Optimal Control and Dynamic Games

Applications in Finance, Management Science and Economics

Christophe Deissenberg
and Richard F. Hartl
(Editors)

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Optimal Control and Dynamic Games

Applications in Finance, Management Science and Economics

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Foreword

I am delighted to be invited to give a few remarks at this workshop honouring Suresh Sethi. He was one of the most hardworking and prolific of any of my (45 or so PhD) students during the course of 42 years of teaching at GSIA. I would like to discuss our interactions both during and after the completion of his doctoral thesis. Suresh entered the PhD program in the fall of 1969 just after I had published a paper on the application of a new mathematical model called Optimal Control Theory which originated in Russia. The paper was called: "The Optimal Maintenance and Sale Date of a Machine". Suresh quickly absorbed the mathematics on which optimal control theory is base. We wrote a joint paper called "Applications of Mathematical Control Theory to Finance: Modeling Simple Dynamic Cash Balance Problems," which was published before the end of 1970. At the same time Suresh wrote nine additional papers by himself (on topics which I have forgotten). He put these nine papers together with the dynamic cash balance paper above to complete his thesis in record time. His PhD was awarded before the end of 1970. The nine additional chapters in his thesis were also published by him in subsequent years.

In 1970 it was uncommon for professors to write joint papers with either their colleagues or with their PhD students. In GSIA we encouraged such joint work and other schools have since imitated this practice. In order to analyze how Suresh has thrived in this environment, I did a quick count of the number of authors in each of the papers listed in the Professional Journal Articles section of his vita, obtaining the following amazing distribution: single author, 37; 2 authors, 96; 3 authors, 113; 4 authors, 46; and 5 authors, 6. Note that three times as many papers having a single author is about the same as the number of papers having 3 authors; half of the 2 authored papers is about the same as the number of 4 authored papers, etc. In order to explain how Suresh could have created an environment in which made these results possible I would like to discuss some of his personal attributes as follows: (a) his congeniality;

(b) his generosity; (c) his breadth of interest; (d) his originality; (e) his creation of new mathematical applications; and (f) his visibility.

1 *Congeniality.* As you know, Suresh is very easy to talk to. One of his favorite questions is, "What are you working on?" When you tell him he will respond by giving you hints and suggestions for directions which you might want to follow in your research on the paper. If you ask him what he is working on, be prepared to listen for a couple of hours.

2 *Generosity.* If you show interest in one of the papers he talks about and you make a suggestion for furthering it, he may invite you to become a coauthor, and assign you a promising direction in which to look for additional results. On the other hand, if he likes what your problem is, he may suggest that he become a coauthor of your paper. If you say yes to either of these suggestions, then be prepared to have him knock on your door a few months later and ask, "How you are getting along with our joint problem."

3 *Breadth of Interest.* In 1970 mathematicians maintained strict control over the kinds of applications which were favourably received in their journals: only mathematical models employing ordinary differential equations or partial differential equations, and applied only to applications involving either physics or engineering problems. What would they say to paper number 5 in Part (ii) Finance and Economics by M. Gordon and S. Sethi, "Consumption and Investment When Bankruptcy is Not a Fate Worse Than Death." Also what would they say to paper 6 in Part(iii) Marketing by E. Haruvey, A. Prasad, and S. Sethi, "Harvesting Altruism in Open Source Software Development." By skimming through his vita, you can see that Suresh knows no bounds on the use of various kinds of theoretical areas such as mathematics, statistics, economics, etc. to analyze a wide range of new application areas.

4 *Originality.* Let me list a few of the new applications areas that appear in his papers: optimal cattle ranching; stochastic manufacturing systems; choosing robot moves in a robotic cell; risk aversion behavior in consumption/investment problems; scheduling of the injection process for golf club head fabrication lines; peeling layers of an onion, an inventory model with multiple delivery modes and forecast horizons; etc. Obviously he enjoys choosing humorous titles for his papers, but each paper contains a serious analysis of an actual real life application.

5 *Visibility.* Besides looking at his publications and working papers, it is possible to measure the extent of the influence of Suresh's work on the fields of Operations Research, Engineering, Economic, etc, by looking at his professional activities which include talks presented at various meeting, invited talks at universities, membership meeting locations in societies, etc. Let S be the set of all these locations that Suresh has attended together with all of the possible such locations that he has not yet attended. If we plotted each of his travels over the years on a globe of the earth they might resemble what is called ergodic (random) motion. A theorem in ergodic theory states that if you let ergodic motion continue long enough, each of the locations in S will be visited with probability one. I propose the following Suresh Ergodic Theorem: If you go to any location in S and wait long enough at that location, you will meet Suresh with probability one.

I would like to wish Suresh Sethi a very happy sixtieth birthday, and I look forward to keeping up with his future publications.

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Introduction

This volume is proudly dedicated by its editors and contributors to Professor Suresh P. Sethi, in recognition of his achievements as a scholar and of his role as a public and private personality.

Since more than thirty years, Professor Sethi has consistently proved one of the foremost personalities in applied control theory. His seminal contributions, which are described in more detail in another chapter, cover such diverse fields as operations management, marketing, finance and economics, forecasting and rolling horizons, sequencing and scheduling, flexible manufacturing systems, hierarchical decision making in stochastic manufacturing systems, and complex inventory problems, among others. This intense research activity has led to the publication of more than one hundred and ninety journal articles, ninety contributions in proceedings and edited books, and nine monographs. Among the latter, the Sethi-Thompson book on Optimal Control Theory can be univocally distinguished for having introduced this at the time still largely unknown subject matter to business schools worldwide.

The nineteen chapters in this volume are closely related to a workshop held in honor of Professor Sethi in Aix en Provence, France, 2-6 June 2005. All chapters are written by internationally recognized specialists in the subject matter, and represent the state of the art in their particular direction, witnessing the importance and popularity of Suresh Sethi in the covered disciplines.

The book is thematically divided in five parts. The first, which is concerned with advertising problems, consists of two chapters.

In their contribution *Advertising Directed Towards Existing and New Customers*, Richard Hartl and Peter Kort start from the classical advertising surveys by Sethi. They propose a specific marketing problem from the class of advertising capital or diffusion models. The extension to the classical models is that they consider two kinds of advertising directed towards new customers and existing customers, respectively. They found that history dependent behavior occurs: if initial goodwill is small then convergence to a saddle point with low goodwill prevails where there is only advertising with the aim to attract new customers. On the other hand, for larger initial goodwill, eventually a steady state with a high goodwill level is reached where both types of advertising are used. In *Advertising and Advertising Claims over Time*, Charles Tapiero considers a stochastic advertising-repeat purchase model in which the advertising policy is essentially defined by two factors: the advertising budget and the content of a statement on the product characteristics. The statement may not necessarily reflect the true characteristics of the product, leading to the following dilemma. Advertising claims that underestimate the product's characteristics do not entice first time purchasers to try the product, but insure that buyers will typically not be disappointed by the discrepancy between the advertised and the real characteristics of their purchase. Overly optimistic advertising messages entice first time purchases but may generate dissatisfaction and induce a switch to competing brands. The chapter provides a theoretical approach to deal with this issue.

The second part of the book is concerned with environmental problems and includes five chapters. In *Capital Resource Substitution, Overshooting, and Sustainable Development,* Hassan Benchekroun, Seiichi Katayama, and Ngo Van Long study, under the utilitarian criterion, the optimal path for an economy that produces a final good using capital and an input extracted from a natural resource. Capital and resource are substitutable inputs in the production of the final good but, contrary to the reference work in the domain, the resource stock is assumed renewable. The authors show that there exists a unique steady state with positive consumption and that, starting from low levels of capital stock and resource stock, the optimal policy consists of three phases. Initially, it is optimal to build up the stock of capital above its steady state level, and to keep the resource stock below its steady state level. That is, there is overshooting. In a second phase, the optimal capital stock declines steadily, while the optimal resource stock continues to grow, until the steady state is reached. In the third and final phase, the economy stays at the steady state. The next chapter *Common Property Resource and Private Capital Accumulation with Random Jump*, by Masatoshi Fujisaki, Seiichi Katayama, and Hiroshi Ohta, studies the existence of a control solution in a model of optimal exploitation of a non-renewable common property resource under the new and natural hypothesis that the process of extraction can be affected by sudden shocks, such as technological problems or social hazards. These shocks are modeled as random jumps in the stock of the resource. The economic agents can invest in private and productive capital. This capital is a substitute to the natural resource. Thus, its accumulation can be steered in order to optimally mitigate the welfare impacts of the resource shocks. In *Hierarchical and Asymptotic Optimal Control Models for Economic Sustainable Development*, Alain Haurie investigates the relevance of asymptotic control theory to the study of economic sustainable development and proposes a modeling framework where sustainable economic development is represented through a paradigm of optimal stochastic control with two time scales. The chapter conclusively shows that several contributions of Sethi in the domain of finance, manufacturing and resource management can also serve to better understand the stakes of sustainability in economic growth and to assess long term environmental policies. A further chapter *Transfer Mechanisms Inducing a Sustainable Forest Exploitation* by Guiomar Martín-Herrán and Mabel Tidball is concerned with the important and topical problem of deforestation as a global environmental issue. The authors consider a Stackelberg differential game played over an infinite horizon between a group of developed countries as the leader and a forestry country as the follower. In that game, the developed countries use financial transfers to improve forest conservation. The chapter investigates the impact of alternative transfer schemes on the optimal deforestation rate paths and forest stocks, and on the countries' revenues, thus allowing policy conclusion on the most efficient transfer modalities. The last chapter of this part, *Characterizing Dynamic Irrigation Policies via Green's Theorem* by Uri Shani, Yacov Tsur, and Amos Zemel, addresses another important policy-making problem: How to derive irrigation management schemes accounting for the dynamic response of biomass yield to salinity and soil moisture as well as for the cost of irrigation water? To that purpose, the authors carry out an original extension the standard Green's Theorem analysis (that, interestingly, was used by Sethi to solve for optimal advertising expenditures in a much cited early paper) to more complex situations with arbitrary end conditions. A numerical application to a concrete problem shows that significant savings on the use of freshwater can be achieved with negligible loss of income.

