degree of homogeneity $d = \dim \Sigma$ has been used by itself as a means of classification, assumptions about d are usually coupled with other hypotheses when 4-dimensional stable planes are considered, see 5.20. It is known that for a plane S with point set homeomorphic to $\mathbb{R}^4$ and simply connected lines, $d \geqslant 10$ implies that $S \cong \mathrm{A}_2\mathbb{R}$, see Löwen [1976b]. Probably, the sharp bound is 9 rather than 10. The hyperbolic plane $\mathrm{IH}(\mathbb{C})$ has degree of homogeneity $d = 8$.

Most of the subsequent results depend on the theory of symmetric planes (see 5.27).

5.20. THEOREM. *Let S be a 4-dimensional locally compact stable plane with $\Sigma = \operatorname{Aut} S$. Assume that either $\dim \Sigma \geqslant 6$ and Σ contains a compact quasisimple group, or Σ contains a semisimple group of dimension > 3.*

Then S is isomorphic to one of eleven open subplanes of $\mathrm{P}_2\mathbb{C}$ invariant under the standard action of $\mathrm{SU}_2\mathbb{C}$ or $\mathrm{SO}_3\mathbb{R}$, or to one of two invariant subplanes of the $\mathrm{SL}_2\mathbb{C}$-plane mentioned in 5.6.

The proof is given in Löwen [1986a,d]. If Σ contains $\mathrm{SO}_3\mathbb{R}$, then the bound $d = \dim \Sigma \geqslant 5$ suffices. On the other hand, there are planes with $d = 5$ admitting the simply connected covering group $\mathrm{Spin}_3 = \mathrm{SU}_2\mathbb{C}$; examples have been constructed by Sperner [1990].

5.21. THEOREM. *Let S be a 4-dimensional locally compact stable plane with connected point set. Assume that every point is the centre of a nontrivial central collineation. Then S is isomorphic to one of the following.*

(a) *An almost projective translation plane, i.e. a projective translation plane with some closed subset of a translation axis deleted.*

(b) *A connected complex Hermitian plane* (see 5.2).

(c) *The exceptional* 3*-symmetric plane of Seidel* (see 5.33).

This is a remarkable result because of the distinguished position assigned to the exceptional 3-symmetric plane. The proof given by Seidel [1990b] first reduces the problem to the cases where every point is the centre of a central collineation of fixed order k. The reduction uses the techniques mentioned in 3.16 to obtain coaffine elements if the given central collineations depend too wildly on their centres. Roughly speaking, it turns out that they generate an almost transitive group. If only $k = \infty$ is possible, then again one obtains coaffine elements and almost projective planes, cf. 3.42. For $k < \infty$, one essentially has a k-symmetric plane (see 5.32) by a result analogous to 5.29. Finally, Seidel's classification of k-symmetric planes is used, see 5.34. The plane (c) appears in the case $k = 3$ only.

The preceding result yields a characterization of 4-dimensional compact projective translation planes. Moreover, $P_2\mathbb{C}$ is characterized as the only 4-dimensional compact projective plane such that every point is the centre of a nontrivial collineation of finite order.

If we use reflections in place of arbitrary central collineations, then the other assumptions of 5.21 can be relaxed; in particular, connectedness is not needed. According to Löwen [1981c], the locally compact stable planes of dimension 2 or 4, such that the set of all centres of reflections contains a nonempty open set of points, are precisely the almost projective translation planes, the real and complex Hermitian planes, and the modified Hermitian planes $H_t(\mathbb{R})$, $UH_t(\mathbb{R})$, $MC(\mathbb{R})$, see 5.3.

Löwen [1982a] proves the following assertions. If every line of a 2- or 4-dimensional locally compact stable plane is the axis of a reflection, then the plane is a real or complex Hermitian or dual Hermitian plane. The complex oval plane, among others, shows that the converse is not true. So in order to obtain a characterization of Hermitian and dual Hermitian planes, a weaker assumption about existence of axial reflections is needed. Such a condition can be formulated using the notion of *twin reflections*. An axial reflection $\sigma \in \Sigma_{[A]}$ is called a twin reflection if for every point $a \in A$, the unique fixed line $L_a \neq A$ of σ passing through a is the axis of some other reflection. L_a exists by the proof of 3.40. Now Löwen proves

5.22. THEOREM. *Let $S = (P, \mathcal{L})$ be a* 2*- or* 4*-dimensional locally compact stable plane. Then S is a real or complex Hermitian or dual Hermitian plane, see* 5.2 *and* 5.3, *if and only if the set of all axes of twin reflections of S contains a nonempty open subset of $\mathcal{L}$.*

8*-dimensional planes*

5.23. *Structure of the automorphism group.* Let $S = (P, \mathcal{L})$ be an 8-dimensional stable plane with automorphism group Σ. In contrast to the 4-dimensional case, we have no sufficiently general criterion ensuring that Σ is a Lie group. It is possible, however, to approximate Σ by Lie groups, i.e. to work with epimorphisms $\Sigma \to \Lambda$ with 0-dimensional kernels, where Λ is a Lie group.

According to Stroppel [1992c], the solvable radical of Λ has dimension 18 at most. For projective planes, it is known that 17 is the sharp bound, see Lüneburg [1992]. (For 16-dimensional stable planes, Stroppel obtains the bound 40, which probably is far from sharp.)

The semisimple part of an approximating Lie group Λ is not understood as thoroughly as in the 4-dimensional case. However, all sufficiently large quasisimple groups that can occur are known (Stroppel [1991]). First one remarks that for every quasisimple Lie group $\Phi \leqslant \Lambda$, the group Σ contains a subgroup Ψ locally isomorphic to Φ, see Löwen [1983a], 3.9, and Stroppel [1991], 2.16. The group Ψ admits a Lie topology, which may be finer than the topology induced by Σ; still, this Lie topology makes Ψ a transformation group of P and $\mathcal{L}$. In the sequel, Ψ will be considered in this topology.

Now Stroppel shows that S is isomorphic to some open subplane of the quaternion plane $P_2\mathbb{H}$ in each of the following cases:

(1) if $\dim \Psi > 16$ (Stroppel [1991]),

(2) if Ψ is compact and $\dim \Psi > 10$ (Stroppel [1993d]),

(3) if Ψ is locally isomorphic to the 15-dimensional anti-unitary group of $\mathbb{H}^3$ (Stroppel [1991]), or

(4) if Ψ is isomorphic to the 15-dimensional group $SL_2\mathbb{H}$ (Stroppel [1990]).

On the other hand, the 16-dimensional group $SL_3\mathbb{C}$ acts on the compact Hughes planes over the Kalscheuer nearfields, see 4.21, and leaves some proper open subplane invariant. There are no other 8-dimensional stable planes admitting an effective action of $SL_3\mathbb{C}$ (Stroppel [1993e]).

A closed subgroup $\Delta \leqslant \operatorname{Aut} S$ is said to be *planar* if its fixed points form a nondegenerate subplane of positive dimension. The 3-dimensional group $\operatorname{Aut}\mathbb{H} \cong SO_3\mathbb{R}$ acts on the projective plane $P_2\mathbb{H}$ as a planar group. This shows that we cannot expect the stiffness properties of Σ to be as simple as in the 4-dimensional case, where a planar group has order at most 2. The following result of Stroppel [1992a] shows that the group $\operatorname{Aut}\mathbb{H}$ represents the worst case.

5.24. THEOREM. *A planar group Δ of automorphisms of an* 8*-dimensional locally compact stable plane is at most* 3*-dimensional.*

More generally, Stroppel shows that $\dim \Delta + \dim F \leqslant 5$ if Δ is a group whose fixed point set F is a nondegenerate, nondiscrete proper subplane. He also proves an analogous result for 16-dimensional planes; in particular, $\dim \Delta \leqslant 14$ for planar groups. Compare 4.18 for the projective case.

Following Baer, a subgroup $\Delta \leqslant \operatorname{Aut} S$ will be called *quasiperspective* if every orbit of Δ on the point set is contained in some line. Stroppel [1993d] remarks that the centralizer of a nontrivial quasiperspective group contains rather large planar subgroups. Moreover, Stroppel [1993c] proves the following result, which is valid for planes of arbitrary finite dimension.

5.25. THEOREM. *Let Δ be a quasiperspective group of automorphisms of a 2l-dimensional locally compact stable plane. Then* $\dim \Delta \leqslant 3l$. *If, moreover, Δ fixes all points in some nonempty open subset of some line, then* $\dim \Delta \leqslant l$.

Symmetric planes

5.26. *Symmetric spaces.* For our purposes, it is best to define a connected *symmetric space* as a connected manifold M endowed with a transitive and effective action of a Lie group Δ and with an involution $\sigma_o \in \Delta$ which has an isolated fixed point $o \in M$ and centralizes the stabilizer Δ_o. For $\delta \in \Delta$, the involution $\sigma_x = \delta\sigma_o\delta^{-1}$ is called the *symmetry* at $x = o^\delta$. We may assume that Δ is generated by all conjugates σ_x of σ_o. In this case, Δ is called the *motion group* of M. Sometimes, this name is used for the subgroup of index two generated by all products of two symmetries. For specialists, we remark that we consider affine (as opposed to Riemannian) symmetric spaces, i.e. we merely require an invariant affine connection instead of an invariant metric. The Riemannian symmetric spaces are those with a compact stabilizer Δ_o. An interesting axiomatic definition has been given by Loos; it includes disconnected symmetric spaces. For references to the literature, and for a quick introduction based on the theory of Lie groups, see Löwen [1979a].

Symmetric spaces behave very much like Lie groups. There is a substitute for the Lie algebra, the so-called *Lie triple system* or *infinitesimal model* of M. It consists of a trilinear multiplication, defined on the tangent space T_oM, and it governs the local structure of the symmetric space M. There is a structure theory just as for Lie groups (cf. 5.19): e.g., T_oM splits into a solvable radical and a Levi complement, symmetric subspaces correspond to subsystems, etc.

5.27. *Symmetric planes.* A stable plane $S = (P, \mathcal{L})$ is called a *symmetric plane* if the point space P is a symmetric space whose motion group Δ is contained in $\operatorname{Aut} S$. Then the symmetry σ_x is a reflection at x by 3.40. Thus, x is uniquely determined by σ_x, and the stabilizer Δ_x is equal to the centralizer of σ_x. Since every line L is invariant under all symmetries at points $x \in L$, a theorem of Loos implies that L is a symmetric subspace of P, see Löwen [1979a]. Hence, T_oL is a sub-Lie-triple system of T_oM for $o \in L$. Moreover, Löwen proves that the subsystems T_oL for all lines $L \in \mathcal{L}_o$ form a compact spread in T_oM and define a topological translation plane (see 4.10). The Lie triple system together with this spread is called the *infinitesimal model* T_oS of the symmetric plane S (or the *tangent translation plane*); it determines S up to local isomorphism, often even up to global isomorphism.

5.28. PROBLEM. *Integration of infinitesimal symmetric planes.* It would be very helpful if one could prove that conversely, every Lie triple system endowed with a compact spread consisting of subsystems is the infinitesimal model of some (unique?) symmetric plane. However, it is not even clear how one should choose the right symmetric space among those having the given triple system. Note added in proof: Löwe has a counterexample.

5.29. THEOREM. *Let $S = (P, \mathcal{L})$ be a locally compact, locally connected stable plane, and let $\Delta \leqslant \operatorname{Aut} S$ be a closed subgroup. If Δ acts transitively on P and contains a unique reflection σ_x at each point x, then S is a symmetric plane, and its motion group is a closed normal subgroup of Δ.*

See Löwen [1979a], 2.1, and note that Δ is a Lie group, cf. 3.38.

5.30. *Symmetric planes: Examples.* The following examples are obtained by applying the last theorem. Every locally compact, connected affine translation plane is a symmetric plane in a trivial way; the motion group is the extension of the translation group by a reflection, and the Lie triple product is the zero product. Note that the motion group of the symmetric plane $A_2\mathbb{F}$ is not the Euclidean motion group defined in 5.2.

The Hermitian planes (see 5.2) are nontrivial examples of symmetric planes. This follows from existence and uniqueness of Hermitian reflections together with the transitivity properties of the unitary group. The motion groups are determined in Löwen [1979a]. For the elliptic and hyperbolic planes, we obtain the motion groups defined in 5.2.

No other examples exist in dimensions 2 and 4, see 5.31 below. In dimension 8, Löwe recently found a one-parameter family of examples that do not embed in the quaternion plane. These examples are shear planes as described in 3.14; they can be obtained as open subplanes of the duals of Kalscheuer's nearfield planes (see 4.16).

5.31. THEOREM. *The only symmetric planes of dimension 2 and 4 are the Hermitian planes and the affine translation planes.*

PROOF. The proof mainly works with the infinitesimal model, see Löwen [1979b]. □

As the examples of Löwe show, it will probably be much more difficult to obtain a similar classification for dimension 8 or 16, yet Löwe has results in this direction.

Roughly speaking, symmetric planes of dimension 2 or 4 can be characterized by the property that every point is the centre of a reflection, cf. 5.17 and the remarks after 5.21. The method of proof has been described in connection with 5.21. The analogous question in dimensions 8 and 16 remains open.

5.32. *Generalized symmetric planes.* If we replace the symmetries of order 2 appearing in the definition of a symmetric space (see 5.26) by symmetries of prime order $k > 1$, then we end up with the definition of a *k-symmetric space* and of a *k-symmetric plane*. If we do not want to specify k, then we speak of *generalized symmetric spaces* and planes. See Seidel [1990a], 2.14 and 3.1, for some subtleties in the definition of generalized symmetric spaces. Of course, '2-symmetric' means the same thing as 'symmetric'. The methods available for generalized symmetric planes are only slightly less powerful than those for symmetric planes.

The real line admits no homeomorphism of finite order greater than 2. This carries over to central collineations of 2-dimensional stable planes, and thus it is not necessary to study 2-dimensional generalized symmetric planes.

5.33. *Generalized symmetric planes: Examples.* All complex Hermitian planes also carry a k-symmetric structure for every prime number $k > 2$, because there are complex numbers of multiplicative order k. The kernel of a 4-dimensional proper translation plane is isomorphic to $\mathbb{R}$, so that the affine plane has no central collineations of finite order > 2. Hence, by 5.31, the planes of dimension 2 or 4 that carry a 2-symmetric structure, but no k-symmetric structure for $k > 2$, are precisely the 2-dimensional Hermitian planes and the (4-dimensional) non-Desarguesian translation planes.

Seidel [1991] constructs an additional example of a 3-symmetric plane, which we call the *exceptional* 3-*symmetric plane*; it is an open subplane of the dual translation plane with irreducible $\mathrm{SL}_2\mathbb{R}$-action found by Betten [1973b]. In this way, Seidel gave the first global description of the underlying 3-symmetric space, which had previously been known only in the shape of an infinitesimal model. This plane belongs to the class of shear planes that was later discovered by Löwe (see 3.14), and it does not admit a 2-symmetric structure. In fact, Seidel [1991] proves the following result. The method of proof is similar to that of 5.31.

5.34. THEOREM. *The only locally compact, connected,* 4-*dimensional stable plane that carries a k-symmetric structure for $k > 2$, but not a* 2-*symmetric structure, is Seidel's exceptional* 3-*symmetric plane with $k = 3$.*

Generalized symmetric planes of dimension 8 or 16 have not been studied. H. Klein [1993] investigates symmetric stable n-spaces. He characterizes them in a manner analogous to 5.29 and starts a classification programme. Compare also 7.6 for a related notion.

6. Generalized polygons and buildings

For definitions of generalized polygons and buildings, see 9.10 and Chapters 11, 12. We would like to mention some recent results on topological generalized polygons and topological buildings. These results suggest that it might be possible to embed the theory of topological projective planes (spaces) into a theory of topological generalized polygons (spherical buildings). At present, we have only a glimpse of this more general theory, but the area is active and developing rapidly.

6.1. *Definition of topological generalized polygons.* Such a definition should require that suitable geometric operations be continuous with respect to given topologies on the set of points and on the set of lines. For generalized 3-gons, i.e. projective planes, there is general agreement that joining of points and intersection of lines are the right geometric operations, compare 4.1. For generalized quadrangles $(P, \mathcal{L}, I)$, with incidence relation I, continuity of the mapping

$$(P \times \mathcal{L})\backslash I \to P \times \mathcal{L}, \ (p, L) \mapsto (q, M), \qquad \text{with } p\,I\,M\,I\,q\,I\,L$$

appears to be the correct requirement, see Forst [1981], Grundhöfer and Knarr [1990], Jäger [1994]. For generalized n-gons, one can find a definition in Grundhöfer and Van

Maldeghem [1990], and another definition in Jäger [1994]; the latter definition requires continuity of projections and is possibly stronger for $n \geqslant 5$. In the compact case, one can use the following very simple definition:

Let $n \geqslant 3$. A *compact* (*connected*) *n-gon* is a generalized n-gon $(P, \mathcal{L}, I)$ with compact (connected) topologies on P and on $\mathcal{L}$ such that I is closed in $P \times \mathcal{L}$.

This implies continuity of various geometric operations, see Grundhöfer and Van Maldeghem [1990], 2.1. See also 6.4.

The following remarkable result, due to Knarr and Kramer, is a topological analogue of the Feit–Higman theorem 2.1 in Chapter 9. See also the Tits–Weiss theorem in Chapter 9, Section 10.

6.2. THEOREM. *Let $(P, \mathcal{L}, I)$ be a compact connected n-gon such that the point space P has finite topological dimension* (see 3.21). *Then $n \in \{3, 4, 6\}$. Furthermore, P, $\mathcal{L}$ and I are generalized manifolds, and the point rows and line pencils are generalized manifolds which are homotopy equivalent to spheres of dimension p and q, respectively, with the following restrictions on p and q:*

if $n = 3$, then $p = q \in \{1, 2, 4, 8\}$;
if $n = 4$ and if $p, q \geqslant 2$, then $p + q$ is odd or $p = q \in \{2, 4\}$;
if $n = 6$, then $p = q \in \{1, 2, 4\}$.

Most of these assertions were proved by Knarr [1990] under the additional hypothesis that point rows and line pencils are topological manifolds; the version above can be found in Kramer [1994]. The hypothesis $\dim P < \infty$ is possibly superfluous; no examples of compact n-gons with infinite topological dimension are known (for $n = 3$, cf. the remarks after 4.7).

The arguments of Knarr and Kramer draw mainly on algebraic topology. The contractibility properties of point rows and line pencils imply that these spaces are generalized manifolds homotopy equivalent to spheres (see Löwen [1983b]); thus P and $\mathcal{L}$ are generalized manifolds, too. The double mapping cylinder of the two projections $I \to P$, $I \to \mathcal{L}$ is another generalized manifold homotopy equivalent to a sphere and contains I as a 'hypersurface'. Now Münzner [1981] has classified the cohomology rings arising in these situations; his classification gives $n \in \{3, 4, 6\}$. The restrictions on p and q follow from results of Adams and Atiyah (for $n = 3$, see also 3.29).

6.3. *Compact connected quadrangles.* Many examples of compact connected quadrangles (4-gons) can be obtained as 'Lie geometries' of locally compact connected Laguerre planes, see Chapter 24 or Schroth [1992b, 1993a], Forst [1981], 5.8, 5.10; the point space of the quadrangle is the one-point compactification of the (suitably topologized) union of the sets of points and circles of the Laguerre plane. In fact, Schroth [1990] proves that, up to duality, every compact quadrangle with topological parameters $p = q = 1$ (as in 6.2) arises in this way from a locally compact Laguerre plane of topological dimension 2, see Chapter 24, 5.13. Recently, Schroth has also constructed compact connected quadrangles from Möbius or Minkowski planes. See also Löwen [1994] for a construction of elation generalized quadrangles from pseudo-ovals via Laguerre planes. The connection between generalized quadrangles and circle geometries also helps to classify compact connected quadrangles with large automorphism groups, as work of Schroth shows.

As an aside, we mention that symplectic quadrangles are characterized by the property that for every point p the derivation

$$\left(p^{\perp}, \{p^{\perp} \cap q^{\perp} \colon q \in p^{\perp}, q \neq p\}, \in\right)$$

is a projective plane, see Schroth [1992a], Lemma 9; see also Chapter 9, 8.2, for the finite case.

The compact connected Moufang quadrangles are enumerated in Grundhöfer and Knarr [1990]; apart from one pair of dual examples related to the exceptional real Lie group $\mathsf{E}_6(-14)$, all of them arise from sesquilinear forms of Witt index 2 on real, complex or quaternion vector spaces.

Using representations of real Clifford algebras, Ferus, Karcher and Münzner [1981] show that many spheres contain isoparametric hypersurfaces with precisely 4 principal curvatures. According to Thorbergsson [1992], each of these hypersurfaces is the flag manifold of a compact connected quadrangle; the point space and the line space are obtained as focal manifolds, cf. also 6.7 below. This construction is most interesting because it yields many compact connected quadrangles which are not Moufang quadrangles. In fact, some of these examples have 'topological parameters' p, q (as in 6.2) different from those of the compact connected Moufang quadrangles; other examples have parameters p, q like a Moufang quadrangle without being homeomorphic to that Moufang quadrangle, see Wang [1988], Theorem 1. This means that 'classical domination' as explained in Sections 3, 4.8 does not hold for compact connected quadrangles: there are many compact connected quadrangles which are quite unlike any Moufang quadrangle, even if only the topology is considered. In addition, there exist non-Moufang examples with automorphism groups acting transitively on the point sets (see Ferus et al. [1981], 6.4); thus the analogue of 4.13 for compact connected quadrangles is not true.

6.4. *Definition of topological buildings.* There are several possibilities to define topological buildings in general, see Burns and Spatzier [1987a], Kühne [1994], Jäger [1994]. For projective spaces, the definitions of Jäger and Kühne agree with the usual definition as given in 1.1; see also 6.1 for buildings of rank 2.

We consider a building of rank r as an incidence structure $\Delta = (V_1, \ldots, V_r; *)$, where V_i is the set of vertices of type i, and $*$ is the (symmetric) incidence relation on

$$V = \bigcup_{i=1}^{r} V_i,$$

see Chapters 3 and 11. The requirement of Burns and Spatzier [1987a] that V is a Hausdorff space such that $*$ is a closed subset of V^2 is too weak in general (one can replace the topology on V by an arbitrary finer topology). Only in the compact case that definition appears to be strong enough. Thus we may use the following definition:

Let $r \geqslant 2$. A *compact* (*connected*) building is a building $\Delta = (V_1, \ldots, V_r; *)$ with compact (connected) topologies on each set V_i such that $*$ is closed in the space

$$\left(\bigcup_{i=1}^{r} V_i\right)^2.$$

Compare also Mitchell [1988], §2, for a definition of topological buildings belonging to BN-pairs in topological groups.

For a compact building Δ, we denote by Σ the group of all automorphisms (continuous collineations) of Δ. The following two theorems have been proved by Burns and Spatzier [1987a].

6.5. THEOREM. *Let Δ be an irreducible spherical compact metrizable building. Then the automorphism group Σ of Δ is locally compact in the compact-open topology.*

This generalization of 3.34 is a basic tool for classifications of homogeneous buildings; it is used in the proof of the following result.

6.6. THEOREM. *Let Δ be an irreducible spherical compact metrizable building which is infinite and locally connected. Assume that the automorphism group Σ of Δ acts strongly transitively* (see Chapter 11, 4.3.1) *on Δ. Then the connected component Σ^1 of Σ is a simple noncompact real Lie group with trivial centre, and Δ is the classical (Moufang) building associated with Σ^1 via parabolic subgroups.*

The transitivity condition in this theorem is equivalent to the 'topological Moufang condition' of Burns–Spatzier; it means that the action of Σ on Δ is given by a Tits system (BN-pair) in Σ, see Chapter 11, 4.3.3. For a generalized n-gon Δ, that transitivity condition is equivalent to the transitivity of Σ on the set of ordered ordinary n-gons in Δ.

Let M be a complete Riemannian manifold of bounded nonpositive sectional curvature and finite volume. A theorem of Ballmann says that M is locally symmetric, provided that M is irreducible and has rank at least 2. Burns and Spatzier [1987b] give a new proof for this theorem, by attaching a building to M and applying 6.6.

The theory of isoparametric submanifolds provides another link between differential geometry and topological buildings. Relying on the local approach to buildings (see Chapter 11, Section 6), Thorbergsson [1991] proved the following fundamental result:

6.7. THEOREM. *Let M be an irreducible compact full isoparametric submanifold of codimension $r \geqslant 3$ in $\mathbb{R}^m$. Then suitable focal manifolds V_i of M form an irreducible spherical compact building Δ of rank r.*

Together with 6.6 this leads to a classification: M is isometric to a principal orbit of the isotropy representation of some Riemannian symmetric space, see Thorbergsson [1991].

At present, it is not known whether 6.7 holds for isoparametric submanifolds M of codimension $r = 2$. The known examples show that a classification of these manifolds M will be more complicated, see 6.3.

The algebraic loop group of a compact symmetric space contains a natural BN-pair; the corresponding affine building allows to derive various results of Bott and Samelson on the topology of loop spaces of compact symmetric spaces, see Mitchell [1988].

See also Chapter 20, Section 7.

7. Some related topics

We supplement our discussion by mentioning a few subjects, which belong to the general area of linear topological geometries, but do not naturally fit into any of the preceding sections.

7.1. *Ovals, ovoids and polarities.* Topologically closed ovoids do not exist in complex projective spaces $\mathrm{P}_n\mathbb{C}$ with $n \geqslant 3$ nor in quaternion projective spaces $\mathrm{P}_n\mathbb{H}$; furthermore, topologically closed ovals do not exist in compact projective planes of dimension 8 or 16, see Buchanan, Hähl and Löwen [1980] for a thorough treatment. All closed ovals in the complex projective plane $\mathrm{P}_2\mathbb{C}$ are conics; this surprising topological analogue of a combinatorial result of B. Segre (see Chapter 4, Theorem 2.1) was proved by Buchanan [1979c]. For homogeneous ovals, see Groh [1971], Löwen [1984a].

Derivation of a (locally compact connected) circle plane leads to (compact connected) projective planes endowed with many (closed) ovals. Therefore, the results mentioned above have strong consequences for topological circle planes, see Chapter 24, Section 2.4. See also Löwen [1994] for pseudo-ovals and their relations with elation Laguerre planes and elation generalized quadrangles.

Some ovals in projective planes arise as the sets of absolute points of polarities. Note that every shift plane (see 4.21) admits a polarity which defines an oval. See Salzmann [1966], Bedürftig [1974a,b], Buchanan [1979b], 9.24, Buchanan et al. [1980] for results on polarities.

7.2. *Differentiable and algebraic planes.* A real (or complex) differentiable projective plane is a projective plane with real (or complex) differentiable structures on the point set and on the line set such that joining and intersecting are real (or complex) differentiable maps.

According to Breitsprecher [1967b], [1971], 1.5.7, the complex projective plane $\mathrm{P}_2\mathbb{C}$ is the only complex differentiable projective plane. See Betten [1971] for examples of two-dimensional compact projective planes which are not real differentiable.

The conjecture that the four classical planes over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only real differentiable projective planes was recently settled by Otte [1993]: the conjecture is true for the special case of translation planes (see also Grundhöfer and Hähl [1990]), but it is false in general. Otte constructs many counterexamples by locally disturbing the classical ternary operations $ax + b$ of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ in a differentiable fashion.

For connections between differential geometry and stable planes, see the subsection on symmetric planes in Section 5.

Strambach [1975b] considers affine (or projective) planes $(P, \mathcal{L})$, where P consists of the algebraic points of an algebraic variety over an algebraically closed field F. He assumes that the lines are subvarieties, and that all parallel projections (in affine parts) of $(P, \mathcal{L})$ are birational maps. These assumptions imply that $(P, \mathcal{L})$ is algebraically and geometrically isomorphic to the Pappian affine (or projective) plane over F. Similar results hold for Möbius and Laguerre planes.

7.3. *Uniform structures.* For the notion of uniform projective spaces and pertaining results see Breitsprecher [1967a], Szambien [1981, 1988, 1989], Kalhoff [1987d], Liepold and Prieß-Crampe [1989].

7.4. *Ordered planes.* For ordered space geometries, see 2.14, Szambien [1988]. Here we restrict attention to ordered planes; this still is a huge topic, see 3.31 and Prieß-Crampe [1983], Kalhoff and Prieß-Crampe [1990], Tecklenburg [1992]; see also Karzel and Kroll [1988], Chapter II.

Every ordered projective plane is a topological projective plane with respect to the order topology, see Wyler [1952]. According to Joussen [1966, 1981], every (finitely generated) free projective plane admits an (Archimedean) ordering. If an ordered projective plane is Archimedean, then its completion is a two-dimensional compact projective plane, see Prieß-Crampe [1967, 1983]. For further results on Archimedean orderings, see also Kalhoff [1986a,b, 1987a,b,c].

Every stable plane with lines homeomorphic to $\mathbb{R}$ has a unique ordering, which can be described in terms of the incidence (essentially by using Pasch configurations), see Menger [1971], Skala [1992]. As a consequence, every collineation of such a plane is continuous.

The Artin–Schreier characterization of ordered fields and the theory of Witt rings have been extended to ternary fields and projective planes, see Kalhoff [1989a,b, 1990, 1992]. A basic ingredient for this is Kalhoff's concept of a radical of a ternary field.

For various weaker notions of ordering, due to Sperner, Karzel, Junkers, Misfeld and others, refer to Karzel and Kroll [1988], II, §4, and Tecklenburg [1992].

7.5. *Hjelmslev planes.* For a definition of topological projective Hjelmslev planes, see the survey by Lorimer [1985], §§1,4 see also Chapter 19, Section 9, and Chapter 3, Section 5.2. Each plane $\mathcal{P}$ of this type is endowed with a continuous open epimorphism onto a topological (possibly discrete) projective plane $\mathcal{P}'$, see Lorimer [1981], 1.9, [1985], 4.21.

We consider locally compact connected projective Hjelmslev planes $\mathcal{P}$. For every $n \in \mathbb{N}$, there exist (Desarguesian) planes of this type with lines of dimension n, see Lorimer [1978a], §9; this is a major difference from stable planes (see 3.29). In fact, the classification of the Desarguesian planes $\mathcal{P}$ already is a formidable task, see Lorimer [1981, 1990, 1992]. According to Lorimer [1985], 7.12, Hjelmslev's classical planes over the topological rings $\mathbb{R}[x]/(x^n)$ are precisely the Pappian planes $\mathcal{P}$ with $\mathcal{P}' \cong \mathrm{P}_2\mathbb{R}$; see also Lorimer [1983] for topological characterizations of the plane over the dual numbers $\mathbb{R}[x]/(x^2)$.

Lorimer [1981], §6, proves an equivalence between a topological and a purely geometrical condition: a locally compact Desarguesian Hjelmslev plane is of finite level in the sense of Artmann if and only if its point space is either connected or totally disconnected (hence zero-dimensional).

Non-Desarguesian locally compact connected affine Hjelmslev planes are constructed in Baker, Lane and Lorimer [1988]. We remark that biternary rings are a basic tool for these investigations.

See Baker, Lane and Lorimer [1982] for the relation between order and topology in Hjelmslev planes.

7.6. *Lie–Johnsen groups.* We mention a related theory, which was initiated by Strambach [1976, 1985] and continued by Plaumann [1991]. It is close in spirit to Bachmann's geometry of reflections. In the category of Lie groups (Strambach) or algebraic groups (Plaumann), one considers Johnsen groups, which are defined by algebraic conditions on their conjugacy classes of involutions. These conditions express the ideas that all involutions are reflections either at points or at lines, that some products of two reflections at lines are reflections at points, etc. According to Strambach, the only connected semisimple Lie–Johnsen groups are the odd covering groups of $\mathrm{PSU}_3(\mathbb{C}, 1)$. For disconnected, or connected nonsolvable Lie groups, necessary and sufficient conditions characterizing Johnsen groups are given.

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CHAPTER 24

Topological Circle Geometries

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Contents

Introduction 1327
1. Benz planes and the embeddable models 1327
2. Topological Benz planes, topological ovals, and the classical Benz planes 1331
3. Möbius planes 1336
4. Minkowski planes 1339
5. Laguerre planes 1343
References 1351

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Introduction

The geometry of all circles on the real 2-sphere in 3-dimensional Euclidean space (i.e. the intersection of the sphere with Euclidean planes) has since long been investigated under various aspects, see, e.g., Wilker [1982], Beardon [1983]. Another model of the same geometry is obtained by stereographic projection from one point of the sphere onto a plane not passing through the point of projection. This leads to the geometry of Euclidean lines extended by an infinite point and Euclidean circles in the real plane. Similarly, one considers extensions of the geometry of Euclidean lines and Euclidean parabolae with given direction of axis or Euclidean hyperbolae with given direction of asymptotes. An algebraic description of these geometries leads to chain geometries, which have been treated in Chapter 14. Here we restrict ourselves to the planar situation and we take a more geometric point of view in this chapter; compare Chapter 6, Section 5. We give an axiomatization and thus a generalization of these geometries which leads to so-called circle planes. The basic definitions are given in Section 1. In the next section we impose topological conditions, which make up for the paucity of geometric conditions and which eventually allow us to classify the most homogeneous planes among these topological circle planes. Since affine and projective planes occur as derived incidence structures, the structure theory and classification of topological locally compact connected finite-dimensional affine and projective planes plays a crucial role. The additional circles give rise to ovals in these derived projective planes, which severely restricts the possible dimensions. In this regard the investigation of circle planes is the study of projective planes with sufficiently many ovals of a certain kind. Therefore, this chapter should be read in conjunction with Chapter 23. On the other hand, locally compact connected antiregular topological generalized quadrangles can be constructed from circle planes and topological locally compact connected circle planes can be studied entirely from this point of view. In the last three sections the three types of circle planes are considered separately in more detail.

1. Benz planes and the embeddable models

Benz planes comprise three types of geometries: Möbius planes, Laguerre planes, and Minkowski planes. These are fully treated from a geometrical point of view in Chapter 6, Section 5. The basic geometric objects are points and circles. Therefore Benz planes are also referred to as circle planes, but see the beginning of Section 3 for a special meaning of this term. We always consider circles as subsets of the collection of all points with incidence being inclusion.

1.1. DEFINITION. *Benz plane.* A *Benz plane* (or *circle plane*) consists of a set of points P, a set of at least two circles, $\mathcal{K}$, and a collection $\mathcal{E}$ of at most two equivalence relations on P, called *parallelisms*. We consider circles as subsets of P. We say that two points are nonparallel if and only if they are not in relation for any of the equivalence relations in $\mathcal{E}$. The identity relation is called a trivial parallelism. Three (or more) points are called *concircular* if they are on a common circle. For a Benz plane $\mathcal{B} = (P, \mathcal{K}, \mathcal{E})$ the following axioms must be satisfied.

(1) *Joining*: Three pairwise nonparallel points can be uniquely joined by a circle.

(2) *Touching*: To every circle $K \in \mathcal{K}$ and any two nonparallel points p, q, where $p \in K$ and $q \notin K$, there is precisely one circle L passing through q which touches K at p, i.e. $K \cap L = \{p\}$.

(3) *Parallel projection*: Parallel classes with respect to a nontrivial parallelism and circles intersect in a unique point.

(4) Parallel classes with respect to different nontrivial parallelisms intersect in a unique point.

(5) *Richness*: Each circle contains at least three points.

If $\mathcal{E}$ consists of two nontrivial parallelisms, written as $\parallel_+$ and $\parallel_-$ and called (+)- and (−)-parallelism, one obtains a *Minkowski plane*.

If $\mathcal{E}$ consists of one nontrivial parallelism $\parallel$, one obtains a *Laguerre plane*. Note that axiom (4) does not apply to Laguerre planes.

If $\mathcal{E}$ consists of the identity relation (= trivial parallelism) one obtains a *Möbius plane* (or *inversive plane*). Note that axioms (3) and (4) do not apply to Möbius planes and that nonparallel simply means distinct.

Note that in the case of Laguerre and Minkowski planes axiom (5) can be replaced by the weaker axiom that there is at least one circle that contains at least three points.

1.2. *Miquelian and embeddable Benz planes.* The *Miquelian* Möbius, Laguerre, and Minkowski planes are obtained as the geometries of nontrivial plane sections of an elliptic quadric, an elliptic cone with its vertex removed, or a ruled quadric, respectively, in 3-dimensional projective space over some field; algebraically one obtains the description as in Chapter 14, 3.6.6(i). Generalizing the notion of a conic, one defines an *oval* to be a subset of points $\mathcal{O}$ of a projective plane such that no line has more than two points in common with it and such that for each point p of $\mathcal{O}$ there is precisely one line, called the *tangent* at p, that meets $\mathcal{O}$ at p only; similarly, an *ovoid* is a subset of points of a 3-dimensional projective space such that no line has more than two points in common with it and such that the collection of all tangents at a point fills a plane, called the *tangent plane* at that point. Then the model for the Miquelian Möbius plane and the Miquelian Laguerre plane can be generalized to an *ovoidal* (or *egglike*) Möbius plane or *ovoidal* Laguerre plane where one respectively takes an ovoid or a cone over an oval instead of an elliptic quadric or an elliptic cone respectively. These ovoidal Möbius and Laguerre planes obviously comprise the Miquelian planes. Since they and the Miquelian Minkowski plane are embeddable in 3-dimensional projective space we call these planes *embeddable* Benz planes.

