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Harmonic Mappings Between Riemannian Manifolds

Jürgen Jost

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PREFACE

These notes originated from a series of lectures I delivered at the Centre for Mathematical Analysis at Canberra. The purpose of the lectures was to introduce mathematicians familiar with the basic notions and results of linear elliptic partial differential equations and Riemannian geometry to the subject of harmonic mappings. I selected some topics to the presentation of which I felt I could contribute something, while on the other hand it was possible to provide complete and detailed proofs of them during these lectures. Thus, these notes are not meant to cover all that is known about harmonic maps, but nevertheless I believe that they give a good account of many of the interesting aspects of the subject and a fair idea of the variety of techniques used in the field.

After some introductory material in chapter 1, we present useful geometric constructions in chapter 2. In particular, we introduce almost linear functions on Riemannian manifolds and prove some properties of approximate fundamental solutions and harmonic coordinates. Most of this material originated from [JK1]. In chapter 3, we present the heat flow method to obtain existence, regularity, and uniqueness properties of harmonic maps into nonpositively curved manifolds. This covers the basic results of Eells-Sampson [ES]. Our approach also uses some ideas as presented by Hartman [Ht], von Wahl [vW], and Jost [J4].

In chapter 4, we prove the existence (due to Hildebrandt-Kaul-Widman [HKW3]) and uniqueness (due to Jäger-Kaul [JäK2]) of harmonic maps with image contained in a strictly convex ball, which solve a Dirichlet problem. The a-priori estimates based on the work of Hildebrandt-Widman [HW2] will be simplified by using the results of chapter 2. In chapter 5, we finally are concerned with harmonic maps between surfaces. We prove the existence result of Lemaire ([L1], [L2]) and Sacks-Uhlenbeck [SkU], as well as a result of Jost [J7] and Brézis-Coron [BC2] yielding the existence of two homotopically distinct solutions for nonconstant Dirichlet boundary data in S^2. We then turn to the question of the existence of harmonic diffeomorphisms, proving the results of Jost [J3] and Jost-Schoen [JS]. They are based on the deep estimates of E. Heinz [Hz5] for the Jacobian of univalent harmonic maps from below. These estimates, however, will not be proved in the present notes. We refer to [JK1] instead. Moreover, we show how a simple variational procedure can produce conformal diffeomorphisms between spheres as well as a version of the Riemann mapping theorem. For more details on harmonic mappings between surfaces, we refer to the author's notes [J8].

I thank Stefan Hildebrandt for his continuous advice and support over many years and for making the financial support of the SFB 72 at the University of Bonn available to me, and Richard Schoen for a fruitful collaboration and interesting discussions about harmonic maps between surfaces. To Hermann Karcher I owe many insights into the geometric aspects of the theory, and what I learned from him or what evolved during our collaboration is not only represented in chapter 2, but also penetrates chapter 4, and I regret that we did not find the opportunity to work out these notes together.

Moreover, I am grateful to Leon Simon for inviting me to Canberra and to the colleagues who attended my lectures for their interest and their stimulating queries and comments and to the Centre for Mathematical Analysis for its support of my work. Finally, I thank Dorothy Nash and Norma Chin for typing these notes with great care and patience.

CHAPTER 1

INTRODUCTION

1.1 A SHORT HISTORY OF VARIATIONAL PRINCIPLES

Among the first persons to realize the importance of variational problems and the physical significance of their solutions was G.W. Leibniz (1646-1716). In his work, however, mathematical and physical reasoning was closely interwoven with philosophical and theological arguments. One of the aims of his philosophy was to solve the problem of theodizee, i.e. to reconcile the evil in the world with God's goodness and almightiness (cf. (Lz]). Leibniz' answer was that God has chosen from the innumerable possible worlds the best possible, but that a perfect world is not possible. (This infinite multitude can only be conceived by an infinite understanding, which provided a proof of the existence of God for Leibniz.) This best possible world is distinguished by a pre-established harmony between itself, the realm of nature, on one hand and the heavenly realm of grace and freedom on the other hand. Through this the effective causes unite with the purposive causes. Thus bodies move due to their own internal laws in accordance with the thoughts and desires of the soul. In this way, the contradiction between the predetermination of the physical world following strict laws and the constantly experienced spontaneity and freedom of the individual is removed. The best possible world must here obey specific laws since an ordered world is better than a chaotic one. This proves therefore the necessity of the existence of natural laws. The contents of the natural laws, however, are not completely determined as is the case for geometric laws but are only determined in a moral sense, since they must satisfy the criteria of beauty and simplicity in the best of all possible worlds. This leads Leibniz even to variational principles. This is because if a physical process did not yield an extreme value, a maximum or minimum, for a particular energy or action integral, the world could be improved and would therefore not be the best possible one. Conversely, Leibniz also uses the beauty and simplicity of natural laws as evidence for his thesis of pre-established harmony. (The notion that we live in the best possible world was frequently rejected and even ridiculed by subsequent critics, in particular Voltaire, on account of the apparent flaws of this world, but Leibniz' point that a perfectly good world is not possible was beyond reach of these arguments.)

Leibniz, however, did not elaborate his argument concerning variational principles in his publications, but only in a private letter. Thus, it happened that a principle of least (and not only stationary) action was later rediscovered by Maupertuis (1698-1759), without knowing of Leibniz' idea. When S. König (1712-1757) then claimed priority for Leibniz on account of his letter that he was not able to show however to the Prussian Academy of Sciences (whose president was Maupertuis) this led to one of the most famous priority controversies in scientific history in which even Voltaire, Euler, and Frederick the Great became involved. It was also pointed out that Maupertuis' principle of least action should be replaced by a principle of stationary action since physical equilibria need only be stationary points but not necessarily minima of variational problems.

1.2 THE CONCEPT OF GEODESICS

One of the variational problems of most physical importance and mathematical interest was the problem of geodesics, i.e. to find the shortest (or at least locally shortest) connections between two points in a metric continuum, e.g. a Riemannian manifold. Geodesics are critical points of the length integral

$$\int_0^1 \left| \frac{\partial}{\partial t} c \right| dt$$

where $c : [0,1] \to N$ is the parametrization, as well as, if they are parametrized proportionally to arclength, of the energy integral

$$\int_0^1 \left| \frac{\partial}{\partial t} c \right|^2 dt \ .$$

Here, unfortunately, we find some ambiguity of terminology, since the mathematical term "energy" corresponds to the physical concept of "action", while in physics "energy" has a different meaning.

Because of the many applications of geodesics, it was rather natural to generalize this concept. While minimal surfaces are critical points of a twodimensional analogue of the length integral, namely the area integral, the generalization of the energy integral for maps between Riemannian manifolds led to the concept of harmonic maps. They are critical points of the corresponding integral where the squared norm of the gradient or energy density has to be defined in terms intrinsic to the geometry of the domain and target manifold and the map between them.

1.3 DEFINITION AND SOME ELEMENTARY PROPERTIES OF HARMONIC MAPS

Suppose that X and Y are Riemannian manifolds of dimensions n and N, resp., with metric tensors $(\gamma_{\alpha\beta})$ and (g_{ij}), resp., in some local coordinate charts $x = (x^1,\ldots,x^n)$ and $f = (f^1,\ldots,f^N)$ on X and Y, resp. Let $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$. If $f : X \to Y$ is a C^1-map, we can define the energy density

$$e(f) := \frac{1}{2}\,\gamma^{\alpha\beta}(x)\; g_{ij}(f)\, \frac{\partial f^i}{\partial x^\alpha}\,\frac{\partial f^j}{\partial x^\beta}$$

where we use the standard summation convention (greek minuscules occurring twice are summed from 1 to n , while latin ones are summed from 1 to N) and express everything in terms of local coordinates. Then the energy of f is simply

$$E(f) = \int_X e(f)\,dX \ .$$

If f is of class C^2 and $E(f) < \infty$, and f is a critical point of E, then it is called *harmonic* and satisfies the corresponding Euler-Lagrange-equations. These are of the form

$$(1.3.1)\qquad \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} f^i\right) + \gamma^{\alpha\beta}\, \Gamma^i_{jk} \frac{\partial}{\partial x^\alpha} f^j \frac{\partial}{\partial x^\beta} f^k = 0$$

in local coordinates, where $\gamma = \det(\gamma_{\alpha\beta})$ and the Γ^i_{jk} are the Christoffel symbols of the second kind on Y .

(1.3.1) is proved as follows. If f is critical, then for all admissible variations ϕ (e.g. $\phi \in C^\infty_c(X)$, and $\phi|\partial X = 0$ if $\partial X \neq \emptyset$)

$$\frac{d}{dt} E(f+t\phi)_{|t=0} = 0 \ .$$

and thus

$$0 = \int_X \left(\gamma^{\alpha\beta}(x) g_{ij}(f(x)) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial \phi^j}{\partial x^\beta} + \frac{1}{2} \gamma^{\alpha\beta} g_{ij,k} \phi^k \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}\right) \sqrt{\gamma}\, dx$$

$$= - \int_X \frac{\partial}{\partial x^\beta} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial f^i}{\partial x^\alpha}\right) g_{ij}\ \phi^j\, dx - \int_X \gamma^{\alpha\beta}(x) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} g_{ij,k} \phi^j \sqrt{\gamma}\, dx$$

$$+ \int_X \frac{1}{2} \gamma^{\alpha\beta}\, g_{ij,k}\, \phi^k \frac{\partial f^i}{\partial x^\beta} \frac{\partial f^j}{\partial x^\alpha} \sqrt{\gamma}\, dx$$

since ϕ is compactly supported

and from this, putting $\eta^i = g_{ij}\phi^j$, i.e. $\phi^j = g^{j\ell}\eta^\ell$, and using the symmetry of $\gamma^{\alpha\beta}$ in the second integral,

$$0 = -\int_X \frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\frac{\partial f^i}{\partial x^\alpha}\right)\eta^i\,dx - \int_X \frac{1}{2}\gamma^{\alpha\beta} g^{\ell j}(g_{ij,k}+g_{kj,i}-g_{ik,j})\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\eta^\ell\sqrt{\gamma}\,dx$$

which implies (1.3.1) by the lemma of Du Bois-Raymond.

We thus obtain a nonlinear elliptic system of partial differential equations, where the principal part is the Laplace-Beltrami operator on X and is therefore in divergence form, while the nonlinearity is quadratic in the gradient of the solution.

We now want to look at the definition of harmonic maps from a more intrinsic point of view. The differential df of f, given in local coordinates by

$$df = \frac{\partial f^i}{\partial x^\alpha}\,dx^\alpha\,\frac{\partial}{\partial f^i}$$

can be considered as a section of the bundle $T^*X \otimes f^{-1}TY$. Then

$$\begin{aligned} e(f) &= \frac{1}{2}\gamma^{\alpha\beta}\left\langle \frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta}\right\rangle_{f^{-1}TY} \\ &= \frac{1}{2}\left\langle df, df\right\rangle_{T^*X\otimes f^{-1}TY} \end{aligned}$$

i.e. $e(f)$ is the trace of the pullback via f of the metric tensor of Y. In particular, $e(f)$ and hence also $E(f)$ are independent of the choice of local coordinates and thus intrinsically defined. f is harmonic, if

(1.3.2) $$\tau(f) = 0\ ,$$

where $\tau(f) = \text{trace}\,\nabla df$, and ∇ here denotes the covariant derivative in the bundle $T^*X \otimes f^{-1}TY$.

Let us quickly show, why (1.3.1) and (1.3.2) are equivalent (cf. [EL 4]).

$$\begin{aligned}\nabla_{\partial/\partial x^\beta}(df) &= \nabla_{\partial/\partial x^\beta}\left(\frac{\partial f^i}{\partial x^\alpha}\, dx^\alpha\, \frac{\partial}{\partial f^i}\right)\\ &= \frac{\partial}{\partial x^\beta}\left(\frac{\partial f^i}{\partial x^\alpha}\right) dx^\alpha\, \frac{\partial}{\partial f^i} + \left(\nabla^{T^*X}_{\partial/\partial x^\beta}\, dx^\alpha\right) \frac{\partial f^i}{\partial x^\alpha}\, \frac{\partial}{\partial f^i}\\ &\quad + \left(\nabla^{f^{-1}TY}_{\partial/\partial x^\beta}\, \frac{\partial}{\partial f^i}\right) \frac{\partial f^i}{\partial x^\alpha}\, dx^\alpha\end{aligned}$$

$$= \frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}\, dx^\alpha\, \frac{\partial}{\partial f^i} - {}^X\Gamma^\alpha_{\beta\gamma}\, dx^\gamma\, \frac{\partial f^i}{\partial x^\alpha}\, \frac{\partial}{\partial f^i} + {}^Y\Gamma^k_{ij}\, \frac{\partial}{\partial f^k}\, \frac{\partial f^j}{\partial x^\beta}\, \frac{\partial f^i}{\partial x^\alpha}\, dx^\alpha \quad ^{1)}$$

and thus, since $\tau(f) = \text{trace } \nabla df$,

$$\tau^k(f) = \gamma^{\alpha\beta}\, \frac{\partial^2 f^k}{\partial x^\alpha \partial x^\beta} - \gamma^{\alpha\beta}\, {}^X\Gamma^\gamma_{\alpha\beta}\, \frac{\partial f^k}{\partial x^\gamma} + \gamma^{\alpha\beta}\, {}^Y\Gamma^k_{ij}\, \frac{\partial f^i}{\partial x^\alpha}\, \frac{\partial f^j}{\partial x^\beta} \quad ,$$

and we see that (1.3.1) and (1.3.2) are equivalent.

From the preceding calculation, we see that the Laplace-Beltrami operator is the contribution of the connection in T^*X , while the connection in $f^{-1}TY$ gives rise to the nonlinear term involving the Christoffel symbols of the image.

With the preceding notations, we can also calculate the Hessian of a harmonic map f for vector fields v , w along f (i.e. v and w are sections of $f^{-1}TY$) . For this purpose, we consider a two-parameter variation f_{st} with

$$v = \left.\frac{\partial f_{st}}{\partial s}\right|_{s,t=0} \quad , \quad w = \left.\frac{\partial f_{st}}{\partial t}\right|_{s,t=0} \; .$$

---

1) Here, we distinguish the Christoffel symbols of X and Y by the superscript X or Y , resp.

We then want to calculate

$$H_f(v,w) := \frac{\partial^2 E(f_{st})}{\partial s \partial t}\Big|_{s,t=0}$$

We have, writing f instead of f_{st}, and taking scalar products $\langle\cdot,\cdot\rangle$ in $T^*X \otimes f^{-1}TY$, if not otherwise indicated,

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\frac{1}{2}\left\langle \frac{\partial f}{\partial x^\alpha} dx^\alpha, \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$= \frac{\partial}{\partial t}\left\langle \nabla_{\partial/\partial s} \frac{\partial f}{\partial x^\alpha} dx^\alpha, \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$= \frac{\partial}{\partial t}\left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \left(\frac{\partial f}{\partial s}\right) dx^\alpha, \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$= \left\langle \nabla_{\partial/\partial t} \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \left(\frac{\partial f}{\partial s}\right) dx^\alpha, \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \left(\frac{\partial f}{\partial s}\right) dx^\alpha, \nabla^{f^{-1}TY}_{\partial/\partial x^\beta} \left(\frac{\partial f}{\partial t}\right) dx^\beta \right\rangle$$

$$= \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \nabla_{\partial/\partial t} \left(\frac{\partial f}{\partial s}\right) dx^\alpha, \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle R^N \left(\frac{\partial f}{\partial x^\alpha} dx^\alpha, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial s}, \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} v\, dx^\alpha, \nabla^{f^{-1}TY}_{\partial/\partial x^\beta} w\, dx^\beta \right\rangle .$$

Now

$$\int_X \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} \nabla_{\partial/\partial t} \frac{\partial f}{\partial s} dx^\alpha, \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle dX$$

$$= \int \frac{\partial}{\partial x^\alpha}\left(\gamma^{\alpha\beta} \left\langle \nabla_{\partial/\partial t} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial x^\beta} \right\rangle_{f^{-1}TY} \sqrt{\gamma}\right) dx^1 \ldots dx^n$$

$$- \int \left\langle \nabla_{\partial/\partial t} \frac{\partial f}{\partial s} dx^\alpha , \nabla_{\partial/\partial x^\alpha} \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$= - \int \left\langle \nabla_{\partial/\partial t} \frac{\partial f}{\partial s} , \gamma^{\alpha\beta} \nabla_{\partial/\partial x^\alpha} \frac{\partial f}{\partial x^\beta} \right\rangle_{f^{-1} TY}$$

by Stokes' Theorem

$$= 0 , \quad \text{since} \quad \gamma^{\alpha\beta} \nabla_{\partial/\partial x^\alpha} \frac{\partial f}{\partial x^\beta} = \text{trace} \ \nabla \, df = 0 , \quad \text{as } f \text{ is harmonic.}$$

Thus

$$H_f(v,w) = \int_X \gamma^{\alpha\beta} \left\langle \nabla^{f^{-1}TY}_{\partial/\partial x^\alpha} v , \nabla^{f^{-1}TY}_{\partial/\partial x^\beta} w \right\rangle_{f^{-1} TY}$$

$$- \int_X \gamma^{\alpha\beta} \left\langle R^N \left(\frac{\partial f}{\partial x^\alpha} , v \right) \frac{\partial f}{\partial x^\beta} , w \right\rangle_{f^{-1} TY}$$

$$= \int_X \left\langle \nabla^{f^{-1}TY} v , \nabla^{f^{-1}TY} w \right\rangle_{f^{-1} TY}$$

$$- \int_X \text{trace}_X \left\langle R^N(df,v) \, df , w \right\rangle_{f^{-1} TY} .$$

For the preceding calculations cf. also [EL4].

We now want to look at the definition of harmonic maps from a somewhat different point of view. By the famous embedding theorem of Nash ([Na]), Y can be isometrically embedded in some Euclidean space $\mathbb{R}^\ell$. We define the Sobolev space

$$W^1_2(X,Y) = \{ f \in W^1_2(X,\mathbb{R}^\ell) : f(x) \in Y \quad \text{a.e.} \}$$

Since $W^1_2(X,\mathbb{R}^\ell) = H^1_2(X,\mathbb{R}^\ell)$ by a well-known theorem of Meyers and Serrin (cf. [MS], p.52; we can assume X to be a compact manifold (possibly with boundary), since we always can localize the problem in the domain. Namely, if

f is a critical point of E on X, then it is also critical on any subdomain) every element in $W_2^1(X,Y)$ can be approximated with respect to the W_2^1 norm by smooth mappings, namely from $C^\infty(X,\mathbb{R}^\ell)$, although the corresponding equality $W_2^1(X,Y) = H_2^1(X,Y)$ does not hold in general, cf. [SU2]. In particular, if we compose an element from $W_2^1(X,Y)$ with a smooth mapping, we can apply a chain rule.

In this Sobolev space, we can still define the energy functional by

$$E(f) = \frac{1}{2}\int |df(x)|^2 \, dX(x)$$

and look for critical points of E in $W_2^1(X,Y)$.

Assume that $f \in W_2^1(X,Y)$ is a critical point of E which maps X into a compact part Y_0 of Y. Y_0 has a uniform neighbourhood in $\mathbb{R}^\ell$ on which the projection π, mapping a point in $\mathbb{R}^\ell$ to the closest point in Y, is smooth.

Thus, if $\phi : X \to \mathbb{R}^\ell$ is smooth and $\phi|\partial X = 0$ and t is sufficiently small, $(f+t\phi)(x)$ lies in this neighbourhood for a. a. $x \in X$. Since f is critical

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} E(\pi(f+t\phi))\big|t = 0 \\ &= \int_X \left\langle D^2\pi(f) \cdot \phi D_\alpha f , \; d\pi(f) D_\alpha f \right\rangle dX \\ &+ \int_X \left\langle d\pi(f) D_\alpha \phi , \; d\pi(f) D_\alpha f \right\rangle dX \end{aligned}$$

applying the chain rule,

where $D_\alpha f = e_\alpha(f)$ and e_α is a moving orthonormal frame on X, $\alpha = 1,\dots n$

$$\begin{aligned} &= \int_X \left\langle D^2\pi(f) \cdot \phi D_\alpha f , \; d\pi(f) D_\alpha f \right\rangle dX \\ &+ \int_X \left\langle D_\alpha \phi , \; d\pi(f) D_\alpha f \right\rangle dX \end{aligned}$$

since π is a projection

$$= \int_X \left\langle D^2\pi(f)\cdot\phi\cdot D_\alpha f \;,\; D_\alpha f \right\rangle \, dX$$

$$+ \int_X \left\langle D_\alpha\phi \;,\; D_\alpha f \right\rangle \, dX$$

since $\pi \circ f = f$ and consequently $d\pi \cdot D_\alpha f = D_\alpha f$ by the chain rule. Thus, f is a weak solution of

$$0 = \Delta f - D^2\pi(f)(df,df) \;, \tag{1.3.3}$$

where Δ is the Laplace-Beltrami operator on X (cf. [SU1] for somewhat different calculations). (1.3.1) and (1.3.3) are equivalent, since they both are the Euler-Lagrange equations of the energy functional E. The point of view leading to (1.3.3) was different, however. Here, the energy was minimized among all maps $u : X \to \mathbb{R}^\ell$ of class $H_2^1 \cap L^\infty(X,\mathbb{R}^\ell)$ satisfying a nonlinear constraint $u(x) \in Y_0$ (for almost all $x \in X$). Since the Dirichlet integral is lower semicontinuous w.r.t. weak H_2^1-convergence we also get

LEMMA 1.3.1 *The energy integral is lower semicontinuous w.r.t. weak* H_2^1*-convergence.*

Finally, let Σ_1 and Σ_2 be surfaces with conformal metrics

$$\sigma^2 dz\, d\bar{z} \qquad\qquad (z=x+iy)$$

and

$$\rho^2 du\, d\bar{u} \qquad\qquad (u=u^1+iu^2) \quad \text{resp.}$$

For a C^1-map $f : \Sigma_1 \to Y$, the energy is then given by

$$E(f) = \frac{1}{2}\int_{\Sigma_1} g_{ij}(u_x^i u_x^j + u_y^i u_y^j)\, dx\, dy$$

in those coordinates. Hence

LEMMA 1.3.2 *If* $k : \Sigma_0 \to \Sigma_1$ *is a conformal map between surfaces, then*

$$E(f \circ k) = E(f) .$$

This means that the energy is conformally invariant.

Moreover, the Laplace-Beltrami operator of Σ_1 in our coordinates is given by $\frac{1}{4\sigma^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$, and (1.3.1) hence takes the form

$$\frac{1}{\sigma^2} u^i_{z\bar{z}} + \frac{1}{\sigma^2} \Gamma^i_{jk} u^j_z u^k_{\bar{z}} = 0$$

(where $u_z := \frac{1}{2} (\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y})$, $u_{\bar{z}} := \frac{1}{2} (\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y})$) .

In the case the image is the surface Σ_2 , this in turn reads as

$$\frac{1}{\sigma^2} u_{z\bar{z}} + \frac{1}{\sigma^2} \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0 . \tag{1.3.4}$$

Thus, the harmonicity of u does not depend on the special metric of Σ_1 , but only on its conformal structure, since we can simply multiply the equation by σ^2 . Hence

LEMMA 1.3.3 *Suppose* $u : \Sigma_1 \to Y$ *is harmonic, and* $k : \Sigma_0 \to \Sigma_1$ *is a conformal map between surfaces. Then* $u \circ k$ *is also harmonic. In particular, in two dimensions conformal mappings are harmonic.*

The harmonicity of u does depend, however, on the image metric, unless $u_z \equiv 0$ or $u_{\bar{z}} \equiv 0$, i.e. u is conformal or anticonformal. (Note that this distinction is only meaningful for oriented surfaces.)

We also note the following

LEMMA 1.3.4 *If* $u : \Sigma_1 \to \Sigma_2$ *is a harmonic map between surfaces, then*

$$\phi = \left\{ |u_x|^2 - |u_y|^2 - 2i \langle u_x, u_y \rangle \right\} dz^2 \qquad (z = x+iy)$$

$$= 4\rho^2 u_z \bar{u}_z \, dz^2$$

is a holomorphic quadratic differential.

Proof Multiplying (1.3.4) by the conformal factor σ^2, we obtain

$$\tilde{\tau}(u) := u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0 .$$

Thus,

$$\phi_{\bar{z}} = 2\rho\rho_u u_{\bar{z}} u_z \bar{u}_z + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} u_z \bar{u}_z + \rho^2 u_{z\bar{z}} \bar{u}_z + \rho^2 u_z \bar{u}_{z\bar{z}}$$

$$= \rho^2 (\bar{u}_z \tilde{\tau}(u) + u_z \bar{\tilde{\tau}}(u)) = 0$$

q.e.d.

We also observe, that if ϕ is holomorphic then $\tau(u) = 0$ with the possible exception of points where $|\bar{u}_z| = |u_z|$, i.e. where the Jacobian $|u_z|^2 - |u_{\bar{z}}|^2$ vanishes. This was actually used by Gerstenhaber and Rauch [GR] as a definition of harmonic maps between surfaces.

We note moreover, that ϕ is just the $(2,0)$ part of the differential form $u^*(4\rho^2(u)du d\bar{u})$, i.e. the pull-back of the image metric under u.

Finally, of course $\phi \equiv 0$ if and only if u is conformal or anti-conformal. Therefore, Lemma 1.3.4, together with the observation that by Liouville's Theorem $\phi \equiv 0$ is the only holomorphic quadratic differential on S^2, shows that any harmonic map from S^2 is conformal or anticonformal.

1.4 MATHEMATICAL PROBLEMS ARISING FROM THE CONCEPT OF HARMONIC MAPS

From 1.3, one sees that new mathematical difficulties arise compared to the case of geodesics. Here, critical points lead to systems of non-linear partial differential equations, while geodesics lead only to systems of ordinary differential equations. The natural space to look for critical points of E is the Sobolev space $W^1_2(X,Y) \cap L^\infty(X,Y)$, since the equations for weak solutions of (1.3.1), namely

$$(1.4.1) \qquad 0 = \int \gamma^{\alpha\beta} \left(\frac{\partial f^i}{\partial x^\alpha} \frac{\partial \phi^i}{\partial x^\beta} - \Gamma^i_{jk} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \phi^i \right) dx$$

make sense only for test functions $\phi \in \mathring{W}^1_2(X,\mathbb{R}^N) \cap L^\infty(X,\mathbb{R}^N)$.

From an analytical point of view, it is not surprising that the equations (1.3.1) turned out to be rather difficult to handle, since the nonlinearity is quadratic in the gradient of the solution. Such systems may have nonsmooth weak solutions. This phenomenon can even occur in the present situation. Namely, mapping the unit ball D^n of dimensions $n \geqq 3$ onto its boundary via radial projection, can be interpreted as a weakly harmonic map (i.e. a solution of (1.4.1)) $f : D^n \to S^{n-1}$, cf. [HKW3].

In order to verify this, we first show that $\frac{x}{|x|}$ has finite energy for $n \geqq 3$.

For $x \in D^n$, $f(x) = \frac{x}{|x|}$,and hence for $x \neq 0$

$$(1.4.2) \qquad \frac{\partial}{\partial x^\alpha} \frac{x}{|x|} = \frac{e_\alpha}{|x|} - \frac{x \cdot x^\alpha}{|x|^3}$$

(here, e_α is a unit vector, and $x = x^\alpha e_\alpha$)

and

$$(1.4.3) \qquad \left| d \frac{x}{|x|} \right|^2 = \frac{(n-1)}{|x|^2}$$

(1.4.3) clearly implies that $\frac{x}{|x|}$ has finite energy for $n \geq 3$ (and also, that the energy is infinite for $n = 2$).

$\frac{x}{|x|}$ is smooth for $x \neq 0$, and we shall verify now, that $\frac{x}{|x|}$ satisfies equation (1.3.3) for $x \neq 0$.

We note $\pi(f) = \frac{f}{|f|}$, and from (1.4.2) thus

$$\frac{\partial}{\partial f^\alpha} \pi(f) = \frac{e_\alpha}{|f|} - \frac{f}{|f|^3} f^\alpha$$

and moreover

$$(1.4.4) \qquad \frac{\partial^2}{\partial f^\alpha \partial f^\beta} \frac{f}{|f|} = -\frac{e_\alpha f^\beta}{|f|^3} - \frac{e_\beta f^\alpha}{|f|^3} - \frac{f\delta_{\alpha\beta}}{|f|^3} + \frac{3ff^\alpha f^\beta}{|f|^5} .$$

Since $|f|^2 = 1$ implies $f^\alpha \dfrac{\partial f^\alpha}{\partial x^\gamma} = 0 \quad (\gamma = 1,\ldots n)$, (1.4.4) yields

$$(1.4.5) \qquad D^2\pi(f)(df,df) = \frac{\partial^2}{\partial f^\alpha \partial f^\beta}\left(\frac{f}{|f|}\right) \frac{\partial f^\alpha}{\partial x^\gamma}\frac{\partial f^\beta}{\partial x^\gamma} = -f\,|df|^2 .$$

Hence the equation for a harmonic map from D^n into S^{n-1} is by (1.3.3) and (1.4.5)

$$(1.4.6) \qquad \Delta f + f|df|^2 = 0 .$$

$f = \dfrac{x}{|x|}$ now satisfies this equation, since by (1.4.4)

$$\Delta \frac{x}{|x|} = \frac{-(n-1)x}{|x|^3}$$

and by (1.4.3)

$$\left|d\frac{x}{|x|}\right|^2 \frac{x}{|x|} = \frac{(n-1)x}{|x|^3} .$$

The following lemma then implies that $\dfrac{x}{|x|} : D^n \to S^{n-1}$ indeed is a weak solution of (1.4.1).

LEMMA 1.4.1 *If* $f : X \to Y$ *is a map of finite energy which is smooth and harmonic outside a subset of* X *of capacity zero, then* f *is weakly harmonic on* X .

For simplicity, we shall show this only for $\dim X \geq 3$ and the case where f is not smooth only at one isolated point. This suffices for our application.

We have to show that

$$\int (\gamma^{\alpha\beta} D_\alpha f^i D_\beta \phi^i - \gamma^{\alpha\beta} \Gamma^i_{jk} D_\alpha f^j D_\beta f^k \phi^i) \sqrt{\gamma}\, dx = 0$$

for all $\phi \in H^1_2 \cap L^\infty(X,Y)$. Let us choose the local coordinates in such a way that 0 is the singular point of f . We define

$$\eta_m := \begin{cases} \frac{1}{2^{m-1}} \left(\frac{1}{|x|} - 2^{m-1}\right) & \text{if} \quad 2^{-m} \le |x| \le 2^{-m+1} \\ 0 & \text{if} \quad 2^{-m+1} \le |x| \\ 1 & \text{if} \quad |x| \le 2^{-m} \end{cases}$$

Clearly, η_m is Lipschitz continuous.

We write

$$\phi = (1-\eta_m)\phi + \eta_m \phi .$$

Since f is harmonic for $x \neq 0$, $f \in H^1_2$ and $\phi \in H^1_2 \cap L^\infty$, it suffices to show

$$\text{(1.4.7)} \qquad \int \gamma^{\alpha\beta} D_\alpha f^i (D_\beta \eta) \phi^i \sqrt{\gamma}\, dx \to 0 \qquad \text{as} \quad m \to \infty \quad .$$

However,

$$D_\beta \eta_m = \begin{cases} \frac{x^\beta}{|x|^3} \; 2^{1-m} & \text{for} \quad 2^{-m} \le |x| \le 2^{-m+1} \\ 0 & \text{otherwise} . \end{cases}$$

Hence

$$|D_\beta \eta_m| \le \frac{2}{|x|} ,$$

and (1.4.7) follows from Hölder's inequality, since we assumed $n \ge 3$.

q.e.d.

It might be worth pointing out that the regularity problem for weakly harmonic maps actually has two inherent nonlinearities, one being the nonlinearity of the equations, i.e. arising from the local geometry of the image, and the other one coming from the fact that in general the target space itself does not have a linear structure, i.e. arising from the global topology of the image.

In these notes, we shall first be concerned with the local regularity problem for solutions of the equations, i.e. the first nonlinearity, in chapters 3 and 4, and then deal with the global topological difficulties only in two dimensions, where the regularity theory is easier.

1.5 SOME EXAMPLES OF HARMONIC MAPS

The variational problem for harmonic maps seems to be the most natural such problem one can pose for mappings between manifolds, and hence it is not surprising that many other canonical or natural maps turn out to be harmonic. In the sequel, we shall list some examples:

- isometries of Riemannian manifolds
- harmonic functions on Riemannian manifolds
- geodesics as maps $S^1 \to M$
- minimal immersions and parametric minimal surfaces
- Hopf maps $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$
- conformal maps on two-dimensional domains (cf. Lemma 1.3.3) (in higher dimensions, they are in general not harmonic, however)
- holomorphic maps between Kähler manifolds (Holomorphic maps between arbitrary complex manifolds are in general not harmonic. This is not surprising, since the Kähler condition just means that the metric and the complex structure of the manifold agree. The definition of harmonic maps was given in terms of the metric structure, and when deriving the Euler-Lagrange equation for stationary points of the energy integral, we tacitly used the fact that the manifold is endowed with the Levi-Civita connection. Otherwise, as is already the case for geodesics, those two concepts - minimizing the energy or length

integral on one hand and being autoparallel on the other hand for geodesics - would not agree. On the other hand, holomorphic maps are defined in terms of the complex structure, and as mentioned above, the Kähler condition means that the complex connection, i.e. the unique torsionfree connection for which the complex structure is parallel, and the Levi-Civita connection, i.e. the unique torsionfree connection for which the metric is parallel, do agree.)

- Gauss maps of minimal submanifolds of Euclidean space, or more generally, of submanifolds with parallel mean curvature vector. This is a theorem of Ruh and Vilms [RV]. With the help of this theorem, one can prove Bernstein type theorems for minimal submanifolds of Euclidean space by proving Liouville type theorems for harmonic maps, since, if the Gauss map is constant, the submanifold has to be a linear subspace. We shall come back to this point in chapter 4.

1.6 SOME APPLICATIONS OF HARMONIC MAPS

We want to calculate for a harmonic map f

$$\Delta e(f)$$

i.e.

$$\Delta \frac{1}{2} \gamma^{\alpha\beta}(x) g_{ij}(f(x)) f^i_{x^\alpha} f^j_{x^\beta} .$$

In order to do this, it will be convenient to introduce normal coordinates at the points x and $f(x)$, i.e. $\gamma_{\alpha\beta}(x) = \delta_{\alpha\beta}$ and $g_{ij}(f(x)) = \delta_{ij}$ and all Christoffel symbols vanish at x and $f(x)$, so that we only have to take derivatives of the Christoffel symbols into account which will yield curvature terms eventually.

First of all, we write the equation for harmonic maps in the form

$$(1.6.1) \qquad 0 = \gamma^{\alpha\beta} f^i_{x^\alpha x^\beta} - \gamma^{\alpha\beta}\, {}^X\Gamma^\eta_{\alpha\beta}\, f^i_{x^\eta} + \gamma^{\alpha\beta}\, {}^Y\Gamma^i_{k\ell}\, f^k_{x^\alpha}\, f^\ell_{x^\beta} .$$

Differentiating this equation at x w.r.t. x^ε, we obtain

(1.6.2)
$$f^i_{x^\alpha x^\alpha x^\varepsilon} = \frac{1}{2}(\gamma_{\alpha\eta,\alpha\varepsilon} + \gamma_{\alpha\eta,\alpha\varepsilon} - \gamma_{\alpha\alpha,\eta\varepsilon}) f^i_{x^\eta}$$
$$- \frac{1}{2}(g_{ki,\ell m} + g_{\ell i,km} - g_{k\ell,im}) f^m_{x^\varepsilon} f^k_{x^\alpha} f^\ell_{x^\alpha} ,$$

using of course that by our choice of coordinates all first derivatives of the metric tensors vanish, and the Christoffel symbols are given by, e.g. $\Gamma^i_{k\ell} = \frac{1}{2} g^{im}(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m})$.

Furthermore, in our coordinates

(1.6.3)
$$\gamma^{\alpha\beta}{}_{,\sigma\sigma} = - \gamma_{\alpha\beta,\sigma\sigma}$$

and by the chain rule

(1.6.4)
$$\Delta g_{ij}(f(x)) = g_{ij,k\ell} f^k_{x^\sigma} f^\ell_{x^\sigma} .$$

From (1.6.2) - (1.6.4) we obtain

(1.6.5)
$$\Delta \frac{1}{2} \gamma^{\alpha\beta}(x) g_{ij}(f(x)) f^i_{x^\alpha} f^j_{x^\beta}$$
$$= f^i_{x^\alpha x^\sigma} f^i_{x^\alpha x^\sigma}$$
$$- (\gamma_{\alpha\beta,\sigma\sigma} + \gamma_{\sigma\sigma,\alpha\beta} - \gamma_{\sigma\alpha,\sigma\beta} - \gamma_{\sigma\alpha,\sigma\beta}) f^i_{x^\alpha} f^i_{x^\beta}$$
$$+ (g_{ij,k\ell} + g_{k\ell,ij} - g_{ik,j\ell} - g_{j\ell,ik}) f^i_{x^\alpha} f^j_{x^\alpha} f^k_{x^\sigma} f^\ell_{x^\sigma}$$
$$= f^i_{x^\alpha x^\sigma} f^i_{x^\alpha x^\sigma} + R^X_{\alpha\beta} f^i_{x^\alpha} f^i_{x^\beta} - R^Y_{ikj\ell} f^i_{x^\alpha} f^j_{x^\alpha} f^k_{x^\sigma} f^\ell_{x^\sigma} ,$$

where $R^X_{\alpha\beta}$ is the Ricci tensor of X and $R^Y_{ikj\ell}$ is the curvature tensor of Y .

In arbitrary coordinates, this formula is of course transformed into

$$\Delta e(f) = g_{ij}(f(x))\gamma^{\alpha\beta}(x)\gamma^{\sigma\eta}(x)\left(f^i_{x^\alpha x^\sigma} + \Gamma^i_{k\ell}\frac{\partial f^k}{\partial x^\alpha}\frac{\partial f^\ell}{\partial x^\sigma}\right)\left(f^j_{x^\beta x^\eta} + \Gamma^j_{mn}\frac{\partial f^m}{\partial x^\beta}\frac{\partial f^n}{\partial x^\eta}\right)$$

$$+ g_{ij}(f(x))R^X_{\alpha\beta}(x)f^i_{x^\alpha}f^j_{x^\beta} - \gamma^{\alpha\beta}(x)\gamma^{\sigma\eta}(x)R^Y_{ikj\ell}(f(x))f^i_{x^\alpha}f^j_{x^\beta}f^k_{x^\sigma}f^\ell_{x^\eta}$$

and in invariant notation, if e_α is an orthonormal frame at x ,

$$\Delta e(f) = |\nabla df|^2 + \langle df\circ \mathrm{Ric}^X(e_\alpha), df\circ e_\alpha\rangle - \langle R^Y(df\circ e_\alpha, df\circ e_\beta)\, df\circ e_\alpha, df\circ e_\beta\rangle$$

(1.6.5) immediately yields the following

COROLLARY 1.6.1 ([ES]) *Suppose* $f : X \to Y$ *is a harmonic map,* X *is compact,* $\mathrm{Ric}^X \geq 0$ *, and the sectional curvature of* Y *is nonpositive.*

Then f *is totally geodesic and has constant energy density. If the Ricci curvature of* X *is positive at one point of* X *at least, then* f *is constant.*

If the sectional curvature of Y *is negative, then* f *is either constant or maps* X *onto a closed geodesic of* Y .

Proof Since $\int_X \Delta e(f)\,dX = 0$, the integral over the right hand side of (1.6.5) has to vanish. Since the integrand is pointwise non-negative by assumption, it has to vanish identically. In particular, $|\nabla df| \equiv 0$, and thus f is totally geodesic. Furthermore $\Delta e(f) \equiv 0$, and since harmonic functions on compact manifolds are constant, $e(f) \equiv$ const.

If at $x \in X$, $R^X_{\alpha\beta}(x)$ is positive definite, then

$$R^X_{\alpha\beta}(x)f^i_{x^\alpha}\, f^i_{x^\beta} = 0$$

implies that at x and hence everywhere $e(f) = 0$, and f is constant.

If Y has negative sectional curvature, then in the same way we see that

$$\dim(df(T_x X)) \leq 1 \qquad \text{for any} \quad x \in X \ .$$

If the dimension is zero somewhere, then $e(f) = 0$ at this point and hence everywhere. Otherwise, f as a totally geodesic map has to map X onto a closed geodesic.

We now want to apply Cor. 1.6.1 in conjunction with the following basic existence and uniqueness theorem of Eells-Sampson(existence) and Hartman (uniqueness) which will be proved in chapter 3 in order to reprove some well known theorems about nonpositively curved manifolds by using harmonic maps.

THEOREM 1.6.1 *If* X *and* Y *are compact Riemannian manifolds and* Y *has nonpositive sectional curvature, then every homotopy class of maps from* X *to* Y *contains a harmonic map. If the curvature of* Y *is negative, then this harmonic map is unique unless its image is a single point or contained in a closed geodesic in which case every other homotopic harmonic map can differ from the given one only by a rotation of this closed geodesic.*

We first deduce Preissmann's Theorem:

THEOREM 1.6.2 *If* Y *is a compact Riemannian manifold of negative sectional curvature, then every Abelian subgroup of the fundamental group is cyclic.*

Proof Suppose a and b are commuting elements of $\pi_1(Y)$. The homotopy between ab and ba allows us to construct a map g from the twodimensional torus T^2 into Y. By Thm. 1.6.1 g is homotopic to a harmonic map $f : T^2 \to Y$, and the image of f is contained in a closed geodesic by Cor. 1.6.1. Hence both a and b are homotopic to some multiple of this geodesic.

q.e.d.

Furthermore, we can prove the following consequence of the Hadamard-Cartan theorem.

THEOREM 1.6.3 *If* Y *is a nonpositively curved compact Riemannian manifold, then all homotopy groups* $\pi_m(Y)$ *vanish for* $m \geq 2$ *, i.e.* Y *is a* $K(\pi,1)$ *manifold.*

Proof We have to show that every map g from a sphere S^m , $m \geq 2$, into Y is homotopic to a constant. By Thm. 1.6.1, g is homotopic to a harmonic map $f : S^m \to Y$, and f is constant by Cor. 1.6.1.

q.e.d.

Finally, we deduce

THEOREM 1.6.4 *If* Y *is a negatively curved Riemannian manifold, then every isometry of* Y *homotopic to the identity coincides with the identity, and the isometry group of* Y *is discrete.*

Proof This follows from the uniqueness part of Thm. 1.6.1, since isometries are harmonic.

q.e.d.

The preceding argument can be generalized to show that the larger the isometry group of a compact manifold is, the more restrictions exist for mappings of this manifold into negatively curved ones, since composing a harmonic map with an isometry again yields a harmonic map. Cf. [SY3] for more details.

While in the preceding part of this section, we have used harmonic maps to reprove some elementary theorems merely for the sake of illustration, we now want to briefly mention some more difficult applications most of which we shall not prove in these notes.