The book's third part is devoted to economics and finance. In *Volatility Forecasts and the Profitability of Automated Trading Strategies*, Engelbert J. Dockner and Günter Strobl take up the approach proposed in 1994 by Noh et al., that is, predict the volatility of asset return with the help of a GARCH model and use the volatility forecasts together with an option pricing formula to calculate future option prices. They apply it to Bund future options, first presenting several volatility models theoretically and then using these specifications to empirically evaluate the efficiency of the Bund future options market at LIFFE. It turns out that their option strategy can outperform the market if there is sufficient volatility clustering, so that a GARCH model accurately predicts conditional variance. In *Two-Part Tariff Pricing in a Dynamic Environment*, Gila E. Fruchter considers non-linear pricing techniques in a dynamic and competitive environment in the case when, as it is often the case in telecommunication services, the price of a product or service is composed of two parts: An entrance fee and a charge per unit of consumption. Managerial guidelines are suggested, which imply that a firm with a small network should focus on acquiring new customers through a low membership fee. As its network grows, the firm should turn more attention to customer retention by offering a higher network-based price discount. The author also shows that the dynamic network-based prices are lower than their static counterparts. Consequently, the network-based price discount is smaller in the dynamic case than in the static one. Jacek B. Krawczyk, in *Numerical Solutions to Lump-Sum Pension Fund Problems that Can Yield Left-Skewed Fund Return Distributions*, proposes approximately optimal solutions to pension funds problems when the underlying performance measure is asymmetric with respect to risk. In such cases the pension fund problems, where a profit-maximizing manager receives an initial amount of money against the obligation to repay an agreed upon sum at a later date, typically do not admit analytical solutions. The author converts the problems into Markov decision chains solvable through approximation, thus obtaining computable solution schemes for more general and more realistic performance criteria than usually studied. In particular, a couple of problems with a non-differentiable asymmetric utility function are solved, for which left-skewed fund-return distributions are reported. Such distributions give more probability to higher payoffs than the right-skewed ones that are common among analytical solutions. In *Differentiated Capital and the Distribution of Wealth*, Gerhard Sorger investigates, for the case where the number of patient households is small, the Ramsey conjuncture that in a stationary equilibrium of the standard neoclassical growth model only the most patient households would own capital. Using a one-sector growth model with finitely many households who differ from each other with respect to their endowments, their preferences, and the type of capital supplied to firms, and assuming monopolistic competition à la Dixit on the capital market and perfect competition on all other markets, the author shows that there exists a unique stationary equilibrium where, contrary to the Ramsey conjuncture, all households have strictly positive wealth. The impact of diverse parameter variations on this equilibrium and its stability are analyzed. In *Optimal Firm Contributions to Open Source Software*, finally, Rong Zhang, Ernan Haruvy, Ashutosh Prasad, and Suresh P. Sethi use a differential game framework to tackle an enduring puzzle: Why do firms increasingly support open source software development, although in many cases this also profits to their direct competitors? The analysis focuses on some important aspects previously neglected in the literature, namely the competitivity aspect already mentioned; the complementarity of the efforts of developers and users, that may result in strong externalities; and the fact that the firm’s contributions to open source development may be steered to generate advances that are more compatible with one’s own products than with products from the competition. The authors find that both the degree of user involvement and the lack of compatibility with a rival's product positively affect profits. However, free-riding may result in reduced incentives for smaller firms to invest and in reluctance by larger firms to share their technologies.

The fourth part of the volume is devoted to production and maintenance. In the first chapter *The Impact of Dynamic Demand and*

Dynamic Net Revenues on Firm Clockspeed, Janice E. Carrillo considers a firm's new product development clockspeed, defined as the frequency of new product introductions to the marketplace. Using a simple analytic model, the author derives the optimal firm clockspeed which is driven by several external market factors and internal organizational related factors: (i) average demand forecasts, (ii) dynamic profits earned over time, (iii) cannibalization of older products, and (iv) organizational constraints limiting the pace of new product development, thus offering managerial insights concerning the dynamics of new product development activities on the firm level. In *Hibernation Durations for Chain of Machines with Maintenance under Uncertainty*, Ali Dogramaci considers the classic problem of the maintenance of a single machine and of its possible replacement over time at given regeneration points, when the probability distribution of machine failure can be improved by predictive or preventive maintenance, adding in the analysis an important but previously neglected aspect: If e.g. the retirement date of a machine is not constrained to be equal to the installment date of its successor, hibernation (i.e., selling the machine or stopping using it) may be profit increasing. In addition to proposing a solution procedure for the optimal hibernation scheduling, the paper has deep reaching implications for the realignment of the calendar for the regeneration points, for company policies on borrowing versus use of internal funds, and for the possible modification of machine replacement time windows. The paper *Self-Organized Control of Irregular or Perturbed Network Traffic* by Dirk Helbing, Stefan Lämmer, and Jean-Patrick Lebacque presents a fluid-dynamic model for the simulation of urban traffic networks with road sections of different lengths and capacities. By simulating the transitions between free and congested traffic, taking into account adaptive traffic control, the authors observe dynamic traffic patterns which significantly depend on the respective network topology. In this connection, they discuss adaptive strategies of traffic light control which can considerably improve throughputs and travel times, using self-organization principles based on local interactions between vehicles and traffic lights. Potential applications of this principle to other queuing networks such as production systems are outlined. The fourth chapter *A Stochastic Optimal Control Policy for a Manufacturing System on a Finite Time Horizon* by Eugene Khmelnitsky and Gonen Singer considers a continuous-time problem of optimal production control of a single reliable machine when the demand is given by a discrete-time stochastic process. The objective is to minimize the linear inventory/backlog costs over a finite time horizon. The paper focuses on using the optimality conditions of stochastic optimal control to develop a computational procedure for finding the optimal control policy over each interval between demand realizations. The procedure is implemented both in the case when the demand distribution is stationary and when it changes over time. Numerical examples are given. The last chapter of this part, *On a State-Constrained Control Problem in Optimal Production and Maintenance* by Helmut Maurer, Jang-Ho Robert Kim, and Georg Vossen, is concerned with the advanced numerical study of a dynamic production/maintenance control problem originally investigated by Cho et al. In the model investigated, the performance of the production process, measured in terms of non-defective units produced, normally declines over time. However, preventive maintenance can be used to counteract this tendency. A recently developed second order sufficiency test is applied to prove the optimality of the computed controls, for which no analytical sufficiency conditions are known in the literature. This test enables the authors to calculate the sensitivity derivatives of switching times with respect to perturbation parameters. Numerical results are also given for the case when there is an additional constraint on number of units produced.

The fifth and final part of the book is methodologically oriented. In *Reliability Index*, Alain Bensoussan considers a method originally introduced by B.M. Ayyub to compute the failure probability of an element subject to several random inputs. Contrary to most of the related literature, the approach is analytical and a rigorous treatment of the main results is provided, offering a potentially more powerful way of addressing reliability problems. In *The Direct Method for a Class of Infinite Horizon Dynamic Games*, Dean Carlson and George Leitmann extend their recent work on the use of Leitmann's direct method to efficiently solve open-loop variational games to the case of an infinite horizon. The basic idea of the direct method, which has been successfully applied to a variety of problems, is to use a coordinate transformation to transform the original problem of interest into another, equivalent problem that is (hopefully) simple to solve. The extension presented here should distinctly increase its interest for such fields as economics, where the standard analyses are regularly carried out over an infinite horizon.

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APPLICATIONS TO MARKETING

Chapter 1

ADVERTISING DIRECTED TOWARDS EXISTING AND NEW CUSTOMERS

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Abstract This paper considers a specific marketing problem based on a model by Gould (1970). The extension is that we have two kinds of advertising directed towards new customers and existing customers, respectively. We found that history dependent behavior occurs: if initial goodwill is small then it does not pay to spend a lot of money on advertising towards existing customers. Consequently convergence to a saddle point with low goodwill prevails where there is only advertising with the aim to attract new customers. On the other hand, for larger initial goodwill, eventually a steady state with a high goodwill level is reached where both types of advertising are used.

1. Introduction

Dynamic advertising models are among the first applications of Pontryagin's maximum principle in the economics and management area. The first comprehensive survey of the dynamic advertising literature was given by Sethi (1977a). It was devoted to determining optimal advertising expenditures over time subject to some dynamics that defines how advertising expenditures translate into sales and in turn into profits for

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a firm or a group of firms under consideration. More than fifteen years later, this survey was updated by Feichtinger, Hartl and Sethi (1994).

The surveys by Sethi (1977a) and Feichtinger, Hartl and Sethi (1994) were organized in four and five model categories, respectively, the first two of which were *advertising capital models* and *sales-advertising response models*. Advertising capital models considered advertising as an investment in the stock of *goodwill* as in the model of Nerlove and Arrow (1962). Sales-advertising response models are characterized by a direct relation between the rate of change in sales and advertising and represent various generalizations of the descriptive model due to Vidale and Wolfe (1957).

Advertising capital models typically are extensions and/or modifications of the early seminal dynamic advertising model due to Nerlove and Arrow (1962). They consider a stock of *advertising goodwill*, which summarizes the effects of current and past advertising expenditures by a firm on the demand for its products. The advertising capital changes over time according to "investments" by current advertising and by a constant proportional depreciation rate. The objective of the monopolistic firm is to maximize the present value of net revenue stream discounted at a fixed interest rate. Since the price does not enter the system dynamics in these models, it can be determined by static maximization of the profit function so that the resulting optimal control model has only advertising as a single control variable. In case revenue is proportional to goodwill, the optimal advertising policy in this linear problem is characterized by a *most rapid approach* to a singular goodwill level; see e.g. Sethi (1977b), and Hartl and Feichtinger (1987). Several nonlinear and other extensions have been proposed. In the model by Gould (1970) revenue is a concave function of goodwill which leads to a smooth optimal advertising policy and an *asymptotic convergence* to an equilibrium advertising capital stock.

The second class of models are *sales-advertising response* models. These models are characterized by a direct relation between the rate of change in sales and advertising in the form of a differential equation. The basic advertising model by Vidale-Wolfe (1957) assumes that increases in sales are proportional to advertising expenditure, u, the captured fraction of the market potential, x, and the remaining market potential, $1 - x$. As in the goodwill models, a constant decay rate is assumed. The dynamics are fundamentally different from the advertising capital dynamics because of the presence of the terms $u(1 - x)$ and $ux(1 - x)$ in place of the term u. While Gould (1970) had analyzed the problems both with diffusion dynamics in the presence of convex advertising cost, his treatment was not exhaustive. More specifically, he obtained a single long run optimal sales level in each of the problems, which the system would converge to from some initial sales level. As it turns out, the second diffusion model with the word of mouth term $ux(1-x)$ admits multiple stable equilibria and convergence to a particular equilibrium depending on the initial level of sales. In the linear objective function case this was shown by Sethi (1979a) while the nonlinear case was treated by Feichtinger and Hartl (1986, Section 11.1.2). The consequences of having multiple equilibria are important to the firm. It means that the advertising policy depends critically on the initial sales level. Moreover, a firm with sufficiently small initial sales level would never reach a high sales level in the long run.