The Miquelian Benz planes are geometrically characterized by Miquel's configuration and can algebraically be represented as chain geometries (cf. Chapter 14, 7.3.1 and 7.3.2). The embeddable planes are geometrically characterized by the bundle condition (cf. Chapter 14, 7.3.7 and 7.3.9, see also Kahn [1980]) which plays an equivalent role as Desargues' theorem for projective planes. Both configurations in their generic form involve eight points and six circles. When the eight points are identified with the vertices of a cube, all four points on a face of the cube being pairwise nonparallel, then in Miquel's configuration one considers the six faces of the cube; if the points of five of the faces are concircular, then the points of the sixth face also are on a circle. In the bundle condition one similarly considers the six planes determined by two of the four parallel edges along a given perimeter of the cube; if the points of five of these planes are concircular, then the points of the sixth face also are on a circle.

1.3. DEFINITION. *Derived plane.* The *internal incidence structure* at a point p of a Benz plane consists of all points not parallel to p and, as lines, all circles passing through p and all parallel classes not passing through p. This is an affine plane, called the *derived affine plane* at p. We call its projective extension the *derived projective plane* at p.

The axioms of a Benz plane are equivalent to each internal incidence structure being an affine plane. A circle K not passing through the point of derivation p induces an oval in the derived projective plane by $(K \setminus |p|) \cup \Omega$, where Ω denotes the set of infinite points of lines that stem from parallel classes of the Benz plane and $|p|$ denotes the set of all points parallel to p. A Benz plane can thus be described in one derived projective plane $\mathcal{P}$ by the lines of $\mathcal{P}$ and a collection of ovals that intersect the infinite line L_∞ in Ω. For Möbius planes $\Omega = \varnothing$ and L_∞ is an exterior line to each such oval. For Laguerre planes $\Omega = \{\omega\}$ and L_∞ is a tangent line to each such oval. For Minkowski planes $\Omega = \{\omega_+, \omega_-\}$ and L_∞ is a secant line to each such oval.

The spatial description of an embeddable Benz plane as the geometry of plane sections of a quadratic set is related to the planar description in one derived plane by stereographic projection from one point of the quadratic set onto a plane not passing through the point of projection.

In a Benz plane one naturally has two kinds of pencils. Two nonparallel points determine the pencil of all circles passing through both points; an incident point-circle pair (p, C) determines the pencil of all circles through p that touch C at p (tangent pencil). There is a third kind of collections of circles, the so-called flocks. A flock is formed by a collection of pairwise disjoint circles that cover the whole point set, two points being excluded in the case of a Möbius plane (see Chapter 6, Section 5). Examples of flocks of embeddable Benz planes are obtained by taking the intersection with planes passing through an exterior line of the underlying quadratic set in 3-dimensional projective space. Flocks are important for finite Möbius and Laguerre planes but they have not been widely used in topological planes. In fact, flocks cannot exist in 4-dimensional Benz planes as there are no disjoint circles, cf. Section 2.4.

1.4. DEFINITION. *Automorphism.* An *automorphism* of a Benz plane is a permutation of the point set which maps circles into circles. In particular, parallel points are mapped to parallel points. The collection of all automorphisms of a Benz plane forms a group with respect to composition, called the *automorphism group*.

The automorphism group of a Benz plane with a nontrivial parallelism has a distinguished normal subgroup: for each nontrivial parallelism the collection of all automorphisms that map each point to one parallel to it forms a normal subgroup, called the *kernel* with respect to the parallelism.

Each automorphism of an embeddable Benz plane is induced by a collineation of the surrounding 3-dimensional projective space that leaves the underlying quadratic set (i.e. the point set of the Benz plane) invariant; cf. Mäurer [1967]. The automorphisms in a kernel are induced by central collineations. The Miquelian planes are the most homogeneous planes and have the 'largest' automorphism groups.

1.5. *Classification of Benz planes with respect to central automorphisms.* Similar to the Lenz–Barlotti classification of projective planes with respect to central collineations, Benz planes can be classified with respect to *central automorphisms*, i.e. automorphisms that fix at least one point and induce central collineations in the derived projective plane at that fixed point (compare Chapter 6, Section 5). This was carried out by Hering [1965], Kleinewillinghöfer [1979, 1980] and Klein and Kroll [1989] for Möbius planes, Laguerre planes, and Minkowski planes respectively; for similar results in this direction see Hartmann [1982a,b]. More precisely, one considers subgroups of central automorphisms which are linearly transitive, i.e. the induced groups of central collineations are transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis. For all three types of Benz planes, the Miquelian planes are those of highest type, where all admissible subgroups of central automorphisms are linearly transitive.

Hering studied two types of central automorphisms in Möbius planes: these are automorphisms that fix precisely one or two points (except the identity) and that induce a translation or homothety in the derived projective plane at each of these fixed points. This leads to 18 different possible types of Möbius planes, some of which are known to be empty, cf. Krier [1973] and Yaqub [1978]. Since the axis of an induced central collineation can stem from a circle or a parallel class of the Laguerre plane or be the infinite line, Kleinewillinghöfer studied three types of central automorphisms in Laguerre planes. Accordingly, the classification became more complicated and she obtained 45 different types of Laguerre planes. Embeddable Laguerre planes always admit all automorphisms that are induced by central collineation of the surrounding 3-dimensional projective space with centre being the vertex of the cone that defines the point set of the Laguerre plane. Hence embeddable Laguerre planes have type I or type at least VI. Klein and Kroll [1989] considered the more general hyperbola structures (see Section 4) and classified those with respect to automorphisms that fix one parallel class pointwise; they obtained six different types. Furthermore, Minkowski planes were classified with respect to automorphisms that fix one point and induce translations in the derived affine plane at that point. This yields seven different types. Combining both classifications, nine different types resulted.

The classification of Benz planes with respect to central automorphisms has not been applied to topological Laguerre and Minkowski planes so far; for topological Möbius planes compare Section 3. In the topological setting instead, Miquelian Benz planes have been characterized by the size (dimension) of their automorphism groups, cf. Section 2.9.

1.6. *Point stabilizer and collineations of the derived plane.* If an automorphism γ fixes a point p, one obtains a collineation of the derived affine plane at p which can be extended to a collineation $\widetilde{\gamma}$ of the derived projective plane at that point. $\widetilde{\gamma}$ obviously fixes the infinite line and each point of Ω. In particular, an automorphism in the kernel with respect to a nontrivial parallelism induces a central collineation in the derived projective plane at a fixed point of it; the centre is the infinite point of lines that come from parallel classes to that parallelism. This relation between the stabilizer of a point and collineations of the derived projective plane at that point allows to exploit the rich and well developed theory of projective planes and their collineation groups in the study and classification of Benz planes.

2. Topological Benz planes, topological ovals, and the classical Benz planes

In the Miquelian Benz planes over $\mathbb{R}$ or $\mathbb{C}$, the set of points and the set of circles carry topologies that are induced from the surrounding 3-dimensional real or complex (topological) projective space; see Chapter 23. We refer to these Benz planes as *classical Benz planes* in order to emphasize that we not only consider the geometries of these Miquelian planes but that they also carry an additional topological structure; see Definition 2.1 below. Since the operation of joining three noncollinear points of the projective space by a plane is continuous (cf. Chapter 23, Section 1), one obtains the continuity of joining in the Benz plane with respect to the induced topologies. The continuity of the other geometric operations in the Benz plane analogously follows from the continuity of related operations in the projective space and the 'smoothness' of the underlying quadratic set. This can even be extended to embeddable Benz planes where the underlying quadratic set has certain smoothness properties. i.e. one has a topological Benz plane in the following sense.

2.1. DEFINITION. *Topological Benz plane.* A *topological* Benz plane is a Benz plane where the set of points and the set of circles carry nonindiscrete topologies such that the geometric operations of joining, touching, parallel projection, intersecting parallel classes with respect to different parallelisms, and intersecting distinct circles are continuous on their domains of definition.

Heise [1969] and Iversen [1970] proposed a weaker definition of a topological Möbius and Laguerre plane, respectively. They only required the continuity of perspectivities and the continuity of intersection in each derivation. Löwen [1981] and Förtsch [1982] used a stronger definition of a topological Benz plane. They also required that all derived affine planes are topological planes. Petkantschin [1941], Kroll [1970, 1971, 1977] and Meyer [1987] were concerned with ordered Benz planes.

Under the weak conditions in the definition of a topological Benz plane above it can be shown by methods similar to those for topological projective planes that any two parallel classes are homeomorphic and that any two circles are homeomorphic. Moreover, the point space is regular. The point space, circles, and parallel classes are either all connected or all totally disconnected.

2.2. *Coherence problem.* A fundamental question is whether or not the continuity of the geometric operations in the Benz plane gives rise to continuous geometric operations in derived planes, i.e. whether the derived planes are topological. To answer this question a number of coherence axioms have been considered. One of these coherence axioms, which is common to all three types of Benz planes, reflects the fact that induced ovals must be topological (see Definition 2.3 below). In Benz planes this becomes that touching is the limit of proper intersection. More precisely, let D be a directed set, let $(C_d)_{d\in D}$ be a net of circles, and let $(x_d)_{d\in D}$, $(y_d)_{d\in D}$, $(z_d)_{d\in D}$ be three nets of points such that the following conditions are satisfied: $x_d, y_d \in C_d$; x_d, y_d, z_d are pairwise nonparallel; $(C_d)_{d\in D}$ converges to a circle C; $(x_d)_{d\in D}$, $(y_d)_{d\in D}$, $(z_d)_{d\in D}$ converge to x, y, z respectively with $x = y$ nonparallel to z. Then the net of circles joining x_d, y_d, z_d converges to the circle passing through z that touches C at x. Specializing points and circles yields the continuity of forming parallel lines in derived affine planes. Other coherence axioms deal with the intersection of circles and parallel classes as the circles pass through points that converge to parallel points. One calls a topological Benz plane *coherent* if all those coherence axioms are satisfied.

As for topological projective planes, one requires ‘good’ topologies in order to obtain better results on the geometry of a topological Benz plane and also to be able to answer the above question in the affirmative. Analogously, a topological Benz plane is called (locally) compact, connected, or finite-dimensional, if the point space has the respective topological property. Here, as in Chapter 23, Section 3.21, the dimension refers to the topological dimension of a space. Furthermore, a locally compact connected topological Benz plane is metrizable so that coherence properties can be expressed by using sequences; cf. Wölk [1966], Groh [1970], and Schenkel [1980].

CONVENTION. We are only concerned with locally compact finite-dimensional topological Benz planes. For brevity, a locally compact finite-dimensional topological Benz plane will be called a *finite-dimensional* Benz plane. Note that such a plane is connected if the point set has positive dimension.

As for topological projective planes it is still an open question whether locally compact infinite-dimensional Benz planes exist. As shown by Strambach [1970a], Groh [1970], and Steinke [1989] for Möbius, Laguerre, and Minkowski planes respectively, connected finite-dimensional Benz planes are coherent. Therefore each derived affine plane of a connected finite-dimensional Benz plane is a topological affine plane with a locally compact, connected, and finite-dimensional point set. This affine plane extends to a compact connected finite-dimensional projective plane. Not surprisingly, many topological results about compact connected finite-dimensional projective planes carry over to connected finite-dimensional Benz planes. Moreover, each oval induced by a circle not passing through the point of derivation is a closed subset and thus is a topological oval, compare Theorem 2.4.

2.3. DEFINITION. *Topological oval.* Let $\mathcal{O}$ be an oval and let $\mathcal{L}_\mathcal{O}$ be the collection of all lines that intersect $\mathcal{O}$. Furthermore, $\mathcal{O} * \mathcal{O}$ denotes the symmetric product of $\mathcal{O}$, i.e. $\mathcal{O} \times \mathcal{O}$ with (x, y) identified with (y, x). We define a map $\psi_\mathcal{O}$: $\mathcal{L}_\mathcal{O} \to \mathcal{O} * \mathcal{O}$ by $\psi_\mathcal{O}(L) = L \cap \mathcal{O}$. This is a bijection between the two sets.

A *topological* oval in a topological projective plane is an oval $\mathcal{O}$ where the map $\psi_\mathcal{O}$ is continuous with respect to the induced topologies.

Buchanan, Hähl and Löwen [1980] showed that a topological oval in a compact connected projective plane is compact and that the following coherence condition is satisfied: if x, y are points of $\mathcal{O}$ and if y tends to x, then the line passing through x and y tends to the tangent to $\mathcal{O}$ at x. Furthermore, the collection of tangents of a topological oval forms a topological oval in the dual plane.

The existence of topological ovals severely restricts the possible dimensions of projective planes and consequently the dimensions of Benz planes. Buchanan, Hähl and Löwen [1980] proved the following

2.4. THEOREM. *An oval in a compact connected finite-dimensional projective plane is topological if and only if it is a closed set. Furthermore, such ovals only exist in planes of dimension* 2 *and* 4. *Topological ovals in a* $2n$*-dimensional projective plane,* $n = 1, 2$, *are homeomorphic to the* n*-sphere* $\mathbb{S}_n$ *and a topological oval in a* 4*-dimensional plane possesses no exterior line.*

2.5. COROLLARY. *A connected finite-dimensional Benz plane is of dimension* 2 *or* 4 *and there is no* 4*-dimensional Möbius plane. Circles in a* $2n$*-dimensional Benz plane,* $n = 1, 2$, *are homeomorphic to* $\mathbb{S}_n$. *There are no disjoint circles in* 4*-dimensional planes.*

Consequently, many of the difficulties one faces with compact projective planes of dimension > 4 do not occur in connected finite-dimensional Benz planes. In the case of Laguerre planes similar results in this direction were obtained by Groh [1968].

Topological ovals in compact totally disconnected Pappian projective planes, i.e. projective planes over a finite extension of the p-adics $\mathbb{Q}_p$ or a field of Laurent series over a finite field GF(p^n), $p \neq 2$, were investigated by Tillmann [1991].

Together with the emerging theory of compact totally disconnected projective planes this could be the starting point in the study of 0-dimensional Benz planes.

A remarkable result of Buchanan [1979] characterizes the topological ovals in the Desarguesian complex projective plane. This result is a topological analogue to the determination of ovals in finite Desarguesian projective planes of odd order by Segre [1955].

2.6. THEOREM. *The topological ovals in the Desarguesian complex projective plane are precisely the conics.*

As a consequence of Theorem 2.6, there is a topological analogue to a result of Chen and Kaerlein [1973] which says that a finite Laguerre or Minkowski plane of *odd* order is Miquelian if at least one derived affine plane is Desarguesian. This is also true for finite Möbius planes of odd order q, $q \notin \{11, 23, 59\}$; see Thas [1990]. Comparing circles not passing through the point of a Desarguesian derivation with conics in the Desarguesian complex projective plane that intersect the infinite line in Ω, Löwen [1981] characterized the classical 4-dimensional Benz planes in terms of a single derivation.

2.7. THEOREM. *A* 4*-dimensional Benz plane (Laguerre or Minkowski plane) is isomorphic to the corresponding classical complex Benz plane if and only if at least one derived* *affine plane is Desarguesian. In particular, every embeddable 4-dimensional Benz plane is Miquelian.*

The description of the classical Benz planes over $\mathbb{C}$ in terms of a single derivation proves to be an important tool in the classification of 4-dimensional Benz planes, because often the dimension and action of the stabilizer of a point ensure that the derived plane at this point is Desarguesian.

Theorem 2.7 is not true for 2-dimensional Benz planes: each derived affine plane of an embeddable Möbius or Laguerre plane is Desarguesian; 2-dimensional Minkowski planes with sufficiently many Desarguesian derived affine planes were classified by Steinke [1990c], and most of the examples of Hartmann [1981] have a Desarguesian derivation at precisely one special point. It is an open problem for 2-dimensional Möbius and Laguerre planes whether or not these planes must be embeddable if all their derived affine planes are Desarguesian.

2.8. *Locally classical Benz planes.* Being classical is a local property of a finite-dimensional Benz plane, see Steinke [1983b]. That is, if a connected finite-dimensional Benz plane looks like the corresponding classical Benz plane around each point, then the plane is classical. A global isomorphism is constructed via a monodromy argument. For 2-dimensional planes one can even use Miquel's configuration: if each point possesses a neighbourhood in which Miquel's theorem is valid when all points occurring in the configuration are in that neighbourhood (locally Miquelian Benz plane), then the Benz plane is locally classical and hence classical; compare Chapter 23, Section 5.18, for locally Desarguesian planes. The proof of this result, as given in Torrechante [1980] for Möbius planes and Steinke [1984a] for all Benz planes, is a local version of Van der Waerden and Smid [1935] and heavily relies on order properties of $\mathbb{R}$ for the coordinatization of sufficiently small neighbourhoods of points. It thus cannot be directly carried over to 4-dimensional planes. We conjecture however that the corresponding result holds for 4-dimensional Benz planes, too.

2.9. *Semiclassical Benz planes.* Semiclassical Benz planes comprise a rather large class of examples of Benz planes. The point sets of these planes are the union of two open connected subsets and certain circles and parallel classes which form the common boundary of both open sets and such that as few as possible circles and parallel classes are used. The induced topology and geometry on each open subset is isomorphic to the topology and geometry of the corresponding classical Benz plane on a corresponding set. So the construction of semiclassical planes can be imagined as two halves of a classical plane being pasted together along certain circles or parallel classes. Since both classical subgeometries can be uniquely embedded, up to automorphisms of the classical plane, into the corresponding classical Benz plane (see Steinke [1983a]), automorphisms of semiclassical Benz planes can be explicitly determined. This construction extends a corresponding method for projective planes, see Steinke [1985a].

2-dimensional semiclassical Möbius planes were classified by Steinke [1986a]. Both open connected subsets are hemispheres, and their common boundary consists of precisely one circle. 2- and 4-dimensional semiclassical Laguerre planes were classified by Steinke

[1987b, 1988, 1990a]. 2-dimensional semiclassical Laguerre planes can be constructed in two different ways by pasting along two parallel classes (cf. Section 5.3) or along one circle. There is only one one-parameter family of 4-dimensional semiclassical planes, see the paragraph following Section 5.8. Here two halves of the classical complex Laguerre plane are glued together along a 1-sphere of parallel classes. The 3-dimensional set of gluing imposes much stronger restrictions on the possible Laguerre planes compared to the situation of 2-dimensional planes.

Although there are many ways to remove the points of some circles and parallel classes from the point set of a 2-dimensional Minkowski plane in order to obtain two connected components, only one one-parameter family of semiclassical Minkowski planes results, see Section 4.4. Not surprisingly, there are no semiclassical 4-dimensional Minkowski planes except the classical complex Minkowski plane.

2.10. *Recognition of topological Benz planes.* The topologies of the point set and the circle set determine each other, i.e. for a given topology on the point set there is at most one topology on the circle set for which the Benz plane becomes topological, and vice versa. In order to obtain a topological Benz plane with a 2-dimensional point set it suffices that circles and parallel classes are closed subsets and are respectively homeomorphic to $\mathbb{S}_1$ and to $\mathbb{R}$ and $\mathbb{S}_1$, depending on the type of Benz plane; compare Wölk [1966], 8.1, Groh [1968], 3.10, Schenkel [1980], 4.4. Using this characterization Polster and Steinke showed that a Benz plane is topological when each derived affine plane is topological and 2-dimensional. In the case of Laguerre and Minkowski planes it suffices to have topological derived affine planes at all points of at least one parallel class. No corresponding result is known for 4-dimensional Benz planes where the verification of the continuity of the geometric operations is much more complicated. For 4-dimensional Minkowski planes this problem can be reduced to the continuity of joining, cf. Section 4.6; for a similar result in this direction for 4-dimensional elation Laguerre planes, see Section 5.12 where the verification of the continuity of the geometric operations can be reduced to the verification of a limit condition on certain describing matrices.

Having topologies on the point set and circle set, one is naturally concerned with *continuous* automorphisms. Each continuous automorphism of a locally compact Benz plane is a homeomorphism of the point set, cf. Chapter 23, Section 3.33. Furthermore, if an automorphism γ fixes a point, then the induced collineation $\tilde{\gamma}$ of the derived projective plane at that fixed point is also continuous. In the sequel the term automorphism group of a topological Benz plane refers to the collection of all continuous automorphisms. The topological structure of these groups was investigated by Strambach [1970a], Förtsch [1982], Schenkel [1980], Steinke [1984b, 1986b]. The automorphism group of a locally compact connected topological Benz plane is a locally compact, second countable (i.e. with a countable base), topological transformation group of the point space and of the circle space with respect to the compact-open topology, cf. Chapter 23, Section 3.35. For connected finite-dimensional Benz planes one even has

2.11. THEOREM. *The collection Γ of all continuous automorphisms of a $2n$-dimensional Benz plane ($n = 1, 2$) is a Lie group with respect to the compact-open topology (or, equivalently, the topology of uniform convergence on compact sets) of dimension at most 6 (Möbius plane), $7n$ (Laguerre plane), $6n$ (Minkowski plane), respectively.*

Each automorphism of a 2-dimensional Benz plane is continuous because such an automorphism induces an isomorphism between derived projective planes which is continuous (see Salzmann [1967]; also cf. Strambach [1970a] for a direct proof for 2-dimensional Möbius planes). 4-dimensional Benz planes may admit discontinuous automorphisms. For the Miquelian Laguerre and Minkowski plane each field automorphism of $\mathbb{C}$ yields an automorphism of the Benz plane. Since there are many discontinuous automorphisms of $\mathbb{C}$, these Benz planes admit discontinuous automorphisms. As for 4-dimensional projective planes, one may conjecture that the classical complex planes are the only ones that admit discontinuous automorphisms; cf. Chapter 23, Section 4.11. In fact, the conjecture in the present context is weaker due to the additional structure constituted by the ovals in a derivation at a point.

Following Von Staudt's point of view, the group of all projectivities of a circle was investigated by Strambach [1977] and Löwen [1977, 1981]. Here, as usual, a projectivity of a fixed circle C is a finite composition of perspectivities between circles, the first and last circle being C; perspectivities between circles are defined in the familiar manner by the correspondence between the points of two circles via a pencil of circles with the obvious modifications for parallel points or points of tangency. These maps are continuous by coherence, cf. Section 2.2. Pursuing this line of investigation the following characterization of classical Benz planes was obtained by Strambach [1977] and Löwen [1981].

2.12. THEOREM. *Let $\mathcal{B}$ be a locally compact, connected, topological Benz plane and let Π be the group of projectivities of a circle C. Then $\mathcal{B}$ is classical, if*

- *Π or its closure in the group of all homeomorphisms of C with respect to the compact-open topology τ is locally compact with respect to τ, or*
- *$\mathcal{B}$ is finite-dimensional and Π acts ω-regularly on C, i.e. there exists a finite set $F \subseteq C$ such that the subgroup in Π fixing F elementwise is discrete with respect to τ.*

One can impose much stronger conditions on the topology and the geometric operations of 4-dimensional Benz planes. For example, if the point set and circle set are complex analytic manifolds and the geometric operations are analytic, then the Benz plane must be classical. This was shown by Scholz [1980] by using the group of projectivities.

3. Möbius planes

Finite Möbius planes of order n (i.e. each circle has precisely $n + 1$ points) are combinatorially characterized as 3-$(n^2 + 1, n + 1, 1)$-designs. In particular, the axiom of touching follows. This is one reason why more general incidence structures have been investigated: here one requires the axiom of joining and the richness axiom but not the axiom of touching. These incidence structures are also called *circle planes* or *circular planes*. However, there shall be no confusion, as we use the term *Benz plane* and *circle plane* interchangeably and use the name *circular plane* for the generalization of Möbius planes.

3.1. EXAMPLE. *Embeddable Möbius planes.* Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\{(x, y, f(x, y))\colon x, y \in \mathbb{R}\}$ extended by the infinite point of the third coordinate axis describes an ovoid $\mathcal{O}(f)$ in 3-dimensional real projective space. Stereographic projection from that infinite point onto the xy-plane yields the following planar description of the ovoidal Möbius plane $\mathcal{M}(f)$ over $\mathcal{O}(f)$. The point set of $\mathcal{M}(f)$ is $\mathbb{R}^2 \cup \{\infty\}$; circles of $\mathcal{M}(f)$ are the Euclidean lines

$$\{(x, y)\colon\ x, y \in \mathbb{R},\ ax + by + c = 0\} \quad (a, b, c \in \mathbb{R},\ (a, b, c) \neq (0, 0, 0))$$

extended by ∞ and circles of the form

$$\{(x, y)\colon\ x, y \in \mathbb{R},\ f(x, y) + ax + by + c = 0\} \quad (a, b, c \in \mathbb{R})$$

provided that set contains at least three points.

Topological Möbius planes were first investigated by Wölk [1966] and Strambach [1967]. Later, Strambach [1970a] studied the more general circular planes. A 2-dimensional circular plane has point space homeomorphic to $\mathbb{S}_2$ – therefore these planes are also known as *spherical* circular or Möbius planes – and the circles form a system of Jordan curves on $\mathbb{S}_2$. The set of circles of a spherical Möbius plane is homeomorphic to the real 3-dimensional projective space with one point deleted; cf. Strambach [1974b]. The complement of each circle has two connected components by the Jordan–Schoenflies theorem, and the collection of these components forms a basis of the point topology. Furthermore, the stabilizer of four nonconcircular points is 0-dimensional. From this and the fact that each circle C admits at most one *inversion* with axis C (i.e. an automorphism that fixes precisely the points of C) one obtains that the automorphism group of a 2-dimensional circular plane has dimension at most 6. The connected component of the identity in the automorphism group of the classical 2-dimensional Möbius plane is isomorphic and acts equivalently to $\mathrm{PSL}(2, \mathbb{C})$ in its natural action on $\mathbb{S}_2 \simeq \mathbb{C} \cup \{\infty\}$.

For spherical Möbius planes the number of Hering types reduces to eight, see Strambach [1970b]: these are the types I.1, I.2, II.1, II.2, III.1, III.2, IV.1, and VII.2. The classical real Möbius plane has Hering type VII.2. Planes of class IV have a distinguished circle C such that all central automorphisms with centre $p \in C$ and co-centre the tangent pencil of C at p exist; no group of central automorphisms with two fixed points is linearly transitive in planes of Hering type IV.1. Planes of class III have a distinguished point p such that the derived plane at p is a translation plane and such that all translations are induced by automorphisms of the Möbius plane. Each group of central automorphisms with centres p and x is transitive for all $x \neq p$ in planes of Hering type III.2; no such group is linearly transitive in planes of Hering type III.1. Examples of spherical Möbius planes of Hering type I.2, whose existence was not assured in Strambach's paper, were constructed in Steinke [1986a] by pasting together two halves of the classical Möbius plane along one circle (semiclassical Möbius plane, cf. Section 2.9). This construction mostly yields nonembeddable Möbius planes. It can also be used to explicitly obtain Möbius planes which have no automorphism other than the identity. Ovoidal Möbius planes with trivial automorphism group were constructed by Ewald [1967].

The first examples of nonembeddable spherical Möbius planes were given by Ewald [1960] although the concept of a topological Möbius plane was not developed at that time. Later Strambach classified spherical Möbius planes, in fact the more general circular planes, admitting an automorphism group of dimension at least 3 in a series of papers, see Strambach [1972, 1973, 1974a,b]; in most cases Möbius planes are obtained. We summarize his results.

3.2. THEOREM. *A spherical circular plane is isomorphic to the classical real Möbius plane if and only if one of the following holds:*

(1) *the automorphism group Γ contains a compact subgroup of dimension* > 1;

(2) *Γ is point-transitive*;

(3) *Γ is circle-transitive*;

(4) *the plane is flexible, i.e. Γ is transitive on the set of incident point-circle pairs*;

(5) *each circle is the axis of an inversion*;

(6) *each pair of distinct points occurs as centres of a nontrivial central automorphism*;

(7) *Γ is at least* 4*-dimensional.*

For the proof, see Strambach [1967, 1970a,b]. One important step is to show that the existence of a subgroup isomorphic to SO(3, $\mathbb{R}$) leads to the classical real Möbius plane. The method described in Section 1.6 and the classification of 2-dimensional affine planes and their collineation groups was successfully exploited here for the first time.

3.3. THEOREM. *Let Σ be a connected* 3*-dimensional group of automorphisms of a spherical circular plane. If Σ is nonsimple, then Σ fixes precisely one point, the derived affine plane at that point is Desarguesian, and Σ induces the full translation group in this derivation. Furthermore, Σ has precisely two orbits on the set of circles. Depending on the form of a* 1*-dimensional complement of the translation group three different families of planes are obtained. When Möbius planes result, these are of Hering type* III.1, III.2, *or* VII.2.

If Σ is simple, then Σ is isomorphic to PSL(2, $\mathbb{R}$). *All such planes are Möbius planes and are of Hering type* IV.1 *or* VII.2. *Σ fixes precisely one circle C and the subgeometry of all circles intersecting C in at most one point is as in the classical real Möbius plane. The resulting Möbius planes split into three distinct uncountable families according to the action on the set of circles through two points of C.*

A 2*-dimensional Möbius plane is of Hering type* IV *if and only if the connected component of the identity in the full automorphism group is isomorphic to* PSL(2, $\mathbb{R}$). *The Möbius plane then is of Hering type* IV.1.

For the proof, see Strambach [1972, 1973].

3.4. THEOREM. *Let Σ be a group of automorphisms of a spherical circular plane. Suppose that Σ is of dimension at least* 2 *or that the connected component Σ^1 of the identity contains nontrivial compact subgroups. Then Σ splits over Σ^1 and Σ can be abstractly embedded in the group of affinities of the real Desarguesian affine plane or in the full automorphism group of the classical real Möbius plane. If Σ^1 is not isomorphic to* $\mathbb{R}^2$, *then Σ can be even topologically embedded in one of the above two groups.*

For the proof, see Strambach [1974a].

3.5. *Laguerre planes constructed from Möbius planes.* 2-dimensional Laguerre planes can be constructed from spherical Möbius planes as follows; cf. Groh [1974a]. Let p be a point of the Möbius plane $\mathcal{M}$. The point set of the Laguerre plane $\mathcal{L}$ is the collection of all oriented circles of $\mathcal{M}$ (an *oriented circle* is a circle together with one of the two connected components of its complement in the point space) passing through p (this is a double covering of the set of lines of a 2-dimensional affine plane); the set of circles of the Laguerre plane consists of all points of the Möbius plane different from p and all oriented circles not passing through p. A point of $\mathcal{L}$, i.e. an oriented circle $C \ni p$, is incident with a circle of $\mathcal{L}$ stemming from a point $q \neq p$ of the Möbius plane if and only if $q \in C$; the point is incident with a circle stemming from an oriented circle $C' \not\ni p$ if and only if C and C' touch each other at some point so that their orientations agree. This construction generalizes the 'spear-cycle' model of the classical real Laguerre plane in terms of oriented Euclidean lines ('spears') and points and oriented Euclidean circles ('cycles'); compare Benz [1973], I.2.1.

Conversely, a 2-dimensional Laguerre plane can be obtained from a 2-dimensional Möbius plane in this way if and only if it admits an involutory fixed-point-free automorphism γ and a circle K such that $\gamma(K)$ does not intersect K; see Groh [1974b]. The corresponding Möbius plane then can be reconstructed as follows. The point set P is the one-point compactification of the set $F \simeq \mathbb{R}^2$ of all circles fixed by γ, i.e. $P = F \cup \{p\}$. The circles of the Möbius plane are of the form $F \cap \mathcal{K}(C)$ for each circle C not fixed by γ or of the form $(F \cap \mathcal{K}_q) \cup \{p\}$ for each point q, where $\mathcal{K}(C)$ consists of all circles touching C and $\mathcal{K}_q$ consists of all circles passing through q. Incidence is the natural one.

Forming the associated Lie geometry of a Laguerre plane, see Section 5.13 or Forst [1981], 5.15, eventually yields a 3-dimensional topological generalized quadrangle. More precisely, one obtains a 3-dimensional antiregular generalized quadrangle that admits an involution whose set of fixed points is homeomorphic to $\mathbb{S}_2$. Exactly those generalized quadrangles can be generated by spherical Möbius planes.

4. Minkowski planes

Similarly as for Möbius planes, deleting the axiom of touching from the axioms of a Minkowski plane gives rise to a more general incidence structure, called a *hyperbola structure*. These hyperbola structures are equivalent to sharply 3-transitive sets of permutations, see Section 4.1 below. In the finite case the axiom of touching follows and one has a Minkowski plane.

4.1. *Planar representation.* Let $\mathcal{M}$ be a Minkowski plane and let F be the ternary field coordinatizing the derived affine plane $\mathcal{A}$ at a point p with respect to two parallel classes of different types as coordinate axes. We define $\overline{F} = F \cup \{\infty\}$. Then $\mathcal{M}$ can be represented as follows. The point set of $\mathcal{M}$ is $\overline{F} \times \overline{F}$, parallel classes are of the form $\{x_0\} \times \overline{F}$ and $\overline{F} \times \{y_0\}$ for $x_0, y_0 \in \overline{F}$, and the point of derivation p becomes the point (∞, ∞). Each circle C of $\mathcal{M}$ can be described by a function $f_C\colon \overline{F} \to \overline{F}$ as

$$C = \big\{\big(x, f_C(x)\big)\colon\ x \in \overline{F}\big\}.$$

The axiom of parallel projection shows that each function f_C is a permutation of $\overline{F}$. The axiom of joining implies that the collection of all those permutations f_C is a sharply

3-transitive set of permutations of $\overline{F}$. Conversely, each such incidence structure constructed from a sharply 3-transitive set of permutations is a hyperbola structure. If we use this representation for a Minkowski plane, we say that the Minkowski plane is in *standard representation*.

One obtains the classical $2n$-dimensional Minkowski plane, $n = 1, 2$, if the functions f_C are precisely the linear rational functions over $\overline{F}$, $F = \mathbb{R}, \mathbb{C}$, in their natural actions on $\overline{F}$, i.e. the sharply 3-transitive set of permutations is the group PGL(2, $\mathbb{R}$) or PGL(2, $\mathbb{C}$) for $n = 1$ and $n = 2$, respectively.

Topological Minkowski planes were first investigated by Schenkel [1980], who mainly dealt with coherence problems and 2-dimensional planes. Löwen [1981] and Förtsch [1982], who investigated 4-dimensional planes, used a stronger definition of a topological Minkowski plane; they also required that all derived affine planes are topological planes. For connected finite-dimensional Minkowski planes, however, this follows from the weaker definition we use here; cf. Steinke [1989].

The point space of a finite-dimensional Minkowski plane is homeomorphic to the product of two parallel classes. Each circle and parallel class of a $2n$-dimensional Minkowski plane is homeomorphic to $\mathbb{S}_n$. Hence the point space is homeomorphic to $\mathbb{S}_n \times \mathbb{S}_n$. Furthermore, the functions describing circles in the standard representation of a Minkowski plane are continuous and thus homeomorphisms of $\mathbb{S}_n$. Fixing three parallel classes of the same kind one has a homeomorphism of the collection of triples of pairwise nonparallel points on these parallel classes to the set of circles. Since the group PGL(2, $\mathbb{R}^n$) (where PGL(2, $\mathbb{R}^2$) stands for PGL(2, $\mathbb{C}$)) acts sharply transitively on the triples of distinct points of $\mathbb{S}_n \approx \mathbb{R}^n \cup \{\infty\}$, one sees that the set of circles of a $2n$-dimensional Minkowski plane is homeomorphic to PGL(2, $\mathbb{R}^n$).

It follows from the classification of sharply 3-transitive transformation groups of $\mathbb{S}_n$ that a $2n$-dimensional Minkowski plane is classical if and only if the functions describing circles form a group; cf. Hartmann [1982a] for Minkowski planes related to groups.

The automorphism group Γ of a Minkowski plane possesses two distinguished normal subgroups, called *kernels*, consisting of all those automorphisms that map each point to one (+)-parallel (resp., (−)-parallel) to it. We denote these subgroups by T^+ and T^- respectively. Since the stabilizer of a quadrangle in compact 2- or 4-dimensional projective planes is trivial or has order at most two respectively, see Chapter 23, Section 4.18, one obtains the following

4.2. LEMMA ON STABILIZERS. *In the automorphism group, the stabilizer of three pairwise nonparallel points contains at most two automorphisms. In the kernel, the stabilizer of three pairwise nonparallel points is trivial.*

From this lemma one finds that the dimension of the automorphism group of a $2n$-dimensional Minkowski plane is at most $6n$ and that each kernel has dimension at most $3n$. These dimensions are attained in the classical $2n$-dimensional Minkowski planes: The connected component of the identity in the automorphism group of the classical $2n$-dimensional Minkowski plane is isomorphic to PSL(2, $\mathbb{R}^n$) $\times$ PSL(2, $\mathbb{R}^n$) and has index eight or four in the full automorphism group for $n = 1, 2$, respectively; each kernel is isomorphic to PGL(2, $\mathbb{R}^n$).

2-dimensional Minkowski planes
Circles form a system of Jordan curves on the torus $\mathbb{S}_1 \times \mathbb{S}_1$. The set of circles is homeomorphic to PGL(2, $\mathbb{R}$) and has two connected components.