- One can prove rigidity theorems for certain classes of nonpositively curved Kähler manifolds, i.e. that the topological type already determines the complex structure, by showing that a suitable harmonic map is actually a holomorphic diffeomorphism. Such results were obtained by Siu [Si], Jost-Yau [JY], Jost-Mok-Yau.

- One can easily prove many results of Teichmüller theory using harmonic maps, for example that Teichmüller space is contractible or even a cell (details can be found in [EE], [Tr], and [J8].) Also, one can recover the Weil-Petersson metric of Teichmüller space from the second variation formula for harmonic maps.

- One can reduce boundary regularity for the minima of certain quadratic functionals to the nonexistence of nontrivial solutions for a certain Dirichlet problem for harmonic maps, cf. [JM] and [SU2].

- As was pointed out by Eells-Wood [EW], harmonic maps can provide an analytic proof of the Theorem of Kneser, that a continuous map ϕ between closed orientable surfaces Σ_1 and Σ_2 has to satisfy the inequality

$$|d(\phi)|\ \chi(\Sigma_2) \geq \chi(\Sigma_1)$$

between its degree and the Euler characteristics of Σ_1 and Σ_2, in case $\chi(\Sigma_2) < 0$ (cf. chapter 5).

- As we shall show in chapter 4, harmonic maps can be used to prove Bernstein type theorems.

1.7 COMPOSITION PROPERTIES OF HARMONIC MAPS

In this section, we shall display an elementary composition property which shall be useful in the sequel. First of all, if $u \in C^2(X,Y)$ is a map between Riemannian manifolds, and $h \in C^2(Y,\mathbb{R})$ is a function, then the following Riemannian chain rule is valid.

(1.7.1) $$\Delta(h \circ u) = D^2 h(u_{e_\alpha}, u_{e_\alpha}) + \langle (\text{grad } h)\circ u, \tau(u)\rangle_Y ,$$

where e^{α} is an orthonormal frame on X. In particular, if u is harmonic, i.e. $\tau(u) = 0$, this reads as

$$\Delta(h \circ u) = D^2 h(u_{e^\alpha}, u_{e^\alpha}) \tag{1.7.2}$$

or in local coordinates

$$\Delta(h \circ u) = \gamma^{\alpha\beta} D^2 h(u_{x^\alpha}, u_{x^\beta}) .$$

Thus

LEMMA 1.7.1 *If* h *is a (strictly) convex function on* Y *and* u *is harmonic, then* $h \circ u$ *is a subharmonic function on* X.

We note the following consequence (cf. Gordon [Go]).

COROLLARY 1.7.1 *Suppose* X *is a compact manifold, possibly with boundary, and* $u : X \to Y$ *is harmonic. If there exists a strictly convex function on* $u(X)$, *and* $u(\partial X)$ *is constant in case* $\partial X \neq \emptyset$, *then* u *is a constant mapping.*

Proof From the maximum principle for subharmonic functions, it follows that $h \circ u$ is constant, and since h has definite second fundamental form, (1.7.2) implies that u itself is constant.

In section 2.3, we shall see that the assumptions of Cor. 1.7.1 are in particular satisfied, if $u(X)$ is contained in a ball $B(p,M)$ which is disjoint to the cut locus of p and satisfies $M < \frac{\pi}{2\kappa}$, where κ^2 is an upper curvature bound on this ball, because in this case $d^2(\cdot,p)$ is strictly convex.

Another consequence is

COROLLARY 1.7.2 *Suppose* X *is a compact manifold with* $\pi_1(X) = 0$ *and the* *sectional curvature of* Y *is nonpositive. Then any harmonic map* $u : X \to Y$ *is constant, provided* $u(\partial X)$ *is constant in case* $\partial X \neq 0$.

Proof By the homotopy lifting theorem, we can lift u to a harmonic map $\tilde{u} : X \to \tilde{Y}$ into the universal covering of Y . The required strictly convex function is then $d^2(\cdot,p)$, where p is any point in $\tilde{Y}$.

If instead of a real-valued function, h is a map from Y into some other Riemannian manifold, then instead if (1.7.1) we get

$$(1.7.3) \qquad \Delta(h\circ u) = \nabla dh(u_{e_\alpha}, u_{e_\alpha}) + (dh)\circ u \cdot \tau(u) \ .$$

In particular

LEMMA 1.7.2 *If* h *is totally geodesic and* u *is harmonic, then* $h\circ u$ *is again harmonic.*

CHAPTER 2

GEOMETRIC PRELIMINARIES

Almost linear functions, approximate fundamental solutions, and representation formulae. Harmonic coordinates.

2.1 OUTLINE OF THE CHAPTER

This chapter begins with a collection of basic estimates for Jacobi fields and some convexity results. We mostly follow the elegant presentation in [BK].

We then introduce the notion of almost linear functions on a manifold, the main technical innovation of [JK1]. Whereas standard coordinate functions, e.g. Riemannian normal coordinates, have only rather poor regularity properties (cf. the example in 2.8) due to the fact that they involve not only the distance function but also angular terms, almost linear functions will be constructed by only using the distance function, which admits a sufficient control through Jacobi field estimates. The basic idea is to use the Euclidean identity $2\langle x,p-q\rangle = |x-q|^2 - |x-p|^2$ $(p = -q)$ as a definition. These functions satisfy almost, i.e. up to a small error term, the usual characterizations of linear functions in Euclidean space, e.g. that the first derivatives are constant, the second ones vanish, or the Taylor expansion terminates after the second term. These error terms are inevitable due to the presence of curvature, conceptually considered as a measure of deviation from Euclidean space. Such error terms, however, generally are of lower order than the other terms which appear already in the Euclidean versions of the formulae and hence can be easily absorbed. In particular, we discuss approximate fundamental solutions of the Laplace and heat equation on manifolds and derive representation formulae. Almost linear functions permit to gain one order of differentiation in such formulae by enabling us to also approximate the derivatives of fundamental solutions.

Another application of almost linear functions is the construction of harmonic coordinates on manifolds with the help of a perturbation argument. They possess even better regularity properties, since, for instance, we can derive C^α-bounds for the corresponding Christoffel symbols in terms of curvature bounds only, not involving any curvature derivatives. They therefore seem to be optimally adapted to the concept of manifolds of bounded geometry. In the present notes, they will play an important role in the derivation of higher order a-priori estimates for harmonic maps.

Starting with section 2.6, all the results of this chapter are either taken from or inspired by [JK1].

2.2 JACOBI FIELD ESTIMATES

Let $c(s,t) = c_t(s)$ be a family of geodesics parametrized by t. s usually will be taken as the arc length parameter on each geodesic. $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ is then a Jacobi field. It satisfies the equation

$$\frac{D}{\partial s}\frac{D}{\partial s} J_t(s) + R\left(\frac{\partial c}{\partial s}, J_t\right)\frac{\partial c}{\partial s} = 0 \tag{2.2.1}$$

which easily follows from $\frac{D}{\partial s}\frac{\partial}{\partial s} c = 0$ and the definition of the curvature tensor.

From (2.2.1) we see that the tangential component of a Jacobi field J, $J^{\tan} = \langle J, \frac{\partial c}{\partial s}\rangle J$ satisfies

$$\frac{D}{\partial s}\frac{D}{\partial s} J^{\tan} = 0$$

and is hence independent of the metric. In particular, $J^{\tan}$ is linear. In order to incorporate the tangential component in the estimates, we have to assume that we have curvature bounds

(2.2.2) $$\lambda \le K \le \mu\ , \quad \lambda \le 0\ , \quad \mu \ge 0$$

i.e. a nonpositive lower and a nonnegative upper bound, or else to assume $J^{\tan} = 0$.

We need some definitions:

$'$ always denotes a derivative with respect to s, while $\cdot$ is the differentiation with respect to t.

We put

$$c_\rho(s) = \begin{cases} \cos(\sqrt{\rho}\, s) & \text{if } \rho > 0 \\ 1 & \text{if } \rho = 0 \\ \cosh(\sqrt{-\rho}\, s) & \text{if } \rho < 0 \end{cases}$$

and

$$s_\rho(s) = \begin{cases} \frac{1}{\sqrt{\rho}} \sin(\sqrt{\rho}\, s) & \text{if } \rho > 0 \\ s & \text{if } \rho = 0 \\ \frac{1}{\sqrt{-\rho}} \sinh(\sqrt{-\rho}\, s) & \text{if } \rho < 0 \end{cases}$$

Both functions solve the Jacobi equation for constant sectional curvature ρ, namely

(2.2.3) $$f'' + \rho f = 0$$

with initial values $f(0) = 1$, $f'(0) = 0$, or $f(0) = 0$, $f'(0) = 1$, resp.

c will always be a geodesic arc parametrized by s proportionally to arclength, and usually $|c'| = 1$ for simplicity.

LEMMA 2.2.1 *Assume* $K \le \mu$ *and* $|c'| = 1$, *and either* $\mu \ge 0$ *or* $J^{\tan} \equiv 0$.

Let $f_\mu := |J(0)|\, c_\mu + |J|'(0)\, s_\mu$ *be the solution of* $f'' + \mu f = 0$ *with the same initial conditions as* $|J|$.

If $f_\mu(s) > 0$ *for* $s \in (0,\sigma)$, *then*

(2.2.4) $$\langle J,J'\rangle f_\mu \geq \langle J,J\rangle f'_\mu \quad \text{on} \quad (0,\sigma)$$

(2.2.5) $$1 \leq \frac{|J(s_1)|}{f(s_1)} \leq \frac{|J(s_2)|}{f(s_2)} \qquad \text{if} \quad 0 < s_1 \leq s_2 < \sigma$$

(2.2.6) $$|J(0)|\, c_\mu(s) + |J|'(0)\, s_\mu(s) \leq |J(s)| \qquad \text{for} \quad s \in (0,\sigma) .$$

Proof $$|J|'' + \mu|J| = |J|^{-1}(-\langle R(c',J)\ c',J\rangle + \mu\langle J,J\rangle)$$

$$+ |J|^{-3}(|J'|^2|J|^2 - \langle J,J'\rangle^2) \geq 0 .$$

Hence

$$(|J|'f_\mu - |J|f'_\mu)' = |J|''f_\mu - |J|f''_\mu \geq 0 .$$

Since $|J|(0) = f_\mu(0)$, $|J|'(0) = f'_\mu(0)$, (2.2.4) follows. Then

$$\left(\frac{|J|}{f_\mu}\right)' = \frac{1}{f_\mu^2}\,(|J|'f_\mu - |J|\cdot f'_\mu) \geq 0 ,$$

since it vanishes at 0 and has nonnegative derivative.

(2.2.5) again follows from the initial conditions, and (2.2.5) implies (2.2.6).

LEMMA 2.2.2 *Assume* $K \leq \mu$, *and either* $\mu \geq 0$ *or* $J^{\tan} = 0$, *and* $|K| \leq \Lambda^2$, $J(0) = 0$, $|c'| = 1$, $c_\mu \geq 0$ *on* $(0,\sigma)$.

Then

(2.2.7) $$|J(s) - sJ'(s)| \leq |J(t)|\cdot\frac{1}{2}\Lambda^2 s^2 .$$

Proof Let P be a parallel vector field along c , and $s \in (0,\sigma)$.

$$|\langle J(s) - sJ'(s),\ P(s)\rangle'| = |s\langle R(c',J)c',P\rangle(s)|$$

$$\leq \Lambda^2\, s|J(s)|$$

$$\leq \Lambda^2\, s|J(\sigma)|\,\frac{s_\mu(s)}{s_\mu(\sigma)} \qquad \text{by (2.2.5)}$$

$$\leq \Lambda^2 \, s|J(\sigma)| \ , \quad \text{since} \quad c_\mu \geq 0 \quad \text{on } [0,\sigma]$$

and (2.2.7) follows by integration of this inequality.

q.e.d.

Instead of prescribing $J(0)$ and $J'(0)$, one can also prescribe $J(0)$ and $J(\rho)$ for $\rho < \pi/\sqrt{\mu}$. For example, since we showed in the proof of Lemma 2.2.1 that $|J|'' + \mu|J| \geq 0$, we conclude, assuming $|c'| = 1$ again,

$$(2.2.8) \quad \sin(\sqrt{\mu}\rho) \ |J(s)| \leq \sin(\sqrt{\mu}s) \ |J(\rho)| + \sin(\sqrt{\mu}(\rho - s)) \ |J(0)| \ .$$

We shall also need the following estimate of Jäger-Kaul [JäK2].

LEMMA 2.2.3 *Suppose* $K \leq \mu$, $|c'| = 1$, *and* $0 < \rho < \pi/\sqrt{\mu}$ *in case* $\mu > 0$. *If* X *is a Jacobi field along* c *with*

$$\langle X, c' \rangle = 0 \ ,$$

then

$$(2.2.9) \quad \langle X, X' \rangle \Big|_0^\rho \geq \frac{s_\mu'(\rho)}{s_\mu(\rho)} \left(|X(0)|^2 + |X(\rho)|^2\right) - \frac{2}{s_\mu(\rho)} |X(0)| \cdot |X(\rho)| \ .$$

Proof Let

$$s(t) := \frac{1}{s_\mu(\rho)} \cdot \left(|X(0)| s_\mu(\rho - t) + |X(\rho)| s_\mu(t)\right) \ .$$

Then s solves

$$(2.2.10) \qquad s'' + \mu s = 0 \ , \qquad s(0) = |X(0)| \ , \qquad s(\rho) = |X(\rho)| \ ,$$

and

$$s \geq 0 \qquad \text{on} \quad [0,\rho]$$

and

$$(2.2.11) \qquad \begin{aligned} s'(0) &= \frac{1}{s_\mu(\rho)} \left(|X(\rho)| - s_\mu'(\rho) |X(0)|\right) \\ s'(\rho) &= \frac{1}{s_\mu(\rho)} \left(s_\mu'(\rho) |X(\rho)| - |X(0)|\right) \ . \end{aligned}$$

Then the function

$$g := s|X|' - s'|X|$$

is differentiable where $|X| \neq 0$. (Note that the zeros of X are isolated, since X solves the Jacobi equation

$$X'' + R(c',X)c' = 0 , \tag{2.2.12}$$

which is a linear second order equation.) Moreover

$$g' = s|X|'' - s''|X| = s\left(\frac{\langle X,X'\rangle}{|X|}\right)' + \mu s|X|$$

$$= s\frac{1}{|X|^3}(|X|^2|X'|^2 - \langle X,X'\rangle^2) - s\cdot\frac{1}{|X|}\langle X,R(c',X)c'\rangle + \mu s|X|$$

$$\geq 0 ,$$

since by assumption $\langle X,R(c',X)c'\rangle \leq \mu|X|^2$. Thus g is not decreasing on those intervals where it is differentiable. As was noted above, points τ where g' does not exist, i.e. $|X(\tau)| = 0$ are discrete, and moreover

$$g(\tau+0) - g(\tau-0) = 2s(\tau)|X'(\tau)| \geq 0 .$$

Thus, g is not decreasing on $[0,\rho]$, and defining

$$|X|'(\rho) = \lim_{\varepsilon\downarrow 0}|X|'(\rho-\varepsilon) , \qquad |X|'(0) = \lim_{\varepsilon\downarrow 0}|X|'(\varepsilon) ,$$

we conclude

$$0 \leq g(\rho) - g(0) = s(\rho)|X|'(\rho) - s'(\rho)|X(\rho)| - s(0)|X|'(0) + s'(0)|X(0)|$$

$$= \langle X,X'\rangle(\rho) - \langle X,X'\rangle(0) - \frac{s_\mu'(\rho)}{s_\mu(\rho)}(|X(0)|^2 + |X(\rho)|^2)$$

$$+ \frac{2}{s_\mu(\rho)}|X(0)|\cdot|X(\rho)| ,$$

by (2.2.11).

q.e.d.

We now turn to describe the effect of a lower curvature bound on Jacobi field estimates.

LEMMA 2.2.4 *Assume* $\lambda \le K \le \mu$ *, and either* $\lambda \le 0$ *or* $J^{\tan} \equiv 0$ *,* $|K| \le \Lambda^2$ *,* $|c'| \equiv 1$ *, and in addition that* $J(0)$ *and* $J'(0)$ *are linearly dependent. For a parameter* τ *, we define again* $f_\tau = |J(0)|c_\tau + |J|'(0)s_\tau$ *. If* $f_{\frac{1}{2}(\lambda+\mu)} > 0$ *on* $(0,\rho)$ *, then*

$$|J(s)| \le |J(0)|\ c_\lambda(s) + |J|'(0)\ s_\lambda(s)\ , \tag{2.2.13}$$

and in any case, if P_s *denotes parallel translation along* c

$$\begin{aligned}|J(s) - P_s(J(0) + sJ'(0))| \le{}& |J(0)|(\cosh(\Lambda s) - 1) \\ &+ |J|'(0)\left(\frac{1}{\Lambda}\sinh(\Lambda s) - s\right) .\end{aligned} \tag{2.2.14}$$

Proof Let τ be a parameter, and $\eta = \max(\mu-\tau, \tau-\lambda)$. Let A be the vectorfield along c that satisfies

$$\frac{D}{ds}\frac{D}{ds}A + \tau A = 0\ , \qquad A(0) = J(0)\ , \qquad A'(0) = J'(0)\ .$$

Let a be the solution of

$$a'' + (\tau-\eta)a = \eta|A|\ , \qquad a(0) = a'(0) = 0$$

and b the solution of

$$b'' + \tau b = \eta|J|\ , \qquad b(0) = b'(0) = 0\ .$$

If P is a unit parallel field

$$|\langle J - A,P\rangle'' + \tau\langle J - A,P\rangle| = |\langle J'' - \tau J,P\rangle| \le \eta|J|\ .$$

Hence

$$d := \{\langle J - A,P\rangle - b\}''\ s_\tau - \{\langle J - A,P\rangle - b\}\ s_\tau'' \le 0$$

and

$$\left(\frac{1}{s_\tau}\{\langle J - A, P\rangle - b\}\right)'(s) = \frac{1}{s_\tau^2(s)}\int_0^s d \le 0 .$$

Thus $\frac{1}{s_\tau}\{\langle J - A, P\rangle - b\} \le 0$, since it vanishes at $s = 0$. If $s_\tau > 0$ on $(0,\rho)$, then this implies

(2.2.15) $$|J - A| \le b \quad \text{on} \quad (0,\rho)$$

and

$$b'' + \tau b \le \eta b + \eta|A| .$$

In a similar way

$$\frac{1}{s_\tau}(b - a) \le 0 ,$$

(2.2.16) i.e. $$b \le a$$

(2.2.15) and (2.2.16) give

(2.2.17) $$|J - A|(s) \le a(s) \quad \text{for} \quad s \in (0,\rho) .$$

Now

(2.2.18) $$(\langle A',A'\rangle\langle A,A\rangle - \langle A,A'\rangle\langle A,A'\rangle)' = 0$$

and thus

$$\langle A',A'\rangle\langle A,A\rangle - \langle A,A'\rangle\langle A,A'\rangle \equiv 0 ,$$

since it vanishes at $s = 0$, as $A(0)$ and $A'(0)$ are linearly dependent. This in turn implies

$$|A|'' + \tau\cdot|A| = 0 ,$$

i.e. $$|A| = f_\tau$$

and hence

$$a = f_{\tau-\eta} - f_\tau$$

and from (2.2.17)

$$|J| \leq f_{\tau-\eta} \; .$$

Choosing $\tau = \frac{1}{2}(\mu+\lambda)$, i.e. $\tau-\eta = \lambda$, then proves (2.2.13).

(2.2.18) also implies that $(A/|A|)' = 0$, i.e. $A/|A|$ is parallel, and choosing $\tau = 0$ then proves (2.2.14).

2.3 APPLICATIONS TO GEODESIC CONSTRUCTIONS

We let $c(s,t) = \exp_p(s\cdot(v+tw))$ be a family of geodesics radially emanating from the point p .

Then

$$J(s) = \frac{\partial}{\partial t} c(s,t)\Big|_{t=0} = (d \exp_p)_{sv}\cdot sw \tag{2.3.1}$$

is a Jacobi field with

$$J(0) = 0 \; , \qquad J'(0) = w \; .$$

If we put $v = w$, then J is tangential to $c(s,0)$ and hence linear, i.e. $J(s) = sv$, which implies

$$|(d \exp_p)_v\cdot v| = |v|$$

or in other words, that $\exp_p : T_pM \to M$ is an isometry in the radial direction.

If w and v are orthogonal, then (2.2.6) and (2.2.13) imply

LEMMA 2.3.1 *If* $w \perp v$, $\lambda \leq K \leq \mu$, *then, if* $s \leq \frac{\pi}{\sqrt{\mu}}$ *in case* $\mu > 0$,

$$|w| \cdot \frac{s_\mu(s)}{s} \leq |(d \exp_p)_{sv}\cdot w| \leq |w| \frac{s_\lambda(s)}{s} \; . \tag{2.3.2}$$

LEMMA 2.3.2 *Let* $B(m,\rho) := \{x \in M : d(m,x) \le \rho\}$ *be a ball in some manifold* M *which is disjoint to the cut locus of its centre* m . *We assume for the sectional curvatures* K *in* $B(m,\rho)$

$$-\omega^2 \le K \le \kappa^2 \qquad \text{and} \qquad \rho < \frac{\pi}{2\kappa}$$

We define $r(x) := d(x,m)$ *and* $f(x) := \frac{1}{2} d(x,m)^2$. *Then* $f \in C^2(B(m,\rho),\mathbb{R})$ *and*

$$|\text{grad } f(x)| = r(x) \tag{2.3.3}$$

$$\kappa r(x) \text{ ctg}(\kappa r(x)) \cdot |v|^2 \le D^2 f(v,v) \tag{2.3.4}$$

$$\le \omega r(x) \coth(\omega r(x)) \cdot |v|^2$$

for $x \in B(m,\rho)$ *and* $v \in T_x M$.

Proof grad $f(x) = - \exp_x^{-1} m$ which implies (2.3.3).

Let $q(t)$ be a curve in M with $q(0) = x$ and $\dot{q}(0) = v$ and

$$c(s,t) = \exp_{q(t)}(s \exp_{q(t)}^{-1} m) \ .$$

Then grad $f(q(t)) = - \frac{\partial}{\partial s} c(s,t)|_{s=0}$, and hence

$$D_v \text{ grad } f(x) = - \frac{D}{\partial t} \frac{\partial}{\partial s} c(s,t)|_{s=0,t=0}$$

$$= - \frac{D}{\partial s} \frac{\partial}{\partial t} c(s,t) \ .$$

For fixed t , $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ is the Jacobi field along the geodesic from m to $q(t)$ with $J_t(0) = \dot{q}(t)$ and $J_t(1) = 0 \in T_m M$. Hence D_v grad $f(x) = D_{J_o(0)}$ grad $f(x) = -J_o'(0)$. Since

$$D^2 f(v,v) = \langle D_v \text{ grad } f, v\rangle = - \langle J_o'(0), J_o(0)\rangle \ ,$$

(2.3.4) follows from (2.2.6) and (2.2.13) (since $J_t(1) = 0$, $J_t(1)$ and

$J_t'(1)$ are linearly dependent).

q.e.d.

2.4 CONVEXITY OF GEODESIC BALLS

The following convexity result was proved in [J2] and [BK], Prop. 6.4.6.

PROP. 2.4.1 *Suppose the ball* $B(m,\rho)$ *is disjoint to the cut locus of* m *, and* $\rho < \frac{\pi}{2\kappa}$ *, where* κ^2 *is an upper bound for the sectional curvature of* $B(m,\rho)$ *. Then any two points in* $B(m,\rho)$ *can be joined in* $B(m,\rho)$ *by a unique geodesic arc. This arc is the shortest connection between its end points and thus in particular does not contain a pair of conjugate points.*

Proof Since the cut locus of a point m is closed, we can find some ρ' , $\rho < \rho' < \frac{\pi}{2\kappa}$, for which $B(m,\rho')$ is still disjoint to the cut locus of m . For any two points p and $q \in B(m,\rho)$, we can find a shortest connection $\gamma(t)$ in $B(m,\rho')$ by the standard Arzela-Ascoli argument. Let $\gamma(0) = p$, $\gamma(1) = q$, and let $c(\cdot,t)$ be the family of geodesics with $c(0,t) = m$, $c(1,t) = \gamma(t)$. The Jacobi fields $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ are monotonically increasing in $s \in [0,1]$ by (2.2.5). Hence, in case γ leaves $B(m,\rho)$ somewhere between p and q , we can project it onto $B(m,\rho)$, i.e. take

$$\tilde{\gamma}(t) = \exp_m\left(\exp_m^{-1} \gamma(t) \cdot \min\left(1 , \frac{\rho}{d(\gamma(t),m)}\right)\right)$$

and obtain a shorter comparison curve in contradiction to the choice of γ . Hence γ is contained in $B(m,\rho)$ and hence in particular in the interior of $B(m,\rho')$ and is therefore geodesic. Furthermore, clearly $\text{length}(\gamma) \le 2\rho$.

The exponential map has maximal rank along any geodesic in $B(m,\rho)$ of length $\le 2\rho$ by Lemma 2.3.1. In particular, they do not contain pairs of conjugate points and are locally unique. Hence, the set of pairs $(p,q) \in B(m,\rho) \times B(m,\rho)$ with two geodesic connections is compact, since two geodesics cannot collapse in the limit into a single one with conjugate points. Thus, if this set were non empty, we could find such a pair (p,q) of minimal distance with two minimal geodesic connections γ_1 and γ_2. γ_1 and γ_2 then have to form a closed geodesic. Namely, otherwise, if they would form an angle $< \pi$ at p for example, then moving a little bit along the geodesic which bisects this angle, we could find a point $\tilde{p}$ which is closer to q and still has two different connections to q, in contradiction to the choice of p and q. (For more details on this argument, cf. [GKM]). On the other hand, by Lemma 2.3.2, $d^2(\cdot,m)$ is strictly convex on $B(m,\rho)$, and therefore the existence of a closed geodesic in $B(m,\rho)$ contradicts Cor. 1.7.1.

If now $p,q \in B(m,\rho)$ would have two geodesic connections, one of which, called γ, is longer than 2ρ, then γ ceases somewhere between p and q to be the shortest connection of its endpoints, and hence we could again find two minimal geodesics, in contradiction to what we already proved.

q.e.d.

This result can be somewhat improved in two dimensions. First of all, we have

LEMMA 2.4.1 *Let* S *be a compact surface, possibly with boundary. If the boundary* γ *is not empty, it is assumed to be convex, i.e. that through every point* $\tilde{q}$ *of* γ *there goes a geodesic arc which is disjoint to* S *in a neighbourhood of* $\tilde{q}$. *Let* $p,q \in S$. *Assume that there are two distinct homotopic geodesic arcs joining* p *and* q. *Then each of the points* p *and* q *has a conjugate point in* S, *and this point is conjugate to* p *or* q, *resp., with respect to a geodesic arc which is the shortest connection in its homotopy class.*

Proof We denote the two geodesic arcs by γ_1 and γ_2. We can assume w.l.o.g. that γ_1 and γ_2 are shortest connections in their homotopy class between p and q, since otherwise, starting e.g. from p and moving on γ_1, we would find a point q_1 which would either be conjugate to p or would have a connection in S to p in the same homotopy class and of equal length as the segment of γ_1 between p and q_1. (At this point, for the existence of such a connection, we have to use the convexity of γ). Since γ_1 and γ_2 are homotopic and distinct, because we could assume that they are shortest connections, they bound a set B of the topological type of the disc.

We now look at a geodesic line emanating from p into B. As γ_1 and γ_2 are shortest, this line has to cease somewhere in B to be shortest connection to p. Repeating the argument, if we have not yet found the desired conjugate point, we get a nested sequence of geodesic two-angles, i.e. configurations consisting of two homotopic geodesic arcs of equal length which furthermore are shortest possible in their homotopy class. In the limit, this construction has to yield a geodesic arc covered twice. The endpoint q_2 therefore is homotopic to p, and furthermore, the geodesic arc is the shortest connection in its homotopy class from p to q_2.

q.e.d.

LEMMA 2.4.2 *Suppose* $B(p,R) := \{q \in \Sigma : d(p,q) \le R\}$, *where* Σ *is a surface, is topologically a disc for some* $r < \frac{\pi}{\kappa}$ $(K \le \kappa^2)$. *Then* $\exp_p\{v : |v| = r\} = \partial B(p,r)$ *for all* $r \le R$, *where* $\exp_p : T_p\Sigma \to \Sigma$ *is the exponential map. Furthermore,* $\partial B(p,r)$ *is convex, if* $r \le \frac{\pi}{2\kappa}$.

Proof Clearly, $\partial B(p,r) \subseteq \exp_p\{v : |v| = r\} \subseteq B(p,r)$. We assume now that

$$\exp_p\{v : |v| = r\} \cap \mathring{B}(p,r) \neq \phi . \tag{2.4.1}$$

$\exp_p$ is a local diffeomorphism on $\{v : |v| < \frac{\pi}{\kappa}\}$ by Lemma 2.3.1, and therefore

$\exp_p\{v : |v| = r\}$ is an immersed smooth curve for $r < \frac{\pi}{\kappa}$. Since $\exp_p\{v : |v| = r\}$ is compact, we can find some $q \in \exp_p\{v: |v| = r\}$ with minimal distance to p. Consequently, the shortest geodesic γ from p to q is orthogonal to $\exp_p\{v : |v| = r\}$ at q and has length $< r$. On the other hand, $q = \exp_p w$, $|w| = r$, and the geodesic $\gamma' = \exp_p tw$, $t \in [0,1]$, is also orthogonal to $\exp_p\{v : |v| = r\}$ at q and different from γ, since its length is precisely r. Thus, γ and γ' have an angle of π at q and match together to a geodesic loop with corner at p. It is not difficult to see that every point inside this geodesic loop can be joined to p by a shortest geodesic, in spite of the fact that this loop might not be convex at p. Thus, we can carry over the argument of Lemma 2.4.1 to assert the existence of a point p' inside this loop which is conjugate to p w.r.t. a shortest geodesic γ''. Since $p' \in B(p,r)$ and $r < \frac{\pi}{\kappa}$, this is in contradiction to Lemma 2.3.1. This proves the first claim. Furthermore, since $\exp_p$ has maximal rank on $\{v \in T_p\Sigma : |v| < \frac{\pi}{\kappa}\}$, as noted above, we infer that every $v \in T_p\Sigma$ with $|v| = r$ has a neighbourhood V which is mapped under $\exp_p$ injectively onto its image (cf. [Kl], p.108f.). From this, we easily see that we may apply the estimate of Lemma 2.3.2. Therefore, if $r \leq \frac{\pi}{2\kappa}$, then h is a convex function on $B(p,r)$, and consequently, $\partial B(p,r) = \exp_p\{v : |v| = r\}$ is convex as a level set of a convex function.

PROP. 2.4.2 *Suppose now, that* $B(p,r)$ *is a geodesic disc on a surface, and* $r < \frac{\pi}{2\kappa}$ $(K \leq \kappa^2)$. *Then each pair of points* $q_1, q_2 \in B(p,r)$ *can be joined by a unique geodesic arc in* $B(p,r)$*, and this arc is free of conjugate points.*

Proof By virtue of Lemma 2.4.2, we could apply Lemma 2.4.1, if there would exist two geodesic arcs joining q_1 and q_2. Consequently, we would find a point q_3 conjugate to q_1 w.r.t. a shortest geodesic arc, i.e. an arc of length $\le 2r < \frac{\pi}{\kappa}$. This would contradict Lemma 2.3.1.

q.e.d.

2.5 THE DISTANCE AS A FUNCTION OF TWO VARIABLES

We suppose again that the ball $B(p,M) \subset N$ is disjoint to the cut locus of p and that $M < \frac{\pi}{2\kappa}$, where $-\omega^2 \le K \le \kappa^2$ are curvature bounds. We define

$$q_\kappa(t) = \begin{cases} \frac{1}{\kappa^2}(1 - \cos \kappa t) & \text{if } \kappa > 0 \\ \frac{t^2}{2} & \text{if } \kappa = 0 \end{cases}$$

and note that

$$q_\kappa(t) = \int_0^t s_{\kappa^2} \ .$$

By assumption and 2.4, any two points $y_1, y_2 \in B(p,M)$ can be joined by a unique minimal geodesic in $B(p,M)$, and we can measure the distance between y_1 and y_2 by the length of the geodesic arc between them. We denote this (possibly modified) distance function again by $d(y_1,y_2)$. Then

$$Q_\kappa(y_1,y_2) := q_\kappa(d(y_1,y_2))$$

defines a C^2 function on $B(p,M) \times B(p,M)$, since $q_\kappa'(0) = 0$. Moreover, we note that

$$T_y(N\times N) = T_{y_1} \oplus T_{y_2} N \quad \text{(isometrically)}$$

for $y = (y_1,y_2) \in N\times N$.

In the following lemma, we shall estimate the Hessian of Q_κ on $B(p,M) \times B(p,M)$, using the Jacobi field estimate of Lemma 2.2.3. This result is again due to Jäger-Kaul [JäK2].

LEMMA 2.5.1 *If* $y_1 \neq y_2$ *, then for all*

$$v \in T_y(N\times N)\ , \qquad y = (y_1,y_2)\ , \qquad y_1,y_2 \in B(p,M)$$

$$D^2Q_\kappa(v,v) \geq \frac{\langle \text{grad } Q_\kappa(y), v\rangle^2}{2Q_\kappa(y)} - \kappa^2 Q_\kappa(y)\,|v|^2\ . \tag{2.5.1}$$

If v *has the special form* $0 \oplus u$ *or* $u \oplus 0$ *, then*

$$D^2Q_\kappa(v,v) \geq (1 - \kappa^2 Q_\kappa(y))\,|u|^2\ , \tag{2.5.2}$$

and this also holds for $y_1 = y_2$.

Proof First some definitions

$\rho := d(y_1,y_2)$

$v =: v_1 \oplus v_2 \in T_{y_1}N \oplus T_{y_2}N$,

$c : [0,\rho] \to B(p,M)$ is the unique geodesic arc from y_1 to y_2 with $|c'| = 1$,

$e_1(y) := -c'(0)$

$e_2(y) := c'(\rho)$

$v_i^{\tan} := \langle v_i, e_i(y)\rangle\, e_i(y)$

$v_i^{\text{nor}} := v_i - v_i^{\tan} \qquad (i = 1,2)$.

Then, since $\rho > 0$,

$\text{grad } d(y) = e_1(y) \oplus e_2(y)$,

$\text{grad } Q_\kappa(y) = s_{\kappa^2}(\rho)(e_1(y) \oplus e_2(y))$, and

$D^2Q_\kappa(y)(v,v) = \langle D_v \text{ grad } Q_\kappa, v\rangle$

$$= s'_{\kappa^2}(\rho)\langle e_1(y) \oplus e_2(y), v_1 \oplus v_2\rangle^2 + s_{\kappa^2}(\rho) D^2 d(v,v)\ . \tag{2.5.3}$$

If $c_t(s)$ is the geodesic arc with

$$c_t(0) = \exp_{y_1}(tv_1^{nor}) \ , \qquad c_t(\rho) = \exp_{y_2}(tv_2^{nor})$$

(note that c_t is unique, if $t \geq 0$ is small enough), then

(2.5.4) $$J(s) := \frac{\partial}{\partial t} c_t(s)_{|s=0}$$

is a Jacobi field along c with

$$J(0) = v_1^{nor} \ , \qquad J(\rho) = v_2^{nor} \ .$$

By Synge's formula (cf. [GKM], §4.1),

(2.5.5) $$\begin{aligned} D^2 d(v,v) &= \frac{\partial^2}{\partial t^2} \operatorname{length}(c_t)_{|t=0} \\ &= \int_0^\rho (|J'|^2 - \langle J, R(c',J)c'\rangle)\,ds \end{aligned}$$

(note that there is no boundary term, since

$$\langle J, c'\rangle = 0 \ )$$

We can apply Lemma 2.2.3 to obtain

$$\begin{aligned} D^2 d(v,v) &= \int_0^\rho (|J'|^2 + \langle J, J''\rangle)\,ds \\ &= \langle J, J'\rangle \Big|_0^\rho \\ &\geq \frac{s'_{\kappa^2}(\rho)}{s_{\kappa^2}(\rho)} (|v_1^{nor}|^2 + |v_2^{nor}|^2) - \frac{2}{s_{\kappa^2}(\rho)} |v_1^{nor}|\cdot|v_2^{nor}| \ , \end{aligned}$$

and thus with (2.5.3)

(2.5.6) $$\begin{aligned} D^2 Q_\kappa(v,v) \geq S'_{\kappa^2}(\rho)(\langle e_1 \oplus e_2,\ v_1 \oplus v_2\rangle^2 + |v_1^{nor}|^2 + |v_2^{nor}|^2) \\ - 2|v_1^{nor}||v_2^{nor}| \ . \end{aligned}$$

If $v = 0 \oplus u$, (2.5.6) implies

$$D^2 Q_\kappa(v,v) \geq s'_{\kappa^2}(\rho) \langle e_2(y), u\rangle^2 + s'_{\kappa^2}(\rho)\ |u^{nor}|^2$$

$$= s'_{\kappa^2}(\rho)\,|u|^2$$

$$= (1 - \kappa^2 Q\ (y))\,|u|^2 \ ,$$

while in the general case, we only have

$$\langle e_1 \oplus e_2, v_1 \oplus v_2\rangle^2 \leq 2(|v_1^{\tan}|^2 + |v_2^{\tan}|^2) \ ,$$

and

$$|v_i|^2 = |v_i^{\tan}|^2 + |v_i^{\mathrm{nor}}|^2 \ ,$$

and therefore from (2.5.6),

$$\begin{aligned} D^2 Q_\kappa(v,v) &\geq s'_{\kappa^2}(\rho)\,\langle e_1 \oplus e_2, v_1 \oplus v_2\rangle^2 - (1 - s'_{\kappa^2}(\rho))(|v_1^{\mathrm{nor}}|^2 + |v_2^{\mathrm{nor}}|^2) \\ &\geq \tfrac{1}{2}(1 + s'_{\kappa^2}(\rho))\,\langle e_1 \oplus e_2, v_1 \oplus v_2\rangle^2 - (1 - s'_{\kappa^2}(\rho))(|v_1|^2 + |v_2|^2) \\ &= \frac{1}{2Q_\kappa(y)}\langle \mathrm{grad}\ Q_\kappa(y), v\rangle^2 - \kappa^2 Q\ (y)(|v_1|^2 + |v_2|^2) \ . \end{aligned}$$

q.e.d.

2.6 ALMOST LINEAR FUNCTIONS

We are now ready to introduce almost linear functions, one of the main tools of [JK1].

Let $B(m,\rho)$ be again a ball in some n-dimensional Riemannian manifold M which is disjoint to the cut locus of m, and assume curvature bounds

$$-\omega^2 \leq K \leq \kappa^2 \ , \qquad |K| \leq \Lambda^2$$

and

$$\rho < \frac{\pi}{2\kappa} \ .$$

We put $r(x) = d(m,x)$, $f(x) = \frac{1}{2}d^2(m,x)$.

DEFINITION 2.6.1 Let $u \in T_m M$ be a unit vector, i.e. $|u| = 1$, and put $p(x) = \exp_m(r(x)u)$, $q(x) = \exp_m(-r(x)u)$. Then

$$\ell(x) := \frac{1}{4r(x)} \left(d(x,q(x))^2 - d(x,p(x))^2\right)$$

is called an *almost linear function.*

We observe that in the Euclidean case, this notion yields precisely the linear functions, because of Pythagoras' theorem. We furthermore note that

(2.6.1) $$-r(x) \le \ell(x) \le r(x) .$$

The estimates of [JK1] for almost linear functions are contained in

THEOREM 2.6.1 *Suppose* $B(m,\rho)$ *is disjoint to the cut locus of* m *,* $-\omega^2 \le K \le \kappa^2$ *,* $|K| \le \Lambda^2$ *on* $B(m,\rho)$ *, and* $\rho < \frac{\pi}{2\kappa}$ *. Let* $u \in T_mM$ *,* $|u| = 1$ *,* $\ell(x)$ *the associated almost linear function, and* $u(x)$ *the radially parallel vector field on* $B(m,\rho)$ *with* $u(m) = u$ *. Then*

(2.6.2) $$|\operatorname{grad} \ell(x) - u(x)| \le 2\kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \cdot r^2(x)$$

(2.6.3) $$|D^2\ell(x)| \le |9\kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \omega r \operatorname{ctgh}(\omega r))\, r(x)$$

(2.6.4) $$|\ell(x) - \langle \operatorname{grad} \ell(x), -\exp_x^{-1} m\rangle| \le \left(\frac{9}{2} \kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \omega r \operatorname{ctgh}(\omega r)\right) r^3(x) .$$

Proof Let $\gamma(t)$ be a geodesic with $\gamma(0) = x$. We then look at the following families of geodesics, joining $\gamma(t)$ with $p(\gamma(t))$ or $q(\gamma(t))$, resp.,

$$c_1(s,t) = \exp_{\gamma(t)}(s \cdot \exp^{-1}_{\gamma(t)} p(\gamma(t)))$$

$$c_2(s,t) = \exp_{\gamma(t)}(s \cdot \exp^{-1}_{\gamma(t)} q(\gamma(t))) .$$

$J_i(\cdot,t) = \frac{\partial}{\partial t} c_i(\cdot,t)$ are Jacobi fields with

$$J_i(0,t) = \dot\gamma(t)$$

$$J_1(1,t) = \dot r u(t)$$

$$J_2(1,t) = -\dot r u(t)$$

where we have abbreviated $r(\gamma(t)) = r(t)$, $u(\gamma(t)) = u(t)$, etc. We also write again $c' = \frac{\partial}{\partial s} c$, $\dot{c} = \frac{\partial}{\partial t} c$. We note that

$$d^2(p(\gamma(t)), \gamma(t)) = c_1'(s,t)^2$$

$$d^2(q(\gamma(t)), \gamma(t)) = c_2'(s,t)^2 .$$

Now

$$\frac{d}{dt} \ell(\gamma(t)) = -\frac{c_2'^2 - c_1'^2}{4r^2} \dot{r} + \frac{1}{2r} \int_0^1 \{<c_2', \frac{D}{\partial t} c_2'> - <c_1', \frac{D}{\partial t} c_1'>\} ds$$

$$(2.6.5) \qquad = -\frac{c_2'^2 - c_1'^2}{4r^2} \dot{r} + \frac{1}{2r} \int_0^1 \{<c_2', J_2>' - <c_1', J_1>'\} ds$$

$$= -\frac{c_2'^2 - c_1'^2}{4r^2} \dot{r} - \frac{\dot{r}}{4r^2} <c_2' + c_1', 2ru>_{s=1} - \frac{1}{2r} <c_2' - c_1', \dot{\gamma}>_{s=0}$$

In order to control $c_1' - c_2' - 2ru$ which vanishes in the Euclidean case, we need the following result which follows from [BK].