Our contribution is to extend the bulk of literature mentioned, by distinguishing between two types of advertising controls: towards existing customers and towards new customers. Advertising towards *existing customers* tries to prevent that these existing customers forget about the product and move to other brands. Advertising towards *new customers* is different in the sense that it provides more information about the underlying product in order to convince new customers. We focus only on the advertising capital or goodwill models but of course this idea can be applied to the other stream, the diffusion models, as well.

We should mention that our model is not the first to consider different types of advertising in one model. The advertising models mentioned above consider the flow from potential adopters to current adopters. Some attempts have been made to extend these two-stage models to incorporate a possible multistage nature of the diffusion process. For instance, Muller (1983) presents a dynamic model of a new product introduction based on a diffusion process, which makes the distinction between two types of advertising objectives: increasing awareness and changing predisposition to buy. Our "advertising towards new customers" is related to his "awareness advertising", which informs prospective customers about the product and thus transfers them from the "unaware group" to the "potential group" . His second control instrument "trial advertising" persuades potential customers to purchase the product. Our model is different here since we do not distinguish between "potential group" and buyers. Consequently, our second control instrument "advertising towards existing customers" is aimed at preventing existing customers or buyers to become non-buyers.

We obtain two long run equilibria and it depends on the initial level of goodwill which one of them will be reached. In the lower steady state only one type of advertising is employed, namely the one which is directed towards new customers. In the larger steady state, i.e. the one with higher goodwill level, also the other type of advertising is applied on order to keep the existing customers. As usual, the two long run equilibria are separated by an unstable steady state which appears to be a focus for most of the parameter values. It should be noted, that we do not need the more complicated diffusion dynamics to identify multiple equilibria. Rather, we observe this interesting phenomenon already in the simpler goodwill model.

The structure of the paper is as follows. The model is presented in Section 1.2. Section 1.3 contains a general mathematical analysis while Section 1.4 specifies the functional forms to achieve more detailed results.

2. The Model

Consider a firm that has to decide about its advertising policy. The firm's sales $R(G)$ are completely dependent on the stock of goodwill (G). Goodwill can be controlled by two types of advertising. We have N which is advertising that aims to attract new customers. Its effectiveness $\eta(N)$ is independent of the current goodwill stock. Furthermore the firm can also choose towards keeping existing customers in the house by reducing the decay rate $\delta(A)$. This type of advertising is thus denoted by A.

The model of the case flow maximizing firm now follows directly:

$$\max_{N,A} \int_0^\infty e^{-rt} \left(R(G) - aN - vA\right) dt \tag{1.1}$$

s.t.

$$\dot{G} = \eta(N) - \delta(A)G \tag{1.2}$$

$$A \geq 0 \tag{1.3}$$

$$N \geq 0 \tag{1.4}$$

Sales are increasing in goodwill with diminishing returns:

$$R' > 0, \quad R'' < 0. \tag{1.5}$$

The effectiveness of advertising towards new customers is a concavely increasing function of N, so that

$$\eta\prime > 0, \quad \eta'' < 0. \tag{1.6}$$

Finally, the decay rate $\delta(A)$ is decreasing in A in a convex way:

$$\delta' < 0, \quad \delta'' > 0. \tag{1.7}$$

3. General Analysis

We first employ the maximum principle to derive the necessary optimality conditions. Then we investigate the stability of the canonical system.

3.1 Maximum Principle

The Hamiltonian (see e.g. Sethi and Thompson, 2000, or Feichtinger and Hartl, 1986) is

$$H = R(G) - aN - vA + \lambda\left(\eta(N) - \delta(A)G\right),$$

where λ is the shadow price of goodwill

Advertising directed towards existing customers, $A = A(\lambda, G)$, is determined by

$$\frac{\partial H}{\partial A} = -v - \lambda\delta'(A)G = 0,$$

from which it is obtained that

$$\begin{aligned}
\frac{\partial A}{\partial G} &= -\frac{\lambda\delta'(A)}{\lambda\delta''(A)G} = -\frac{\delta'(A)}{\delta''(A)G} > 0, \qquad (1.8)\\
\frac{\partial A}{\partial \lambda} &= -\frac{\delta'(A)G}{\lambda\delta''(A)G} = -\frac{\delta'(A)}{\lambda\delta''(A)} > 0.
\end{aligned}$$

So this type of advertising increases with goodwill which makes sense because its effectiveness is proportional to this stock.

Advertising directed towards new customers, $N = N(\lambda)$, is determined by

$$\frac{\partial H}{\partial N} = -a + \lambda\eta'(N) = 0.$$

This first order condition implies that advertising towards new customers does not depend on goodwill, but only on its shadow price:

$$\frac{\partial N}{\partial \lambda} = -\frac{\eta'(N)}{\lambda\eta''(N)} > 0.$$

The adjoint equation is

$$\dot{\lambda} = \left(r + \delta\left(A(\lambda, G)\right)\right)\lambda - R'(G), \qquad (1.9)$$

which, together with the state equation

$$\dot{G} = \eta\left(N(\lambda)\right) - \delta\left(A(\lambda, G)\right)G \qquad (1.10)$$

form the canonical system to be analyzed.

We investigate whether the necessary optimality conditions are also sufficient. To do so we check whether the maximized Hamiltonian

$$H^o(G, \lambda) = R(G) - aN(\lambda) - vA(\lambda, G) + \lambda\left(\eta\left(N(\lambda)\right) - \delta\left(A(\lambda, G)\right)G\right)$$

is concave in G for all possible values of λ that can occur along the costate trajectory $\lambda(t)$; see Seierstad and Sydsaeter (1977).

PROPOSITION 1.1 The maximized Hamiltonian is strictly concave iff

$$R''(G) + \lambda \frac{(\delta'(A))^2}{\delta''(A)\, G} < 0.$$

Proof. It is straightforwardly obtained that

$$\frac{\partial H^0}{\partial G} = R'(G) - v\frac{\partial A(\lambda, G)}{\partial G} - \lambda G \delta'(A(\lambda, G)) \frac{\partial A(\lambda, G)}{\partial G} - \lambda \delta(A(\lambda, G)).$$

Because of the envelope theorem (see Derzko, Sethi and Thompson, 1984) this can be rewritten into

$$\frac{\partial H^0}{\partial G} = R'(G) - \lambda \delta(A(\lambda, G)), \tag{1.11}$$

so that

$$\frac{\partial^2 H^0}{\partial G^2} = R''(G) - \lambda \delta'(A(\lambda, G)) \frac{\partial A(\lambda, G)}{\partial G}.$$

Employing (1.8), we obtain that

$$\frac{\partial^2 H^0}{\partial G^2} = R''(G) + \lambda \frac{(\delta'(A))^2}{\delta''(A)\, G} < 0.$$

■

We separately investigate the case where the control A is at its lower boundary, i.e., $A = 0$. We have

$$H^o(G, \lambda) = R(G) - aN(\lambda) + \lambda(\eta(N(\lambda)) - \delta(0)\, G),$$

which implies that

$$\frac{\partial H^0}{\partial G} = R'(G) - \lambda \delta(0). \tag{1.12}$$

This gives

$$\frac{\partial^2 H^0}{\partial G^2} = R''(G) < 0,$$

so that $H^0(G, \lambda)$ is strictly concave in G.

Furthermore, by comparing (1.11) and (1.12) it is clear that $H^0(G, \lambda)$ does not exhibit a kink in G when the $A = 0$ boundary is reached.

3.2 Stability Analysis

The Jacobian of the canonical system (1.9) and (1.10) is

$$J=\left[\begin{array}{cc} -\delta\left(A\right)-\delta'\left(A\right)G\frac{\partial A}{\partial G} & \eta'\left(N\right)\frac{\partial N}{\partial \lambda}-\delta'\left(A\right)G\frac{\partial A}{\partial \lambda} \\ \delta'\left(A\right)\lambda\frac{\partial A}{\partial G}-R''\left(G\right) & r+\delta\left(A\right)+\delta'\left(A\right)\lambda\frac{\partial A}{\partial \lambda} \end{array}\right].$$

The determinant of this matrix is

$$\begin{aligned} \det J &= \left(-\delta-\delta' G\frac{\partial A}{\partial G}\right)\left(r+\delta+\delta'\lambda\frac{\partial A}{\partial \lambda}\right) \\ &\quad -\left(\eta'\frac{\partial N}{\partial \lambda}-\delta' G\frac{\partial A}{\partial \lambda}\right)\left(\delta'\lambda\frac{\partial A}{\partial G}-R''\right) \\ &= -\delta\left(r+\delta\right)-\delta'\left(\left(\delta\lambda+GR''\right)\frac{\partial A}{\partial \lambda}+\left(r+\delta\right)G\frac{\partial A}{\partial G}\right) \\ &\quad -\eta'\frac{\partial N}{\partial \lambda}\delta'\lambda\frac{\partial A}{\partial G}+\eta'\frac{\partial N}{\partial \lambda}R''. \end{aligned}$$

This can further be simplified by inserting the expressions for $\frac{\partial N}{\partial \lambda}, \frac{\partial A}{\partial G}, \frac{\partial A}{\partial \lambda}$:

$$\begin{aligned} \det J &= -\delta\left(r+\delta\right)+\delta'\left(\left(\delta\lambda+GR''\right)\frac{\delta'}{\lambda\delta''}+\left(r+\delta\right)\frac{\delta'}{\delta''}\right) \\ &\quad -\delta'\lambda\frac{\delta'}{\delta'' G}\frac{\left(\eta'\right)^2}{\lambda\eta''}-R''\frac{\left(\eta'\right)^2}{\lambda\eta''} &(1.13) \\ &= \underbrace{-\delta\left(r+\delta\right)}_{<0}+\underbrace{\frac{\left(\delta'\right)^2}{\delta''}}_{>0}\left(\underbrace{\left(r+2\delta\right)}_{>0}+\underbrace{\frac{GR''}{\lambda}}_{<0}\right) &(1.14) \\ &\quad \underbrace{-\frac{\left(\delta'\right)^2\left(\eta'\right)^2}{\delta''\eta'' G}}_{>0}\underbrace{-\frac{\left(\eta'\right)^2R''}{\lambda\eta''}}_{<0} &(1.15) \end{aligned}$$

Since there are conflicting terms, both stable and unstable equilibria can occur.

4. Analysis with specified functions

In the first two subsections we concentrate on the implications of different specifications of our new function $\delta\left(A\right)$. In Subsection 1.4.3 we specify the other functional forms as well in order to perform the complete analysis.

