Logic, Epistemology, and the Unity of Science 35

Mark van Atten

Essays on Gödel's Reception of Leibniz, Husserl, and Brouwer

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Logic, Epistemology, and the Unity of Science

VOLUME 35

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To the memory of Leen Stout

Preface

This is a collection of most of the essays on Kurt Gödel that I have authored or co-authored. In their publication over the past decade, these essays have been dispersed, as they address various topics for various primary audiences: philosophers of logic and mathematics, phenomenologists interested in science, and historians of modern philosophy. The rationale for bringing them together here is that, not so much in spite as because of this variety, they show a coherence predicated on that of the many-sided project of Gödel's that they collectively analyse: the project of using Husserl's phenomenology to reconstruct and develop Leibniz' monadology as an axiomatic metaphysics, and then to provide a Platonistic foundation for classical mathematics starting from the metaphysics thus obtained. Brouwer's intuitionism serves as a foil. In choosing the title of this book, I have preferred descriptive accuracy to other, perhaps greater, qualities.

At times these essays go into issues internal to Leibniz', Husserl's, or Brouwer's thought that Gödel made few explicit comments on, or none at all. But in an evaluation of Gödel's reception of other philosophers this is only to be expected: A judgement to what extent the various ideas Gödel appeals to fit together and suit the purpose he has for them depends not on the web of Gödel's remarks, but on the web of ideas to which those remarks point.

I have chosen to leave the papers in their original form, and not to rework them into one continuous narrative. Naturally, this entails some repetition and overlap, but I hope that this at the same time facilitates access to the book as a whole. My consideration has been twofold. First, the continuous narrative would have been the form of choice for the more organic and comprehensive analysis that an intellectual biography of Gödel calls for; but, although the essays collected here may be read as preparatory steps for a biography of that type, they cannot, in their limitation to this one particular project, be more than that, and more should not be suggested. (In the Introduction, I give some examples of how these essays refrain from establishing connections to other parts of Gödel's life of the mind.)

Second, it is intrinsic to Gödel's project to be of interest from different perspectives and to different audiences; a form of presentation that explicitly responds to these differences is therefore not inappropriate.

For the occasion of their reprint in this volume, the essays have been recast in a uniform format, including uniform bibliographical references, and citations and translations have silently been added where missing. Spelling and punctuation have been standardised to British where appropriate. But otherwise I have followed what I consider to be the good practice of not revising papers when collecting them. Occasionally I have added a content footnote to these reprints, flagged as such; and the citation footnotes that were required by some journals have been deleted in favour of citations in the main text. As a consequence, the footnote numbering in these reprints in general diverges from that in the original publications. To be able to make the chapters available separately, as is required for the electronic edition, each comes with its own list of references; with an eye on the paper edition, a cumulative bibliography has been added, as well as a subject index and an index of authors and citations. In quotations, translations are my own, while emphasis stems from the author quoted, except where noted otherwise.

The acknowledgements specific to each chapter, the details of its original publication, and an acknowledgement of the permission to reprint it in this volume are included in the chapters themselves. Words of thanks that do not have a natural place there are those to my co-authors Robert Tragesser and Juliette Kennedy; it was a pleasure to write the two respective joint papers included here with them. Also, I am indebted to William Howard for his generous and good-spirited email letters about Gödel and related topics, in an exchange that occurred after most of the essays had been written.

More generally, I wish to thank the following persons for frequent or occasional, but in any case extensive, discussion of Gödel and Gödeliana over the years: Eric Audureau, Matthias Baaz, Paul Benacerraf, Julien Bernard, Marc Bezem, Paola Cantù, Pierre Cassou-Noguès, Thierry Coquand, Gabriella Crocco, Dirk van Dalen, John and Cheryl Dawson, Michael Detlefsen, Igor Douven, Jacques Dubucs, Eva-Maria Engelen, Fernando Ferreira, Juliet Floyd, Jaime Gaspar, Warren Goldfarb, Yannick Grannec, Leon Horsten, Piet Hut, Shinji Ikeda, Nuno Jerónimo, Aki Kanamori, Juliette Kennedy, Roman Kossak, Georg Kreisel, Nico Krijn, Paolo Mancosu, Per Martin-Löf, Amélie Mertens, Mitsu Okada, Marco Panza, Charles Parsons, Jan von Plato, Adrian Rezuş, Robin Rollinger, the late Gian-Carlo Rota, Philippe de Rouilhan, Rudy Rucker, Wilfried Sieg, Hourya Sinaceur, Göran Sundholm, Steven Tainer, William Tait, Richard Tieszen, Robert Tragesser, Anne Troelstra, Jouko Väänänen, Albert Visser, and Palle Yourgrau. (In spite of its compactness, this list may not be complete; I apologise to anyone I may have failed to include.)

At the Historical Studies-Social Science Library of the Institute for Advanced Study in Princeton, Marcia Tucker, Christine Di Bella and Erica Mosner have always been most helpful and forthcoming in all matters concerning the Gödel Papers. Likewise, I thank the staff of the Department of Rare Books and Special Collections at the Firestone Library of Princeton University, where the Gödel Papers are actually held, for their efficiency and kindness. Gabriella Crocco gave access to her copy of the microfilm edition of the Gödel Papers, while Cheryl Dawson, Robin Rollinger, and Eva-Maria Engelen provided me with numerous transcriptions from Gödel's shorthand. Without their generosity, the research reported here would never have been. My gratitude to them is profound.

Generous institutional support during the writing of these essays came from the Department of Philosophy at the University of Leuven; the Institut d'Histoire et de Philosophie des Sciences et des Techniques (CNRS/Paris I/ENS), Paris; the Institute for Advanced Study, Princeton; and Sciences, Normes, Décision (CNRS/Paris IV), Paris.

Many thanks are due to Shahid Rahman and John Symons for accepting this volume in their series *Logic, Epistemology, and the Unity of Science*. At Springer, Ties Nijssen and Christi Lue were helpful and efficient editors. Springer also engaged a reader who commented on the manuscript as a whole, which I much appreciated. I prepared the manuscript for printing using Donald Knuth's typesetting system TEX and Leslie Lamport's extension LATEX, editing my files with Rob Pike's editor Acme. I am grateful to the authors of these very useful and interesting programs, and to the internet communities dedicated to them for their advice and examples. Giuseppe Primiero, Richard Shore, and Robert Thomas kindly made available the publishers' files of the original publications of chaps. 3, 6, and 8, respectively. This saved me much time.

Without the love and patience of my wife and son, this book could not have been completed.

This volume is dedicated to the memory of Leen Stout, who, in his history class at the Erasmiaans Gymnasium in Rotterdam, oversaw my first writing on Gödel.

Saint-Germain-en-Laye, France
May 2014

Mark van Atten

Contents

Chapter 1
Introduction

Mark van Atten

I have been trying first to settle the most general philosophical and epistemological questions and then to apply the results to science.

Gödel to Cohen, 1967[1]

There is no definite knowledge in human affairs. Even science is very prejudiced in one direction. Knowledge in everyday life is also prejudiced. Two methods to transcend such prejudices are: (1) phenomenology; (2) going back to other ages.

Gödel to Wang, early 1970s[2]

Abstract After a statement of the subject and aim of the book, three aspects of Gödel's philosophical methodology are discussed: Gödel's commitment to phenomenology from about 1959 to the end of his life, the religious component in phenomenology, and the pragmatic value of Husserl's and Gödel's historical turns in philosophy. Finally, an overview is provided of the essays that follow.

Keywords L.E.J. Brouwer • Foundations • Kurt Gödel • History • Edmund Husserl • Gottfried Wilhelm Leibniz • Phenomenology • Religion

[1]Gödel (2003, 386).

[2]Wang (1996, 308). Wang does not give the exact date; p. 326 suggests it is 1971. But if comments 9.3.22 and 9.3.23 (the one quoted here) were made in the same session, then it seems it should be 1972, as Gödel made the suggestion to change 'structural factualism' to 'factual substantialism' in 1972 and on p. 144 Wang says these comments were made after that renaming. Of course, the appropriateness of the motto does not hinge on this.

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1.1 Subject and Aim

Far from considering past philosophers irrelevant to actual systematic concerns, Kurt Gödel[3] embraced the use of historical authors to frame his own philosophical perspective and work. The subject of this book is a project of his defined by reference to Leibniz and Husserl, consisting of two stages:

1. Use Husserl's transcendental phenomenology to reconstruct and develop Leibniz' monadology into an axiomatic metaphysics,
2. Apply the metaphysics thus obtained to develop a Platonistic foundation for classical mathematics.

By 'Platonism', I here mean the view that, to adopt one of Gödel's own formulations, 'mathematical objects and facts (or at least something in them) exist independently of our mental acts and decisions'[6]; I will not be concerned with the familiar question to what extent Plato's own views were Platonistic in this sense.[7]

The aim of this book is to analyse historical and systematic aspects of this project of Gödel's, and to assess its feasibility. The emphasis is on its second stage, and, to that end, also the discussion of the first stage is oriented towards pure mathematics; correspondingly, my assessment will be a partial one, as I will have relatively little to say about the feasibility of the first stage.

Towards the end of his life, Gödel was willing to admit in print that he had not succeeded in completing the first stage. In a text of 1976 called 'Some facts about Kurt Gödel', Hao Wang's written record of an account that Gödel gave him of his own intellectual development, and that Gödel permitted Wang to publish after his death, we read that

> In philosophy Gödel has never arrived at what he looked for: to arrive at a new view of the world, its basic constituents and the rules of their composition. (Wang 1987, 46)

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[3]For readers wishing to form a picture of Gödel as a person, I refer to the biographies Dawson (1997) and Yourgrau (2005); the accounts by Kreisel (1980) and Rucker (1983, 164–171) of their respective personal contacts with him[4]; and the extant snippet of film footage of Gödel at https://www.youtube.com/watch?v=aq9X-ERgnuY (Gödel appears from 1:05 to 1:25).[5] There are now also various dramatisations in novels or theatre plays; the one whose portrayal of Gödel I prefer is Grannec (2012) (whose main character is actually Gödel's wife, Adele).

[4]In his blog, Rudy Rucker has now made available scans of his handwritten conversation notes: http://www.rudyrucker.com/blog/2012/07/31/conversatons-with-kurt-godel/ (note the mistyped 'conversatons'). There is also a reprint of the text of his 1983 account at http://www.rudyrucker.com/blog/2012/08/01/memories-of-kurt-godel/

[5]According to the text at http://www.cosmolearning.com/videos/einstein-dirac-godel-selberg-harish-chandra-in-princeton-1947-1125/ (paired to a likewise impressive, but different video), the film was made in 1947 by the mathematician Abe Gelbart. Gelbart was a member of the IAS in 1947–1948.

[6]Gödel (*1951, 311). For a discussion of Gödel's Platonistic views throughout his career, see Parsons (1995).

[7]For discussion of that question, see, e.g., Burnyeat (1987), Moravcsik (1992), Chap. 7, and Pritchard (1995). A wider-ranging book-length discussion of Platonism in contemporary mathematics is Panza and Sereni (2013).

Moreover, Wang reports elsewhere that

> Gödel did not think that he himself had come close to attaining the ideal of an axiomatic theory of metaphysics. He said several times that he did not even know what the primitive concepts are. (Wang 1996, 294)

The fact that, at a given point, one has not attained the ideal, is of course no argument that this cannot be done. As Gödel had remarked to Wang a few years before, in 1972:

> It is not appropriate to say that philosophy as a rigorous science is not realizable in the foreseeable future. Time is not the main fact [factor]; it can happen any time when the right idea appears. (Wang 1996, 143)

Indeed, perhaps the right idea will appear, and the first stage of Gödel's project will be completed. The upshot of my final chapter, however, will be that the second stage is bound to fail. The reasons for that failure will turn out to be internal to transcendental phenomenology as Husserl developed it, rather than due to some twist that Gödel put on it. This means that, if my argument there is correct, its range of application is wider than just the case of Gödel's project.

The essay in this volume that would serve well as an extended introduction to Gödel's project, and hence also to this book, is 'On the philosophical development of Kurt Gödel' (written with Juliette Kennedy). The reader looking for such a longer introduction is advised to start there. If I have nevertheless not put that paper before the others, it is because I have preferred the alternative of an arrangement congruent with the inner logic of Gödel's project: Start with Leibniz (Part I), modify and develop using Husserl (Part II), compare with Brouwer (Part III), and, finally, assess (Part IV). But not much is at stake here, as the essays can be read in different orders.

The comparison with Brouwer is occasioned by the fact that his intuitionism is, to my mind, the principal foil for Gödel's project: the close affinities between phenomenology and intuitionism – conceptual affinities, and to some extent also historical ones – set the bar for Gödel's attempt to use phenomenology in quite the opposite, non-constructivistic direction. Ample attention is given therefore not only to Gödel's reception of Leibniz and Husserl, but also to his lifelong, vivid interest in Brouwer's intuitionism and the challenge that that alternative foundation poses to his project.

Gödel obviously had many other philosophical interests besides the project under discussion, e.g., Plato,[8] medieval philosophy,[9] Kant,[10] other varieties of post-Kantian German Idealism than transcendental phenomenology, the relation

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[8]See, e.g., Toledo's notes of her conversations with Gödel (Toledo 2011) and the comments on them in Franks (2011). Also Yourgrau (1989, 394–395, 397–403, 405, 407–408; 1999, 196–200).

[9]See, e.g., Engelen (2013). One particular aspect of Gödel's interest in medieval philosophy will play a role in 'Monads and Sets' (Chap. 3, Sect. 3.4.3).

[10]See, e.g., Yourgrau (1999, Chap. 5), Kovač (2008), and Parsons (2010). A work that one would have expected to find notes to in Gödel's archive is *Husserl und Kant* (Kern 1964), which appeared as Vol. 16 of the series *Phaenomenologica*, available at the Princeton University Library. (Gödel's archive does contain reading notes to Vols. 2 and 4 of that series, together with a note on Vols. 1–23; Gödel Papers 9c/22.)

of the monadology to (modern) physics and biology, and Sheldon, Royce, and Hartshorne.[11] I make no attempt here to analyse any of these interests; in a more general work on Gödel and philosophy that would of course be required.

And although I occasionally make essential use of material from Gödel's pre-phenomenological *Max-Phil* notebooks (Gödel Papers 6b/63–72), filled from the late 1930s to 1946 and then, much more sporadically, until about 1955, I make no systematic effort to relate these notebooks to the project under discussion. It is clear from the partial transcription of *Max-Phil* that is presently available that various ideas recorded there remained dear to Gödel, who for example in the 1970s repeated them in conversations with Hao Wang and with Rudy Rucker. But it is, certainly in hindsight, evident that Gödel had grown discontented with the philosophical approach he had taken in those notebooks, and that to some extent the turn to phenomenology was meant as a new start.[12] These issues can probably not be tackled before the transcription of *Max-Phil* has been completed.[13]

Finally, I should mention Gödel's philosophical correspondence with Gotthard Günther (Gödel 2003, 456–535). It took place precisely in between the period of the *Max-Phil* notebooks and Gödel's turn to phenomenology (1954–1961; Gödel's last letter is from 1959). In his Introduction to that correspondence in the *Collected Works*, Charles Parsons arrives at the following conclusion:

> [Gödel] was evidently prepared to entertain the possibility that post-Kantian idealism, to which he had apparently not had a lot of exposure, would be a source of illumination. He found Günther a clear expositor of ideas from that tradition. But he does not seem to have been disposed to work out himself a line of thought in which self-consciousness is a central concept, and when Günther did not pursue what Gödel thought the most promising direction, he lost interest. Not long after his last letter he began his study of Husserl, whose version of idealism he seems to have found much more satisfactory. (Gödel 2003, 475–476)

As I think that that conclusion is correct, and that Günther's thought did not significantly contribute to Gödel's project either as a source or as a foil, I will not treat of it here. That is not to say, however, that I think that Günther's work and his exchange with Gödel are without systematic and historical interest for idealistic philosophy.

[11] These are the three he mentions when asked by Wang 'to name some recent philosophers whom he found congenial' (Wang 1996, 141).

[12] Gödel had clearly been looking for such a new start. In a letter of April 20, 1967, Kreisel said to Gödel that the 'pregnancy' of the latter's formulations in a letter on Feferman's work had reminded him of a conversation in 1956, in which Gödel had mentioned to him that he was going to write a book on philosophy (Gödel Papers 01/90, 011233).

[13] That work is currently being done in a group led by Gabriella Crocco (Université d'Aix-Marseille), of which I am a member.

1.2 Gödel's Commitment to Phenomenology

The project described here was central to Gödel's philosophical thought from about 1959, when he began his serious study of Husserl,[14] until the end of his active career. The best known of Gödel's (implicit or explicit) recommendations of phenomenology is of course the posthumously published essay of 1961, 'The modern development of the foundations of mathematics in the light of philosophy' (Gödel *1961/?). Others are:

1. His remark in a letter to Bernays of August 11, 1961, about Kreisel's work purporting to show that ϵ_0 is the exact limit of finitary mathematics:

   > I find this result very beautiful, even if it will perhaps require a phenomenological substructure in order to be completely satisfying. (Gödel 2003, 193)[15]

2. Gödel's remark in a draft for the supplement to the 1964 reprint of his Cantor paper, left out from the published version:

   > Perhaps a further development of phenomenology will, some day, make it possible to decide questions regarding the soundness of primitive terms and their axioms in a completely convincing manner. (Gödel Papers 4/101, 040311, 12)

3. His recommendation to some logicians in the 1960s, reported by Wang, 'that they should study the sixth investigation in *Logical Investigations* for its treatment of categorial intuition' (Wang 1996, 164).
4. Kreisel's strongly expressed wish, in a letter to Gödel of June 12, 1969 (Gödel Papers 1/92, 011266), that, in one of their future conversations, Gödel give him examples of detailed phenomenological analyses. This indicates that in their exchanges at the time Gödel had continued to advocate phenomenology.[16]
5. Gödel's arrival at the view of the Dialectica Interpretation as an application of phenomenology in the late 1960s. Details are presented in 'Gödel and intuitionism' (Chap. 11). The work on the revision of the Dialectica paper is

[14] That is the year he mentions to Wang in 1976 (Wang 1987, 46; 1996, 88). Among the first items Gödel studied was the 1959 volume of the *Zeitschrift für philosophische Forschung*, of which the first two issues contained a number of contributions on Husserl, on the occasion of the latter's 100th birthday. The library request slip in Gödel's archive is stamped October 21, 1960 (Gödel Papers 5/22, 050111).

[15] Ich finde dieses Resultat sehr schön, wenn es auch vielleicht eines phänomenologischen Unterbaus bedürfen wird, um voll zu befriedigen. (Gödel 2003, 192)

[16] An advocacy that had not been lost on Kreisel. In that same year, Kreisel published his one recommendation of phenomenology that I know of:

> What this shows is, at most, that the notions considered [of subset and powerset] are difficult to analyze, not that they are dubious . . . Coming back to set theory, probably the first step is: to recognize the objectivity of the basic notions (subset, powerset) mentioned above; and then, if possible, to give a phenomenological analysis of these notions. (Kreisel 1969b, 97)

It is clear that in fact the whole paper is strongly influenced by Gödel.

surely Gödel's deepest response to his reading of Husserl. (Note that this work had begun before Kreisel's letter mentioned in the previous item; apparently, but if so, not uncharacteristically, Gödel had refrained from bringing it up in their conversations.)

6. Gödel's statement in a draft letter to Gian-Carlo Rota of 1972 that

> I believe that his [i.e., Husserl's] transc[endental] phen[omenology], carried through, would be nothing more nor less than Kant's critique of pure reason transformed into an exact science, except for the fact that the result (of the 'critique') would be far more favourable for human reason. (Gödel Papers 1/141, 012028.7)

7. Wang's report that 'in his discussions with me in the 1970s he repeatedly urged me to study Husserl's later work' (Wang 1996, 164).[17]
8. Gödel's statement to Wang in these same discussions that

> Husserl's is a very important method as an entrance into philosophy, so as finally to arrive at some metaphysics. Transcendental phenomenology with epoche as its methodology is the investigation (without knowledge of scientific facts) of the cognitive process, so as to find out what really appears to be – to find the objective concepts. (Wang 1996, 166)

Even when Gödel acknowledged to Wang that

> Phenomenology is not the only approach. Another approach is to find a list of the main categories (e.g., causation, substance, action) and their interrelations,

he continued

> which, however, are to be arrived at phenomenologically. The task must be done in the right manner.

It is true that Gödel, like many, had some qualms with Husserl's writings. For example, Wang recounts (Wang 1996, 320):

> Even though Gödel usually praised Husserl's work, he did occasionally express his frustration in studying it. I have a record of what he said on one of these occasions.[18]
>
>> I don't like particularly Husserl's way: long and difficult. He tells us no detailed way about how to do it. His work on time has been lost from the manuscripts.

And a conversation note by Sue Toledo from 1975 reports that Gödel said about the Husserliana volume *Analysen zur passiven Synthesis*:

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[17]In *From Mathematics to Philosophy* (Wang 1974, 189), Wang also writes:

> With regard to the task of setting up the axioms of set theory (including the search for new axioms), we can distinguish two questions, viz. (1) what, roughly speaking, the principles are by which we introduce the axioms, (2) what their precise meaning is and why we accept such principles. The second question is incomparably more difficult. It is my impression that Gödel proposes to answer it by phenomenological investigations.

I have not included this passage in the list above because Wang here only reports his impression, not what Gödel said. Its content is similar to that of item 2 in the list.

[18]*Note MvA*. The date is 1976; see Wang (1996, 168).

> Material of Vol. XI of Husserliana (passive constitution) should have been interesting but doesn't appear to be so.
>
> Work published during Husserl's lifetime appears more interesting. (Toledo 2011, 206)

But such qualms are perfectly compatible with a commitment to phenomenology as a body of thought. Also the occasional criticism of Husserl does not change this, as when, for example, Gödel says to Toledo in 1972 that

> His analysis of the objective world (e.g., p. 212 of *From Formal to Transcendental Logic* [sic]) is in actuality universal subjectivism, and is not the right analysis of objective existence. It is rather an analysis of the natural way of thinking about objective existence. (Toledo 2011, 202)[19]

Likewise, the fact that Gödel never published anything on (or using) phenomenology does not, by itself, indicate a reservation on Gödel's part about the validity of phenomenology. Kennedy (2013) sees the fact that the 1961 essay does not appear on either of Gödel's two lists 'What I could publish' (*Was ich publizieren könnte*) found in the archive[20] as one step towards the view that 'judgement on this point [i.e., Gödel's commitment to phenomenology] should perhaps be left open'. But that fact is wholly consistent with a characteristic trait of Gödel's, here exemplified for his views on mind and matter (Wang 1996, 5):

> In commenting on a draft of this paper,[21] Gödel asked me to add the following paragraph:
>
>> Gödel told me that he had certain deep convictions regarding mind and matter which he believed are contrary to the commonly accepted views today. The reasons for his convictions are of a very general philosophical nature and the arguments he possessed are not convincing to people with different convictions. Hence, he had chosen to state only those parts or consequences of his convictions which are definite even without reference to his general philosophy.

All things considered, then, there is no question but that from 1959 until the end of his active career Gödel was not only studying phenomenology, but moreover was committed to it, seeing it not as a finished doctrine laid down in any single text, but as a research program, to be developed, applied, and modified in the light of further reflection and experience. This is of course the only way in which commitment to a

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[19] *Note MvA*. Husserl's analysis on p. 212 (in the pagination of the original edition of *Formale und transzendentale Logik*, Husserl (1929); p. 240 in the translation Husserl (1969)) begins as follows: 'Let us start from the fact that for us – stated more distinctly: for *me* qua ego – the world is constituted as "Objective" (in the above-stated sense: there for everyone), showing itself to be the way it is, in an intersubjective cognitive community.' ('Gehen wir davon aus, daß die Welt für uns, deutlicher gesprochen, daß sie ja für *mich* als Ego konstituiert is als "objektive", in jenem Sinn der für Jedermann daseienden, sich als wie sie ist in intersubjektiver Erkenntnisgemeinschaft ausweisenden.')

[20] Gödel Papers 4/108, 040360 and 040361. These have not yet been published, but Cheryl Dawson has transcribed them. Gödel is reported to have sent a third list of this type to Oskar Morgenstern (Gödel 1995, v note a).

[21] *Note MvA*. Wang (1978).

body of thought can make philosophical sense. Husserl himself had seen it that way, as Gödel, and any attentive reader of Husserl's work, was well aware:

> According to Gödel, Husserl just provides a program to be carried out. (Wang 1996, 164)

Given Gödel's strong commitment, it is, much as Gödel scholarship owes to Hao Wang,[22] regrettable that Gödel did not, as far as I have been able to determine, discuss phenomenology also with someone who was more interested in it and better prepared. I am thinking of William Howard in particular, who recounts:

> In the fall of 1972 I am having lunch in the Institute cafeteria, and in walks Hao Wang. We know each other from ASL meetings in past years.
>
> Hao Wang: 'I come in once every two weeks, from New York, for a meeting with Gödel. He is making me read various parts of Husserl's writings, which I don't particularly want to do,[23] and then, at the meetings, he makes me discuss what I have read.'
>
> I told this to my friend Tennenbaum, and he said, 'Gödel is one of the greatest living authorities on Husserl'. I decided that I should take advantage of this, so I went to the Princeton University bookstore and looked for books by Husserl. There was one called *Cartesian Meditations*. I decided that, since I was an expert on meditation, this was down my alley. I was right. So I studied this and also the two books by Husserl that Hao Wang told me Gödel was making him read.[24] When I felt I was sufficiently prepared (early spring of 1973), I tried to get Gödel to talk about Husserl. No dice! Gödel had decided that Husserl was not on the agenda for any of our meetings, and that was that. (Howard, story 1, p. 80)[25]

One may wonder what Gödel's intentions were at the occasion. Wang has made the following observation on the dynamics of (his) conversation with Gödel:

> Now and then Gödel mentioned things of interest to me which seemed related to what we had discussed on some previous occasion. When I asked him why he had not said these things before, he would reply, 'But you did not ask me'. I interpret this response to imply that, since he had so many ideas on so many things, he preferred to limit his remarks to what was strictly relevant to the immediate context. One consequence of this was that he avoided topics and views on which he did not believe there was a shared interest, or even some empathy.

Gödel's reply to Wang reminds one of a dictum of the fourteenth-century monk Kenkō, which suggests a different explanation than Wang's, or, depending on Gödel's psychology, a complementary one:

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[22] For a comprehensive appraisal of Hao Wang as a logician and as a philosopher, see the collection Parsons and Link (2011).

[23] *Note MvA*. Hao Wang writes, for example: 'Gödel had recommended Husserl's *Ideas* to me, and I tried to read it. Not being sufficiently motivated, I found it too long-winded.' (Wang 1996, 142)

[24] *Note MvA*. According to prof. Howard (in the email referred to in the next footnote, and a second one of March 9, 2013), the two texts in question were the two that, in Quentin Lauer's English translation, are included in Husserl (1965): 'Philosophy as a Rigorous Science' and 'Philosophy and the Crisis of European Man'. The latter is Husserl's Vienna Lecture of 1935, 'Die Philosophie in der Krisis europäischen Menschheit' (Husserl 1954, 314–348).

[25] As related, with minor editorial changes, in an email from William Howard to MvA, March 7, 2013. Later, in a letter of August 2, 1973, in which he requested a fourth meeting, Howard asked Gödel specific questions about intentionality and also about the relation of phenomenology to Indian philosophy.

> It is impressive when a man is always slow to speak, even on subjects he knows thoroughly, and does not speak at all unless questioned. (Kenkō [1330–1332] 1967, 69)

Howard's case, however, was the opposite of Wang's, for he was interested in phenomenology, and he did ask. Gödel's refusal must have been a frustrating experience for Howard, as it is, indirectly, for me today.[26]

More generally, the regrettable fact seems to be that there is no record left of anyone's conversation with Gödel on phenomenology at the level of expertise that his remaining reading notes and bibliographical memoranda on the topic indicate he was capable of.

1.3 The Religious Component in Phenomenology

In *Reflections on Kurt Gödel*, Hao Wang suggests a difference between Gödel and Husserl that, to my mind, would strongly limit the extent to which Husserl's phenomenology could be used to realise the first stage Gödel's project, if it indeed exists:

> In addition, G looks for an exact or axiomatic theory in philosophy and thinks that it is also Husserl's aim. But G's conception of metaphysics as first philosophy includes centrally the concepts of God and soul.[27] It appears clear that this religious component is not part of Husserl's conception of philosophy. (Wang 1987, 161)

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[26] In an email of March 7, 2013, William Howard adds:

> I did not feel that he was brushing me aside; it was just that he had a list of topics that he wanted to discuss with me, and Husserl was not on the list. When I say 'list', I mean it literally: When I arrived for our meetings, he would have a sheet of paper on his desk before him, a sort of memorandum to himself concerning the topics for our meeting. I had a list of topics that I wanted to discuss (not on a sheet of paper but firmly in my mind); but he had his own questions, which he would ask me one after the other; it was hard for me to get any of my questions in edgewise!

[27] *Note MvA*. There is of course Gödel's remark in 1970, reported by Oskar Morgenstern in a diary note for August 29, 1970, that he feared that publishing his ontological proof of God's existence would lead people to think 'that he actually believes in God, whereas he is only engaged in a logical investigation (that is, showing that such a proof with classical assumptions (perfection, etc.), correspondingly axiomatized, is possible)' (Gödel 1995, 388, translation modified).[28] But, besides to Wang (e.g., Wang 1996, 88), over the years Gödel expressed an unequivocal belief in God in a number of places. To mention three: the *Max-Phil* notebooks; a series of letters to his mother in 1961 (Gödel 2003, 428–439); and a draft reply of 1975 to a questionnaire of the sociologist Burke Grandjean where he specified that 'My belief is theistic not pantheistic (following Leibniz rather than Spinoza)' (Gödel 2003, 448). It seems safe to say, then, that Gödel believed in God more often than not. See also Chap. 10, Footnote 4 in this volume.

[28] 'Über sein ontologischen Beweis – er hatte das Resultat vor einigen Jahren, ist jetzt zufrieden damit aber zögert mit der Publikation. Es würde ihm zugeschrieben werden daß er wirkl[ich] an Gott glaubt, wo er doch nur eine logische Untersuchung mache (d.h. zeigt, daß ein solcher Beweis mit klassischen Annahmen (Vollkommenheit usw.), entsprechend axiomatisiert, möglich sei)', as quoted in Dawson (1997, 307).

Does the last sentence of this quotation reflects Gödel's view or Wang's?[29] It is, in any case, not difficult to see what may have suggested that view. In his published works Husserl speaks about God very rarely, and when he does so in Sects. 51 and 58 of *Ideas I* (Husserl 1976a), it is to say that God can neither be a mundane object nor have His being as an episode in consciousness (Sect. 51), that God transcends both the world and absolute consciousness (Sect. 58), and that God therefore falls outside the scope of phenomenology as Husserl defines it there.

Yet, in a letter to William Ernest Hocking of July 7, 1912, so from the time Husserl was writing *Ideas I*, Husserl writes that

> Even if I have made it my life's task to found a philosophy 'from below' at least for myself, to my satisfaction (which is very difficult to gain!), I nevertheless strive unceasingly from this 'below' upwards into the heights. In the last years, metaphysical considerations, and especially the idea of God, have entered ever more powerfully into the horizon of my studies. (Brainard 2002, 251–252n80)[30]

Indeed, in research manuscripts and correspondence from 1908 (predating *Ideas I*) until the end of his life,[31] Husserl kept reflecting on God and metaphysics in relation to phenomenology, against the background of Leibniz. (Gödel would have been sensitive to this contrast between Husserl's published and unpublished writings.) For example, a text from 1908 has the title 'Teleology, God, the possibility of an all-consciousness, transcendental-phenomenologically founded metaphysics and teleology'.[32] In it, Husserl presents a conception of God as the universal consciousness, which creates the finite monads and unifies all the contents of their

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[29] In the articles on Husserl in the volume of the *Zeitschrift für philosophische Forschung* that Gödel borrowed in 1960, mentioned in Footnote 14 above, it is made very clear that Husserl believed in God and that this plays a central role in his later philosophy. See in particular Diemer (1959, 248–250) (who also notes the contrast between Husserl's published work and his correspondence) and Ingarden (1959, 462). There are various other such places in the early literature on Husserl after 1945 that Gödel is likely to have seen, but the present example is documented and already strongly suggests that the view Wang states in this quotation is not Gödel's. Note that, in a perceptive comment on Gödel's 1961 essay, Wang remarks that 'His proposed solution appears to be Husserl's phenomenology, and he says nothing explicitly about its relation to religious concepts . . . Elsewhere he suggests that Husserl's method may be applicable to metaphysical or religious concepts as well' (Wang 1996, 162).

[30] Habe ich es mir zur Lebensaufgabe gemacht eine Philosophie 'von unten' mindestens für mich, zu meiner (sehr schwer zu gewinnenden!) Befriedigung zu begründen, so strebe ich doch unablässig von dem 'Unten' hinauf in die Höhen. In den letzten Jahren sind metaphysische Erwägungen und ist insbesondere auch die Gottesidee immer stärker in den Kreis meiner Studien getreten. (Husserl 1994b, 3:160)

[31] A highly interesting report on conversations with Husserl on religion in the last years of his life is Jaegerschmid (1981a,b). For a biographical perspective on Husserl's religiosity, see Karl Schuhmann's introduction to Husserl's correspondence (Husserl 1994b, 10:33–36). For systematic considerations, the most important reference here is of course part III of Husserl (2013), 'Metaphysik: Monadologie, Teleologie und Philosophische Theologie'. See also Hart (1986), Iribarne (2000), Lo (2008), and Ales Bello (2009).

[32] Teleologie, Gott, Möglichkeit eines All-Bewusstseins. Transzendentalphänomenologisch fundierte Metaphysik und Teleologie (Husserl 2013, 160–168).

consciousnesses. That can certainly be seen as an interpretation of Leibniz' concept of the central monad; similarly, a later text, probably from 1922, considers, as its title indicates, 'The possibility of fusion of monads; the possibility of a highest (divine) monad'.[33]

But it is important to note that for Husserl these are questions, possibilities and convictions that he ponders as such; he does not present full phenomenological analyses leading to conclusions. And to the philosopher Husserl, as distinct from the faithful Christian that he also was,[34] it remained essential to follow the right methodology.[35] As he writes in a letter of 1933,

> The philosophical problems disclose themselves in their genuine meaning as transcendental-phenomenological ones in an essential systematic series of steps. On these occasions it becomes manifest that the religious-ethical problems are problems of the highest level. ... This is precisely the reason why in my writings I kept silent about the problems of philosophy of religion. (Spiegelberg 1981, 182)[36,37]

[33] Möglichkeit der Verschmelzung von Monaden. Möglichkeit einer (göttlichen) Übermonade (Husserl 1973a, 300–302).

[34] Husserl was a Jew by birth, but was not raised as a practising one. As a student he read the New Testament, decided to convert to Christianism, and was baptised Lutheran (as Gödel would be, and Leibniz had been). But Husserl (again like Gödel, and like Leibniz) was not a churchgoer. Brouwer (presumably) was baptised Dutch Reformed, as this was the denomination of his parents. When he had just turned 17, he decided to enter the Remonstrant Church, a more progressive variety of Protestantism, and wrote a highly personal profession of faith for the occasion. He was no churchgoer either, but while Leibniz, Husserl, and Gödel liked to read the Bible, we do not have such evidence in Brouwer's case. In one of his student notebooks he even claims that 'one's conscience ... is not nourished by Plato or the Bible, but it is by Kant' ('het geweten ... van Plato en de bijbel wordt het niet gevoed, wel van Kant'). – For these facts on Brouwer, see van Dalen (1999, 17–22), which includes a full translation of the profession, and Brouwer Archive, Notebook III, 31; on Gödel, Wang (1996, 27) and Dawson (1997, 4–6); on Husserl (1994b, 3:432); on Leibniz, Guhrauer (1846, 1:1, 2:332).

[35] When Husserl received a copy of Rudolf Otto's *The Holy* (*Das Heilige*, Otto 1918), he wrote in a letter to its autor (and his friend and former colleague in Göttingen) of March 5, 1919, that he much appreciated the book for its description of religious phenomena, but criticised its philosophical elaboration as follows: 'The metaphysician (theologian) in Mr Otto has carried, so it seems to me, the phenomenologist Otto away on his wings and for an image I think here of the angels who with their wings cover the eyes' (Husserl (1994b, 7:207); 'Der Metaphysiker (Theologe) in Herrn Otto hat scheint es mir den Phänomenologen Otto auf seinen Schwingen davongetragen u[nd] ich denke dabei als Gleichnis an die Engel, die mit ihren Schwingen die Augen verdecken.'). Husserl's reference is to Isaiah 6:2.

[36] Husserl to E.P. Welch, June 17/21, 1933; 'Die philosophischen Probleme erschliessen sich mit ihrem echten Sinn als transcendental-phänomenologische in einer wesensmässigen systematischen Stufenfolge. Es zeigt sich dabei, dass die ethisch-religiösen Probleme solche der höchsten Stufe sind. ... Eben darum schwieg ich mich in meinen Schriften über religionsphilosophische Probleme aus' (Husserl 1994b, 6:459).

[37] A similar attitude is found in the work of Michael Dummett; see for example his introduction to *The Logical Basis of Metaphysics* (Dummett 1991).

Moreover, in a letter of a few months earlier, Husserl did estimate that he had made progress in being able to frame the questions in the right manner. About the question of God as 'indeed the "highest and final question" in the system-building of the phenomenological method'[38] he writes:

> I am grateful enough that I have been able to develop the method and explicitly carry on with it to see the theoretical locus of the problem as a phenomenological one: first of all as the problem of the possibility of the transcendental Totality.[39]

Indeed, Husserl wrote in probably Summer 1934, this is where philosophy intrinsically leads:

> An autonomous philosophy, such as the Aristotelian was and such as remains an eternal demand, will necessarily arrive at a philosophical teleology and theology – as a non-confessional way to God.[40]

The purpose of presenting the quotations from Husserl above is to show that, contrary to what the passage from Wang suggests, a religious component was very much present in Husserl's conception of philosophy; from a systematic point of view there is, ultimately, no mismatch between Gödel's and Husserl's aims in philosophy on this account.

I had hoped to find in Gödel's archive reading notes to Dietrich Mahnke's 'Eine neue Monadologie' (1917), essentially a rewriting of Leibniz' tractate in phenomenological terms. Gödel thought highly of this work by Mahnke, assessing it as 'sensible!' (*vernünftig!*; Gödel Papers 5/25, 050120.1); the paragraphs on God in it are closely related to (but not the same as) Husserl's ideas on the topic.[41] The task of broadening the contextualisation of Gödel's project as presented here should start, I believe, with an analysis of Mahnke's work on the monadology. It would be very interesting if it could be determined whether Gödel's thoughts on the matter were, in effect, closer to Husserl's or to Mahnke's.[42]

Likewise, one could reflect on Gödel's version of the ontological proof in this context. A key question here is whether Gödel saw that proof as making merely a logical point[43] or took that proof indeed to establish the existence of God; as Parsons observes,

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[38] die in der Tat im Systembau der phänomenologischen Methode 'höchste und letzte Frage'

[39] Husserl to Father Daniel Feuling, March 30, 1933, in Husserl (1994a, 7:87f). 'Ich bin dankbar genug, dass ich die Methode soweit durchbilden und explizit fortführen konnte, um den theoretischen Ort des Problems als eines phänomenologischen zu sehen: zunächst als des Problems der Möglichkeit der transcendentalen Totalität.'

[40] Eine autonome Philosophie, wie es die aristotelische war und wie sie eine ewige Forderung bleibt, kommt notwendig zu einer philosophischen Teleologie und Theologie – als inkonfessioneller Weg zu Gott. (Husserl 2013, 259)

[41] See also Chap. 6, Sect. 6.4.1 in this volume.

[42] For a list of differences between their positions, prepared by Husserl's assistant Eugen Fink in 1933, see Husserl (1994b, 3:519–520).

[43] As the remark taken down by Morgenstern suggests; see Footnote 27 above.

> For that, it would be necessary for him to have confidence in the specific conceptual apparatus and premises of the proof. I suspect that if questioned about that, he would have said that he had not developed his philosophical views to a sufficient extent to have that level of certainty. (Parsons 2010, 186)

One way of trying to obtain that level of certainty would be to develop a phenomenological critique of that conceptual apparatus and the premises. It is not clear to me whether or not Husserl at some point meant to go that way. In 1892/93, Husserl lectured on proofs of God's existence (Schuhmann 1977, 34), but no lecture notes seem to remain, and in any case this was long before Husserl's development of the transcendental phenomenology that interested Gödel. A relevant reminiscence about the transcendental Husserl, one of the few pieces of evidence on the topic, can be found in the memoirs of the biologist and philosopher Hans Driesch. On a very long conversation he had with Husserl at a conference in April 1914, he notes:

> Of particular interest to me was a specific point in our conversation. I asked H[usserl] whether I was right to see the ontological proof of God – which from the 'essentia', i.e., the conceptual 'essence' of God, wants to conclude to his 'existentia' – as the final goal of his logical investigations. He answered yes to this question, but he has, as far as I know, never gone into it in his writings.[44]

The qualification 'as the final goal' is crucial; certainly Husserl would not, after his transcendental turn, accept, in a philosophical sense, God's existence without further phenomenological ado. A forceful statement to this effect in relation to proofs of God's existence was made by Husserl almost twenty years later, in a letter of 1932 to Father Erich Przywara. Having stated his methodological priority of an exhibition and transcendental critique of the evidence again, and having confirmed that this also applies to religious evidence, Husserl continues:

> That will help the theologians some day, although at first it will seem as if this results in bad heresies.[45]

[44]Translated from the quotation in Schuhmann (1977, 186): 'Von besonderem Interesse war mir ein bestimmter Punkt unserer Unterhaltung. Ich frug H[usserl], ob ich im Recht sei, wenn ich den ontologischen Gottesbeweis – (der aus der "essentia", d.h. dem begrifflichen "Wesen" Gottes, seine "existentia" ableiten will) – als letztes Ziel seiner logischen Untersuchungen ansahe. Er bejahte diese Frage, ist aber meines Wissens nie schriftstellerisch auf sie eingegangen.' The original is in the posthumous (Driesch 1951, 153–154).

[45]*Note MvA*. See also this passage in *Formal and Transcendental Logic*, written three years earlier: 'Even God is for me what he is, in consequence of my own productivity of consciousness; here too I must not look aside lest I commit a supposed blasphemy, rather I must see the problem. Here too, as in the case of the other ego, productivity of consciousness will hardly signify that I invent and make this highest transcendency.' (Husserl 1969, 251) ('Auch Gott ist für mich, was er ist, aus meiner eigenen Bewußtseinsleistung, auch hier darf ich aus Angst vor einer vermeinten Blasphemie nicht wegsehen, sondern muß das Problem sehen. Auch hier wird wohl, wie hinsichtlich des Alterego, Bewußtseinsleistung nicht besagen, daß ich diese höchste Transzendenz erfinde und mache', Husserl 1974, 258).

> Any wish to commit myself to theism in the Scholastic tradition – in the usual interpretations of its intentions – I decidedly dismiss. Of course this is said against Mr Keilbach and his proofs of God's existence that I (according to things I am supposed to have said in conversation) allegedly hope for.[46,47]

In absence of evidence on specifically phenomenological-theological ideas such as Gödel may have had, I cannot attempt to develop this theme any further.[48]

1.4 The Pragmatic Value of Husserl's and Gödel's Historical Turn

Not addressed in these essays is an aspect of transcendental phenomenology that is nevertheless of direct importance to Gödel's project: the later Husserl's insistence that a turn to history is not only of pragmatic value to systematic philosophy, but is necessary to it, without philosophy thereby becoming a form of historicism. Husserl argues for this position in *The Crisis of European Sciences and Transcendental Phenomenology* (Husserl 1954), a work that Gödel owned and knew well; besides to the *Crisis*, I refer the interested reader to the analyses by Carr (1987, Chaps. 3 and 4) and Hopkins (2010). I do not have much to add to their discussion, and, as far as I can see, neither would Gödel have had. But I should like to make some comments here on what the pragmatic value of a historical turn consists in; to Gödel, interested in the history of philosophy from his student days,[49] Husserl's discussion in the *Crisis* and related texts will have been a reinforcement and a development of a view he already held.

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[46]Husserl to Przywara, June 15, 1932, in Husserl (1994b, 7:237). 'Den Theologen wird das einmal helfen, obschon es zunächst scheinen wird, dass dabei arge Ketzereien resultieren. Mich auf den Theismus der Schultradition – in den üblichen Auslegungen seines Sinnes – festlegen zu wollen, lehne ich entschieden ab. Natürlich ist das gegen Herrn Keilbach gesagt und dessen vermeintlich von mir (nach angeblichen Gesprächsäusserungen) erhoffte "Gottesbeweise".'

[47]Husserl here has in mind this passage in Keilbach (1932, 213):

> We also know for a fact, that Husserl in 1926 made the following oral profession: ' . . . That the solution to the teleological problem can only be found in the theological conception, is something I too believe. But it will take a 100 years before my school can carry through an exact proof of God's existence. ('Wir wissen auch genau, daß Husserl im Jahre 1926 mündlich folgendes Bekenntnis ablegte: " . . . Daß die Lösung des teleologischen Problems nur im theologischen Begriff gefunden werden kann, das glaube ich auch. Aber es wird noch 100 Jahre dauern, bis meine Schule einen exakten Beweis für das Dasein Gottes wird führen können." ')

Keilbach does not cite a source. Also noteworthy is that there is no offprint of Keilbach's article in Husserl's personal library at the Husserl Archive in Leuven.

[48]Wang says that he is 'sure' that 'Gödel's tentative thoughts about religious metaphysics . . . did not . . . make much use of Husserl's method' (Wang 1996, 163).

[49]In 1925, Gödel attended Heinrich Gomperz' course 'Übersicht über die Geschichte der europäischen Philosophie'; the notes he took have been preserved (Gödel Papers 3/72.5, 030100.4).

The study of earlier positions and their development may clarify and sharpen our own ideas in various ways:

1. They may show interesting contrasts to our own position that allow us to arrive at a richer articulation of it;
2. They may reveal presuppositions of our own position;
3. They may enrich our own position by showing us motivations, arguments and approaches that we had not, or not sufficiently, been aware of may be available to us as well.

The study of an earlier philosophical system may turn not only to the questions and arguments that that system has, as a matter of historical fact, dealt with, but also to alternative questions and alternative arguments, which, for one reason or another, were not actually taken up. (Such reasons may themselves have been philosophical ones, but need not.) This study of the systematical possibilities and limits of a historical position is nowadays known as 'doing philosophy historically'.[50] It leads to a better understanding of a historical position and, through comparison, of our own. When doing philosophy historically, one may, and indeed should, freely use philosophical and other (e.g., mathematical) knowledge that was developed only after the historical position being studied. As Parsons has written to justify his use of modern knowledge of the foundations of logic and mathematics in a paper on Kant,

> Experience shows that one does not get far in understanding a philosopher unless one tries to think through the problems on their own merits, and in this one must use what one knows; second, if one is today to take Kant seriously as a philosopher of mathematics, one must confront him with modern knowledge. (Parsons [1969] 1983, 110–111)

Taking Leibniz and Husserl seriously as philosophers of mathematics this way is part of the very design of Gödel's project. (For an anecdote on how Gödel integrated the monadology and modern set theory into one topic, see Footnote 1 of Chap. 3.)

1.5 Overview of the Essays

The four central essays are 'Monads and Sets' (Chap. 3), 'On the philosophical development of Kurt Gödel' (Chap. 6), 'Gödel and intuitionism' (Chap. 11), and 'Construction and constitution in mathematics' (Chap. 12).

'Monads and Sets' analyses and criticises Gödel's attempt to justify, by an argument from analogy with the monadology, the reflection principle in set theory. The direct importance of that chapter for my present purpose is that my counterargument proceeds in such a way that it at the same time lends support to the belief embodied in the first stage of Gödel's project, the belief that the monadology needs to be reconstructed phenomenologically.

[50] See, e.g., Piercey's paper of that title (2003), and Ameriks' monograph *Kant and The Historical Turn* (2006) – with emphasis on Karl Leonhard Reinhold as a pioneer of this approach.

'On the philosophical development of Kurt Gödel' studies Gödel's reading of Husserl, its relation to Leibniz' monadology, and its influence on his published writings. A much greater influence of phenomenology, however, was overlooked when writing 'On the philosophical development'; this is addressed in the following paper.

'Gödel and intuitionism' discusses how on various occasions Brouwer's intuitionism actually inspired Gödel's work, in particular the Dialectica Interpretation, which Gödel in an unpublished note once characterised as 'a new intuitionistic insight . . . based on phenomenological reflection'. Although we will see (Chap. 11) that Gödel abandoned this particular attempt to construe the Dialectica Interpretation as intuitionistic in the noetic sense, the shift to the notion of reductive proof employed in the even further and better known revision still depended on phenomenology, and still marked a rapprochement to Brouwerian intuitionism. The work on a revision of the Dialectica paper shows that Gödel was not only studying Husserl and recommending his work to others, but also tried to advance the phenomenology of mathematics by working on a concrete problem that was at the same time of great technical interest. For reasons I explain in the next paper I will mention, I do not think it is a coincidence that Gödel deepest response to his reading of phenomenology lies in an elaboration not of Platonistic, but of constructivistic ideas.

'Construction and constitution in mathematics' addresses the question whether classical mathematics admits of the phenomenological foundation that Gödel envisaged. It proceeds by arguing that phenomenology rather leads to intuitionistic mathematics. That view is not only contrary to Gödel's, but also to that of recent authors on the topic like Føllesdal, Hartimo, Hauser, Liu, Rosado Haddock, and Tieszen.[51] What I find wanting in Gödel as well as in the authors mentioned is, briefly, an appreciation of the way the transcendental Husserl developed his thought about categorial objects and categorial intuition in *Formal and Transcendental Logic*, *Experience and Judgement*, and related manuscripts from that period, as compared to the earlier *Logical Investigations* and *Ideas I*. In this paper, I describe that later doctrine in detail, drawing particular attention to its contentions that the objects of pure mathematics are productions in the ontic sense, and that all possible rational subjects can in principle produce the same purely categorial objects. This means that the kind of mathematics compatible with Husserl's variety of transcendental idealism is constructive mathematics, and that classical mathematics cannot be conceived of as constructive mathematics for a higher (in particular: ideal) mind. Anyone concerned with developing a Platonistic foundation for classical mathematics under reference to Husserl's later works therefore should argue that Husserl was wrong to develop his doctrine of categorial objects and intuitions that way, and strive to give a detailed phenomenological account of Platonism based on a new, alternative development of the doctrine of categorial intuition. None of this

[51] E.g., Føllesdal on p. 372 of his introduction to Gödel (*1961/?), Hartimo (2012), Hauser (2006), Liu (2010), Rosado Haddock (1987), and Tieszen (2011).

is to be found in Gödel or the mentioned recent authors; but to use the later Husserl in support of a metaphysics of mathematical objects that exist independently of our mental constructions is perverse. Clearly, exegesis of Husserl is no idle matter at this point. I will leave in the middle here whether the conclusion of this chapter should be taken as an objection to classical mathematics or to phenomenology. In either case it entails that Gödel's project cannot succeed.

The essays surrounding those four central ones have been included for further details and context.

In the 'Note on Leibniz and infinite wholes', I defend the claim that Leibniz' famous argument against the existence of infinite wholes is not only incorrect, as Russell has shown, but incorrect even on Leibniz' own terms. This is relevant to Gödel's project because it shows that, should there be an obstacle to integrating Cantorian set theory within a Leibnizian philosophy, as Gödel wished to do, it will not be this.

'Gödel's Dialectica Interpretation and Leibniz' shows that there was a direct influence of Leibniz' ideas about proofs on Gödel's revision of his Dialectica Interpretation. Seen together with his description of that work, quoted above, as 'a new intuitionistic insight ... based on phenomenological reflection', we see that Gödel at that late stage of his career was willing to experiment with elements adopted and adapted from each of Leibniz, Husserl, and Brouwer.

'Mathematics' is a general discussion, with examples, of the phenomenology of mathematics.

'Gödel, mathematics, and possible worlds' provides a phenomenological unification of Gödel's Platonism and the Leibnizian idea of possible worlds, thus rejecting Hintikka's view on the motivation of Gödel's Platonism. Although this chapter belongs just a much in the part on Gödel and Leibniz, I have chosen to put it in the part on Gödel and Husserl: the question it addresses has its home in the Leibnizian context, but the argument it develops wholly depends on transcendental phenomenology.

'Two draft letters from Gödel on self-knowledge of reason' discusses unpublished remarks from the 1960s in which Gödel connects his Incompleteness Theorem to idealistic philosophy, of which transcendental phenomenology is a particular form. As far as their content is concerned (not necessarily Gödel's intentions), these remarks could be read as notes for a continuation of his 1961 essay.

'Rivalling brothers' looks at Gödel's relation to Brouwer and shows that, besides deep disagreements, there are also deep agreements between their philosophical ideas.

'Mysticism and mathematics' (written with Robert Tragesser) compares Gödel's and Brouwer's explorations of mysticism and its relation to mathematics. It is, of course, the essay farthest removed from the discussion of Gödel's project as such, in which mysticism plays no rôle at all; it has been included because it complements 'Rivalling brothers' and 'Gödel and intuitionism'.

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Part I
Gödel and Leibniz

Chapter 2
A Note on Leibniz's Argument Against Infinite Wholes

Mark van Atten

Abstract Leibniz had a well-known argument against the existence of infinite wholes that is based on the part-whole axiom: the whole is greater than the part. The refutation of this argument by Russell and others is equally well known. In this note, I argue (against positions recently defended by Arthur, Breger, and Brown) for the following three claims: (1) Leibniz himself had all the means to devise and accept this refutation; (2) This refutation does not presuppose the consistency of Cantorian set theory; (3) This refutation does not cast doubt on the part-whole axiom. Hence, should there be an obstacle to Gödel's wish to integrate Cantorian set theory within Leibniz' philosophy, it will not be this famous argument of Leibniz'.

Keywords Georg Cantor • Kurt Gödel • Gottfried Wilhelm Leibniz • Part-whole axiom • Bertrand Russell • Set theory

2.1 Introduction

Leibniz had a well-known argument against the existence of infinite wholes that is based on the part-whole axiom: the whole is greater than the part.[1] The refutation of this argument by Russell and others is equally well known. In this note, I argue (against positions recently defended by Arthur, Breger and Brown) for the following three claims:

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[1]The part-whole axiom is also referred to as the 'Aristotelian principle' (e.g., Benci et al. 2006) or 'Euclid's Axiom'. The former label is justified as it follows from what Aristotle says at Metaphysics 1021a4: 'That which exceeds, in relation to that which is exceeded, is "so much" plus something more' (Aristotle 1933, 263); the latter, to the extent that it figures as Common Notion 5 in Book I of Euclid's *Elements* from (at the latest) Proclus on (Euclid 1956, 232). See also Leibniz's 'Demonstratio Axiomatum Euclidis' (1679), Leibniz (1923–, 6,4:167).

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1. Leibniz himself had all the means to devise and accept this refutation.[2]
2. This refutation does not presuppose the consistency of Cantorian set theory.
3. This refutation does not cast doubt on the part-whole axiom.

A note on sources: although Leibniz's texts used below range from 1672 to after 1714, they all express the same view on the issue at hand. Of each the year will be given where possible. Emphasis in quotations from Leibniz is his. Translations without a reference are mine.

2.2 Leibniz's Argument and Its Refutation

The following presentation of Leibniz's argument and its refutation is meant to establish claim (1) and to set the stage for the defence of claims (2) and (3).

Leibniz denied the existence of infinite wholes of any kind.[3] For example, while he acknowledges that there are infinitely many numbers

> For it cannot be denied that the natures of all possible numbers are really given, at least in God's understanding, and that as a consequence the multitude of numbers is infinite.[4]

he denies that this multitude forms a whole:

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[2]This contributes to showing that there is no intrinsic obstacle in Leibniz's philosophy to combining it with Cantorian set theory, as Kurt Gödel wished to do. For further details on this aspect of Gödel's thought, which provided the motivation for writing the present note, see van Atten (2009a). Neither Gödel's published papers, nor, as far as I can tell from the currently existing partial transcriptions from Gabelsberger shorthand, his notebooks contain a direct comment on Leibniz's argument. However, in his paper on Russell from 1944, he wrote:

> Nor is it self-contradictory that a proper part should be identical (not merely equal) to the whole, as is seen in the case of structures in the abstract sense. The structure of the series of integers, e.g., contains itself as a proper part. (Gödel 1944, 139)

Among other things, Gödel says here that it is consistent that an equality relation holds between a proper part and the whole. This entails a rejection of Leibniz's argument.

[3]Friedman (1975, 338) suggests that even so, Leibniz might have been willing to accept the for him inconsistent concept of an infinite whole as a fiction that may prove useful in calculations, on a par with his acceptance of imaginary roots in algebra. To illustrate this point, Friedman refers to Leibniz (1705) 1882, 145. See also Leibniz's letter to Des Bosses of 1 September 1706: 'properly speaking, an infinity consisting of parts is neither one nor a whole, and can only be conceived of as a quantity by a mental fiction.' ('proprie loquendo, infinitum ex partibus constans neque unum esse neque totum, nec nisi per fictionem mentis concipi ut quantitatem', Leibniz 1875–1890, 2:314)

[4]'Neque enim negari potest, omnium numerorum possibilium naturas revera dari, saltem in divina mente, adeoque numerorum multitudinem esse infinitam.' (Leibniz 1875–1890, Leibniz to Des Bosses, 11/17 March 1706; 2:305)

> I concede [the existence of] an infinite multitude, but this multitude forms neither a number nor one whole. It only means that there are more terms than can be designated by a number; just as there is for instance a multitude or complex of all numbers; but this multitude is neither a number nor one whole. (Leibniz 1849–1863, Leibniz to Joh. Bernoulli, 21 February 1699; 3:575)[5,6]

The point of departure for Leibniz's argument that an infinity cannot be a whole is the part-whole axiom, which he justifies by the following definitions and argument[7]:

> If a part of one thing is equal to the whole of another, the former is called greater, the latter less. Hence the whole is greater than a part. For let the whole be A, the part B. Then A is greater than B, because a part of A (namely, B) is equal to the whole of B. This can be expressed in a syllogism whose major proposition is a definition, its minor an identity:
>
>> Whatever is equal to a part of A is less than A, by definition. But B is equal to a part of A (namely, to B), by hypothesis. Therefore B is less than A.
>
> (Leibniz 1969, 668 (after 1714))[8,9]

The major proposition shows that for Leibniz, 'part' means 'proper part', i.e., a part which is not equal to the whole. His argument from this axiom against infinite wholes runs as follows[10]:

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[5] 'Concedo multitudinem infinitam, sed haec multitudo non facit numerum seu unum totum; nec aliud significat, quam plures esse terminos, quam numero designari possint, prorsus quemqdmodum datur multitudino seu complexus omnium numerorum; sed haec multitudo non est numerus, nec unum totum.'

[6] Also: 'PH. . . . Nothing is clearer than the absurdity of an actual idea of an infinite number. TH. I agree. But this is not because one couldn't have the idea of the infinite, but because an infinity cannot be a true whole' ('PH. . . . rien n'est plus sensible que l'absurdité d'une idée actuelle d'un nombre infini. TH. Je suis du même avis. Mais ce n'est pas parcequ'on ne sauroit avoir l'idée de l'infini, mais parcequ'un infini ne sauroit estre un vrai tout', Leibniz (1705) 1882, 146)

[7] Although an axiom is a proposition that is evident, Leibniz sees two uses for a demonstration, that is, a reduction to A = A: it contributes to the unification of the sciences and to the analysis of ideas. See Couturat (1901, 200ff.).

[8] 'Si pars unius sit aequalis alteri toti, illud vocatur Minus, hoc Majus. Itaeque Totum est majus parte. Sit totum A, pars B, dico A esse majus quam B, quia pars ipsius A (nempe B) aequatur toti B. Res etiam Syllogismo exponi potest, cujus Major propositio est definitio, Minor propositio est identica:

> Quicquid ipsius Q parti aequale est, id ipso A minus est, ex definitione, B est aequale parti ipsius A, nempe sibi, ex hypothesi, ergo B est minus ipso A.'
>
> (Leibniz 1849–1863, 7:20)

[9] See also Leibniz (1903, 518)/Leibniz (1969, 267) (around 1686), Leibniz (1923–, 6/4:167 (1679)), Leibniz (1875–1890, 7:300), Leibniz (1849–1863, 7:274) (1695, according to de Risi 2007, 82), and Leibniz (1849–1863, 3:322 (1696)).

[10] Leibniz presented the same argument on various occasions: see Leibniz (1923–, 2,1:226 and 228 (1672)); Leibniz (1923–, 4,3:403 (1675)), Leibniz (1923–, 6,3:463 (1675)), Leibniz (1923–, 6,3:168 (1676)), and Leibniz (1923–, 6,3:550–3 (1676)); his draft letter to Malebranche of 22 June 1679, Leibniz 1875–1890, 1:338. See also the reference to it in a letter to Johann Bernoulli of 1698, Leibniz (1849–1863, 3:535).

> There is no maximum in things, or what is the same thing, the infinite number of all unities is not one whole, but is comparable to nothing. For if the infinite number of all unities, or what is the same thing, the infinite number of all numbers, is a whole, it will follow that one of its parts is equal to it; which is absurd. I will show the force of this consequence as follows. The number of all square numbers is a part of the number of all numbers: but any number is the root of some square number, for if it is multiplied into itself, it makes a square number. But the same number cannot be the root of different squares, nor can the same square have different roots. Therefore there are as many numbers as there are square numbers, that is, the number of numbers is equal to the number of squares, the whole to the part, which is absurd. (Leibniz 2001, 13)[11]

The reductio argument that Leibniz presents here can be reconstructed as follows:

1. The infinite multitude of the numbers forms a whole. (Assumption)
2. Every square is a number, but not vice versa. (Premise)
3. The multitude of the squares is equal to a part of the whole of the numbers. (1, 2)
4. There exists a bijection between the multitude of the numbers and the multitude of the squares. (Premise)
5. The multitude of the squares is equal to the whole of the numbers. (1, 4)
6. A part of the whole of the numbers is equal to the whole of the numbers. (3, 5)
7. The whole is greater than its parts. (Premise)
8. Contradiction. (6, 7)
9. Therefore, the infinite number of all numbers do not form a whole. (1, 8)

Leibniz holds the propositions in lines 2, 4 and 7 to be true; the source of the contradiction that arises in line 8 for him is the assumption made in line 1.

Russell and others have observed that Leibniz's argument is not correct because it rests on an equivocation on the concept of equality.[12] Clearly, in line 3 'is equal to' means 'is identical to', while in line 5 it means 'can be put in a bijection with'. While in their application to finite multitudes the concepts of equality given through these senses are equivalent, this is not so in the infinite case, for if we substitute the sense of 'is equal to' in line 3 for that of 'is equal to' in line 5, we obtain the falsehood that the multitude of the squares is identical to that of the numbers. Leibniz (who, at times, used the word 'term' instead of 'concept'[13]) defined

[11] 'Nullum datur Maximum in rebus, vel quod idem est Numerus infinitus omnium unitatum non est unum totum, sed nihilo aequiparatur. Nam si numerus infinitus omnium unitatum, seu quod idem est, Numerus infinitus omnium numerorum, sequetur aliquam eius partem esse ipsi aequalem. Quod est absurdum. Consequentiae vim ita ostendo. Numerus omnium Numerorum Quadratorum est pars Numeri omnium Numerorum: at quilibet Numerus est radix alicuius Numeri quadrati, nam si in se ducatur, fiet aliquis numerus quadratus; nec idem numerus potest esse radix diversorum quadratorum, nec idem quadratus diversarum radicum, tot ergo sunt Numeri, quot Numeri quadrati, seu Numerus Quadratorum aequalis est Numeri Numerorum, totum parti, quod est absurdum.' (Leibniz 1923–, 6,3:98 (1672–3))

[12] Russell (1919, 80–81). See also Benardete (1964, 47–48), and Levey (1998, 61–62).

[13] 'By a term I do not understand the word but the concept or that which the word signifies, you could also say the notion or the idea.' ('Per Terminum non intellego nomen sed conceptum seu id quod nomine significatur, possis et dicere notionem, ideam', Leibniz 1903, 243 (c. 1680))

> Same or coincident terms are those which can be substituted for each other anywhere without affecting truth . . . Diverse terms are those which are not the same or in which substitution sometimes does not work. (Leibniz 1969, 371 (early 1690s))[14]

Therefore, by Leibniz's own criterion, the concepts of equality in lines 3 and 5 are diverse or different, as substitution of the one for the other does not preserve truth here; but then, from lines 3 and 5, one cannot infer to line 6, for that inference presupposes that the two concepts are the same. Hence, Leibniz was in a position to see that the inference was incorrect.

Earlier in the text in which he states the proof of the part-whole axiom, Leibniz defines 'Equals are things having the same quantity' (Leibniz 1969, 667 (after 1714))[15,16] About the concept of quantity, Leibniz there only remarks that it is essentially comparative: 'Quantity or magnitude is that in things which can be known only through their simultaneous compresence – or by their simultaneous perception' (Leibniz 1969, 667).[17,18] However, that leaves unaddressed the fact that there are essentially different ways of comparing, resulting in correspondingly different concepts of equality.[19]

Breger has recently attempted to meet this objection to Leibniz's argument by suggesting that Leibniz was working strictly within a theory of finite multitudes: 'The fact that one finds objects outside the theory examined here for which both notions [of equality] are not equivalent is of no importance within the theory' (Breger 2008, 314). However, if that was what Leibniz was doing, then he could

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[14] 'Eadem seu coincidentia sunt quorum alterutrum ubilibet potest substitui alteri salva veritate . . . Diversa sunt quae non sunt eadem, seu in quibus substitutio aliquando non procedit.' (Leibniz 1875–1890, 7:236)

[15] Aequalia sunt ejusdem quantitatis. Leibniz (1849–1863, 7:19)

[16] See also Leibniz (1923–, 6,4:165 (1679)) and (1923–, 6,4:406 (1680(?))).

[17] Quantitas seu Magnitudo est, quod in rebus sola compraesentia (seu perceptione simultanea) cognosci potest. (Leibniz 1849–1863, 7:18 (after 1714))

[18] See also Leibniz (1923–, 6,4:168 (1679)).

[19] The example that Leibniz goes on to give in the same text, that of measuring either in inches or in feet, is not a case of two essentially different ways of comparing, because, as Leibniz himself points out (for a different purpose), they are interdefinable. Cantor remarks, after having shown that extending an infinite line (with one endpoint) by a finite line does not lead to a new line with more points, because there will be a 1–1 mapping:

> Who here, and in the case of any actually infinite quantity, sees a violation of the principle of contradiction, is quite mistaken, by losing sight of the abstractive character of 'magnitude' and wrongly identifying it with the substantial entity of the quantity at hand.[20]

Evidently, which abstractions are legitimate in defining a specific concept of magnitude depends on the context: there are perfectly good uses for the concept of magnitude that Cantor in this context rightly rejects. See also my concluding remark.

[20] Wer hier wie überhaupt bei aktual-unendlichen Quantitäten einen Verstoß gegen das Widerspruchsprinzip findet, irrt durchaus, indem er den abstraktiven Charakter der 'Größe' aus dem Auge verliert und sie fälschlich mit der substanziellen Entität des vorliegenden Quantums identifiziert. (Cantor 1887–1888, 393)

not have devised his argument against infinite wholes in the first place, for he then would have had no theory to apply to the assumption by which he begins his argument. Leibniz is, on the contrary, working in a theory of parts, wholes and finite as well as infinite multitudes, and within that theory attempts to show that infinite multitudes cannot be wholes. Breger, moreover, holds that the refutation depends on a perspective that was developed only in the second half of the nineteenth century and that for Leibniz 'it would have been absurd, absolutely unthinkable, to reject the equivalence [of the different notions of equality]' (Breger 2008, 315). However, as the above reconstruction emphasises, the refutation uses no concept or technique that was not available to Leibniz.

2.3 The Consistency of Cantorian Set Theory

Arthur, Breger and Brown hold that the refutation of Leibniz's argument depends on whether Cantor's theory of infinite sets is, in fact, consistent:

> [The] argument (like those of Cantor, Russell, and Rescher before it) reduces to this: if with Cantor one assumes ... [the proposition C] that an infinite collection (such as the set of all numbers) is a whole or unity, then one can establish a consistent theory of infinite number; therefore Leibniz's argument against it is unsound ... To say that Leibniz's argument is unsound on the basis of the success of Cantor's theory is to assume the truth [and hence the consistency] of C, and thus to beg the question (unless one has an independent argument for C, which Cantor does not) (Arthur 2001, 105).[21]

> I suppose that one might argue that, for all we know and he knew, Leibniz's argument against infinite number and wholes might be sound; for despite the fact that most mathematicians now seem to assume that Cantorian set theory is consistent in light of its long record of success and the absence of any proof of its inconsistency, it remains true that neither does there exist a general consistency proof for that theory. It remains at least possible, I suppose, that Leibniz's argument against infinite number and wholes is sound, and that there really is some inconsistency lying dormant and undiscovered in the assumption that the part-whole axiom fails in the case of the infinite-and ultimately some inconsistency lying dormant and undiscovered in the Cantorian definitions of 'less than', 'greater than', and 'equal to' for infinite sets-so that infinite wholes are indeed impossible. (Brown 2005, 486)

> It is striking that one already needs the existence (free of contradiction) of an infinite totality, which is precisely what is supposed to be proved or refuted, to show the non-equivalence of the two notions of 'having the same number' ... The two expressions are equivalent if and only if they are applied to finite multitudes. In other words: one can demonstrate the non-equivalence of the two notions if and only if one assumes the existence of infinite multitudes that as objects free of contradiction constitute a whole and are thus sets(as happens in the

[21] Arthur (Leibniz 2001, 407n41) adds to this that 'the paradoxes of the infinite still [beset] set theory'. But in the iterative concept of set, which has become the standard understanding of Cantorian set theory, no paradox has yet been found. See Gödel (1947, 518–519) and Wang (1974, 181–193).

> Zermelo-Fraenkel theory) or if one has already demonstrated this in some other way. As long as this has not happened, the objection that Leibniz is using two non-equivalent notions is false. (Breger 2008, 313–314)

However, it is Leibniz who makes the assumption at the beginning of his argument that an infinite whole exists. Of course, for him that is only an assumption towards a reductio ad absurdum, where the absurdity will arise when the part-whole axiom is brought in. The equivocation on the concept of equality occurs already before that stage of the argument is reached. When Leibniz makes this equivocation, the assumption has not yet been cancelled. Therefore, the need to make that assumption in order to be able to distinguish the two concepts of equality in play is no objection to the refutation of Leibniz's argument. Moreover, as the assumption is part of the argument itself, and not a presupposition of its refutation, the latter does not depend on whether Cantorian set theory in fact is consistent. (This also means that this refutation by itself does not show that there can be no correct arguments against infinite wholes; only that Leibniz's argument is not one.)

2.4 The Part-Whole Axiom

It is often held that the distinction between different concepts of equality on which the refutation of Leibniz's argument depends also serves to show that the existence of infinite wholes and the part-whole axiom are incompatible.[22] Here, too, means available to Leibniz can be indicated that enable one to see that, logically speaking, there is no such incompatibility. This turns on the fact that Leibniz defines the concept of proper part in terms of equality: a proper part is a part that is not equal to the whole. The idea now is that, as there are (extensionally) different concepts of equality, concepts whose definition involves the concept of equality may turn out to be equivocal, too. For example, a part of a whole may be proper with respect to one concept of equality and not with respect to another. Likewise, a part may be greater than another one with respect to one concept of equality and not with respect to another. If one accepts this view, then one can argue as follows.

Given that the part-whole axiom relates two concepts (proper part and being greater than) to each other that both involve the equivocal concept of equality, in any particular application of that axiom the same concept of equality should be used throughout. Now consider the axiom in its more explicit form: 'For all x and y, if x is a proper part of y, then y is greater than x'. (Leibniz recognised explicitation of this kind: 'A is B, that is the same as saying that if L is A, it follows that L also is B'[23]; and 'The affirmative universal proposition Every b is c can be reduced to this

[22] See, for example, the quotation from Brown (2005) in Sect. 2.2, and Leibniz (2001, 406n41).

[23] A est B, idem est ac dicere si L est A sequitur quod et L est B. (Leibniz 1903, 260)

hypothetical one: If a is b, then a is c'.[24]) Instantiating the axiom in this form by taking for x the multitude of the squares and for y the whole of the numbers yields the simple conditional 'If the multitude of squares is a proper part of the whole of the numbers, then the whole of the numbers is greater than the multitude of the squares'. Any choice of concept of equality that, via the concept of 'proper part' it induces, makes the antecedent come out true, will, by the definition of 'greater than', make the consequent come out true as well; for example, the concept of equality defined in terms of elementhood. If, on the other hand, the chosen concept of equality makes the antecedent false, such as that defined in terms of a bijection, then it would be open to Leibniz to accept the conditional as vacuously true: the principle that conditionals with false antecedents are true can be proved in (a rational reconstruction of) the logical calculus that Leibniz devised in 1690 (Leibniz 1903, 421–423).[25] In neither case does the conditional come out false. This shows that if one defines the concept of proper part in terms of equality, the part-whole axiom and the existence of infinite wholes are not logically incompatible.

2.5 Concluding Remark

In Cantorian set theory, concepts of size are defined ones, and there is nothing against defining alternatives to Cantor's own, or to working with several at the same time. In general, with different concepts of size will come different principles of arithmetic. (Indeed, Cantor himself has two different concepts of size, cardinality and ordinality, with different arithmetics.) For a mathematical exploration of Cantorian set theory equipped with a concept of size that respects the part-whole axiom non-vacuously for infinite sets, see Benci et al. (2006), or, for an introduction with philosophical and historical background, Mancosu (2009).[26]

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[24] Propositio Universalis affirmativa Omne b est c reduci potest ad hanc hypotheticam Si a est b, a erit c. (Leibniz 1923–, 6,4:126 (1678/9(?)))

[25] 'Fundamenta calculi logici'. With minor modifications, this system is equivalent to classical propositional logic. For an axiomatisation and a completeness proof, see Castañeda (1976).

[26] ⟦A critical view on the epistemic usefulness of such theories has now been developed in Parker (2013).⟧

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Chapter 3
Monads and Sets: On Gödel, Leibniz, and the Reflection Principle

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Voor Göran, in dank en vriendschap

Abstract Gödel once offered an argument for the general reflection principle in set theory that took the form of an analogy with Leibniz' monadology. I discuss the mathematical and philosophical background to Gödel's argument, reconstruct the proposed analogy in detail, and argue that it has no justificatory force. The paper also provides further support for Gödel's idea that the monadology needs to be reconstructed phenomenologically, by showing that the unsupplemented monadology is not able to found mathematics directly.

Keywords Analogy • Georg Cantor • Idealism • Kurt Gödel • Gottfried Wilhelm Leibniz • Metaphysics • Monadology • Reflection principle • Set theory • Transcendental notions

3.1 Introduction

Gödel described his general philosophical theory to Hao Wang as 'a monadology with a central monad ... like the monadology of Leibniz in its general structure' (Wang 1996, 0.2.1). At the same time, he believed that Cantorian set theory is a true theory, which describes some 'well-determined reality' (Gödel 1990, 181). I will first discuss the embedding of Cantorian set theory in a Leibnizian metaphysics that the combination of these two beliefs of Gödel's requires.[1] Then I turn to an attempt

[1]Paul Benacerraf kindly allowed me to relate the following. At a dinner in 1974 or 1975, Gödel had conversations with Gerald Sacks on large cardinals and with Benacerraf on the mind-body

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by Gödel to justify (a particular form of) the reflection principle in set theory by drawing an analogy to the monadology. Of this attempt I will argue that, although its success might not depend on whether the monadology is true or not, it fails. More generally, I defend the claim that while a Leibnizian metaphysics is compatible with Cantorian set theory, by itself it provides no clues that can be used in justifying set-theoretical principles, be it by analogy or directly.[2]

3.2 Fitting Cantor's Sets into Leibniz' Metaphysics

One immediate obstacle to the project of relating Cantorian set theory to Leibniz' metaphysics in any positive way would seem to be this. Cantor defines a set as a 'each many, which can be thought of as a one' and as 'each gathering-together M into a whole of determined well-distinguished objects m of our intuition or of our thought (which are called the "elements" of M)'.[3] Cantorian set theory being largely about infinite sets, it is a theory of certain infinite wholes. But Leibniz denies the existence of infinite wholes of any kind.[4] For example, he says that one has to

problem. In the latter, he made reference to 'monads'. Gödel carried on these two conversations *simultaneously*, turning from left to right and back. (One argument advanced by Gödel was this: (1) the monads that our minds are have unambiguous access to the full set-theoretic hierarchy; (2) the full set-theoretic hierarchy cannot be adequately represented physically; therefore, (3) the mind cannot be reduced to a physical structure.)

[2]A monograph on the monadology in relation to Cantorian set theory is Osterheld-Koepke (1984). However, the reflection principle is not discussed there. On another note, it is argued there (p. 128) that on monadological grounds we can never decide the Continuum Hypothesis; one may well doubt that Gödel's understanding of the monadology and its relation to set theory would have had such a consequence. Gödel paired his belief in the monadology to a conviction that in principle a rational mind could decide every mathematical proposition. (He believed that 'Leibniz did not in his writings about the Characteristica universalis speak of a utopian project' and that this would provide a means 'to solve mathematical problems systematically' (Gödel 1990, 140). He realised that, because of his own Incompleteness Theorem, such a Characteristica could not assume the form of an entirely formal system.) In particular, he worked hard (but unsuccessfully) at deciding the Continuum Hypothesis. For further discussion of Gödel's belief in the solvability of all mathematical problems, see Kennedy and van Atten (2004).

[3]'jedes Viele, welches sich als Eines denken läßt' (Cantor [1883] 1932, 204n1) and 'jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen' (Cantor [1895] 1932, 282). The English translation of the latter is that of Grattan-Guinness (2000, 112).

[4]Friedman (1975, 338) suggests that even so, Leibniz might have been willing to accept the for him inconsistent notion of infinite whole as a fiction that may prove useful in calculations, on a par with his acceptance of imaginary roots in algebra. To illustrate this point, Friedman refers to Leibniz ([1705] 1882, II, Chap. 17, Sect. 3).

acknowledge that there are infinitely many numbers,[5] but he denies that they can be thought of as forming a unity:

> I concede [the existence of] an infinite multitude, but this multitude forms neither a number nor one whole. It only means that there are more elements than can be designated by a number, just as there is a multitude or complex of all numbers; but this multitude is neither a number nor one whole.[6]

The distinction Leibniz draws between aggregates that are unities and aggregates that are mere multitudes is somewhat similar to the one Cantor would later draw between sets and proper classes, but their reasons are very different. Leibniz arrives at this distinction by a general argument that would rule out any infinite set altogether. He argues that there can be no infinite wholes or unities of any kind. It is not the notion of infinity as such that poses the problem for him, as is clear from this exchange between Philalèthe and Théophile (who represents Leibniz) in the *New Essays*:

> PH: We have no idea of an infinite space, and nothing is clearer than the absurdity of an actual idea of an infinite number.
> TH: I agree. But the reason for this is not that one could have no idea of the infinite, but that an infinity cannot be a true whole.[7]

Specifically, Leibniz holds that the notion of an infinite whole contradicts the axiom that the whole is greater than the part. It is well known that Leibniz' argument is not sound and rests on an equivocation on 'greater than', once defined in terms of the notion of proper superset and once defined in terms of the notion of non-surjective injection.[8] It can be shown, although for limitations of space I will not do so here, that Leibniz himself had all the means to see that his argument is not sound. The importance of that fact is that it shows that Leibniz' denial of infinite wholes does not reflect a limitation intrinsic to his philosophical system.

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[5] Leibniz to Des Bosses, March 11/17, 1706: 'One cannot deny that the natures of all possible numbers are indeed given, at least in God's mind, and that as a consequence the multitude of numbers is infinite.' ('Neque enim negari potest, omnium numerorum possibilium naturas revera dari, saltem in divina mente, adeoque numerorum multitudinem esse infinitam.' Leibniz 1875–1890, 2:305)

[6] Leibniz to Joh. Bernoulli, February 21, 1699: 'Concedo multitudinem infinitam, sed haec multitudo non facit numerum seu unum totum; nec aliud significat, quam plures esse terminos, quam numero designari possint, prorsus quemadmodum datur multitudino seu complexus omnium numerorum; sed haec multitudo non est numerus, nec unum totum.' (Leibniz 1849–1863, 3/2:575)

[7] Leibniz (1875–1890, 5:146): 'PH : Nous n'avons pas l'idée d'un espace infini, et rien n'est plus sensible que l'absurdité d'une idée actuelle d'un nombre infini. TH : Je suis du même avis. Mais ce n'est pas parcequ'on ne sauroit avoir l'idée de l'infini, mais parcequ'un infini ne sauroit estre un vrai tout.'

[8] See, for example, the refutation in Benardete (1964, 47–48).

In Gödel's notebooks, I have so far not found a specific comment on Leibniz' argument that there can be no infinite wholes. But in the Russell paper from 1944 he wrote:

> Nor is it self-contradictory that a proper part should be identical (not merely equal) to the whole, as is seen in the case of structures in the abstract sense. The structure of the series of integers, e.g., contains itself as a proper part. (Gödel 1990, 130)

Among other things, Gödel says here that it is consistent that an equality relation holds between proper part and the whole. This entails a rejection of Leibniz' argument. And in a very similar note from 1944, again without mentioning Leibniz, Gödel adds: 'the same can be contained as a part in 2 different ways'.[9] That same consideration can be used to show that Leibniz' argument is not valid. Of course, the incorrectness of Leibniz' argument against infinite wholes implies nothing as to whether its conclusion is true or false. But clearly it will not be this argument that poses an obstacle to combining, as Gödel did, a belief in monadology with a belief in Cantorian set theory.

I now turn to the status of pure sets in Leibniz' calls collections 'aggregates' or 'multitudes'. In his philosophical remarks on them, he usually discusses aggregates of objects in the world; but from these remarks together with what he says about pure numbers, one can derive what his philosophical views on pure sets would have been.[10]

In a letter to De Volder of 1704, Leibniz writes that

> Whatever aggregates out of pluralities there are, they are *unities* only in thought. They have no other reality than a borrowed one or that of the things out of which they are composed.[11]

Note the similarity with Cantor's definitions of a set that were quoted in Sect. 3.2; there with an emphasis on sets being a 'one' or a 'whole', here on the fact that for Leibniz, the unity of an aggregate consists its elements being thought or considered together. Therefore, Leibniz says, an aggregate has the character of a relation:

> Being and one are reciprocal notions, but where a being is given by aggregation, we also have one being, even though that entity and that unity are semi-mental.