Schenkel [1980] classified 2-dimensional Minkowski planes with an automorphism group of dimension at least 4 and those with a 3-dimensional kernel $T^{\pm}$.

4.3. *Planes with 3-dimensional kernel.* A 3-dimensional kernel $T^{\pm}$ must act doubly transitively on each ($\mp$)-parallel class. Hence the connected component of the identity in $T^{\pm}$ must be isomorphic and act equivalently to PSL(2, $\mathbb{R}$). Furthermore, this component acts transitively on each of the two connected components of the circle set. Now if we let f be an orientation-reversing homeomorphism of $\mathbb{S}_1$, then PSL(2, $\mathbb{R}$) $\cup f \cdot$ PSL(2, $\mathbb{R}$) is a sharply 3-transitive set of permutions of $\mathbb{S}_1 = \mathbb{R} \cup \{\infty\}$ which in turn even yields a 2-dimensional Minkowski plane $\mathcal{M}(f)$; cf. Section 4.1. This plane is nonclassical if and only if f does not belong to PGL(2, $\mathbb{R}$) (in its natural action on $\mathbb{S}_1$). The planes $\mathcal{M}(f)$ are precisely the 2-dimensional Minkowski planes having a 3-dimensional kernel.

The above construction was generalized by Jakóbowski [1993]. Let g be an orientation-preserving homeomorphism of $\mathbb{S}_1$, then PSL(2, $\mathbb{R}$)$\cup f\cdot$PSL(2, $\mathbb{R}$)$\cdot g$ yields a 2-dimensional Minkowski plane. Choosing f and g suitably one can obtain Minkowski planes with no automorphism other than the identity, cf. Steinke [1994a].

Other examples of 2-dimensional Minkowski planes with a 3-dimensional automorphism groups were constructed by Hartmann [1981], Steinke [1985b], and Artzy and Groh [1986].

4.4. *Planes with 4-dimensional automorphism groups.* For $r, s \in \mathbb{R}^+$ let the orientation-reversing homeomorphism $f_{r,s}$ of $\mathbb{S}_1 \simeq \mathbb{R} \cup \{\infty\}$ be defined by

$$f_{r,s}(x) = \begin{cases} x^{-r} & \text{if } x \in \mathbb{R},\ x > 0, \\ -s|x|^{-r} & \text{if } x \in \mathbb{R},\ x < 0, \end{cases}$$

extended by $f_{r,s}(0) = \infty$ and $f_{r,s}(\infty) = 0$. A 2-dimensional Minkowski plane admitting a 4-dimensional automorphism group is isomorphic to $\mathcal{M}(f_{r,1})$ for some $r > 0$, see Section 4.3 for this type of planes, or it is isomorphic to a Minkowski plane $\mathcal{M}(r_1, s_1; r_2, s_2)$, where $r_1, s_1, r_2, s_2 \in \mathbb{R}^+$, of the following form: the plane is in standard representation and the circle set of $\mathcal{M}(r_1, s_1; r_2, s_2)$ consists of all Euclidean lines extended by (∞, ∞) and the sets $\{(x, af_{r_1,s_1}(x - b) + c)\colon\ x \in \mathbb{S}_1\}$ and $\{(x, -af_{r_2,s_2}(x - b) + c)\colon\ x \in \mathbb{S}_1\}$ for all $a \in \mathbb{R}^+$, $b, c \in \mathbb{R}$, where we use the convention that $a \cdot \infty + b = \infty$ for all $a, b \in \mathbb{R}$, $a \neq 0$. These examples generalize Hartmann's [1981] construction of 2-dimensional Minkowski planes, which in the above notation are the planes $\mathcal{M}(r_1, 1; r_2, 1)$.

The examples in Section 4.3 also comprise all semiclassical 2-dimensional Minkowski planes. These are precisely the planes $\mathcal{M}(f_{1,s})$ for some $s > 0$. They are obtained by pasting along two parallel classes of the same type.

The classical real Minkowski plane was characterized by Schenkel [1980] by the following

4.5. THEOREM. *A 2-dimensional Minkowski plane $\mathcal{M}$ is isomorphic to the classical real Minkowski plane if one of the following holds:*

(1) *one of the normal subgroups $T^{\pm}$ is 3-dimensional and at least one derived plane is Desarguesian;*

(2) *the automorphism group of $\mathcal{M}$ is at least 5-dimensional.*

The only embeddable 2-dimensional Minkowski plane is the Miquelian plane. Each point has a Desarguesian derivation. However, to obtain this plane it suffices to have one parallel class of points and three pairwise nonparallel points at which the derived affine planes are Desarguesian; see Steinke [1990c]. Moreover, all 2-dimensional Minkowski planes with two parallel classes π_+ and π_- of points at which derived affine planes are Desarguesian are classified there. They are precisely the Minkowski planes $\mathcal{M}(r, 1; r, 1)$ with $r \in \mathbb{R}^+$, $r \neq 1$, and π_+ and π_- being the (+)- and (–)-parallel class of the point (∞, ∞); cf. Section 4.4 for this kind of Minkowski planes.

4-dimensional Minkowski planes

The set of circles $\mathcal{K}$ of a 4-dimensional Minkowski plane in standard representation can be identified with a collection of homeomorphisms of $\mathbb{S}_2$, cf. Section 4.1. The geometric operation of intersecting two circles is related to the inverse and composite of the associated representing homeomorphisms. We endow $\mathcal{K}$ with the compact-open topology, with respect to which the set of all homeomorphisms of $\mathbb{S}_2$ is a topological group, so that $\mathcal{K}$ becomes a topological subspace thereof. It can be shown, see Steinke [1993a], that the continuity of joining implies the continuity of the other geometric operations.

4.6. THEOREM. *Let $\mathcal{M}$ be a Minkowski plane with parallel classes homeomorphic to $\mathbb{S}_2$ in standard representation. The point set $P \simeq \mathbb{S}_2 \times \mathbb{S}_2$ is endowed with the product topology τ_P of $\mathbb{S}_2 \times \mathbb{S}_2$. We assume that circles are closed subsets of P with respect to τ_P and that joining is continuous with respect to the induced topology on the set of all pairwise nonparallel triples of points (a subset of P^3 with the product topology τ_P^3) and the compact-open topology $\tau_{\mathcal{K}}$ on the circle set $\mathcal{K}$, where $\mathcal{K}$ is considered as a subset of the collection of all homeomorphisms of $\mathbb{S}_2$, see above. Then $\mathcal{M}$ is a topological (locally compact, 4-dimensional) Minkowski plane.*

The following theorem, obtained by Steinke [1992], is an important tool in the characterization of the classical complex Minkowski plane and the future classification of 4-dimensional Minkowski planes with large automorphism groups.

4.7. THEOREM. *A locally compact connected group Σ of automorphisms of a 4-dimensional Minkowski plane is semisimple with trivial centre or Σ fixes at least one parallel class.*

There are only a few semisimple Lie groups with trivial centre of dimension at most twelve and much is known about their possible actions on $\mathbb{S}_2 \times \mathbb{S}_2$. If one parallel class is fixed, one has a topological transformation group of $\mathbb{S}_2$ and the kernel of this action is contained in a kernel group $T^{\pm}$. With such methods and information about 4-dimensional projective planes Förtsch [1982] and Steinke [1992, 1993b] obtained

4.8. THEOREM. *A 4-dimensional Minkowski plane is isomorphic to the classical complex Minkowski plane if and only if one of the following holds:*

(1) *one of the normal subgroups $T^{\pm}$ is at least 4-dimensional;*

(2) *the automorphism group Γ is at least 8-dimensional;*

(3) *Γ contains a closed connected 7-dimensional subgroup that fixes no point.*

A 4-dimensional Minkowski plane admitting a 7-dimensional connected group of automorphisms Σ is either classical or Σ fixes a distinguished point at which the derivation is a translation plane with a 7-dimensional collineation group, cf. Steinke [1993b]. The classification of such translation planes by Betten [1976] can be used, in theory, to find all 4-dimensional Minkowski planes with a 7-dimensional automorphism group. Although promising candidates have been constructed in this way, in practice however, the verification of the axioms of a topological Minkowski plane for the resulting candidates poses, so far, insurmountable problems. It is therefore still an open problem whether or not there are nonclassical 4-dimensional Minkowski planes.

4.9. *Minkowski planes and generalized quadrangles.* Similarly to the construction in Section 3.5 of topological generalized quadrangles from spherical Möbius planes, Schroth recently obtained an antiregular $3n$-dimensional generalized quadrangle by a suitable 2-fold covering of one connected component of the circle space of a $2n$-dimensional Minkowski plane. These quadrangles are precisely those antiregular $3n$-dimensional generalized quadrangles that admit an involution whose sets of fixed points and fixed lines form a thin generalized quadrangle.

5. Laguerre planes

The general theory of Laguerre planes differs in many respects from the theory of Möbius and Minkowski planes. Laguerre planes are the Benz planes most closely related to generalized quadrangles. Further differences will become clear in this section.

5.1. *Planar representation.* Each Laguerre plane $\mathcal{L}$ can be represented in one derived affine plane $\mathcal{A}$ extended by an infinite parallel class in the following way. Let F be the ternary field coordinatizing $\mathcal{A}$ with respect to a parallel class as the second coordinate axis. Then the point set of $\mathcal{L}$ is $\overline{F} \times F$, where $\overline{F} = F \cup \{\infty\}$, parallel classes are of the form $\{x_0\} \times F$ for $x_0 \in \overline{F}$, and the point of derivation becomes the point $(\infty, 0)$. Circles of $\mathcal{L}$ then are described by

$$\{(x, f_{a,b,c}(x)) \colon x \in F\} \cup \{(\infty, a)\} \quad (a, b, c \in \mathbb{R})$$

where $f_{a,b,c}$ describes the circle through (∞, a) that is tangent to the line of slope b that intersects the second coordinate axis at $(0, c)$ (i.e. the circle described by $f_{0,b,c}$; here $f_{0,b,c}(x) = \tau(b, x, c)$ where τ is the ternary operation of F).

Topological Laguerre planes were first investigated by Groh [1968, 1970]. Circles and parallel classes of a $2n$-dimensional Laguerre plane are homeomorphic to $\mathbb{S}_n$ and to $\mathbb{R}^n$ respectively. Fixing three distinct parallel classes, one obtains a homeomorphism from the product space of these parallel classes to the set of circles by joining three points by a circle. Hence, the set of circles of a $2n$-dimensional Laguerre plane is homeomorphic to $\mathbb{R}^{3n}$.

The automorphism group Γ of a Laguerre plane possesses a distinguished normal subgroup consisting of all those automorphisms that map each point to one parallel to it. We denote this subgroup by T and call it the *kernel.* Since the stabilizer of a quadrangle in compact 2- or 4-dimensional projective planes is trivial or has order at most two respectively, see Chapter 23, Section 4.18, one obtains the following

5.2. LEMMA ON STABILIZERS. *Let F be a set of four nonconcircular points with at least three of them being pairwise nonparallel. The stabilizer of F in the automorphism group contains at most two automorphisms. The stabilizer of F in the kernel T is trivial.*

This lemma readily yields that the dimension of the automorphism group of a $2n$-dimensional Laguerre plane is at most $7n$ and that the kernel has dimension at most $4n$. These dimensions are attained in the classical $2n$-dimensional Laguerre planes: The kernel is isomorphic to $\mathbb{R} \setminus \{0\} \ltimes \mathbb{R}^3$ and $\mathbb{C} \setminus \{0\} \ltimes \mathbb{C}^3$ for $n = 1$ and $n = 2$ respectively; the connected component of the identity in the automorphism group of the classical 2- an 4-dimensional Laguerre plane is isomorphic to $(\mathbb{R}^+ \times \mathrm{SO}(3, 1, \mathbb{R})) \ltimes \mathbb{R}^3$ and $(\mathbb{C} \setminus \{0\} \times \mathrm{SO}(3, \mathbb{C})) \ltimes \mathbb{C}^3$, respectively.

2-dimensional Laguerre planes

A 2-dimensional Laguerre plane has point set homeomorphic to the cylinder $\mathbb{S}_1 \times \mathbb{R}$. Moreover, the homeomorphism can be chosen in such a way that parallel classes of the Laguerre plane are mapped to fibres $\{x\} \times \mathbb{R}$; cf. Groh [1968], 3.6. Following we give some large families of 2-dimensional Laguerre planes.

5.3. (a) *Embeddable Laguerre planes.* Let f be a real parabolic function, i.e.

$$\mathcal{O}(f) = \{(x, f(x)) \colon x \in \mathbb{R}\} \cup \{\omega\}$$

is an oval in the real Desarguesian plane with the line at infinity as a tangent. Then the ovoidal Laguerre plane $\mathcal{L}(f)$ over $\mathcal{O}(f)$ has the circles

$$\{(x, af(x) + bx + c) \colon x \in \mathbb{R}\} \cup \{(\infty, a)\}$$

for $a, b, c \in \mathbb{R}$. The classical real Laguerre plane is obtained for $f(x) = x^2$.

(b) *Semiclassical Laguerre planes.* Up to isomorphism, all 2-dimensional semiclassical Laguerre planes obtained by pasting along two parallel classes are obtained in the following way; cf. Steinke [1988]. Let f_0, f_1, f_2 be three homeomorphisms of the real line which are either all orientation preserving or all orientation reversing. Then the semiclassical Laguerre plane $\mathcal{L}(f_0, f_1, f_2)$ has the circles

$$\begin{aligned}&\{(x, ax^2 + bx + c)\colon\ x \in \mathbb{R},\ x \geqslant 0\}\\ &\cup \big\{\big(x, f_0^{-1}\big(f_2(a)x^2 + f_1(b)x + f_0(c)\big)\big)\colon\ x \in \mathbb{R},\ x \leqslant 0\big\}\\ &\cup \{(\infty, a)\}\end{aligned}$$

for $a, b, c \in \mathbb{R}$. The collection of finite points $\mathbb{R}^2$ carries the Euclidean topology; neighbourhoods of infinite points (∞, a) consist of infinite points (∞, a') with a' close to a and of finite points (x, y) with $|x|$ sufficiently large and y/x^2 or $f_2^{-1}(f_0(y)/x^2)$ close to a for $x > 0$ and $x < 0$, respectively. Such a plane $\mathcal{L}(f_0, f_1, f_2)$ is embeddable if and only if all three functions f_0, f_1, f_2 are affine mappings. Consequently, this construction mostly yields nonembeddable Laguerre planes. Specialising f_0, f_1, f_2 suitably (more precisely, the triple (f_0, f_1, f_2) is not affinely equivalent to $(f_0^{-1}, f_1^{-1}, f_2^{-1})$, (f_2, f_1, f_0), or $(f_2^{-1}, f_1^{-1}, f_0^{-1})$, which means that corresponding homeomorphisms are not in a relation $f = \alpha\tilde{f}\beta$ for affine maps α, β) one can explicitly obtain Laguerre planes which have no automorphism other than the identity.

5.4. THEOREM. *A 2-dimensional Laguerre plane $\mathcal{L}$ is embeddable if and only if one of the following holds*:

(1) *the kernel T is 4-dimensional*;

(2) *T contains a subgroup isomorphic to $\mathbb{R}^3$*;

(3) *T is transitive on the set of circles.*

$\mathcal{L}$ is classical if and only if the automorphism group Γ is at least 6-dimensional.

For the proof, see Groh [1969], Pfüller [1986], 1.23, Löwen [1994], 6.8, and Löwen and Pfüller [1987b].

5.5. *2-dimensional Laguerre planes with 5-dimensional automorphism groups.* Löwen and Pfüller [1987b] determined all 2-dimensional Laguerre planes with 5-dimensional automorphism groups. These are all embeddable planes over skew-parabolae, i.e. these Laguerre planes are of the form $\mathcal{L}(f)$ as in Section 5.3(a), where

$$f(x) = \begin{cases} x^\alpha & \text{if } x \geqslant 0,\\ \beta|x|^\alpha & \text{if } x \leqslant 0,\end{cases}$$

with $\alpha > 1$, $\beta > 0$; the Miquelian plane is obtained precisely for $\alpha = 2$, $\beta = 1$.

In the same paper all 2-dimensional Laguerre planes admitting a 4-dimensional automorphism group such that the connected component of the identity fixes a point were also determined. At the other end of the possible range of actions of a 4-dimensional group of automorphisms Steinke [1993c] showed that a 2-dimensional Laguerre plane with a 4-dimensional point-transitive automorphism group is classical.

The possible dimensions of the full automorphism group and the kernel of a 2-dimensional Laguerre plane are as follows: (7,4), (5,4), and all pairs (n, m) with $0 \leqslant m \leqslant n \leqslant 4$, $n - m \leqslant 3$. Examples for ten of these 16 possible types can be found among the semiclassical Laguerre planes in Steinke [1987b, 1988]. The type (7,4) describes the classical real Laguerre plane and the types (5,4) and (4,4) describe the embeddable Laguerre planes over proper ovals. At present no examples of 2-dimensional Laguerre planes of type (2,2) and (2,0) are known.

Furthermore, if $n - m = 3$, i.e. if Γ/T is 3-dimensional, then the connected component of the identity in Γ/T is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ and the automorphism group acts transitively on the set of parallel classes. The proof of this and the following theorem can be found in Steinke [1990b].

5.6. THEOREM. *A quasisimple connected Lie subgroup $\Lambda \leqslant \Gamma$ of the automorphism group Γ of a 2-dimensional Laguerre plane is isomorphic to either $\Omega = \mathrm{PSL}(2, \mathbb{R})$ or the simply connected covering group Ω^* of Ω. Moreover, Λ operates transitively on the set of parallel classes.*

4-dimensional Laguerre planes

In the planar description of Laguerre planes the point set has a product structure. This product structure, however, is not compatible with the topology in 4-dimensional planes, i.e. the point space of a 4-dimensional Laguerre plane is not homeomorphic to the product of a circle with a parallel class.

Given a circle in a 4-dimensional Laguerre $\mathcal{L}$ one can consider the point space as a fibre bundle over this circle with fibres being the parallel classes. The projection map of the bundle is given by the parallel projection in the Laguerre plane. Since the point space of a 4-dimensional affine plane is the direct product of two lines, it follows that the point space of $\mathcal{L}$ is a locally trivial fibre bundle over $\mathbb{S}_2$ with fibres homeomorphic to $\mathbb{R}^2$. Considering the characteristic map of the bundle, which can be related to the function describing some circle, Steinke [1991a] proved the following theorem.

5.7. THEOREM. *The point space of a 4-dimensional Laguerre plane is equivalent to the point space of the classical complex Laguerre plane in the category of fibre bundles over $\mathbb{S}_2$ with fibres homeomorphic to $\mathbb{R}^2$. The point space of the classical complex Laguerre plane is bundle-equivalent to the tangent bundle of the 2-sphere.*

This theorem extends the 2-dimensional situation, because the direct product $\mathbb{S}_1 \times \mathbb{R}$ is bundle-equivalent to the tangent bundle of the 1-sphere $\mathbb{S}_1$.

Characterizations of the classical complex Laguerre plane via its automorphism group as the most homogeneous plane were obtained by Förtsch [1982] and Steinke [1987a].

5.8. THEOREM. *A 4-dimensional Laguerre plane is isomorphic to the classical complex Laguerre plane if and only if one of the following holds:*

(1) *the kernel is 8-dimensional;*

(2) *the automorphism group is at least 11-dimensional.*

For a long time, no nonclassical 4-dimensional Laguerre planes were known. In Steinke [1987a] the first nonclassical examples were constructed; these models have a 10-dimensional automorphism group and a 7-dimensional kernel. Furthermore, all such 4-dimensional planes (i.e. with 10-dimensional automorphism group and 7-dimensional kernel) were classified there. The resulting planes are of three types which can be distinguished according to the structure and action of their automorphism groups. There is one one-parameter family of semiclassical planes and two singular planes. Up to isomorphism, one obtains a representation of the semiclassical planes similar to the one in Section 5.3(b) where the two classical halves are given by

$$\{(z,w) \in \mathbb{C}^2\colon \operatorname{Im}(z) \geqslant 0\} \cup \{\infty\} \times \mathbb{C}$$

and

$$\{(z,w) \in \mathbb{C}^2\colon \operatorname{Im}(z) \leqslant 0\} \cup \{\infty\} \times \mathbb{C}.$$

All three describing functions f_0, f_1, f_2 are equal to

$$f(z) = z + q\bar{z}$$

for some real parameter q with $0 \leqslant q < 1$, where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. The classical complex Laguerre plane is obtained precisely for $q = 0$. All planes with 10-dimensional automorphism group and 7-dimensional kernel are elation Laguerre planes.

5.9. DEFINITION. *Elation Laguerre plane.* An *elation Laguerre plane* is a Laguerre plane that admits a subgroup in the kernel, called the *elation group*, which acts sharply transitively on the set of circles.

The name elation group comes from the fact that such an automorphism induces an elation in the derived projective plane at each of its fixed points. (Alternatively, these automorphisms induce elations in the associated Lie geometry, see Section 5.13.) For finite and for 4-dimensional elation Laguerre planes, the collection Δ of all automorphisms in the kernel T that fix no circle plus the identity is a closed normal subgroup of T. Furthermore, in the case of elation Laguerre planes, Δ acts sharply transitively on the set of circles, see Steinke [1991b,c]. So Δ is the elation group in these cases. It is not known at present, whether or not Δ forms a group in 2-dimensional Laguerre planes in any case. The 2-dimensional elation Laguerre planes are precisely the embeddable planes by Theorem 5.4. With this definition of Δ Steinke [1991b] obtained the following description of 4-dimensional elation Laguerre planes; cf. Löwen [1994].

5.10. THEOREM. *For a 4-dimensional Laguerre plane $\mathcal{L}$, the following are equivalent:*

(1) *$\mathcal{L}$ is an elation Laguerre plane;*
(2) *the kernel T is circle transitive;*
(3) *T is at least 7-dimensional;*
(4) *Δ is 6-dimensional;*
(5) *Δ is connected and is isomorphic to $\mathbb{R}^6$.*

Each derived projective plane of an elation Laguerre plane is a dual translation plane with centre ω (the infinite point of lines that come from parallel classes of the Laguerre plane). The stabilizer of a circle is linearly represented on Δ by conjugation. Therefore the well-developed theory of translation planes and, if the stabilizer of a circle is large enough, the representation theory of Lie groups can be applied to classify the most homogeneous elation Laguerre planes. This was carried out by Steinke [1987a] and all 4-dimensional elation Laguerre planes with a 4-dimensional stabilizer of a circle were classified. Those planes can be distinguished according to whether or not a parallel class is fixed and whether a 3-dimensional complement Σ of the kernel in the stabilizer of a circle acts irreducibly on $\Delta \simeq \mathbb{R}^6$. The semiclassical 4-dimensional Laguerre planes are precisely those planes with no fixed parallel class and reducible action of Σ on Δ.

Löwen [1994] gave a reinterpretation of elation Laguerre planes as pseudo-ovoidal Laguerre planes which naturally generalizes the construction of embeddable Laguerre planes. His construction uses the Grassmannian $G_{n,m}(F)$, i.e. the collection of all m-dimensional subspaces of the n-dimensional vector space F^n over a field F, see Chapter 3. In the case $F = \mathbb{R}$ the Grassmannian $G_{n,m}(\mathbb{R})$ can be made into a compact connected manifold, compare Chapter 23, Section 1.4.

5.11. DEFINITION. *Topological pseudo-oval.* A set $\mathcal{O} \subseteq G_{3m,m}(\mathbb{R})$ of m-dimensional subspaces of $\mathbb{R}^{3m}$ is called a *pseudo-oval* if for every element $p \in \mathcal{O}$ there is a $2m$-dimensional subspace t_p ('tangent') such that t_p and the collection of all joins in $G_{3m,m}(\mathbb{R})$ of p with elements of $\mathcal{O}$ distinct from p induce a planar spread in the factor space $\mathbb{R}^{3m}/p$. A *topological pseudo-oval* is one where the map

$$\mathcal{O} * \mathcal{O} \to G_{3m,2m}(\mathbb{R})\colon\ \{p,q\} \mapsto \begin{cases} p \vee q & \text{if } p \neq q, \\ t_p & \text{if } p = q, \end{cases}$$

is a homeomorphism onto its image; compare Definition 2.3 of a topological oval.

A pseudo-oval can be regarded as a sufficiently large family of translation planes sitting on different $2n$-dimensional subspaces of one $\mathbb{R}^{3n}$, such that the intersection of any two translation planes is a line in both of them. Finite analogues were studied by Thas [1971], Payne and Thas [1984], and Casse, Thas and Wild [1985]. For $m = 1$ one obtains a topological oval in the real projective plane. The other interesting case with respect to 4-dimensional Laguerre planes is $m = 2$.

5.12. *Matrix representation of 2-dimensional pseudo-ovals and 4-dimensional elation Laguerre planes.* For each $z \in \mathbb{S}_2 \simeq \mathbb{R}^2 \cup \{\infty\}$ let $D(z) = (A(z)\ B(z)\ C(z))$ be a 2×6 matrix with 2×2 matrices $A(z)$, $B(z)$, $C(z)$ such that $D(\infty) = (I\ 0\ 0)$ and $C(z) = I$ for all $z \in \mathbb{R}^2$, where 0 and I denote the 2×2 zero and identity matrix, respectively.

Each compact 2-dimensional pseudo-oval can be represented by such a mapping D. Moreover, D is continuous in $\mathbb{R}^2$ and $A(z)^{-1}D(z)$ tends to $D(\infty)$ for $z \to \infty$. The 2-dimensional subspaces generated by the rows of $D(z)$ correspond to the elements of the pseudo-oval.

In terms of the planar description of Laguerre planes in Section 5.1, circles are described by $f_c(z) = D(z)c$ for every $c \in \mathbb{R}^6$. The elation group Δ of the corresponding elation Laguerre plane is given by all maps $(z, w) \mapsto (z, w + D(z)c)$ for $c \in \mathbb{R}^6$; the connected component of the identity in the kernel consists of all maps $(z, w) \mapsto (z, rw + D(z)c)$ for $c \in \mathbb{R}^6$, $r \in \mathbb{R}$, $r > 0$. The continuity of the geometric operations in such a Laguerre plane described by a matrix valued mapping D reduces to

$$\lim_{z \to \infty} A(z)^{-1} D(z) = D(\infty)$$

and to the continuity of D in $\mathbb{R}^2$.

Laguerre planes and generalized quadrangles
Laguerre planes themselves occur as derivations of antiregular generalized quadrangles, see Payne and Thas [1984] and Thas [1985]. For the definition of a generalized quadrangle we refer to Chapter 9, Sections 1.3 and 10; see also Chapter 23, Section 6 for generalized quadrangles and generalized polygons in a topological setting. Let $p^\perp$ denote the set of all points that can be joined to the point p. A generalized quadrangle is called *antiregular*, if $a^\perp \cap b^\perp \cap c^\perp$ contains either no point or precisely two points for every triple of pairwise nonjoinable points a, b, c. The derivation of an antiregular generalized quadrangle at a point p is defined as follows: The point set is $p^\perp \setminus \{p\}$. Two such points are parallel if and only if they are on a common line through p. Circles of the derivation are of the form $p^\perp \cap q^\perp$ where q is some point nonjoinable to p. Conversely, combining points and circles of a Laguerre plane in a larger structure (Lie geometry) results in a generalized quadrangle (see also Chapter 9).

5.13. DEFINITION. *Lie geometry, topological generalized quadrangle.* The *Lie geometry* associated with a Laguerre plane $\mathcal{L}$ has point set consisting of the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point at infinity, denoted by ∞. Lines of the Lie geometry are the parallel classes of $\mathcal{L}$ extended by ∞ and the extended tangent pencils of $\mathcal{L}$, i.e. collections of the form

$$\{D\colon\ D \text{ touches } C \text{ at } x\} \cup \{x\}$$

for a circle C and a point $x \in C$. Incidence is the natural one.

A *topological generalized quadrangle* is a (thick) generalized quadrangle where the point set and the set of lines carry Hausdorff topologies such that the mapping that takes an antiflag (p, L) to the unique flag (q, M), where $p \in M$ and $q \in L$, becomes continuous.

The classical generalized quadrangles are associated with classical groups and form a particular class of buildings. For finite examples of generalized quadrangles, see in Chapter 9, Section 3.

The dual of a topological generalized quadrangle is a topological quadrangle, too. Hence each topological property of the point set or of lines (as point rows) is equally valid for the set of lines and for line pencils, respectively. Forst [1981] first investigated topological generalized quadrangles. We list some of his results: Each line and each point set $p^{\perp}$ is closed, any two lines are homeomorphic, each line is 2-homogeneous, the map in Definition 5.13 is open, the operation of joining two points by a line is continuous on its domain of definition, the point set is locally homeomorphic to the product of two lines and a line pencil, and dually. Later Grundhöfer and Knarr [1990] studied nondiscrete, locally compact generalized quadrangles and showed with methods developed for stable planes that the point set, line set, every line and line pencil have countable basis and are separable, σ-compact, metrizable spaces. These spaces are either all totally disconnected or all connected. Furthermore, in locally compact connected generalized quadrangles all these spaces are compact, locally and globally arcwise connected, and locally contractible. If the point set has finite topological dimension, then all spaces are absolute neighbourhood retracts and integral homology manifolds, lines and line pencils are homotopy equivalent to spheres. Furthermore, when lines are connected and at most 2-dimensional, then lines are homeomorphic to spheres. This is the situation that occurs in Lie geometries obtained from 2- or 4-dimensional Laguerre planes $\mathcal{L} = (P, \mathcal{K}, \|)$. The point set $\infty \cup P \cup \mathcal{K}$ of the Lie geometry carries a compact topology such that the topologies induced on P and on $\mathcal{K}$ are the given ones and such that $\mathcal{K}$ is open. Indeed, Schroth [1993, 1994a] proved

5.14. THEOREM. *For every $2n$-dimensional Laguerre plane, $n = 1, 2$, the associated Lie geometry with respect to a suitable topology on the point set and the topology on the line set induced by the Hausdorff metric is a topological antiregular $3n$-dimensional generalized quadrangle with lines and line pencils homeomorphic to $\mathbb{S}_n$.*

Conversely, each derivation of such a $3n$-dimensional generalized quadrangle is a $2n$-dimensional Laguerre plane.

In 3-dimensional generalized quadrangles, up to duality, $p^{\perp}$ is homeomorphic to the one-point compactification of the real cylinder, see Forst [1981], 3.20. A remarkable result of Schroth [1990] even shows that every 3-dimensional generalized quadrangle can be constructed from a 2-dimensional Laguerre plane: either the derivation at every point of the quadrangle or the derivation at every point of the dual quadrangle yields a 2-dimensional Laguerre plane. Equivalently, every 3-dimensional generalized quadrangle is antiregular up to duality. Recent work of Schroth extends this result to 6-dimensional generalized quadrangles: Every 6-dimensional generalized quadrangle with point rows and line pencils homeomorphic to $\mathbb{S}_2$ is antiregular up to duality. Using this correspondence Schroth [1991] solved the Apollonius problem in 2-dimensional Laguerre planes; cf. Groh [1971b] for a partial solution. (The Apollonius problem asks how many circles touch three given circles, which may be degenerated to points.) To prove Theorem 5.14 for 4-dimensional Laguerre planes, Schroth [1994c] first solved the corresponding Apollonius problem. The Apollonius problem in 2-dimensional Möbius planes was partially solved by Groh [1972]; see also Coxeter [1968].

Theorem 5.14 further allows to construct new Laguerre planes from the 4-dimensional elation Laguerre planes with 10-dimensional automorphism groups by first passing over to the associated Lie geometry and then deriving at a point $\neq \infty$. When deriving at two points of the Lie geometry of a $2n$-dimensional Laguerre plane, $n = 1, 2$, the two derived Laguerre planes are isomorphic if and only if there is an automorphism of the Lie geometry that maps one point of derivation to the other point.