4.1 Exponential decay function $\delta\left(A\right)$

As a first specification consider the following exponential function:

$$\delta(A) = e^{-\gamma A} \quad \text{with} \quad \gamma > 0 \tag{1.16}$$

which gives $\delta'(A) = -\gamma e^{-\gamma A}$ and $\delta''(A) = \gamma^2 e^{-\gamma A}$. This leads to the following proposition.

PROPOSITION 1.2 *Consider the case where $\delta(A) = e^{-\gamma A}$ with $\gamma > 0$. Then, a steady state is a saddle point ($\det J < 0$) if and only if the maximized Hamiltonian is locally concave in this steady state. Instability ($\det J > 0$) is equivalent to the maximized Hamiltonian being convex there.*

Proof. Specification (1.16) implies that

$$\frac{(\delta')^2}{\delta''} = \delta. \tag{1.17}$$

By Proposition 1.1 and equation (1.17), concavity of the maximized Hamiltonian is equivalent to

$$R''(G) + \frac{\delta\lambda}{G} < 0.$$

From (1.14) and equation (1.17) we obtain that

$$\begin{aligned} \det J = & -\delta(r+\delta) + \delta(r+2\delta) + \delta\frac{GR''}{\lambda} - \frac{\delta(\eta')^2}{\eta'' G} - \frac{(\eta')^2 R''}{\lambda\eta''} && (1.18) \\ = & \ \delta^2 + \delta\frac{GR''}{\lambda} - \frac{\delta(\eta')^2}{\eta'' G} - \frac{(\eta')^2 R''}{\lambda\eta''} && (1.19) \\ = & \left(\frac{\delta\lambda}{G} + R''\right)\left(\frac{\delta G}{\lambda} - \frac{(\eta')^2}{\lambda\eta''}\right). && (1.20) \end{aligned}$$

Since the second term is always positive, it is clear that local concavity (convexity) of H^0 is equivalent to $\det J < 0$ (and $\det J > 0$, respectively). ■

In order to investigate whether instability can go along with the maximized Hamiltonian being concave, in which case the unstable steady state would be a node with continuous policy function (see Hartl, Kort, Feichtinger, and Wirl, 2004) we consider another specification for $\delta(A)$.

4.2 Decay function $\delta(A)$ specified as power function

Here we specify

$$\delta(A) = (1+A)^{-\gamma} \quad \text{with} \quad \gamma > 0 \tag{1.21}$$

which implies that $\delta'(A) = -\gamma(1+A)^{-\gamma-1}$ and $\delta''(A) = \gamma(1+\gamma)(1+A)^{-\gamma-2}$. This gives the following result.

PROPOSITION 1.3 *Consider the case where $\delta(A) = (1+A)^{-\gamma}$ with $\gamma > 0$. Then, (local) concavity of the maximized Hamiltonian implies that the steady state is a saddle point.*

Proof. Specification (1.21) implies that

$$\frac{(\delta')^2}{\delta''} = \frac{\gamma}{\gamma+1}\delta. \tag{1.22}$$

By Proposition 1.1 and equation (1.17), concavity of the maximized Hamiltonian is equivalent to

$$R''(G) + \frac{\gamma}{\gamma+1}\frac{\delta\lambda}{G} < 0.$$

From (1.14) and equation (1.17) we have that

$$\begin{aligned}
\det J = & -\delta(r+\delta) + \frac{\gamma}{\gamma+1}\delta(r+2\delta) \\
& + \frac{\gamma}{\gamma+1}\delta\frac{GR''}{\lambda} - \frac{\gamma}{\gamma+1}\frac{\delta(\eta')^2}{\eta''G} - \frac{(\eta')^2 R''}{\lambda\eta''} \\
= & \frac{\delta^2(\gamma-1) - r\delta}{\gamma+1} + \frac{\gamma}{\gamma+1}\delta\frac{GR''}{\lambda} - \frac{\gamma}{\gamma+1}\frac{\delta(\eta')^2}{\eta''G} - \frac{(\eta')^2 R''}{\lambda\eta''} \\
= & \left(\frac{\gamma}{\gamma+1}\frac{\delta\lambda}{G} + R''\right)\left(-\frac{(\eta')^2}{\lambda\eta''}\right) + \frac{\delta}{\gamma+1}\left(\delta(\gamma-1) - r + \gamma\frac{GR''}{\lambda}\right) \\
= & \left(\frac{\gamma}{\gamma+1}\frac{\delta\lambda}{G} + R''\right)\left(-\frac{(\eta')^2}{\lambda\eta''}\right) \\
& + \frac{\delta G\gamma}{(\gamma+1)\lambda}\left(\frac{\delta(\gamma-1)\lambda}{G\gamma} - \frac{\gamma}{\gamma+1}\frac{\delta\lambda}{G} - \frac{r\lambda}{G\gamma} + \frac{\gamma}{\gamma+1}\frac{\delta\lambda}{G} + R''\right) \\
= & \left(\frac{\gamma}{\gamma+1}\frac{\delta\lambda}{G} + R''\right)\left(\frac{\delta G\gamma}{(\gamma+1)\lambda} - \frac{(\eta')^2}{\lambda\eta''}\right) - \frac{\delta}{\gamma+1}\left(\frac{\delta}{\gamma+1} + r\right)
\end{aligned}$$

It is clear that local concavity of H^0 implies that $\det J < 0$. ■

So this specification does not help to identify instability while H^0 is concave.

4.3 Analysis with all functions specified

Again, let δ be specified as (1.16). In order to complete the analysis, we specify

$$\begin{aligned} R(G) &= G^{\alpha} \quad \text{with} \quad \alpha < 1 \\ R'(G) &= \alpha G^{\alpha-1} \\ R''(G) &= \alpha(\alpha-1)G^{\alpha-2} \end{aligned} \tag{1.23}$$

$$\begin{aligned} \eta(N) &= N^{\beta} \quad \text{with} \quad \beta < 1 \\ \eta'(N) &= \beta N^{\beta-1} \\ \eta''(N) &= \beta(\beta-1)N^{\beta-2} \end{aligned} \tag{1.24}$$

The maximum principle conditions now can be solved explicitly as

$$-a + \lambda\beta N^{\beta-1} = 0 \qquad \Longrightarrow \qquad N = \left(\frac{\lambda\beta}{a}\right)^{\frac{1}{1-\beta}} \tag{1.25}$$

and

$$-v + \lambda\gamma e^{-\gamma A}G = 0 \qquad \Longrightarrow \qquad A = \frac{1}{\gamma}\ln\frac{\lambda\gamma G}{v}. \tag{1.26}$$

The latter equation implies that

$$A \geq 0 \quad \text{iff} \quad \lambda > \frac{v}{\gamma G}. \tag{1.27}$$

It is clear that we have to distinguish between positive and zero A.

4.3.1 Analysis for $A > 0$. Let us first check concavity:

$$\begin{aligned} H^0_{GG} &= \alpha(\alpha-1)G^{\alpha-2} + \lambda\frac{e^{-\gamma A}}{G} \\ &= \alpha(\alpha-1)G^{\alpha-2} + \frac{v}{\gamma G^2} \\ &= \frac{1}{G^2}\left(\alpha(\alpha-1)G^{\alpha} + \frac{v}{\gamma}\right) < 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \alpha(\alpha-1)G^{\alpha} + \frac{v}{\gamma} &< 0 \\ G &> \left(\frac{v}{\gamma\alpha(1-\alpha)}\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{1.28}$$

We proceed by presenting the canonical system:

$$\dot{G} = \left(\frac{\lambda\beta}{a}\right)^{\frac{\beta}{1-\beta}} - \frac{v}{\lambda\gamma}, \tag{1.29}$$

$$\dot{\lambda} = r\lambda + \frac{v}{\gamma G} - \alpha G^{\alpha-1}. \tag{1.30}$$

The isocline $\dot{\lambda} = 0$ is given by

$$\lambda = \frac{\alpha G^{\alpha} - \frac{v}{\gamma}}{rG}, \tag{1.31}$$

which achieves its maximum value for

$$G = \left(\frac{v}{\alpha\gamma(1-\alpha)}\right)^{\frac{1}{\alpha}} \qquad \Longrightarrow \qquad \lambda = \frac{v\alpha}{\gamma r(1-\alpha)}\left(\frac{\alpha(1-\alpha)\gamma}{v}\right)^{\frac{1}{\alpha}}$$

The isocline $\dot{G} = 0$ is given by

$$\left(\frac{\lambda\beta}{a}\right)^{\frac{\beta}{1-\beta}} - \frac{v}{\lambda\gamma} = 0$$

$$\lambda = \left(\frac{v}{\gamma}\right)^{1-\beta}\left(\frac{a}{\beta}\right)^{\beta}. \tag{1.32}$$

Equations (1.31) and (1.32) imply that an equilibrium must satisfy

$$\alpha G^{\alpha} - \frac{v}{\gamma} = rG\left(\frac{v}{\gamma}\right)^{1-\beta}\left(\frac{a}{\beta}\right)^{\beta}. \tag{1.33}$$

This cannot be solved analytically except for $\alpha = 0.5$, which leads to

$$G_{1,2} = \frac{1}{2\left(\frac{v}{\gamma}\right)^{2(1-\beta)}\left(\frac{a}{\beta}\right)^{2\beta} r^2\gamma^2}\left(\alpha\gamma \pm \sqrt{\alpha^2\gamma^2 - 4\left(\frac{v}{\gamma}\right)^{1-\beta}\left(\frac{a}{\beta}\right)^{\beta} rv\gamma}\right)^2. \tag{1.34}$$

Returning to the general case, the Jacobian determinant reduces to

$$\begin{aligned} \det J = \quad & -\frac{v}{\lambda\gamma G}\left(r + \frac{v}{\lambda\gamma G}\right) + \frac{v}{\lambda\gamma G}\left(r + 2\frac{v}{\lambda\gamma G} + \frac{\alpha(\alpha-1)}{\lambda G^{1-\alpha}}\right) \\ & - \frac{\beta v\left(\frac{\lambda\beta}{a}\right)^{\frac{\beta}{1-\beta}}}{\lambda\gamma(\beta-1)G^2} - \frac{\beta\alpha(\alpha-1)\left(\frac{\lambda\beta}{a}\right)^{\frac{\beta}{1-\beta}}}{(\beta-1)\lambda G^{2-\alpha}} \\ \det J = \quad & \left(\frac{v}{\gamma} - \left(\frac{\beta}{a}\right)^{\frac{\beta}{1-\beta}}\frac{\beta}{\beta-1}\lambda^{\frac{1}{1-\beta}}\right)\frac{\frac{v}{\gamma} + \alpha(\alpha-1)G^{\alpha}}{G^2\lambda^2}. \end{aligned}$$

After inserting the value $\lambda = \left(\frac{v}{\gamma}\right)^{1-\beta}\left(\frac{a}{\beta}\right)^{\beta}$ of the $\dot{G} = 0$ isocline from (1.32) we get that

$$\det J = \frac{1}{1-\beta}\frac{\frac{v}{\gamma} + \alpha\left(\alpha - 1\right)G^{\alpha}}{G^{2}\left(\frac{a}{\beta}\right)^{2\beta}\left(\frac{v}{\gamma}\right)^{1-2\beta}}. \tag{1.35}$$

This implies that the steady state is a saddle point iff

$$\frac{v}{\gamma} < \alpha\left(1-\alpha\right)G^{\alpha}$$

$$G > \left(\frac{v}{\alpha\left(1-\alpha\right)\gamma}\right)^{\frac{1}{\alpha}}. \tag{1.36}$$

In case of instability, either a focus or a node is possible. A focus occurs when (see Feichtinger and Hartl, 1986, p. 105):

$$r^{2} < 4\det J.$$

We return to this issue later.