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[9] dasselbe [kann] auf 2 verschiedene Weisen als Teil enthalten sein. (Gödel Papers, 6b/70, 030097 (*Max* XI), 18)

[10] Gödel makes some remarks on monads and sets in Wang (1996, 296), but not so much on the relation between them.

[11] January 21, 1704: 'quaecunque ex pluribus aggregata sunt, ea non sunt unum nisi mente, nec habent realitatem aliam quam mutuatam seu rerum ex quibus aggregantur' (Leibniz 1875–1890, 2:261). Also: 'This unity of the idea of aggregates exists truly enough, but at bottom we must admit that this unity of collections is nothing but a connection or relation whose foundation is in that which lies within each of the individual substances taken by themselves. Thus, these beings by aggregation have no full unity but a mental one; and hence their objecthood is also in a way mental or phenomenal, like that of the rainbow' ('Cette unité de l'idée des Aggregés est tres veritable, mais dans le fonds il faut avouer que cette unité des collections n'est qu'un rapport ou une relation dont le fondement est dans ce qui se trouve en chacune des substances singulieres à part. Ainsi ces Estres par Aggregation n'ont point d'autre unité achevée que la mentale ; et par consequent leur Entité aussi est en quelque façon mentale ou de phenomene, comme celle de l'arc en ciel', (Leibniz [1705] 1882, 133).)

Numbers, units, and fractions have the nature of relations. And to that extent, they may in a sense be called beings.[12]

Leibniz here qualifies a unified aggregate as a semi-mental entity because he is thinking of aggregates of objects in the world. But an aggregate of mental objects would be entirely mental. The pure sets as we know them from Cantor's would fundamentally be pure relations that are entirely in the mind. Not in the human mind, but in God's mind, for, as Leibniz writes in the *New Essays*:

> The relations have a reality that is dependent on the mind, as do truths; but not on the human mind, as there is a supreme intelligence that determines all of them at all times.[13]

Correspondingly, the truths about these pure relations have their existence in God's mind:

> One must not say, with some Scotists, that the eternal verities would exist even though there were no understanding, not even that of God. For it is, in my judgement, the divine understanding which gives reality to the eternal verities, albeit God's will have no part therein. All reality must be founded on something existent. It is true that an atheist may be a geometrician: but if there were no God, geometry would have no object. And without God, not only would there be nothing existent, but there would be nothing possible. That, however, does not hinder those who do not see the connexion of all things with one another and with God from being able to understand certain sciences, without knowing their first source, which is in God. (Leibniz 1991, 158)[14]

And, in 'On the radical origination of things' from 1697:

> Neither these essences nor the so-called eternal truths about them are fictitious but exist in a certain region of ideas, if I may so call it, namely, in God himself, who is the source of all essence and of the existence of the rest . . . and since, furthermore, existing things come into being only from existing things, as I have also explained, it is necessary for eternal truths to have their existence in an absolutely or metaphysically necessary subject, that is, in God, through whom those possibilities which would otherwise be imaginary are (to use an outlandish but expressive word) realised. (Leibniz 1969, 488)[15]

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[12]Leibniz to Des Bosses, March 11, 1706: 'Ens et unum convertuntur, sed ut datur Ens per aggregationem, ita et unum, etsi haec Entitas Unitasque sit semimentalis. Numeri, Unitates, fractiones naturam habent Relationum. Et eatenus aliquo modo Entia appellari possunt' (Leibniz 1875–1890, 2:304).

[13] Les rélations ont une réalité dépendante de l'esprit commes les Verités ; mais non pas de l'esprit de l'homme, puisqu'il y a une suprême intelligence, qui les détermine toutes en tout temps. (Leibniz [1705] 1882, II, Chap. 30, Sect. 4)

[14]'Il ne faut point dire avec quelques Scotistes, que les verités eternelles subsisteroient, quand il n'y auroit point d'entendement, pas même celuy de Dieu. Car c'est à mon avis l'entendement Divin qui fait la realité des Verités eternelles : quoyque sa volonté n'y ait point de part. Toute realité doit être fondée dans quelque chose d'existant. Il est vray qu'un Athée peut être Geometre. Mais s'il n'y avoit point de Dieu, il n'y auroit point d'objet de la Geometrie. Et sans Dieu, non seulement il n'y auroit rien d'existant, mais il n'y auroit rien de possible. Cela n'empêche pas pourtant que ceux qui ne voyent pas la liaison de toutes choses entre elles et avec Dieu, ne puissent entendre certaines sciences, sans en connoitre la premiere source qui est en Dieu.' (Leibniz [1710] 1885, Sect. 184). See also Leibniz ([1705] 1882, II, 25, Sect. 1; 1875–1890, 7:111).

[15] Neque essentias istas, neque aeternas de ipsis veritates quas vocant, esse fictitias, sed existere in quadam ut sic dicam regione idearum, nempe in ipso Deo, essentiae omnis

Leibniz even explicitly draws the conclusion that the eternal truths are invariant with respect to possible worlds:

> And these [propositions] are of eternal truth, they will not only obtain as long as the world will remain, but they would even have obtained, if God had created the world in another way.[16]

As Robert Adams has pointed out, Leibniz' thesis that mathematical objects have their existence in God's mind might well be acceptable to a mathematical Platonist, given the necessary existence of God, given the independence of God's thought from, in particular, human thought, and given the independence of eternal truths of God's will (Adams 1983, 751).[17] It is therefore not surprising to see the Platonist Gödel remark in a notebook from 1944, at the time, that is, when he was studying Leibniz intensely (1943–1946), that 'the ideas and eternal truths are somehow parts of God's substance', that 'one cannot say that they are created by God', and that they rather 'make up God's essence'.[18] Gödel also writes that, of the mappings from propositions to states of affairs 'the correct one' is 'the one which is realised in God's mind'.[19]

This aspect of Leibniz' views on mathematical objects therefore will have provided an additional interest for Gödel in a Leibnizian proof of God's existence: a corollary of such a proof for him would be that a single, fixed universe of all sets

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> existentiaeque caeterorum fonte ... existentia autem non possint esse nisi ab existentibus, ut jam supra monuimus; oportet aeternas veritates existentiam habere in quodam subjecto absolute vel Metaphysice necessario, id est in Deo, per quem haec, quae alioqui imaginaria forent (ut barbare sed significanter dicamus) realisentur. (Leibniz 1875–1890, 7:305)

[16] Et hae sunt aeternae veritatis, nec tantum obtinebunt, dum stabit Mundus, sed etiam obtinuissent, si Deus alia ratione Mundum creasset. (Leibniz 1903, 18)

[17] For Descartes, in contrast, mathematical truth is a matter of God's will, and hence on a Cartesian conception God could *choose* to make reflection true, perhaps for similar reasons as why according to Leibniz (1991, Sect. 46; [1710] 1885, Sect. 380), God favours reflection in the physical world. See also Footnote 42 below. A particularly interesting comment by Leibniz on the relation between God's will, mathematics, and creation is found in Leibniz ((1695) 1994, 57). He there says that, although irrational numbers are to some extent imperfect because they cannot be expressed as fractions, this imperfection 'comes from their own essence and cannot be blamed on God' (*cette irrégularité des lignes incommensurables vient de l'essence même des figures, et ne doit point être imputée à Dieu*); and that, although God could have avoided creating objects (in the world) with irrational measures, if He has nevertheless done so, it is because it results in a universe with a greater variety of forms.

[18] 'Die Ideen und ewigen Wahrheiten sind irgendwie Teile der göttlichen Subst[anz]. Daher kann man nicht sagen, daß sie von Gott geschöpft wurden (denn Gott wurde nicht von Gott geschöpft), sondern sie machen das Wesen Gottes aus' (Gödel Papers, 6b/70, 030097 (*Max* XI), 31). Compare Leibniz (1875–1890, 7:305, lines 1–4), which Gödel copied in a note (Gödel Papers, 10a/35, 050130), Leibniz ([1710] 1885, Sects. 80, 335), and the passage in Leibniz' letter to Wedderkopf, quoted in Sect. 3.4.2 below.

[19] Daß eine gewisse Kombination von Begriffen oder Symbolen 'wahr' ist, bedeutet, daß sie ein adäquates Bild von etwas Existierendem ist, hängt also von der Abbildungsrelation ab. Manche Abbildungsrelationen können wir selbst konstruieren, manche (und insbesondere 'die richtigen', nämlich die im Verstand Gottes realisierten) finden wir vor. (Gödel Papers, 6b/72, 030099 (*Phil* XIV), 7; July 1946 or later)

V indeed exists, and hence that there is a privileged model for the axioms of set theory. Gödel describes his belief in such a privileged model in, for example, his Cantor paper from 1947 (Gödel 1990, 181).

3.3 The Reflection Principle

There is an attempt of Gödel's to justify, by drawing an analogy to Leibniz' monadology, the reflection principle in set theory. Gödel never published the argument but he did present it to Wang (1996, 8.7.14); here it will be quoted in Sect. 3.4.1 below.

The basic idea behind the reflection principle is that the universe V of all sets is in some sense too large to be adequately conceivable or definable in set-theoretic terms. From this observation, one concludes to

(1) If a clearly conceived, set-theoretical property holds of V, this property cannot be unique to V and will also characterise a set contained in it.

With respect to that property, that set is then said to 'reflect' the universe.[20] (Again by reflection one then also sees that that set is not the only one to reflect the universe in that way, and that there are many more.)

Well-known applications of this informal principle are the following. The universe contains (set-theoretic encodings of) the natural numbers, hence there is also a set that contains the natural numbers (and so, by separation, there exists a set that contains nothing but the natural numbers). This use of reflection is already found in Cantor.[21] Or: of any given set, the universe contains all its subsets, hence there is also a set that contains all subsets of the given set (and so, by separation, there exists a set that contains nothing but the subsets of the given set). Or: the universe is inaccessible, hence there is an inaccessible cardinal.[22]

The first two of these applications yield justifications of two axioms of Zermelo-Fraenkel set theory, the axiom of infinity and the axiom of the powerset. Regarding the latter, note that it is not particularly clear (although for Gödel himself it

[20]E.g., Lévy (1960a, 228; 1960b, 1). For two recent monographs on the reflection principle, diametrically opposed to one another in their philosophical approach, see (2002) and Arrigoni (2007). The former corresponds more closely to Gödel's view as described here.

[21]In note 2 to his paper from 1883, 'On infinite, linear point manifolds 5': 'Whereas, hitherto, the infinity of the first number class ... has served as [a symbol of the Absolute], for me, precisely because I regarded that infinity as a tangible or comprehensible idea, it appeared as an utterly vanishing nothing in comparison with the absolutely infinite sequence of numbers', Hallett (1984, 42). ('Die absolut unendliche Zahlenfolge erscheint mir daher in gewissem Sinne als ein geeignetes Symbol des Absoluten; wogegen die Unendlichkeit der ersten Zahlenklasse (I), welche bisher dazu allein gedient hat, mir, eben weil ich sie für eine faßbare Idee (nicht Vorstellung) halte, wie ein ganz verschwindendes Nichts im Vergleich mit jener vorkommt', Cantor 1932, 205n2.) See also Hallett (1984, 116–117).

[22]A cardinal κ is inaccessible if it is regular (i.e., not the supremum of k ordinals all smaller than k) and a limit (i.e., not the next cardinal greater than some cardinal λ.)

apparently was) that, as the standard iterative concept of set has it, the collection of all subsets of an infinite set is a set as opposed to a proper class.[23] The informal reflection principle is a means to provide the justification needed. It is of course not excluded that alternative ways to convince ourselves of the truth of these (and other) axioms exist. Regarding the justification of the existence of inaccessibles, Gödel stated his preference for reflection over other methods in a letter to Paul Cohen of August 13, 1965:

> As far as the axiom of the existence of inaccessibles is concerned I think I slightly overstated my view.[24] I would not say that its evidence is due *solely* to the analogy with the integers. But I do believe that a clear analogy argument[25] is much more convincing than the quasi-constructivistic argument in which we imagine ourselves to be able somehow to reach the inaccessible number. On the other hand, Lévy's principle[26] might be considered more convincing than analogy. (Gödel 2003, 386)

Indeed, as Wang said that the justification of axioms by an appeal to reflection is the fundamental one:

> All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now. (Wang 1996, 8.7.9)[27]

Ackermann had stated that the notion of set is open-ended and that therefore the universe of all sets does not admit of a sharp definition (and is in that sense unknowable) (Ackermann 1956, 337). (This is a reflection principle because it means that if we do find a set-theoretic property of V, this cannot be a definition of it, and hence there is a set that shares the property.) He also took this to be in accord with Cantor's had used to justify the existence of the *set* of all natural numbers (see Footnote 21 above).

[23] For Gödel's justification of the power set axiom on the iterative conception of set (not by reflection), see Wang (1974, 174, 1996, 220). For criticism, see e.g., Parsons ((1977) 1983, 277) and Hallett (1984, 236–238). Gödel's comment on an early version of Parsons ((1977) 1983) seems to me to be instructive but also indicative of a weakness of Gödel's own use of idealisation: 'he does not understand "idealization" broadly enough' (Gödel 2003a, 390). On a different occasion, Gödel acknowledged that there are cases where idealisation is understood too broadly to be very convincing; see the quotation from his letter to Cohen that follows in the main text.

[24] Given the beginning of the preceding paragraph in the letter, 'When we spoke about the power set axiom . . .' (p. 385), presumably Gödel here refers to that same conversation.

[25] *Note Gödel.* Such as, e.g., the one obtained if an inaccessible α is defined by the fact that sums and products of fewer than α cardinals $< \alpha$ are $< \alpha$.

[26] A formulation of the idea of the unknowability of V that one also finds in Cantor and Ackermann (quoted elsewhere in this paper); in Lévy's words, 'the idea of the impossibility of distinguishing, by specified means, the universe from partial universes', Lévy (1960b, 1). Lévy in that paper studies four specific versions of that principle.

[27] See also Wang (1996, 8.7.16).

In some sense one could say that, if Gödel's belief in this reducibility of the principles for setting up the axioms to reflection is correct, then the informal reflection principle captures the concept of set. Note that the reflection principle that Gödel has in view here is not to be confused with reflection principles that are provable in a particular formal system, such as the Montague-Lévy reflection theorem in *ZF*.[28] By Gödel's Incompleteness Theorem, no single formal system for set theory can be complete, and the reflection principle Gödel is speaking about is precisely meant as the fundamental way to arrive at further axioms to extend any given system. His principle therefore has to be, and to remain, informal. Its strength increases with every application because the resulting stronger system in turn gives rise to the formulation of stronger properties to reflect.

In its fully general form (1), the principle of course cannot be upheld. For example, the property of containing every set in the universe is not reflected by any set contained in it, as such a set would have to contain itself. Reflection principles will therefore have to be precise or restrictive about the properties for which they are supposed to hold. Gödel suggested that reflection holds for *structural* properties.[29] The property of containing all sets is not structural, because it does not specify a property of all sets that might define a structure that they instantiate or exemplify. A sufficiently rich positive characterisation of the notion of structural property is still wanting, but the present consideration illustrates why Gödel included it in the reflection principle that I will discuss here (the label is mine):

> (2) A *structural* property, possibly involving V, which applies only to elements of V, determines a set; or, a subclass of V thus definable is a set. (Wang 1996, 8.7.10)

Gödel's realist conception of V permits him to look for properties of V directly; this marks a deep difference with the kind of thinking about reflection that had been introduced by Zermelo (1930). Zermelo saw set theory as describing rather an open-ended, always extendable series of ever larger universes. Like Gödel, he accepted a version of the reflection principle, but, because of his different idea of what set theory is about, his principle is justified and used in a somewhat different way.[30] According to Zermelo, V does not really exist and hence there are no literal truths to be found about it. Talk of properties of V must really be talk about the limited set of principles used in the construction of some initial segment of the open-ended series of universes.[31] This limited set of principles remains available in

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[28]In the context of a particular formal system, the properties of V that can be reflected are of course limited by what can be expressed and defined in that system. That should contribute much to the principle's being provable, in case it is.

[29]See Wang (1977, Sect. 3; 1996, 283–285), and Reinhardt (1974, 189n1).

[30]On their differences, see also the extensive discussion (from a somewhat different perspective) by Tait (1998).

[31]'Construction' in the sense that the existence of this segment is derived from specific axioms by specific principles. In classical set theory, such axioms and principles will themselves generally not be 'constructive' in the sense in which that term is used to characterise varieties of mathematics such as intuitionism.

the construction of any longer segment of the series, and this is why the property in question will persist. In other words, we have a justification of Zermelo's reflection principle by a continuity argument.[32] Gödel, on the other hand, is not forced to construe talk of properties of V as talk about something limited; hence, reflection as exemplified by Gödel's principle has been characterised as 'top-down', Zermelo's as 'from below'.[34] Potentially, top-down reflection is the more powerful of the two. But in its use the principle is correspondingly more difficult, as it requires one to sort out those properties of V that are not reflectable from those that are; hence Gödel's quest for 'structural properties'. Moreover, one might think that Zermelo's conception is to be preferred on philosophical grounds, as by accepting it, one is freed from the demands for an argument for the existence of V and for an account of how can we come to know truths about it.[35]

But it is precisely here that Gödel will have seen an advantage for his view. As Hellman, who supports and develops Zermelo's conception, has noted, that conception requires that one accepts a notion of possible objects that does not imply the existence of possibilia (Hellman 1989, 57, 8). But as we saw in Sect. 3.2, from a Leibnizian point of view such a notion of possibility cannot be accepted, and talk of possibilities that are not grounded in something existent is ultimately unintelligible. The same criticism would be applied to any other interpretation of set theory in which commitment to the existence of V is avoided by resorting to modal notions.[36] According to Gödel, the open-endedness of the notion of set that motivates resorting to notions of possibility is not the correlate of an

[32]The logic of open-ended series is intuitionistic rather than classical. This type of reasoning we will see again later on in this paper, see Footnote 70. For more on this type of argument and its justifications, see van Atten and van Dalen (2002). Georg Kreisel wrote to me in a letter of March 7, 2006, that in the period that he knew Gödel, the latter was 'sympathetic to a justification by intuitionistic logic (in terms of not necessarily constructive knowledge)' of set-theoretic reflection principles.[33]

[33]⟦In the meantime, I found this passage in Kreisel (1969b, 99–100):

> Let us note that, in terms of knowledge of the unbounded hierarchy, reflection principles are quite evident: since the hierarchy is unattained, the only way I can know that an assertion A is valid for the hierarchy is to have attained a stage α such that A is valid for V_α and the proof of validity uses only closure conditions that hold for the hierarchy. (Of course the argument justifies the formal statement of the reflection principle only for the new interpretation of the logical operations.)

In his *Autorreferat* of this paper in the *Zentralblatt*, Kreisel (1973, 12) calls this 'the author's new contribution' to the discussion of similar ideas at the time. Also Tait (1998, 478) argues that 'the logic that applies to arbitrary formulas of set theory, when these are interpreted in the universe of all sets, should be constructive, not classical logic.'⟧

[34]E.g., Hellman (1989, 90).

[35]From the point of view of constructive mathematics in the sense explained in Footnote 31, what remains to be accounted for in Zermelo's conception would of course still be far too much.

[36]Yourgrau's criticism of Parsons' position is of the same type. See Parsons ((1977) 1983, 268–297) and Yourgrau (1999, 177–185).

ontological fact: 'To say that the universe of all sets is an unfinishable totality does not mean objective undeterminedness, but merely a subjective inability to finish it' (Wang 1996, 8.3.4).[37] (Here, 'subjective' seems to refer to the act, however idealised, of obtaining a collection by putting it together from elements which are considered to be given prior to that act. Cantor's notion of set (quoted above) contains a subjective element in just this particular sense. The universe V can never be obtained in such an act, as V cannot be a set.) A closely related Leibnizian observation is made by Mugnai:

> In man's limited intellect there is a distinction between the 'capacity to think' and the 'actual exercise' of that capacity. This distinction is not met within God. If the ideas *in Mente Dei* are conceived as 'dispositional properties' then we must also postulate a 'state' of the divine intellect in which it carries out a limited activity, during which all the totality of ideas are never present all at once. This is surely unacceptable from the theological point of view, however, since it limits the divine powers and assimilates the psychological and reasoning activity of God to the example of human activity. (Mugnai 1992, 24)[38]

3.4 Gödel's Analogy Argument for the Reflection Principle

3.4.1 Presentation of the Argument

Gödel's argument for principle (2) that I should like to analyse (not his only one) consists in drawing an analogy to Leibniz' monadology. Here I will present that argument, try to fill in the details, consider the question whether it is a good argument, and conclude that it is not. In doing so, I will not be arguing that the alternative arguments that Gödel had for the validity of reflection principles are incompatible with a Leibnizian metaphysics. What I am going to argue is that the one argument we know of in which Gödel explicitly tries to argue from a Leibnizian metaphysics to a form of the reflection principle in set theory does not work.

A note on the sources that will be used here: as yet, Gödel's philosophical notebooks have been transcribed only partially. For all I know there may be material in those untranscribed parts that is relevant to the matter at hand. As a principle of interpretation, I will assume that the argument that Gödel in the 1970s, when he had perfect access to his notebooks from the 1940s (except for the one from 1945–1946 that he reported lost), is the version that he considered best. As for Leibniz, I have tried to use, whenever possible, writings from 1686 and later, as that is the phase in the development of Leibniz' philosophy that in 1714 culminated in the *Monadology*. But in particular cases earlier texts may be relevant as well.

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[37]Tait (1998, 478) wishes to leave open the same possibility of objective undeterminedness that Gödel denies.

[38]Similarly, Jolley (1990, 138) notes: 'Now Leibniz might be more reluctant than Mates to allow that divine ideas are dispositions, for this may be difficult to reconcile with the traditional view that God is pure act.'

What might motivate one to draw an analogy between monadology and set theory is that in both cases we have a universe of objects, the objects resemble in some sense the whole, and the actual universe is in some sense the best out of a collection of possible universes. In the monadology, God chooses a universe or world to actualise from out of the collection of possible worlds, according to some criteria for which one is best; in set theory, models for *ZFC* are known which are generally not believed to correspond to set-theoretical reality (e.g., the so-called 'minimal model' is considered not to be the 'best' model because it is too small). The themes of reflection and mirroring occur often in Leibniz' writings. A typical example is Leibniz' formulation of his Principle of Harmony in Sect. 56 of the *Monadology*[39]:

> Now this interlinkage or accommodation of all created things to each other, and to each of all the others, brings it about that each simple substance has relations that express all the others, and is in consequence a perpetual living mirror of the [whole] universe. (See *Theodicy*, secs. 130, 360.) (Leibniz 1991, Sect. 56; amendment Rescher)[40]

One could use the monadology as a means to generate structural principles for monads and their relations, substitute in such a principle the notion of set for that of monad, and then seek independent reasons why the set-theoretical principle thus obtained should be true. The justification one might then come up with will not depend on an analogy between the universes of monads and sets. This merely heuristic approach was followed by Joel Friedman in his paper 'On some relations between Leibniz' monadology and transfinite set theory' (1975) where he obtained maximising principles in set theory on the basis of maximising principles of harmony in the monadology. A similar somewhat loose (but not necessarily less fruitful) approach was taken by Wim Mielants in his paper 'Believing in strongly compact cardinals', where 'Leibniz's philosophy is only a source of inspiration for the maximization properties we use here' (Mielants 2000, 290). One conclusion that may be drawn from the present paper is that such a heuristic approach will probably be more fruitful than an analogy of the type Gödel wished to draw.

Gödel's analogy is one that he takes to be by itself a justification of a form of the reflection principle, without the need to adduce independent reasons. As will be discussed later, a convincing analogy argument does not always require that the situation with which an analogy is drawn is, in its full extent, actual or real. But, to look ahead a bit, Gödel's use of his analogy as a sufficient justification is based on the idea that the reflection principle is true in set theory for exactly the same reason why a certain monadological proposition is true. As long as it is not clear that such a general reason, should it exist at all, would involve no specifically monadological notions, it is not clear whether here, too, justification can be treated independently of a justification of the monadology. For the moment I will leave it an open question

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[39] The actual name is given to it in Sect. 78.

[40] Or cette liaison ou cet accommodement de toutes les choses creées à chacune et de chacune à toutes les autres, fait que chaque substance simple a des rapports qui expriment toutes les autres, et [elle] est par consequent un mirroir vivant perpetuel de l'univers. (*Theodicée*, secs. 130, 360) (Leibniz 1991, Sect. 56)

whether one has to accept the monadology as the true metaphysics in order to be convinced by Gödel's argument, and concentrate rather on the prior task of filling in the details of the analogy that he indicates.

Hao Wang argument in item 8.7.14 of his *Logical Journey*. For clarity, I quote the preceding item as well:

> 8.7.13 ... Consider a property $P(V, x)$, which involves V. If, as we believe, V is extremely large, then x must appear in an early segment of V and cannot have any relation to much later segments of V. Hence, within $P(V, x)$, V can be replaced by some set in every context. In short, if P does not involve V, there is no problem; if it does, then closeness to each x helps to eliminate V, provided chaos does not prevail.
>
> 8.7.14 There is also a theological approach, according to which V corresponds to the whole physical world, and the closeness aspect to what lies within the monad and in between the monads. According to the principles of rationality,[41] sufficient reason, and preestablished harmony, the property $P(V, x)$ of a monad x is equivalent to some *intrinsic* property of x, in which the world does not occur. In other words, when we move from monads to sets, there is some set y to which x bears intrinsically the same relation as it does to V. Hence, there is a property $Q(x)$, not involving V, which is equivalent to $P(V, x)$. According to medieval ideas, properties containing V or the world would not be in the essence of any set or monad. (Wang 1996, 8.7.14)

So in the case for sets, the claim is that $P(V, x) \equiv Q(x)$, where $Q(x) = \exists y P(y, x)$ and x and y are sets. (Certainly, the fact that $Q(x)$ is a one-place predicate does not suffice to make it express a non-relational property. See Ishiguro 1990, Chap. 6.)

The approach is 'theological' because in the monadological setting, it is a central monad or God who creates a universe of objects.[42] To make Gödel's analogy more explicit, I propose to put it in a slightly different form, the rationale of which will be explained as we go along. As Gödel adds the explanation that 'according to medieval ideas, properties containing V or the world would not be in the essence of any set or monad'. it is clear that he in this analogy argument considers only essential properties. He first presents, in effect, the following monadological proposition:

> Essential, relational properties of (created) monads are intrinsic properties in which the universe as a whole does not occur but part of it does.

[41] By this, I take it, Gödel means the principle of contradiction.

[42] A curious example of a theological approach by Gödel to a mathematical question is found in his notebook Max X of 1943–1944 (Gödel Papers, 6b/70, 030096, 18): 'Does the commandment that one shall make neither likeness nor image perhaps also mean, that type theory must be accepted and that any formalisation of the all leads to a contradiction?' ('Bedeutet vielleicht das Gebot, du sollst dir kein Gleichnis noch Bildnis machen, auch, daß die Typentheorie anzunehmen ist und jede Formalisierung des Alls zu einem Widerspruch führt?'). The inference from a commandment to a mathematical truth would seem to fit a Cartesian view of the relation between God and mathematics better than a Leibnizian one. For Descartes, mathematical truth was determined by God's will; Leibniz contested this. For an analysis of this difference between Descartes and Leibniz, see Devillairs (1998). More positive statements by Gödel on type-free logic occur in, for example, his correspondence with Gotthard Günther, see Gödel (2003, 527, 35).

'Part' here is meant in the proper sense according to which no part of the universe expresses the whole universe perfectly; this is in fact implied by the condition that in the properties in question 'the universe as whole does not occur'. The notion of expression Leibniz describes as follows:

> That is said to express a thing in which there are relations which correspond to the relations of the thing expressed. (Leibniz 1969, 207)[43]
>
> It is sufficient for the expression of one thing in another that there should be a certain constant relational law, by which particulars in the one can be referred to corresponding particulars in the other. (Rutherford 1995, 38)[44]
>
> One thing expresses another (in my terminology) when there exists a constant and fixed relationship between what can be said of one and of the other. (Mates 1986, 38n11)[45]

Clearly, a perfect expression of x by y requires a 1-1 correspondence between all properties of x and (some) properties of y.

Let us call the above monadological proposition the 'reflection principle for (created) monads'. Gödel then proposes that we move from monads to sets and obtain from this, by analogy, the reflection principle for sets:

> Essential, relational properties of sets are intrinsic properties in which V does not occur but a set does.

In the move from monads to sets, the immediate analogue of a part (in the strong sense) of the universe of monads (a collection of monads) is a part of the universe of sets, hence a collection of sets and not an individual set. But this actually suffices, because of the following principle that Gödel accepted: any collection that is properly contained in V and that cannot be mapped 1-1 to it (and in that sense cannot perfectly 'express' V), is not a proper class but a set. This is known as 'Von Neumann's axiom'.[46] So although the immediate analogue of a collection of monads that does not perfectly express the universe of monads is a collection of sets that does not perfectly express V, by Von Neumann's axiom analogy argument concludes to.

Gödel commented on Von Neumann's axiom:

> As has been shown by Von Neumann, a multitude is a set if and only if it is smaller than the universe of all sets. (Wang 1996, 8.3.7)

43 Exprimere aliquam rem dicitur aliud, in quo habentur habitudines, quae habitudinibus rei exprimendae respondent. (Leibniz 1875–1890, 7:263)

44 Sufficit enim ad expressionem unius in alio, ut constans quaedam sit lex relationum, qua singula in uno ad singula respondentia in alio referri possint. (Leibniz 1903, 15)

45 Une chose exprime une autre (dans mon langage) lorsqu'il y a un rapport constant et reglé entre ce qui se peut dire de l'une et de l'autre. (to Arnauld, October 9, 1687; Leibniz 1875–1890, 2:112)

46 The idea had already been formulated by Cantor in a letter to Dedekind of July 28, 1899, first published in Cantor (1932), 7 years after Von Neumann's paper (1925). For a clear and detailed discussion of this axiom, see Hallett (1984, Sect. 8.3).

> The great interest which this axiom has lies in the fact that it is a maximum principle, somewhat similar to Hilbert's axiom of completeness in geometry. For, roughly speaking, it says that any set which does not, in a certain well defined way, imply an inconsistency exists. (Wang 1996, 8.3.8)[47]

This fits well into Leibniz' picture according to which mathematical existence is equivalent to mathematical possibility, and the latter is wholly determined by a (global) principle of non-contradiction; we will come back to this later.

3.4.2 The Analogy Is Ineffective

The conception of analogy arguments I will use here is Kant's, who in Sect. 58 of the *Prolegomena* writes: 'Such a cognition is one by analogy, which does not signify for example, as the word is commonly understood, an imperfect similarity of two things, but a perfect similarity of two relations between entirely dissimilar things.'[48] If the similarity in question is perfect, it will be embodied in a general principle that governs both of the domains involved in the analogy. Only the existence of such an underlying general principle can give an analogy argument genuine force. Of course, once such a general principle has been identified, it can be used to construct a direct argument for the desired conclusion, and the analogy is no longer necessary. The function of the analogy will then have been to have pointed to the relevant general principle.[49]

So in order to show that the similarity claimed by Gödel is not arbitrary or superficial, but does indeed carry argumentative weight, it would have to be shown that the reflection principle holds for monads because they instantiate a more general principle that implies reflection for universes of objects satisfying certain conditions. Applying that same more general principle to the universe of sets should then yield the reflection principle for sets.[50]

But such a principle, I claim, cannot exist. In a first step, I argue that it is consistent with the purely metaphysical principles of the monadology to assume that

[47]The inconsistency Gödel refers to here is the inconsistency arising from conceiving of a particular kind multitude as set. As we saw above, for Gödel V genuinely exists, but as a mere multitude and not as a set.

[48]'Eine solche Erkenntnis ist die nach der Analogie, welche nicht etwa, wie man das Wort gemeiniglich nimmt, eine unvollkommene Ähnlichkeit zweier Dinge, sondern eine vollkommene Ähnlichkeit zweier Verhältnisse zwischen ganz unähnlichen Dingen bedeutet.' (Kant [1783] 1965b, 124) Gödel will surely have known this passage; but in his copy of the Reclam 1888 edition of the *Prolegomena*, there are no reading marks to it. (I am grateful to Marcia Tucker at the Historical Studies-Social Science Library of the IAS for having verified this.)

[49]To emphasise that this is the function of an analogy, St. Augustine classified it with the signs, cf. Maurer (1973).

[50]In his formulation of reflection principle (2) in Sect. 3.3 above, Gödel mentions a restriction on the properties that can be reflected, saying that they should be 'structural'. I will come back to the possible role of this restriction in the analogy later.

the reflection principle for monads holds but the reflection principle for sets fails. In the second step, I explain why this entails that Gödel's analogy is ineffective, whether the monadology is true or not.

That in the monadology the reflection principle for monads is consistent follows from the fact that, as I will now argue, in the monadology that principle is true.

As a preliminary, the meaning of the term 'essence' has to be clarified. Leibniz uses it in different ways. Sometimes he defines the essence of a monad as simply the collection of all its properties, considered in abstraction from the existence of that monad. As he holds that each monad expresses the whole universe or world, by this definition it is trivially false that the essence of a monad does not involve the world.[51] But Leibniz also has another notion of essence, which is the one that will be relevant here. This notion is defined as the collection of all the necessary properties of that substance. For example, in 1676 Leibniz first defines an 'attribute' as 'a necessary predicate conceived through itself, or that cannot be analysed into several others' and then 'an *essence* is ... the aggregate of all the attributes (of a thing)' (Adams 1994, 127).[52] In 1678 he defines the 'essence of a thing' as 'the specific reason of its possibility' and specifies that what is true in the region of essences is 'unconditionally, absolutely and purely true' (Adams 1994, 136, 38).[53] This definition he repeats two decades later, in 1701, 'the essence of the thing being nothing but that which makes its possibility in particular'.[54] Of particular interest for its idealistic content is Leibniz' remark in the *New Essays* (1705) that possibility is the same as being distinctly intelligible (which intelligibility is ruled out for contingent properties).[55] Finally, in 1714, he writes that

[51] While reading Leibniz (1903), Gödel noted: 'The proposition that every thing involves all others, can be understood purely logically. Namely: It involves all accidents, among these however also the relations to all other things; these however involve the other things. But that is only an accidental, no necessary involvement. But to the extent that to the essence belongs the reaction in arbitrary situations, it also involves essentially – also through knowledge (mirror) – accidental involvement.' (Gödel Papers, 6b/70, 030096 (*Max* X), 70–71. 'Die Aussage, daß jedes Ding alle andere involviert, kann rein logisch verstanden werden. Nämlich: Es involviert alle Acc[identia], unter diesen aber auch die Beziehungen zu allen anderen Dingen; diese involvieren aber die anderen Dinge. Das ist aber nur ein accident[elles], kein notwendiges Involvieren. Aber insofern zum Wesen die Reaktion in beliebigen Lagen gehört, involviert [es?] sie auch essentiell – auch durch Erkenntnis (Spiegel) – acci[dentelles] Involvieren.') Here Gödel must be referring to Leibniz' statement on p. 521 of that edition, 'Every singular substance involves in its perfect notion the whole universe' ('Omnis substantia singularis in perfecta notione sua involvit totum universum.')

[52] Attributum est praedicatum necessarium quod per se concipitur, seu quod in alia plura resolvi non potest ... Essentia est id omne quod in re per se concipitur, id est aggregatum omnium attributorum. (Leibniz 1923–, 6,3:574)

[53] 'nam essentia rei, est specialis ratio possibilitatis' and 'nulla conditione facta, absolute et pure verum est' (Leibniz 1923–, 2,1:390 and 392)

[54] l'essence de la chose n'étant que ce qui fait sa possibilité en particulier. (Leibniz 1875–1890, 4:406)

[55] 'But whether they depend on the mind or not, it suffices for the reality of their ideas, that these modes are *possible* or, which is the same thing, distinctly intelligible.' ('Mais soit qu'ils dependent ou ne dependent point de l'esprit, il suffit pour la realité de leur idées, que ces Modes soyent

> I consider possible everything that is perfectly conceivable, and which therefore has an essence, an idea; without taking into consideration whether the other things allow for it to come into being.[56]

With this notion of essence in place, the argument for Reflection for created monads proceeds as follows:

1. All properties of monads consist in their own perceptions; this does not rule out relational properties as these are intrinsic too. (Premise)
2. Essential properties correspond to distinct perceptions. (Premise)
3. No created monad can distinctly perceive the whole universe. (Premise)
4. Essential, relational properties of (created) monads are intrinsic properties in which the as a whole does not occur but part of it does. (From 1, 2 and 3)

In the opening sections of the *Monadology*, Leibniz says that monads are the ultimate constituents of reality. They are simple in the sense that they are not composed out of parts (Sect. 1). Elsewhere, Leibniz also says that the monads are not in space and time, but that space and time are rather phenomena that depend on the way monads represent reality to themselves. Although monads are simple, they do have inner states, and these can change. This does not contradict the fact that they have no parts, if this is understood to mean (in terms of Husserl's third *Logische Untersuchung*) that they have no independent parts but only dependent ones, like a continuum.[57] The changes arise within the monad itself and do not come from outside, for monads have no parts that can be acted upon from outside; they 'have no windows' (Sect. 7). Only God can be said to act upon the created monads directly. Leibniz identifies the specification and variety of simple substances with the internal complexity of these inner states (Sect. 12), and calls these transitory states 'perceptions' (Sect. 14). The properties of a monad consist in its proper perceptions.[58] Perceptions 'enfold and represent a multiplicity in a unity, or in the simple substance' (Sect. 14), and in fact each monad perceives or represents

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possibles ou, ce qui est la même chose, intelligibles distinctement', Leibniz 1875–1890, 5:246) Gödel noted this one, see Gödel Papers, 10a/36, 050131.

[56]To Bourguet, December 1714, Leibniz (1875–1890, 3:573–574): 'J'appele possible tout ce qui est parfaitement concevable, et qui a par consequent une essence, une idée : sans considerer, si le reste des choses luy permet de devenir existant.' See also Leibniz (1991, Sect. 43, [1710] 1885, Sect. 390).

[57]Leibniz used the absence of independent parts as an argument against the conception of the mind as a machine or mechanism: the mind is a unity, whereas a machine has (independent) parts, e.g., in his *New System of the Nature and Communication of Substances* from 1695, Leibniz (1969, 456). Gödel appealed to the very same argument: 'Consciousness is connected with one unity. A machine is composed of parts' (Wang 1996, 6.1.21).

[58]The special case of reflexive knowledge or consciousness that some monads sometimes have of their inner states, apperception, plays no role in Gödel's analogy.

the whole universe.[59] Various crucial points for Gödel's analogy are now made in Sect. 60:

> For in regulating the whole, God has had regard for each part, and in particular for each monad, which, its very nature being representative, is such that nothing can restrict it to representing only part of things. To be sure, this representation is only confused regarding the detail of the whole universe. It can only be distinct in regard to a small part of things, namely those that are nearest or most extensively related to each monad. Otherwise each monad would be a deity. It is not in their object [namely the whole universe], but in the particular mode of knowledge of this object that the monads are restricted. They all reach confusedly to the infinite, to the whole; but they are limited and differentiated by the degrees of their distinct perceptions. (Leibniz 1991, Sect. 60)[60]

If monads did not differ this way, they would all be one and and the same, by identity of indiscernibles (which is a consequence of Sufficient Reason). For the only properties monads have are perceptual, and perceptions differ only in degree of distinctness.[61] Only the monad which is God perceives the whole universe perfectly; the perception of the universe by created monads necessarily is (partly) confused, because their receptivity is necessarily limited (section 47).[62] It follows that the perceptions of no created monad can exhaust the universe. This precludes that the perceptions of a created monad stand in 1-1 relation to the elements of the universe, and therefore no created monad expresses the universe perfectly.

Note in passing how the fact that monads have no windows and only God acts directly upon them explains, when combined with the idea that sets are objects in God's mind, Gödel's that the monads have unambiguous access to the full set-theoretic hierarchy.[63] As Leibniz wrote around 1712:

[59] Compare also the earlier *On Nature's Secrets* from around 1690: 'Indeed, the multiple finite substances are nothing other than diverse expressions of the same universe according to diverse respects and each with its own limitations', Leibniz (1991, 217) ('Quin imo substantiae finitae multiplices nihil aliud sunt quam diversae experssiones ejusdem Universi secundum diversos respectus et proprias cuique limitationes', Leibniz 1875–1890, 7:311n).

[60] Parce que Dieu en reglant le tout a eu égard à chaque partie, et particulierement à chaque Monade, dont la nature étant representative, rien ne la sauroit borner à ne representer qu'une partie des choses ; quoy qu'il soit vrai que cette representation n'est que confuse dans le detail de tout l'Univers, et ne peut être distincte que dans une petite partie des choses, c'est à dire, dans celles, qui sont ou les plus prochaines ou les plus grandes par rapport à chacune des Monades. Autrement chaque Monade seroit une Divinité. Ce n'est pas dans l'objet, mais dans la midification de la connoissance de l'objet, que les Monades sont bornées. Elles vont toutes confusement à l'infini, au tout ; mais elles sont limitées et distinguées par les degrés des perceptions distinctes. (Leibniz 1991, Sect. 60)

[61] Gödel writes in his Notebook Max X (1943–1944): 'Almost any property can be had to different degrees' (Gödel Papers, 6b/70, 030096, 20. 'Man kann fast alle Eigenschaften in verschiedenen Graden haben.')

[62] Necessarily, for by identity of indiscernibles God is unique; Sect. 39 cites, alternatively, the principle of sufficient reason.

[63] See Footnote 1 above; this part of the anecdote is also reported in Maddy (1990, 79).

> I am convinced that God is the only immediate external object of souls, since there is nothing except him outside of the soul which acts immediately upon it. Our thoughts with all that is in us, in so far as it includes some perfection, are produced without interruption by his continuous operation. So, inasmuch as we receive our finite perfections from his which are infinite, we are immediately affected by them. And it is thus that our mind is affected immediately by the eternal ideas which are in God, since our mind has thoughts which are in correspondence with them and participate in them. It is in this sense that we can say that our mind sees all things in God. (Leibniz 1969, 627)[64]

The fact that all of a monad's properties are internal to it might seem to rule out relational properties, in which case Gödel's analogy argument would not work, for if there are no relations between monads then there is no basis for an analogy concluding to the existence of relations between sets. In fact, on Leibniz' understanding of relations, relational properties are not at all ruled out: a monad x will have a relational property P if x expresses the relata in the way characteristic for P. But to express other monads this way is an entirely internal property; it does by itself not guarantee that these other monads indeed exist. This is indeed what Leibniz meant, as he makes clear in his reply to an objection made by his correspondent Des Bosses. Des Bosses had written to Leibniz (April 6, 1715):

> If the monads of the universe get their perceptions out of their own store, so to speak, and without any physical influence of one upon the other; if, furthermore, the perceptions of each monad correspond exactly to the rest of the monads which God has already created, and to the perceptions of these monads, and are harmonised so as to represent them; it follows that God could not have created any one of these monads which thus exist without constructing all the others which equally exist now, for God can by no means bring it about that the natural perception and representation of the monads should be in error; their perception would be in error, however, if it were applied to nonexistent monads as if they existed. (Leibniz 1969, 611)[65]

64 'Entretien de Philarète et Ariste' (one of the direct forerunners of the *Monadology*). 'Je suis persuadé que Dieu est le seul objet immediat externe des ames, puisqu'il n'y a que luy hors de l'ame qui agisse immediatement sur l'Ame. Et nos pensées avec tout ce qui est en nous, entant qu'il renferme quelque perfection, sont produites sans intermission par son operation continuée. Ainsi, entant que nous recevons nos perfections finies des siennes qui sont infinies, nous en sommes affectés immediatement, et c'est ainsi que notre esprit est affecté immediatement par les idées eternelles qui sont en Dieu, lorsque notre esprit a des pensées qui s'y rapportent, et qui en participent. Et c'est dans ce sens que nous pouvons dire, que notre Esprit voit tout en Dieu' (Leibniz 1875–1890, 6:593–594). Gödel seems to have had this or a similar passage, e.g., Leibniz ((1686) 1880, Sect. 28), in mind when he remarked in his letter to Gotthard Günther of April 4, 1957: 'That abstract conceptual thought enters individual monads only through the central monad is a truly Leibnizian thought' (Gödel 2003, 527).

65 Si monades universae ex propria penu, ut sic loquar, et sine ullo physico unius in aliam influxu perceptiones suas habent, si praeterea cujuslibet monadis perceptiones caeteris quae nunc a Deo creatae sunt monadibus earumque perceptionibus praecise respondent et attemperantur eas repraesentando, non potuit ergo Deus ullam ex his quae modo existunt monadibus creare quin alias omnes quae nunc pariter existunt conderet, Deus enim nullo pacto efficere potest ut naturalis monadum perceptio ac repraesentatio fallatur, falleretur autem si ferretur in monadas non existentes tanquam existentes. (Leibniz 1875–1890, 2:493)

And Leibniz replied:

> He can do it absolutely [i.e., as far as logic is concerned]; he cannot do it hypothetically [i.e., when also God's will is taken into account], because he has decreed that all things should function most wisely and harmoniously. There would be no deception of rational creatures, however, even if everything outside of them did not correspond exactly to their experiences, or indeed if nothing did, just as if there were only one mind; because everything would happen just as if all other things existed, and this mind, acting with reason, would not charge itself with any fault. For this is not to err. . . . Not from necessity, therefore, but by the wisdom of God does it happen that judgements formed upon the best appearances, and after full discussion, are true. (Leibniz 1969, 611; Leibniz to Des Bosses, April 29, 1715)[66]

So in what Leibniz calls an 'absolute' sense, a monad can have a relational property without that relation obtaining in the world. But in the actually created world this is excluded, for in choosing that world God sees to it that the perceptions of its monads are in harmony with one another.[67] This depends on God's will instead of logic and that is why Leibniz says that it is not 'absolutely' but 'hypothetically' necessary that relational properties express relations that indeed obtain. In the presence of this principle of harmony, the circumstance that a monad x in a world truly has relational property P not only implies, but is equivalent to, the circumstance that it has an appropriate intrinsic property. This explains why Gödel mentions the principle harmony in his analogy argument: as he wishes to reason by analogy that there exists a set y that is related to the set x by $P(y, x)$, he needs, in the domain to which the analogy is drawn, the existence of a monad (or collection of monads; see below) y for the monad x to relate to. Without a principle of harmony, that existence would not be guaranteed.

The following step is to see that, more specifically, properties that are essential correspond to perceptions that are distinct. Leibniz understands by necessary properties those that admit of finite analysis into primitive ones (Sect. 33). They cannot involve confused perceptions, as those combine many perceptions into one in such a way that there is no complete, finite analysis into distinct perceptions. In the *Monadology*'s twin, the paper *Principles of Nature and Grace* from the same year, 1714, Leibniz states in Sect. 13 that 'Our confused perceptions are the result

[66]'Potuit absolute, non potuit hypothetice, ex quo decrevit omnia sapientissime agere et ἁρμονικωτάτως. Deceptio autem creaturarum rationlium nulla foret, esti Phaenomenis earum non omnia extra ipsas exacte responderent, immo si nihil: veluti si mens aliqua sola esset; quia omnia perinde evenirent, ac si essent alia omnia, neque illa cum ratione agens sibi damnum accerseret. Hoc enim est non falli . . . Non igitur ex necessitate, sed ex sapientia Dei fit, ut judicia ex maxime verisimilibus post plenam discussionem formata sint vera.' (Leibniz 1875–1890, 2:496). See also Leibniz ((1686) 1880, Sect. 14; [1710] 1885, Sect. 37; 1875–1890, 4:530).

[67]Also to Arnauld, April 30, 1687: 'It can be said that God arranges a real connection by virtue of that general concept of substances which implies perfect interrelated expressions between all of them, though this connection is not immediate, being based on what God has wrought in creating them.' (Rutherford 1995, 146) ('On peut dire que Dieu fait qu'il y a une connexion reelle env ertu de cette notion des substances, qui porte qu'elles s'entrexpriment parfaitement toutes, mais cette connexion n'est pas immediate, n'estant fondée que sur ce que Dieu a fait en les creant', Leibniz 1875–1890, 2:95–96.)

of the impressions which the whole universe makes upon us'.[68] They therefore correspond to, or express, contingent truths (*Monadology*, Sect. 36). God knows contingent truths a priori, but not by demonstration. An infinite demonstration is impossible according Leibniz, as such an object would form an infinite whole, which he believed could not exist; rather, God knows contingent truths by a (direct) 'infallible vision'.[69] There is a continuum of qualities of perception, of which complete distinctness is one extreme. The more distinct a perception is, the more it contributes to the individuality of a monad, to the point where complete distinctness corresponds to essential properties.

In particular, a relational property of a monad that is part of its essence demands that its expression of all relata is clear and distinct. It follows that, as Gödel says, it cannot be an essential property of any monad x to stand in a relation P to the universe. A monad may well stand in a relation P to the universe but this will then not be an essential property of the monad. Suppose that one finds a necessarily true proposition A that says of a created monad x that it stands in a relation P to the universe. For Leibniz, that A is a necessary truth means that A expresses an essential property of x. For the reason just given, what specifically makes A true cannot involve the whole universe but only a proper part of it. Hence, A is equivalent to a proposition B that says that x stands in a necessary relation P to just part of the universe. By the principle of harmony, a part of the universe such as perceived by x indeed exists. Thus we have arrived at what we have called a 'reflection principle for (created) monads'.[70] As noted above (Sect. 3.4.1), this argument does not yield the conclusion that there is a monad to which x is related, but that there is a part of the universe (in the sense of a collection of monads that does not express the universe perfectly) to which it is related; we also saw why, in the presence of Von Neumann's axiom, this suffices for Gödel's analogy. If an individual monad z such that x stands in the same relation to z as it does to the collection of monads y is

[68] Nos perceptions confuses sont le resultat des impressions que tout l'univers fait sur nous. (Leibniz 1875–1890, 6:604)

[69] 'On Freedom' (1689), Leibniz (1973, 111).

[70] This principle is of course closely related to the ancient and medieval idea that things are known according to the capacities of the knower, and that hence a lower being's knowledge of a higher being is necessarily incomplete. A difference between that idea and reflection is that only the latter explicitly concludes to the existence of a third object (with a certain property). But in a formulation of Odo Reginaldus from around 1243–1245, that conclusion is more or less present: 'How can a finite being reach the infinite? About this, some others have said that God will present himself to us moderated, and that he will show himself not in his essence, but in a creature'. ('Quomodo potest finitum attingere ad infinitum? Propter hoc dixerunt alii quod deus contemperatum se exhibebit nobis, et quod ostendet se nobis non in sua essentia, sed in creatura' (Coté 2002, 78). Odo then comments that this opinion has fallen from favour ('Sed hec opinio recessit ab aula'), which, theologically, is not surprising.). From here it is only a small step to: 'Suppose creature A has a perception of God. Then God is capable of making a creature B such that A's perception cannot distinguish between God and B.' The argumentation here is reminiscent of continuity arguments. Côté's monograph (2002) is an invaluable analysis of the medieval discussion of finite beings' knowledge of an infinite God.

possible, then it could be argued that God would go on actually to create that monad z, on the ground of a principle of maximality or plenitude (which is a form of the principle of harmony).[71]

At this point, the following might seem to be a quick argument against Gödel's analogy. Reflection for monads depends on God's will (namely, on His choice to create a universe that is harmonious), and is in that sense contingent; reflection for sets, on the other hand, is supposed to be a necessary principle. But then these two forms of reflection cannot be true on the ground that both instantiate one and the same general principle. However, this argument does not succeed, because the general principle might be (or could be made) conditional on harmony: 'For all harmonious universes, . . .' In the case at hand, all that harmony amounts to is the requirement that, if an object in a universe has a relational property, the relata also exist in that universe. For the universe of monads this needs, as we saw, some argument, while for the universe of sets it seems trivial. But for the applicability of the general principle the reason why a universe is harmonious would not matter, only that it is.

Instead, the argument against Gödel's analogy proceeds from the fact that, in contrast to reflection for monads, it is consistent with the monadology that no reflection principle for sets holds. This is because the monadology poses no metaphysical constraints on the essential mathematical properties that a set (or any object of pure mathematics) can have. The explanation for this is as follows.

As we saw in Sect. 3.2, the objects of pure mathematics are, for Leibniz, entirely mental objects, and have their primary and original existence in God's mind. As a consequence, the existence of relations between pure sets or collections (in particular, V) will have no foundation in a created monad. Relations between pure sets or collections are, ontologically, relations between God and himself. Relations have their ultimate reality in God's being able to think them. But, contrary to the case of created substances, for Leibniz there are no intrinsic limitations to God's thinking other than non-contradiction. 'Possible things are those which do not imply a contradiction,' he says,[72] and God thinks all possibilities:

> The infinity of possibles, however large it may be, is not larger than that of the wisdom of God, who knows all possibles.[73]

What is true in mathematics, and in particular what relations can obtain between mathematical objects, depends only on the Principle of Contradiction. The Principle

[71] 'After due consideration I take as a principle the Harmony of things, that is, that the greatest amount of essence that can exist does exist.' (Mercer 2001, 413–414) ('Recte expensis rebus pro principio statuo, Harmoniam rerum, id est quantum plurimum essentiae potest existat', Leibniz 1923–, 6,3:472 (1671))

[72] Leibniz to Joh. Bernoulli, February 21, 1699, Leibniz (1849–1863, 3:574): 'Possibilia sunt quae non implicant contradictionem.'

[73] L'infinité des possibles, quelque grande qu'elle soit, ne l'est pas plus que celle de la sagesse de Dieu, qui connaît tous les possibles. (Leibniz [1710] 1885, Sect. 225)

of Sufficient Reason and its consequences have no influence on what is or is not the case in mathematics. Leibniz explains this in his second letter to Clarke, from 1715:

> The great foundation of mathematics is the *principle of contradiction or identity*, that is, that a proposition cannot be true and false at the same time and that therefore A is A and cannot be non-A. This single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles. But in order to proceed from mathematics to natural philosophy, another principle is requisite, as I have observed in my *Theodicy*; I mean the *principle of a sufficient reason* ... Now by that single principle, viz., that there ought to be a sufficient reason why things should be so and not otherwise, one may demonstrate the being of a God and all the other parts of metaphysics or natural theology. (Leibniz 1969, 677–678)[74,75]

Leibniz says this to support his contention that the mathematical principles of the materialist philosophers are the same as those of Christian mathematicians, the difference between them rather being the metaphysical one that the Christians admit immaterial substances. As Leibniz sees it, the truths of metaphysics (i.e., the principles specifically about monads and their relations to one another) all follow from the principle of sufficient reason (together with the principle of contradiction), but the principle of contradiction is prior to the principle of sufficient reason. In particular, then, sufficient reason and its consequences are compatible with any relation that obtains between pure possibilities. As a special case, no metaphysical principle constrains what is true in pure mathematics. This idea one finds in both early and late Leibniz. For example, the young Leibniz wrote to Magnus Wedderkopf (May 1671),

> No reason can be given for the ratio of 2 and 4 being the same as that of 4 and 8, not even in the divine will. This depends on the essence itself, or the idea of things. For the essences of things are numbers, as it were, and contain the possibility of beings which God does not make as he does existence, since these possibilities or ideas of things coincide rather with God himself. (Leibniz 1969, 146)[76]

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74 Le grand fondement des Mathematiques est le *Principe de la Contradiction*, ou de l'*Identité*, c'est à dire, qu'une Enontiation ne sauroit etre vraye et fausse en même temps, et qu'ainsi A est A, et ne sauroit etre non A. Et ce seul principe suffit pour demontrer toute l'Arithmetique et toute la Geometrie, c'est à dire tous les Principes Mathematiques. Mais pour passer de la Mathematique à la Physique, il faut encor un autre Principe, comme j'ay remarqué dans ma Theodicée, c'est le *Principe du besoin d'une Raison suffisante* ; ... Or par ce principe seul, savoir : qu'il faut qu'il y ait une raison suffisante, pourquoy les choses sont plutost ainsi qu'autrement, se demonstre la Divinité, et tout le reste de la Metaphysique ou de la Theologie Naturelle (Leibniz 1875–1890, 7:355–356).

75 Also Sect. 9 in the fifth letter to Clarke (Leibniz 1875–1890, 7:390; 1969, 697), and Leibniz ([1710] 1885, Sect. 351).

76 Per exemplum quod ea ratio est 2 ad 4 quae 4 ad 8, ejus reddi ratio nulla potest, ne ex voluntate quidem divina. Pendet hoc ex ipsa Essentia seu Idea rerum. Essentiae enim rerum sunt sicut numeri, continentque ipsam Entium possibilitatem quam Deus non facit, sed existentiam: cum potius illae ipsae possibilitates seu Ideae rerum coincidant cum ipso Deo. (Leibniz 1923–, 2,1:117)

And, much later, in a letter to Pierre Varignon of June 20, 1702,

> Between you and me, I believe that Mr de Fontenelle, who is of a courteous and beautiful spirit, wanted to make fun of us when he said that he wanted to make elements of metaphysics out of our calculus.[77]

As Michel Fichant has concluded,

> The idea of a metaphysics of the calculus of the infinite, or of a metaphysical transposition of a consideration on the calculus of the infinite, is entirely alien to Leibniz; whenever someone ventured in that area, he has always objected to it.[78]

This absence of a metaphysical constraint on mathematical truth implies that no description or reasoning in purely metaphysical terms can lead us to the discovery of an underlying general principle that would imply that reflection holds for sets, too, as such a metaphysical description will be equally compatible with the falsehood of reflection for sets. Yet, Gödel's analogy argument in effect precisely attempts to draw attention to a general principle in this way. Gödel first describes a purely metaphysical fact, namely the reflection principle for monads, and then arrives at the desired mathematical conclusion by, as he says, 'moving from monads to sets'. This analogy argument therefore fails. Of the reason for this failure, i.e., the fact that metaphysical principles do not constrain pure possibilities,[79] two further consequences should be noted. First, adding, in particular, a metaphysical principle that somehow corresponds to the restriction in Gödel's formulation of reflection principle (2) in Sect. 3.3 above that the set-theoretical properties to be reflected should be *structural* will not help making the analogy work. A second consequence

[77] Entre nous je crois que Mons. de Fontenelle, qui a l'esprit galant et beau, en a voulu railler, lorsqu'il a dit qu'il vouloit faire des elemens metaphysiques de nostre calcul. (Leibniz 1849–1863, 4:110)

[78] 'L'idée d'une métaphysique du calcul de l'infini ou d'une transposition métaphysique d'une réflexion sur le calcul de l'infini est totalement étrangère à Leibniz ; il l'a toujours récusée chaque fois que quelqu'un s'est aventuré dans ces parages.' (Fichant 2006, 29–30). A few lines further on, he writes: 'It is true that he says in a famous letter to Varignon that "the real never fails to be perfectly governed by the ideal and the abstract", on account of which, in effect, mathematical calculations are applicable to nature, but, basically, to nature inasmuch as the real in question is at the level of the phenomena, not at that of the substances.' ('Il est vrai qu'il dit dans une lettre célèbre à Varignon, que "le réel ne laisse pas de se gouverner parfaitement par l'idéal et l'abstrait", ce qui fait que, effectivement, les calculs mathématiques sont applicables à la nature, mais, au fond, à la nature pour autant que le réel dont il est alors question se situe sur le plan du phénomène, et non sur celui des substances.') A curious exception to Leibniz' advocated practice of keeping metaphysical principles out of purely mathematical arguments occurs in his attempts to show that absolute space is Euclidian, which appeal to the principle of sufficient reason. For a full discussion of this exception, see de Risi (2007, 252–264).

[79] One of Leibniz' manuscripts of around the time of the *Monadology* is titled 'The metaphysical foundations of mathematics', Leibniz (1969, 666–674). But in it, Leibniz actually proceeds by defining and developing pure concepts; it is metaphysical on account of the great generality of them. An example is his argument for the proposition that the whole is always greater than the part. But there is no mention whatsoever of monads and the principles governing them and their relations, and therefore not metaphysical in that sense.

is that the monadology will not suggest a *disanalogy* with the reflection principle for sets either. No truth about monads and their relations can contradict mathematics, for on Leibniz' conception, God's acts of creation are (voluntary) acts of applying mathematics.[80]

The argument against Gödel's analogy does not depend on Leibniz' specific construal of mathematical possibility in terms of non-contradiction; what it depends on is the more general condition that the notion of pure possibility that defines mathematical truth is a boundary condition on the possible worlds out of which the metaphysical principles select one. Any notion of mathematical possibility that guarantees invariance of mathematical truth with respect to possible worlds will satisfy this condition.[81]

Generally speaking, a successful analogy from a state of affairs in one domain to a state of affairs in another may or may not presuppose that the first domain actually exists. The only function of the description of a state of affairs in the first domain is to suggest to us the relevant general principle governing it, so that we can apply that to the second domain. Such a principle may well hold in merely possible or fictional domains as well as in actual ones. Gödel's analogy argument may or may not presuppose that the monadology is true. That would seem to depend on whether the general principle required should involve notions specific to the monadology or not. The reason just presented why the analogy is ineffective does not turn on the answer to this question, however, for it was argued that there can be no such

[80]'God makes the world while calculating and exercising knowledge' ('Cum Deus calculat et cogitationem exercet, fit mundus'), Leibniz (1875–1890, 7:191n), and 'Necessity in geometry is absolute, but it follows that this is also the case in physics, because the supreme Wisdom, who is the source of things, acts as the most perfect geometer and observes harmony' ('Absolutae est necessitatis in Geometria, sed tamen succedit et in Physica, quoniam suprema Sapientia, quae fons est rerum, perfectissimum Geometram agit et Harmoniam observat'), Leibniz (1849–1863, 6:129). (The notion of absoluteness here must be a wider one than the one in Leibniz' letter to Des Bosses, quoted above, from which harmony is explicitly excluded.) For general discussion of this idea, see Osterheld-Koepke (1984, 138–144). The idea also contributes to an explanation of the following observation by Gödel from 1942 (Gödel Papers, 6b/68, 030092, 380): 'The principle that every math[ematical? metaphysical?] proposition has a generalisation for arbitrary higher cardinality (but not the other way around) expresses one of the most general properties of the structure of the world. Namely: Everything is mirrored in everything. (The symbol and the reference are structur[ally] the same?) God created man to His likeness. The same thing appears at different levels. Here we have an "unfolding".' ('Das Prinzip, daß jeder math[ematische? metaphysische?] Satz eine Verallgemeinerung für beliebig höhere Mächtigkeit hat (aber nicht umgekehrt) drückt eine der allgemeinsten Eigenschaften des Aufbaus der Welt aus. Nämlich: Alles spiegelt sich in allem. (Das Symbol und die Bedeutung sind struktur[ell] gleich?) Gott schuf den Menschen sich zum Bild. Dasselbe erscheint auf verschiedene Niveaus. Es handelt sich um eine "Entfaltung".') Compare *Monadology*, Sect. 83.

[81]Note that for Leibniz, what makes mathematical truths true has nothing to do with possible worlds, only with the principle of contradiction. For an argument that the notion of possibility that defines mathematical truth in Husserl's transcendental idealism satisfies the condition mentioned, see van Atten (2001). The particular relevance of this fact is that after 1959 Gödel adopted Husserl's transcendental idealism as a means to develop Leibniz' monadology scientifically. See the Concluding remark, below.

principle anyway. This also means that, if one makes the assumption that Leibniz' monadology (or something sufficiently close to it) is the true metaphysics, there is no direct argument either: knowing the details of exactly how sets fit into this metaphysics yields no additional means to determine the truth value of the reflection principle. Both the analogy argument and a direct argument will fail for the same reason, namely, that in Leibniz' system the specifically metaphysical principles do not imply constraints on what can be true about pure sets and collections. More generally, as we have seen, Leibniz' specifically metaphysical principles do not imply constraints on what can be true in any part of pure mathematics. The present considerations on Gödel's analogy argument and on the possibility of a direct argument are therefore not really specific to sets and reflection, and can be expected to have wider application.

In the light of the absence of implied metaphysical constraints on mathematics, it is not surprising that when Leibniz attempts to show that there can be no infinite wholes, he proceeds from logical truths and not from metaphysics or properties of minds. Contrast this to, for example, Brouwer, who based his idea that in mathematics there exist only potentially infinite constructions (and hence no constructed infinite wholes) not on a conceptual argument but on an observation about the human mind.

3.4.3 'Medieval Ideas'

After having presented the analogy with the monadology, Gödel adds that 'according to medieval ideas, properties containing V or the world would not be in the essence of any set or monad'. As the reflection principle for monads follows from the monadology itself, and the analogy should then directly lead to the reflection principle for sets, this remark on medieval ideas does not seem to play a role in the argument. It seems rather an afterthought, a corroboration of the argument and its conclusion from medieval quarters.

A characteristically medieval idea (in the Christian world) is that the world and its creation are radically contingent. If the essence of any object in the world would involve the whole world, that essence might be taken to put limits on that contingency, and hence on God's freedom in creating the world. A related point is that if the essence of an object would involve the world, understood as the totality of all actual objects, it would in particular involve its own existence, but for the medievals this is only the case for God. To the extent however that one is looking for medieval ideas that could be applied to set theory, where truths are necessary and contingency plays no role whatsoever, this seems not the right suggestion for what Gödel may have had in mind.

The only idea I have been able to find that does not depend on contingency would be the idea that 'being' is what medieval philosophers called a transcendental notion. This means that the notion of 'being' (*ens*), and for example others such 'one' or 'true' transcend the categories into which reality can be classified because they are too general notions to define a category. The extension of the concept of being coincides with, or (if one assumes God exists but does not fall under the categories) even properly includes, the extensions of the categories combined. Aristotle already recognised the existence of such notions (*Metaphysics* 1003b25, 1061a15). The idea is therefore not medieval in the sense of having been introduced in the Middle Ages; but it is typically medieval in that the development of theories about transcendentals did not begin until then. The first systematic treatment of transcendentals is taken to be *Summa de Bono* by Philip the Chancellor, written between 1228 and 1236; but the best known passages dealing with this notion are those in Aquinas' *De Veritate* (1256–1259) and the *Summa Theologica* (1265–1272).[82]

Aquinas specifies that 'the individual essence of an object is what is given by the definition [of that object]'.[84] In turn, that definition consists in a specification of the genus of the object and of the specific differences that distinguishes it from other objects of the same genus. The argument that being cannot be a genus is the following: 'Every genus has differences distinct from its generic essence. Now no difference can exist distinct from being; for non-being cannot be a difference.'[85] The idea is that genera and differences serve to distinguish the objects that exist from one another, and hence correspond to asymmetries between them; however, no two objects that both have being can be related to being asymmetrically. Therefore, on the Aristotelian model of definitions, the concept of being cannot contribute to the definition of any object. If one understands by 'the world' 'all that has being', this means that the essence of no object involves the world. Indeed, Aquinas calls the multitude that results from dividing being according to all its forms the 'transcendent multitude', points out that like being itself it is not a genus, and distinguishes this from 'numerical multitudes' (Aquinas [1265–1274] 1888–1906, I, 30, a.3).

Leibniz also recognises that 'being' is a transcendental notion. Usually he refers to the characteristic property of transcendentals that they are all convertible with being: that is, the transcendental terms (e.g., being, one, true) differ from one another intension but not in extension. To Des Bosses, Leibniz wrote on February 14, 1706: 'I agree with you that being and one are convertible;'[86] and some 20 years earlier, on April 30, 1687, to Arnauld:

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[82]From notes in his archive, it is known that Gödel read works of Aquinas.[83]

[83]⟦For discussion of an example, see Engelen (2013).⟧

[84] essentia proprie est id quod significatur per definitionem. (Aquinas [1265–1274] 1888–1906, I, 29, a.2 ad 3)

[85]'omne enim genus habet differentias quae sunt extra essentiam generis; nulla autem differentia posset inveniri, quae esset extra ens; quia non ens non potest esse differentia.' (Aquinas [1265–1274] 1888–1906, I, 3, a.5) See also Aristotle's argument in *Metaphysics*, Aristotle (1933, 998b21–28).

[86] Ens et unum converti tecum sentio. (Leibniz 1875–1890, 2:300)

> I regard as an axiom this proposition of which the two parts differ only by their emphasis, namely, that what is not really *one* being is not really one *being* either. It has always been believed that one and being are reciprocal.[87]

In this last sentence, Leibniz makes an implicit reference to Aristotle and the scholastics. I do not know whether it was this reference that led Gödel to consider medieval philosophy in this context. Be that as it may, Leibniz' reason for considering 'being' a transcendental was different from that of the scholastics.[88] Where the scholastics considered the notion of being as it applies to an object in the actual world, Leibniz considered the notion of being as it applies to a possible object. This notion corresponds to that of being one, as a possible object is determined by one complete concept. Leibniz' conception in terms of purely possible as opposed to actual beings (in the world) comes closer to what Gödel says when he invokes these 'medieval ideas', as he wants to include sets, which for Leibniz are always possible but, being 'incomplete' (i.e., never concrete), never actual objects.

The conception of being (or the world) as a transcendental, whether construed in the scholastic or in the Leibnizian sense, would indeed have the consequence that Gödel mentions, namely that no essence of a substance involves the world. But the reason why this is so would hardly be suggestive of the reflection principle. The argument from the transcendental nature of being would go through regardless of the exact properties of the universe (or of the realm of possible objects), for it depends only on an intrinsic characteristic of Aristotelian definitions. No aspect of inexhaustibility or inconceivability of the universe plays a role in it. It would seem, then, that the transcendental nature of being is compatible with both the failure and the correctness of the reflection principles for sets and for monads.

3.5 Concluding Remark

As we have seen, Leibniz' monadology is compatible with whatever the truths of pure mathematics may turn out to be. A positive consequence of this fact is that, should a purely conceptual or internal justification for the reflection principle be found[89] this will fit into the monadology immediately. But Gödel was also interested in yet another approach. The idea here is to deepen Leibniz' monadology by considering that concepts and possibilities, though not created by God, are constituted in his mind. To Hao Wang Gödel once complained that 'some of the concepts, such as that of possibility, are not clear in the work of Leibniz' (Wang

[87] Je tiens pour un axiome cette proposition identique qui n'est diversifiée que par l'accent, sçavoir que ce qui n'est pas veritablement *un* estre, n'est pas non plus veritablement un *estre*. On a tousjours crû que l'un et l'estre sont des choses reciproques. (Leibniz 1875–1890, 2:97)

[88] See also Kaehler (1979, 119n39).

[89] James van Aken (1986, 1001) observes that such an internal argument would be 'a coup'.

1996, 310), and he stressed that 'Leibniz had not worked out the theory' (Wang 1996, 87). As a means to develop Leibniz' philosophy, Gödel came to embrace and recommend Husserl's transcendental phenomenology from 1959 onward.[90] The suggestion, then, is that a phenomenological analysis of the types of acts and powers involved in the constitution of possibilities may lead to sufficient clarification of the notion of mathematical possibility to lead to a (direct) justification of the reflection principle.[91]

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[90] For an analysis of Gödel's turn to phenomenology, see van Atten and Kennedy (2003). He praised Dietrich Mahnke's *Neue Monadologie* (1917), a version of Leibniz' monadology written from a largely phenomenological point of view, as *vernünftig!* (van Atten and Kennedy 2003, 457).

[91] As Gödel (*1961/?, 383–385) suggests for mathematical axioms in general.

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Chapter 4
Gödel's Dialectica Interpretation and Leibniz

Mark van Atten

Abstract In an envelope of material relating to his work on the translation and revision of the Dialectica paper in 1968, Gödel kept a note that is in shorthand but in which one immediately notices the longhand name 'Leibniz'. When transcribed and put into context, the note allows one to show that Leibniz was a source of inspiration for Gödel's revision of the Dialectica Interpretation.

Keywords Analogy • Analysis • Definition • Dialectica Interpretation • Kurt Gödel • Gottfried Wilhelm Leibniz • Proof • Reductive proof • Subject-predicate • Truth

The writing is hard to read; by a coincidence, the parts that could be transcribed are the two parts whose importance Gödel indicated with vertical bars:

> Ph[ilosophy] between 1 and 4/68
>
> Leibniz says that [there is] an analogy between[?]
>
> | | | |
> |---|---|---|
> | fundamental concept | – | fundamental principle (should not be complicated, and[?] as many as possible) |
> | def[inition] | – | proof |
> | defined concept | – | theorem |

. . .

> ? Consider[?] this: The th[eorem] [is not just] composed[?] starting from uncertain[1] Ax[ioms], but contains[?] moreover parts that are not essential, which in the proof are obtained by 'weakening' and often are the reason for the ease with which the th[eorem] can be formulated.[2]

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[1]*Note MvA.* Gödel writes 'ungewissen'. If one understands 'axiom' in its foundational meaning of a statement of an immediate evidence (about the primitive terms), this qualification will be oxymoronic. But I take it that Gödel is here referring to its laxer use in ordinary mathematics (even when directed at truth).

[2]Gödel Papers 9b/148, 040495. Transcription Eva-Maria Engelen, Robin Rollinger, and MvA; translation MvA. The bars and underlinings are Gödel's. We have worked from the microfilm

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Note first the dates Gödel wrote above this note, January to April 1968. In his letter to Bernays of December 20, 1967, Gödel had announced he would have the translation of the 1958 version of Dialectica typed early January; but in his letter of May 16, 1968, Gödel explained why this did not happen:

> As I took up the manuscript in order to have it typed, I found in the philosophical introduction (i.e., in the first $3\frac{1}{2}$ pages in *Dialectica*) much even in the original text [that was] presented in such an unsatisfying and fragmentary way that I considered numerous supplementary remarks and changes as absolutely necessary. In the end, then, I completely rewrote that part and doubled its size. Then at the beginning of April I fell ill ... (Gödel 2003, 261)[4]

Hence the period in which the note on Leibniz was written coincides with the period during which Gödel tried to rewrite the philosophical introduction to the Dialectica paper intuitionistically, a revision that, after this letter to Bernays, was only to be abandoned again; Gödel then decided to add further, explicatory footnotes (letter to Bernays, December 17, 1968, Gödel 2003, 264); this led to a new version in 1970, which Bernays received from Dana Scott (Gödel 2003, 280/281). (Bernays would remark on a wrong preposition in note k, but in the remainder of their correspondence, he never commented on the notion of reductive proof itself. It is also clear that he did not actually get to see the later version of 1972 anymore, in which note k was developed further; see also Feferman's remarks in Gödel 2003, 73.) It seems, then, that the idea to justify the Dialectica Interpretation

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edition of the Gödel papers. Consultation of the original document might allow full transcription of the middle part.

Ph[ilosophie] zwischen I und IV 68

Leibniz sagt, dass eine Analogie [ist] zwischen[?]

Grundbegriff	–	Grundsatz (sollen unkomp[li]z[iert] sein und[?] möglichst viele)
Def[inition]	–	Bew[eis]
definier[ter] Begriff	–	Theorem

...

? Das betrachten[?][3]: Das Th[eorem] [ist nicht nur] zusammengesetzt[?] aus ungewissen Ax[iomen], sondern es enthält[?] außerdem unwesentliche Bestandteile, welche beim Beweis durch 'Abschwächung' erhalten werden und welche oft der Grund für 'einfache Formulierbarkeit' des Th[eorems] sind.

[3]*Note MvA*. Eva-Maria Engelen remarks that, although it is not excluded that instead of the shorthand for 'betrachten', Gödel wrote that for 'bestimmen', the former makes better sense.

[4] Als ich das Manuskript vornahm, um es abtippen zu lassen, fand ich in der philosophischen Einleitung (d.h. den ersten $3\frac{1}{2}$ Seiten in den Dialectica) vieles schon im Urtext so unbefriedigend u. lückenhaft dargestellt, dass ich zahlreiche Ergänzungen u. Änderungen für unbedingt nötig hielt. Schliesslich habe ich dann diesen Teil gänzlich umgearbeitet u. den Umfang auf mehr als das doppelte vermehrt. Anfangs April wurde ich dann krank ... (Gödel 2003, 260)

in specifically intuitionistic terms – documented in van Atten (2014) – and the idea to justify it in terms of reductive proof both occurred to Gödel in early 1968, and that he decided to work on the former first. When that attempt did not lead to a satisfactory result, he returned to the development of the notion of reductive proof.

Leibniz' (development of an) account of truth is rich and intricate, and there is a wealth of literature; for extensive (recent) discussion and further references, I refer to Ishiguro (1990), Grosholz and Yakira (1998), Rauzy (2001), and Lenzen (2004). The following is only meant to recapitulate the minimum needed to exhibit the connection between Leibniz' account of truth and Gödel's notion of reductive proof.

The two fundamental tenets of Leibniz' account can be stated as follows:

1. 'Praedicatum inest subjecto': Propositions of the form 'A is B', if true, are true on the ground that the notion (or concept) of the predicate B is in (*inest*) the notion of the subject A; the latter contains (*continet*) the former.[5]
2. To propositions that are not of that first form correspond propositions that are; in such a way that truth of the former is accounted for in terms of truth of the latter.

That the notion of A is contained in that of B is shown by unfolding their definitions, until a so-called identical proposition is reached, and this simultaneously demonstrates the corresponding proposition 'A is B'.[6] Hence the analogy that Gödel mentions between primitive concept and axiom, defined concept and theorem, and between definition and proof. As identical propositions Leibniz recognised all instances of 'A is A', 'A is not non-A', and the like; more generally, 'propositiones per se notas' (Leibniz 1903, *Generales Inquisitiones*, Sect. 61, 372).[7]

[5] This idea goes back to Aristotle, but in Leibniz' logic and metaphysics takes on a systematic and pivotal role. See Grosholz and Yakira (1998, Sect. II.2.1) for differences between Aristotle's and Leibniz' views on predication.

[6] A notorious case of such a demonstration by unfolding definitions is Leibniz' alleged proof that an infinite whole cannot exist because the whole is greater than the part. In van Atten (2011) I argue that not only is that argument incorrect, as Russell has shown, but Leibniz had all the means to see this. In his Russell paper of 1944, Gödel wrote:

> Nor is it self-contradictory that a proper part should be identical (not merely equal) to the whole, as is seen in the case of structures in the abstract sense. The structure of the series of integers, e.g., contains itself as a proper part. (Gödel 1944, 139)

Among other things, Gödel says here that it is consistent that an equality relation holds between a proper part and the whole. That entails a rejection of Leibniz's argument.

[7] Of course, Gödel was able to give precise reasons why primary truths need to be recognised whose truth is not a matter of their syntax.

For the present purpose, a particularly important passage can be found in Leibniz' letter to Conring of March 19, 1678:

> But we know that identical propositions are necessary propositions without any understanding or analysis of their terms, for I know that *A* is *A*, whatever may be understood by *A*. All propositions, however, whose truth must be shown by further analysing and understanding their terms are demonstrable by such analysis, that is, by definitions. So it is clear that demonstration is a chain of definitions [catenam definitionem]. For in the demonstration of any proposition, nothing is used but definitions, axioms (with which I here include postulates), theorems which have been demonstrated previously, and observations. Since the theorems again must themselves be demonstrated, and axioms, except for identities, can also all be demonstrated, it follows that all truths can be resolved into definitions, identical propositions, and observations – though purely intelligible truths do not need observations. After the analysis has been completed, it will become manifest that the chain of demonstration begins with identical propositions or observations and ends in a conclusion but that the beginning is connected with the conclusion through intervening definitions. In this sense I said that a demonstration is a chain of definitions. (Leibniz 1956, 187)[8]

Leibniz does not use the term 'analogy' here, but he does in related passages, e.g., Leibniz (1903, 377), and a conspicuous use of it (which Gödel had seen – Gödel Papers 10a/30, 050125 'gelesen') is in a footnote by Couturat to a passage at p. 352 of Leibniz (1903): 'Remarquer l'analogie établie ici entre les concepts et les propositions, ou entre les propositions et les inférences'.

According to Leibniz, in the case of truths of reason, these chains of definition are finite (Leibniz 1903, 1, 19, 408), while contingent truths only admit of infinite analysis.[9]

The notion of 'reductive proof' that Gödel introduced in the revisions of Dialectica is clearly modelled after Leibniz' notion of demonstration just explained:

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[8] Propositiones autem identicas necessarias esse constat, sine omni terminorum intellectu sive resolutione, nam scio *A* esse *A*, quicquid demum intelligatur per *A*. Omnes autem propositiones quarum veritatem ex terminorum demum resolutione et intellectu patere necesse est, demonstrabiles sunt per eorum resolutionem, id est per definitionem. Hinc patet, Demonstrationem esse catenam definitionum. Nam in demonstratione alicujus propositionis non adhibentur nisi definitiones, axiomata (ad quae hoc loco postulata reduco), theoremata jam demonstrata et experimenta. Cumque theoremata rursus demonstrata esse debeant, et axiomata omnia exceptis identicis demonstrari etiam possint, patet denique omnes veritates resolvi in definitiones, propositiones identicas et experimenta (quanquam veritates pure intelligibiles experimentis non indigeant) et perfecta resolutione facta apparere, quod catena demonstrandi ab identicis propositionibus vel experimentis incipiat, in conclusionem desinat, definitionum autem interventu principia conclusioni connectantur, atque hoc sensu dixeram Demonstrationem esse catenam definitionum. (Leibniz 1875–1890, 1:194)

[9] For a connection between Leibniz' notion of infinite analysis and Gödel's 1944 paper on Russell, see Charles Parsons' introduction to the latter (Gödel 1990, 115–116).

> A narrower concept of proof [than Heyting's], which may be called 'reductive proof' and which, roughly speaking, is defined by the fact that, up to certain trivial supplementations, the chain of definitions of the concepts occurring in the theorem together with certain axioms about the primitive terms forms by itself a proof, i.e., an unbroken chain of immediate evidences. (Gödel 1972, 275n(h1))

Gödel characterises proof in this sense as a chain of definitions, which is Leibniz' 'catenam definitionem' in the letter to Conring. Moreover, also Gödel's terminological choice to call these proofs 'reductive' seems to go back to Leibniz[10,11]:

> Analysis is of two kinds. The common type advances by leaps and is used in algebra. The other is special and far more elegant but less well known; I call it 'reductive' analysis. (Leibniz 1956, 233)[12]

Leibniz does not explain that term further there, but elsewhere he writes:

> Analysis is through a leap, when we begin to solve the problem itself, with no other assumptions. In the same way also synthesis is through a leap, when at the very beginning we pass from all necessary truth to our problem. But analysis is by degrees when we reduce [revocamus] the proposed problem to an easier one, and this to an easier still, etc., until we arrive at one which is within our power.' (Leibniz 1956, 234, supplemented)[13]

[10]To be distinguished from 'reductive' as in 'reductive proof theory', which studies particular relations between formal systems. Closely related, on the other hand, is the use of 'reduction' to indicate the reduction of terms, in a formal system, to normal form. In his introduction to the Dialectica paper in the *Collected Works*, Troelstra writes that 'In view of Gödel's choice of terminology ("reductive proof") in note n1 [h1], it is tempting to think that he had something like a term model, defined via reductions, in mind. But there is no conclusive evidence for this.' (Gödel 1990, 234) Indeed, for the reason explained in the main text I believe that the primary reason for Gödel's choice of that term is the use Leibniz made of it.