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Author Index

Abbiw-Jackson, D.K. 772
Abels, H. 518, 548, 589
Abramenko, P. 589, 590
Abramenko, P., *see* Abels, H. 589
Adams, J.F. 1278
Adamson, I.T. 1038
Ahrens, R.W. 394
Aigner, M. 195, 219, 224, 227, 228
Ajoodani-Namini, S., *see* Khosrovshahi, G.B. 373
Albert, A.A. 157
Alexandrov, A.D. 925, 928, 929, 936–938
Alltop, W.O. 353
Alpers, B. 934, 976, 996, 997, 1000
Anderson, F.W. 1119, 1139
André, J. 97, 98, 124, 143, 146, 151, 152, 325, 764
Andreescu, D., *see* Radó, F. 934
Anisov, S.S. 1287
Arasu, K.T. 115, 356
Artin, E. 667, 717, 886, 927, 954, 1052, 1058
Artmann, B. 99, 1065, 1118, 1119, 1122, 1123
Artzy, R. 213, 243, 246, 249, 756, 771, 829, 969, 1341
Arumugam, S. 1283
Aschbacher, M. 80, 84, 547, 653, 667, 687, 691, 696, 697, 705, 726, 727, 729, 1154, 1166, 1167, 1180, 1183
Ash, A. 1107
Ash, C.J. 220
Aslaksen, H. 1285
Assmus, E.F. 365
Assmus Jr, E.F. 128, 371
Assouad, P. 914
Astrakov, S.N. 929
Atiyah, M.F. 5

Babai, L. 267, 268
Bachem, A. 202, 267
Bachmann, F. 16, 953, 958, 978, 980, 988, 1002, 1003
Bader, L. 168, 334, 335, 337, 338, 392, 399
Baer, R. 8, 110, 117, 124, 948, 950, 951, 955, 980, 987, 988, 1048, 1064, 1080, 1117, 1122, 1269
Bagchi, B. 252, 361, 363, 369, 370, 375
Bagchi, B., *see* Bagchi, S. 125
Bagchi, S. 125
Baily, W.L. 697
Baily Jr, W.L. 1105
Baker, C.A. 141, 1312
Baker, H.F. 401
Baker, R.D. 167, 335
Bannai, E. 729, 860, 867, 869, 871, 874, 899, 900
Baptist, P. 980
Barbilian, D. 1035
Barlotti, A. 116, 119, 139, 156, 246, 254, 309, 311, 312, 755, 761, 762, 764, 767, 770–772, 965, 967
Barriga, O.E. 162, 181, 182
Bartolone, C. 165, 182, 824, 826, 832, 834, 838
Bartolozzi, F., *see* Bartolone, C. 832, 834, 838
Barwise, J. 742, 743, 773
Bass, H. 1038, 1039
Basterfield, J.G. 226
Bastida, J.R. 219
Batten, L.M. 73, 131, 231, 245
Batten, L.M., *see* Baker, C.A. 141
Baumeister, B. 1217
Baumert, L.D. 356
Beardon, A.F. 1327
Beckman, F.S. 930–932
Bedürftig, T. 1310
Behr, H. 1107
Benabou, J. 1018
Bénéteau, L. 40
Benkart, G.M. 1288
Benson, C.T. 412
Benz, W. 237, 243, 245, 623, 783, 795, 800, 804, 813, 818, 819, 822–824, 826, 827, 829, 831, 834, 838, 923, 924, 927, 930–934, 961, 962, 980, 981, 983, 986–988, 994, 997, 999, 1004, 1339

Berardi, L. 126
Berlekamp, E.R. 452
Bernardi, M. 1288
Beth, T. 99, 351, 356
Betten, D. 162, 1265, 1268, 1269, 1288, 1290–1293, 1296, 1299, 1306, 1310, 1343
Betten, D., *see* Salzmann, H. 1258, 1263, 1271, 1278, 1279, 1281, 1283–1292, 1295–1297
Beutelspacher, A. 12, 125, 131, 230, 232, 323, 326, 328, 329
Beutelspacher, A., *see* Batten, L.M. 131
Biacino, L. 1022
Bichara, A. 408, 718
Biggs, N.L. 216, 328, 360
Biliotti, M. 96, 99, 163, 180
Bingen, F. 1035
Birkhoff, G. 74, 75, 195, 199, 201, 212, 653, 1048, 1119
Biscarini, P. 240
Bixby, R.E. 260
Björner, A. 220, 226, 517, 519, 589
Black, S.C. 367
Blackburn, J.E. 224
Blanck, R. 980
Blass, A. 653, 721
Block, R.E. 252, 355
Blokhuis, A. 126, 127, 303–305, 317, 328, 363, 369, 898–900, 902
Blokhuis, A., *see* Bannai, E. 899, 900
Blokhuis, A., *see* Bruen, A.A. 302
Blumenthal, L. 5
Blumenthal, L.M. 847, 879
Blunck, A. 788, 813, 832
Bödi, R. 1290
Boerner-Lantz, V. 155
Boffa, M. 73
Bogart, K.P., *see* Bonin, J.E. 215
Böhm, J. 896
Bonin, J.E. 215
Bonisoli, A. 241, 243, 249
Borchers, H.G. 925, 928
Borel, A. 549, 567, 568, 571, 572, 635, 704, 1071, 1087, 1094, 1100, 1101, 1103–1107, 1109–1111
Borel, A., *see* Baily Jr, W.L. 1105
Borisov, Y.F. 929
Boros, E. 123, 127, 301
Borovik, A.V. 716, 729
Bose, R.C. 7, 126, 139, 298, 309, 311, 322, 356, 389, 439, 441, 442, 453, 463, 464
Bose, R.C., *see* Bruck, R.H. 124, 143, 144, 326
Bourbaki, N. 3, 480, 502–504, 517, 518, 543, 548, 551, 559, 563, 564, 572, 579, 679, 681, 700, 710, 729, 905, 912, 948, 951, 954, 955, 958, 960, 976, 1136, 1172
Brauer, R. 354
Brauner, H. 965, 974, 975, 990
Brehm, U. 1118, 1119, 1125–1127, 1136, 1137
Breitsprecher, S. 1286, 1288, 1310, 1311
Brickell, E.F. 195, 256
Bridges, W.G. 363
Brieskorn, E. 5, 6
Briginshaw, A.J. 936
Brion, M. 714
Britten, D.J., *see* Benkart, G.M. 1288
Bröcker, L. 813, 1300
Brouwer, A.D. 1225
Brouwer, A.E. 125, 236, 331, 357, 359, 360, 371, 373, 403, 427, 438–441, 453, 459–464, 519, 545, 548, 616, 677, 678, 684, 685, 688, 712, 720, 722, 723, 729, 867, 912, 1164, 1165
Brouwer, A.E., *see* Bagchi, B. 370
Brouwer, A.E., *see* Blokhuis, A. 328, 369
Brouwer, A.E., *see* Wilbrink, H.A. 456
Brown, J.M.N. 113
Brown, K.S. 481, 589, 1107, 1109
Brown, K.S., *see* Abels, H. 589
Brown, R. 697
Bruck, R.H. 40, 124, 128, 143, 144, 148, 246, 253, 326, 327, 353, 441, 766, 803
Bruen, A.A. 126–128, 168, 302, 308, 323, 326, 328, 333, 362, 452, 456
Bruen, A.A., *see* Blokhuis, A. 303–305
Bruhat, F. 481, 482, 591, 1096–1100
Bryant, V. 97
Brylawski, T. 264
Buchanan, T. 121, 1286, 1288, 1292, 1310, 1333
Buekenhout, F. 12, 22, 40, 46, 70, 73–76, 79–82, 84, 85, 89, 90, 92–96, 124, 125, 198, 200, 218, 229, 231, 237, 238, 246, 249, 250, 253, 271, 272, 317, 374, 408, 437, 444, 448, 464, 466, 468, 469, 485, 556, 617, 649, 652, 654, 656, 659, 662, 663, 668, 669, 672, 674, 676, 688, 691, 705, 717, 721, 722, 724, 727, 822, 824, 887, 965–967, 969, 996, 1145, 1150, 1153, 1155, 1163, 1184, 1191, 1192, 1194, 1198, 1199, 1202, 1203, 1209, 1211, 1214–1217, 1219, 1223, 1224, 1227–1229, 1231–1233, 1235, 1243, 1244

Bulitko, V.K. 723
Burau, W. 801, 835, 836
Burmeister, M.V.D. 153
Burn, R.P. 969
Burns, K. 1111, 1280, 1308, 1309
Burton, R.C., *see* Bose, R.C. 322
Burton, R.H., *see* Bose, R.C. 126
Buset, D., *see* Buekenhout, F. 80
Bussemaker, F.C. 884
Bussey, N.J., *see* Veblen, O. 139, 297
Bussey, W.H., *see* Veblen, O. 10, 1117

Calderbank, A.R. 362, 363, 372, 445, 446, 452
Calderbank, A.R., *see* Blokhuis, A. 363
Cameron, P.J. 40, 58, 99, 181, 212, 223, 228, 232, 267–271, 273, 276, 278, 351, 361, 376, 392, 405, 420, 425, 439, 442, 445, 449, 452, 453, 459, 460, 677, 678, 698, 722, 866, 880, 882, 889, 904, 907, 1161, 1199
Camillo, V.P. 1140
Cantwell, J. 1269, 1279
Cartan, E. 18, 517
Carter, R.W. 392, 548, 572, 591, 667, 703, 705, 707
Cartier, P. 1087
Casse, L.R.A. 302, 303, 1348
Ceccherini, P.V. 230
Chang, C.C. 743, 773
Chang, L.C. 440
Charnes, C. 168
Chee, Y.M. 373
Chen, Y. 246, 247, 311, 1004, 1333
Cherlin, G. 270
Cherowitzo, W. 122, 300
Cheung, A.L.C. 202
Chevalley, C. 21, 698, 701, 703, 957
Choe, T.H. 1259
Chowla, S. 353, 372
Clarke, B.L. 1017, 1022
Clebsch, A. 440
Cofman, J. 173, 249, 250, 272
Cohen, A.M. 402, 444, 461, 518, 519, 547, 616, 676, 677, 679, 685, 687, 688, 691, 696, 701, 707, 715, 717–722, 725, 729, 1164, 1195, 1211
Cohen, A.M., *see* Brouwer, A.E. 357, 359, 360, 371, 438, 460–464, 519, 545, 548, 616, 677, 678, 684, 685, 688, 712, 720, 722, 723, 729, 867, 912, 1164, 1165
Cohen, A.M., *see* Buekenhout, F. 75, 96, 649, 656, 662, 663, 668, 669, 672, 691, 705, 1184
Cohen, A.M., *see* Van Leeuwen, M.A.A. 697
Cohen, S.D. 156, 168, 183
Cohn, P.M. 42, 1038, 1040, 1075, 1131
Colbourn, Ch.J., *see* Chee, Y.M. 373
Cole, F.N. 328
Conway, J.H. 331, 332, 362, 466, 661, 677, 726, 861, 869, 871, 908, 910, 912, 1145, 1153, 1180
Conwell, G.M. 339
Cooperstein, B.N. 332, 684, 687, 697, 717, 718, 720, 722, 724
Cooperstein, B.N., *see* Cohen, A.M. 688, 691, 696, 719, 720, 722, 1164
Cossu, A. 120
Coxeter, H.S.M. 440, 452, 500, 501, 517, 846, 849, 855, 870, 889, 891, 893–897, 905, 996, 1158, 1188, 1231, 1238, 1350
Crapo, H.H. 13, 195, 219, 256, 259
Crapo, H.H., *see* Blackburn, J.E. 224
Crawley, P. 218, 255, 1119, 1126
Crismale, M. 168
Curtis, C.W. 519, 704, 1087, 1089, 1092
Curtis, R.T., *see* Conway, J.H. 466, 661, 677, 726, 1145, 1153, 1180
Cushman, R.H., *see* Cohen, A.M. 715
Cuypers, H. 456, 457, 668, 725–728, 1196, 1208, 1212, 1225, 1230
Czerwinski, T. 168, 181

Damerell, R.M. 438, 463
Damerell, R.M., *see* Bannai, E. 867
Damiani, E. 224
D'Antona, O., *see* Damiani, E. 224
Danzer, L. 73, 85, 518, 1188
Darafsheh, M.R. 729
Davenport, D.M., *see* Brickell, E.F. 195, 256
Davis, E.H. 168
Davis, M.W. 506, 518
Davis, P.J. 5
Day, A. 1118, 1122, 1123, 1132
De Bruijn, N.G. 12, 130, 353
De Clerck, F. 337–339, 436, 442–444, 447, 448, 451, 453, 457
De Clerck, F., *see* Thas, J.A. 335, 337, 436, 447, 457
De Finis, M. 418
De Laguna, T. 1017, 1020
De Resmini, M.J. 161

De Soete, M. 418
De Soete, M., *see* De Clerck, F. 448
De Vries, H., *see* Freudenthal, H. 519, 590, 705, 712
Debroey, I. 448–451, 456, 457
Debroey, I., *see* Thas, J.A. 436, 457
Dehon, M. 72, 236, 353, 368, 1244
Dekster, B.V. 934
Del Fra, A. 85, 278, 1158, 1197–1199, 1203, 1244
Delandtsheer, A. 236, 272–276, 1194
Delandtsheer, A., *see* Buekenhout, F. 271, 374, 469, 1155, 1194
Delgado, A.L. 618
Delsarte, P. 360, 365, 368, 369, 371, 857, 863, 867, 874
Delsarte, P., *see* Bannai, E. 899, 900
Delsarte, P., *see* Calderbank, A.R. 372
Delsarte, P., *see* Goethals, J.M. 365
Demazure, M. 1071
Dembowski, P. 7, 13, 56, 57, 99, 109, 111, 114, 124, 140, 149, 151, 154–156, 163, 165, 172, 173, 180, 210, 242, 246, 249, 251, 254, 308, 310, 351, 352, 356, 375, 388, 389, 393, 745, 763, 764, 771, 1165, 1193
Dempwolff, U. 113, 168, 178, 179
Denniston, R.H.F. 120, 128, 247, 311, 314, 327
Deodhar, V. 688
Depunt, J., *see* Willems, M.L.H. 254
Dey, A. 351
Deza, M. 40, 87, 235, 276, 278, 914
Deza, M., *see* Assouad, P. 914
Deza, M., *see* Cameron, P.J. 223, 271, 276
Deza, M., *see* Terwilliger, P. 916
Dickson, L.E. 151, 694
Dicuonzo, V. 242, 246
Dienst, K.J. 242, 249, 254, 426, 669, 988, 1004
Dieudonné, J. 5, 6, 57, 59
Dieudonné, J.A. 664, 667, 952, 957
Dilworth, R.P. 226
Dilworth, R.P., *see* Crawley, P. 218, 255
Dilworth, R.R., *see* Crawley, P. 1119, 1126
Divinsky, N.J. 1038
Dixmier, S. 401, 402
Doignon, J.-P. 97, 1259, 1263, 1270
Dorn, G., *see* Artmann, B. 99
Dowling, T.A. 213, 226, 227
Dowling, T.A., *see* Bose, R.C. 463, 464
Doyen, J. 232, 368, 1193
Doyen, J., *see* Buekenhout, F. 271, 374, 469, 1155, 1194
Drake, D.A. 68, 99, 100, 1062, 1065
Drake, D.A., *see* Artmann, B. 99
Dress, A.W.M. 260, 497, 499, 518, 573, 577, 616, 850
Du Val, P. 904
Duchamp, A. 201, 259, 260
Dugas, M. 1119
Dugundji, J. 1280
Dumer, I.I. 371
Dunbar, J., *see* Laskar, R. 86
Durgaprasad, K.V. 180
Dye, R.H. 331, 332
Dye, R.H., *see* De Clerck, F. 444
Dynkin, E. 19

Ealy Jr, C.E. 424
Ebbinghaus, H.D. 1263
Ebert, G.L. 168
Ebert, G.L., *see* Baker, R.D. 167, 335
Edelman, P. 74
Edmonds, J., *see* Murty, U.S.R. 223, 224
Edmonds, J., *see* Young, H.P. 223
Egawa, Y. 461
Ehresmann, C. 1018
Eich, M.M. 400
Einstein, A. 925
Eisele, E. 1266, 1285
Ellard, C.A. 720–722
Ellers, E.W. 115, 956, 957
Engelking, R. 1276
Erdős, P. 130
Erdős, P., *see* Chowla, S. 372
Erdős, P., *see* De Bruijn, N.G. 12, 130, 353
Esser, M. 1045
Eugeni, F., *see* Berardi, L. 126
Evans, D.M. 270
Ewald, G. 232, 242, 251, 822, 958, 980, 988, 1004, 1337

Faigle, U. 1118, 1134
Faina, G. 242, 315
Faltings, K. 1118, 1119, 1124, 1125
Fano, G. 139, 297
Farrahi, B. 931
Faulkner, J.R. 697, 1035, 1036, 1065, 1067–1069, 1071, 1073
Feit, W. 380, 385, 1153

Fellagara, G. 310
Ferrar, J.C. 1035, 1053–1056, 1062
Ferrar, J.C., *see* Faulkner, J.R. 697, 1065
Ferus, D. 1308
Fick, E. 1266
Figueroa, R. 112
Fink, J.B. 163, 170, 171
Fischer, B. 726, 727, 1199
Fisher, J.C. 123, 167, 301, 334, 336
Fisher, J.C., *see* Bruen, A.A. 127, 128
Fisher, P.H. 247
Flato, M. 936
Flatto, L. 869, 870
Fon-Der-Flaas, D.G., *see* Brouwer, A.D. 1225
Fong, P. 423, 425, 591
Forst, M. 426, 1306, 1307, 1339, 1349, 1350
Förtsch, A. 1331, 1335, 1340, 1343, 1346
Foulser, D.A. 149, 160, 161, 169, 170, 177, 181, 182
Frank, R. 234, 1268
Frankl, P., *see* Calderbank, A.R. 362, 363
Frankl, P., *see* Cameron, P.J. 276
Frankl, P., *see* Deza, M. 276
Freeman, J.W. 328
Freudenthal, H. 12, 250, 252, 401, 519, 572, 573, 590, 639, 694, 697, 705, 707, 712, 767, 1257, 1269, 1272, 1283, 1286
Fritsch, R. 770
Frohardt, D.E. 661
Frucht, R. 775
Frumkin, A. 365
Fuji-Hara, R. 328
Fuller, K.R., *see* Anderson, F.W. 1119, 1139
Funk, M. 745, 750–753, 755, 757, 758, 764, 766, 768, 770, 772–777

Gabriel, P., *see* Demazure, M. 1071
Gamba, A. 937
Ganley, M.J. 124, 155, 162, 179, 182, 398
Ganley, M.J., *see* Cohen, S.D. 156, 168, 183
Gelbgras, B. 78
Gerards, A.M.H. 260
Gerla, G. 1020, 1024
Gerla, G., *see* Biacino, L. 1022
Gerner, E. 89, 92
Gevaert, H. 140, 167, 337, 398, 448
Gevaert, H., *see* De Clerck, F. 338, 442, 444, 448
Gewirtz, A. 313, 333, 440, 441
Gheorghe, C. 938
Ghinelli, D. 1203
Ghinelli, D., *see* Del Fra, A. 85, 278, 1158, 1197–1199, 1203, 1244
Gibert, K.Ya. 836, 838
Gilkey, P.B. 701
Gindikin, S.G., *see* Vinberg, È.B. 1269
Glock, E. 772
Glynn, D.G. 121, 251, 299, 304, 320, 328, 334
Glynn, D.G., *see* Casse, L.R.A. 302, 303
Gochet, P. 5
Godsil, C.D. 867, 913
Goethals, J.M. 361, 365, 442, 860, 862, 868, 869, 873, 874, 885, 887, 888, 1198, 1203
Goethals, J.M., *see* Cameron, P.J. 442, 866, 889, 904, 907
Goethals, J.M., *see* Delsarte, P. 365, 368, 863, 867
Goldschmidt, D.M. 618
Goodaire, E. 140, 141, 177
Goodman, N., *see* Leonard, H.S. 1022
Gorenstein, D. 22
Gorn, S. 212
Götzky, M. 957
Graf, R. 823
Grams, G. 1217
Grätzer, G. 1264
Greene, C. 226
Greferath, M. 1119, 1128–1135
Grieser, D. 213
Griess Jr, R.L. 729
Griess Jr, R.L., *see* Cohen, A.M. 701, 729
Grinberg, E. 228
Gröbner, W. 804
Grochowska, M. 98
Groh, H. 242, 1257, 1265–1267, 1271, 1287, 1298, 1299, 1310, 1332, 1333, 1335, 1338, 1339, 1344, 1345, 1350
Groh, H., *see* Artzy, R. 1341
Groh, H., *see* Choe, T.H. 1259
Gross, H. 664, 667, 689
Gruber, P.M. 854, 860
Grünbaum, B. 5, 589, 865, 1188, 1231
Grundhöfer, T. 112, 116, 152, 156, 1263, 1266, 1280, 1281, 1284, 1285, 1293, 1306–1308, 1350
Grundhöfer, T., *see* Salzmann, H. 1258, 1263, 1271, 1278, 1279, 1281, 1283–1292, 1295–1297
Grüning, K. 125
Grünwald, J. 1035

Grzegorczyk, A. 1017, 1021
Guillitte, St. 507
Guts, A.K. 928, 929

Haagerup, U. 896
Haantjes, J. 879
Haemers, W.H. 356, 357, 388, 389, 442, 444, 446
Haemers, W.H., *see* Brouwer, A.E. 453
Hähl, H. 1287, 1288, 1291, 1292, 1310
Hähl, H., *see* Buchanan, T. 1288, 1292, 1310, 1333
Hähl, H., *see* Salzmann, H. 1258, 1263, 1271, 1278, 1279, 1281, 1283–1292, 1295–1297
Hahn, A.J. 1076
Halder, H.R. 827, 965, 967
Hale, M. 87
Hall, J.I. 230, 457, 458, 653, 721, 724, 727, 1201, 1208, 1231
Hall, J.I., *see* Brouwer, A.E. 462, 677
Hall, J.I., *see* Cuypers, H. 726, 727
Hall, M. 115, 122, 129, 139, 158, 741, 754, 762, 769, 1062
Hall, M., *see* Aschbacher, M. 727
Hall Jr, M. 40, 257, 270, 313, 351, 354, 356, 394, 724
Hall Jr, M., *see* Bridges, W.G. 363
Halsey, M.D. 217
Hamada, N. 362, 363, 365, 368
Hamblin, C.L. 1018
Hanani, H. 228, 372
Hanke, K.U. 1284
Hanssens, G. 99, 100, 688, 720–722
Harary, F., *see* Blass, A. 721
Harder, G. 1105, 1107, 1109, 1110
Haris, S.J. 697
Harrington, L., *see* Cherlin, G. 270
Hartman, A. 372
Hartmanis, J. 208
Hartmann, E. 242–244, 249, 251, 252, 254, 1330, 1334, 1340, 1341
Hartmann, P. 1259, 1284
Hasse, H. 297
Haupt, O. 1257, 1270
Hausknecht, K. 1269
Havel, T.F., *see* Dress, A.W.M. 850
Havlicek, H. 802, 803, 820, 952, 965
Hayden, J.L., *see* Bridges, W.G. 363
Heden, O. 328
Hegerfeldt, G.C., *see* Borchers, H.G. 925, 928
Heise, W. 238, 240, 241, 243, 246, 253–255, 748, 753, 755, 783, 824, 829, 1259, 1331
Heise, W., *see* Halder, H.R. 965, 967
Heiss, S. 1178, 1179, 1189, 1239, 1240, 1242
Heller, I. 261
Hering, C. 112, 113, 161, 165, 169, 172, 174–176, 180, 181, 251, 252, 375, 1330
Hermand, M., *see* Buekenhout, F. 79, 95
Heron, A.P. 211, 226
Herrmann, C., *see* Faigle, U. 1118, 1134
Hersh, R., *see* Davis, P.J. 5
Herssens, C., *see* De Clerck, F. 337, 338
Herssens, C., *see* Thas, J.A. 337
Herstein, I.N. 832
Hertel, E., *see* Böhm, J. 896
Herzer, A. 99, 117, 118, 237, 245, 796, 803, 808, 810–812, 818, 825, 826, 829, 831–833, 838, 1004
Herzer, A., *see* Biliotti, M. 99
Hessenberg, G. 118, 139, 297
Higgs, D.A., *see* Blackburn, J.E. 224
Higman, D.G. 388, 442, 458, 651, 712, 721, 729, 1153, 1170, 1179–1181
Higman, G., *see* Feit, W. 380, 385
Hilbert, D. 6, 11, 14, 139, 987, 988, 1257, 1269, 1286, 1287
Hill, R. 307, 312, 313, 452
Hillebrandt, G. 80, 362
Hiller, H. 714
Hiramine, Y. 158, 168, 183
Hirsch, G. 5
Hirschfeld, J.W.P. 7, 13, 57, 120, 122, 297, 298, 300, 301, 304, 309, 311, 312, 315, 318–321, 325–327, 330, 390, 401, 454
Hirschfeld, J.W.P., *see* Bruen, A.A. 333, 452
Hirschfeld, J.W.P., *see* Fisher, J.C. 123, 301
Hjelmslev, J. (=J. Petersen) 1035
Hobart, S. 1244
Hodge, W.V.D. 392
Hoffman, A.J. 246, 253, 440, 462
Hoggar, S.G. 846
Hölz, G. 254, 808
Horstmann, C., *see* Betten, D. 1268
Hotje, H. 243, 813, 817, 838, 1004
Hua, L.-K. 925, 936
Huang, H. 180
Hubaut, X. 360, 810, 838
Hubaut, X., *see* Buekenhout, F. 70, 85, 654, 721, 1145, 1198, 1199, 1202, 1203
Hubaut, X., *see* Doyen, J. 232, 368, 1193

Hubaut, X., *see* Hirschfeld, J.W.P. 320
Hubaut, X.L. 440, 446
Huber, M. 322, 323
Hubig, M. 1291, 1300
Huckenbeck, U. 929, 934
Hughes, D.R. 34, 87, 109, 110, 112, 140, 153, 155, 159, 272, 355, 358, 425, 1051, 1052, 1062, 1158, 1193, 1215–1217, 1222, 1244
Hughes, D.R., *see* Burmeister, M.V.D. 153
Hughes, D.R., *see* Cameron, P.J. 1199
Hughes, D.R., *see* Del Fra, A. 1244
Hughes, D.R., *see* Dembowski, P. 242
Hughes, D.R., *see* Hobart, S. 1244
Humphreys, J.E. 518, 519, 679, 704, 705
Husemoller, D., *see* Milnor, J. 905
Huybrechts, C. 1164

Iden, O. 243, 753, 755, 756, 761–764, 771
Im Hof, H.-C. 896, 897
Inaba, E. 1117, 1122, 1133
Ingleton, A.W. 220, 263
Ivanov, A.A. 453, 1232–1235, 1239, 1240
Iversen, U. 1331
Iwahori, N. 481
Iwahori, N., *see* Curtis, C.W. 519

Jackson, W.A. 376
Jacobson, N. 692, 953, 967, 1038, 1040, 1058, 1061, 1066
Jäger, M. 1306–1308
Jakóbowski, J. 1341
James, D.G. 667
Jamison, R.E. 323, 324
Jamison, R.E., *see* Edelman, P. 74
Janko, Z. 128
Jantosciak, J., *see* Prenowitz, W. 97
Jantzen, J.C. 712
Jha, V. 140, 156, 166, 168, 169, 171, 178, 179, 182, 183
Jha, V., *see* Biliotti, M. 180
Jha, V., *see* Cohen, S.D. 183
Jha, V., *see* Ganley, M.J. 182
Johnson, N.L. 140, 141, 156–158, 160–162, 168, 171, 176, 177, 179, 182, 183, 332, 334, 335, 337, 339, 423
Johnson, N.L., *see* Biliotti, M. 180
Johnson, N.L., *see* De Clerck, F. 447
Johnson, N.L., *see* Fink, J.B. 163
Johnson, N.L., *see* Foulser, D.A. 161
Johnson, N.L., *see* Gevaert, H. 140, 167, 337, 398
Johnson, N.L., *see* Huang, H. 180
Johnson, N.L., *see* Jha, V. 140, 156, 166, 168, 178, 179, 182
Johnson, P.M. 652, 662, 669, 676, 721, 722
Johnson, P.M., *see* Cuypers, H. 668
Johnstone, P.T. 1019
Jónsson, B. 1117, 1120, 1122, 1133, 1191
Jordan, C. 297, 716
Joussen, J. 763, 768, 1287, 1311
Juel, C. 835, 836
Jungnickel, D. 115, 356
Jungnickel, D., *see* Arasu, K.T. 356
Jungnickel, D., *see* Beth, T. 99, 351, 356
Jungnickel, D., *see* Drake, D.A. 99, 1062, 1065

Kac, V. 702
Kaerlein, G. 245
Kaerlein, G., *see* Chen, Y. 247, 1333
Kahn, J. 212, 233, 245, 261, 262, 264, 265, 273, 1328
Kalhoff, F. 1311
Kallaher, M.J. 140, 150, 156–158, 169, 171, 176, 177, 181
Kallaher, M.J., *see* Fink, J.B. 170, 171
Kallaher, M.J., *see* Foulser, D.A. 182
Kallaher, M.J., *see* Goodaire, E. 140, 141, 177
Kallaher, M.J., *see* Jha, V. 171, 183
Kallaher, M.J., *see* Johnson, N.L. 141, 176, 177
Kalmbach, G. 1264
Kalscheuer, F. 1288–1290
Kaneta, H. 303
Kantor, W.M. 85, 114, 140, 154, 162–164, 210, 211, 223–225, 230, 232, 233, 253, 256, 265–267, 269–271, 273, 331, 332, 334, 365, 373, 374, 376, 395–399, 403, 426, 444, 464, 467, 612, 617, 691, 708, 1153, 1168, 1179, 1180, 1182, 1184, 1193, 1268
Kantor, W.M., *see* Calderbank, A.R. 445, 452
Kantor, W.M., *see* Cameron, P.J. 270, 273, 392, 420, 425
Kaplansky, I. 154
Kaplinskaya, I.M., *see* Vinberg, È.B. 907
Karcher, H., *see* Ferus, D. 1308
Karpelevic, F.I. 20
Karrass, A., *see* Magnus, W. 771
Karzel, H. 7, 15, 71, 99, 100, 115, 767, 783, 813, 829, 836, 958, 964, 965, 969, 988, 1257, 1311
Karzel, H., *see* Ellers, E.W. 115

Karzel, H., *see* Heise, W. 240, 241, 243, 246, 254, 255, 829
Kats, V.G., *see* Vinberg, È.B. 1269
Katz, N.M. 58
Kay, D.C., *see* Cantwell, J. 1269
Kegel, O.H. 113, 773, 777
Kegel, O.H., *see* Funk, M. 745, 750–753, 755, 764, 773–777
Keisler, H.J., *see* Chang, C.C. 743, 773
Kelly, G. 376, 763
Kelly, J.B. 914
Kelly, L.M., *see* Basterfield, J.G. 226
Kempf, G. 1105
Keppens, D. 99, 100, 788, 1062
Kern, W. 234
Kern, W., *see* Bachem, A. 202, 267
Kestenband, B. 123, 301
Key, J.D. 252, 369
Key, J.D., *see* Assmus, E.F. 365
Khalifah, M.F. 795
Khosrovshahi, G.B. 373
Killing, W. 18, 20
Kilmer, D. 117
Kilmoyer, R. 548
Kilmoyer, R., *see* Curtis, C.W. 519
Kiltinen, J.O. 1260, 1284
King, O.H. 1233
King, O.H., *see* Buekenhout, F. 1233
Kirkpatrick, P.B. 158
Kist, G., *see* Karzel, H. 71, 100, 836, 988
Kitcher, P. 5
Klee, V. 199, 200, 203
Kleidman, P.B. 332, 729
Kleidman, P.B., *see* Buekenhout, F. 271, 374, 469, 1155
Kleidman, P.B., *see* Conway, J.H. 331, 332
Klein, F. 1035, 1268
Klein, H. 1269, 1306
Klein, M. 252, 1330
Kleinewillinghöfer, R. 250, 251, 1330
Kleinfeld, E. 139, 156
Kleinfeld, E., *see* Hughes, D.R. 155
Kline, M. 5, 9
Klingenberg, W. 1035, 1053, 1058, 1060, 1063, 1064, 1074, 1257
Klopsch, P. 1003
Knarr, N. 399, 1285, 1288, 1291, 1292, 1307
Knarr, N., *see* Betten, D. 1292
Knarr, N., *see* Grundhöfer, T. 1306, 1308, 1350
Kneser, M. 911, 957, 1100, 1101
Knörrer, H., *see* Brieskorn, E. 6
Knudsen, F., *see* Kempf, G. 1105
Knüppel, F. 1035, 1080
Knus, M.-A. 572
Knuth, D.E. 155, 229
Koch, H. 910
Koehler, P. 1181, 1182
Köhler, P. 614, 617
Kolmogoroff, A.N. 7, 1257, 1263, 1269
Kołodziejczyk, K. 226
Komissartschik, E.A. 1237, 1244
König, M., *see* Karzel, H. 988
Kopejkina, L.I. 763
Korchmáros, G. 122
Korchmáros, G., *see* Bonisoli, A. 243
Korchmáros, G., *see* Faina, G. 315
Korchmáros, G., *see* Segre, B. 126
Kosheleva, O.M., *see* Vroegindewey, P.G. 937
Koszul, J.L. 506, 518
Kottwitz, R.E. 1109, 1110
Kramer, E.S. 370
Kramer, L. 1307
Krebs, R. 251
Kreher, D.L. 371
Kreher, D.L., *see* Chee, Y.M. 373
Kreinovic, V.Ja., *see* Vroegindewey, P.G. 937
Kreuzer, A. 97, 99, 100, 234, 235, 1064, 1133
Krier, N. 242, 251, 1330
Kroll, H.J. 251–254, 808, 988, 1331
Kroll, H.J., *see* Karzel, H. 7, 71, 99, 100, 767, 783, 836, 1257, 1311
Kroll, H.J., *see* Klein, M. 252, 1330
Krull, W. 1061
Krusemeyer, M.I. 1039
Kühne, R. 1260–1262, 1308
Kung, J.P.S. 224, 226–228, 261, 262
Kung, J.P.S., *see* Kahn, J. 212, 262
Künneth, H., *see* Haupt, O. 1257, 1270
Kuppuswamy Rao, K., *see* Durgaprasad, K.V. 180
Kuz'minykh, A.V. 931, 932, 934

Lachlan, A.H., *see* Cherlin, G. 270
Lam, C.W.H. 128, 354, 361, 443
Lander, E.S. 351, 355, 356, 363, 369
Lane, N.D., *see* Baker, C.A. 1312
Lane, R., *see* Hall Jr, M. 313
Lang, K. 788
Lang, S. 219, 264

Laskar, R. 86
Laurent, M., *see* Deza, M. 87, 276, 278
Leader, I. 263
Leavitt, D.W. 370
Leavitt, D.W., *see* Kramer, E.S. 370
Lee, J. 260
Lefèvre, C., *see* Buekenhout, F. 408, 448
Lefèvre-Percsy, C. 87, 315, 376, 408, 457, 1215
Lefèvre-Percsy, C., *see* Buekenhout, F. 669
Lefèvre-Percsy, Ch., *see* Guillitte, St. 507
Lehrer, G.I., *see* Curtis, C.W. 1087, 1089, 1092
Leissner, W. 988
Leißner, W. 1035, 1074, 1079, 1080
Leißner, W., *see* Benz, W. 838
Lekkerkerker, C.G., *see* Gruber, P.M. 854, 860
Lemmens, P.W.H. 858, 884, 898, 905
Lenstra, A.K. 370
Lenstra Jr, H.W., *see* Lenstra, A.K. 370
Lenz, H. 139, 156, 664, 930, 931, 988, 1259, 1268, 1270
Lenz, H., *see* Beth, T. 99, 351, 356
Leonard, D.A. 236
Leonard, H.S. 1022
Lester, J.A. 929, 930, 933, 934, 937
Li, H. 274, 1179, 1180
Li, H., *see* Delandtsheer, A. 274
Libois, P. 12, 129
Lichtenstein, W. 714
Lieb, E. 224
Liebeck, M.W. 1153
Liebeck, M.W., *see* Buekenhout, F. 271, 374, 469, 1155
Liebeck, M.W., *see* Cohen, A.M. 707, 729
Liebeck, M.W., *see* Kantor, W.M. 271
Liebeck, M.W., *see* Kleidman, P.B. 729
Liebler, R. 1168
Liebler, R., *see* Kantor, W.M. 1182
Liebler, R.A. 86, 153, 154, 163, 165
Liebler, R.A., *see* Kantor, W.M. 612, 617
Liebler, R.A., *see* Krier, N. 251
Liepold, F. 1311
Liersch, A., *see* Joussen, J. 763
Lim, M.H. 206, 274
Limaye, B.V. 834
Limaye, N.B., *see* Limaye, B.V. 834
Lindström, B. 220, 256, 263, 264
Lingenberg, R. 958, 988
Linial, N. 369
Lippert, M.F., *see* Groh, H. 1299
Lisser, B., *see* Cohen, A.M. 701, 729
Lisser, B., *see* Van Leeuwen, M.A.A. 697
List, R.J., *see* Black, S.C. 367
Lobačevskij, N.I. (Lobachevsky) 1017
Logothetti, D. 5
Long, C.T., *see* Johnson, N.L. 141
Loos, O. 1067
Lorimer, J.W., *see* Baker, C.A. 1312
Lorimer, J.W. (Mike) 1311, 1312
Lorimer, P. 159
Lovász, L., *see* Björner, A. 220
Lovász, L., *see* Lenstra, A.K. 370
Löwe, H. 1274
Löwen, R. 1262, 1266, 1268, 1271–1283, 1286, 1288, 1289, 1291, 1293–1298, 1300–1305, 1307, 1310, 1331, 1333, 1336, 1340, 1345, 1347, 1348
Löwen, R., *see* Buchanan, T. 1310, 1333
Löwen, R., *see* Hausknecht, K. 1269
Löwen, R., *see* Kühne, R. 1260–1262
Löwen, R., *see* Salzmann, H. 1258, 1263, 1271, 1278, 1279, 1281, 1283–1292, 1295–1297
Lozanov, C. 246
Lucas, D., *see* Brylawski, T. 264
Lück, H.H. 1060
Lunardon, G. 140, 156, 547, 549, 687, 810, 811, 1167–1169
Lunardon, G., *see* Bader, L. 168, 334, 335, 337, 338, 392, 399
Lunardon, G., *see* Johnson, N.L. 168, 334, 337
Lüneburg, H. 115, 140, 143, 144, 146, 149, 150, 161, 163, 168, 173, 174, 176, 181, 182, 252, 331, 403
Lüneburg, M. 1291, 1303
Lunelli, L. 299
Luzzatto, G., *see* Gamba, A. 937

Ma, S.L. 356
Mac Gregor, H.H. 835
Mac Lane, S. 195, 219, 256, 263
MacDuffee, C.C. 363
Mackenzie, K., *see* Key, J.D. 369
MacWilliams, F.J. 13, 128, 307, 360, 361, 371
MacWilliams, F.J., *see* Delsarte, P. 365, 368
Maeda, F. 1117, 1119, 1121, 1122
Maeda, S., *see* Maeda, F. 1117, 1119
Magaard, K. 729
Magliveras, S.S., *see* Kramer, E.S. 370
Magliveras, S.S., *see* Leavitt, D.W. 370

Magnus, W. 771
Mahoney, C.R. 227
Main, R.A., *see* Ingleton, A.W. 220
Mainzer, K. 1257
Mallows, C.L. 880
Maneri, C. 307
Manin, Y.I. 855
Mann, H.B. 356
Marchi, M., *see* Karzel, H. 71
Marcus, M. 846
Margulis, G.A. 1111
Mars, J.G.M. 696, 697, 705
Maruta, T., *see* Kaneta, H. 303
Mason, G. 168, 170, 171
Mason, G., *see* Barriga, O.E. 162
Mason, G., *see* Foulser, D.A. 170
Mason, J.H. 227
Mathon, R. 125, 443
Mathon, R., *see* Bussemaker, F.C. 884
Matsumoto, E., *see* Iwahori, N. 481
Matsumoto, H. 503, 518
Matsumoto, M., *see* Hiramine, Y. 168
Mattson Jr, H.F., *see* Assmus Jr, E.F. 128, 371
Mäurer, H. 218, 242, 244, 248–254, 256, 783, 824, 826, 834, 969, 988, 1004, 1330
Mäurer, H., *see* Benz, W. 826, 838
Mäurer, H., *see* Dienst, K.J. 242
Mazzocca, F. 323, 412, 413, 416, 417
McDonald, B.R. 834, 1040
McFarland, R.L. 356
McKay, J. 860
McKay, J., *see* Lam, C.W.H. 128, 443
McLaughlin, J. 452
McLaughlin, J., *see* Higman, D.G. 1180
McMullen, P. 518, 1188, 1189, 1237
McPherson, A.D., *see* Kantor, W.M. 271
Meierfrankenfeld, U. 618
Meixner, T. 96, 417, 613, 617, 1178, 1181, 1182, 1184, 1190, 1197–1199, 1202–1204, 1206, 1207, 1219, 1220, 1229
Meixner, T., *see* Del Fra, A. 85, 1197–1199, 1203
Meixner, T., *see* Grams, G. 1217
Meixner, T., *see* Kantor, W.M. 617, 1182
Meixner, T., *see* Koehler, P. 1181, 1182
Meixner, T., *see* Köhler, P. 614, 617
Melchior, U. 788, 826
Mendelsohn, E. 146, 775, 776
Menger, K. 653, 847, 849, 879, 1311
Menichetti, G. 154
Menichetti, G., *see* Biliotti, M. 163, 180
Meschiari, M. 252
Mesner, D.M. 328
Mesner, D.M., *see* Kramer, E.S. 370
Metsch, K. 130, 131, 226, 234
Metz, R. 124, 125, 315, 809
Metz, R., *see* Buekenhout, F. 237
Metz, R., *see* Mäurer, H. 834
Meyer, R. 1331
Meyerowitz, A. 86
Meyerowitz, A., *see* Liebler, R.A. 86
Mielants, W., *see* Willems, M.L.H. 254
Mihul, E., *see* Gheorghe, C. 938
Miller, Z., *see* Blass, A. 721
Milnor, J. 905, 1261
Milnor, J.W. 846, 896
Minc, H., *see* Marcus, M. 846
Misfeld, J. 1259, 1260, 1263, 1269
Miskimins, R., *see* Meyerowitz, A. 86
Mitchell, S.A. 1309, 1310
Modenov, P.S. 931, 932
Moe, J.G., *see* Iden, O. 243, 771
Monk, G., *see* Jónsson, B. 1117, 1122, 1133
Monson, B. 1189
Montgomery, D.C. 351
Moorhouse, G.E. 368
Morley, F. 996
Morley, F.V., *see* Morley, F. 996
Morozov, V.V. 20
Moser, W.O.J., *see* Coxeter, H.S.M. 1158, 1188
Motzkin, T.S. 226
Moufang, R. 1268
Moulton, F.R. 11
Mühlherr, B. 589, 590, 636, 698
Mullin, R.C., *see* Erdős, P. 130
Mumford, D., *see* Kempf, G. 1105
Munemasa, A. 723
Munkholm, H.J.M. 896
Munkholm, H.J.M., *see* Haagerup, U. 896
Münzner, H.F. 1307
Münzner, H.F., *see* Ferus, D. 1308
Murty, U.S.R. 223, 224
Mylle, F. 338