4.3.2 Analysis for $A = 0$. We already know from Section 1.3.1 that the maximized Hamiltonian H^{0} is strictly concave here. We proceed by presenting the canonical system:

$$\dot{G} = \left(\frac{\lambda\beta}{a}\right)^{\frac{\beta}{1-\beta}} - G$$

$$\dot{\lambda} = \left(r+1\right)\lambda - \alpha G^{\alpha-1}.$$

The $\dot{\lambda} = 0$ isocline is given by

$$\lambda = \alpha\frac{G^{\alpha-1}}{r+1}, \tag{1.37}$$

while the $\dot{G} = 0$ isocline is given by

$$\lambda = \frac{a}{\beta}G^{\frac{1-\beta}{\beta}}. \tag{1.38}$$

Hence, a steady state in this region satisfies

$$\frac{\alpha\beta}{\left(r+1\right)a} = G^{\frac{1-\alpha\beta}{\beta}}.$$

For $A = 0$, the Jacobian (1.14) reduces considerably to

$$\det J = -\delta\left(r+\delta\right) + \eta'\frac{\partial N}{\partial\lambda}R'' < 0.$$

The conclusion is that the steady state is always a saddle point in this region, provided that it exists.

4.3.3 Numerical example. To perform the numerical analysis we specify the parameters as:

$$\beta = 0.9, \quad \gamma = 3, \quad a = 3, \quad v = 1, \quad r = 0.03.$$

In Figure 1.1 we check whether the occurrence of an unstable node is possible. For $\alpha = 0.5$ we vary r between 0.01 and 0.07 and plot the values of the stable steady state (dotted upper curve), the unstable steady state (bold curve), the determinant of the Jacobian in the unstable steady state (dashed, positive) and $r^2 - 4 \det J$ (dashed-dotted) which is negative in the plotted region showing that always a focus occurs.

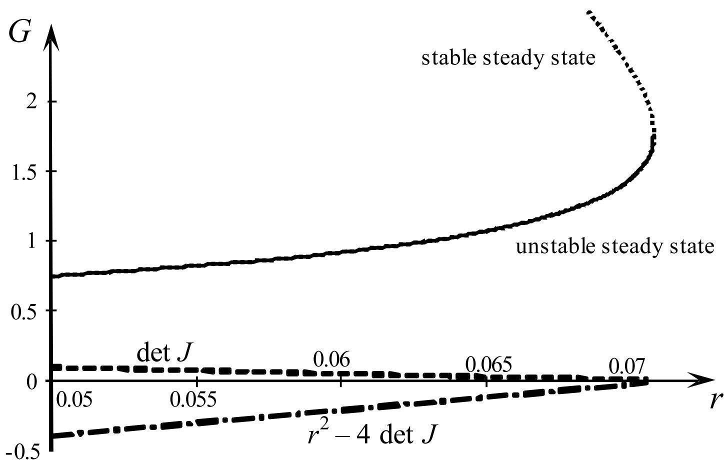

Figure 1.1. Stable and unstable steady state as a function of the discount rate r.

We are now in a position to plot the phase diagram. We choose the above parameter values. Figure 1.2 shows the state-costate phase diagram from which the region $A > 0$ can be seen clearly. There is a saddle point stable "large" equilibrium with goodwill $G_l = 30.66$ and an unstable focus at $G_u = 0.59$. From the orientation of the vector fields it is clear that the saddle point path converging to the larger equilibrium is downward sloping. Also, it lies in the region $A > 0$ so that both types of advertising are used to reach this large value of goodwill.

In order to see the region to the left of the $A = 0$ curve more clearly, we zoom into Figure 1.2 for small goodwill levels. This yields Figure 1.3 which shows the unstable (focus) G_u and a "small" equilibrium with goodwill level $G_s = 0.043$. This equilibrium lies in the region $A = 0$, so that only some moderate advertising towards new customers is applied which is sufficient to approach or maintain this small goodwill stock. Also here the saddle point path is downward sloping.

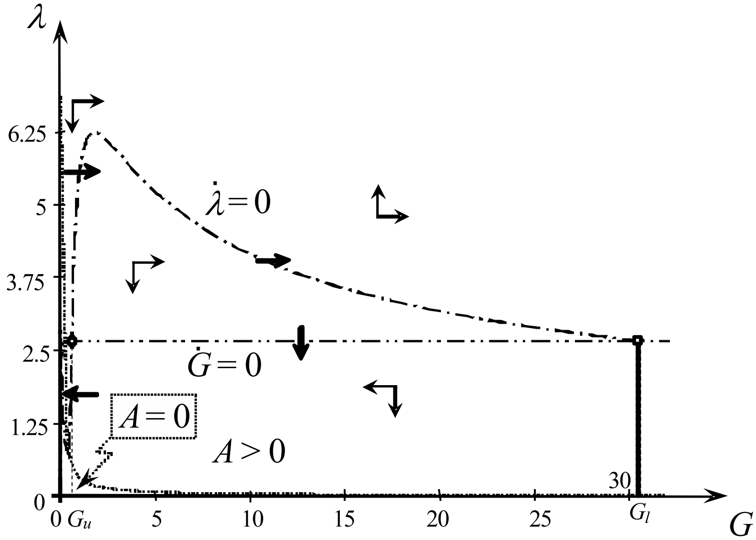

Figure 1.2. The phase diagram.

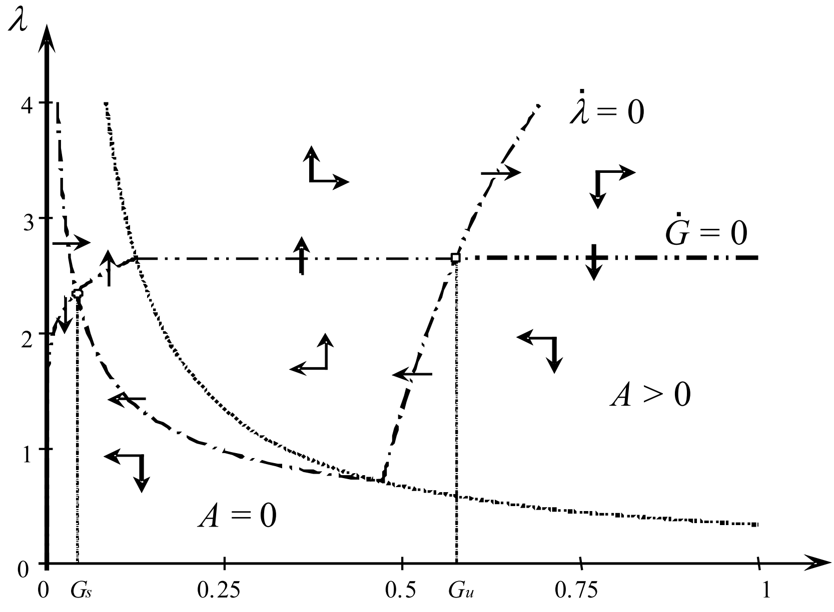

Figure 1.3. The phase diagram.

Somewhere in between G_s and G_l there must be a Skiba point. It is not necessarily close to the unstable steady state G_u but it must be in the overlap region of the two saddle point paths emerging from the unstable focus at G_u and converging to G_s and G_l, respectively.

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Chapter 2

ADVERTISING AND ADVERTISING CLAIMS OVER TIME

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Abstract Advertising budget allocation with carryover effects over time is a problem that was treated extensively by economists. Additional developments were carried out by Sethi who has also provided some outstanding review papers. The model treated by Sethi were essentially defined in terms of optimal control problems using deterministic advertising models while my own were essentially sales response stochastic models with advertising budget determined by stochastic control problems. These problems continue to be of academic and practical interest. Issues relating to the "advertising message" such as truthful claims advertising directed to first time buyers has not attracted much attention however.

The purpose of this paper is to address issues relating to advertising and their messages by suggesting a stochastic advertising-repeat purchase model. In this model, advertising directed to first time buyers is essentially defined by two factors: the advertising budget and the advertising message (such as statement regarding the characteristics of a product, its lifetime etc.). Consumers experience in case they buy the product will define the advertising message "reliability", namely that the probability that advertised message are confirmed or not. Repeat purchasers, however, are influenced by two factors, on the one hand the advertising messages that are directed to experienced consumers and of course the effects of their own experience (where past advertising claims whether truthful, or not, interact with customers' personal experience). Advertising claims that underestimate products characteristics might be "reliable" but then they might not entice first time purchasers, while overly optimistic advertising messages might entice first time purchasers but be perceived as unreliable by repeat purchasers who might switch to other competing brands. In this sense, the decision to advertise is necessarily appended by the decision to "what to advertise", which may turn out to be far more important for a firm. This paper provides a theoretical approach to deal with this issue.