LEMMA 2.6.1 *Put* $\varepsilon(r) := \frac{2}{3} \kappa\Lambda r^3 \ \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}$

$$(2.6.6) \qquad |c_1' - (\exp_x^{-1} m + ru)| \ (x) \le \varepsilon(r)$$

$$(2.6.7) \qquad |c_2' - (\exp_x^{-1} m - ru)| \ (x) \le \varepsilon(r)$$

$$(2.6.8) \qquad |-c_1' - (\exp_m^{-1} x - ru)| \ (p(x)) \le \varepsilon(r)$$

$$(2.6.9) \qquad |-c_2^{-1} - (\exp_m^{-1} x + ru)| \ (q(x)) \le \varepsilon(r) .$$

Proof of Lemma 2.6.1 Let $v \in T_x M$, $c(t) = \exp tv$, $c(1) = q$, where q is some point in M . Let $w \in T_x M$ and $w(t)$ be the parallel vector field along $c(t)$.

We first want to estimate $d(F(w), G(w))$, where

$$F(w) = \exp_x(v + w)$$

$$G(w) = \exp_q(w(1)) .$$

We consider the family of geodesics

$$c(s,t) = \exp_{c(t)}(s \cdot (w(t) + (1-t)\,\dot{c}(t)))$$

and the corresponding Jacobi fields

$$J_t(x) = \dot{c}(s,t) .$$

The initial conditions are

$$(2.6.10) \qquad \begin{aligned} J_t(0) &= \dot{c}(t) \\ \frac{D}{\partial s} J_t(0) &= \frac{D}{\partial t}\frac{\partial}{\partial s} c(0,t) = -\dot{c}(t) . \end{aligned}$$

We let $J_t^{norm}(s)$ be the component of $J_t(s)$ which is orthogonal to $c'(s,t)$.

Since the curve $c(1,t)$ joins $F(w)$ and $G(w)$ and has tangent vector $J_t(1) = J_t^{norm}(1)$, because $J_t^{tan}(1) = 0$ (this follows from (2.6.10))

$$(2.6.11) \qquad d(F(w),\, G(w)) \le \int_0^1 |J_t^{norm}(1)|dt .$$

We now want to apply (2.2.14). Since $|c'|$ is not necessarily equal to 1, we have to rescale $c(\cdot,t)$, i.e. to look at the geodesics $\gamma(s,t) = c(\frac{s}{|c'|},t)$ and the Jacobi Fields $\tilde{J}(s,t) = J(\frac{s}{|c'|},t)$. This amounts to replacing Λ by $\Lambda|c'|$ in (2.2.14).

Since by (2.6.10) $J_t(0) + J_t'(0) = 0$, (2.2.14) yields, putting $\rho = \max(|w|,\ |v+w|)$, and using $\cosh x - \frac{\sinh x}{x} \le \frac{1}{3} x \sinh x$,

$$(2.6.12) \qquad |J_t^{norm}(1)| \le |J_t^{norm}(0)| \cdot |c'| \cdot \frac{1}{3}\Lambda \sinh(\Lambda\rho) .$$

Moreover,

$$\begin{aligned} |J_t(0)^{norm}|^2 \cdot \left|\frac{\partial c}{\partial s}\right|^2 &= \left|\frac{\partial c}{\partial t}\right|^2 \cdot \left|\frac{\partial c}{\partial s}\right|^2 - \left\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\right\rangle^2 \\ &= |v|^2\ |w + (1-t)\ v|^2 - \langle v, w + (1-t)\ v\rangle^2 \\ &= |v|^2\ |w|^2 - \langle v,w\rangle^2 . \end{aligned}$$

Therefore, (2.6.11) and (2.6.12) imply

$$(2.6.13) \quad d(F(w), G(w)) \le \frac{1}{3}|v|\cdot|w|\cdot\Lambda \sinh(\Lambda(|v|+|w|))\cdot \sin \measuredangle(v,w) .$$

In (2.6.13), we then put $v = \exp_x^{-1} m$, $w = \pm ru$.

Then

$$F(w) = \exp_x(\exp_x^{-1} m \pm ru)$$

$$G(w) = \exp_m(\pm ru) = p(x) \quad \text{or} \quad q(x) \quad \text{resp.}$$

$$= \exp_x c_1' \quad \text{or} \quad \exp_x c_2' \quad \text{resp.}$$

Therefore, (2.6.6) and (2.6.7) follow from (2.6.13) and (2.3.2). (2.6.8) and (2.6.9) follow in a similar manner.

q.e.d.

We now continue the proof of Thm. 2.6.1:

(2.6.6) and (2.6.7) yield

$$(2.6.14) \qquad |c_1' - c_2' - 2ru|(x) \le 2\varepsilon(r) ,$$

and similarly from (2.6.8) and (2.6.9), if p denotes parallel transport along radial geodesics

$$(2.6.15) \qquad |pc_1' - pc_2' - 2ru|(m) \le 2\varepsilon(r) .$$

(2.6.15) and $|c_1' + c_2'| \le 4r$ imply

$$(2.6.16) \qquad |c_2'^2 - c_1'^2 + \langle pc_2' + pc_1', 2ru\rangle| \le 8r\varepsilon(r) .$$

Since $|\dot{r}| \le |\dot{\gamma}|$, (2.6.5), (2.6.14), and (2.6.16) then yield

$$\langle \operatorname{grad} \ell - u, \dot{\gamma}\rangle \,| \le \frac{3}{r}\varepsilon(r)|\dot{\gamma}|$$

i.e. (2.6.2).

Differentiating (2.6.5), we get

$$(2.6.17)\quad \frac{d^2}{dt^2}\,\ell(\gamma(t)) = \langle c_2' + c_1',\ c_2' - c_1' + 2ru\rangle \left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right)$$

$$+ \frac{\dot{r}}{2r^2}\langle c_2' - c_1',\ \dot{\gamma}\rangle_{s=0} - \frac{\dot{r}}{4r^2}\langle c_2' + c_1',\ 2\dot{r}u\rangle_{s=1}$$

$$- \frac{1}{2r}\langle J_2' - J_1',\ \dot{\gamma}\rangle_{s=0} - \frac{\dot{r}}{4r^2}\langle J_2' + J_1',\ 2ru\rangle_{s=1}$$

$$- \frac{\dot{r}}{4r^2}\,\frac{d}{dt}\,(c_2'^2 - c_1'^2)\ .$$

In the course of (2.6.5), we obtained

$$\frac{d}{dt}\,(c_2'^2 - c_1'^2) = -\langle c_2' + c_1',\ 2\dot{r}u\rangle_{s=1} - \langle c_2' - c_1',\ 2\dot{\gamma}\rangle_{s=0}\ .$$

Hence

$$(2.6.18)\qquad \frac{d^2}{dt^2}\,\ell(\gamma(t)) = \langle c_2' + c_1',\ c_2' - c_1' + 2ru\rangle \left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right)$$

$$+ \frac{1}{2r}\Bigg(\frac{2\dot{r}}{r}\langle c_2' - c_1',\ \dot{\gamma}\rangle_{s=0}$$

$$+ \langle J_1', J_1\rangle(0) - \langle J_2', J_2\rangle(0) - \langle J_1', J_1\rangle(1) + \langle J_2', J_2\rangle(1)\Bigg)$$

Since $\ddot{f} = r\ddot{r} + \dot{r}^2$, with (2.3.4)

$$\left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right) \le \frac{|\dot{\gamma}|^2}{4r^3}\,(3 + \omega r\ \mathrm{ctgh}(\omega r))\ .$$

(2.6.14) then gives

$$(2.6.19)\quad \langle c_2' + c_1',\ c_2' - c_1' + 2ru\rangle \left(\frac{-\ddot{r}}{4r^2} + \frac{\dot{r}^2}{2r^3}\right) \le \frac{2\varepsilon(r)}{r^2}\,(3 + \omega r\ \mathrm{ctgh}(\omega r))\ |\dot{\gamma}|^2$$

Furthermore, since

$$(J(s) - p\circ J(0) - sJ'(s))' = sR(c', J)\ c'\ ,$$

$$|J(s) - p\,J(0) - sJ'(s)| \le \Lambda^2\ |c'|^2 \int_0^s \sigma|J(\sigma)|\ .$$

Using

$$|J(s)| \le |J(1)|\,\frac{\sin(\kappa|c'|s)}{\sin(\kappa|c'|)} + |J(0)|\,\frac{\sin(\kappa|c'|(1-s))}{\sin(\kappa|c'|)}$$

$$\le 2\,\max(|J(0)|,\ |J(1)|)\circ\frac{\sin(\kappa r)}{\sin(2\kappa r)}\ ,$$

which follows from (2.2.8), we get

$$(2.6.20) \qquad |J(s) - p\,J(0) - s\,J'(s)| \le \frac{\Lambda^2 r^2 s^2 \sin(\kappa r)}{\sin(2\kappa r)} \max(|J(0)|, |J(1)|)$$

and similarly

$$|J(1-s) - p\,J(1) + (1-s)\,J'(1-s)|$$

is estimated by the same quantity.

We are now ready to control the second term of (2.6.18). First

$$(2.6.21) \qquad \left|\frac{2\dot r}{r} \langle c_2' - c_1', \dot\gamma\rangle + \langle 4\dot r u, \dot\gamma\rangle\right| \le \frac{4|\dot r|}{r} |\dot\gamma|\ \varepsilon(r)\ .$$

Next

$$(2.6.22) \qquad \langle p\,J_1(1) - J_1(0), J_1(0)\rangle - \langle p\,J_2(1) - J_2(0), J_2(0)\rangle$$
$$- \langle J_1(1) - p\,J_1(0), J_1(1)\rangle + \langle J_2(1) - p\,J_2(0), J_2(1)\rangle$$
$$- 4\,\langle \dot r u, \dot\gamma\rangle = 0\ ,$$

since $J_i(0) = \dot\gamma$, $J_1(1) = \dot r u$, $J_2(1) = -\dot r u$.

Since $|\dot r| \le |\dot\gamma|$, (2.6.20), (2.6.21), and (2.6.22) then give

$$(2.6.23) \qquad |\langle J_1'(0), J_1(0)\rangle - \langle J_2'(0), J_2(0)\rangle - \langle J_1'(1), J_1(1)\rangle$$
$$+ \langle J_2'(1), J_2(1)\rangle + \frac{2\dot r}{r}\langle c_2' - c_1', \dot\gamma\rangle|$$
$$\le \left(\frac{4\varepsilon(r)}{r} + 4\Lambda^2\, \frac{r^2 \sin(\kappa r)}{\sin(2\kappa r)}\right) |\dot\gamma|^2$$

(2.6.18), (2.6.19), and (2.6.23) finally yield

$$\left|\frac{d^2}{dt^2}\,\ell(\gamma(t))\right| \le \left(\frac{8\varepsilon(r)}{r^2} + \frac{2\varepsilon(r)}{r^2}\,\omega r\ \mathrm{ctgh}(\omega r) + 2\Lambda^2 r \frac{\sin(\kappa r)}{\sin(2\kappa r)}\right) |\dot\gamma|^2$$
$$\le \left(9\kappa\Lambda\,\frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}\ \omega r\ \mathrm{ctgh}(\omega r)\right)\cdot r \cdot |\dot\gamma|^2\ .$$

Thus, (2.6.3) is proved.

For any geodesic c

$$\frac{d}{dt}\,(\ell(c(t)) - t \langle \operatorname{grad} \ell,\ \dot{c}(t)\rangle) = -t\, D^2\ell(\dot{c},\dot{c}) \ .$$

Taking the radial geodesic from m to x, we then see that (2.6.4) follows from (2.6.3).

q.e.d.

For later purposes, we also need to investigate how almost linear functions depend on the base point m. To emphasize this dependence, we now use a subscript m, i.e. write $\ell_m(x)$ for the corresponding almost linear function. Let now $\gamma(t)$ be a geodesic arc, $u(t)$ a parallel unit vector field along $\gamma(t)$ and $\ell_{\gamma(t)}(x)$ the corresponding almost linear functions.

LEMMA 2.6.2 *For* $z \in B(\gamma(t),\rho)$, $\rho < \min(i(\gamma(t)), \pi/2\kappa)$

$$\left|\frac{d}{dt}\,\ell_{\gamma(t)}(z)\right| \le (5 + c\,\Lambda^2\rho^2) \ . \tag{2.6.24}$$

Proof Let

$$\rho(t) = d(\gamma(t), z)$$

$$p(t) = \exp_{\gamma(t)}(\rho(t)\ u(t))$$

$$q(t) = \exp_{\gamma(t)}(-\rho(t)\ u(t)) \ .$$

Then

$$\ell_{\gamma(t)}(z) = \frac{1}{4\rho(t)}\,(d^2(z,q(t)) - d^2(z,p(t))) \ . \tag{2.6.25}$$

We look at the family of geodesics

$$c(s,t) = \exp_{\gamma(t)}(s\rho(t)\ u(t)) \ .$$

The corresponding Jacobi field $J_t(s) = \frac{\partial}{\partial t}\,c(s,t)$ then satisfies

$$J_t(0) = \dot{\gamma}(t)$$

$$\frac{D}{\partial s}\,J_t(0) = \dot{\rho}(t)\ u(t) \ , \qquad \text{since } u(t) \text{ is parallel along } \gamma$$

$$J_t(1) = \dot{p}(t) \ .$$

In particular, $\frac{D}{\partial s} J_t(0)$ is tangential to the geodesic $c(\cdot,t)$. Thus, $J_t^{norm}(0)$ and $J_t^{norm'}(0)$ are linearly dependent, and (2.2.13) implies

$$|\dot{p}| \le |\dot{\rho}| + \cosh(\Lambda\rho)\,|\dot{\gamma}| \,, \tag{2.6.26}$$

and the same inequality holds for $|\dot{q}|$.

(2.6.24) then follows from (2.6.26), $|\dot{\rho}| \le |\dot{\gamma}|$, and $d(z,q(t))$, $d(z,p(t)) \le 2\rho(t)$.

q.e.d.

Actually, one can even show the stronger estimate

$$\left|\frac{d}{dt}\, \ell_{\gamma(t)}(z) - \langle u(t), \dot{\gamma}\rangle\right| \le c\,\Lambda^2\rho^2 \,. \tag{2.6.27}$$

The proof is rather tedious, however, and hence left out, since we do not need (2.6.27) in the sequel.

2.7 APPROXIMATE FUNDAMENTAL SOLUTIONS AND REPRESENTATION FORMULAE

We first apply Lemma 2.3.2 to construct approximate fundamental solutions of the Laplace and the heat equation on manifolds.

LEMMA 2.7.1 *Let* $B(m,\rho)$ *be as in Lemma 2.3.2.* $\Lambda^2 := \max(\kappa^2, \omega^2)$, *and let* Δ *be the Laplace-Beltrami operator on* M, *and* $n = \dim M$, $h(x) := d(x,m)^2$.

$$|\Delta \log r(x)| \le 2\Lambda^2 \qquad \textit{for } x \ne m \textit{ if } n = 2 \tag{2.7.1}$$

$$|\Delta\, r(x)^{2-n}| \le \frac{n-2}{2}\,\Lambda^2\, r^{2-n}(x) \qquad \textit{for } x \ne m \textit{ if } n \ge 3 \tag{2.7.2}$$

and

$$\left|\left(\Delta - \frac{\partial}{\partial t}\right)\left(t^{-n/2}\exp\left(-\frac{h(x)}{4t}\right)\right)\right| \le 2\Lambda^2\,\frac{h(x)}{4t}\,t^{-n/2}\exp\left(-\frac{h(x)}{4t}\right) \tag{2.7.3}$$

for $(x,t) \ne (m,0)$.

The proof follows through a straightforward computation from Lemma 2.3.2.

q.e.d.

We now derive approximate versions of Green's representation formula, first in the elliptic case.

LEMMA 2.7.2 *Let* $B(m,\rho)$ *be as above,* $h(x) = d(x,m)^2$. *Let* ω_n *denote the volume of the unit sphere in* $\mathbb{R}^n$. *If* $\phi \in C^2(B(m,\rho), \mathbb{R})$, *then*

(2.7.4) *if* $n = 2$
$$\left|\omega_2 \phi(m) + \int_{B(m,\rho)} \Delta\phi \cdot \log\frac{r(x)}{\rho} - \frac{1}{\rho}\int_{\partial B(m,\rho)} \phi\right| \le 2\Lambda^2 \int_{B(m,\rho)} |\phi|$$

(2.7.5) *if* $n \ge 3$
$$\left|(n-2)\omega_n \phi(m) + \int_{B(m,\rho)} \Delta\phi \left(\frac{1}{r(x)^{n-2}} - \frac{1}{\rho^{n-2}}\right) - \frac{(n-2)}{\rho^{n-1}}\int_{\partial B(m,\rho)} \phi\right| \le \frac{n-2}{2}\Lambda^2 \int_{B(m,\rho)} \frac{|\phi|}{r(x)^{n-2}} .$$

We note that the error term is of lower order than the other two terms which are the same as in the Euclidean version of the Green representation formula.

Proof We shall prove only (2.7.5) for simplicity. We put

$$g(x) = r(x)^{2-n} - \rho^{2-n} .$$

Then for $\varepsilon > 0$

$$\int_{B(m,\rho)\backslash B(m,\varepsilon)} (g\Delta\phi - \phi\Delta g) = \int_{\partial(B(m,\rho)\backslash B(m,\varepsilon))} \langle g \operatorname{grad}\phi - \phi \operatorname{grad} g, d\vec{O}\rangle .$$

Now

$$\left|\int_{B(m,\rho)\backslash B(m,\varepsilon)} \phi\Delta g\right| \le \frac{n-2}{2}\Lambda^2 \int_{B(m,\rho)} \frac{|\phi|}{r^{n-2}(x)} \qquad \text{by (2.7.2)}$$

$$g|\partial B(m,\rho) = 0$$

$$\int_{\partial B(m,\rho)} \phi \langle \operatorname{grad} g, d\vec{O}\rangle = \frac{n-2}{\rho^{n-1}}\int_{\partial B(m,\rho)} \phi$$

$$\lim_{\varepsilon\to 0}\int_{\partial B(m,\varepsilon)} g\langle \operatorname{grad}\phi, d\vec{O}\rangle = 0$$

$$\lim_{\varepsilon\to 0} \int_{B(m,\varepsilon)} \phi \langle \operatorname{grad} g, \vec{dO}\rangle = (n-2)\omega_n \phi(m)$$

and (2.7.5) follows.

q.e.d.

In the parabolic case, the corresponding version is

LEMMA 2.7.3 *Let* $B(m,\rho)$ *be as above,*

$$B(m,\rho,t_0,t) := \{(x,\tau) \in B(m,\rho) \times [t_0,t]\},$$

$$\phi(\cdot,\tau) \in C^2(B(m,\rho), \mathbb{R}), \quad \phi(x,\cdot) \in C^1([t_0,t], \mathbb{R}) .$$

Then

$$(2.7.6) \qquad \Big| (\sqrt{4\pi})^n \phi(m,t) + \int_{B(m,\rho,t_0,t)} \left(\Delta - \frac{\partial}{\partial t}\right) \phi(x,\tau)(t-\tau)^{-n/2} \left(\exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) - \exp\left(\frac{-\rho^2}{4(t-\tau)}\right)\right) dx\, d\tau \Big|$$

$$\le \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} |\phi| + \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} |\phi(x,\tau)|$$

$$+ (t-t_0)^{-n/2} \int_{B(m,\rho)} |\phi(x,t_0)|\, dx$$

$$+ 2\Lambda^2 \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)| \frac{r^2(x)}{(t-\tau)} (t-\tau)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right).$$

Here, c_n *is a constant depending only on* n .

Proof We put

$$g(x,\sigma) = \sigma^{-n/2}\left(\exp\left(-\frac{r^2(x)}{4\sigma}\right) - \exp\left(-\frac{\rho^2}{4\sigma}\right)\right) .$$

Let $\varepsilon > 0$. Then

$$\int_{B(m,\rho,t_0,t-\varepsilon)} \left\{ g(x,t-\tau)\left(\Delta - \frac{\partial}{\partial\tau}\right)\phi(x,\tau) - \phi(x,\tau)\left(\Delta + \frac{\partial}{\partial\tau}\right) g(x,t-\tau)\right\} dx\, d\tau$$

$$= \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t-\varepsilon}} \langle g(x,t-\tau) \operatorname{grad} \phi(x,\tau) - \phi(x,\tau) \operatorname{grad} g(x,t-\tau), \vec{dO}\rangle\, d\tau$$

$$+ \int_{\substack{\tau=t-\varepsilon \\ r(x)\le\rho}} g(x,\varepsilon)\ \phi(x,t-\varepsilon)\ dx - \int_{\substack{\tau=t_0 \\ r(x)\le\rho}} \phi(x,t_0)(t-t_0)^{-n/2}$$

$$\left(\exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) - \exp\left(-\frac{\rho^2}{4(t-t_0)}\right)\right) dx .$$

Now

$$\int_{B(m,\rho,t_0,t-\varepsilon)} \phi(x,\tau)\left(\Delta + \frac{\partial}{\partial\tau}\right) g(x,t-\tau)\ dx\ d\tau$$

$$\le 2\Lambda^2 \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)|\ \frac{r^2(x)}{(t-\tau)}\ (t-\tau)^{-n/2}\ \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) dx\ d\tau$$

by (2.7.3)

$g(x,t-\tau) = 0$ if $r(x) = \rho$

$$\int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} \phi(x,\tau)(t-\tau)^{-n/2}\ \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) \frac{2r(x)}{4(t-\tau)} \langle \text{grad } r(x),\ d\vec{0}\rangle$$

$$\le \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} |\phi(x,\tau)|$$

since

(2.7.7) $\qquad \exp(-y) \le c_\alpha\ y^{-\alpha} \quad$ for $y > 0$, $\alpha \ge 0$.

$$\int_{B(m,\rho,t_0,t)} \phi(x,\tau)\ \frac{\partial}{\partial\tau}\left((t-\tau)^{-n/2}\ \exp\left(-\frac{\rho^2}{4(t-\tau)}\right)\right) dx\ d\tau$$

$$= \int_{B(m,\rho,t_0,t)} \phi(x,\tau)\left((t-\tau)^{-n/2-1}\ \exp\left(-\frac{\rho^2}{4(t-\tau)}\right)\right)\left(-\frac{n}{2}+\frac{\rho^2}{4(t-\tau)}\right) dx\ d\tau$$

$$\le \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)|\ dx\ d\tau \qquad \text{by (2.7.7) again}$$

$$\int_{r(x)\le\rho} \phi(x,t_0)(t-t_0)^{-n/2}\left(\exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) - \exp\left(-\frac{\rho^2}{4(t-t_0)}\right)\right) dx$$

$$\le (t-t_0)^{-n/2} \int_{\substack{\tau=t_0 \\ r(x)\le\rho}} |\phi(x,t_0)|\ dx$$

$$\int_{r(x)\le\rho} \phi(x,t-\varepsilon)\ \varepsilon^{-n/2}\left(\exp\left(-\frac{r^2(x)}{4\varepsilon}\right)\right)\left(-\exp\left(-\frac{\rho^2}{4\varepsilon}\right)\right) dx$$

$$\to (\sqrt{4\pi})^n\ \phi(m,t) \qquad \text{as} \quad \varepsilon \to 0$$

and (2.7.6) follows.

q.e.d.

For a later purpose, we also note the following formula

$$(2.7.8)\qquad \Bigg| (\sqrt{4\pi})^n \, \phi(m,t) + \int_{B(m,\rho,t_0,t)} \left(\Delta - \frac{\partial}{\partial\tau}\right) \phi(x,\tau)(t-\tau)^{-n/2} \left(\exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) - \exp\left(-\frac{\rho^2}{4(t-\tau)}\right)\right) dx\, d\tau$$

$$- \int_{B(m,\rho)} \phi(x,t_0)(t-t_0)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) dx \Bigg|$$

$$\leq \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} |\phi| + \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0\leq\tau\leq t}} |\phi(x,\tau)|$$

$$+ c_n \int_{B(m,\rho)} |\phi(x,t_0)| \, dx$$

$$+ 2\Lambda^2 \int_{B(m,\rho,t_0,t)} |\phi(x,\tau)| \frac{r^2(x)}{(t-\tau)} (t-\tau)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-\tau)}\right) dx\, d\tau \, .$$

(2.7.8) also follows from the preceding proof by handling the boundary term at $t = t_0$ in a different way.

We now use almost linear functions in order to also obtain an approximate version of the derivative of Green's function. This is important for obtaining derivative estimates for functions on manifolds.

LEMMA 2.7.4 *Let* $B(m,\rho)$ *be as before. For* $x \in B(m,\rho)$, $x \neq m$, *we define*

$$a(x) = \ell(x)(r(x)^{-n} - \rho^{-n}) \, ,$$

where $\ell(x)$ *is an almost linear function.*

Then

$$(2.7.9) \qquad |\Delta a| \leq 9n^2 \kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)} \omega r \operatorname{ctgh}(\omega r)\, r^{-n+1} \qquad \textit{for} \quad x \neq m .$$

Proof

$$(2.7.10) \quad \operatorname{grad} a = \operatorname{grad} \ell (r^{-n} - \rho^{-n}) - n \cdot \ell\, r^{-n-2} \operatorname{grad} f \qquad (f = \tfrac{1}{2} d(\cdot, m)^2)$$

and

$$\Delta a = -2nr^{-n-2} \langle \operatorname{grad} f, \operatorname{grad} \ell \rangle + \Delta\ell \cdot (r^{-n} - \rho^{-n})$$

$$- n\ell r^{-n-2} \Delta f + n(n+2)\ell\, r^{-n-4} |\operatorname{grad} f|^2$$

and hence

$$|\Delta a| \leq |\Delta\ell| r^{-n} + 2n\, r^{-n-2} |\ell - \langle \operatorname{grad} f, \operatorname{grad} \ell \rangle| + n|\ell|\, r^{-n-2} |\Delta f - n|$$

since $\operatorname{grad} f = -\exp_x^{-1} m$ and $|\operatorname{grad} f| = r$, cf. (2.3.3).

(2.7.9) then follows from (2.6.3), (2.6.4), and (2.3.4).

q.e.d.

We now can prove that the gradient bound that is obtained in the Euclidean case by differentiating Green's representation formula, again holds on Riemannian manifolds up to a small error term.

LEMMA 2.7.5 *Suppose* $h \in C^2(B(m,\rho), \mathbb{R})$, *where* $B(m,\rho)$ *satisfies the same assumptions as before.*

Then

$$(2.7.11) \qquad \omega_n |\operatorname{grad} h(m)| \leq \frac{n}{\rho^n} \int_{\partial B(m,\rho)} |h(\cdot) - h(m)| + \int_{B(m,\rho)} \frac{|\Delta h|}{r^{n-1}}$$

$$+ c\, \Lambda^2 \int_{B(m,\rho)} \frac{|h(\cdot) - h(m)|}{r^{n-1}(\cdot)} .$$

Here c *is a constant which depends only on* n *and* $\Lambda\rho$.

Proof For simplicity, we assume $h(m) = 0$.

Let ℓ be an almost linear function with

$$\text{(2.7.12)} \qquad \langle \operatorname{grad} \ell(m), \operatorname{grad} h(m)\rangle = |\operatorname{grad} h(m)|$$

and let $a(x) = \ell(x)(r(x)^{-n} - \rho^{-n})$. Then for $\varepsilon > 0$

$$\int_{B(m,\rho)\setminus B(m,\varepsilon)} (a\cdot\Delta h - h\cdot\Delta a) = \int_{\partial(B(m,\rho)\setminus B(m,\varepsilon))} \langle a \operatorname{grad} h - h \operatorname{grad} a, \vec{dO}\rangle$$

Now

$$\int_{B(m,\rho)} |a\cdot\Delta h| \le \int_{B(m,\rho)} \frac{|\Delta h|}{r(x)^{n-1}} \qquad \text{since} \quad |\ell(x)| \le r(x)$$

$$\int_{B(m,\rho)} |h\cdot\Delta a| \le c\cdot\Lambda^2 \int_{B(m,\rho)} \frac{|h|}{r(x)^{n-1}} \qquad \text{by (2.7.9)}$$

$$a|\partial B(m,\rho) = 0$$

$$\int_{\partial B(m,\rho)} |\langle h \operatorname{grad} a, \vec{dO}\rangle| \le \frac{n}{\rho^n} \int_{\partial B(m,\rho)} |h| \quad \text{by (2.7.10)} .$$

Furthermore by (2.6.4) and since $\vec{dO} = \frac{1}{r} \operatorname{grad} f\cdot|\vec{dO}|$

$$\left|\frac{1}{r^n}\langle \ell\cdot\operatorname{grad} h, \vec{dO}\rangle - \frac{1}{r}\langle \operatorname{grad} \ell, \operatorname{grad} f\rangle \cdot \frac{1}{r}\langle \operatorname{grad} h, \operatorname{grad} f\rangle \cdot \frac{|\vec{dO}|}{r^{n-1}}\right|$$

$$\le c_1 \cdot r^3 \cdot \frac{1}{r^n} |\operatorname{grad} h| \cdot |\vec{dO}|$$

and hence, using (2.7.12),

$$\lim_{\varepsilon\to 0} \int_{\partial B(m,\varepsilon)} \langle a \operatorname{grad} h, \vec{dO}\rangle = |\operatorname{grad} h(m)| \cdot \int_{S^{n-1}} \cos^2\theta \, d\omega^{n-1}$$

$$=: \alpha_n |\operatorname{grad} h(m)| .$$

Finally, since $h(x) = \langle \operatorname{grad} h, \operatorname{grad} f\rangle + O(r(x)^2)$, using (2.7.10)

$$\lim_{\varepsilon\to 0} \int_{\partial B(m,\varepsilon)} \langle h \operatorname{grad} a, \vec{dO}\rangle = \lim_{\varepsilon\to 0} \int_{\partial B(m,\varepsilon)} \langle \operatorname{grad} h, \operatorname{grad} f\rangle$$

$$\langle \operatorname{grad} \ell \cdot r^{-n} - n\cdot\ell\cdot r^{-n-2} \operatorname{grad} f, \vec{dO}\rangle$$

$$= \alpha_n (1-n) \ |\operatorname{grad} h(m)|, \text{ using (2.6.4) as before.}$$

The preceding estimates easily imply (2.7.11), noting $\omega_n = n\alpha_n$.

q.e.d.

2.8 REGULARITY PROPERTIES OF COORDINATES. HARMONIC COORDINATES

In this section, we are concerned with regularity properties of coordinates on manifolds. Eventually, we shall show that harmonic coordinates, i.e. ones for which the coordinate functions are harmonic, possess best possible regularity properties.

We start by noting that Riemannian normal coordinates have rather poor regularity properties. Namely, in [JK1] there was displayed the following example of a two-dimensional metric with Hölder continuous curvature which itself is only Hölder continuous in normal coordinates, but not better:

$$ds^2 = dr^2 + G^2(r,\phi)\, d\phi^2$$

with

$$G^2(r,\phi) = \begin{cases} r^2(1 + r^2 \sin^\alpha \phi)^2 & \text{for } 0 \le \phi \le \pi \qquad (0 < \alpha < 1) \\ r^2 & \text{for } \pi \le \phi \le 2\pi . \end{cases}$$

For this metric

$$K = -\frac{G_{rr}}{G} = \begin{cases} -\dfrac{6 \sin^\alpha \phi}{1 + r^2 \sin^\alpha \phi} & \text{for } 0 \le \phi \le \pi \\ 0 & \text{for } \pi \le \phi \le 2\pi . \end{cases}$$

The reason for this phenomenon is that the formula for K in normal coordinates does not involve any derivatives of G with respect to ϕ .

Our aim is to construct coordinates for which we can control - in contrast to normal coordinates - the Christoffel symbols in terms of curvature bounds.

Let us first derive some general identities for any coordinate map $H = (h^1,\dots,h^n) : (B,\langle\cdot,\cdot\rangle) \to \mathbb{R}^n$, where B is the coordinate domain and

$\langle\cdot,\cdot\rangle$ the Riemannian metric. If $v \in T_pB$, then its coordinates are $v^i = dh^i(p)v$. Thus $\langle v,w\rangle = g_{ij}\, v^i w^j$, and choosing $v = w = e_k$, where (e_k) is an orthonormal basis of T_pB , we get

$$(2.8.1) \qquad g^{jk} = \langle \text{grad } h^j, \text{grad } h^k\rangle = dh^j \text{ grad } h^k .$$

Moreover

$$(2.8.2) \qquad \begin{aligned} D^2_{v,w}H = \langle D_v \text{ grad } H,w\rangle &= v(dH\cdot w) - dH\cdot D_v w \\ &= v(dH\cdot w) - dH\, d_v w - dH\cdot\Gamma(v,w) \\ &= - dH\cdot\Gamma(v,w) \end{aligned}$$

since $dH = id$ is linear.

Hence we see that the Christoffel symbols Γ are given by the second derivatives of the coordinate functions. Thus, we have to control those second derivatives for suitable coordinates.

We first construct coordinates by almost linear functions. Let $U = \{u_1,\ldots,u_n\}$ be an orthonormal basis of T_mM , and $\ell_1,\ldots,\ell_n$ the corresponding almost linear functions.

We define $L : B(m,\rho) \to T_mM \cong \mathbb{R}^n$ via

$$(2.8.3) \qquad L(x) = \ell_i(x)\cdot u_i(x) .$$

Then, if P denotes parallel transport along radial geodesics, from Thm. 2.6.1

$$(2.8.4) \qquad |dL - P(u)| \le 2\sqrt{n}\, \kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}\, r^2(x)$$

$$(2.8.5) \qquad |D^2L(x)| \le 9\sqrt{n}\, \kappa\Lambda \frac{\sinh(2\Lambda r)}{\sin(2\kappa r)}\, \omega r \text{ ctgh}(\omega r)\cdot r(x)$$

Note that the injectivity radius of p also enters, namely by restricting the size of the domain of definition of L . (2.8.4) implies that L is invertible on some ball $B(m,\delta)$, where δ depends on Λ , n, and the injectivity radius. Hence L defines coordinates on this ball, and the corresponding Christoffel symbols are bounded because of (2.8.2) and (2.8.5).

If we average this construction over all orthonormal bases U of $T_m M$, then the coordinates become canonical, since independent of a particular choice of U, while keeping the estimates (2.8.4) and (2.8.5).

We call these coordinates *almost linear coordinates.*

Let now $L : B(p,R) \to T_p M = \mathbb{R}^n$ be almost linear coordinates. We then take the harmonic map

$$H : B(p,R) \to \mathbb{R}^n$$

with

$$H|\partial B(p,R) = L|\partial B(p,R) .$$

We want to show that for some suitably chosen R, H is injective, i.e. a coordinate map.

THEOREM 2.8.1 *For each* $p \in M$ *there exists some* $R > 0$, *depending only on* $\Lambda^2 = \max(|K|)$ *(* K *is the sectional curvature of* M *),* $i(p)$ *(the injectivity radius of* p *), and* $n = \dim M$, *with the property that on* $B(p,R)$ *there exist harmonic coordinates.*

Proof Let ℓ be almost linear on some ball $B(p,R)$. We solve

$$\Delta h = 0 \quad \text{in} \quad B(p,R)$$

$$h|\partial B(p,R) = \ell|\partial B(p,R) .$$

Assuming $R < \frac{\pi}{2\Lambda}$ and putting $k = h-\ell$, (2.6.3) implies

$$(2.8.6) \qquad |\Delta k| \le 9n\Lambda^2 \cdot \Lambda d(x,p) \ \mathrm{ctgh}(\Lambda d(x,p)) \cdot \frac{\sinh(\Lambda d(x,p))}{\sin(\Lambda d(x,p))} \cdot d(x,p) .$$

On the other hand, for

$$\phi(x) = c_0\Lambda^2 (d^3(x,p) - R^3)$$

by Lemma 2.3.2

$$\Delta\phi(x) \geq c_0\Lambda^2(3d^2(x,p)(n-1)\Lambda \operatorname{ctg}(\Lambda d(x,p)) + 6d) .$$

For given $R \leq R_0 < \frac{\pi}{2\Lambda}$, we can calculate $c_0 = c_0(\Lambda \cdot R_0, n)$ for which $k \pm \phi$ is sub- or superharmonic, resp. Since $k \pm \phi | \partial B(p,R) = 0$, the maximum principle implies

$$|k(x)| \leq |\phi(x)| \leq c_0\Lambda^2 R^3 \tag{2.8.7}$$

and for $x_1 \in \partial B(p,R)$, $x_2 \in B(p,R)$

$$\frac{|k(x_1) - k(x_2)|}{|x_1 - x_2|} \leq \frac{c_0|\phi(x_2)|}{|x_1 - x_2|} \leq 3c_0\Lambda^2 R^2 \tag{2.8.8}$$

or

$$|k(x_2)| \leq 3c_0\Lambda^2 R^2 d(x_2, \partial B(p,R)) . \tag{2.8.9}$$

Let $x \in B(p,R)$, $\rho := d(x,\partial B(p,R))$. Lemma 2.7.5, applied to $B(x,\rho)$ yields

$$\omega_n|\operatorname{grad} k(x)| \leq \frac{n}{\rho^n}\int_{\partial B(x,\rho)} |k(y) - k(x)|\, dy + \int_{B(x,\rho)} \frac{|\Delta k(y)|}{d(x,y)^{n-1}}\, dy$$
$$+ c_1(\Lambda\rho, n) \int_{B(x,\rho)} \frac{|k(y) - k(x)|}{d(x,y)^{n-1}}\, dy$$

and hence with (2.8.6) and (2.8.9)

$$|\operatorname{grad} k(x)| \leq c_2\Lambda^2 R^2 .$$

Here $c_2 = c_2(\Lambda R_0, n)$ remains bounded for fixed n and $R_0 \to 0$.

(2.6.2) then implies

$$|\operatorname{grad} h(x) - u(x)| \leq c_3\Lambda^2 R^2 , \tag{2.8.10}$$

$c_3 = c_3(\Lambda \cdot R_0, n)$.

Let $\{e_i\}$ be an orthonormal basis of T_pM, ℓ^i corresponding almost linear functions and h^i harmonic functions with $h^i|\partial B(p,R) = \ell^i|\partial B(p,R)$.

Putting $H(x) = h^i(x)e_i$, (2.8.10) implies

(2.8.11) $$|dH - id| \le c_3 \sqrt{n}\, \Lambda^2 R^2 \quad \text{on} \quad B(p,R) .$$

We then average again over orthonormal bases of T_pM .

As for almost linear coordinates, we see that harmonic coordinates exist on fixed balls, the radius of which depends only on $i(p)$ (since $R < i(p)$ is necessary for the above constructions), Λ^2 , and n .

q.e.d.

If (g_{ik}) is the metric tensor for the harmonic coordinates constructed above, then from (2.8.1) and (2.8.10)

(2.8.12) $$\begin{aligned}|g^{ik} - \delta^{ik}| &= |\langle \operatorname{grad} h^i - u^i, \operatorname{grad} h^k\rangle - \langle u^i, \operatorname{grad} h^k - u^k\rangle| \\ &\le (2 + c_3\Lambda^2 R^2)\, c_3\Lambda^2 R^2 = c_4\Lambda^2 R^2 .\end{aligned}$$

(2.8.12) implies

$$\|g_{ik}\|_\infty \le \frac{1}{1 - c_4 n\Lambda^2 R^2}$$

and hence

(2.8.13) $$|g_{ik} - \delta_{ik}| \le c_4\Lambda^2 R^2 \, \|g_{ik}\|_\infty \le \frac{c_4\Lambda^2 R^2}{1 - c_4 n\Lambda^2 R^2} .$$

We now want to estimate the Christoffel symbols for harmonic coordinates.

LEMMA 2.8.1 *Let* $H = (h^1,\dots,h^n)$ *be harmonic coordinates. Then, if* (e_i) *is an orthonormal frame, satisfying* $\nabla_{e^i}(e_j) = 0$ *at* x

(2.8.14) $$\begin{aligned}\Delta g^{ik} &= \Delta\langle \operatorname{grad} h^i, \operatorname{grad} h^k\rangle \\ &= 2\, R_{mn}\, h^i_{e^m} h^k_{e^n} + 2\, h^i_{e^j e^\ell} h^k_{e^j e^\ell} ,\end{aligned}$$

where R_{mn} *is the Ricci tensor.*

The proof uses the calculations presented in 1.6.

LEMMA 2.8.2 *There exists some* $R_0 > 0$ *, depending only on* n *,* Λ^2 *,* $i(p)$ *, with the property that for all* $R \le R_0$ *on* $B(p,R)$ *there exist harmonic coordinates the metric tensor* g *of which satisfies*

$$|dg(x)| \le \frac{c_5 \Lambda^2 R^2}{d(x, \partial B(p,R))} \quad \text{for } x \in B(p,R) , \tag{2.8.15}$$

where $c_5 = c_5(n, \Lambda R_0)$.

Proof Since

$$e_\ell \langle \operatorname{grad} h^i, \operatorname{grad} h^k \rangle = h^i_{e_j e_\ell} h^k_{e_j} + h^i_{e_j} h^k_{e_j e_\ell} \tag{2.8.16}$$

in normal coordinates, (2.8.10) and (2.8.14) imply

$$|\Delta g| \le 2 \|\mathrm{Ric}\| (1 + c_3 \Lambda^2 R^2)^2 + \frac{9}{2} (1 + c_3 \Lambda^2 R^2) |dg|^2 . \tag{2.8.17}$$

We now use a method of Heinz [Hz1] to obtain (2.8.15).

Let $\mu := \max\limits_{x \in B(p,R_0)} d(x, \partial B(p,R_0)) \, |dg(x)|$.

Then there is some $x_1 \in B(p,R_0)$ with

$$\mu = d(x_1, \partial B(p,R_0)) \, |dg(x_1)| , \tag{2.8.18}$$

and

$$|dg(p)| \le \frac{\mu}{R_0} . \tag{2.8.19}$$

Let $d := d(x_1, \partial B(p,R_0))$, i.e. $\frac{\mu}{d} = |dg(x_1)|$.

By Lemma 2.7.5, applied to $B(x_1, d\theta)$, $0 < \theta < 1$

$$\frac{\mu}{d} \le \frac{c_5}{d^n \theta^n} \int_{d(x,x_1)=d\theta} |g(x) - g(x_1)| + c_6 \int_{B(x_1,d\theta)} \frac{|\Delta g(x)|}{d(x,x_1)^{n-1}} \tag{2.8.20}$$

$$+ c_7 \Lambda^2 \int_{B(x_1,d\theta)} \frac{|g(x) - g(x_1)|}{d(x,x_1)^{n-1}}$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III} .$$

By (2.8.12)

$$I \le \frac{c_8 \Lambda^2 R^2}{d\theta} ,$$

by (2.8.17)

$$II \le c_9 \, d\theta(\|Ric\| + |dg|^2) \le c_9 \|Ric\| \, d\theta + 2c_9 \, d\theta \, \frac{\mu^2}{d^2} ,$$

if we choose $\theta \le \frac{1}{2}$, since then for $x \in B(x_1, d\theta)$ $d(x, \partial B(p,R_0)) \ge d(1-\theta) \ge \frac{d}{2}$ and by (2.8.12) again

$$III \le c_{10} \Lambda^4 \, R^2 \, d\theta .$$

Hence

(2.8.21) $$\mu \le \frac{1}{\theta} \, (c_8 \Lambda^2 R^2 + c_9 \|Ric\| \, d^2\theta^2 + c_{10} \Lambda^4 R^2 \, d^2\theta^2) + 2c_9 \theta \mu^2$$

$$=: \frac{1}{2\theta} \, a\Lambda^2 R^2 + b\theta \, \frac{\mu^2}{2} .$$

a and b depend only on n and ΛR_0 (for $R \le R_0$) .