[11]In what is probably the first draft of note h of Gödel (1972) ('k' in Gödel's original marking), Gödel had written 'reductive or analytical provability', and then crossed out 'or analytical' (Gödel Papers 9b/142, 040452, 2). Similarly, there is a draft for that footnote in which it is said of 'reductively provable' that it is 'a concept which closely approaches Kant's meaning of "analytic" ' (Gödel Papers 9b/145, 040458, 2 for the reference to the insertion and item 040462, k(2) ∘ for its text.). Gödel also claimed this in conversation with Kreisel, as recalled in Kreisel's letter to Gödel of February 19, 1972 (Gödel Papers 2a/94, 011289), and in Kreisel (1987, 118) – in both cases followed by Kreisel's objection that in proofs of propositions $\forall x A(x)$ may occur functions of unbounded type that are not contained in the definition of A.

[12] Duplex est analysis, una communis per saltum qua utuntur in Algebra, altera peculiaris quam voco reductricem, quae longe elegantior est, sed parum cognita. (Leibniz 1875–1890, 7:297)

[13] idque in analysi per saltum, cum ipsa problema solvere ordimur nullis aliis praesuppositionis. Eodem modo et synthesis est per saltum cum a primis oriendo omnia necessaria percurrimus ad nostrum usque problema. Sed per gradum Analysis est, cum problema propositum revocamus ad facilius et hoc rursus ad facilius, et ita porro, donec veniamus ad id quod est in potestate. (Leibniz 1903, 351)

and

> Pure analysis, which contains no syntheses, is anagogical,[14] in which we always proceed backward through the unknown, reducing [reducendo] the problem that was proposed to another, simpler one, and that again to another.[15]

Naturally, both Leibniz and Gödel take care of the condition that, to be able to conceive of chains of definitions as proofs of theorems, one has to ensure that the objects defined exist.

On Leibniz' understanding of essential propositions, this means that it has to be shown that the object in question is possible (exists in some possible world).[16] Thus, in the *Nouveaux Essais* he writes:

> Two is one and one, Three is two and one, four is three and one, and so on. It is true that there is a statement hidden in there which I have already remarked on,[17] namely that these ideas are possible: and that is known here intuitively, so that one can say that intuitive knowledge is contained in definitions if their possibility appears first.[18] And in this way, all adequate definitions contain primitive truths of reason and hence intuitive knowledge. Finally, one can say in general that all primitive truths of reason are immediate through an immediation of ideas.[19]

Gödel puts it as follows:

> Note that in this context a definition is to be considered as a theorem stating the existence and unicity of an object satisfying certain conditions. (Gödel 1972, 275n(h1))

He had already remarked on this use of definition in 1943 or early 1944, in notebook *Max* X (12.III.1943–27.I.1944):

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[14]*Note MvA.* Indeed, 'reduco' is the standard Latin translation of Aristotle's 'ἀνάγω' (here in its meaning of to 'lead back', 'to refer back') when he speaks of transforming an imperfect syllogism into one in which all information needed to see its validity has been made explicit (*Prior Analytics* 29b1), or of the transformation of an argument into syllogistic form (ibid., 46b40).

[15]
> Analysis pura quae nihil syntheseos habet, est Anagogica, in qua semper procedimus per incognita retro, nempe reducendo problema propositum ad aliud facilius, et hoc iterum ad aliud. (Leibniz 1903, 558)

[16]See, besides the quotation following in the main text, *Generales Inquisitiones*, Sects. 144 and 146 (Leibniz 1903, 391, 392), and, for comments, Ishiguro (1990, 183–187).

[17]*Note MvA.* See *Nouveaux Essais*, book II, Chap. 32, Sect. 1 (Leibniz 1875–1890, 5:250).

[18]Note that Leibniz does not characterise the identical propositions purely formally. To a rare text in which Leibniz does say that the unprovability of an axiom is seen by the senses (Leibniz 1903, 186), Couturat adds the footnote 'Cet appel à l'évidence sensible n'est guère conforme au rationalisme leibnitien'.

[19]*Nouveaux Essais*, book IV, Chap. 2, Sect. 1: 'Deux est un et un, Trois est deux et un, Quatre est trois et un, et ainsi de suite. Il est vray qu'il y a là-dedans une enonciation cachée que j'ay déja remarquée, savoir que ces idées sont possibles : et cela se connoist icy intuitivement, de sorte qu'on peut dire, qu'une connoissance intuitive est comprise dans les definitions lorsque leur possibilité paroist d'abord. Et de cette maniere toutes les definitions adequates contiennent des verités primitives de raison et par consequent des connoissances intuitives. Enfin on peut dire en general que toutes les verités primitives de raison sont immediates d'une immediation d'idées' (Leibniz 1875–1890, 5:347).

Remark (Grammar): There are 3 ways to conceive of definitions:

1. As propositions of the form: a has the same sense and reference as b (where by using variables in a and b infinitely many cases can be subsumed), i.e., as merely typographical abbreviations.
2. As propositions of the form: a has the same reference as b and is a name of the object described by b. |
3. As propositions of the form $\phi(a)$, i.e., descriptions (but then existence and uniqueness have to be proved). In this sense the axioms of geometry could be definitions of the basic concepts.[20]

This was written only 2 years after the Yale lecture of 1941, and further on in the same notebook there are a number of remarks on Leibniz,[21] occasioned, as Gödel annotates, by his reading of Couturat's edition Leibniz (1903), and including glosses on Leibniz' notions of 'inesse' and of truth.[22] But the context is not that of the functional interpretation, and it seems Gödel did not connect proofs in that interpretation to ideas about definitions before the note of 1968 quoted at the beginning of this note.

Another point that both Leibniz and Gödel make is that, for the notion of reductive proof to be of epistemic use, we do not have to insist on actually producing reductive proofs, but may instead reason about them in ways that assure that such a proof can be produced. In Leibniz, this is stated as follows:

Truth in general I define as follows: A is true, if when putting a value for A, and treating whatever is contained in the value of A itself in turn like A, if this can be done, B and non-B, that is, contradiction, never occurs. From this follows that to be certain of a truth, either the resolution must be continued until first truths (or at least to those that have already been dealt with in such a process, or of which it has been established that they are true), or

[20]Gödel Papers, 6b/70, 030096, 25–26. Transcription Robin Rollinger, Eva-Maria Engelen, in collaboration with other members of Gabriella Crocco's group; based on earlier work by Cheryl Dawson.

Bem[erkung] (Gr[ammatik]): Es gibt 3 Arten Def[initionen] aufzufassen:

1. Als Aussagen der Form: a ist sinn- und bedeutungsgleich mit b (wobei durch Variablen in a und b unendlich viele Fälle zusammengefaßt werden können), d.h. als bloß typogr[aphische] Abkürzungen.
2. Als Aussagen der Form: a ist bedeutungsgleich mit b und ist ein Name des mit b Beschriebenen. |
3. Als Aussagen der Form $\phi(a)$, d.h. Beschreibungen (dann muß aber Existenz und Eindeutigkeit bewiesen werden). In diesem Sinn könnten die Ax[iome] der Geometrie Def[initionen] der Grundbegriffe sein.

[21]Gödel Papers, 6b/70, 030096 (*Max* X), 70–73 and 79–85.

[22](For the transcribers, see Footnote 20.) 71: '2. Every proposition expresses a containment, analytic ones the containment of the predicate in the subject, synthetic ones the containment of "Being" in the combination subject-predicate.' ('2. Jeder Satz drückt ein Enthaltensein aus, bei analytischen das Enthaltensein des Präd[ikats] im Subjekt, bei synth[etischen] das Enthaltensein des "Seins" in der Kombination Subj[ekt]-Präd[ikat]'); 73: 'Truth (according to Leibniz) = Relation between subject and predicate, more precisely an "inesse".' ('Wahrheit (nach Leibniz) = Verhältnis des Subj[ekts] und Präd[ikats], genauer ein "inesse".')

> it must be demonstrated from the progression itself of the resolution, that is from a certain general relation between the preceding resolutions and the next, that such a thing will never occur, in whatever way the resolution is continued.[23]

Gödel writes:

> Note that it is *not* claimed that the proofs of T' *are* reductive. This is true only in certain cases, in particular for the proofs of the axioms of T and of the individual cases of the rules of T ... What is claimed is only that no other concept of proof than that of reductive proof occurs in the propositions and proofs of T', except, of course, insofar as any theorem P in intuitionism means: A proof of P has been given. (Gödel 1972, 276n(h3))

An essential step in obtaining a reductive proof, if there is one, will be to apply the steps of the soundness proof of HA under the Dialectica Interpretation (sketched by Gödel on p. 280 and its note n) to the non-reductive proof at hand, thereby obtaining the witnessing term and reducing the use of logic to the minimum that is reductive.[24] In such a case, Gödel's soundness proof precisely plays the role of what Leibniz called a 'demonstration from the progression itself of the resolution' that this resolution would, if continued in full detail, terminate successfully.

In *The Philosophy of Leibniz*, Russell objected to what I above identified as the second tenet in Leibniz' account of truth; Russell argued that Leibniz' insistence on the subject-predicate form as the fundamental one makes it impossible to account for the truth of relational propositions, and indeed for mathematical truth, to the extent that numerical propositions involve a plurality of subjects (Russell 1900, 12ff.). Russell's objections are unfounded, as for example Ishiguro has shown (Ishiguro 1990, Chap. 12, in particular 102–103), and there is no indication that Gödel was concerned about them.[25]

However, Leibniz did recognise that an affirmative particular proposition (e.g., 'Someone is learned') cannot be correlated to a subject-predicate proposition without further ado:

> But in an affirmative particular proposition it is not necessary that the predicate is in the subject of itself and considered absolutely, or that the notion of the subject of itself contains the notion of the predicate, but it suffices that the predicate be contained in some species of the subject or that the notion of some example or species of the subject contains the notion of the predicate; although what sort of species that is, is not expressed.[26]

[23] *Generales Inquisitiones*, Sect. 56: 'Verum in genere sic definio, Verum est A, si pro A ponendo valorem, et quodlibet quod ingreditur valorem ipsius A rursis ita tractando ut A, si quidem id fieri potest, numquam occurat B et non-B seu contradictionem. Hinc sequitur ut certi simus veritatis vel continuandam esse resolutionem usque ad primo vera aut saltem jam tali processu tractata, aut quae constat esse vera, vel demonstrandum esse ex ipsa progressione resolutionis, seu ex relatione quadam generali inter resolutiones praecedentes et sequentem, nunquam tale quid occursurum, utcunque resolutio continuetur' (Leibniz 1903, 370–371).

[24] The soundness proof is given in full detail in Troelstra (1973, Sect. 3.5.4).

[25] Gödel's reading notes to Russell's book can be found in Gödel Papers, 10a/27. (NB The entry in the Finding aid for 10a/38 (Gödel 2003, 544) mistakenly states 'Bertrand Russell' where the Leibniz translator Charles William Russell is meant.)

[26] 'Sed in Propositione affirmativa particulari non est necesse ut praedicatum in subjecto per se et absolute spectato insit seu ut notio subjecti per se praedicati notionem contineat, sed sufficit

In the case of Heyting Arithmetic, Gödel's Dialectica Interpretation precisely supplies a witness for propositions of the form $\exists x P(x)$, predicate logic's rendition of the affirmative particular proposition with an indefinite term in Leibniz' term logic. In a note of as late as 1974, without mentioning Leibniz' name, Gödel lists as one of the advantages of his Dialectica Interpretation that 'the problem of being and having for existential propositions is solved'.[27] In light of the above, this should be read as a reference to Leibniz' theory of truth; taking 'being' and 'having' here to be references to the notions of *inesse* and *continere*.

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praedicatum in aliqua specie subjecti contineri seu notionem alicujus [exempli seu] speciei subjecti continere notionem praedicati; licet qualisnam ea species sit, non exprimatur.' (Leibniz 1903, 55)

[27] 'das Probl[em] von Sein und Haben für Ex[istenz]sätze wird gelöst.' For further details on that note, see van Atten (2014).

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Part II
Gödel and Husserl

Chapter 5
Phenomenology of Mathematics

Mark van Atten

to philosophise in regard to their mathematics (a hard task!)...

Kant, Critique of Pure Reason, B753

Abstract This is an introduction to the phenomenology of mathematics, written for phenomenologists.

Keywords L.E.J. Brouwer • Consciousness • Foundations • Kurt Gödel • Edmund Husserl • Idealism • Informal rigour • Phenomenology • Practice • Realism

5.1 Connecting Phenomenology and Mathematics

Three different ways of connecting Husserlian phenomenology and mathematics come to mind at once: (1) One can study the role of mathematics in the historical development of phenomenology; (2) One can study differences between phenomenology and mathematics as sciences, and (3) One can apply phenomenology to mathematics considered as a performance of consciousness.

5.1.1 Mathematics as Part of Husserl's Motivation to Develop Phenomenology

Husserl obtained his PhD in mathematics in 1883 (under supervision of Leo Königsberger), and became interested in the philosophy of mathematics. The

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more he thought problems in the philosophy of mathematics through, the more he realised that he would need to address more general problems of mind and language first. It is quite probable that certain mathematical developments and conceptions of mathematics made it easier (heuristically) for Husserl to arrive at some of the fundamental tenets of his phenomenology, such as the primacy of intuition, and, arguably, his method of eidetic variation. In the period up till the *Logical Investigations*, Husserl read widely in the literature on the foundations of mathematics of the time. In particular, he was well acquainted with the work and thought of Cantor, Frege, Hilbert, Kronecker, Schröder, and Weierstrass. In 1906, Husserl described the question that had been central to his philosophical development so far as follows:

> I was tormented by those incredibly strange realms: the world of the purely logical and the world of actual consciousness – or, as I would say now, that of the phenomenological and also the psychological. I had no idea to unite them; and yet they had to interrelate and form an intrinsic unity. (Husserl 1994a, 490–491)[1]

Gradually, Husserl came to loose contact with mathematics and the contemporary discussion of its foundations, although, especially in the early 1920s, he would hear of certain developments from students and friends such as Hermann Weyl and Oskar Becker. In 1927, he wrote on the eve of his visit to Amsterdam, where he would meet L.E.J. Brouwer, that he would certainly disappoint Brouwer as he no longer could speak about the foundations of mathematics (letter to H.J. Pos, December 26, 1927).[2] In spite of that confidence, Husserl shortly after published his *Formal and Transcendental Logic*; conceivably, an effort to catch up with recent developments in the foundations of mathematics might have resulted in a somewhat less programmatic work.

[1]⟦‘Und während ich mich mit den Entwürfen zur Logik des mathematischen Denkens und insbesondere des mathematischen Kalküls abmühte, peinigten mich die unbegreiflich fremden Welten: die Welt des rein logischen und die Welt des Aktbewußtseins, wie ich heute sagen würde, des Phänomenologischen und auch Psychologischen. Ich wußte sie nicht in eins zu setzen, und doch mußten sie zueinander Beziehung haben und eine innere Einheit bilden.’ (Husserl 1956b, 294)⟧

[2]⟦Husserl (1994b, 4:442): ‘Mr. Brouwer, whom I look forward to meeting, I will certainly disappoint. For at present the philosophical-mathematical is somewhat distant from my mind and I would not like to speak about that from behind the lectern. It would take me to much time to make myself familiar with it again – much though I have worked on it in the past.’ (‘Herrn Brouwer, den ich kennen zu lernen mich freue, werde ich freilich enttäuschen. Denn mir liegt z.Z[t]. das philosophisch-Mathematische etwas fern u. ich möchte darüber nicht vom Katheder sprechen, ich würde zu viel Zeit brauchen mich wieder hineinzufinden – so viel ich darüber früher gearbeitet habe.”)⟧

5.1.2 Mathematics and Phenomenology can be Described as Two (Different) Types of Science, with Correspondingly Different Types of Knowledge and of Reasoning

Husserl offers such descriptions in, for example, Sects. 71 and 75 of *Ideas I* (Kant also discusses the topic, e.g., on B740–766 of the *Critique of Pure Reason*). One function of such a description is to show that it is harmful to try to model philosophy too closely after mathematics (see also 'The pernicious influence of mathematics on philosophy', in Rota (1997).) But seeing the difference may also serve to open up a possibility:

> In terms used by Kant (A713) – philosophy analyses and mathematics builds up concepts – Gödel looked for a combination (where Kant saw only a distinction): for a given problem one may have the choice between a solution by means of philosophical analysis and easy mathematics and one by elaborate or otherwise subtle constructions. The simplest example is a solution by new axioms, discovered and justified by means of philosophical analysis. (Kreisel 1980, 150).

This brings us to a third way of connecting mathematics and philosophy.

5.1.3 Phenomenology of Mathematics

From a phenomenological perspective, mathematics is a performance of consciousness; we constitute our awareness of the objects and their relations. Thus, one can study types and degrees of evidence in mathematics, phenomena of presence and absence of mathematical objects, mathematical intuition, idealisation, the arithmetical versus the geometrical, the noetic-noematic correlation in mathematics (as Tragesser (1973, 293) put it, 'Something is recognizable as being a mathematical object if it can be recognized that it can be completely thought through mathematically'). The first contributions to the phenomenology of mathematics were made by Husserl himself (e.g., in his *Philosophy of Arithmetic*, Husserl 1970); but, as mentioned above, Husserl gradually lost contact with developments in (the foundations of) mathematics.

As the phenomenologically fundamental concepts of intentionality and constitution both have to do with the notion of object, it lies close to hand that phenomenological investigations of mathematics will also be (not exclusively, but primarily) concerned with mathematical objects. In this respect, phenomenology's concerns are very similar to the traditional ('foundationalist') concerns in the philosophy of mathematics – say, as practiced in the first decades of the twentieth century: what is the nature of mathematical objects, and how we can have knowledge of them? But after a few decades of focussing on these questions, more recent work in the philosophy of mathematics has seen a shift of emphasis to a number of other aspects of (succesfull) mathematical experience; for example, applicability (What explains what has been called the unreasonable success of pure mathematics in physics?), explanation (What distinguishes an explanatory proof from an uninformative one?), probabilistic reasoning (What role does or can it have in a deductive science?), visualisation (What is the role and epistemological status of diagrams and pictures?), beauty (What is its relation to mathematical truth and to insight?), and reasoning by analogy. (For a number of eloquent and insightful essays based on a phenomenological view of mathematics that pays attention to those aspects and is not overly concerned with the more traditional questions, see Rota (1997).)

According to a view congenial to me, shifts of emphasis (here, away from foundationalist concerns) are tantamount to shifting one's grounds. This will be readily (and probably enthusiastically) admitted by those thus shifting, but a further point to be made is that the different grounds are not always related symmetrically. Two ways to make this clear are the following. First, an unenlightening proof is still a proof; a theorem that is not beautiful is still a theorem; a mathematical theory without applications is still a mathematical theory. This means that for a complete account of mathematics, at some point one will have to address some of the more traditional questions. Second, as a matter of conceptual priority one needs to have a (at least tentative) idea of what mathematics consists in before one can begin to investigate the role of, say, beauty in mathematics. Otherwise, how would one confidently select the data for such an investigation? For example, someone who does not accept certain infinite sets that are talked about in Cantor's set theory will not include such sets (and their relations) among the data in an investigation of mathematical beauty (although for such a person they still may exemplify other types of beauty, in the way fictional objects can).

The rest of this chapter will be limited to (3), the phenomenology of mathematics, with an emphasis on its foundations.

5.2 Transcendental Phenomenology as a Foundation of Mathematics

The phenomenology of mathematics can be viewed from at least two perspectives, according as to whether one refers to phenomenology as an ongoing tradition or, more abstractly, as a particular approach to philosophical problems. Obviously, the two perspectives are largely overlapping; but distinguishing them points to the possibility of investigations that, although neither their author nor the text explicitly identifies with phenomenology and therefore cannot be said to be part of the historical movement, can profitably be read as exercises in phenomenology all the same. The second perspective also serves as a reminder that study of writings on mathematics in the phenomenological tradition may give rise to second thoughts about (aspects of) that tradition. We will be concerned with this second perspective.

Phenomenology gives an analysis of (certain aspects of) mathematical thought, but does not bring in (alleged) aspects of consciousness that are essentially different from those that mathematicians (with or without being aware) use already. A common objection to phenomenological analyses of mathematics, in particular with its elements of (categorial) intuition and eidetic variation, is that this simply does not seem an accurate description of one's experience when doing mathematics. A response one finds already in early Husserl is to invoke the difference between having an ability and being able to describe that ability (see for example p. 57 of his review of Ernst Schröder's *Vorlesungen über die Algebra der Logik* in Husserl (1994a)). One may be a good mathematician and at the same time have no or mistaken ideas about philosophical accounts of mathematics. In other words, what, philosophically speaking, we do when we do mathematics need not be immediately transparent to our awareness. By (systematised, developed) introspection, phenomenology analyses the way in which we form our conscious ideas on the basis of what is truly given to us. (Gödel once remarked, 'Both Husserl and Freud considered – in different ways – subconscious thinking' (Wang 1996, 167).) Thus, phenomenological analysis is not an alternative or competitor to mathematical analysis, but rather a deepening of it (at the level of the fundamental concepts).

How does transcendental phenomenology position itself relative to traditional philosophies of mathematics? One idea is that the latter, who are all in conflict with each other, may be (over)emphasising different aspects of mathematical thinking at the cost of others:

> In his publications Gödel used traditional terminology, for example, about conflicting views of 'realist' or 'idealist' philosophies. In conversation, at least with me, he was ready to treat them more like different branches of the subject, the former concentrating on the things considered, the latter on the processes of acquiring knowledge about these objects or processes . . . Naturally, for a given question, a 'conflict' remains: Which branch studies the aspects relevant to solving that question? (Kreisel 1980, 209)

The question posed by Kreisel here presupposes an answer to another question: what aspects of our experiences when we are doing mathematics are actually relevant to mathematics, be it from an 'idealist' or from a 'realist' perspective? This question can of course only be answered if one has a prior conception what mathematics is. Gödel indicated a characteristic feature of mathematical objects when he pointed out 'they can be known (in principle) without using the senses (that is, by reason alone) for this very reason, that they don't concern actualities about which the senses (the inner sense included) inform us, but possibilities and impossibilities' (Gödel 1995, 312n3). Phenomenologically, this characterisation can be rendered and amplified by saying that the objects of mathematics are purely categorial objects (pure forms or structures); these are studied not to the extent that they are (or have been) factually given but with respect to their possibility or impossibility. Paul Bernays has described mathematics as 'the science of possible idealized structures'

(Wang 1996, 336ff). To these possibilities and impossibilities of categorial objects correspond possibilities and impossibilities of categorial intuition:

> The ideal conditions of the possibility of categorial intuition as such are correlated to the conditions of possibility of the objects of categorial intuition and of categorial objects as such.[3]

By the fundamental principle of transcendental idealism, as formulated by Husserl in Sect. 142 of his *Ideas I* (Husserl 1976a), the existence of an object (of any kind) is equivalent to the existence of a possible consciousness to which this object is given originally and adequately; so as a special case, a mathematical object can be said to exist if it can be given adequately and originally with purely categorial evidence to a possible mind. Because of this straightforward relation between existence in mathematics and the conditions of possibility of categorial intuition, mathematics is particularly close, and therefore particularly interesting, to phenomenology; and there is a sense in which mathematics is part of phenomenology. A pertinent question, also for transcendental idealism in general, is what exactly the notion of possibility amounts to here.

Note that Husserl's transcendental idealism combines (selected aspects of) traditional 'realist' and 'idealist' philosophies, and thereby allows one, as Gödel did, to think of them as branches of the same subject. (In fact, Gödel told Kreisel that he had formed this view while reading Husserl. Two places in Husserl that Gödel may have been referring to are Sects. 18–23 of *Ideas I* (Husserl 1976a), and Sect. 16 of the *Britannica* article (Husserl 1962). See van Atten and Kennedy 2003, 464–465 for further discussion.[4])

The question what aspects of our experiences when doing mathematics are actually relevant to mathematics as such can now be answered by saying that the relevant aspects are exactly those that, ultimately, pertain to the constitution of purely categorial objects. (For an elaboration of this view on mathematics, see Tragesser 1973; Rosado Haddock 1987; van Atten 2002). From the point of view of transcendental phenomenology, all reality, including its abstract aspects or components, is constituted reality; the position of transcendental phenomenology relative to traditional or non-phenomenological foundations of mathematics then is one of a standard to which the latter may be tested. In particular to the extent that the latter may be in conflict with each other, phenomenology will view each of them as an exaggeration of one or more particular aspects of mathematical experience

[3] Die idealen Bedingungen der Möglichkeit kategorialer Anschauung überhaupt sind korrelativ die Bedingungen der Möglichkeit der Gegenstände kategorialer Anschauung und der Möglichkeit von kategorialen Gegenständen schlechthin. (Husserl 1984b, 718–719)

[4] ⟦See Chap. 6, Sect. 6.6.1 in this volume.⟧

at the cost of others. Any genuine insight on the nature of mathematics obtained on the basis of one of those non-phenomenological philosophies of mathematics should, precisely to the extent that it us genuine, admit of a phenomenological reconstruction or, perhaps more aptly, rediscovery. Conversely, one may try to be efficient and look at questions in the foundations of mathematics directly from a phenomenological perspective (see Gödel 1995, 374–387; Rota 1973). A particularly attractive topic is what has been called, by Palle Yourgrau (2005, 182), 'the Gödel Program'. While Hilbert focused on formal aspects of mathematics, and Brouwer on intuitive aspects, Gödel focused on the interplay between these two aspects: his 'Program' (or what in any case can, with hindsight, be presented as such) consisted in the investigation of the limits of formal methods in capturing intuitive concepts (besides logic and mathematics, also in relativity theory).

To hold that a phenomenological investigation cannot contribute to the foundations of mathematics for the reason that formal rigour is the only type of rigour possible is question-begging (Gödel 1995, 383; Rota 1997, Chap. 7). When formalising a certain subject and its intuitive notions, surely some other kind of rigour must have been at work in devising the axioms and in verifying that they hold for the intended meaning. Kreisel has called this other kind 'informal rigour', and has emphasised this is often sufficient to settle mathematical questions (Kreisel 1967b). It is not to be thought of as a new kind of rigour, or as a new method coming in from outside mathematics (competing with mathematician's methods); it rather is simply the kind of rigour that mathematicians have always been applying when not working formally (and unlike rigour, formal systems are a late arrival in the history of mathematics). This includes the correction of earlier mistakes by the application of more informal rigour (see also Sect. 24 of Husserl's *Ideas I* Husserl 1976a). Of course, both kinds of rigour may be combined by first analysing a mathematical problem informally but rigorously, and then formalising the solution; and formal problems may often be due to insufficient prior informal analysis. Husserl's *Wesensschau* is an instance of informal rigour; and *Wesensschau* as such is not a radically new technique but rather a common and age-old method made fully conscious and then developed further. Husserl has described the great care that has to be taken in performing eidetic variations (e.g., in Sects. 86–93 of *Experience and Judgement*, Husserl 1973e); similarly, Gödel has observed that intuition requires more, not less, caution and experience than (formal) proofs (Wang 1996, 301). On the other hand, the rewards are proportionally higher. None of this is to belittle or deny the virtues and uses of formalisation, which is a practial necessity; but it is to set straight the relation between the formal and the intuitive. In formalisation, generally a particular kind of rigour is gained, but (part of the) evidence is lost. In the specific case of arithmetic, this can even be demonstrated by formal means, by Gödel's Incompleteness Theorem (see example 5 below). That theorem also shows that occasionally a formal result suffices to refute a philosophical position, in this case Hilbert's formalism in its original sense. Another example is the completeness of the formal theory of elementary geometry (Tarski), which proves that Kant was wrong when he said (*Critique of Pure Reason*, B743–745) that in doing geometrical proofs one will always need geometrical intuitions. Such intuitions will of course have been used in setting up the axioms of the appropriate formal system, but Tarski showed that no more of it is needed to arrive at the propositions that follow from these axioms.

There are three different ways to see the relation between philosophy and the (actual practice of) mathematics: philosophy describes, but has no right to impose limits or other changes on mathematical practice; philosophy tests, according to some philosophical standard, what is acceptable in mathematical practice; philosophy has the right to impose, according to some philosophical standard, limits but also extensions of mathematical practice. These three conceptions may be called non-revisionism, weak revisionism, and strong revisionism, respectively. Husserl repeatedly has claimed that (a) mathematics without a philosophical foundation is not a science but a mere technique; (b) philosophical considerations may lead to the rejection of parts of mathematical practice; but (c) they cannot lead to mathematical innovations. So he certainly saw a phenomenological foundation of mathematics as a form of weak revisionism. In *Ideas III* (Husserl 1971), e.g., Sect. 14, Husserl introduced the idea that transcendental phenomenology provides the universal ontology (see also versions 3 and 4 of the *Encyclopædia Britannica* article[5]), that is, the idea that transcendental phenomenology is the science of all possible being. As such, it plays a founding role in any eidetic science (see also Rota 1973). Moreover, it can be argued that from the point of view of transcendental phenomenology Husserl's claim (3) is not correct. Clearly, Husserl had no aim of revising classical logic or mathematics; it can however be argued that this has more to do with Husserl's own background and psychology than with what his philosophical principles imply, and that these principles warrant a strong revisionism (van Atten 2002). If that is correct, this means that mathematics is exceptional in science in that it is possible to argue from (structures of) consciousness to objects in mathematics (phenomenology would have this in common with the traditional philosophy of mathematics of intuitionism). Such arguments from philosophy to mathematics are likely to find their motivation, however, in a certain mathematical problem. Whether Husserl's claim (c) is correct or not, in either case transcendental phenomenology treats the mathematical tradition, including traditional foundations of mathematics, with a certain reserve. It asserts a certain autonomy.

Directly opposed to such a (weak or strong) revisionism is the currently popular form of non-revisionism known as naturalism, according to which no point of view external to the practice of mathematics, in particular no philosophical one, can demand changes in mathematical practice. A naturalist may well acknowledge that it is possible to exaggerate the agreement between practicing mathematicians; for example, within the one field of contemporary set theory there is much disagreement over the acceptability of various principles and of certain (very large) objects, and the small but significant minority of constructive mathematicians persists in rejecting various principles from classical mathematics. But a naturalist might go

[5]⟦Version 4 is included integrally in Husserl (1962, 277–301); for the relevant part of version 3, see Husserl 1962, Beilage XXX, 519–526.⟧

on to say that the disagreement is considered not to be amenable, or not in need of, philosophical adjudication; rather, mathematics with its methodologies that have been developed through the ages could take care of itself. A problem with that line is that those disagreements are motivated (and kept alive) by philosophical considerations of the mathematicians themselves.

Note that Husserl once radically changed his ontological account of mathematics for a reason that had nothing to do with the content of mathematics: when instead of saying that mathematical objects are outside of time (*unzeitlich*; Husserl 1984a, 129) he came to hold that they are in all time (*allzeitlich*; Husserl 1985b, Sect. 64), this was not at all motivated by anything in mathematics specifically, but Husserl's new insight on that constitution of objects of any kind presupposes the constitution of time. This also serves to illustrate a point that is not always appreciated by mathematicians: mathematical objects may have properties that, though not of mathematical interest, are of philosophical (in particular, metaphysical) interest.

The two (overlapping) perspectives mentioned above – phenomenology as an ongoing tradition and as an abstract system of thought – suggest various (overlapping) tasks. One task is to arrive at a historically responsible and coherent reading of texts on mathematics in the phenomenological tradition; another, to take as point of reference the debate in the foundations of mathematics (as it raged in the first decades of the twentieth century – note that the reasons why the debate came to an end are mostly of a sociological and psychological nature, rather than philosophical: from the latter point of view, many of the issues are still open) and try to determine Husserl's position (or what would or should have been his position) with respect to the schools participating in that debate (in the widest sense): logicism, Platonism, formalism, intuitionism. For a phenomenological reflection on problems and issues in the various traditional schools, see Tieszen (1995). At present, different positions in the philosophy of mathematics have gained clarification from phenomenological investigations; but it is likely that instead of eventually coming to favour one of them over the others, further phenomenological investigations will clarify and corroborate the picture sketched above of one subject with different branches, where the branches differ according to aspects taken into account and according to degrees of evidence and idealisation accepted. Below, attention will be drawn to examples of cases where explicit considerations on the notion(s) of evidence in mathematics have led to concrete mathematical developments. In other words, some cases are touched on in which what was (either explicitly or in effect) adopting a phenomenological perspective led to a deeper understanding of a mathematical issue.

5.3 Examples

Six examples of (explicitly or implicitly) phenomenological thought in contemporary foundations of mathematics are given. They are mostly concerned with (varieties of) constructive mathematics; this is not a necessity but is easily explained by the fact that classical mathematics involves idealisations that go much farther than those in constructive mathematics (see, in this context, Husserl's discussion of the Principle of the Excluded Middle in his *Formal and Transcendental Logic*[6]); to that extent, these idealisations are harder to evaluate, and less work has been done on specifically classical topics. That is not to say, of course, that such work cannot be done, and such work would constitute considerable progress in the phenomenology of mathematics.

5.3.1 Intuitionistic Logic

Brouwer's 'intuitionism' views mathematics as an activity of making constructions in the mind, the fundamental construction material being the intuition of the passage of time. Intuitionistic logic is just the logic of those constructions. A proposition p is true if we have (or have a method to obtain) a mental construction that is correctly described by p. On this view, that the proposition p='2+2=4' is true is accounted for as follows: Construct the number 2 (by abstracting all the content from the experience of the passage of time from one 'now' to another, leaving the pure form); construct it again; put the two results together. Now construct the number 4; comparison with the result of 2+2 shows they are the same, and therefore p is true. In the case of the proposition q='2+2=5', a similar procedure would have led us to see that 4 and 5 cannot be made to coincide; hence q is false, or, put differently, $\neg p$ is true. The proposition $p \vee q$ is true, because I have a construction that makes one of the two disjuncts true. Similarly, the proposition $p \wedge q$ is false.

One's actual, essentially languageless activity of making constructions can be described, and logic is taken to be a description of observed regularities in these descriptions. Logic, according to Brouwer, is thus first of all an empirical science. However, it is possible to establish essential laws on the basis of eidetic insight into the nature of constructions. In this way (but without explicitly invoking phenomenology), Brouwer's (former) student Heyting formalised intuitionistic logic (see Heyting 1930a,b,c); after that, he gave an explicit interpretation, and time he did invoke phenomenology (Heyting 1931). He knew the work of Husserl's student Oskar Becker and corresponded with him (van Atten 2005); he adopted Becker's idea that intuitionistic logic can be thought of as a logic of intentions:

> We here distinguish between propositions and assertions. An assertion is the affirmation of a proposition. A mathematical proposition expresses a certain expectation. For example, the proposition, 'Euler's constant C is rational', expresses the expectation that we could find two integers a and b such that $C = a/b$. Perhaps the word 'intention', coined by the phenomenologists, expresses even better what is meant here. . . . The affirmation of a proposition means the fulfillment of an intention. (Benacerraf and Putnam 1983, 58–59)[7]

[6] ⟦Husserl 1974, Sects. 77–80.⟧

[7] ⟦'Ich unterscheide zwischen Aussagen und Sätzen: ein Satz ist die Behauptung einer Aussage. Eine mathematische Aussage drückt eine bestimmte Erwartung aus; z.B. bedeutet due Aussage "Die Eulersche Konstante C ist rational" die Erwartung, man könne zwei ganze Zahlen a und b

Thus, the intention expressed by a proposition is fulfilled exactly if we know a mathematical construction that shows that things are the way the proposition at which the intention is directed says they are. From this general principle of interpretation, the following explanations of the meaning of the logical constants are derived (for brevity, only the clauses for propositional logic are given):

conjunction	(the intention expressed by) $p \wedge q$ is fulfilled exactly when p is fulfilled and q is fulfilled.
disjunction	$p \vee q$ is fulfilled exactly when at least one of p, q is fulfilled.
implication	$p \rightarrow q$ is fulfilled exactly when the subject has a construction that transforms any construction (proof) of p into one of q.
negation	$\neg p$ is fulfilled exactly when the subject has a construction that transforms any proof of p into a proof of a contradiction.

Intuitionistic logic is different from classical logic. Classically, $p \vee \neg p$ is always true (the principle of the excluded middle, abbreviated PEM); intuitionistically, the intention $p \vee \neg p$ is fulfilled exactly when at least one of p, $\neg p$ is fulfilled. But while it is true that not both can be fulfilled, it is not the case that for every p, we have, at any given moment, either a construction fulfilling p or a construction fulfilling $\neg p$. For example, p may be a still open problem. Therefore, intuitionistically the PEM is not valid. (Which is not to say that it is false; it just is not always true.)

Because to have a construction that fulfills an intention directed at a proposition is to have a proof of that proposition, Heyting's interpretation can also be stated in terms of having proofs; and in that more widespread form, it has become known as 'the proof interpretation'.

5.3.2 Choice Sequences

It has been known since Aristotle that treating a line (a continuum) as a set of discrete points cannot do justice to its continuous nature, as on that conception the points are isolated from one another. Brouwer showed how the continuum can be dealt with in a more satisfactory way if one introduces so-called choice sequences into mathematics (Troelstra 1985; van Atten et al. 2002). A choice sequence is a potentially infinite sequence of (say) numbers, chosen at will, one after the other, by the individual mathematician. At any particular moment, only finitely many choices will have been made, and therefore a choice sequence is always becoming and never finished. Without going into the technical details of how choice sequences are used to arrive at an alternative theory of the continuum, it can be indicated how Brouwer's work in two ways depends on (what is in effect) phenomenological

finden, derart, daß $C = a/b$. Vielleicht noch besser als das Wort "Erwartung" drückt das von den Phänomenologen geprägte Wort "Intention aus, was hier gemeint wird. . . . Die Behauptung einer Aussage bedeutet die Erfüllung der Intention.' (Heyting 1931, 113).]]

analysis. First, the recognition that a continuum is a whole in a different sense than a set is. A line is continuous through and through: each of its parts exhibits this property just as the whole does. (In Sect. 19 of his third *Logical Investigation*, Husserl calls wholes of this type ‘extensive’.) Second, an analysis of the notion of mathematical construction is needed to show that choice sequences are a genuine type of object (that can be constituted with evidence, and have identity conditions) and moreover are genuinely mathematical objects, in spite of the fact that classical mathematicians do not accept them as such.

Another aspect of choice sequences that is of phenomenological interest is that they force intuitionistic logic for theories about them. For example, if one begins a choice sequence by choosing 0 three times in a row, then there is (assuming one has not imposed certain conditions or restrictions on one’s own choices) no fact of the matter whether all numbers in the sequence will be 0 or not. So, by intuitionistic logic (see above), we have no sufficient evidence to assert that either all numbers are 0 or they are not. Thus, the principle of the excluded middle does not generally hold for choice sequences. This illustrates a theme from Husserl’s genetic analysis of judgment (in *Formal and Transcendental Logic*, and in *Experience and Judgment*) that has been developed further in Chap. 4 of Tragesser (1977): with different domains of objects, different logics are associated. Minimal experience with choice sequences shows that, whatever a universally valid logic may look like, it will not be classical logic.

5.3.3 The Bar Theorem

A full explanation of Brouwer’s ‘Bar Theorem’ from 1927 cannot be given here, but for the purpose of getting an idea of its philosophical interest it suffices to know that it is an implication that roughly states something of the form: if in a tree structure particular kind of subset of nodes (a ‘bar’) exists, then this subset contains a small tree that can be constructed in an ordered way. (Brouwer applied this theorem to trees in which the branches are choice sequences.)

In Brouwer’s intuitionism, as described above, to say that something exists is to say that we have a construction method for it. Correspondingly, to say that a proposition is true is to say that we know a proof of it. The truth of a proposition is evidenced by, and only by, a proof. Brouwer’s argument for the Bar Theorem starts by assuming that the antecedent is true. He then proceeds by asking what a proof of the antecedent of the theorem can possibly be like. He then argues that any such proof can be analysed into a particular canonical form; finally, he shows how that canonical form of the proof can be transformed into a construction of the small tree, which proves the consequent.

The phenomenologically interesting step here is Brouwer’s analysis what proofs of the antecedent can be like. For here he reasons that mathematical proofs are (in general) infinite, mental objects of a certain structure. (At the time, Brouwer saw this as his main argument against Hilbert's formalism.[8]) It is the structure of proofs considered as mental objects that will be the basis for the construction of the tree mentioned in the consequent. Brouwer's analysis of the nature of mental proofs can be accounted for in terms of introspection, the notions of horizon and intentional implication, and the temporal structure of (mathematical) acts; such an account is given, together with a full explanation of the theorem, in Chap. 4 of van Atten (2004a).

5.3.4 Hilbert's Program

How much evidence is needed to be convinced of (not the truth, but) the consistency of classical analysis? David Hilbert hoped to show that the answer was: as much evidence as is needed to be convinced of the truth of finitary mathematics. This was one of the most important parts of what is known as 'Hilbert's Program',[9] one of the efforts to provide a foundation for classical mathematics. Generally, Hilbert's Program consisted in the effort to show that all of classical mathematics could be captured in one formal system, and that that system could be shown to be consistent by finitary means.

The 'finitary mathematics' that Hilbert wanted to use deals with only finitely many, concrete objects, objects that one can survey. One never makes actual use of an infinity of objects, or of objects that are infinitely complex, or of objects that are abstract and cannot be made concrete or visualised. Finitary mathematics is therefore constructive. (A good example of finitary mathematics is the arithmetic one learns in primary school.)

Because it stays away from the infinite and from abstract objects, finitary mathematics was (and in some corners still is) taken to be especially clear and secure. The idea, then, was to show by finitary means that formal systems for classical analysis are consistent, that is, that in these systems it is not possible to arrive both p and $\neg p$. The rationale was that these formal systems can themselves be reasoned about finitarily.

There are a number of aspects to Hilbert's Program that are of phenomenological interest: there is the question of the exact nature and range of Hilbert's notions of intuition and evidence, which go back to Kant; the question which of the known principles of mathematics can be (re)interpreted so as to be acceptable in finitary mathematics; the question as to the relation between language and objects in mathematics. From a historical point of view, there is a close relation between Hilbert's method of ideal elements and Husserl's 'Double Lecture' in Göttingen in 1901 (Schuhmann and Schuhmann 2001).

The final two examples are both, in different ways, reactions to Hilbert's Program.

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[8]⟦Brouwer 1927B, 64n8.⟧

[9]⟦A recent study is Sieg 2013.⟧

5.3.5 Incompleteness and Intuition

One formulation of the results Gödel published in 1931 (Gödel 1986, 144–195) is:

First Incompleteness Theorem: Any formal system (satisfying some mild conditions) that contains arithmetic is either incomplete (i.e., leaves certain of its propositions undecided) or inconsistent. Moreover, for any consistent formal system of the appropriate type, undecidable propositions (are not merely known to exist but) can actually be specified.

Second Incompleteness Theorem: Among the undecidable propositions is one that (formally) expresses the consistency of the system.

But these propositions that are undecidable in the system are decidable by evident (informal) inferences which, firstly, cannot (on pain of inconsistency) be reflected formally in the original system and, secondly, are exactly as evident as the inferences possible in the original system. The second Incompleteness Theorem shows one example is a proposition is that (formally) expresses the consistency of the original system: any ground for doubting that proposition is a ground for doubting the original system, and to the extent that one believes that the original system is sound, one believes that it is consistent.

Both Incompleteness Theorems serve to refute Hilbert's Program in its original sense. The first, because it shows that it is not possible to capture all of mathematics in one system, as soon as it contains arithmetic, there are propositions it cannot decide; the second, because it shows that a formal demonstration of the consistency of a given system needs means at least as strong as those supplied by the system itself.

Another philosophical doctrine refuted by Gödel, again using his Incompleteness Theorems, is Carnap's position that mathematics is merely the (conventional) syntax of a certain language and has no content of its own (Gödel 1995, 334–362). On that conception, only empirical propositions have content. (This of course would be contested also by Husserl, on phenomenological grounds.) One of Gödel's arguments against this position is that, even though rules of syntax may be chosen arbitrarily, they stand in need of justification to the extent that one has to ensure that they are consistent. For if they are not, then they will allow one to deduce, from any given (empirical) proposition accepted as true, every other possible proposition (the mathematical as well as the non-mathematical ones), including the incorrect ones. But the second Incompleteness Theorem shows that a proof of the consistency of the syntactical rules will require a mathematical intuition of the same power as those rules; therefore, non-empirical content cannot be eliminated.

5.3.6 The Dialectica Interpretation

How much evidence is needed to be convinced of (not the truth, but) the consistency of classical arithmetic? By Gödel's Incompleteness Theorems, more evidence would be needed than can be provided by the finitary mathematics from Hilbert's Program (see above). In particular, this means that abstract evidence is needed. But how much? The Dialectica Interpretation, which was published in 1958 (in the journal Dialectica, hence the name) under the significant title 'On a hitherto unutilized extension of the finitary standpoint', is an attempt to answer that question (Gödel 1990, 240–251).

The first step is to invoke Gödel's translation from 1933 (Gödel 1986, 286–295) of classical arithmetic into intuitionistic arithmetic, also known as Heyting Arithmetic: it consists of Peano's axioms for arithmetic joined not to classical logic but to Heyting's system of intuitionistic logic. In the translation the interpretation of the statements is changed, and from an intuitionistic point of view the translations obtained are weaker than the original statements (classically they are equivalent). However, the translation of a statement like $1 = 0$ is $1 = 0$. Hence, if a contradiction can be derived in classical arithmetic, then a contradiction can be derived in intuitionistic arithmetic (the converse is immediate). In other words, classical arithmetic is consistent if and only if intuitionistic arithmetic is consistent.

In the second step, the Dialectica Interpretation proper provides an interpretation of Heyting Arithmetic. As we saw above, intuitionists interpret intuitionistic logic in terms of proofs; what Gödel wanted to show was that, in the context of arithmetic, it can also be interpreted in terms objects that are, like proofs, abstract, but (in a sense that can be made more precise), less so. The objects that Gödel suggests are computable functionals of finite type; roughly, functionals of type 0 are the natural numbers, and functionals of higher type are defined by constructive operations that assign to finite tuples of functionals of already defined types another functional of already defined type.

The Dialectica Interpretation may by itself constitute a significant advantage from an epistemological point of view, but what Gödel had in mind is the following application. The interpretation shows that intuitionistic arithmetic indeed is consistent. Combined with the translation, one obtains a consistency proof for classical arithmetic. So the answer the Dialectica paper provides to the question how much evidence is needed to establish the consistency of classical arithmetic is: more than for Hilbert's finitary reasoning, but less than for the intuitionistic notion of proof.

However, for technical reasons, a necessary condition for this epistemological advantage is that the notion of a computable functional of finite type is taken as immediately intelligible, or primitive. A comparative phenomenological investigation of these functionals and the intuitionistic notion of proof may be required to determine to what extent the advantage is real. An interesting remark by Gödel in this respect is that 'One may doubt whether we have a sufficiently clear idea of the content of this notion [of computable functional of finite type], but not that the axioms [given in this paper] hold for it. The same apparently paradoxical situation also obtains for the notion, basic to intuitionistic logic, of a proof that is informally understood to be correct' (Gödel 1990, 245n5).

It may be considered odd that a proof of the consistency of arithmetic, that is, of a theory of the natural numbers, should be attempted in terms of a notion that is more complicated, that of computable functional of finite type, which even includes the natural numbers as type 0. But both classical and intuitionistic arithmetic appeal to non-constructive or abstract notions via their logic, which is after all also part of these systems. This appeal has so to speak to be made up for, and this is exactly what the functionals are used for: to reduce logical complexity.

A revised version from 1972 (published posthumously in Gödel 1990, 271–280), adds, among other things, more details on the notion(s) of evidence in play; it is not impossible that Gödel had further developed his sensitivity to such aspects during his wide reading in phenomenology that he begun shortly after completing the original version from 1958.[10]

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[10]⟦Indeed, it has now been documented that, for the revision of the Dialectica paper, Gödel had a phenomenological reading in mind. See Chap. 11 of this volume.⟧

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Chapter 6
On the Philosophical Development of Kurt Gödel

Mark van Atten and Juliette Kennedy

Dedicated to the memory of Karl Schuhmann (1941–2003)

Abstract Gödel first advocated the philosophy of Leibniz and then, since 1959, that of Husserl. Based on research in Gödel's archive, from which a number of unpublished items are presented, we argue that (1) Gödel turned to Husserl in search of a means to make Leibniz' monadology scientific and systematic, and (2) This explains Gödel's specific turn to Husserl's transcendental idealism as opposed to the realism of the earlier Logical Investigations. We then give three examples of concrete influence from Husserl on Gödel's writings.

Keywords Consciousness • Foundations • Kurt Gödel • Edmund Husserl • Idealism • Gottfried Wilhelm Leibniz • Metaphysics • Monadology • Phenomenology • Primitive terms • Rationalism • Realism • Set theory

6.1 Introduction

It is by now well known that Gödel first advocated the philosophy of Leibniz and then, since 1959, that of Husserl.[1] This raises three questions:

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[1]As introductions to phenomenology, we recommend Husserl's *The Idea of Phenomenology* (Husserl 1999) – as we will see, a 'momentous lecture' according to Gödel – and Kockelmans' annotated edition of Husserl's article for the *Encyclopædia Britannica* (Kockelmans 1994), an article that Gödel also studied. The first is work from before, the second from after the appearance

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1. How is this turn to Husserl to be interpreted? Is it a dismissal of the Leibnizian philosophy, or a different way to achieve similar goals?
2. Why did Gödel turn specifically to the later Husserl's transcendental idealism?
3. Is there any detectable influence from Husserl on Gödel's writings?

Regarding the first question, Wang (1987, 165) reports that Gödel '[saw] in Husserl's work a method of refining and consolidating Leibniz' monadology'. But what does this mean? In what for Gödel relevant sense is Husserl's work a refinement and consolidation of Leibniz' monadology?

The second question is particularly pressing, given that Gödel was, by his own admission, a realist in mathematics since 1925.[2] Wouldn't the uncompromising realism of the early Husserl's *Logical Investigations* have been a more obvious choice for a Platonist like Gödel?

The third question can only be approached when an answer to the second has been given, and we want to suggest that the answer to the first question follows from the answer to the second. We begin, therefore, with a closer look at the actual turn towards phenomenology.

Some 30 years before his serious study of Husserl began, Gödel was well aware of the existence of phenomenology. Apart from its likely appearance in the philosophy courses that Gödel took,[3] it reached him from various directions.

First, Gödel will have heard about phenomenology at the meetings of the Vienna Circle. Members and other attendants of the Circle's meetings critically discussed Husserl's epistemology, in particular his conception of a priori knowledge. Several of them had been, or were, in closer contact with phenomenology. Carnap studied with Husserl in 1924–1925 (Schuhmann 1977, 281), Husserl's work had been discussed and criticised by Schlick in his *Allgemeine Erkenntnislehre* (*General Theory of Knowledge*) of 1918[4]; and Felix Kaufmann was at the same time a practicing phenomenologist who moreover was personally close to Husserl.[5]

of the less transparent *Ideas I* (Husserl 1976a). For an accessible historical treatment, see Spiegelberg (1983), a work of which Gödel owned and read the earlier, second edition from 1965.

[2]The Grandjean questionnaire, printed in Wang (1987, 18, 20).

[3]'Die Systeme der großen Denker', taught by Moritz Schlick, WS 24/25; 'Übersicht über die Hauptprobleme der Philosophie I' ('Overview of the main problems in philosophy I'), by Heinrich Gomperz, WS 24/25, and its sequel in SS 25, 'Übersicht über die Hauptprobleme der Philosophie II' ('Overview of the main problems in philosophy II'); 'Philosophische Übungen' ('Philosophical exercises'), by Rudolf Carnap, SS 28; no data on SS 29 and WS 29/30 available. Source: Universitätsarchiv Wien, as reprinted in Schimanovich-Galidescu (2002, 145–146). These are the 'inskribierte Vorlesungen'; he may have attended courses for which he did not register, such as Schlick's 'Logik und Erkenntnistheorie' ('Logic and Theory of knowledge'), WS 25 (Schimanovich-Galidescu 2002, 135).

[4]Husserl criticised Schlick's discussion in the introduction to the second edition of the *Logical Investigations* (Husserl 1984b, 535–536); Schlick replied in the second edition of the *Erkenntnislehre* of 1925 (Schlick 1925, 127–128n3).

[5]On Kaufmann, see Stadler (1997). His correspondence with Husserl is in Husserl (1994, 4).

Second, within the specific context of mathematics, Gödel attended Heyting's talk at the Königsberg conference in September 1930. Brouwer's foremost student presented the intuitionistic point of view there and explained the concepts of proposition and proof in terms of Husserl's concepts of intention and fulfilment. Gödel will have read the relevant remarks again in the published version of Heyting's talk (Heyting 1931), of which he published a review (Gödel 1932f). In between, he encountered phenomenology when reviewing a paper by Husserl's student Oskar Becker of 1930, 'Zur Logik der Modalitäten' ('On the logic of modalities') (Becker 1930); Gödel's review in Gödel (1931e).

There is a possibility that Gödel heard Husserl lecture in those years, when on May 7, 1935, Husserl gave a lecture in Vienna which, because of the large audience it drew, he was asked to repeat on May 10.[6] Wang (1987, 97–98) mentions that Gödel was in Vienna then, but goes on to say 'Presumably G did not attend these lectures, since his interest in certain aspects of Husserl's work came much later'. However, in the Gödel archive there is a library slip dated July 27, 1935 (Gödel Papers, 10c/62),[7] requesting Husserl's *Vorlesungen zur Ph.[*änomenologie*] des Zeitbewusstseins* (*Lectures on the Phenomenology of Inner Time Consciousness*) (Husserl 1928). This slip suggests a transitory interest in phenomenology on Gödel's part. Perhaps Husserl's presence in Vienna had made Gödel curious, whether or not he heard the lectures.

Despite these various contacts in the 1930s, Gödel did not come to feel close to phenomenology. In 1975, responding to a question in a letter from Barry Smith (Gödel Papers, 3a/167, 012358), Gödel wrote that

> I can say that my conceptual realism, which I am holding since about 1925, was in no way brought about by phenomenology. I have a high regard for Husserl, but I did not get acquainted with his writings before many years after I emigrated to the U.S. (Gödel Papers, 3a/167, 012359)

Indeed, around 1959, Gödel's distance from phenomenology made place for a strong interest in the work of Husserl. It is not so clear, however, what influence on his published writings this interest had. Wang, for example, writes

> G's own interest in Husserl's work probably derives from his belief that Husserl's 'methods' will play an important role toward realising his goal. However, I have not been able to detect Husserl's influences in G's available philosophical work. (Wang 1987, 221)

A few years later, however, Wang wrote that 'there are traces of Husserl's influence in some of Gödel's very limited number of available writings after 1959' (Wang 1996, 61); but Wang does not give any examples. We believe that such influences as Wang meant indeed can be shown in Gödel's published work. In order to adduce these, we first have to come to a concrete understanding of the goal Gödel hoped

[6] Edmund and Malvine Husserl to Husserl's former student Roman Ingarden, July 10, 1935 (Husserl 1994, 3:302).

[7] References of the form 'Gödel Papers x/y, z' refer to item z in folder y in box x of the Gödel Papers in the Firestone Library, Princeton.

to reach using Husserl's methods. With this understanding in place, we will then present three examples of Husserl's influence that can be detected in Gödel's published work.

With our discussion, we hope to complement work by Wang and by Parsons (1995), to build further on Tieszen (1998), and to cast some doubt on interpretations of Gödel that downplay or dismiss the direct influence of Husserl.[8] A different case is that of Maddy (1990). She lets herself be inspired by Gödel's philosophical remarks, but is explicit that her naturalistic project is not Gödel's (Maddy 1990, 75–80, 78); it therefore falls outside the scope of our attempt at reconstructing Gödel's thinking behind his words.

6.2 Gödel's Position in the 1950s: A Stalemate

6.2.1 Inconclusive Arguments

We now note some of the broader themes appearing in Gödel's general position in the 1950s, just before his turn to Husserl in 1959. In 1951, Gödel delivered the Gibbs lecture, titled 'Some basic theorems of the foundations of mathematics and their implications' (Gödel *1951). Charles Parsons has remarked of it that it 'seems to complete for Gödel the process of avowing his Platonistic position' (Parsons 1995, 55). It is true that all the basic elements of Gödel's Platonism are in place, yet at the end of that lecture Gödel concedes that the arguments he gives in favour of that position are not conclusive:

> Of course I do not claim that the foregoing considerations amount to a real proof of this view [i.e., Platonism] about the nature of mathematics. The most I could assert would be to have disproved the nominalistic view, which considers mathematics to consist solely in syntactical conventions and their consequences. Moreover, I have adduced strong arguments against the more general view that mathematics is our own creation. There are however, other alternatives to Platonism … In order to establish Platonic realism, these theories would have to be disproved one after the other, and then it would have to be shown that they exhaust all possibilities. I am not in a position to do this now. (Gödel *1951, 35)

By end of that decade, nothing had changed; Gödel is still 'not in a position to establish Platonic realism', in spite of his years-long efforts, as he explains in a letter to Schilpp of February 3, 1959. He there comments on his inability to finish his paper 'Is mathematics syntax of language?' that he had begun 6 years before:

> The fact is that I have completed several different versions, but none of them satisfies me. It is easy to allege very weighty and striking arguments in favor of my views, but a complete elucidation of the situation turned out to be more difficult than I had anticipated, doubtless in consequence of the fact that the subject matter is closely related to, and in part identical with, one of the basic problems of philosophy, namely the question of the objective reality of concepts and their relations. On the other hand, because of widely held prejudices, it may do more harm than good to publish *half done* work.' (Gödel 2003a, 224, emphasis ours)

[8]For example, those of Köhler (2002b, 341–386) and Hintikka (1998). For specific criticism of the latter interpretation, see van Atten (2001).

Lurking behind Gödel's dissatisfaction with the paper was the fact that a standard of philosophical argumentation to which he adhered, was not met in the Syntax paper: this standard of 'rigour' in philosophy we will discuss below (Sect. 6.2.2). For now, we note that Gödel knew that, in this case, he did not have such a rigorous argument and rather had to settle for partial justifications: negative ones in the form of arguments against alternative views, and positive ones in the form of plausibility arguments.

In particular, as Warren Goldfarb has noted in his introduction to the Syntax paper, there is no positive epistemological account:

> On the question of how we gain knowledge of the mathematical realm, Gödel has little to say except that we do it by our faculty of mathematical intuition, which he also calls 'mathematical reason' and sometimes simply 'reason' ... He gives no further details about its structure; nor does he consider whether aspects of it are involved in other regions of our cognition, as would certainly be suggested by calling it 'reason'. Here perhaps it is perhaps most glaring that Gödel failed to arrive at the sort of 'complete elaboration of the situation' that he sought. (Gödel 1995, 333–334)

Equally lacking is a positive ontological account of the objects. This is related to the epistemological shortcoming, as the ontological status of the objects and the nature of mathematical intuition mutually constrain each other in so far as the objects are to be accessible to intuition. The less we can say about the one, the less we can say about the other. An example of a certain ontological wavering can be seen when, having given one of his arguments that 'the objects and theorems of mathematics are as objective and independent of our free choice and our creative acts as is the physical world', he qualifies the argument by adding that

> It determines, however, in no way what these objective entities are – in particular whether they are located in nature or in the human mind or in neither of the two. These three views about the nature of mathematics correspond exactly to the three views about the nature of concepts, which traditionally go by the names of psychologism, Aristotelian conceptualism and Platonism. (Gödel *1951, 312n17)

This underdeterminacy is typical of the arguments in the 1950s.[9]

6.2.2 *Realism and Rationalism*

Even discounting the self-described lack of rigour (Gödel *1951, 311) in Gödel's arguments in the Gibbs lecture, there is a more serious conflict in Gödel's position in the 1950s, arising from his being committed to two theses, seemingly opposed: (Platonic) realism, or Platonism, and rationalism. There is much to say about what these terms precisely meant to Gödel, as well about what the nature and strength of his commitments to these were. We will not pursue many of these larger issues here

[9] For further discussion of problems with Gödel's arguments, see Köhler (2002b, 341–386). We entirely disagree, however, with the positive interpretation developed there, in Sect. 4.3, of Gödel as a conventionalist of a particular kind: we claim this is neither historically nor systematically correct.

(though we will discuss the nature of Gödel's rationalism below), but simply take note of Gödel's position at this point, which was, roughly, as follows: while he did not want to abandon realism, it did not appear possible to give a proof of the validity of realism, on any rational understanding of the term 'proof'. In the letter to Schilpp of 1959, the project announced in the Gibbs lecture of 1951 had come to a halt.

An early symptom of this perceived impossibility may have been Gödel's enigmatic remark in a lecture in 1933 (enigmatic because of all of Gödel's remarks on Platonism, this is the single one which is negative):

> The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind. (Gödel *1933o, 50)

By the early 1950s, Gödel evidently thought the situation had changed somewhat, as he then, in the letter to Schilpp, says that he does have arguments in favour of realism that are 'weighty' and 'striking'. So they presumably would go a certain way to satisfying a critical mind. Still the situation had not essentially changed – his arguments evidently do not meet his own standard of rationalism, the kind of standard indicated by his remark, at the end of the Russell paper, that Leibniz' project of the characteristica universalis was not utopian. That this standard has not been lowered in the meantime is clear from Gödel's list of 14 items that comprise his 'philosophical viewpoint', which was probably drawn up around 1960. One there finds beliefs such as '3. There are systematic methods for the solution of all problems (also art, etc.).' and '13. There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness' (Wang 1996, 315–316).

Before turning to the nature of Gödel's rationalism, we note that it is clear that at every stage of his thought, rationalism was a core belief of Gödel's – it emerges in almost all of his philosophical conversations, and extensively in his writings, both published and unpublished. Even so, he never argued for it, nor did he go any distance toward offering a critique of the attitude as a whole. If only from his reading of Kant, Gödel will have been aware of the questions concerning the general validity (the transcendental possibility) of science. But pre-phenomenologically, these do not seem to be questions with which Gödel was explicitly preoccupied. In contrast to Platonic realism, which was a conviction (doxa) of Gödel's and therefore in need of proof if it is to count as established (episteme), for Gödel, rationalism as such seems to have required no proof of its own validity.

So what did rationalism amount to for Gödel in the 1950s? In the Gibbs lecture of 1951, when turning to the philosophical implications of his Incompleteness Theorems, he gives as the reason why his audience 'must not expect these inferences to be drawn with mathematical rigour' that this is 'a consequence of the undeveloped state of philosophy in our days' (Gödel *1951, 311).[10] Also, Gödel explains at the end of the lecture, 'after sufficient clarification of the concepts in question it will be possible to conduct these discussions with mathematical rigour' (Gödel *1951, 322).

[10]This remark echoes the final paragraph of the Russell paper, e.g., 'How can one expect to solve mathematical problems systematically by mere analysis of the concepts occurring if our analysis so far does not even suffice to set up the axioms?' (Gödel 1994, 152).

So Gödel's model for philosophical argumentation was that of mathematics, as it had been for Leibniz. Philosophy should be a demonstrative or deductive science. But then what could a 'philosophical theorem' be like? If the mathematical standards are paradigm, as the language Gödel uses here suggests, then two prominent features of mathematical propositions should be cited. The first is the feature that all the terms occurring in mathematical propositions either have a fixed definition within a (more or less) fixed language, or are primitive terms like 'set'. The other important property mathematical propositions enjoy is that they are attached to transparent verification procedures. Simply put, they are provable in a more or less precise sense of the term 'proof'. Thus, by analogy, philosophical propositions will involve primitive terms, to be arrived at, undoubtably, by a kind of conceptual analysis; moreover these propositions are 'provable' or verifiable via some inferential means. Rationalism demands that such a notion of philosophical proof be developed; philosophical propositions can then be treated exactly, and conclusions become 'results'.

Much later, Gödel gave the basic picture to Wang in *A Logical Journey*, naming the important desiderata that the primitive terms should be 'simple', and few in number [11]:

> The beginning of physics was Newton's work of 1687, which needs only very simple primitives: force, mass, law. I look for a similar theory for philosophy or metaphysics. Metaphysicians believe it possible to find out what the objective reality is; there are only a few primitive entities causing the existence of other entities. (Wang 1996, 167, 5.3.11)

And:

> Philosophy aims at a theory ... In a theory concepts and axioms must be combined, and the concepts must be precise ones. – Genetics is a theory. Freud only gives a sketch of a developing theory; it could be presented better. Marx gives less of a theory. (Wang 1996, 306, 9.3.10)[12]

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[11] Gödel's emphasis on simplicity has to do with a particular moral and aesthetic view of the world which finds its inspiration in Leibniz, an important issue which we have not pursued in this paper.

[12] The phrase we have omitted from the quotation, 'Phenomenology does not give a theory', we take to mean that phenomenology is primarily a method (to isolate what is intuitively given), and not itself a theory. As Husserl said,

> If philosophy has any stock whatever of 'essentially necessary' fundamentals in the genuine sense which, according to their essence, can therefore be grounded only by an immediately presentive intuition, then the controversy concerning them is decided not only independently of any philosophical science, but of the idea of such a science and of the latter's allegedly legitimated theoretical content. (Husserl 1983, 34) ('Hat überhaupt Philosophie einen Bestand an "prinzipiellen" Grundlagen in dem echten Sinne, die also ihrem Wesen nach nur durch unmittelbar gebende Anschauung begründet werden können, so ist ein Streit, der diese betrifft, in seiner Entscheidung unabhängig von aller philosophischen Wissenschaft, von dem Besitz ihrer Idee und ihres angeblich begründeten Lehrgehaltes', Husserl 1950c, 41/Husserl 1976a, 40. Gödel owned Husserl 1950c, but the preferred edition [now] is its revision Husserl 1976a. For convenience, we will always mention the page numbers in both editions.)

Of course, when applying this method one obtains propositions, and it is in this sense that Husserl presents, in Sect. 60 of the *Cartesian Meditations*, 'metaphysical results' (*'metaphysische Ergebnisse'*). See also Brainard's discussion of this point (Brainard 2002, Sect. 2.1).

To Gödel, then, the (main) difficulty in carrying out the project of philosophy as an exact theory may have been specifying the primitive terms. To Sue Toledo he said in 1972:

> There is a certain moment in the life of any real philosopher when he for the first time grasps directly the system of primitive terms and their relationships. This is what had happened to Husserl ... The analytic philosophers try to make concepts clear by defining them in terms of primitive terms. But they don't attempt to make the primitive terms clear. Moreover they take the wrong primitive terms, such as 'red' etc., while the correct primitive terms would be 'object', 'relation', 'will', 'good', etc. (Toledo n.d., 3/24/72:1–2)[13]

These remarks to Wang and Toledo were made in the 1970s. It is to be noted that at the earlier stage which is our present concern, the pre-phenomenological stage of the 1950s, Gödel had no concrete method for arriving at what the primitive terms should be; nor is there evident an explicit notion of what a conceptual treatment of them would be like. This had to await 'sufficient clarification'.

Gödel was taken to task over this conception of philosophy by Boolos in his introduction to this lecture:

> The suggestion with which he closes the lecture may seem utterly strange ... Gödel's idea that we shall one day achieve sufficient clarity about the concepts involved in *philosophical* discussion of mathematics to be able to prove, mathematically, the truth of some proposition in the philosophy of mathematics, however, appears significantly less credible at present than his Platonism. (Gödel 1995, 303–304)

An assessment of Gödel's suggestion (particularly in relation to Leibniz' universal characteristic) would have to consider what exactly he meant by 'mathematical rigour'. This will certainly not have been the type of rigour exhibited by formal systems. At the very least such systems will be subject to Gödel's own Incompleteness Theorems (as Gödel of course knew). Indeed Wang reported Gödel as saying:

> The universal characteristic claimed by Leibniz (1677) does not exist. Any systematic procedure for solving problems of all kinds would have to be nonmechanical. (Gödel Papers, 3c/209, 013184, 1)

Gödel, in pencil, amended the first sentence to read

> The universal characteristic claimed by Leibniz (1677) if interpreted as a formal system does not exist.[14]

One might rather think that 'informal rigour', as Kreisel called it, was what Gödel had in mind. And this does not mean the impossibility of Leibniz' characteristic; it just means that, as Gödel said to Carnap in 1948, while the system cannot be completely specific, it may still give sufficient indications as to what is to be done (Wang 1987, 174). This suggests that the system is 'mathematically rigorous' to the extent that its indications are 'sufficient'. Boolos seems to suggest that the characteristica should 'mathematically prove', but that would be asking for too much.

[13]⟦Addition MvA: Now Toledo (2011, 200).⟧

[14]Note that this alteration is not present in the version Wang gives in Wang (1996, 202). However, Wang mentions the idea in Wang (1987, 174).

Boolos may have been premature in dismissing Gödel's rationalism, but we suggest that in the years following the Gibbs lecture, Gödel himself became aware that he needed a different, deeper notion of rationality. Although Gödel anticipated Husserl in a strong sense in arriving at a version of rationality on his own, still Husserl's notion of 'philosophy as a strict science' (Husserl (1911) 1981) went further than the notion Gödel had in the 1950s. Nevertheless it is clearly the 'mature' version of Gödel's view.[15]

6.2.3 *Epistemological Parity*

We now note another theme underlying Gödel's thinking overall, before as well as after the turn to phenomenology, which functions as a regulative principle and, in particular, qualifies and complicates any 'naïve' realist beliefs he may have had. We will call it here 'epistemological parity': the idea that, regarding physical objects on the one hand and abstract or mathematical objects on the other, from the point of view of what we know about them, there is no reason to be more (or less) committed to the existence of one than of the other. After his turn to Husserl, Gödel will have found Husserl's version of epistemological parity in *Ideas I*:

> No conceivable theory can make us err with respect to the principle of all principles: that every originary presentive intuition is a legitimizing source of cognition, that everything originarily (so to speak, in its 'personal' actuality) offered to us in 'intuition' is to be accepted simply as what is presented as being, but also only within the limits in which it is presented there. We see indeed that each ‹Theory› can only again draw its truth itself from originary data. Every statement which does no more than confer expression on such data by simple explication and by means of significations precisely conforming to them is . . . actually an absolute beginning called upon to serve as a foundation, a principium in the genuine sense of the word. (Husserl 1983, 44)[16]

Gödel's most striking formulation of the view appears in a footnote to the Syntax paper, in the context of arguing against empiricism: 'It seems arbitrary to me to

[15] A concise encapsulation of how Gödel's concept of rationalism had evolved under Husserl's influence (from 'deciding' philosophical propositions to 'clarifying' them) can be seen from the following note to himself (Gödel Papers, 12/43, 060571), likely from after 1961: 'Perhaps phenomenology combined with foundational research will someday ~~decide~~ clarify those questions in an absolutely convincing manner.'

[16] Am Prinzip aller Prinzipien: daß jede originär gebende Anschauung eine Rechtsquelle der Erkenntnis sei, daß alles, was sich uns in der 'Intuition' originär (sozusagen in seiner leibhaften Wirklichkeit) darbietet, einfach hinzunehmen sei, als was es sich gibt, aber auch nur in den Schranken, in denen es sich da gibt, kann uns keine erdenkliche Theorie irre machen. Sehen wir doch ein, daß eine jede ihre Wahrheit selbst wieder nur aus den originären Gegebenheiten schöpfen könnte. Jede Aussage, die nichts weiter tut, als solchen Gegebenheiten durch bloße Explikation und genau sich anmessende Bedeutungen Ausdruck zu verleihen, ist also wirklich . . . ein absoluter Anfang, im echten Sinne zur Grundlegung berufen, principium. (Husserl 1950c, 52/Husserl 1976a, 51)

consider the proposition "This is red" an immediate datum, but not so to consider the proposition stating modus ponens' (Gödel *1951, 347n34).

But the principle, in another form, is already in the Russell paper from 1944: 'It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence.' (Gödel 1990, 137). It also continued to preoccupy Gödel through the 1960s, e.g., the idea in some form occurs in a note to himself in the folder titled 'Phil[osophische] Varia' (mostly after 1961):

> It should also be noted that even a statement like 'this is red' *if there is to be a valid motive for making it* presupposes that something besides the independent sense experience is given. (Gödel Papers, 12/43, 060572; emphasis ours)

As Gödel explains in the footnote from the Syntax paper that we just quoted, there are varieties of evidence – in the two cases, different types of relations between different types of relata are perceived – but each should be taken as or for what it is. According to Gödel, notwithstanding all the differences, e.g., of meaning, between the two propositions, au fond, they both express perceptions. There is no good reason, then, to accept one as given and not the other. That is to say, the epistemological problems generated by the assumption of the existence of physical objects are neither less nor more problematic than those generated by the assumption of the existence of abstract or mathematical objects. If one adheres to the principle of epistemological parity, then the question how the varieties of evidence and, correspondingly, the varieties of objects they are evidence for, are connected, becomes secondary (a theme echoed much later in Wang's 'substantial factualism', when he tries to make a case for the 'overwhelming importance of existing knowledge for philosophy' by pointing out, among other things, that 'We know more about what we know than how we know what we know' Wang 1974, 1).

Epistemological parity pulls in the direction of realism as well as of rationalism: acknowledgement of abstract evidence pushes one a certain extent in the direction of realism about abstract objects, and it would be irrational to neglect evidence.

Epistemological parity is related to skepticism. If 'skepticism wishes to avoid dogmatism, the hasty conviction that goes beyond what can legitimately be asserted' (Sokolowski 1996, 46), then there are two ways of going beyond what can be legitimately asserted: by asserting too much or by being too restrictive (which is just making too many negative assertions). In Gödel's thinking, both corresponding skeptical forces are at work: against, e.g., empiricism for asserting too little; against naïve Platonism, for asserting more than it can hope to justify rationally. A principle of epistemological parity such as Gödel holds serves as an antidote to both. But precisely to the extent that the principle of epistemological parity is an antidote to various excesses, it is intrinsically a limitative principle: while making the question of a philosophical account of what actually is given more urgent, it does not suggest how to answer it. For that, Gödel had to look elsewhere.[17]

[17] As Kant wrote in the *Critique of Pure Reason* (A761/B789), 'Scepticism is thus a resting-place for human reason, where it can reflect upon its dogmatic wanderings and make survey of

As an aside, we note that epistemological parity must have placed Gödel at odds with the pragmatic and naturalistic turns of philosophy that were made around him. In particular, the principle goes against causal accounts of knowledge as well as of reference, accounts which came to exercise much influence in the philosophy of mathematics (the locus classicus is Benacerraf's 'Mathematical truth', Benacerraf 1973). This influence put realism (in mathematics) under attack. For one may correctly observe that we do not interact causally with mathematical objects; but then, on such accounts, if realism were correct, we could have no knowledge of such objects – which amounts to having no mathematical knowledge. But that would contradict our intuition that we clearly do have such knowledge. Thus realism is untenable. Moreover, Benacerraf asks, if we take causal accounts to be correct for empirical knowledge, how could an account of mathematical knowledge be continuous with our account of empirical knowledge?

It is not the purpose of the present paper to judge causal accounts on their merits; instead, we note that the environment that gave rise to such programs was an uncongenial one for Gödel, and moreover, one which sometimes led to serious misunderstanding of Gödel's outlook. Not only did Gödel not participate in the late twentieth century project to carry out the reduction of mathematical terms to the allegedly epistemologically prior domain of empirical terms, because of epistemological parity it could never have been a legitimate project for him in the first place. Also, if it is 'arbitrary' – if there is no good reason, that is – to consider empirical propositions as being more secure than principles like modus ponens, on what basis should we insist that the accounts for the latter kind of knowledge be uniform with accounts of the former?[18]

In any case, that is Gödel's contemporaries notwithstanding, the letter to Schilpp from 1959 marks a kind of stalemate in Gödel's philosophical evolution: he wishes to retain as much as he can of his earlier realist views, however this became increasingly difficult in view of his other, core, commitments to rationalism and to epistemological parity.

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the region in which it finds itself, so that for the future it may be able to choose its path with more certainty. But it is no dwelling-place for permanent settlement' (Kant (1781–1787a) 1965a); 'So ist der Skeptizismus ein Ruheplatz für die menschliche Vernunft, da sie sich über ihre dogmatische Wanderung besinnen und den Entwurf von der Gegend machen kann, wo sie sich befindet, um ihren Weg fernerhin mit mehrerer Sicherheit wählen zu können, aber nicht ein Wohnplatz zum beständigen Aufenthalte' (Kant (1781–1787b) 1996).

[18]Tait (1986, Sect. 7) raises the same objection. Maddy's set theoretic realism, on the other hand, is one attempt to bring mathematical objects 'into the world we know' (Maddy 1990, 48) by reconstructing the perception of mathematical objects along causal lines. Thus 'set(s) participate in the generation of ... perceptual beliefs in the same way that my hand participates in the generation of my belief that there is a hand before me when I look at it in a good light' (Maddy 1990, 58). Whether or not Maddy is proposing the reduction of mathematical terms to physical ones in that work – and we think it can be argued that she is – we note that her proposal falls into the category of moves to which, we are suggesting, Gödel would have been opposed. We hasten to add that Maddy readily acknowledges this point of disagreement between herself and Gödel, and she is aware that her naturalist interpretation of Gödel's remarks would not have been his own, as we have noted above.

6.2.4 A Way Out?

Gödel sensed fairly early that there may be a way out of the conflict we have attributed to him. In a letter dated June 30, 1954, to the German immigrant philosopher in the United States, Gotthard Günther, Gödel explained

> The reflection on the subject treated in idealistic philosophy (that is, your second topic of thought), the distinction of levels of reflection, etc., seem to me very interesting and important. I even consider it entirely possible that this is 'the' way to the correct metaphysics. However, I cannot go along with the denial of the objective meaning of thought that is connected with it, ⟦although⟧ it is really entirely independent of it. I do not believe that any Kantian or positivistic argument, or the antinomies of set theory, or quantum mechanics has proved that the concept of objective being (no matter whether for things or abstract entities) is senseless or contradictory.[19] When I say that one can (or should) develop a theory of classes as objectively existing entities, I do indeed mean by that existence in the sense of ontological metaphysics, by which, however, I do not want to say that abstract entities are present in nature. They seem rather to form a second plane of reality, which confronts us just as objectively and independently of our thinking as nature. (Gödel 2003, 503, 505)[20]

We will come back to this letter more than once. Here, we wish to draw the following conclusion from it: Gödel obviously believes that Platonism should be a consequence of 'the correct metaphysics', and he in addition believes that some form of idealistic philosophy will lead to the correct metaphysics. We reconstruct his reasoning behind this latter suggestion as follows. In so far as a Platonic realm of objects is thought to be problematic, this will not least of all be because of the concomitant epistemological problem how our subjective thought grasps these objective realities. The question of the exact sense of the notion of a Platonic realm is therefore just as much a question of the exact sense of the notion of subjectivity. From this perspective, it is not surprising that Gödel turned to idealism.

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[19]*Note Gödel.* Of course I don't wish by that to claim that naïve thought already grasps objective being correctly on all points, as ontological metaphysics often seems to suppose.

[20] Die in der idealist. Phil. behandelte Reflexion auf das Subjekt (d.h. Ihr II Thema d. Denkens), die Unterscheidung von Reflexionsstufen etc. scheint mir sehr interessant u. wichtig. Ich halte es sogar für durchaus möglich, daß dies 'der' Weg zur richtigen Metaphysik ist. Die damit verbundene (in Wahrheit aber davon ganz unabhängige) Ablehnung der objektiven Bedeutung des Denkens kann ich aber nicht mitmachen. Ich glaube nicht, daß irgend ein Kantsches oder positivistisches Argument oder die Antinomien d. Mengenl., oder die Quantenmechanik bewiesen hat, daß der Begriff des objektiven Seins (gleichgültig ob für Dinge oder abstrakte Wesenheiten) sinnlos oder widerspruchsvoll ist. [footnote: Damit will ich natürlich nicht behaupten, daß schon das naive Denken das objektive Sein in allen Punkten richtig erfaßt, wie die ontol. Metaphysik vielfach anzunehmen scheint.] Wenn ich sage, daß man eine Theorie der Klassen als objektiv existierender Gegenstände entwickeln kann (oder soll), so meine ich damit durchaus Existenz im Sinne der ontol. Metaphysik, womit ich aber nicht sagen will, daß die abstrakten Wesenheiten in der Natur vorhanden sind. Sie scheinen vielmehr eine zweite Ebene der Realität zu bilden, die uns aber ebenso objektiv. u. von unserem Denken unabhängig gegenübersteht wie die Natur. (Gödel 2003, 502, 504)

Yet in the Gibbs lecture and the Syntax papers, he did not make any effort to argue from idealistic premises. Indeed, at one point in the Gibbs lecture, he says that

> This whole consideration incidentally shows that the philosophical implications of the mathematical facts explained do not lie entirely on the side of rationalistic or idealistic philosophy, but that in one respect they favor the empiricist viewpoint. (Gödel *1951, 313)

But it should be noted that this remark is put in a different perspective in the footnote added to it:

> To be more precise, it suggests that the situation in mathematics is not so very different from that in the natural sciences. As to whether, in the last analysis, apriorism or empiricism is correct is a different question. (Gödel *1951, 313n20)

Unlike the main text, this footnote does not suggest that the philosophical implications support empiricism rather than rationalism or idealism. Be that as it may, the letter to Günther shows that even in 1954 Gödel had yet to find a version of idealism that he found congenial. Pending that, he had to settle for (or continue to employ) a notion of rationality that is not idealistically informed. In this light, Gödel's failure during the 1950s to produce a rational account of Platonism might be taken to indicate the failure of his notion of rationality at the time to be sensitive to the role of subjectivity. What Gödel needed, then, to bridge the gap between his realist convictions and the rational arguments he was able to find to support them, was an account of subjectivity that integrates rationality and Platonism. We will suggest below that this was the issue that provoked his 'conversion' to phenomenology.

6.3 Gödel's Turn to Husserl's Transcendental Idealism

We want to substantiate the claim that what attracted Gödel so much in Husserl's philosophy is the doctrine of transcendental idealism, which Husserl developed after his *Logical Investigations*.

6.3.1 Varieties of Idealism

As a first characterisation,[21] idealism holds that there exists a non-physical realm of 'the ideal' on which everything that does not belong to it in some sense depends.[22] Idealism, thus understood, is not incompatible with realism and in fact requires

[21] In this section we are much indebted to Beiser (2002), Wartenberg (1992), and the various entries on idealism in Ritter (1971).

[22] In the literature, there are two terminological traditions. One uses 'ideal' to qualify the objects in the independent realm; the other uses it to qualify the objects that are dependent on that realm. We will adhere to the first usage. Compare, for example, Beiser (2002, 6) and Wartenberg (1992, 104).

it with respect to the realm of the ideal (for example, Plato's idealism involves a realism about Ideas). The notion of the ideal can be construed in a number of ways. Usually, it is related to the mind and its contents; but there are exceptions to this, such as linguistic idealism. Likewise, there are various possibilities for the dependence relation: it may be ontological, epistemological, or conceptual. Thus one obtains a number of varieties of idealism. Our first characterisation should be taken as a formal one and is not meant to suggest that these various idealisms are all developments of one basic truth. In order to get a clear view of Husserl's particular version, and on Gödel's approach to it as compared to other idealisms he knew, we provide the following simple taxonomy of idealism as background:

Problematic or skeptical idealism. (Descartes) 'the ideal': the ideas in one's mind. They certainly exist, but we cannot be certain that anything external to them exists.