Narasimha Sastri, N.S., *see* Bagchi, B. 252
Narasimha Sastry, N.S., *see* Bagchi, B. 252, 369, 375
Narayana Rao, M.L. 168, 180
Narayana Rao, M.L., *see* Durgaprasad, K.V. 180

Nesbitt, C.J., *see* Artin, E. 1058
Neumaier, A. 84, 89, 547, 874, 904, 907, 913, 1179, 1196, 1209, 1223–1225
Neumaier, A., *see* Brouwer, A.E. 357, 360, 371, 438, 460–464, 677, 684, 685, 688, 712, 723, 729, 867, 912
Neumann, P.M. 270, 354
Neutzler, B. 828
Nicod, J. 1017
Nikitin, A.A. 764
Nikitin, A.A., *see* Shirshov, A.I. 764
Nikulin, V.V. 911
Niles, R. 613
Nizette, N. 99, 201
Noll, W. 938
Nolte, W. 958, 988, 1003
Nolte, W., *see* Mäurer, H. 834
Norton, S.P. 727
Norton, S.P., *see* Conway, J.H. 466, 661, 677, 726, 1145, 1153, 1180

Oakden, D.J. 168
Odlyzko, A.M. 860
Ohmori, H., *see* Hamada, N. 368
Ohn, C. 74, 200
O'Keefe, C. 122, 300, 310
O'Keefe, C.M. 247
Olanda, D. 254
Olanda, D., *see* Mazzocca, F. 416, 417
O'Meara, O.T., *see* Hahn, A.J. 1076
Orlik, P. 716
Orr, W.F. 167, 246, 334, 339
Osborn, J.M., *see* Benkart, G.M. 1288
Ostrom, T.G. 114, 140, 146, 150, 151, 156, 160, 163, 168–172, 174, 176, 177, 184
Ostrom, T.G., *see* Bartolone, C. 165
Ostrom, T.G., *see* Jha, V. 169
Ostrom, T.G., *see* Johnson, N.L. 157, 171, 183
Ostrom, T.G., *see* Kallaher, M.J. 169, 171, 177, 181
Ostrom, T.G., *see* Mason, G. 168, 171
Ott, U. 114, 161, 405, 548, 988, 1168
Ott, U., *see* Bruen, A.A. 362
Otte, J. 1310
Ovchinnikova, V.V., *see* Alexandrov, A.D. 936
Oxley, J.G. 201, 216, 260, 262, 265
Oxley, J.G., *see* Kung, J.P.S. 261
Oyama, T., *see* Hiramine, Y. 168

Panella, G. 246, 309
Papert, D. 1018
Papert, S., *see* Papert, D. 1018
Parker, R.A., *see* Conway, J.H. 466, 661, 677, 726, 1145, 1153, 1180
Parkhomenko, A.S., *see* Modenov, P.S. 931, 932
Pasch, M. 5, 6, 10, 987, 1268, 1270
Pasechnik, D. 1158, 1199, 1202, 1222, 1225, 1234
Pasini, A. 75, 77, 81, 92, 93, 96, 276, 425, 547, 616, 682, 1152, 1164–1170, 1184–1187, 1190, 1191, 1193, 1195, 1196, 1198, 1199, 1202–1205, 1211, 1212, 1214, 1217–1219, 1225, 1226
Pasini, A., *see* Biliotti, M. 96
Pasini, A., *see* Cameron, P.J. 1199
Pasini, A., *see* Cuypers, H. 668, 725, 1196, 1208
Pasini, A., *see* Del Fra, A. 85, 278, 1197–1199, 1203
Pasini, A., *see* Deza, M. 276
Pasini, A., *see* King, O.H. 1233
Pasini, A., *see* Lunardon, G. 547, 549, 687, 1167–1169
Pasini, A., *see* Meixner, T. 1190, 1199, 1206, 1207
Passman, D.S. 151
Patterson, N. 1198
Pavicić, M. 1264
Payne, S.E. 121, 167, 168, 244, 331–334, 337–339, 385, 388, 390, 394–406, 411, 412, 415, 417, 418, 420–423, 441, 760, 1348, 1349
Payne, S.E., *see* Cameron, P.J. 405
Payne, S.E., *see* Eich, M.M. 400
Payne, S.E., *see* Thas, J.A. 331, 415, 423, 468
Payne, S.E., *see* Van Maldeghem, H. 418, 424
Pedoe, D. 996
Pedoe, D., *see* Hodge, W.V.D. 392
Pedrini, C. 243
Pellegrino, G. 313, 452
Pentilla, T. 122
Pentilla, T., *see* O'Keefe, C. 122
Penttila, T. 253, 299, 300
Penttila, T., *see* Fisher, P.H. 247
Penttila, T., *see* O'Keefe, C. 300, 310
Penttila, T., *see* O'Keefe, C.M. 247
Percsy, N. 70, 77, 243, 246, 249, 255, 662
Perkel, M. 86
Petersen, J., *see under* Hjelmslev, J.
Petkantschin, B. 823, 1331
Pfüller, U. 1345

Pfüller, U., *see* Löwen, R. 1345
Pica, G. 1165
Pickering, D., *see* Day, A. 1118, 1122, 1123, 1132
Pickert, G. 109, 116, 118, 140–142, 744, 754, 762, 969, 1284, 1290
Piff, M.J. 228, 259, 263
Pinneri, I., *see* Pentilla, T. 122
Pinneri, I., *see* Penttila, T. 299, 300
Piper, F.C. 125, 376
Piper, F.C., *see* Hughes, D.R. 34, 87, 109, 110, 112, 140, 153, 358, 425, 1051, 1052, 1062
Pjateckiĭ-Šhapiro, I.I., *see* Vinberg, È.B. 1269
Platonov, V.P. 1110
Plaumann, P. 1288, 1292, 1312
Pless, V. 372
Pohl, H.-J. 1291
Pohl, H.-J., *see* Groh, H. 1299
Poincaré, H. 16
Polley, C. 1279, 1300
Pomareda, R., *see* Barriga, O.E. 181, 182
Pomareda, R., *see* Johnson, N.L. 339
Pontrjagin, L.S. 1263, 1285
Popovici, I. 929
Pott, A. 356
Pott, A., *see* Arasu, K.T. 356
Praeger, C.E., *see* Fisher, P.H. 247
Praeger, C.E., *see* Liebeck, M.W. 1153
Prasad, G. 1110
Prasad, G., *see* Borel, A. 1110
Prenowitz, W. 97
Priess-Crampe, S. 11
Prieß-Crampe, S. 1279, 1287, 1311
Prieß-Crampe, S., *see* Kalhoff, F. 1311
Prieß-Crampe, S., *see* Liepold, F. 1311
Proctor, R.A. 519
Prohaska, O. 176
Pták, P. 1264
Pudlák, P. 213, 255
Pulmannová, S., *see* Pták, P. 1264
Pultr, A. 1023

Quaisser, E. 958, 988
Quaisser, E., *see* Rautenberg, W. 988
Quarles Jr, D.A., *see* Beckman, F.S. 930–932
Quattrocchi, P. 241
Quattrocchi, P., *see* Bonisoli, A. 249
Quattrocchi, P., *see* Meschiari, M. 252
Qvist, B. 121, 309

Radó, F. 931, 933, 934, 1035, 1074, 1079, 1080
Rado, R. 200
Rădulescu, D.C., *see* Popovici, I. 929
Radziszowski, S.P., *see* Kreher, D.L. 371
Raghavarao, D. 351
Raghunathan, M.S. 1105
Rahilly, A.J. 159
Raphael, R. 1081
Rätz, J. 924, 927
Rautenberg, W. 988
Ray-Chaudhuri, D.K. 273, 353
Rees, S. 80, 339, 1165
Regonati, F., *see* Damiani, E. 224
Reid, T.J., *see* Oxley, J.G. 260
Reifart, A., *see* Dempwolff, U. 168
Rice, B.F., *see* McFarland, R.L. 356
Richardson, R. 706
Riesinger, R. 1288
Rinaldi, G. 254
Rinaldi, G., *see* Quattrocchi, P. 241
Rinauro, S. 1204, 1221
Rink, R. 150, 1284
Robertson, A.P. 200
Rogers, L.A. 423
Rogers, L.A., *see* Payne, S.E. 398
Rohlfs, J. 1111
Rohlfs, J., *see* Margulis, G.A. 1111
Röhrle, G., *see* Richardson, R. 706
Romanowicz, Z., *see* Kołodziejczyk, K. 226
Ronan, M.A. 70, 77, 84, 417, 419, 420, 480, 481, 488, 499, 532, 580, 587, 588, 598, 614, 615, 617, 619, 625, 626, 629, 631, 634, 637, 638, 654, 661, 678, 713, 720, 722, 765, 1100, 1173, 1179, 1181, 1182, 1189, 1191, 1196, 1199, 1211, 1215, 1222–1224, 1226, 1239
Roos, C. 371, 463
Roos, C., *see* Damerell, R.M. 463
Roos, C., *see* Haemers, W.H. 388
Rosenfeld, B.A. 846
Rosenthal, J.W., *see* Ash, C.J. 220
Rostom, N.K.A., *see* Key, J.D. 252
Rota, G.C. 228
Rota, G.C., *see* Crapo, H.H. 13, 195, 219, 256, 259
Roth, R.M., *see* Seroussi, G. 305, 309
Rothaus, O.S. 938
Rothschild, B.L., *see* Linial, N. 369
Rowley, P. 1239, 1240
Royle, G.F., *see* Fisher, P.H. 247

Royle, G.F., *see* O'Keefe, C. 310
Royle, G.F., *see* Pentilla, T. 122
Royle, G.F., *see* Penttila, T. 300
Rudolph, R., *see* Ash, A. 1107
Ryba, A.J.E., *see* Griess Jr, R.L. 729
Ryba, A.J.E., *see* Kleidman, P.B. 729
Ryser, H.J. 308, 356
Ryser, H.J., *see* Bruck, R.H. 128, 353
Ryser, H.J., *see* Chowla, S. 353
Ryser, H.J., *see* Hall Jr, M. 356

Sabidussi, G. 775
Sabidussi, G., *see* Deza, M. 40, 235
Sachar, H.E. 363, 368
Sah, Chih Han 716
Saint-Donat, B., *see* Kempf, G. 1105
Salzmann, H. 1257, 1258, 1263, 1266, 1270, 1271, 1278–1281, 1283–1292, 1295–1299, 1310, 1336
Salzmann, H., *see* Grundhöfer, T. 1263, 1285
Salzmann, H., *see* Löwen, R. 1289
Samaga, H.-J. 246, 803, 827–829, 933, 988, 999
Samaga, H.-J., *see* Benz, W. 804, 813, 829, 831, 834, 838
Sandler, R. 771
Sarli, J. 709
Sasaki, U. 230
Satake, I. 572, 707
Satyanarayana, K., *see* Narayana Rao, M.L. 168
Saxl, J., *see* Buekenhout, F. 271, 374, 469, 1155
Saxl, J., *see* Cohen, A.M. 707, 729
Saxl, J., *see* Liebeck, M.W. 1153
Sce, M., *see* Lunelli, L. 299
Schaeffer, H. 246, 833, 933, 988, 989, 1004
Schaeffer, H., *see* Benz, W. 804, 813, 829, 831, 834, 838
Schaeffer, H.J. 161
Schaeffer, H.J., *see* Hering, C. 112, 113
Schäffer, J.J. 938
Schäffer, J.J., *see* Noll, W. 938
Scharlau, R. 94, 497, 499, 547, 558, 573, 577, 589, 605, 615, 682, 1162, 1163
Scharlau, R., *see* Dress, A.W.M. 573, 577, 616
Scharlau, W. 572, 635, 667, 953
Schellekens, G.L. 690
Schellhammer, I. 1296
Schenkel, A. 1332, 1335, 1340, 1341
Schilling, O.F.G. 1061
Schleiermacher, A. 116, 242, 744, 748, 752–756, 761, 763, 768, 770, 772
Schleiermacher, A., *see* Kegel, O.H. 113, 773, 777
Schmidt, H.J. 1018, 1025
Schmidt, S.E. 1118, 1128–1130
Schmidt, S.E., *see* Greferath, M. 1119, 1128–1131, 1133–1135
Schnabel, R. 988
Schneider, J.E. 716
Schoenberg, I.J. 914
Scholz, W. 1336
Schreiber, E. 764
Schreiber, E., *see* Barlotti, A. 767, 770–772
Schreiber, U., *see* Weihrauch, K. 1018, 1023, 1024
Schrijver, A., *see* Van Lint, J.H. 445
Schröder, E.M. 820, 829, 838, 930, 931, 933, 934, 947, 958–962, 966, 967, 969, 973–976, 979–981, 983, 986–992, 994, 1002–1004
Schröder, E.M., *see* Alpers, B. 934
Schröder, E.M., *see* Baptist, P. 980
Schröder, E.M., *see* Blanck, R. 980
Schroth, A.E. 426, 1307, 1308, 1350
Schulte, E. 85, 518
Schulte, E., *see* Danzer, L. 73, 85, 518, 1188
Schulte, E., *see* McMullen, P. 518, 1188, 1189, 1237
Schultz, R.H. 181
Schultze, K.-P. 254
Schur, F. 1268
Schützenberger, M.P. 353, 1117
Schwabhäuser, W. 987
Schwerdtfeger, H. 996
Schwermer, J. 1109
Segal, G. 702
Segre, B. 7, 57, 121–124, 126, 246, 298–303, 310–312, 325, 332, 452, 967, 1333
Segre, C. 835, 1035
Seib, M. 119, 124, 358
Seidel, H.-P. 1301, 1302, 1305, 1306
Seidel, J.J. 401, 439, 440, 446, 458, 849, 857, 874, 875, 882, 886, 887, 896–898
Seidel, J.J., *see* Bannai, E. 899, 900
Seidel, J.J., *see* Berlekamp, E.R. 452
Seidel, J.J., *see* Blokhuis, A. 899
Seidel, J.J., *see* Bussemaker, F.C. 884
Seidel, J.J., *see* Cameron, P.J. 442, 866, 889, 904, 907

Seidel, J.J., *see* Delsarte, P. 863, 867, 874
Seidel, J.J., *see* Goethals, J.M. 361, 442, 860, 862, 868, 869, 873, 874, 885, 887, 888, 1198, 1203
Seidel, J.J., *see* Haantjes, J. 879
Seidel, J.J., *see* Lemmens, P.W.H. 858, 884, 898, 905
Seidel, J.J., *see* Neumaier, A. 874, 904, 907
Seidel, J.J., *see* Van Lint, J.H. 439, 880
Seier, W. 788
Seifert, H. 498
Seitz, G.M. 612, 728, 729, 1167, 1180, 1181
Seitz, G.M., *see* Cohen, A.M. 707, 729
Seitz, G.M., *see* Fong, P. 423, 425, 591
Seitz, G.M., *see* Gilkey, P.B. 701
Seroussi, G. 305, 309
Serre, J.-P. 573, 589, 618, 908, 1098, 1108, 1109
Serre, J.-P., *see* Borel, A. 1094, 1100, 1101, 1103, 1105–1107
Seshadri, C.S. 715
Severin, R., *see* Leißner, W. 1080
Seybold, H., *see* Heise, W. 238
Seymour, P.D. 216, 224, 260, 261, 265, 867
Shad, S.A. 459, 462
Shearer, J.B., *see* Brouwer, A.E. 373
Shephard, G.C., *see* Grünbaum, B. 5, 865
Sherk, F.A. 154
Sherman, B.F. 242, 321
Shirshov, A.I. 764
Shpectorov, S.V. 1231, 1233, 1239, 1240
Shpectorov, S.V., *see* Brouwer, A.D. 1225
Shpectorov, S.V., *see* Ivanov, A.A. 453, 1232–1235, 1239
Shpectorov, S.V., *see* Munemasa, A. 723
Shrikhande, S.S. 353
Shrikhande, S.S., *see* Bose, R.C. 389, 453
Shult, E. 886, 887, 1201
Shult, E., *see* Buekenhout, F. 887
Shult, E., *see* Cohen, A.M. 1195, 1211
Shult, E., *see* Hall, J.I. 1201
Shult, E.E. 331, 332, 401, 458–461, 656, 676–678, 688, 718, 719, 721, 722, 724
Shult, E.E., *see* Buekenhout, F. 46, 556, 652, 662, 676, 722
Shult, E.E., *see* Cameron, P.J. 889, 904, 907
Shult, E.E., *see* Cohen, A.M. 676, 725
Shult, E.E., *see* Cooperstein, B.N. 724
Shult, E.E., *see* Cuypers, H. 728
Shult, E.E., *see* Ellard, C.A. 720–722
Shult, E.E., *see* Hall, J.I. 721, 724
Shult, E.E., *see* Johnson, P.M. 652, 721, 722
Shult, E.E., *see* Mason, G. 171
Shult, E.E., *see* Shad, S.A. 459, 462
Siebenmann, E.C. 761
Siegel, C.L. 1105
Silverman, R., *see* Bruen, A.A. 308
Silverman, R., *see* Maneri, C. 307
Silvester, J.R. 1076
Singer, J. 114, 356
Singhi, N.M., *see* Cameron, P.J. 276
Singleton, R.R. 412, 462
Singleton, R.R., *see* Hoffman, A.J. 440, 462
Sinkhorn, R. 895
Skala, H.L. 1311
Skornjakov, L.A. 1119, 1140, 1257, 1266, 1271, 1283–1286
Sloane, N.J.A., *see* Bannai, E. 860
Sloane, N.J.A., *see* Brouwer, A.E. 373
Sloane, N.J.A., *see* Calderbank, A.R. 372
Sloane, N.J.A., *see* Conway, J.H. 362, 861, 869, 871, 908, 910, 912
Sloane, N.J.A., *see* MacWilliams, F.J. 13, 128, 307, 360, 361, 371
Sloane, N.J.A., *see* Mallows, C.L. 880
Sloane, N.J.A., *see* Odlyzko, A.M. 860
Smid, L.J., *see* Van der Waerden, B.L. 237, 245, 783, 823, 1004, 1334
Smit-Ginelli, D.G. 356
Smith, H.J.S. 363
Smith, J.T. 988
Smith, K.J.C. 365
Smith, S.D., *see* Aschbacher, M. 1180
Smith, S.D., *see* Frohardt, D.E. 661
Smith, S.D., *see* Ronan, M.A. 84, 617, 713, 1182, 1189, 1222, 1224
Smith, W.D., *see* Brouwer, A.E. 373
Snapper, E. 927, 928, 963, 979
Sobolev, S.L. 868
Soicher, L. 1224, 1225
Solitar, D., *see* Magnus, W. 771
Solomon, L. 543, 548, 573
Solomon, L., *see* Kilmoyer, R. 548
Solomon, L., *see* Orlik, P. 716
Sörensen, K. 230, 235, 988, 1259, 1263, 1266, 1268, 1269
Sörensen, K., *see* Heise, W. 243, 748, 753, 755, 1259
Sörensen, K., *see* Karzel, H. 99, 964, 965, 969, 988

Sörensen, K., *see* Kroll, H.J. 988
Sós, V.T., *see* Erdős, P. 130
Spanier, E.H. 498
Spatzier, R., *see* Burns, K. 1111, 1280, 1308, 1309
Spence, E. 442
Spera, A.G., *see* Bartolone, C. 826
Sperner, E. 201, 958, 988
Sperner, P. 1292, 1297, 1298, 1301
Spivak, M. 6
Sprague, A. 86–88, 447, 718, 1207
Sprague, A., *see* Buekenhout, F. 662, 1163
Sprague, A.P. 51, 254
Springer, T.A. 573, 690–692, 694, 702, 704, 707, 1035, 1065, 1067
Springer, T.A., *see* Van der Blij, F. 1066
Stanik, R., *see* Karzel, H. 988
Stanley, R.P. 227, 589
Stasheff, J., *see* Milnor, J. 1261
Stein, E.M. 851
Steinberg, R. 703, 704, 707
Steinberg, R., *see* Richardson, R. 706
Steinke, G.F. 1280, 1287, 1332, 1334, 1335, 1337, 1340–1348
Stellmacher, B. 618, 727
Stellmacher, B., *see* Delgado, A.L. 618
Stephenson, W. 1136, 1140
Stern, M. 1264
Sternheimer, D., *see* Flato, M. 936
Stettler, R. 927
Stevenson, F.W. 1123
Stinson, D.R., *see* Erdős, P. 130
Storme, L. 302, 304, 305
Storme, L., *see* Hirschfeld, J.W.P. 315
Strambach, K. 776, 1293, 1297, 1299, 1311, 1312, 1332, 1335–1338
Strambach, K., *see* Barlotti, A. 755, 761, 762, 764, 767, 770–772
Strambach, K., *see* Freudenthal, H. 250, 252, 767
Strambach, K., *see* Funk, M. 745, 750–753, 755, 757, 758, 764, 766, 768, 770, 772–777
Strambach, K., *see* Grundhöfer, T. 1293
Strambach, K., *see* Plaumann, P. 1288, 1292
Strambach, K., *see* Schleiermacher, A. 242, 744, 748, 752–756, 761, 763, 768, 770, 772
Straume, E. 506, 518
Straus, E.G., *see* Chowla, S. 372
Street, A.P. 351
Street, D.J., *see* Street, A.P. 351
Stroppel, M. 1268, 1280, 1283, 1294, 1295, 1297, 1298, 1300, 1301, 1303
Stroppel, M., *see* Blunck, A. 788
Stroppel, M., *see* Grundhöfer, T. 1280, 1281
Stroppel, M., *see* Salzmann, H. 1258, 1263, 1271, 1278, 1279, 1281, 1283–1292, 1295–1297
Stroth, G. 93, 613, 1161, 1178, 1179, 1181, 1183, 1184, 1189
Stroth, G., *see* Ronan, M.A. 84, 617, 1179, 1189, 1211, 1239
Stroth, G., *see* Shpectorov, S.V. 1239, 1240
Study, E. 20, 1035
Suetake, C. 165
Suzuki, M. 1203
Swiercz, S., *see* Lam, C.W.H. 128, 354, 361, 443
Szambien, H. 1259, 1264, 1284, 1311
Szekeres, G., *see* Ahrens, R.W. 394
Szenthe, J. 1282, 1289
Szmielew, W. 98
Szmielew, W., *see* Schwabhäuser, W. 987
Szönyi, T. 123
Szönyi, T., *see* Blokhuis, A. 317
Szönyi, T., *see* Boros, E. 123, 301

Tallini, G. 51, 126, 412, 414, 718, 933, 967
Tallini, G., *see* Bichara, A. 718
Tallini, G., *see* Ceccherini, P.V. 230
Tallini, G., *see* Mazzocca, F. 323
Tallini Scafati, M. 316, 318, 320
Tamagawa, T. 1110
Tarski, A., *see* Schwabhäuser, W. 987
Taylor, D.E. 667, 858, 882
Taylor, D.E., *see* Seidel, J.J. 882, 886
Tecklenburg, H. 98, 119, 1311
Teirlinck, L. 69, 70, 72, 201, 210, 212, 229–231, 234, 235, 237, 267, 353, 368, 373, 654, 656
Terwilliger, P. 916
Testerman, D. 729
Thas, J.A. 120, 122, 140, 167, 235, 247, 248, 253, 254, 265, 301–305, 310, 311, 313–317, 330–339, 397, 398, 401, 403–408, 412–419, 423, 424, 426, 436, 443–445, 447, 448, 452, 454–457, 468, 838, 1056, 1199, 1333, 1348, 1349
Thas, J.A., *see* Bader, L. 168, 334, 337, 338, 399

Thas, J.A., *see* Blokhuis, A. 303–305
Thas, J.A., *see* Bruen, A.A. 126, 127, 302, 323, 328, 456
Thas, J.A., *see* Cameron, P.J. 405
Thas, J.A., *see* Casse, L.R.A. 1348
Thas, J.A., *see* De Clerck, F. 338, 339, 436, 442, 444, 448, 451, 457
Thas, J.A., *see* De Soete, M. 418
Thas, J.A., *see* Debroey, I. 448–450, 456, 457
Thas, J.A., *see* Fisher, J.C. 123, 167, 301, 334, 336
Thas, J.A., *see* Gevaert, H. 140
Thas, J.A., *see* Hanssens, G. 720, 722
Thas, J.A., *see* Hirschfeld, J.W.P. 7, 13, 297, 311, 312, 315, 318, 319, 325, 326, 330, 454
Thas, J.A., *see* Payne, S.E. 168, 331–333, 337–339, 385, 388, 390, 394–396, 398, 400–403, 405, 406, 411, 412, 415, 417, 420–422, 441, 760, 1348, 1349
Thas, J.A., *see* Shult, E.E. 724
Thas, J.A., *see* Storme, L. 302, 304, 305
Thas, J.A., *see* Van Maldeghem, H. 418, 424
Thas, J.A., *see* Willems, M.L.H. 254, 308
Thiel, L., *see* Lam, C.W.H. 128, 354, 361, 443
Thom, R. 5
Thomas, S. 230
Thompson, J.G., *see* MacWilliams, F.J. 128, 361
Thomsen, M.J., *see* Karzel, H. 15
Thorbergsson, G. 1112, 1308, 1309
Thrall, R.M., *see* Artin, E. 1058
Threlfall, W., *see* Seifert, H. 498
Tillmann, J. 1284, 1333
Timm, J., *see* Heise, W. 253
Timmesfeld, F.G. 545, 613, 614, 617, 726–728, 1178, 1181, 1183, 1184, 1186, 1187
Timmesfeld, F.G., *see* Meixner, T. 96
Timmesfeld, F.G., *see* Stellmacher, B. 618
Tits, J. 3, 7, 12, 17, 18, 20, 22, 45, 46, 49, 70, 76–82, 88, 90, 93–96, 163, 244, 251, 267, 310, 385, 387, 391, 392, 404, 423, 425–427, 453, 481, 482, 488, 497, 499, 500, 512, 515, 518, 520, 521, 523, 532, 538, 539, 546–550, 558, 560, 561, 563, 568, 569, 571–573, 577, 585, 586, 589–592, 605, 606, 608, 612, 614–619, 633–639, 662–668, 670, 676, 677, 681, 682, 684, 686, 688, 691, 698, 704, 709, 717, 722, 724, 725, 754, 823, 967, 1110, 1145, 1162–1166, 1172, 1186, 1203, 1222, 1290, 1295
Tits, J., *see* Borel, A. 571, 572, 1103
Tits, J., *see* Bruhat, F. 481, 482, 591, 1096–1100
Tits, J., *see* Cohen, A.M. 402, 461
Tits, J., *see* Curtis, C.W. 1087, 1089, 1092
Tits, J., *see* Kantor, W.M. 612, 617, 1182
Tits, J., *see* Ronan, M.A. 481, 615, 619, 625, 626, 631, 634, 637, 638, 765
Todd, J.A. 255
Tonchev, V.D. 368, 444
Tonchev, V.D., *see* De Clerck, F. 442
Törner, G. 100, 1064
Törner, G., *see* Artmann, B. 99
Torrechante, C.E. 1334
Tortora, R., *see* Gerla, G. 1020
Totten, J. 131
Troyer, R.J., *see* Snapper, E. 927, 928, 963, 979
Truemper, K. 261
Tsaranov, S.V. 1160, 1191, 1236
Tsaranov, S.V., *see* Komissartschik, E.A. 1237, 1244
Tsaranov, S.V., *see* Pasini, A. 1191
Tůma, J., *see* Pudlák, P. 213, 255
Tutte, W.T. 195, 216, 260
Tymoczko, T. 5
Tzanakis, N. 452

Valcan, D., *see* Radó, F. 934
Valette, A. 76, 80
Valette, G. 241
Vámos, P. 260, 262
Van, Trung Tran, *see* Janko, Z. 128
Van Bon, J. 726, 727, 1199, 1202, 1228, 1229
Van der Blij, F. 1066
Van der Kallen, W. 1039, 1040
Van der Waerden, B.L. 237, 245, 783, 823, 849, 1004, 1334
Van Leeuwen, M.A.A. 697
Van Lint, J.H. 361, 439, 442, 445, 880, 896
Van Lint, J.H., *see* Berlekamp, E.R. 452
Van Lint, J.H., *see* Brouwer, A.E. 360, 439, 440
Van Lint, J.H., *see* Cameron, P.J. 58, 351, 439, 445
Van Lint, J.H., *see* Haemers, W.H. 444
Van Maldeghem, H. 100, 418, 420, 424–426
Van Maldeghem, H., *see* Buekenhout, F. 464, 466, 468, 1150, 1153
Van Maldeghem, H., *see* De Clerck, F. 453

Van Maldeghem, H., *see* Grundhöfer, T. 1284, 1307
Van Maldeghem, H., *see* Hanssens, G. 99, 100, 721
Van Maldeghem, H., *see* Thas, J.A. 418, 423, 426, 468
Van Nypelseer, L. 1215
Van Nypelseer, L., *see* Lefèvre-Percsy, C. 1215
Van Zanten, A.J., *see* Damerell, R.M. 463
Van Zanten, A.J., *see* Roos, C. 463
Vanden Cruyce, P. 1238
Vandensavel, M., *see* Doyen, J. 368
Vanstone, S.A. 130
Vanstone, S.A., *see* Fuji-Hara, R. 328
Varadarajan, V.S. 705, 712
Vasershtein, L.N. 1039
Vavilov, N.A. 519
Veblen, O. 8, 10, 13, 14, 35, 99, 139, 141, 229, 297, 722, 724, 835, 1117, 1191
Vedder, K., *see* Jungnickel, D. 115
Veldkamp, F.D. 70, 638, 654, 656, 663, 668, 676, 722, 1035, 1038–1040, 1045–1055, 1059, 1061, 1062, 1064, 1067, 1080, 1081
Veldkamp, F.D., *see* Ferrar, J.C. 1035, 1053–1056, 1062
Veldkamp, F.D., *see* Springer, T.A. 1035, 1065, 1067
Venkov, B.B. 910
Venkov, B.B., *see* Koch, H. 910
Vinberg, È.B. 518, 896, 907, 908, 911, 1269
Voloch, J.F. 301
Voloch, J.F., *see* Hirschfeld, J.W.P. 315
Von Neumann, J. 1117, 1119, 1121, 1123, 1129
Von Staudt, Ch. 116, 783, 835
Von Staudt, K.G.C. 139, 297
Vroegindewey, P.G. 928, 937

Wagner, A. 114
Wagner, A., *see* Betten, D. 1293
Wagner, A., *see* Dembowski, P. 56, 210, 375
Wagner, A., *see* Ostrom, T.G. 114
Wähling, H. 988
Wähling, H., *see* Hanke, K.U. 1284
Wales, D.B., *see* Calderbank, A.R. 446
Wales, D.B., *see* Cohen, A.M. 729
Wales, D.B., *see* Hall Jr, M. 313
Walker, M. 140, 159, 162, 163, 167, 183, 334, 424, 425
Walker, M., *see* Foulser, D.A. 170
Walker, R.J. 156
Wallach, N., *see* Borel, A. 1109
Wallis, W.D. 443
Wang, Q.-M. 1308
Wanka, A., *see* Bachem, A. 202
Wanner, T. 257
Warner, S. 1284
Webster, R., *see* Bryant, V. 97
Wedderburn, J.H.M., *see* Veblen, O. 13, 139, 141
Weetman, G. 1158, 1159
Weetman, G.M. 722, 723
Weigand, C. 1292
Weigand, C., *see* Betten, D. 1293
Weigand, C., *see* Knarr, N. 1285
Weihrauch, K. 1018, 1023, 1024
Weil, A. 297
Weiss, A.I. 1189
Weiss, A.I., *see* Coxeter, H.S.M. 1238
Weiss, A.I., *see* Monson, B. 1189
Weiss, G., *see* Stein, E.M. 851
Weiss, R. 426, 586, 1198, 1199, 1203, 1227–1229
Weiss, R., *see* Stroth, G. 93
Weiss, R., *see* Van Bon, J. 726, 727, 1199, 1202
Weiss, R., *see* Van Maldeghem, H. 425
Wells, A.L. 661
Welsh, D.J.A. 30, 74, 195, 201, 203, 216, 219, 223, 227, 258, 260, 264
Wenzel, W. 202, 260
Wenzel, W., *see* Dress, A.W.M. 260
Werner, M. 810, 811, 813, 814, 836, 838
Wester, M. 1179
Wester, M., *see* Kantor, W.M. 617, 1182
Wester, M., *see* Koehler, P. 1181, 1182
Wester, M., *see* Köhler, P. 614, 617
Weston, J.D., *see* Robertson, A.P. 200
White, N.L. 74, 195, 206, 210, 216, 218, 219, 224, 228, 257–261, 264
Whitehead, A.N. 1017, 1020, 1022
Whitesides, S.H. 128
Whitman, P.M. 255
Whitney, H. 12, 195, 257
Wilbrink, H.A. 98, 115, 243, 252, 356, 456
Wilbrink, H.A., *see* Bagchi, B. 370
Wilbrink, H.A., *see* Blokhuis, A. 126, 328, 369
Wilbrink, H.A., *see* Brouwer, A.E. 459–462, 677, 722

Wild, P.R. 87
Wild, P.R., *see* Casse, L.R.A. 1348
Wilke, F.W., *see* Fink, J.B. 163
Wilke, F.W., *see* Jha, V. 182
Wilke, F.W., *see* Johnson, N.L. 168, 182, 334, 337
Wilker, J.B. 783, 895, 1327
Wille, R. 218, 232
Willems, M.L.H. 245, 254, 308
Willems, M.L.H., *see* Thas, J.A. 254
Wilson, R.A., *see* Conway, J.H. 331, 332, 466, 661, 677, 726, 1145, 1153, 1180
Wilson, R.M. 369, 372
Wilson, R.M., *see* Dowling, T.A. 226, 227
Wilson, R.M., *see* Ray-Chaudhuri, D.K. 273, 353
Windelberg, D., *see* Karzel, H. 964, 965, 988
Winkler, P.M. 723
Witt, E. 268, 311, 313, 517, 1145
Wolf, K., *see* Leißner, W. 1080
Wolff, H. 988
Wolfskill, J., *see* Tzanakis, N. 452
Wölk, D. 1332, 1335, 1337
Wong, S., *see* Patterson, N. 1198
Wong, S., *see* Stroth, G. 1189
Woodall, D.R. 226
Wyler, O. 231, 232, 1311

Yaglom, I.M. 983, 996
Yakir, A. 365
Yakir, A., *see* Frumkin, A. 365
Yang, C.T. 329
Yanushka, A. 419
Yanushka, A., *see* Shult, E.E. 459–461, 676–678, 722
Yaqub, J.C.D.S. 181, 242, 251, 1330
Yates, F. 7
Yoshiara, S. 1182, 1198
Yoshiara, S., *see* Pasini, A. 1190, 1199, 1202, 1205, 1211, 1212, 1214, 1217–1219
Yoshiara, S., *see* Weiss, R. 1198, 1199, 1203
Young, H.P. 40, 222, 223, 235
Young, H.P., *see* Murty, U.S.R. 223, 224
Young, J.W. 835
Young, J.W., *see* Veblen, O. 8, 14, 35, 99, 229, 722, 724, 835, 1117, 1191

Zanella, C. 1259, 1262, 1266
Zara, F. 727
Zara, F., *see* Dixmier, S. 401, 402
Zaslavsky, T. 226
Zaslavsky, T., *see* Seymour, P.D. 867
Zassenhaus, H. 151
Zeeman, E.C. 935
Zemmer, J.L. 141
Zhao, C.K. 227
Ziegler, G.M., *see* Wanner, T. 257
Zil'ber, B.I. 270