We now choose R_0 so small that

(2.8.22) $$ab\Lambda^2 R_0^2 < 1 .$$

Then (2.8.21) implies that for each $\theta \le \frac{1}{2}$ either

$$\mu \le \frac{1 - \sqrt{1 - ab\Lambda^2 R^2}}{b\theta}$$

or

$$\mu \ge \frac{1 + \sqrt{1 - ab\Lambda^2 R^2}}{b\theta} \ge 2 \, \frac{1 + \sqrt{1 - ab\Lambda^2 R^2}}{b}$$

$$=: \mu_0 .$$

On the other hand, for each $\mu_1 > \mu_0$ there is some $\theta_1 < \frac{1}{2}$ with

$$\frac{1 - \sqrt{1 - ab\Lambda^2 R^2}}{b\theta_1} < \mu_1 < \frac{1 + \sqrt{1 - ab\Lambda^2 R^2}}{b\theta_1} .$$

Hence the second possibility cannot hold for any $\theta \le \frac{1}{2}$, and the first one therefore is valid for each $\theta \le \frac{1}{2}$, in particular for $\theta = \frac{1}{2}$, and

$$\mu \le 2a\Lambda^2 R^2 .$$

(2.8.15) then follows from the definition of μ.

q.e.d.

Lemmata 2.8.1 and 2.8.2 now imply in conjunction with linear elliptic theory, that dg^{ij} is Hölder continuous on balls $B(p,R)$, $R < R_0$ with any exponent $\alpha \in (0,1)$. We only have to observe that the Laplace-Beltrami operator, written in harmonic (or almost linear) coordinates, now is a divergence type elliptic operator with C^1-coefficients while the right-hand side of (2.8.14) is bounded since the Christoffel symbols can be expressed in terms of dg^{ik}. The corresponding estimates for the Green's functions of Δ can be found in [GW]. The important point is that even the Hölder norm of dg^{ik} for harmonic coordinates depends only on the dimension, the injectivity radius, and curvature bounds, but does not involve any curvature derivatives.

We want to present a simple proof of this result for $\alpha = \frac{2}{3}$, using almost linear functions.

Let us first define the notion of Hölder continuity in a way which is invariant under renormalizations. A map $f : B(p,R) \to Y$ is called Hölder continuous with exponent α, if for all $x,y \in B(p,R)$

$$d(f(x), f(y) \le \text{const. } R^{1-\alpha}\, d(x,y)^{\alpha} .$$

Similarly, the k-th derivative of f is Hölder continuous, if

$$|D^k f(x) - D^k f(y)| \le \text{const. } R^{1-(k+\alpha)}\, d(x,y)^{\alpha} .$$

THEOREM 2.8.2 *Let* $p \in X$. *There exists* $R_0 > 0$, *depending solely on the injectivity radius of* p, *the dimension* n *of the considered manifold* X *and bounds for the sectional curvature on* $B(p,R_0)$ *with the property that for*

$R \le R_0$ *there exist harmonic coordinates on* $B(p,R)$ *the metric tensor* $g = (g_{ij})$ *of which satisfies on each ball* $B(p,(1-\delta)R)$

$$(2.8.23) \qquad |dg|_{C^{2/3}} \le \frac{c(\Lambda R_0,n)}{\delta^2} \Lambda^2 R^2 .$$

In particular, the Hölder norms of the corresponding Christoffel symbols are bounded in terms of ΛR_0 *and* n .

Proof Let x be a basepoint, $U = (u^1,\ldots,u^n)$ be an orthonormal base of $T_x X$, and denote by $L_x(z) = (\ell_x^1(z),\ldots,\ell_x^n(z))$ the corresponding vector valued almost linear function. Finally, put

$$b_x(z) = L_x(z) \cdot d(x,z)^{-n} .$$

We now want to estimate $|\text{grad } v(x) - \text{grad } v(y)|$ for $v(z) = g^{ij}(z)$. The claim then follows from (2.8.12) and Lemma 2.8.2.

Let $x,y \in B(p,R)$, m be the average of x,y , i.e. that point on the geodesic arc joining x and y with equal distance to both of them, and $\rho = C \cdot d(x,y)^{1/3} \cdot R^{2/3}$, where C will be chosen later.

As in the proof of Lemma 2.7.5, we obtain

$$\begin{aligned}
(2.8.24) \quad \omega_n |\text{grad } v(x) - \text{grad } v(y)| &\le \lim_{\varepsilon\to 0} \Big| \int_{B(m,\rho)\setminus B(m,\varepsilon)} \{(v(z) - v(x))\, \Delta b_x(z) \\
&\quad - (v(z) - v(y))\, \Delta b_y(z)\}\, dz \Big| + \Big| \int_{B(m,\rho)} (b_x(z) - b_y(z))\, \Delta v(z)\, dz \Big| \\
&\quad + \Big| \int_{\partial B(m,\rho)} (b_x(z) - b_y(z)) \langle \text{grad } v(z), d\vec{O} \rangle \Big| \\
&\quad + \Big| \int_{\partial B(m,\rho)} \{(v(z) - v(x)) \langle \text{grad } b_x(z), d\vec{O} \rangle - (v(z) - v(y)) \\
&\qquad \cdot \langle \text{grad } b_y(z), d\vec{O} \rangle\} \Big| \\
&=: I + II + III + IV .
\end{aligned}$$

First of all, by Lemmata 2.7.4 and 2.8.2

$$I \le \frac{c_{11}\Lambda^2 R^2}{\delta R} \Lambda^2 \rho^2 . \tag{2.8.25}$$

(Note that we do not exploit the difference $\Delta b_x - \Delta b_y$ in I, since we control only the absolute value of Δb, as we do not want to admit dependence of the estimates on curvature derivatives.)

Choosing w.l.o.g. x and y close together and C suitably, we can assume

$$5d(x,y) \le \rho = C\cdot d(x,y)^{1/3} R^{2/3} \le \delta R . \tag{2.8.26}$$

We then split II into

$$\int_{B(m,\rho)} = \int_{B(m,5d(x,y))} + \int_{B(m,\rho)\setminus B(m,5d(x,y))} \tag{2.8.27}$$

$$= II_a + II_b .$$

(2.8.15), (2,8.17) and the definition of b give

$$II_a \le c_{11}\Lambda^2 \; d(x,y)\left(1 + \frac{\Lambda^2 R^2}{\delta R}\right)^2 . \tag{2.8.28}$$

For II_b, we write

$$b_x(z) - b_y(z) = \frac{\ell_x(z) - \ell_y(z)}{d(x,z)^n} + \ell_y(z)\left(\frac{1}{d(x,z)^n} - \frac{1}{d(y,z)^n}\right) \tag{2.8.29}$$

and use Lemma 2.6.2 and (2.8.15), (2.8.17) to get

$$II_b \le \frac{c_{12}}{1-\alpha} \frac{\Lambda^2 R^2}{(\delta R)^2} d(x,y)^\alpha \rho^{1-\alpha} \tag{2.8.30}$$

taking $d(x,z)$, $d(y,z) \ge d(x,y)$ on $B(m,\rho)\setminus B(m,5d(x,y))$ into account.

Similarly, we get

$$III \le \frac{c_{13}\Lambda^2 R^2}{\delta R} d(x,y)\cdot\rho^{-1} . \tag{2.8.31}$$

Finally, we write the integrand of IV as

$$(v(z) - v(x))(\text{grad } b_x z - \text{grad } b_y z) - (v(x) - v(y)) \text{ grad } b_y(z) .$$

If we use the splitting of (2.8.29), then the only nontrivial expression to estimate is

$$|\operatorname{grad} \ell_x(z) - \operatorname{grad} \ell_y(z)| .$$

For this purpose, let $\gamma(t)$ be the geodesic arc from x to y and let P_t be the parallel transport along geodesics emanating from $\gamma(t)$. Then from (2.6.2)

$$|d\ell_{\gamma(t)}(z) - P_t \circ u(t)(z)| \le c_{14}\, d(\gamma(t),z)^2 .$$

Moreover

$$|P_t \circ u(t)(z) - P_\tau \circ u(\tau)(z)| \le c_{15}\, d(\gamma(t),z) \cdot d(\gamma(t),\gamma(\tau)) .$$

Thus

$$|\operatorname{grad} \ell_x(z) - \operatorname{grad} \ell_y(z)| \le c_{16}\, \rho^2 \qquad \text{for} \quad z \in \partial B(m,\rho) .$$

Altogether, we get

$$\text{IV} \le \frac{c_{17}\Lambda^2 R^2}{\delta R} (\Lambda^2 \rho^2 + d(x,y) \cdot \rho^{-1}) . \tag{2.8.32}$$

Putting everything together, and using $\rho = C d(x,y)^{1/3} R^{2/3}$

$$\text{I} + \text{II} + \text{III} + \text{IV} \le \frac{c_{18}\Lambda^2 R^2}{\delta^2} \left(\Lambda^2 R^2 C^2 + \frac{1}{C}\right) R^{-5/3}\, d(x,y)^{2/3} .$$

This is just the right power of R, since $\operatorname{grad} v$ contains the second derivatives of the coordinate functions h^i. This finishes the proof.

q.e.d.

Moreover, we note that once having proved Thm. 2.8.2 or Lemma 2.8.2, (2.8.14) in conjunction with linear elliptic theory implies

THEOREM 2.8.2 *Let* $R \le R_0$ *, where* R_0 *is chosen as in Thm. 2.8.2, and let* $g = (g_{ij})$ *be the metric tensor of the corresponding harmonic coordinates on* $B(p,R)$ *. If the Riemann curvature tensor on* $B(p,R)$ *is of class* C^k *or* $C^{k+\beta}$ $(k \in \mathbb{N},\ \beta \in (0,1)$ *, then* $g \in C^{k+1+\alpha}$ *(for every* $\alpha \in (0,1)$*) or* $g \in C^{k+2+\beta}$ *, resp., in the interior of* $B(p,R)$ *. The corresponding estimates* *depend in addition to the quantities mentioned in Thm. 2.8.2 on the* C^k *or* $C^{k+\beta}$*-norm, resp., of the curvature tensor.*

That harmonic coordinates possess best possible regularity properties was first pointed out by de Turck-Kazdan [dTK]. The explicit construction implying the existence of harmonic coordinates on fixed (curvature controlled) balls and the explicit estimates of this section are due to Jost-Karcher [JK1].

Finally, for later purposes, we need still another construction of coordinates. We want to introduce coordinates with curvature controlled Christoffel symbols in a neighbourhood of a point $q \in B(p,M)$, without using any information of the geometry outside $B(p,M)$. We suppose again that $M < \frac{\pi}{2\kappa}$, $M < i(p)$. In case $d(p,q) \le \frac{1}{2}M$, we taken an arbitrary orthonormal base $e_1,\dots e_n$ of T_qY $(B(p,M)\subset Y, \dim Y = n)$. If $d(p,q) > \frac{1}{2}M$, we choose $e_1,\dots,e_n$ in such a way that $\Sigma\, e_i$ is tangent to the geodesic from q to p. We now want to show that the geodesics $\exp_p(t\cdot e_i)$ stay inside $B(p,M)$ for $t \le t_0$, where $t_0 > 0$ can be estimated from below in terms of ω, M, and n. Indeed, by the Rauch-Toponogow Comparison Theorem (cf. [GKM], p.194f),

$$d(p, \exp_q t\cdot e_i) \le d^\omega(\tilde p, \exp_{\tilde q} t\cdot \tilde e_i) ,$$

where the right hand side is the distance in the comparison triangle in the plane of constant curvature $-\omega^2$, with $d^\omega(\tilde p,\tilde q) = d(p,q)$, $\tilde e_i$ having the same angle with the geodesic form $\tilde q$ to $\tilde p$ as e_i has with the geodesic from q to p. Consequently

$$\cosh(\omega d(p, \exp_q te_i)) \le \cosh \omega t \cdot \cosh(\omega d(p,q)) - \frac{1}{n} \sinh \omega t \cdot \sinh(\omega d(p,q))$$

$$\le \cosh \omega t \cdot \sinh \omega M - \frac{1}{n} \sinh \omega t \cdot \sinh \omega M ,$$

if $t \le \frac{1}{2}M$

$$\le \cosh \omega M ,$$

if $t \le \bar t$, say.

Then, for $t \le t_0 = \min(\bar{t}, \frac{1}{2}M)$, $d(p, \exp_q te_i) \le M$, and consequently the geodesics $\exp_q te_i$ stay inside $B(p,M)$ for $t \le t_0$.

LEMMA 2.8.4 *In a neighbourhood* $B(q,\tau) \cap B(p,M)$ *of* $q \in B(p,M)$ *, we can define local coordinates for which the Christoffel symbols are bounded in absolute value and* $\tau > 0$ *is bounded from below, both in terms of* ω *,* κ *,* n *,* M *only, via*

$$k_i(s) := \frac{1}{2t_0} (d^2(s, \exp_q t_0 e_i) - d^2(s,q)) .$$

Proof By Lemma 2.3.2

$$|D^2 k_i(s)| \le \frac{\omega M}{t_0} \coth \frac{\omega M}{2} \tag{2.8.33}$$

if $d(s,q) \le \frac{1}{2}M$, and

$$dk|_q \quad \text{is an isometry ,} \tag{2.8.34}$$

where $k = (k_1, \ldots, k_n) : B(p,M) \to \mathbb{R}^n$.

This easily implies a lower bound τ for the radius of the set on which k is injective. Furthermore, the Christoffel symbols are given by $D^2 k$ (cf. (2.8.2)), and hence the bound on the Christoffel symbols follows from (2.8.33).

q.e.d.

CHAPTER 3

THE HEAT FLOW METHOD

Existence, regularity, and uniqueness results for a nonpositively curved image

3.1 APPROACHES TO THE EXISTENCE AND REGULARITY QUESTION

There are four different approaches to the existence and regularity theory of harmonic maps available. The first one is the so-called heat flow method. In order to find a harmonic map homotopic to a given map $g : X \to Y$, one investigates the parabolic system

$$(3.1.1) \qquad \begin{aligned} \frac{\partial f(x,t)}{\partial t} &= \tau(f(x,t)) \qquad \text{for } x \in X \text{ and } t \geq 0 \\ f(x,0) &= g(x) \qquad \text{for } x \in X \end{aligned}$$

and one tries to prove that a solution of (3.1.1) exists for all $t \geq 0$ and that $f(\cdot,t)$ converges to a harmonic map f as $t \to \infty$. That means one tries to deform g into a homotopic harmonic map by an analogue of heat dispersion on manifolds. One should compare this method with the gradient flow descent method common in Morse theory. Whereas this method in our case would lead to an ordinary differential equation for a mapping from X into the Sobolev space $W_2^1(X,Y)$, i.e. an infinite dimensional target space, and follow the gradient lines of the energy functional, the heat flow method instead leads to a partial differential equation for a mapping from X into the finite dimensional manifold Y.

The second approach tries to establish regularity (and a-priori estimates) for weak solutions f of the elliptic system

$$(3.1.2) \qquad \int_X \{\gamma^{\alpha\beta}(x)\, g_{ij}(f(x))\, \frac{\partial f^i}{\partial x^\alpha} \frac{\partial \phi^j}{\partial x^\beta} - \gamma^{\alpha\beta}(x)\, \Gamma^i_{jk}(f(x))\, \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta}\, \phi^i\}\, dX$$

$$\text{for all } \phi \in W_2^1 \cap L^\infty .$$

In case this approach works, it implies in particular the regularity of an energy minimizing map and hence establishes the existence by a variational method. Alternatively, it can be used in conjunction with Leray-Schauder degree theory to assert the existence of a solution.

The third approach uses perturbed energy functionals which satisfy the compactness condition (C) of Palais-Smale. It can reprove the results obtained by the first approach, requiring much deeper estimates, however.

The fourth approach is the so-called method of partial regularity. It tries to characterize the possible singularities of energy minimizing maps and then to show that under appropriate conditions those singularities cannot exist and that an energy minimizing map is hence regular. In contrast to the other approaches, here the techniques so far are restricted to energy minimizing maps. Nevertheless, a posteriori this method comprises the results obtained by the other ones, since in all cases, where those methods work, one can prove a uniqueness result with the implication that in those cases any harmonic map is energy minimizing.

The first method was initiated by Eells-Sampson [ES], the second one by Hildebrandt-Kaul-Widman [HKW3], the third one by Uhlenbeck [U] and the fourth one by Schoen-Uhlenbeck [SU1] and Giaquinta-Giusti [GG1], [GG2].

In the present notes, we shall only develop the first two methods. We believe that our presentations have some advantages compared to the ones existing in the literature, as either the estimates are more precise, the proofs are shorter, or the arguments are more elementary. In particular, all proofs are self-contained.

We start with the heat-flow method in the present chapter, and shall develop the second one in chapter 4.

During all of chapter 3, the manifold X will be assumed to be *compact*.

The results of this chapter are due to Eells-Sampson [ES] and Hartman [Ht]. We shall also use some ideas as presented by von Wahl [vW] and Jost [J4]. Similar arguments were also known to R. Schoen.

3.2 SHORT TIME EXISTENCE

The parameter t will be referred to as time parameter, while $x \in X$ is the space variable, according to the thermodynamic interpretation of the present method.

We shall start by proving the existence of a solution of (3.1.1) for small time.

LEMMA 3.2.1 *Suppose* $g \in C^{2+\alpha}(X,Y)$. *Then there is some* $\varepsilon > 0$ *depending only on the geometry of* X *and* Y *and on* g *with the property that (3.1.1) has a solution* $f(x,t)$ *for* $0 \le t < \varepsilon$.

Proof The linearization of the operator $\left(\frac{\partial}{\partial t} - \tau\right)$ at f is computed in local coordinates as

$$L_f^i(\phi) = \frac{\partial \phi^i}{\partial t} - \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial \phi^i}{\partial x^\beta}\right) - \gamma^{\alpha\beta}\, \Gamma^i_{jk,\ell}\, \phi^\ell \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta}$$

$$- \gamma^{\alpha\beta}\, \Gamma^i_{jk}\left(\frac{\partial \phi^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} + \frac{\partial f^j}{\partial x^\alpha}\frac{\partial \phi^k}{\partial x^\beta}\right) .$$

By the theory of linear parabolic equations, the system

$$L_f(\phi) = h(x,t)$$

$$\phi(x,0) = g(x)$$

for given h of class C^α in x and t and g of class $C^{2+\alpha}$ in x , has a unique solution $\phi(x,t)$ of class $C^{2+\alpha}$ in x and $C^{1+\alpha}$ in t .

Moreover, the corresponding a-priori estimates[1)] imply that L_f is a continuous bijective linear operator between the corresponding mapping spaces. The implicit function theorem then implies Lemma 3.2.1.

q.e.d.

COROLLARY 3.2.1 *The set of* $T \in (0,\infty)$ *for which the solution of (3.1.1) exists for* $t \in [0,T]$ *is open.*

This follows by taking $f(\cdot,T)$ as initial values in Lemma 3.2.1.

q.e.d.

Note that in contrast to the results in the following sections, for the small time existence of the solution of (3.1.1) we do not have to require any curvature assumptions for Y .

3.3 ESTIMATES FOR THE ENERGY DENSITY OF THE HEAT FLOW

We first show that the energy $E(f(\cdot,t))$ is a decreasing function of t . For,

$$(3.3.1)\quad \frac{d}{dt} E(f(\cdot,t)) = \frac{d}{dt}\frac{1}{2}\int \langle df,df\rangle = \int \left\langle \frac{\partial}{\partial t} df, df\right\rangle = \int \left\langle d\frac{\partial}{\partial t} f, df\right\rangle$$
$$= -\int \left\langle \frac{\partial}{\partial t} f, \tau(f)\right\rangle = -\int \left|\frac{\partial}{\partial t} f\right|^2$$

since f satisfies the equation (3.1.1), i.e. $\frac{\partial}{\partial t} f = \tau(f)$.

It is also interesting to compute the second time derivative of $E(f(\cdot,t))$, although this formula is not needed in the sequel. As in (1.6.5), we compute

$$\frac{\partial}{\partial t}\left|\frac{\partial}{\partial t} f\right|^2 = \Delta\left|\frac{\partial}{\partial t} f\right|^2 - \left|\nabla \frac{\partial f}{\partial t}\right|^2 + \left\langle R^Y\left(df\cdot e_\alpha, \frac{\partial f}{\partial t}\right) df\cdot e_\alpha, \frac{\partial f}{\partial t}\right\rangle ,$$

and hence, since X is compact

---

1) Note that since X is compact, $g(X)$ is bounded in Y .

$$(3.3.2) \qquad \frac{d^2}{dt^2} E(f(\cdot,t)) = \int_X \left| \nabla \frac{\partial f}{\partial t} \right|^2 - \int_X \left\langle R^Y\left(df\cdot e_\alpha, \frac{\partial f}{\partial t}\right) df\cdot e_\alpha, \frac{\partial f}{\partial t} \right\rangle .$$

We note that, in case Y is nonpositively curved,

$$\frac{d^2}{dt^2} E(f(\cdot,t)) \geq 0 .$$

From now on, we shall assume for the rest of this chapter, that Y has *nonpositive sectional curvature.*

As in 1.6, we look at the energy density of $f(x,t)$

$$e(f) = \tfrac{1}{2} \gamma^{\alpha\beta}(x)\, g_{ij}(f(x,t)) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} .$$

If $f(x,t)$ is a solution of (3.1.1), the calculations of 1.6 imply

$$(3.3.3) \qquad \Delta e(f) - \frac{\partial}{\partial t} e(f) = |\nabla df|^2 + {}^X R_{\alpha\beta}(x) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} g_{ij} - \gamma^{\alpha\beta}\gamma^{\delta\eta}\, {}^Y R_{ikj\ell} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\delta} \frac{\partial f^j}{\partial x^\beta} \frac{\partial f^\ell}{\partial x^\eta} .$$

Since X is a compact manifold of class C^3, its Ricci tensor is bounded. Since we assume that Y has nonpositive sectional curvature, (3.3.3) implies

$$(3.3.4) \qquad \Delta e(f) - \frac{\partial}{\partial t} e(f) \geq -c\, e(f) .$$

The constant c may still depend on t, since as $t \to \infty$, the image of $f(x,t)$ may become unbounded since we did not assume so far that Y is compact. This does not matter, however, since we shall see in 3.5 that for any $T < \infty$ and $t \in [0,T]$, $f(x,t)$ remains in a bounded subset of Y, possibly depending on T.

We now want to use (3.3.4) to derive estimates for $e(f)$.

For a given point $m \in X$, we choose a ball $B(m,\rho)$ satisfying the assumptions of Lemma 2.3.2. We note that $\rho > 0$ can be chosen uniformly for $m \in X$, since X is compact.

Plugging (3.3.4) into (2.7.6) and using (2.7.7), we obtain

$$(3.3.5)\quad e(f)(m,t) \le c_1 \int_{B(m,\rho,t_0,t)} e(f)(x,\tau)(t-\tau)^{-\frac{1}{2}}\, r(x)^{-n+1}\, dx\, d\tau$$
$$+ \frac{c_n}{\rho^{n+2}} \int_{B(m,\rho,t_0,t)} e(f) + \frac{c_n}{\rho^{n+1}} \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} e(f)$$
$$+ (t-t_0)^{-n/2} \int_{B(m,\rho)} e(f)(x,t_0)\, dx\ .$$

Here, c_1 depends on n and Λ^2, a bound for the sectional curvature of X.

First of all, we observe that if $i(X) > 0$ is the injectivity radius of X, $\rho_0 = \min\left(i(X), \frac{\pi}{2\Lambda}\right)$, we can choose $\rho \in [\rho_0/2, \rho_0]$ with

$$(3.3.6)\quad \int_{\substack{r(x)=\rho \\ t_0\le\tau\le t}} e(f) \le \frac{2}{\rho} \int_{B(m,\rho,t_0,t)} e(f)\ .$$

We define

$$g_1(m,p,t) = t^{-\frac{1}{2}} \cdot d(m,p)^{-n+1}$$

$$g_k(m,p,t) = \int_{\substack{t_0\le\tau\le t \\ d(x,p)\le\rho}} g_{k-1}(m,x,t-\tau)\, g_1(x,p,\tau)\, dx\, d\tau$$

and choose $\rho = \rho(p)$ in the definition of g_k in such a way that (3.3.6) is satisfied for p instead of m. We observe that

$$g_k(m,p,t) \le c_2 (t-t_0)^{\frac{1}{2}}\, d(m,p)^{-n+k}$$

and hence g_k is bounded for $k > n$.

Thus, if we iterate (3.3.5), using (3.3.5) again for $e(f)(x,\tau)$ in the first integral in (3.3.5), we obtain after a finite number of steps

$$(3.3.7)\quad e(f)(m,t) \le c_3\, \rho^{-n-2} \int_{B(m,n\rho,t-n(t-t_0),t)} e(f)$$
$$+ c_4 (t-t_0)^{-n/2} \int_X e(f)(x,t-n(t-t_0))\, dx\ .$$

In order to locate the last integral at $t-n(t-t_0)$, we have used the fact that the energy decreases in time by (3.3.1).

Choosing $t_0 \geq 0$ in such a way that $t \geq n(t-t_0) \geq \varepsilon$ and using (3.3.1) again

$$e(f)(m,t) \leq c_5(t\rho^{-n-2} + \varepsilon^{-n/2}) \int_X e(f)(x,0)\, dx \, . \tag{3.3.8}$$

If we want to avoid the term with $\varepsilon^{-n/2}$, we can use (2.7.8) instead of (2.7.6) and obtain in a similar way

$$e(f)(m,t) \leq c_6\, \rho^{-2} \sup_{x\in X} e(f)(x,0) \, . \tag{3.3.9}$$

Namely, we then have the term

$$\int_{B(m,\rho)} e(f)(x,0)(t-t_0)^{-n/2} \exp\left(-\frac{r^2(x)}{4(t-t_0)}\right) dx$$

which is an approximate solution of the heat equation with initial values $e(f)(x,0)$, and we use that by the maximum principle the supremum over the space variables of a solution of the heat equation is nonincreasing in time.

We collect these estimates in

LEMMA 3.3.1 *Suppose* f *is a solution of (3.1.1) on* $[0,t]$. *If* $t \geq \varepsilon$ *and* $0 < R < \min\left(i(x), \frac{\pi}{2\Lambda}\right)$

$$e(f)(m,t) \leq c_5(tR^{-n-2} + \varepsilon^{-n/2}) \int_X e(f)(x,0)\, dx$$

Furthermore, for any $t_0 < t$, *in particular* $t_0 = 0$,

$$e(f)(m,t) \leq c_6\, R^{-2} \sup_{x\in X} e(f)(x,t_0) \, .$$

3.4 THE STABILITY LEMMA OF HARTMAN

We now let $f(x,t,s)$ be a smooth family of solutions of (3.1.1) depending on a parameter s and having initial values $f(x,0,s) = g(x,s)$, $0 \le s \le s_0$.

LEMMA 3.4.1 (Hartman [Ht]) *Suppose again, that* Y *has nonpositive sectional curvature.*

For every $s \in [0,s_0]$

$$\sup_{x\in X}\left(g_{ij}(f(x,t,s)) \cdot \frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right)$$

is nonincreasing in t . *Hence also*

$$\sup_{x\in X, s\in[0,s_0]}\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right)$$

is a nonincreasing function of t .

Proof As in 1.6, one calculates in normal coordinates

$$(3.4.1)\quad \left(\Delta - \frac{\partial}{\partial t}\right)\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right) = \frac{\partial^2 f^i}{\partial x^\alpha \partial s}\frac{\partial^2 f^j}{\partial x^\alpha \partial s} - R_{ikj\ell}\frac{\partial f^i}{\partial s}\frac{\partial f^k}{\partial x^\alpha}\frac{\partial f^j}{\partial s}\frac{\partial f^\ell}{\partial x^\alpha},$$

and since Y has nonpositive sectional curvature, hence

$$\left(\Delta - \frac{\partial}{\partial t}\right)\left(g_{ij}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial s}\right) \ge 0 .$$

The lemma then follows from the maximum principle for parabolic equations.

q.e.d.

We now assume that f_1 and f_2 are smooth homotopic maps from X to Y , and $h : X \times [0,1] \to Y$ is a smooth homotopy with $h(x,0) = f_1(x)$, $h(x,1) = f_2(x)$.

Since $h(x,s)$ is smooth in x and s , the curve $h(x,\cdot)$ connecting $f_1(x)$ and $f_2(x)$ depends smoothly on x . We let $g(x,\cdot)$ be the geodesic from $f_1(x)$ to $f_2(x)$ which is homotopic to $h(x,\cdot)$ and parametrized proportionally to arc length. Since Y is nonpositively curved, this geodesic arc is unique and hence depends smoothly on x . We define $\tilde{d}(f_1(x),f_2(x))$ to be the length of this geodesic arc.

We then put $f(x,0,s) = g(x,s)$.

COROLLARY 3.4.1 *Suppose, as before, that* Y *is nonpositively curved. Assume that the solution* $f(x,t,s)$ *of (3.1.1) exists for all* $s \in [0,1]$ *and* $t \in [0,T]$. *Then*

$$\sup_{x\in X} \tilde{d}(f(x,t,0),\ f(x,t,1))$$

is nonincreasing in t *for* $t \in [0,T]$.

Proof By construction, at $t = 0$

$$\sup_{x\in X, s\in[0,1]} \left\{g_{ij} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s}\right\} = \sup_{x\in X} \tilde{d}^2(g(x,0),\ g(x,1)) \ .$$

On the other hand, for any $t \in [0,T]$

$$\tilde{d}^2(f(x,t,0),\ f(x,t,1)) \le \sup_{s\in[0,1]} g_{ij}(f(x,t,s)) \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s}$$

since $f(x,t,\cdot)$ is a curve joining $f(x,t,0)$ and $f(x,t,1)$ in the homotopy class chosen for the definition of $\tilde{d}$. The claim then follows from Lemma 3.4.1.

q.e.d.

3.5 A BOUND FOR THE TIME DERIVATIVE

Our first application of Lemma 3.4.1 will be a bound for the time derivative of a solution of (3.1.1).

LEMMA 3.5.1 *Suppose* $f(x,t)$ *solves (3.1.1) for* $t \in [0,T)$ *and* Y *has non-positive sectional curvature. Then for all* $t \in [0,T)$ *and* $x \in X$

$$\left|\frac{\partial f(x,t)}{\partial t}\right| \le \sup_{x\in X} \left|\frac{\partial}{\partial t} f(x,0)\right| \ . \tag{3.5.1}$$

Proof This follows by putting

$$f(x,t,s) = f(x,t+s)$$

and applying Lemma 3.4.1 at $s = 0$.

q.e.d.

LEMMA 3.5.2 *Suppose* $f(x,t)$ *solves (3.1.1) for* $t \in [0,T)$ *and* Y *has nonpositive sectional curvature. Then for every* $\alpha \in (0,1)$

$$|f(\cdot,t)|_{C^{2+\alpha}(X,Y)} + \left|\frac{\partial f}{\partial t}(\cdot,t)\right|_{C^{\alpha}(X,Y)} \le c_7 . \tag{3.5.2}$$

c_7 *depends on* α *,* T *(only in case* $f(\cdot,t)$ *becomes unbounded, but anyway it will be finite for any finite* T *), the initial values* $g(x) = f(x,0)$ *, and the geometry of* X *and* Y *, or more precisely on curvature bounds, injectivity radii and dimensions of* X *and* Y .

Proof We write (3.1.1) in the following way

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta}\right) = -\gamma^{\alpha\beta}\,\Gamma^i_{jk}\,\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} + \frac{\partial f^i}{\partial t} . \tag{3.5.3}$$

If we centre our coordinate charts on X and Y at m and $f(m,t_0)$, then for a fixed neighbourhood $B(m,\rho) \times [t_0,t_1]$ of (m,t_0), $f(x,t)$ will stay inside this coordinate chart in Y by Lemmata 3.3.1 and 3.5.1. Furthermore, those lemmata also imply that the right hand side of (3.5.3) is bounded. This first implies a bound for $|f(\cdot,t)|_{C^{1+\alpha}(X,Y)}$ by elliptic regularity theory. But then the right hand side of (3.1.1) is bounded in $C^\alpha(X,Y)$, and (3.5.2) now follows from parabolic regularity theory.

The statements concerning the dependence of the estimates on the geometry follow from the results of section 2.8, where we constructed local coordinates for which the Hölder constants of the Christoffel symbols are bounded in terms of the quantities appearing in the statement of the lemma (cf. Thm. 2.8.2).

q.e.d.

LEMMA 3.5.3 *The solution of (3.1.1) exists for all* $t \in [0,\infty)$ *, if* Y *has nonpositive sectional curvature.*

Proof Lemma 3.2.1 shows that the set of $T \in [0,\infty)$ with the property that the solution exists for all $t \in [0,T]$ is open and nonempty, while Lemma 3.5.2 implies that it is also closed.

q.e.d.

3.6 GLOBAL EXISTENCE AND CONVERGENCE TO A HARMONIC MAP (THEOREM OF EELLS-SAMPSON)

We assume now, that $f(x,t)$ remains in a compact subset of Y for all t . This is trivially the case, if Y itself is compact.

If we use the energy decay formula (3.3.1), namely

$$\frac{\partial}{\partial t} E(f(\cdot,t)) = - \int_X \left|\frac{\partial f(x,t)}{\partial t}\right|^2 dX ,$$

observe that $E(f(\cdot,t))$ is by definition always nonnegative, and use the time independent C^{α}-bound for $\left|\frac{\partial f}{\partial t}\right|$, we obtain

LEMMA 3.6.1 *If* $f(x,t)$ *remains in a bounded subset of* Y *, then there exists a sequence* (t_n) *,* $t_n \to \infty$ *as* $n \to \infty$ *, for which* $\frac{\partial f}{\partial t}(x,t_n)$ *converges to zero uniformly in* $x \in X$ *as* $n \to \infty$.

Now using the $C^{2+\alpha}$-bounds for $f(\cdot,t)$ of Lemma 3.5.2, we can assume, by possibly passing to a subsequence, that $f(x,t_n)$ converges uniformly to a harmonic map $f(x)$ as $t_n \to \infty$. In Cor. 3.4.1 which we may apply because of Lemma 3.5.3, we then put

$$g(x,0) = f(x,0,0) = f(x,t_n)$$

$$g(x,s_0) = f(x,0,s_0) = f(x)$$

By uniform convergence, some $f(\cdot,t_n)$ (and hence all, since $f(x,t)$ is continuous in t) are homotopic to f .

Since $f(x)$ as a harmonic map is a time independent solution of (3.1.1), $f(x,t,s_0) = f(x)$ for all t . Cor. 3.4.1 then implies

$$d(f(x,t_n+t),\ f(x)) \leq d(f(x,t_n),\ f(x)) \qquad \text{for all} \quad t \geq 0 .$$

Hence it follows that the selection of the subsequence is not necessary and that $f(x,t)$ uniformly converges to $f(x)$ as $t \to \infty$.

We thus have proved the existence theorem of Eells-Sampson [ES] with the improvements by Hartman [Ht].

THEOREM 3.6.1 *Suppose* Y *is nonpositively curved. Then the solution of (3.1.1) exists for all* $t \in [0,\infty)$. *If the solution remains in a bounded subset of* Y *, in particular if* Y *is compact, then it converges uniformly to a harmonic map as* $t \to \infty$. *In particular, any map* $g \in C^{2+\alpha}(X,Y)$ *is homotopic to a harmonic map.*

Remarks 1) The result also holds, if we merely assume $g \in C^0$. A suitable modification of Lemma 3.2.1 pertains to this case, and we choose some $t_0 \in (0,\varepsilon)$, where ε is the time-range of Lemma 3.2.1. Then $f(x,t_0)$ is of class $C^{2+\alpha}$ in x and can be chosen as new initial values for the heat flow, and we apply the arguments of the preceding sections to these initial values.

2) If we take one branch of the curve $y = \frac{1}{x}$ and rotate it around the x-axis, we obtain a negatively curved surface of revolution. The image of a point on $y = \frac{1}{x}$ under this rotation yields a closed homotopically nontrivial curve which is not homotopic to any closed geodesic. It is not difficult to see that as $t \to \infty$ the solution of the heat equation with those initial values will disappear at infinity, not converging to anything. From this we see that the hypothesis in Thm. 3.6.1 that the solution remains in a bounded set is necessary for the existence of a harmonic map.

On the other hand, noncompactness of the target space does not inevitably prevent the solution of the heat flow from converging to a harmonic map as is seen by rotating the curve $y = x^2 + 1$ instead of $y = \frac{1}{x}$ around the x-axis.

3.7 ESTIMATES IN THE ELLIPTIC CASE

We now want to derive estimates for a harmonic map $f : X \to Y$. Since Y is nonpositively curved, (1.6.5) implies

$$\Delta e(f) \geq -c\, e(f) \,. \tag{3.7.1}$$

For simplicity, we assume $n = \dim X \geq 3$. We put $\rho_0 = \min\left(i(X), \frac{\pi}{2\Lambda}\right)$. By a suitable choice of $\rho \in (\frac{1}{2}\rho_0, \rho_0)$, (3.7.1) in conjunction with the representation formula (2.7.5) yields

$$e(f)(m) \leq \frac{c_8}{\rho^2} \int_{B(m,\rho)} \frac{e(f)(x)}{r(x)^{n-2}}\, dx \,. \tag{3.7.2}$$

Iteration of (3.7.2) yields as in 3.3

$$e(f)(m) \leq \frac{c_9}{\rho^2} \int_{B(m,\frac{n}{2}\rho)} e(f)(x)\, dx \,. \tag{3.7.3}$$

THEOREM 3.7.1 *If* $f : X \to Y$ *is harmonic,* X *compact and* Y *nonpositively curved,*

$$|f|_{C^{2+\alpha}(X,Y)} \leq c_9 \,,$$

where c_9 *depends on the energy* $E(f)$ *and on curvature bounds, injectivity radii and dimensions of* X *and* Y.

Proof We again look at the equation

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} f^i\right) = -\gamma^{\alpha\beta}\, \Gamma^i_{jk} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \,.$$

(3.7.3) implies that the right hand side is bounded and that for every $m \in X$, a uniform neighbourhood $B(m,\rho)$ is mapped into the same coordinate chart on the image. Elliptic regularity theory implies $f \in C^{1+\alpha}$, which in turn implies that the right hand side is of class C^{α} and hence $f \in C^{2+\alpha}$.

The assertions about the dependence of c_9 on the geometry of X and Y follow, if we choose harmonic coordinates at m and $f(m)$. For those coordinates, the Christoffel symbols have the required regularity properties, as is shown in 2.8 (cf. Thm. 2.8.2).

q.e.d.

3.8 THE UNIQUENESS RESULTS OF HARTMAN

In this section, we shall be concerned with uniqueness properties of harmonic maps into nonpositively curved manifolds.

THEOREM 3.8.1 (Hartman [Ht]) *Let* $f_1(x)$, $f_2(x)$ *be two homotopic harmonic maps from* X *into the nonpositively curved manifold* Y. *For fixed* x, *let* $f(x,s)$ *be the unique geodesic from* $f_1(x)$ *to* $f_2(x)$ *in the homotopy class determined by the homotopy between* f_1 *and* f_2, *and let the parameter* $s \in [0,1]$ *be proportional to arc length.*

Then, for every $s \in [0,1]$, $f(\cdot,s)$ *is a harmonic map with* $E(f(\cdot,s)) = E(f_1) = E(f_2)$. *Furthermore, the length of the geodesic* $f(x,\cdot)$ *is independent of* x.

Hence any two harmonic maps can be joined by a parallel family of harmonic maps with equal energy.

Proof We let $f(x,t,s)$ be the solution of (3.1.1) with initial values $f(x,0,s) = f(x,s)$. $f(x,t,s)$ exists for all time by Lemma 3.5.3.

By Cor. 3.4.1, for any $s \in [0,1]$ and $t \in (0,\infty)$

$$(3.8.1) \qquad \sup_{x\in X} \tilde{d}(f(x,t,s),\ f_1(x)) \leq \sup_{x\in X} \tilde{d}(f(x,s),\ f_1(x))$$

$$\leq \sup_{x\in X} d(f_2(x),\ f_1(x)) \ .$$

Hence, $f(x,t,s)$ stays in a bounded subset of Y as $t \to \infty$.

Thm. 3.6.1 implies that $f(x,t,s)$ converges to a harmonic map $f_0(x,s)$ as $t \to \infty$.

We choose $x_0 \in X$ with

$$\tilde{d}(f_2(x_0),\ f_1(x_0)) = \sup_{x\in X} \tilde{d}(f_2(x),\ f_1(x))$$

and by construction therefore

$$\tilde{d}(f(x_0,s),\ f_1(x_0)) = \sup_{x\in X} \tilde{d}(f(x,s),\ f_1(x)) \quad \text{for all } s \ .$$

From (3.8.1)

$$(3.8.2) \qquad \tilde{d}(f(x_0,t,s),\ f_1(x_0)) \leq \tilde{d}(f(x_0,s),\ f_1(x_0))$$

and similarly

$$(3.8.3) \qquad \tilde{d}(f(x_0,t,s), f_2(x_0)) \leq \tilde{d}(f(x_0,s),\ f_2(x_0)) \ .$$

Note that all distances are measured by the length of that geodesic which is mentioned in the statement of the theorem.

Then (3.8.2) and (3.8.3) imply

$$(3.8.4) \qquad f(x_0,t,s) = f_0(x_o,s) = f(x_0,s) \quad \text{for all } s \ .$$

We now look at

$$e_s(f)(x,t,s) = g_{ij}(f(x,t,s)) \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s} \ .$$

By Lemma 3.4.1

$$(3.8.5) \quad \sup_{x\in X} e_s(f)(x,t,s) \leq \sup_{x\in X} e_s(f)(x,0,s) \qquad \text{for every } s \in [0,1],\ t \in (0,\infty)$$

On the other hand, from (3.8.4)

$$(3.8.6) \qquad e_s(f)(x_0,t,s) = e_s(f)(x_0,0,s) = \sup_{x\in X} e_s(f)(x,0,s) \ .$$

Hence for all t, the supremum in (3.8.5) is attained at $x = x_0$ and is independent of t. Since by (3.4.1)

$$(3.8.7) \qquad \left(\Delta - \frac{\partial}{\partial t}\right) e_s(f) \geq 0 \ ,$$

the strong maximum principle implies that $e_s(f)(x,t,s)$ is independent of x and t, i.e.

$$e_s(f)(x,t,s) = e_s(f)(x_0,0,s) \qquad \text{for all } s \ .$$

Since s is the arc length parameter on the geodesic $f(x_0,\cdot)$, $e_s(f)(x_0,0,s)$ and hence $e_s(f)(x,t,s)$ is also independent of s. Thus for every x and t, $f(x,t,\cdot)$ is a curve of equal length from $f_1(x)$ to $f_2(x)$ parametrized proportionally to arc length. Since $f(x,0,\cdot)$ was a minimal geodesic, all $f(x,t,\cdot)$ are minimal geodesics and independent of t. In particular $f(x,t,s)$ is time independent for every s, and hence $f(x,0,s) = f(x,s)$ is harmonic, since $f(x,t,s)$ solves (3.1.1).