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1. Introduction

Advertising budget allocation with carryover effects over time is a problem treated extensively by economists (Dorfman and Steiner (1954); Nerlove and Arrow (1962); Gould (1970); Nelson (1970); Nelson (1974); Schmalensee (1972); Katowitz and Mathewson (1979)), marketers (Vidale and Wolfe (1957); Ehrenberg (1972); Feichtinger (1982); Feichtinger et al. (1988); Schmittlein et al. (1985)). Additional developments were carried by both Sethi (Sethi (1973); Sethi (1974); Sethi (1975); Sethi (1977a); Sethi (1981); Sethi (1983a); Sethi (1983b)), who has also provided some outstanding review papers, Sethi (1977b), Feichtinger, Hartl and Sethi (1994) and myself (Tapiero (1975a); Tapiero (1975b); Tapiero (1977); Tapiero (1978); Tapiero (1979); Tapiero (1981); Tapiero (1982a); Tapiero (1982b); Tapiero (1982c); Tapiero (1982d)) as well as Farley and Tapiero (1981); Farley and Tapiero (1982) and Tapiero, Elyashberg and Wind (1987). As the Sethi, Feichtinger and Hartl papers attest, the number of references related to these problems is indeed extremely large. The models treated by Sethi are essentially defined in terms of deterministic optimal control problems while my own were essentially sales response stochastic models defining the optimal advertising policy in terms of stochastic control problems. A number of such studies value advertising expenses in terms of their contribution to the firm profit objectives or their effects on competitive posture and market structure such as the market response, the effects of memory, competition and other important topics that differentiate advertising models by the hypotheses they make about the sales response to advertising (through goodwill-capital accumulation, word of mouth and their like).

These problems continue to be of academic and practical interest both by raising new hypotheses regarding the effects of carry-over (memory) effects of advertising and the market competition structure (leading thereby to differential games for example, Tapiero (1978)). Issues relating to advertising claims (such as truthfulness in advertising) and the effects of experience and advertising efficiency on repeat purchasers has attracted relatively little attention however. This is in contradiction to strong empirical evidence that advertising weights (quantities) do not always matter while advertising copy may have a greater effect on sales response (Lodish et al. (1995)). Explicitly, Lodish et al. (1995), using extensive and shared data on advertising on TV claim that increasing advertising budgets in relation to competitors does not increase sales in general. However, changing brand, copy and media strategies in categories can in many cases lead to a sales response to advertising. Furthermore, they conclude that "New brands or line extensions tend to be more responsive to alternative T.V. advertising plans than established products". Such a claim supports the hypothesis that advertising acts essentially on "first time purchasers" and less on purchasers of well established brands that derive their essential sales from repeat purchasers.

The purpose of this paper is to address issues relating to the relationships between advertising budgets and advertising claims based on a stochastic model of advertising-quality and quantity (see also Tapiero (2000)). We clearly distinguish between first time purchasers and repeat purchasers, the former determined in terms of a stochastic "Capital Goodwill Model" (Nerlove and Arrow (1962); Tapiero (1978)) while the latter is based on a probability response to ex-post product consumption and satisfaction. Further, unlike previous optimal and stochastic control models of advertising, we recognize that in many products a firm derives benefits not only from product sales to customers but also from the services they sell to these customers. Such an assumption is increasingly realistic and emphasizes as well the importance of repeat purchasing management in firms' marketing and advertising strategies.

Consumers experience compared to advertising claims defines the advertising claim "reliability", namely the probability that an advertised claim is confirmed or not by the experienced purchaser. Of course, "confirmation" of an advertising claim by an experienced client contributes to repeat purchase while, "disappointment" contributes (in probability) to a consumer switching to some other firms. Since true products characteristics are necessarily random (due to the production process, use and misuse of the product) advertising claims truthfulness is inherently random as well. Thus, there is always a probability that advertising claims are not met. Advertising claims that underestimate product characteristics might be "reliable", namely mostly true, but then they might not entice first time purchasers, while overly optimistic advertising claims might entice first time purchasers but be perceived as unreliable by repeat purchasers who might switch to other competing products. In this sense, the decision to advertise is necessarily concurrent to the decision to "what to advertise". Such decisions are compounded by the fact that in prevalent marketing philosophy, a consumer is also a consumer of services (such as warranties, product servicing, etc.) and a firm profits not only from the revenues generated at the time of sale but also in derived revenues maintained as long as the customer remains a client of the firm. This paper provides some preliminary approaches to dealing with these issue by developing a sales/repeat purchase response stochastic model that defines these characteristics and by considering a decision model that raises a number of issues such as "how much to advertise" conjointly with "what to advertise". We begin by considering a long run (average based) decision model and continue with a dynamic (optimal control) model.

2. Advertising Quantity and Advertising Reliability: Stationary Model

Let first time purchasers' sales $S(p,x)$, be determined by the price $p(t)$ and a "Goodwill-Capital" $\{x, t \geq 0\}$. "Goodwill" is a random function, expressed by a non-homogenous non-stationary Poisson Process with mean λ(t) and generated by advertising efforts $a(t)$ carried over time with an exponential memory and a forgetting rate parameter m (Nerlove and Arrow (1962); Tapiero (1975a); Tapiero (1977); Tapiero (1978)):

$$P(x(t) = i) = \frac{\left[\lambda(t)^i e^{-\lambda(t)}\right]}{i!} \tag{2.1}$$

$$d\lambda(t)/dt = -m\lambda + qa(t), \;\; \lambda(0) = \lambda_0, q > 0$$

where q is a parameter expressing the advertising efficiency. For simplicity we set $S(p,x) = x$. Now let there be an advertising claim α. The "better" this claim, the larger the effects of advertising on new purchasers. This means that $q = q\,(\alpha)\,, q'\,(\alpha) > 0$, thereby generalizing the stochastic goodwill model in (2.1). In addition, an advertising claim confirmed by consumption experience will induce a repurchase and vice versa, non confirmation will induce disloyalty. Over the long run, assuming stationary policies (which are starred), we have: $\lambda^* = q\,(\alpha^*)\,a^*/m$. Thus, in a stationary state, the rate of incoming first time purchasers has a Poisson distribution whose mean is λ^*, a function of both the advertising rate and the advertised claim.

An experienced unsatisfied customer may be a lost customer while a satisfied one may repeat purchase. Let α^* be the advertising claim and let $\tilde{\alpha}$ be the true product characteristic, a random variable, with cumulative distribution $F(\tilde{\alpha})$. If $\tilde{\alpha} \geq \alpha^*$, the probability of a repeat purchase after a unit consumption is $1 - F(\alpha^*)$, meaning that the customer experience is better than the advertised claim. If a client remains loyal for $\tilde{k}_1$ units, till a unit is found to be non-conforming, then the total number of units purchased is $1 + \tilde{k}_1$. This can be specified by a geometric distribution, or:

$$g_1(\tilde{k}_1 : 1) = \left[1 - F_\alpha(\alpha^*)\right]^{\tilde{k}-1} F_\alpha(\alpha^*) \tag{2.2}$$

If the customer remains loyal until r units are found to be non-conforming, the number of units acquired by a customer is $1 + \tilde{k}_r$, given by the negative binomial distribution:

$$g_r(\tilde{k}_r : r) = C_{r-1}^{\tilde{k}_r - 1}\left[1 - F_\alpha(\alpha^*)\right]^{\tilde{k}_r - r} F_\alpha(\alpha^*)^r \tag{2.3}$$

with mean and variance:

$$\begin{aligned} E\left(k_r\right) &= r\theta^* \\ var\left(k_r\right) &= \frac{r\theta^*}{1 - F_\alpha(\alpha^*)} \\ \theta^* &= \frac{F_\alpha(\alpha^*)}{1 - F_\alpha(\alpha^*)} \end{aligned} \tag{2.4}$$

where θ^* is the odd that the advertising claim turns out to be false. Estimate for these parameters can be determined based on historical data of the number of units bought by individual customers while they were loyal customers. Such data also reveals customers' "impatience". For example, "an impatient customer" will not repeat purchase as soon as a non-conforming unit is experienced. In this case, the total number of units bought by the customer would be only $1 + \tilde{k}_1$ as stated earlier. A patient customer "might give a second a chance" to the firm and remain loyal if the quantity $\tilde{k}_1$ consumed thus far is greater than some parameter k_1^* etc. Therefore, customers impatience can be denoted in terms of probability parameters χ_j , expressing the proportion willing to accept j non-conforming claims before resigning and not repeat purchase. In this case, the probability distribution for the number of units bought by a customer is given by the mixture distribution:

$$\tilde{k} = \tilde{k}_i \text{ wp } \chi_i,\ \tilde{k}_i = g_i(. : i),\ i = 1, 2, ..., n \tag{2.5}$$

where $\tilde{k}_i,\ i = 1, 2, ...$ are denoted by the negative binomial distribution (2.2). For example, say that in a heterogeneous population a proportion χ_1 is impatient while the remaining is "patient" willing to give a second chance to non-conforming experienced units. Then, the probability distribution of the number units sold in repeat purchase is:

$$\tilde{k} = \tilde{k}_i \text{ wp } \chi_i,\ \tilde{k}_i = g_i(. : i),\ i = 1, 2 \tag{2.6}$$

where $\chi_2 = 1 - \chi_1$. As a result, the mean number of units that are repeat purchased and their variance are given by:

$$\begin{aligned} E\left(\tilde{k}\right) &= \sum_{j=1}^{\infty} E\left(\tilde{k}_j\right)\chi_j \\ var\left(\tilde{k}\right) &= \sum_{j=1}^{\infty} E\left(\tilde{k}_j^2\right)\chi_j - \left(\sum_{j=1}^{\infty} E\left(\tilde{k}_j\right)\chi_j\right)^2 \end{aligned} \tag{2.7}$$

If the underlying characteristic of the product advertised is its lifetime, a random variable $\tilde{\tau}_i$, identically and independently distributed, and if the customer repeat purchases $\left(\tilde{k}\right)$ units, then the time a customer remains loyal to the firm is given by the Compound random variable $\tilde{T}$

$$\tilde{T} = \sum_{i=1}^{1+\tilde{k}} \tilde{\tau}_i \tag{2.8}$$

whose mean and variance can be calculated explicitly by:

$$\begin{aligned} E\left(\tilde{T}\right) &= E\left(1+\tilde{k}\right) E\left(\tilde{\tau}_i\right); \\ var\left(\tilde{T}\right) &= E\left(1+\tilde{k}\right) var\left(\tilde{\tau}_i\right) + var\left(1+\tilde{k}\right) E\left(\tilde{\tau}_i\right)^2 \end{aligned} \tag{2.9}$$

In this sense, equations (2.7) and (2.9) provide an estimate for the number of units bought by an individual customer and the amount of time the customer remains loyal (and thereby, consuming the firm services associated to the product).