Dogmatic idealism. (Berkeley) 'the ideal': the ideas in one's mind. Only they exist, and the existence of anything external to them is illusory.

Subjective or formal idealism. (Kant, Fichte) 'the ideal': the forms of our experiences. The mind imposes these forms on the matter of our experiences, and therefore we cannot come to know things in an unmediated way, as things in themselves. Moreover, whatever does not lend itself to being given through those forms, may exist but is beyond our cognitive reach. Kant's specific version is known as 'transcendental idealism'.

Objective or absolute idealism. (Schelling, Hegel) 'the ideal': the forms of our experiences. Unlike in subjective idealism, these forms are neither contributed by our minds, nor by the objects of our experiences, but rather are all-pervasive aspects of reality.

Linguistic idealism. (Wittgenstein, Quine) 'the ideal': language (or 'grammar'), understood as inherently public. What exists, and what does not, is determined only relative to a language, and in that sense there is no reality independently from language.

Husserl's transcendental idealism. 'the ideal': the realm of possible consciousness. Existence (of any particular object) is equivalent to accessibility to a possible subject. We will describe this type of idealism in detail in Sect. 6.3.3. (Husserl also formulated a stronger version. As we will see in Sect. 6.3.4, Gödel found that one objectionable.)

6.3.2 Gödel and German Idealism

A letter from Kreisel to Gödel is a good starting point for obtaining further clarification of the differences between subjective idealism and objective idealism (historically these two positions are known as 'German Idealism'), their position relative to realism, and why they were important to Gödel. On September 6, 1965, Kreisel sent Gödel the manuscript of his contribution to *Bertrand Russell,* *Philosopher of the Century* (Kreisel 1967). Concerning a passage in the manuscript on the distinction between realism and objective idealism, in the accompanying letter (Gödel Papers, 2a/87, 011187.5) he recalls (presumably from their conversations) Gödel's view that the distinction between objective and subjective idealism is of greater philosophical consequence.[23] The passage that Kreisel is referring to is the following:

> There is a *broad* distinction between views which hold that all mathematics is purely 'conventional' or, more precisely, depends on human reactions incapable of *a priori* explanations and those which hold that there is something 'objective' about mathematics ... It would seem that there are quite basic distinctions between different views of the second kind, for instance those that stress the objective aspects which are *external* to ourselves and those which do not. (Trivially, everything has an aspect not external to ourselves simply by virtue of being perceived and understood.) It might well be that the methods needed to extend known axioms depend on whether or not mathematical objects are (primarily) external to ourselves. (Kreisel 1967, 221)

Kreisel does not use the terms 'realism' and 'objective idealism' here, but he does employ them in his letter when describing the passage. As Frederick Beiser explains it,

> The basic difference between [subjective and objective idealism] is quite straightforward. While subjective idealism attaches the forms of experience to the transcendental subject, which is their source and precondition, objective idealism detaches them from that subject, making them hold for the realm of pure being as such. (Beiser 2002, 11)

In subjective idealism there is no guarantee that what is a property of our minds also characterises anything in objective reality outside of it. Objective idealism, on the other hand, agrees with naïve realism to the extent that it locates the forms outside the subject and in that sense conceives of them as aspects of objective reality. This explains Kreisel's observation that for Gödel, the distinction between subjective and objective idealism was more important than that between objective idealism and realism. The two latter strands of thought both have room for the objectivity of mathematics in the strong sense that Gödel cared about. From his point of view, it would not suffice to say, as one should, that mathematical objects are invariants in our (mathematical) experience; for a subjective idealist could say the same. What Gödel wants to accommodate is the idea that these invariants are objective and in no way subjective.[24] From his letter to Günther (quoted in Sect. 6.2.4), it is clear that,

[23]Given the explanation of the contrast that follows, it is not likely that 'objective idealism' here specifically means that of Schelling and Hegel, it seems also to include Husserl's transcendental idealism.

[24]William Howard tells the following story from 1972 or 1973 (Shell-Gellasch 2003, 40–41):

> Because Gödel had repeatedly asked me to describe my experiences during meditation, I finally suggested that maybe he would like to learn how to do T[ranscendental] M[editation]. He said no, and I asked why not. Gödel replied, 'The goal of Maharishi's system of meditation is to erase thoughts, whereas the goal of German Idealism is to construct an object.'

of the two choices mentioned, Gödel opts for a form of idealism, not for realism (in the sense in which it is opposed to idealism as described by Gödel).

According to the letter to Günther, what Gödel finds agreeable in idealism is that it takes the subject into account, but he refrains from embracing it when it denies truth or sense to the notion of objective existence (it is relevant that Gödel explicitly includes abstract objects here). Hegel once stated the problem very clearly:

> But also the objectivity of thought, in Kant's sense, is again itself subjective, in the following way. Thoughts, according to Kant, although universal and necessary determinations, are 'only our' thoughts – separated by an unbridgeable gap from the thing as it exists 'in itself'.

> In saying this, he became quite forceful, holding out his hand – palm and fingers upward – as if he were grasping an object. I think what Gödel meant was that the goal was to build a mental structure. I tried to explain that it was not the purpose of TM to erase thoughts; but his mind was made up.

The explanation of Gödel's phrase 'constructing an object' here is a delicate matter, and Howard's explanation may not be sufficient;[25] but Gödel's gesture is very telling.

[25]⟦Addition MvA: In an email of March 15, 2013, William Howard explains to me:

> When I said, in my interview with Amy, that I thought that what Gödel meant was that the goal was to build a mental structure, what I had in mind was a combination of two things:
>
> 1. what I had been able to learn about German Idealism (including some passages from Schelling, and what I had been able to get out of Kant's *Critique of Pure Reason*,
> 2. what I had learned from studying the writings of Gurdjieff and Ouspensky (this preceded my TM adventure; it is where I first learned about the various states of consciousness and is the reason that I took up TM).
>
> ...
>
> When I said 'mental structure', I had in mind what Gurdjieff described as a product of 'inner work'. He certainly regarded such a structure as being objective. Not that I consider myself an expert on Gurdjieff. Also, I would not normally cite him in a philosophical discussion, but at least you can see what was on my mind at the time.

As prof. Howard mentioned, Gödel did not want to learn TM. But, Howard tells me in an email of March 9, 2013,

> I had been going to TM retreats every few weeks, and I was asked if I would give a talk. This was early in 1973. I prepared a 2-page outline, to be handed out at the talk. Caroline Underwood, the secretary at the Institute, typed it up and made mimeographed copies for me;
>
> ...
>
> Caroline Underwood (above) was so fascinated with the material that she had typed up for me, and my explanation of it, that she joined the TM organization a couple of months later. I was pretty surprised, since she was a 'no nonsense' type of person (had been an officer in the Navy, I think).

(Quoted by permission of William Howard.)⟧

> But the true objectivity of thinking consists in the thoughts being not merely ours, but at the same time being the 'in themselves' of the things and whatever is an object.[26]

Hegel introduced the term 'subjective idealism' to describe Kant's position. In a note on a discussion with Wang (Gödel Papers, 3c/208, 013171), Gödel contrasted the 'realism' of 'objectivity, Platonism' with 'Kantianism', which is 'only subj[ective] id[ealism]'.

Gödel considered this denial of objectivity (in this strong sense) the reductio ad absurdum of Kantian idealism, as he wrote in one of his papers on the theory of relativity in the late 1940s :

> Unfortunately, whenever this fruitful viewpoint of a distinction between subjective and objective elements in our knowledge (which is so impressively suggested by Kant's comparison with the Copernican system, see below, p. ⟦29⟧)[27] appears in the history of science, there is at once a tendency to exaggerate it into a boundless subjectivism, whereby its effect is annulled. Kant's thesis of the unknowability of the things in themselves is one example. (Gödel *1946/9-C1, 257–258n27)

There are in fact various interpretations of Kant's notion of the 'thing in itself', according as it is thought of as an independent object or, for example, as a mere (negative) idea of a limit to our knowledge. We see here that Gödel was inclined to accept the first interpretation of Kant. What Kant's transcendental idealism, thus understood, has in common with, say, Berkeley's dogmatic idealism, is that according to both there are no things in themselves to be known, because they are out of our cognitive system's reach, or, respectively, because they do not even exist. A decade later, upon his reading of Husserl, Gödel came to see an alternative. At the end of his paper from around 1961, 'The modern development of the foundations of mathematics in the light of philosophy',[28] he writes:

> On the other hand, however, just because of the lack of clarity and the literal incorrectness of many of Kant's formulations, quite divergent directions have developed out of Kant's thought – none of which, however, really did justice to the core of Kant's thought.

[26] Ferner ist nun aber auch die kantische Objektivität des Denkens insofern selbst nun wieder subjektiv, als nach Kant die Gedanken, obschon allgemeine und notwendige Bestimmungen, doch 'nur unsere' Gedanken und von dem, was das Ding 'an sich' ist, durch eine unübersteigbare Kluft unterschieden sind. Dagegen ist die wahre Objektivität des Denkens diese, dass die Gedanken nicht bloss unsere Gedanken, sondern zugleich das Ansich der Dinge und des Gegenständlichen überhaupt sind. (Hegel (1830) 1906, IV, 2, Sect. 41, Zusatz 2)

[27] The annotation between brackets is by the editors of Gödel 1995 and refers to p. 29 of Gödel's manuscript, which is printed on p. 258 of Gödel 1995.

[28] As an aside, we suggest that the views Gödel expounds in this paper on the history of philosophy since the Renaissance and the division of world views according to their optimism or pessimism, where idealism and theology are on the optimistic side, may have been inspired by a paper by Heimsoeth that he knew, 'Leibniz' Weltanschauung als Ursprung seiner Gedankenwelt', Heimsoeth 1916, esp. 370, 72, 76, and by Heimsoeth's book *Die sechs großen Themen der abendländischen Metaphysik* (Heimsoeth 1934), on which Gödel made 29 pages of notes, perhaps in 1962 (see p. 27), in which one finds 'p. 19–70 = Opt[imismus]-Pess[imismus]'. The notes and the reference to the paper on Leibniz are in 'idealis[ische] Ph[ilosophie].' (Gödel Papers, 9c/23)

> This requirement seems to me to be met for the first time by phenomenology, which, entirely as intended by Kant, avoids both the death-defying leaps [salto mortale] of idealism into a new metaphysics as well as the positivistic rejection of all metaphysics. (Gödel *1961/?, 387)[29]

The 'death-defying leap'[30] is the jump into a boundless subjectivism with, correlatively, an overblown conception of the subject, a conception which has not been arrived at scientifically, that is, a conception that is not founded on intuition (in a technical sense of the word, as in Kantian philosophy or in phenomenology). On the other hand, the 'positivistic rejection of all metaphysics' is motivated by the, according to Gödel, mistaken idea that there is no scientific way to arrive at metaphysics in the first place. Gödel's qualm here is about arriving at metaphysics by a leap of faith, not about metaphysics (in the form of idealism) as such.

6.3.3 The Turn to Husserl's Transcendental Idealism

The position that Gödel found himself in, then, as far as metaphysics is concerned, was the following. On the one hand, as the letter to Günther shows, Gödel believed that idealism may well be the correct way to metaphysics, and that Kant had the right intentions; on the other hand, he thought that Kant did not work out this intention correctly, and that the forms of idealism he had seen so far were beset by problems of being unscientific and boundlessly subjectivist. These problems are responsible for what we have called the stalemate of the 1950s. We suggest that after that difficult decade he became convinced that there was a cure for these problems, Husserl's phenomenology. Indeed, Gödel's reason for this conviction may well have been that his views on idealism and on the place of Kant were very much those that Husserl had arrived at some five decades before; Husserl had been facing the same problems with them, and seemed to have found a solution.

Husserl confessed in 1915, a decade after his transcendental turn, that 'Mir ist der ganze deutsche Idealismus immer zum K... gewesen' – 'German Idealism has

[29] Anderseits haben aber eben wegen der Unklarheit und im wörtlichen Sinn Unrichtigkeit vieler Kantscher Formulierungen sich ganz entgegengesetzte philosophische Richtungen aus [dem] Kantschen Denken entwickelt, von denen aber keine dem Kantschen Denken in seinem Kern wirklich gerecht wurde. Dieser Forderung scheint mir erst die Phänomenologie zu genügen, welche ganz im Sinne Kants sowohl dieselben Salto mortale des Idealismus in eine neue Metaphysik als auch die positivistische Ablehnung jeder Metaphysik vermeidet. (Gödel *1961/?, 386)

[30] Gödel's term in the German original, 'Salto mortale' will have been a reference to Friedrich Heinrich Jacobi (1743–1819), who argued that in the end all knowledge can only be grounded by making an un-reasonable 'salto mortale' or leap of faith, a leap which he valued positively and considered the proper response to scepticism. The term became famous because of the ensuing 'Pantheism controversy' in which, among others, Jacobi, Mendelssohn, Kant, Herder, Goethe and Hamann took part. See Beiser (1987, Chap. 2) for further discussion.

always made me want to throw up' (Boehm 1968, 28). But Husserl also found a way to appreciate it,[31] and by 1919 he had developed the view that, although the idealists lacked the required attitude and methods to turn the fruits of their labour into something of scientific value, they were looking in the right direction:

> Although Kant and the other German Idealists hardly have anything satisfying and defensible to offer by way of a scientifically strict treatment of the motives of inquiry that move them so much, anyone who can really follow and understand these motives and immerse himself into their intuitive content, can be sure that in the idealist systems completely new dimensions of problems come to the fore that are the most radical in philosophy. Only by their clarification and by the development of the specific method required for them can philosophy open the way to its final and highest goals. [33]

The 'dimensions of problems that are the most radical in philosophy' refer to Kant's Copernican turn which characterises German Idealism, and which according to Husserl ultimately leads to his notion of transcendental subjectivity; and by the 'specific method required for them' Husserl of course meant his own transcendental phenomenology. A formulation by Husserl that Gödel will have seen before 1961, because it is in *Ideas I*, which we think is among the first of Husserl's works that he read,[34] is

> Accordingly, it is understandable that phenomenology is, so to speak, the secret nostalgia of all modern philosophy . . . And then the first to correctly see it was Kant, whose greatest intuitions become wholly understandable to us only when we have obtained by hard work a fully clear awareness of the peculiarity of the province belonging to phenomenology.

[31]⟦Addition MvA: Revealing about Husserl's personal situation and the mentality with which he faces it is his letter to Mahnke of April 23, 1921 (Husserl 1994, 3:430): 'Unfortunately I (who as you know am Lutheran, but of Jewish origin) could not become member of the Fichtegesellschaft[32] and I deplore the intrusion of anti-semitism into philosophy(!). But no one can keep me from working, according to my weak powers, in Christian spirit and for Fichte and for German Idealism.' ('Leider konnte ich (der ich wie Sie wissen zwar Lutheraner, aber jüdischer Abstammung bin) der Fichtegesellschaft nicht beitreten und beklage das Eindringen des Antisemit[ismus] in die Philosophie (!). Niemand kann mich aber hindern, nach schwächen Kräften im christlichen Geist und für Fichte und den deutschen Idealismus zu wirken.')⟧

[32]⟦Addition MvA: The Fichtegesellschaft was a political-philosophical society founded in 1914. In November 1917, Husserl gave three appreciative lectures on Fichte's ideal of humanity (Husserl 1987, 267–293, English translation Husserl 1995); he repeated those in January and November 1918.⟧

[33] Mochten Kant und die weiteren deutschen Idealisten für eine wissenschaftlich strenge Verarbeitung der sie machtvoll bewegenden Problemmotive auch wenig Befriedigendes und Haltbares bieten: die diese Motive wirklich nachzuverstehen und sich in ihren intuitiven Gehalt einzuleben vermögen, sind dessen sicher, daß in den idealistischen Systemen völlig neue, und die allerradikalsten Problemdimensionen der Philosophie zutage drängen und daß erst mit ihrer Klärung und mit der Ausbildung der durch ihre Eigenart geforderten Methode der Philosophie ihre letzten und höchsten Ziele sich eröffnen. (Husserl 1987, 309)

[34]From his notes, it is clear that Gödel read Husserl's texts mainly in the Husserliana series; by 1961, eight volumes had appeared, of which *Ideas I* was the third (Husserl 1950c).

> It then becomes evident to us that Kant's mental regard was resting on that field, although he was still unable to appropriate it or recognize it as a field of work pertaining to a strict science proper. (Husserl 1983, 142, translation modified)[35]

And Gödel probably also saw p. 287 of Vol. 1 of *Erste Philosophie* (*First Philosophy*) (Husserl 1956a), on which there are reading notes in Gödel (Papers, 9c/22), and which appeared in the Husserliana in 1956, where transcendental phenomenology is described as 'an attempt to realise the deepest meaning of Kant's philosophising';[36] and p. 181 of the second volume (Husserl 1959), which appeared in 1959, where Husserl, speaking of transcendental idealism, writes that 'phenomenology is nothing but the first strictly scientific form of this idealism'.[37] It was only after his transcendental turn that Husserl came to say that Kant and he share deep philosophical intentions[38]; so the fact that Gödel in his paper from around 1961 (Gödel *1961/?, 374–387), writing just 2 or 3 years after beginning his study of phenomenology, describes the relation between Kant and Husserl in a very similar way, namely by saying that the latter is the first to realise the true intentions of the former, shows his specific interest in Husserl's transcendental idealism.

What Husserl found particularly troublesome in the German Idealists' methodology is that they allowed themselves to build castles prior to checking the ground. Their philosophies are grand systems, yet not systematic in the required sense, as their beginnings and first principles, even if there is a grain of truth in them, come out of the blue. According to Husserl, all knowledge, however, should be rooted in intuition; it should not be somehow 'deduced' from principles that are not themselves given in intuition. Husserl here employs Kant's dictum 'thoughts without content are empty' (Kant (1781–1787a) 1965a, A51/B75)[39] against Kant himself. Gödel agreed with Husserl here; when Wang, in one of the drafts for his book *From Mathematics to Philosophy*, wrote

> One aspect of structural factualism is to take the idea of reflection seriously. It seems to follow that we should pay sufficient attention to the data on which we are to reflect and not to philosophize over thin air (Gödel Papers, box 20: '*From Mathematics to Philosophy*, II Fassung, 1–30 and Introduction', 6)

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35 So begreift es sich, daß die Phänomenologie gleichsam die geheime Sehnsucht der ganzen neuzeitlichen Philosophie ist . . . Und erst recht erschaut sie Kant, dessen größte Intuitionen uns erst ganz verständlich werden, wenn wir uns das Eigentümliche des phänomenologischen Gebietes zur vollbewußten Klarheit erarbeitet haben. Es wird uns dann evident, daß Kants Geistesblick auf diesem Felde ruhte, obschon er es sich noch nicht zuzueignen und es als Arbeitsfeld einer eigenen strengen Wesenswissenschaft nicht zu erkennen vermochte. (Husserl 1950c, 148/Husserl 1976a, 133)

36 ein Versuch . . ., den tiefsten Sinn Kant'schen Philosophierens wahrzumachen.

37 die ganze Phänomenologie [ist] nichts anderes als die erste streng wissenschaftliche Gestalt dieses Idealismus.

38 For examples from correspondence and work, see Kern (1964, 28–33).

39 Gedanken ohne Inhalt sind leer. (Kant (1781–1787b) 1996, A51/B75)

Gödel added a footnote to 'over thin air' saying that Kant 'gets the categ[ories] out of thin air', and wrote next to Wang's last line, 'Husserl'.

It is now clear that Gödel's criticism of Kant, 'A thorough beginning is better than a sloppy architectonic' (Wang 1996, 171), is equally a recommendation of Husserl. And Gödel probably thought the same about the other idealists.[40]Indeed, again in full agreement with Husserl, he said that 'Idealistic philosophers are not able to make good ideas precise and into a science' (Wang 1996, 168).

Gödel reaffirms his belief in transcendental idealism in a draft letter to Gian-Carlo Rota from 1972 [43]:

> I believe that his [i.e., Husserl's] transc[endental] phen[omenology], carried through, would be nothing more nor less than Kant's critique of pure reason transformed into an exact science, except for the fact that[V] the result (of the 'critique') would be far more favourable for human reason (Gödel Papers, 2c/141, 012028.7)

adding in a footnote that

> [V] Kant's subjectivism & negativism for the most part would be eliminated

[40]It seems that Gödel made a much more intensive study of German Idealism than Husserl ever had;[41] compare the amount of material in the relevant folders in the Gödel *Nachlaß* (Gödel Papers, 9b/16, 9b/17, 9b/18, and 9c/23) with the remark by Boehm (Boehm 1968, 50n1).[42] (Boehm is of course right when he goes on to point out that lack of direct acquaintance with a body of philosophical work does not imply a judgment on the quality of one's critique of it.) In passing, we note that Gödel closely studied the section '‹Kant und die Philosophie des Deutschen Idealismus›' ('‹Kant and the philosophy of German Idealism›') in Husserl's *First Philosophy* (Husserl 1956a, 395ff.). Conversely, Husserl was well-read on British empiricism, of which in particular Hume was important to him: Hume's *Treatise on Human Nature* is the most heavily annotated book in Husserl's library, and Husserl, in his transcendental phase, has a very high opinion of Hume (e.g., Husserl 1954, 91; Husserl 1956a, 156–157). Gödel, on the other hand, seems never to have studied the British empiricists that carefully. Robin Rollinger pointed out to us that this means that, via Husserl, Hume had a considerable *indirect* influence on Gödel. Indeed, Köhler (2002b, 359–360) has drawn attention to the similarity of two passages in Gödel and Hume.

[41]⟦Addition MvA: Husserl wrote to E.P. Welch on June 17/21, 1933 that 'even the great Idealists after Kant I have come to know only through fragments, hence I have never studied them intensively' (Spiegelberg 1981, 183) ('auch die grossen Idealisten nach Kant habe ich nur in Bruchstücken kennen gelernt, also nie eingehend studiert', Husserl 1994, VI:460).⟧

[42]⟦Addition MvA: Boehm remarks there that Husserl had read in detail of Kant only the main works, of Fichte only the 'popular' writings, of Schelling perhaps nothing, of Hegel some 50 pages.⟧

[43]This draft is a reply to Rota's review of Husserl's *The Crisis of the European Sciences and Transcendental Phenomenology* (Husserl 1954), intended for Scientific American, sent to Gödel by Rota on July 11, 1972. It must have been written somewhere in the period July–September 1972, as Gödel sent the final version (omitting the passages we quote) on September 15, 1972 (Gödel Papers, 2c/141, 012030). Incidentally, that final version differs in letter but not in spirit from Rota's rendition of it in Rota (2000), a circumstance that Rota hints at by next presenting as a quotation what is really a gloss on the final paragraph of Gödel's paper from around 1961 (Gödel *1961/?, 374–387).

We already saw that, according to Gödel, this 'negativism' – the unknowability of things in themselves – and 'subjectivism' – resulting in a notion of objectivity as contingent upon the peculiarities of the human mind – do not lead to the correct metaphysics. Husserl's transcendental idealism, on the other hand, he thinks will lead to a 'perfectly consistent metaphysics', and in another footnote in this draft letter to Rota, Gödel says that 'There is no reason why this metaph[ysics] should not be objectively true'. Such a metaphysics is exactly the goal Husserl set in his lecture series from 1907 that introduced his transcendental idealism and was titled *The Idea of Phenomenology*:

> What is required is a science of what exists in the absolute sense. This science, which we call metaphysics, grows out of a 'critique' of positive knowledge in the particular sciences. It is based upon the insight acquired by a general critique of knowledge into the essence of knowledge and known objectivity according to its various basic types, that is, according to the various basic correlations between knowledge and known objectivity. (Husserl 1999, 19)[44]

Gödel was acquainted with the text and considered it, as one can see on one of his bibliographical *Zettel*, a 'momentous lecture' (Gödel Papers, 9c/22, 050110). This assessment is indicative of the relief Gödel must have felt at finding that, instead of seeing his philosophical game end in a stalemate, there was a move to be made after all.

What Husserl had come to see in those lectures of 1907 is that there is a necessary correlation between things in themselves and consciousness, in such a way that the latter is open to the former. We take the key formulation to be the one in *Ideas I* of 1913, which was a further development of the 1907 lectures:

> Of essential necessity (in the Apriori of the unconditioned eidetic universality) to every 'truly existing' object there corresponds the idea of a possible consciousness in which the object itself is seized upon originarily and therefore in a perfectly adequate way. Conversely, if this possibility is guaranteed, then eo ipso the object truly exists. (Husserl 1983, 341)[45]

This correspondence is at the heart of Husserl's version of idealism. The thinking that led him to assert this correlation may be summarised as follows.

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[44] Es bedarf einer Wissenschaft vom Seienden in absolutem Sinn. Diese Wissenschaft, die wir Metaphysik nennen, erwächst aus einer 'Kritik' der natürlichen Wissenschaften auf Grund der in allgemeinen Erkenntniskritik gewonnenen Einsicht in das Wesen der Erkenntnis und der Erkenntnisgegenständlichkeit nach ihren verschiedenen Grundgestaltungen, in den Sinn der verschiedenen fundamentalen Korrelationen zwischen Erkenntnis und Erkenntnisgegenständlichkeit. (Husserl 1950b, 23)

[45] Prinzipiell entspricht (im Apriori der unbedingten Wesensallgemeinheit) jedem 'wahrhaft seienden' Gegenstand die Idee eines möglichen Bewusstseins, in welchem der Gegenstand selbst originär und dabei volkommen adequat erfaßbar ist. Umgekehrt, wenn diese Möglichkeit gewährleistet ist, ist eo ipso der Gegenstand wahrhaft seiend. (Husserl 1950c, 349/Husserl 1976a, 329)

Husserl held that it makes no sense to assert the existence of a certain object if one at the same time holds that this object is in no way accessible to any possible consciousness. Kant in fact agreed with this. The specific ingredient that gives Husserl's transcendental idealism its power in ontological matters (or, alternatively, its hubris) is that it holds that there is an essence of mind. Husserl's idea here is that minds in their full concreteness may be very different from each other, but they are essentially the same or they wouldn't all be minds. This means that if, by reflecting on our human minds, we come to know certain essential properties of them, we can then make judgments that hold for any kind of mind, even God's. Running ahead of things, we mention here that Leibniz, in the preface to his *Theodicy*, says that 'the perfections of God are those of our souls, but he possesses them in boundless measure' (Leibniz 1991, 111).[46] Here Kant disagreed, he thought that different minds need not be essentially the same; in particular, he postulated the existence of a mind called the intellectus archetypus – God's mind – which was able to see things as they are in themselves, which, according to Kant, we humans cannot. This move was unacceptable to Husserl, as it is not possible for us to have intuitions of such different minds, precisely because of the fundamental difference. Again, we see Husserl proving himself to be more Kantian than Kant. This must have been one of Gödel's reasons to conclude, in his paper from around 1961, that Husserl is the real Kant (Gödel *1961/?, 387).

For Husserl, then, when I correctly take something to be objectively existing, I grasp the accessibility of that object to a possible mind, and, Kant notwithstanding, 'a possible mind' here must mean 'a variation on my mind such that the essential properties are the same'. In other words, a possible mind in the required sense is a modalisation of my own concrete subjectivity. Thus understood, objectivity ultimately refers back to my own concrete subjectivity; while I do not create the object, I do create my awareness of it as something objective. Notice how Gödel already in the letter to Günther (quoted in Sect. 6.2.4), so before studying Husserl, said of reflection on the subject that he considered it 'entirely possible that this is "the" way to the correct metaphysics'; and he expressed a conviction that, in choosing this way to metaphysics, one does not prejudge the issue of objective existence, even though some well-known varieties of idealism, such as Kant's, do: 'the denial of the objective meaning of thought that is connected with [idealistic philosophy] ... is really entirely independent of it'. For in Husserl, Gödel found an idealistic philosophy that does not deny the objective meaning of thought. In his draft letter to Rota, Gödel strengthened his claim for the powers of reflection by asserting that 'by introspective analysis of the principles of our thinking one arrives by nec[essity] to [sic] a certain *perfectly consistent* metaphysics' (emphasis Gödel's).

[46] Les perfections de Dieu sont celles de nos ames, mais il les possede sans bornes. (Leibniz (1710) 1885, 27)

Husserl seems to have found a way to do justice both to what is valid in idealism and to what is valid in realism; he denied the validity of each as a whole, but claimed that his transcendental idealism includes what is correct in them. From realism, he accepted that objects are not created by our consciousness, but rejected the idea that the objects are independent in the strong sense of being unknowable things in themselves. Instead of the principled disconnectedness implied by that notion, Husserl distances himself from Kant and offers his correlation thesis, which is the specifically idealistic element in Husserl's transcendental idealism. However, at the same time he rejects the idea that accessibility must mean accessibility to any particular subject at any particular time, because the thesis is formulated in terms of a *possible* consciousness; thus he avoids Berkeleyan idealism as well.

We are aware that Husserl in 1934 wrote to Abbé Baudin that 'No ordinary "realist" has ever been as realistic and concrete as I, the phenomenological "idealist" (a word which by the way I no longer use)' (Gödel 1995, 369n(d); trans. Føllesdal)[47] We interpret this, following Kern (1964, 276), as follows. Husserl repeatedly saw himself confronted with assimilations of his type of idealism to one of the more traditional ones. To discourage that, in the 1930s he came to discontinue his use of that term; however, it would still be applicable.

Husserl's claim in his letter to Baudin is that, by seeing the correlation that defines his idealism, he has a fuller and in that sense more realistic picture of reality than ordinary realists have. This fuller picture embodied for Gödel a solution of scientific value to a problem of long-standing interest to him. As we suggested above, Gödel needed, in order to bridge the gap between his realist convictions and the rational arguments he was able to find in support of them, an account of subjectivity that integrates rationality and Platonism; put differently, he needed a rational account of how abstract objects are accessible to our consciousness. Transcendental phenomenology bridges this gap. Through its correlation thesis it connects consciousness and existence of the objects; at the same time, it connects consciousness and rationality, by conceiving of rationality (or reason) as a predicate earned by consciousness when it proceeds correctly, that is, motivated by evidence of the kind appropriate to the objects it is investigating, thereby obtaining further evidence. In Sect. 23 of the *Cartesian Meditations*, Husserl says that 'Reason refers to possibilities of verification; and verification refers ultimately to making evident and having evident' (Husserl 1973d, 57).[48] As Marcus Brainard has put it, for Husserl, 'reason is nothing separate from consciousness; it is not a subject or a substance, but rather something that belongs intimately to consciousness. But then not as a faculty in the classical sense. Rather, as something predicated of consciousness. Hence it is more appropriate to speak of "rationality" (*Vernünf-*

[47] Kein gewöhnlicher 'Realist' ist je so realistisch und so concret gewesen als ich, der phänomenologische 'Idealist' (ein Wort, das ich übrigens nicht mehr gebrauche). (Husserl 1994, 7:16)

[48] Vernunft verweist auf Möglichkeiten der Bewährung, und diese letztlich auf das Evident-Machen und Evident-Haben. (Husserl 1950a, 92)

tigkeit) than of "reason" [*Vernunft*] inasmuch the former term emphasizes the predicative nature of reason in *Ideas I*' (Brainard 2002, 203). By thus relating, on the one hand, existence to consciousness and, on the other, consciousness to reason, transcendental phenomenology by transitivity establishes the connection between existence and reason which had been missing in Gödel's notion of rationality in the 1950s, a lack that had led him to what we have called, in Sect. 6.2, the stalemate.

Husserl saw his philosophy as the 'method by which I want to establish, against mysticism and irrationalism, a kind of super-rationalism which transcends the old rationalism as inadequate and yet vindicates its inmost objectives' (Spiegelberg 1983, 78).[49] (Compare Gödel's use of the term 'superknowledge' as the goal of phenomenology (Wang 1996, 167), and of 'superscience' to describe knowledge that he thought Kant privately had (Wang 1996, 166).) What enables Husserl to transcend the old rationalism is his insight how transcendental subjectivity relates existence to reason, in the way that we just sketched:

> Phenomenology as eidetic is ... rationalistic; it overcomes restrictive and dogmatic rationalism, however, through the most universal rationalism of inquiry into essences, which is related uniformly to transcendental subjectivity, to the I, consciousness, and anything objective of which I am conscious. (Kockelmans 1994, 321, translation modified)[50]

We have argued that Gödel's aim in his study of Husserl was to find, by such 'inquiry into essences', a deeper notion of rationality. This thesis is corroborated in Gödel's draft letter of 1969 to I. Shenker of the New York Times. He there says that his 'primary interest' from 1959 – the year he begun his study of Husserl – to 1969 has been 'the problem of reason & the philosophical preparations for it' (Gödel Papers, 4c/50, 020971). In particular, Gödel saw he could work on this problem from the point of view of transcendental idealism, as opposed to Husserl's earlier version of phenomenology. In fact, Husserl himself made his breakthrough to transcendental phenomenology after a period of severe depressions during which he confessed, in posthumously published personal notes, that

> Until I have sorted out, in outline, the sense, essence, methods, and main points of a critique of reason, until I have devised, designed, established and founded a general blueprint of it, I will not be able to live genuinely and truthfully.[51]

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[49] Gödel will have read this in the 1965 edition of Spiegelberg (1983), where it occurs on p. 84. The passage is from a letter from Husserl to Lévy-Bruhl, March 11, 1935: 'die Methode ... durch die ich gegen den schwächlichen Mystizismus und Irrationalismus eine Art Überrationalismus begründen will, der den alten Rationalismus als unzulänglich überschreitet und doch seine innerste Intentionen rechtfertigt' (Husserl 1994, 7:164).

[50] Die Phänomenologie als Eidetik ... ist rationalistisch; sie überwindet aber den beschränkten dogmatischen *Rationalismus* durch den universalsten der auf die transzendentale Subjektivität, auf Ich, Bewußtsein und bewußte Gegenständlichkeit einheitlich bezogenen Wesensforschung. (Husserl 1962, 301)

[51] Ohne in allgemeinen Zügen mir über Sinn, Wesen, Methoden, Hauptgesichtspunkte einer Kritik der Vernunft ins Klare zu kommen, ohne einen allgemeinen Entwurf für sie ausgedacht, entworfen, festgestellt und begründet zu haben, kann ich wahr und wahrhaftig nicht leben. (Husserl 1956b, 297)

These lines were written on September 25, 1906; the lecture series *The Idea of Phenomenology*, held between April 26 and May 2 of the next year, marked the beginning of transcendental phenomenology and, as part of that, of a critique of reason, which had been lacking in Husserl's earlier work, notably the *Logical Investigations*.

In the Fall of 1909 Husserl discovered the 'absolute time-constituting flow' of consciousness and, correlatively, the 'absolute self'. With this he arrived at a thick notion of self, one that in the *Logical Investigations* he had still claimed he could not find.[52] It allowed him to fill in some of the details of the notion of subjectivity implied by the transcendental turn as documented in *The Idea of Phenomenology*. To Sue Toledo, Gödel commented

> Husserl's philosophy is very different before 1909 from what it is after 1909. At this point he made a fundamental philosophical discovery, which changed his whole philosophical

[52]Concerning this, Gödel marked some of the relevant passages in his own copy of the *Logical Investigations*. This was a 1968 reprint of the second (B1) edition; it had not appeared in the Husserliana series yet. Before 1968 he must have studied a copy of another edition. In his own copy, the footnote on p. 354 to the last line of Sect. 4, is marked by a large exclamation mark on the left, and underlined as follows:

> Die sich in diesem Paragraphen schon aussprechende Opposition gegen die Lehre vom 'reinen' Ich billigt der Verf., wie aus den oben zitierten *Ideen* ersichtlich ist, nicht mehr. (vgl. a.a.O., §57, S. 109; §80, S. 159.) (Husserl 1973c, 542n1)[53]

Correspondingly, on the preceding page (353),

> Das phänomenologisch reduzierte Ich ist also nichts Eigenartiges, das über den mannigfaltigen Erlebnissen schwebte, sondern es ist einfach mit ihrer eigenen Verknüpfungseinheit identisch. (Husserl 1973c, 542n1)[54]

is marked by a large question mark in the margin. And a passage in *Ideas I* is underlined as follows:

> In den 'Log. Unters.' vertrat ich in der Frage des reinen Ich eine Skepsis, die ich im Fortschritte meiner Studien nicht festhalten konnte. Die Kritik, die ich gegen Natorps gedankenvolle 'Einleitung in die Psychologie' richtete (II^1, S. 340f.), ist also in einem Hauptpunkte nicht triftig. (Husserl 1950c, 138/Husserl 1976a, 124)[55]

On p. 4 of the first four inside pages of his copy of *Ideas I*, among the notes is one in which Gödel refers to this footnote.

[53] The opposition to the doctrine of a 'pure' ego, already expressed in this paragraph, is one that the author no longer approves of, as is plain from his *Ideas* cited above.

[54] The phenomenologically reduced ego is therefore nothing peculiar, floating above many experiences: it is simply identical with their own interconnected unity.

[55] In the *Logische Untersuchungen* [*Logical Investigations*] I advocated a skepticism with respect to the question about the pure Ego, but which I could not adhere to asmy studies progressed. The criticism which I directed against Natorp's thoughtful *Einleitung in die Psychologie* [*Introduction to Psychology*] is, as I now see, not well-founded in one of its main contentions. (Husserl 1983, 133)

> outlook and is even reflected in his style of writing. He describes this as a time of crisis in his life, both intellectual and personal. *Both* were resolved by his discovery. (Toledo n.d., 3/24/72, 1)[56]

In a similar comment to Wang, Gödel concludes that

> At some point in this period, everything suddenly became clear to Husserl, and he did arrive at some absolute knowledge.[57]

In these comments to Toledo and Wang, Gödel must be reporting on his reading of Husserl's posthumously published notes.[58]

Husserl indeed came out of his deep depression and in 1911 (10 years after the publication of the *Logical Investigations*) wrote, in a letter to the neo-Kantian Hans Vaihinger,

> I am working, for the tenth year already, with all my powers, on a systematic foundation of phenomenology, or rather, on the phenomenological theory and critique of all reason. I believe I have overcome the main difficulties.[59]

Consistent with Gödel's high opinion of *The Idea of Phenomenology* and his conviction that with the subsequent discovery of the absolute self, Husserl had arrived at some absolute knowledge, Husserl's key publications, according to Gödel, are from after Husserl's transcendental turn. Wang reports

> Gödel told me that the most important of Husserl's published works are *Ideas* and *Cartesian Meditations* (the *Paris Lectures*): 'The latter is closest to real phenomenology – investigating how we arrive at the idea of self' (Wang 1996, 164).[60]

Husserl found the absolute self when he took up his earlier analyses of time again, now from his recently won transcendental point of view. Accordingly, Gödel suggested to Toledo

[56]⟦Addition MvA: Now Toledo 2011, 200.⟧

[57]On the other hand, at one point Gödel said to Wang that 'Husserl aimed at absolute knowledge, but so far this has not been attained.' (Wang 1996, 291; see also 169)

[58]For an intellectual-psychological biography of Husserl, see Wetz (1995).

[59]
> Ich arbeite, nun schon das zehnte Jahr, mit Aufwand aller Kräfte an einer systematischen Begründung der Phänomenologie, bzw. der phänomenologischen Theorie und Kritik der gesammten Vernunft. Ich glaube, die wesentlichen Schwierigkeiten überwunden zu haben. (Husserl 1994, 5:206)

[60]Incidentally, the *Paris Lectures* and the *Cartesian Meditations* are not the same work; the latter is a much worked out version of the former. They are, however, published together in Husserliana, *Cartesianische Meditationen und Pariser Vorträge* (Husserl 1950a), which Gödel owned. Perhaps Gödel mentioned this title to Wang and the 'und' got lost in the note-taking.

> Perhaps the best would be to repeat [Husserl's] investigation of time. (Toledo n.d., 3/24/72:3)[61,63]

6.3.4 Gödel's Criticisms of Husserl's Idealism

Given the above, at first sight Wang's (early) suggestion that Gödel 'probably did not accept Husserl's emphasis on subjectivity' (Wang 1987, 122) seems implausible. However, there is a grain of truth in it. When Husserl reached the view that the objective for its existence is dependent on the subjective, he introduced an ontological asymmetry that Gödel did not approve of. Here are two examples of Gödel's objecting to it.

First, a remark recorded by Sue Toledo in 1972. It concerns Husserl's *Formal and Transcendental Logic* (Husserl 1974), which appeared in between the two works he recommended to Wang as Husserl's most important, *Ideas I* and the *Cartesian Meditations*. Gödel said:

[61] Gödel continues:

> At one point there existed a five hundred page manuscript on this investigation (mentioned in letters to Ingarden, with whom he wished to publish the manuscript). This manuscript has apparently been lost, perhaps when Husserl's works were taken to Louvain in 1940. It is possible that this and other works were removed.

(See also Wang 1996, 320.) Gödel has made similar remarks about manuscripts disappearing, e.g., some of Leibniz', which have sometimes been dismissed as symptoms of a possible mental instability on Gödel's part. But in this case, Gödel was completely correct, and by way of proof he pointed Sue Toledo (n.d., 9)[62] to the following statement of Husserl's former student Roman Ingarden from 1962:

> Thus in 1927 Husserl proposes to me also that I should 'adjust' a great bundle of manuscripts (consisting probably of 600–700 sheets of paper) on the original constitution of time, which he had written in Bernau in 1917–1918. He gave me a completely free hand with the editing of the text, his only condition being that the work should be published under our two names. I could not, however, accept his proposition, first of all because I was convinced that Husserl would have done the work much better himself at the time. To tell the truth, I now regret my decision. Judging by what he told me on the context of his study, it was certainly his most profound and perhaps most important work ... As it happened, the work has not been edited at all, and what is worse nobody seems to know where the manuscript is. (Ingarden 1962, 157n4)

Apparently unbeknown to Ingarden, after his declining Husserl's proposition, the task was accepted by Husserl's assistant Eugen Fink. However, he hardly worked on it and in 1969 he gave the manuscript to the Husserl Archive in Leuven. It was published only in 2001 (Husserl 2001a).

[62] ⟦Addition MvA: Now Toledo (2011, 200).⟧

[63] ⟦Addition MvA: Now Toledo (2011, 200).⟧

> [Husserl's] analysis of the objective world (e.g., p. 212 of *From Formal to Transcendental Logic* [sic]) is in actuality universal subjectivism, and is *not* the right analysis of objective existence. It is rather an analysis of the natural way of thinking about objective existence. (Toledo n.d., 3/24/72, 6)[64]

On the page mentioned (p. 212 in the original publication, pp. 246–247 in Husserl 1974), Husserl explains objectivity as 'what is there for everyone, legitimated as what it is in an intersubjective sharing of knowledge'.[65] It is possible to read this as suggesting that the notion of objectivity is exhausted by that of (maximal) intersubjective agreement. According to Gödel, however, there should also be something which is given, something which goes beyond the merely subjective. As he said to Wang,

> What is subjective, even with agreement, is different from what is objective, in the sense that there is an outside reality corresponding to it. One should distinguish questions of principle from questions of practice: for the former, agreement is of no importance. (Wang 1996, 171)

We note that on p. 233 (p. 270 in Husserl 1974) on the other hand, Husserl speaks of objectivity purely in terms of the correlation thesis; and from Gödel's reading notes (Gödel Papers, 9c/22, 050099) it is clear that he knew that passage as well.

The second example concerns a passage in Husserl's *Ideas I* that is quoted by Weyl on p. 292 of his paper 'Insight and reflection' (Weyl 1969).[66] Gödel marked and partly underlined it on the reprint that he owned (Gödel Papers, box 20, no separate folder):

> All <u>real</u> entities are entities of the intellect. Intellectual <u>entities</u> <u>presuppose the existence of a consciousness which assigns them</u> <u>their meaning</u> and which, in turn, exists absolutely and not as the result of assigned meaning.

Gödel may have noted that this translation of the beginning of Sect. 55 of *Ideas I* is not particularly accurate. The German reads:

> Alle realen Einheiten sind 'Einheiten des Sinnes'. Sinneseinheiten setzen ... sinngebendes Bewußtsein voraus, das seinerseits absolut und nicht selbst wieder durch Sinngebung ist. (Husserl 1950c, 134/Husserl 1976a, 120)

In the original German article, of which Gödel read an English translation, Weyl quotes this correctly. The more careful translation by F. Kersten reads

> All real unities are 'unities of sense'. Unities of sense presuppose ... a sense-bestowing consciousness which, for its part, exists absolutely and not by virtue of another sense-bestowal. (Husserl 1983, 128–129)

Be that as it may, his remark to Toledo just quoted convinces us that, when Gödel annotated, in Gabelsberger, the quotation in Weyl's paper by 'falsch' ('wrong'), this was not on account of the translation. As remarked by Iso Kern (1964, 280),

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[64]⟦Addition MvA:Now Toledo (2011, 202).⟧

[65] 'objektive', in jenem Sinn der für Jedermann daseienden, sich als wie sie ist in intersubjektiver Erkenntnisgemeinschaft ausweisenden. (Husserl 1974, 247)

[66]For an account of the later Weyl's disagreements with Husserl, see Bell (2003).

while Husserl's thesis of a correlation of being and consciousness does mean that all being is in principle accessible to consciousness, it does not imply that being is ontologically dependent on consciousness. To assent to that dependence is a further step that Husserl took. With this step, Alfred Schutz noted, 'the idea of constitution has changed from a clarification of sense structures, from an explanation of the sense of being, into the foundation of the structure of being; it has changed from explication to creation' (Schutz 1966, 83). In terms of our characterisation of idealism in Sect. 6.3.1, Kern and Schutz point out that Husserl changed his dependence relation. It is outside the scope of this paper to discuss Husserl's reasons for doing this, and we confine ourselves to noticing that Gödel was not prepared to follow Husserl in that direction. In this respect, Gödel will have preferred Husserl's conception in *The Idea of Phenomenology* over that in *Ideas I*.

6.4 How Is the Turn Related to Leibniz?

6.4.1 Phenomenology as a Methodical Monadology

Where Gödel's fascination with Leibniz originated is hard to gauge. The earliest evidence of Gödel's study of Leibniz that we have found is a library slip from 1929 (Gödel Papers, 10b/54, 050173) requesting volumes of Gerhardt's edition of Leibniz (1875–1890).[67] Karl Menger mentions that Gödel had begun to concentrate his philosophical studies on Leibniz in the early 1930s (Menger 1981, 69). On the other hand, in one version of his reply to the Grandjean questionnaire, Gödel says that 'the greatest phil[osophical] infl[uence] on me came from Leibniz which I studied about 1943–1946' (Wang 1987, 19).[68] Perhaps, on Gödel's view, some of his encounters with Leibniz had more influence on him than others. The first (and only) published avowal of his Leibnizian views are the striking remarks at the end of the Russell paper from 1944. The fact that Gödel had come to accept, well before he took up the study of phenomenology, fundamental tenets of Leibniz' philosophy, will provide, when considered in combination with his view on the relation between Leibniz and Husserl, an important clue as to what Husserl had to offer him.

We begin by repeating this remark of Gödel's to Wang:

> Gödel told me that the most important of Husserl's published works are *Ideas* and *Cartesian Meditations* (the *Paris lectures*): 'The latter is closest to real phenomenology – investigating how we arrive at the idea of self' (Wang 1996, 164).

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[67]⟦Correction MvA: The request, dated December 18, 1929, is for Vol. 4, which includes the *Dissertatio de arte combinatoria* of 1666.⟧

[68]In a letter to Gödel of July 26, 1954 (Gödel Papers, 2c/141, 011919), Nicholas Rescher recalls that as a Princeton graduate student he had noticed that Gödel borrowed most of the books on Leibniz available at the Firestone Library at some point between 1946 and 1948.

The emphasis that Gödel put here on the idea of self points to his interest in Leibniz, in particular the latter's 'monadology', a term that Gödel borrowed to describe his own philosophy to Wang as late as 1976 (Wang 1996, 309).

It has been reported by Wang (1996, 166) that 'Gödel's own main aim in philosophy was to develop metaphysics – specifically, something like the monadology of Leibniz transformed into an exact theory – with the help of phenomenology'. Gödel considered such a reading of phenomenology to a large extent historically justified: 'Metaphysics in the form of something like the Leibnizian monadology came at one time closest to Husserl's ideal' (quoted in Wang 1996, 170). The quotation continues, 'Baumgarten [1714–1762] is better than Wolff [1679–1754]'. This probably refers to the fact that the Leibnizian philosopher Christian Wolff and his influential school downplayed the importance of the monadology in its original sense and revised it into oblivion, the exception being his student Alexander Gottlieb Baumgarten, who tried to restore and develop it (Casula 1975). Thus there arose two tendencies of unequal strength, one to take the monadology as an attempt at the true metaphysics, the other to take it as misguided or, at best, an attempt at poetry. Husserl and Gödel clearly showed the first tendency.[69]

Gödel's claim that something like the Leibnizian monadology came at one time closest to Husserl's ideal is easily supported. In a letter of January 5, 1917, to his former student, the Leibniz scholar Dietrich Mahnke,[70] Husserl confessed that 'I am, in fact, a monadologist myself'.[71] When Husserl made this remark, he had just begun to think of his own phenomenology as a monadology (1914 (van Breda 1967, 143)), and he continued to do so for the rest of his life.[72] In his lectures from 1923, *First Philosophy*, he said:

> Leibniz, in his brilliant insight of a theory of monads meant that everything that is can in the final analysis be reduced to monads . . . It may well be that in the end, a world view founded by a transcendental philosophy simply demands exactly such an interpretation.[73,74]

In passing, we mention that the 'world view' (*'Weltbetrachtung'*) is echoed in Gödel's statement 'Husserl used Kant's terminology to reach, for now, the founda-

[69] We are thankful to Arthur Collins for urging us to be explicit about these two contrary tendencies.

[70] Mahnke had studied mathematics with Hilbert and philosophy with Husserl in Göttingen from 1902 till 1906. He obtained his Doktorat in 1922 with Husserl in Freiburg, and his Habilitation in Greifswald in 1926. In 1927 he succeeded Heidegger in Marburg, when the latter came to Freiburg to succeed Husserl. In 1939, he died in a car accident. (Cristin and Sakai 2000, 323–325; Husserl 1994, 3:453, 57)

[71] Ich selbst bin eigentlich Monadologe. (Husserl 1994, 3:408)

[72] Of the literature on this connection, we would in particular like to mention van Breda (1967), Ehrhardt (1967), Cristin (1990) and Cristin and Sakai (2000).

[73] ⟦Correction of the translation, MvA: after 'such an interpretation', add 'or a similar one'.⟧

[74] Leibniz meinte in seinem genialen Aperçu einer Monadenlehre: nach seinem letzten wahren Sein reduziere sich alles Seiende auf Monaden . . . Es könnte am Ende sein, daß eine transzendental-philosophisch begründete Weltbetrachtung gerade eine solche oder ähnliche Interpretation als schlechtsinnige Notwendigkeit forderte. (Husserl 1956a, 71–72)

tions and, afterwards, used Leibniz to get the world picture' (quoted in Wang 1996, 166). We also note that Gödel made detailed notes on these lectures (Gödel Papers, 9c/22), of which the historical part was published in 1956. To return to Husserl, at the end of the same lectures, he said

> This way, phenomenology leads to the monadology that Leibniz in a stroke of genius anticipated.[75]

Although Husserl credits Leibniz for his insights, he faults him for not working them out systematically. To Mahnke, Husserl writes about Leibniz, 'He is truly a seer, but unfortunately, detailed theoretical analysis, without which what one has seen cannot become science, is missing everywhere.'[76] (Note that Husserl's criticism of Leibniz is of the same type as that of the German Idealists that we saw before.)

Husserl centred his systematic introduction to phenomenology of 1929, although titled *Cartesian Meditations*, around a version of the Leibnizian notion of the monad. The importance of that notion is brought out in the fourth of these *Meditations*, where it is explained that all of phenomenology in the end is a study of the monadic ego:

> Since the monadically concrete ego includes also the whole of actual and potential conscious life, it is clear that the problem of explicating this monadic ego phenomenologically (the problem of his constitution for himself) must include all constitutional problems without exception. Consequently the phenomenology of this self-constitution coincides with phenomenology as a whole. (Husserl 1973d, 68)[77]

This is the phenomenological interpretation of Leibniz' claim that each monad mirrors the whole universe (*Monadology*, Sect. 60). In fact, Dietrich Mahnke managed to rewrite Leibniz' *Monadology*, some 200 years after its conception, paragraph by paragraph, from a phenomenologically informed point of view, in an essay named 'Eine neue Monadologie' ('A new monadology') (Mahnke 1917).[78]

Gödel read Mahnke's essay and commented, in a stenographical note on one of his bibliographical 'Zettel', 'vernünftig!' – 'sensible!' (Gödel Papers, 9c/25, 050120.1). There have been other further developments of Leibniz' monadology than Husserl's and Mahnke's, and Gödel for example was aware (Gödel Papers,

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[75] So führt die Phänomenologie auf die von Leibniz in genialem *aperçu* antizipierte Monadologie. (Husserl 1959, 190) See also Husserl (1956a, 196–197) and Husserl (1973b, 7).

[76] Er selbst ist ja durchaus ein Schauer, nur daß leider überall die theoretische Einzelanalyse und Einzelausführung fehlt, ohne die Geschautes eben nicht zur Wissenschaft werden kann. Husserl to Mahnke, January 5, 1917 (Husserl 1994, 3:407–408).

[77] Da das monadisch konkrete ego das gesamte wirkliche und potentielle Bewußtseinsleben mit befaßt, so ist es klar, daß das Problem der phänomenologischen Auslegung dieses monadischen ego (das Problem seiner Konstitution für sich selbst) alle konstitutiven Probleme überhaupt in sich befassen muß. In weiterer Folge ergibt sich die Deckung der Phänomenologie dieser Selbstkonstitution mit der Phänomenologie überhaupt. (Husserl 1950a, 102–103)

[78] There are differences between Husserl's interpretation of the monad and Mahnke's. Husserl's assistant Eugen Fink wrote a draft of a discussion of these, see Husserl (1994, 3:519–520).

10c/62) of the *Nouvelle monadologie* (*New Monadology*) of Renouvier and Prat (1899). But the absence of such non-phenomenological developments from Gödel's drafts and programmatic statements indicates that they did not lend themselves to his philosophical purposes as well as the phenomenological approach.[79]

6.4.2 Searching for the Primitive Terms

In a Leibnizian philosophy that takes its bearings from a mathematical conception of rationality – and we have been describing Gödel's philosophy that way –, one of the most important tasks is to find the right primitive terms or categories. How is this related to phenomenology? Gödel once said to Wang that

> Phenomenology is not the only approach. Another approach is to find a list of the main categories (e.g., causation, substance, action) and their interrelations, which, however, are to be arrived at phenomenologically. The task must be done in the right manner. (Wang 1996, 166)

[79] Perhaps of relevance for the question whether Husserl ever imagined the possibility of incompleteness as proved, during his lifetime, by Gödel, a fact never known to Husserl, is the following passage in Mahnke's book from 1917, which Husserl marked in his copy:

> That not all, even rather few, manifolds in the outside world have the property of being definite, is obvious. But also concerning formal mathematics it is still a great question whether its totality is a heap of infinitely many different and unrelated theories of manifolds, or rather can be organised into one big, definite system. The concept of mathematics seems to demand that the latter is the case; yet, a proof is still to be found. ('Dass nicht alle, ja sogar nur herzlich wenige Mannigfaltigkeiten der wirklichen Welt diese Eigenschaft der Definitheit haben, liegt auf der Hand. Aber auch in der formalen Mathematik ist noch eine grosse Frage, ob ihre Gesamtheit ein beziehungsloses Nebeneinander von unendlich vielen verschiedenen Mannigfaltigkeitslehren ist oder vielmehr selbst in ein einziges, grosses definites System geordnet werden kann. Der Begriff der Mathematik scheint zu erfordern, dass das letztere der Fall ist. Doch steht der Nachweis dafür noch aus', Mahnke 1917, 32.)

There has been much discussion whether Gödel's incompleteness theorems are fatal to Husserl's philosophy of mathematics as expounded in *Formal and Transcendental Logic* (Husserl 1974). Cavaillès (Cavaillès 1947, 71ff.) may have been the first to raise that question. However, the surprising thing is that Husserl's former student Felix Kaufmann did not raise it in 1931. On January 15, 1931, Kaufmann participated in a meeting of the Wiener Kreis where Gödel presented his Incompleteness Theorems. Exchanges between Kaufmann and Gödel on that occasion have been recorded as well (Stadler 1997, 278–280). At that very time, Kaufmann was corresponding with Husserl, reporting on his study of *Formal and Transcendental Logic*, and on his work on an article 'Logische Prinzipienfragen in der mathematischen Grundlagenforschung' ('Principal questions of logic in foundational research in mathematics') (Husserl 1994, 4:179–181). If Kaufmann had been aware of the potential problem for *Formal and Transcendental Logic*, he surely would have told Husserl. For a convincing reply to the charge that the Incompleteness Theorems are fatal to Husserl's philosophy of mathematics, see Lohmar 1989, Chap. 11.

Yet, on other occasions Gödel suggested that phenomenology and trying to find the primitive terms are one and the same project. For example, to Sue Toledo he said in 1972:

> Husserl never mentions that his goal for phenomenology is finally to come to an understanding of the primitive terms themselves. (Toledo n.d., 3/24/72:4)[80]

And he also told her,

> Following Husserl's program with diligence could lead one finally to a grasping of the primitive terms (although there are other ways and perhaps quicker ways). (Toledo n.d., 3/24/72:6)

Indeed, Leibniz himself saw a turn towards the subject as the correct way to arrive at the primitive terms of metaphysics. In a letter to Queen Sophie Charlotte of Prussia of 1702, he formulated the idea in the following way:

> The thought of *myself*, who perceives sensible objects, and the thought of the action of mine that results from it, adds something to the objects of the senses. To think of some color and to consider that one thinks of it are two very different thoughts, just as much as color itself differs from the 'I' who thinks of it. And since I conceive that other beings can also have the right to say 'I', or that it can be said for them, it is through this that I conceive what is *substance* in general. It is also the consideration of myself that provides me with the other notions of *metaphysics*, such as cause, effect, action, similarity, etc., and even those of *logic* and *ethics*. Thus it can be said that there is nothing in the understanding that did not come from the senses, except the understanding itself, or that which understands. (Leibniz 1989, 188)[81]

(One notices the similarity to Gödel's argument in the supplement to the Cantor paper (Gödel 1964, 271–272).) And in Sect. 30 of the *Monadology*, Leibniz writes:

> It is also through the knowledge of necessary truths and through their abstraction [from merely sensuous matters] that we are raised to *Reflexive Acts*, which enable us to think of what is called I and to consider that this or that lies within *ourselves*. And, it is thus that in thinking of ourselves we think of being, of substance, of the simple and compound, of the immaterial, and of God himself, by conceiving that what is limited in us is unlimited in him. And these reflexive acts furnish the principal objects of our [present metaphysical] reasonings. (See *Theodicy*, Preface.) (Leibniz 1991, 110–111; amendments Rescher)[82]

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[80] ⟦Addition MvA: Now Toledo (2011, 201).⟧

[81] Cette pensée de moy, qui m'apperçois des objets sensibles, et de ma propre action qui en resulte, adjoute quelque chose aux objets des sens. Penser à quelque couleur et considerer qu'on y pense, ce sont deux pensées tres differentes, autant que la couleur même differe de moy qui y pense. Et comme je conçois que d'autres Estres peuvent aussi avoir le droit de dire moy, ou qu'on pourroit le dire pour eux, c'est par là que je conçois ce qu'on appelle la substance en general, et c'est aussi la consideration de moy même, qui me fournit d'autres notions de metaphysique, comme de cause, effect, action, similitude, etc., et même celles de la Logique et de la Morale. Ainsi on peut dire qu'il n'y a rien dans l'entendement, qui ne soit venu des sens, excepté l'entendement même, ou celuy qui entend. (Leibniz 1875–1890, 6:502)

[82] C'est aussi par la connoissance des vérités necessaires et par leurs abstractions que nous sommes élevés aux *Actes Reflexifs*, qui nous font penser à ce qui s'appelle *Moy* et a considérer que ceci ou cela est en nous : et c'est ainsi qu'en pensant à nous, nous pensons

We suggest that Gödel had passages such as these in mind when he wrote in the letter to Günther (quoted in Sect. 6.2.4),

> The reflection on the subject treated in idealistic philosophy (that is, your second topic of thought), the distinction of levels of reflection, etc., seem to me very interesting and important. I even consider it entirely possible that this is 'the' way to the correct metaphysics.

And similarly, Leibniz' passages give a more specific meaning to Gödel's statement to Wang,

> If you know everything about yourself, you know everything of philosophical interest. (Gödel Papers, 3c/209, 013184)

(The qualification 'of philosophical interest' was added by Gödel in Wang's typescript, but is not included in Wang's quotations (Wang 1987, 210; 1996, 298).)

In a draft letter from (June?) 1963 from Gödel to Time Inc., regarding the upcoming publication *Mathematics* in the Life Science Library, he connects his phenomenological program to his famous 'disjunctive conclusion' that either the human mind infinitely surpasses the powers of any finite machine, or there exist absolutely unsolvable diophantine problems (Gödel *1951, 310). In that draft letter, he mentions the disjunction again, with the disjuncts in reverse order, and then comments:

> I believe, on ph[ilosophical] grounds, that the sec[ond] alternative is more probable & hope to make this evident by a syst[ematic] developm[ent] & verification of my phil[osophical] views. This dev[elopment] & ver[ification] constitutes the primary obj[ect] ~~matter~~ of my present work. (Gödel Papers, 4b/30, 020514.7)

And another version of that passage reads

> I conj[ecture] that the sec[ond] altern[ative] is true & perhaps can be verified by a phenomenol[ogical] investigat[ion] of the processes of ~~thin[king]~~ reasoning.

We will leave a discussion of Gödel's efforts on the question of minds and machines for another time,[83] noting for now that these remarks show Gödel's optimism about phenomenology in the early 1960s. In the Life book, which was published in 1963, he even allowed the following to be reported on his intentions:

> 'Either mathematics is too big for the human mind,' he says, 'or the human mind is more than a machine.' He hopes to prove the latter. (Bergamini and Life 1963, 53)

> à l'Etre, à la Substance, au simple et au composé, à l'immateriel et à Dieu même ; en concevant que ce qui est borné en nous, est en lui sans bornes. Et ces Actes Reflexifs fournissent les objets principaux de nos raisonnemens. (Théodicée, Préface, 4a.) (Leibniz 1991, 111)

[83] van Atten, Horsten, and Rucker, 'Evolving a mind', in progress.[84]

[84] ⟦Addition MvA: I hope that one day we will finish that paper.⟧

6.5 Comparison with Earlier Interpretations

To say that Gödel's interest in transcendental phenomenology arose first of all from the need for a method is not to suggest that he thought that Husserl's results, in contrast, were wide of the mark, but rather that he considered the methodology the original part. For the general picture Husserl arrived at, Gödel already knew from Leibniz: 'Husserl used Kant's terminology to reach, for now, the foundations and, afterwards, used Leibniz to get the world picture' (quoted in Wang 1996, 166).

It is often remarked that Gödel turned to phenomenology for its method of clarification of meanings and concepts, and that he considered such clarification necessary to justify the axioms of mathematics. Indeed, this is the motive Gödel himself gives in his 'The modern development of the foundations of mathematics in the light of philosophy' (Gödel *1961/?, 383).

Analysis of meanings by itself, however, is not enough to justify one view on the foundations of mathematics over another. To do that, one would also have to analyse which meanings can be fulfilled. One not only wants to know what the terms mean, but also if they correspond to something in reality.

For example, Gödel used phenomenology to support classical mathematics, but Becker and Weyl have used it to support intuitionism and other varieties of constructivism. (Weyl in the preface to *Das Kontinuum* (*The Continuum*) (Weyl 1918) refers to Husserl's *Ideas I* as his philosophical framework, the same book that Gödel saw as essential to his own conception of phenomenology.) To resolve this discrepancy phenomenologically, one would need not just an analysis of meaning but also a particular theory of constitution and evidence. This means that to explain Gödel's choice for phenomenology over another framework as a foundation of mathematics, it is not enough to point out that the other framework could not do justice to the meaning of concepts in classical mathematics (e.g., Tieszen 1998, 200). The very point of, say, the intuitionist's arguments in favour of revising classical mathematics is that he thinks there is something amiss with those classical meanings, and that these should be supplanted by meanings which are not thus defective. Put into phenomenological terms, the intuitionist claims that some of the meanings that play a role in classical mathematics will never lead to fulfilled intentions. This is illustrated by the two stages of Brouwer's criticism of Cantor's hierarchy beyond the first number class (Brouwer 1975, 80): he first analyses what Cantor means, and then, while leaving open the possibility that Cantor's theory is consistent, argues that to such meanings can correspond no mathematical reality.

This is where the correlation thesis of Husserl's transcendental idealism becomes relevant: to say that something exists is to say that intentions directed at it can ideally be fulfilled, and vice versa. Being is always open to consciousness. Only if one goes beyond mere analysis of meaning, and analyses the possibility of fulfillability as well, can one come to distinguish the true from the merely consistent. This means that, from the transcendental phenomenological point of view, conflicts over mathematical ontology are conflicts not primarily over meaning but over what meanings correspond to intentions that can be fulfilled, in other words, over what mathematical objects can be constituted with full evidence.

A particular aspect of the correlation thesis that Husserl studied is the 'noetic-noematic correlation'. This is the correlation between the structure of acts (noeses) and the structures of the objects intended in them (noemata), the functional relationship, or rather functional relationships, between the acts in which an object is intended and the way that object appears to us in those acts. Already in the lecture from 1907, Husserl had announced 'the various basic correlations between knowledge and known objectivity' (see Sect. 6.3.3) as a central theme in his new metaphysics, and a significant portion of *Ideas I* (Part III, Chap. 4) is devoted to detailed analyses of them.

It is the noetic-noematic correlation that Gödel is thinking of when, in his paper from around 1961, he proposes to extend our knowledge of abstract concepts by directing our attention from the concepts to the acts in which we perceive them (Gödel *1961/?, 383); for without this specific correlation, there would be no reason to suppose that thus redirecting our attention could teach us anything about the concepts themselves.[85]

In his copy of *Ideas I*, Gödel marked the passages where Husserl criticised his own earlier *Logical Investigations* for not yet having developed the noetic-noematic correlation (and, hence, for not having worked out the correlation thesis that defines his transcendental idealism). Here is an example of such a passage, with Gödel's underlinings:

> Dies ist noch die Einstellung der 'Log. Unters.' In wie erheblichem Maße auch die Natur der Sachen daselbst eine Ausführung noematischer Analysen erzwingt, so werden diese doch mehr als Indices für die parallelen noetischen Strukturen angesehen; der wesensmäßige Parallelismus der beiden Strukturen ist dort noch nicht zur Klarheit gekommen. (Husserl 1950c, 315n1/Husserl 1976a, 296)[86]

In translation:

> That is still the focus of the *Logical Investigations*. However great the extent to which the nature of the matters themselves compels the carrying out of noematic analyses, the noemas are nevertheless regarded more as indices for the parallel noetic structures; the essential parallelism of the two structures has not yet attained clarity there. (Husserl 1983, 308)

[85]See also the paragraph on p. 189 of Wang 1974, that ends 'It is my impression that Gödel proposes to answer it by phenomenological considerations', and Wang's draft for *From Mathematics to Philosophy*, 'II Fassung Further revisions of the chapter on set theory', p. 3 (Gödel Papers, box 20), where Gödel rewrote a passage as follows: 'The observations in this and the last paragraph are meant to be the beginning of a descriptive analysis of our in part subconscious thinking process about sets and *thereby* of the objective ideas we have in mind when we use the term set' [emphasis ours]. For further discussion of the relation between Husserl's correlation thesis and the ontology of mathematics, see van Atten (2001) and van Atten (2002).

[86]Similarly, Gödel marked, by a stenographical note 'wichtig' ('important'), underlining, and vertical lines in the margin, the paragraph where Husserl reaffirms the noetic-noematic correlation and its universal importance beginning, in the edition that he owned (Husserl 1950c), on p. 330, line 31 and continuing on 331 until line 17 (Husserl 1976a, 311 line 25–312 line 11; 1983, 323 line 26–324 line 10).

Just as it is not sufficient to point to the method of meaning analysis, one cannot fully explain the concrete form Gödel's interest in phenomenology took by pointing to phenomenology's realism about the abstract or conceptual, and to the possibility of categorial intuition or intuition of essences.[87] These certainly were essential determinants of Gödel's choice.[88] But an additional element is needed to explain why Gödel opted for the transcendental Husserl instead of the 'realist' or 'ontological' phenomenology of the *Logical Investigations*, in which this realism and intuition figure as well. In fact, Wang (1996, 165) mentions, in addition, the belief shared by Husserl and Gödel, in 'the one-sidedness of what Husserl calls "the naïve or natural standpoint"'; this of course points to the other side, i.e., transcendental subjectivity. Wang concludes that

> For Gödel, the appeal of Husserlian phenomenology was, I think, that it developed the transcendental method in a way that accommodated his own beliefs in intellectual intuition and the reality of concepts. (Wang 1996, 165)

But the question is why Gödel was interested in the transcendental method in the first place. After all, the beliefs that Wang mentions were also shared by followers of Husserl who refused to take the transcendental turn. We are thinking here of phenomenologists in Munich and Göttingen such as Johannes Daubert, Adolf Reinach, Alexander Pfänder, and Hedwig (Conrad-) Martius, who chose rather to develop the framework of the *Logical Investigations*. They said they were not able to understand Husserl's transcendental turn, and argued that a turn towards an alleged transcendental subjectivity was eo ipso a turn away from 'the things themselves' and thus, they continued, defeated the purpose of phenomenology.[89]

From his wide reading in phenomenology (as witnessed by the memoranda and reading notes in Gödel Papers, 9c/22 and 10a/41), Gödel will have been aware of these realist phenomenologists, but he disagreed with them. He held that Husserl's turn to transcendental subjectivity was a turn for the better. Once, when he said to Sue Toledo that 'There are also some detailed phenomenological analyses in the *Logical Investigations*', he added, 'which were made, *however*, before 1909' (Toledo n.d., 3/24/72, 7)[90]; as we saw, 1909 was for Gödel the year Husserl discovered the true notion of self and thereby completed his transcendental turn. And from Gödel's notes in his copy of Husserl's *Ideas I*, it is clear that he seconds Husserl's self-criticisms of the *Logical Investigations* for their lack of what we above have called the correlation thesis and their lack of a rich notion of

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[87]Føllesdal stresses these in his introduction to Gödel's paper from around 1961, although he certainly also touches on themes in transcendental phenomenology on p. 369 and 372 (Gödel 1995).

[88]In recent years, Richard Tieszen has done much to make these Husserlian themes accessible to those interested in Gödel. See, e.g., Tieszen (2002).

[89]See, for example, the account by Conrad-Martius in van Breda and Taminiaux (1959), a book that Gödel knew, as there are reading notes in Gödel (Papers, 9c/22, 050111) (although not to this paper in particular). See also Kuhn et al. (1975).

[90]⟦Addition MvA: Now Toledo (2011, 202).⟧

the self or subject.[91] Only these discoveries enabled Husserl to pick the fruits of Leibniz' suggestions. For Gödel, to adopt the alternative 'realist' or 'ontological' phenomenology of the *Logical Investigations* would have required the sacrifice of the Leibnizian framework that he had made his own early on and would hold on to till the end of his life.

6.6 Influence from Husserl on Gödel's Writings

6.6.1 On the Schools in the Foundations of Mathematics

In 1965, Kreisel published the suggestion that 'what characterises the difference between e.g., the idealist and the realist view is what aspects of (crude) experience (in this case, mathematical experience) are regarded as significant and suitable for study' (Kreisel 1965, 190). On this view, the different traditional schools in the foundations of mathematics each are an exaggeration of a particular aspect of mathematical experience at the cost of others.

Kreisel related this view to Robert Tragesser at Stanford in 1966 (see also Tragesser's reformulation Tragesser 1973, 294), and told him that he had learned this view from Gödel; this is our reason to count it among Gödel's published views, in a way similar to the passages that ended up in Wang (1974), uncredited, at Gödel's wish (see Charles Parsons' introduction to the Gödel-Wang correspondence Gödel 2003a, esp. 395). Kreisel acknowledged his debt to Gödel on this point in print in 1980:

> In his publications Gödel used traditional terminology, for example, about *conflicting* views of 'realist' or 'idealist' philosophies. In conversation, at least with me, he was ready to treat them more like different *branches* of the subject, the former concentrating on the things considered, the latter on the processes of acquiring knowledge about these objects or processes . . . Naturally, for a given question, a 'conflict' remains: Which branch studies the aspects relevant to solving that question? (Kreisel 1980, 209)[93]

Moreover, Tragesser informed us, Kreisel added that Gödel told him that he had formed this view while reading Husserl. What makes Kreisel's account very plausible is that there are two obvious places in Husserl's work that express the

[91]On p. 237 of Gödel's copy of the *Crisis* (Husserl 1954), there are many underlinings in the passage where Husserl describes how already in the 5th and 6th *Logical Investigations* the problematic of the noetic-noematic correlation comes close to the surface. Husserl therefore concludes: 'Thus, in that work[92] lie the first, albeit very imperfect, beginnings of "phenomenology".' ['So liegen in diesem Werke in der Tat die ersten, freilich sehr unvollkommenen Anfänge der "Phänomenologie".']

[92]⟦Correction of the translation, MvA: insert 'indeed'.⟧

[93]The conversations Kreisel is here referring to will have been the same ones as those in the background of his letter to Gödel of September 6, 1965 that we mentioned in Sect. 6.3.2.

very idea in question,[94] both of which Gödel had surely read (as is evidenced by reading notes and excerpts), the one surely, the other most likely, before Kreisel's publication of 1965.

The first is Sects. 18–23 of *Ideas I*, the second Sect. 16 of the 'Britannica article' (to which there are reading notes in Gödel Papers, 9c/22). Of the latter, evidence that Gödel may have read it before 1962, when it was republished in the Husserliana series that Gödel used, consists in a library slip (Gödel Papers, 9c/22, 050103) requesting the relevant volume (17) of the 14th edition of the *Britannica*.

In the 'Britannica article' in particular there is a passage that comes very close to what Kreisel writes. Husserl there speaks of

> oppositions such as between rationalism (Platonism) and empiricism, relativism and absolutism, subjectivism and objectivism, ontologism and transcendentalism, psychologism and anti-psychologism, positivism and metaphysics, teleological and causal interpretations of the world. (Kockelmans 1994, 319, translation modified)[95]

on which he then comments

> Throughout all of these, [one finds] justified motives, but throughout also half-truths or impermissible absolutizing of only relatively and abstractively legitimate one-sidedness. (Kockelmans 1994, 319)[96]

6.6.2 *The Given*

Among Gödel's most famous philosophical passages is that occurring in the 1964 supplement to his Cantor paper:

> That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create

[94] Robert Tragesser suggested to us that Husserl in turn may well have taken Kant's discussion in the 'Amphiboly' section of the *Critique of Pure Reason* as a model, for example the comparison of Leibniz with Locke on B327.

[95] Gegensätze wie die zwischen Rationalismus (Platonismus) und Empirismus, Relativismus und Absolutismus, Subjektivismus und Objektivismus, Ontologismus und Transzendentalismus, Psychologismus und Antipsychologismus, Positivismus und Metaphysik, teleologischer und kausalistischer Weltauffassung. (Husserl 1962, 300)

[96] Überall berechtigte Motive, überall aber Halbheiten oder unzulässige Verabsolutierungen von nur relativ und abstraktiv berechtigten Einseitigkeiten. (Husserl 1962, 300)

Robert Sokolowski has remarked: 'Husserl acknowledges a debt to Leibniz in regard to *mathesis universalis* [and much more, as we have seen above], but Leibniz's hope of reconciling conflicting points of view, in science and politics, may also be at work in phenomenology, with similar deep-seated limitations.' In a footnote to this passage, he refers to Mahnke (Mahnke 1925) and adds: 'Probably the greatest weakness is the conviction that agreement of minds pacifies human affairs' (Sokolowski 1973, 320). At the same time, we are reminded of Borges: 'The metaphysicians of Tlön . . . know that a system is naught but the subordination of all the aspects of the universe to one of those aspects – *any* one of them. (Borges 1998, 74)

> any qualitatively new elements, but only reproduce and combine those that are given. Evidently the 'given' underlying mathematics is closely related to the abstract elements in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted.[97] (Gödel 1964, 271–272)

This remark echoes the ones by Leibniz quoted above (Sect. 6.4.2) and has, in a more systematic context, been connected to Husserl already in 1977, by Robert Tragesser in his book *Phenomenology and Logic* (Tragesser 1977) (and others have done so later). Yet Gödel in the Cantor paper does explicitly address the relation of his views to those of Kant, but not to those of Husserl. However, in a draft of the supplement there is an additional final paragraph that starts:

> Perhaps a further development of phenomenology will, some day, make it possible to decide questions regarding the soundness of primitive terms and their axioms in a completely convincing manner. (Gödel Papers, 8c/101, 040311, 12)

Why did Gödel decide to leave this paragraph out? We suggest that Gödel felt safe enough to make, in print, the negative point about Kant, but not to make the positive point about Husserl.[98] Wang (1996, 80) mentions that in the 1960s, Gödel advised logicians to read Husserl's 6th *Logical Investigation* for its treatment of 'categorial intuition', and we add that the passage in the Cantor paper easily lends itself to interpretation according to the theory Husserl expounds there. Gödel's recommendation shows that he *was* willing to call attention to Husserl in private communication.[99]

Given Gödel's preference for transcendental phenomenology over the *Logical Investigations*, it should be added that Husserl was convinced that in particular the 6th *Logical Investigation* could be raised to the transcendental level; see his foreword to the second edition of that *Investigation* from 1921 (Husserl 1984b, 534). One of the reasons why Husserl did not actually do this was his tendency to get lost in his present research rather than revisit and integrate earlier manuscripts. His incomplete drafts however show, for example, that in the new text he took the noetic-noematic correlation into account. For a discussion of the many issues involved, we refer to Ulrich Melle's introduction to Husserl (2002a).

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[97] *Note MvA and JK.* Concerning this last remark, see our discussion of Gödel's attitude toward Kant's idealism in Sect. 6.3.2.

[98] ⟦Addition MvA: In the postscript to a recent reprint of Parsons (1995), Parsons refers to this suggestion and comments that 'their conjecture fits well with what we know about Gödel's temperament, but I think it likely that he also had not, to his own satisfaction, sufficiently integrated what he was deriving from Husserl with his earlier ideas' (Parsons 2014, 196). Gödel's draft remark is rather tentative, and it seems to me that, to make it, one needs not so much to have achieved sufficient integration as to harbour enough hope for it. But I certainly agree with Parsons' suggestion as such; see Sect. 6.7 of the present chapter.⟧

[99] Warren Goldfarb (Gödel 1995, 324) thinks that Gödel was overestimating the extent to which positivist dogmas remained orthodoxy in 1959, but even so it is not likely that a phenomenological view would have been welcomed by philosophers of mathematics at the time.

In the history of ideas, Husserl's theory of categorial intuition may be seen as an alternative to another anti-Kantian account of intellectual intuition, the one given in a defence of Leibniz' philosophy by Johann August Eberhard.[100] In a series of papers from 1789, Eberhard had claimed that everything of value in Kant's *Critique of Pure Reason* had already been said by Leibniz, and better (e.g., Kant (1790) 1998, Stück III, Nr.2, S.289). Exceptionally, Kant chose to defend his system against this attack and in 1790 this resulted in a paper called 'On a discovery according to which all new critique of pure reason should be made superfluous by an older one'.[101] It is not quite clear who the winner of the Kant-Eberhard controversy is Gawlina (1996, Chap. 7), but surely Kant managed to raise many questions concerning this account of intellectual intuition. Gödel was aware of the Kant-Eberhard controversy: on a bibliographical note (Gödel Papers, 10c/62, 050191; 1959–?) one can find the Princeton library call number 6174.667 of Eduard Ferber (1871–?), *Der philosophischen Streit zwischen I. Kant und Johann Aug. Eberhard*, Berlin: Itzkowski, 1894. Gödel did not pursue Eberhard's line, preferring Husserl's construal.[102]

6.6.3 Revisions in the Main Text of the Cantor Paper

But not only the Supplement to the Cantor paper shows the influence of studying Husserl. Some of the revisions in the main text do, too. Compare the following paragraph from the 1947 text with its 1964 counterpart (we have emphasised the differences in the 1964 version that we want to comment on):

> The negative attitude towards Cantor's set theory, however, is by no means a necessary outcome of a closer examination of its foundations, but only the result of certain philosophical conceptions of the nature of mathematics, which admit mathematical objects only to the extent in which they are (or are believed to be) interpretable as acts and constructions of our own mind, or at least completely penetrable by our intuition. For someone who does not share these views, there exists a satisfactory foundation of Cantor's set theory in its whole original extent, namely, axiomatics of set theory, under which the logical system of Principia mathematica (in a suitable interpretation) may be subsumed. (Gödel 1947, 518)
>
> However, this negative attitude towards Cantor's set theory, and toward classical mathematics, of which it is a natural generalization, is by no means a necessary outcome of a closer examination of their foundations, but only the result of a certain philosophical conception of the nature of mathematics, which admits mathematical objects only to the

[100]Eberhard defends 'nicht sinnliche Anschauung' ('non-sensuous intuition'). That he means 'intellektuelle Anschauung' ('intellectual intuition') is clear from his explanation (Kant (1790) 1998, Stück III, Nr.2, S. 281–282). See also Gawlina (1996, 193–194).

[101] Über eine Entdeckung nach der alle neue Kritik der reinen Vernunft durch eine ältere entbehrlich gemacht werden soll (Kant (1790) 1998).

[102]Husserl never discussed Eberhard in his manuscripts (including his published work), although he has read at least Kant's side of the polemic, as is witnessed by his pencil lines and marginal comments in one of his editions of Kant's writings. We are grateful to Robin Rollinger at the Husserl Archive in Leuven for investigating this for us.

> extent in which they are interpretable as our own constructions of our own mind, or at least, *can be completely given in mathematical intuition. For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them*, there exists, I believe, a satisfactory foundation of Cantor's set theory in its whole original extent and meaning, namely axiomatics of set theory interpreted in the way sketched below. (Gödel 1964, 262)

First, Gödel trades in the phrase 'penetrable by our intuition' for 'being given in mathematical intuition'. Charles Parsons has raised the question whether 'Gödel saw this as more than a stylistic change' (Parsons 1995, 57n25); it can be answered that the latter phrase is highly idiomatical in Husserl's work, while the former is not.