Symbol Index

$<_Q^M \{k,l\}$ 999
$\widehat{-}_Q^M$ 999
$\cdot 1$ 171, 1198, 1224, 1225, 1239, 1240, 1243
$\cdot 2$ 171, 1146, 1211, 1212, 1225, 1233, 1240, 1243
$\cdot 3$ 171, 465, 855, 858, 1146, 1198
$2\mathsf{M}_{12}$ 1220
$3 \cdot (\mathsf{Suz} \cdot 2)$ 1225
$3^{23}\mathsf{Co}_2$ 1241
${}^2\mathsf{B}_2(q)$ 711
${}^2\mathsf{E}_6$ 569
${}^2\mathsf{E}_6(q)$ 711
${}^2\mathsf{F}_4$ 466, 613
${}^2\mathsf{F}_4(2)$ 661
${}^2\mathsf{F}_4(q)$ 392, 711
${}^2\mathsf{G}_2(q)$ 711
${}^3\mathsf{D}_4(2)$ 661
${}^3\mathsf{D}_4(K)$ 54
${}^3\mathsf{D}_4(q)$ 711
${}^3\mathsf{D}_4(\sqrt[3]{q})$ 391, 392
${}^3\boldsymbol{R}_Q$ 959
$({}^i\mathsf{X}_n^m,\ \mathsf{Y}_r)$ 569

$A(K,R)$ 796
$A(n,K)$ 33
$A_2(R,B)$ 1076
$A_{n,k}(\boldsymbol{D})$ 653
$\mathrm{A}_2\mathbb{F}$ 1295
$\mathrm{AG}(n,q)$ 1193
$\mathrm{Alg}(K/F)$ 219
$\mathrm{Ann}(U)$ 43
$\mathrm{AS}(q)$ 394, 1199
$\mathrm{Aut}(K)$ 41
$(\boldsymbol{A},\boldsymbol{L},\mathrm{I},\|)$ 794
$(\boldsymbol{A},\widehat{-}_Q)$ 993
$\boldsymbol{A}(V,\mathbb{F})$ 971
$\mathcal{A}=(P,L,\not\simeq,\|)$ 1079
A_1 580
$\mathsf{A}_1 \times \mathsf{A}_1$ 21
A_2 21, 612
A_3 598, 603, 612, 626, 632
A_5 84
A_d 870, 888
$\mathsf{A}_d \cdot \mathsf{Af}^* - \mathsf{AF} \cdot \mathsf{A}_{n-d-1}$ 1201
A_n 19, 48, 501, 515, 522, 539, 543, 572, 578, 580, 633, 634, 1159
$\mathsf{A}_n\mathsf{P}$ 1230
$\mathsf{A}_{n,1}$ 722, 724
$\mathsf{A}_{n,k}$ 722, 724
$\mathsf{Af} + n\infty$ 239
Af, C 84
$\mathsf{Af} \cdot \mathsf{C}_{n-1}$ 1195
$\mathsf{Af} \cdot \mathsf{D}_{n-1}$ 1204

$\overline{\mathsf{A}}_d$ 889
$\widetilde{\mathsf{A}}_1$ 1172
$\widetilde{\mathsf{A}}_d$ 906
$\widetilde{\mathsf{A}}_m$ 1172
$\widetilde{\mathsf{A}}_n$ 633

(B,N) 563
$B(k,l)$ 997
$B(V,K,F)$ 837
$B^{(p)}$ 904
$\mathcal{B}(G)$ 1087
$\mathcal{B}(G,k)$ 1087
$\mathcal{B}_a(G,k)$ 1096
$\widetilde{\mathcal{B}}_a$ 1101
B_2 21
B_3 612
B_d 870, 888
B_n 19, 501, 569, 1159
$\mathsf{B}_{n,1}$ 722, 724
$\mathsf{B}_{n,k}$ 722
$\mathsf{B}_{n,n-2}$ 722
$\mathsf{B}_{n,n}$ 722
$\mathsf{B}_{n,n}(q)$ 724
BM, *see* F_2
$\overline{\mathsf{B}}_d$ 889
$\widetilde{\mathsf{B}}_m$ 1172

$(C,\Gamma)^J$ 828
$C(n,s)$ 367
$C(q)$ 51
$C(Q)(+,\circ)$ 958
$C(u,w)$ 789
$C(\Pi)$ 145
$\mathrm{C}(\mathbb{F})$ 1295
Cl(16) 440, 450, 469
$C(R)$ 790
$\boldsymbol{C}(U)$ 806
$\boldsymbol{CG}(\boldsymbol{P},\boldsymbol{V})$ 218
$\mathcal{C}(\mathrm{St}_\Delta A)$ 575
$\mathcal{C}(\Delta(\mathcal{C}))$ 492
$\mathcal{C}(\Gamma)$ 678
$\mathcal{C}(\Gamma(\mathcal{C}))$ 679
$\mathcal{C}^\dagger(\boldsymbol{a})$ 936
$\mathsf{c}(\mathsf{D})_n$ 1206
$\mathsf{c}^1 \cdot \mathsf{C}_{n-1}$ 1195
$\mathsf{c}^2 \cdot \mathsf{c}^*$ 1219
$\mathsf{c}^{n-1} \mid \mathsf{I}_2(m)$ 1226
$\mathsf{c}^{n-2} \cdot \mathsf{C}_2$ 1195
$\mathsf{c}^{n-2} \cdot \mathsf{I}_2(2m)$ 1225
$\mathsf{c}^{n-3} \cdot \mathsf{C}_3$ 1195
C_2 612
C_3 523, 598, 603, 614, 626, 633
C_d 888
C_n 19, 48, 501, 513, 515, 522, 539, 544, 553, 569, 572, 578, 580, 606, 635, 1159
$\mathsf{C}^*_{n-1} \cdot \mathsf{L}$ 1220
$\mathsf{C}^*_{n-1} \cdot \mathsf{L}^{(\omega)}$ 1220
Co_1, *see* $\cdot 1$
Co_2, *see* $\cdot 2$
Co_3, *see* $\cdot 3$
$\overline{\mathsf{C}}_d$ 889
$\widetilde{\mathsf{C}}_m$ 1172

$D(Q)$ 147
$D^{(p)}$ 904
$\mathrm{DA}_2\,\mathbb{F}$ 1295
$\boldsymbol{D}^{\sigma,\varepsilon}$ 669
$\boldsymbol{D}_{\sigma,\varepsilon}$ 669
D_4 568, 569, 612
D_5 859
D_d 870, 888
D_n 19, 53, 501, 514, 515, 539, 544, 553, 572, 578, 580, 606, 634, 1159
$\mathsf{D}_{n,n-2}$ 722
$\mathsf{D}_{n,n}$ 722, 724
$\mathsf{D}^*_{n-1} \cdot \mathsf{L}$ 1220
$\mathsf{D}^*_{n-1} \cdot \mathsf{L}^{(\omega)}$ 1220
$\overline{\mathsf{D}}_d$ 889

$\widetilde{\mathsf{D}}_d$ 906
$\widetilde{\mathsf{D}}_m$ 1172

$E(\mathfrak{J})$ 753
E^{q-1} 849
E_{k+1} 1260
$\mathrm{E}_n(A)$ 1067
$\mathrm{E}_n(R)$ 1038, 1074
$\mathrm{EH}(\mathbb{F})$ 1294
$\mathrm{EH}(\mathbb{O})$ 1295
$\mathrm{EH}_t(\mathbb{R})$ 1296
$\mathrm{End}(S)$ 52
E_6 19, 501, 516, 543, 578, 580, 606, 634, 691, 710, 859, 870, 888, 1065, 1160
$\mathsf{E}_{6,1}$ 722, 724
$\mathsf{E}_{6,2}$ 722
$\mathsf{E}_6/\mathsf{D}_5$ 516
E_7 19, 501, 543, 569, 578, 580, 606, 607, 634, 697, 710, 859, 870, 888, 1160
$\mathsf{E}_{7,1}$ 722
$\mathsf{E}_{7,4}$ 722
$\mathsf{E}_{7,7}$ 722
E_8 20, 444, 501, 543, 569, 578, 580, 606, 607, 634, 710, 859, 861, 870, 888, 1160
$\mathsf{E}_{8,1}$ 722
$\mathsf{E}_{8,4}$ 722
E_n 539
$\overline{\mathsf{E}}_8$ 889
$\widetilde{\mathsf{E}}_6$ 906, 1172
$\widetilde{\mathsf{E}}_7$ 906, 1172
$\widetilde{\mathsf{E}}_8$ 906, 1172

$F(K,R)$ 833
$(F_\lambda;+,\circ)$ 149
$F(\mathfrak{J})$ 761
$F(\mathfrak{J}_0)$ 754
$F(\mathfrak{J}(n))$ 761
$\mathrm{FAlg}(d,F)$ 219
$\mathrm{Flag}(X,\leqslant)$ 485
$\mathrm{Flag}(X_\Delta,\leqslant)$ 486
$\boldsymbol{F}(\alpha,\boldsymbol{P})$ 112
$\boldsymbol{F}_Q$ 964, 990
$\boldsymbol{F}_Q^\infty$ 972
$((\dot{F},\mathcal{L}_r),\mathcal{C}_r)$ 1002
$\boldsymbol{FS}(\boldsymbol{P},\boldsymbol{V})$ 218
F_1 1211, 1224, 1239, 1240, 1242
F_2 1146, 1214, 1233, 1240, 1242
F_3 444, 1243
F_4 20, 501, 516, 539, 569, 578, 580, 606, 607, 636, 696, 710, 870, 888, 1065, 1160
$\mathsf{F}_{4,1}$ 722
$\mathsf{F}_{4,1}(2)$ 709
F_5 1243
Fi_{22} 726, 1146, 1198, 1211
Fi_{23} 726, 1146, 1198
Fi_{24} 1146
Fi'_{24} 726, 1198, 1211, 1224
$\overline{\mathsf{F}}_4$ 889
$\widetilde{\mathsf{F}}_4$ 1172

$(G,\vee,\wedge)$ 1265
$g(r,\mathfrak{l})$ 809
$G(\Pi)$ 145
$G_K(R,R)$ 811
$G_K(U,R)$ 810
$G_{H,d}(n,K)$ 1201
$G_{n,m}(F)$ 1348
$\mathrm{Gew}(56)$ 441, 450, 469
$\mathrm{GE}_n(R)$ 1074
$\mathrm{GF}(q)$ 55
$\mathrm{GL}(n,K)$ 41, 111
$\mathrm{GL}(V)$ 563, 1261
$\mathrm{GL}(V)_f$ 716
$\mathrm{GL}_n(R)$ 1037
$\mathrm{GM}_d(s,t,c)$ 463
$\mathcal{G}(A,B)$ 529

$\mathcal{G}_{\Delta}(A,B)$ 529
G_2 20, 569, 612, 710, 888
$\mathsf{G}_2(2)$ 461
$\mathsf{G}_2(4)$ 461
$\mathsf{G}_2(K)$ 54
$\mathsf{G}_2(q)$ 391, 392
$\overline{\mathsf{G}}_2$ 889
$\widetilde{\mathsf{G}}_2$ 617, 1172

$H(3,q^2)$ 413
$H(4,q^2)$ 414, 454
$H(d,q^2)$ 390
$H(F,K)$ 111
$H(n,q^2)$ 329
$H(q)$ 391
$H(q,\sqrt[3]{q})$ 391
$h\gamma_d$ 889
$H^i\mathcal{B}(G,M)$ 1091
H^q 848
$H^{(p)}$ 904
$H_3(C)$ 1066
H_q^{n+1} 447
$H_q^{(n+1)*}$ 451
$\mathrm{Harm}_k(\mathbb{R}^d)$ 850
$\mathrm{Hom}_k(\mathbb{R}^d)$ 850, 899
HoS(50) 440, 446, 450, 469, 470
HS(100) 440, 450, 469
HS(77) 441, 450, 469
H_3 501, 578, 580, 586, 598, 605, 626, 870, 1160, 1237
H_4 501, 580, 586, 605, 870, 1160
He 1232, 1239, 1243
He · 2 1228
HN, *see* F_5
HS 465, 469, 858, 1146, 1198, 1234

$I(Q)$ 816, 955
I^{p-1} 847
IH($\mathbb{F}$) 1294
IH($\mathbb{O}$) 1295
$\mathfrak{I}(n)$ 761
IP(q) 241
I_2 578, 580, 1160
$\mathsf{I}_2(m)$ 501
I_2^p 870

$J(K,R)$ 833
J_1 86, 465, 1146, 1236, 1237, 1244
J_2 85, 461, 465, 1146, 1216, 1228
$\mathsf{J}_2 \cdot 2$ 1225
J_3 1222
J_4 1233, 1242

$k(\Delta)$ 636
$K(\Delta)$ 636
$K(\Pi)$ 142
$k_{m,n,q}$ 316, 318
$((\boldsymbol{K}, \boldsymbol{L_K}), \mu_Q|_{{}^3K})$ 1000

$L(Q)$ 145
$L(U,W)$ 49
$L(\Pi)$ 146
$L({}_RM)$ 1120
$L_r(V)$ 805
LP(n,q) 450
LP(q) 241
LPG 1052
L_3 959
$\mathsf{L(D)}_n$ 1200
L_n 1147
$\mathsf{L}_{n-1} \cdot \mathsf{C}_2$ 1195
$\mathsf{L}_{n-1} \cdot \mathsf{L(D)}_2$ 1200
$\mathsf{L}_{n-1} \cdot \mathsf{L}^*$ 1219
Ly 617, 1182
$(\mathbb{L}, \mathbb{F}, Q_{\mathbb{L}})$ 960

$M(C_n)$ 216
$M_2(K)$ 814
M($\mathbb{F}$) 1295
MC($\mathbb{R}$) 1296
MDC($\mathbb{R}$) 1276, 1296

$\mathrm{Mod}(T)$ 743
$\mathrm{MP}(q)$ 241
$\mathrm{MP}(\Gamma, \Omega)$ 243
$\boldsymbol{M}(Q)$ 992
$\boldsymbol{M}(\overline{Q})$ 997
$\boldsymbol{M}_{11}$ 209
$\boldsymbol{M}_{12}$ 209
$\boldsymbol{M}_{22}$ 209
$\boldsymbol{M}_{23}$ 209
$\boldsymbol{M}_{24}$ 209
$\boldsymbol{M}_{n+1}(q)$ 262
$\boldsymbol{M}_Q$ 975
$\boldsymbol{M}_Q(a)$ 977
M, *see* F_1
M_{11} 209, 469, 1146, 1192, 1219, 1232, 1234, 1235
M_{12} 85, 209, 370, 465, 1146, 1192, 1217, 1229, 1234
M_{22} 209, 446, 469, 1146, 1192, 1207, 1214, 1216, 1233, 1235, 1241, 1243
M_{23} 209, 1146, 1192, 1233, 1241
M_{24} 209, 370, 1146, 1192, 1222, 1235, 1239, 1240, 1242, 1243
McL 465, 1146, 1179, 1198, 1227, 1234
MS 1215
$\overline{M(k)}$ 450

$N(Q)$ 148
$N_m(Q)$ 146
$N_r(Q)$ 146
$\mathrm{NQ}^{\pm}(2n-1, 2)$ 451

$O(n, m, q)$ 421
$\mathrm{O}(5, 3)$ 1197
$\mathrm{O}(7, 2)$ 1197
$\mathrm{O}(7, 3)$ 1197
$\mathrm{O}(9, 3)$ 1197
$\mathrm{O}(\mathbb{C})$ 1295
$\mathrm{O}^+(10, 3)$ 1197
$\mathrm{O}^+(6, 3)$ 1197
$\mathrm{O}^+(8, 3)$ 1197
$\mathrm{O}^-(10, 3)$ 1197
$\mathrm{O}^-(4, q)$ 1194
$\mathrm{O}^-(6, 2)$ 1197
$\mathrm{O}^-(6, 3)$ 1197
$\mathrm{O}^-(8, 3)$ 1197
$\mathrm{O}_7(q)$ 1180
$\mathrm{O}_6^+(q)$ 1179
$\mathrm{Orifl}(S)$ 558
$\boldsymbol{O}_{n+1}(q)$ 262
O'N 1234, 1235

$(P, H, |, \not\approx)$ 1135
(P, C) 793
$(P, \boldsymbol{L}, \boldsymbol{C})$ 837
$(P, \mathcal{K}, \mathcal{E})$ 1328
$(P, \Gamma | \Delta, \mathrm{I})$ 829
$P(2, K)$ 35
$P(n, K)$ 31, 33
$P(R)$ 785
$P(V, K)$ 805
$P^n(R)$ 835
$P_2(C)$ 1066
P_M 540
$P_M^{\leqslant d}$ 541
$P_n(R)$ 1038, 1042
$P_n(R)^*$ 1042
$P_R(U)$ 806
Pe(10) 440, 450, 469, 470
PG 1052
$\mathrm{PG}(2, 2)^*$ 203
$\mathrm{PG}(n, 1)$ 262
$\mathrm{PG}(n, q)$ 262, 298, 1193
$\mathrm{PGL}(V)$ 563
Pn(5) 469
$\mathrm{Pol}(\mathbb{R}^{p+q})$ 899
$\mathrm{Pol}_k(\mathbb{R}^d)$ 850
$\mathrm{PQ}^+(4n-1, 2)$ 444
$\mathrm{PQ}^+(4n-1, 3)$ 445
$\mathrm{PQ}^+(4n-1, q)$ 443

$\mathrm{PSL}(2, q^2)$ 1194
$\mathrm{PSL}(n, q)$ 667
$\mathrm{PSL}(n+1, k)$ 705
$\mathrm{PSL}(V)$ 563
$\mathrm{PSL}(V)_f$ 666
$\mathrm{PSO}(2n, \boldsymbol{F})$ 705
$\mathrm{PSO}(2n+1, q)$ 667
$\mathrm{PSO}(2n+1, \boldsymbol{F})$ 705
$\mathrm{PSO}^+(2n, q)$ 667
$\mathrm{PSO}^-(2n, q)$ 667
$\mathrm{PSO}^\varepsilon(n, q)$ 666
$\mathrm{PSp}(2n, q)$ 667
$\mathrm{PSp}(2n, \boldsymbol{F})$ 705
$\mathrm{PSp}(n, q)$ 667
$\mathrm{PSU}(n, q)$ 666, 667
$\mathrm{PU}(f)$ 1294
$\mathrm{P\Gamma L}(n, K)$ 41
$\mathcal{P}(r)$ 1002
$\mathcal{P}(\boldsymbol{a})$ 928
$\mathcal{P}^\uparrow(\boldsymbol{a})$ 936
$\mathcal{P}^\downarrow(\boldsymbol{a})$ 936
$((\mathcal{P}_r, \mathcal{L}_r), \mathcal{C}_r)$ 1002
PA_n 1230
pG_α 1244

$Q(2n, q)$ 329
$Q(4, q)$ 389, 411
$Q^+(2n+1, q)$ 329
$Q^+(3, q)$ 389
$Q^-(2n+1, q)$ 329
$Q^-(5, q)$ 390, 413, 454
$Q_6(2)$ 1197
$Q_k(t)$ 853, 900
$\boldsymbol{Q}_n(G)$ 213
$\boldsymbol{Q}_{n+1}(G)$ 262
$\mathcal{Q}(\boldsymbol{a})$ 928
$\mathcal{Q}^\uparrow(\boldsymbol{a})$ 936

$(R, \leqslant, \delta, \|)$ 1023
$R(r, m)$ 367
R_{n-1} 319
R^+_{n-1} 319
R^-_{n-1} 319
$\boldsymbol{R}_{10}$ 261
$\mathcal{R}(\boldsymbol{a})$ 933
Ru 1146, 1229
$(\mathbb{R}^n, \mathbb{R}, Q_e)$ 948
$(\mathbb{R}^n, \mathbb{R}, Q_m)$ 948
$\mathbb{R}^{p,q}$ 846

$s(B)$ 973
$(s(M))^*$ 994
$S(C)$ 509
$S(G, J)$ 395
$S(t, k, v)$ 208
$S(t, k, \nu)$ 351
S^{p-1} 847
$S^{p,q}$ 901
$S^r(Q)$ 977
$S_d(n, K)$ 1201
$S_\lambda(t, k, \nu)$ 351
$\mathrm{SC}(\mathbb{H})$ 1295
$\mathrm{SH}(\mathbb{H})$ 1295
$\mathrm{Sh}_{I_0}(B)$ 604
$\mathrm{SL}(n, K)$ 42
$\mathrm{SL}(V)$ 563
SL_n 1098
$\mathrm{SL}_n(k)$ 1099
$\mathrm{Spin}_{l+1}\,\mathbb{R}$ 1296
$\mathrm{srg}(v, k, \lambda, \mu)$ 438
$\mathrm{St}_\Delta\, A$ 484
$\mathrm{SU}_2\,\mathbb{F}$ 1296
$\mathbf{sl}_n$ 704
$\mathcal{S}(G, \Omega)$ 276
$\boldsymbol{\mathcal{S}}(\mathcal{K})$ 443
$\mathcal{S}(C, J)$ 491
$\mathcal{S}(\mathcal{M})$ 144
Suz 437, 1146, 1182, 1198, 1228

t-(ν, K, λ) 352
$T(n)$ 440, 450
$T(n, m, q)$ 421
$T(\mathcal{U})$ 153
$T(\mathcal{V})$ 153
$T(\Pi)$ 142
$T(\sigma, \varepsilon)$ 665
$T_2(O)$ 393
$T_3(O)$ 393, 453
$T_2^*(O)$ 394, 1199
$T_2^*(\mathcal{K})$ 443
$T_2^*(\mathcal{U})$ 454
$T_3^*(\mathcal{O})$ 452, 453, 469
$T_3^*(\mathcal{Q})$ 469
$T_4^*(\mathcal{K})$ 452
$T_5^*(\mathcal{K})$ 469
$\mathrm{TD}[k; n]$ 369
T_n 1239
Th, *see* F_3

$U_{2,3}(n)$ 450
$U_H(q)$ 1155
$U_R(q)$ 1155
$\mathrm{U}(4,2)$ 1148, 1197
$\mathrm{U}(6,2)$ 1145
$\mathrm{U}_3(5)$ 1179
$\mathrm{UC}(\mathbb{F})$ 1295
$\mathrm{UH}(\mathbb{F})$ 1295
$\mathrm{UH}_t(\mathbb{R})$ 1296
$\mathrm{Unf}(\Gamma)$ 1200

$(V\Gamma, E\Gamma)$ 651
$(V\Gamma, \mathcal{L}(\Gamma))$ 651

(W, R) 679
$W(M)$ 502
$W(q)$ 411, 1155
$W(\Sigma)$ 509
$W_3(q)$ 390
$W_n(q)$ 329
$\overline{W(2n+1, q)}$ 450

$(X, \leqslant)$ 485
$(X, \{R_0, \ldots, R_d\})$ 359
$[x_1, \ldots, x_n \colon \mathcal{R}]$ 742
$X^*(S)$ 1087
$X_*(S)$ 1087

$\mathsf{Y}_{n,j}$ 683

β_d 889
$\Gamma = (X, *, t)$ 75
$\Gamma(\mathcal{C})$ 678
$\Gamma(\mathcal{C}(\Gamma))$ 679
$\Gamma\mathrm{L}(n, K)$ 41
γ_d 889
$\Delta(\mathcal{C}(\Delta))$ 492
$\Delta(G, G^i)$ 494
$\Delta(G, k)$ 568
$\Delta(k, K)$ 636
(Δ, Γ) 1147
(Δ, Γ, G) 1147
(Δ, Γ, G, p) 1148
μ_Q 959
(Π, μ_Q) 990
$\Pi(b)$ 766
$\Pi(K, S)$ 160
$\Pi(Q)$ 141
$\Pi(S)$ 152
$\Pi(S)$ 143
$\Pi(V, \mathbb{F})$ 990
$\boldsymbol{\Pi}_n$ 213
$\pi(\mathcal{C}, \mathcal{C}_0)$ 593
$\Sigma(2m, 2)$ 887, 886
$\Sigma(\mathsf{A}_n)$ 550
$\Sigma(\mathsf{C}_n)$ 554
$\Sigma(\mathsf{D}_n)$ 555
$\Sigma(K, R)$ 790
$\Sigma_K(U, R)$ 806
(Ω, Δ) 881
$\Omega^\varepsilon(2m, 2)$ 886, 887

Subject Index

(0,1,all)-property 69
$(0, \alpha)$-geometry 436
0-diameter 82
1-diameter 82
2-cover, universal 77
2-design 71
– symmetric 56
2-dimensional Minkowski planes 1341
2-simple-connectedness 537
2-truncation 196
3-manifolds, hyperbolic 896
3-reflection lemma 958, 963
– affine 978
– projective 968
3-reflection theorem 958
– affine 978
– projective 968
3-regularity 413
4-gonal family 396
600-cell 871

absolute 78
absolute bound for strongly regular graphs 439
absolute space 663
addition 819
– of free angles 819
addition formula for Gegenbauer polynomials 852, 900
ADJACENCY 65, 483
– between chambers 484, 488, 1157
– i- 75, 484, 488
– $\bar{i}$- 1163
– in graphs 436
– of chambers 65, 678
– thin 1162
adjacency matrix 856, 879
– $(0, 1)$ 439
adjoint of DLS 202
adjoint map 702
admissible 664
affine diagram 1172
affine elation 147, 173
affine geometry 6
affine homology 147, 173
affine incidence structures 744
affine line 1274
affine metric planes 980
affine perspectivity 147, 173
AFFINE PLANE 34, 70, 83, 744, 750, 1147, 1279
– completely projectively inflated 131
– derived 1329
– Desarguesian 467, 770, 971
– free 762
– free extension process 754
– nondegenerate 754
– obtained from projective plane 34
– over associative rings 1074
– Pappian 770, 971, 1311
– partial 744
– perspectivities 767
– projection pencils 767
– topological 1266
– Σ-free constructions of 750

AFFINE PLANE (*cont'd*)
– Σ-hyperfree elements in a partial 752
– Σ-open 763
affine ring plane 1076
– coherent 1077
affine root 1095
affine section 1204, 1210
AFFINE SPACE 9, 33, 195, 470, 724, 796, 1147
– corresponding to vector space 971
– Desarguesian 971, 1156
– missed 197, 271
– nearfield 201
– Pappian 971, 989
– partial 794
– topological 1266
– underlying strong chain space 826
– weak 795
affine triple system 235
affinity 798, 1338
affino-projective space 234
Alexandrov neighbourhood 937
Alexandrov theorem 924, 925
algebra 788, 813
– associated with quadratic form 960
– inseparable 960
– kinematic 813
– local 788
– quadratic 813, 817
– quadratic $\mathbb{F}$- 960
– semilocal 834
– semiprimary 789, 797
– separable 960
algebraic group, split 634
algebraic groups, representation of 712
algebraic space 219
– full 219, 270
all modular hyperplanes theorem 212
alphabet 307
amalgam 617, 773
– classical 613
– noncollapsing 611
amalgamated sum 610
amalgamation property for partial models 774
ambient space 407, 408
ample connectedness 71, 656
André geometry 97
André plane 151
André system 151
angle 819, 974, 999
– free 819
– oriented 974, 980, 999
– Q-conform 974
– Q-relation 974
anisotropic kernel 635
anisotropic space 561
anisotropy 1100
annihilator 43, 1120
anti-automorphism 559
– of matrix ring 962
– of quadratic algebra 813
anti-Cauchy–Schwarz inequality 848, 896
anti-Cauchy–Schwarz property 891
anticollineation 835
anticorrelation 835
antiflag 435
antiflag-transitivity 67
apartment 21, 88, 479, 526, 1088, 1095, 1189, 1242
– diameter 91
– large 90
Apollonius problem 1350
approximation of unit sphere 874
arc plane 1298, 1299
Archimedean solid 604
Arguesian law 1122
arithmetic group, parabolic 1110
Artin ring 1039
Artin–Schreier characterization 1311
Artinian plane 932
– space 928
Artinian vector space 954
assignment 742
association of $(d, d-1)$-DLS's 204
association scheme 359
asymptote 973
AUTOMORPHISM 32, 41, 77, 355, 771, 833, 837
– axial 355
– causal 935, 937
– central 248, 355, 1330
– continuous 1335

AUTOMORPHISM (*cont'd*)
– discontinuous 1336
– F-semilinear 1261
– linearly induced 997
– of 2-design 354
– of abstract projective space 1261
– of affine metric plane 987
– of affine metric space 974, 996
– of affine plane 771
– of affine polar space 1195
– of affine ring plane 1078
– of Benz plane 248, 1329, 1335
– of building 1088
– of Burau geometry 837
– of chain geometry 791, 796, 806, 810
– of chamber system 1185
– of DLS 267
– of extension 757
– of free geometry 771
– of full matrix ring 1057
– of (g, d_p, d_l)-gon 466
– of generalized Hughes plane 111
– of geometry 17, 77
– of icosahedron 878
– of integral lattice 911
– of Klein space 1002
– of Laguerre plane 771
– of lattice 1026, 1267
– of Minkowski plane 771
– of Minkowski product 828
– of module 784
– of Möbius geometry 824, 833
– of Möbius plane 771, 1337
– of projective metric space 990, 996
– of projective plane 771
– of projective space 32
– of quadratic circle geometry 992
– of space 650
– of spherical circular plane 1338
– of stable plane 1279
– of strong chain space 826
– of switching class 883
– of vector space 955
AUTOMORPHISM GROUP 826
– acting transitively on independent m-sets 275
– classical 1180
– exceptional 1181
– locally classical 1181
– locally exceptional 1181
– of compact (connected) projective plane 1288, 1291
– of DLS 268
– of Minkowski plane 1340
– of Möbius plane 1338
– of Laguerre plane 1346
– of the first kind 826
– positive 276
– topology of 1280
autotopism 148
autotopism group 148
axiom 5
– maximality 1257
– of joining 1336, 1339
– of linearity 544
– of parallel projection 1339
– of touching 1336
– richness 1336
– set of axioms 743
AXIOM SYSTEM
– affine plane 34, 744
– affine polar space 724
– affine space 724
– apartment 88
– Bachmann place 1003
– Barbilian domain 1075
– Benz plane 746, 1328
– block space 13
– blocking set 321
– BN-pair, *see* – Tits system
– building 90
– buildings, geometrical 606
– chain space 792, 820
– circular space 792
– Clarke 1022
– closure space 198
– connectedness relation 1021
– connection structure 1020
– contact space 793
– coordinatizable lattice 1125
– dimensional linear space 196
– Euclidean geometry 6

AXIOM SYSTEM (*cont'd*)
– Faulkner plane 1070
– generalized n-gon 387, 426, 520, 745
– generalized quadrangle 46, 387
– generalized regulus 807
– geometric lattice 1264
– Grzegorczyk 1021
– hyperplane of lattice 1124
– isoparametric map 1111
– Jordan homomorphism 831
– k-net 745
– Klingenberg space 1060
– Laguerre plane 240, 746
– Leißner plane 1079, 1080
– m-spread 144
– metasymplectic space 719
– metric product 1004
– Minkowski plane 240, 746
– Möbius plane 746
– Möbius space 793
– partial affine space 792
– physical geometry 1026
– point system 1124
– pointless metric space 1023
– polar space 45, 556
– principal denominator set 1054
– projective lattice geometry 1128, 1131
– projective plane 34, 744
– projective space 10, 39
– quadratic form 947
– quadratic set 815
– quasifield 141
– root system 699
– semilinear bijection 955
– stable n-space 1265
– Steiner system 748
– structured complex 526
– Tits system 563
– uniformity 1027
– V-plane 1051
– V-n-space 1045
– weak chain space 791
– weak Minkowski chain space 826
– Whitehead 1020
axis 968, 1049, 1282
– of inversion 249
axis C 249

Bachmann plane 1003
Baer cone 322
– of type (r, d) 322
Baer group 179
Baer involution 169, 835
Baer p-element 177
Baer ring 1137
Baer subplane 115, 125, 158, 316, 835, 1286
Baer subspace 322, 452
Baer theorem 169
Banach spaces, reflexive 1264
Barbilian domain 1075, 1078, 1080
Barbilian plane 1036
Barbilian space 1036, 1045, 1135
– affine 1050
– dual affine 1050
– empty 1135
Bartolone–Ostrom plane 165
base of cone 318
base point 420
base-line 417
basic graph 485, 490
basis 200, 552, 784, 1097, 1129
– affine 976
– homogeneous 129, 1121, 1123
– max 73
– min 73
– of flat 1047
– of $P^n(R)$ 835
– orthogonal 954
basis number, invariant 1040
Beckman–Quarles theorem 930
beetle figure 795
Benz plane 218, 229, 238, 250, 748, 767, 795, 821, 823, 1327
– classical 1331
– coherent 1332
– complete 237
– connected finite-dimensional 1333
– coordinatizable 242
– egglike 242
– embeddable 242, 1328
– finite-dimensional 1332
– free extension process 755