Returning to (3.4.1), since $g_{ij} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial s}$ is constant and Y is nonpositively curved,

$$(3.8.8) \qquad \frac{\partial^2 f^i}{\partial x^\alpha \partial s} = 0$$

in normal coordinates, or in invariant notation

$$\nabla_{\frac{\partial}{\partial s}} \left(\frac{\partial f^i}{\partial x^\alpha} \frac{\partial}{\partial f^i}\right) = 0$$

where ∇ now is the covariant derivative in the bundle $f^{-1}(x,\cdot)TY$. This implies that the energy density

$$e(f)(x,s) = \gamma^{\alpha\beta}(x) \ g_{ij}(f,x,s)) \ \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta}$$

is independent of s.

In particular, all the harmonic maps $f(\cdot,s)$ have the same energy.

q.e.d.

THEOREM 3.8.2 (Hartman [Ht]) *If* Y *has negative sectional curvature, then a harmonic map* $f : X \to Y$ *is unique in its homotopy class, unless it is constant or maps* X *onto a closed geodesic. In the latter case, nonuniqueness can only occur by rotations of this geodesic.*

Proof In this case, we see from (3.4.1), that since

$$R_{ikj\ell} \frac{\partial f^i}{\partial s} \frac{\partial f^k}{\partial x^\alpha} \frac{\partial f^i}{\partial s} \frac{\partial f^\ell}{\partial x^\alpha} = 0 \tag{3.8.9}$$

by the previous proof, either $\frac{\partial f}{\partial s} \equiv 0$ which means that the family $f(\cdot,x)$ is constant in s and hence consists of a single member, i.e. the harmonic map is unique, or the image of T_xS under df is a one-dimensional subspace of $T_{f(x)}Y$. Furthermore, if the harmonic map is not unique, then $f(x,s)$ for any $x \in X$ is a geodesic arc by the construction of the preceding proof. (3.8.9) implies again that df maps T_xX onto the tangent direction of this geodesic. This easily implies that X is mapped onto this geodesic.

We now have to show that this geodesic arc extends to a closed geodesic which is covered by $f(X)$.

Since X is compact, $f(X)$ is closed and hence covers some geodesic arc γ. Suppose this arc has an endpoint $p = f(x)$ for some $x \in X$. We choose $q \in \gamma$ within the injectivity radius of p. Then $d^2(q,f(y))$ is a subharmonic function on a suitable neighbourhood of $x \in X$ (by Lemmata 2.3.2 and 1.7.1) and has a local maximum at x which is a contradiction, unless $f(y) \equiv p$ for $y \in X$. Thus, if f is not constant, it has to cover a closed geodesic.

q.e.d.

Different proofs of Thms. 3.8.1 and 3.8.2 were obtained by Schoen-Yau

[SY3]. They show that under the hypotheses of those theorems, after lifting to suitable covers, the squared distance between two homotopic harmonic maps is a well defined smooth subharmonic function, in case Y is nonpositively curved, from which the argument proceeds in a similar way as above.

3.9 THE DIRICHLET PROBLEM

One can also solve the Dirichlet problem for harmonic mappings into nonpositively curved manifolds.

THEOREM 3.9.1 (Hamilton [Hm]) *Suppose* X *is a compact manifold with nonempty boundary* ∂X *,* Y *is complete (without boundary) and has nonpositive sectional curvature. If* $g : X \to Y$ *is a continuous map, then the parabolic system*

$$(3.9.1) \qquad \begin{aligned} \frac{\partial f}{\partial t}(x,t) &= \tau(f)(x,t) && \text{for } (x,t) \in X \times (0,\infty) \\ f(x,0) &= g(x) && \text{for } x \in X \\ f(y,t) &= g(y) && \text{for } y \in \partial X \end{aligned}$$

has a smooth solution $f(x,t)$ *for all* $t \to (0,\infty)$. *As* $t \in \infty$ *,* $f(x,t)$ *converges to the unique harmonic map homotopic to* g *with the same boundary values as* g *on* ∂X .

Instead of extending the Hölder estimates of the previous section to the boundary, Hamilton develops an L^p-regularity theory for harmonic maps for the proof of Thm. 3.9.1. Since the boundary values are fixed, $f(x,t)$ remains always in a bounded subset of Y as $t \to \infty$, even if Y is noncompact.

In case Y is simply connected, a simpler proof of Thm. 3.9.1 was obtained by Hildebrandt-Kaul-Widman [HKW1].

As an application of the maximum principle, Hamilton also showed that convex sets provide barriers for solutions of the heat equation.

THEOREM 3.9.2 *Suppose* $C \subset Y$ *is a convex set and* f *solves (3.9.1). If* $g(X) \subset C$, *then* $f(x,t) \subset C$ *for all* $t \in [0,\infty)$.

3.10 AN OPEN QUESTION

A difficult open problem is to determine whether a solution of the heat equation (3.1.1) or (3.9.1) exists for all $t > 0$ without any curvature assumptions on Y .

Since there are manifolds X and Y and homotopy classes in $[X,Y]$ which do not contain harmonic representatives, as we shall see in chapter 5, even if the solution of the heat equation exists for all $t > 0$, in general it cannot converge uniformly to a harmonic map as $t \to \infty$.

There seems to be some indication that if one maps the unit ball D^n homotopically nontrivial onto the sphere S^n with constant boundary values, then the solution of (3.9.1) may cease to exist after a finite time, at least for large n .

Besides the results of this chapter and the case of "warped products" (cf. Lemaire [L3]), the existence of a solution of (3.9.1) for all time is only known in case $g(X)$ is contained in a ball $B(p,M) \subset Y$ which is disjoint to the cut locus of its centre p with $M < \frac{\pi}{2\kappa}$, where κ^2 is an upper bound for the sectional curvature on $B(p,M)$. This was carried out in [J4], combining some arguments of the present chapter with a result from elliptic regularity theory as shown in the next chapter and a stability inequality of (JäK2] analogous (but more difficult) to 3.4. A more general approach to long-time existence of solutions of nonlinear parabolic systems without divergence or variational structure by using stability inequalities was developed by von Wahl [vW]. For arbitrary Y , however, such stability inequalities do not hold, and von Wahl's approach is mainly aiming at applications different from harmonic maps.

Simon [Sm] showed that if f is a locally energy minimizing map between real analytic manifolds, then a solution of (3.1.1) exists for all time and converges to a harmonic map with the same energy as f, provided the initial values are already close to f in some high C^k-norm. It is not known whether the assumption that the manifolds involved are real analytic is necessary for Simon's theorem.

CHAPTER 4

REGULARITY OF WEAKLY HARMONIC MAPS

Regularity, existence, and uniqueness of solutions of the Dirichlet problem, if the image is contained in a convex ball

4.1 THE CONCEPT OF WEAK SOLUTIONS

We first want to discuss the concept of stationary points of the energy integral or of weak solutions of the corresponding Euler-Lagrange equations. In the present chapter, the image Y will always be covered by a single coordinate chart so that we can define the Sobolev space $W_2^1(\Omega,Y)$ unambiguously with the help of this chart, without having to use the Nash embedding theorem as in 1.3.

Ω will be an open bounded set in some Riemannian manifold with boundary $\partial\Omega$.

In the sequel, we shall use some of the notations of [EL4].

If $u \in W_2^1(\Omega,Y)$, then du is an almost everywhere on Ω defined 1-form with values in $u^{-1}TY$. The energy of u is

$$E(u) = \frac{1}{2}\int_\Omega \langle du,du\rangle \, d\Omega \ ,$$

where the scalar product is taken in $T^*\Omega \otimes u^{-1}TY$.

We let $\phi \in C_0(\bar{\Omega},u^{-1}TY)$ be a section along u which vanishes on $\partial\Omega$. This means $\phi(x) \in T_{u(x)}Y$. We want to construct a variation of u with tangent field ϕ .

Since we assume that Y is covered by a single coordinate chart, we can simply represent everything in those coordinates and denote the representations in these coordinates by $\tilde{}$ and define

$$\tilde{u}_t(x) = \tilde{u}(x) + t\tilde{\phi}(x) .$$

These coordinates also identify each tangent space T_uY with $\mathbb{R}^n$ ($n = \dim Y$) . Hence $\tilde{\phi}$ is a map from Ω into $\mathbb{R}^n$. This allows us to define $d\tilde{\phi}$ and hence via this identification also $d\phi$. (Note that it is not obvious how to define $d\phi$ intrinsically, since $\phi(x) \in T_{u(x)}Y$, and as u is not necessarily continuous, the base point of ϕ may vary in a noncontinuous way.) We then suppose that

$$\int_\Omega \langle d\phi, d\phi\rangle < \infty \tag{4.1.1}$$

and show that the Euler-Lagrange equations, if u is a critical point of E ,

$$\int\left\{\gamma^{\alpha\beta}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial \psi^i}{\partial x^\beta} - \gamma^{\alpha\beta}\,\Gamma^i_{kj}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta}\psi^i\right\}\sqrt{\gamma}\,dx = 0 \qquad \text{for } \psi \in W^1_2 \cap L^\infty(\Omega, \mathbb{R}^n) \quad (\psi|\partial\Omega = 0) \tag{4.1.2}$$

are equivalent to

$$\int \langle du, d\phi\rangle = 0 \qquad \text{for all bounded } \phi \text{ satisfying (4.1.1) and } \phi|\partial\Omega = 0 . \tag{4.1.3}$$

Proof Let

$$\phi = \phi^i(x)\frac{\partial}{\partial u^i} .$$

Then

$$d\phi = \nabla_{\frac{\partial}{\partial x^\alpha}}\left(\phi^i \frac{\partial}{\partial u^i}\right) dx^\alpha = \frac{\partial \phi^i}{\partial x^\alpha}\frac{\partial}{\partial u^i} + \phi^i\,\Gamma^k_{ij}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial}{\partial u^k} .$$

Hence

$$\langle du, d\phi\rangle = g_{ij}\,\gamma^{\alpha\beta}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial \phi^j}{\partial x^\beta} + \gamma^{\alpha\beta}\,g_{ik}\,\phi^\ell\,\Gamma^k_{\ell j}\frac{\partial u^j}{\partial x^\beta}\frac{\partial u^i}{\partial x^\alpha} . \tag{4.1.4}$$

On the other hand, we choose $\psi^i = g_{ij}\phi^j$ as a test vector in (4.1.2). Then the integrand of (4.1.2) becomes

$$\gamma^{\alpha\beta}\,g_{ij}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial \phi^j}{\partial x^\beta} + \gamma^{\alpha\beta}\,g_{kj,\ell}\frac{\partial u^\ell}{\partial x^\beta}\frac{\partial u^k}{\partial x^\alpha}\phi^j - \gamma^{\alpha\beta}\,g_{ij}\,\Gamma^i_{k\ell}\frac{\partial u^k}{\partial x^\alpha}\frac{\partial u^\ell}{\partial x^\beta}\phi^j$$

$$= \gamma^{\alpha\beta}\,g_{ij}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial \phi^j}{\partial x^\beta} + \gamma^{\alpha\beta}\,\tfrac{1}{2}(g_{kj,\ell} + g_{k\ell,j} - g_{j\ell,k})\frac{\partial u^\ell}{\partial x^\beta}\frac{\partial u^k}{\partial x^\alpha}\phi^j$$

which after changing some indices, is the same as (4.1.4).

q.e.d.

Remark If one wants to define $d\phi$ also if the image is not necessarily contained in a single coordinate chart, one can use the Nash embedding theorem as in section 1.3.

In the following sections, we want to provide conditions which ensure that a weak solution $u \in H_2^1 \cap L^\infty$ of (4.1.2) or (4.1.3) is continuous (which then in turn will also imply higher regularity of u).

We have already seen in 1.4 that $\frac{x}{|x|} : D^n \to S^{n-1}$ for $n \geq 3$ is a discontinuous weak solution. One might think that the discontinuity in this case is caused by the global topology of the image. We can however take the totally geodesic embedding $i : S^{n-1} \to S^n$ which maps S^{n-1} onto the equator of S^n. By Lemma 1.7.2, $i \circ \frac{x}{|x|}$ then is harmonic for $x \neq 0$ and hence weakly harmonic by the argument of 1.4. The image of $i \circ \frac{x}{|x|}$, however, is contained in a closed hemisphere, so that there is no longer a topological obstruction to regularity, and the discontinuity has to be caused by the geometry of the image.

As pointed out, in this case the image is contained in a geodesic ball of radius $\frac{\pi}{2}$ in S^n. In the following sections, we shall see that the radius $\frac{\pi}{2}$ is precisely the limiting case for regularity, i.e. that any weakly harmonic map with image contained in a geodesic ball of radius $< \frac{\pi}{2}$ actually is regular. (We shall of course consider more general image manifolds than only spheres.)

Finally, we remark that in many cases $i \circ \frac{x}{|x|}$ even minimizes energy w.r.t. its boundary values, as was demonstrated by Jäger-Kaul [JäK3] and Baldes [Ba].

In the following sections, we assume w.l.o.g. that the dimension n of the domain Ω is at least 3, because otherwise we can simply look at the map

$\tilde{u} : \Omega \times S^1 \to Y$, $\tilde{u}(x,t) = u(x)$ which satisfies the same assumptions as u .

4.2 A LEMMA OF GIAQUINTA-GIUSTI-HILDEBRANDT

The following lemma is due to Giaquinta-Hildebrandt [GH].

LEMMA 4.2.1 *Suppose* $u : \Omega \to Y$ *is weakly harmonic,* $f : Y \to \mathbb{R}$ *is strictly convex on* $u(\Omega)$. *Then for every ball* $B(x_0, 2R_0) \subset \Omega$

(4.2.1) $$\int_{B(x_0,R_0)} d(x,x_0)^{2-n} \, |du|^2 \le c_1 < \infty .$$

Furthermore, for any $\varepsilon > 0$ *and* $R_0 > 0$ *we can calculate* $R_1 > 0$ *, independent of* x_0 *and* u *with the property that for some* R *,* $R_1 \le R \le R_0$

(4.2.2) $$R^{2-n} \int_{B(x_0,R)} |du|^2 \le \varepsilon .$$

c_1 *and* R_1 *depend on the supremum of* f *and on a lower bound* $\lambda > 0$ *for the eigenvalues of its Hessian and on the geometry of* Ω *(curvature bounds, injectivity radius, dimension).*

Proof One idea is taken from [JK1], p.11, the other from [GG1], p.

We put $h = f \circ u$. By (1.7.2)

(4.2.3) $$\Delta h \ge \lambda |du|^2 .$$

Let $r(x) = d(x,x_0)$ and $g_\rho(x) = \min\left\{ r(x)^{2-n} - \rho^{2-n}, \left(\frac{\rho}{2}\right)^{2-n} - \rho^{2-n} \right\}$ on $B(x_0,\rho)$. Then

(4.2.4) $$\lambda \int_{B(x_0,\rho)} g_\rho(x) \, |du|^2 \le \int_{B(x_0,\rho)} g_\rho(x) \, \Delta h(x) \qquad \text{by (4.2.3)}$$

$$= - \int_{B(x_0,\rho)\setminus B(x_0,\rho/2)} \langle \mathrm{grad}\, g_\rho , \mathrm{grad}\, h \rangle$$

$$= \int_{B(x_0,\rho)\setminus B(x_0,\rho/2)} h \, \Delta g_\rho - \int_{\partial(B(x_0,\rho)\setminus B(x_0,\rho/2))} h \, \langle \mathrm{grad}\, g_\rho , \vec{dO} \rangle$$

$$\le c_2 \rho^2 + \frac{n-2}{\rho^{n-1}} \int_{\partial B(x_0,\rho)} h - \frac{n-2}{(\rho/2)^{n-1}} \int_{\partial B(x_0,\rho/2)} h$$

by Lemma 2.7.1, if ρ satisfies the assumptions of this lemma.

Now on $B(x_0,R_0\cdot 2^{-i})\backslash B(x_0,R_0\cdot 2^{-i-1})$, $r(x)^{2-n} \le (R_0\cdot 2^{-i-1})^{2-n}$ and thus

$$(4.2.5)\quad \int_{B(x_0,R_0)} r(x)^{2-n}\,|du|^2 \le \sum_{i=0}^{\infty} 2^{n-2}(R_0\cdot 2^{-i})^{2-n} \int_{B(x_0,R_0\cdot 2^{-i})} |du|^2 .$$

Since $2^{2-n}((\frac{1}{2})^{2-n}-1)\,r^{2-n} = r^{2-n}-(2r)^{2-n}$, from (4.2.5), defining $g_i = g_{R_0\cdot 2^{-i+1}}$

$$\int_{B(x_0,R_0)} r(x)^{2-n}\,|du|^2 \le c_n \sum_{i=0}^{\infty} \int_{B(x_0,R_0\cdot 2^{-i+1})} g_i\,|du|^2$$

where c_n depends only on n.

(4.2.4) then implies

$$(4.2.6)\quad \int_{B(x_0,R_0)} r(x)^{2-n}\,|du|^2 \le \frac{2c_2}{\lambda} c_n R_0^2 + \frac{c_n}{\lambda}\sum_{i=0}^{\infty}$$

$$\left\{(R_0\cdot 2^{-i+1})^{1-n}\int_{\partial B(x_0,R_0\cdot 2^{-i+1})} h - (R_0\cdot 2^{-i})^{1-n}\int_{\partial B(x_0,R_0\cdot 2^{-i})} h\right\}$$

$$=: c_3 R_0^2 + c_4 \sum_{i=0}^{\infty} (\mu_{i-1}-\mu_i) .$$

Hence

$$(4.2.7)\quad \int_{B(x_0,R_0)} r(x)^{2-n}\,|du|^2 \le c_3 R_0^2 + c_4\,\mu_0 .$$

This implies (4.2.1), noting that $\mu_0 \le \sup\limits_{u(\Omega)} f\cdot \mathrm{vol}\,\partial B(x_0,2R_0)R_0^{1-n}$. From (4.2.4)

$$(R_0\cdot 2^{-i})^{2-n}\int_{B(x_0,R_0\cdot 2^{-i})} |du|^2 \le c_5((R_0\cdot 2^{-i})^2 + \mu_{i-1}-\mu_i) .$$

We first choose i_0 so large that $c_5\left(R_0\cdot 2^{-i_0}\right)^2 \le \frac{\varepsilon}{2}$. For every $m \in \mathbb{N}$, we can find j, $i_0 \le j \le m+i_0$, with

$$\mu_j - \mu_{j+1} \le \frac{1}{m}\mu_{i_0} \le \frac{1}{m}(\mu_0+c_6R_0^2)$$

(for the last inequality, note that h is subharmonic and see the proof of (2.7.5)).

Hence choosing $m \geq \dfrac{2c_6(\mu_0 + c_6R_0^2)}{\varepsilon}$ and $R_1 = R_0 \cdot 2^{-i_0 - m}$, $R = R_0 \cdot 2^{-j}$, (4.2.2) follows.

q.e.d.

4.3 CHOICE OF A TEST FUNCTION

Suppose $B(x_0,2R) \subset \Omega$ for some $R > 0$. Let $\eta \in \mathrm{Lip}(B(x_0,2R))$ be the standard localizer, i.e. $\eta \equiv 1$ on $B(x_0,R)$, $|\nabla\eta| \leq \frac{c}{R}$, supp $\eta \subset\subset B(x_0,2R)$. Suppose there exists a strictly convex function on $u(B(x_0,2R))$, i.e. the assumptions of Lemma 4.2.1 are satisfied.

Suppose f is a C^2-function on $u(B(x_0,2R))$, and g is a Lipschitz function on $B(x_0,2R)$, so we can choose $\nabla f \cdot \eta \cdot g$ as a test vector ϕ in (4.1.3).

If e_α is an orthonormal frame on Ω , ω^α the dual coframe, then $du = u_{e_\alpha}\,\omega^\alpha$, and (4.1.3) yields

$$(4.3.1) \qquad 0 = \int_{B(x_0,2R)} \eta\, g \,\langle d(\nabla f),\, u_{e_\alpha}\,\omega^\alpha\rangle + \int_{B(x_0,2R)} g\, \eta_{e_\alpha}\, f(u)_{e_\alpha} + \int_{B(x_0,2R)} \eta\, g_{e_\alpha}\, f(u)_{e_\alpha} .$$

Now

$$\begin{aligned}
\langle d(\nabla f),\, u_{e_\alpha}\,\omega^\alpha\rangle_{T^*\Omega\otimes u^{-1}TY} &= \langle d(\nabla f)e_\alpha,\, u_{e_\alpha}\rangle_{u^{-1}TY} \\
&= \langle \nabla^{u^{-1}TY}_{e_\alpha}\nabla f,\, u_{e_\alpha}\rangle_{u^{-1}TY} \quad \text{by definition of } d \\
&= \langle \nabla^{Y}_{u_*(e_\alpha)}\nabla f,\, u_*(e_\alpha)\rangle_{TY} \\
&= D^2 f(du,du)
\end{aligned}$$

where $D^2 f$ is the Hessian of f .

Hence from (4.3.1)

$$(4.3.2)\quad \int_{B(x_0,2R)} g_{e_\alpha}(\eta f(u))_{e_\alpha} = -\int_{B(x_0,2R)} \eta g\, D^2 f(du,du) - \int_{B(x_0,2R)} g\eta_{e_\alpha} f(u)_{e_\alpha} + \int_{B(x_0,2R)} f(u)\, g_{e_\alpha} \eta_{e_\alpha} .$$

Remark If one is not familiar with the notation employed in the derivation of (4.3.2), one can alternatively insert the test vector ψ given by $\psi^i = \eta\cdot g \frac{\partial f}{\partial u^i}$ in (4.1.2) and carry out the calculations in local coordinates.

For $y \in B(x_0,R/2)$, $x \in B(x_0,2R)$, we now put

$$g(x) = g^\nu(x,y) = \min(d(x,y)^{2-n}, \nu) \qquad \text{for } \nu \in \mathbb{N} .$$

Writing $D(x_0,\nu,R) = \{x \in B(x_0,R) : d(x,y)^{2-n} < \nu\}$, (4.3.2) yields

$$(4.3.3)\quad \int_{D(x_0,\nu,R)} g^\nu(\cdot,y)_{e_\alpha} (\eta\, f(u))_{e_\alpha} = -\int_{B(x_0,2R)} \eta g^\nu(\cdot,y)\, D^2 f(du,du) - \int_{B(x_0,2R)} g^\nu(\cdot,y)\eta_{e_\alpha} f(u)_{e_\alpha} + \int_{D(x_0,\nu,R)} f(u)\, g^\nu(\cdot,y)_{e_\alpha} \eta_{e_\alpha} .$$

We write (4.3.3) as

$$I_\nu = II_\nu + III_\nu + IV_\nu .$$

Then with $D'(x_0,\nu,R) = \{x \in B(x_0,2R) : d(x,y)^{2-n} \geq \nu\}$

$$(4.3.4)\quad I_\nu = \int_{D(x_0,\nu,R)} \Delta(d(\cdot,y)^{2-n})\eta f(u) - \int_{\partial D'(x_0,\nu,R)} \eta f(u) \langle \operatorname{grad} g(\cdot,y), d\vec{O}\rangle ,$$

since η has compact support in $B(x_0,2R)$.

By (2.1.4), for sufficiently small R (depending on the injectivity radius and an upper curvature bound on Ω)

$$(4.3.5)\quad \int_{D(x_0,\nu,R)} \Delta(d(\cdot,y)^{2-n})\ \eta f(u) \leq c_7 R^2 < \varepsilon , \qquad \text{if } R \leq R_1(\varepsilon) ,$$

where c_7 depends on $n = \dim \Omega$, a curvature bound on Ω , and on $\sup f$.

If we choose for $f\circ u$ its Lebesgue representative, then we can find a subsequence of the ν's for which

$$(4.3.6) \quad \lim_{\nu\to\infty} \int_{\partial D'(x_0,\nu,R)} \eta f(u) \langle \text{grad } g(\cdot,y),\, d\vec{O}\rangle = -(n-2)\omega_n\, f(u(y))$$

(note that $\eta(y) = 1$, since $y \in B(x_0,R/2)$) .

Furthermore

$$(4.3.7) \qquad II_\nu = -\int_{B(x_0,R)} - \int_{T(x_0,R)} ,$$

where $\qquad T(x_0,R) := B(x_0,2R) \setminus B(x_0,R)$.

Since $y \in B(x_0,R/2)$, we infer from Lemma 4.2.1

$$(4.3.8) \qquad \int_{T(x_0,R)} \eta g^\nu(\cdot,y)\, D^2 f(du,du) < \varepsilon(n-2)\omega_n$$

for prescribed $\varepsilon > 0$ and some R , $R_2(\varepsilon) \le R \le R_1(\varepsilon)$, where $R_2 = R_2(\varepsilon) > 0$ can be calculated explicitly in terms of ε . It depends on the Hessian of f , but is independent of ν and y and u .

Since $\eta_{e_\alpha} \equiv 0$ outside $T(x_0,R)$

$$(4.3.9) \quad III_\nu \le \frac{c_8}{R}\int_{T(x_0,R)} g^\nu(\cdot,y)\,|du| \le \frac{c_9}{R}\left(\int_{T(x_0,R)} g^\nu(\cdot,y)\right)^{\frac{1}{2}} \left(\int_{T(x_0,R)} g^\nu(\cdot,y)\,|du|^2\right)^{\frac{1}{2}} \le (n-2)\omega_n\varepsilon ,$$

again for some suitable R which we can choose to be the same one as in (4.3.8). Here, the quantities depend on $|\nabla f|$.

In order to estimate IV_ν, let u_R be the mean value of u on $T(x_0,R)$. u_R can be defined with the help of our coordinates. We write

$$u_R = ⨍_{T(x_0,R)} u .$$

We now write $f(u) = f(u_R) + (f(u) - f(u_R))$. Similar as in (4.3.5) and

(4.3.6), we obtain

$$(4.3.10)\qquad \lim_{\nu\to\infty} IV_\nu \le (n-2)\omega_n\ f(u_R) + c_{10}\ R^2$$

$$+ \int_{T(x_0,R)} (f(u) - f(u_R))(d(\cdot,y)^{2-n})_{e_\alpha}\ \eta_{e_\alpha}\ .$$

Furthermore

$$\int_{T(x_0,R)} (f(u) - f(u_R))\ d(\cdot,y)^{2-n}_{e_\alpha}\ \eta_{e_\alpha} \le \frac{c_{11}}{R^n}\int_{T(x_0,R)} |f(u) - f(u_R)|$$

$$\le \frac{c_{12}}{R^n}\ R^{n/2}\ \sup|\nabla f|\ (\int_{T(x_0,R)} d(u,u_R)^2)^{\frac{1}{2}}$$

$$\le c_{13}\ R^{-n/2}\ (c_{14}\ R^2 \int_{T(x_0,R)} |du|^2)^{\frac{1}{2}}$$

by the Poincaré inequality, where c_{13} and c_{14} are independent of R.

Combined with (4.3.10), the preceding inequality yields

$$(4.3.11)\ \lim_{\nu\to\infty} IV_\nu \le (n-2)\omega_n\ f(u_R) + c_{10}\ R^2 + c_{15}(R^{2-n}\int_{T(x_0,R)} |du|^2\ )^{\frac{1}{2}}$$

$$\le (n-2)\omega_n\ f(u_R) + \varepsilon(n-2)\omega_n$$

(w.l.o.g. we can assume that (4.3.11) again is satisfied for the same R as in (4.3.8) and (4.3.9)).

From (4.3.3)-(4.3.11), we obtain for $y \in B(x_0,R/2)$, using Lebesgue's Theorem on dominated convergence

$$(4.3.12)\ f(u(y)) \le f(u_R) + 4\varepsilon - \{(n-2)\omega_n\}^{-1}\int_{B(x_0,R)} d(\cdot,y)^{2-n}\ D^2 f(du,du)$$

for some $R \ge R_3(\varepsilon)$ where $R_3(\varepsilon) > 0$ is independent of u and x_0.

4.4 AN ITERATION ARGUMENT. CONTINUITY OF WEAK SOLUTIONS

In this section, we want to use an iteration argument based on (4.3.12) to prove continuity of a weakly harmonic map with image in a convex ball. This result appeared explicitly for the first time in [HJW], but the method of proof in a somewhat different setting was already developed in [HW2]. The present proof (4.2-4.4) uses ideas of Wiegner, Hildebrandt, Widman, Kaul, Jost, Giaquinta, and Karcher, cf. [Wi], [HW2], [HKW3], [HJW], [GH], and [JK].

THEOREM 4.4.1 *Suppose* $u : \Omega \to B(p,M)$ *is weakly harmonic, that* $-\omega^2 \le K \le \kappa^2$ *are curvature bounds on* $B(p,M) \subset Y$ *,* $M < \min\left(\frac{\pi}{2\kappa}, i(p)\right)$ *, where* $i(p)$ *is the injectivity radius of* p *, and* $x_0 \in \Omega$ *.*

Then for each $\tau > 0$ *one can calculate* $\rho > 0$ *with*

$$\operatorname*{osc}_{B(x_0,\rho)} u < \tau .$$

ρ *depends only on* $\tau, d(x_0, \partial\Omega)$ *, curvature bounds and the injectivity radius of* Ω *,* $\dim \Omega$ *,* $\dim Y$ *,* M *,* ω *,* κ *.*

In particular, u *is continuous in* Ω *.*

Proof Let

$$h_0 = \min\left(\frac{\pi}{2M\kappa} - 1, 1\right) .$$

Then there exists ε' , $0 < \varepsilon' < 1$, with

$$h' = \frac{\pi}{2\kappa} - \{(1-h_0)^2 M^2 + \varepsilon'\}^{\frac{1}{2}} > 0 .$$

Let

$$h = \min\left(\frac{2h'\kappa}{\pi}, h_0\right)$$

and

$$0 < \varepsilon'' < \tau^2 .$$

Let ε in (4.3.12) be taken as

$$\varepsilon = \frac{1}{8} h(2-h) \min\left(\varepsilon', \frac{\varepsilon''}{8}\right)$$

and s be the smallest positive integer with

$$(1-h)^{2s} < \frac{\varepsilon''}{8M^2} .$$

The assumptions of Lemma 4.2.1 are satisfied, because $r^2(q) = d^2(q,p)$ is strictly convex on $B(p,M)$ by Lemma 2.3.2.

We start with $R_0 = \frac{1}{2}d(x_0, \partial\Omega)$, $p_0 = p$. On $B(p,M)$, we initially take normal coordinates centred at $p = p_0$. They cover $B(p,M)$, since $B(p,M)$ by assumption is disjoint to the cut locus of p.

Let $\bar{u}_{R_0}$ be the mean value of u on $T(x_0,R_0) = B(x_0,2R_0) - B(x_0,R_0)$ taken with respect to these coordinates:

$$\bar{u}_{R_0} = \fint_{T(x_0,R_0)} u(x)\, dx .$$

Let c_0 be the unique geodesic arc from p_0 to $\bar{u}_{R_0}$, and let p_1 be the point on c_0 with

$$d(p_1,p_0) = h_0\, d(\bar{u}_{R_0},p_0) .$$

Now for $q \in B(p,M)$

$$\begin{aligned} d(q,p_1) &\le d(q,p_0) + d(p_1,p_0) \\ &\le M + h_0 M \\ &\le \frac{\pi}{2\kappa} \qquad\qquad \text{by choice of } h_0 . \end{aligned}$$

Hence, by Lemma 2.3.2, $d^2(\cdot,p_1)$ is convex on $B(p,M)$. Thus, for $y \in B(x_0,R_1)$, where $2R_1$ is the radius $R \le R_0$ of (4.3.12), (4.3.12) implies for $f = d^2(\cdot,p_1)$

$$\begin{aligned} (4.4.1) \quad d^2(u(y),p_1) &\le d^2(\bar{u}_{R_0},p_1) + 4\varepsilon \\ &\le (1-h_0)^2 \sup_{x\in B(x_0,2R_0)} d^2(u(x),p_0) + 4\varepsilon \end{aligned}$$

by choice of p_1.

Let $j \in \mathbb{N}$.

Suppose now that we have found points $p_i \in B(p,M)$ and radii R_i for $i \leq j-1$ with the property that for $y \in B(x_0,R_i)$

$$(4.4.2) \qquad d^2(u(y),p_i) \leq (1-h_0)^2 M^2 + \varepsilon'$$

and

$$(4.4.3) \qquad d^2(u(y),p_i) \leq (1-h)^2 \sup_{B(x_0,2R_{i-1})} d^2(u(x),p_{i-1}) + 4\varepsilon .$$

We then want to prove (4.4.2) and (4.4.3) for $i = j$ and suitably chosen p_j and R_j .

First of all, by (4.4.2)

$$d(u(y),\ p_{j-1}) \leq \frac{\pi}{2K} - h' \qquad \text{for} \quad y \in B(x_0,R_{j-1}) .$$

If we choose normal coordinates on $B(p,M)$ centred at p_{j-1} which is possible by Prop. 2.4.1, and take $\bar{u}_{R_{j-1}}$ as being the mean value of u over $T(x_0,R_{j-1})$ with respect to these coordinates, then again by Prop. 2.4.1, there is a unique geodesic arc c_{j-1} in $B(p,M)$ from p_{j-1} to $\bar{u}_{R_{j-1}}$. We choose p_j as that point on c_{j-1} with

$$d(p_j,p_{j-1}) = h\ d(\bar{u}_{R_{j-1}},p_{j-1}) .$$

Then for $y \in B(x_0,R_{j-1})$

$$\begin{aligned} d(u(y),p_j) &\leq d(u(y),\ p_{j-1}) + d(p_j,p_{j-1}) \\ &\leq ((1-h_0)^2 M^2 + \varepsilon')^{\frac{1}{2}} + hM \qquad \text{by (4.4.2)} \\ &\leq \frac{\pi}{2K} - h' + hM \\ &\leq \frac{\pi}{2K} . \end{aligned}$$

Hence, $d^2(\cdot,p_j)$ is convex on $u(B(x_0,R_{j-1}))$, and from (4.3.12) for $y \in B(x_0,R_j)$, taking $2R_j = R \leq R_{j-1}$ in (4.3.12)

$$d^2(u(y),p_j) \le d^2(p_j,\bar{u}_{R_{j-1}}) + 4\varepsilon$$

$$\le (1-h)^2 \; d^2(\bar{u}_{R_{j-1}},p_{j-1}) + 4\varepsilon$$

$$\le (1-h)^2 \sup_{x\in B(x_0,2R_{j-1})} d^2(u(x),p_{j-1}) + 4\varepsilon \; .$$

Thus (4.4.3) is also satisfied for $i = j$.

Iterating (4.4.3), we obtain

$$(4.4.4) \qquad \sup_{y\in B(x_0,R_j)} d^2(u(y),p_j) \le (1-h)^{2j} \sup d^2(u(y),p_0) + 4\varepsilon \frac{1}{1-(1-h)^2} \; .$$

For $j > 0$, $$\frac{1}{1-(1-h)^2} + (1-h)^{2j} \le \frac{2}{h(2-h)} \; ,$$

and thus from (4.4.4) and (4.4.1), since $d^2(u(x),p_0) \le M^2$,

$$(4.4.5) \qquad \sup_{y\in B(x_0,R_j)} d^2(u(y),p_j) \le (1-h)^{2j} \; (1-h_0)^2 \; M^2 + \min\left(\varepsilon', \frac{\varepsilon''}{8}\right) \; .$$

In particular, (4.4.2) holds for $i = j$. Moreover, (4.4.5) implies

$$\left(\operatorname*{osc}_{B(x_0,R_j)} u\right)^2 \le 4 \sup_{y\in B(x_0,R_j)} d^2(u(y),p_j) \le 4(1-h)^{2j} \; M^2 + \frac{\varepsilon''}{2}$$

and hence

$$\operatorname*{osc}_{B(x_0,R_s)} u < \sqrt{\varepsilon''} < \tau \; .$$

R_s can be computed explicitly, since the radius $R_3(\varepsilon)$ in (4.3.12) can be computed from the geometric quantities of the statement of the theorem by Lemma 4.2.1. Note in particular, that the strictly convex function required in Lemma 4.2.1 is $d^2(\cdot,p)$ and that all choices of f in (4.3.12) are likewise given by squared distance functions. Hence their gradients and Hessians are controlled by the geometry of the image through Lemma 2.3.2.

q.e.d.

4.5 HÖLDER CONTINUITY OF WEAK SOLUTIONS

We now want to prove Hölder continuity of u.

THEOREM 4.5.1 *Suppose that the assumptions of Thm. 4.4.1 hold. Let* $B(x_1,2d) \subset \Omega$ *be a ball which is disjoint from the cut locus of its centre. Furthermore, suppose that* $-\sigma^2 \le K \le \tau^2$ *for the curvature on* $B(x_1,2d)$ *and* $d < \frac{\pi}{4\tau}$. *Then for all* $x,y \in B(x_1,d)$

$$d(u(x),u(y)) \le c\, d(x,y)^\beta$$

where $\beta \in (0,1)$ *and* c *depend only on* $\dim \Omega$, $\dim Y$, σ, τ, ω, κ, d, *and* M.

Proof By Thm. 4.4.1, we can find ρ, $0 < \rho < d$, with

$$\underset{B(x,\rho)}{\operatorname{osc}} u < M \qquad \text{for all } x \in B(x_1,d) . \tag{4.5.1}$$

We choose an arbitrary $x_0 \in B(x_1,d)$ and R with $0 < R \le \frac{\rho}{2}$, and define again $T(x_0,2r) = B(x_0,2r) \setminus B(x_0,r)$ and moreover

$$T^*(x_0,2r) = B\left(x_0, \frac{7R}{4}\right) \setminus B\left(x_0, \frac{5R}{4}\right) .$$

We let q be the point, where

$$H(q) = \int_{T(x_0,2R)} d^2(u(x),q)\, dx$$

achieves its minimum. (That we can find a unique such q, follows from (4.5.1) and (2.3.4)). Then

$$\int_{T(x_0,2R)} \nabla_q\, d^2(u(x),p) = 0$$

$$\Leftrightarrow \int_{T(x_0,2R)} \exp_q^{-1} u(x) = 0 .$$

That means that if we choose normal coordinates centred at q and denote the corresponding coordinate representation by v, then

$$\int_{T(x_0,2R)} v(x) = 0 \tag{4.5.2}$$

and hence by the Poincaré inequality

$$(4.5.3) \qquad \int_{T(x_0,2R)} v^2 \le c_{15} R^2 \int_{T(x_0,2R)} |\nabla v|^2$$

where c_{15} like the following constants $c_{16},\dots$ is independent of R.

η from 4.3 will now be required to satisfy $\eta \equiv 1$ on $B\left(x_0, \frac{5R}{4}\right)$ and $\eta \equiv 0$ on $\Omega \backslash B\left(x_0, \frac{7R}{4}\right)$.

In (4.3.3) we now take $f(u) = d^2(u,q)$ and $y = x_0$. Then from (4.3.4)-(4.3.6)

$$(4.5.4) \qquad \lim_{\nu\to\infty} I_\nu \ge (n-2)\omega_n d^2(u(x_0),p) - c_{16} R^2 .$$

Furthermore

$$D^2 f(du,du) \ge 2\kappa M \operatorname{ctg} \kappa M \, |du|^2$$

by Lemma 2.3.2 and (4.5.1) and hence

$$(4.5.5) \quad c_{17} \int_{B(x_0,2R)} \eta g^\nu(\cdot,x_0)\, D^2 f(du,du) \ge \int_{B(x_0,2R)} \eta g^\nu(\cdot,x_0)\, |du|^2 .$$

By choice of η, the integral III_ν extends only over $T(x_0,2R)$, and taking $\nu > R^{2-n}$, noting $f(u)_{e_\alpha} = 2vv_{e_\alpha}$ $(f(u(x)) = v^2(x))$,

$$(4.5.6) \qquad |III_\nu| \le c_{18}\left(R^{2-n} \int_{T(x_0,2R)} |\nabla v|^2 + R^{-n} \int_{T(x_0,2R)} |v|^2\right)$$

$$\le c_{19} R^{2-n} \int_{T(x_0,2R)} |\nabla v|^2 \qquad \text{by (4.5.3)} .$$

Now

$$(4.5.7) \quad |IV_\nu| \le c_{20}\left(R^{-2} \delta^{-1} \int_{T(x_0,2R)} |v|^2 + \delta \int_{T^*(x_0,2R)} |v|^2 \, |\nabla d(\cdot,x_0)^{2-n}|^2\right.$$

for each $\delta > 0$

$$\int_{T(x_0,2R)} \phi^2 v^2 \, d(\cdot,x_0)^{2-n} \, \Delta d(\cdot,x_0)^{2-n} = \int \phi^2 v^2 \, |\nabla d(\cdot,x_0)^{2-n}|^2$$

$$+ \int 2\phi\nabla\phi v^2 \, d(\cdot,x_0)^{2-n} \, \nabla d(\cdot,x_0)^{2-n} + \int \phi^2 2v\nabla v \, d(\cdot,x_0)^{2-n} \, \nabla d(\cdot,x_0)^{2-n}$$

if $\operatorname{supp}\phi \subset T(x_0,2R)$, $\phi \equiv 1$ on $T^*(x_0,2R)$, $|d\phi| \le \frac{c}{R}$.

Using Lemma 2.7.1 and Hölder's inequality, this implies

$$(4.5.8)\quad \int_{T^*(x_0,2R)} v^2 |\nabla d(\cdot,x_0)^{2-n}|^2 \le c_{21}\, R^{2(2-n)} \int_{T(x_0,2R)} v^2$$

$$+ c_{22}\Big(R^{-2} \int_{T(x_0,2R)} |v|^2\, |d(\cdot,x_0)^{2-n}|^2 + \int_{T(x_0,2R)} d\,(\cdot,x_0)^{2(2-n)}\, |\nabla v|^2\Big) .$$

Choosing $\delta = R^{n-2}$ in (4.5.7) and using (4.5.8) and (4.5.3),

$$(4.5.9)\qquad |IV_\nu| \le c_{23}\, R^{2-n} \int |dv|^2 .$$

From (4.5.3), (4.5.4), (4.5.5), (4.5.6), and (4.5.9) and letting $\nu \to \infty$

$$(4.5.10)\quad \int_{B(x_0,R)} d(\cdot,x_0)^{2-n}\, |du|^2 \le c_{24}\, R^{2-n} \int_{T(x_0,2R)} |dv|^2 + c_{26}\, R^2$$

$$\le c_{25} \int_{T(x_0,2R)} d(\cdot,x_0)^{2-n}\, |du|^2 + c_{26}\, R^2 .$$

(Note that $\int |dv|^2 = \int |du|^2$, since the energy is invariant under coordinate transformations.)