Second, in the 1964 version, Gödel seems to insist on intuition (of the concepts), while at the same time denying the necessity of intuition of each individual mathematical object. Both the positive and the negative claim go with Husserl well. Husserl admitted the need for ideas in a Kantian sense.[103] In particular, in Sects. 143 and 144 of *Ideas I* he points out that not even individual *physical* objects can be given completely (adequately) in intuition. We always only see a side (*Abschattung*) of a physical object. 'But,' Husserl writes, 'perfect givenness is nevertheless predesignated as "Idea" (in the Kantian sense)' (Husserl 1983, 342).[104] This Idea is a system that prescribes in what ways the object can appear to us, and depends on the type (or essence) of the object. This Idea as such, unlike the infinite series it determines, can be given in intuition. And Husserl explains in the next section,

> When the presentive intuition is one of something transcendent to it, then something objective cannot become adequately given; only the idea of that something objective can be given, or rather of its sense and its 'epistemic essence',[105] and consequently there can be given an a priori rule for law-conforming infinities of inadequate experiences. (Husserl 1983, 343, translation modified)[106]

In mathematics, uncountability and, in a sense, even countability beyond certain bounds, imposes limits on intuitive accessibility in an entirely analogous way. Idealisations will be involved; they are conspicuous in the familiar explanations of

[103]⟦Addition MvA: I have now come to think that, on the one hand, a foundation of classical mathematics in which intuition of the objects is central, such as Gödel's, is indeed forced to take recourse to Kantian ideas also in that (non-physical) context; but that, on the other hand, Husserl's doctrine of purely categorial objects as developed after *Ideas I* precludes doing just that. See van Atten (2010, 78–79) for further discussion (Sect. 12.3 of the reprint in this volume).⟧

[104] Aber als 'Idee' (im Kantischen Sinn) ist gleichwohl die volkommene Gegebenheit vorgezeichnet. (Husserl 1950c, 351/Husserl 1976a, 331)

[105]*Note MvA and JK.* The 'epistemic essence' is like the sense but includes aspects of evidence.

[106] Wo die gebende Anschauung eine transzendierende ist, da kann das Gegenständliche nicht zu adäquater Gegebenheit kommen; gegeben sein kann nur die Idee eines solchen Gegenständlichen, bzw. seines Sinnes und seines 'erkenntnismäßigen Wesens' und damit eine apriorische Regel für die eben gesetzmäßigen Unendlichkeiten inadäquater Erfahrungen. (Husserl 1950c, 352/Husserl 1976a, 332)

the iterative concept of set. Disputes will then be about what idealisations make sense and are admissible (which brings us back to the above discussion of the schools in the foundations of mathematics).

Finally, in the new version Gödel specifies that the intuition he is discussing is ‘mathematical’ intuition. This is significant in the light of the phenomenological doctrine that to each realm of objects is associated a different kind of intuition; this doctrine is just a consequence of the noetic-noematic correlation. In his copy of *Ideas I*, Gödel underlined and marked with a vertical line in the margin the following passage:

> We have had to emphasize many times that each species of being has, owing to its essence, *its* modes of givenness and with that its own cognitive method. It is countersensical to treat their essential peculiarities as deficiencies, let alone to count them among the sort of adventitious, factual deficiencies pertaining to ‘our human’ cognition. (Husserl 1983, 187)[107]

Such an ‘essential peculiarity’ may for example be the need for certain idealisations involving Kantian ideas; moreover, such an idealisation may be stronger than a constructivist would allow for. In his recent discussion of Gödel’s Platonism, Eckhart Köhler (2002a, 104) claims that Husserl, like Brouwer, limited his notion of intuition to what is constructive. Although some followers of Husserl, such as Oskar Becker, indeed at some point did so, it is not at all clear that Husserl did, and in fact Dieter Lohmar has convincingly argued that he did not (Lohmar 1989, 194–195n15). Husserl never aimed at revising classical mathematics.[108] Gödel recognised this; in a draft for a letter to Gian-Carlo Rota, dated September 7, 1972, he wrote:

> As far as Husserl’s reform of logic is concerned I don’t think he aimed at the rejection of anything in today’s mathematical logic, but rather at supplementing it and laying its foundations deeper. (Gödel Papers, 2c/141, 012029)

6.7 Gödel’s Assessment of His Philosophical Project

We have argued that Gödel resorted to Husserl’s transcendental phenomenology as a systematic means to combine the two strands of thought he had adopted earlier, his strong realist view of mathematics and the Leibnizian framework that put subjectivity in central position (monadology). On a very general level this

[107] Jede Seinsart, wir haben das schon mehrfach betonen müssen, hat wesensmäßig *ihre* Gegebenheitsweisen und damit ihre Weisen der Erkenntnismethode. Wesentliche Eigentümlichkeiten derselben als Mängel behandeln, sie gar in der Art zufälliger, faktischer Mängel ‘unserer menschlichen’ Erkenntnis anrechnen, ist Widersinn. (Husserl 1950c, 191/1976a, 176)

[108] A different question is whether Husserl’s intended non-revisionism is indeed forced by his philosophical premises; for an argument that it is not, see van Atten (2002).

attempt at integration may or may not have succeeded.[109] Yet, in spite of Gödel's optimism in the early 1960s, concrete results did not come quickly, if at all.[110] We are not aware of any specific contribution of Gödel to phenomenology,[111] other than, arguably, his contributions to the chapter on sets in Wang's *From Mathematics to Philosophy* (Wang 1974, Chap. 6) (see our Footnote 85). Typical of Gödel's view of his philosophical project in the second half of the 1960s is a passage in his letter to Paul Cohen of April 27, 1967, in which Gödel explains why he declines Cohen's invitation to speak at a conference:

> For many years, my own thinking has moved along lines entirely different from those of the conference and even of a, perhaps envisaged, philosophical section of it. Namely, I have been trying first to settle the most general philosophical and epistemological questions and then to apply the results to science. On the other hand I have not yet advanced far enough to make such applications. For this reason I have not participated actively in the recent most interesting developments and am, at the present moment, not in a position to participate in their continuation. (Gödel Papers, 1b/32, 010417.6)

At the same time, he did not think that the proof of the independence of CH posed a threat to his realist program by suggesting a certain relativism in set theory. To Church, who did interpret the independence proof along relativistic lines, he wrote on September 29, 1966:

> You know that I disagree about the philosophical consequences of Cohen's result. In particular I don't think realists need expect any permanent ramifications (see bottom of p. 8)[112] as long as they are guided, in the choice of the axioms, by mathematical intuition and by other criteria of rationality. (Gödel Papers, 1b/26, 010334.36)[113]

Indeed, in the 1970s Gödel reaffirmed his belief in phenomenology. In a draft letter to Rota from 1972 that we already quoted from, Gödel says that 'his [i.e., Husserl's] transc[endental] phen[omenology], carried through, would be nothing more nor less than Kant's critique of pure reason transformed into an exact science', which 'far from destroying trad[itional] metaph[ysics] ... would rather prove a solid foundation for it' (Gödel Papers, 2c/141, 012028.7).[114] Gödel recognised the

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[109]As far as the integration of phenomenology with monadology is concerned, one finds optimism about it in the essays by Mahnke (1917) and Cristin (1990), and those by Cristin and Poser in Cristin and Sakai (2000) and Mertens (2000), on the other hand, has argued that a phenomenological monadology is impossible in principle.

[110]Kreisel complains about this in his letter to Gödel of April 12, 1969 (Gödel Papers, 2a/92, 011266).

[111]⟦Addition MvA: But now see van Atten (2014) (Sect. 11.3.5.6 in this volume).⟧

[112]*Note MvA and JK.* Of Church's manuscript for his talk at the International Congress of Mathematicians, Moscow 1966, published in its proceedings in 1968 (Church 1968).

[113]⟦Addition MvA: In the context of Husserl's transcendental idealism, the role of these other criteria is, and can only be, to inform provisional choices, until intuitiveness is arrived at or, in a higher-order intuition, seen to be unattainable. See also Chap. 12, Sect. 12.3 in this volume.⟧

[114]Combined with Gödel's conviction, expressed in the draft letter to Rota quoted at the end of Sect. 6.6.3, that Husserl was not a revisionist in mathematics, and moreover with Gödel's continued attempts at defending realism in his conversations with Wang, this is our reason not to believe

difficulty of this task and was aware of his lack of results. But, he pointed out to Wang in 1972, there is no reason to abandon the project:

> It is not appropriate to say that philosophy as a rigorous science is not realizable in the foreseeable future. Time is not the main fact; it can happen any time when the right idea appears. (Gödel Papers, 3c/209, 013184)

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that, rather than a phenomenologically grounded realism, the 1972 version of the Dialectica paper should be considered Gödel's final philosophical view, a possibility that Sol Feferman suggested to us. We think that in that paper, Gödel is showing his talent for penetrating a philosophical position that is not his own, as he had done before in his papers on Kant.

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Chapter 7
Gödel, Mathematics, and Possible Worlds

Mark van Atten

Abstract Hintikka has claimed that Gödel did not believe in possible worlds and that the actualism this induces is the motivation behind his Platonism. I argue that Hintikka is wrong about what Gödel believed, and that, moreover, there exists a phenomenological unification of Gödel's Platonism and possible worlds theory. This text was written for a special issue of *Axiomathes* on the philosophy of Nicolai Hartmann, which explains the two introductory paragraphs.

Keywords Actualism • Existence • Kurt Gödel • Nikolai Hartmann • Jaakko Hintikka • Edmund Husserl • Gottfried Wilhelm Leibniz • Platonism • Possibility • Possible worlds • Transcendental subjectivity

On a cold but sunny afternoon a few years ago, I was sipping a hot chocolate in Gian-Carlo Rota's office, holding my office hours as TA in the course on Heidegger that he was giving in MIT's math department. As it began to look as if no student would stop by that day, I haphazardly reached for one of the books on the shelf nearest to me, and began to leaf through it. I stopped when I hit upon a passage where it was explained that for mathematical objects, possibility and actuality are equivalent. That I found an intriguing thought, and as it could readily be connected to work on Husserl I was then doing, I read on till it was getting dark. The next time I saw Gian-Carlo, he said that the book in question was one of his favourite philosophy books, and that nowadays its author did not get the attention he deserved. The book was *Möglichkeit und Wirklichkeit*, the author, Nicolai Hartmann.[1]

The argument I would like to put forward here is not immediately concerned with Hartmann's own writings; my excuse is that it nevertheless is based entirely on the thought of his that I learned on that afternoon. The theme, therefore, should not be untypical.

[1] ⟦This book was published in 1937; I will be citing the second edition, Hartmann 1949.⟧

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What I would like to discuss is a claim made by Hintikka in his article 'On Gödel's philosophical assumptions' (Hintikka 1998), and repeated in his little book *On Gödel* (Hintikka 2000), where it is worded thus:

> This is the motivation of Gödel's Platonism. He has to construe the meaning of all the crucial concepts by reference to the actual world which in effect means finding a slot for logical and mathematical objects in the actual world and construe logical and mathematical truths as being about this world of ours. (Hintikka 2000, 50)

Hintikka reasons that Gödel did not believe in possible worlds and therefore had to hold that there is only one valid interpretation of mathematical language:

> Leibniz was Gödel's favorite philosopher. But one central idea of Leibniz's was never taken up by Gödel: the idea of possible worlds. This is one of the many indications of Gödel's actualism. He agreed for instance in so many words with Russell's statement that logic deals with the real world quite as much as zoology, albeit with its more abstract features. (Hintikka 2000, 47–48)

Hintikka says that Gödel was an 'actualist' or 'one-world theorist'.[2] But this claim about Gödel's attitude towards possible worlds is simply false. In his paper 'A remark about the relationship between relativity theory and idealistic philosophy' from 1949, Gödel presents a modal argument against the objectivity of change. He reasons from the absence of objective lapses of time in 'certain possible worlds' to the same absence in 'our world' (Gödel 1949a, 562). For an extended discussion of Gödel's argument here, I refer to Yourgrau's monograph (Yourgrau 1999).

The reference to Russell is of course to the statement quoted by Gödel in his paper 'Russell's mathematical logic' (Gödel 1990, 120): 'Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features' (Russell 1919, 169). To agree that logic (which, on Russell's understanding, is identical to mathematics) deals with more abstract features of the real world does not, by itself, imply actualism, for one could hold that there are different possible worlds but that their abstract features are all the same. (In other words, Gödel may be thinking of 'reality' in a more encompassing sense than Hintikka has use for.) Hintikka thinks this is not the right way to understand Gödel's position, as is clear from the following passage:

> Gödel's philosophical beliefs implied that model theory was, if not impossible, then largely irrelevant to philosophy and for the purposes of deeper theoretical understanding of the foundations of mathematics. The reason is that to do model theory is to consider a variety of different interpretations of the nonlogical concepts one is dealing with. But according to Gödel there is in effect only one relevant interpretation of mathematical language, viz. the one in which mathematical terms refer to the citizens of the Platonic realm of objects within our actual world. (Hintikka 2000, 49)

Hintikka's own position seems to be the following. Think of a world as comprising both empirical and abstract objects. Different possible worlds may have different

[2]The term 'actualism' is ambiguous, depending on whether one makes a distinction between the existence of a world and its obtaining. If one does, one could be an actualist and still believe in possible worlds by holding that possible worlds exist but only the actual world obtains. Evidently, this is not Hintikka's use here.

abstract regions. Thus, mathematical language may receive different but equally valid interpretations, according to the world one is referring to, and Platonism is false. A mathematical object may be possible, yet not actual: in that case, it belongs to a possible world that does not coincide with the actual.

Hartmann rejects this idea. He holds that for mathematical objects, possibility (*Möglichkeit*) and actuality (*Wirklichkeit*) are equivalent notions. In this note, I want to explain in what sense Gödel seems to share this view of Hartmann's. In particular, when Hintikka writes,

> Gödel's actualism is a case in point. Gödel does not dare to leave the familiar ground of the actual world and venture to speculate about unrealized possibilities. (Hintikka 2000, 50)

I claim that it wouldn't have made a difference if Gödel had 'dared to leave the familiar ground of the actual world': his mathematics would have remained the same.[3]

To set the stage for a justification of this claim, consider Gödel's statement, in a lecture from 1951, that one of the distinguishing features of mathematical objects is that

> they can be known (in principle) without using the senses (that is, by reason alone) for this very reason, that they don't concern actualities about which the senses (the inner sense included) inform us, but possibilities and impossibilities. (Gödel *1951, 312n3)

To make a connection between mathematical objects and modality is a familiar theme in the philosophy of mathematics. One finds it, for example, in Kant, Husserl, Parsons, and indeed Hintikka himself. Their positions differ from one another in their answer to the question what the possibilities and impossibilities pertain to.

Gödel, at the time of the lecture from which I just quoted, found it difficult to articulate an answer to this question. If he had been able to, he would have found a positive answer to the question what mathematics is. But in February 1959, in a letter to Paul Arthur Schilpp, he names his inability to come up with a satisfactory answer as the reason to withdraw his contribution to the Carnap volume in the *Library of Living Philosophers* (Gödel 1995, 324). He started to read Husserl carefully in the same year (Wang 1987, 121), and from his 1961 essay on the modern development of the foundations of mathematics (Gödel *1961/?), it is clear that he hoped to find an answer by phenomenological methods. Of particular relevance in this respect is that in the 1960s he advised logicians to study Husserl's sixth *Logical Investigation* (Wang 1996, 80). It is there that Husserl deals with the relation between the possibility and actuality of purely categorial objects. They are a subset of the ideal objects, and they are the objects of (pure) mathematics.

[3]Yourgrau (1999, 44) writes that 'for a mathematical Platonist like Gödel, the mere possibility of a formal structure would, I take it, imply its actual (mathematical) existence'. I am in broad agreement with this statement but think 'possibility' needs qualification; moreover, if we take Gödel's turn to phenomenology into account, there is room for a substantial argument for the position thus qualified.

When it comes to ideal objects, Hartmann and Husserl both distinguish logical possibility from conceptual possibility.[4] In modern language, they can be defined as follows.

Logical possibility. An object is logically possible exactly if a corresponding existence statement has a model. Hartmann's term is 'logische Möglichkeit' (Hartmann 1949, Sect. 41); Husserl's, 'formale Möglichkeit' (Husserl 1984a, Sect. 14).

Hartmann mentions a square circle as an example of an object that is logically possible in this sense (provided we take 'square' and 'circle' as primitive terms). 'There is a square circle' has the form $\exists x(Px \wedge Qx)$, which has a model. This will not be the intended model, but that is no longer a matter of just logic. There is no contradiction in the logical form. Similarly, an example of an object that is not logically possible is something that is round and not round: $\exists x(Px \wedge \neg Px)$ has no models, it is a formal contradiction.

Conceptual possibility. An object is conceptually possible exactly if it is logically possible and moreover there is no material contradiction involved. Hartmann's term is 'Wesensmöglichkeit' (Hartmann 1949, Sect. 41); Husserl's, 'materiale Möglichkeit' (Husserl 1984a, Sect. 14).

In this sense, a square circle is not possible. As we saw, logically it is possible, but it is a materially contradictory concept. As Hartmann puts it:

> Much is possible logically which is not possible conceptually. Logic must take 'impossible objects' into account, that is, conceptually impossible objects (a square circle); depending on the pre-given properties it recognises (in this case, of a circle) such objects are logically possible or not. The impossibility of such objects however is an impossibility of being (in this case, geometrically), not a logical impossibility.[5]

Hartmann's claim is that for ideal entities, the notions of conceptual possibility and conceptual existence (*Wesenswirklichkeit*) are equivalent:

> [Reality and unreality] do not play a distinctive role, as regards ideal being, next to the possibility of being and not being. They come together, as a matter of course, and do not add anything to the possibility of being and not being.[6]

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4 This brief discussion of notions of possibility is adapted from van Atten (2002).

5 Sehr vieles ist logisch möglich, was nicht wesensmöglich ist. Die Logik muß mit 'unmöglichen Gegenständen' rechnen, d.h. mit wesensunmöglichen (viereckiger Kreis); je nachdem, was sie an vorgegebenen Merkmalen (etwa des Kreises) anerkennt, sind solche Gegenstände für sie möglich oder nicht. Die 'Unmöglichkeit' solcher Gegenstände ist eben eine Seinsunmöglichkeit (etwa eine geometrische), nicht eine logische. (Hartmann 1949, 323)

6 [Wirklichkeit und Unwirklichkeit] spielen keine eigene Rolle im idealen Sein neben der Möglichkeit des Seins und Nichtseins. Sie sind mit ihr gesetzt, sind ein Selbstverständliches, besagen nicht 'mehr' als das Seinkönnen und Nichtseinkönnen. (Hartmann 1949, 318)

But in Husserl's transcendental idealism, a third notion is introduced, which is absent from Hartmann (as to the reason why, see van der Schaar 2001). The object must admit of being constituted with evidence by the transcendental subject. We may call this transcendental possibility:

Transcendental possibility. An object is transcendentally possible exactly if it is conceptually possible and moreover can (ideally) be constituted with full evidence.

For example, the number 3 and the set of all even natural numbers. Conceptually possible, but transcendentally impossible entities are, it would seem, non-wellfounded sets. From the point of view of Husserl's transcendental idealism, they could therefore not be accepted as existing objects. Hartmann, on the other hand, would ascribe mathematical existence to them, as long as they are not internally inconsistent.

In Hartmann's philosophy there is neither room nor need for this third notion, because he holds that the object of knowledge is not dependent on consciousness. Speaking of ideal objects in particular, Hartmann writes,

> Knowledge can in no way be implied by the existence of its object. Ideal existence is no less indifferent to knowledge of ideal truths than real existence is to knowledge of real truths.[7]

Compare this to a fundamental principle of Husserl's transcendental idealism:

> Of essential necessity (in the Apriori of unconditioned eidetic universality), to every 'truly existing' object there corresponds the idea of a possible consciousness in which the object itself is seized upon originarily and therefore in a perfectly adequate way. Conversely, if this possibility is guaranteed, then eo ipso the object truly exists' (Husserl 1983, 341).[8]

This principle clearly states a mutual dependence between object and consciousness, something that Hartmann wants to steer clear of. If one does recognise this third notion of possibility, and in a moment we will see that there are good reasons to assume that Gödel did, the way is open to the following argument to the conclusion that mathematics is the same in all possible worlds. I will first give the three premises and the conclusion, and then comment on them:

P1. For mathematical objects, transcendental possibility is equivalent to existence.
P2. To a possible world is correlated a possible subject.
P3. Transcendental subjectivity is invariant across possible subjects.
C. All possible worlds have exactly the same realm of mathematical objects.

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7 Denn Erkenntnis kann vomSein ihres Gegenstandes aus überhaupt nicht impliziert werden. Das ideale Sein ist an sich nicht weniger indifferent gegen die Idealerkenntnis, als das reale Sein gegen die Realerkenntnis. (Hartmann 1949, 464)

8 Prinzipiell entspricht (im Apriori der unbedingten Wesensallgemeinheit) jedem 'wahrhaft seienden' Gegenstand die Idee eines möglichen Bewußtseins, in welchem der Gegenstand selbst originär und dabei volkommen adäquat erfaßbar ist. Umgekehrt, wenn diese Möglichkeit gewährleistet ist, ist eo ipso der Gegenstand wahrhaft seiend. (Husserl 1976a, 329)

ad P1. Elsewhere, I have argued in detail that for Husserl's transcendental subject, transcendental possibility and actuality of mathematical objects coincide (van Atten 2002). A difference with Hartmann's principle is that the additional condition of transcendental possibility is likely to affect the range of possibility for mathematical objects, which would make Hartmann's mathematical universe richer than Husserl's. In any case, P1 does not by itself seem to contradict Hintikka's claim, for one would also have to take into consideration the phenomenology of possible worlds; this is the function of P2 and P3. That it is relevant to do this, in the case of Gödel, is clear from the following facts. From the 1961 essay, we know that Gödel advocated Husserl's phenomenology; from what he told to Wang, we know that it was in particular the later Husserl's transcendental phenomenology, introduced in *Ideas* and elaborated in later works such as the *Cartesian Meditations*:

> Gödel told me that the most important of Husserl's published works are *Ideas* and *Cartesian Meditations* (the *Paris Lectures*): 'The latter is closest to real phenomenology – investigating how we arrive at the idea of self'. (Wang 1996, 164)[9]

And Gödel therefore was aware of the problem of constitution. He said to Wang:

> The way how we form mathematical objects from what is given – the question of constitution – requires a phenomenological analysis. But the constitution of time and of mathematical objects is difficult. (Wang 1996, 301)

(Note that Gödel mentions the constitution of time and the constitution of mathematical objects in one breath; this is typical for the later Husserl but not for the early.) So an interpretation of what Gödel meant by 'possibility' of mathematical objects, in the strongest sense, in terms of transcendental phenomenology is more likely to be faithful to his intentions than one along the lines of Hartmann.

ad P2. In *Ideas I*, Sect. 47, Husserl describes the relation between the actual world and possible worlds. From an eidetic point of view, he says, we see the following:

> then the result is the correlate of our factual experience, called 'the actual world', as one special case among a multitude of possible worlds and surrounding worlds which, for their part, are nothing else but the correlates of essentially possible variants of the idea, 'an experiencing consciousness', with more or less orderly concatenations of experience. (Husserl 1983, 106)[10]

Logically, we may posit a possible world in abstraction from any possible subject, but according to Husserl the notion of such a world is empty. To abstract from any possible subject in that world is to abstract from any possible constituting consciousness; but then, as follows from what above was identified as the main

[9]The *Paris Lectures* and the *Cartesian Meditations* are not the same work, but the latter is an elaboration of the former.

[10] dann ergibt sich das Korrelat unserer faktischen Erfahrung, genannt 'die wirkliche Welt', als Spezialfall mannigfaltiger möglicher Welten und Unwelten, die ihrerseits nicht anderes sind als Korrelate wesensmöglicher Abwandlungen der Idee 'erfahrendes Bewußtsein' mit mehr oder minder geordneten Erfahrungzusammenhängen. (Husserl 1976a, 100)

principle of transcendental idealism, it no longer makes sense to say that in that possible world certain objects exist. In any meaningful possible world, a possible subject must be able to constitute its objects:

> The idea of such transcendence is therefore the eidetic correlate of the pure idea of this demonstrative experience. This is true of any conceivable kind of transcendence which could be treated as either an actuality or a possibility. (Husserl 1983, 106)[11]

From Husserl's point of view, a possible world conceived in abstraction from any possible subject surely would not provide a relevant interpretation of mathematical language. This is where Husserl's position differs from Hartmann's. In Hartmann's conception, there is no analogue of the main principle of transcendental idealism, and the notion of a possible world need not be paired to a notion of a possible subject in order to do its work.

In a well-documented pas de deux with Mohanty, Hintikka came to acknowledge that, phenomenologically, a possible world must be understood in the context of constitution and modalisation by the actual subject, and hence as correlated to a possible subject (Hintikka 1975; Mohanty 1981, 1982, 1984, 1990; Harvey and Hintikka 1991). To the extent that Gödel ascribed to transcendental phenomenology, for him the situation is the same.

ad P3. Transcendental subjectivity first of all refers to a set of eidetic features of subjectivity. A subject cannot be a subject without having these features. Husserl developed his notion of transcendental subjectivity, sometimes speaking of individual transcendental subjects and their histories, but at root always remained an anonymous core. The laws of purely categorial formation are part of that core; so at least in this respect, the transcendental subject is the same with respect to every possible world.

ad C. It follows that the possibilities and impossibilities to constitute mathematical objects are the same in every possible world. Therefore, Hintikka's interpretation of Gödel would have been unacceptable to Gödel. The reason why Gödel thinks, in Hintikka's words, 'that there is in effect only one relevant interpretation of mathematical language' is not because of actualism, but because for Gödel it makes no sense to think of mathematics as admitting of a non-trivial possible-worlds semantics (i.e., one where at least two possible worlds differ in their mathematical objects). Hintikka is of course right when he points out that in Wang's writings on Gödel, the concept of possible worlds is introduced by Wang himself and not by his subject (Hintikka 1998, 13). However, Gödel's silence on this point is no less compatible with the phenomenological interpretation offered here than with Hintikka's interpretation of Gödel as an actualist. I do not know whether Gödel was concerned with the relation between mathematics and possible worlds before his turn to phenomenology (which is not mentioned by Hintikka 1998, 2000); but there would not have been much reason to be afterward.

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[11] Die Idee dieser Transzendenz ist also das eidetische Korrelat der reinen Idee dieser ausweisenden Erfahrung. Das gilt für jede erdenkliche Art von Transzendenz, die als Wirklichkeit oder Möglichkeit soll behandelt werden können. (Husserl 1976a, 101)

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Chapter 8
Two Draft Letters from Gödel on Self-Knowledge of Reason

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Abstract In his text 'The modern development of the foundations of mathematics in the light of philosophy' from around 1961, Gödel announces a turn to Husserl's phenomenology to find the foundations of mathematics. In Gödel's archive there are two draft letters that shed some further light on the exact strategy that he formulated for himself in the early 1960s. Transcriptions of these letters are presented, together with some comments.

Keywords Kurt Gödel • Idealism • Intuition • Machine • Mind • Non-mechanical procedures • Phenomenology • Reason self-knowledge • Paul Tillich • Undecidable propositions

In his text 'The modern development of the foundations of mathematics in the light of philosophy' from around 1961, Gödel announces a turn to Husserl's phenomenology to find the foundations of mathematics (Gödel 1995, 382ff). In Gödel's archive,[1] there are two draft letters that shed some further light on the exact strategy that he formulated for himself in the early 1960s.

In the transcriptions below, all marks are Gödel's own, except for the double square brackets ⟦ ⟧, which indicate the editor's amendments, and the numerical footnote marks, which refer to the editor's footnotes. Insertions to be made (which Gödel indicated by the marks ⋁, ⩛, $\forall\!\!\!/$, $\not\forall$ and •) have not actually been carried out but left as they appear on the document. Words or phrases crossed out by Gödel are mentioned only where considered to make a relevant difference for the content. The occasional grammatical inconsistency has been left intact.

The first draft letter (Gödel Papers, 4b/30, 020514.7) is rather rough and does not mention an addressee; however, it is clearly related to correspondence with Time,

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Inc., kept in the same archive folder, about Gödel's contribution to their forthcoming book 'Mathematics' in the *Life* science library (Bergamini and Life 1963). From some of this correspondence, which we will turn to below, one gathers that this draft letter is probably from June 1963. It begins as follows:

> Before my results had been obt⟦ained⟧ it was conject⟦ured⟧ that any precisely form⟦ulated⟧ math⟦ematical⟧ yes or no quest⟦ion⟧ can be decided by the mechanical rules of logical inf⟦erence⟧ on the basis of a few math⟦ematical⟧ ax⟦ioms⟧. In 1931 I proved that this is not so. i.e.: No matter what & how many ax⟦ioms⟧ are chosen there always exist nu⟦mber⟧ th⟦eoretical⟧ yes or no questions which cannot be decided from these ax⟦ioms⟧. Comb⟦ining⟧ the proof of this result with Turing's theory of comp⟦uting⟧ mach⟦ines⟧ one arrives at the following concl⟦usion⟧: Either there exist inf⟦initely⟧ many nu⟦mber⟧ theor⟦etical⟧ qu⟦estions⟧ which the hum⟦an⟧ mind is unable to answer or the hum⟦an⟧ mind [is someth⟦ing⟧] contains an element totally different from a finite comb⟦inatorial⟧ mechanism, such as a nerve net acting like an electronic computer. [I hope I shall be able to prove on math⟦ematical,⟧ phil⟦osophical,⟧ & psychol⟦ogical⟧ grounds that the sec⟦ond⟧ alternative (is realized) holds] 1.2.3.4.5.6.7

In this draft, 'contains an element' is actually written above the bracketed 'is something' and is obviously meant as a replacement for the latter; the phrase 'on math⟦ematical,⟧ phil⟦osophical,⟧ & psychol⟦ogical⟧ grounds' is written above 'or to make it probable' which was first bracketed and then crossed out. The numbers at the end, of which '6' is circled red, refer to a list of alternative formulations of the sentence between brackets. In fact, Gödel goes on to list ten alternatives:

1. I conjecture the second alternative can be proved or be made very probable and am hoping that the work I am now engaged in will lead to a sol⟦ution⟧ of this probl⟦em⟧ ⋁ ⋁ to a verification of this conjecture
2. I believe, on ph⟦ilosophical⟧ grounds, that the sec⟦ond⟧ alternative is more probable & hope to make this evident by a syst⟦ematic⟧ developm⟦ent⟧ & verification of my phil⟦osophical⟧ views. This dev⟦elopment⟧ & ver⟦ification⟧ constitutes the primary obj⟦ective⟧ of my present work
3. I conj⟦ecture⟧ that the sec⟦ond⟧ altern⟦ative⟧ is true & that the transformation of certain aspects of trad⟦itional⟧ phil⟦osophy⟧ into an exact science will lead to its proof. I am now working toward such a transformation.
4. I conj⟦ecture⟧ that the sec⟦ond⟧ altern⟦ative⟧ is true & perhaps can be verified by a phenomenol⟦ogical⟧ investigat⟦ion⟧ of the processes of[2] reasoning
5. I conj⟦ecture⟧ that the sec⟦ond⟧ altern⟦ative⟧ is true & hope that I shall succeed in proving it or making it probable
6. I conjecture that a deep-going invest⟦igation⟧ of the[3] working of human reason will show that [not the first but] the sec⟦ond⟧ altern⟦ative⟧ holds
7. I hope to show by an invest⟦igation⟧ of the $\not\forall$ basic ideas[4] underlying all[5] our thinking ⋁[6] that the second alternative holds $\not\forall$

[2]After this word, crossed out: 'thin⟦king⟧'.

[3]After this word, crossed out: 'action'.

[4]Stricken insertion: 'and realities'.

[5]After this word, crossed out: 'reality &'.

[6]Above this mark, Gödel had written and then crossed out: 'about reality and all being'. Also 'reality' and 'about reality' occur at this position and are crossed out. See also Footnote 7.

$\forall\!\!\!|$ our thinking [& at the same time all reality about which we think][7]
$\forall\!\!\!\neq$ & that systematic but nonmechanical methods for the dec⟦ision⟧ of math⟦ematical⟧ questions exist which make it probable that any math⟦ematical⟧ yes or no question <u>can</u> be answered by the human mind
$\forall\!\!\!\not\equiv$ content & origin of the $\bigvee$[8] & of the manner in which they work in our minds

8. I hope to show that nonmechanical systematic met⟦hods⟧ for the sol⟦ution⟧ of mathematical probl⟦ems⟧ exist & thereby to make it probable that not the first but the second of the two aforementioned alternatives holds true.
9. I conjecture that not the first but the second alternat⟦ive⟧ holds & that this will become apparent[9] by a suff⟦icient⟧ly wide extension of our phil⟦osophical⟧ & math⟦ematical⟧ knowledge. I am working toward this goal
10. I con⟦jecture⟧ that it will be possible by developing nonmech⟦anical⟧ syst⟦ematic⟧ methods for the sol⟦ution⟧ of math⟦ematical⟧ problems to make it probable that not the first but the sec⟦ond⟧ alt⟦ernative⟧ holds

The main text of the draft continues:

It follows from my results that a consistency proof for any part of math⟦ematics⟧ containing nu⟦mber⟧ th⟦eory⟧ is impossible if in this proof one confines himself to <u>concrete</u> (*i.e.*, emp⟦irically⟧ meaningful) combinatorial prop⟦erties⟧ & relations of formulæ considering formulæ to be[10] finite strings of symb⟦ols⟧ without meaning.[11]

Rather[12] the use of[13] <u>abstract</u> math⟦ematical⟧ concepts (which have no [im⟦mediate⟧] emp⟦irical⟧ meaning) • which is necessary in order to succeed[14]

• $\forall\!\!\!|$ & certain immediate insights about them

$\forall\!\!\!|$ such as the concept of a 'rightly convincing proof' or of a '<u>procedure</u>' which applied to such a proof (or to a 'procedure of the I order') yields another such proof (or procedure of the I order) or the 'existence' of an integer with a given prop⟦erty⟧ no matter whether there is a way of finding it [irrespective of it's 'givenness']

An also handwritten but much neater draft of a full letter – nothing is crossed out, there are no abbreviations, and only one insertion – is Gödel Papers, 4b/30, 020517; it is addressed to Mrs. Beatrice Matthews, 'Science Reporter' at Time. The part corresponding to the rougher draft given above is only slightly different. Gödel introduces it as follows:

Having seen some of the books on science which Life publishes it occurred to me that the formulation of my results which you read to me over the phone is perhaps a little too crude

[7] Although the insertion mark $\forall\!\!\!|$ does not occur in version 1, it seems this text should be inserted where that version has the mark $\bigvee$.

[8] No separate text to be inserted here was found, but it would fit if Gödel here meant to have the phrase 'basic ideas' that appears immediately after the mark '$\forall\!\!\!\not\equiv$' in the main part.

[9] After this word, crossed out: 'or even demonstrable'.

[10] After this word, crossed out: 'merely'.

[11] 'Finite strings' is a replacement for '[shapes]'. The final part originally read: 'formulæ⟦,⟧ which are considered as physical shapes of symb⟦ols⟧ without meaning'.

[12] After this word, crossed out: 'refere⟦nce⟧'.

[13] After this word, crossed out: 'certain', then 'some'.

[14] Here the draft breaks off in mid-sentence.

for the readers of these books. In particular the term ‘machine’ should be made more precise and Turing’s name should be mentioned, who first developed a general theory of computing machines.

Then follows the formulation Gödel proposes, which is very similar to that in rough draft above. But neither the final, bracketed, sentence from that draft nor one of its 10 alternative formulations is used; now he suggests only ‘Gödel hopes it will be possible to prove that the second alternative holds’ (to which version 5 above comes closest). Presumably this draft is (nearly) identical to the letter Gödel finally sent. On June 28, 1963, Beatrice Matthews wrote back to Gödel:

We appreciate your interest in the text which will accompany your picture, and I want to thank you for taking the time to write us with your suggestions. However, by the time I received your letter the book had already gone to the printers, and it was impossible to make any alterations. (Gödel Papers, 4b/30, 020518)

It is the date of this letter that suggests that item 020514.7 above is also from June. In his reply of July 8 Gödel expressed his regret in particular over the fact that Turing would not be named (Gödel Papers, 4b/30, 020521). The actual contribution by Gödel to the book as found in the caption on p. 53 is:

‘Either mathematics is too big for the human mind,’ he says, ‘or the human mind is more than a machine.’ He hopes to prove the latter. (Bergamini and Life 1963, 53)

This is likely to have been the formulation that Beatrice Matthews had read to Gödel over the phone (see above).

The second draft letter was annotated by Gödel ‘Tillich ca. June $\overline{63}$’ (Gödel Papers, 3b/188, 012868.5), and would consequently be from around the same time as the correspondence with Time. The text, with Gödel’s underlinings and including an occasional irregularity, reads as follows:

<u>Dear Prof Tillich</u>
It occurred to me that I in our conversation of last Sunday I answered one of your questions incompletely. I said that in math⟦ematical⟧ reasoning the non-comput⟦ational⟧ (*i.e.* intuitive) element consists in intuitions of[15] higher & higher infinities. This is quite true but[16] it this situation can be further analysed & then it turns out that they result (as becomes perfectly clear when these things are carried out in detail) from a deeper & deeper self knowledge of reason [to be more precise from a more & more complete rational knowledge of the <u>essence</u> of reason (of which essence the fac⟦ulty⟧ of self knowledge is itself a constituent part)] [I believe that comput⟦ational⟧ reason also results from self knowledge of reason but not from essential but factual knowledge] It seems to me that this is a verification (in the field of math⟦ematics⟧) of some tenets of idealistic philosophy.[17]

The envisaged recipient may well have been the Protestant theologian Paul Tillich, then professor of theology at the University of Chicago. Tillich was at the

[15] After this word, ‘various’ has been crossed out.

[16] After this word, ‘moreover these intuitions are not ineducable’ has been crossed out.

[17] After this, various attempts at a closing formula have all been crossed out. The last and most complete one of them reads ‘I’ll be very glad to give you further explan⟦ations⟧ as far as I am able. Sinc⟦erely⟧’.

height of his fame, travelled widely, and had contacts in Princeton. Later that year, Gödel wrote in a letter to his mother dated October 20, 1963: 'It was in any case to be expected that sooner or later use of my proof would be made for religion, since it is indeed justified in a certain sense' (Wang 1996, 45). Could this remark reflect an earlier conversation with Tillich? However, I have not yet been able to verify that the Tillich in question was indeed Paul Tillich. (If this is the case, then the difference in style between this draft and the draft letter to Time would appropriately match the difference between their intended audiences.)

As in the draft to Time, the key idea is the second incompleteness theorem. But here a slightly different aspect is thematised, namely, exactly how knowledge of consistency is arrived at. This is through self-knowledge (or reflection), in particular reflection which leads to knowledge of essential properties of reason. (See also Sects. 75 and 77–79 of Husserl's *Ideas*, Husserl 1976a). This presupposes that by reflecting on itself, reason can come to obtain mathematical knowledge. What may have been Gödel's rationale for thinking so? Gödel was an enthusiastic reader of idealistic philosophy, in particular of German Idealism and of Husserl's transcendental idealism; he considered Husserl's idealism a scientific form of the former, and as a way to a correct metaphysics (for a further analysis of this theme, see van Atten and Kennedy 2003). Tenets of idealistic philosophy that Gödel thought of as having been verified by his Incompleteness Theorems (as he says at the end of the draft to Tillich) may have been the following two: (1) There are aspects of reality that cannot be reduced to material configurations; (2) These (abstract) aspects are accessible to, and, moreover, constituted, by the mind. (In the Gibbs lecture, Gödel had already argued that they are not *created* (Gödel 1995, 314–315).) This second aspect would explain why self-reflection of reason leads to knowledge of the mathematical realm and also seems to have been on Gödel's mind when writing alternative seven in the list in the draft to Time; see also van Atten (2002). As mentioned in Footnote 16, Gödel thought of writing that 'these intuitions are not ineducable'; that idea is a consequence of his belief, expressed in for example the draft to Time, in the existence of 'systematic but nonmechanical methods' to solve mathematical problems. Something similar is operative in his manuscript from around 1961, where he thinks of phenomenology as a 'systematic method' or 'conscious procedure' for the clarification of meaning that should lead to the axioms (Gödel 1995, 383, 385).

In 1972, Gödel returned to the theme of self-knowledge of reason, this time in a way reminiscent of Kant: 'If it were true [that there exist number-theoretical questions undecidable for the human mind] it would mean that human reason is utterly irrational by asking questions it cannot answer, while asserting emphatically that only reason can answer them' (Wang 1974, 324–325).[18]

[18]See also Wang (1996, 317). For a discussion of the development of Gödel's views on the possibility of absolutely undecidable statements in the particular case of set theory, see Kennedy and van Atten (2004).

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Part III
Gödel and Brouwer

Chapter 9
Gödel and Brouwer: Two Rivalling Brothers

Mark van Atten

Abstract I look at Gödel's relation to Brouwer and show that, besides deep disagreements, there are also deep agreements between their philosophical ideas. This text was originally written in French and published in a special issue on logic of *Pour la Science*, the French edition of *Scientific American*. This accounts for its introductory character and the absence of references and footnotes. The translation and slight revision are my own.

Keywords L.E.J. Brouwer • Foundations • Kurt Gödel • Grundlagenstreit • Hilbert Program • Incompleteness theorem • Intuitionism • Language • Platonism

Luitzen Egbertus Jan Brouwer (1881–1966) and Kurt Friedrich Gödel (1906–1978) were the two deepest thinkers about the foundations of mathematics of the twentieth century. Gödel initiated the mathematical study of logic as it is practiced today, and was in that sense the first modern logician. Brouwer developed the first, coherent system of logic and mathematics that is not compatible with classical logic and mathematics, and emphasised its philosophical advantages. In doing so, both were strongly driven by philosophical convictions concerning mathematics. Their convictions differed, were not popular at the time, and are still not popular today.

It is of course a philosophical question whether mathematics needs a philosophical foundation at all; in particular the American philosophers Willard Quine and Hilary Putnam have argued, in the 1960s and after, that it does not. Today that is considered almost a commonplace. But for Brouwer, Gödel, and most of their contemporaries, it was important to have a philosophical foundation. Both Brouwer and Gödel had arrived at the essentials of their strong philosophical views

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when they were still young students, and these were highly determinative of their mathematical careers. We will see that, different though their views were, they are in some important respects very similar; and we will see that this explains the fact that Brouwer was one of Gödel's main inspirations for his celebrated Incompleteness Theorem.

Much of the background to both Brouwer's and Gödel's careers is formed by the then popular and widespread views of a German mathematician of one generation before Brouwer and two before Gödel, David Hilbert (1862–1943). According to his position called 'formalism' (a name given to it by Brouwer), mathematics could be grounded on manipulations with signs and formulas that have no intrinsic meaning; mathematics is not about any particular kind of objects that exist independently of the formulas. Certainly the formalists did not think that in practice mathematical meaning could be dispensed with, but they did think that mathematics admits of a theoretical reconstruction in terms of purely formal systems. Brouwer eventually convinced Hilbert that a theory of formal systems would have to acknowledge at least the natural numbers as genuinely mathematical content and in particular the principle of induction: without them, hardly any property of formal systems can actually be demonstrated. Particularly desirable properties for a formal system are completeness (roughly, that it can prove every formula in the language of the system that would be valid when interpreted) and consistency (that it cannot prove contradictions). Hilbert's idea was to use a simple, philosophically unproblematic part of mathematics, a part which is constructive and is called 'finitary', to prove the completeness and consistency of suitable formal systems for ever larger fragments and eventually all of classical mathematics. This project became known as 'Hilbert's Program'.

Brouwer, born in Overschie (nowadays part of Rotterdam), was of the same generation as Gödel's mother and as Gödel's closest friend later in life, Albert Einstein (both born 1879). At the age of 17, Brouwer wrote a declaration of his Protestant faith, in which one already sees the outlines of his philosophical view: only the self and its experiences really exist, without there being a reality that is independent of them (God is thought of as the origin of the self). In 1905 he elaborated this idea into a detailed world view in a little book called *Life, Art and Mysticism* (*Leven, kunst en mystiek*); in 1907, he defended a dissertation at the University of Amsterdam in which he showed how to develop mathematics on the basis of these views. As a central role is played by the pure intuition of time (as in Kant), Brouwer came to call his foundational program 'intuitionism'. The objects of mathematics are constructions that are made in the mind, starting from the pure intuition of time; no mathematical objects exist independently of the mind, and nothing outside thought determines mathematical truth. The basic intuition of time and the mental constructions based on it provide mathematics with content of its own, and this is the main difference between intuitionism and Hilbert's formalism. Moreover, these constructions are not of a linguistic nature, and this is a second difference.

In his dissertation, Brouwer made explicit the distinction between mathematics and metamathematics, as well as that between language and metalanguage. He also emphasised that the language of mathematics can be considered in isolation from its interpretation and can itself be subjected to mathematical study (something that he himself had no interest in at all, and therefore did not do). Brouwer explained these ideas to Hilbert during a shared holiday at the beach of Scheveningen in 1909; Hilbert would later make these distinctions central to his program for the foundations of mathematics. The mathematical study of the language of mathematics is also central to both Gödel's completeness and Incompleteness Theorems.

As a pause in his intuitionistic program, in the period 1909–1913 Brouwer created modern topology (resulting in for example his famous fixpoint theorem and the first correct definition of dimension). On the strength of these results he was appointed full professor (ordinarius) in Amsterdam in 1913. He then returned, for the rest of his life, to his intuitionism. A few years later he refused offers from the two top departments at the time, Göttingen and Berlin. He explained: 'I rather live here between Dutch friends to enjoy, and Dutch enemies to see through, than far away among strangers'. In the 1920s, Brouwer became one of the main participants in the *Grundlagenstreit*, the public battle over the foundations of mathematics between (mainly) formalism and intuitionism.

In 1906, while Brouwer was writing his doctoral dissertation, Gödel was born in Brünn as a citizen of the Empire of Austria-Hungary. After the First World War became Austrian and in 1924 moved to Vienna to study physics. He quickly changed to mathematics, became interested in philosophical questions, and, as he later said, around 1925 became a Platonist. This means that he came to hold the view that the objects of mathematics exist independently of our minds and of constructions made in the mind. Whereas for Brouwer mathematical truths are constructed, for Gödel they are discovered; but like Brouwer, Gödel too held that mathematics and its objects are not of a linguistic nature. Here Gödel disagreed not only with Hilbert but also with the Vienna Circle, the meetings of which he began to attend in 1926, when still a student. Several of its members were Gödel's teachers, such as Rudolf Carnap, Moritz Schlick, and Hans Hahn. These 'logical positivists' thought of logic and mathematics as languages without an independent meaning that were simply useful in organising the natural sciences. Because of his Platonist convictions, Gödel hardly ever agreed with the logical positivism that reigned in Vienna, but later in life he said that he valued these meetings because they brought him in contact with the philosophical problems and literature of the day.

Gödel's first result was the Completeness Theorem for first-order logic, first presented in his dissertation from 1929: every statement of first-order logic that is valid can be derived purely formally. His approach in the proof of this theorem exemplifies what made him distinctly modern (as Aki Kanamori has pointed out): Gödel was not only willing to make a clear separation between formal systems on the one hand and their mathematical interpretations on the other – this was a distinction that had become familiar in the preceding two decades – but he was also willing to study the formal systems themselves set-theoretically, freely using non-constructive methods when necessary. This set Gödel apart from Hilbert as well as from contemporaries such as Skolem, Bernays and Herbrand.

By the time the Completeness Theorem was published in a journal (1930), the seeds had already been sown of Gödel's next and even more famous result, the Incompleteness Theorem: 'In every consistent formal system that includes arithmetic, there are statements that the system can neither prove nor disprove'. In March 1928, at the height of the *Grundlagenstreit*, Gödel heard the two lectures that Brouwer gave in Vienna on the intuitionistic foundations of mathematics and on intuitionistic analysis. (Also in the audience, at least at the first lecture, was Wittgenstein; it is said that it made Wittgenstein return to philosophy.) A year and a half later, while drinking coffee with Rudolf Carnap in a Viennese *Konditorei*, Gödel began to explain that mathematics is inexhaustible. As Carnap reports in his diary entry for that day, December 12, 1929: '[Gödel] was stimulated to this idea by Brouwer's Vienna lecture. Mathematics is not completely formalizable. He appears to be right'. The rest of Carnap's entry shows that what Gödel had in mind was an argument from the second of Brouwer's lectures: on the one hand, the full continuum is given to us in a priori intuition, while on the other hand, it cannot be exhausted by a language with countably many expressions (because there are more than countably many real numbers or points on the line). For this reason no formalisation can exhaust mathematics, as any formalisation will leave out truths about the continuum that require further intuitions of the continuum to arrive at.

To accept this argument one has to acknowledge that there is something in the foundations of mathematics that is not of a formal nature. More generally, one has to accept that there is something in the foundations of mathematics that is not of a linguistic nature. This posed no obstacle to Brouwer, who saw mathematics as languageless constructions based on the intuition of time. And it did not pose an obstacle to Gödel either, who saw mathematics as descriptions of a Platonic realm, a realm that exists independently of the human mind and its creations (of which language is one).

Brouwer had presented ideas of the type that Gödel picked up from his Vienna lectures already in his dissertation from 1907. This goes some way to explaining why Brouwer was not surprised when more than 20 years later Gödel's Incompleteness Theorems were published. But that reaction was not particularly generous on Brouwer's part, for Gödel's actual theorems go considerably beyond Brouwer's expectations. As Gödel stressed in a letter to Ernst Zermelo from 1931 (without reference to Brouwer's lectures): 'I would still like to remark that I see the essential point of my result not in that one can somehow go outside any formal system (that follows already according to the diagonal procedure), but that for every formal system of metamathematics there are statements which are expressible within the system but which may not be decided from the axioms of that system, and that those statements are even of a relatively simple kind, namely, belonging to the theory of the positive whole numbers.'

Brouwer and Gödel agreed that the development of mathematics will always require new appeals to mathematical intuition, but they disagreed over the exact nature of what is given to us in that intuition. In other words, they disagreed over the nature of the objects that are studied in mathematics.

One might think that such a disagreement is of philosophical but no mathematical importance, and justify this by saying that what counts in mathematics is not the nature of its objects as such, but rather the relations in which these objects stand to each other. After all, $2+2=4$, whether 2, 4, the addition function, and the equality predicate are mental constructions, Platonic objects, or yet something else.

But that would be correct only up to a point. Specific views on the nature of mathematical objects may introduce, or, on the contrary, rule out, specific constraints on what mathematical objects can exist. If the Platonist's realm of objects really exists, there will be objects in it that we cannot also construct in the mind, even if we would abstract from limitations on our memory and time. As an example, one may think of infinite sets that are larger than the set of the natural numbers. The possibility that such objects exist is introduced precisely by the independence of that Platonic realm from our minds. Conversely, Brouwer came to realise, there are mental constructions that would have no equivalent among the objects in a Platonic realm (for example, choice sequences, which for their existence depend on an individual's choices in time). Moreover, it turns out that certain principles of reasoning are acceptable on the one view of mathematics but not on the other. Famously, on Brouwer's view the principle of the excluded middle loses its status as a universal law. Because of differences such as this one, of Gödel's three most famous results (completeness, incompleteness, and consistency of the axiom of choice) only the Incompleteness Theorem counts as acceptable and meaningful mathematics to an intuitionist.

Thus, with different philosophical views may come different kinds of mathematics. Gödel's view of mathematical objects is compatible with classical mathematics and in particular with set theory as founded by Cantor (also a Platonist); Brouwer's view is not. This means that acceptance of Brouwer's intuitionism obliges one to give up various parts of classical mathematics. Intuitionists will have to reconstruct analogues to these rejected parts according to their own principles. At the same time, intuitionism allows one to develop kinds of mathematics that classically are not even possible. The relation between classical mathematics and intuitionistic mathematics is a different one from the relation between classical physics and relativistic physics: Einstein's theory is an improvement over Newton's that includes the latter as a limiting case. But classical mathematics is not a special case of intuitionistic mathematics, and intuitionistic mathematics is not a special case of classical mathematics.

By the time Gödel's Incompleteness Theorems were published (1931), Brouwer's career had already been brought to a halt. Several large conflicts at the end of the 1920s had taken their mental and emotional toll: there had been battles with Hilbert over the Bologna conference, again with Hilbert over the journal Mathematische Annalen, and with Karl Menger over the priority for the correct definition of dimension. In addition, in 1929 Brouwer's mathematical notebook was stolen from him on a tram in Brussels, and he despaired of ever being able to reconstruct its contents. Brouwer later said that this loss was instrumental in the shift of his interests from mathematics to philosophy. Only after the Second World War Brouwer resumed his mathematical work again. He wrote a series of papers that exploited one of the most radical, classically unacceptable aspects of intuitionism, the notion of the 'creating subject' (*scheppend subject*). But the times had changed, and these papers hardly drew the attention they would have drawn in the 1920s. Ironically, the significant influence that Brouwer's intuitionism has in some parts of mathematics and computer science today comes from the area that he himself considered to be the least important, namely, intuitionistic logic.

Gödel's major results after the Incompleteness Theorems were in set theory. At the end of the 1930s, he showed that a disputed axiom, the Axiom of Choice, and a famous hypothesis, the Continuum Hypothesis, are both consistent with the accepted axioms for set theory (known as ZF, for Zermelo-Fraenkel). Gödel then tried to show that their negations too were consistent with ZF, which would establish that from ZF one can derive neither these two formulas nor their negations. In that case one would say that these axioms are 'independent' from ZF. But Gödel kept failing to demonstrate these independence results. In 1942 he had become so frustrated by this that (like Brouwer a decade before) he decided to stop working in mathematical logic and switched to philosophy. His particular interests were Leibniz and, from 1959 onwards, Husserl's phenomenology. The independence results that he had been seeking would eventually be established in 1963 by Paul Cohen. To some set theorists this independence suggested that there is no absolute truth in set theory: one can either accept the Axiom of Choice or accept its negation, and neither option will lead to a contradiction with ZF. Gödel, on the other hand, said that what this independence shows is just that we have to find a system for set theory that describes the Platonic realm in greater detail than ZF does. His suggestion therefore was to look for axioms to add to ZF; in recent years, this approach has gained in popularity (e.g., Woodin's work). Gödel spent most of his time in the 1960s and 1970s on philosophy, but in the late 1960s once more returned to a technical problem, that of deciding the Continuum Hypothesis; his efforts were again not successful.

In 1940, Gödel and his wife had emigrated to the United States and a few years later had become Americans. Gödel worked at the Institute for Advanced Study in Princeton, where he became Einstein's closest friend. Gödel's mother, who had remained in Vienna, was particularly proud of her son's friendship with the famous physicist, and in her letters to Kurt she frequently inquired about him.

In 1953, Brouwer made a lecture tour through Canada and the USA which included a talk in Princeton. Gödel invited Brouwer for lunch and for tea. From a letter to his mother, one gathers that Gödel did not like Brouwer at all, and this impression is confirmed by some of Gödel's friends. Gödel also told his mother that Brouwer's lectures had not been a great success, and he commented, 'for good reason!'. Nevertheless, in several respects the philosophical ethos of Brouwer and Gödel were the same: proudly at odds with the *Zeitgeist*, highly independent, stressing intuition, sceptical of the powers of language, and rejecting naturalistic, materialistic, and mechanistic philosophies. They also shared an active interest in mysticism.

Let me repeat a point I made earlier on, but now with a different emphasis. As I have mentioned, in their technical work Brouwer and Gödel were driven by their respective, equally unpopular philosophical convictions. A difference is that Gödel's convictions were convictions in support of the existing, classical mathematics. The technical results that he obtained were to a large extent motivated by Platonistic heuristics yet also acceptable to the majority of mathematicians, who did not share Gödel's philosophical ideas or who even held that philosophy cannot provide a foundation for mathematics. Brouwer's intuitionistic mathematics, on the other hand, goes against the tradition and seems to demand a philosophical motivation. To accept intuitionism means to abandon traditional practices, and this will certainly have been an obstacle to the popularisation of intuitionism, and continues to be so. Yet, Brouwer succeeded much better than Gödel in actually integrating his philosophy and his technical work. Gödel developed a fundamental world view, departing from Leibniz and using methods from Husserl, but he never quite managed to make the connection with mathematics so tight, let alone so fruitful, as Brouwer.

One's philosophical ideas may or may not be partly determined by one's psychological characteristics; but there is no doubt that a given psychological disposition goes better with certain philosophical positions than with certain others. The intuitionist Brouwer looked for the truth within himself and then imposed what he had found on the outside world; the Platonist Gödel looked for the truth outside himself and then subordinated his life to what he had found. As a consequence, Brouwer engaged with the outside world much more energetically than Gödel did. Brouwer is, and already was in his lifetime, notorious for the many scientific and political conflicts that he enthusiastically fought with fellow mathematicians and others; he travelled often and widely. Gödel, on the other hand, generally avoided political involvement, did not like to travel, respected authorities, and shunned public discussion.

Neither Brouwer nor Gödel passed away quietly. On an evening in 1966, when crossing the street, Brouwer was hit by several cars; in 1978 Gödel, after several years of increasing mental disturbance, essentially starved himself to death.

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Chapter 10
Mysticism and Mathematics: Brouwer, Gödel, and the Common Core Thesis

Mark van Atten and Robert Tragesser

Abstract We compare Gödel's and Brouwer's explorations of mysticism and its relation to mathematics.

Keywords L.E.J. Brouwer • Common Core Thesis • Kurt Gödel • The Good • Mathematics • Mysticism • Philosophy • Time awareness

10.1 Introduction

David Hilbert opened 'Axiomatic Thought' (Hilbert 1918) with the observation that 'the most important bearers of mathematical thought,' for 'the benefit of mathematics itself have always . . . cultivated the relations to the domains of physics and the [philosophical] theory of knowledge.' We have in L.E.J. Brouwer[1] and Kurt Gödel[2] two of those 'most important bearers of mathematical thought' who cultivated the relations to philosophy for the benefit of mathematics (though not only for that). And both went beyond philosophy, cultivating relations to mysticism for the benefit of mathematics (though not for that alone).

There is a basic conception of mysticism that is singularly relevant here. ('Mysticism' labels that.) That corresponds to a basic conception of philosophy ('Philosophy'), also singularly relevant here. Both Mystic and Philosopher begin

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[1] 1881–1966. For his biography, see van Stigt (1990) and van Dalen (1999, 2001b, 2012).

[2] 1906–1978. For his biography, see Wang (1987, 1996) and Dawson (1997).[3]

[3] ⟦Addition MvA: Now also Yourgrau 2005.⟧

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in a condition of seriously unpleasant, existential unease, and aim at a condition of abiding ease. For Mystic and Philosopher the way to that ease is through being enlightened about the real and true good of all things. Thus Mysticism and Philosophy are triply optimistic: there is a real, true good of all things, the Philosopher and Mystic can become enlightened about it, and being thus enlightened would give them ease.

That Enlightenment sought comes from some sort of cognitive or intelligent engagement with what we will here call 'the Good'. Some use 'the Absolute' when it seems important to emphasise that 'the Good' is unconditioned – there is nothing behind it, nothing above it. Others use 'the One'; still others, 'God'.[4] It is natural to regard the Good as somehow mind-like, or like something (permanently) in mind. It should in either case be in some way homogeneous with, or in sympathy with, our minds, for the Good must attract and support the intelligent engagement of it by our minds. In that way it can enlighten us.

The distinction between Philosophy and Mysticism is a matter of degree. Philosophy is dominated by the intention to articulate and rationally proof all claims and insights. Mysticism is not so dominated. But nevertheless, perhaps at some point close to the Good, where every step so far has been rationally proofed, the Philosopher could well have the final and most sublime enlightenment, but find that it is beyond his power or interest to articulate and rationally proof the content of that. It could be beyond his interest because the massive insight is so bright and sharp, so ineluctably clear and certain, that any rational proof at its best could only yield something comparatively darker and less compelling. Gödel attributed such an experience to various philosophers:

[4] ⟦Addition MvA: In an email to me of March 9, 2013, William Howard related the following reminiscences to me:

> Typically, we talked about Maharishi's conception of states of consciousness; namely, waking (i.e., ordinary) consciousness, transcendental consciousness, cosmic consciousness, God consciousness, and unity consciousness. Cosmic consciousness involves a separation of subjective consciousness from the Absolute. This is a matter of cognition. But then a development of the emotions takes place, in which one feels grateful to God for the gift of life, and the subjective consciousness wants to merge with the Absolute. If this merging takes place, the result is unity consciousness (a state attained by very few people, and certainly not yours truly).
>
> ...
>
> As soon as I told Gödel about God consciousness, he asked me, 'Does Maharishi believe in God?' I replied to the effect that the Vedic (i.e., ancient Indian) conception of God is not the same as the Christian conception of God, and we talked about what I knew of the matter.
>
> ...
>
> Gödel: 'I don't know about cosmic consciousness or unity consciousness, but I am in favor of God consciousness.'
>
> ...
>
> In two or three of our meetings, he expressed his strong approval of Maharishi's idea of God consciousness. I wondered whether Gödel believed in God but never felt it was appropriate to ask him. Also, he never asked *me* whether *I* believed in God. (Howard, stories 8 and 9)

⟧

> I myself never had such an experience. For me there is no absolute knowledge: everything goes only by probability. Both Descartes and Schelling explicitly reported an experience of sudden illumination when they began to see everything in a different light. (Wang 1996, 170)

Where we encounter a Philosopher making claims from out of such a moment, but without any successful rational proofing of those claims, then we can regard what he has given us as Mysticism. In what follows, we will think of the practice of Mysticism as trying to find ways to experience this Good directly. The practice of Philosophy is the attempt to describe this Good intelligently.

Below, we will describe how Brouwer and Gödel each relate mysticism and mathematics, and make a comparison. On the basis of that, we then present a partial argument against what is known as 'the common (or universal) core thesis' (CCT). CCT says that the various mysticisms in the end all are just different ways to express the same core of truths. It seems to us that the common core thesis can be analysed into two propositions:

(a) Mysticism holds that Reality is Good. Mystical practice aims to perceive this Good.
(b) This Good is objective, i.e., the same for all varieties of mysticism.

Of course (a) is only a minimal characterisation of mysticism. It leaves out most aspects of mysticism (e.g., feelings of bliss), but it seems extensionally adequate. We take it to be empirically adequate. 'Reality' with a capital 'R' has a different meaning than 'all-inclusive reality'. The latter surely is one, in the tautological sense that there can be nothing outside of it. But the Good known from mystical traditions has more meaning attached to it.

What makes CCT prima facie implausible is that among the various mystical traditions, and even within each tradition, we find so much disagreement on explicit doctrine and methodology. However, the interest of the common core thesis depends on the existence of such disagreements, for in the absence of that it would be almost trivially true.

There is a somewhat analogous case in the philosophy of science: scientific realists hold to a common core thesis with respect to scientific theories through the ages. These theories show massive disagreements; still, the realist holds, they all try to express the same objective reality.

One is reminded of a metaphor that Leibniz used: The same city may look very different depending on what direction you approach it from.

Of course, neither the analogy to scientific realism nor Leibniz' metaphor adds support to CCT. They just suggest how the thesis might be true in spite of prima facie evidence against it. The argument we want to suggest aims to weaken the case for CCT. It attempts to show that the references of Gödel's and Brouwer's terms for the Good cannot possibly be the same (Gödel speaks of 'the Absolute', Brouwer does not have a term but speaks of a return of consciousness to 'its deepest home'). This leaves open the possibility that at least one of them even does not refer at all. The Good as conceived of by Brouwer may not exist, and the Good as conceived of by Gödel may not exist. One can intend, but not establish, reference to something that doesn't exist. So an argument from the assumption that at least one of these does not exist to the conclusion that Gödel and Brouwer cannot be referring to the same thing is trivial. The case that remains is to assume that both do exist and see if you can then also reach the conclusion that they cannot in fact be the same. Therefore we will consider the latter case.

10.2 Brouwer's Mysticism

Brouwer thought that there was a 'deepest home' of consciousness (Brouwer 1949). In the deepest home, our experience oscillates between stillness and having sensations. There is no subject-object distinction there. This state Brouwer identifies with wisdom (compare Brouwer 1908C, 108; 1949, 1240). Our awareness of objects and other people arises in various stages on what he calls an 'exodus' of consciousness from the deepest home. The first step of this exodus is the result of a free-will act that introduces an awareness of time. In fact, time consciousness is a prerequisite for the awareness of objects and people (including oneself as an embodied person) and everything else in the exterior world. It is time awareness that introduces a distance between the experiencing 'I' and what is experienced. The latter recedes into the past, as a memory, while the former remains in the streaming 'now'. This is the genesis of intentionality (a word that Brouwer does not use).

Brouwer calls consciousness in so far as it exhibits intentionality 'mind' (we will, after the discussion of this particular aspect of Brouwer's philosophy, not use that word in his technical sense). Once that is in place, the mind further develops with consciousness indulging itself in organising sensations into complexes, in particular into 'causal chains' and 'things': the former being vehicles for empowering the will to control the latter. Whatever hold all of these mind-particularising contents have on the particular self, it is a hold that self maintains; the self could in principle and in practice free itself from that hold by as it were disclaiming all the relevant sensation complexes, for those complexes were adopted on the foundation of absolutely free will intrinsic to consciousness:

> Everyone can have the inner experience, that he can at will dream himself to be without time awareness and without the separation of the I and the world of perceptions, or bring about this latter separation and the condensation of individual things by his own effort.[5] (Brouwer 1929A, 154)

Mathematics, Brouwer says, is also built up from our experience of time, as in Kant – hence the name 'intuitionism' for Brouwer's philosophy of mathematics,

[5] Es kann jedermann die innere Erfahrung machen, daß man nach Willkür entweder sich ohne zeitliche Einstellung und ohne Trennung zwischen Ich und Anschauungswelt verträumen, oder die letztere Trennung aus eigener Kraft vollziehen und in der Anschauungwelt die Kondensation von Einzeldingen hervorrufen kann.

referring to the pure intuition of time. The discrete (the natural numbers) arises from our awareness of successive 'nows', the continuous (e.g., the straight line) from our awareness that time is a flow and hence there is something 'in between' the discrete 'nows'. In what Brouwer calls the unfolding of this basic intuition, all of mathematics is created. On this picture, mathematics is a creation of the individual mind. It does not describe an independent reality. It comes into being in an act of the will. In formalising mathematics, on the other hand, any possible volitional elements are precisely shut out. This is why Brouwer kept clear from formalising intuitionistic logic (as his student Heyting did), and from setting up a formal theory of the role of the subject in mathematics. As regards the latter endeavour, Stanley Rosen has aptly remarked that

> Analytical philosophy . . . objectifies the subject, or overlooks the presence of the subject in the structure of the proposition . . . This tendency is illustrated in the attempt by Kreisel and others to mathematize Brouwer's conception of the creative subject as expressing the force of mathematics, a force that cannot itself be expressed in mathematical terms. (Rosen 1980, 186)

According to Brouwer, if you look at it from the philosophical and not merely technical point of view, engaging in mathematics is one of the first things that lead consciousness out of its deepest home. Consciousness builds up its world by starting an 'exodus from the deepest home. We saw that he thinks of these building processes as operating on sequences of sensations, and that is where mathematics comes in. It is not only when doing technical work, but when just constructing 1 and 2 that you are on the wrong track, according to Brouwer! In his notebooks in which he conceived his 1907 thesis, there are many astonishing remarks on how destructive he thinks mathematics is. For example, one there finds gems such as 'One's aim in life could be the abolition of and the deliverance from all mathematics' (van Stigt 1990, 399).[6] And he meant it: in writings all through his career, Brouwer comments on how mathematics (and, based on that, the natural sciences) introduces great unhappiness in our lives and keeps us away from attaining wisdom again (by returning to the deepest home). Without time awareness, there can be no mathematics. But to be free of mathematics is exactly what we should aim for in our pursuit of the deepest home. And there is even a chance of using language to indicate mystical experiences; but not in the form of analytical (i.e., mathematically structured) prose[7]:

> Perhaps the greatest merit of mysticism is its use of language independent of mathematical systems of human collusion, independent also of the direct animal emotions of fear and desire. If it expresses itself in such a way that these two kinds of representations cannot be detected, then the contemplative thoughts – whose mathematical restriction appears as the

[6] Als levensdoel zou kunnen worden gezien: Afschaffing en verlossing van alle wiskunde. (van Stigt 1990, 394)

[7] ⟦Addition MvA: The first two passages were meant to be included in Brouwer's dissertation, but rejected by his adviser, D.J. Korteweg.⟧

> only live element in the mathematical system – may perhaps again come through without obscurity, since there is no mathematical system that distorts them. (van Stigt 1990, 409, modified)[8]
>
> The mystical writer will even be careful to avoid anything that smacks of mathematics or logic: weak minds might otherwise be easily made to believe and act mathematically outside the domain where this is required either by the community or their own struggle for life and end up in all kinds of follies. (van Stigt 1990, 409–410)[9]
>
> Nowhere in mysticism is there a thread or appropriate sequence; every sentence stands by itself and does not need another to precede or follow it. (van Stigt 1990, 122)[10]

As examples of such language, Brouwer quotes, in 1905, from Meister Eckhart and Jacob Boehme (Brouwer 1905A); and in 1948, from the Bhagavad-Gita (Brouwer 1949). The intellect has nothing to do with it. Access to the Good is only possible when the intellect is switched off. In a review (1915) of a book called *Geometry and Mysticism*,[11] Brouwer wrote:

> As the making and observing of mathematical forms in the *Anschauungs*-world is a preparation for, and a consequence of, the intellectual self-preservation of man, and since theoretical mathematics can only be defined as the activity of the intellect in isolation, and since furthermore, mystical vision only begins after the intellect has gone to sleep,[12]

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[8] En misschien is de beste qualificatie van *mystiek* een gebruik van de taal, onafhankelijk van de wiskundige systemen der verstandhouding, maar ook onafhankelijk van directe dierlijke aandoeningen van vrees of begeerte. Kleedt zij zich zodanig in, dat het lezen van voorstellingen van de beide zooeven genoemde groepen, onmogelijk is, dan kunnen misschien die contemplatieve gedachten, waarvan de in het wiskundig systeem levende, de wiskundige vereenzijdigingen zijn, weer ongetroebeld doorbreken, daar er geen wiskundig systeem is, dat ze verwringt. (Brouwer 1981B, 28)

[9] En zelfs zal de mystieke schrijver alles wat naar wiskunde of logica zweemt, zorgvuldig trachten te vermijden; anders worden zwakke geesten er allicht door gebracht tot wiskundig gelooven en wiskundig handelen buiten het gebied, waar hetzij de gemeenschap, hetzij hun persoonlijke levensstrijd het eischt, en komen zoo tot allerlei dwaasheden. (Brouwer 1981B, 29)

[10] Nergens heeft mystiek een draad of passende volgorde: elke sententie staat op zich zelf, en behoeft geen andere om vooraf te gaan of te volgen, zooals begrijpelijk voor iets wat begeleidt wat buiten den tijd is. (Brouwer 1905A, 76)

[11] ⟦Addition MvA: *Meetkunde en mystiek*, Naber 1915.⟧

[12] ⟦Addition MvA: In an email of March 7, 2013, William Howard comments: ‘One reason I am struck by Brouwer’s use of the phrase “intellect has gone to sleep” is that Gödel, during our conversations, used essentially the same phrase. In trying to understand my description of my own experience during meditation, he would say:

> Now, when the mind goes to sleep. . .

At this point, I would interrupt him and say:

> No, no, Professor Gödel, when we go to sleep, awareness decreases; whereas during meditation, awareness increases.

I have sometimes wondered whether whether I misunderstood Gödel when he said ‘when the mind goes to sleep’. Maybe it was his shorthand way of saying ‘when mental activity decreases and awareness increases’? I tend to doubt it; cf. his remark, ‘The goal of Maharishi’s system of

> practical nor theoretical geometry can have anything to do with mysticism. (van Dalen 1999, 287)[14]

We note that for Brouwer, mystical practice was a serious and solitary affair. When visiting Krishnamurti, Brouwer said to a friend: 'Oh my, this is the baby room of philosophy' (van Dalen 2001b, 324).

For Brouwer the intellect plays a negative role in spiritual life. Mathematics is a necessary step away from apprehending mystical truth to apprehending the outside world, one's own body, one's fellows.

10.3 Gödel's Mysticism

Rudy Rucker (1983, 182–183) has reported on his conversations on mysticism with Gödel. Gödel's philosophy of mathematics is called Platonism. He held that mathematical objects are part of an objective reality, and that what the mathematician has to do is perceive and describe them. Gödel once published some very brief remarks on how we have a perception of the abstract objects of mathematics in a way that is analogous to our perception of concrete objects (Gödel 1964). Rucker, seeking elucidation of these remarks, asked Gödel 'how best to perceive pure abstract possibility'. Gödel says that, first, you have to close off the other senses, for instance, by lying down in a quiet place, and, second, you have to seek actively. Finally,

> The ultimate goal of such thought, and of all philosophy, is the perception of the Absolute . . . When Plato could fully perceive the Good, his philosophy ended.[15]

Therefore, according to Gödel, doing mathematics is one way to get into contact with that Absolute. Not so much studying mathematics as such, but studying it in a particular frame of mind. This is how we interpret Gödel's remark about Plato. There is, then, no break between mathematical and mystical practice. The one is part of the other, and the good of mathematics is part of the Good.

Gödel also talked about his interest in perceiving the Absolute with his Eckermann, Hao Wang. Wang reports:

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meditation is to erase thoughts . . . '.[13] My impression is that in his view, the Good was to be attained by rational thought, not by a decrease in mental activity. In this respect, Gödel and Brouwer were polar opposites.'⟧

[13]⟦Addition MvA: See Footnote 24 in Chap. 6 in this volume.⟧

[14] Daar het maken en het bemerken van mathematische vormen in de aanschouwingswereld voorbereiding en gevolg zijn van de intellectueele zelfhandhaving der menschen, en theoretische mathesis slechts kan worden gedefinieerd als werkzaamheid van het intellect in isolement; daar verder mystieke aanschouwing eerst aanvangt, nadat het intellect is ingeslapen, kan practische noch theoretische meetkunde met mystiek iets hebben uit te staan. (Brouwer 1915, 6)

[15]The original incorrectly has 'Plautus' instead of 'Plato', but Rucker confirmed to us that this is a misprint.

> One of Gödel's recurrent themes was the importance of experiencing a sudden illumination – like a religious conversion – in philosophy. (This theme, by the way, reminds me of the teachings of Hui Neng's 'sudden school' of Zen (Chan) Buddhism in China.) In particular, Gödel believed that Husserl had such an experience at some point during the transition between his early and later philosophy. (Wang 1996, 169–170)

In the late 1950s, Gödel began to develop an interest in Edmund Husserl's phenomenology. Besides a specific application of phenomenology to the foundations of mathematics, Gödel had a broader interest. This is again related to the Absolute. To Wang he said,

> At some time between 1906 and 1910 Husserl had a psychological crisis. He doubted whether he had accomplished anything, and his wife was very sick. At some point in this period, everything suddenly became clear to Husserl, and he did arrive at some absolute knowledge. But one cannot transfer absolute knowledge to somebody else; therefore, one cannot publish it. A lecture on the nature of time also came from this period, when Husserl's experience of seeing absolute knowledge took place. I myself never had such an experience. For me there is no absolute knowledge: everything goes only by probability. Both Descartes and Schelling explicitly reported an experience of sudden illumination when they began to see everything in a different light. (Wang 1996, 169–170)

and, as we saw above,

> Later, Husserl was more like Plato and Descartes. It is possible to attain a state of mind to see the world differently. One fundamental idea is this: true philosophy is [arrived at by] something like a religious conversion. (Wang 1996, 293)

It is likely that Gödel tried to experience such an illumination or conversion. In this connection, we mention that besides books on Christianity and Islam, introductions to Buddhism, Watchtower publications, works on theosophy and some on spiritism, Gödel's personal library also contained *The Physiological Effects of Transcendental Meditation* (Wallace [1970] 1973).[16]

Rudy Rucker asked Gödel if he believed that there is a single Mind behind all the various appearances and activities of the world (Rucker 1983, 183). Gödel assented: 'yes, the Mind is the thing that is structured, but the Mind exists independently of its individual properties'. When Rucker then went on to ask Gödel if he believed that the Mind is everywhere, as opposed to being localised in the brains of people, Gödel again assented, saying, 'Of course. This is the basic mystic teaching'.[17]

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[16] ⟦Addition MvA: This had been given to him by William Howard (email of William Howard to MvA, March 16, 2013).⟧

[17] ⟦Addition MvA: William Howard wrote to me in an email of March 16, 2013, upon reading this passage:

> I was pretty interested in Gödel's reply when Rudy Rucker asked him the question about a single Mind ... Gödel agreed that Mind is everywhere. Here is a story ... It concerns a passage in Gödel's 1964 Supplement to his article on the continuum problem (Gödel 1990, 268):
>
> > Evidently the 'given' underlying mathematics is closely related to the abstract elements contained in our empirical ideas. ... but, as opposed to the sensations,

Gödel was convinced that 'the world is rational', and that this rationality can be grasped by the mind: 'There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; an this is also most highly fruitful for science' (Wang 1996, 316). For Gödel, then, the intellect has a positive role to play in spiritual life.

10.4 Comparison of Brouwer and Gödel: Mathematics and the Good

We have seen that both Gödel and Brouwer were looking for mystical experiences, in which an openness of the mind to the Absolute is operative. What is disclosed in such experiences has the air of being something imparted to the person. The imparting is preceded by a preparation or transformation of the person. The self must be brought into a condition to receive, support, and appreciate what is to be disclosed. This preparation we see mentioned by both Brouwer (the abandonment of mathematics) and Gödel (closing off the senses, etc.)

However, they made very different claims as to how what is disclosed in such experience is related to mathematics. What strikes us is how the bond between mathematics and mysticism is equally tight in Gödel and Brouwer, but that the

> their presence in us [i.e., of this data of the second kind] may be due to another kind of relationship between ourselves and reality.

Some years before I talked to Gödel, my friend Stanley Tennenbaum insisted that this was a very significant passage. As for myself, I did not, at that time, know what to make of it. After getting into T[ranscendental] M[editation], I heard Maharishi talk (on a video tape) about how, during meditation, one could tap into the Being ... that level of reality that underlies everything else. Well, here was 'another kind of relationship between ourselves and reality'. So I mentioned, to Gödel, the passage in his article, then explained Maharishi's viewpoint. To make this more vivid, I said, half humorously:

> It is as if one were able to 'plug in' to the Being, rather like accessing the electrical power supply by plugging into an electrical outlet.

I pointed to one in his office.

> Could that be what you meant?

Rather to my surprise, he said:

> Yes; that would be an acceptable way of putting it.

After the interview, I pondered his reply. I wondered if he was just humoring me. After all, what I had proposed was pretty 'far out'. In the light of what Gödel said to Rudy Rucker, it now appears to me that his remark was serious.

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signs are different so to speak. According to both, mathematics relates individual thought to ultimate reality, but Gödel thinks of a positive relation and Brouwer of a negative one.

For Gödel, doing mathematics is a way of accessing the Absolute. For Brouwer, doing mathematics precisely prohibits access to the Absolute.

Put differently, according to Gödel, mathematical experience reveals (part of) Reality; according to Brouwer, mathematical experience conceals Reality.

A mystical disclosure in the relevant sense has about it the phenomenological character of being a form of knowing or enlightened understanding; it discloses the Good, the significant, the important, fundamental values. Therefore, we may try to formulate the difference between Brouwer and Gödel in terms of the Good. What do they think about the relation of the Good and mathematics, and what do they think is the good of mathematics itself?[18]

Let us call historical mathematics, mathematics as it is now and has been standardly practiced, H-mathematics. We can speak of the good of the good of such hammers as the one that we happen to find in our toolbox. But for that good, this hammer may not be the best we could make. Similarly, it could very well happen that the good of mathematics is not best served by any H-mathematics. That is to say, mathematics at its best (given what is the good of mathematics) may be rather different than H-mathematics.

Brouwer's intuitionistic mathematics is often construed as merely an epistemological or semantical affair. But it might be better understood as a reform of H-mathematics in the direction of better serving its good. Brouwer tried to to realise a mathematics which is at once a creation of the free will for the sake of the fullest, most free, and most concrete exercise of the will. Our will and inner time are coeval, and inner time is where the will meets causality. Definitely controlling the structuring of time is the finest possible preparation for the will to exercise itself on causality through those temporal structures, which are the structures of intuitionistic mathematics. The good of mathematics, on this picture, is that it facilitates our will to power. Brouwer's program shows how a particular understanding of the good of mathematics can have a revisionary effect on mathematical practice.

Notice also that we can speak of the good of something without that good being ultimately beneficial to us, and in that way not part of the Good. This is how Brouwer speaks of the good of mathematics. It facilitates our will to power, but thereby collaborates in furthering our Fallen Condition, and in that sense is an evil. The good of mathematics does not coincide with any absolute good. As Brouwer wrote in the notebooks already mentioned, 'Mathematics and its application are sinful because of the intuition of time which is directly experienced as sinful' (van Stigt 1990, 400).[19] What Brouwer means is that it is the move of time which leads to the way out of the

[18]G.H. Hardy (1940) aims at evaluating the good of mathematics and the good of mathematics in relation to himself. But he certainly is straining to avoid mystic ways. This gives a contrast between mystical and non-mystical evaluations.

[19] Dat de wiskunde en haar toepassing zondig zijn volgt uit de direct als zondig gevoelde oerintuïtie. (van Stigt 1990, 395)

deepest home; anything that keeps you from returning there is defined as 'sinful'. However, Brouwer did acknowledge a more intrinsic yet limited or conditional goodness of parts of mathematics. For example, about classical logic he says,

> Fortunately classical algebra of logic has its merits quite apart from the question of its applicability to mathematics. Not only as a formal image of the technique of common-sensical thinking has it reached a high degree of perfection, but also in itself, as an edifice of thought, it is a thing of exceptional harmony and beauty. Indeed, its successor, the sumptuous symbolic logic of the twentieth century which at present is continually raising the most captivating problems and making the most surprising and penetrating discoveries, likewise is for a great part cultivated for its own sake. (Brouwer 1955, 116)

A yet higher beauty is that found in intuitionistic mathematics:

> But the fullest constructional beauty is the introspective beauty of mathematics, where ... the basic intuition is left to free unfolding. This unfolding is not bound to the exterior world, and thereby to finiteness and responsibility; consequently its introspective harmonies can attain any degree of richness and clearness. (Brouwer 1949, 1239)

But neither classical nor intuitionistic mathematics has a share in the ultimate or highest beauty. The best would be to abandon logic and mathematics in order to return to the deepest home: 'In wisdom, there is no logic' (Brouwer 1975, 110).[20] For Brouwer, the worth of philosophical investigation of mathematics is shown by its disclosing the relationships between the good of intuitionistic and classical mathematics, and between the good of mathematics and the Good. 'Research in foundations of mathematics is inner inquiry with revealing and liberating consequences, also in non-mathematical domains of thought' (Brouwer 1949, 1249).

We saw that for Gödel, on the other hand, the good of mathematics is part of the Good. This allows him to form, as projections from mathematical knowledge, expectations about the Good:

> One uses inductive evidence. It is surprising that in some parts of mathematics we get complete developments (such as some work by Gauss in number theory). Mathematics has a form of perfection. In mathematics one attains knowledge once for all. We may expect that the conceptual world is perfect and, furthermore, that objective reality is beautiful, good, and perfect. (Wang 1996, 316)

Such an induction would have been unacceptable to Brouwer.