Further, since the number of first time purchasers over a given period of time has a Poisson distribution with mean rate λ^* while "loyalty time" is a random variable $\tilde{T}$ whose mean and variance are given by (2.9), we recognize this type of model as an M/G/Infinity queue (Gross and Harris (1985)). Thus, in a stationary state, the number of customers, paying a service fee of w per unit time has also a Poisson distribution with mean:

$$\rho = \lambda^* E\left(\tilde{T}\right) = \frac{q\left(\alpha^*\right) a^*}{m} E\left(1+\tilde{k}\right) E\left(\tilde{\tau}_i\right) \tag{2.10}$$

The expected number of sales per unit time is given by ψ, which is equated here to the total number of units bought in a time interval by customers arriving at a specific instant of time divided by this time interval, or

$$\psi = \lambda^* \frac{E\left(1+\tilde{k}\right)}{E(\tilde{T})} = \frac{\lambda^*}{E\left(\tilde{\tau}_i\right)} \tag{2.11}$$

Setting to $C(a^*)$ the advertising cost per unit time, the firm profit per unit time is:

$$\begin{aligned} \underset{\alpha^*, a^*}{Max}\, \pi &= w\rho + p\psi - C(a^*) \\ &= \frac{q\left(\alpha^*\right) a^*}{m} \left(wE\left(1+\tilde{k}\right) E\left(\tilde{\tau}_i\right) + \frac{p}{E\left(\tilde{\tau}_i\right)} \right) - C(a^*) \end{aligned} \tag{2.12}$$

An optimization of (2.12) with respect to the advertising budget and the advertising claim will provide a solution to this problem. Explicitly, a partial derivative with respect to advertising yields:

$$\pi_a = \frac{q(\alpha^*)}{m}\left(wE\left(1+\tilde{k}\right)E(\tilde{\tau}_i) + p\frac{1}{E(\tilde{\tau}_i)}\right) - C'(a^*) \tag{2.13}$$

At the optimum however, π_a=0 and therefore:

$$C'(a^* = \frac{q(\alpha^*)}{m}\left(wE\left(1+\tilde{k}\right)E(\tilde{\tau}_i) + p\frac{1}{E(\tilde{\tau}_i)}\right) \tag{2.13a}$$

And the marginal cost of advertising equals the marginal profit from both sales and derived profits as given by (2.13a). Similarly, a partial derivative with respect to the advertising claim yields:

$$\pi_{\alpha^*} = (\ \pi + C(a^*))\frac{q_\alpha(\alpha^*)}{q(\alpha^*)} + \frac{q(\alpha^*)\,a^*}{m}\left(wE(\tilde{\tau}_i)\left(E(1+\tilde{k})\right)_\alpha\right) \tag{2.14}$$

At optimality π_{α^*}=0 we have:

$$\frac{m(\ \pi + C(a^*))}{a^* wE(\tilde{\tau}_i)} = \frac{\left(\left(E(1+\tilde{k})\right)_\alpha\right)}{(1/q(\alpha^*))_\alpha} \tag{2.15}$$

However, at optimality, $\pi_a a + aC'(a^*) = \pi + C(a^*),\ \pi_a = 0$ and therefore,

$$\frac{mC'(a^*)}{wE(\tilde{\tau}_i)} = \frac{\left(\left(E(1+\tilde{k})\right)_\alpha\right)}{(1/q(\alpha^*))_\alpha} \tag{2.16}$$

Inserting, (2.13) in (2.16), we obtain at last an expression in the advertising claim only:

$$1 + \frac{p}{w(E(\tilde{\tau}_i))^2}(\ln(q(\alpha^*)))_\alpha = -\left(E(\tilde{k})\right)_\alpha - E\left(\tilde{k}\right) \tag{2.17}$$

Explicitly, if we let $q(\alpha^*) = e^{\varepsilon\alpha^*}$, then we have $q_\alpha = \varepsilon q$,and:

$$1 + \frac{p\varepsilon}{w(E(\tilde{\tau}_i))^2} = -\left(E(\tilde{k})\right)_\alpha - E\left(\tilde{k}\right) \tag{2.18}$$

Where $\left(E(1+\tilde{k})\right)_\alpha$ <0 (since the more we claim, the smaller the number of units repurchased), and the optimal advertising claim is obtained by a solution of (2.18). Assume again for simplicity that a consumer remains loyal until a unit consumed does not conform to advertised claims, then the number of units repeat purchased is a geometric random variable given by: $g_1(\tilde{k}_1 : 1) = [1 - F_\alpha(\alpha^*)]^{\tilde{k}-1} F_\alpha(\alpha^*)$ and $E(\tilde{k}_1) = \theta^*$ where θ^* is the odd that an advertised claim is not met, or

$\theta^* = F_\alpha(\alpha^*)/\left[1 - F_\alpha(\alpha^*)\right]$. As a result, $\left(E(1+\tilde{k})\right)_\alpha = \theta_\alpha$ and therefore,

$$1 + \frac{p\varepsilon}{w\left(E\left(\tilde{\tau}_i\right)\right)^2} = -\theta_\alpha - \theta > 0 \tag{2.19}$$

which can be solved for the advertised claim. For simplicity say that the odds of an advertised claim is met is given by a logistic regression of the type $\ln\theta^* = a_0 - a_1\alpha$, then $\theta_\alpha = -a_1\theta,\ a_1 > 1$ and therefore:

$$1 + \frac{p\varepsilon}{w\left(E\left(\tilde{\tau}_i\right)\right)^2} = \left(a_1 - 1\right)e^{a_0 - a_1\alpha} \tag{2.20}$$

We note in particular:

$$\begin{aligned} \frac{d\vartheta}{d\alpha} &= \frac{\varepsilon}{a_1\left(a_1 - 1\right)\theta\left(E\left(\tilde{\tau}_i\right)\right)^2} \\ 1 + \vartheta\frac{\varepsilon}{\left(E\left(\tilde{\tau}_i\right)\right)^2} - \left(a_1 - 1\right)e^{a_0 - a_1\alpha} &= 0 \\ \vartheta &= p/w \end{aligned} \tag{2.21}$$

Thus, the larger the advertising efficiency, the greater the claim and the greater the relative price p/w. And vice versa, when the odds that an advertised claim is not met by a consumption experience the price p/w is smaller. Interestingly, the larger the product expected life, the smaller the relative price p/w. The optimal claim, in this particular case is then specified by (2.20), or:

$$\alpha^* = \ln\left(\frac{\left(a_1 - 1\right)w\left(E\left(\tilde{\tau}_i\right)\right)^2}{w\left(E\left(\tilde{\tau}_i\right)\right)^2 + p\varepsilon}\right)^{\frac{a_0}{a_1}} \tag{2.22}$$

The corresponding advertising budget is then determined by equation (2.16). If an advertising claim is set to a conservative low figure, $q\left(\alpha^*\right)$ will be smaller reducing the advertising efficiency but maintaining customers as repeat purchasers. Further, the higher the advertising claim, the higher the advertising budget (i.e. the firm has an essentially first purchaser dominant strategy). We can also note that the larger the advertising forgetting rate and the smaller the service payment fee, the more we advertise and as a result, the more we shall claim. Our conclusions, based on a simple and theoretical analysis confirms some hypotheses set by Lodish et al. (1995) stating that in well established brands (where a major part of sales are generated by loyal repeat purchasers), the tendency will be to advertise less.

3. The Non-stationary Advertising-Claims Model

Let $y(t)$ be the number of first purchasers at time t with $y(s) = x$. Now assume that at new purchase events occur at the random (Poisson) times $\tau_i, s < \tau_1 < \tau_2 < ... < \tau_i < ...$, with a non-stationary Poisson process given by:

$$\frac{d\lambda(t)}{dt} = -m\lambda(t) + q\,(\alpha)\,a(t),\ \ \lambda(0) = \lambda_0 \tag{2.23}$$

For generality, we let ξ_i be the number of new first purchasers, *a random quantity.* This means that at timeτ_i, at which time new purchasers come, we have:

$$\begin{aligned} y(t+dt) &= y(t) \text{ at } t \in (\tau_i, \tau_{i+1}) \\ y(\tau\,{}^{+}_{i}) &= y(\tau\,{}^{-}_{i}) + \xi_i \text{ at } t = \tau_i \\ i &= 1, 2, 3, ... \end{aligned} \tag{2.24}$$

Or,

$$y(t) = y(0) + \int\limits_{0}^{t} \int_{\Re_n} z\mu(dx, dz) \tag{2.25}$$

where $\mu\,(dx, dz)$is a function denoting the number of jumps of first purchases process *y(t)* in the time interval $(0, t]$ defined as follows:

$$\mu\,(\Delta, A) = \mu\,(t + \Delta t, A) - \mu\,(t, A) \tag{2.26}$$

and A is a Borel subset on $\Re^n$. The measure $\mu(\Delta, A)$ is called the jump measure of the process $\{y(t), t \geq 0\}$. If first time events occur one at a time, then we have a simple nonstationary Poisson process with,

$$\begin{aligned} Prob\,[\tau_{i+1} \geq s\,|\tau_i] &= \exp\left[-\int\limits_{\tau_i \wedge s} \lambda(x)dx\right] \text{and} \\ \mu(dx, dz) &= \mu(dx)\delta(z-1) \end{aligned} \tag{2.27}$$

as indicated in this paper where $\mu(t)$ is a point process whose jumps equals one or such that:

$$\begin{aligned} P\,(\mu(dt) = 1) &= \lambda(t)dt + 0(dt) \\ P\,(\mu(dt) = 0) &= 1 - \lambda(t)dt + 0(dt) \\ P\,(\mu(dt) \geq 2) &= 0(dt) \end{aligned} \tag{2.28}$$

Hence, $\mu(t)$ is a random variable distributed according to a Poisson law whose mean (parameter) is:

$$\Lambda(t) = \int_0^t \lambda(x)dx; \frac{d\Lambda(t)}{dt} = \lambda(t);\ \Lambda(0) = 0 \tag{2.29}$$

In summary, first time purchases are generated by a non-homogenous Poisson process with mean $\lambda(t)$ –the goodwill equation, a function of advertising $a(t)$ at time t and the advertising claim α. The implications of this observation is that at time t, the total number of cumulative first purchases is given by *N(t)* which has a Poisson distribution with parameter $\Lambda(t)$ given by equation (2.29). If the amount of time a customer remains loyal to the firm (namely remains a repeat purchaser) has a probability density function $b\left(\tilde{T}\right)$, then the total number of customers that have experienced the firm's product and are no longer customers is given by the event *(M(t)|N(t)=n)*. Thus, the current number of customers the firm has (and paying for derived services and charges associated to the product consumption) is *n-(M(t)|N(t)=n)*. First, note the probability distribution of the conditional event is given by the binomial distribution which is specified as follows:

$$\begin{aligned} P[M(t)|N(t) = n] = \binom{n}{M(t)} \pi(t)^{M(t)} \left[1 - \pi(t)\right]^{n-M(t)}, \\ \pi(t) = \int_0^t \frac{\lambda(u)b(t-u)}{\Lambda(t)} du \end{aligned} \tag{2.30}$$