Benz plane (*cont'd*)
– locally classical 1334
– Miquelian 241, 250, 793, 802, 813, 821, 1328
– nondegenerate 756
– projective 748
– pseudo-pencil 767
– pseudo-perspectivity 767
– semiclassical 1334
– topological 1331
– valency of element 753
– Σ-free construction for 751
Benz space 237, 254
Bernoulli numbers 1109
Bezout domain 1062
Bézout ring 1131
– proper right 1131
biassociativity 1263
bijection
– ρ-linear 955
– linear 955
– semilinear 955
bimodule 1136
– isomorphism 1139
biplane 1147
Birkhoff–Stone theorem 1019
block design 13
– balanced incomplete 351
Block lemma 252
block space 4, 13, 66, 67
– topological 4
blocking set 68, 126, 321
– irreducible 127
BLOCKS 66, 86, 351
– base 356
– in Laguerre plane 746
– in Minkowski plane 746
– in Möbius plane 746
– in Steiner system 748
– initial 356
– of design 351
– of locally classical geometry 1226
– of semibiplane 86
– repeated 351
blueprint 481, 625, 626, 629, 637
– realizable 627, 629
BN-pair 385, 479, 549, 563, 688, 1309, *see also* Tits system
– split 591
BNB condition 90
Boerner semifield 154
Bolyai–Lobachevsky geometry 891, 895
Boolean space 211, 262
Borel measure 851, 874
Borel subgroup 20, 79, 479, 563, 567, 712, 1093, 1100, 1101, 1153, 1184, 1185
– finite 1184
Borel–Serre compactification 1101
Bose–Mesner algebra 360
boundary 504
Brianchon theorem 970
broken grid 413
Bruck net 441
Bruck–Chowla–Ryser theorem 354
Bruck–Kleinfeld theorem 638
Bruhat cell 519
Bruhat decomposition 564, 587
– refined 587, 629
Bruhat order 517
Bruhat ordering 519
Bruhat poset 715
Bruhat–Tits building 612
Buekenhout theorem 967
Buekenhout–Mäurer theorem 969
Buekenhout–Metz Hermitian arcs 315
Buekenhout–Metz unitals 125
BUILDING 15, 21, 69, 70, 88, 90, 526, 585
– affine 1096
– at infinity 1173
– compact (connected) 1309
– conforming to blueprint 627
– exceptional 592
– of algebraic group 568
– of spherical type 528
– of spherical type, spherical 22, 88, 578
– topological 1306
bundle 245, 997
bundle condition 233, 245, 822, 1328
bundle space 234
bundle theorem 233, 245, 986, 999
bundle with three property 220
bundle-equivalency 1346
Burau geometry 837

calotte 965
canonical space 1023
carrier 245, 248, 413, 765
– of flock 333
Cartan apartment 93
Cartan decomposition 1099
Cartan involution 1101
Cartan subalgebra 18
Cartan subgroup 1109
category 3
Cauchy filters 1026
– infinitely close 1026
Cauchy sequence 1024
causality 935
Cayley algebra 479, 572, 636
Cayley numbers 638
Cayley–Dickson process 1262
Cayley–Menger distance matrix 849, 895, 914, 915
Cayley-distance 990
centre 786, 968, 972, 993, 1049, 1282
– of homology 425
– of projection pencil 765
central projection 9
– centre of 9
centralizer, connected 1088
chain 790, 806
– 2- 882
– affine 796
– affine n- 836
– in lattice 1121
– K- 789
– m- 835
– of shape $[v]$ 1027
chain domain 1061
chain geometry 790, 1328
– affine 796, 817
– direct products of 830
– Miquelian 822
– n- 835
– over quadratic algebras 813
– restricted affine 800
– trivial 831
chain group 218, 260
chain ring 1132
– left 1132
– right 1132
chain space 794, 821, 825
– associated to quadratic set 816
– connected 793
– Miquelian 821
– nontrivial 793
– strong 825
– weak 793
chamber 17, 65, 75, 96, 479, 483, 488, 678, 1098
chamber system 65, 96, 488, 489, 509, 678
– connected 489, 593
– locally classical 1185
– locally finite 1185
– pointed 593
– simple m-connectedness 509
– strongly connected 489
– with group action 494
chamber system correspondence 679
chambers opposite 580
Chang graphs 443
Chang–Łoś–Suszko theorem 743
character 1087
character of semi-oval 317
characteristic 948, 990, 1148, 1169
characteristic set 263
– algebraic 264
Chevalley basis 702
Chevalley group 466, 479, 702, 703, 710
– twisted 711
circle 83, 249, 836, 981
– belonging to bundle 997
– Euclidean 1337
– in Burau geometry 836
– in Laguerre plane 240
– in Minkowski plane 240
– oriented 1339
– proper 991
– r- 1002
– regular 981, 992, 994, 996
– singular 981, 994
circle plane 237, 1327, 1336
circles opposite 249
– symmetric 249
– touching 992

circuit 436, 520
– geodesic 88
– in graph 386
– optimal 88
circular extension 1202
circular plane 237, 824, 1336
circular space 208, 794
– locally affine 795
– sharply locally affine 253
– strongly locally affine 253
– weak locally affine 253, 272, 1194, 1195, 1205
clan 398
class 261
– hereditary 262
– of 3-transpositions 726
– of k-root subgroups 727
– of k-transvections 727
– subgeometry-closed 261
classical domination 1270, 1286, 1308
claw bound 439
Clebsch graph 440, 463, 856, 905
Clebsch quartic surface 440
Clifford algebra 51, 698, 958, 1308
Clifford theorem 983
clique 651
closed set 416
closure 198, 416
– circle geometric 997
– lemma 650
closure space 73, 195, 198
– graded 73
– induced by restriction 199
– simple 198
– simplified 198
co-amalgam 773
coaffine plane 1275
cocentre 173
cocharacter 1087
coclique 275
cocycle, 2- 882
code 128, 307, 361
– GDRS 309
– GRS 309
– of affine geometry 365
– of projective geometry 365
– self-dual 362
– spherical A- 854
– tight 854
codeword 128
codimension 202, 483, 950
coding theory 13
cohomological dimension 1106
cohomology 882
coline 202
collar 669
collinear 414
collinearity 68, 435, 650
collinearity graph 68, 437, 651
COLLINEATION 41, 43, 113
– affine 1048, 1078
– axis 1282
– central 35, 1282, 1330
– centro-axial 1282
– continuous 1335
– fixing Von Staudt chain 837
– h-affine 1050
– in Leißner plane 1079
– of affine metric plane 987
– of affine metric space 974, 976
– of affine plane 43
– of Burau geometry 837
– of chain geometry 810
– of compact (connected) projective plane 1288
– of derived affine plane 1330
– of derived projective plane 1330
– of Figueroa plane 113
– of generalized quadrangle 420, 424
– of L-plane 1079
– of projective space 1136
– of quadric 815
– of stable plane 1279
– of translation plane 172
– of Veldkamp space 1048
– planar 177
– semilinear induced 965
COLLINEATION GROUP 42, 145, 183
– planar 177
– semitransitive relative to Π_0 183
– tangentially transitive relative to Π_0 183
– transitive relative to Δ 183
comet 1173

commutator map 631
commuting tori graph 729
compactification 1105, 1263
– theorem 1100
compactness in lattices 1120
– local 1262
comparison of parabolics 1183
complement 1120
– of graph 436
complete set of invariants 898
COMPLEX 65, 96, 483, 504, 521, 539
– admitting parameters 539
– belonging to M 521
– locally finite 539
– of Coxeter type 522
– over I 484
– pure 483
– realization of 96
– simplicial 482
– strongly connected 484
– thin 504
– thin-with-boundary 504
components
– of spread 143
– of translation plane 143
composition of paths 599
condition SR_n 1039
cone 318, 847, 956, 961, 965
– elliptic 1328
– future- 936
– null 926
– of norm 0 847
– of singularity 797
– open 1269
– quadratic 237, 814
conference matrix 885
configurational postulate 823
configuration
– induced 389
– tactical 820
conformal transformation 929
congruence 143
– B- 1080
congruence order 879
congruence relation 972, 988, 989
congruency 819
conic 57, 121, 802, 816, 1333
– irreducible 298
– on quadric 822
conjugations 1262
connected 436
connectedness 76, 1024, 1262
– relation 1021
– residual 76
connection relation 1020
connection structure 1020
connectivity, finite 1279
contact relation 795
– geometric 811
– natural 795, 799, 811
contact space 795
containment relation 1128
continuation along geodesics 587
continuity
– of intersection 1265
– of join 1265
contractibility, local 1272
contraction 1272
– of closure space 203
convex hull 589
convexity 507
– condition 89
– space 97
coordinatizability 232
– F-unique 264, 266
– of lattice 215
coordinatization 37, 256
– F- 256
– of DLS 256
– of lattice 1125
– of projective plane 35
coplane 202
copoint 202
copolar space 458
corank 75
– of subset 75
corner associated to parabolic subgroup 1103
corner condition 823
correlation 43, 77, 78, 359
– of designs 359

coset geometry 17, 20, 79
coset, left 17
cotriangle space 457, 458
cotype 75, 484
countability
– first 1262
– second 1262
cover 77
– 2- 77
– proper 77
– universal 2- 1161, 1178, 1187
– universal m- 77
COVERING 324, 1171
– between abstractive sets 1020
– minimal 324
– of Barbilian planes 1071
– of Barbilian planes, universal 1071
– of closed set by closed set 199
– of $PSL_2 \mathbb{R}$ 1301
– of $PSU_3(\mathbb{C}, 1)$ 1312
– of $SO_{l+1} \mathbb{R}$ 1296
– reduced, *see* – minimal
Coxeter complex 1171
Coxeter diagram 83, 500, 1160
– spherical 1160
Coxeter geometry 22
Coxeter graph 905, 912
Coxeter group 502, 517, 563, 679
Coxeter matrix 500
Coxeter part of diagram 1148
Coxeter stroke 1148
Coxeter system 502
Coxeter–Killing transformation 870
Coxeter–Tits complex 480, 500, 518, 624, *see also* Coxeter complex
– belonging to M 503
Cremona transformation 804
Cremonian geometry 804
cross of lines 965
cross ratio 59, 787, 818
cross-polytope 867, 904
– truncated 904
cube 877
– twisted 256
cubic
– lattice 865
– twisted 298
cuboctahedron 866
curve 829
– Hermitian 123
– rational 801
curve of degree r
– rational normal 801
curve space 829
cycles 1339
cylinder
– oval 242
– semi-oval 242
cylinder plane 1268, 1274, 1295, 1299, 1300
– dual 1299
– modified real 1296
– modified real dual 1276
– skew 1295

Danzer's problem 85
deck transformations 594
decomposition of motions, canonical 979
deep holes 912
deficiency 328, 1149
– 0- 1149
– 1- 1149
degree 438
– of homogeneity 1281, 1301
– of representation 812
Dembowski semi-$(i+1)$-spaces 254
Dembowski semiplane 254
Dembowski–Wagner theorem 210
Dempwolff group 179
Dempwolff plane 161, 162
denominator
– planar 1054
denominator set
– planar, *see* – principal
– principal 1054
dependence 849
derivable plane 157
derivation 156, 168, 338, 702
– inner 883
derived plane 157
– topological 1332
derived quasifield 157
Desargues axiom 11, 69

Desargues condition 1079
Desargues configuration 91, 213
Desargues postulate 1122, 1130
Desargues theorem 31, 37, 109
– dual 220
Desarguesian plane 969
– locally 1300
Desarguesian projective space 236
Desarguesian space 270
DESIGN 13, 351, 352, 360, 387
– affine 2- 56
– affine resolvable 2- 56
– antipodal spherical 867
– block schematic 360
– derived 352
– divisible 372
– Euclidean t- 875
– Hadamard 3- 56
– matroid 195
– quasi-symmetric 360
– regular geometric 195
– residual 352
– resolvable 352
– spherical t- 875
– square 352
– symmetric 352
– symmetric 2- 56
De Sitter space 848
De Sitter space-time 930, 937
diagonal axiom 451
diagonal clique 451
diagonal points property 220
diagonal subtorus 1096
DIAGRAM 48, 81
– basic 81
– hyperbolic 1171
– linear 85
– maximal 1148
– minimal circuit 1148
– of almost finite type 1157
– of Coxeter type 1149
– of finite type 1157
– of infinite type 1157
– of singular type 1149
– of strongly finite type 1161
– of strongly infinite type 1161
– special 82, 1148
diameter 386, 437, 520, 688, 1023, 1157
– of building 528
– of shadow space 688
Dickson nearfield 151
Dickson–Knuth semifield 155
difference families 356
difference set 355
digon 48
dilatation 250, 786, 798, 979, 1049
– affine 1050
– dual a-affine 1050
Dilworth completion 217
Dilworth truncation 217
DIMENSION 196
– affine 971
– of affine space 971
– of affino-projective space 234
– of chain space 795, 797
– of closed set 416
– of flat 1047
– of lattice of varieties 200
– of poset 485
– of projective lattice geometry 1129
– of projective space 552
– of representation 812
– of translation plane 146
– of variety 196
– projective 964, 1259
– small inductive 1276
dimension formula 1281
dimension function 1121
dimension inequality 1264
dimension lemma 211
dimensional linear space, *see* DLS
direct product 652
– of chamber systems 1185
direct sum 80
– of DLS's 204
– of geometries 38
direct sum theorem 81, 1160
direction 798
– of Q-symmetry 976
disconnectedness, total 1262
discreteness 1262

discriminant 911
disjoint relation in lattices 1120
distance 923, 1023, 1027
– elliptic 875
– hyperbolic 891
– in Barbilian space 1135
– in physical space 1027
– Q- 972
distance function 807
distance matrix 913
– hypermetric 915
– of negative type 913
distant relation 1043, 1076, 1079, 1135
distributor 147
division ring 8, 30, 559
– locally compact 1263
– topological 1260
DLS 196, 198, 206, 208, 216, 223, 260, 263, 1191, 1265
– algebraic 263
– basis-transitive 273, 275
– Boolean 208, 224
– circular 208
– completely trivial 208
– connected 206
– coordinatizable 263
– design 209, 223
– F-algebraic 263
– finitary 200
– flag-transitive 271
– highly transitive 267
– hypercircular 223
– hypercircular t- 208
– hyperplane-antiflag-transitive 273
– $(i+1)$-fold circular n- 222
– irreducible 206
– Jordan-transitive 270
– k-connected 206
– m-closed 216
– n- 196, 257
– n-homogeneous 268
– n-modularity complemented 257
– noncircular 208
– nonregular 223
– nonregular hypercircular 209
– point-antiflag-transitive 273
– point-primitive 275
– regular 223, 260
– regular n- 221
– regular hypercircular n- 209
– sharply locally projective 211
– strongly locally affine n- 265
– strongly locally projective 211
– strongly locally projective n- 265
– supersolvable n- 257
– thick 208
– top-thick 266
– trivial 223
– trivial n- 208
– type x-transitive 268
– unimodular 260
– voltage-graphic 262
dodecahedron 869, 890
domination by classical planes, *see* classical domination
dot product 923
double coset 564
double space 99
Dowling lattice 212, 213
Dowling space 262
duad 401
DUAL 72
– affine Barbilian space 1050
– algebraic hypersurface 303
– basis 1041
– basis of flat 1047
– code 308
– cylinder plane, modified real 1296
– degree 371
– design 448
– distance 371
– eigenmatrix 360
– Euclidean plane 1295
– grid 393
– Hermitian plane 1295
– lattice 911
– linear space 465
– Minkowski plane 1295
– net 465
– of classical generalized quadrangle 633
– of complex 636

DUAL (*cont'd*)
– of design 351
– of DLS 203
– of function space 218
– of generalized polygon 387
– of hyperplane space 72
– of lattice of n-DLS 202
– of modular lattice 202
– of projective space 202
– of projective space over ring 1042
– of topological projective plane 1274
– pairing 1036
– partial quadrangle 333
– polar graph 676
– polar space 676
– statement 1045
duality 43, 78, 117
– for k-arcs 305
– for projective spaces 1044
– H- 118
– orthogonal 202
Dynkin diagram 19, 571, 700, 905
– extended 905
– relative 550, 1101

eigenvalue 356, 439, 442
Einstein's cylinder universe 237, 929, 937
elation 35, 42, 376, 420
elation generalized quadrangle 420
elation group 420, 1347
element 785
– Euclidean 952
– geometrical 1021
– n-generated 1124, 1129
– singular 952
– unimodular 785
– zero 849
– $\wedge$-irreducible 1019
elementary divisor 363
elementary divisor theorem 1098
elementary group 1038
elementary-homotopic paths 599
elementary homotopy 512
– of labels 627
– of sequences 628
elementary Weyl homotopy 525
ellipse 802
ellipsoid 886
elliptic plane 875, 1294
elliptic quadric 309, 451, 455, 469
elliptic space 847, 879, 949
embeddability 125
– F-unique 264
– universal 235
embedding 78, 255, 436, 654, 743, 773
– C- 265
– canonical 789
– F-rigid 266
– F-unique 266
– F-universal 266
– flat 436
– full 436
– generalized projective 256
– joint 773
– lattice 255
– of chain geometry 831
– of geometry 78
– of linear space 232
– of negative definite space 914
– of planar space 232
– projective 256
– strong 267, 1268
embedding theorem 951, 1268
enough modular hyperplanes theorem 212
entourages 1027
– fundamental system of 1027
epimorphism 77, 425
– of generalized 3-gon 425
equiangular lines 858
equidistant sets of elliptic points 881
equimetry 955, 961, 975, 991
equivalence
– affine 799
– thin 1162
erection 202
Erlangen Program 16, 845
Euclid's axiom 845
Euclid's postulate 794
Euclidean plane 295, 819, 964, 1003
– dual 1295, 1299
Euclidean projections 911

Euclidean space 849, 948, 1029, 1327
Euler characteristic 1092
Euler graph 880
– labelled 881
Euler–Poincaré characteristic 1108
Euler–Poincaré measure 1108, 1109
eumorphism 1053
eutactic star 864, 889, 890
exceptional 3-symmetric plane 1302, 1306
exchange axiom 199
exchange condition 503, 536
extension 753, 1145
– étale 1099
– free 759
– generated over model 753
– i-fold c- 209
– i-fold L- 209
– n-times linear 1224
– of design 352
– totally ramified 1263
– Σ-free generated 754
extension process 772
– faithful 772
extension series 753
– free 759
– Σ-hyperfree 754
extension theorem 619, 637, 959
– for twin buildings 637
exterior algebra 52
exterior hyperbolic space, *see* De Sitter space
exterior product 808
exterior set 338

F-universality 267
face 96, 483
– associated to parabolic subgroup 1103
– of complex 96
factorization of groups 1153
failed bundle DLS 258
failed Desargues space 258
failed Pappus space 258, 263
fan 245
Fano plane 197
– free 765
Faulkner plane 1036, 1070
FDLS, supersolvable 217
Feit–Higman theorem 539, 1149, 1307
fibre bundle 1346
field 55
– alternative 631
– Galois 55
– mereological 1021
– of left quotients 1125
– ternary 1271
fifteen schoolgirls problem 327
Figueroa planes 112
filter, completely prime 1019
filtration 1092
finite character property 73
firmness 491
first main characterization of buildings 532, 548, 595
Fischer group 726
Fischer plane 166, 167, 235
Fischer space 229, 235, 236, 677, 727
– generalized 728
Fisher inequality 353
flag 17, 47, 65, 75, 271
– opposite 418
– reduction of 94
flag complex 96, 485
flag graph 437
flag space 1293
flag-diameter 437
flag-transitive linear space 469
flag-transitivity 17, 66, 67
flat 30, 277, 913
– bases of 1047
– distant 1048
– dual 1047
– dually distant 1048
– empty 1047
– k- 1264
– of affine space 33
– spanned by independent points 1047
– transversal 1047
flexibility 1338
flexible plane 1290
flock 72, 166, 248, 333, 397
– linear 167, 333
flocks 336, 337, 338, 340, 1329
– derived from generalized quadrangles 398

flocks (*cont'd*)
– exceptional 335
– of Bader, Lunardon and Thas 337
– of cones 336
– of Fisher 337
– of Fisher, Thas and Walker 336
– of Ganley 337
– of hyperbolic quadrics 334
– of Johnson and Payne 337
– of Kantor 336, 337
– of ovoids 334
– of Payne 337, 338
– of Payne, Thas, Johnson, Lunardon and Wilke 337
– of Thas 335
– sporadic 340
folding 506, 1200, 1226
– belonging to panel 508
– opposite 578
forbidden subgeometry theorems 259
FORM
– 2-singular 952
– alternating 665, 886
– anti-Hermitian 665
– Artinian 954, 961
– bilinear 44, 803, 948
– – alternating 44
– – symmetric 927
– – symmetric self-dual 362
– – symmetric self-orthogonal 362
– elliptic 948, 952
– Euclidean 948, 952, 953, 990
– Galilean 952
– Hermitian 559, 635, 665, 670, 728
– – alternating 560
– – trace valued 560
– hyperbolic 948, 952
– irregular 950
– isotropic 950
– K-linear 813
– linear 949
– Minkowskian 948, 952, 961, 965, 990
– nondegenerate 664
– orthogonal 886
– pseudoquadratic 635, 670
– quadratic 45, 52, 363, 560, 670, 690, 813, 947, 948, 1197
– – Hermitian 522
– – nonsingular 45
– – split 52
– reflexive 664
– regular 950
– sesquilinear 44, 1308
– – Hermitian 44
– singular 950, 952
– symmetric 665, 1294
– unitary 886
– σ-anti-Hermitian 665
– σ-Hermitian 665
– σ-quadratic 670
– σ-sesquilinear 664
– – associated to polarity 665
– – proportional 665
– – reflexive 559
– – trace-valued 666
– (σ, ε)-Hermitian 665
– (σ, ε)-quadratic 670
forms
– equimorphic 955
– geometrically equivalent 955
– isometric 955
– isomorphic 955
– proportional 671
formula 742
– atomic 742
– universal 743
Foulser planes 160
foundation 481, 625
– supporting building 626
frame 1018, 1052
– H- 714
– m- 835
– orthogonal 864
– polar 557
– spatial 1019
free extension 742
free set 73, 74
Frobenius automorphism 706
Frobenius–Burnside lemma 354
fully projective set 657

function space 218
function, Γ-invariant 868
functional space 97
functor 3
functoriality 1088
fundamental group 600
fundamental representation of G 713
fundamental theorem
– of affine geometry 925
– of affine metric geometry first 974
– of affine metric geometry second 976
– of projective geometry 562, 965
– of projective metric geometry first 990
– of projective metric geometry second 991
– of quadric circle geometry 992
– on stereographic projection 995

(g, d_0, d_1)-gon 82
(g, d_p, d_l)-gon 437
– regular 464
(g, d_p, d_l)-pair
– distance transitive 466
– line distance transitive 466
– line geodesic transitive 466
– locally i-arc transitive 466
– point distance transitive 466
– point geodesic transitive 466
– weakly distance transitive 466
– weakly geodesic transitive 466
gallery 483, 489
– closed 483
– connected 483
– simple 483
Galois automorphism 690
Galois cohomology 1101
Galois extension 1089, 1099
Galois field 933
Galois geometry 57
Galois group 571, 1089, 1099
gamma space 651, 654
gamma space lemma 651
Ganley semifield 155
gate property 537, 576, 589
Gauss–Bonnet theorem 1108
Gegenbauer coefficients 857, 858, 859, 860
Gegenbauer polynomial 853, 862, 900, 901, 902
generalized ∞-gon 521
generalized $2m$-gon, pseudo-classical 1226
generalized André plane 149
generalized André system 149
generalized digon 81, 387, 467, 485, 521
generalized g-gon 82
generalized Hall plane 158
generalized Hall system 158
generalized hexagon 54, 387, 460, 661, 689, 1230
– classical 391
– Moufang 424
generalized homology 420
generalized Hughes plane 112, 115
generalized m-gon 21, 520, 585
generalized m_{ij}-gon 625
generalized n-gon 426, 745, 750, 768
– free 385
– free extension process 754
– groups of projectivities 768
– nondegenerate 755
– order 387
– perspectivities 768
– projection pencils 768
– Σ-hyperfree chain in a partial 752
generalized octagon 387, 661, 709
– classical 392
– Moufang 425
generalized plane, Σ-free construction of 750
generalized polygon 387, 464, 466, 520
– Moufang 426
– topological 1306
generalized quadrangle 46, 334, 369, 387, 441, 453, 460, 465, 467, 660, 768, 1196, 1230, 1240, 1339, 1343, 1349
– $3n$-dimensional 1350
– antiregular 1349
– classical 389, 448
– derivation of 1349
– Desarguesian 417
– elation 1307, 1310
– embedded in affine space 408
– embedded in projective space 407
– Moufang 423
– $\{p, L\}$-transitive 424
– perspectivities 768
– projection pencils 768

generalized quadrangle (*cont'd*)
– projective 407
– Roman 399
– topological 1349
– weak projective 408
generalized triangle 387, 465, 521
geodesic 507, 628, 1090
– based at point 437
– maximal 437
geodesic action 1102
geodesic condition 88
geodesic transitivity 1150
geodesically closed 652
geometric group 271, 276
geometric lattice 12, 1147
geometric realization 1089
geometric t-spread 325
geometrical realization 498, 555
geometrically irreducible subgroup 171
geometrically primitive subgroup 172
geometries, homogeneous classes of 777
GEOMETRY 47, 75, 76, 222, 1166, 1226
– affine-dual-affine 1215, 1218, 1223
– affine metric 970, 971
– Alt(7) 89, 523, 687, 1166, 1178, 1203, 1211
– at infinity 992
– B^*- 240
– (B_n^*)- 829
– belonging to diagram 48, 81
– $\mathrm{c}^{i+1}\mathrm{L}^{n-i-2}$- 222
– chain, *see* chain geometry
– circle 470, 988
– classical projective 1129
– combinatorial 195, 201
– continuous 1121
– coset, *see* coset geometry
– cycle 254
– d-injection 241
– Desarguesian 1130
– dual 82
– elliptic 845
– Euclidean 6, 845
– F-squashed 278
– finite 13
– finite projective 13
– finitely generated 1129
– firm 47, 76
– flag- 1226
– flat 49
– flat anomalous 1167
– hyperbolic 845
– inner 991
– (k, m, n)- 764
– locally classical 1166, 1226
– locally finite 1157
– locally pseudo-classical 1226
– minor 1129
– module-induced 1129
– $(n$-)cluster 1129
– noncommutative 764
– non-Euclidean 845
– nonparabolic 728
– of Coxeter type 547
– of grade n 218
– ordered space 1269
– partial 1244
– permutation 271, 276
– permutation of rank $s + 1$ 278
– physical 1025
– plane Euclidean 980
– point-irreducible 1134
– point-uniform 1132
– pointless 1017
– projective 1117, 1208
– projective (lattice) 1119, 1128
– projective metric 990
– pseudo-affine 1208
– pure 81
– quadric circle 992
– quotient 1129
– reduced 1129
– reflection 988
– spherical 845
– squashed 241, 271
– squashed of rank n 278
– stable 1135
– thick 47, 76, 222
– thin 76
– tilde 1239
– topological 7
– transversal 47

GEOMETRY (*cont'd*)
– with group universality 775
– with homomorphism 1060
– without points 1017
– Z- 1130
Gewirtz graph 313, 333, 441, 463
girth 437, 1158
– geometric 437
Gödel's completeness theorem 743
Golay code 854, 859
– extended binary 361
– extended ternary 372
gonality 82, 437
Gosset polytope 500, 855, 871
Gram matrix 846, 864, 866, 878, 894, 898, 902, 910
– doubly stochastic 895
GRAPH 386, 436, 651, 721, 855, 1159
– associated to model 749
– bipartite 386
– complete 436
– distance regular 438
– generalized line 904
– geometric 442
– half dual polar 678
– locally Petersen 1231
– locally Δ 721
– pseudo-geometric 442
– reduced 652
– reflexive 907
– regular 438, 856, 1158
– regular two- 878, 885, 905
– simple 216
– strongly regular 438, 856, 1159
– two- 881
graphs, switching equivalent 880
Grassmann coordinates 392, 1261
Grassmann geometry 1204
– affine 1208
– affine d- 1201
Grassmann line 49
Grassmann manifold 1261
Grassmann map 809, 810, 814
Grassmannian 46, 49, 653, 808, 836, 1260, 1348
Grassmannian characterisation 718
grid 393, 721
GROUP 16, 374
– affine 374
– almost simple 374, 469, 667
– anti-unitary 1295
– classical 18, 667
– extraspecial 171
– of Lie–Chevalley type 22
– of type (FL) 1106
– planes 1298
– projective 18
– quasiperspective 1298
– simple 18
– symplectic 18
– topological transformation 1280
– unitary 1294
group schemes 1100
group universality 775
– for geometries 775
– theorem 773

H-configuration 118
Hadamard 3-design 376
Hadamard design 275, 369, 467
Hadamard matrix 352, 860
Haemers partial geometry 446
Haemers–Roos inequality 388
hair 1173
half Moufang property 468
half ordered plane 100
half way increase 224
half-plane 1279
half-spinor geometry 53
half-spinor space 53
Hall plane 125, 158
Hall system 158
Hall triple system 230, 235, 1192
Hamada's conjecture 368
Hamming distance 307, 361
Hamming graph 461
harmonic polynomial 850
Hartmanis partition lattice 224
Hasse diagram 715
Hasse–Weil theorem 58
Hausdorff space 1266, 1276
height 199
Helmholz–Lie problem 12, 1269

hemisystem 332
Hering plane 161, 165, 374, 470, 1156
Hering space 374, 470, 1156
Hering types of Möbius planes 1337, 1338
Hering–Ott–Schaeffer plane 161
Hermite 2-graph 886
Hermitian arc 314, 316, 317, 323
Hermitian curve, nonsingular 315
Hermitian matrix 1066
Hermitian plane 1294, 1302, 1305, 1306
– connected complex 1302
– modified 1295, 1302
– quaternion 1297
Hermitian space 1268
– nondegenerate 561
Hermitian unital 275, 352, 374, 470, 1155
Hermitian variety 369, 390, 456, 1169, 1198, 1199
– nonsingular 320
Heron's theorem 210
hexagon 905
– inscribed 969
– Pascalian 969
Higman inequality 388, 442
Higman–Sims family 440
Higman–Sims graph 440
Hilbert space 1264
Hilbert–Smith conjecture 1282
Hjelmslev diagram 99
Hjelmslev geometry 1063, 1132, 1133
– point-Noetherian weak 1133
– weak 1132
Hjelmslev plane 100, 933, 1311
– locally compact connected 1311
Hjelmslev ring 788, 1064, 1133
– Artinian 1133
– left-weak 1132
– Noetherian 1133
Hjelmslev–Klingenberg geometry 99, 1063
Hjelmslev–Moufang plane 1065
Hoffman–Singleton graph 440, 463, 466, 1149, 1150, 1154, 1238
homeomorphism
– of circle 1336
– of topological projective spaces 1262
– of vector spaces 1262
homogeneity 776
– flag- 1293
– line- 1294, 1298
– point- 1268, 1294, 1298
– theorem 776
homology 35, 43
– generalized 425
homomorphism 77
– full 1053
homothety 250
homotopy 512, 599
– of stable planes 1278
homotopy group 593
homotopy type of building 1091
host space 256
Hughes plane 110, 1292, 1303
Hughes–Kleinfeld semifield 155, 159
Hughes–Kleinfeld semifield planes, dual 160
hull
– affine 971
– convex 574
hyperbola 802
hyperbola structure 1330, 1339
hyperbolic plane 891, 1258, 1279, 1300
– exterior 1294
– interior 1294, 1299
– modified real 1296
– united 1295
hyperbolic quadric 443, 451
hyperbolic space 848, 949, 1268
hypercube 555, 889
– halved 889
hypergraph 66, 351
hyperline 606
hyperoctahedron 889
hyperoval 120, 298
hyperovals of Cherowitzo 300
HYPERPLANE 30, 71, 200, 212, 323, 654, 816, 836, 949, 950, 1124
– affine 70
– at infinity 972
– dually distant 1045
– geometric 70
– in Barbilian space 1045, 1135
– in geometric lattice 1264

HYPERPLANE (*cont'd*)
– in Klingenberg space 1060
– in projective space 964
– modular 212
– of modular lattice 1124
– of projective space over ring 1042
– osculating 298, 811
– projective 70
– regular 1124
– tangent 316, 317
hyperplane space 71

i-order 83
i-secant 318
I_0-shadow 604
icosahedron 857, 866, 867, 869, 871, 877, 883
icosidodecahedron 869
ideal 698
identities 764
INCIDENCE 6, 109, 112, 124
– deduced from t-spread 325
– in affine plane 744
– in Barbilian space 1045, 1135
– in DLS 198
– in Figueroa plane 112
– in general hexagon 391
– in generalized n-gon 745
– in k-net 745
– in projective plane 109, 744
– in projective space over ring 1042
– in shadow space 686
– in Steiner system 748
– in topological projective plane 1284
incidence complex 85
incidence geometry 3
– metric 3
– ordered 3
– topological 3
incidence graph 68, 386, 437, 749
incidence group 100
– topological sliced 1269
incidence matrix 352
incidence number 436
incidence relation 47, 75
incidence structure, internal 1329
incidence system 684
inclusion 1018, 1020
independence space 195, 201
independence with finite character 200
independent family 1121
independent sets 74
index 568
infinity
– hyperplane at 33
– line at 9
– plane at 9
– point at 9
– subspace at 9
inherited group 157
inner distribution 360
inner product 851
– indefinite 891
inner product space 846
inscribed angles theorem 981
instant 1018
interior construction 749
– restricted 749
interlacing 357
internal plane 310
interpretation function 742
intersection 1259
intersection array 438
intersection property 93, 95, 682
– for flags 605
– for objects 605
intersection set 419
interval causal 937
interval of x over y 1120
invariant 16
– basic 869
– factor 363
inverse path 599
inversion 249, 800, 824, 895, 996, 1337
– B- 249
– elliptic 996
– hyperbolic 996
inversion principle 228
inversive plane, *see* Möbius plane
involution 787
– Baer 1282
– free 1282

irreducible chain 745
– closed 745
isometry 928, 955, 961, 975
– in elliptic space 877
ISOMORPHISM 33, 77, 832, 833
– between angle group and $W \to R^*/K^*$ 819
– between Sp(4,3) and U(4,2) 1197
– between spaces of polynomials 851
– between Sym(6) and Sp(4,2) 1196
– fundamental 832
– induced by K-Jordan isomorphism 833
– of affine chain spaces 832
– of affine metric spaces 974, 976
– of affine ring planes 1078
– of affine spaces 33
– of bimodules 1138
– of chain geometries 806
– of chain spaces 817
– of closure spaces 199
– of collinearity graphs 663
– of (combinatorial) simplicial complexes 555
– of generalized quadrangles 400
– of geometries 77
– of lattices 1019, 1124, 1126, 1127, 1140
– of metric vector spaces 962
– of models 776
– of modules 1139
– of ppm-spaces 1025
– of projective metric spaces 990, 995
– of quadratic algebras 960
– of quadratic circle geometries 962, 992
– of spaces 650
– of spherical buildings 619
– of stable planes 1279
– of structured complexes 527
– of submodule lattices 1119
– of two affine planes 147
– of vector spaces 955
isoparametric foliation 1112
isoparametric map 1111
isoparametric subvariety 1112
isosceles triangle theorem 986
isotopism 147
isotropic vector 561
isotropy group 1112
Iwahori subgroup 1098, 1101
Iwasawa decomposition 1099

J-residue 489
J-shadow 682
J-star 489, 1185
Jacobi identity 698
Jacobi vector field 1111
Jacobson radical 789, 813, 1038, 1122
Johnson plane 159
Johnson scheme 360
Johnson–Walker plane 159
join 483, 652, 1120, 1259
join geometry 97
join-irreducibility 1121
Jordan algebra 1066
Jordan automorphism, K- 832
Jordan curve 1337, 1341
Jordan decomposition 1092
Jordan group 270
Jordan homomorphism, K- 832
Jordan isomorphism, K- 832
Jordan pair 1067
Jordan product 1066
Jordan set 270
– improper 270
Jordan theorem 1279
Jordan–Dedekind chain condition 201
Jordan–Dedekind condition 73
Jordan–Schoenflies theorem 1337
Jordan-transitivity 270

k-arc 119, 298, 301
– complete 298
– embedding 301
$\{k;n\}$-arc 119, 120
– complete 120
– maximal 120
– trivial 120
k-cap 309
– complete 309
k-group 567
k-net 745, 767
– free extension process 755
– nondegeherate 755
– partial 746
– satisfying Thomsen condition 770

k-net (*cont'd*)
– Σ-free construction 751
– Σ-hyperfree elements in a partial 753
(k, n)-Steiner system, Σ-hyperfree elements of a partial 753
k-parabolic 568
Kac–Moody group 591, 618
– split 592
Kalscheuer nearfield 1290, 1291, 1303
Kalscheuer nearfield plane 1305
Kantor embedding theorems 266
Kantor inflation trick 1157
Kantor plane 164
– flag-transitive 164
– of type I 164
– of type II 165
Kantor semifield 154
Kantor theorem 211
Kantor uniqueness lemma 1268
kern 142
kernel 142, 298, 421, 1026, 1329, 1340, 1344
– anisotropic 568
– of normal rational curve 304
– of quadratic form 951
– of quasifield 142
– of region 1026
– of translation plane 142, 146
Killing form 1110
kinematic space 817
Klein correspondence 327, 403
Klein model 892, 895
Klein plane, real 1003
Klein quadric 51, 53, 558, 810
Klein space 1000
Klingenberg plane 1060
Klingenberg space 1058, 1060
Knarr generalized quadrangle 399
knot, *see* nucleus
Knuth semifield 154, 155
Kostant's conjecture 729
Krein conditions 439, 442
Krein inequalities, *see* Krein conditions
Kurosh–Ore dimension 1121

labelling 586, 626, 627
– consistent 627
Laguerre algebra 788, 793, 813, 826
Laguerre geometry 792, 826
Laguerre i-structures 254
– special 254
Laguerre plane 195, 237, 238, 239, 240, 251, 252, 746, 767, 795, 802, 814, 1307, 1327, 1328
– 2-dimensional 1344, 1350
– 4-dimensional 1346, 1347
– classical complex 1347
– complete 238
– constructed from Möbius plane 1338
– elation 1310, 1350
– embeddable 251
– Miquelian 252, 1333, 1345
– nonembeddable 1345
– ovoidal 1328, 1344
– perspectivities of 767
– semiclassical 1335, 1344
– topological 1344
– with 5-dimensional automorphism group 1345
Laguerre space 195, 826
language, first-order 742
Laplace equation 850
Laplace operator 850, 899
Laplacian 1111
large set of designs 352
large set of t-designs 373
Latin square 369, 372
LATTICE 31, 34, 39, 199, 201, 1098, 1117, 1259
– algebraic 1120
– Arguesian 1120, 1122, 1130
– atomic 31, 39, 199, 552
– atomic semimodular 195
– atomistic 1121, 1264
– Boolean 208
– bounded 1123
– complemented 201, 1120
– complete 199
– cubic 86
– cyclic 1122
– Euclidean 910
– even 861, 916
– geometric 195, 199, 201, 1264
– graded 201
– indefinite 910
– integral 861, 908

LATTICE (*cont'd*)
– integral unimodular 909
– irreducible 1121
– locally modular 34
– matroid 850
– modular 31, 39, 552, 1117, 1120
– – with point-system 1118
– module-induced 1124
– of finite length 1121
– of geometric lattice 1264
– of order n 1121
– of type I 908
– of type II 908
– ordered 31
– partition 213, 224
– primary 1122
– ranked 39
– reflexive 911
– relatively irreducible 1121
– root 700, 911
– semimodular 1264
– semiprimary 1122
– topological 1259
– unimodular 861, 908
– – integral 911
– upper continuous 1120
– upper semimodular 199
– weakly distributive 1026
– weight 700
Laurent series 1263, 1333
law alternative 1065
Leech lattice 854, 861, 871, 890, 903, 910, 912
Leißner plane 1036
– affine 1079
– c-transitive 1079
– ℓ-transitive 1079
length of vector 949
Lenz axiom 794
Lenz–Barlotti class 139
Lenz–Barlotti classification 1289
Levi complement 1301
Levi subgroup 706, 713, 1090, 1102
Lie algebra 519, 698, 699, 709, 1087, 1269, 1281
– simple 699, 700
– special 709
Lie bracket 1111
Lie geometry 1307, 1347
– associated to Laguerre plane 1339, 1349
Lie group 13, 18, 1281, 1301
– exceptional 479
– linear 18
– orthogonal 18
– quasisimple 1301
– semisimple 1342
– simple 18
– simple complex 18
Lie topology 1303
Lie transvection F-plane 1073
– coordinate ring of 1073
Lie triple system 1304
Lie–Chevalley group 15
Lie–Johnsen group 1312
lifting 78, 509
light cone 913
likeable mapping 162
likeable plane 162, 398
LINE 10, 94, 109, 111, 124, 325, 606, 651, 816
– absolute 118
– affine 724
– as manifold 1277
– at infinity 141
– coaffine 1275
– coherent 1077
– concurrent 387, 435
– cross 1076, 1079
– elliptic 875
– equiangular 855, 902
– Euclidean 959, 1337
– extended 997
– external to k-cap 309
– fixed 391
– hyperbolic 58, 395, 891
– improper 141
– in affine ring plane 1076
– in Alt(7)-geometry 1166
– in Barbilian space 1135
– in block space 210
– in design 56, 375
– in DLS 196
– in generalized n-gon 745
– in geometric lattice 1264