In Brouwer, then, mathematics is action for volition, while in Gödel, mathematics is contemplation. Hence Brouwer's disinterest in theoretical values in mathematics (values furthering contemplative knowledge, understanding), and hence Gödel's obsession with the theoretical (contemplative) form of mathematics.[21] The contrast between the two attitudes is well illustrated by comparing this quote from Brouwer,

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[20]⟦Addition MvA: 'In wijsheid is geen logica' (Brouwer 1908C).⟧

[21]Incidentally, this distinction between views of mathematics pegged on contemplation and on volition also bears on the old issue whether mathematics is an 'art' or a 'science'. The former correlates mathematics with activities, actions, controlled volitions, cunningly skilled doings; the latter correlates mathematics with demonstration, exhibition, insightful seeing and understanding.

> Strictly speaking the construction of intuitive mathematics in itself is an action and not a science; it only becomes a science . . . in a mathematics of the second order, which consists of the mathematical consideration of mathematics or of the language of mathematics. (Brouwer 1975, 61)[22]

with Plato's statement in the *Republic*, 527a6–b1,

> They [i.e., geometers] speak in a way which is ridiculous and compulsory; for they are always talking about squaring and applying and adding as if they were doing things and were developing all their propositions for the sake of action; but, in fact, the whole subject is pursued for the sake of understanding. (Mueller 1992, 185)[23,24]

Brouwer sees the good of mathematics in its power to facilitate the action of the will in 'the world', as only – given his conception of the world as something we have constructed from organised sensations, his 'unbound by concept' world – and on a higher level, one step up, mathematics reveals the freedom of the will and its power over all presumed logical Apriori. The free becoming of mathematics instructs us in the power of free will. It can shake the supposedly logic a priori, get around even such supposed universal and necessary laws. Brouwer's debt to Schopenhauer is fully manifest (Koetsier 1998). For both, Will is prior to Intellect. The Will in its freedom can slay the 'brain children' of intellect, it can slay the very laws of logic.

Gödel sees the good of doing mathematics conceptually. It reveals the power of the logical Apriori, its universality. It pervades all of Reality, and therefore Mind cannot free itself from it. In view of this, and in stark contrast to Brouwer, Gödel plays down the freedom of the will considerably, in the following sense. To Rucker he said,

> It should be possible to form a complete theory of human behavior, i.e., to predict from the hereditary and environmental givens what a person will do. However, if a mischievous person learns of this theory, he can act in such a way so as to negate it. Hence I conclude that such a theory exists, but that no mischievous person will learn of it. . . . The a priori is greatly neglected. Logic is very powerful. (Rucker 1983, 181)

This is most revealing about the depth of the Goodness of things. It means that even though there are in principle deep freedoms, they are kept from those who would use them for evil purposes or mischief.

Brouwer and Gödel would agree that the Good is to be sought; but they would disagree on the role that mathematics could play in that search.

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22 Eigenlijk is het gebouw der intuïtieve wiskunde zonder meer een daad, en geen wetenschap; een wetenschap . . . wordt zij eerst in de wiskunde der tweede orde, die het wiskundig bekijken van de wiskunde of van de taal der wiskunde is. (Brouwer 1907, 98n1)

23 Λέγουσι μέν που μάλα γελοίως τε καὶ ἀναγκαίως· ὡς γὰρ πράττοντές τε καὶ πράξεως ἕνεκα πάντας τοὺς λόγους ποιούμενοι λέγουσι τετραγωνίζειν τε καὶ παρατείνειν καὶ προστιθέναι καὶ πάντα οὕτω φθεγγόμενοι, τὸ δ' ἔστι που πᾶν τὸ μάθημα γνώσεως ἕνεκα ἐπιτηδευόμενον. (Plato 1905, 527a6-b1)

24 ⟦Addition MvA: In the original publication, we had forgotten to include the credit for this translation, for which we apologise.⟧

10.5 A Partial Argument Against CCT

We now suggest that, if you believe strongly enough in the stability of mathematics to recognise that in spite of their differences – the differences between classical and intuitionistic mathematics – , Gödel and Brouwer are both dealing with the same subject matter (i.e., mathematics), then their two cases taken together function as an argument against the common core thesis; for what is there left for a common core of truths if according to Gödel, mathematics leads you to the Absolute, whereas according to Brouwer, the same thing leads you away from it? What gives one access to the Good according to Gödel, denies this access according to Brouwer. But if both are speaking the truth, which we assumed for the sake of argument, then this must mean that by 'the Good' they mean different things. Therefore, Brouwer and Gödel cannot be referring to the same when they speak about the Good.

Note that the argument does not show that CCT is false; but, if correct, it shows the following: as long as we don't know that Gödel's or Brouwer's position is false, there is no argument for CCT.

So we hold that an argument for CCT would have to show that the positions of Gödel and Brouwer cannot both be true. To establish that would actually be a stronger conclusion than our one, which it implies, but is not implied by. However, it seems much simpler to point out a difference in methods of access such that it precludes sameness of reference, than directly to establish the truth or falsity of these mystical positions.

We are thus trying to say something about CCT while avoiding having to make doctrinal comments about what the Absolute is really like. We refrain from that (at the cost of not being able to say something about the truth of CCT directly) and focus on methods of access to the Absolute. Given that in history there has been much doctrinal as well as methodological disagreement, we see no reason why in general a method of access argument should fare better than a doctrinal argument. The relations between alternative (alleged) methods may be so unclear or loose as to yield no argument. What makes the Gödel/Brouwer case different is that their particular implicit disagreement on method admits formulation as a sharp antithesis, and what they are disagreeing about, mathematics, is itself something very stable.

10.6 Closing Remarks

We would like to end by making the following two remarks.

First, of course one could, and usually does, engage in mathematics for its own sake, without any interest in relating it, be it positively or negatively, to mysticism. From Gödel's and Brouwer's point of view, that would probably be not unlike the possibility to perform a hymn for its own sake, without any interest in the religious meaning it may have.

The second remark is related to the first. In spite of the incommensurability of Brouwer's and Gödel's positions, their respective motivations to take the mystical turn may have much in common. Both were disgruntled with the materialistic and formalistic philosophies prevalent at their times; both thought that these philosophies could not do justice to the Good.

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Chapter 11
Gödel and Intuitionism

Mark van Atten

Abstract After a brief survey of Gödel's personal contacts with Brouwer and Heyting, examples are discussed where intuitionistic ideas had a direct influence on Gödel's technical work. Then it is argued that the closest rapprochement of Gödel to intuitionism is seen in the development of the Dialectica Interpretation, during which he came to accept the notion of computable functional of finite type as primitive. It is shown that Gödel already thought of that possibility in the Princeton lectures on intuitionism of Spring 1941, and evidence is presented that he adopted it in the same year or the next, long before the publication of 1958. Draft material for the revision of the Dialectica paper is discussed in which Gödel describes the Dialectica Interpretation as being based on a new intuitionistic insight obtained by applying phenomenology, and also notes that relate the new notion of reductive proof to phenomenology. In an appendix, attention is drawn to notes from the archive according to which Gödel anticipated autonomous transfinite progressions when writing his incompleteness paper.

Keywords Autonomous transfinite progressions • L.E.J. Brouwer • Creating subject • Dialectica Interpretation • Finitary mathematics • Kurt Gödel • Arend Heyting • Edmund Husserl • Intuitionism • Gottfried Wilhelm Leibniz • Non-mechanical procedures • Phenomenology • Proof Explanation • Proof Interpretation

11.1 Introduction

The principal topics are (1) personal contacts Gödel had with Brouwer and Heyting; (2) various influences of intuitionism on Gödel's work, in particular on the introduction of computable functional of finite type as a primitive notion; (3) archive material in which Gödel describes the Dialectica Interpretation as based on an

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intuitionistic insight obtained by an application of phenomenology; (4) archive material around the notion of reductive proof and its relation to phenomenology; and, in an appendix, (5) archive material according to which Gödel anticipated autonomous transfinite progressions when writing his incompleteness paper. A short companion paper describes archive material documenting the influence of Leibniz on the revision of the Dialectica paper (van Atten Forthcoming).

11.2 Personal Contacts

11.2.1 Gödel and Brouwer

According to Wang (1987, 80), 'it appears certain that Gödel must have heard the two lectures' that Brouwer gave in Vienna in 1928; and in fact in a letter to Menger of April 20, 1972, Gödel says he thinks it was at a lecture by Brouwer that he saw Wittgenstein (Gödel 2003a, 133). But it is not likely that on that occasion Brouwer and Gödel had much, or indeed any, personal contact. In a letter of January 19, 1967, to George Corner of the American Philosophical Society, who had solicited a biographical piece on the then recently deceased Brouwer,[1] Gödel wrote that 'I have seen Brouwer only on one occasion, in 1953, when he came to Princeton for a brief visit' (Gödel Papers, 4c/64, 021257; carbon copy)[2]; this is consistent with the above if Gödel meant that he had never actually talked to Brouwer before 1953.

In 1975 or 1976, Gödel stated that the first time he studied any of Brouwer's works was 1940 (see Gödel's draft replies to Grandjean's questionnaire, Gödel 2003, 447, 449; also Wang 1987, 17, 19). A letter to his brother Rudolf in Vienna of September 21, 1941 (Gödel Sammlung, item LQH0236598)[3] documents an attempt to buy two of Brouwer's publications, his dissertation *Over de Grondslagen der Wiskunde* (*On the Foundations of Mathematics*) and the collection of three articles *Wiskunde, Waarheid, Werkelijkheid* (*Mathematics, Truth, Reality*).[4] The attempt is of some additional interest because of the situation in which it was written:

> Now I have a big favour to ask: could you order the following two books by L.E.J. Brouwer for me at Antiquarium K.F. Koehler (Leipzig, Täubchenweg 21)? 1. Over de Grondslagen

[1] December 2, 1966.

[2] More on that visit below.

[3] Hence, after his Princeton lecture course on intuitionism of Spring 1941; according to Gödel's letter to Bernays of February 6, 1957, these were held at the Institute for Advanced Study (i.e., not in the Mathematics Department) (Gödel 2003, 144). IAS Bulletin no. 9, of April 1940, gives as dates for the Spring Term of the academic year 1940–1941 February 1 to May 1. In the letter to Bernays, Gödel mentions that there exists no transcript of the course. However, his own lecture notes still exist, and are kept in the archive in Gödel Papers, 8c/121, 040407 and 8c/122, 040408. There are related notes in 8c/123, 040409. Also the notes in 6a/54, 030077, 'Beweis d[er] Gültigkeit d[er] int[uitionistischen] Ax[iomen]' belong with these.

[4] Brouwer (1907, 1919B). The latter is a combined reprint of Brouwer (1908C, 1909A, 1912A). The one place in Brouwer's papers between 1919 and 1941 where Brouwer (1907, 1919B) are referred to together is Footnote 1 of Brouwer (1922A) (and its Dutch version Brouwer 1921).

> der Wiskunde Katalog 115 No 487 2. Wiskunde, Waarheid, Werkelijkheid Groningen 1919. They are small books, which will cost only a few Marks. I am told that German bookstores ship books to foreign addresses without ado (probably at the risk of the recipient), if they are ordered and paid for by a resident. On the other hand, from here nothing can be ordered from Germany through bookstores. Of course I make this order only in case the books are in stock. To have them searched for would come too expensive.[5]

The correspondence of the two brothers was then interrupted by the World War. Were the books sent? They are not in Gödel's personal library (although they might have disappeared from it). In the letters after the war, Gödel did not repeat the request.[6] Be that as it may, Gödel did at some point before 1952 make a detailed study of *Over de Grondslagen der Wiskunde*, as witnessed by the 13 pages of reading notes in his archive (Gödel Papers, 10a/39, 050135; on the envelope Gödel wrote '$< \overline{52}$').

When Brouwer visited Princeton in 1953, Gödel invited him twice: once for lunch and once for tea. From Gödel's remarks in a letter to his mother dated October 31, 1953 (Schimanovich-Galidescu 2002, 197), one gathers that Gödel did this because he felt obliged to. Indeed, Kreisel (1987, 146) reports that 'Gödel was utterly bored by Brouwer', in spite of the latter's 'probably genuine exuberance'. Brouwer, in turn, in a letter to Morse of January 4, 1955 (van Dalen 2011, 455), sent his best wishes to several named people at the Institute, but did not include Gödel. A more positive, though less direct, connection between Brouwer and Gödel is that the author of the monumental handbook on English grammar (Poutsma 1914–1929) that Gödel considered authoritative (Gödel 2003a, 303) was a maternal uncle of Brouwer's.

11.2.2 Gödel and Heyting

Gödel began to correspond with Brouwer's former student and then foremost follower, Arend Heyting, immediately after the Königsberg conference in 1930 that they had both attended. Like Brouwer, Heyting will not have been surprised by the incompleteness of formal systems for arithmetic, but Heyting acknowledged more explicitly the work behind it. Plans in the early 1930s for a joint book by Heyting and Gödel, which was to present an overview of contemporary research

[5]Gödel Sammlung, LQH0236598. Translation MvA. 'Jetzt habe ich noch eine grosse Bitte an Dich: Könntest Du die folgenden beiden Bücher von L.E.J. Brouwer beim Antiquarium K.F. Koehler (Leipzig, Täubchenweg 21) für mich bestellen? 1. Over de Grondslagen der Wiskunde Katalog 115 No 487 2. Wiskunde, Waarheid, Werkelijkheid Groningen 1919. Es sind kleine Bücher, die bloss ein paar Mark kosten werden. Man sagt mir dass Deutsche Buchhandlungen ohne weiteres Bücher an ausländische Adressen (wahrscheinlich auf Gefahr des Empfängers) versenden, wenn Sie von einem Inländer bestellt u[nd] bezahlt werden. Andrerseits kann man von hier aus durch Buchhandlungen nichts aus Deutschland bestellen. Natürlich mache ich die Bestellung bloss für den Fall, dass die Bücher vorrätig sind. Sie suchen zu lassen käme zu teuer.'

[6]In a letter of August 3, 1947 (Gödel Sammlung, item LQH0237199), he does ask Rudolf to find out in a bookstore whether anything had been published since 1941 by or about Leibniz.

in the foundations of mathematics, never quite materialised. Eventually, Heyting published his part separately (Heyting 1934), and Gödel never completed his.[7]

In December 1957, Gödel and Heyting met again. The occasion was a lecture tour that Heyting was making from the East to the West coast of the United States. In Princeton, Heyting gave two lectures at the Institute for Advanced Study, on Gödel's invitation (Gödel to Heyting, October 7, 1957; Heyting Papers, V57E-b-6), and one at Princeton University. The titles and dates were 'Intuitionistic theory of measure and integration', IAS, December 9; 'The interpretation of intuitionistic logic', IAS, December 10; and 'On the fundamental ideas of intuitionism', Princeton University, December 11.[8]

William Howard was at the lecture on measure theory and integration, and recalls:

> I was working at Bell Labs at the time and it was only a short drive to Princeton. Nerode mentioned that Heyting was going to give a lecture, so we went. Gödel sat in the back. Then, during the question period at the end of the lecture, he got up with his little notebook in hand and started reading out a series of questions (criticisms). The only one I remember was as follows: Can a proposition which is neither true nor false today become true tomorrow (i.e., if someone proves it tomorrow)? Gödel really did not like the idea that truth could vary from one day to the next. He really went after Heyting, who seemed to be rather taken aback. (Email William Howard to MvA, January 25, 2013.)

It is somewhat curious that Gödel should have chosen the lecture on measure theory rather than the one on the interpretation of logic, which of course he knew would be held the next day, to voice this particular criticism; perhaps Gödel wanted to make his opening shot at the earliest possible occasion.

In that lecture on logic, Heyting restricted himself to discussing what he calls the 'originally intended interpretation', i.e., what has become known as the Proof Interpretation, but should really be called Proof Explanation[9]; in particular, he did not discuss Gödel's work from the 1930s.

Gödel, in his invitation letter, had expressed the hope that Heyting would 'be able to stay in Princeton for some days in addition to those when you will be giving the lectures so that we may discuss foundational questions with you' (Gödel to Heyting, October 7, 1957; Heyting Papers, V57E-b-6). Because of Heyting's further commitments, he actually had to leave Princeton immediately. But according to a

[7]See the Gödel-Heyting correspondence, and Charles Parsons' introduction to it, in Gödel 2003a. Draft notes by Gödel for this joint project are in Gödel Papers, 7a/10, 040019.

[8]The texts of Heyting's lectures are held in the Heyting papers at the Rijksarchief Noord-Holland in Haarlem, items V57 (December 11), V57A (December 10), and V57B (December 9).

[9]As Sundholm (1983, 159) points out, in logical-mathematical contexts, 'interpretation' has come to refer to the interpretation of one formal theory in another. In contrast, the so-called Proof Interpretation (also known as BHK-Interpretation) is not an interpretation in this mathematical sense, but a meaning explanation. Gödel's Dialectica Interpretation, on the other hand, indeed is one. Note that this immediately shows that the Proof Explanation and the Dialectica Interpretation differ in kind. Of course, a mathematical interpretation may devised because one has a particular meaning explanation in mind for the formulas it yields; this was Gödel's foundational aim with the Dialectica Interpretation.

little diary that he kept of his American tour (Heyting Papers, V57E-r), he did have an otherwise unspecified discussion with Gödel on December 6 or 7, lunch with Gödel on December 8, and a discussion with Gödel on impredicative definitions on December 9.

The latter discussion may well have touched on the clause for implication in Heyting's Proof Explanation, but no notes on its actual contents seem to exist.[10] It may also have included Gödel's distinction between 'predicative intuitionism' and 'impredicative intuitionism', which later led Kreisel, Myhill and others to develop the latter (Kreisel 1968, Sect. 5; Myhill 1968, 175). Perhaps they talked about their shared conviction that constructive mathematics (understood as a foundational program) is not contained in classical mathematics, and is an altogether different subject (see Sect. 11.3.5.3 below). Another possible topic in this conversation may have been Markov's Principle.[11] Not long before Heyting's visit, Gödel had shown that if one can establish completeness of intuitionistic predicate logic relative to a so-called internal interpretation, this entails the validity of Markov's Principle.[12] Markov's Principle is rejected by most, though not all, intuitionists. Gödel's argument, which also goes through for the notions of validity defined by Beth and Kripke, therefore seems to show that the intuitionist cannot hope ever to establish completeness of intuitionistic predicate logic. This result was a motivation for Veldman and De Swart to develop alternative semantics, relative to which completeness does not entail Markov's Principle (Veldman 1976; de Swart 1976a, 1977). They treat negation not as absence of models but as arriving at a falsehood. In particular, De Swart (1977) presented a semantics that, within the limits of formalisation, seems to mirror Brouwer's conception of mathematical activity quite faithfully.

The last contact between Heyting and Gödel seems to have been in 1969, when Heyting inquired if Gödel were interested, as was rumoured, in publishing his collected works. If true, Heyting continued, he would very much like to have them appear in the series *Studies in Logic* (North-Holland), of which he was one of the editors. But Gödel replied that he actually had no such interest, and that he considered such a project not very useful, as his important papers were all readily available (Gödel 2003a, 74–75).

[10]William Howard (email to MvA, February 1, 2013) recalls:

> Re the issue of impredicativity in BHK: Gödel and I did not discuss this issue explicitly, but it was implicit in some of our discussions of my little theory of constructions (the formulæ-as-types paper [Howard (1980)], which then existed in the form of handwritten document, which I had sent to Gödel as part of my application for my sabbatical at the IAS, 1972–1973). Yes, he had obviously read the little paper.

[11]⟦The relevant form here is $\forall\alpha_{0,1}\neg\neg\exists x A(\alpha, x) \rightarrow \forall\alpha_{0,1}\exists x A(\alpha, x)$ where A is primitive recursive, and $\alpha_{0,1}$ ranges over choice sequences (not: lawless sequences) chosen from $\{0, 1\}$.⟧

[12]This result was published by Kreisel (1962, 142), who specifies that Gödel had obtained it in 1957. See for a discussion of the notion of internal validity (Dummett 2000b, Sect. 5.6).

11.3 Philosophical Contacts

Gödel recognised the epistemological advantages of constructivism, and looked for interpretations of formal systems for intuitionistic logic and arithmetic. A characteristic feature of Gödel's technical results in this area is that none of them is concerned with the intuitionists' intended interpretations, except, perhaps, in the negative sense of avoiding them. To Sue Toledo he said (at some point in the period 1972–1975) that 'intuitionism involves [an] extra-mathematical element. Namely, the mind of the mathematician + his ego', and he described intuitionism to her as 'essential a priori psychology' (Toledo n.d., 42).[13] (I will come back to that characterisation below.) This did not keep Gödel from studying specifically Brouwerian topics closely[14]; moreover, on various occasions Gödel has shown that he knew how to let (ideological) intuitionism inspire him in his own work. These will be commented on, or, in the last case, discussed at length, in the following sections:

- The Incompleteness Theorem (Sect. 11.3.1)
- Weak counterexamples (Sect. 11.3.2)
- Intuitionistic logic as a modal logic (Sect. 11.3.3)
- Continuity arguments in set theory (Sect. 11.3.4)
- Around the Dialectica Interpretation (Sect. 11.3.5)

11.3.1 The Incompleteness Theorem

According to an entry in Carnap's diary for December 23, 1929,[15] Gödel talked to him that day

> about the inexhaustibility of mathematics (see separate sheet). He was stimulated to this idea by Brouwer's Vienna lecture. Mathematics is not completely formalizable. He appears to be right. (Wang 1987, 84)[16]

On the 'separate sheet', Carnap wrote down what Gödel had told him:

> We admit as legitimate mathematics certain reflections on the grammar of a language that concerns the empirical. If one seeks to formalize such a mathematics, then with each formalization there are problems, which one can understand and express in ordinary language, but cannot express in the given formalized language. It follows (Brouwer) that mathematics is inexhaustible: one must always again draw afresh from the 'fountain of intuition'. There

[13]⟦Now Toledo 2011, 206.⟧

[14]See the index to his *Arbeitshefte* (Gödel Papers, 5c/12, 030016) and the headings in the *Arbeitshefte*, both published in English in Dawson and Dawson (2005, 156–168), as well as the remarks on Gödel and Brouwer's Bar Theorem further down in the present paper, and Footnote 109.

[15]In line 14 on p. 498 of van Atten and Kennedy 2009, read '23' for '12'.

[16] 5 3/4–8 1/2 Uhr Gödel. Über Unerschöpflichkeit der Mathematik (siehe besonderes Blatt). Er ist durch Brouwers Wiener Vortrag zu diesen Gedanken angeregt worden. Die Mathematik ist nicht restlos formalisierbar. Er scheint recht zu haben. (Köhler 2002a, 92)

> is, therefore, no characteristica universalis for the *whole* mathematics, and no decision procedure for the whole mathematics. In each and every *closed language* there are only countably many expressions. The *continuum* appears only in 'the whole of mathematics' . . . If we have *only one language*, and can only make 'elucidations' about it, then these elucidations are inexhaustible, they always require some new intuition again. (Wang 1987, 50, trans. Wang?)[17]

Brouwer's argument in Vienna had been that no language with countably many expressions can exhaust the continuum, hence one always needs further appeals to intuition (Brouwer 1930A, 3, 6). Of course, the theorems that Gödel went on to demonstrate are of a different and much more specific nature.[18]

11.3.2 Weak Counterexamples

Also in Brouwer's Vienna lectures, Gödel will have noticed Brouwer's technique of the weak counterexamples Brouwer (1929A). Gödel used this technique shortly after, and most effectively, when in 1930 he refuted Behmann's claim that classical existence proofs (not involving the uncountable infinite) can always be made constructive. (See Gödel 2003, 17; Gödel 2003a, 565–567; and Mancosu 2002.)

11.3.3 Intuitionistic Logic as a Modal Logic

Another case is Gödel's translation of 1933 of intuitionistic propositional logic into the modal logic S4 (Gödel 1933f). Troelstra (Gödel 1986, 299) has pointed out that this translation was very likely inspired by Heyting's talk at the Königsberg conference, which Gödel attended and of which he reviewed the published version (Heyting 1931; Gödel 1932f). Heyting introduced a provability operator, but chose not to develop its logic. As he explained, on the intuitionistic understanding of mathematical truth, an explicit provability operator is redundant. Gödel's idea of truth, of course, was different.

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[17] 'Wir lassen als legitime Mathematik gewisse Überlegungen über die Grammatik einer Sprache, die vom Empirischen spricht, zu. Wenn man eine solche Math[ematik] zu formulieren versucht, so gibt es bei jeder Formalisierung Probleme, die man einsichtig machen und in gewöhnlicher Wortsprache ausdrücken, aber nicht in der betroffenen formalisierten Sprache ausdrücken kann. Daraus folgt (Brouwer), dass die *Math[ematik] unerschöpflich* ist: man muss immer wieder von neuem aus dem "Born der Anschauung" schöpfen. Es gibt daher keine Characteristica universalis für die *gesamte* Math[ematik], und kein Entscheidungsverfahren für die gesamte Math. In irgend einer *abgeschlossenen Sprache* gibt es nur abzählbar viele Ausdrücke. Das *Kontinuum* tritt nur in der "gesamten Math." auf. . . . Wenn wir *nur eine Sprache* haben, und über sie nur "*Erläuterungen*" machen können, so sind diese Erläuterungen unausschöpfbar, sie bedürfen immer wieder neuer Anschauung.' (Carnap and Gödel 2002, 110) Note that Köhler, unlike Wang, does not explicitly identify this as the 'separate sheet' mentioned in the diary note; but both give the same date for it.

[18] For Brouwer's reaction to the Incompleteness Theorems, the reader is referred to Sect. 3.5 of the on-line article van Atten (2012).

11.3.4 Continuity Arguments in Set Theory

A use of intuitionistic ideas that goes beyond the heuristic is found in Gödel's work in set theory. In conversation with Hao Wang, Gödel claimed, 'In 1942 I already had the independence of the axiom of choice [in finite type theory]. Some passage in Brouwer's work, I don't remember which, was the initial stimulus' (Wang 1996, 86).[19] One can see what idea of Brouwer's Gödel was probably referring to by consulting Gödel's *Arbeitsheft* 14 which contains his notes on the proof dated 'ca. Ende März 1942'. There, Gödel uses Brouwer's continuity principle for choice sequences to define a notion of 'intuitionistic truth' for propositions about infinite sequences (e.g., (Gödel Papers, 5c/26, 030032), pp. 14–16). The principle states that if to every choice sequence a natural number is assigned, then for each sequence this number is already determined by an initial segment. By 1942, Gödel may have seen it in Brouwer's papers 1918B, 1924D1, 1924D2, and 1927B; (Gödel Papers, 9b/13, 050066) contains shorthand notes to the latter two. Unfortunately I have not been able to determine the date of these notes.[20] Gödel also described to Wang the method he had used as 'related' to Cohen's (Wang 1996, 251). In Cohen's forcing, too, truth values for propositions about certain infinite objects (generic sets) are always already determined by information about a finite part of such an object. This is of course not to suggest that Gödel invented forcing before Cohen: much more than the idea of finite approximations is needed to arrive at that.[21]

11.3.5 Around the Dialectica Interpretation

By far the closest rapprochement of Gödel to intuitionism, however, is seen in the change over the years in Gödel's conception of constructivity. It would probably be one-sided to consider this change part of intuitionism's legacy on Gödel, yet it is inextricably intertwined with his ponderings on the Proof Explanation from the early 1930s onward. Moreover this change was such that, as we will see, it actually brought Gödel closer to the Proof Explanation that he otherwise always criticised.'

[19]Gödel did not publish this result; he states his reasons in a letter to Church of September 29, 1966 (Gödel 2003, 372–373) and in a letter to Rautenberg of June 30, 1967 (Gödel 2003a, 182–183).

[20]On the otherwise empty back, Gödel wrote 'Brouwer bar theorem'; that English term was introduced only in Brouwer (1954A). But it is not excluded that Gödel made these notes before or in 1942 and then added that jotting on the back later.

[21]For a detailed analysis of the analogy between forcing and intuitionistic logic, see Fitting (1969). In fact, Cohen's development of forcing after his initial discovery was influenced by this analogy, when Dana Scott pointed out to him how it could be used to simplify his treatment of negation; see Scott's foreword to Bell (1985). Scott there also mentions the anticipation of forcing in Kreisel (1961).

11.3.5.1 Early Qualms About the Proof Interpretation

Gödel's qualms with Heyting's Proof Explanation seem to have arisen as soon as it was devised. The problem, as Gödel voices it in his Cambridge lecture in 1933, is that the clause for negation (more generally, the clause for implication) involves the notion of arbitrary intuitionistic proof in an essential way, and that this notion is too indeterminate. It does not comply with a condition that Gödel at the time posed on constructivity:

> Heyting's axioms concerning absurdity and similar notions . . . violate the principle . . . that the word 'any' can be applied only to those totalities for which we have a finite procedure for generating all their elements . . . The totality of all possible proofs certainly does not possess this character, and nevertheless the word 'any' is applied to this totality in Heyting's axioms . . . Totalities whose elements cannot be generated by a well-defined procedure are in some sense vague and indefinite as to their borders. And this objection applies particularly to the totality of intuitionistic proof because of the vagueness of the notion of constructivity. (Gödel *1933o, 53)

Gödel says – at this point – that a general notion of intuitionistic proof would only be constructively acceptable if it forms a totality that can be generated from below. An intuitionist might reply that this is the wrong demand to make. What matters to the intuitionist is that 'we recognise a proof when we see one' (Kreisel). The clause for implication (and hence that for negation) is not to be understood as quantifying over a totality of intuitionistic proofs – something that for a principled intuitionist like Brouwer or Heyting does not exist. Rather, the clause should be understood as expressing that one has a construction that, whenever a proof is produced that one recognises as a proof of the antecedent, can be used to transform that proof into a proof of the consequent. Although an intuitionist believes the notion of proof to be open-ended, this understanding of implication can be expected to work because in proofs of implications usually nothing more is assumed about a proof of the antecedent than that it indeed is one.

The prime example of an intuitionistic theorem that goes beyond that assumption is Brouwer's proof of the Bar Theorem (Brouwer 1924D1, 1924D2, 1927B, 1954A). This (classically trivial, but constructively remarkable) theorem basically says that, if a tree contains a subset of nodes such that every path through the tree meets it (a 'bar'), then there is a well-ordered subtree that contains a bar for the whole tree. Brouwer's extraction of additional information from the hypothesis that we have obtained a proof of the antecedent (i.e., that we have obtained a proof that the tree contains a bar) is based on his analysis of proofs as mental structures, and of mathematical objects as mentally constructed objects (a view wholly opposed to Gödel's). These analyses enable Brouwer to formulate a necessary condition for having a proof of the antecedent, namely, that it admit of being put into a certain canonical form Brouwer (1927B, 64); on the basis of that canonical form, a proof of the consequent is obtained. So what enables Brouwer to say something about all proofs of the antecedent is not the availability of a method that generates exactly these, but an insight into their mental construction that yields a necessary condition on them. In effect, Brouwer deals with 'such vast generalities as "any proof"' by presenting a transcendental argument; for more on this, see Sundholm and van Atten (2008).[22]

In notes of 1938 for his 'Lecture at Zilsel's', Gödel states that 'Heyting's system [for intuitionistic arithmetic] violates all essential requirements on constructivity' (Gödel *1938a, 99). In his Yale lecture of April 15, 1941, 'In what sense is intuitionistic logic constructive?', he phrases his objection to Heyting's Proof Explanation by saying that the clause for implication requires that

> the notion of derivation or of proof must be taken in its intuitive meaning as something directly given by intuition, without any further explanation being necessary. This notion of an intuitionistically correct proof or constructive proof lacks the desirable precision. (Gödel *1941, 190)

Then Gödel goes on to present, in an informal manner, an interpretation of HA in a system of higher types Σ, and explains the motivations behind it. This use of functionals for a consistency proof of arithmetic is the main step forward compared to the discussion of functionals in the Lecture at Zilsel's. It seems Gödel had found the heuristic how to do this on January 1, 1941.[23] In the lecture, Gödel proposes schemata for defining the constructive operations used in the definition of computable function. He admits, however, that 'a closer examination of the question in which manner the functions obtained by these two schemes are really calculable is pretty complicated' (Gödel *1941, 195); the formal proof he had depends on Heyting arithmetic and hence is, foundationally speaking, no progress.

In the Dialectica paper of 1958, Gödel decides to take the notion of computable functional as primitive. Philosophically, this marks a sea change.

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[22]In a draft note for the revision of the Dialectica paper (Gödel Papers, 9b/148.5, 040498.59), Gödel wrote: 'Finally I wish to note that the definition of a proof as an unbroken chain of immediate evidences should be useful also for Heyting's interpretation of logic. In particular $A \supset B$ can then be defined simpler, namely by requiring that a proof of $A \supset B$ is a finite sequence P_i of propositions ending with B and such that each $P_i \neq A$ is immediately evident, either by itself, or on the basis of some of the preceding propositions.' A proof of the Bar Theorem based on that explanation of $\supset$ has not yet been found; compare Gödel's Footnote d (Gödel 1990, 272).

[23]Gödel Papers, 5c/19, 030025, 12–15 in the backward direction (Arbeitsheft 7). This note is labeled 'Gentzen'. In the index to the *Arbeitshefte* (Gödel Papers, 5c/12, 030016) the reference to this note is the first entry under the heading 'Interpr[etation] d[er] int[uitionistischen] Logik', and has '(heur[istisch])' written after it. It contains, for example, a version of the proof for the validity of modus ponens interpreted by functionals. This must be the note that Wang describes, without a specific reference, in note 13 on p. 47 of Wang (1987). The date '1./I.1941' is on top of p. 12 (backward direction) of *Arbeitsheft* 7. On the same page, before the note 'Gentzen', there is one on 'Rosser Wid[erspruchs]fr [eiheits] Bew[eis]', with a horizontal line in between; that the date also holds for the second item on the page is very likely because it is also the date of another note, headed 'Jede F[u]nct[ion] d[es] eigentl[ich] intuit[ionistischen] Systems ist berechenbar'. That note is in the notebook *Resultate Grundlagen III* (Gödel Papers, 6c/85, 030118, 188–191) and states the date '1./I.1941'. It begins with a reference to p. 34 (backward direction) of *Arbeitsheft* 7, which is where the formal system Σ is defined.

11.3.5.2 The Shift to the Intensional

Kreisel writes that 'In the first 20 minutes of our first meeting, in October 1955, [Gödel] sketched some formal work he had done in the forties, and later incorporated in the so-called Dialectica interpretation (with a total shift of emphasis)' (Kreisel 1987, 104). I take the 'shift of emphasis' to be what I here call the sea change. (Note also that on p. 110, Kreisel says that, when he saw the Dialectica paper in 1958, 'the principal novelty – both absolutely speaking, and to me personally' was 'the primitive notion of effective rule ... Gödel had never breathed a word to me about his project of exploiting such a notion'.) I therefore do not agree with Feferman who comments on Kreisel's report that 'Evidently Gödel misremembered: there is really no significant difference in emphasis, though the 1941 lecture mentions a few applications that are not contained in the 1958 Dialectica article' (Feferman 1993, 220).

In the Dialectica paper, Gödel defends his new approach by pointing to a similar case:

> As is well known, A.M. Turing, using the notion of a computing machine, gave a definition of the notion of computable function of the first order. But, had this notion not already been intelligible, the question whether Turing's definition is adequate would be meaningless. (Gödel 1990, 245n6)[24]

Mutatis mutandis, Gödel could have written this about his former self. He could have written: 'In 1941 I tried to give a definition of the constructive operations used in the definition of computable functional of finite type. But, had this notion not already been intelligible, the question whether my definition of 1941 is adequate would be meaningless.' In Gödel's view, what Turing had done was to define (and in that sense see sharper) an objective concept that we had been perceiving all along, albeit less sharply (Wang 1974, 84–85). Similarly, Gödel holds in 1958, there is an objective concept of computable functional of finite type, which we may not yet (and possibly never) be able to make completely explicit, but which at the same time we see enough of to determine some of its properties:

> One may doubt whether we have a sufficiently clear idea of the content of this notion [of computable functional of finite type], but not that the axioms given [in this paper] hold for it. The same apparently paradoxical situation also obtains for the notion, basic to intuitionistic logic, of a proof that is informally understood to be correct. (Gödel 1990, 245n5)[25,26]

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[24] A.M. Turing hat bekanntlich mit Hilfe des Begriffs einer Rechenmaschine eine Definition des Begriffs einer berechenbaren Funktion erster Stufe gegeben. Aber wenn dieser Begriff nicht schon vorher verständlich gewesen wäre, hätte die Frage, ob die Turingsche Definition adäquat ist, keinen Sinn. (Gödel 1958, 283n2)

[25] Man kann darüber im Zweifel sein, ob wir eine genügend deutliche Vorstellung vom Inhalt dieses Begriffs haben, aber nicht darüber, ob die weiter unten angegebenen Axiome für ihn gelten. Derselbe scheinbar paradoxe Sachverhalt besteht auch für den der intuitionistischen Logik zugrunde liegenden Begriff des inhaltlich richtigen Beweises. (Gödel 1958, 283n1)

[26] Compare Gödel's claim, in the Lecture at Zilsel's from 1938, that the axioms of the subsystem of Heyting's logic presented there are, when interpreted intuitionistically, 'actually plausible' (*tatsächlich plausibel*; (Gödel *1938a, 100/101)).

The point Gödel makes in this footnote is reminiscent of paragraph XXIV in Leibniz' *Discours de Métaphysique* (Leibniz 1875–1890, 4:449), where attention is drawn to the fact that there are situations in which we are able to classify certain things correctly and perhaps moreover explain the grounds on which we do this, yet without having at our disposal a complete analysis of the notion of those things into primitive terms. Whether Gödel, who surely knew that passage at the time,[27] also had it in mind when writing his footnote, remains an open question.[28]

Gödel's point here goes well with the disappearance in 1958 of his earlier denial that the Proof Explanation is genuinely constructive. Indeed, Gödel's own earlier objections to the intuitionistic notion of proof would equally apply to the primitive notion he substitutes for it in 1958. Since he considers the latter to be constructive (given his by then widened notion of constructivity), these earlier objections could no longer be used to support a claim that the Proof Explanation is not at all constructive; and the 1958 paper proposes no replacements for them. The difference has become one of degree, not of kind.[29]

What might have been Gödel's reason for changing his tune on the conditions of constructivity? And when did he do this? Two as yet unpublished sources that are relevant here are Gödel's notes for the Princeton lecture course on intuitionism of Spring 1941 and his philosophical notebook *Max* IV (May 1941 to April 1942).[30] From a letter to his brother Rudolf of March 16, 1941 (Gödel Sammlung, item LQH0236556), we know that Gödel worked on the Princeton lecture course and the Yale lecture at the same time:

> Now I have again many things to do, because I give a lecture course and in addition have again[31] been invited to give a lecture, where in both cases the topic is my most recent work, which I haven't put to paper in exact form yet even for myself.[32]

[27]Notes he wrote to it when reading it in Gerhardt's edition can be found in Gödel Papers, 10a/35. As Sundholm reminded me, the same distinctions are explained by Leibniz also in an earlier text of the same period (published in the same volume of Gerhardt's edition), the 'Meditationes de cognitione, veritate, et ideis' of 1684 (Leibniz 1875–1890, 4:422–426). That text actually served as a basis for the section in the *Discours*. Leibniz takes up the theme again in the *Nouveaux Essais* of 1704 (but published posthumously, in 1765), Book II, Chap. XXXI, Sect. 1 (Leibniz 1875–1890, 5:247–248).

[28]There is a direct and documented relation between ideas of Leibniz and the revisions of the Dialectica Interpretation; see van Atten (Forthcoming).

[29]This is most explicit in the 1972 version, for the more specific interpretation in terms of reductive proof, which Gödel says is 'constructive and evident in a higher degree than Heyting's' (Gödel 1972, 276n(h)).

[30]For the archive numbers of the lecture course, see Footnote 3 above. The notebook is in (Gödel Papers, 6b/67, 030090).

[31]*Note MvA*. On November 15, 1940, Gödel had lectured at Brown University on the consistency of the Continuum Hypothesis.

[32] Ich habe jetzt wi[eder] eine Menge zu tun, da ich eine Vorlesung halte u[nd] ausserdem wieder zu einem Vortrag eingeladen bin wobei in beiden Fällen das Thema meine allerletzte Arbeiten sind, die ich noch nicht einmal für mich selbst genau zu Papier gebracht habe.

(By the end of the lecture course, there were only three students left[33]; it would be interesting to know who they were.) In his notes for the lecture course, Gödel writes, when he arrives at the concrete question of the computability of the functionals of his system Σ:

> I don't want to give this proof in more detail because it is of no great value for our purpose for the following reason. If you analyse this proof it turns out that it makes use of logical axioms, also for expressions containing quantif[iers] and since [sic] it is exactly these ax[ioms] that we want to deduce from the system Σ. (Gödel Papers, 8c/122, 040408, 61)[34]

Then follow two alternative continuations for this passage; here labelled (A) and (B). (B) is also quoted in Vol. III of the *Collected Works*, but (A) is not. (A) is written immediately below the previous quotation and reads:

> (A) So our attitude must be this that the ax[ioms] of Σ (in part[icular] the schemes of def[inition]) must be admitted as constructive without proof and it is shown that the ax[ioms] of int[uitionistic] logics[35] can be deduced from them with suitable def[initions]. This so it seems to me is a progress (Gödel Papers, 8c/122, 040408, 61–62)

Gödel crossed out (A). (B) follows immediately after it, and is not crossed out:

> (B) There exists however another proof. Namely it is possible instead of making use of the log[ical] operators applied to quantified expression[s] to use the calculus of the ordinal nu[mbers] (to be more exact of the ord[inal] nu[mbers] $< \epsilon_0$). I shall speak about this proof later on. (Gödel Papers, 8c/122, 040408, 62)[36]

Gödel then introduces (pp. 62 and 63) the idea of an ordinal assignment to terms such that with a reduction of a term comes a decrease of the ordinal. In an alternative version of (B), on p. 63[iii], here labelled (C), Gödel writes:

> (C) However it seems to be possible to give another proof which makes use of transfinite induct[ion] up to certain ord[inals] (probably up to [the] first ϵ-nu[mber] would be sufficient). (Gödel Papers, 8c/122, 040408, 63[iii])[37]

It seems that Gödel wrote (A) before (B) and (C) and that he preferred the possible solution described in the latter two. This preference for (B) and (C) however does not seem to indicate a categorical rejection of (A), for on p. 63[iv], which follows right after (C), Gödel goes on to comment:

> (D) Of course if you choose this course then the question arises in which manner to justify the inductive inference up to the certain ordinal nu[mber] and one may perhaps be of the

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[33]Letter to Rudolf Gödel, May 4, 1941 (Gödel Sammlung, item LQH0236557): 'Hier ist jetzt das Semester zu Ende u[nd] ich bin froh dass mit meiner Vorlesung Schluss ist, ich hatte zum Schluss nur mehr 3 Hörer übrig.' As mentioned in Footnote 3 above, the Spring Term had ended on May 1.

[34]See also Gödel (1995, 188).

[35]*Note MvA*. Perhaps Gödel uses the plural here because he is thinking of intuitionistic logic as it figures in different theories.

[36]See also Gödel (1995, 189).

[37]See also Gödel (1995, 189).

> opinion that the ax[ioms] of Σ are simpler as a basis than this transfinite m[ethod], by ...[38] to justify them. But whatever the opinion to this question may be in any case it can be shown that int[uitionistic] logic if applied to nu[mber] theory (and also if applied in this whole system Σ) can be reduced to this system Σ. (Gödel Papers, 8c/122, 040408, 63[iv])

(Like (A), this passage is not quoted in the *Collected Works*.)

On the one hand, as Troelstra remarks on (B) and (C), 'Since the notes do not contain any further particulars, it is not likely that Gödel had actually carried out such a proof in detail' (Gödel 1995, 189).[39] On the other hand, (A), even crossed out, and the somewhat less emphatic (D), show that Gödel already around the time of the Yale lecture, in which there is no mention of the possibility of accepting the notion of computable functional as primitive, had considered doing just that.

Gödel's philosophical notebook *Max* IV, which covers the period from May 1941–April 1942, that is, the period immediately after the Princeton course and the Yale lecture, contains the following remark:

> Perhaps the reason why no progress is made in mathematics (and there are so many unsolved problems), is that one confines oneself to ext[ensions] – thence also the feeling of disappointment in the case of many theories, e.g., propositional logic and formalisation altogether.[40]

The disappointment in the case of propositional logic that Gödel speaks of here may well be a reference to the fact that the difficulties he ran into when attempting a (foundationally satisfying) formal reconstruction of intuitionistic logic within number theory in the system Σ of the Yale lecture appeared already with the propositional connectives; his disappointment with formalisation (an act that pushes one to the extensional view) in general may have found additional motivation in his Incompleteness Theorems.

This remark from 1941–1942, with its implicit recommendation to shift emphasis to the intensional, strongly suggests that by then Gödel had indeed come to accept the solution proposed in passage (A) of the Princeton lecture course. In print, this would become clear of course only in 1958. Perhaps this view on the development of the Dialectica Interpretation would need some refinement in light of Kreisel's report that

[38]Two or three words that are difficult to read; perhaps 'which we try'?

[39]I have not attempted to reconstruct, from the *Arbeitshefte*, how far Gödel got. But he evidently did not succeed: in conversation with Kreisel in 1955, he mentioned the assignment of ordinals as an open problem (Kreisel 1987, 106), and, although it was solved for a special (but in a sense sufficient) case in Howard (1970), he did so again in a telephone conversation with Tait in 1974. But, as Tait remarks, to exploit such an assignment in a proof of normalisation, PRA together with induction up to ϵ_0 are required, so it could not serve Gödel's foundational aim (Tait 2001, 116 and its n39). (See Kanckos 2010 for a version of Howard's proof in the setting of Natural Deduction.)

[40]'Vielleicht kommt man in der Math[ematik] deswegen nicht weiter (und gibt es so viele ungelöste Probl[eme]), weil man sich auf Ext[ensionen] beschränkt – daher auch das Gefühl der Enttäuschung bei manchen Theorien, z.B. dem Aussagenkalkül und der Formalisierung überhaupt.' (Gödel Papers, 6b/67, 030090, 198) Transcription Cheryl Dawson and Robin Rollinger.

> Gödel made a point of warning me [in 1955] that he had not given any thought to the objects meant by (his) terms of finite type. The only interpretation he had in mind was formal, as computation rules obtained when the equations are read from left to right. (Kreisel 1987, 106).

But in the light of (A) and (D), it seems to me that the claim ascribed to Gödel in the first sentence here cannot be quite correct; unfortunately, we do not have Gödel's own words. (In the notes for the Princeton lectures, Gödel also defined and used a model in terms of what became known as the hereditarily effective operations HEO (Gödel Papers, 8c/123, 040409, 109ff; see also Gödel 1995, 187–188); but unlike the primitive notion of computable functionals and the method of assigning ordinals, HEO has, because of the logic in its definition, no significance for the foundational aim that Gödel hoped to achieve.)

Before continuing the discussion of the shift to the intensional, it is worth noting that, as for the purely proof-theoretical applications of the interpretation described in the Yale lecture, according to Kreisel Gödel 'dropped the project after he learnt of recursive realizability that Kleene found soon afterwards' (Kreisel 1987, 104). (Kleene told Gödel about realisability in the summer of 1941 (Kleene 1987, 57–58).) In contrast to realisability, the functional interpretation lends itself to an attempt to make the constructivity of intuitionistic logic (within arithmetic) more evident, and, as I have tried to show, that was Gödel's purpose for it already by 1941 or 1942. This raises the question why Gödel waited until 1958 to publish these ideas. In an undated draft letter to Frederick W. Sawyer, III (written after February 1, 1974, the date of Sawyer's letter to which it is a reply), Gödel says:

> It is true that I first presented the content of my Dialectica paper in a course of lectures at the Institute in Spring   and in a talk at Yale in   . There were several reasons why I did not publish it then. One was that my interest shifted to other problems, another that there was not too much interest in Hilbert's Program at that time. (Gödel 2003a, 211; spaces left open by Gödel)[41]

The shift to the intensional had its first effect in print soon, in Gödel's remarks on analyticity in the Russell paper of 1944 (Gödel 1944, 150–153), in particular in the last sentence of its Footnote 47:

> It is to be noted that this view about analyticity [i.e., truth owing to the meaning of the concepts] makes it again possible that every mathematical proposition could perhaps be reduced to a special case of $a = a$, namely if the reduction is effected not in virtue of the definitions of the terms occurring, but in virtue of their meaning, which can never be completely expressed in a set of formal rules.

There are a number of later echoes of the remark of 1941 or 1942 in Gödel's writings, published and unpublished. I mention three that are directly related to Gödel's development of the Dialectica Interpretation. The first is Kreisel's report that, when in October 1955 Gödel explained the formal part of the functional

[41] At the beginning of the Yale lecture, Gödel said that 'the subject I have chosen is perhaps a little out of fashion now' (Gödel 1995, 189); and he told Wang in April 1977 that at the Yale lecture, 'nobody was interested' (Wang 1996, 86).

interpretation to him, Gödel added a warning about the '*Aussichtlosigkeit*, that is, hopelessness of doing anything decisive in foundations by means of mathematical logic' (Kreisel 1987, 107; 104 for the date). The relation between (existing) mathematical logic and extensionality that one must see in order to connect the remark of 1941 and this warning is made explicit in the second echo. It occurs in Gödel's letter to Bernays of July 14, 1970, concerning the revision of the Dialectica paper: 'The mathematicians will probably raise objections against that [i.e., the decidability of intensional equations between functions], because contemporary mathematics is thoroughly extensional and hence no clear notions of intensions have been developed' (Gödel 2003, 283). The third echo also has to do with that revision, and is a draft for part of note k (the later note h in the *Collected Works* (Gödel 1990, 275–276), but in the version published there Gödel had decided not to include this):

> This note (and also some other parts of this paper) constitutes a piece of 'meaning analysis', a branch of math[ematical] logic which, although it was its very starting point, today is badly neglected in comparison to the purely math[ematical] branch which has developed amazingly in the past few decades. (The reason of this phenomenon doubtless is the antiphil[osophical] attitude of today's science.) (Gödel Papers, 9b/145, 040462)[42]

Indeed, Kreisel has observed that, compared to the 1930s, 'later Gödel became supersensitive about differences in meaning'. He illustrated this, appropriately, by the contrast in attitude between Gödel's remark in 1933 that intuitionistic arithmetic involves only 'a somewhat deviant interpretation' from its classical counterpart and the caveat in 1958 that 'further investigation is needed' to determine to what extent the Dialectica Interpretation can replace the intuitionistic meanings (Gödel 1933e, 37/1986, 295, 1958, 286/1990, 251; Kreisel 1987, 82, 104–105, 159; see also Footnote 22 above).

The shift in Gödel's view described here constitutes a remarkable *rapprochement* with intuitionism, which by its very nature takes intensional aspects to be the fundamental ones in mathematics. On both Gödel's new view and the intuitionistic one, foundational progress will therefore have to come mainly from informal analysis of intuitive concepts. To that end, Gödel around 1959 made an explicit turn to phenomenology as a method (Gödel *1961/?; van Atten and Kennedy 2003),[43]

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[42]Compare also Gödel to Bernays, September 30, 1958: 'Kreisel told me that in your lectures in England you discussed the combinatorial concept of set in detail. I very much regret that nothing about that has appeared in print. Conceptual investigations of that sort are extremely rare today.' (Gödel 2003, 157) ('Kreisel erzählte mir, dass Sie in Ihren Vorträgen in England den kombinatorischen Mengenbegriff näher besprochen haben. Ich habe sehr bedauert, dass darüber nichts in Druck erscheinen wird. Begriffliche Untersuchungen dieser Art sind ja heute äusserst selten', Gödel 2003, 156.)

[43]In a letter to Gödel of June 17, 1960, written after a visit to him, Sigekatu Kuroda wrote: 'It was my great pleasure also that I heard from you that you are studying Husserl and you admired his philosophy, which was the unique philosophy that I devoted rather long period and effort in my youth. I hope I have a chance some day to speak with you about Husserl. As you are doing now, I would like to recollect Husserl's philosophy after returning to my country.' (Gödel Papers, 01/99, 011378) Note that by that time Kuroda had published philosophical and technical work on intuitionistic logic, notably Kuroda (1951), in which he moreover says (p. 36) that he shares

and encouraged Kreisel's developing and advocating the notion of informal rigour (Kreisel 1967b) (at the same time warning him that mathematicians would not be enamoured of the idea).[44] Brouwer did not make an explicit turn to phenomenology, but his work lends itself to phenomenological reconstruction (van Atten 2004a, 2007, 2010). They of course differed (in effect) on *which* phenomenological aspects are relevant to pure mathematics, because they had different conceptions of what pure mathematics, as a theory, consists in. Both took it to give an ontological description, but where for Gödel the domain described is a Platonist realm, for Brouwer it is that of our mental constructions.[45] A telling anecdote related to this was related to me by William Howard:

> I don't remember the context, but I started to talk about 'Brouwer's Bar Theorem', for which Brouwer gave a sort of justification but certainly not a proof. As soon as I got the words 'Brouwer's bar theorem' out of my mouth, Gödel interrupted me, saying,
> 'But he did not provide a PROOF!'.[46]
> This was delivered with strong emotion and quite aggressively. I sat there thinking: well, it's not *my* fault. But I replied,
> 'Well, yes, I agree,'
> and then went on with whatever I had been saying.[47]

The drafts for the revised version of the Dialectica paper (those that so far have remained unpublished) shed further light on Gödel's views on intuitionism and relate them to his turn to phenomenology. There are four versions to consider (together with the notes Gödel wrote when working on these):

D67 The translation of Gödel (1958) by Leo Boron (in collaboration with William Howard), revised by Gödel but without substantial additions (Gödel Papers, 9b/141, 040449 and 9b/142, 040451).

D68 A version that is essentially D67 with a rewritten, longer philosophical introduction (Gödel Papers, 9b/141, 040450).

Brouwer's view that mathematics is an activity of thought that is independent of logic and based on immediate evidence that is intuitively clear. Without further sources it is of course impossible to tell whether Gödel and Kuroda discussed phenomenology and intuitionism in relation to one another, but Kuroda's letter gives the impression that they had not.

[44] Personal communication from Georg Kreisel, letter to MvA, January 10, 2005.

[45] For more on ontological descriptivism, Brouwer's exploitation of it, and its contrast to meaning-theoretical approaches to mathematics such as Dummett's or Martin-Löf's, see Sect. 5 of Sundholm and van Atten (2008).

[46] *Note MvA.* See also Gödel (1972, 272n(d)): 'Unfortunately, however, no satisfactory constructivistic proof is known for either one of the two principles [i.e., Brouwer's bar induction and Spector's generalisation to finite types]'.

[47] Howard, story 20. In an email to me of January 26, 2013, William Howard adds that this was the only occasion during his conversations with Gödel (which took place during Howard's year at the IAS, 1972–1973) that the topic of the Bar Theorem and of bar induction came up.

D70 A version that is essentially D67 with an additional series of notes a–m (Gödel Papers, 9b/144, 040454). Circulated on Gödel's request by Dana Scott.[48] Galley proofs exist (Gödel Papers, 9b/149, 040456 and 040459).

D72 A revised version of D70. Last version available, published in *Collected Works* as Gödel (1972). (See the material for D70 (and the revisions on it), and, for the revisions of notes c and k, Gödel Papers, 9b/145, 040560 and 040457, respectively.).

As is to be expected from what for Gödel was work in progress, the drafts and notes D68–D72 are homogeneous in neither form nor content. Although written with an eye on publication, they can of course not be granted the same status as Gödel's published work. But various passages in them are coherently related to each other and to remarks Gödel has made elsewhere, and in any case allow one to document his thinking on these matters. For that reason, I should like to discuss a (necessarily, limited) selection of these materials, in particular from the new introduction in D68.[49] That new introduction exists in the archive as a set of pages in longhand numbered 1–26 and 1F–12F (for footnotes; Gödel Papers, 9b/141, 040450). As Gödel wrote to Bernays on May 16, 1968 that it is 'essentially finished', and on December 17, 1968 that 'in the end I liked the new one as little as the old' (Gödel 2003, 261, 265), I will give 1968 as the date of these drafts.

11.3.5.3 The Relation of Constructive Mathematics to Classical Mathematics

If constructive mathematics is conceived in such a way as to involve reference to properties of the mathematician's mental acts, then this explains Gödel's view, reported by Shen Yuting in a letter to Hao Wang of April 3, 1974, that 'classical mathematics does not "include" constructive mathematics' (Gödel Papers, 3c/205, item 013133).[50] For a classical mathematician holding this view, considerations about, and results from, constructive mathematics (in the sense described) have mathematical significance only when they can be 'projected into the mathematically objective realm'. The choice of words here is Bernays', who uses it to define the distinction between a 'reserved' and a 'far-reaching' intuitionism in a letter of March 16, 1972 (Gödel 2003, 295). Perhaps Gödel had a similar distinction in mind when, in a draft

[48] I thank Dirk van Dalen for letting me photocopy the purple-ink duplicate he received from Scott in Oxford.

[49] The material is rich, and should also be studied with other questions in mind, and from other perspectives. To my mind, in particular D68 would have deserved to be included in the *Collected Works* as well.

[50] This was also Heyting's view: 'I must protest against the assertion that intuitionism starts from definite, more or less arbitrary assumptions. Its subject, constructive mathematical thought, determines uniquely its premises and places it beside, not interior to classical mathematics, which studies another subject, whatever subject that may be' (Heyting 1956, 4).

note for the revision of the Dialectica paper, he sees a need to give a characterisation of finitism the form of a 'translation' of its traditional conception, as follows:

> Translating the definition of finitism given above into the language of modern mathematics (which does not consider spacetime intuition to belong to its field) one may say equivalently: The objects of finitary mathematics are hereditarily finite sets (i.e., sets obtained by iterated formation of finite sets beginning with a finite number of individuals or the 0-set); and finitary mathematics is what can be made evident about these sets and their properties, relations, and functions (definable in terms of the ∈-relation) without stepping outside this field of objects, and using from logic only propositional connectives, identity, and free variables for hereditarily finite sets. Clearly on this basis recursive definitions (proceeding by the 'rank' of the sets) are admissible as evidently defining well-determined functions without the use of bound variables. (Gödel Papers, 9b/141, 040450, 2F (1968))

It is, a fortiori, not surprising then that Gödel never came to accept in his own work on constructive mathematics the objects and techniques that are typical for Brouwer's 'far-reaching' intuitionism, such as choice sequences, Brouwer's proof of the Bar Theorem, and creating subject arguments.

11.3.5.4 Effective But Non-recursive Functions

This distinction between a reserved and a far-reaching intuitionism is also important for the question whether there exist effective but non-recursive functions.

This particular question had gained importance for Gödel by the time he came to revise the 1958 paper, as is clear from a comparison of the two versions of his footnote on Turing. While in 1958 he had written,

> As is well-known, A.M. Turing, using the notion of a computing machine, gave a definition of the notion of computable function of the first order. But, had this notion not already been intelligible, the question whether Turing's definition is adequate would be meaningless. (Gödel 1990, 245n6)[51,52]

in the 1972 version this became

> It is well known that A.M. Turing has given an elaborate definition of the concept of a *mechanically* computable function of natural numbers. This definition most certainly was

[51] A.M. Turing hat bekanntlich mit Hilfe des Begriffs einer Rechenmaschine eine Definition des Begriffs einer berechenbaren Funktion erster Stufe gegeben. Aber wenn dieser Begriff nicht schon vorher verständlich gewesen wäre, hätte die Frage, ob die Turingsche Definition adäquat ist, keinen Sinn. (Gödel 1958, 283n2)

[52] Moreover, Gödel will have known the observation by Skolem, Heyting, and Péter that in constructivism, 'computable function' cannot be taken to *mean* 'recursive function'. See Skolem (1955, 584), a paper to which my attention was drawn by Coquand (2014); Heyting (1958, 340–341), which appeared in the same special issue of *Dialectica* as Gödel's paper; Péter (1959). Heyting is the one who emphasises the alternative of taking that notion as primitive. Tait (2006, 212–213) holds that the fact that a definition would be circular shows that there is a problem with the idea of constructive evidence for the computability of a function. To my mind, that is not correct, but I will not develop this point here. See also Kreisel's review Kreisel (1969a) of Tait (1967).

> not superfluous. However, if the term 'mechanically computable' had not had a clear, although unanalyzed, meaning before, the question as to whether Turing's definition is adequate would be meaningless, while it undoubtedly has an affirmative answer. (Gödel 1972, 275n5)

In the latter version, the mechanical character of Turing's notion is made explicit and is emphasised. It might have been natural then also to ask about constructively evident but non-mechanical computability. Gödel chose not to do so in D72. He had in D68:

> In my opinion there are no sufficient reasons for expecting computability by thought procedures to have the same extension [as mechanical computability], in spite of what Turing says in Proc. Lond. Math. Soc. 42 (1936), p. 250. However, it must be admitted that, even in classical mathematics, the construction of a welldefined thought procedure which could actually be carried out and would yield a numbertheoretic function which is not mechanically computable would require a substantial advance in our understanding of the basic concepts of logic and mathematics and of our manner of conceiving them. (Gödel Papers, 9b/141, 040450, 20–21 (1968))

In 1972, Gödel prepared a slightly different version of this remark for publication outside the Dialectica paper, Gödel 1972a, p. 306 (remark 3).[53]

However, in 'far-reaching' intuitionistic mathematics, Kripke has devised (but not published) an example of just such a function. The presentation I will follow here is that of van Dalen (1978, 40n3), which owes its elegance to its explicit use of the so-called Theory of the Creating Subject, CS (Kreisel 1967b, 159–160).[54] Write $\Box_n A$ for 'The creating subject has at time n a proof of proposition A'. Let K be a set that is r.e., but not recursive. Define

$$f(n,m) = \begin{cases} 0 \text{ if } & \Box_m n \notin K \\ 1 \text{ if not } & \Box_m n \notin K \end{cases}$$

For the creating subject, f is effectively computable, as at any given moment m, it is able to determine whether $\Box_m n \notin K$. By the standard principles governing the creating subject,[55] we have $n \notin K \leftrightarrow \exists m f(n,m) = 0$; this means that, if f were

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[53] Yet another version was published, with Gödel's approval, in Wang (1974, 325–326) (reprinted in Gödel 2003a, 576).

[54] It is also possible to avoid CS, formally, by using the Brouwer-Kripke Schema BKS instead, usually formulated as $\exists\alpha(\exists n\alpha(n) = 1 \leftrightarrow A)$ (but the parenthetical qualification in Footnote 55 below also holds here: BKS should really be formulated as two rules with parameters P and $\alpha = \alpha_P$). However, from the intuitionistic point of view, the known justification of BKS also justifies CS. Versions using BKS were given by Gielen (as quoted in de Swart 1976b, 35) and Dragálin (1988, 134–135); Gielen's construction is closest to Van Dalen's. The (weaker) point that BKS and Church's Thesis are incompatible was first made in print by Myhill (1966, 296–297), and taken up in the influential (Troelstra 1969, 100).

[55] E.g., Troelstra and van Dalen (1988, 1:236), in particular: $A \leftrightarrow \exists n(\Box_n A)$. Intuitionistically, this is not difficult to justify; see the discussions of the topic in Dummett (2000b, Sect. 6.3), and van Atten (2004a, Chap. 5). (By the considerations in Sundholm and van Atten (2008), and also in Sundholm (2014), the principle cited should in fact be presented as a pair of (proof, not inference)

moreover recursive,[56] then the complement of K would be r.e., which contradicts the assumption.

According to Van Dalen (in conversation), at the Summer Conference on Intuitionism and Proof Theory, SUNY at Buffalo, 1968, the example was considered common knowledge. Kreisel learned the example before that conference, perhaps from Kripke himself when the latter visited Stanford somewhere between 1963 and 1965 (Letter from Kreisel to MvA, August 19, 2006). Kreisel presented the theory CS to Gödel in a letter of July 6, 1965 (Gödel Papers, 01/87, 11182). I have not yet been able to determine yet whether Gödel came to know Kripke's function as well. But it seems likely that he did, given his close contact with Kreisel at the time.

11.3.5.5 Choice Sequences

While rejecting choice sequences from his own point of view, in his reflections on finitism in Hilbert's sense Gödel was led to conclude that choice sequences should be acceptable on that position. In a draft letter to Bernays of July 1969, he wrote:

> it now seems to me, after more careful consideration, that choice sequences are something concretely evident and therefore are finitary in Hilbert's sense, even if Hilbert himself was perhaps of another opinion. (Gödel 2003, 269)[57]

and with that draft he included the text of a footnote for the revision of the Dialectica paper, in which he stated:

> Hilbert did not regard choice sequences (or recursive functions of them) as finitary, but this position may be challenged on the basis of Hilbert's own point of view. (Gödel 2003, 270)[58]

In the letter he actually sent at the end of that month, he did not include the text for the footnote, and wrote 'Hilbert, I presume, didn't want to permit choice sequences? To me they seem to be quite concrete, but not to extend finitism in an essential way' (Gödel 2003, 271).[59] However, in D70 (which Bernays would still see) and D72, he chose the slightly weaker formulation 'a closer approximation to Hilbert's finitism

rules, rather than as a bi-implication as understood in Natural Deduction. Note that the explanation usually given of the principle as cited is in effect that of the rules.)

[56]The equivalence would be best understood as an extensional one, so as to forestall paradoxes that might appear if one would straightforwardly render the sentential operator $\Box_n$ by a provability predicate. Alternatively, one could use BKS instead of CS to construct the function, as mentioned in Footnote 54. I thank Albert Visser for raising this issue and for his *Répondez !*.

[57] es scheint mir jetzt, nach reiflicher Überlegung, dass die Wahlfolgen etwas Anschauliches u[nd] daher im Hilbertschen Sinn Finites sind, wenn auch Hilbert selbst vielleicht anderer Meinung war. (Gödel 2003, 268)

[58]This corresponds to Gödel Papers, 9b/148, 040498.

[59] Hilbert wollte Wahlfolgen wohl nicht zulassen? Mir scheinen sie durchaus anschaulich zu sein, aber den Finitismus nicht wesentlich zu erweitern. (Gödel 2003, 270)

[than using the notion of accessibility] can be achieved by using the concept of free choice sequences' (Gödel 1972, 272n(c)).

11.3.5.6 1968: The Dialectica Interpretation as a Phenomenological Contribution to Intuitionism

Archive material shows that the foundation of the Dialectica Interpretation on a notion of 'reductive proof', well known from the publication of D72 in the *Collected Works*, was preceded by an attempt to construe the Dialectica Interpretation as a specifically intuitionistic result in the sense of Brouwer, and that both attempts were meant as applications of Husserl's phenomenology.[60]

Documentation of Gödel's phenomenological but not specifically intuitionistic approach to reductive proof is presented in Sect. 11.3.5.8.

The manuscript D68 contains various references to phenomenology. It is referred to as a possible method for developing a wider, yet no less convincing notion of constructivism than that of the formalists' (and, implicitly, of Gödel's former self of the 1930s). Having stated the intuitionistic conception, he adds this footnote:

> This explanation describes the standpoint taken, e.g., in A. Heyting's development of intuitionistic logic (see footn[ote] 13).[61] Formalists in their consistency proofs are aiming at a stricter version of constructivism, which however has never been precisely defined. The most important additional requirement would no doubt be that the use of the term 'any' is restricted to totalities for which procedures for constructing all their elements are given. Also (which should be a consequence of this requirement) the conceptual selfreflexivities occurring not only in classical, but also in intuitionistic mathematics (e.g., that a numbertheoretic proof may contain the concept of numbertheoretic proof) are to be avoided. The hierarchy mentioned in footn[ote] 6 is an example of this stricter constructivism, possibly even in case it is extended beyond ϵ_0, which can be done by treating as one step any sequence of steps which has been recognized as permissible (e.g., any ϵ_0 sequence of steps). The concepts of 'accessible', Brouwer's ordinals, and similarly defined classes would seem to need further analysis (perhaps in terms of the just mentioned hierarchy) in order to be strictly constructivistic.
>
> Not even the PFN functions[62] (if defined as below on p.) are strictly constructivistic (see p.). This makes one suspect that the aforementioned requirements of strict constructivism are too restrictive. Perhaps confining the extensions of concepts to sets that can somehow be 'overlooked' and avoiding selfreflexivities in the primitive terms are not the only means of reaching completely convincing proofs. Phenomenological clarification of the basic elements of our thinking should be another very different, and perhaps less restrictive, possibility. (Gödel Papers, 9b/141, 040450, 9F, 9.1F, 9.2F (1968))[63]

[60]These phenomenological projects were overlooked in the research for van Atten and Kennedy (2003), to which this part of the present paper should be considered an addendum.

[61]*Note MvA.* In his Footnote 13, Gödel refers to Heyting (1934, 14).

[62]*Note MvA.* 'Primitive recursive functions of finite type over the natural numbers' (Gödel Papers, 9b/141, 040450, 24 (1968))

[63]*Note MvA.* References left open by Gödel.

There is also some hesitation:

> As far as obtaining incontrovertible evidence [as the basis of a consistency proof of classical analysis] is concerned, what is needed would be phenomenological analysis of mathematical thinking. But that is a rather undeveloped field and there is no telling what future work in it may bring to light. (Gödel Papers, 9b/141, 040450, 12 (1968))

But later on in the same set of draft pages, in a passage here labelled (I), further-going claims are made:

> (I) On the other hand the interpretation of T used in this paper yields a consistency proof based on a new intuitionistic insight, namely the immediate evidence of the axioms and rules of inference of T for the computable functions defined above. Note that, as our analysis has shown, this insight is based on psychological (phenomenological) reflection, whose fruitfulness for the foundations of mathematics is thereby clearly demonstrated. (Gödel Papers, 9b/141, 040450, 21–22 (1968))

The following four (overlapping) topics evoked by or related to (I) will be commented on and illustrated by other passages:

I1. Psychological and logical reflection,
I2. Psychology and intuitionism,
I3. Psychology and phenomenology,
I4. The Dialectica Interpretation as an application of phenomenology,
I5. Gödel's 'analysis'.

I1. Psychological reflection is contrasted with logical reflection:

> We comprise both kinds of concepts (i.e., those obtained by logical and those obtained by psychological reflection under the term 'abstract', because the thoughts in question always contain abstract elements, either as their object or at least as being used. However, finer distinctions are of course possible. E.g., the concept of idealized finitary intuitions (see p.   above) evidently is formed by psychological reflection. (Gödel Papers, 9b/141, 040450, 6.1F (1968))[64]

The contrast is again described in the notes towards D70:

> Husserl Note that conc[erning] abstr[act] concepts[65] one has to distinguish thoughts & their content (obtained by psychol[ogical] & log[ical] reflection resp[ectively]) The former (to which int[uitionists] try to confine themselves[66]) are occurrences in the real world & therefore are in a sense just as concrete as . . . [67] of symbols which should make them all the more acc[eptable] to finitists. (Gödel Papers, 9b/148.5, 040498.60)

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[64]*Note MvA*. Reference left open by Gödel.

[65]*Note MvA*. Above 'concepts', Gödel wrote: 'entities'.

[66]*Note MvA*. Also: 'speaking (as intuitionists . . . do) of thoughts as occurrences in spacetime reality (instead of their content) the objectivation (in the statements of the theory) of abstract entities and existential assertions about them are avoided and, moreover, the content of the thoughts to be admitted, although itself something abstract, always refers to something concrete, namely other thoughts or symbols or actions' Gödel Papers, 9b/148.5, between 040498.39 and 040498.43.

[67]Unreadable word; 'comb[inations]'?

Husserl discusses this distinction in, for example, in Sects. 41 and 88 of *Ideas I*, a work that Gödel owned in its first edition (Husserl 1950c) and knew well; these are titled 'The really inherent composition of perception and its transcendent object' and 'Real and intentional components of mental processes. The noema', respectively.[68] It is the distinction between (mental) acts as concrete occurrences in time and their intended objects as such. The distinction applies to all thoughts, but Gödel's concern is with those that are in some sense abstract:

> We comprise both kinds of concepts (i.e. those obtained by logical and those obtained by psychological reflection) under the term 'abstract', because the thoughts in question always contain abstract elements, either as their object or at least as being used. However, finer distinctions are of course possible. E.g., the concept of idealized finitary intuition (see p. above) evidently is formed by psychological reflection. (Gödel Papers, 9b/141, 040450 (1968))[69]

An example of a thought that is not directed at an abstract object but nevertheless uses an abstract element would be an insight about infinitely many concrete acts. As an infinity of acts cannot actually be carried out by us, they cannot all be concretely represented in a thought, and we have to represent them abstractly. According to Gödel, both finitary mathematics and intuitionistic mathematics arise from psychological reflection; the difference between them is that in the latter, the abstract elements that can be used in thoughts about the concrete acts themselves also become objects of the theory.

I2. Gödel emphasises that the kind of psychology of which in (I) he considers intuitionism to be a form is not empirical psychology:

> Of course, in order to carry through this interpretation accurately and completely, a much more careful examination of the situation would be necessary. In particular the question would have to be answered why intuitionistic mathematics does not become an empirical science under this point of view. Roughly speaking, the answer is the same as that to a similar question about metamathematics as the science of handling physical symbols (although the situation is much more involved in our case). The relevant considerations in both cases are these: 1. There exist necessary propositions about concrete objects, e.g., that parts of parts are parts. 2. Mathematical propositions (in particular existential propositions) in this interpretation may be looked upon as implications whose hypotheses are certain (evidently possible) general empirical facts. 3. Instead of speaking of the occurrences (in reality) of mental acts or physical symbols one may speak of their individual forms (which determine their qualities in every relevant detail). In the special intuitionistic considerations given in the present paper the psychological interpretation has not been used throughout, e.g., we speak of rules (in the sense of procedures decided upon as to be followed) governing mental activity, not only of mental images of such rules. (Gödel Papers, 9b/141, 040450, 9.2 (1968))

Hence he could say to Toledo, as we saw above, that intuitionism is 'essential a priori psychology'.

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[68] 'Der reelle Bestand der Wahrnehmung und ihr transzendentes Objekt' and 'Reelle und intentionale Erlebniskomponenten. Das Noema'. Translations taken from Husserl (1983); the second one is modified.

[69] *Note MvA.* Page reference left open by Gödel.

I3. It is clear that Gödel, when in (I) he sees intuitionism as 'psychological (phenomenological) reflection' which is at the same time 'a priori', he is speaking of what Husserl called 'phenomenological psychology'. Husserl wrote extensively about this in two places that Gödel knew well: the *Encyclopædia Britannica* article[70] and in the last part of the *Krisis* (Husserl 1954), titled 'The way into phenomenological transcendental philosophy from psychology'.[71] And, as it happened, the Husserliana volume with Husserl's 1925 lectures *Phänomenologische Psychologie* came out in 1968; but I don't know whether Gödel got to see that when working on the new introduction to the Dialectica paper in the first months of that year.[72]

Phenomenological psychology describes mental phenomena and unlike empirical psychology is not concerned with individual concrete facts but with invariant forms they instantiate and which delineate the range of possible concrete facts. In other words, it deals with the essence of our psychology.[73]

Note that Gödel's conception of the intuitionistic subject here as a subject in a psychological and hence mundane sense can be challenged, on grounds that are no doubt clearer in Brouwer's writings than in Heyting's. For an argument that the intuitionistic subject is better understood as a transcendental subject in Husserl's sense, see van Atten (2004a, Chap. 6; 2010, 66–68).

I4. Given the above, the Dialectica Interpretation is in D68 meant to be an application of phenomenology that moreover belongs to intuitionism, because it is based on an insight into mathematical procedures understood as acts carried out in thought over time (noeses), which are then, in acts of reflection, objectified as such to become objects of the theory.[74]

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[70] Gödel may have read this before 1962, the year the original German manuscript was reprinted in the Husserliana edition (Husserl 1962); there is a library slip (Gödel Papers, 9c/22, 050103) requesting the relevant volume (17: 'P to Planting of Trees') of the 14th edition of the *Britannica* of 1929. There are also some reading notes in the same folder. For a different connection between Gödel and the Britannica article, see van Atten and Kennedy 2003, Sect. 6.1 (Sect. 6.6.1 in this volume).

[71] Der Weg in die phänomenologische Transzendentalphilosophie von der Psychologie aus.

[72] 1968 is the copyright year. That is not necessarily the year the book became available.

[73] The project of a non-empirical (e.g., 'a priori', 'rational' or 'transcendental') psychology has a long tradition (e.g., Wolff, Kant); for Gödel, Husserl's version will have been attractive because it is closely related to transcendental phenomenology, to which Husserl considered it to be propaedeutic.

[74] 'Strictly speaking the construction of intuitive mathematics in itself is an action and not a science; it only becomes a science ... in a mathematics of the second order, which consists of the mathematical consideration of mathematics or of the language of mathematics' (Brouwer 1975, 61n1). ('Eigenlijk is het gebouw der intuïtieve wiskunde zonder meer een daad, en geen wetenschap; een wetenschap ... wordt zij eerst in de wiskunde der tweede orde, die het wiskundig bekijken van de wiskunde of van de taal der wiskunde is.' (Brouwer 1907, 98n))

I5. Among the notes Gödel made in 1968 in preparation for D68 is the following:

> Foundations: it is really incredible, how all important philosophical and psychological problems are actualised in a rigorous treatment of my system T, and how many important distinctions become clear. For example: evocation of the image of a procedure and application of the procedure; image of a rule and rule (one sees how 'flimsy'[75] the former is, and how 'iron' the latter); results of the intermediate steps and the operations of the intermediate steps; operation in the sense of a mental act and of a mathematical object (briefly: rule, image of a rule, application of a rule, image[76] of the application of a rule); definitional procedure to obtain the functions of T, procedure to compute the individual functions of T;[77,78]

However, although D68 itself does contain what could be considered to be preliminary remarks to a (phenomenological) analysis, e.g.,

> By 'procedure' we mean here 'mental' or 'thought' procedures, i.e., the steps are (intuitionistically meaningful) ideas or mental images formed by mental acts on the basis of the preceding steps according to the rule of the procedure. Also 'starting with' or 'terminating with' means: starting or terminating with a mental image of .... For practical reasons the writing down and 'reading' of symbols (used only for denoting well determined thoughts) are also to be admitted as steps of the procedures. Of course the rules of the procedures are supposed to be such that each step is (in a repeatable manner) uniquely determined by the previous steps. (Gödel Papers, 9b/141, 040450, 17)

the body of a detailed analysis, in terms of the concepts and distinctions mentioned in the previous quotation, is not to be found in it. There is further archive material

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[75]*Note MvA.* In English in the original.

[76]*Note MvA.* I translate 'Vorstellung' as 'image' here, because that is the term Gödel uses in these manuscripts when writing in English. Spiegelberg (1965), a work that Gödel owned (2nd ed.) and knew well, translates it as 'representation', and a popular alternative is 'presentation'. (NB Cairns' recommendation, published in 1973, for the broadest Husserlian sense is '(mental) objectivation' (Cairns 1973, 131).) I take it that Gödel's choice of 'image' is motivated by a wish to avoid special terminology as much as possible, so as to avoid making his philosophical remarks seem more dependent on a particular philosophy than they are. To Wang he said, 'I am cautious and only make public the less controversial parts of my philosophy' (Wang 1996, 235). Similary, Wang remarks that 'Gödel's desire to shun conflict also affected his published work. He would make great efforts to present his ideas in such a form that people with different perspectives could all appreciate them (in different ways)' (Wang 1996, 235). (I thank Nuno Jerónimo for locating these comments.)

[77]*Note MvA.* Here the list stops, at the bottom of the left half of the page, and the right half of the page begins with a new remark.

[78](Gödel Papers, 9b/148, 040492). Transcription Eva-Maria Engelen; translation MvA. 'Gr[undlagen]: Es ist unglaublich, wie sämtliche wichtigen ph[ilosophischen] und psych[ologischen] Probleme bei genauer Behandlung meines Systems T aktualisiert [werden] und wie viele wichtige Distinct[ionen] klar werden: zum Beispiel: Evokation der Vorstellung eines Verfahrens und Anwendung des Verfahrens; Vorstellung einer Regel und Regel (man sieht wie 'flimsy' die erstere und wie 'ehern' die letztere ist); Resultate der Zwischenschritte und Operationen der Zwischenschritte; Operation im Sinn einer geistigen Handlung und eines mat[hematischen] Objekts (kurz: Regel, Vorstellung der Regel, Anwendung der Regel, Vorstellung der Anwendung der Regel); Def[initions-]Verfahren, um die Funktionen von T zu erhalten, Verfahren um die einzelnen Funktionen von T zu berechnen;'

around D68 awaiting transcription; but I assume that, if it contained substantial further analysis, that would have been included in the longhand draft. Without such an analysis, it is not clear that there will be any advantage in shifting to a specifically intuitionistic (noetic) perspective (as, in contrast, there is, for Brouwerians, when demonstrating the Bar Theorem). This will undoubtedly have played a major role in Gödel's eventual dissatisfaction with D68.

This ends my discussion of I1–I5.

In spite of the above, perhaps one doubts Gödel's description in (I) of the reflection that led to the fundamental insight of the Dialectica Interpretation as 'phenomenological', on the ground that he had obtained that insight long before his turn to phenomenology around 1959.[79] But I don't think that Gödel here is making an implicit historical claim, but rather is using, on the occasion of a new presentation of his earlier insight, the framework that by then he had come to see as the best one for its philosophical reconstruction and explication. Gödel's philosophical remarks in the introduction to (both versions of) the Dialectica paper comfortably fit the description he had given of phenomenology and its use in his earlier text *1961?:

> Here clarification of meaning consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc. But one must keep clearly in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is (or in any case should be) a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us. (Gödel *1961/?, 383)[81]

However, having written (I), at some point Gödel put question marks next to it; recall his remark to Bernays of December 1968, quoted in Sect. 11.3.5.2, that he

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[79] As Feferman points out in his introduction to the Gödel-Bernays correspondence, it is noteworthy that in this philosophically rich exchange, Husserl is never discussed (Gödel 2003, 66n(ax)). (Gödel mentions phenomenology once, on August 11, 1961 (Gödel 2003, 193).) Bernays will have known of Gödel's enthusiasm for Husserl early on: Bernays was in Princeton from November 1959 to April 1960, and came back for shorter visits around Easter 1961, in May 1961, and in the Spring of 1965.[80] A short text presented in 1963 – in the middle of this period of visits – and published the next year, 'Begriffe des Phänomenologischen und das Programm der phänomenologischen Philosophie' (Bernays 1964), shows Bernays rather critical of Husserlian phenomenology, in particular of epoché, the possibility to see essences, and its foundational character. In his letters to Gödel, Bernays never mentions this text or the objections formulated in it; one possible explanation is that they had dealt with the topic in their conversations.

[80] See letters 26–31; 34–36; 38–39; and 52–53, respectively.