If we now add $c_{25} \int_{B(x_0,R)} d(\cdot,x_0)^{2-n}\, |du|^2$ to both sides of (4.5.10), i.e. we fill the hole (that explains why this device introduced by Widman is called the hole filling technique), we obtain with $\theta = \dfrac{c_{25}}{1 + c_{25}} < 1$

$$(4.5.11)\quad \int_{B(x_0,R)} d(\cdot,x_0)^{2-n}\, |du|^2 \le \theta \int_{B(x_0,2R)} d(\cdot,x_0)^{2-n}\, |du|^2 + c_{27}\, R^2$$

or, using the notation $\Phi(R) := \int_{B(x_0,R)} d(\cdot,x_0)^{2-n}\, |du|^2 + c_{27}\, R^2$

$$(4.5.12)\qquad \Phi(R) \le \theta_0\, \Phi(2R) \qquad \text{with} \quad \theta_0 = \max(\theta,\tfrac{1}{2}) .$$

LEMMA 4.5.1 (de Giorgi) *For* $\alpha = \log_2(\theta_0^{-1})$ *and all* $r < R$

$$(4.5.13)\qquad \Phi(r) \le 2^\alpha \left(\frac{r}{R}\right)^\alpha \Phi(R) .$$

Proof of the lemma If $2^{-k-1} R \le r \le 2^{-k} R$,

$$\Phi(r) \le \Phi(2^{-k} R) \le \theta_0^k \, \phi(R) \qquad \text{by (4.5.12)} .$$

Writing $\theta_0^k = (2^{-k}) \log_2(\theta_0^{-1})$ and using $2^{-k} \le 2 \frac{r}{R}$,

$$\Phi(r) \le 2^{\alpha} \left(\frac{r}{R}\right)^{\alpha} \Phi(R)$$

which proves the lemma.

Since $0 < \theta_0 < 1$, $\alpha > 0$, and hence Thm. 4.5.1 will follow from (4.5.13) in conjunction with the following well-known Dirichlet growth theorem of Morrey, noting that the right hand side of (4.5.13) is finite by (4.2.1) or by (4.5.11)

THEOREM 4.5.2 (Morrey) *If* $f \in H_2^1(B(x_1,d)$ *satisfies*

$$\int_{B(x_1,d) \cap B(x_0,\rho)} |\nabla f|^2 \le M^2 \rho^{n-2+2\beta}$$

for all $x_0 \in B(x_1,d)$ *and all* $\rho > 0$ *for some positive constants* β *and* M *, then* $f \in C^{0,\beta}(B(x_1,d))$ *, and*

$$|f(x) - f(y)| \le c_n M \, |x - y|^{\beta}$$

for all $x,y \in B(x_1,d)$ *, where* c_n *depends only on* n .

For a proof, cf. e.g. [M3] .

The preceding proof of Thm. 4.5.1 was taken from [HJW]. It uses the method of [HW1]. Different proofs of Thm. 4.5.1 were obtained by Eliasson [Es], Sperner [Sp], and Tolksdorf [To].

4.6 APPLICATIONS TO THE BERNSTEIN PROBLEM

Actually, the dependence on the geometry of the domain in Thm. 4.5.1 can be considerably weakened. In [HJW], the following result is shown.

THEOREM 4.6.1 *Let again* $B(p,M) \subset Y$ *be a geodesic ball, disjoint to the cut locus of* p *, with* $M < \frac{\pi}{2\kappa}$ *, where* $-\omega^2 \le K \le \kappa^2$ *are curvature bounds on* $B(p,M)$.

Let $D(0,2d) = \{x \in \mathbb{R}^n : |x| < 2d\}$ *be a coordinate chart on the domain with metric tensor* $\gamma_{\alpha\beta}(x)$ *satisfying*

$$\lambda|\xi|^2 \le \gamma_{\alpha\beta}(x)\ \xi^\alpha\xi^\beta \le \mu|\xi|^2\ , \qquad 0 < \lambda \le \mu \tag{4.6.1}$$

for all $x \in D(0,2d)$ *and all* $\xi \in \mathbb{R}^n$.

If $u : D(0,2d) \to B(p,M)$ *is harmonic, then for all* $x,y \in D(0,d)$

$$d(u(x),\ u(y)) \le \frac{c}{d^\beta}\ d(x,y)^\beta$$

for some $\beta \in (0,1)$ *and* $c > 0$ *, depending only on* n *,* $\dim Y$ *,* ω *,* κ *,* M *,* λ *, and* μ *, but not on* d *.*

In the proof of Thm. 4.6.1, one has to use the Green function of the Laplace-Beltrami operator of the domain instead of the approximate fundamental solutions we use in the proof of Thms. 4.4.1 and 4.5.1. The truncated functions $g^\nu(x,y)$ of section 4.3 have to be replaced by mollifications of the Green function. The proof then yields the desired result because one can control the Green function only in terms of the ellipticity constants of the differential operator, i.e. by (4.6.1). The required estimates for the Green function depend on Moser's Harnack inequality and are carried out in [GW]. Also, Lemma 4.2.1 has to be proved in a different way to get the stronger estimate, again using Moser's Harnack inequality, cf. e.g. [GH].

Thm. 4.6.1 has the following

COROLLARY 4.6.1 *Let the manifold* X *be diffeomorphic to* $\mathbb{R}^n$ *, with a metric tensor* $\gamma_{\alpha\beta}(x)$ $(x \in \mathbb{R}^n)$ *satisfying*

$$\lambda|\xi|^2 \le \gamma_{\alpha\beta}(x)\ \xi^\alpha\xi^\beta \le \mu|\xi|^2 , \qquad 0 < \lambda \le \mu$$

for all $\xi \in \mathbb{R}^n$ *and* $x \in \mathbb{R}^n$,

Suppose $u : X \to Y$ *is harmonic and* $u(X) \subset B(p,M)$ *where* $B(p,M)$ *again satisfies the assumptions of Thm. 4.6.1.*

Then u *is constant.*

Cor. 4.6.1 in turn can be used to prove Bernstein type theorems for minimal submanifolds of Euclidean space, when combined with the following result of Ruh and Vilms [RV].

THEOREM 4.6.2 *Suppose* $F : M \to \mathbb{R}^{n+p}$ *is of class* C^3 *and immerses the* n-*dimensional manifold* M *into Euclidean* (n+p)-*space. Then its Gauss map* $G : F(M) \to G(n,p)$ *into the Grassmannian manifold of* n-*planes in* (n+p)-*space endowed with its standard Riemannian metric is harmonic if and only if* M *is immersed with parallel mean curvature field. This in particular is the case, if* $F(M)$ *is a minimal submanifold of* $\mathbb{R}^{n+p}$.

Cor. 4.6.1 and Thm. 4.6.2 yield the following Bernstein type theorem of [HJW].

THEOREM 4.6.3 *Suppose* $F : \mathbb{R}^n \to \mathbb{R}^{n+p}$ *is a* C^3-*immersion and* $X = F(\mathbb{R}^n)$ *is minimal or has parallel mean curvature field. Suppose there exists a fixed oriented* n-*plane* P_0 *, and a number* α_0

$$\alpha_0 > \cos^m\left(\frac{\pi}{2\kappa\sqrt{m}}\right) , \qquad m = \min(n,p) , \qquad \kappa^2 = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } m \ge 2 \end{cases} \tag{4.6.2}$$

and

$$\langle P, P_0\rangle \ge \alpha_0 \tag{4.6.3}$$

holds for all oriented tangent planes P *of* X .

Suppose also that the metric

$$\gamma_{\alpha\beta}(x) = F_{x^\alpha}(x)\, F_{x^\beta}(x) \qquad (x \in \mathbb{R}^n)$$

of X *is uniformly equivalent to the Euclidean metric in the sense of (4.6.1).*

Then, X *is an affine linear subspace of* $\mathbb{R}^{n+p}$.

The conditions (4.6.2) and (4.6.3) guarantee that the image of the Gauss map of X is contained in a ball in $G(n,p)$ which satisfies the assumptions of Thm. 4.6.1, cf. [HJW].

If $p = 1$, then $m = \kappa = 1$ in (4.6.2), and hence Thm. 4.6.3 implies Moser's weak Bernstein theorem:

An entire solution of the minimal surface equation

$$\operatorname{div}\left\{\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right\} = 0$$

with $\sup|\nabla f| < \infty$ is linear.

Note that in the strong Bernstein theorem the assumption $\sup|\nabla f| < \infty$ is not necessary. On the other hand, this stronger version is only true for $n \le 7$, whereas Thm. 4.6.3 requires no restriction on the dimension.

The results of Thm. 4.6.3 seem to be also interesting, although probably not optimal, in codimension $p \ge 2$.

4.7 ESTIMATES AT THE BOUNDARY

In this section, we want to prove a-priori estimates at the boundary for weak solutions whose image is contained in a convex ball.

The following result can be found, e.g., in [GH].

THEOREM 4.7.1 *Suppose* $u : \Omega \to B(p,M)$ *is harmonic, where* $B(p,M)$ *again is a ball with* $M < \frac{\pi}{2\kappa}$ *and disjoint to the cut locus of* p . *Suppose* $\partial\Omega$ *is of class* C^2 *, and* $|K| \le \Lambda^2$ *for the sectional curvature of* Ω . *If* $g = u|\partial\Omega$ *is continuous, then for every* $\varepsilon > 0$ *we can find some* $\delta > 0$ *, depending on* ω *,* κ *,* M *,* Λ *,* $i(\Omega)$ *,* $\dim \Omega$ *,* $\partial\Omega$ *, the modulus of continuity of* g *, and on* ε *, for which*

$$d(u(y),\ u(x_0)) \le \varepsilon \qquad \text{for } y \in \Omega \cap B(x_0,\delta) \ . \tag{4.7.1}$$

If g *is Hölder continuous with some exponent* β *, then*

$$d(u(y),\ u(x_0)) \le c_\alpha |y - x_0|^\alpha \qquad \text{for } y \in \Omega \cap B(x_0,\delta) \tag{4.7.2}$$

where α *and* c_α *depend on* ω *,* κ *,* M *,* β *,* Λ *,* $i(\Omega)$ *,* $\partial\Omega$ *,* $\dim \Omega$*, and* $|g|_{C^\beta}$.

Proof W.l.o.g. $n \ge 3$. We need some definitions:

$$D(x_0,R) := \Omega \cap B(x_0,R) \ .$$

If $x_0 \in \partial\Omega$, let $c : [0,1] \to B(p,M)$ be the geodesic with $c(0) = p$, $c(1) = g(x_0)$, parametrized proportionally to arc length, and

$$p_t := c(t) \ ,$$

$$v_t := d^2(u(x),p_t) \ .$$

Furthermore, let $w_{t,R}$ be the solution of

$$\Delta w_{t,R} = 0 \qquad \text{on } D(x_0,R)$$

$$w_{t,R}|\partial D(x_0,R) = v_t|\partial D(x_0,R) \ .$$

As in the proof of Lemma 2.1.3 we derive for $y \in D(x_0,\frac{R}{2})$,

$\rho \le \min\left(\frac{R}{2}, \frac{\pi}{2\Lambda}, i(\Omega)\right)$

$$(4.7.3)\qquad \frac{1}{(n-2)\omega_n}\int_{D(y,\rho)} \Delta v_t\left(\frac{1}{d(y,z)^{n-2}} - \frac{1}{\rho^{n-2}}\right)dz \le w_{t,R}(y) - v_t(y) + \frac{\Lambda^2}{2\omega_n}\int_{D(y,\rho)} \frac{|w_{t,R} - v|}{d(y,z)^{n-2}}\,dz$$

using the fact that the boundary term on $\partial\Omega$ vanishes by definition of $w_{t,R}$. From the definition of v_t and $w_{t,R}$, we have

$$(4.7.4)\qquad v_t(y) = d^2(u(y),p_t) \le (1+t)^2 M^2$$

and

$$(4.7.5)\qquad w_{t,R}(x_0) = v_t(x_0) = d^2(g(x_0),p_t) \le (1-t)^2 M^2 .$$

We now want to exploit that the boundary values of $w_{t,R}$ on $\partial\Omega \cap D(x_0,R)$ are given by $d^2(g(x),p_t)$, i.e. controlled by assumption. Namely, given $\varepsilon' > 0$ and $R > 0$, $R \le R_0$, there exists some number $r = r(\varepsilon',R)$ (depending on ε', R, M, $\partial\Omega$, and the modulus of continuity of $d^2(g(x),p_t)$ and $\partial\Omega \cap D(x_0,R_0)$ with the property that

$$(4.7.6)\qquad w_{t,R}(y) \le w_{t,R}(x_0) + \frac{\varepsilon'}{2}$$

for all $y \in D(x_0,r)$. This is a result from potential theory (and can be found, e.g., in [GT], Thm. 8.27).

If $d^2(g(x),p_t)$ is Hölder continuous, we even have

$$(4.7.7)\qquad w_{t,R}(y) \le w_{t,R}(x_0) + \bar{c}|y - x_0|^{2\alpha} \qquad \text{for } y \in D(x_0,r) ,$$

where α, $\bar{c}$ depend ω, κ, M, β, $\partial\Omega$, and $|g|_{C^\beta}$.

We now want to apply an iteration procedure, and put

$$\bar{t} := \frac{\pi}{2M\kappa} - 1 ,$$

$$\varepsilon' := \min(M^2(1-(1-\bar{t})^2),\varepsilon)$$

$$t_i := i\bar{t} \qquad \text{for} \quad 1 \le i \le \mu-1$$

and $t_\mu := 1$, where μ is the smallest integer with $\mu\bar{t} \ge 1$. Furthermore, we start with some radius $R_0 < 1$ and define

$$R_i = \min\left(\frac{R_{i-1}}{2},\ r(\varepsilon', R_{i-1})\right) \qquad (i = 1,\dots,\mu) ,$$

where r is the same r as in (4.7.6).

Then, with

$$m_i := \max_{x \in D(x_0, R_{i-1})} (v_{t_i}(x))^{\frac{1}{2}} ,$$

by Lemma 2.1.1, (1.7.2), (4.7.3), (4.7.6) for $y \in D(x_0, R_i)$, $\rho_{i-1} = \frac{1}{2}R_{i-1}$

$$(4.7.8) \quad 2\kappa m_i \operatorname{ctg}(\kappa m_i) \int_{D(y,\rho_{i-1})} |du|^2 \left(\frac{1}{r(\cdot)^{n-2}} - \frac{1}{\rho_{i-1}^{n-2}} \right) + v_{t_i}(y)$$

$$\le w_{t_i, R_i}(y) + c_{26} R_0^2 \le (1-t_i)^2 M^2 + \varepsilon' \le M^2$$

choosing R_0 so small that $c_{26} R_0^2 \le \frac{\varepsilon'}{2}$. Furthermore,

$$m_1 \le \frac{\pi}{2\kappa} \qquad \text{by (4.7.4)} ,$$

and if $m_i \le \pi/2\kappa$, then by (4.7.8)

$$(v_{t_{i+1}})^{\frac{1}{2}} \le (v_{t_i})^{\frac{1}{2}} + \bar{t}M \le \frac{\pi}{2\kappa} ,$$

i.e. $m_{i+1} \le \pi/2\kappa$.

Therefore, by induction,

$$m_\mu \le \frac{\pi}{2\kappa} ,$$

and again from (4.7.8) and (4.7.6)

$$v_1(y) \le w_{1,R_\mu}(y) + c_{26} R_0^2 \le w_{1,R_\mu}(x_0) + \varepsilon = \varepsilon$$

for all $y \in D(x_0, R_\mu)$.

This gives the desired estimate of the modulus of continuity at the boundary, putting $\delta = R_\mu$.

In case the boundary data are Hölder continuous, we use (4.7.7) to get

$$d(u(y),u(x_0)) \le (v_1(y))^{\frac{1}{2}} \le c|y - x_0|^\alpha .$$

q.e.d.

4.8 C^1-ESTIMATES

Having established Hölder continuity of weakly harmonic maps in Thms. 4.5.1 and 4.7.1, it is well known that these maps are actually of class C^1 (and hence of class $C^{2,\alpha}$). Proofs of this assertion can be found in [LU] and [G], and more specifically for harmonic maps in [GH] and [Sp]. Instead of repeating those proofs, we contend ourselves to derive a-priori estimates for the gradient of harmonic maps (i.e. already assuming that the map is regular) which can be obtained in a very easy way following [JK1].

THEOREM 4.8.1 *Let* X *and* Y *be Riemannian manifolds,* $B(x_0,R_0) \subset X$, $R_0 < \min\left(i(x_0), \frac{\pi}{2\kappa_X}\right)$ *, where* $-\omega_X^2 \le K_X \le \kappa_X^2$ *are curvature bounds on* $B(x_0,R_0)$ *, and* $B(p,M) \subset Y$ *,* $M < \min\left(i(p), \frac{\pi}{2\kappa_Y}\right)$ *, where* $-\omega_Y^2 \le K_Y \le \kappa_Y^2$ *are curvature bounds on* $B(p,M)$ *. If* $u : X \to B(p,M)$ *is harmonic, then for all* $R \le R_0$

$$|\nabla u(x_0)| \le c_0 \cdot \max_{x\in B(x_0,R)} \frac{d(u(x),\ u(x_0))}{R} \tag{4.8.1}$$

where $c_0 = c_0(R_0,\omega_X,\kappa_X,\dim\ X,\ M,\ \omega_Y,\kappa_Y,\ \dim Y)$.

Proof The proof is based on an idea of E. Heinz [Hz1] and similar to the one of Lemma 2.8.3. Let $\dim X = n$, $\dim Y = N$. We define

$$\mu := \max_{x\in B(x_0,R_0)} (R_0 - d(x,x_0))\ |du(x)| .$$

Then there exists $x_1 \in B(x_0, R_0)$ with

$$\mu = (R_0 - d(x_1, x_0)) \cdot |du(x_1)|$$

and

$$|du(x_0)| \le \frac{\mu}{R_0} . \tag{4.8.2}$$

We put

$$d := R_0 - d(x_1, x_0) .$$

We shall prove

$$\mu \le \frac{\delta(\theta_0)}{2\theta} + \frac{b\theta}{2} \mu^2 \qquad \text{for all } \theta \le \theta_0 \tag{4.8.3}$$

where θ_0 can be chosen so small (with the help of Thm. 4.5.1) that

$$\delta(\theta_0) \cdot b < 1 .$$

Then (4.8.1) follows as in the proof of Lemma 2.8.3.

We now use the functions k_i of Lemma 2.8.4 for $q = u(x_1)$. Then

$$\frac{\mu}{d} = |du(x_1)| = |d(k \circ u)(x_1)| . \tag{4.8.4}$$

Moreover

$$|D^2 k| \le c_1 \cdot \frac{1}{t_0} , \qquad \text{where } c_1 = c_1(\omega_Y, M, N) \quad \text{(cf. (2.8.33)} \tag{4.8.5}$$

and hence

$$|\Delta(k \circ u)| \le \frac{c_1}{t_0} |du|^2 \qquad \text{(cf. (1.7.2)) .} \tag{4.8.6}$$

Furthermore, dk is an isometry at $u(x_1)$, and hence from (4.8.5)

$$|dk(y)| \le c_2 , \qquad c_2 = c_2(\omega_Y, M, N, \kappa_Y) \qquad \text{(cf. Lemma 2.8.4) .} \tag{4.8.7}$$

We put

$$\delta = \delta(\theta) := \max_{x \in B(x_1, d\theta)} d(u(x), u(x_1)) .$$

By Thm. 4.5.1, δ can be made arbitrarily small by choosing θ sufficiently small. At the moment, we need only

$$\delta \le M .$$

By Lemma 2.7.5, putting $\Lambda_X = \max(\omega_X, \kappa_X)$

(4.8.8) $$\frac{\mu}{d} = |dk\circ u(x_1)| \le \frac{c_3}{d^n\theta^n}\int_{d(x,x_1)=d\theta} |k(u(x)) - k(u(x_1))|$$

$$+ c_4 \int_{d(x,x_1)\le d\theta} \frac{|\Delta k\circ u|}{d(x,x_1)^{n-1}} + c_5\,\Lambda_X^2 \int_{d(x,x_1)\le d\theta} \frac{|k(u(x)) - k(u(x_1))|}{d(x,x_1)^{n-1}} .$$

By (4.8.7), $|k(u(x)) - k(u(x_1))| \le c_2\delta$, and by (4.8.6),

$$|\Delta(k\circ u)| \le \frac{c_1}{t_0}\cdot |du|^2 \le \frac{c_1}{t_0}\frac{\mu^2}{d^2(1-\theta)^2} .$$

Estimating the integrals, we also get volume factors

$$\left(\frac{\sinh(\Lambda_X d\theta)}{\Lambda_X d\theta}\right)^{n-1} ,$$

which will be included in the constants $(c_i \to c_i', \ i = 3,4,5)$. Hence

$$\frac{\mu}{d} \le \left(\frac{c_3' c_2 \delta}{d\theta} + \frac{c_4' c_1 \cdot \mu^2}{t_0 d(1-\theta)^2}\,\theta + c_5' \Lambda_X^2 c_2 \delta\cdot d\theta\right) \mathrm{vol}(S^{n-1}) ,$$

or, assuming $\theta \le \frac{1}{2}$ w.l.o.g.,

$$\mu \le \frac{\delta(\theta_0)}{2\theta} + \frac{b\theta}{2}\,\mu^2 \qquad \text{for all } \theta \le \theta_0 ,$$

i.e. (4.8.3). By definition of $\delta(\theta)$ and Thm. 4.5.1, $\delta(\theta_0)$ can be made arbitrarily small by choosing θ_0 sufficiently small, and the result follows as in the proof of Lemma 2.8.3.

q.e.d.

At the boundary, we have

THEOREM 4.8.2 *Let* Ω *be a bounded domain in some Riemannian manifold,* $\partial\Omega$ *of class* C^2 *, and let* $u : \Omega \to B(p,M)$ *be harmonic, where* $B(p,m)$ *satisfies the same assumptions as in Thm. 4.8.1. Suppose* $u|\partial\Omega = \phi \in C^2$ *. Then* $|u|_{C^1(\bar\Omega)}$ *can be bounded in terms of the geometric quantities of Thm. 4.8.1,* *bounds for the principal curvatures of* $\partial\Omega$ *,* $|\phi|_{C^2}$ *, and a lower bound for a number* τ *satisfying* $0 < \tau < \frac{\pi}{2\kappa_Y} - S$ *,* $\dot{B}(p,M+\tau)$ *disjoint to the cut locus of* p *.*

Proof The proof is again taken from [JK1] and refines an argument of [HKW1]. Let $d(x_0,\partial\Omega) = R_0$. By Thm. 4.8.1, it suffices to show

$$\max_{x\in B(x_0,R_0)} d(u(x),\ u(x_0)) \le cR_0 .$$

This in turn follows, if

$$d(u(x_2),\ u(x_1)) \le cR_0 \tag{4.8.9}$$

in case $x_1 \in \partial\Omega$, $d(x_0,x_1) = R_0$, $d(x_0,x_2) \le R_0$.

We choose some number $\tau > 0$ as described in the statement of the theorem; w.l.o.g.

$$\tau \le \frac{\pi}{4\kappa_Y} . \tag{4.8.10}$$

By Lemma 2.4.1, any two points in $B(p,M+\tau)$ can be joined by a unique geodesic arc inside $B(p,M+\tau)$.

By Thm. 4.7.1, we can calculate $R_1 > 0$ with the property that for all $R_0 \le R_1$ and $x \in \Omega \cap B(x_0,2R_0)$, x_1 as above

$$d(u(x),\ u(x_1)) \le \frac{\tau}{2} . \tag{4.8.11}$$

If $u(x) \neq u(x_1)$, we connect $u(x)$ to $u(x_1)$ by a geodesic arc and continue this arc beyond $u(x_1)$ until a distance τ . We thus reach some point $q(x) \in B(p,M+\tau)$.

By Thm. 4.7.1 again, we can find some subdomain $\Omega_0 \subset \Omega$ satisfying

$$B(x_0,R_0) \subset \Omega_0 \tag{4.8.12}$$

(4.8.13) $$\Omega \cap B(x_1,\delta) \subset \Omega_0 \quad \text{for some} \quad \delta > 0$$

(4.8.14) $$u(\Omega_0) \subset B\left(q(x), \frac{\pi}{2\kappa_Y}\right) \quad \text{for all} \quad x \in B(x_0,R_0) \quad (\text{cf. } (4.8.10))$$

(4.8.15) $$\partial\Omega_0 \in C^2 .$$

We then fix $x_2 \in B(x_0,R_0)$, assume $u(x_1) \neq u(x_2)$ w.l.o.g., and put $q = q(x_2)$.

By (4.8.14)

$$v(x) := d^2(u(x),q)$$

is subharmonic in Ω_0 .

Let h be the harmonic function on Ω_0 with the same boundary values, i.e.

(4.8.16) $$\begin{aligned} \Delta h &= 0 \quad \text{in } \Omega_0 \\ h(x) &= d^2(u(x),q) \quad \text{for } x \in \partial\Omega_0 . \end{aligned}$$

By the maximum principle

(4.8.17) $$v \le h \quad \text{in } \Omega_0 .$$

Now

$$\begin{aligned} d(u(x_1), u(x_2)) &= d(u(x_2),q) - d(u(x_1),q) \qquad \text{by choice of } q \\ &\le \frac{1}{2\tau}\left(d^2(u(x_2),q) - d^2(u(x_1),q)\right) \\ &\le \frac{1}{2\tau}\left(h(x_2) - h(x_1)\right) \qquad \text{by (4.8.16) and (4.8.17)} . \end{aligned}$$

Thus, (4.8.9) follows from a Lipschitz bound for the harmonic function h at the boundary, which in turn follows from standard barrier arguments, taking (4.8.12), (4.8.13), and (4.8.15) into account, cf. [GT], chapter 13.

q.e.d.

Different gradient estimates were provided by Giaquinta-Hildebrandt [GH], Sperner [Sp], and Choi [Ci] (only interior estimates). The latter two papers employ an auxiliary function introduced by Jäger-Kaul [JäK2], cf. 4.11.

4.9 HIGHER ESTIMATES

If we write the equations

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} u^i \right) + \gamma^{\alpha\beta} \Gamma^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0$$

in terms of harmonic coordinates on domain and image, then the regularity properties of harmonic coordinates (cf. section 2.8) immediately imply $C^{2,\alpha}$-estimates for harmonic maps, again depending only on curvature bounds, injectivity radii, and dimensions, using standard results from potential theory. We have the following result of [JK1].

THEOREM 4.9.1 *Suppose that the assumptions of Thm. 4.8.1 hold and* τ *is chosen as in Thm. 4.8.2. Then the* $C^{2+\alpha}$*-norm of* u *on* $B\left(x_0, \frac{R_0}{2}\right)$ *is bounded in terms only of the quantities appearing in Thm. 4.8.1 and* τ *. A corresponding result holds at the boundary, provided* $\partial\Omega$ *and* $u|\partial\Omega$ *are of class* $C^{2+\alpha}$ *(for all* $\alpha \in (0,1)$ *). Similarly if* $u|\partial\Omega$ *is only of class* $C^{1+\alpha}$*, then* $u \in C^{1+\alpha}(\bar{\Omega})$ *with appropriate estimates.*

Finally, Thm. 2.8.3 implies

THEOREM 4.9.2 *If under the assumptions of Thm. 4.9.1 the Riemann curvature tensors of domain and image are of class* C^k *or* $C^{k+\beta}$ *(* $k \in \mathbb{N}$*,* $\beta \in (0,1)$ *), then* u *is of class* C^{k+2} *or* $C^{k+3+\beta}$*, resp., and the corresponding estimates depend in addition on the* C^k *or* $C^{k+\beta}$*-norm, resp., of the curvature tensors. A similar statement holds at the boundary, provided* $\partial\Omega$ *and* $u|\partial\Omega$ *are sufficiently regular.*

4.10 THE EXISTENCE THEOREM OF HILDEBRANDT-KAUL-WIDMAN

In this section, we shall establish the existence of a weakly harmonic map with given boundary data contained in a convex ball which admit an extension with finite energy. This map will be obtained as the minimum of energy among maps with image in this ball. The results of the preceding sections then imply regularity of this map, and hence we can solve the Dirichlet problem.

A useful tool will be the following maximum principle for energy minimizing maps which is taken from [J6] and based on the same idea as the one in [H1], Lemma 6.

LEMMA 4.10.1 *Suppose that* B_0 *and* B_1 , $B_0 \subset B_1$, *are closed subsets of a Riemannian manifold* N . *Suppose that there exists a projection map*

$$\pi : B_1 \to B_0 ,$$

which is the identity on B_0 *and which is of class* C^1 *and distance decreasing outside* B_0 , *i.e.*

$$|d\pi(v)| < |v| \qquad \text{if} \quad v \in T_x N , \quad v \neq 0 , \quad x \in B_1 \backslash B_0 .$$

If $h : \Omega \to B_1$ *is an energy minimizing* W_2^1 *mapping with respect to fixed boundary values which are contained in* B_0 , *i.e.*

$$h(\partial\Omega) \subset B_0 , \tag{4.10.1}$$

then we also have

$$h(\Omega) \subset B_0 ,$$

if we choose a suitable representation of the Sobolev mapping h .

Proof Since $|d\pi(v)| < |v|$ for every nonzero $v \in T_x N$, $x \in B_1 \backslash B_0$, and since $\pi \circ h \in W_2^1(\Omega, N)$, we would have

$$E(\pi\circ h) < E(h) ,$$

contradicting the minimality of h, unless $dh = 0$ a.e. on $h^{-1}(B_1\setminus B_0)$. Thus $dh = d\pi\circ h$ a.e. on Ω, and since h and $\pi\circ h$ agree on $\partial\Omega$ by (4.10.1), we conclude from the Poincaré inequality that $\pi\circ h = h$ a.e. on Ω, which easily implies the claim.

LEMMA 4.10.2 *Suppose that* B_0 *and* B_1, $B_0 \subset B_1$, *are compact subsets of a Riemannian manifold* N, *and that every point in* $B_1\setminus B_0$ *can be joined to* ∂B_0 *by a unique geodesic normal to* ∂B_0, *and that the distance between every pair of such geodesics normal to* ∂B_0 *is in* $B_1\setminus B_0$ *always bigger than on* ∂B_0. *Then the same conclusion as in Lemma 4.10.1 holds.*

Proof We project $B_1\setminus B_0$ along normal geodesics onto ∂B_0 and apply Lemma 4.10.1.

q.e.d.

We shall see another useful consequence of Lemma 4.10.1 in chapter 5.

We are now ready to prove the existence of a weakly harmonic map.

LEMMA 4.10.3 *Suppose* $B(p,M)$ *is disjoint to the cut locus of* p, *and* $M < \frac{\pi}{2\kappa}$, *where, as usual,* κ^2 *is an upper curvature bound.*

If $g : \Omega \to B(p,M)$, Ω *being a bounded domain in some Riemannian manifold, has finite energy, then there exists a weakly harmonic map* $u : \Omega \to B(p,M)$ *with* $u-g \in \overset{\circ}{H}{}^1_2(\Omega,B(p,M))$. u *minimizes the energy among all such maps.*

Proof Since the cut locus of a point p is a closed set, we can find M^1, $M < M^1 < \frac{\pi}{2\kappa}$, for which $B(p,M^1)$ is still disjoint to the cut locus of p. We take a minimizing sequence for the energy in $V := \{v \in H^1_2(\Omega,B(p,M^1))$, $v-g \in \overset{\circ}{H}{}^1_2\}$. Note that $g \in V$ and hence $V \neq \emptyset$. Such a sequence has a subsequence converging weakly in H^1_2, and the limit, denoted by u, minimizes energy in V because of the lower semicontinuity of the energy integral (cf. Lemma 1.3.1).

We then put $B_0 = B(p,M)$ and $B_1 = B(p,M^1)$. If $c(\cdot,t)$ is a smooth family of geodesics with $c(0,t) = p$, $c(1,t) \in \partial B(p,M^1)$, then (2.2.5) implies that the Jacobi fields $J_t(s) = \frac{\partial}{\partial t} c(s,t)$ are monotonically increasing for $s \in [0,1]$. Hence the assumptions of Lemma 4.10.2 are satisfied. Therefore, $u(\Omega) \subset B(p,M)$.

We identify $B(p,M^1)$ with its image in $\mathbb{R}^N$ under normal coordinates centred at p. If $\eta \in \mathring{H}^1_2 \cap L^\infty(\Omega,\mathbb{R}^N)$, we infer that for sufficiently small $|t| > 0$, $u + t\eta$ still maps Ω into $B(p,M^1)$. Hence $u + t\eta$ is a valid comparison map, and since u was minimizing, differentiating $E(u + t\eta)$ w.r.t. t at $t = 0$ implies (4.1.2), i.e. that u is weakly harmonic.

q.e.d.

Remark As easy examples show the map u constructed in Lemma 4.10.3 need not be minimizing among all maps $v : \Omega \to Y$ with $v-g \in \mathring{H}^1_2$ (Y is a target manifold containing $B(p,M)$), not even among maps which are homotopic to u. Hence, u in general is only a local minimum of energy.

Lemma 4.10.3 together with the regularity results of the previous sections imply the existence theorem of Hildebrandt-Kaul-Widman [HKW3].

THEOREM 4.10.1 *Suppose again that* $B(p,M)$ *is disjoint to the cut locus of* p *and* $M < \frac{\pi}{2\kappa}$, *where* κ^2 *is an upper bound for the sectional curvature of* $B(p,M)$. *If* Ω *is a bounded domain in some Riemannian manifold and* $g : \Omega \to B(p,M)$ *has finite energy, then there exists a harmonic map* $u \in C^{2,\alpha}(\Omega,B(p,M))$ $(0 < \alpha < 1)$ *with* $u-g \in \mathring{H}^1_2(\Omega,B(p,M))$. *At* $\partial\Omega$, u *is as regular as* g *and* $\partial\Omega$ *permit.*

Actually, one can solve the Dirichlet problem for any continuous map

$w : \partial\Omega \to B(p,M)$, i.e. find a harmonic map $u \in C^{2,\alpha}(\Omega, B(p,M)) \cap C^0(\bar{\Omega}, B(p,M))$ with $u|\partial\Omega = w$, without assuming that w admits an extension of finite energy. In order to achieve this, one has to combine the a-priori estimates of the preceding sections with Leray-Schauder degree theory instead of using variational methods. For this, one first deforms w into constant boundary values, mapping $\partial\Omega$ onto p and then multiplies the nonlinearity in (1.3.1) by a parameter λ , $\lambda \in [0,1]$. Such a twofold deformation process was applied in [HKW2], for instance.

4.11 THE UNIQUENESS THEOREM OF JÄGER-KAUL

In this section, we want to prove the uniqueness and stability theorem of Jäger-Kaul [JäK2] for solutions of the Dirichlet problem with image contained in a convex ball.

THEOREM 4.11.1 *Suppose that* $u_i : \bar{\Omega} \to Y$, $i = 1,2$, *are harmonic maps of class* $C^0(\bar{\Omega}, Y) \cap C^2(\Omega, Y)$, Ω *is a bounded domain in some Riemannian manifold, and* $u_i(\bar{\Omega}) \subset B(p,M)$, *where* $B(p,M)$ *is a geodesic ball in* Y , *disjoint to the cut locus of* p *and with radius* $M < \frac{\pi}{2\kappa}$ (κ^2 *is an upper bound for the sectional curvature of* $B(p,M)$).

Then the function θ ,

$$\theta(x) := \frac{q_\kappa(d(u_1(x), u_2(x)))}{\cos(\kappa d(p, u_1(x))) \cdot \cos(\kappa d(p, u_2(x)))}$$

$$(q_\kappa(t) := \begin{cases} \dfrac{1}{\kappa^2}(1 - \cos \kappa t) , & \text{if } \kappa > 0 \\ \dfrac{t^2}{2} , & \text{if } \kappa = 0 \,) , \end{cases}$$

satisfies the maximum principle

$$\sup_{\Omega} \theta \le \sup_{\partial\Omega} \theta . \tag{4.11.1}$$

In particular, if $u_1|\partial\Omega = u_2|\partial\Omega$ *, then*

$$u_1 \equiv u_2 .$$

The proof of Thm. 4.11.1 will actually show that we have strict inequality in (4.11.1) unless $\theta \equiv$ const. Furthermore, Thm. 4.11.1 also holds for weakly harmonic maps (cf. [JäKl]).

Proof We assume that θ has a positive maximum at some interior point $x_0 \in \Omega$. Then, θ is positive in a neighbourhood of x_0 , and $\log \theta > -\infty$ in this neighbourhood.

We define

$$\psi(x) := Q_\kappa(u_1(x), u_2(x)) = \begin{cases} \frac{1}{\kappa^2}(1 - \cos \kappa d(u_1(x), u_2(x))) & \text{if } \kappa > 0 \\ \frac{1}{2} d^2(u_1(x), u_2(x)) & \text{if } \kappa = 0 \end{cases}$$

$$\phi_i(x) = \cos(\kappa d(p, u_1(x))) , \qquad i = 1,2 .$$

Then $\theta = \dfrac{\psi}{\phi_1 \cdot \phi_2}$, and consequently

$$\text{grad} \log \theta = \frac{\text{grad } \psi}{\psi} - \frac{\text{grad } \phi_1}{\phi_1} - \frac{\text{grad } \phi_2}{\phi_2} , \tag{4.11.2}$$

and

$$\Delta \log \theta = \frac{\Delta\psi}{\psi} - \frac{|\text{grad } \psi|^2}{\psi^2} - \frac{\Delta\phi_1}{\phi_1} + \frac{|\text{grad } \phi_1|^2}{\phi_1^2} - \frac{\Delta\phi_2}{\phi_2} + \frac{|\text{grad } \phi_2|^2}{\phi_2^2} . \tag{4.11.3}$$

Since $x \to u(x) = (u_1(x), u_2(x)) \in B(p,M) \times B(p,M)$ is also harmonic, we can make use of the chain rule (1.7.2) in order to apply Lemma 2.5.1. This yields

$$\Delta\psi \geq \frac{|\text{grad } \psi|^2}{2\psi} - \kappa^2\psi(|du_1|^2 + |du_2|^2) , \tag{4.11.4}$$

since

$$|\text{grad } \psi|^2 = \sum_\alpha \langle(\text{grad } Q_\kappa) \circ u, \ du(e_\alpha)\rangle^2 ,$$

where e_α is an orthonormal frame on Ω .

Similarly, from (2.5.2), since

$$\phi_i(x) = 1 - \kappa^2 Q_\kappa(p, u_i(x)) \ ,$$

we obtain

$$\Delta\phi_i(x) \le -\kappa^2 \phi_i |du_i|^2 \ . \tag{4.11.5}$$

Finally, by (4.11.2),

$$-\tfrac{1}{2}\frac{|\text{grad } \psi|^2}{\psi^2} + \frac{|\text{grad } \phi_1|^2}{\phi_1^2} + \frac{|\text{grad } \phi_2|^2}{\phi_2^2} \ge -\left\langle \text{grad log } \theta, \ \tfrac{1}{2} \text{ grad log } \theta + \frac{\text{grad } \phi_1}{\phi_1} + \frac{\text{grad } \phi_2}{\phi_2} \right\rangle \ . \tag{4.11.6}$$

Putting

$$k(x) := \tfrac{1}{2} \text{ grad log } \theta + \frac{\text{grad } \phi_1}{\phi_1} + \frac{\text{grad } \phi_2}{\phi_2}$$

and plugging (4.11.4), (4.11.5), and (4.11.6) into (4.11.3), we obtain

$$\Delta \log \theta + \langle \text{grad log } \theta, \ k(x) \rangle \ge 0 \ .$$

Therefore, the assumption that θ has a positive maximum in the interior contradicts E. Hopf's maximum principle, and Thm. 4.11.1 is proved.

CHAPTER 5

HARMONIC MAPS BETWEEN SURFACES

5.1 NONEXISTENCE RESULTS

In this chapter, we want to present the existence theory for harmonic maps between closed surfaces, possibly with boundary. In the two-dimensional case, the regularity theory for minimizing maps is very easy, and the local geometry of the image does not lead to any difficulties in contrast to the situation we encountered in chapter 4 (cf. the example in section 4.1). This allows us to investigate in more detail what obstructions for the existence of harmonic maps are caused by the global topology of the image.

We first want to show some instructive nonexistence results which illustrate the difficulties we shall encounter later on when we try to prove existence results by variational methods.

Lemaire [L1] showed

PROPOSITION 5.1.1 *There is no nonconstant harmonic map from the unit disc* D *onto* S^2 *mapping* ∂D *onto a single point.*

Proof Suppose $u : D \to S^2$ is harmonic with $u(\partial D) = p \in S^2$. Since the boundary values of u are constant, u is also a critical point with respect to variations $u \circ \psi$, where $\psi : D \to D$ is a diffeomorphism, mapping ∂D onto itself, but not necessarily being the identity on ∂D .

Thus, one can use a standard argument to show that u is a conformal map (cf. [L1] or [M3], pp.369-372). Since u is constant on ∂D one can extend it by reflection as a conformal map on the whole of $\mathbb{R}^2$. But then this conformal map is constant on a curve interior to its domain of definition, namely ∂D , and thus has to be constant itself.

q.e.d.

The same argument was used independently and in a different context by H. Wente [Wt].

One can obtain examples of homotopy classes which do not contain energy minimizing maps by making use of the following special case of a result of Morrey [M2].

LEMMA 5.1.1 *For every* $\varepsilon > 0$ *there exists a map* $k : D \to S^2$ *of degree 1, mapping* ∂D *onto some point* $p \in S^2$ *and satisfying*

$$E(k) \le \text{Area}(S^2) + \varepsilon \,. \qquad (5.1.1)$$

Such a map k *is called* ε*-conformal.*

Proof of Lemma 5.1.1 We divide S^2 into $B(p,\delta)$ and $S^2 \backslash B(p,\delta)$.

All the maps to follow will be understood to be equivariant w.r.t. the rotations of D and to those of S^2 leaving p fixed.

First of all, for sufficiently small δ , we can map $\{z \in \mathbb{C} : \frac{1}{2} \le z \le 1\}$ onto $B(p,\delta)$, $\{|z| = \frac{1}{2}\}$ going onto $\partial B(p,\delta)$ and $\{|z| = 1\}$ going onto p with energy smaller than ε . On the other hand, $\{z \in \mathbb{C} : |z| \le \frac{1}{2}\}$ can be mapped conformally onto $S^2 \backslash B(p,\delta)$, $\{|z| = \frac{1}{2}\}$ going again onto $\partial B(p,\delta)$, and the energy of this map, since conformal, equals the area of its image and is hence smaller than the area of S^2 . This proves the claim.

q.e.d.

It is quite instructive to look at the second map of the proof more closely. If we stereographically project S^2 onto $\mathbb{C}$, choosing the antipodal point $\bar{p}$ of p as the origin, $S^2 \backslash B(p,\delta)$ is mapped onto

$\{|z| \leq N\}$ with $N \to \infty$ as $\delta \to 0$. The conformal map used above is then just given by $z \mapsto 2Nz$. Thus, the preimage of $\{|z| \leq 1\}$, which corresponds to the hemisphere centred at $\bar{p}$, under this map is $\{|z| \leq \frac{1}{2N}\}$, i.e. shrinks to a single point as $N \to \infty$. In this way, we see how a singularity is created in the limit of an energy minimizing sequence of degree 1 from D onto S^2 , mapping ∂D onto p .