LINE (*cont'd*)
– in group coset geometry 395
– in Laguerre plane 240
– in Leißner plane 1079
– in Minkowski plane 240
– in projective plane 109
– – over octonion algebra 1067
– in projective space 551, 964
– in stable plane 1270
– in vector space 950
– interior 111
– isotropic 1294, 1297
– K- 1000
– $\mathfrak{l}$-distance 810
– null 925
– of affine plane 744
– of affine space 971
– of block space 68
– of Desarguesian topological projective 1263
– of embedded shadow space 715
– of Euclidean space 849
– of generalized Hughes plane 111
– of geometry quadrangle 521
– of inner product space 847
– of $k_{m,n,q}$ 318
– of k-net 745
– of Klein space 1000
– of Lie geometry 1349
– of locally classical geometry 1226
– of perspectivity 417
– of projective plane 744
– of space 649
– of Sperner plane 1004
– of translation plane 141
– on Barbilian space 1046
– passant 436
– passing 965
– projective 59, 785, 1273
– Q-regular 972
– r- 1002
– regular 797, 994
– secant 965
– – to k-arc 300
– – to k-cap 309
– singular 238, 797, 972, 994
– spacelike 925
– tangent 317, 436, 669, 800, 816, 965
– – to Hermitian arc 314
– – to k-arc 300
– – to k-cap 309
– thick 208, 651, 1162
– thin 435, 651
– timelike 925
– transversal 325, 805
line graph 1159
line space 68
line-diameter 437
linear algebra 14
linear combination 215
linear fractional transformations 59
linear group, general 1037
linear part of similarity 975
linear programming 857
– bound 360
linear representation 436
LINEAR SPACE 12, 70, 83, 129, 195, 198, 352, 435, 464, 465, 1003, 1147
– embeddable 129
– nondegenerate 129
– partial 70, 435
– residual 241
– restricted 131
– subspaces of 12
– with dimension 70, 87
– with dimension n 1147
linear variety 414
lineation 1280
lines orthogonal 387
– parallel 1274
linked threes 465
local diameter 437
local field 612
locally compact field 1094
– non-Archimedean 612
locally symmetric space 1111
locals 1018
logarithmic concavity 224
Lorentz space 901
Lorentz transformation 925, 935
– orthochronous 935
Lorentz–Minkowski geometry 964

Lorimer–Rahilly plane 159
Łoś–Tarski theorem 743
Lucas theorem 364
Lüneburg plane 163, 374, 1156
Lunelli–Sce hyperoval 122, 1199

m-covering 509, 592
– universal 593
m-homotopy 593
– elementary 593
m-spread 144
– proper 144
m-truncation 197
(m, n)-plane 764
Mac Lane theorem 219
Mac Lane–Steinitz axiom 199
MacWilliams transform 371
manifold 18
– analytic 1336
– integral homology 1350
– topological n- 1276
map
– adjacency preserving 620
– canonical 1088
– strong 259
mappings, equiforme 820
maps, gluing 626
Matchstick space, full 236, 262
Mathieu–Witt design 208, 212, 232, 253, 255, 268, 270, 272, 273, 275, 370, 1192, 1208, 1216
matrices 846
– congruent 846
– elementary 1038
– totally unimodular 261
matrix representation of 4-dimensional elation Laguerre planes 1348
matroid 12, 30, 195, 216, 416
– circuit 216
– cographic 216
– cycle 216
– free 208
– graphic 216
– paving 208
– polygon 216
– simple 195, 201
– uniform 208
– unimodular 216
maximal arc 313
maximal exterior set 339
– linear 339
maximal intersection, principle of 75
maximal parabolic subgroup, standard 713
maximal subspace 1169
maximal torus 694
McLaughlin graph 442, 855
MDS codes 307
– linear 308
measure function 980
meet 1120
metasymplectic space 636, 639, 696, 709, 719, 1209, 1210, 1214
metric 927
metric product 959, 968, 1004
metric space 5
– affine 974
– elliptic 990
– hyperbolic 990
– irregular 990
– pointless 1023
– projective 990
– regular 990
– singular 990
metric vector space 927, 948
minimality condition 88
Minkowski chain space, weak 827
Minkowski geometry 793, 807, 811, 828
Minkowski plane 195, 238, 239, 240, 252, 277, 746, 767, 795, 802, 814, 1295, 1327, 1328
– 4-dimensional 1342
– classical real 1342
– finite-dimensional 1340
– Miquelian 252, 770, 829, 1333
– perspectivities 767
– pseudo-pencils 767
– semiclassical 1335
– with 3-dimensional kernel 1341
– with 4-dimensional automorphism groups 1341
– with group transitive concircular points 252
Minkowski product 828, 831
– symmetric 828

Minkowski space 195
Minkowski space-time 237, 924
Minkowskian plane 952, 953
Minkowskian space 948, 963
minor 203, 259
Miquel condition 246, 821
Miquel configuration 1328
Miquel theorem 245, 999, 1004
– first 983
– second 983
– third 984
Miquelian inversive plane 369
Miquelian plane 252
Möbius 2-graph 886
Möbius function 228
Möbius geometry 792, 807, 811, 964
– characterization 823
– d-dimensional 218, 253
Möbius plane 195, 237, 238, 239, 275, 375, 746, 767, 802, 814, 964, 1147, 1194, 1327, 1328, 1336
– egglike 310, 1328
– embeddable 1337
– Miquelian 310, 820
– nonembeddable 1337
– ovoidal 310, 1328, 1337
– pencils 767
– perspectivities of 767
– semiclassical 1334
– spherical 1337
– with 2-transitive group 252
– with point-transitive group 252
Möbius space 195, 218, 253, 794, 808, 816, 895
– strongly Miquelian 824
Möbius strip 1279, 1299
Möbius-closure 997
model 742
– full 741
– infinitesimal 1304
– parabola 1287, 1292
– partial 743
– plane 242
– residual plane 240
– spear-cycle 1339
– Σ-confined 752
– Σ-free 754
– Σ-open 752
modular identity 1120
modular law 1048
module 805
– faithful 805
– free 784, 1036
– left 784
– torsion free 1125
– with high weight ρ^{w_0} 713
Molien–Poincaré series 869
– harmonic 870, 871
monomials standard 714
monomorphism 743
Moore geometry
– generalized 463
– proper 465
Moore graph 450, 462, 1154
Moorhouse conjecture 368
Morita equivalence 1139, 1137
MORPHISM 3, 77, 78, 489
– distant-preserving 1053
– neighbour-preserving 1053
– of affine chain spaces 832
– of bimodules 1139
– of chamber systems 489
– of complexes 483
– of curve spaces 829
– of frames 1018
– of geometric lattices 255
– of geometries 77, 78
– of lattices 1136, 1139
– of models 743
– of modules 784
– of projective spaces 661, 664
– of rings 832
– of rings, surjective 1040
– of S-modules 1139
– of spaces 650
– of structured complexes 527
– of transformation groups 1028
– of V-spaces 1053
– special, *see* – type preserving
– type preserving 489
Mostow's rigidity theorem 896

motion 820, 928, 975, 976, 995
– Q- 991, 996
– $\overline{Q}$- 995
motion group 1304
– elliptic- 1294
– Euclidean 1295
– hyperbolic 1294
Moufang blueprint 630, 632
Moufang building 585, 591, 629
– natural labelling of 630
Moufang condition 46
– topological 1309
Moufang conditions 423, 424
Moufang generalized m-gon 585
Moufang loop 40
Moufang metasymplectic space 707
Moufang oval 251
Moufang plane 153, 662, 969, 1289
– compact projective 1285
Moufang property 468
Moufang quadrangle 633, 1308
Moulton plane 11, 35, 1287, 1289, 1291, 1296, 1299, 1300
multiplicity freeness 1149, 1152
mutation 1287, 1291

n-dimensional universal model 262
n-gon, compact (connected) 1307
n vectors, equiangular 879
natural labelling 589
near n-gon 458, 460, 465
– classical 459
– ovoidal 459
– regular 460
– sporadic 459
near hexagon, regular 460
near octagon 661
– regular 461
near polygon 676
near-pencil 130
nearfield 15, 111, 151, 1288
– irregular 152
– locally compact connected 1288
– kernel 111
– regular 151
nearfield plane 1156
neighbour 1043
– dually 1043
neighbour relation 1043, 1135
neighbourhood 1158
net 465, 794
– Desarguesian 369
– double 764
– of circles 1332
– of points 1332
Netto circular space 269, 272
Netto system
– classical locally 1194, 1205
– generalized 1156
– triple 1156
Neumaier geometry 1196, 1202
Newton numbers 854, 860
Niemeier lattice 910
nilpotent element 702
non-Desarguesian plane 13
nontangential inclusion 1020
norm, additive 1097
normal rational curve 298
nucleus 121, 126, 148, 298, 1000
– left 147
– middle 146
– of normal rational curve 304
– right 146
numbered complex 623
– with group action 494
numbering 484, 527
– of building 527

o-polynomial 121
obstructions to GF(q)-coordinatizability 260
octahedron 871
octaves 690
octonion 690, 1263
octonion algebra 1065
– split 1066
octonion division algebra 1066
opposite plane 1274
opposition 417
optimal condition 89
orbit 868, 1028, 1281
– on DLS 269

ORDER 124, 1121, 1168, 1169
– feasible 1169
– i- 221
– in lattice 1121
– known type 1168
– of affino-projective space 234
– of Benz plane 238
– of chain space 793
– of flag 221
– of partial linear space 435
– of unital 124, 125
– partial 1018
– temporal 1018
Ore conditions 1061
– left 1054
– right 1054
Ore domain, left 1125, 1138
Ore geometry 1133
oriflamme complex 558
origami space, full 262
orthogonal decomposition 954
orthogonal sum 954
orthogonality 202, 949, 950
– of hyperbolic lines 892
orthogonality relation 988
– affine 972
orthoscheme 896
– complete 896
– doubly truncated 897
– ordinary 897
– simply truncated 897
Ostrom dimension 146
Ostrom–Foulser plane 160
Ott–Schaeffer plane 161
oval 57, 119, 120, 298, 317, 965, 1021, 1328
– closed 1310
– complete 298
– symmetric with respect to point 969
– topological 1332
– translation 251
oval plane, complex 1295
overlapping relation 1020
ovoid 124, 166, 242, 309, 323, 330, 402, 816, 965, 1328, 1337
– closed 1310
– of generalized quadrangles 403
– of polar spaces 330

p-adic number 1263
packing 327
pair
– complementary 1267
– regular 395
– skew 1267
Paley 2-graph 886
Paley design, *see* Hadamard design
Pappian space 965
Pappus configuration 91
Pappus theorem 31, 117, 970
Pappus–Pascal theorem 969, 983
parabola 802
– skew 1292
parabolic subgroup 79, 392, 479, 564, 567, 1087, 1089, 1185
– maximal 1183
– minimal 1183
– standard 689, 707
– theorem 565
parallel 33
parallel axiom 1274
parallel classes 762, 792
parallel relation 827
PARALLELISM 40, 97, 98, 724, 1327
– in affine ring plane 1076
– in affine space 33, 724
– in block space 98
– in design 352
– in Laguerre plane 746
– in Leißner plane 1079
– in Minkowski plane 746
– in Möbius plane 746
– in nets 368
– in quasi-affine space 97
– of hyperbolic lines 892
parallelodromy condition 1079
parameter system 539, 625
parameters 548
– topological 1308
parapolar space 688, 693, 717, 719, 720
paraprojective space 654
partial flock 339

partial geometry 465
– embedded in projective space 448
– parameters 441
– proper 441
partial linear gamma space 684, 722
partial linear space 435, 650
– of order (s,t) 435
– thick 435
– thin 435
partial model
– degenerate with respect to some extension process 760
– Σ-degeneration 760
partial plane, *see* partial linear space
partial quadrangle 449
– thin 450
partial spread 327
partial t-spread 324
– maximal 324
partition
– partial G- 213
– t- 208
partition axiom 199
partition space 224
Pascal theorem 970, 983
Pasch axiom 10, 36, 87, 447, 653, 1223, 1270, 1279
Pasch configuration 213
Pasch–Veblen axiom 68
path 520, 956, 975
– in graph 386
– reduced 520
pencil 72, 245, 300, 978, 1271, 1329
– line 1277
– of hyperplanes 304
– of points 765, 768
– tangent 762, 1329
Penrose tiling 865
pentagon 719, 857
pentagram 275
periods of time 1018
permutation group 16, 268
– n-DLS primitive 268
– n-homogeneous 268
– of quadric 815
permutation rank 728
perpendicular bisector 989
perspective 30
perspective closedness 674
perspective relation in lattices 1120
perspectivity 60, 115, 766, 1271
Petersen graph 84, 91, 440, 463, 466, 856, 890, 904, 907, 1149, 1150, 1154
Pfaffian 51
Pickert–Buekenhout theorem 969
pigeonhole principle, topological 1277
planar condition 141
planar difference sets 356
planar representation
– of Laguerre plane 1343
– of Minkowski plane 1339
planar space 230
– L^0- 235
– locally affine 230
– locally projective 230
– sharply locally projective 230
– strongly locally affine 254
– strongly locally projective 230
– weakly locally projective 230
PLANE 816, 950
– Desarguesian L- 1080
– F-planes, connected 1070
– F-planes, transvection 1070
– in Alt(7)-geometry 1166
– in design 375
– in projective space 964
– model 244
– null 925
– of affine space 971
– of inner product space 847
– Q-admissible 816
– spacelike 925
– tangent 309
– timelike 925
– translation L- 1080
Plücker coordinates 715
Plücker relations 715
Poincaré model 893, 895
Poincaré series 540
Poincaré space 1002

POINT 6, 10, 86, 93, 109, 111, 124, 351, 606, 785, 790, 791, 793, 797, 811, 815, 835, 1021, 1022, 1026, 1294
– 3-regular 413
– 4-th Q- 968
– absolute 118, 663
– adherent to region 1021
– affine 1274
– at infinity 972, 997
– belonging to region 1019, 1021, 1022
– coaffine 1275
– cocatenal 790, 793
– collinear 387, 968
– concircular 238, 1328
– conjugate 811, 815
– dependent 958
– distant 785, 793, 797, 818, 996, 1045
– double 814, 816, 965
– elliptic 875
– Euclidean 968
– exterior hyperbolic 892
– free 1128
– Hermitian 1294
– hyperbolic 891
– ideal 848, 893
– – hyperbolic 892
– in affine ring plane 1076
– in Alt(7)-geometry 1166
– in Barbilian space 1045, 1135
– in Benz plane 1327
– in closure set 198
– in DLS 196
– in generalized n-gon 745
– in geometric lattice 1264
– in group coset geometry 395
– in Klingenberg space 1060
– in Laguerre plane 240, 746
– in lattice 1128
– in Leißner plane 1079
– in Minkowski plane 240, 746
– in Möbius plane 746
– in physical space 1026
– in projective plane 109
– – over octonion algebra 1067
– in projective space 551, 964
– in shadow space 93
– in Steiner system 748
– independent 1044, 1045
– interior 111
– isotropic 58, 1294, 1296, 1297
– near 996
– nonsingular 318
– of affine plane 744
– of affine space 971
– of chain geometry 806
– of curve space 829
– of De Sitter space 848
– of design 351
– of elliptic space 847
– of Euclidean space 849
– of finite classical polar space 330
– of generalized Hughes plane 111
– of generalized quadrangle 521
– of hyperbolic space 848
– of inner product space 847
– of k-net 745
– of Klein space 1000
– of Lie geometry 1349
– of locally classical geometry 1226
– of Minkowski plane 1339
– of $P^n(R)$ 835
– of polar space 556
– of projective line 785
– of projective plane 744
– of projective space over ring 1042
– of semibiplane 86
– of space 649
– of Sperner plane 1004
– of spherical space 847
– of translation plane 141
– of $\Sigma(\mathsf{A}_n)$ 550
– opposite 1090
– orthogonal 387
– parallel 791, 811, 815
– Q-regular 968
– Q-singular 968
– r- 1002
– R-rational 567
– regular 395
– self-conjugate 391
– semiregular 370, 415

POINT (cont'd)
– simple 965
– singular 58, 318
– singular ideal 239
– singularly collinear 238
– skew-Hermitian 1294
– unimodular 1128
– with complement 1128
point graph 386, 437
– of cyclotomic type 445
– of partial geometry 441
point sets, equidistant 880
point system 1124
point-diameter 437
POLAR SPACE 45, 49, 58, 329, 455, 553, 556, 621, 649, 662, 685, 717, 719, 968, 1161, 1202, 1203
– abstract 45
– affine 724, 1195
– classical 675
– classification 58
– classification of nondegenerate 676
– dual 698
– embeddable 675
– half dual 678, 698, 719
– lower truncation of 1223
– Moufang 675, 698
– of rank n 662
– orthogonal 49
– symplectic 49
– thick-lined 1161
– thin 557
– thin from above 685
– truncation of 1222
– unitary 49
polar transvection 663, 705, 726
– group 728
polarity 43, 78, 119, 124, 562, 658, 663, 772, 968, 990
– classification 58
– fundamental 811
– induced by quadratic form 968
– nondegenerate 562
– of generalized quadrangles 404
– orthogonal 119, 124, 666
– symplectic 450, 666
– unitary 124, 666
polarization identity 923
pole 968
polyhedral group 871
polynomial
– annihilator 901, 902
– harmonic 869
– homogeneous 812, 900
– minimal 813
– p- 220
Pontrjagin theorem 1263
poset 75, 485
– pure 485
ppm-space, canonical 1025
preapartment 88
pregeometry 76
primitive concepts 5, 1017
product set 849
product space 849
projection 576
– canonical 77
projection map 576
projection of quadric 319
projection pencil 765
projection pseudo-pencil 765
projection set 765
– pseudo- 765
projective n-space
– topological 1265
projective closure 972
projective group of Barbilian space
– full 1052
– little 1052
PROJECTIVE PLANE 11, 34, 109, 170, 385, 465, 744, 750, 1279
– almost 1275, 1298, 1300
– Archimedean 1311
– classical 389, 1270
– corresponding to metric vector space 969
– derived 1329
– Desarguesian 109, 115, 964, 1333
– – complex 1333
– free 762
– free extension process 754
– modified real 1296, 1300
– Moufang 631

PROJECTIVE PLANE (*cont'd*)
– non-Desarguesian 11, 1286
– nondegenerate 754
– obtained from projective plane 34
– over Cayley algebra 572
– over Cayley octonions 1285
– over split octonions 1066
– Pappian 770, 964, 1333
– perspectivities 767
– projection pencils 767
– proper 110
– punctured 1275
– real 1300
– topological 1266, 1283, 1332
– Σ-free constructions of 750
– Σ-hyperfree elements in a partial 752
– Σ-open 763
projective set 657
PROJECTIVE SPACE 9, 30, 39, 70, 195, 551, 816, 1129
– associated to projective lattice geometry 1129
– associated to vector space 964
– Desarguesian 550, 964, 967, 1155, 1261
– generalized 207
– missed 197
– of infinite dimension 1263
– over full matrix ring 1056
– over ring 1038, 1042
– over topological division ring 1260
– Pappian 964
– possibly reducible 39
– reducible 1161
– thick 551
– thin 551
– topological 1259
– which corresponds to vector space 964
projectivity 60, 115, 766, 786, 806, 968, 1271
– affine 767
– of Möbius geometry 824
– Q- 991
– $\overline{Q}$- 995
proof 743
PROPERTY
– A 717
– B 989
– **B** 233
– **BB** 245
– C_m 520
– C_n 1131
– CS_m2 524
– CS_m3 524
– CS1 524
– D 1130
– D_m 520
– D1,2,3 538
– Deg 1152
– F 521
– Fth 1152
– hereditary 76, 1262
– HH 607
– IC 95
– Int-F 605
– Int-O 605
– IP 95, 1152
– $(IP)_s$ 1152
– L1,L2 1004
– LH 607
– Lin 606
– LL 95, 607, 1149
– LS 95
– **M** 246
– **M8** 246
– MF 1149, 1152
– NT 1152
– O 607, 1152
– OO 95
– Pri 1152
– Q_C 537
– QS 1069
– R0 1124
– R2 1152
– Sph 1152
– Sph Cox 1152
– Th 1150
– TMF 1152
– U 95
– UC 1152
– X 199
pseudo-isometry 928
pseudo-modularity 220

pseudo-oval 1310, 1348
– topological 1348
pseudo-perspectivity 766
pseudo-similarity 929
pseudomanifold 504
pseudometric space, pointless 1023
pseudoquadratic form 46
– trace-valued 46
pseudoquadric 671
Pythagorean field 933

Q-conformity 999
Q-distance 990
Q-knot 968
Q-measure 999
Q-sphere 973
– proper 973
Q-symmetry 968
quad 459
quadrangle 677
– property 1137
– section condition 1069
– symplectic 1308
quadratic cone 167
quadratic set 242, 816, 965
– degenerate 816
– ruled 965, 967
quadratic space, nondegenerate 561
quadratic surface
– hyperbolic 166
– elliptic 166
quadric 389, 822, 990
– affine 972
– associated with quadratic form 964
– at infinity 972, 980
– elliptic 237, 814, 1328
– hyperbolic 237, 814
– ovoidal 965
– perfect σ- 244
– ruled 965, 1328
quadric circle geometry 991
quadrilateral 417
quadrilaterals in perspective 417, 418
quadruple 60
– equi-anharmonic 60
– harmonic 60
quad 677
quasi-design 195, 208
quasi-inversion 250
quasifield 141
– locally compact connected 1288
quasipolarity 663
– determined by σ-sesquilinear form 665
quasiprojective set 657
quasisimplicity 1100
quasisplit group 568
quaternion 43, 813, 1263
quaternion algebra 636
– split 1065
quaternion division ring 673
quotient 77
– 2- 77
– of block space 67
– of geometry 77
quotient of affine polar space
– improper 1196
– standard 1196, 1202, 1203

R-module flat 1136
r-net 86
R-space 499
radical 237, 567, 650, 664, 950
– of planar space 237
– solvable 1301
radius 972
Radon transform 228
RANK 30, 39, 75, 552, 662
– in lattice 1121
– of building 1308
– of chamber system 488, 678
– of complex 483
– of finite partial model 762
– of flat 277
– of free module 784, 1041
– of geometry 47
– of hyperbolic diagram 1172
– of incidence matrix 362
– of integral lattice 908
– of J-star 1185
– of lattice 39
– of parabolic subgroup 1183
– of poset 485

RANK (*cont'd*)
– of projective space 552
– of projective subspace 964
– of Q-sphere 973
– of subset 75
– of vector space 30
– polar 688
– singular 652
– stable 785, 1039
– uniform 1121
rapid growth 262
rational representation of chain geometry 831
realization of blueprint 627
reciprocity 78
rectangle, (i, j)- 828
rectangle property 243
reduced expression 503
reducibility 749
reductive group 567
Ree group 466, 470
Ree unital 374, 470, 1155
Reed–Muller code 367
– punctured 367
reflection 43, 505, 699, 1282
– affine 1096
– fundamental 699
– with respect to circle C 249
reflection group 869
region
– bounded 1023
– empty 1023
regions 1018, 1029
– r-approximately overlapping 1029
regular action 114
regular space 561
regular system of order m 332
regularity condition on projectivities 770
regulus 51, 97, 327, 805
– generalized 809
– opposite 51
– SPG 454
regulus condition 419
regulus net 447
representation 769
– algebraic 256
– irreducible 769, 872
– linear 256
– of group 872
– projective 256
– rational 812
– smooth 812
residual connectedness 679
residual space 794
residual trace 794, 796
residually connected 47
RESIDUE 47, 76
– bottom 202
– of flag 76
– top 197, 202
residue geometry 20
resolution 352
restricted $D(i + 1)$-spaces 254
restriction of closure space 199
retraction 527
retract, absolute neighbourhood 1350
rhomboedra 865
Riemannian manifold 1111
rigid plane 1293
rim 216
ring 1064
– alternative 1067
– biternary 1312
– chain 1060
– completely primary and uniserial 1122
– full matrix 1039, 1056
– GE_2 1076
– local 788, 791, 833, 1058, 1060
– of double numbers 960
– of dual numbers 960
– oppositive 1036
– planar sexternary 1062
– planar ternary 1062
– primitive 834
– regular 1121
– semilocal 834
– semiprimary 789, 1039
– semisimple 811
– two-sided units 1035, 1070, 1077
– valuation 1061
– Von Neuman regular 1134
– weakly quadratic 822

Robertson–Walker space-time 930, 937
roof 276
root 506, 567, 583, 699
– affine 1096
– in building 583
– opposite 508
root group 584, 703, 727
– exceptional 728
root group space 708
– exceptional 720
root system 18, 699, 854, 861, 870, 888, 1096
– relative 1089
– simple 19
rotation 979

s-property, see splitting property 1137
Sandler semifield 154
satisfaction relation 742
scalar
– spacelike 934
– timelike 934
scalar product 1111
Schläfli configuration 855
Schläfli graph 856, 905
Scum theorem 259
secant 300
– m- 436
secant segments theorem 984
second main characterization of buildings 598
section 994
– affine-dual-affine 1214
segment, straight 1021
Segre manifold 811
Segre product 831
Segre theorem 57, 247
Segre variety 325
semi-affine plane 235
semi-affinity 929
semi-building 91
semi-oval 242, 317
– regular 317
semi-ovoid 242, 317
semibiplane 86, 217, 1147
semidirect product, almost 1301
semifield 152, 1287
semifield plane 152, 1291
– proper 153
semilinear space, *see* partial linear space
semilinear transformation, θ-semilinear 41
semipartial geometry 403, 448
– proper 449
semisimple algebraic group 567
semisimple group 567
sentence 742
separateness relation 1021
separation 927
– k- 206
set 808, 811
– abstractive 1020
– affinely dependent 971
– affinely independent 971
– closed 198
– – under reguli 808
– – with respect to r-distance lines 811
– concordant 228
– dependent 200
– few-distance 888
– independent 200, 964, 994
– of odd type 320
– of type $(1, m, q+1)$ 316, 318
– orthogonal 886
– partially ordered 75
– quadratic 1331
– representative 1021
– skew characteristic 263
– spanning 200
– standard 835
– symplectic 886
set of permutations, geometric 277
set of representatives 323, 324
– minimal 324
– reduced, *see* – minimal
Seymour decomposition theorem 261
shadow 93, 499
– geometry 615
– poset 615
– space 93, 604, 682
– – embeddings of 712
– – over j 682
shape 1026

shear 173
– plane 1274
shellability 589
shift plane 1287, 1292
signature 928
– of inner product 846
– of integral lattice 908
– of symmetric matrix 846
similarity 820, 928, 975, 977, 995
– of Minkowskian plane 979
simple connectedness, residual 77
simple design 351
simplex 42, 483
– empty 483
– hyperbolic 895, 896
– opposite 580
– regular 867
Singer group 114
singular space 318
singular stroke 1148
singular subspace 444
size
– i- 221
– of i-variety 221
skew affine space 97
skew field, *see* division ring
skew-hyperbolic plane, quaternion 1295
Skolemization 773
sliced space 1268
Smith graph 866
Smith normal form 363
socle 181
space 6, 649, 650
– complete 650
– LP- 764
– m-closure 216
– nondegenerate 650
– paraprojective 684
– rich n- 1265
– singular 650
– stable n- 1265
– strongly parapolar 688
– topological n- 1265
– with homomorphism 1058
spanning diamond 1123
spanning family 1120
Sperner space 98, 795
sphere
– algebraic n- 811
– unit 901
spherical design 863
spherical harmonics, zonal 852
spherical space 847, 1002
spin group 635
spinor 51, 52
– pure 52
split group 568
splitting property 1137
spoke 216
sporadic group 1243
spread 124, 143, 327, 330, 402, 1288
– of generalized quadrangle 403
– partial 1274
– planar regular 807
– proper 143
– regular 124
– subregular of index k 327
square classes 1000
square ratio 979
stability 1265
stabilizer 17, 1281
– lemma 1340, 1344
– of quadrangles 1290
stable plane 1258, 1266, 1270
– homogeneous 1293
– locally compact 1273
– locally connected 1273
– proper 1271
stable space, sliced 1269
standard representation of Minkowski plane 1340
star 483, 1173
Steinberg relations 703
Steinberg representation 1092, 1094
STEINER SYSTEM 13, 56, 124, 208, 351, 353, 373, 1149
– 3-(22, 6, 1) 908
– 4-(23,7,1) 903
– 5-(24,8,1) 854, 860, 887
– free extension process 756
– (k, n)- 748
– nondegenerate (k, n)- 756

STEINER SYSTEM (*cont'd*)
– partial (k,n)- 748
– quadruple 40, 230, 1192
– triple 370
– Σ-free constructions of (k,n)- 752
stereographic projection 241, 816, 817, 893, 910, 993, 1327, 1337
stiffness 1269, 1281
Stirling number 224
Stokes law 882
strain 173
Strambach's $SL_2\,\mathbb{R}$-plane 1297, 1299
strength
– of measure 874
– of spherical design 863
string 486, 1173
strong B-group 179
strong exchange axiom 198
strong regularity 360
strong transitivity 563
structure 4
– $\mathcal{L}$- 742
structure constants 701, 803
structured complex 526
sub-n-gon, proper 405
subalgebra 698
subcomplex 574
– convex 574
subdiagram 1152
subgeometry 76, 204, 259
– induced 76
– of x in G 1129
– Σ-free 761
subgroup 1104
– anisotropic 571
– bounded 1097
– maximal arithmetic 1110
– maximal parabolic 20
– nonsemi-simple 20
– parabolic hyperspecial 1110
– parahoric 1101
– planar 1303
– quasiperspective 1303
– S-arithmetic 1104
– t-homogeneous 869
– torsion-free 1106
submanifolds, isoparametric 1309
submodel 749, 774
– Σ-closed 774
– Σ-hyperfree 752
suboctagon 407
subquadrangle 405
subregular sequence of spreads of length k 327
subring
– admissible 1054, 1055
– maximal for morphism 1055
subset 216, 219, 576, 598, 1120
– algebraically independent over F 219
– determining Barbilian domain 1075
– flat 533
– gated 576
– geodetically closed 459
– independent 552
– m-closed 216
– prelinear 557
– proper 965
– proper m-closed 216
– regular 1019
– singular 650
– spherical 598
SUBSPACE 36, 45, 70, 71, 73, 663, 676, 821
– absolute 663
– adjacent 676
– affine 9, 971, 975
– complementary 950
– distant 1043
– dually distant 1043
– generated by subset 650
– geodetically closed 684
– in projective space 551
– irregular 951
– isoclinic 897
– isotropic 951
– linear 70
– maximal singular 952, 954
– maximal totally isotropic 330
– nonsingular 994
– of chain space 821
– of closure space 73
– of free module 1037
– of hyperplane space 71

SUBSPACE (*cont'd*)
– of line space 70
– of point-line space 650
– of polar space 556
– of projective space 964
– of projective space over ring 1042
– of $\Sigma(\mathsf{A}_n)$ 550
– open 1267
– parallel 971
– planar 231
– projective 36, 964
– regular 951
– singular 70, 951, 952, 994
– spanned by vectors 949
– totally isotropic 45, 70, 561
– totally singular 45
– transversal 1043
substructure, $\mathcal{L}$- 742
subtorus 1087
Suetake plane 165
sum 551
– of flats 1048
– orthogonal 851
supersum of DLS's 204
support 773
Süss group 820
Suzuki chain 1224
Suzuki group 54
Suzuki–Tits ovoid 54, 244, 469
Suzuki–Tits plane 252
switching of graphs 878
symmetric group 3
symmetric plane 1280, 1298, 1304
– generalized 1306
– k- 1305
symmetric space
– generalized 1305
– isotropic 1297
– k- 1305
– Riemannian 1309
symmetry 420, 1304
– affine Q- 976
– Q- 957
symplectic polarity 390
symplecton 636, 688, 1170
syntheme 401
system of apartments 90, 526
system of blocks of imprimitivity 99
system of groups 566
– satured 566
system of large apartments 90
system of regular apartments 92

t-design 13, 351
– spherical 868
t-design code 371
t-property, *see* triangle property
t-regulus 325
t-spread 324
– regular 324
t-wise balanced design 352
(t, s)-blocking set 321, 323
– minimal 323
– reduced, *see* – minimal
tactical decomposition 352
tail 1173
Tamagawa numbers 1109, 1110
tangent 300, 973, 1328
– plane 1328
– to $k_{m,n,q}$ 318
tangent bundle plane 1071
tangent curve 305
tangent space 421, 811, 816, 1071, 1073
– to Veldkamp plane 1071, 1073
tangent theorem 983
tangent vector 1071, 1073
– to Veldkamp plane 1071, 1073
Teirlinck theorem 210
term 742
ternary field 1284
– topological 1271
tetrahedron 871
– hyperbolic 895
theory 743
– closed 743
– equivalent 743
– inductive 743
– reproducible 760
– universal 743

thick support 1162
thickness 387
tiling
– aperiodic 865
– icosahedron 865
Timmesfeld diagram 1174, 1178
Tits generalized quadrangle 393
Tits group 709
Tits ovoid 310, 370
Tits property 468
Tits system 479, 549, 563, 572, 688, 1098
– saturated 564
– split 591
Tits theorem 967
Tits–Weiss theorem 1307
top heaviness 224
topology 1025
– compact-open 1280
– Lie group 1111
– product 1090
– quotient 1088
torus
– k-split 567, 1087
– maximal 567
– split 1087
totally singular subspaces, maximal 330
trace set 665
transformation, semilinear 929
transitivity 109
– 3-|| 825
– F- 251
– of Q-symmetries 977
transitivity theorem 976, 987
translation 141, 250, 798, 975, 1078, 1291
– group 43, 421
– hyperoval 121
– ℓ- 1079
– ovoid 244
– plane 141, 334, 369, 770, 988, 1291, 1337
– – affine 1156, 1305
– – almost projective 1302
– – non-Desarguesian 1306
– – nontopological 1288
– – tangent 1304
– – topological 162
– vector 975
translation complement 145
– linear 145
translation generalized quadrangle 420, 421, 423
transvection 42, 726
– central 1049
– group, h- 1051
– h- 1051
trial solid 391
triality 53, 78, 391, 690
triangle axiom 1231
triangle, ideal 894
triangle property 887, 1136
triangular lattice 865
triple
– balanced 1128
– directly balanced 1128
– flat 880
– odd 880
truncated icosahedron 869
truncation
– $\{1^+, \ldots, h^+, 1^-, \ldots, k^-\}$ 1201
– hyperplane- 209
– lower 1221, 1222
– n- 202
– point- 209
– point hyperplane- 201
– upper 202
truncational multiplicity freeness 1152
Tutte group 260
Tutte homotopy theorem 260
twin buildings 591
twin reflections 1302
twisted cubic 161
twisted field, generalized 154
two-distance set, spherical 854
type 21, 47, 75
– distinguished 569
– of gallery 484
– of reality 836
type function 75
type set, opposite 581

U-property 1131
unfolding 1162, 1226
uniform element 1121

uniformity 1027
unimodular element 1037
unisecant 318
unit vector 1075
unital 124, 125, 358, 452, 886
– classical 124
– Hermitian 273
universal part of theory 743
universe 742

V-plane 1051
– Desarguesian 1051
– transvection 1051
valency 438
Vámos DLS 256
Van Lint–Schrijver geometry 445
variable 742
variety 208, 212
– Hermitian 242
– improper 196
– in DLS 196
– modular 212
– projective 811
– proper 196
– quasiprojective 810, 831
– thin 208
Veblen axiom, *see* Pasch axiom
Veblen–Wedderburn system 141
Veblen–Young axiom 48, 551, 1117, 1129
Veblen–Young theorem 36, 229, 653
vector 846
– 4-th Q- 958
– balanced 864
– isotropic 846
– negative 846
– null 927
– of minimum norm 916
– orthogonal 927
– positive 846
vector space 4
– anisotropic 927
– nonsingular 927
Veldkamp plane 1036
Veldkamp space 656, 1036, 1045, 1135
– affine 1050
– (c, h)-transitive 1050
– dual affine 1050
– dually (c, h)-transitive 1050
Veronese manifold 801, 836
vertex 318, 483
– of cone 318
Von Neumann geometry 1134
Von Staudt chain 837
Von Staudt–Hua theorem 834

Walker plane 161, 162
Walker quasifield 162
wall 506
weak ordering 517
Weetman theorems 1158
weight 371, 700, 712, 859
– antidominant 712
– dominant 712
– essential 371
– fundamental 712
– minuscule 714
weight enumerator 540, 548
Weil conjecture 1109, 1110
Weil restriction 1104
Weyl group 385, 505, 568, 700, 870, 871
– absolute 568
– affine 1096, 1098
– relative 1089
Weyl homotopy 525, 596
– elementary 525
wheel 216
whirl 216
Whitney number 224
Wille geometry 195
– of grade n 218
Witt decomposition 956
Witt decomposition theorem 954
Witt design, *see* Mathieu–Witt design
Witt index 363, 666, 927, 954, 967
Witt lattice 854, 861
Witt ring 1311
Witt space 1202
Witt theorem 562, 689
Witt–Bose–Shrikhande space 1155
word 307, 484, 769
– reduced 769

Zeeman theorem 935
zero divisor 1063
– right 1132
zonotope, space-filling 261

Δ space, proper 458
Σ-brick 750, 757
Σ-construction 749
Σ-free construction 750
– compatible with respect to homomorphism 758
– tied 757
(σ, ε)-Hermitian form 559, 633
– associated to (σ, ε)-quadratic form 561
(σ, ε)-quadratic form 561