[81] Die Sinnklärung besteht hier darin, daß man die betreffenden Begriffe schärfer ins Auge faßt, indem man die Aufmerksamkeit in einer bestimmten Weise dirigiert, nämlich auf unsere eigenen Akte bei der Verwendung dieser Begriffe, auf unsere Mächte bei der Vollführung unserer Akte, etc. Man muß sich dabei klar darüber sein, saß diese Phänomenologie nicht eine Wissenschaft im selben Sinn ist wie die andere Wissenschaften. Sie ist vielmehr (oder sollte jedenfalls sein) ein Verfahren oder Technik, welches in uns einen neuen Bewußtseinszustand hervorbringen soll, in dem wir die von uns verwendeten Grundbegriffe unseres Denkens detaillieren oder andere bisher uns unbekannte Grundbegriffe erfassen.' (Gödel *1961/?, 382)

liked the ‘new philosophical introduction’ that he had written ‘as little as the old’. There are various possibilities as to what type of doubt they express. For example, Gödel may have developed doubts whether this was what he could claim, or whether he would be able sufficiently to develop this claim in writing so as to be convincing to others, or whether this claim would be well received given the *Zeitgeist* as he perceived it,[82] or whether to convince others of his consistency proof it was even necessary to make and develop this specific claim. Finally, he may have developed second thoughts about presenting this work as an intuitionistic insight.[83]

If this hesitation is indeed a mark of the same discontent that Gödel expressed in his letter to Bernays of December 1968, then it is no surprise that the next year, he drafted the following (unsent) reply to an inquiry from Van Dalen:

> My relationship with Intuitionism consists primarily in some theorems I proved about certain parts of intuitionistic mathematics in particular that published in Dial[ectica] 12. The question as to whether this paper is important for the foundations of Intuitionism I must leave for Intuitionists to answer. I did not write the paper from this point of view and some supplementation would be necessary in order to clarify it’s [sic] relevance for the foundations of Intuitionism. (Gödel Papers, 01/199, 012891; underlining Gödel’s)

This repeats of course Gödel’s statement towards the end of the 1958 publication,

> Selbstverständlich wird nicht behauptet, dass die Definitionen 1–6 den Sinn der von Brouwer und Heyting eingeführten logischen Partikel wiedergeben. Wieweit sie diese ersetzen können, bedarf einer näheren Untersuchung. (Gödel 1958, 286; translation below)

and moreover remains silent about his recently aborted attempt to develop the Dialectica Interpretation as a specifically intuitionistic insight. Naturally, then, in D70 and D72 Gödel left the content of the statement of 1958 unaltered:

> Of course it is not claimed that Definitions 1–6 express the meaning of the logical particles introduced by Brouwer and Heyting. The question to what extent they can replace them requires closer investigation. (Gödel Papers, D70, 9b/142, 040451, 7; Gödel 1990, 280)

Although Gödel thus abandoned the effort in D68 to give Dialectica a specifically intuitionistic content, he did not abandon his phenomenological approach to the philosophical deepening of that work; see ‘Reductive proof: phenomenology and demonstrations’ (Sect. 11.3.5.8).

[82]That seems to have been the kind of reason why, in the published version of the supplement to the second edition of his Cantor paper, Gödel left out a hopeful reference to phenomenology that is present in the draft (Gödel Papers, 8c/101, 040311), while at the same time recommending Husserl to logicians in conversation. See van Atten and Kennedy (2003, 466); Sect. 6.6 in this volume.

[83]Elsewhere, I have argued that Gödel’s program to employ transcendental phenomenology to found *classical* mathematics is misguided (van Atten 2010); Sect. 12.3 in this volume. But, by the positive argument in the same paper, his attempt to use it to enrich intuitionism, whether eventually successful or not, makes perfectly good sense.

11.3.5.7 Demonstrability and Impredicativity

In D70, Gödel included in his definition of 'computable function of type t_k' that it is 'intuitionistically demonstrable' that it is always performable; in the publication of 1958, this had been 'constructively recognizable'[84] (Gödel 1990, 245). Bernays read the galleys and on July 12, 1970 remarked to Gödel about this definition:

> Here the reader could well be taken aback, since your procedure is surely intended to avoid the concept of intuitionistic proof. It seems to me, however, that in fact you do not need that concept here at all, and that onloy a suitable reformulation is needed in order to make that clear. (Gödel 2003, 281)[85]

In later letters, Bernays proposed two alternatives:

1. Replace 'is intuitionistically demonstrable' by 'follows directly from the definition of the function in question and those of the functions in the k-tuple' (Gödel 2003, 286/287, October 12, 1970)
2. Add a footnote saying that what is meant here is only that 'for the determination in question the methods of proof excluded by intuitionism must not be used' (Gödel 2003, 292–295, March 16, 1972)

Curiously, Gödel in D68 had foreseen Bernays' objection, answered it as he would later do in D72, and then rejected that answer:

> For the special application of the concept of computable function to be made in the present paper it is preferable to replace in its definition the term 'demonstrable' by 'evident on the basis of the definition of the procedure and previous definitions used in it'.
>
> ...
>
> It might be objected that the concept of proof is used also in our interpretation, since 'demonstrability' occurs explicitly in the definition of CFI function.[86] | The answer is that, in constructing a model of T, 'demonstrable', in the definition of CFI function, may be replaced by 'evident without proof on the basis of the structure of the definitions.' (Gödel Papers, 9b/141, 040450, 19 and 23–24)

But Gödel then crossed out the first passage, and wrote 'Wrong' next to the second.[87] I do not know what reason for doing so he had in mind; but it was evidently what motivated him to write 'intuitionistically demonstrable' in D70, which Bernays then got to see – unlike D68. By the time of D72, Gödel had dropped whatever his earlier objection had been, and replaced 'intuitionistically demonstrable' by

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[84] konstruktiv erkennbare. (Gödel 1958, 282)

[85] Hier könnte wohl der Leser stutzen, da doch Ihr Verfahren bezweckt, den Begriff des intuitionistischen Beweises zu vermeiden. Es scheint mir jedoch, dass Sie de facto hier diesen Begriff auch gar nicht brauchen un dass es nur einer geeigneten Umformulierung bedarf, um dieses zum Ausdruck zu bringen. (Gödel 2003, 201)

[86] *Note MvA*. 'CFI' is Gödel's abbreviation in D68 for 'computable functional of finite type' (Gödel Papers, 9b/141, 040450, 23–24 (1968)).

[87] 'Falsch', in shorthand.

'constructively evident'.[88] It is not clear to what extent that was under the influence of Bernays' remarks.

In D68 Gödel already realised that there is an impredicativity in his system T:

> There are functions of lower type which (within T) can only be defined by using functions of much higher types. This 'impredicativity' is perfectly legitimate, also from the constructivist point of view. It was in substance admitted even in Princ[ipia] Math[ematica] 2nd ed. p.   . And indeed the fact that the concept of functions of high type is defined in terms of that of functions of low type in no way precludes an inverse relationship for individual functions; i.e., the chain of definitions of a PCN function may go up and down in the system of types. What can be concluded from this state of affairs is only that the PCN functions (if introduced as above) are not strictly constructivistic in the sense of footn.   (references left open by Gödel. Gödel Papers, 9b/141, 040450, 25–26 (1968))

Note that Gödel here harks back to his Russell paper:[89]

> In the second edition of *Principia*, however, it is stated in the Introduction (pages xl and xli) that 'in a limited sense' also functions of a higher order than the predicate itself (therefore also functions defined in terms of the predicate as, e.g., in p '$\kappa \epsilon \kappa$) can appear as arguments of a predicate of functions; and in Appendix B such things occur constantly. This means that the vicious circle principle for propositional functions is virtually dropped. This change is connected with the new axiom that functions can occur in propositions only 'through their values', i.e., extensionally, which has the consequence that any propositional function can take as an argument any function of appropriate type, whose extension is defined (no matter what order of quantifiers is used in the definition of this extension). There is no doubt that these things are quite unobjectionable even from the constructive standpoint (see page 136), provided that quantifiers are always restricted to definite orders. (Gödel 1944, 134)

The sense in which in D68 the PCN functions are said not to be strictly constructivistic, then, is the sense in which it would be demanded that functions are generated from below exclusively.

In D70, with the introduction of the notion of reductive proof, it was clear to Gödel that notion would not serve to avoid this impredicativity, and the corresponding remark is phrased thus:

> In particular, there exist functions of lower type which, within T, can only be defined in terms of functions of higher types. This is a kind of impredicativity. True, it is only one of those weak impredicativities that are admitted even in Princ[ipia] Math[ematica] 2nd ed. p. [XL],[90] ff. In our proofs of the axioms of T this impredicativity appears in the fact that the concept of reductive proof may itself occur in reductive proofs (just as in Heyting's logic the general concept of proof may occur in a proof).

[88]In note k of D72 (Gödel 1990, 275n(h)), however, he quoted that part of the definition as 'constructively evident or demonstrable'; I assume this was left in inadvertently.

[89]Gödel studied his Russell paper when working on the Dialectica paper. This is clear from the remarks on the loose sheet inserted with one of the four offprints that Gödel owned, offprint D in the 'Textual notes' in the *Collected Works*, Gödel 1990, 315–322; the remarks in question are on p. 320 and p. 321. NB Correction: The note on the title page of D does not say 'gelesen bis p. 135 oben' (Gödel 1990, 320), but 'gelesen bis p. 138 oben'.

[90]*Note MvA*. The typescript erroneously has 'XI', as had the original publication of Gödel's Russell paper (Gödel 1944, 134).

It is the impredicativity mentioned in the final sentence that Gödel refers to as the 'unavoidable self-reflexivities'[91] in his letter to Bernays of December 22, 1970 (Gödel 2003, 290/291). From Gödel's letter to Bernays of 2 years later (December 26, 1972), it is clear that he remained convinced that, all the same, the notion of reductive proof was an epistemic advance over Heyting's Proof Interpretation:

> I also thank you very much for your letter about the question whether the general intuitionistic concept of proof is necessary for the intuitionistic interpretation of my system T (which would make my interpretation of the logical operators epistemologically worthless). I think that is *not* the case, but rather that a *much* narrower and in principle *decidable* concept of proof suffices, which I introduced in note k[92] of the translation of my *Dialectica* paper and called 'reductive provability'. But to carry that through satisfactorily in detail is not all that easy, mainly on account of the *non-eliminable* impredicativity also of this narrower concept of proof, which is closely connected with the impredicativity of the concept of function that you mentioned. It is doubtful whether carrying it through would be worth the trouble. Up to now, therefore, I have not been able to make up my mind to do it, although the further pursuit of that question could perhaps contribute in an essential way to the clarification of the foundations of intuitionism. (Gödel 2003, 301)[93]

An as yet unpublished assessment by Gödel of this situation, dating from 1974, will be presented in Sect. 11.3.5.9.

11.3.5.8 Reductive Proof: Phenomenology and Demonstrations

Although Gödel abandoned the attempt in D68 to construe Dialectica as intuitionistic in the noetic sense, the shift to the notion of reductive proof employed in D70 and D72 still depended on phenomenology, and still marked a rapprochement to Brouwerian intuitionism.

That Gödel could continue to use the phenomenological method is not surprising: as a study of consciousness, phenomenology is, at least at a certain level of

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[91] *Note MvA*. '[die] unvermeidlichen "self reflexivities" '.

[92] *Note MvA*. Presented, in the later version of D72, as note h in Gödel (1972).

[93] Ich danke Ihnen auch bestens für Ihren Brief über die Frage, ob der allgemeine intuition[istischen] Beweisbegriff für die intuition[istischen] Interpretation meines Systems T nötig ist (was meine Interpretationder logischen Operatoren erkenntnistheoretisch wertlos machen würde). Ich glaube, dass das *nicht* der Fall ist, sondern dass ein *viel* engerer (u[nd] im Prinzip) entscheidbarer Beweisbegriff genügt, den ik in Note k der Übersetzung meiner Dialectica [arbeit] eingeführt u[nd] 'reduktive Beweisbarkeit' genannt habe. Aber das im einzelnen befriedigend durchzuführen, ist nicht ganz leicht, hauptsächlich wegen der *nicht eliminierbaren* Imprädikativität auch dieses engeren Beweisbegriffes, welch mit der von Ihnen erwähnten Imprädikativität des Funktionsbegriffes nahe zusammenhängt. Est ist zweifelhaft, ob die Durchführung die Mühe lohnen würde. Ich habe mich daher bis jetzt nicht dazu entschliessen können, obwohl die weitere Verfolgung dieser Fragen vielleicht wesentlich zur Aufklärung der Grundlagen des Intuitionismus beitragen könnte. (Gödel 2003, 300)

generality, compatible with different views as to what mathematics is and how it is related to consciousness.[94] This is clear from the following group of notes:[95]

4. In my interpretation there are …no such iterations of implications, as little as of universal propositions, because the premisse always contains Red[uctive].[96]

noema
/
5. Does in the intentional object of the knowledge of such an imp[lication] …[97] $\supset c = g(d)$ the concept of insight occur? No, because the noema of the cognitional act contains both of these in a certain relation.

….

7. A good example of the distinction between (and transition from) noema to noesis [is] the int[uitionistic] (Heyt[ingian]) interpretation of imp[lication].

….

12. Important:

    ….

    2.. All propositions must be accepted as meaningful objects[98] (= int[entional] Obj[ects] = Noemata) and likewise [the] chains of evidences for them, where the preceding propositions have indeed been seen to be true before, hence no assumption.

    3.. The chains of evidences, which appear when expl[icating] Red[uctive] Proof, are of this kind. That is also exactly as for finitary proofs (this is so for the theorems and inferences in P[99] in general)[100]

[94] I write 'at a certain level of generality', because this compatibility may or may not be preserved when making one's conception of phenomenology more specific. In van Atten (2010) I argue that, in particular, if one's conception of phenomenology is that of the transcendental Husserl (of, roughly, the 1920s and 1930s), then intuitionistic mathematics is compatible with, and moreover part of, phenomenology, whereas classical mathematics is neither.

[95] In the archive, these are not kept with the drafts and galleys for the revised Dialectica paper, but in a folder named 'Dialectica interpretation' under the heading 'Other loose manuscript notes'.

[96] *Note MvA*. There are notes in which Gödel writes antecedents as '$Red(p)$', for 'p is reductively provable' (which for given p is decidable, or should be once the notion of reductive proof has been sufficiently clarified).

[97] *Note MvA*. The antecedent is almost unreadable, but it seems safe to say that Gödel here gives an example of an implication in T.

[98] Ground objects and (x) is therefore an operation binding ground variables that leads to objects.

[99] *Note MvA*. P is the system that was going to be named T' in D72 (see the manuscript for D70, Gödel Papers, 9b/142, 040452, insertion to note k3. In the circulated typescript, this is the footnote on p. 13).

[100] Gödel Papers, 11b/6, 060039. Transcription Eva-Maria Engelen, Robin Rollinger, and MvA. Translation MvA.

4. In meiner Interpretation gibt es …keine solchen Iterationen von Impl[ikationen], ebensowenig wie die von Allsätzen, weil in der Prämisse immer Red[uctive] steht.

When working the ideas into a revision of D70's note k (not included in D72), this became:

> As for item 2. ['the meaning of the implications of the form "If $x, y, \ldots$ have certain types, then ... " occurring implicitly both in the definition of 'computable of type t' and in the axioms and theorems of T'] it is first to be noted that implication occurs only in this form: 'If the procedure A yields the result a, then the procedure B yields the result b' where it need not be known whether procedures A or B yield any result at all, even though they are supposed to be defined with perfect precision.[102] For, also the statement '$(x)\phi(x)$ is reductively provable' (which is the only way in which quantification occurs in the interior of formulas) means that a certain procedure of checking the chain of defi|nitions of the concepts in ϕ yields a certain result. But such implications can be interpreted to mean: 'If I (the reasoning mathematician) carried out the procedure A and obtained the result a then, if I carry out the procedure B, I shall obtain the result b', where the 'ifs' here mean a truthvalue function, i.e., 'either the implicans is false or the implicatum true'. This entails that, in the last analysis, the implication in question means that a certain procedure involving both A and B yields a certain result, whenever carried out.' (Gödel Papers, 9b/145, 040458, 2–3)

In this draft towards a publication, the phenomenological terminology used in Gödel's private notes has disappeared. This seems to me to be intentional; see also Footnote 76 above.

Note how Gödel's explanation of implication (with respect to reductive proof), is given by demonstration-conditions, that is, conditions in terms of procedures that, actually or hypothetically, have been carried out; as opposed to proof-(object-)

Noema
/
5. Kommt im intentionalen Objektder Erkenntnis einer solchen Imp[likation] $\ldots \supset c = g(d)$ der Begriff der Einsicht vor? Nein, | denn der Akt der Erkenntnis hat diese beide im Noema in einem [be]stimmten Zusammenhang.

....

7. Ein gutes Beispiel für Unterscheidung von (und Übergang von) Noema zu Noesis [ist] die int[uitionistische] (Heyt[ingsche]) Interpretation der Imp[likation].

....

12. Wichtig:

   ....

   2.. Alle Sätze müssen als sinnvolle Objekte[101] (= int[entionale] Obj[ekte] = Noemata) anerkannt werden und ebenso [die] Evidenzketten für solche wo die vorausgehenden Sätze vorher tatsächlich eingesehen sind, also keine Annahme.

   3.. Die Evidenzketten, welche bei der Expl[ikation.] von Red[uctive] Proof herauskommen, sind von dieser Art. Das ist also ganz genau so wie bei finiten Beweisen (wie überhaupt die Sätze und Schlüsse in P)

[101]Grundobjekte und (x) ist also eine Grund variablen bindende Operation, welche zu Objekte[n] führt.

[102]*Note MvA.* The part from 'where' to the end is a later insertion: Gödel Papers, 9b/145, 040462, k(2) +.

conditions, which are given in terms of properties of proofs independently of the actual or hypothetical fact that we know this proof-object (Sundholm 2007).

In Sundholm and van Atten (2008) arguments are given why demonstration-conditions, not proof-conditions, are required for a correct reading of Brouwer. On the one hand, then, the Dialectica Interpretation on the basis of reductive proof is much closer to Brouwerian intuitionism than to alternative constructive foundations (see Sect. 11.3.5.9, remark on [A]). On the other hand, the combined effect of items 5 and 7 is to distance the notion of reductive proof from that of intuitionistic proof in the specifically noetic sense of Brouwer and Heyting (see item I4 above, Sect. 11.3.5.6). As a consequence, Brouwer's specific mentalism about mathematics as a whole makes certain types of argument available to him that Gödel cannot use.[103]

11.3.5.9 Gödel's Two 1974 Assessments of the Dialectica Interpretation

The *Collected Works* include a draft letter from Gödel to Frederick Sawyer, already mentioned on Sect. 11.3.5.2, which was probably written not long after February 1, 1974. Gödel there claims that (because of the employment of the notion of reductive proof),

> the implicit use of 'implication' and 'demonstrability' occurring (through the words 'immer ausfuehrbare'[104] and 'erkennbare' in the definition of 'computable function of finite type' on p. 282–283[105] does *not* give rise to any circularity. (Gödel 2003a, 211)

However, the archives also contain the following note, dated February 11, 1974, not long after the day on which Gödel must have received Sawyer's letter (the labels [A], [B], [C] are mine):

> February 11, 1974
>
> [A] My Dialectica paper with the notion of reductive proof does not give an interpretation that excludes the paradoxes (hence the foundation not essentially better than Heyting, namely for this reason, that for example the general concept of computable number-theoretic function occurs and this speaks of a chain of definitions (hence the definition $x \in a \equiv \sim x \in x$ may occur). The difference is only that the concept of evidence is applied only to the correctness of a definitions, not to the correctness of a proof. That is to say, they do not exclude the 'vastness' of the domain in question, as the concepts 'number-theoretic evidence', 'type-theoretical evidence', 'evidence with respect to functional of finite type'

[103]See Sundholm and van Atten (2008, Sect. 6), and the remark on item [C] in Sect. 11.3.5.9 below.

[104]I.e., *if* the arguments are computable.

[105] Wenn die Begriffe 'berechenbare Funktion vom Typus t_0', 'berechenbare Funktion vom Typus t_1', ..., 'berechenbare Funktion vom Typus t_k' (wobei $k \geq 1$) bereits definiert sind, so wird eine berechenbare Funktion vom Typus $(t_0, t_1, \ldots, t_k)$ definiert als eine immer ausführbare (und als solche konstruktiv erkennbare) Operation, welche jedem k-tupel berechenbarer Funktionen der Typen $t_0, t_1, \ldots, t_k$ eine berechenbare Funktion vom Typus t_0 zuordnet. Dieser Begriff ist als unmittelbar verständlich zu betrachten, vorausgesetzt dass man die Begriffe 'berechenbare Funktion vom Typus t_i' $(i = 0, 1, \ldots, k)$ bereits verstanden hat.

etc. do.[106] These concepts as primitive concepts are 'vague'. But perhaps the admissible propositions can be defined precisely (and these would then be a constructed set like the natural numbers, Gentzen), but the concept of number-theoretic meaningful proposition would presuppose the concept of number-theoretic meaningful proof, as it may contain B, hence [is] circular. What is thus accomplished, is threefold:
[B]

1.) 'correct proof' replaced by 'correct Def[inition]',
2.) the proof is mathematically more direct (many 'convolutions' are avoided),
3.) the problem of being and having for existential propositions is solved.

[C] Some normal form theorem for proofs might follow (from 3.)), from which bar induction might follow?? The impossibility to prove something that is absolutely unprovable might follow from an idealisation of proofs using certain primitive notions and then one could define proofs as mathematical proofs by these means and that would suffice for the consistency proof.

But all this, to make sense, presupposes that one has resolved the paradox $\sim x \in x$.[107]

I wish to make the following comments on the parts of this note, starting with [B] and [C].

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[106]*Note MvA*. Because in the first two cases the proofs are generated from below, and in the third case (the notion used in the main text of Gödel 1958, 1972) the evidence is taken to be immediate.

[107]Gödel Papers, 10a/40, 050136. Transcription Eva-Maria Engelen; translation MvA. The bars and underlining are Gödel's.

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[A] Meine Dial[ectica] Arbeit mit dem Begriff des reduktiven Beweis[es] gibt keine die Parad[oxien] ausschließende Interpretation (daher die Fundierung nicht wesentlich besser als Heyting und zwar deswegen, weil zum Beispiel der allgemeine Begriff der berechenbaren zahlentheoretischen Funktion vorkommt und dieser von irgendeiner Def[initions]-Kette spricht (also die Def[inition] $x \in a \equiv \sim x \in x$ kann vorkommen). Der Unterschied ist nur, dass der Begriff Evidenz nur auf Richtigkeit einer Def[inition] nicht auf Richtigkeit eines Beweises angewendet wird. Das heißt also, sie schließen nicht die 'vastness' des betracht[eten] Bereichs aus wie das Begriffe 'zahlentheoretische Evidenz', 'typentheoretische Evidenz', 'Evidenz hinsichtlich Funktion endlichen Typs' etc. tun. Diese Begriffe als Grundbegriffe sind 'vage'. Aber vielleicht kann man präzise die erlaubten Sätze definieren (und diese wären dann eine konstruierte Menge wie die natürlich[en] Zahl[en], Gentzen), aber der Begriff des zahlentheoretisch sinnvollen Satzes würde | den Begriff des zahlentheoretisch sinnvollen Beweis[es] voraussetzen, da er B enthalten kann, also zirkulär [ist]. Was also geleistet wird, ist dreierlei:
[B]

1.) 'richtiger Beweis' ersetzt durch 'richtige Def[inition]',
2.) der Beweis ist mat[hematisch] direkter (es werden viele 'Verschlingungen' vermieden,
3.) das Probl[em] von Sein und Haben für Ex[istenz]sätze wird gelöst.

[C] Es könnte da irgendein Normalform-Th[eorem] für Beweise folgen (aus 3.)), aus welchem der Bar Ind[uktion] folgen könnte ?? Die Unmöglichkeit eine absolute Unbeweisbarkeit zu beweisen, könnte folgen aus einer Idealisierung der Beweise mit gewissen Grundbegriffen und dann könnte man Beweise definieren als mat[hematische] Beweise mit diesen Mitteln und das würde genügen für den Widerspruchsfreiheitsbeweis.

Item 3 in [B] is a reference to Leibniz' theory of truth, in whom Gödel found his inspiration for the notion of reductive proof; I refer to van Atten (Forthcoming) for the argument for this claim, with documentation from the archive. Gödel's adaptation of Leibniz' idea of reductive analysis is indicative of his commitment to Leibnizian ideas even at that late stage of his career.[108]

In [C], 'Bar Induction' is mentioned with an eye on Spector's consistency proof of analysis (Spector 1962), which uses a (generalised) form of Brouwer's principle.[109] The strategy that Gödel proposes here is to find a canonical form for proofs of the antecedent in the principle, like Brouwer; but unlike Brouwer, for Gödel this canonical form cannot be defined in noetic terms, as discussed above.

[A] As documented above (Sect. 11.3.5.7), Gödel knew in D68 that T was impredicative, and also, in D70, that his new notion of reductive proof could not remove this. Gödel was also well aware of alternative foundations of intuitionistic logic and arithmetic that had been proposed from the late 1960s on, in which impredicativity was avoided.

In particular, Gödel of course knew the work by Kreisel and by Goodman on the Theory of Constructions,[110] which however never led to a satisfactory development; Kreisel's version was inconsistent, and in Goodman's version a proof of $A \rightarrow B$ is no longer a construction that is applicable to *any* proof of A. One should also mention here[111] the theory of constructions developed in response to these problems in Scott's 'Constructive validity' (1970): When it ran into problems over decidability, Gödel and Kreisel insisted that one accept abstract proofs and have a 'proof predicate' as a decidable propositional function over the universe of all of them; see the postscript to Scott's paper.[112]

Aber all das, damit es Sinn hat, setzt voraus, dass man der Parad[oxie] $\sim x \in x$ aufgelöst hat.

[108]Given this influence, it would be interesting also to look at Gödel's notion of reductive proof in relation to his remarks on analyticity of mathematics and Leibniz at the end of his Russell paper of 1944 (quoted in Sect. 11.3.5.2), and to the brief exchange on this in the Gödel-Bernays correspondence (Gödel 2003, 194, 200, and also p. 57 of the introduction); but I will not do this here.

[109]As is clear from the subject headings in Gödel's mathematical *Arbeitshefte*, conveniently listed in Dawson and Dawson (2005), Gödel closely studied Brouwer's interpretation of analysis. At the time of writing this, Jan von Plato has announced a talk (at the conference in Aix-en-Provence in July 2013) on these notes in relation to Gödel's thoughts about Gentzen's work, equally documented in these notebooks, in particular with an eye on the question to what extent Gödel may have anticipated Spector's result. I will therefore not attempt to say more about the matter here.

[110]There are reading notes on Kreisel (1965) in Gödel Papers, 11c/28, 060369, and on Goodman 1970 in 10a/40, 050142.

[111]As Sundholm urged me to do.

[112]It is remarkable that Kreisel in his long paper on Gödel and intuitionism (Kreisel 1987) refers to neither his own, nor Goodman's, nor Scott's work on the theory of constructions.

By 1974, Gödel had also studied Howard's seminal manuscript of 1969 (later published as Howard 1980) on what has become known as the Curry-Howard isomorphism.[113] But Gödel wanted to accept abstract proofs as objects in the theory, and, as Artemov (2001, 4) observes, 'as *proof objects* Curry-Howard λ-terms denote nothing but derivations in Int [i.e., formalised intuitionistic propositional logic] itself and thus yield a circular provability semantics for the latter'.

In the notion of reductive proof, Gödel had found, he believed, the right notion that is decidable and narrower than the general notion of intuitionistic proof, with the three advantages listed in [B]. As mentioned above, this depends on understanding 'proof' as demonstration (i.e., acts that have been carried out) instead of (knowable but perhaps unknown) proof-objects. This marks a fundamental difference with the constructive foundations mentioned above and those inspired by them (notably Martin-Löf's Constructive Type Theory).

But the formulation of this view evidently did not lead Gödel finally to publish the revised paper. That is not surprising; already in December 1970 he had written to Bernays that 'The time of publication seems to me to be less important than the

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[113]See Footnote 10 above. In the same email referred to there, William Howard also recalls the following conversation with Gödel, probably in their first meeting during that sabbatical:

> Gödel: 'You should extend your theory of constructions to transfinite types in such a way as to get a functional interpretation of set theory (ZFC).'
> Me: 'I made such an attempt a couple of years ago and concluded that, to carry this out, I would have to learn more set theory.'
> Gödel: 'So, do it.'
> Me: 'Learning a sufficient amount of set theory appears to be a daunting task. There are a lot of papers.'
> Gödel: 'Very little of a *substantial* nature has been done. In fact, if you just read my two papers, that may be sufficient.'
> At that point, he got up, walked across the room to a filing cabinet, pulled out reprints of the two papers (Proc. Nat. Acad. Sciences 1938, 1939) [Gödel 1938, 1939a] and handed them to me, saying, 'Here is what you should read. You may keep these.' Howard, story 5, p. 83

William Howard comments (in the same email):

> Presumably what he had in mind in his first remark was that if my little theory of constructions is extended to transfinite types, in a natural way, as far into the transfinite as possible, the resulting theory would provide an interpretation of a part of ZFC (or a constructive version of a part of ZFC) which would be significantly weaker than ZFC itself. Hence one would have shown an essential limitation on what could be achieved by Brouwer's ideas. In other words, do to Brouwer's program what Gödel had done to Hilbert's program. At least, that was my impression at the time. I seem to recall that he actually said something to that effect, but I don't have any quotation, in my notes for Amy [as part of the preparation for the article Shell-Gellasch 2003], of him saying that to me.

Probably in the Spring of 1973, Gödel encouraged Howard to read Girard's thesis (1972a) to get some ideas towards such an extension to transfinite types. (Details in Howard, story 16, p. 110.) Aczel's interpretation of CZF in Martin-Löf's Constructive Type Theory (Aczel 1978) may be seen as an execution of this project.

improvements to the text' (Gödel 2003, 291), and he certainly didn't have the full details this time either.[114] Also, his bad health at the time may have prevented him from doing substantial further work in any case.

Acknowledgements This is the revised and much extended text of the talk with the same title given at the conference 'Calculability and constructivity: historical and philosophical aspects' of the International Union of the History and Philosophy of Science (Joint Session of the Division of Logic, Methodology and Philosophy of Science and of the Division of the History of Science and Technology), Paris, November 18, 2006. Much of that talk was derived from a manuscript that has in the meantime appeared as part of the present author's contribution to van Atten and Kennedy 2009 (written in 2005). Other versions of that talk were presented at the plenary discussion 'Gödel's Legacy' at the ASL European Summer Meeting in Nijmegen, August 2, 2006 and at seminars in Nancy (2005), Tokyo (2006), Utrecht (2006), and Aix-en-Provence (2007). I am grateful to the respective organisers for the invitations, and to the audiences for their questions, criticisms, and comments.

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Gödel's letters to his brother quoted here are part of a collection of letters that was found in 2006. I am grateful to Matthias Baaz and Karl Sigmund for bringing this correspondence to my attention, and for providing me with photocopies. These letters have been deposited at the Wienbibliothek im Rathaus, Vienna. The quotations appear courtesy of the Kurt Gödel Papers, The Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA.

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[114] 'Der Zeitpunkt des Erscheinens scheint mir weniger wichtig zu sein als die Textverbesserungen.' (Gödel 2003, 290)

Appendix: Finitary Mathematics and Autonomous Transfinite Progressions

Naturally, the draft notes for the revision of the Dialectica paper also contain remarks that are not concerned with intuitionism as such, but with finitary mathematics.

> In support of the admission of abstract objects, note also that it is altogether illusory to try to eliminate abstractions completely, whatever the science in question may be. Even finitism in its strictest form does contain them, since every general concept is an abstract entity (although not necessarily an abstract concept, which term is reserved for concepts referring to something abstract). The difference between finitism and the envisaged extension of it only is that in the former abstractions occurring are only used, but are not made objects of the theory. So the question is not whether abstractions should be admitted, but only which ones and in what sense. It seems reasonable, at any rate, to admit as object of the investigation anything which is admitted for use. This leads to something like the hierarchy described in footn[ote] 7. (Gödel Papers, 9b/148.5, 040498.31)

The example referred to at the end is that of autonomous transfinite progressions, which Gödel describes in Footnote 2 on p. 281 of the 1958 version and Footnotes 4 and f of the 1972 version. On both occasions he refers to the formal work that appeared in print in Kreisel (1960, 1965); but in D68, he moreover writes that he had arrived at this idea when writing his incompleteness paper Gödel (1931), and had considered it finitary:

> That in Mon[ats]H[efte für] Math[ematik und] Phys[ik] 38 (1931), p. 197 I said that finitary mathematics conceivably may not be contained even in formalized set theory is due to the fact that, contrary to Hilbert's conception, I considered systems obtained by reflection on finitary systems to be themselves finitary. (Gödel Papers, 9b/141, 040450, 4F (1968))

and, in a different version with the title 'Kreisel's hierarchy',

> How far in the series of ordinals this sequence of systems reaches is unknown. Evidently it is impossible to give a constructive definition and proof for its precise limit, since this ordinal would then itself be an admissible sequence of steps. When in Mon[ats]H[efte für] Math[ematik und] Phys[ik] 38 (1931) p. 197 I was speaking of 'conceivably' very powerful finitary reasoning, I was really thinking of this hierarchy, overlooking the fact that from a certain point on (and, in fact, already for rather small ordinals) abstract concepts are indispensable for showing that the axioms of the system are valid, even though they need not be introduced in the systems themselves. (Gödel Papers, 9b/146, 040477)

An evaluation of the reliability and importance of these remarks will have to take into account that Gödel is not writing shorthand notes for himself here, but is drafting passages in longhand towards a paper meant for publication. Also, the fact that Gödel did not mention the idea of this hierarchy when he addressed the topic of possible finitary proofs that are not formalizable in Principia in his letter to Herbrand of July 25, 1931 (Gödel 2003a, 22–23), not long after the publication of the incompleteness paper,[115] could well be explained by a quick discovery of his own oversight.

[115] It had appeared in February or March, and by March 25 at the latest (Gödel 1995, 518).

In the 1960s Gödel was inclined to think that the limit of finitary mathematics is ϵ_0. He saw support for this in arguments proposed by Kreisel, Tait, and Bernays; for a discussion of this matter, I refer to Sects. 2.4 and 3.4 of Feferman's introduction to the Gödel-Bernays correspondence in Gödel 2003 and to Tait 2006. Here I add the following element. In D72, Gödel says that Kreisel's 'arguments would have to be elaborated further in order to be fully convincing', and mentions that 'Kreisel's hierarchy can be extended far beyond ϵ_0 by considering as one step any sequence of steps that has been shown to be admissible' (Gödel 1990, 274n(f)). In one of the draft notes he actually endorses that idea:

> Kreisel himself says on p. 177 [of Kreisel 1965] under 3.621: 'the only support for taking ϵ_0 ... as a bound is empirical'. I was formerly myself leaning towards Kreisel's conjecture. But today it seems much more probable to me that the limit of idealized Finitism is quite large. (Gödel Papers, 9b/145, insertion for p. 12)

Feferman has raised the possibility that 'Gödel wanted it seen as one of the values of his work in 1958 and 1972 that the step to the notions and principles of the system T would be just what is needed to go beyond finitary reasoning in order to capture arithmetic' (Gödel 2003, 74). That suggestion finds corroboration in the following passage:

> I do not wish to say that every math[ematical] concept which is non-finitary must nec[essarily] be called abstract, let alone that it must be abstract in the special sense explained below. But I don't think that there is any other ext[ension] of finitism which preserves Hilbert's idea of justifying the infinite of the Platonistic elem[ents] of math[ematics] in terms of what is finite, concretely given & precisely knowable. Note that in contradist[inction] to Plat[onistic] entities, precise thoughts about things that are or can in principle be concretely given & precisely known are themselves something concretely given & precisely knowable.[116] If this ext[ension] of finitism is combined with a training in this kind of int[uition], something in character very close to finitary evidence but much more powerful may result. (Gödel Papers, 9b/147, 040486)

This same passage may also serve to address Tait's suggestion that Gödel, by extending Hilbert's finitary position with thought contents or structures, 'simply doesn't see the "finite" in "finitary"' (Tait 2010, 93). Gödel emphasises that the same criterion that leads Hilbert, who considers only space-time intuition, to a restriction to configurations of a finite number of objects, allows for further, different objects when applied to thoughts, given a correspondingly wider notion of intuition. To hold that everything which is concretely given and precisely knowable is thereby, in a numerical sense or otherwise, finite, is to follow an old tradition.

[116] *Note MvA*. Compare Gödel's formulation in his letter to Constance Reid of March 22, 1966: 'Moreover, the question remains open whether, or to what extent, it is possible, on the basis of a formalistic approach, to prove "constructively" the consistency of classical mathematics, i.e., to replace its axioms about abstract entities of an objective Platonic realm by insights about the given operations of our mind' (Gödel 2003, 187). The quotation marks around the word 'constructively' are there, it seems, to distinguish its sense from that in which a proper part of classical mathematics is constructive; see also Sect. 11.3.5.3.

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Part IV
A Partial Assessment

Chapter 12
Construction and Constitution in Mathematics

Mark van Atten

Abstract I argue that Brouwer's notion of the construction of purely mathematical objects and Husserl's notion of their constitution by the transcendental subject coincide. Various objections to Brouwer's intuitionism that have been raised in recent phenomenological literature (by Hill, Rosado Haddock, and Tieszen) are addressed. Then I present objections to Gödel's project of founding classical mathematics on transcendental phenomenology. The problem for that project lies not so much in Husserl's insistence on the spontaneous character of the constitution of mathematical objects, or in his refusal to allow an appeal to higher minds, as in the combination of these two attitudes.

Keywords Analogy • L.E.J. Brouwer • Categorial intuition • Categorial objects • Classical mathematics • Constitution • Construction • Constructive mathematics • Kurt Gödel • Higher minds • Edmund Husserl • Idealism • Intuitionism • Phenomenology • Platonism • Realism • Time awareness

12.1 Introduction

In the following, I argue that L.E.J. Brouwer's notion of the construction of purely mathematical objects and Edmund Husserl's notion of their constitution coincide. That conclusion will be the combined result of a defence of the following two claims:

1. From a systematical point of view, Brouwer's intuitionistic mathematics should be considered part of Husserl's transcendental-phenomenological foundations of pure mathematics;

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2. Transcendental phenomenology cannot provide a foundation for a pure mathematics that would go beyond intuitionism.

Thus, according to the first claim, Husserl's notion of constitution subsumes Brouwer's notion of construction, but it is, according to the second claim, not a wider one. Already Oskar Becker, in his *Mathematical Existence*, readily assimilated Brouwer's mathematical constructions to Husserl's purely categorial formations; he comfortably writes:

> To the intuitionist, this cognition [that arithmetic cannot be obtained from formal logic] is no surprise, but just a consequence of his view that pure mathematics in general is based on a number of acts and categorial objects that have to be recognised intuitively, such as collecting, ordering, assigning, exchanging, etc., which even allow for an endless expansion.[1]

But he made no attempt at an explicit justification of that assimilation. The second claim, also made by Becker, is worth confronting with (later) efforts, notably by Gödel, to provide a phenomenological foundation of classical mathematics.

Obviously, a defence of these claims requires finding answers to the following questions:

1. Do Husserl and Brouwer deal with consciousness at the same level? Husserl's later phenomenology deals with consciousness at the level of a transcendental subjectivity, but Brouwer's intuitionism is generally seen as a form of (mundane) psychologism.
2. Can any part of phenomenology be mathematics? One might think that phenomenology, which exercises epoché and is purely descriptive of consciousness, has nothing to say on the ontology of mathematics at all.
3. Can Brouwer's notion of the construction of mathematical objects indeed be understood as a form of Husserl's notion of their constitution?

The following discussion will propose answers to these questions, and also address various objections to mathematical intuitionism that have been raised in the recent phenomenological literature, in particular by Claire Ortiz Hill (2010), Guillermo Rosado Haddock (2006, 2010), and Richard Tieszen (1989, 2010), as well as suggestions by Kurt Gödel.

With respect to the physical world, Brouwer was, unlike Husserl, a phenomenalist. In the following, I shall be concerned only with his thoughts on mathematics.

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[1] Für den Intuitionisten ist diese Erkenntnis [daß die Arithmetik nicht aus der formalen Logik herzuleiten ist] nicht verwunderlich, sondern nur eine Folge seiner Auffassung, daß die reine Mathematik überhaupt auf eine Reihe von intuitiv zu erkennenden Akten und kategorialen Gegenständlichkeiten beruht, wie Kolligieren, Ordnen, Zuordnen, Vertauschen usw., die sogar eine Ausdehnung ins Endlose gestatten. (Becker 1927, 23; see also 69 and 196)

12.2 Intuitionistic Mathematics Is Part of Transcendental Phenomenology

12.2.1 Husserl: Pure Mathematics as Formal Ontology

Husserl's introduction of purely categorial objects in the *Logical Investigations* opened the way to a conception of pure mathematics that he never abandoned. Pure mathematics, as a science that aims at knowledge and true judgements, is the theory of the purely categorial formations,[2] which, in turn, he came to identify with formal ontology[3]: 'The whole mathesis universalis is therefore the analytics of possible categorialia, the theory of their essential forms and eidetic laws'[4] And:

> The formal ontology [can] also directly be set as a task from the outset, without taking the idea of a theory of science as the starting point. Its question is then: what can one assert within the empty region 'any object whatever'? Purely a priori, in this formal generality are available the syntactical formations, by means of which from any objects of thought pre-given as possible (anything whatever), always new categorialia can be thought of as

[2]In Sect. 31 of *Formal and Transcendental Logic*, Husserl describes the ideal of the definite manifold, and attempts to characterise it precisely by saying that such a manifold indeed is described by a syntactically complete axiom system. The actual claim that pure mathematics is a definite manifold is implied in a text from 1920–1921: 'The ideal total extension of the purely rational objects and the extension of the objective objects [*objektive Gegenstände*] cognizable to every possible rational I, coincide. It turns out that this extension is a definite manifold, and infinitely constructible system' (Husserl 1974, 388). But Gödel's first Incompleteness Theorem, established shortly after *Formal and Transcendental Logic* was published, shows that in any consistent theory containing arithmetic there are undecidable sentences. A fortiori, the same holds for set theory and the whole of pure mathematics. See also Lohmar (1989, 197n27). Clearly, for Gödel, this fact was not an essential obstacle to embracing Husserl's transcendental-phenomenological approach to the foundations of mathematics, and, more generally, in Gödel's foundational project there hardly seems to have been a role for Husserl's theory of formal systems; given Gödel's realist conception of mathematical objects, this is not surprising. Note that Dietrich Mahnke had raised the question whether pure mathematics is one definite manifold in his *Neue Monadologie* of 1917, and conjectured the answer is yes, while realising that an argument is needed (Husserl marked this passage in his own copy): 'That not all, indeed even rather few, manifolds in the actual world have this property of being definite, is obvious. But also in formal mathematics it is still a big question whether its totality is a heap of infinitely many different and unrelated theories of manifolds, or rather can be organised into one big, definite system. The concept of mathematics seems to demand that the latter is the case. Yet a proof is still to be found' (Mahnke 1917, 32). 'Dass nicht alle, ja sogar nur herzlich wenige Mannigfaltigkeiten der wirklichen Welt diese Eigenschaft der Definitheit haben, liegt auf der Hand. Aber auch in der formalen Mathematik ist noch eine grosse Frage, ob ihre Gesamtheit ein beziehungsloses Nebeneinander von unendlich vielen verschiedenen Mannigfaltigkeitslehren ist oder vielmehr selbst in ein einziges, grosses definites System geordnet werden kann. Der Begriff der Mathematik scheint zu erfordern, dass das letztere der Fall ist. Doch steht der Nachweis dafür noch aus.'

[3]See also Husserl (1974, 82; 1975, 245, 247; 1976a, 26–27; 1979, 166, a text from 1903; 1985a, 52, 55, 61, 78, 167; 2002a, 266).

[4] Die gesamte mathesis universalis ist also Analytik möglicher Kategorialien, Theorie ihrer Wesensformen und Wesensgesetze. (Husserl 1976a, 143)

> produced. One will thereby also come to distinguish possible productions that yield merely distinct meanings, but, being contradictory, can never lead to possible objects themselves, etc. *Obviously, the whole formal mathesis then arises.* (Husserl 1973d, 148, emphasis mine)[5]

The insight that the pure categorialia are available in formal ontology, even though these are not limited to 'any object whatever' (*Etwas-überhaupt*) but are for the most part internally structured objects composed from it, marked an important step for Husserl. In 1910 he wrote:

> A cross to bear for me is the relation between mathematics and ontology as formal ontology. But once I have placed the idea of the 'material' (*des 'sachhaltigen'*), of 'reality' in the sphere of meaning, I obtain the idea of the non-material (*des Nicht-Sachhaltigen*), and that is the ontological in the formal sense.[6]

What changed over the years, however, is that Husserl came to see the difference between the logic of non-contradiction[7] and the logic of truth, and that it is the

[5] Die formale Ontologie [kann] auch von vornherein direkt als Aufgabe gestellt werden, ohne von der Idee einer Wissenschaftslehre auszugehen. Ihre Frage ist dann: was kann man innerhalb der Leerregion Gegenstand-überhaupt aussagen? Rein apriori in dieser formalen Allgemeinheit stehen die syntaktischen Gestaltungen zur Verfügung, durch die aus irgendwelchen als möglich vorgegeben gedachten Gegenständen (Etwas-überhaupt) immer neue Kategorialien erzeugt gedacht werden können. Man wird dabei auch auf den Unterschied möglicher Erzeugungen kommen, die bloß deutliche Meinungen liefern, aber als widerspruchsvolle nicht zu möglichen Gegenständen selbst führen können usw. *Offenbar erwächst dann die ganze formale Mathesis*. (Husserl 1974, 153–154, emphasis mine)

[6] Ein Kreuz ist für mich das Verhältnis von Mathematik und Ontologie als formaler Ontologie. Nachdem ich aber die Idee des 'sachhaltigen', der 'Realität' in der Bedeutungssphäre untergebracht habe, so gewinne ich ja die Idee des Nicht-Sachhaltigen, und das ist das Ont<ologische> im formalen Sinn' (Husserl 1995a, 343). Note the contrast with Alexander Pfänder, who, working in the period 1928–1935, saw mathematics not as part of formal ontology, but as a material region (Pfänder 1973, 42). In his lectures on 'first philosophy', Husserl speaks of both the 'formal region' and the 'formal-ontological quasi-region; (Husserl 1956a, 187n1).

[7] Husserl's description of the mathematics of non-contradiction as 'the mathematics of the mathematicians' (*die Mathematik der Mathematiker*, Husserl 1974, 146) is problematic. On the one hand, it is true that only a mathematician will engage in the study of formal systems for their own sake; on the other hand, after criticism by Henri Poincaré and Brouwer, David Hilbert acknowledged in 1922 that that study requires accepting at least part of pure mathematics as contentual (Hilbert 1922, 165, 174). (Husserl knew this paper; see Lohmar 1989, 216n10.) Roughly, that part is arithmetic including the principle of induction; for elaborate discussion see, e.g., Kreisel (1965), Tait (1981), and Parsons (1998). In other words, pure mathematics cannot be limited to the logic of consequence. James Dodd (in an article that I in many respects agree with) speaks of 'Brouwer's polemical accusation, invited one must say by Hilbert himself, that finitism amounts to a mere empty game with symbols devoid of all sense, and which in the end has no genuine connection to mathematical objectivity at all' (Dodd 2007, 268). (The reference given in Dodd's footnote to this sentence is not, as one would have expected, to a passage where Brouwer makes the alleged accusation but to Husserl 1974, Sects. 33–34.) But Brouwer never did, and never would, equate finitism with empty symbolism devoid of all sense: finitism's characteristic acceptance of the mathematics of natural numbers as contentual is precisely due

latter, not the former, that governs formal ontology; being entirely clear about this distinction is one of the ways in which *Formal and Transcendental Logic* makes significant progress over the *Prolegomena* (Husserl 1974, 15, 76). The mathematics of non-contradiction is not yet formal ontology (Husserl 1974, 150), and not yet formal mathesis in the sense relevant to an interpretation of Brouwer, who recognised the existence of the mathematics of non-contradiction, but was himself exclusively concerned with mathematical truth.[8]

When seen as a positive science, formal ontology is exercised in the natural attitude and, as such, no part of transcendental phenomenology. Yet Husserl came to say that, from the transcendental perspective, it is a part, together with all other a priori ontologies. The rationale of this claim is, briefly, as follows. For the transcendental Husserl, all being, including mathematical being, is constituted being; moreover, every type of existing objects is, as a matter of essence, correlated to the type of act in which they are constituted by the transcendental subject. This is an essential correlation (*Wesensbeziehung*; Husserl 1976a, 159) between transcendental being and transcendent being. It allowed Husserl to hold that transcendental phenomenology is not limited to an eidetic, descriptive study of consciousness as such, but in a sense, namely, through that correlation, comprises all a priori ontologies: 'any clarifying ontological insight obtained within axiomatic clarity, which is not directly phenomenological, becomes so by a mere *change of attitude* [*Blickwendung*], just as, conversely, among all phenomenological insights there must occur ones that by a mere change of attitude become ontological.'[9]

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to Brouwer's influence on Hilbert. (For a discussion of this influence, with full references, see Brouwer (1928A2). Husserl owned a copy of the latter; see van Atten (2007, 128n7).)

[8]In 1908 Brouwer both showed the consistency of the principle of the excluded middle (as its double negation is true, the principle itself cannot be false) and justified his doubts about its truth. 'Consequently the theorems which are usually considered as proved in mathematics, ought to be divided into those that are true and those that are non-contradictory' (Brouwer 1975, 110n2) ('Men behoort dus in de wiskunde de gewoonlijk als bewezen geldende stellingen te onderscheiden in juiste en niet-contradictoire', Brouwer 1908C, 158n2). Brouwer characterises the difference between a true proposition and a merely non-contradictory one by the presence, in the former case, of a mathematical construction that that proposition adequately describes. This corresponds to Husserl's characterisation of truth-logic as the one that, unlike consequence-logic is concerned with the existence of the objects. Note that, before Gödel proved his Incompleteness Theorem, Brouwer was quite optimistic about Hilbert's program to establish the consistency of classical mathematics; in his first Vienna lecture of 1928, he said that 'An appropriate mechanisation of the language of this intuitionistically non-contradictory mathematics should therefore deliver exactly what the formalist school has set as its goal' (Brouwer 1929A, 164). ('Eine geeignete Mechanisierung der Sprache dieser intuitionistisch-nichtkontradiktorischen Mathematik müßte also gerade das liefern, was die formalistische Schule sich zum Ziel gesetzt hat.') Of course, he at the same time insisted that that would have no value for mathematics as such, given its concern, in his view, with constructions.

[9]'alle klärende und im Rahmen der axiomatischen Klarheit vollzogene ontologische Einsicht, die nicht direkt phänomenologisch ist, wird dazu durch eine bloße *Blickwendung*, wie umgekehrt im All der phänomenologischen Einsichten solche auftreten müssen, die durch bloße Blickwendung zu ontologischen werden.' (Husserl 1952, 105) Husserl also comments on the phenomenological

In the 1920s he makes the point by saying that formal logic as formal ontology and formal apophantics is 'a stratum necessarily belonging' to transcendental phenomenology (*eine ihr notwendig zugehörige Schichte*, Husserl 1974, 277). More generally, all possible a priori disciplines are branches (*wesensnotwendige Verzweigungen*, Husserl 1962, 298) of transcendental phenomenology,[10] which is 'as opposed to the only seemingly universal ontology in positivity, the truly universal'.[11] He is explicit about the aptness of transcendental phenomenology to provide the foundations of mathematics: 'For the a priori disciplines that are grounded within phenomenology (for example as mathematical sciences) [there can] be no "paradoxes", no "foundational crises".'[12]

Hill, on the other hand, has recently claimed that

> Transcendental phenomenology has no dealings with a priori ontology, none with formal logic and formal mathematics . . . Transcendental phenomenology is phenomenology of the constituting consciousness, and consequently not a single objective axiom, meaning one relating to objects that are not consciousness, belongs in it, no a priori proposition as truth for objects, as something belonging in the objective science of these objects, or of objects in general in formal universality. (Hill 2010, 62)

In light of the passages from Husserl that we have just seen, as a claim about Husserl's conception of transcendental phenomenology this cannot be right. Admittedly, Husserl does in the same period also say that the universal epoché is exercised 'with respect to all objectivity that is valid for me'[13] which certainly includes mathematics. But there is no contradiction: the point of the passages above, when applied to mathematics, is that the mathematical objectivity known in the natural attitude can be reconstituted, to the extent that this can be done with full evidence, from within the transcendental attitude. An a priori ontology delineates the pure possibilities for objects of a certain type, but in general does not by itself suffice to bring an individual object of that type to intuitive givenness, as in general that depends on the availability of specific material content (*Sachhaltiges*). Purely formal objects are the exception. Their constitution, although ultimately founded on material content, does not depend on any specific material content or type thereof. But then there are no further conditions on the constitution of any particular formal

clarification of geometry by tracing it back to its constitution in nexuses of consciousness: 'That is an application of phenomenology, not phenomenology itself. The predicate 'phenomenological' carries over, of course, to the applications' 'Das ist Anwendung der Phänomenologie, nicht Phänomenologie selbst. Das Prädikat "phänomenologisch" überträgt sich natürlich auf die Anwendungen', (Husserl 1952, 83). As we will see in the main text, later Husserl expressed an even stronger view.

[10] See also Husserl (1988, 18–19; 2002b, 300–301).

[11] gegenüber der nur scheinbar universalen Ontologie in der Positivität die wahrhaft universale (Husserl 1962, 297).

[12] Für die apriorischen Disziplinen, die innerhalb der Phänomenologie zur Begründung kommen (z.B. als mathematische Wissenschaften) [kann es] keine 'Paradoxien', keine 'Grundlagenkrisen' geben. (Husserl 1962, 297)

[13] hinsichtlich aller mir geltenden Objektivität (Husserl 1959, 445).

object than those that exist a priori.[14] As Husserl had already written in the sixth *Logical Investigation*, 'The ideal conditions of the possibility of categorial intuition in general are, correlatively, the ideal conditions of the possibility of the objects of categorial intuition and of the possibility of categorial objects as such'.[15] For purely categorial objects, possibility and being coincide. As a consequence, in this unique case transcendental phenomenology provides the ontology in its whole extension.[16] One can turn immediately from concrete formal-mathematical insights (which are part of positive science) to purely phenomenological ones and vice versa by an appropriate shift of one's regard. As Husserl put it in a text that he dates at the end of the war or perhaps St. Märgen 1921:

> Over all sciences stands a mathesis universalissima, and not as a naive mathematics which, going much farther still than Leibniz' mathesis universalis, constructs the formal-ontological Apriori in a systematical order and develops it in theories, but as a mathematics of cognitive performances, the noetic study of which, carried out in pure subjectivity, comprises the mathematical as a noematic formation of reason and hence as a correlate of consciousness.[17]

That conclusion does not contradict the fact, elaborated on by Husserl in Sects. 72–75 of *Ideas I*, that transcendental phenomenology and mathematics are two different kinds of science, the former being descriptive, the latter deductive. Transcendental phenomenology can describe the constituting performances of the subject's consciousness when it is engaged in doing deductive science. In fact, according to Husserl it is only by giving such descriptions that the axioms and the mediate results of such a science can be philosophically grounded (Husserl 1952, 83; 2002b, 301). The difference in kind is precisely a condition of possibility of such a grounding.[18]

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[14] In a text from 1923, Husserl writes of the formal Apriori and the purely formal objects (Husserl 1959, 225n): 'But here the matter is quite different from that of a geometrical, nature-ontological etc. Apriori; here, on the ontological side the specifications and correlations that have an influence on the sense are lacking, only the general constitutive relation remains' ('Aber hier ist die Sache doch anders als bei einem geometrischen, naturontologischen usw. Apriori; hier fehlen auf ontologischer Seite die sinn-mit-bestimmenden Besonderungen und Korrelationen, es bleibt nur die *allgemeine* konstitutive Korrelation').

[15] Die idealen Bedingungen der Möglichkeit kategorialer Anschauung überhaupt sind korrelativ die Bedingungen der Möglichkeit der Gegenstände kategorialer Anschauung und der Möglichkeit von kategorialen Gegenständen schlechthin. (Husserl 1984b, 718–719)

[16] This point I have argued for in greater detail in van Atten (2002).

[17] Über allen Wissenschaften steht eine Mathesis universalissima, und nicht als eine naive Mathematik, die noch weit über die Leibniz'sche Mathesis universalis hinaus das formalontologische Apriori systematisch geordnet konstruiert und in Theorien entfaltet, sondern als eine Mathematik von Erkenntnisleistungen, deren noetisches und in der reinen Subjektivität vollzogenes Studium das Mathematische als noematisches Gebilde der Vernunft und somit als Bewußtseinskorrelat begreift. (Husserl 1959, 249)

[18] For further discussion of the relation between phenomenology and mathematics, see Gödel (*1961/?) and Yoshimi (2007).

12.2.2 Brouwer: Mathematics as Mental Constructions

Brouwer's intuitionism aims to provide a philosophical foundation for pure mathematics by seeing it as 'an autonomous interior constructional activity' (Brouwer 1981A, 92).[19] In this activity, the subject builds up a stock of mathematical objects. The material out of which these mental constructions are made is provided by the intuition of time, which Brouwer calls 'the basic intuition'. (As I argue below, this sounds more Kantian than it is.) The corresponding notion of existence of a mathematical object then should be defined in terms of mental construction: to say that a mathematical object exists is to say that, in principle, it can be constructed in the mind out of the basic intuition. We will discuss two of Brouwer's own, more detailed characterisations of intuitionism below.

It turns out that various parts of classical mathematics are not constructible according to the principles of intuitionism, and to that extent are, from the intuitionistic point of view, not mathematics proper. For example, intuitionism rejects the universal validity of the principle of the excluded middle, together with most of Cantorian set theory. Conversely, Brouwer introduces objects and principles of reasoning about them that are not acceptable in classical mathematics, such as choice sequences (*Wahlfolgen*), e.g. Brouwer (1918B, 3). It should be emphasised that this revisionism is an outcome, rather than a preset goal, of the intuitionistic reconstruction of mathematics. In principle, it is indifferent to the intuitionist whether the theorems of mathematics as founded on his notion of construction turn out to coincide with those of classical mathematics; what matters is the philosophical foundation.[20] As point of departure, I take two characteristic fragments from Brouwer, which I call A and B, and in which I label various parts for reference in the following discussion.

Fragment A is from the lecture 'Consciousness, Philosophy, and Mathematics' (held in 1948):

A1. First of all an account should be rendered of the phases consciousness has to pass through in its transition from its deepest home to the exterior world in which we cooperate and seek mutual understanding. This account does not imply mutual understanding and in some way may remain a soliloquy. The same can be said of some other parts of this lecture too.

[19]For recent introductions to intuitionism and its history, see Troelstra and van Dalen (1988), Hesseling (2003), and van Atten (2004b). Also the following articles in the on-line *Stanford Encyclopedia of Philosophy*: van Atten (2008, 2009b), Iemhoff (2009), Moschovakis (2008), and Bridges (2009).

[20]Dodd writes that 'Brouwer's revolution [has] run aground on the insuperable technical difficulties in re-establishing classical analysis on exclusively intuitionist principles' (Dodd 2007, 300–301). For an intuitionist, the impossibility to re-establish classical analysis is of no particular philosophical importance. Moreover, a number of theorems of intuitionistic analysis formally contradict classical analysis. That fact also shows that, while previously established theorems of classical analysis may of course serve as a heuristic in the search for new intuitionistic theorems, this is only so up to a point.

A2. *Consciousness* in its deepest home seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation.

A3. And it seems that only the status of sensation allows the initial phenomenon of the said transition. This initial phenomenon is a *move of time*. By a move of time a present sensation gives way to another present sensation in such away that consciousness retains the former one as a past sensation, and moreover, through this distinction between present and past, recedes from both and from stillness, and becomes *mind*.

A4. As mind it takes the function of a subject experiencing the present as well as the past sensation as object. And by reiteration of this two-ity-phenomenon, the object can extend to a world of sensations of motley plurality ...

In the world of sensation experienced by the mind, the free-will-phenomenon of *causal attention* occurs. It performs identifications of different sensations and of different complexes of sensations, and in this way, in a dawning atmosphere of forethought, creates *iterative complexes of sensations* ...

On the other hand there are iterative complexes of sensations whose elements are permutable in point of time. Some of them are completely estranged from the subject. They are called *things*. Forinstance *individuals*, i.e. human bodies, the home body of the subject included, are things ...

The whole of things is called the *exterior world of the subject*. ...

A5. Mathematics comes into being, when the two-ity created by a move of time is divested of all quality by the subject,

A6. And when the remaining empty form of the common substratum of all two-ities, as basic intuition of mathematics, is left to an unlimited unfolding,

A7. Creating new mathematical entities in the shape of *predeterminately or more or less freely proceeding infinite sequences* of mathematical entities previously acquired,

A8. And in the shape of *mathematical species*, i.e., properties supposable for mathematical entities previously acquired and satisfying the condition that if they are realized for a certain mathematical entity, they are also realized for all mathematical entities which have been defined equal to it. (Brouwer 1949, 1235 and 1237)

Fragment B is taken from the lecture ‘Points and Spaces’ (1954):

B1. *The first act of intuitionism* completely separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic. It recognizes that mathematics is a languageless activity of the mind having its origin in the basic phenomenon of the perception of a *move of time*, which is the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory.

B2. If the two-ity thus born is divested of all quality, there remains the common substratum of all two-ities, the mental creation of the *empty two-ity*.

B3. This empty two-ity and the two unities of which it is composed, constitute the *basic mathematical systems*. And the basic operation of mathematical construction is the *mental creation of the two-ity of two mathematical systems previously acquired*, and the consideration of this two-ity as a new mathematical system.

B4. It is introspectively realized how this basic operation, continually displaying *unaltered* retention by memory, successively generates each natural number, the infinitely proceeding sequence of the natural numbers, arbitrary finite sequences and infinitely proceeding sequences of mathematical systems previously acquired, finally a continually extending stock of mathematical systems corresponding to 'separable' systems of classical mathematics.

...

B5. In the edifice of mathematical thought based on the first and second act of intuitionism, language plays no other part than that of an efficient, but never infallible or exact, technique for memorizing mathematical constructions, and for suggesting them to others;

B6. So that the wording of a mathematical theorem has no sense unless it indicates the construction either of an actual mathematical entity or of an incompatibility (e.g., the identity of the empty two-ity with an empty unity) out of some constructional condition imposed on a hypothetical mathematical system.

B7. So that mathematical language, in particular logic, can never by itself create new mathematical entities, nor deduce a mathematical state of things. (Brouwer 1954A, 2–3)

12.2.3 A Systematic Comparison

Given Husserl's conception of pure mathematics as formal ontology, which in turn is a branch of transcendental phenomenology, the two questions whether Brouwer's intuitionism is part of transcendental phenomenology, and, if so, whether it is mathematics as understood in that framework, together amount to the question whether Brouwer's construction of mathematical objects is a form of Husserl's constitution of purely categorial objects by the transcendental subject. These questions can only be answered by taking as concrete an approach as possible: one has to see what the acts have in common in which, according to their writings, Brouwer's constructions and Husserl's purely categorial objects are brought to givenness.[21]

With the two fragments A and B from Brouwer as a guiding thread, I shall now defend the claim that Brouwer's acts of mathematical construction have the

[21]On p. 14 of Eley (1969), there is an interesting reference to 'H.-R. Brennecke, Untersuchungen zur Genealogie der Menge. Das Vorprädikative und Prädikative in der Begründung der Mengenlehre bei E. Husserl und L.E.J. Brouwer (Köln, 1968).' Unfortunately, it turns out that this was a projected dissertation by an assistant at the Husserl Archive in Cologne that was never finished; moreover, at the Archive, not even fragments of it are known. I thank Matthias Wille, Dirk Fonfara, and Dieter Lohmar (all in Cologne) for their clarification of this matter.

following 11 properties in common with Husserl's acts of the constitution of purely categorial objects as described in, notably, the sixth *Logical Investigation*, *Formal and Transcendental Logic*, and *Experience and Judgment*:

C1. These acts result in non-sensuous intuitions of formal objects.
C2. In these acts, the objects are, in the ontic sense, produced.
C3. These acts are active and voluntary.
C4. These acts are prelinguistic.
C5. They are synthetic, objectifying acts that give form to 'matter'.
C6. These acts can be iterated, limitlessly.
C7. These acts are, however, ultimately founded on sensuous intuition.
C8. Although these acts are active and voluntary (C3), the fulfilment of the corresponding intentions is not arbitrary.
C9. The intuition obtained in these acts is closely related to the awareness of inner time.
C10. In the intuitions obtained in these acts, the intuited objects can be adequately given.
C11. The subject that is correlate to these acts is not a psychological, but a transcendental subject, considered in its essential properties; this motivates the introduction of the notion of a single, ideal subject.

Since Husserl's theory of constitution of categorial objects is the better known of the two, references to it in the discussion below sometimes go without further comment. Nor are they meant to be exhaustive.

C1. These acts result in non-sensuous intuitions of formal objects. (Husserl 1984b, Chap. 6)

For Brouwer the result of acts of mathematical construction is an object given in a non-sensuous intuition. The general point was already made in Brouwer's dissertation, where he said that 'in the intellect, mathematical sequences [are] not sensed, yet perceived' (Brouwer 1975, 53).[22] (Note how this shows that Brouwer, like Husserl (e.g., Husserl 1984b, 732) understands perception (*waarneming*) to include the givenness of non-sensuous objects.) In particular, the specification that infinitely proceeding sequences need not be determined by a law (A7) again reveals the intuitive character of these sequences, for it means that even in absence of a (finitely stated) law to go proxy for them, they can be given as themselves. (More on infinite sequences as categorial objects in C10 below.)

C2. In these acts, the objects are, in the ontic sense, produced.

The transcendental Husserl explicitly held that the objects of mathematics are produced by the subject; this means that (ontological) constructivism is not only 'perfectly compatible with transcendental phenomenology', as Rosado Haddock

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[22] in het intellect ... niet gevoelde, doch waargenomen wiskundige volgreeksen (Brouwer 1907, 81).

concedes (Rosado Haddock 2006, 218n144), but transcendental phenomenology leaves, pace Rosado Haddock, no other possibility. How this ontological constructivism of Husserl is related to Brouwer's, and whether it allows Husserl to think of classical mathematics as a constructivism for higher minds, are questions that will be discussed later in the present paper. This view of Husserl on the ontology of mathematical objects is mainly documented in the lecture course *Phenomenological Psychology* of 1925, in *Formal and Transcendental Logic*, and in *Experience and Judgment*.[23] For example, in Sect. 63 of the latter, Husserl draws the contrast between objects of sense perception and categorial objects as follows. Of the former he says: 'What the I in its doing produces is precisely only the representations of [the object of perception], but not the object itself'.[24] Of the latter: 'By contrast, in spontaneous production the state of affairs itself is produced and not a representation of it.'[25] It is clear that Husserl here names the state of affairs just as an example of a categorial formation. A footnote to the first of these two passages emphasises that what is at stake in this discussion is the ontic:

> The fact that from the transcendental point of view also the object [of perception] itself is a product of constitution, can be left out of consideration when drawing this contrast, which concerns an ontic difference.[26]

This stands in sharp contrast to Sects. 22 and 23 in the earlier *Ideas I*. In particular, in Sect. 22 Husserl says, 'I form my representations of numbers … but … a representation of a number is not the number itself',[27] and in Sect. 23, 'In this manner, also in spontaneous abstraction it is not the essence, but the awareness of it that is something produced'.[28] On the basis of such a distinction between the object and our awareness of it, one might begin to develop a phenomenological justification of realism in mathematics.[29] However, Husserl's marginal notes in two of his personal copies of *Ideas I* question and criticise Sects. 22 and 23 on precisely this point (Schuhmann 1973, 161). Notably, regarding Sect. 23 Husserl marked 'wrong, improve!' (*falsch, bessern!*) and

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[23] Husserl (1962, 25–26; 1974, 86, 87, 176, 267, 270, 323; 1985b, 301). Also, e.g., Husserl (1950, 87–88; 1966, 291; 2003b, 147).

[24] Was das Ich hier in seinem Tun erzeugt, sind eben nur die Darstellungen von ihm, nicht aber der Gegenstand selbst. (Husserl 1985b, 301)

[25] Hingegen wird im spontanen Erzeugen der Sachverhalt selbst erzeugt und nicht eine Darstellung von ihm. (Husserl 1985b, 302)

[26] Daß auch der [Wahrnemungs-Gegenstand selbst unter transzendentalem Gesichtspunkt Produkt der Konstitution ist, kann im Rahmen dieser Kontrastierung, die einen ontischen Unterschied betrifft, außer Betracht bleiben. (Husserl 1985b, 301)

[27] Ich bilde meine Zahlvorstellungen … aber … Zahlvorstellung ist nicht Zahl selbst. (Husserl 1976a, 48–49)

[28] So ist auch im spontanen Abstrahieren nicht das Wesen, sondern das Bewußtsein von ihm ein erzeugtes. (Husserl 1976a, 50)

[29] E.g., Tieszen (1989, 178–179; 1992, 189; 2010).

> There are real and ideal products. The production of a 'centaur' in phantasy is production of something ideal and not production of a psychic act (which is confused with its noematic 'object'). Similarly, an essence is an ideal product.[30]

These comments date from the period 1914–1921 (Husserl 1976b, 478). In the 1922–1923 lectures, Husserl puts it thus: 'Every ideal object is what it is only as actus verus, as an everlasting potentiality that I can actualise anytime and perhaps now indeed actualise'.[31] Correspondingly, in *Formal and Transcendental Logic*, Husserl says that the question to be answered is 'how subjectivity *can produce within itself, purely from the sources of its spontaneity, formations* that can have the status of ideal objects of an ideal "world" '.[32]

The view, then, that it is not the mathematical object that is constructed but(only) our awareness of it (Tieszen 1992, 189), or, put differently, that it is in and through the activity of the mind itself that mathematical objects are given to us as mind-independent (Tieszen 2010, Sects. IV and V), finds no support in transcendental idealism as Husserl developed and refined it after *Ideas I*. For the same reason, I see no justification for Rosado Haddock's claims that in *Formal and Transcendental Logic* 'there is no sign of any constructivism' (Rosado Haddock 2006, 218), and that, more generally, not only before but also after the transcendental turn Husserl propounded an 'ontological Platonist mathematics' as opposed to a constructive one (Rosado Haddock 2006, 200n40; 2010, 26–27).[33] In the lectures *Phenomenological Psychology* of 1925, Husserl diagnoses psychologism as a misinterpretation of the exact sense of this production:

> Precisely this circumstance that if the occasion arises, we encounter ideal objects as subjective formations in our forming consciousness and doing, has been the source of the then almost universal psychologisation of the ideal objects. Even if it was made evident that ideal objects, in spite of the fact that they are formed in consciousness, have their own being, being-in-themselves, this posed a large task, never earnestly seen and tackled: namely, that

[30] Es gibt reale und ideale Erzeugnisse. Eine Erzeugung eines 'Kentauren' in der Phantasie ist Erzeugung eines Idealen und nicht Erzeugung des psychischen Aktes (der verwechselt wird mit seinem noematischen 'Gegenstand'). Ebenso ist Wesen ein ideales Erzeugnis. (Husserl 1976b, 482–483)

[31] Jede ideale Gegenständlichkeit ist, was sie ist, nur als actus verus, als immerwährende Potenz, die ich jederzeit aktualisieren könnte une eventuell jetzt wirklich aktualisiere. (Husserl 2002b, 280)

[32] wie die Subjektivität *in sich selbst rein aus Quellen ihrer Spontaneität Gebilde schaffen kann*, die als ideale Objekte einer idealen 'Welt' gelten können (Husserl 1974, 267, emphasis mine).

[33]Note also that if the transcendental Husserl had been a Platonist in any sense that is not compatible with constructive mathematics – there is a compatible sense: one might, perhaps somewhat redundantly, hold that the objects that exist independently correspond exactly to what we can mentally construct – one would not have expected him to be as reserved about the validity of the principle of the excluded middle in truth-logic as he is in *Formal and Transcendental Logic* (Husserl 1974, Sects. 77, 79).

> of making this peculiar correlation between ideal objects of the purely logical sphere and subjective-psychic consciousness as a forming doing into a topic of research.[34]

One important element in the clarification of the *Ansich-sein* of ideal objects is that the way in which they are produced in the acts in which they come to givenness is such that the subject can at any time repeat it and produce identically the same object:

> Indeed, we find ourselves even urged to say: numbers are produced in counting, judgments are produced in judicative doing . . . When I 'repeat' a similar producing, when I carry out once more similar actions of counting, predicating, inferring, then they are, to be sure, psychically a new fact, but I can cognise with evidence that what has come to be there is identically the same pure number, identically the same truth, etc.[35]

And in *Formal and Transcendental Logic*:

> [Ideal objects] are what they are, only 'as arising from' original production. But that does not at all mean that they are what they are only in and during the original production. Their being 'in' the original production means that in it, they are given in consciousness as a certain intentionality of the form of spontaneous activity, in fact in the mode of the original self. This manner of givenness arising from original activity is nothing but the type of 'perception' proper to them.[36]

Similarly, in Sect. 64c of *Experience and Judgment*, Husserl explains in what sense a proposition is omnitemporal, even though it is a 'unity of becoming' (*Werdenseinheit*) and 'the becoming is here, seen from the subject, a being-created' (Husserl 1985b, 309):

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[34] Eben dieser Umstand, daß ideale Gegenstände uns gegebenenfalls als subjektive Gebilde im bildenden Erleben und Tun entgegentreten, war die Quelle der damals fast allgemeinen Psychologisierung der idealen Gegenstände gewesen. Wenn nun auch evident gemacht wurde, daß ideale Gegenstände, trotzdem sie zur Bildung im Bewußtsein kommen, ihr eigenes Sein, Ansich-sein haben, so bestand hier doch eine große und nie ernstlich gesehen und in Angriff genommene Aufgabe: nämlich die, diese eigentümliche Korrelation zwischen idealen Gegenständen der rein logischen Sphäre und subjektiv psychischem Erleben als bildendem Tun zum Forschungthema zu machen. (Husserl 1962, 25–26) See also Husserl (1974, 177–178).

[35] Ja wir sehen uns sogar gedrängt zu sagen: erzeugt werden die Zahlen in Zählen, erzeugt werden die Urteilssätze im urteilenden Tun. . . . 'Wiederhole' ich ein gleiches Erzeugen, übe ich noch einmal gleiche zählende, prädizierende, schließende Aktionen, so sind sie zwar seelisch ein neues Faktum, aber ich kann evident erkennen, daß, was da geworden ist, identisch dieselbe reine Zahl, identisch dieselbe Wahrheit usw. sei. (Husserl 1962, 25–26).

[36] [Ideale Gegenständlichkeiten] sind, was sie sind, nur 'aus' ursprünglicher Erzeugung. Das sagt aber keineswegs, sie sind, was sie sind, nur in und während der ursprünglichen Erzeugung. Sind sie 'in' der ursprünglichen Erzeugung, so sagt das, sie sind in ihr als einer gewissen Intentionalität von der Form spontaner Aktivität bewußt, und zwar im Modus des originalen selbst. Diese Gegebenheitsweise aus solcher ursprünglichen Aktivität ist nichts ander als die ihr eigene Art der 'Wahrnehmung'. (Husserl 1974, 176)

> The irreality of the proposition as the idea of a synthetic unity of becoming is the idea of something that can occur at any point in time, occurs at each of them necessarily temporally and temporally becoming, and yet is 'at all times' the same.[37]

This view of the transcendental Husserl is in fact an adaptation of views he expressed much earlier. Already in *On the Concept of Number* of 1887, he had written:

> The numbers are mental creations inasmuch as they are the results of actions that we perform on concrete contents; but what these actions create are not absolute contents that we can find again somewhere in the 'outside world' but rather peculiar relational concepts that can time and again only be produced, but by no means somewhere be found pre-made.[38]

Correspondingly, in Sect. 27a of *Formal and Transcendental Logic*, Husserl reads his *Philosophy of Arithmetic*, in hindsight, as giving a constitution analysis of categorial objects.

Brouwer always considered mathematics to be a mental creation (A7, B2, B3). Already in his dissertation, he had stated: 'Existence in mathematics means: to be constructed in intuition ... Mathematics is a free creation, independent of experience'.[39] In his own copy of his dissertation, Brouwer later substituted 'constructible' for 'constructed',[40] which is more appropriate; see C11 below. The term 'free' here does not mean 'arbitrary', by C8 below.

In his first Vienna lecture in 1928, Brouwer criticised the tendency both in philosophy and mathematics to introduce 'fetish-like "concepts"' and 'ideal truths' (*fetischartige 'Begriffe'*, *ideale Wahrheiten*) about them that do not originate in the activity of the mind (Brouwer 1929A, 159–160); the point being that it is through such activity that the genuine concepts and truths about them are given in intuition. Unlike Hill (2010, 55), I take this to indicate a broad agreement, not a disagreement, between Brouwer and Husserl.

C3. These acts are active and voluntary (Husserl 1984b, Sect. 62; 1974, Sect. 63; 1985b, Sects. 61 and 63).

This third common property is really an aspect of C2 that is worthy of particular emphasis.

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37 Die Irrealität des Satzes als Idee einer synthetischen Werdenseinheit ist Idee von Etwas, das an jeder Zeitstelle auftreten kann, an jeder notwendig zeitlich und zeitlich werdend auftritt, und doch 'allzeit' dasselbe ist. (Husserl 1985b, 311)

38 Geistige Schöpfungen sind die Zahlen, sofern sie Resultate von Tätigkeiten bilden, die wir an konkreten Inhalten üben; aber was diese Tätigkeiten schaffen, das sind nicht absolute Inhalte, die wir irgendwo in der 'Aussenwelt' wiederfinden können, sondern es sind eigentümliche Relationsbegriffe, die immer wieder nur erzeugt, aber keineswegs irgendwo fertig vorgefunden werden können. (Husserl 1970, 317)

39 bestaan in de wiskunde beteekent: intuïtief zijn opgebouwd ... De wiskunde is een vrije schepping, onafhankelijk van de ervaring. (Brouwer 1907, 177, 79)

40 In the original, *op te bouwen* for *opgebouwd*; see van Dalen (2001a, 134n(g)).

A categorial object is a spontaneous accomplishment of the ego ('in the presence of the I' (*unter Dabeisein des Ich*', Husserl 1985b, 301.)). To each part of a purely categorial object that is given in intuition necessarily corresponds a preceding voluntary act (an 'operation') in which that part was explicitly intended and produced. The syntheses involved are active and carried out in freedom. A consequence that Husserl notes is that to break off the constitution of a particular categorial object at an arbitrary point implies the failure to constitute it at all (Husserl 1985b, 302). For Brouwer, having sensations is passive (A2). But mathematics, he wrote in his dissertation (Brouwer 1907, 179), is 'a free construction' (*een vrije schepping*). In 'Mathematik, Wissenschaft und Sprache', he writes that mathematics is an 'activity of mankind' which has its origin in 'the individual man's will to live'. The perception of a two-ity, which he there calls 'the intellectual ur-phenomenon', is not a 'passive attitude' but an 'act of the will' (Brouwer 1929A, 153–154).[41] A particular role of the will in Brouwer's foundations of mathematics is seen in the construction of infinitely proceeding sequences of elements that the subject freely chooses (A7).

C4. These acts are prelinguistic.

For Husserl, authentic acts of thought, hence in particular also authentic acts of purely categorial thought, are not of a linguistic nature. These intuitive acts can, within certain limits, be expressed in language, but these acts as such do not depend on anything signitive (Husserl 1984b, 672–3, 731; 1985b, Sect. 50b).

Rosado Haddock has argued that the view that for Husserl, linguistic expressions are not essential for meaningful thinking, is, in the case of mathematics, 'questionable, to say the very least': 'A study of Husserl's philosophy of mathematics ... shows very clearly that in his theory of deductive theories in which logic and mathematics build, respectively, the logico-linguistic and the ontological main components, the most basic part, presupposed by the rest of the edifice, is a theory of pure logical grammar, which is essentially a theory of the syntactic categories and the syntactic rules of formation' (Rosado Haddock 1991, 357–358). But as we saw in the quotation at the beginning of Sect. 12.2.1, Husserl explicitly noted the fact that formal ontology can be approached directly as the theory of purely categorial formations, leaving aside the theory of scientific theories. Scientific theories are linguistic objects, but purely categorial formations are not.[42]

Brouwer takes pure mathematics to be 'essentially languageless' (B1). The practical importance of language is recognised (B5), but nothing linguistic enters

[41] *Aktivität der Menschheit*; *der Wille zum Leben des einzelnen Menschen*; *das intellektuelle Urphänomen*; *passive Einstellung*; *Willesakt*.

[42] In this context, it is interesting that Husserl decided to use the term 'syntactic' in two distinct senses: with respect to linguistic form, and with respect to the categorial form of an object: Husserl (1976a, Sect. 11; 1974, Sect. 42b–d, Beilage I; 1985b, 247n).

into the conditions of possibility of acts of mathematical construction as such: 'Formal language accompanies mathematics as a score accompanies a symphony by Bach or an oratorio by Handel.'[43] No linguistic act can bring a mathematical object into existence; that can only be done in a languageless act of construction (C2). 'I can conceive mathematical truth which can never be fixed down in any system of formulas ... But even when the formal system coincides with intuitive mathematics, or expressed more exactly, when they are parallel, then exactness lies in the intuition, never in the formula' (Brouwer 1975, 452).[45] Similarly: 'The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect, the formalist says: on paper' (Brouwer 1913C, 83).