This heuristic reasoning will be made precise in Prop. 5.1.2 below, with the help of the following easily checked

LEMMA 5.1.2 *If* $f : \Sigma_1 \to \Sigma_2$ *is a map between surfaces, then*

$$\text{Area}(f(\Sigma_1)) \leq E(f) \ , \tag{5.1.2}$$

where the area is counted with appropriate multiplicity. Furthermore, equality holds in (5.1.2) if and only if f *is conformal.*

As a consequence, we have for example the following result, again due to Lemaire [L1].

PROPOSITION 5.1.2 *Let* α *be a homotopy class of maps of degree* ± 1 *from a closed surface* Σ *of positive genus onto* S^2 *. Then the minimum of energy is not attained in* α .

Proof Let B be any disc in Σ and let $\varepsilon > 0$. Since B is conformally equivalent to the unit disc D , Lemmata 5.1.1 and 1.3.2 imply that we can find a map $k : B \to S^2$ of degree ± 1 , mapping ∂D onto some point p , and satisfying (5.1.1). If we extend k to all of Σ by mapping $\Sigma \backslash B$ onto p , then $k : \Sigma \to S^2$ still satisfies (5.1.1) and is of degree ± 1 .

If there would be an energy minimizing h in α , then h would have to satisfy consequently

$$E(h) = \text{Area}(S^2)$$

by Lemma 5.1.2, and would hence have to be conformal, by Lemma 5.1.2 again. On the other hand, a conformal map of degree ± 1 has to be a diffeomorphism which is not possible since Σ is by assumption not homeomorphic to S^2 .

q.e.d.

The following example where some homotopy classes contain harmonic representatives, while others do not, is again based on the idea of Lemaire [L1].

Let D be the unit disc in the complex plane, and $k : D \to S^2$ be a conformal map mapping D onto the upper hemisphere and ∂D onto the equator. Furthermore, suppose that k is equivariant with respect to the rotations of D and S^2 (the latter ones leaving the north and south pole of S^2 fixed).

We choose the orientation on S^2 in such a way that the Jacobian of k is positive.

Let $D(0,r)$ be the plane disc with centre 0 and radius r (i.e. $D = D(0,1)$) .

Let h_r be a map from $D(0,r)$ onto S^2 which maps $\partial D(0,r)$ onto the north pole, is injective in the interior of $D(0,r)$ and has a positive Jacobian there, and is ε-conformal. We introduce polar coordinates (ρ,ϕ) on D and define for $0 < r < 1$ the mapping k_r by

$$k_r(\rho,\phi) = \begin{cases} k\left(\frac{1}{1-r}\,\rho + \frac{r}{r-1},\ \phi\right) & \text{if } r \le \rho \le 1 \\ h_r(\rho,\phi) & \text{if } 0 \le \rho \le r \,. \end{cases}$$

Using Lemma 5.1.1 it is easy to see that the energy of k_r can be made arbitrarily close to 6π if we choose $r > 0$ sufficiently small.

On the other hand, 6π is just the area of the image of k_r , counted with multiplicity. Hence, if there is an energy minimizing map homotopic to

k_r , its energy has to be 6π , and it therefore has to be conformal. Since the boundary values are equivariant, this conformal map itself has to be equivariant (otherwise there would exist infinitely many homotopic conformal maps with the same boundary values which is not possible). This, however, implies that it would have to collapse a circle in D to a point which is not possible for a conformal map. Hence there is no energy minimizing map homotopic to k_r .

By letting h_r cover S^2 more than once, we obtain other classes without energy minimizing maps by a similar argument. If h_r , however, has degree -1 , then k_r is homotopic to a map of D onto the lower hemisphere and hence homotopic to an energy minimizing map. Hence, in this example, there are precisely two homotopy classes which contain energy minimizing maps, while all the others do not.

The preceding example is discussed in [BC2] by means of explicit calculations.

While Prop. 5.1.2 only excluded the existence of an energy minimizing map, one can even show

PROPOSITION 5.1.3 *If* Σ_1 *is diffeomorphic to the two-dimensional torus, and* Σ_2 *to* S^2 *, then there is no harmonic map* $h : \Sigma_1 \to \Sigma_2$ *of degree* $d(h) = \pm 1$ *, for any metrics on* Σ_1 *and* Σ_2 .

This result was obtained by Eells-Wood [EW] as a consequence of their

THEOREM 5.1.1 *Suppose that* Σ_1 *and* Σ_2 *are closed orientable surfaces,* $\chi(\Sigma)$ *denotes the Euler characteristic of a surface* Σ *, and* $d(\phi)$ *is the degree of a map* ϕ .

Suppose $h : \Sigma_1 \to \Sigma_2$ *is harmonic with respect to metrics* γ *and* g *on*

Σ_1 and Σ_2, resp. If

$$\chi(\Sigma_1) + |d(h)|\ |\chi(\Sigma_2)| > 0 ,$$

then h *is holomorphic or antiholomorphic relative to the complex structures determined by* γ *and* g .

Thm. 5.1.1, together with the existence theorem of Lemaire and Sacks-Uhlenbeck, to be proved below, also enabled Eeels and Wood to give an analytic proof of the following topological result of H. Kneser [Kn2]

THEOREM 5.1.2 *Suppose again that Σ_1 and Σ_2 are closed orientable surfaces, and furthermore $\chi(\Sigma_2) < 0$. Then for any continuous map $\phi : \Sigma_1 \to \Sigma_2$*

$$|d(\phi)|\ \chi(\Sigma_2) \geq \chi(\Sigma_1) . \tag{5.1.3}$$

Proof of Theorem 5.1.2 We introduce some metrics γ and g on Σ_1 and Σ_2, resp., and find a harmonic map h homotopic to ϕ by Thm. 5.3.1. By Thm. 5.1.1, h is (anti) holomorphic in case $|d(\phi)|\ \chi(\Sigma_2) < \chi(\Sigma_1)$. This, however, is in contradiction to the Riemann-Hurwitz formula, which says $|d(h)|\ \chi(\Sigma_2) = \chi(\Sigma_1) + r$, $r \geq 0$ for an (anti) holomorphic map h . Therefore, (5.1.3) must hold.

q.e.d.

Before proving Thm. 5.1.1, we note another consequence

COROLLARY 5.1.1 *If Σ_1 is diffeomorphic to S^2, then any harmonic map $h : \Sigma_1 \to \Sigma_2$ is (anti) holomorphic (and therefore constant, if $\chi(\Sigma_2) \leq 0$).*

This is due to Wood [W1] and Lemaire [L1].

Cor. 5.1.1 also follows from Lemma 1.3.4, since there are no nonzero holomorphic quadratic differentials on S^2 which easily follows from

Liouville's theorem.

We need some preparations for the proof of Thm. 5.1.1.

We shall make use of some computations of Schoen and Yau [SY]] in the sequel. It is convenient to use the complex notation. If $\rho^2(z)\, dz d\bar{z}$ and $\sigma^2(h)\, dh d\bar{h}$ are the metrics w.r.t. to conformal coordinate charts on Σ_1 and Σ_2, resp., then h as a harmonic map satisfies

(5.1.4) $$h_{z\bar{z}} + \frac{2\sigma_h}{\sigma} h_z h_{\bar{z}} = 0 , \qquad \text{cf. (1.3.4)} .$$

LEMMA 5.1.3 *At points, where* ∂h *or* $\bar{\partial} h$ *, resp., is nonzero*

(5.1.5) $$\Delta \log|\partial h|^2 = K_1 - K_2(|\partial h|^2 - |\bar{\partial} h|^2)$$

(5.1.6) $$\Delta \log|\bar{\partial} h|^2 = K_1 + K_2(|\partial h|^2 - |\bar{\partial} h|^2) ,$$

where K_i *denotes the Gauss curvature of* Σ_i *, and*

$$|\partial h|^2 = \frac{\sigma^2}{\rho^2} h_z \cdot \bar{h}_{\bar{z}} , \qquad |\bar{\partial} h|^2 = \frac{\sigma^2}{\rho^2} \bar{h}_z h_{\bar{z}} .$$

Proof For any positive smooth function f on Σ_1,

(5.1.7) $$\Delta \log f = \frac{1}{f} \Delta f - \frac{1}{f^2} \cdot \frac{1}{\rho^2} f_z f_{\bar{z}} .$$

Furthermore,

(5.1.8) $$\Delta \log \frac{1}{\rho^2} = K_1 .$$

In order to abbreviate the following calculations, we define D as the covariant derivative in the bundle $h^{-1}\, T\Sigma_2$, e.g.

$$D_{\partial/\partial z}\, h_z = h_{zz} + \frac{2\sigma_h}{\sigma} h_z h_z .$$

(5.1.4) then is expressed as

(5.1.9) $$D_{\partial/\partial z}\, h_{\bar{z}} = 0 .$$

Since

$$\sigma^2 h_z \bar{h}_{\bar{z}} = \langle h_z, \bar{h}_{\bar{z}} \rangle_{h^{-1}T\Sigma_2} ,$$

$$(5.1.10) \qquad \Delta\sigma^2 h_z \bar{h}_{\bar{z}} = \frac{1}{\rho^2} \frac{\partial}{\partial \bar{z}} \left\langle D_{\partial/\partial z} h_z, \bar{h}_{\bar{z}} \right\rangle , \qquad \text{using } (5.1.9)$$

$$= \frac{1}{\rho^2} \left\langle D_{\partial/\partial \bar{z}} D_{\partial/\partial z} h_z, \bar{h}_{\bar{z}} \right\rangle + \frac{1}{\rho^2} \left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial \bar{z}} \bar{h}_{\bar{z}} \right\rangle$$

$$= \frac{1}{\rho^2} R\left(h_*\left(\frac{\partial}{\partial \bar{z}}\right), h_*\left(\frac{\partial}{\partial z}\right), h_z, \bar{h}_{\bar{z}}\right) + \frac{1}{\rho^2} \left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial \bar{z}} \bar{h}_{\bar{z}} \right\rangle ,$$

where R denotes the curvature tensor of Σ_2

$$= -K_2 |\partial h|^2 \cdot J(h) + \frac{1}{\rho^2} \left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial \bar{z}} \bar{h}_{\bar{z}} \right\rangle ,$$

where $J(h) = |\partial h|^2 - |\bar{\partial} h|^2$ is the Jacobian of h . Moreover,

$$(5.1.11) \quad \frac{1}{\rho^2} \frac{\partial}{\partial z} \langle h_z, h_{\bar{z}} \rangle \cdot \frac{\partial}{\partial \bar{z}} \langle h_z, h_{\bar{z}} \rangle = \frac{1}{\rho^2} \langle h_z, h_{\bar{z}} \rangle \left\langle D_{\partial/\partial z} h_z, D_{\partial/\partial \bar{z}} h_{\bar{z}} \right\rangle ,$$

using again (5.1.9), and the fact that the complex dimension of Σ_2 is 1 .

(5.1.5) now follows from (5.1.7), (5.1.8), (5.1.10), and (5.1.11), and (5.1.6) can either be calculated in the same way or directly deduced from (5.1.5), since $|\bar{\partial} h|^2 = |\partial \bar{h}|^2$ and complex conjugation on the image can be considered as a change of orientation.

q.e.d.

LEMMA 5.1.4 *If* $h_z(z_0) = 0$, *then*

$$(5.1.12) \qquad |\partial h|^2 = \zeta \cdot |k|^2 \qquad \text{near } z = z_0 ,$$

where ζ *is a nonvanishing* C^2 *function, and* k *is holomorphic. A corresponding result holds for* $h_{\bar{z}}$.

Proof By (5.1.4), $f := h_z$ satisfies

$$|f_{\bar{z}}| \le c|f| .$$

Therefore, we can apply the similarity principle of Bers and Vekua (cf. [B] or [Hz1]), to obtain the representation (5.1.12) with Hölder continuous ζ. An inspection of the proof of the similarity principle shows that in our case $\zeta \in C^2$ (cf. [Hz1], p.210). (We note that a similarity principle can be derived from Cor. 5.5.2 below which also contains the existence of solutions of Beltrami equations, cf. [BJS].)

q.e.d.

Proof of Theorem 5.1.1 Lemma 5.1.4 shows that the zeros z_i of $|\partial h|^2$ are isolated, unless $\partial h \equiv 0$, and that near each z_i.

$$|\partial h|^2 = a_i |z - z_i|^{n_i} + o(|z - z_i|^{n_i}),$$

for some $a_i > 0$ and some $n_i \in \mathbb{N}$.

By Lemma 5.1.3 and the residue formula, unless $\partial h \equiv 0$

$$\int_{\Sigma_1} K_1 - \int_{\Sigma_2} K_2(|\partial h|^2 - |\bar{\partial} h|^2) = -\Sigma n_i . \tag{5.1.13}$$

Similarly, if $\bar{\partial} h \not\equiv 0$,

$$\int_{\Sigma_1} K_1 + \int_{\Sigma_2} K_2(|\partial h|^2 - |\bar{\partial} h|^2) = -\Sigma m_i \tag{5.1.14}$$

where $m_i \in \mathbb{N}$ are now the orders of the zeros of $|\bar{\partial} h|^2$. Thus, since $|\partial h|^2 - |\bar{\partial} h|^2$ is the Jacobian of h,

$$\chi(\Sigma_1) - d(h)\,\chi(\Sigma_2) \le 0, \qquad \text{unless } \partial h \equiv 0$$

and

$$\chi(\Sigma_1) + d(h)\,\chi(\Sigma_2) \le 0, \qquad \text{unless } \bar{\partial} h \equiv 0,$$

and Thm. 5.1.1 follows.

5.2 SOME LEMMATA

In this section, we want to derive some tools for our existence proofs. First of all, we note

LEMMA 5.2.1 *Suppose* B_0 *is a geodesic ball with centre* p *and radius* s, $s \le \frac{1}{3}\min(i(p), \pi/2\kappa)$, *where* κ^2 *is an upper bound for the sectional curvature of* N *and* $i(p)$ *is the injectivity radius of* p. *If* $h : \Omega \to N$ *is energy minimizing among maps which are homotopic to some map* $g : \Omega \to B_0$, *and if* $h(\partial\Omega) \subset B_0$, *then also*

$$h(\Omega) \subset B_0 .$$

(for a suitable representative of h, *again).*

Proof By assumption, we can introduce geodesic polar coordinates (r,ϕ) on $B(p,3s)$ $(0 \le r \le 3s)$.

We define a map π in the following way:

$$\begin{aligned} \pi(r,\phi) &= (r,\phi) && \text{if } r \le s \\ \pi(r,\phi) &= (\tfrac{1}{2}(3s-r),\phi) && \text{if } s \le r \le 3s \\ \pi(q) &= p && \text{if } q \in N\backslash B(p,3s) . \end{aligned}$$

(Here, we have identified a point in $B(p,3s)$ with its representation in geodesic polar coordinates.)

Using Lemma 2.2.1, it is easily seen that π can be approximated by a map satisfying the assumptions of Lemma 4.10.1.

q.e.d.

Moreover, we have the following result, based on an idea of Lebesgue and extensively used by Courant in his study of minimal surfaces (cf. e.g. [Co]).

Suppose Ω is an open subset of some two-dimensional Riemannian manifold Σ of class C^3, while S is any Riemannian manifold.

LEMMA 5.2.2 *Let* $u \in H_2^1(\Omega,S)$, $E(u) \le D$, $x_0 \in \Sigma$, $-\lambda^2$ *a lower bound for the curvature* K *of* Σ, $\delta < \min(1,\ i(\Sigma)^2,\ 1/\lambda^2)$. *Then there exists some* $r \in (\delta,\sqrt{\delta})$ *for which* $u|\partial B(x_0,r) \cap \bar{\Omega}$ *is absolutely continuous and*

$$d(u(x_1),\ u(x_2)) \le 4\pi \cdot D^{\frac{1}{2}} \cdot (\log 1/\delta)^{-\frac{1}{2}}$$

for all x_1, $x_2 \in \partial B(x_0,r) \cap \bar{\Omega}$.

Proof We introduce polar coordinates on $B(x_0,r)$, i.e. $ds^2 = dr^2 + G^2(r,\theta)\, d\theta^2$.

Since $K = -\frac{G_{rr}}{G}$ (cf. [Bl], p.153) and $G(0,\theta) = 0$, we infer

$$G(r,\theta) \le 1/\lambda \sinh \lambda r . \tag{5.2.1}$$

Now for x_1, $x_2 \in \partial B(x_0,r)$ and almost all r, since u is a Sobolev function $u|\partial B(x_0,r)$ is absolutely continuous and

$$\begin{aligned} d(u(x_1),\ u(x_2)) &\le \int_0^{2\pi} |u_\theta(x)|\, d\theta \\ &\le 2\pi \left(\int_0^{2\pi} |u_\theta|^2\, d\theta\right)^{\frac{1}{2}} \end{aligned} \tag{5.2.2}$$

where we assumed w.l.o.g. $B(x_0,r) \subset \Omega$.

The Dirichlet integral of u on $B(x_0,r)$ is

$$E(u;B(x_0,r)) = \frac{1}{2}\int_{B(x_0,r)} \left(|u_r|^2 + \frac{1}{G^2}|u_\theta|^2\right) G\, dr\, d\theta .$$

Thus, we can find some $r \in (\delta,\sqrt{\delta})$ with

$$\int_0^{2\pi} |u_\theta(r,\theta)|^2\, d\theta \le \frac{2D}{\displaystyle\int_\delta^{\sqrt{\delta}} \frac{1}{G(\rho,\theta)}\, d\theta} \le \frac{2D}{\log 1/\rho} \tag{5.2.3}$$

since for $r \leq \sqrt{\delta} \leq 1/\lambda$, $G(r,\theta) \leq 2r$ by (5.2.1).

The lemma follows from (5.2.2) and (5.2.3).

q.e.d.

Finally, we shall need the two-dimensional version of Theorem 4.10.1. This also follows from Morrey's work on the minima of two-dimensional variational problems. We shall present a proof which already illustrates some of the ideas of the arguments in later sections and is based on Lemmata 4.10.2 and 5.2.2.

LEMMA 5.2.3 *Suppose* $\partial\Omega \neq \emptyset$, $B(p,M)$ *is a disc in some surface* Σ *with radius* $M < \frac{\pi}{2\kappa}$, *where* $\kappa^2 \geq 0$ *is an upper bound of the Gauss curvature of* $B(p,M)$, *and* $g : \partial\Omega \to B(p,M)$ *is continuous and admits an extension* $\bar{g} \in H_2^1(\Omega, B(p,M))$.†

Then there exists a harmonic map $h : \Omega \to B(p,M)$ *with boundary values* g , *and* h *minimizes the energy with respect to these boundary values. Vice versa, each such energy minimizing map is harmonic. The modulus of continuity of* h *can be estimated in terms of* λ , $i(\Sigma_1)$, M , κ , *and* $E(\bar{g})$ *and the modulus of continuity of* g .

Proof (The idea is taken from the proof of Thm. 4.1 in [HW1].) As in Lemma 4.10.3, we find a weakly harmonic map which minimizes energy among all maps into $B(p,M)$ with boundary values g .

By Prop. 2.4.2, every two points in $B(p,M)$ can be joined by a unique geodesic arc in $B(p,M)$, and this arc is free of conjugate points. Suppose $q \in B(p,M)$, v_1 and v_2 are unit vectors in $T_q\Sigma$, and c_1, c_2 are the geodesic parametrized by arc length and starting at q with tangent vectors

† Here, we can again define $H_2^1(\Omega, B(p,M))$ unambiguously with the help of the global coordinates on $B(p,M)$ given by $\exp_p$.

v_1, v_2 . By Lemma 2.3.1

$$|v_1 - v_2| \cdot \frac{\sin(t\kappa)}{t\kappa} \le d(c_1(t), c_2(t))$$

as long as $c_1(t), c_2(t) \in B(p,M)$.

Therefore, on $B(p,M)\backslash B(q,\varepsilon)$, with the help of (2.2.4)

$$d(c_1(t), c_2(t)) \ge \min\left(d(c_1(\varepsilon), c_2(\varepsilon)), |v_1 - v_2| \cdot \frac{\sin(2M\kappa)}{2M\kappa}\right) .$$

Consequently, there exists $\varepsilon_0 > 0$ with the property that $B_0 := B(q,\varepsilon) \cap B(p,M)$ and $B_1 := B(p,M)$ satisfy the assumptions of Lemma 4.10.2 for every $q \in B(p,M)$ and every $\varepsilon \le \varepsilon_0$. Lemma 5.2.2 then implies that for each $x \in \Omega$ there exists a sufficiently small $\rho > 0$ with the property that

$$h(B(x,\rho) \cap \Omega) \subset B(q,\varepsilon)$$

for some $q \in B(p,M)$. ρ depends on ε , λ , $i(\Omega)$, the energy of h (which is bounded by the energy of $\bar{g}$), and the modulus of continuity of g .

Therefore, Lemma 4.10.2 implies the continuity of h . Higher regularity then follows as in chapter 4.

q.e.d.

5.3 THE EXISTENCE THEOREM OF LEMAIRE AND SACKS-UHLENBECK

We are now in a position to attack the general existence problem for harmonic maps between surfaces.

For this purpose, let Σ_1 and Σ_2 denote compact surfaces, $\partial\Sigma_2 = \emptyset$, but Σ_1 possibly having nonempty boundary. Let $\phi : \Sigma_1 \to \Sigma_2$ be a continuous map with finite energy. We denote by $[\phi]$ the class of all continuous maps which are homotopic to ϕ and coincide with ϕ on $\partial\Sigma_1$, in case $\partial\Sigma_1 \neq \emptyset$.

We choose $s = \frac{1}{3}\min(i(\Sigma_2), \pi/2\kappa)$, where $\kappa^2 \geq 0$ is an upper curvature bound on Σ_2 , and $i(\Sigma_2)$ is the injectivity radius of Σ_2 .

Let $\delta_0 < \min(1, i(\Sigma_1)^2, 1/\lambda^2)$ ($-\lambda^2$ being a lower bound for the curvature of Σ_1) satisfy

(5.3.1) $$2\pi \cdot E(\phi)^{\frac{1}{2}} (\log 1/\delta_0)^{-\frac{1}{2}} \leq s/2 ,$$

where $E(\phi)$ is the energy of ϕ , and

(5.3.2) $$d(x_1, x_2) \leq \sqrt{\delta_0} \Rightarrow d(\phi(x_1), \phi(x_2)) \leq s/2 \qquad \text{for } x_1, x_2 \in \partial\Sigma_1 .$$

Let $0 < \delta \leq \delta_0$. There exists a finite number of points $x_i \in \Sigma_1$, $i = 1, \ldots, m = m(\delta)$, for which the discs $B(x_i, \delta/2)$ cover Σ_1 .

We let u_n be a continuous energy minimizing sequence in $[\phi]$, $E(u_n) \leq E(\phi)$ w.l.o.g. for all n .

Applying Lemma 5.2.2 and using (5.3.1) and (5.3.2), for every n , we can find $r_{n,1}$, $\delta < r_{n,1} < \sqrt{\delta}$, and $p_{n,1} \in \Sigma_2$ with the property that

(5.3.3) $$u_n(\bar{\partial}B(x_1, r_{n,1})) \subset B(p_{n,1}, s)$$

where we defined $\bar{\partial}B(x,r) = \partial(B(x,r) \cap \Sigma)$.

We now have two possibilities:

either

1) There exists some δ , $0 < \delta \leq \delta_0$, with the property that for any $x \in \Sigma_1$, some r (depending on x and n) with $\delta < r \leq \sqrt{\delta}$ and with $u_n(\bar{\partial}B(x,r)) \subset B(p,s)$ for some $p \in \Sigma_2$, and every sufficiently large n , $u_n|B(x,r)$ is homotopic to the solution of the Dirichlet problem

(5.3.4) $$\begin{aligned} & g : B(x,r) \to B(p,s) \\ & g|\bar{\partial}B(x,r) = u_n|\bar{\partial}B(x,r) \end{aligned} \qquad \text{harmonic and energy minimizing}$$

(The existence of g is ensured by Lemma 5.2.3; g is actually unique by Thm. 4.11.1, but this is not needed in the following constructions.)

or

2) Possibly choosing a subsequence of the u_n, we can find a sequence of points $x_n \in \Sigma_1$, and radii $r_n > 0$, $x_n \to x_0 \in \Sigma_1$, $r_n \to 0$, with $u_n(\bar{\partial} B(x_n, r_n)) \subset B(p_n, \varepsilon_n)$ for some $p_n \in \Sigma_2$, $p_n \to p \in \Sigma_2$, $\varepsilon_n \to 0$ (using Lemma 5.2.2), but for which $u_n | B(x_n, r_n)$ is not homotopic to the solution of the Dirichlet problem (5.3.4).

In case 1), we replace u_n on $B(x_1, r_{n,1})$ by the solution of the Dirichlet problem (5.3.4) for $x = x_1$ and $r = r_{n,1}$. We can assume $r_{n,1} \to r_1$ and, using the interior modulus of continuity estimates for the solution of (5.3.4) (cf. Lemma 5.2.3) that the replaced maps, denoted by u_n^1, converge uniformly on $B(x_1, \delta-\eta)$, for any $0 < \eta < \delta$. By Lemma 5.2.1

(5.3.5) $$E(u_n^1) \le E(u_n) .$$

By the same argument as above, we then find radii $r_{n,2}$, $\delta < r_{n,2} < \sqrt{\delta}$, with

$$u_n^1(\partial B(x_2, r_{n,2})) \subset B(p_{n,2}, s)$$

for points $p_{n,2} \in \Sigma_2$.

Again, we replace u_n^1 on $B(x_2, r_{n,2})$ by the solution of the Dirichlet problem (5.3.4) for $x = x_2$ and $r = r_{n,2}$. We denote the new maps by u_n^2. Again, w.l.o.g., $r_{n,2} \to r_2$.

If we take into consideration that, by the first replacement step, u_n^1 in particular converges uniformly on $B(x_2, r_2) \cap B(x_1, \delta-\eta/2)$, if $0 < \eta < \delta$, we see that the boundary values for our second replacement step converge uniformly on $\bar{\partial} B(x_2, r_{n,2}) \cap B(x_1, \delta-\eta/2)$.

Using the estimates for the modulus of continuity for the solution of (5.3.4) at these boundary points (cf. Lemma 5.2.3) we can assume that the maps u_n^2 converge uniformly on $B(x_1, \delta-\eta) \cup B(x_2, \delta-\eta)$, if $0 < \eta < \delta$.

Furthermore, by Lemma 5.2.1 again and (5.3.5)

$$E(u_n^2) \le E(u_n^1) \le E(u_n) .$$

In this way, we repeat the replacement argument, until we get a sequence $u_n^m =: v_n$, with

$$E(v_n) \le E(u_n) \tag{5.3.6}$$

which converges uniformly on all balls $B(x_i, \delta/2)$, $i = 1, \dots, m$, and hence on all of Σ_1, since these balls cover Σ_1.

We denote the limit of the v_n by u. By uniform convergence, u_n is homotopic to ϕ.

Since $E(v_n) \le E(\phi)$ by (5.3.6), the v_n converge also weakly in H_2^1 to u, and by lower semicontinuity of the energy w.r.t. weak H_2^1 convergence and since the v_n are a minimizing sequence by (5.3.6), u minimizes energy in its homotopy class.

In particular, u minimizes energy when restricted to small balls, and hence it is harmonic and regular by Lemma 5.2.1 and Lemma 5.2.3. Observing that if $\pi_2(\Sigma_2) = 0$, any two maps from a disc into Σ_2 are homotopic, we obtain

THEOREM 5.3.1 *Suppose* Σ_1 *and* Σ_2 *are compact surfaces,* $\partial\Sigma_2 = \emptyset$, *and* $\pi_2(\Sigma_2) = 0$. *If* $\phi : \Sigma_1 \to \Sigma_2$ *is a continuous map with finite energy, then there exists a harmonic map* $u : \Sigma_1 \to \Sigma_2$ *which is homotopic to* ϕ, *coincides with* ϕ *on* $\partial\Sigma_1$ *in case* $\partial\Sigma_1 \neq \emptyset$ *and is energy minimizing among all such maps.*

Theorem 5.3.1 is the fundamental existence theorem due to Lemaire ([L1], [L2]) and Sacks-Uhlenbeck ([SkU], in case $\partial\Sigma_1 = \emptyset$) .

A different proof was given by Schoen-Yau [SY2]. The present proof was taken from [J6].

In the case of the Dirichlet problem, it is actually not necessary that Σ_2 is compact, but only that it it homogeneously regular in the sense of Morrey [M2], cf. [L2], since the boundary values prevent a minimizing sequence from disappearing at infinity.

Furthermore, the image can be of arbitrary dimension, not necessarily 2, for Thm. 5.3.1 to hold. This is also easily seen from the present proof. Finally, if one does not prescribe the homotopy class of u , the existence of a harmonic map was already proved by Morrey [M2].

5.4 THE DIRICHLET PROBLEM IF THE IMAGE IS HOMEOMORPHIC TO S^2 . TWO SOLUTIONS FOR NONCONSTANT BOUNDARY VALUES

In this section, we want to show the following result of Jost [J7] and Brezis and Coron [BC2] (in the latter paper, only simply connected domains are treated).

THEOREM 5.4.1 *Suppose Σ_1 is a compact two-dimensional Riemannian manifold with nonempty boundary $\partial\Sigma_1$, and Σ_2 is a Riemannian manifold homeomorphic to S^2 (the standard 2-sphere), and $\psi : \partial\Sigma_1 \to \Sigma_2$ is a continuous map, not mapping $\partial\Sigma_1$ onto a single point and admitting a continuous extension to a map from Σ_1 to Σ_2 with finite energy. Then there are at least two homotopically different harmonic maps $u : \Sigma_1 \to \Sigma_2$ with $u|\partial\Sigma_1 = \psi$, and both mappings minimize energy in their respective homotopy classes.*

Proof We first investigate more closely case 2) of section 5.3. W.l.o.g.

$B(p_n,\varepsilon_n) \subset B(p,2\varepsilon_n)$ and $\varepsilon_n \le s/2$ for all n, and thus the solution g of (5.3.4) for $x = x_n$, $r = r_n$ is contained in $B(p,2\varepsilon_n)$ by Lemma 5.2.1.

Since $u_n | B(x_n,r_n)$ is not homotopic to g, it has to cover $\Sigma_2 \backslash B(p,2\varepsilon_n)$. If we define

$$\tilde{u}_n = \begin{cases} u_n & \text{on } \Sigma_1 \backslash B(x_n,r_n) \\ g & \text{on } B(x_n,r_n) \end{cases}$$

then we see that

$$\text{(5.4.1)} \qquad \begin{aligned} \lim E(u_n) &\ge \lim E(u_n | \Sigma_1 \backslash B(x_n,r_n)) + \lim E(u_n | B(x_n,r_n)) \\ &\ge \lim E(\tilde{u}_n) + \text{Area}(\Sigma_2) \ , \end{aligned}$$

since $E(g) \to 0$ as $n \to \infty$, because

$$\int_0^{2\pi} |g_\theta(r_n,\theta)|^2 \, d\theta \to 0$$

as $n \to \infty$ (cf. (5.2.3)).

(Furthermore, by Lemma 5.1.2

$$E(v;B) \ge \text{Area}(v(B)) \ ,$$

and equality holds if and only if v is conformal.)

We now define

$$E_\alpha := \inf\{E(v) : v \in \alpha\}$$

for a homotopy class α of maps with $v | \partial\Sigma_1 = \psi$, and

$$E := \min_\alpha E_\alpha \ .$$

We first show the existence of a minimizing harmonic map in any homotopy class α with

$$\text{(5.4.2)} \qquad E_\alpha < E + \text{Area}(\Sigma_2) \ .$$

We choose a minimizing sequence u_n in α with

$$E(u_n) < E + \mathrm{Area}(\Sigma_2) .$$

Assuming that 2) holds, we define $\tilde{u}_n$ as above. Since clearly

$$E(\tilde{u}_n) \geq E ,$$

this would contradict (5.4.1), however. Therefore, as shown above, we obtain an energy minimizing harmonic map in α (cf. [BCl] for a similar argument). Now let $\tilde{\alpha}$ be a homotopy class with

$$E_{\tilde{\alpha}} = E ,$$

and let $\tilde{u}$ an energy minimizing map in $\tilde{\alpha}$, i.e. $E(\tilde{u}) = E$. We want to construct a map v in some homotopy class $\alpha \neq \tilde{\alpha}$ with

$$E(v) < E(\tilde{u}) + \mathrm{Area}(\Sigma_2) . \tag{5.4.3}$$

Then the arguments above show that we can find a harmonic map of minimal energy in α. In order to complete the proof, it thus only remains to construct v.

By Thm. 5.5.1 below, the metric on Σ_2 is conformally equivalent to the standard metric on S^2, and thus, we can use S^2 as a parameter domain for the image. Since ψ is not a constant map, also $\tilde{u}$ is not a constant map, and hence we can find a point x_0 in the interior of Σ_1 for which $d\tilde{u}(x_0) \neq 0$. Rotating S^2, we can assume that $\tilde{u}(x_0)$ is the south pole p_0. We introduce local coordinates on the image by stereographic projection $\pi : S^2 \to \mathbb{C}$ from the south pole p_0. $d\pi(p_0)$ then is the identity map up to a conformal factor. By Taylor's theorem, $\pi\circ\tilde{u}|\partial B(x_0,\varepsilon)$ is a linear map up to an error of order $O(\varepsilon^2)$, i.e.

$$\left|\pi\circ\tilde{u}(x) - d(\pi\circ\tilde{u})(x_0)(x - x_0)\right| = O(\varepsilon^2) \tag{5.4.4}$$

for $x \in \partial B(x_0,\varepsilon)$.

We now look at conformal maps of the form

$$w = az + b/z \,, \qquad a,b \in \mathbb{C} \,, \qquad a = a_1 + ia_2 \,, \qquad b = b_1 + ib_2 \,.$$

The restrictions of such a map to a circle $\rho(\cos\theta + i\sin\theta)$ in $\mathbb{C}$ is given by

$$u = \left(a_1\rho + \frac{b_1}{\rho}\right)\cos\theta + \left(\frac{b_2}{\rho} - a_2\rho\right)\sin\theta$$

$$v = \left(a_2\rho + \frac{b_2}{\rho}\right)\cos\theta + \left(a_1\rho - \frac{b_1}{\rho}\right)\sin\theta \,,$$

where $w = u + iv$.

Therefore, we can choose a and b in such a way that w restricted to this circle coincides with any prescribed nontrivial linear map. This map is nonsingular if

$$\rho^4 \neq \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}$$

W.l.o.g.

(5.4.5) $$\rho^4 \leq \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}$$

(otherwise we perform an inversion at the unit circle).

Hence w can be extended as a conformal map from the interior of the circle $\rho(\cos\theta + i\sin\theta)$ onto the exterior of its image. (If equality holds in (5.4.5), then this image is a straight line covered twice, and the exterior is the complement of this line in the complex plane.)

We are now in a position to define v .

On $\Sigma_1 \backslash B(x_0,\varepsilon)$ we put $v = \tilde{u}$.

On $B(x_0,\varepsilon-\varepsilon^2)$ we choose a conformal map w as above which coincides on the boundary with the linear map $\frac{1}{1-\varepsilon}\cdot d(\pi\circ\tilde{u})(x_0)$, and put $v = \pi^{-1}\circ w$.

On $B(x_0,\varepsilon)\setminus B(x_0,\varepsilon-\varepsilon^2)$ we interpolate in the following way. We introduce polar coordinates r,ϕ and define

$$f(\phi) := (\pi\circ\tilde{u})(\varepsilon,\phi)$$

$$g(\phi) := d(\pi\circ\tilde{u})(x_0)(\varepsilon,\phi) = \frac{1}{1-\varepsilon}\, d(\pi\circ\tilde{u})(x_0)(\varepsilon-\varepsilon^2,\phi)$$

and

$$t(r,\phi) := (f(\phi) - g(\phi))\cdot\frac{r}{\varepsilon^2} + \frac{1}{\varepsilon}\,(g(\phi) - (1-\varepsilon)\, f(\phi)) \; .$$

Thus $t(r,\phi)$ coincides with $f(\phi)$ and $g(\phi)$, resp. for $r = \varepsilon$ and $r = \varepsilon-\varepsilon^2$, resp.

The energy of $t(r,\phi)$ on the annulus $B(x_0,\varepsilon)\setminus B(x_0,\varepsilon-\varepsilon^2)$ is given by

$$E(t) = \int_{r=\varepsilon-\varepsilon^2}^{\varepsilon} \int_{\phi=0}^{2\pi} \left(\frac{1}{\varepsilon^4}\,|f(\phi) - g(\phi)|^2 + \frac{1}{r^2}\,\left|\left(\frac{r}{\varepsilon^2} - \frac{1-\varepsilon}{\varepsilon}\right) f'(\phi) + \left(\frac{1}{\varepsilon} - \frac{r}{\varepsilon^2}\right) g'(\phi)\right|^2\right) r\,dr\,d\phi \; .$$

Using (5.4.4) and $|f'(\phi)| = O(\varepsilon)$, $|g'(\phi)| = O(\varepsilon)$, we calculate

$$E(t) = O(\varepsilon^3) \; ,$$

and hence also

$$E(\pi^{-1}\circ t) = O(\varepsilon^3) \; ,$$

We put $v = \pi^{-1}\circ t$ on the annulus $B(x_0,\varepsilon)\setminus B(x_0,\varepsilon-\varepsilon^2)$. Therefore

$$\begin{aligned} E(v) &= E(\tilde{u}|\Sigma_1\setminus B(x_0,\varepsilon)) + E(\pi^{-1}\circ w|B(x_0,\varepsilon-\varepsilon^2)) + E(\pi^{-1}\circ t|B(x_0,\varepsilon)\setminus B(x_0,\varepsilon-\varepsilon^2)) \\ &\le E(\tilde{u}) - O(\varepsilon^2) + \mathrm{Area}(\Sigma_2) + O(\varepsilon^3) \; , \end{aligned}$$

since $E(\tilde{u}|B(x_0,\varepsilon)) = O(\varepsilon^2)$, because $d\tilde{u}(x_0) \neq 0$, and the energy of $\pi^{-1}\circ w$ is the area of its image, as π and w and hence also $\pi^{-1}\circ w$ are conformal. Thus, for sufficiently small $\varepsilon > 0$, (5.4.3) is satisfied, and the proof is complete.

5.5 CONFORMAL DIFFEOMORPHISMS OF SPHERES. THE RIEMANN MAPPING THEOREM

THEOREM 5.5.1 *Suppose* Σ *is a compact two-dimensional Riemannian manifold diffeomorphic to* S^2 *. Then there is a conformal (and hence harmonic) diffeomorphism* $h : S^2 \to \Sigma$.

This is of course well-known. We want to provide a variational proof of Theorem 5.5.1, in order to illustrate on one hand how one can overcome the difficulties arising from the noncompactness of the action of the conformal group on S^2 , and on the other hand the idea to minimize energy in an a priori suitably restricted subclass of mappings.

Proof of Thm. 5.5.1 We choose three different points z_1 , z_2 , z_3 in S^2 and three different points p_1 , p_2 , p_3 in Σ . Let $\mathcal{D}$ be the class of all diffeomorphisms $v : S^2 \to \Sigma$ satisfying

$$(5.5.1) \qquad v(z_i) = p_i \qquad (i = 1,2,3) ,$$

and let $\bar{\mathcal{D}}$ be the weak H_2^1-closure of $\mathcal{D}$.

We now claim that a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging weakly in H_2^1 is equicontinuous. For each $z \in S^2$ and $\varepsilon > 0$, by Lemma 5.2.2 we can find $\delta > 0$ and for each $n \in \mathbb{N}$ then some $r_n \in (\delta,\sqrt{\delta})$ for which

$$\operatorname{diam}(v_n(\partial B(x,r_n)) \le \varepsilon .$$

Here, δ is independent of z and n , since the energy of a weakly convergent sequence is uniformly bounded. We can choose δ so small that $B(z,\sqrt{\delta})$ contains at most one of the points z_1 , z_2 , z_3 . Now $v_n(\partial B(z,r_n))$ divides Σ into two parts, one of them being $v_n(B(z,r_n))$, since v_n is a diffeomorphism. If ε is chosen small enough, then the smaller part, i.e. the one having diameter at most ε , contains at most one of the points p_1 ,

p_2, p_3 and hence has to coincide with $v_n(B(z,r_n))$. In particular,

$$\operatorname{diam}(v_n(B(z,\delta)) \leq \varepsilon ,$$

and the v_n are equicontinuous as claimed.

We now choose an energy minimizing sequence in $\mathcal{D}$. A subsequence then converges weakly in H_2^1 towards some $v \in \bar{\mathcal{D}}$. Since the energy is lower semicontinuous with respect to weak H_2^1 convergence, v minimizes energy in $\bar{\mathcal{D}}$. We also can find a sequence of diffeomorphisms $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging weakly to v. Since the v_n are equicontinuous as shown above, they converge uniformly to v. In particular, v is continuous and homotopic to the v_n. (We can of course assume that all the v_n are homotopic.)

Moreover, if we have a sequence of diffeomorphisms $(w_n)_{n\in\mathbb{N}}$ from S^2 onto Σ, not necessarily satisfying (5.5.1), and converging uniformly and weakly in H_2^1 towards some w, then we still have

$$E(v) \leq E(w) \tag{5.5.2}$$

since the normalization (5.5.1) can always be achieved by composing w_n with a Möbius transformation, i.e. a conformal automorphism of S^2, without changing $E(w_n)$ (cf. Lemma 1.3.3).

Hence, if $\sigma_t : S^2 \to S^2$ is a family of diffeomorphisms, depending smoothly on t, with $\sigma_0 = \mathrm{id}$, then

$$\frac{d}{dt} E(v\circ\sigma_t)\Big|_{t=0} = 0 , \tag{5.5.3}$$

since $v\circ\sigma_t$ is the uniform and weak H_2^1-limit of $v_n\circ\sigma_t$.

We introduce local coordinates $z = x+iy$ on S^2 by stereographic projection and put

$$E = |v_x|^2 , \qquad F = \langle v_x, v_y \rangle , \qquad G = |v_y|^2$$

(E, F, G are defined almost everywhere, since $v \in H_2^1$) ,

$$\sigma_t = \xi + i\eta$$

(5.5.4) $$\left.\frac{\partial \sigma_t}{\partial t}\right|_{t=0} = \nu + i\omega .$$

Using Lemma 1.3.2, the energy is given by

$$E(v) = \frac{1}{2}\int_{\mathbb{C}} (E + G)\, dx\, dy$$

and

$$E(v\circ\sigma_t) = \frac{1}{2}\int_{\mathbb{C}} \{E(\xi_y^2 + \eta_y^2) - 2F(\xi_x\xi_y + \eta_x\eta_y) + G(\xi_x^2 + \eta_x^2)\}(\xi_x\eta_y - \xi_y\eta_x)^{-1}\, dx\, dy$$

Since $\sigma_0(z) = z$ and hence for $t = 0$ $\xi_x = \eta_y = 1$, $\xi_y = \eta_x = 0$, (5.5.3) then implies

$$\int_{\mathbb{C}} \{(E - G)(\nu_x - \omega_y) + 2F(\nu_y + \omega_x)\}\, dx\, dy = 0 ,$$

Putting $\phi := E - G - 2iF$, this becomes

$$\mathrm{Re}\int_{\mathbb{C}} \phi(\nu + i\omega)_{\bar{z}}\, dx\, dy = 0 .$$

Replacing $\nu + i\omega$ by $\omega - i\nu$, we see that the imaginary part likewise vanishes, and thus

(5.5.5) $$\int_{\mathbb{C}} \phi(\nu + i\omega)_{\bar{z}}\, dx\, dy = 0 .$$

Given ν and ω , we can always find a family of diffeomorphisms (for small t) satisfying (5.5.4), for example

$$\sigma_t(z) = x + t\nu(x,y) + i(y + t\omega(x,y)) .$$

Hence (5.5.5) implies

(5.5.6) $$\phi_{\bar{z}} \equiv 0 ,$$

i.e. that ϕ is holomorphic.

Since ϕ represents a quadratic differential on S^2, in stereographic projection we have $\phi(\infty) = 0$. Hence

$$\phi \equiv 0$$

by Liouville's Theorem, i.e. v satisfies the conformality relations

$$|v_x|^2 \equiv |v_y|^2 \tag{5.5.7}$$

$$\langle v_x, v_y \rangle \equiv 0$$

almost everywhere.

For notational convenience, we introduce local coordinates (v^1, v^2) on Σ. We want to exploit that v is weakly (anti)conformal and the uniform limit of diffeomorphisms in order to show that the Jacobian $v_x^1 v_y^2 - v_y^1 v_x^2$ of v has the same sign almost everywhere in S^2 (cf. 9.3.7 [M3]). Here, additional difficulties arise from the fact that v so far is only known to be of class $C^0 \cap H_2^1$, but these problems can be overcome with the arguments of Lemmata 9.2.4, 9.2.5 of [M3].

DEFINITION 5.5.1 *Suppose* G *is a plane domain of class* C^1, $\phi \in C^1(G, \mathbb{R}^2)$, $z \notin \phi(\partial G)$.

Then $m(z, \phi(\partial G))$ *is defined to be the winding number of the curve* $\phi(\partial G)$ *w.r.t. z.*

If only $\phi \in C^0(G, \mathbb{R}^2)$, *then*

$$m(z, \phi(\partial G)) := \lim_{n \to \infty} m(z, \phi_n(\partial G))$$

for any sequence $\phi_n \in C^1(\partial G, \mathbb{R}^2)$ *which converges uniformly to* ϕ *on* ∂G.

That $m(z, \phi(\partial G))$ is well defined, follows from elementary properties of winding numbers (cf. e.g. [Fe]).

LEMMA 5.5.1 *G a plane domain, $\phi \in C^0 \cap H_2^1(G,\mathbb{R}^2)$. Then for every $x_0 \in G$, there exists a set $C(x_0)$ with $H^1(C(x_0)) = 0$, where H^1 is 1-dimensional Hausdorff measure, such that for all $R \notin C(x_0)$*

$$\int_{B(x_0,R)} J(\phi)\;dx = \int_{\phi(B(x_0,R))} m(z,\phi(\partial B(x_0,R))\;dz$$

if $B(x_0,R) \subset\subset G$

$$(J(\phi) := \phi_x^1\phi_y^2 - \phi_y^1\phi_x^2)$$

Proof We can find a sequence $\phi_n \in C^1(D)$, $D \subset\subset G$, converging uniformly and strongly in H_2^1 to ϕ, so that $\phi_n \to \phi$ strongly in $H_2^1(\partial B(x_0,R))$ on $\partial B(x_0,R)$, if $R \notin C(x_0)$, $H^1(C(x_0)) = 0$.

Since $H_2^1(\partial B(x_0,R))$ functions are absolutely continuous, and the lengths of $\phi_n(\partial B(x_0,R))$ and $\phi(\partial B(x_0,R))$ are uniformly bounded, the two-dimensional measure of $\phi(\partial B(x_0,R))$ vanishes ($R \notin C(x_0)$). Consequently, $z \notin \phi(\partial B(x_0,R))$ for almost all z, and thus

$$(5.5.8) \qquad m(z,\phi_n(\partial B(x_0,R)) \to m(z,\phi(\partial B(x_0,R)) \qquad \text{for these } z.$$

Now

$$\lim_{n\to\infty} \int_{\phi_n(B(x_0,R))} m(z,\phi_n(\partial B(x_0,R))\;dz = \lim_{n\to\infty} \int_{B(x_0,R)} J(\phi_n)\;dx = \int_{B(x_0,R)} J(\phi)\;dx$$

Since

$$\int_I m(z,\phi_n(\partial B(x_0,R))\;dz \le \left(\frac{\text{meas } I}{\pi}\right)^{\frac{1}{2}} \text{length } (\phi_n(\partial B(x_0,R)),$$

for any measurable set I, we can integrate (5.5.8), and the result follows.

q.e.d.

LEMMA 5.5.2 *We suppose that $\phi_n : S^2 \to \Sigma$ are diffeomorphisms, converging uniformly and weakly in H_2^1 to ϕ.*

Then $J(\phi)$ *has the same sign almost everywhere.*

Proof We introduce coordinates on S^2 by stereographic projection. Let $B(x_0,R)$, $R \notin C(x_0)$ satisfy the assumptions of Lemma 5.5.1

$$\varepsilon_n := \max_{x\in\partial B(x_0,R)} |\phi_n(x) - \phi(x)|$$

$$V_n := \{z : d(z,\phi(\partial B(x_0,R)) > \varepsilon_n\} .$$

For $z \in V_n$, $m(z,\phi_n(\partial B(x_0,R)) = m(z,\phi(\partial B(x_0,R))$.

Lemma 5.5.1 therefore implies

$$\lim_{n\to\infty} \int_{\phi_n^{-1}(V_n)\cap B(x_0,R)} J(\phi_n) = \int_{B(x_0,R)} J(\phi) . \tag{5.5.9}$$

Since we can assume w.l.o.g., $J(\phi_n) \geq 0$ in $B(x_0,R)$ for all n , and (5.5.9) holds for almost all discs $B(x_0,R)$, the result follows.

q.e.d.

Thus, v is a weak solution of the corresponding Cauchy-Riemann equations, i.e.

$$\begin{aligned} v_x^2 &= -g_{22}^{-1}(g_{12}v_x^1 + k\sqrt{g}\ v_y^1) \\ v_y^2 &= g_{22}^{-1}(k\sqrt{g}\ v_x^1 - g_{12}v_y^1) \end{aligned} \tag{5.5.10}$$

$(g = g_{11}g_{22} - g_{12}^2)$, where $k = \pm 1$ is constant by Lemma 5.5.2. Since (5.5.10) is a linear first-order elliptic system, v is regular.

LEMMA 5.5.3 v *is a homeomorphism.*

Proof We assume that v is not a homeomorphism. The v is not injective, i.e. there must exist two points z_1 , z_2 , $z_1 \neq z_2$ with $v(z_1) = v(z_2)$. We choose a shortest segment γ_n joining $v_n(z_1)$ and $v_n(z_2)$. Since v_n is a homeomorphism, $\tilde{\gamma}_n := v_n^{-1}(\gamma_n)$ is a curve joining z_1 and z_2.

If $p_{n,\delta}$ is a point on $\partial B(z_1,\delta) \cap \tilde{\gamma}_n$, then for $n \to \infty$ we can find a subsequence of $(p_{n,\delta})$ converging to some point p_δ on $\partial B(z_1,\delta)$. Since the v_n converge uniformly to v, we see that $v(p_\delta) = v(z_1) = v(z_2)$. Thus, a whole continuum is mapped onto the single point $v(z_1) = v(z_2)$ by v.

At interior points, we can choose again local coordinates v^1, v^2. From (5.5.10) we conclude that v^1 and v^2 are harmonic, e.g.

$$\Delta v^1 + \Gamma^1_{11}(v^1_x v^1_x + v^1_y v^1_y) + 2\Gamma^1_{12}(v^1_x v^2_x + v^1_y v^2_y) + \Gamma^1_{22}(v^2_x v^2_x + v^2_y v^2_y) = 0 . \tag{5.5.11}$$

From (5.5.10) and (5.5.11) we obtain

$$|v^1_{z\bar{z}}| \le c|v^1_z| \le K \tag{5.5.12}$$

since $v \in C^2(B)$.

We now use the following result of Hartman-Wintner [HtW] (a proof of the version presented here can also be found in [J8]).

LEMMA 5.5.4 *Suppose* $u \in C^{1,1}(G,\mathbb{R})$, G *a plane domain,* $z_0 \in G$, *and*

$$|u_{z\bar{z}}| \le K(|u_z| + |u|) , \tag{5.5.13}$$

where K *is a fixed constant.*

If

$$u(z) = o(|z - z_0|^n) \tag{5.5.14}$$

for some $n \in \mathbb{N}$ *in a neighbourhood of* z_0, *then*

$$\lim_{z \to z_0} u_z \cdot (z - z_0)^{-n}$$

exists. If (5.5.14) holds for all $u \in \mathbb{N}$, *then*

$$u \equiv 0 .$$

We continue the proof of Lemma 5.5.3.

If now $v_z^1(z_0) = 0$ for some $z_0 \in S^2$, Lemma 5.5.4 gives the asymptotic representation

(5.5.15) $$v_z^1 = a(z - z_0)^n + o|z - z_0|^n)$$

for some $a \in \mathbb{C}$, $z \neq 0$, and some positive integer n, unless $v_z^1 \equiv 0$ in a neighbourhood of z_0. The latter is not possible, however, since it implies that the set where $v_x^1 v_y^2 - v_y^1 v_x^2 = 0$ is nonvoid and open in S^2, and therefore $v_x^1 v_y^2 - v_x^1 v_y^2 \equiv 0$ in S^2 in contradiction to the fact that v is a surjective $C^{2,\alpha}$ map onto Σ. We can choose the local coordinates in such a way that

(5.5.16) $$g_{ij}(v(z_0)) = \delta_{ij} .$$

Using (5.5.16), (5.5.11) and integrating (5.5.15), we infer

$$w(z) := v^1 + iv^2 = \rho(z - z_0)^{n+1} + \sigma(\bar{z} - \bar{z}_0)^{n+1} + o(|z - z_0|^{n+1}) + w_0 ,$$

where ρ, $\sigma \in \mathbb{R}$, $|\rho| + |\sigma| \neq 0$, $w_0 = (v^1 + iv^2)(z_0)$, in a neighbourhood of z_0.

Without loss of generality, by performing homeomorphic linear transformations, we can assume $\rho = 1, \sigma > 0, z_0 = w_0 = 0$, i.e.

(5.5.17) $$w(z) = z^{n+1} + \sigma\bar{z}^{n+1} + o(|z|^{n+1}) .$$

This, however, is in contradiction to the consequence we have obtained from the assumption that v is not injective, namely that a whole continuum of points is mapped to a single point. This proves the lemma. (The application of the Hartman-Wintner formula in the above argument is due to E. Heinz [Hz2]).

LEMMA 5.5.5 v *is a diffeomorphism.*

Proof We want to show that since v is a homeomorphism by Lemma 5.5.3, (5.5.17) cannot hold with $n \geq 1$.

Assume on the contrary, (5.5.17) holds for $n \geq 1$. Then

$$v^1(re^{i\theta}) = (1+\sigma)\, r^{n+1} \cos((n+1)\theta) + o(r^{n+1}) ,$$

and in particular

$$v^1(re^{i\pi k/n+1}) = (1+\sigma)\, r^{n+1} (-1)^k + o(r^{n+1}) \tag{5.5.18}$$

for $k = 0,1,\ldots,2n+1$.

For sufficiently small $\varepsilon > 0$ and $r \leq \varepsilon$, the sign of the left hand side of (5.5.18) is therefore $(-1)^k$.

If z traverses a Jordan curve in $\{z : z \neq 0, |z| \leq \varepsilon\}$, then $v^1(z)$ hence has to change sign at least $2n+2$ times. On the other hand, for sufficiently small $\delta > 0$, since v is a homoemorphism, the preimage of $\{|w| = \delta\}$ is such a curve, but here v^1 changes sign exactly twice. Hence $n = 0$, and the Jacobian of v does not vanish, and the lemma is proved.

q.e.d.

This also finishes the proof of Thm. 5.5.1.

COROLLARY 5.5.1 *Let* Σ *be a surface homeomorphic to* S^2 *with metric tensor given in local coordinates by bounded measurable functions* g_{ij} *, satisfying*

$$g_{11}g_{22} - g_{12}^2 \geq \lambda > 0 \qquad \text{almost everywhere .} \tag{5.5.19}$$

Then there is a homeomorphism $h : S^2 \to \Sigma$ *satisfying the conformality relations*

$$\begin{aligned} g_{ij} \frac{\partial h^i}{\partial x} \frac{\partial h^j}{\partial x} &= g_{ij} \frac{\partial h^i}{\partial y} \frac{\partial h^j}{\partial y} \\ g_{ij} \frac{\partial h^i}{\partial x} \frac{\partial h^j}{\partial y} &= 0 \end{aligned} \tag{5.5.20}$$

almost everywhere.

If $(g_{ij}) \in C^\alpha$ *, then* h *is a diffeomorphism of class* $C^{1,\alpha}$ *,* *satisfying (5.5.20) everywhere.*

Proof We let (g^n_{ij}) be a sequence of $C^{2,\alpha}$ metrics converging to (g_{ij}) pointwise almost everywhere. We denote the corresponding surfaces by Σ^n and let $h_n : S^2 \to \Sigma^n$ be a conformal diffeomorphism constructed in Thm. 5.5.1.

Since the h_n satisfy a system of the type of (5.5.10), elliptic regularity theory implies uniform C^α as well as H^1_2 estimates. Hence a subsequence converges uniformly and weakly in H^1_2 towards a weak solution h of (5.5.10).

Furthermore, since the h_n are diffeomorphisms, their inverses satisfy a system of the same type, namely

(5.5.21)
$$y^n_{v^1} = \frac{g^n_{12}}{\sqrt{g^n}} x^n_{v^1} - \frac{g^n_{11}}{\sqrt{g^n}} x^n_{v^2}$$
$$y^n_{v^2} = \frac{g^n_{22}}{\sqrt{g^n}} x^n_{v^1} - \frac{g^n_{12}}{\sqrt{g^n}} x^n_{v^2} ,$$

where $g^n = g^n_{11} g^n_{22} - (g^n_{12})^2$.

Therefore, also h^{-1}_n satisfies a uniform Hölder estimate by elliptic regularity theory, and thus we see that the limit map h has to be invertible, i.e. a homeomorphism.

In case $\Sigma \in C^{1,\alpha}$, the metrics (g^n_{ij}) can be chosen to converge with respect to the C^α-norm to (g_{ij}). From (5.5.14) we infer that the h^{-1}_n then satisfy uniform $C^{1,\alpha}$ estimates, and consequently the limit map h is a diffeomorphism.

Thus we have found the desired conformal representation of Σ , and the proof of Cor. 5.5.1 is complete.

q.e.d.

We can also derive the following version of the Riemann mapping theorem (cf. e.g. [AB]):

COROLLARY 5.5.2 *Let* S *be a compact surface with boundary, homeomorphic to the unit disc* D *, and a metric tensor* (g_{ij}) *satisfying the assumptions of Cor. 5.5.1.*

Then there is a conformal representation $h : D \to S$ *, satisfying the same conclusions as in Cor. 5.5.1.*

Proof Let S' be an isometric copy of S with opposite orientation; let $i : S \to S'$ be the isometry. Identifying s with $i(s)$ for $s \in \partial S$ gives a surface Σ to which we can apply Cor. 5.5.1 and find a conformal homeomorphism $h : S^2 \to \Sigma$. Then $i \circ h$ is another conformal homeomorphism, and we can find a conformal automorphism k of S^2 satisfying $h \circ k = i \circ h$. (This is clear for smooth metrics on Σ , since then $h^{-1} \circ i \circ h$ is a conformal diffeomorphism of S^2 . The general case follows again by approximation.) The fixed point set of k then is a circle and hence bounds a disc which is conformally equivalent to S .

q.e.d.

Note that our proof immediately yields the one-to-one-correspondence of the boundaries, first proved by Osgood and Caratheodory.

We can again normalize the conformal map by e.g. prescribing the images on ∂S of three distinct points on ∂D .

The preceding result is due to Lichtenstein [Li] (for C^{α}-metrics), Lavrent'ev [Lv] (for continuous metrics), and Morrey [M1].

In a future publication, I shall demonstrate that the preceding methods can also yield conformal representations of surfaces of higher genus. This approach can considerably simplify a large portion of the uniformization theory.

5.6 EXISTENCE OF HARMONIC DIFFEOMORPHISMS, IF THE IMAGE IS CONTAINED IN A CONVEX BALL

THEOREM 5.6.1 *Assume* $u : D \to B(p,M)$ *is an injective harmonic map, where* D *is the unit disc and* $B(p,M)$ *is a disc on some surface with* $M < \frac{\pi}{2\kappa}$ *, where* κ^2 *again is an upper curvature bound. Assume that* $g := u|\partial D$ *is a* C^2*-diffeomorphism onto* $g(\partial D)$ *satisfying*

(5.6.1) $$0 < b \le \left|\frac{dg(\phi)}{d\phi}\right| \qquad \text{for all} \quad \phi \in \partial D .$$

Assume furthermore that $g(\partial D)$ *is strictly convex w.r.t.* $u(D)$ *, the geodesic curvature* κ_g *satisfying*

(5.6.2) $$0 < a_1 \le \kappa_g(g(\partial D))(g(\phi)) \le a_2 \qquad \text{for all} \quad \phi \in \partial D .$$

Then the functional determinant $J(u(x))$ *satisfies for all* $x \in D$

(5.6.3) $$|J(u(x))| \ge \delta_1^{-1} ,$$

where $\delta_1 = \delta_1(\omega,\kappa,M,a_1,a_2,b,|g|_{C^{1,\alpha}})$.

Without assuming (5.6.1) and (5.6.2), on each disc $B(0,r)$ *,* $0 < r < 1$ *,*

$$|J(u(x)| \ge \delta_2^{-1} \qquad \text{for} \quad x \in B(0,r)$$

$$\delta_2 = \delta_2(\omega,\kappa,M,r, \text{ meas } u(D), |g|_{C^\alpha}) ,$$

or

$$|J(u(x)| \ge \delta_3^{-1} \qquad \text{for} \quad x \in B(0,r)$$

where δ_3 *depends on* ω *,* κ *,* M *,* r *,* meas $u(D)$ *,* $E(u)$ *, and on some kind of normalization like fixing the images of three boundary points or of one interior point.*

We omit the proof which can be found in [JK1]. Whereas the boundary estimate basically follows by applying the maximum principle to $d^2(u(x), g(\partial D))$, the interior estimate depends on deep estimates of E. Heinz ([Hz5]).

We can now prove the main result of [J3].

THEOREM 5.6.2 *Suppose* Ω *is a compact domain with* C^2 *boundary* $\partial\Omega$ *on some surface, and that* Σ *is another surface. We assume that* $\psi : \bar{\Omega} \to \Sigma$ *maps* $\bar{\Omega}$ *homeomorphically onto its image, that* $\psi(\partial\Omega)$ *is contained in some disc* $B(p,M)$ *with radius* $M < \frac{\pi}{2\kappa}$ *(where* $\kappa^2 \geq 0$ *is an upper curvature bound on* $B(p,M)$ *) and that the curves* $\psi(\partial\Omega)$ *are of class* C^2 *and convex w.r.t.* $\partial(\Omega)$.

Then there exists a harmonic mapping $u : \Omega \to B(p,M)$ *with the boundary values prescribed by* ψ *which is a homeomorphism between* $\bar{\Omega}$ *and its image, and a diffeomorphism in the interior.*

Moreover, if $\psi|\partial\Omega$ *is even a* C^2*-diffeomorphism then* u *is a diffeomorphism up to the boundary.*

Theorems 5.6.2 and 4.11.1 imply

COROLLARY 5.6.1 *Under the assumptions of Thm. 5.6.2, each harmonic map which solves the Dirichlet problem defined by* ψ *and which maps* Ω *into a geodesic disc* $B(p,M)$ *with radius* $M < \frac{\pi}{2\kappa}$*, is a diffeomorphism in* Ω .

Proof of Theorem 5.6.2 First of all, $\partial\Omega$ is connected. Otherwise, $\psi(\partial\Omega)$ would consist of at least two curves, both of them convex w.r.t. $\psi(\Omega)$. Therefore, we could find a nontrivial closed geodesic γ in $\psi(\Omega) \subset B(p,M)$ with an easy Arzela-Ascoli argument. Since a geodesic can be considered as a special case of a harmonic map and $M < \frac{\pi}{2\kappa}$, Lemmata 1.7.1 and 2.3.2 imply that γ has to be a point, which is a contradiction. Therefore, $\partial\Omega$ is connected, and since Ω is homeomorphic to $\psi(\Omega)$, we conclude that Ω is a disc, topologically.

Therefore, we have to prove the theorem only for the case where Ω is the plane unit disc D, taking the existence (cf. Cor. 5.5.2) of a conformal map $k : D \to \Omega$ and the composition property Lemma 1.3.3 into account.

For the rest of this section, we assume that $\psi : \partial D \to \psi(\partial D)$ is a C^2-diffeomorphism between curves of class $C^{2,\alpha}$, that $\psi(\partial\Omega)$ is not only convex, but strictly convex, and that we have the following quantitative bounds

$$(5.6.4) \qquad \left|\frac{d^2}{d\phi^2}\psi(\phi)\right| \le b_1$$

and $\qquad$ for $\phi \in \partial D$

$$(5.6.5) \qquad \left|\frac{d}{d\phi}\psi(\phi)\right| \ge b_2^{-1}$$

and

$$(5.6.6) \qquad 0 < a_1 \le \kappa_g(\psi(\partial D)) \le a_2 .$$

These assumptions can be removed later on by approximation arguments which we shall indicate below.

By virtue of Cor. 5.5.2 again, there is a conformal map $k : D \to \psi(D)$. By a variation of boundary values, we now want to deform this conformal map into a harmonic diffeomorphism u.

Without loss of generality, we may assume that the boundary value map preserves the orientation of ∂D. Now let γ be the parametrization of the boundary curve of $\psi(D)$ by arc length. We set

$$(5.6.7) \quad \omega(\phi,\lambda) := \gamma(\lambda\gamma^{-1}(k(\phi)) + (1-\lambda)\gamma^{-1}(\psi(\phi))) , \quad \phi \in \partial D , \lambda \in [0,1] .$$

ω deforms the boundary values of k into the boundary values prescribed by ψ .

Since we assumed that (5.6.4) and (5.6.5) hold and that $\psi(\partial D) \in C^{2,\alpha}$, well-known regularity properties of conformal maps imply that

(5.6.8) $$\omega(\phi,\lambda) \ , \quad \frac{\partial}{\partial\phi}\omega(\phi,\lambda) \quad \text{and} \quad \frac{\partial^2}{\partial\phi^2}\omega(\phi,\lambda)$$

are continuous functions of λ ,

(5.6.9) $\frac{\partial}{\partial\phi}\omega(\phi,\lambda)$ does not vanish for any $\phi \in \partial D$ and $\lambda \in [0,1]$.

Let now u_λ denote the harmonic map from D to $B(p,M)$ with boundary values $\omega(\cdot,\lambda)$, (the existence of u_λ follows from Lemma 5.2.3) and let $\lambda_n \in [0,1]$ be a sequence converging to some $\lambda \in [0,1]$.

By Thm. 4.9.1, the Arzela-Ascoli Theorem and the uniqueness theorem 4.11.1, u_{λ_n} converges to the harmonic map u_λ in the $C^{1,\beta}$-topology, $0 < \beta < \alpha$. In particular,

$$p(\lambda) := \inf_{x\in D} \left|J(u_\lambda)(x)\right|$$

depends continuously on λ ($J(u_\lambda)$ denotes the Jacobian of u_λ). We define $L := \{\lambda \in [0,1] : p(\lambda) > 0\}$. By Cor. 5.5.2, $0 \in L$ (u_0 is the conformal map k), and therefore L is not empty. Since we assumed (5.6.5) and (5.6.6), which implied (5.6.8) and (5.6.9) we can apply Thm. 5.6.1 to the extent that

(5.6.10) $$p(\lambda) \geq p_0 > 0 \qquad \text{for } \lambda \in L .$$

Since $p(\lambda)$ depends continuously on λ , (5.6.10) implies $L = [0,1]$. Thus, u_1 is a local diffeomorphism and a diffeomorphism between the boundaries of D and $u_1(D)$, and consequently a global diffeomorphism by the homotopy lifting theorem.

Thus, the proof of Thm. 5.6.2 is complete, except for the approximation arguments.

So far, we have assumed that the boundary of the image is strictly convex, and, in addition, that the boundary values are a diffeomorphism of class C^2. We now have to prove the theorem also for the case that the boundary is only supposed to be convex and that the boundary values are only supposed to induce a homeomorphism of the boundaries.

We shall present only the first approximation argument. It is a modification of the corresponding one given by E. Heinz in [Hz4], pp.178-183. The reasoning for the second case can be taken over from [Hz3], pp.351-352, in case $\partial\psi(D) \in C^{2,\alpha}$.

Therefore, let us suppose that the boundary of the image $\psi(D)$ is only convex, while the boundary values ψ are still assumed to be a diffeomorphism of class C^2. Then we argue in the following way:

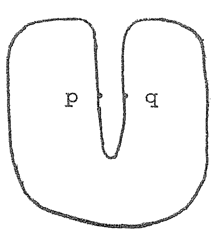
Given a metric g_{ij} on the image with respect to which the boundary of $A := \psi(D)$ is convex, there is a sequence $\{g^n_{ij}\}$ of metrics on A such that ∂A is even strictly convex with respect to g^n_{ij}, according to [Hz4], §4. Moreover, $\{g^n_{ij}\}$ can be chosen to converge uniformly to g_{ij} on A together with their first and second derivatives, as $n \to \infty$. Keeping the boundary values ψ fixed, we consider the map $u_n(x)$ which is harmonic in the metric g^n_{ij} and which solves the Dirichlet problem with boundary values ψ. The existence of u_n is guaranteed by the arguments given above — at least for large values of n, when g^n_{ij} is so close to g_{ij} that the geometric conditions are satisfied.

By virtue of Thm. 5.6.1, on each disc $B(0,r)$, $r < 1$, there is an a-priori bound of the functional determinant of $u_n(x)$ from below. Moreover, by virtue of Thm. 4.9.1, we can choose a subsequence of the functions $u_n(x)$ which converges uniformly on D together with the first derivatives to a map $u(x)$. In particular, the u_n converge to u strongly in H_2^1. Therefore, u is a weakly harmonic map w.r.t. the metric g_{ij}, i.e. a weak solution of the corresponding Euler equations. Since u is also of class C^1, linear elliptic regularity theory implies that u is a classical solution, i.e. harmonic. Moreover, u is a local diffeomorphism in the interior, and since it is the uniform limit of diffeomorphisms, it is a diffeomorphism in the interior.

q.e.d.

Remarks 1) Actually, using a further approximation argument, we do not even have to assume that the boundary values are homeomorphic. We need only that they are continuous and monotonic, i.e. a uniform limit of homeomorphisms. The corresponding harmonic solution of the Dirichlet problem still remains a diffeomorphism in the interior.

2) In the case where both Ω and $\psi(\Omega)$ are bounded simply connected domains in the plane, the assertion of Thm. 5.6.2 was already obtained by Radó and Kneser [Rd], [Kn1], and Choquet [Cq]. Choquet also showed that the convexity of the boundary of the image is necessary for Thm. 5.6.2 to hold. The reason is the following. Suppose the image has the depicted shape. If



the boundary values $\psi(\partial\Omega)$ are concentrated near p and q, then by the mean value property of harmonic functions, some points of Ω will be mapped onto points between p and q not belonging to $\psi(\Omega)$.

This is in essential contrast to the case of conformal maps where convexity of the image is not necessary to guarantee that the solution is a diffeomorphism (cf. Cor. 5.5.2). Note that a conformal map is a solution of a free boundary value problem instead of a Dirichlet problem.

5.7 EXISTENCE OF HARMONIC DIFFEOMORPHISMS BETWEEN CLOSED SURFACES

The main result of this section is

THEOREM 5.7.1 *Suppose that* Σ_1 *and* Σ_2 *are compact surfaces without boundary, and that* $\phi : \Sigma_1 \to \Sigma_2$ *is a diffeomorphism. Then there exists a harmonic diffeomorphism* $u : \Sigma_1 \to \Sigma_2$ *homotopic to* ϕ *. Furthermore,* u *is of least energy among all diffeomorphisms homotopic to* ϕ *.*

Thm. 5.7.1 was proved by Jost-Schoen [JS], but it was first claimed by Shibata [Sh] in 1963. His proof contained several mistakes, however, and was therefore rejected.

H. Sealey then carefully examined Shibata's paper in his thesis [Se] and was able to correct some (but not all) of the mistakes. The proof of [JS], however, proceeds along completely different lines than the Shibata-Sealey approach and depends in an essential way on Thm. 5.6.2.

Thms. 5.7.1 and 4.11.1 immediately imply the following corollary, proved by Schoen-Yau [SY1] and Sampson [Sa].

COROLLARY 5.7.1 *If under the assumptions of Thm. 5.7.1,* Σ_2 *has nonpositive curvature, then every harmonic map homotopic to a diffeomorphism is itself diffeomorphic.*

Furthermore, we have

COROLLARY 5.7.2 *Suppose that* Σ_1 *and* Σ_2 *are compact surfaces without boundary, and that* $\psi : \Sigma_1 \to \Sigma_2$ *is a covering map, i.e. a local diffeomorphism. Then there exists a harmonic covering map* $u : \Sigma_1 \to \Sigma_2$ *, homotopic to* ψ *.*

Proof of Corollary 5.7.2 We can pull back the metric ds^2 of Σ_2 via ψ to obtain a surface Σ_2', diffeomorphic to Σ_1 and with metric ψ^*ds^2. Then $\psi : \Sigma_2' \to \Sigma_2$ is a local isometry. By Thm. 5.7.1, there is a harmonic diffeomorphism $u' : \Sigma_1 \to \Sigma_2'$, homotopic to the identity. $u := \psi \circ u$ then is the desired harmonic covering map.

Proof of Theorem 5.7.1 (following [JS]) If Σ_1 and Σ_2 are homeomorphic to S^2, then we can find a conformal (and hence harmonic) diffeomorphism homotopic to ψ by Thm. 5.5.1. The case where Σ_1 and Σ_2 are homeomorphic to the real projective space is similarly handled by passing to two-sheeted coverings. Thus we can assume w.l.o.g. that $\pi_2(\Sigma_i) = 0 \quad (i = 1,2)$.

We let $\mathcal{D}$ be the class of diffeomorphisms from Σ_1 onto Σ_2 homotopic to ϕ. Since $\pi_2(\Sigma_2) = 0$ a homotopically trivial Jordan curve separates Σ_2 into two topologically different parts, one being a disc and the other one having higher connectivity. Therefore, the argument in the proof of Thm. 5.5.1 gives equicontinuity of a weakly convergent sequence in $\mathcal{D}$ even without a normalization.

We again let $\bar{\mathcal{D}}$ be the weak H_2^1-closure of $\mathcal{D}$, and choose an energy minimizing sequence in $\bar{\mathcal{D}}$. A subsequence then converges weakly in H_2^1 towards some $u_0 \in \bar{\mathcal{D}}$, and u_0 minimizes energy in $\bar{\mathcal{D}}$ by lower semi-continuity again. We also can find a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converging weakly in H_2^1 to u_0. Since the u_n are equicontinuous, they converge also uniformly to u_0, and hence u_0 is continuous and homotopic to ϕ. The u_n, since converging weakly, have uniformly bounded energy,

$$E(u_n) \le K , \quad \text{say} .$$

We want to show that u_0 is a harmonic diffeomorphism. We consider an arbitrary point $x_0 \in \Sigma_1$ and define

$$B_\sigma := \overset{\circ}{B}(u_0(x_0),\sigma)$$

i.e. the open disc in Σ_2 centred at $u_0(x_0)$ with radius σ .

We restrict ourselves in the sequel to values of σ which are smaller than the injectivity radius of Σ_2 and smaller than $\pi/2\kappa$, where κ^2 again is an upper bound for the curvature of Σ_2 . We define

$$\Omega_0 := u_0^{-1}(B_\sigma)$$

$$\Omega_n := u_n^{-1}(B_\sigma) \qquad (n \in \mathbb{N})$$

W.l.o.g., we can assume $x_0 \in \Omega_n$ for all n , since the u_n converge uniformly to u_0 . Let D be the unit disc in the complex plane and

$$F_n : D \to \bar{\Omega}_n$$

be a conformal mapping which maps 0 to x_0 .

The proof of the existence of F_n is the same as that of Cor. 5.5.2 since instead of fixing three boundary points, we can fix an interior point (and a tangent direction at this point, but that is not necessary for the proof) in order to guarantee the equicontinuity of a minimizing sequence as in 5.5.

Since $\Gamma_n := \partial\Omega_n$ is a Jordan curve of class C^1 (because u_n is a diffeomorphism), F_n is a homeomorphism of D onto $\bar{\Omega}_n$, and therefore $u_n \circ F_n$ maps ∂D homeomorphically onto ∂B_σ . By Thm. 5.6.2 and Cor. 5.6.1, there exists a unique harmonic mapping $v_n : D \to B_\sigma$ which assumes the boundary values prescribed by $u_n \circ F_n$, and v_n minimizes energy in its homotopy class and is a diffeomorphism.

In particular

(5.7.1) $$E_D(v_n) \le E_D(u_n \circ F_n) = E_{\Omega_n}(u_n) \le K$$

by Lemma 1.3.2 ($E_S(f)$ is the energy of the mapping f over the set S). Since the u_n converge uniformly to u_0 , we can assume that $u_n \circ F_n(0)$ stays in an arbitrarily small neighbourhood of $u_0(x_0)$. Therefore, we can again apply the argument of section 5.5 to show that the maps $u_n \circ F_n$ are equicontinuous on D . In particular, the boundary values of v_n , namely $u_n \circ F_n|\partial D$, are equicontinuous. By Thms. 4.9.1 and 4.7.1, we can therefore assume that the v_n converge uniformly on D to a map v_0 which is harmonic in the interior of D . Using Thm. 5.6.1, we see furthermore that v_0 is a diffeomorphism in the interior of D .

We define now

$$\tilde{u}_n = \begin{cases} v_n \circ F_n^{-1} & \text{in } \Omega_n \\ u_n & \text{in } \Sigma_1 \setminus \Omega_n . \end{cases}$$

Clearly, $\tilde{u}_n$ is a Lipschitz map and lies in H_2^1 and $E(\tilde{u}_n) \le K$. We can also assume w.l.o.g. (by approximation) that the u_n are of class $C^{1,\alpha}$. Then, for each n , the functional determinant of $\tilde{u}_n$ is defined and bounded from below on $\Sigma_1 \setminus \Omega_n$ by Thm. 5.6.1. It is easily seen by an approximation argument that $\tilde{u}_n \in \bar{\mathcal{D}}$.

Using Lemma 5.2.2 as before, we can assume again w.l.o.g. that the $\tilde{u}_n$ converge on Σ_1 weakly in H_2^1 and uniformly to a map $\tilde{u}_0 \in \bar{\mathcal{D}}$ and that the F_n converge uniformly on compact subsets to a conformal map F . Since $E_D(F_n) = \mathrm{Area}(\Omega_n) \le \mathrm{Area}(\Sigma_1)$, F maps $\mathring{D}$ diffeomorphically onto some open set $\Omega \subset \Sigma_1$, and 0 is mapped to x_0 . F is not necessarily smooth on ∂D , but that does not affect the following arguments.

$u_0 \circ F$ is the uniform limit of $u_n \circ F_n$ and thus extends continuously to D . Since $u_n \circ F_n$ and v_n coincide on ∂D , it follows that also $u_0 \circ F$ and

v_0 coincide there, and since v_0 is harmonic and therefore energy minimizing (by Theorem 4.11.1) in its homotopy class,

$$E_D(v_0) \le E_D(u_0 \circ F) \ .$$

Since conformal maps preserve energy by Lemma 1.3.2, this implies

$$E_\Omega(\tilde{u}_0) \le E_\Omega(u_0) \ . \tag{5.7.2}$$

We now want to show that

$$E_{\Sigma_1 \setminus \Omega}(\tilde{u}_0) = E_{\Sigma_1 \setminus \Omega}(u_0) \ . \tag{5.7.3}$$

For this, it is sufficient to show that u_0 and $\tilde{u}_0$ coincide almost everywhere outside Ω. We claim that

$$\Sigma_1 \setminus \Omega \subset u_0^{-1}(\Sigma_2 \setminus B_\sigma) \ . \tag{5.7.4}$$

We define

$$\rho_n(x) := d(u_n(x), \ u_0(x_0))$$

$$\rho_0(x) := d(u_0(x), \ u_0(x_0))$$

for $x \in \Sigma_1$. Let $x \in \Sigma_1 \setminus \Omega$. If

$$\rho_0(x) = \lim_{n\to\infty} \rho_n(x) \ge \sigma \ ,$$

then

$$x \in u_0^{-1}(\Sigma_2 \setminus B_\sigma) \ .$$

Since the $\rho_n \circ u_n \circ F_n$ are equicontinuous and equal to σ on ∂D, $\rho_0(x) < \sigma$ implies that

$$d(F_0^{-1}(x), \ \partial D) \ge \delta > 0$$

for sufficiently large n.

Since on the other hand, the F_n converge uniformly to F on compact subsets of D, this would imply $x \in F(D) = \Omega$ which contradicts the assumption $x \in \Sigma_1 \sim \Omega$. This proves (5.7.4).

We also have

$$u_0^{-1}(\Sigma_2 \backslash B_\sigma) = u_0^{-1}(\partial B_\sigma) \cup u_0^{-1}(\Sigma_2 \backslash \bar{B}_\sigma) ,$$

and since the sets $u_0^{-1}(\partial B_\sigma)$ cover a neighbourhood of x_0 and are disjoint, we can assume w.l.o.g. that the two-dimensional measure of $u_0^{-1}(\partial B_\sigma)$ vanishes for our chosen σ. If

$$x \in u_0^{-1}(\Sigma_2 \backslash \bar{B}_\sigma) ,$$

then

$$\lim_{n\to\infty} \rho_n(x) = \rho_0(x) > \sigma$$

and because of the equicontinuity of the functions ρ_n, there exists an open neighbourhood U of x such that $\rho_n|U > \sigma$ for sufficiently large n. This implies

$$\tilde{u}_0 = \lim_{n\to\infty} \tilde{u}_n = \lim_{n\to\infty} u_n = u_0 \quad \text{on } U .$$

Therefore $u_0 = \tilde{u}_0$ almost everywhere on $u_0^{-1}(\Sigma_2 \backslash B_\sigma)$, and (5.7.3) now follows from (5.7.4). By the choice of u_0, we have on the other hand

$$E_{\Sigma_1}(u_0) \le E_{\Sigma_1}(\tilde{u}_0) .$$

Thus, we conclude from (5.7.2) and (5.7.3) that

$$E_\Omega(\tilde{u}_0) = E_\Omega(u_0)$$

and consequently

$$E_D(v_0) = E_D(u_0 \circ F) .$$

Since v_0 and $u_0 \circ F$ coincide on ∂D, we conclude from the uniqueness of energy minimizing maps (Thms. 4.11.1 and Lemma 5.2.3) that v_0 and $u_0 \circ F$ coincide on D. Therefore $u_0 \circ F$ and consequently also u_0 is a harmonic diffeomorphism, the latter in Ω, which is a neighbourhood of an arbitrarily chosen point $x_0 \in \Sigma_1$. This finishes the proof of Theorem 5.7.1.

q.e.d.

With the same method, we can also improve Thm. 5.6.2.

THEOREM 5.7.2 *Let* $\Omega \subset \Sigma_1$ *be a two-dimensional domain with nonempty boundary* $\partial\Omega$ *consisting of* C^2 *curves, and let* $\psi : \bar{\Omega} \to \Sigma_2$ *be a homeomorphism of* $\bar{\Omega}$ *onto its image* $\psi(\bar{\Omega})$ *, and suppose that the curves* $\psi(\partial\Omega)$ *are of class* C^2 *and convex with respect to* $\psi(\Omega)$.

Then there exists a harmonic diffeomorphism $u : \Omega \to \psi(\Omega)$ *which is homotopic to* ψ *and satisfies* $u = \psi$ *on* $\partial\Omega$. *Moreover,* u *is of least energy among all diffeomorphisms homotopic to* ψ *and assuming the same boundary values.*

This result is again taken from [JS]. The case of non-positive image curvature was solved in [SY1].

Proof We assume first that $\partial\Omega$ and $\psi(\partial\Omega)$ are of class $C^{2+\alpha}$ and that ψ gives rise to a diffeomorphism between $\partial\Omega$ and $\psi(\partial\Omega)$ and that $\psi(\partial\Omega)$ is strictly convex with respect to $\psi(\Omega)$.

In this case, the proof proceeds along the lines of the proof of Theorem 5.7.1 with an obvious change of the replacement argument at boundary points involving the first estimate of Thm. 5.6.1. The general case now follows by approximation arguments as in 5.6.

q.e.d.

5.8 SOME REMARKS

We want to indicate briefly which of the results of this chapter can be generalized to higher dimensions.

Prop. 5.1.1 was extended to arbitrary dimensions by Wood [W2], Karcher-Wood [KW], and Schoen-Uhlenbeck [SU2]. This result can be used to prove complete boundary regularity of energy minimizing maps, cf. [SU2] and [JM].

As was observed by Morrey (cf. [ES]), the minimum of energy is attained in no nontrovial homotopy class for maps from S^n onto itself, if $n \geq 3$.

It is not known whether Prop. 5.1.3 can be generalized, i.e. whether for example there is a harmonic map of degree 1 from the three-dimensional torus onto S^3 or not.

As already pointed out the existence question becomes quite different in higher dimensions, and thus it is not likely that Thm. 5.3.1 can be fully generalized. For known existence results beyond those of chapters 3 and 4, see [SU1], [SU2], [E], [J6]. An interesting non-existence result was derived by Baldes [Ba].

Thm. 5.7.1 fails in higher dimensions; even Cor. 5.7.1 does not extend, as was pointed out by Eells-Lemaire in [EL2], based on a result of Calabi [Ca].

There are, however, some interesting results about harmonic diffeomorphisms between certain classes of Kähler manifolds, cf. [Si] and [JY].

For a more complete guide to the literature on harmonic maps, we refer to the excellent survey articles by Eells and Lemaire [EL1-4]).

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