For one specific type of mathematical object and one specific mathematical object Brouwer gave arguments to the effect that not only are they originally non-linguistic, but they cannot be properly represented in language. Of mathematical demonstrations understood as mental objects, he said:

> These mental mathematical demonstrations, which in general have infinitely many parts, should not be confused with their linguistic accompaniments, which are finite, necessarily inadequate, and hence not belonging to mathematics.[46]

And he goes on to call this 'my main argument against the aspirations of Hilbert's metamathematics'.[47] And of the continuum, he remarked that it is given in a priori intuition, yet cannot be exhausted by a language with denumerably many

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[43] 'de formuletaal, die de wiskunde begeleidt als het notenschrift een symfonie van Bach of een oratorium van Händel' (Brouwer et al. 1937, 262). For a reason unknown to me, the (partial) translation 'Signific Dialogues' (Brouwer 1975, 447–52), which as far as I know is not by Brouwer, substitutes 'Formal language accompanies mathematics as the weather-map accompanies the atmospheric processes' (Brouwer 1975, 451); that simile is not inappropriate, but the original is better, because it is a richer analogy.[44]

[44] ⟦Here I was mistaken. As is now evident to me, the translation in Brouwer 1975 was made from the publication of the 'Signifische Dialogen' in book form (Brouwer et al. 1939), not from the prior publication in the journal *Synthese* (Brouwer et al. 1937); the bibliographical details as stated in Brouwer 1975, p. x, do not mention there are differences between the two. But already in the book, the musical image had been replaced by '... de formuletaal, die de wiskunde begeleidt als het weerkaartje het atmosferisch gebeuren' – which the English quoted above renders correctly. In the preface to the book, Mannoury says that this edition is 'more complete', and he thanks the editorial board of *Synthese*, but he does not say that changes have been made, let alone by whom or why. Be that as it may, I apologise to the translators.⟧

[45] kan ik mij wiskundige juistheid denken, die nooit in enig formulesysteem kan worden vastgelegd ... Maar ook áls de intuïtieve wiskunde en de formalistiek overeenstemmen, of juister uitgedrukt: parallel lopen, dan ligt het exakte in de intuïtie, maar nooit in de formule. (Brouwer et al. 1937, 262–263)

[46] Diese gedanklichen, im allgemeinen unendlich viele Glieder aufweisenden mathematischen Beweisführungen dürfen mit ihren endlichen, notwendigerweise inadäquaten, mithin nicht zur Mathematik gehörenden sprachlichen Begleitungen nicht verwechselt werden. (Brouwer 1927B, 64n8)

[47] mein Hauptargument gegen die Anspruche der Hilbertsche Metamathematik.

expressions, as the real numbers are not denumerable (Brouwer 1930A, 3, 6). Thus, Husserl and Brouwer agree that there is prelinguistic thought, and that this is moreover the authentic mode of thinking, to which other subjects have only indirect access, through the mediation of language. This renders authentic thought (and hence ideal cognition) to a certain extent private (Bernet et al. 1989, 177), and this is the sense in which Brouwer argues that there is no 'plurality of mind' (Brouwer 1949, 1239–1240). For Brouwer, the construction of mathematics can, ideally, wholly take place in authentic thought. As argued below (C11), the constructions in question are based on essential properties of the mind, which are therefore shared by any other subject. There is no reason, then, to think that Brouwer's epistemological solipsism entails a denial of the intersubjective validity of mathematics, nor that Husserl, in so far as he too holds that mathematics can ideally wholly be done in authentic thought, would have an essentially different view on this matter.[48]

Logic, on Brouwer's understanding of the term, is an application of mathematics to the language of mathematicians. It is the study of patterns in linguistic recordings of our activities of mathematical construction, and in particular the patterns that characterise validity of inferences, defined as the preservation of constructibility. This understanding is, in one sense, much closer to modern mathematical logic than Husserl's, for whom it also included formal ontology. But the difference is only terminological: Husserl would call Brouwer's logic 'formal apophantics', whereas logic in Husserl's wide sense corresponds to Brouwer's logic and pure mathematics combined.

C5. They are synthetic, objectifying acts that give form to 'matter' (Husserl 1984b, Sect. 40). Husserl formulates this idea particularly clearly in his lectures of 1906–1907:

> But as a matter of principle, mathematics and pure logic in general ... are a fund of purely intellectual truths ... They are thoroughly and purely grounded in the pure 'forms of thought', in the essence of formal thoughts about meanings and objects, which resemble templates in which first a material must be poured in order that material thoughts relating to material objects can result.[49]

In Brouwer, creating the two-ity of two mathematical systems previously acquired is to synthesise the latter into a new object of which they are parts, and thereby to give a form to the matter which these systems are (B3). In A3 and A4, where the passage

[48]For a different view, see Tieszen (1995, 453). For extensive discussion of intersubjectivity and intuitionism, see Placek (1999) and van Atten (2004b, Chap. 6).

[49] Prinzipiell aber bildet die Mathematik und reine Logik überhaupt ... ein[en] Fonds reiner Verstandeswahrheiten ... Sie gründen vielmehr durchaus und rein in den bloßen 'Denkformen', in dem Wesen der formalen Bedeutungs- und Gegenstandsgedanken, die Schablonen gleichen, in die erst ein Stoff eingefüllt sein muß, damit sachhaltige Gedanken mit Beziehung auf sachhaltige Gegenstände resultieren können. (Husserl 1985a, 61)

of sensations comes to found a new act (see Husserl 1984b, 679), one recognises the steps of preconstitution and subsequent objectivation that according to Husserl are characteristic for the constitution of syntactical objects (Husserl 1985b, Sects. 58 and 61).

The same matter may admit of different categorial formations (Husserl 1984b, Sect. 62; 1985b, 303), in which case the subject is free to choose which one to make. Correspondingly, Brouwer writes that this is how the basic intuition gives rise to a number of different fundamental notions:

> There are elements of mathematical construction that in the system of definitions must remain irreducible, and which therefore, in communication, must be understood from a single word, sound or symbol; they are the elements of construction that are immediately read off from the basic intuition or intuition of the continuum; notions such as continuous, unity, once more, and so on are irreducible.[50]

In a handwritten note in his own copy of his dissertation, Brouwer said that these notions are each different 'polarisations' (*polarizeringen*) of the basic intuition (van Dalen 2001a, 136).

C6. These acts can be iterated, limitlessly (Husserl 1984b, Sect. 60; 1974, Beilage I 6b; 1985b, Sect. 61).

For Brouwer, the object that is the result of combining mathematical systems in a two-ity is itself again a mathematical system (B3). Mathematical objects, once constructed, can be used as parts in the construction of further mathematical objects (B4); this iteration is the 'unlimited unfolding' mentioned in A6. Note that Husserl, similarly, speaks of 'the ideally iterative production of forms in open infinity' (Husserl 1974, 305).

The status of infinitely proceeding sequences as mathematically constructible objects, and thereby given in intuition (A7, B4; also the quotation from Brouwer's dissertation in C1 above) depends on the givenness of this iterative form itself. In the early development of intuitionism, Brouwer had not yet recognised the possibility of choice sequences, but he later commented that 'the extension [of mathematics with choice sequences] is an immediate consequence of the self-unfolding' (Brouwer 1981A, 93n). Husserl recognises this iteration as a categorial form in *Experience and Judgment* (Husserl 1985b, 258–259); the matter is discussed further in (van Atten 2007, Sect. 6.2).

Rosado Haddock sees in the iterative character of purely categorial formation an affinity with classical set theory as opposed to constructivism:

> His epistemology of mathematics of the second part of the sixth *Logical Investigation* . . . offers an iterative constitution of mathematical objects in categorial intuition. Such a

[50] Er zijn elementen van wiskundige bouwing, die in het systeem der definities onherleidbaar moeten blijven, dus bij mededeeling door een enkel woord, klank of teeken, weerklank moeten vinden; het zijn de uit de oer-intuïtie of continuumintuïtie afgelezen bouwelementen; begrippen als *continu*, *eenheid*, *nog eens*, *enzoovoort* zijn onherleidbaar. (Brouwer 1907, 180)

> view is clearly related to the views of his friends Cantor and Zermelo on the iterative notion of set, which is not to be related with constructivisms of Kantian or Brouwerian, or any other sort. (Rosado Haddock 2006, 219)

But we see that iteration is as prominently present in Brouwer's notion of construction. More generally, iteration is such a conspicuous phenomenon in both classical and constructive mathematics, that Husserl's emphasis on it can, as such, hardly be taken to indicate a proximity to the one rather than the other.

C7. These acts are, however, ultimately founded on sensuous intuition (Husserl 1984b, Sects. 48, 60).

The basic intuition of the empty two-ity is generated by emptying any concrete two-ity of its sensuous content, and hence depends on it for its existence (A5; B2). As any further mathematical objects are constructed from that basic intuition, by transitivity these objects are likewise founded on sensuous intuition.

C8. Although these acts are active and voluntary (C3), the fulfilment of the corresponding intentions is not arbitrary (Husserl 1984b, Sect. 62).

The fact that constructions are voluntary but not arbitrary, that there is an interplay of the active and the passive, is explicitly recognised in intuitionism:

> The only possible foundation of mathematics must be sought in this construction under the obligation carefully to watch which constructions intuition allows and which not. [51,52]

Similarly, note the 'limits' that Heyting invokes when he writes that

> [Brouwer's] construction of intuitionist mathematics is nothing more nor less than an investigation of the utmost limits which the intellect can attain in its self-unfolding (Heyting 1968, 314).

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[51] . . . dat dus in dezen opbouw, onder de verplichting, zorgvuldig acht te geven, wat de intuïtie veroorlooft te stellen en wat niet, de eenig mogelijke grondvesting der wiskunde is te zoeken (Brouwer 1907, 77).

[52] This also provides an answer to the following objection to intuitionism, formulated by Lohmar (1989, 212): 'Other doubts in turn are directed at the view that the objects of mathematics are produced in the mathematician's acts. However understandable this view is as a counter-reaction to Platonism, and in spite of its pointing to the contribution of actions to the constitution of mathematical objects, it conceals the fact that cognition and itself-givenness of mathematical connections are founded on something which, in the activity that leads up to them, occurs passively. In mathematics, too, all we can do is to bring ourselves to the point where cognition either takes place or not.' ('Andere Bedenken richten sich wiederum auf die Ansicht, daß die Gegenstände der Mathematik im Handeln des Mathematikers erzeugt werden. So verständlich dies als Gegenreaktion zum Platonismus ist und so klarsichtig hiermit auf den Anteil an Handlungsaktivität hingewiesen wird, der in der Konstitution mathematischer Gegenständlichkeiten enthalten ist, so wird damit doch überdeckt, daß Erkennen und Selbstgegebenheit mathematischer Zusammenhänge auf etwas beruht, das sich in der Aktivität des Heranführens passiv einstellt. Auch in der Mathematik gilt, daß wir nur an den Punkt heranführen können an dem sich Erkennen einstellt oder nicht.')

C9. The intuition obtained in these acts is closely related to the awareness of inner time.

In transcendental phenomenology, there is an intimate relation between time and formal ontology, as the awareness of inner time constrains the form of all particular objects and multiplicities of objects:

> While time consciousness is the primal site of the constitution of the unity of identity or of objecthood [*Gegenständlichkeit*], and then of the combination-forms of coexistence and succession of all objects [*Gegenständlichkeiten*] that come to givenness in consciousness, it is nevertheless no more than the consciousness that produces a general form. Mere form is of course an abstraction, and hence the intentional analysis of time consciousness and its accomplishment is from the outset an abstractive one. It apprehends, is only interested in the necessary temporal form of all singular objects and pluralities of objects, or correlatively in the form of the manifolds that constitute the temporal [object].[53,54]

Anticipating on Sect. 12.3 below, I remark that, in particular, the form of time constrains the purely categorial formations, and thereby constrains the cardinality of mathematical multiplicities that can ideally be given in intuition. (Outside a strictly phenomenological context, the relation between the form that time has for a subject and the cardinality of the mathematical constructions that this subject can carry out has been remarked on by Charles Parsons, in a discussion of the iterative concept of set. More on that below.)

In Brouwer, the constraint of the form of time on the mathematical objects is immediate (A3–A5, B1–B2). Moreover, like transcendental phenomenology (Mensch 1996, 109), intuitionism recognises the fundamental role of time awareness in the genesis of intentionality itself (A3). In this context, Brouwer takes care not to understand time as mundane:

> And since in this intuition [i.e., the basic intuition of mathematics] we become conscious of time as change per se, we can state:
> The only a priori element of science is time. (Brouwer 1975, 61)[55]

[53] 'Ist nun das Zeitbewusstsein die Urstätte der Konstitution von Identitätseinheit oder Gegenständlichkeit, und dann der Verbindungsformen der Koexistenz und Sukzession aller bewusst werdenden Gegenständlichkeiten, so ist es doch nur das eine allgemeine Form herstellende Bewusstsein. Bloße Form ist freilich eine Abstraktion, und so ist die intentionale Analyse des Zeitbewusstseins und seiner Leistung von vornherein eine abstraktive. Sie erfasst, interessiert sich nur für die notwendige Zeitform aller einzelnen Gegenstände und Gegenstandsvielheiten, bzw. korrelativ für die Form der Zeitliches konstituierenden Mannigfaltigkeiten.' (Husserl 1966, 128) Also Husserl (1966, 312; 1974, Beilage II 2c, 318; 1950, 99).

[54] ⟦For a detailed discussion of the role of inner time awareness in the constitution of sets, see van Atten (2015).⟧

[55] En daar deze samenvalt met de bewustwording van den tijd als verandering zonder meer, kunnen we ook zeggen:
Het eenige aprioristische element in de wetenschap is de tijd. (Brouwer 1907, 99)

to which he adds in a footnote:

> Of course we mean here intuitive time which must be distinguished from scientific time. By means of experience and very much a posteriori it appears that scientific time can suitably be introduced for the cataloguing of phenomena, as a one-dimensional coordinate having a one-parameter group. (Brouwer 1975, 61n2)[56]

C10. In the intuitions obtained in these acts, the intuited objects can be adequately given.

Husserl writes in the sixth *Logical Investigation*, 'The ideal conditions of the possibility of categorial intuition in general are, correlatively, the ideal conditions of the possibility of the objects of categorial intuition and of the possibility of categorial objects as such'.[57] The justification of this claim would seem to be that, because of the wholly spontaneous character of its constitution (C3), a purely categorial object can only have a certain part if an act of the will has been carried out in which that part arose by giving form to some matter at hand. This means that if we make the idealisation of a perfect memory, all parts of a purely categorial object are necessarily simultaneously and adequately perceivable. Husserl explicitly denies the possibility of ideal objects that cannot, in principle, be adequately given, and he therefore says that ideal objects are 'ideally immanent' (Husserl 1974, 389–90). Since for Brouwer, too, these acts are wholly voluntary, the same argument applies. The emphasis on 'unaltered retention by memory' (B4) reflects a point made in a lecture of 1932, 'Will, Knowledge, Speech':

> The languageless constructions which arise from the self-unfolding of the basic intuition are, on the sole basis of their presence in memory, exact and correct, but the human faculty of memory which must survey these constructions is, even when it seeks the support of linguistic signs, by its nature limited and fallible. (Brouwer 1975, 443, translation modified)[58]

With an eye on Brouwer's infinitely proceeding sequences (A7, B4), it should be noted that ideal, adequate givenness of a potentially infinite sequence does not consist in its being given as an actually infinite sequence, for that would contradict the essence of the object qua potentially infinite. (On this see also Dummett 2000b, 41–43.) Rather, it consists in the givenness of the whole finite initial segment

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[56] Natuurlijk wordt hier bedoeld de intuïtieve tijd, wel te onderscheiden van de wetenschappelijke tijd, die, wel zeer a posteriori, eerst door de ervaring blijkt, als met een eenledige groep voorziene eendimensionale coördinaat geschikt te kunnen ingevoerd tot het katalogizeeren der verschijnselen. (Brouwer 1907, 99n1)

[57] Die idealen Bedingungen der Möglichkeit kategorialer Anschauung überhaupt sind korrelativ die Bedingungen der Möglichkeit der Gegenstände kategorialer Anschauung und der Möglichkeit von kategorialen Gegenständen schlechthin. (Husserl 1984b, 718–719)

[58] ...dat de door de zelfontvouwing der oerintuïtie ontstaande taallooze constructies, uit kracht van hun in de herinnering aanwezig zijn alleen, exact en juist zijn, dat echter het menschelijk herinneringsvermogen, dat deze constructies heeft te overzien, ook als het linguïstische teekens te hulp roept, uit den aard der zaak beperkt en feilbaar is (Brouwer 1933A2, 58).

generated so far, however large the number of its elements may be, together with the open horizon that adequately indicates the ever present possibility to construct additional elements of the sequence. The absence of such further elements from an intuition of the sequence at a given moment does not render that intuition inadequate, because they do not yet even exist. In contrast, the reason why our intuition of a physical object at a given moment is necessarily inadequate is precisely that, as a matter of three-dimensional geometry, any concrete view of the object hides parts that do at that moment exist. Such perspectival givenness is not a feature of purely mathematical objects. Like Brouwer, Husserl holds that an infinite sequence can only be given as a potentially infinite sequence, for example in Sect. 143 of *Ideas I*, and again in *Nature and Spirit* of 1927:

> How can an infinity be experienced? Only in this way, that a finite stock at a time directly falls within experience, and is at the same time the carrier of a horizonal presumption, a reference to a subjectively possible progress to new experience, etc.[59]

To the extent that Husserl is here describing consciousness eidetically, and therefore is not describing just human consciousness, it follows from this passage together with his observation that the ideal conditions of categorial intuition are, correlatively, the conditions of possibility of categorial objects (quoted above), that for him infinite sequences are possible only as potentially infinite sequences. But perhaps one considers the view Husserl here expresses on the experience of infinite sequences to be valid for beings like us, but not necessarily for higher minds, who might be able to experience actually infinite sequences. The question whether in Husserl's phenomenology there indeed is room for a consideration of this type is dealt with in Sect. 12.3 below, where it will be answered in the negative.

That potentially infinite sequences are indeed categorial objects is stated explicitly in *Formal and Transcendental Logic*:

> I mention here only the fundamental form, never stressed by the logicians, of the 'and so on', of the iterative 'infinity',[60] which has its subjective correlate in 'one can again and again' ... Mathematics is the realm of infinite constructions, a realm of ideal existences, not only in a 'finite' sense, but also of constructive infinities. Obviously the problem of the subjective constitutive origins repeats itself here as the hidden construction method which should be

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[59] Wie kann eine Unendlichkeit erfahren sein? Nur so, dass ein endlicher Bestand jeweils direkt in die Erfahrung fällt und zugleich Träger ist einer Horizontpräsumtion, einer Verweisung auf einen subjektiv möglichen Fortgang zu neuer Erfahrung usw. (Husserl 2001b, 107)

[60] Husserl's historical claim here is not quite correct: e.g., Brouwer in his dissertation (1907) and Ludwig Wittgenstein in his *Tractatus logico-philosophicus* (1921) had thematised the notion 'and so on'. Brouwer (1975, 80n): 'The expression "and *so* on" means the indefinite repetition of *one and the same* object or operation, even if that object or that operation is defined in a rather complex way' ('Waar men zegt "en *zoo* voort", bedoelt men het onbepaald herhalen van *eenzelfde* ding of operatie, ook al is dat ding of die operatie tamelijk complex gedefinieerd', Brouwer 1907, 143n); Wittgenstein: 'The concept of the successive application of an operation is equivalent to the concept "and so on".' ('Der Begriff der successiven Anwendung der Operation ist äquivalent mit dem Begriff "und so weiter".', Wittgenstein (1921) 2013, 52 (5.2523))

> revealed and reconstructed as norm, the method in which the 'and so on' in various senses and the infinities as categorial formations of a new kind become evident.[61,62]

(I think Husserl's references to construction and method here should indeed be interpreted as the constructive mathematics it suggests. But as Husserl does not quite make explicit how he understands the notions of construction and method, I prefer to let this reading be justified by the arguments in the rest of this paper, instead of using this quotation itself as evidence for Husserl's constructivism.)

C11. The subject that is correlate to these acts is not a psychological, but a transcendental subject, considered in its essential properties; this motivates the introduction of the notion of a single, ideal subject.

Like phenomenology, intuitionism studies essential, structural properties of consciousness, not those of any particular individual's consciousness. Brouwer characterised intuitionism as 'inner architecture' (Brouwer 1949, 1249), and was interested in the question what mathematical constructions this inner architecture in principle allows, given unlimited memory, time, and so on. Among the various indications of that fact is Brouwer's claim that the subject can construct infinitely proceeding sequences; any factual notion of subject would have ruled out that claim, as factual humans are limited to the construction of finite sequences before they pass away. To use Noam Chomsky's distinction: Intuitionism does not study the performance of human consciousness in making certain constructions, but its competence.[63] Its focus on essential properties of the mind also allows one to account for the intersubjective validity of intuitionistic mathematics. Brouwer's statement that, for the subject, language serves to suggest mathematical constructions to others (B5) implicitly contains the claim that other subjects exist and have the same constructional means at their disposal.

As Brouwer wrote, 'The stock of mathematical entities is a real thing, for each person, and for humanity' (Brouwer 1981A, 90).

Like phenomenology, intuitionism recognises that the fundamental notion of subject is not psychological but transcendental. Brouwer's notion, as described in fragment A, does not presuppose the construction of the world. As a comparison of A4 and A5 shows, the construction of the world, including the subject as a mundane subject, requires preserving the sensations that make up the concrete two-ities, while

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61 Ich erinnere nur noch an die von den Logikern nie herausgehobene Grundform des 'Und so weiter', der iterativen 'Unendlichkeit', die ihr subjektives Korrelat hat im 'man kann immer wieder'. ... Die Mathematik ist das Reich unendlicher Konstruktionen, ein Reich von idealen Existenzen, nicht nur 'endlicher' Sinne, sondern auch von konstruktiven Unendlichkeiten. Offenbar wiederholt sich hier das Problem der subjektiven konstitutiven Ursprünge als der verborgenen, zu enthüllenden und als Norm neu zu gestaltenden Methode der Konstruktionen, der Methode, in der das 'und so weiter' verschiedenen Sinnes und die Unendlichkeiten als neuartige kategoriale Gebilde ... evident werden. (Husserl 1974, 196)

62 See also Husserl (1985b, 258–259).

63 For Brouwer's rejection of psychological interpretations of intuitionism in a letter to Van Dantzig from 1949, see van Atten (2004b, 75–76).

the construction of mathematics requires exactly the opposite, namely, abstracting from all qualitative aspects of the two-ities. The notion of subject that is correlate to Brouwer's acts of mathematical construction is therefore not that of a subject in the world, and cannot be psychological. 'Mathematics is certainly completely independent of the material world'[64]; moreover, for Brouwer, the constitution of the mundane presupposes the mathematical, because he conceives of the mundane as a mathematical construction out of sense data (Brouwer 1929A, 153; 1949, 1235). Lohmar has objected to intuitionism that its notion of the subject is mundane (Lohmar 1989, 211; 2004, 63); while this seems to be the case for Heyting, to whom Lohmar refers in this context, it is not the case for Brouwer.

Given that Brouwer and Husserl are interested in the essential possibilities the subject has, it is no surprise to see both introduce in their writings the notion of a possible ideal subject who suffers from no limitations on the practical realisation of these possibilities due to non-essential factors (e.g., limits of time, space, attention, memory), and to whom therefore any object that can ideally be given in these acts can indeed be given adequately. Husserl points to the correlation between, on the one hand, ideal objects, ideal possibilities, ideal necessities, and, on the other hand, ideally possible subjects ('ideal mögliche Subjekte', Husserl 2002b, 279) and speaks of the ideal cognising subject (*ideales Erkenntnissubjekt*, Husserl 1974, 383–387).[65] That it is a justifiable idealisation to hold that this ideal knowing subject can always continue the constitution of a potentially infinite sequence has its ground in a structural property of the transcendental subject's time awareness. As Husserl says, the transcendental subject is immortal, it is a subject for whom 'the future means infinite time', and who is 'an eternal being in becoming'.[66]

Brouwer first thematises the notion of an ideal subject in the form of 'hypothetical human beings with an unlimited memory' (Brouwer 1975, 443),[67] and later calls it 'the creating subject' (Brouwer 1975, 478–479).[68] That such a hypothetical being is conceived as having an infinite future is implied by the fact that Brouwer's mathematics requires it to be able to extend, for example, the sequence of the natural numbers and also choice sequences arbitrarily far (the latter is necessary for a choice sequence of rationals to converge to an irrational value). For both Husserl and Brouwer, then, to say that a certain mathematical object exists is to say that it can be brought to givenness by their respective ideal subjects. Whether this also means that they arrive at the same mathematics is discussed in Sect. 12.3.

These common points C1–C11 express how constitution (of purely categorial objects) and construction function in Husserl and Brouwer, and hence how these

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[64] De wiskunde is zeker geheel onafhankelijk van de materieele wereld. (Brouwer 1907, 177)

[65] See on this idealisation also Husserl (2001b, 200–201; 1975, 188–189; 1984b, Sect. 64); and Becker (1927, 285, 287, 292f, 304, 320f).

[66] *unsterblich*, *die Zukunft unendliche Zeit bedeutet*, *ein ewiges Sein im Werden* (Husserl 1966, 378, 379, 381).

[67] *hypothetische mensen met onbeperkt herinneringsvermogen* (Brouwer 1933A2, 59).

[68] *het scheppend subject* (Brouwer 1948A).

notions are to be understood. They agree on the relevant aspects of the ego (C11), the cogito (C3–C7, C9), and the cogitatum (C1, C2, C8, C10). Such, then, are my grounds for the claim that Brouwer's intuitionism can be interpreted as belonging to Husserl's theory of purely categorial formation. Some objections to this claim will be discussed below, after the following elucidations.

Given Husserl's view on the relation between transcendental phenomenology and the a priori sciences, discussed in Sect. 12.2.1 above, intuitionism is therefore both mathematics and part of phenomenology. For this reason, I do not think that Lothar Eley's comment on the *Philosophy of Arithmetic* also holds for the transcendental Husserl: 'Husserl's approach to philosophy recognises a thematising subjectivity, however not a constructive one in the intuitionists' sense.'[69]

The common points C1–C11 also show that Brouwer is, systematically speaking, much closer to Husserl than to Kant, who did not acknowledge categorial intuition,[70] and for whom the flow of time is not self-given to us in a mode of intuition proper to it. In Brouwer, it is essentially an application of categorial intuition to the intuition of the flow of time that allows him to accept potentially infinite sequences as objects given in intuition. There is no parallel possibility in Kant: For him the result of acts of mathematical construction is a sensuous intuition (an image) in which a mathematical concept is instantiated by actual or possible empirical objects, but in a determinate image there is no intuitive givenness of the open horizon and the categorial form 'and so on'. This seems to me to be the reason behind Kant's refusal, in his letter to Rehberg from Autumn 1790 (Kant 1900–, 11:207–210), to identify the square root of 2 with a potentially infinite sequence of ever closer rational approximations, an identification that Brouwer on the other hand did make. Although Brouwer in his writings never remarks on it, his acceptance of such sequences as objects constructed in intuition is the strongest indication that his notion of intuition is in fact different from Kant's.[71]

In his inaugural lecture 'Intuitionism and formalism' (1912), Brouwer referred to Kant to characterise his own position as 'abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time' (Brouwer 1913C, 85).[72] Perhaps Brouwer here had Kant's full doctrine of time in mind, perhaps only the apriority it ascribes to time. Be that as it may, in his later writings

[69] 'Der Husserlsche Ansatz der Philosophie kennt eine thematisierende Subjektivität, hingegen nicht eine konstruktive im Sinne der Intuitionisten.' (Husserl 1970, xxvi) Note that Eley, in his *Metakritik der formalen Logik*, is sympathetic with Brouwer's thought (Eley 1969, 14, 15, 64, 264n1, 329n2, 332).

[70] 'Admittedly, in Kant's thought the categorial (logical) functions play an important role; but he does not arrive at the fundamental extension of the concepts of perception and intuition over the categorial realm' ('In Kants Denken spielen zwar die kategorialen (logischen) Functionen eine große Rolle; aber er gelangt nicht zu der fundamentalen Erweiterung der Begriffe Wahrnehmung und Anschauung über das kategoriale Gebiet.', (Husserl 1984b, 732)).

[71] A detailed discussion of this matter can be found in van Atten (2012).

[72] van de theorie van Kant de aprioriteit der ruimte prijs te geven, doch aan de aprioriteit van de tijd des te vastberadener vast te houden (Brouwer 1912A, 11).

this characterisation no longer occurs, and in particular it is absent from his later historical introductions to intuitionism (Brouwer 1952B, 1954A).[73] Brouwer introduced choice sequences soon after his inaugural lecture (Troelstra 1982). For the reasons explained in the previous paragraph, from a systematical point of view it would be no coincidence that Kant disappears as choice sequences appear. That this was also historically Brouwer's motivation no longer to present his intuitionism as a direct modification of Kant's position can, at present, only be conjectured.

12.2.4 Discussion of Some Remaining Objections

But against the background of this agreement, perhaps one sees a difference between the temporal characteristics of Husserl's purely categorial objects and Brouwer's mental constructions. As Tieszen formulates his objection to identifying these notions:

> Constructions are given as having temporal duration, for example, but numbers and finite sets are not. As Husserl puts it in [*Experience and Judgment*], intuitions are temporal processes but objects like numbers and finite sets that are given in these acts are given as 'omnitemporal'. (Tieszen 1989, 178)

But for Husserl there is no contradiction here. As we saw in C2 above, Husserl in Experience and Judgment and other texts holds that objects like numbers ontically indeed are constructions.

This does not contradict their omnitemporality, because by that Husserl means that this ontical construction is of such a kind that the subject can, in principle, repeat it at any later time and could have made it at any earlier time. The intuitionist can readily acknowledge that many of his constructions also have this property, namely, those that do not depend on choices. When a mathematical construction depends on free choices, we have to reckon with the fact that, when at different times the subject is confronted with the same situation in which to choose, it can do so differently each time, and hence these objects cannot be omnitemporal. Indeed, Brouwer thematises acts that satisfy properties C1–C11, yet result in the construction of such objects: these are the 'more or less freely proceeding infinite sequences' (A7) or 'arbitrary infinitely proceeding sequences' (B4), now better known as 'choice sequences'. They come into being when the subject begins making a series of choices, and they grow in time with each choice the subject makes.

It can be argued that, phenomenologically, a choice sequence is an object in the emphatic sense of a 'correlate of an identification to be accomplished in an open, endless, and free repetition' (Husserl 1985b, 64; see also van Atten 2007,

[73] In 1929, in his second Vienna lecture, Brouwer expresses his 'fundamental [*im wesentlichen*] agreement' with Kant and Schopenhauer on the specific point of taking the continuum to be given in a priori intuition (Brouwer 1930A, 1, 6). That formulation does not imply agreement on the details.

Sect. 6.2). In each of these repeated identifications, which can take place once the choice sequence has been begun, this potentially infinite sequence is given originally (see C10 above). Moreover, it is a specifically mathematical object (van Atten 2007, Sect. 6.3). Besides the fact that the acts in which choice sequences are constructed share the characteristics C1–C11, they preserve the intersubjectivity[74] and monotonicity of mathematical truth. The burden of proof that in spite of having all these properties, choice sequences are not mathematical objects, clearly lies with those phenomenologists who insist that omnitemporality is essential to mathematical objects as such. It is in the theory of choice sequences that theorems are obtained that formally contradict classical mathematics. Husserl's attitude towards revisionism in logic and mathematics seems to have been negative. But, as I have argued elsewhere, that was more a matter of Husserl's psychology than an intrinsic characteristic of transcendental phenomenology (van Atten 2002, 14).

As another argument against the identification of purely categorial objects and mental constructions, Tieszen (1989, 178) cites the following passage from *Experience and Judgment* in order to support his claim that 'Husserl offers arguments in a number of places in his later writings to show that mathematical objects could be said to 'exist' independently of our minds, or of our constructions':

> 'there are' mathematical and other irreal objects which no one has yet constructed. Their existence, to be sure, is revealed only by their construction (their 'experience'), but the construction of those already known opens in advance a horizon of objects capable of being further discovered, although still unknown. As long as they are not discovered (by anyone), they are not actually in spatiotemporality; and as long as it is possible (how far this is possible, there is no need to decide here) that they never will be discovered, it may be that they will have no world-reality (Husserl 1973e, 260).[75]

However, this passage equally admits of an intuitionistic interpretation. Perhaps one already finds sufficient ground for such a reading in the infinite iterability of the operation of forming two-ities out of previously constructed objects. Actual human beings certainly cannot concretely overview the constructions implied in the possibility of that iteration. In Husserl's passage, it is clearly constructions that the horizon is opening up to, yet he also says the mathematical objects are 'capable of being discovered.' But this need not be taken in the sense of discovering an object that exists independently of our acts; there is also an epistemic sense in which a constructional possibility can appropriately be said to be discovered.

[74]It is true that in a context of multiple subjects, a choice sequence is owned by the subject that creates it. But all that this subject knows about the sequence can be communicated by it to other subjects.

[75]That is the translation Tieszen uses. 'Mathematische und sonstige irreale Gegenstände "gibt es", die noch niemand konstruiert hat. Ihr Dasein erweist freilich erst ihre Konstruktion (ihre "Erfahrung"), aber die Konstruktion der schon bekannten eröffnet voraus einen Horizont weiter entdeckbarer, wenn auch noch unbekannter. Solange sie nicht entdeckt sind (von niemandem), sind sie nicht faktisch in der raum-Zeitlichkeit, und sofern es möglich ist (darüber wie weit dies möglich ist, braucht nicht entschieden zu werden), daß sie nie entdeckt worden wären, hätten sie überhaupt keine Weltwirklichkeit. (Husserl 1985b, 312).

A closely related argument depends on Brouwer's 'denumerably unfinished sets' (*aftelbaar onaffe verzamelingen*). These are sets such that 'only denumerable subsets of it can be indicated in a well-defined way, but from each such denumerable subset we can immediately obtain, following some previously defined mathematical process, new elements which are considered also to belong to the set in question' (Brouwer 1975, 82, modified).[76] Brouwer gave as examples the definable real numbers and the ordinals of Georg Cantor's second number class (respectively, Brouwer 1907, 82; 1908A, 570). He then observed that, a fortiori, the totality of all intuitionistic constructions is denumerably unfinished. In effect, he thereby confirmed the existence of such a horizon as Husserl is speaking of here; note that this horizon exists for both actual humans and the ideal subject (see C11 above).

As for the second half of Husserl's passage ('As long as ...'), Brouwer defines mathematical existence in terms of constructibility by an ideal subject (C11 above). In that sense, there may well exist mathematical objects of which a given actual human or group of humans will, as a matter of fact, never discover their constructibility. But it would be contradictory to assume that a mathematical object can exist without it being in principle discoverable for humans, because the ideal subject, and hence mathematical existence, is defined by what is in principle constructible by the human mind. (The question whether the same would also be true according to Husserl will be addressed in Sect. 12.3 below.)

Hill (2010, 64–65) sees a contrast between Brouwer's view that mathematics has its origin in the perception of a move of time and Husserl's rejection of theories of number based on intuitions of time in *On the Concept of Number* and the *Philosophy of Arithmetic*. But there is no such contrast. What Husserl criticises are views according to which time enters into the content of the concept of number, but it is not Brouwer's view that it does.[77] For him, the concept of ordinal number is defined in terms of nested empty two-ities, which are purely formal (categorial) objects.

[76] waarvan niet anders dan een aftelbare groep welgedefinieerd is aan te geven, maar waar dan tevens dadelijk volgens een of ander vooraf gedefinieerd wiskundig proces uit elke zoodanige aftelbare groep nieuwe elementen zijn af te leiden, die gerekend worden eveneens tot de verzameling in kwestie te behooren (Brouwer 1907, 148–149).

[77] Nor, for that matter, is it Kant's. E.g., in his letter to Johann Schultz of November 25, 1788, there is this well-known passage: 'Time has, as you very well remark, no influence on the properties of numbers (as pure determinations of magnitude), as it does for example on the property of any alteration (as of a quantum), which itself is possible only relative to a specific property of the inner sense and its form (time), and the science of number is, regardless of the succession that any construction of magnitude requires, a pure intellectual synthesis, which we represent to ourselves in thought' ('Die Zeit hat, wie Sie ganz wohl bemerken, keinen Einflus auf die Eigenschaften der Zahlen (als reiner Größenbestimmungen), so wie etwa auf die Eigenschaft einer jeden Veränderung (als eines Qvanti), die selbst nur relativ auf eine specifische Beschaffenheit des inneren Sinnes und dessen Form (die Zeit) möglich ist, und die Zahlwissenschaft ist, unerachtet der Succession, welche jede Construction der Größe erfodert, eine reine intellectuelle Synthesis, die wir uns in Gedanken vorstellen', Kant 1900–, 10:556–557). I take it that when in the *Critique of Pure Reason*, A142–43/B182, Kant speaks of number as a schema, he is speaking of number in so far as that concept is constructible by us.

The order of the ordinal numbers is certainly founded on the order that is intrinsic to time, but time as such does not enter into the content of the concept of ordinal number. The same holds for cardinal numbers, which are defined in terms of ordinal numbers and abstraction from ordering. It is rather in the genesis of the two-ities that the intuition of time plays its role; hence, Brouwer's reference to that intuition as the 'origin'. Brouwer's position actually agrees with (a transcendental reading of) Husserl's conclusion of his discussion of this matter in relation to cardinal numbers in the *Philosophy of Arithmetic*:

> Thus we see that time only plays the role of a psychological precondition for our concepts, and that in a two-fold manner:
>
> 1) It is essential that the partial representations united in the representation of the multiplicity or number be present in our consciousness simultaneously.
> 2) Almost all representations of multiplicities – and, in any case, all representations of numbers – are results of processes, are wholes originated gradually out of their elements. Insofar as this is so, each element bears in itself a different temporal determination.
>
> But we found that neither simultaneity nor successiveness in time enters in any way into the (logical) content of the representation of the multiplicity; and so, likewise, into that of the representation of number. (Husserl 2003a, 33)[78]

One readily finds agreement on related points in Husserl's discussion: like Husserl, Brouwer does not hold that the mere noticing of a temporal sequence of contents suffices to mark out a determinate multiplicity; to mark those contents as a determinate multiplicity, they have to be combined, and this happens by repeatedly forming, in spontaneous acts, two-ities out of the present content and the preceding ones. (See also Brouwer 1907, 179n.) This operation is also central to the formation of 'complexes of sensations' (A4 above). For the same reason, Brouwer would agree with Husserl that to perceive temporally successive contents does not yet mean to perceive contents as temporally successive (Husserl 1970, 29). The latter requires framing the contents in a categorial structure.

Finally, in his concern with essential properties of the mind (C11), and his recognition of inner time as opposed to physical time as the time that is relevant for the constitution of pure mathematics, Brouwer agrees with Husserl that mathematics is not a chapter of psychology; Brouwer is not concerned, as Hill (2010, 64) thinks he is, with 'what happens in or to real temporal matters of fact that we call mental experiences of experiencing individuals'.

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[78] Wir sehen also, die Zeit spielt für unsere Begriffe nur die Rolle einer psychologischen Vorbedingung und dies in doppelter Weise:

1) Es ist unerläßlich, daß die in der Vorstellung der Vielheit bzw. Anzahl geeinigten Teilvorstellungen zugleich in unserem Bewußtsein vorhanden sind.
2) Fast alle Vielheitsvorstellungen und jedenfalls alle Zahlvorstellungen sind Resultate von Prozessen, sind aus den Elementen sukzessive entstandene Ganze. Insofern trägt jedes Element eine andere zeitliche Bestimmtheit an sich.

Wir erkannten aber, daß weder die Gleichzeitigkeit noch die Aufeinanderfolge in der Zeit in den Inhalt der Vielheits- und somit auch der Zahlvorstellungen irgendwie eintreten. (Husserl 1970, 32)

More generally, Brouwer can readily acknowledge that even though the construction of any mathematical object requires the perception of the move of time, this does not by itself mean that time enters into the content of mathematical concepts (the concept of number being an example). When the later Husserl (around 1917) abandoned the view that purely ideal objects are atemporal, and came to hold that they are omnitemporal (because their constitution presupposes that of time), he must have relied on a very similar observation (Lohmar 1993; van Atten 2007, Sect. 5.4.1). But it was a fundamental insight of Brouwer that the possibility that (some notion of) time does enter into some mathematical concept is not excluded either. So far, no other example than that of choice sequence (or structures based on it, such as the 'spread') has been found. Husserl, who knew about choice sequences through works by Becker and by Hermann Weyl that he read, never directly discussed the question whether choice sequences are mathematical objects or not.[79]

A real contrast between Husserl's account of number in the *Philosophy of Arithmetic* and Brouwer's lies elsewhere: Brouwer holds that cardinal numbers genetically depend on ordinal numbers, while Husserl thinks it is the other way around and that his analyses will confirm this (Husserl 1970, 11, 13). Elsewhere, I have argued at length that Brouwer's view here is the phenomenologically more accurate one, and I will not repeat that discussion here (van Atten 2004c).[80]

Rosado Haddock sees in Husserl's distance from Kant also a distance from Brouwer:

> Moreover, Husserl never seemed to have retracted of his classification of Kant's views in Chap. VII of LU I as a sort of specific relativism. Hence, one should not press too much the affinities between Husserl's and Kant's transcendental philosophies, and beware of assessing Husserl's views as a foundation of Brouwer's Fichtean mathematical subject. (Rosado Haddock 2006, 200n40)[81]

For the transcendental Husserl, the relevant contrast drawn in that chapter of the *Logical Investigations* is that between an individual or species on the one hand and the ideal cognising subject (see C11 above) on the other. Husserl rejects the necessity of a correspondence between the subjective capacities of the former and the laws of logic and mathematics; but he says such a correspondence does exist with the latter. As it is the latter that serves, as argued above, as a foundation of Brouwer's mathematical subject, and not an individual or species, the arguments in Chap. 7 of the *Prolegomena* are no obstacle to a transcendental-phenomenological justification of intuitionism.

[79]See van Atten (2007, 72–74) for further discussion.

[80]A somewhat revised version has been published as an appendix in van Atten (2007).

[81]Note that the qualification 'Fichtean' is not Brouwer's. It is of course a separate question to what extent that qualification is applicable; I will not go into it here.

12.3 Beyond Intuitionistic Mathematics?

So far, the fundamental constructive principles in intuitionism have been limited to the 'two acts' described in fragment B; and eventually, Brouwer came to see that the second act is a special case of the first (Brouwer 1981A, 93n). However, it is also a basic tenet of intuitionism to be prepared to extend these two acts whenever further descriptive analysis of consciousness warrants this; see the first quotation at C8 above.[82] In particular, had Becker's attempt in *Mathematische Existenz* at a constructive, phenomenological foundation of Cantor's ordinals in terms of structures of self-reflection succeeded beyond the denumerable, this would have readily provided an intuitionistically acceptable extension of the two acts; but, in fact, it did not, as Becker acknowledges.[83] In his book, he admits that he can only give a clear phenomenological interpretation of ordinals 'up into the first epsilon numbers' (Becker 1927, 129),[84] which are still countable. Brouwer, in the notebooks for his dissertation from 1904–1907, had also briefly attempted to get the whole second and further number classes by appropriately differentiating the meaning of 'and so on' for each class (thus, like Becker, referring to differently structured horizons; Brouwer Archive, Notebook VI, 38). But he noticed that one cannot give a reasonable constructive sense to such differentiations, as witnessed by the explanation in his dissertation: 'The expression "and so on" means the indefinite repetition of one and the same object or operation, even if that object or that operation is defined in a rather complex way' (Brouwer 1975, 80n).[85]

A more general question is whether Husserl's phenomenology can provide a foundation also for nonconstructive, classical mathematics, as notably

[82] Also Heyting: 'It is, as a matter of principle, impossible to devise a system of formulas that would be equivalent to intuitionistic mathematics, because the possibilities of thinking do not admit a reduction to a finite number of rules that can be set up in advance.' ('Es ist prinzipiell unmöglich, ein System von Formeln aufzustellen, das mit der intuitionistischen Mathematik gleichwertig wäre, denn die Möglichkeiten des Denkens lassen sich nicht auf eine endliche Zahl von im voraus aufstellbaren Regeln zurückführen', Heyting 1930a, 3.)

[83] Note that Becker in his *Grundlagen der Mathematik in geschichtlicher Entwicklung* of 1954 describes the horizon phenomenon and the possibilities of reflection, but without reference to his work of 1927. He says: 'And thus one can endlessly proceed through the series of indices numbered by transfinite ordinal numbers – in so far as the ordinal numbers used can be defined constructively, and hence can be univocally and exactly named.' ('Und so kann man unbegrenzt fortfahren in der Reihe der durch transfinite Ordnungszahlen numerierten Indices – soweit sich die verwendeten Ordnungszahlen konstruktiv definieren und infolgedessen eindeutig und exakt bezeichnen lassen', Becker 1954, 386.) For a clear recent note on constructive transfinite ordinals, see Jervell (2006).

[84] These numbers are the infinite ordinals ϵ such that $\epsilon = \omega^{\epsilon}$. The smallest is $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$

[85] Waar men zegt 'en *zoo* voort', bedoelt men het onbepaald herhalen van *eenzelfde* ding of operatie, ook al is dat ding of die operatie tamelijk complex gedefinieerd. (Brouwer 1907, 143n)

Gödel hoped.[86] Is there a phenomenological basis for claiming that the mathesis universalis, understood as the theory of purely formal objects (see the citation from *Formal and Transcendental Logic* at the beginning of Sect. 12.2.1), is richer than Brouwer's intuitionism? In other words: are there purely categorial formations that go beyond the ones that Brouwer indicates or could have indicated?

Becker and Weyl held the view that constructive mathematics admits of a phenomenological foundation, but classical does not. The question has been raised to what extent their conceptions of phenomenology coincide with Husserl's, and to what extent Husserl (would have) agreed with them (Lohmar 1989, 195). The discussion below is meant to emphasise that in the transcendental Husserl one indeed finds all the elements needed to arrive at the same conclusion. Some remarks on mathematics and minds that Gödel made after his turn to transcendental phenomenology will serve as a foil, and reveal a significant tension in his position.

The starting point is the voluntariness of categorial formation. Our experience shows that voluntary acts of human consciousness are, structurally, limited to the denumerable, in the sense that any series of them that we can carry out is either finite or potentially infinite (see C10 above); and Brouwer's two acts of intuitionism cover precisely that. There is in our experience no motivation whatsoever to think that, if only we could abstract from certain empirical limitations on our memory, lifetime, and so on, we could, on the basis of our intuition of time, make either the uncountable or the actual infinite intuitive.

In recognition of the limitation of our mental acts to the denumerable,[87] Gödel suggested that it is not necessary to insist that, ideally, mathematical objects can be completely given in our intuition. In his paper on Cantor (1964 version), he wrote:

> This negative attitude toward Cantor's set theory [is] only the result of a certain philosophical conception of the nature of mathematics, which admits mathematical objects only to the extent to which they are interpretable as our own constructions or, at least, can be completely given in mathematical intuition. For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them, there exists, I believe, a satisfactory foundation of Cantor's set theory in its whole original extent and meaning (Gödel 1964, 262).

Note that by the time Gödel wrote this passage, he had already made his turn to transcendental phenomenology (Wang 1987, 121–122); and although it is written

[86]For Gödel's philosophical views, in particular in their relation to phenomenology, see, e.g., Tragesser (1977), Tieszen (1992), Føllesdal (1995), Parsons (1995), and van Atten and Kennedy (2003).

[87]'It has some plausibility that all things conceivable by us are denumerable' (Gödel 1946, 152). In his introduction to that paper, Parsons notes that this is ambiguous between 'For any x, if x is conceivable by us, then x is denumerable' and 'Only denumerably many things are conceivable by us', and says that the latter reading seems more likely (Gödel 1990, 148). I agree: at the point in the text where Gödel makes this remark, it is used to support an objection to accepting 'undenumerably many sets.'

in such a way as to admit different interpretations, evidence from Gödel's archive strongly suggests that he wrote it with the phenomenological perspective in mind. In a draft of Gödel's paper, there is an additional final paragraph that starts: 'Perhaps a further development of phenomenology will, some day, make it possible to decide questions regarding the soundness of primitive terms and their axioms in a completely convincing manner.'[88] Clearly, Gödel saw the Continuum Hypothesis[89] as a question about (or very close to) the primitive terms and their axioms: the problem shows (at least from a realist perspective) that we do not yet have a full grasp of the relation between the basic notions of exponentiation and cardinality. Therefore, I do not agree with Rosado Haddock (2010, 25) that the question is not about the very basis of classical set theory. Similarly, and again unlike Rosado Haddock, I would say that questions about large cardinals lie at the heart of classical set theory because they show that we do not yet have a sufficient grasp of the concept of set and of the universe of sets to decide what basic objects there are in the universe.

A Husserlian objection to Gödel's suggestion in the passage just quoted is that, for a given mathematical concept to be admitted into truth-logic, we should be able to constitute at least one object falling under it with evidence. It is true that Husserl says that the starting point of eidetic variations may as well be an imagined object as one given in experience (Husserl 1985b, 411–412), which suggests that we can also constitute essences without having constituted with evidence an object that falls under it. But in the case of purely categorial objects, the distinction between constitution with evidence and imagination collapses: an object can only be clearly imagined to the extent that it is a possible object, and for purely categorial objects, possibility and existence coincide.[90]

Without having constituted a mathematical object that falls under the concept in question, the concept remains empty, a mere meaning but not an essence (Husserl 1952, Sect. 16). Remaining at the level of mere meanings, without intuitions of the objects meant, genuine knowledge of these objects cannot be obtained, and no eidetic variations can be performed on them. At best, we might be able to obtain purely analytic knowledge about the concepts involved. From an ontological perspective, this would not get us beyond if-thenism.

Although it has sometimes been said that a distancing from questions of truth and ontology is characteristic of the practice of modern pure mathematics,[91] that is certainly not the conception of anyone who, like Gödel and Brouwer, holds that

[88] As quoted on p. 466 of van Atten and Kennedy (2003).

[89] This is Cantor's hypothesis that the power of the (classical) continuum, $2^{\aleph_0}$, is equal to the first uncountable cardinal, $\aleph_1$.

[90] See Sect. 12.2.1 above, and van Atten (2002).

[91] In *Formal and Transcendental Logic*, Husserl says that the then prevailing understanding of purely formal mathematics among mathematicians is that of 'a pure analytics of consistency', and adds that on that understanding, the concept of truth remains unthematised (Husserl 1974, 15–16).

mathematics describes a domain of objects (be they Platonic or mental).[92] If, on the other hand, a mathematical concept is not empty, and a purely categorial object can be given to us that falls under it, then, according to Husserl, that object can in principle be given adequately, it can, to borrow Gödel's expression, be 'completely given in mathematical intuition' (see C10 above). Gödel, in the passage quoted here, is concerned with the part of Cantorian set theory that is not constructive. But given that the human mind seems, as Becker's failed attempts emphasise, unable to constitute even a single uncountable ordinal, it seems that we cannot come to 'recognize the soundness and the truth of the axioms' concerning uncountable ordinals, for the reason given in the previous paragraph. A phenomenological foundation for Cantor's set theory will then not be forthcoming.

There seem to be two ways out of this difficulty that still respect in some sense the idealistic tenet in transcendental phenomenology: an appeal to ideas in the Kantian sense, and an appeal to higher minds.

In Sects. 142–44 of *Ideas I*,[93] Husserl invokes ideas in the Kantian sense to resolve the following apparent contradiction. On the one hand, the fundamental principle of his idealism equates existence of an object with its being given adequately to a possible consciousness (Husserl 1976a, 329). On the other hand, there clearly exist objects (such as physical objects) that can never be given adequately. His solution is to say that in the latter case, adequate givenness of the object is nevertheless the correlate of an incompletable, infinite series of acts, in each of which a further partial determination of the same object obtained. Husserl says that such a series is lawlike or governed by a rule that depends on the type of the object. To the extent that we have evidence that we can actually begin such a series of experiences and then in principle can always continue it according to that rule, the positing of an object that is the correlate of the infinite series is rationally motivated. Husserl does not discuss whether ideas in the Kantian sense can play a role in the constitution of objects of pure mathematics, but Lohmar (1989, 194–195n15)[94] suggests that they can, and sees it as a means for Husserl to accept mathematical objects that do not satisfy the intuitionistic existence criteria. While Lohmar incorrectly holds that intuitionists accept only finite constructions, and correspondingly sees a use for Kantian ideas in attempts to go beyond the finite, a valid question is whether Kantian ideas provide a way to accept more infinite objects than the intuitionist can.

However, it seems to me that, if the suggestion to accept Kantian ideas in the account of mathematics works, it will not motivate positing anything beyond the denumerable. As soon as we try to be precise about the law or rule that governs the process of generating the infinite series, we run into the fact that humans can only follow it if it prescribes no more than a finite or potentially infinite number of steps.

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92 With respect to mathematics, Gödel and Brouwer are both what may be called ontological descriptivists; see Sundholm and van Atten (2008, 71).

93 See also Husserl (1974, 66–67n1; 1985b, 346; 2009, 217).

94 See also Lohmar (1989, 141–143).

As noted above, it was because of this limitation that Becker's attempt at founding Cantor's ordinals failed for the non-denumerable. Moreover, it can be argued that this suggestion cannot work to begin with. For Husserl, purely categorial objects are immanent and can in principle be given adequately (C10 above), so the notion of a purely categorial object that as a matter of principle cannot be adequately given is contradictory. But only for objects that cannot be adequately given does it make sense to introduce adequate givenness as a Kantian idea (Husserl 1966, 21; 1976a, 332).[95] Conceptually at least, this would also explain the fact that Husserl does not give examples from pure mathematics (formal ontology) to illustrate the notion of a Kantian idea.[96] (It is important to recall here that the inapplicability of Kantian ideas here does not mean that all purely categorial formations must be finite: as argued above, a potentially infinite sequence can also be adequately given.)

Turning now to the appeal to higher minds, the idea behind it is that, if a categorial object cannot be constituted by our minds, perhaps it can by minds of a higher type. So it is, perhaps, no coincidence that we find Gödel saying, to Hao Wang, 'For every set there is some mind which can overview it in the strict sense' (Wang 1996, 260). and

> What this idealization [to the integers as a totality, and also with arbitrary omissions] – realization of a possibility – means is that we conceive and realize the possibility of a mind which can do it. (Wang 1996, 220)

Indeed, Gödel believed that higher minds exist: in a private note (ca. 1960) headed 'Meine philosophische Ansicht[en?]', Gödel stated, 'There are other worlds and

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[95]Note that my objection does not apply to Husserl's appeal to Kantian ideas in the context of geometry (which is not part of purely formal mathematics), e.g., in Sect. 74 of *Ideas I*. There, in an act of ideation an ideal geometrical notion is given through the elements of an incompletable infinite series of sensuous intuitions, as its ideal limit. But as that notion is not a (mereological) composition out of these elements, for our present purpose there is no relevant analogy with the constitution of higher-order categorial objects.

[96]⟦An exception would seem to be a passage in a manuscript probably from the early 1920s:

> Finally, there is the concept of a Kantian idea, which needs its own clarification. This comprises real and 'ideal', irreal objects, e.g., the number sequence '1, 2, 3 and so on' is a truly existing object, seeable as such, with this 'and-so-on'. A law of iteration and iterative construction is given to me with evidence, together with the indeterminate 'idea' of an open multiplicity. ('Endlich haben wir den Begriff der Kantischen Idee, der seiner eigenen Klärung bedarf. Umspannt sind hier reale und 'ideale', irreale Gegenstände, z.B. die Zahlenreihe '1, 2, 3 usw.' ist ein wahrhaft seiender, als das erschaubarer Gegenstand, mit diesem 'Und-so-Weiter'. Ich habe ein Gesetz der Iteration und iterativen Konstruktion einsichtig gegeben, neben der unbestimmten 'Idee' einer offenen Vielheit', Husserl 2012, 79.)

But I argue that the sequence as a potentially infinite one is given adequately here; it is only the sequence as an actually infinite one that might be given as a Kantian idea, inadequately. But in that case, it would be inconsistent with Husserl's ideas about categorial objects to think of this sequence as one of them. Be that as it may, it should be noted that, in its dependence on an iterative construction that is given to us with evidence, Husserl's use of the notion of Kantian idea is not one that would generalise to the non-denumerable.⟧

rational beings of a different and higher kind' (Wang 1996, 316).[97] In particular, Gödel appeals to a higher mind in an argument for the existence of the classical power set of ℕ (the set of the natural numbers), based on the idea that a subset of ℕ may be characterised by specifying what members of the number series are not elements of it:

> To arrive at the totality of integers involves a jump. Overviewing it presupposes an [idealized] infinite intuition. In the second jump we consider not only the integers as given but also the process of selecting integers as given in intuition. 'Given in intuition' here means [an idealization of] concrete intuition. Each selection gives a subset as an object. Taking all possible ways of leaving elements out [of the totality of integers] may be thought of as a method for producing these objects. What is given is a psychological analysis, the point is whether it produces objective conviction. This is the beginning of analysis [of the concept of set]. (Wang 1996, 220, emendations by Wang)

And Gödel elucidates: 'What this idealization [to the integers as a totality, and also with arbitrary omissions] – realization of a possibility – means is that we conceive and realize the possibility of a mind which can do it' (Wang 1996, 220). From other remarks to Wang, it is clear that Gödel not only accepts the psychological analysis but it also gives him the objective conviction: 'We can form the power set of a set, because we understand the selection process (of singling out any subset from the given set) intuitively, not blindly' (Wang 1996, 259), and 'From the idealized subjective view, we can get the power set' (Wang 1996, 260). This kind of reasoning leads to the idea of classical mathematics as constructive mathematics for higher minds,[98] with, perhaps, God as the highest mind.[99] The intuitionist, on the other hand, will see Gödel's psychological analysis as the analysis of a mistake: to speak of 'all possible ways of leaving elements out' of ℕ is to speak of something that is not denumerable, and hence not constructible in the intuitionistic sense.[100]

Objections to such quasi-constructive interpretations of axioms of classical set theory are of course well-known outside the phenomenological literature, from the discussions of the so-called iterative concept of set,[101] notably by Parsons ([1977] 1983) and by Hallett (1984, Chap. 6). Indeed, the point noted above that a higher being would need to have a time awareness with a far richer structure than ours

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[97] 'Es gibt andere Welten und vernünftige Wesen einer anderen und höheren Art.' Transcription Robin Rollinger, according to whom the word 'einer' is actually hard to make out on the microfilm.

[98] Compare Gödel's remark on Frank Ramsey's idea of using propositions of infinite length to provide a foundation of a classical theory of classes: 'Ramsey's viewpoint is, of course, everything but constructivistic, unless one means constructions of an infinite mind' (Gödel 1944, 145; see also 142).

[99] In the period of his intensive study of Leibniz in the 1940s, one idea of Leibniz that appealed to Gödel was that the objects of mathematics exist in God's mind; see van Atten (2009a, 7–8).

[100] Gödel's argument was first published in Wang (1974, 182); for criticism, see Parsons ([1977] 1983) and Hallett (1984, 220).

[101] For expositions and defences of that conception see, e.g., Boolos (1971), Wang (1974, Chap. 6), and Shoenfield (1977, 322–327).

was made in that context by Parsons ([1977] 1983, 273). When Wang in a letter drew Gödel's attention to Parsons' paper, Gödel wrote a shorthand remark, 'Often [he] continually confuses concepts and sets, and moreover he does not understand "idealization" broadly enough' (Gödel 2003a, 390, trans. Parsons)[102]; according to Parsons, it seems clear that that remark indeed refers to his paper. Gödel apparently thought that such idealisations as he made posed, or would be shown to pose, no obstacle to the phenomenological foundation of classical set theory that, as is known from other sources, he hoped to find (e.g., Gödel *1961/?; van Atten and Kennedy 2003, esp. 466, 69).[103]

But also on the basis of Husserl's texts an objection to Gödel's proposal can be developed, starting from the question: Can we say, with evidence, what categorial objects a certain mind higher than ours can constitute? It would not suffice if we know about this greater ability because it follows analytically from the concept of the specific higher being we are imagining, for then the evidence would still be lacking that the existence of such a mind is not a mere empty or a problematic

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[102] Oft verwechselt fortwährend concepts und sets und ausserdem versteht er 'Idealisierung' nicht genug weit

[103] It should be noted that, on another occasion, Gödel admitted that arguments that depend on idealising human mental (constructive) capacities are not always the most evident ones. In a letter to Paul Cohen of August 13, 1965, he wrote:

> As far as the axiom of the existence of inaccessibles[104] is concerned I think I slightly overstated my view. I would not say that its evidence is due solely to the analogy with the integers.[105] But I do believe that a clear analogy argument is much more convincing than the quasi-constructivistic argument in which we imagine ourselves to be able somehow to reach the inaccessible number. On the other hand, Lévy's principle[106] might be considered more convincing than analogy. (Gödel 2003, 386)

[104] *Note MvA*. A cardinal number κ is (strongly) inaccessible if it is uncountable and neither the sum nor the product of κ numbers smaller than κ. Such an inaccessible, itself a set, can be used to build a model of the axioms of ZFC, and thereby establish the consistency of that theory. From Gödel's second Incompleteness Theorem it follows that, if ZFC is consistent, it cannot prove the existence of inaccessible numbers, because otherwise ZFC could prove its own consistency. The question of the existence of inaccessibles therefore concerns the acceptability of a new axiom.

[105] *Note MvA*. All numbers smaller than the countable infinite cardinal $\aleph_0$ are finite, but $\aleph_0$ is neither the sum nor the product of finitely many finite numbers. If one believes that $\aleph_0$ is not very special among the infinite cardinals, then there should exist an uncountable cardinal with the analogous property.

[106] *Note MvA*. This is a specific form of the general reflection principle. The latter (roughly) says that, if the universe of all sets has a certain property, then there is a set in the universe that also has it; the property of the universe is reflected in that set. Note that the reflection principle does not yield a construction of that set. The argument Gödel refers to uses reflection (roughly) as follows: If the cardinality of the universe is inaccessible, then by reflection the same is true of a set in the universe. For a discussion of Gödel's justification of the reflection principle by an analogy to Leibniz' monadology, see van Atten (2009a).

possibility, but one that we could say with insight could indeed obtain, and hence one that we could appeal to in order to increase our knowledge about categorial objects.[107]

On the other hand, if our judgement on the higher mind's capacities is not analytic, but is arrived at by making its categorial formations intuitive to ourselves (perhaps after appropriate idealisations), then we are, in principle, able to do what that mind can, which is therefore, after all, not essentially higher than ours. In fact, in a text of 1909, Husserl emphatically denies that the notion of an essentially different knowing subject whose performance cannot even ideally be matched by ours makes any sense:

> One can entertain the thought that, just as man stands higher intellectually than minerals or [the] jellyfish, there might actually be beings that, compared to man, are more highly developed intellectually, in this sense, that they have fundamentally new ways of knowing at their disposal ... The common talk about possible cognitive natures that are not at all ours and have nothing to do with ours, is pointless, indeed nonsensical: for [then] there is nothing to sustain the unity of the concept of cognition. For there to be a point to talk of such possibilities, it should be about ways of knowing that are discernibly essentially identical to ours according to their generic character; and even when on factual grounds, on empirical psychological ones, such ways of knowing will never be present in our psyche and will never really be conceivable, a priori there should exist the possibility of an extension of our cognition as an ideal possibility, by which our cognition would itself become the cognition of that higher intellect that we are thinking of in an indirect and empty representation.[108]

Gödel, in the list referred to earlier, stated that 'The higher beings are connected to the others by analogy, not by composition' (Wang 1996, 316),[109] which strongly suggests that, correspondingly, at least part of our knowledge about these higher beings would be obtained by drawing analogies. But Husserl's insistence on the unity of knowledge rules out an appeal to analogy where idealisation falls short, and

[107]Errett Bishop, who saw constructive mathematics as the mathematics that finite beings are capable of, said: 'If God has mathematics of his own that needs to be done, let him do it himself' (Bishop 1967, 2).

[108] Mann kann den Gedanken erwägen, dass wie der Mensch intellektuell höher steht als die Mineralien oder <die> Qualle, so es in Wirklichkeit Wesen geben mag, die dem Menschen gegenüber intellektuell höher entwickelt sind, und zwar so, dass sie über ganz neue, prinzipiell neue Erkenntnisarten verfügen. ... Die allgemeine Rede von möglichen Erkenntniswesen, die durchaus nicht unsere sind und mit unseren gar nichts zu tun haben, ist sinnlos, ist in der Tat widersinnig: da nichts vorhanden ist, was die Einheit des Begriffs der Erkenntnis aufrecht erhält. Soll von solchen Möglichkeiten sinnvoll die Rede sein, so muss es sich um Erlebnisarten handelen, die einsehbar wesensidentisch ihrem Gattungscharacter nach sind mit den unseren; und wenn auch aus faktischen Grunden, aus empirisch psychologischen, in unserer Seele nie solche Erkenntnisarten auftreten und wirklich vorstellbar sein können, so müsste es a priori die Möglichkeit einer Erweiterung unserer Erkenntnis bestehen als ideale Möglichkeit, durch die unsere Erkenntnis selbst zu der Erkenntnis jener in indirekt-leerer Vorstellung gedachten höheren Intellekte würde. (Ms. K II 4, 109a/b, October 1909; quoted from Kern 1964, 129–130.)

[109]The original reads: 'Die höheren Wesen sind durch Analogie, nicht durch Komposition mit den anderen verbunden.' Transcription Robin Rollinger.

indeed should keep us from accepting the possibility of such beings altogether.[110] Thus we find a tension in Gödel's position: he wishes to use transcendental phenomenology to provide a foundation for classical mathematics, but the reference to higher beings that this seems to require has no phenomenological support. When I spoke to Robert Tragesser about these issues, it reminded him of Kierkegaard's *Fear and Trembling*: How unsatisfactory it is, when trying to make rational sense of the story of Abraham and Isaac, to be told, 'Surely no one was as great as Abraham. Who is able to understand him?' (Kierkegaard [1843] 2006, 11).

In various texts by the transcendental Husserl, we find this idea of the unity of knowledge applied to (knowledge of) categorial objects, and to mathematics in particular:

> Speaking in eidetic generality, should every subject as such be capable of seeing every eidetic object [*Gegenständlichkeit*] (and likewise every object of sense, every ideal object in the widest sense)? The question should be answered affirmatively. This is a matter of an a priori [truth]. Of course, not every subject (not every actual or, in the attitude of the pure consideration of possibilities, every possible [subject], that is, posited as possible actuality) needs be thought of as actually cognising every eidetic object (or be thought of as cognising it).[111]

> The ideal total extension of the purely rational objects and the extension of the objective objects [*objektive Gegenstände*] cognizable to every possible rational I, coincide.[112]

> Any mathematical step that someone else makes, I should be able to repeat originaliter in myself ... It should be observed that a monosubjective [*einzelsubjektive*] mathematics is eo ipso intersubjective, and that, conversely, none is intersubjectively possible that is not already fully and completely founded monosubjectively.[113]

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[110] For further discussion of Husserl's denial of essentially higher minds, see Kern (1964, 125–134); also of interest here is the correspondence between Becker and Mahnke (2005), together with Mancosu's introduction to it (Mancosu 2005).

[111] Muss jedes Subjekt überhaupt, in eidetischer Allgemeinheit gesprochen, jede eidetische Gegenständlichkeit (und ebenso jede Sinnesgegenständlichkeit, jede ideale Gegenständlichkeit im weitesten Sinn) erschauen können? Die Frage ist zu bejahen. Es handelt sich hier um ein Apriori. Nicht jedes Subjekt braucht natürlich (nicht jedes wirkliche oder, in der Einstellung reiner Möglichkeitsbetrachtung, jedes mögliche, d.i. als mögliche Wirklichkeit angesetzte) jeden eidetischen Gegenstand wirklich zu erkennen (oder als ihn erkennend gedacht zu werden). (Husserl 2003b, 147, from 1918 at the latest)

[112] Der ideale Gesamtumfang der rein rationalen Gegenstände und der Umfang der einem jeden möglichen Vernunft-Ich erkennbaren objektiven Gegenstände deckt sich. (Husserl 1974, 388, from 1920–1921)

[113] Jeden mathematischen Schritt, den ein anderer macht, muß ich in mir selbst originaliter nachmachen können. ... Die Feststellung [ist] zu machen, dass eine einzelsubjektive Mathematik eo ipso intersubjektiv sei und umgekehrt keine intersubjektiv möglich ist, die nicht schon voll und ganz einzelsubjektiv begründet ist. (Husserl 1974, 344, November 1926)

On Husserl's view, if one being is able actually to achieve less in mathematics than another, this is due only to contingent limitations:

> It should be realised that, what an I thinks (given the matter of thought), any I could think. This 'could' implies, however, that with every logical formation and with every grounding of truth, every possible inhibition is compatible, and that no essential law can reach into a logical one, as was shown in the *Logical Investigations*: just as every matter is freely variable, every inhibition is 'variable.' That means: the essential laws of freedom do presuppose an unfreedom ('lower' Psyche), but not one arising from positing and intervening.[114]

It seems to me that in Husserl's transcendental idealism, the fundamental 'unfreedom' is that imposed by the basic structure of inner time consciousness (C9 above); in Brouwer, this is wholly explicit. In the considerations so far, I have taken intuitive givenness as the criterion for accepting certain mathematical objects and axioms about them; but that need not be the only rationally motivated criterion. Examples of additional criteria would be the inductive one of the success of an axiom in deciding a number of important open questions (Gödel 1964, 265), or the cognitive-aesthetic one of the extent to which the introduction of certain objects simplifies a theory. But in the context of Husserl's transcendental idealism, if we are not to break the tie to truth and evidence, decisions based on such additional criteria can only be provisional, to be replaced, ideally, by the insight that the objects and axioms thus introduced can be constituted by the ideal subject. Such additional criteria can therefore only play a heuristic role. Finally, it should be remarked that the transcendental Husserl not only saw a correlation between consciousness and being, according to which existence is equivalent to accessibility to a possible consciousness, but moreover held that the existence of consciousness is absolute, while that of all beings constituted by it is not Husserl (1976a, 120). Without going into systematic issues concerning the meaning and strength of this second claim, I note that at least one occasion can be documented on which Gödel was critical of it.[115] The discussion of Gödel's view in the present section, however, has only been concerned with the correlation as such, and in particular with the notion of 'possible consciousness' that occurs in it; my argument that at this point there is an incompatibility between Gödel's view and Husserl's transcendental idealism does therefore not depend on the correctness of Husserl's further-going claim.

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[114] Es muß eingesehen werden, daß, was ein Ich denkt (die Denkmaterie aber vorausgesetzt), jedes Ich denken könnte. Dieses Könnte besagt aber, daß mit jedem logischen Gebilde und jeder Wahrheitsbegründung jede mögliche Hemmung verträglich ist und daß kein Wesensgesetz in ein logisches hineingreifen kann, wie in den Logischen Untersuchungen gezeigt ist: Wie jede Materie frei variabel ist, so ist auch jede Hemmung 'variabel'. Das sagt: Die Wesensgesetze der Freiheit setzen zwar eine Unfreiheit ('niedere' Psyche) voraus, aber keine durch Setzen und Eingreifen. (Husserl 1974, 386n1, from 1920–1921)

[115] For a brief discussion of this matter, see van Atten and Kennedy (2003, 454–455).

12.4 A Historical Note

The claim defended above that Brouwer's intuitionistic mathematics should be considered part of Husserl's transcendental phenomenology is not meant in a historical sense. In contrast to other mathematicians who reflected phenomenologically on mathematics, Brouwer did not take Husserl's (or any other phenomenologist's) writings as his point of departure.

He certainly could have done so. For example, by the time Brouwer began to develop intuitionism as a student, around 1904, Husserl had published two major works on the philosophy of mathematics and logic: *Philosophy of Arithmetic* in 1891 and *Logical Investigations* in 1900–1901. But Brouwer seems to have been quite unaware of them. In the nine notebooks in which he prepared his dissertation during the period 1904–1907 (Brouwer Archive, Notebooks I–IX) Husserl is not mentioned at all. (Incidentally, in his dissertation he did not refer to Gottlob Frege either; but in the notebooks there are some references to Frege's discussion with Hilbert on axiomatic geometry. Brouwer agrees with Frege's criticism (Brouwer Archive, Notebook VIII, 29, 32). The philosophers of logic and mathematics whom the young Brouwer did study extensively were Cantor, Louis Couturat, Richard Dedekind, David Hilbert, Kant, Felix Klein, Henri Poincaré, and Bertrand Russell.)

There is in Brouwer's archive and writings no evidence that he read Husserl's work at a later stage either; so we must suppose that, when the two met in person and had discussions, in Amsterdam in 1928, Brouwer's intuitionism had already become mature independently. Martin Heidegger, in his 1925 lectures entitled *History of the Concept of Time*, claims that intuitionism, of which he names Brouwer and Weyl as representatives, was 'essentially influenced by phenomenology' (*wesentlich von der Phänomenologie beeinflusst*, Heidegger 1979, 4); in Weyl's case, this is clear from his preface to *The Continuum* (Weyl 1918), but in Brouwer's case there is no evidence whatsoever for this. It is, on the other hand, known that Husserl owned a copy of Brouwer's paper 'Intuitionistische Betrachtungen über den Formalismus' (Brouwer 1928A2), which appeared in the year before Husserl's *Formal and Transcendental Logic*, and is highly relevant to the themes of that book (see van Atten 2007, 128n7).

12.5 Concluding Remark

Rather than a direct influence from Husserl (or other phenomenologists) on Brouwer, it seems there was a close intellectual kinship between them. The object and methods of Brouwer's intuitionism have sufficiently much in common with Husserl's phenomenology as to be interpretable in the latter. Compared to Brouwer's own explicitly formulated philosophy (Brouwer 1949), Husserl's is the broader and more detailed framework. Moreover, Brouwer's many philosophical passages about his own position tend to state conclusions rather than give analyses that lead to them.

His original descriptions and explanations are not explicitly phenomenological. Phenomenology can clarify, explicate and deepen the intuitionistic position; and this allows us to see intuitionism as an example of a fruitful application of phenomenological analysis to mathematics. Philosophically, of the applications of transcendental phenomenology to mathematics so far, intuitionistic mathematics has been the most successful one. It has led to phenomenologically grounded foundations that are coherent, detailed and comprehensive: it includes not just arithmetic, but also analysis, measure theory, topology, and algebra (Heyting 1956; Troelstra and van Dalen 1988). To show that one can go beyond this and that classical mathematics is a true mathematics in the sense of *Formal and Transcendental Logic*, general considerations about intentionality, meanings, essences, idealisations, and, perhaps, non-revisionism, will not do: what is still wanting is a concrete and detailed phenomenological foundation of even just one characteristically classical alleged truth, such as the existence of the power set of $\mathbb{N}$. On the basis of the foregoing considerations, the prospects for such a foundation within transcendental phenomenology seem to me to be dim. The reason for this lies not so much in Husserl's insistence on the spontaneous character of the constitution of mathematical objects, or in his refusal to allow an appeal to higher minds, as in the combination of these two attitudes.

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Appendix: Null on Choice Sequences

Gilbert Null's review (2008) of my *Brouwer Meets Husserl* is centred around his attribution to me of two claims (Null 2008, 119, 128):

1. Choice sequences are objects without identity criterion;
2. Choice sequences are real (*reell*) parts of inner time.

Not only are these claims made nowhere in the book, they are directly and explicitly contradicted in it.The passage that Null quotes from p. 36 of my book to support the attribution of claim (1) denies only that (non-lawlike) choice sequences have an identity criterion that is extensional. It does not at all say that choice sequences can have no identity criterion whatsoever. Indeed, as I argue extensively in Chap. 6, in particular on pp. 92–93 (which Null nowhere refers to), the principle of individuation (in terms of the moment of the beginning of a choice sequence) provides the identity criterion, and this criterion I identify explicitly as intensional (e.g., p. 90).[116]

Null further maintains that my book 'notes ... but leaves the need for a closer analysis of choice event protentions ... unfulfilled' (Null 2008, 129). Not only is their role made fully explicit in the analysis on pp. 92–93, but this role is discussed at many other places as well because, as is explicit on p. 92, protentions are essential to the constitution of choice sequences as open-ended (pp. 6, 15, 36, 92, 97, 98, 105). In this light, it is incomprehensible that Null makes it seem as if my book, and 'Brouwerians' in general, neglect 'any present (actual) choice event's inner horizon of openly possible contrary futures' (Null 2008, 129).[117]

Claim (2) is denied outright on p. 92 of my book, for reasons given on p. 91. Null's mistaken attribution of claim (2) to me may well stem from my expressing agreement with Guido Küng's thesis (Küng 1975) that the noema is a real (*reell*) moment (p. 70 and note 162). But whatever the merits of the thesis of the noema as a real moment, I do consider it a thesis about the noema in its fullest extent; and I state just as explicitly (p. 71 and pp. 89–90) that I consider the noematic essence and the noematic nucleus as ideal objects that are omnitemporal.[118] On p. 128 of his review, Null says that I am also committed to the thesis that a number can occur in a choice sequence only once. But since it is in virtue of the noematic nucleus that an intentional act has whatever object it has, also on my conception of the noema it is perfectly possible to choose the same number more than once in a choice sequence. One might think that, if the full noema is taken to be a real (*reell*) moment of the noesis, then the noematic nucleus cannot be omnitemporal, as the concrete noesis can, by definition, not be repeated in time. But it does not follow from this that the noematic nucleus cannot be omnitemporal. According to Sect. 64c of *Experience and Judgment*, Husserl calls an object 'omnitemporal' (*allzeitlich*) if in its constitution, which always takes place at a specific moment in time, this moment in time does not 'enter into' (*eingehen*) that object, that is, this temporal determination is not part of what makes the object the object it is. But then it is not at all excluded that some appropriate part of a given noesis can be constituted at a different time as identically the same: namely, if in the constitution of that part the moment in time does not 'enter into it', is not part of what makes it the object it is

[116]In the intuitionistic literature, 'intensional identity' (and the related 'intensional equivalence') is a standard notion, both in philosophical and mathematical discussions (e.g., Dummett 2000a, 16–17; Troelstra 1977, 5, respectively). Applied to choice sequences, 'extensional' means 'in terms of the numbers chosen in the sequence', 'intensional' means 'in terms of other aspects of how the sequence is given to us'. In particular, the moment at which a sequence was begun is such an intensional aspect. So are any of the restrictions that we may have imposed on our choices.

[117]It is worth pointing out that the identity criterion for choice sequences that Null himself suggests ('CIP', Null 2008, 127), is a criterion in which protentions play no role whatsoever: for $n < \omega$, CIP operates only on initial segments (which are finite sequences and as such given to us without an open horizon (van Atten 2007, 90)), and for $n = \omega$, we are dealing with sequences of infinitely many given numbers, which are not given as open-ended either. So CIP is not suitable as an identity criterion for Brouwer's choice sequences; they are not even among the objects it operates on.

[118]On this difference see also the remarks in Bernet et al. (1989, 94–95).

(discussed in van Atten 2008, 70–71). Not only is this not at all excluded, it is exactly parallel to Husserl's explanation how categorial objects are, ontically, productions, yet omnitemporal (Husserl 1985b, 311), quoted in van Atten (2008, 11–12); see the discussion of point C2 above.

This also bears on page 124 of Null's review. He there says that a choice sequence contains either (i) the senses or (ii) the referents of choice events in an associated choice process.[119] The problem with (ii), according to Null, is that in that case 'the available Husserlian approach [sic] leaves Van Atten's characterization of choice sequences as intratemporal prima facie unsupported', because 'numbers are omnitemporal if anything is' (Null 2008, 124). If this argument is to work, then Null must hold that, if a higher-order object (here, a choice sequence) is founded on omnitemporal objects (here, numbers), then that higher-order object must also be omnitemporal. As Null does not supply any argument for this idea, he is begging the question against those who defend that choice sequences are intratemporal objects.

One may ask, of course, whether an argument such as Null fails to supply nevertheless exists. This is not the case, because there are counterexamples (independently from the one that, as I argue, choice sequences are). A specific act in which I judge, with full evidence, that the number 2 is an omnitemporal object, is founded on the number 2: for if the number 2 did not exist, neither could this specific act of judgement with full evidence. But this act, when objectified, is not an omnitemporal object, as it exists only for a certain stretch of time.

Husserl makes a closely related point: 'That a subject conceives a proposition with evidence, lends the proposition locality,[120] and, as the thought of this thinker etc. a unique one, but not to the proposition as such, which would be the same when thought at different times'.[121] In Husserl's example, the particular thinker's particular thought episode of the proposition as evident is founded on the proposition, but does not share the temporal characteristics of the latter.

More generally, Husserl had already remarked, speaking of the constitution of a higher-order object on the foundation of lower-order ones, 'And even when the time-constituting acts of the lower level also enter, they need not do this in such a way that the times enter, like the objects themselves, into the objects constituted at the higher level'.[122] (Recall that Husserl says that 'such an irreality has the temporal being of supertemporality, of omnitemporality, which however is a mode of temporality'.[123])

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[119]Incidentally, characterisation (ii) is explicitly stated in my book to be the appropriate one (e.g., van Atten 2008, 1, 24).

[120]*Note MvA*. Husserl here means spatio-temporal locality; see Husserl (1985b, 311).

[121] 'Daß ein Subjekt ein Satz evident denkt, das gibt dem Satz Lokalität, und als gedachtem dieses Denkers etc. eine einzige, aber nicht dem Satz schlechthin, der derselbe wäre als zu verschiedenen Zeiten etc. gedachter.' (Husserl 1985b, 312–313)

[122] Und gehen auch die zeitkonstituierenden Akte der Unterstufe mit ein, so brauchen sie es doch nicht so zu tun, dass die Zeiten wie der Gegenständlichkeiten selbst in die höher konstituierten Gegenständlichkeiten eingehen. (Husserl 1985b, 310)

[123] eine solche Irrealität [hat] das zeitliche Sein der Überzeitlichkeit, der Allzeitlichkeit, die doch ein Modus der Zeitlichkeit ist. (Husserl 1985b, 313)

In view of the above, Null's ascription of claims (1) and (2) to me is entirely mistaken, and, since his review turns on them, his discussion is, in effect, not about my book.[124]

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[124]There are other infelicitous claims in Null's review, of which I here briefly mention the following two (which, moreover, are incompatible with each other). (1) Discussing the tree he set up on p. 126 ('Fig. 1a'), Null claims of the paths through it that 'each of these 2 to the $\aleph_0$ many sequences is an ideal object of the sort accepted by Husserl, Weyl, Becker, and Kaufmann' (Null 2008, 127). This is not correct. On the one hand, these sequences are all predeterminate (in each sequence the n-th element is fixed from the beginning for each n); on the other hand, by laws one can specify at most denumerably many sequences. As a consequence, most of these $2^{\aleph_0}$ many sequences are predeterminate but not given by a law. But that combination is possible in neither of the respective varieties of constructivism by which Weyl, Becker and Kaufmann defined their philosophical positions. Husserl is not explicit about the matter, but, as I argue in the present paper, accepting non-constructible objects in mathematics is not an option in his framework either. (2) Null claims that Husserl's formal objects should be 'countable', which, as defined by Null, implies decidability of equivalence. But although classical mathematics, intuitionistic mathematics, recursive analysis, Bishop's constructive mathematics all have different conceptions of real number, in none of them is equality of real numbers decidable. This would leave Husserl's phenomenology incapable of founding any of them. But I do not think that Husserl anywhere actually poses or implies Null's countability condition.

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ERRATUM

Essays on Gödel's Reception of Leibniz, Husserl, and Brouwer

Mark van Atten

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Due to error during the production of this book, the name of the co-author was mistakenly omitted from the chapter heading.

Under the title of chapter 6,

"On the Philosophical Development of Kurt Gödel", should be inserted "and Juliette Kennedy".

Under the title of chapter 10,

"Mysticism and Mathematics: Brouwer, Gödel, and the Common Core Thesis" should be inserted "and Robert Tragesser".

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Except where required otherwise,

1. Brouwer's writings are referred to according to the scheme in Van Dalen's bibliography (van Dalen 2008);
2. Gödel's writings are referred to according to the scheme in the Collected Works (Gödel 1986, 1990, 1995, 2003, 2003a)
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