Since n has a Poisson distribution, $M(t)$ has also a Poisson distribution with mean $\Phi(t) = \Lambda(t)\pi(t)$, explicitly given by equation (2.31),

$$\begin{aligned} P(M(t) = j) = \frac{\Phi(t)^j \exp(-\Phi(t))}{j!} \\ \Phi(t) = \int_0^t \lambda(u)b(t-u)du) \end{aligned} \tag{2.31}$$

As a result, the number of clients that have left the firm is given by the difference of two non-homogenous Poisson processes, one with mean given by equation (2.29) and the other by (2.31). Of course, when the first purchase rate is constant, and for t large, the amount of time a customer remains loyal is independent of time with $M(t)$ a Poisson distribution with mean:

$$\Lambda^\infty = \lambda \int_0^\infty b(t-u)du = \lambda \int_0^\infty dB(u) = \lambda E(\tilde{T}) \tag{2.32}$$

where $E(\tilde{T})$ is the mean loyalty time. From a modeling point of view, it is seen that $a(t)$, the advertising budget, determines the mean rate of new sales while the density function $B(.)$ is defined by the advertising claim. First note that revenues derived from current customers has a mean given by $w\left(\Lambda(t)-\Phi(t)\right)$. Product sales revenues from both first and repeat purchasers are calculated as follows. Consider a new client arriving at time u and buying at price p. Such a client will repeat purchase $\tilde{k}$ units at times $u+\tilde{\tau}_1,\ u+\tilde{\tau}_1+\tilde{\tau}_2,\ ,\ldots\ u+\sum_{i=1}^{\tilde{k}}\tilde{\tau}_i$ and since the arrival rate is Poisson and given by $\lambda(u)$ at this time, the discounted value of all future purchases (to time u) by the client arriving at time u and discounted at the discount rate r is:

$$E\left\{p\lambda(u)E\left(1+\sum_{j=1}^{\tilde{k}}e^{-r\sum_{i=1}^{j}\tau_i}\right)\right\} \tag{2.33}$$

Let $E\left(e^{-r\tilde{\tau}}\right)=L_\tau^*(r)$ be the Laplace Transform of an individual unit life time. Note that if the client repeats purchase one unit only $(k=1)$, then $E\left(e^{-r\tilde{\tau}}\right)=L_\tau^*(r)$. For two units (k=2), with independent product lifetimes, $E\left(e^{-r\tilde{\tau}}\right)+E\left(e^{-r(\tilde{\tau}_1+\tilde{\tau}_2)}\right)=L_\tau^*(r)+\left(L_\tau^*(r)\right)^2$ etc. for a larger number of units. As a result, if the customer buys j units in repeat purchase, we have:

$$\begin{aligned}
&L_\tau^*(r)+\left(L_\tau^*(r)\right)^2+\ldots+\left(L_\tau^*(r)\right)^j=\\
&=\sum_{i=0}^{j}\left(L_\tau^*(r)\right)^i-1\\
&=\tfrac{1-\left(L_\tau^*(r)\right)^j}{1-L_\tau^*(r)}-1\\
&=\tfrac{L_\tau^*(r)\left(1-\left(L_\tau^*(r)\right)^{j-1}\right)}{1-L_\tau^*(r)}
\end{aligned}$$

As a result, the expected discounted value of all repeat purchases by clients that have had their first purchases at time u is:

$$\begin{aligned}
p\lambda(u)E\left(\sum_{j=1}^{\tilde{k}} e^{-r\sum_{i=1}^{j}\tau_i}\right) &= p\lambda(u)\sum_{j=1}^{\infty} P_j \frac{L_\tau^*(r)\left(1-(L_\tau^*(r))^{j-1}\right)}{1-L_\tau^*(r)} \\
&= p\lambda(u)\frac{L_\tau^*(r)}{1-L_\tau^*(r)}\sum_{j=1}^{\infty} P_j\left(1-(L_\tau^*(r))^{j-1}\right) \\
&= p\lambda(u)\frac{L_\tau^*(r)}{1-L_\tau^*(r)} \\
&\quad\times\left(\sum_{j=1}^{\infty} P - \frac{1}{L_\tau^*(r)}\sum_{k=1}^{\infty} P_j\ \ (L_\tau^*(r))^j\right) \\
&= p\lambda(u)\frac{L_\tau^*(r)}{1-L_\tau^*(r)}\left(1-\frac{\prod_P\left(L_\tau^*(r)\right)}{L_\tau^*(r)}\right)
\end{aligned} \tag{2.34}$$

where P_j is the probability that there are j such repeat purchases and $\prod_P$ is the probability generating function of the number of units repeat purchased, then equation (2.34) can be summarized by:

$$p\lambda(u)E\left(\sum_{j=1}^{\tilde{k}} e^{-r\sum_{i=1}^{j}\tau_i}\right) = p\lambda(u)\frac{L_\tau^*(r)}{1-L_\tau^*(r)}\left(1-\frac{\prod_P\left(L_\tau^*(r)\right)}{L_\tau^*(r)}\right) \tag{2.35}$$

For example if a customer does not repeat purchase as soon as consumption experience does not conform to advertising claim, we have:

$$P_k = g_1(k_1:1) = \left[1-F_\alpha(\alpha^*)\right]^{k-1} F_\alpha(\alpha^*) \tag{2.36}$$

whose probability generating function is:

$$\begin{aligned}
\prod_P &= \sum_{k=1}^{\infty} P_k z^k = \\
&= zF_\alpha(\alpha^*)\sum_{k=1}^{\infty}\left[1-F_\alpha(\alpha^*)\right]^{k-1} z^{k-1} = \\
&= \frac{zF_\alpha(\alpha^*)}{1-z\left[1-F_\alpha(\alpha^*)\right]}
\end{aligned} \tag{2.37}$$

and therefore,

$$
\begin{aligned}
& p\lambda(u)E\left(\sum_{j=1}^{\tilde{k}} e^{-r\sum\limits_{i=1}^{j}\tau_i}\right) = \\
& = p\lambda(u)\frac{L_\tau^*(r)}{1-L_\tau^*(r)}\left(\frac{1-L_\tau^*(r)\left[1-F_\alpha(\alpha^*)\right]-F_\alpha(\alpha^*)}{1-L_\tau^*(r)\left[1-F_\alpha(\alpha^*)\right]}\right)
\end{aligned}
\tag{2.38}
$$

If for simplicity we keep this expression to calculate the discounted value of a customer's repeat purchases, we have the following optimal control problem:

$$
\underset{a_1(t)\geq 0,\alpha}{Max}\int_0^\infty e^{-ru}\left[p\lambda(u)G(\alpha)+w\left(\Lambda(u)-\Phi(u)\right)-C(a)\right]du
$$

Subject to:

$$
\begin{aligned}
& d\lambda(u)/dt = -m\lambda + q(\alpha)\,a(u),\ \ \lambda(0)=\lambda_0 \\
& d\Lambda(u)/dt = \lambda(u),\ \ \Lambda(t)=\Lambda_0 \\
& \Phi(u) = \int_0^u \lambda(v)b(u-\nu)dv \\
& G(\alpha) = 1+\frac{L_\tau^*(r)}{1-L_\tau^*(r)}\left(1-\frac{\prod_P\left(L_\tau^*(r)\right)}{L_\tau^*(r)}\right)
\end{aligned}
\tag{2.39}
$$

The solution of this problem, a three states variables control problem, can be solved by an application of Pontryagin Maximum Principle. For discussion purposes, if we set the advertising claim constant and let the advertising policy be defined a-priori in terms of some function of time such as $C(a) = C(\bar{a}, t)$ (or as a feedback function of the means of first time purchasers, repeat purchasers and lost clients for example), we have by simple Laplace transform techniques that:

$$
s\lambda^*(s)-\lambda(0) = -m\lambda^*(s)+q(\alpha)\,a^*(s) \tag{2.40}
$$

$$
,\ \ s\Lambda^*(s)-\Lambda(0)=\lambda^*(s),\ \ \Phi^*(s)=\lambda^*(s)b^*(s) \tag{2.41}
$$

where

$$\lambda^*(s) = \int_0^\infty e^{-s\lambda(t)}dt,$$
$$\Lambda^*(s) = \int_0^\infty e^{-s\Lambda(t)}dt, \tag{2.42}$$
$$\Phi^*(s) = \int_0^\infty e^{-s\Phi(t)}dt$$

As a result,

$$\begin{aligned} \lambda^*(s) &= \frac{q(\alpha)\, a^*(s) + \lambda(0)}{(s+m)}, \\ \Lambda^*(s) &= \frac{q(\alpha)\, a^*(s) + \lambda(0)}{s\,(s+m)} + \frac{\Lambda(0)}{s}, \\ \Phi^*(s) &= \frac{q(\alpha)\, a^*(s) + \lambda(0)}{(s+m)} b^*(s). \end{aligned} \tag{2.43}$$

where $b(t) = \sum_{i=1}^{\tilde{k}} \tau_i$ and therefore $b^*(s) = \prod_P (L_\tau(s))$ where $L_\tau(s)$ is the transform of an individual unit life time while $\prod_P(.)$ is the probability generating function of the number of units bought by a customer (and thereby a function of the advertising claim). Inserting in our objective function, we have:

$$\begin{aligned} J &= \int_0^\infty e^{-ru}\Pi du = \\ &= p\lambda^*(r)G(\alpha) + w\left(\Lambda^*(r) - \Phi^*(r)\right) - C^*(\bar{a}, r) \end{aligned} \tag{2.44}$$

For a linear advertising cost, this is reduced to:

$$\begin{aligned} J &= \frac{a^*(r)}{(r+m)}\left\{pq(\alpha)G(\alpha^*) + wq(\alpha)(\frac{1}{r} - \prod_P (L_\tau(r))) - 1\right\} \\ &+p\frac{\lambda(0)}{(r+m)}G(\alpha^*) + w\left(\frac{\lambda(0)}{r\,(r+m)} + \frac{\Lambda(0)}{r} - \frac{\lambda(0)}{(r+m)}b^*(r)\right) \end{aligned} \tag{2.45}$$

which is linear in the advertising policy transform. For example, if the advertising policy consists of a fixed quantity $\bar{a}$, then $a^*(r) = \bar{a}/r$ and the optimal advertising allocation and claim policy if given by maximizing (2.44) with respect to $(\bar{a}, \alpha)$.

$$\frac{\partial J}{\partial \alpha} = \frac{a^*(r)}{(r+m)} \begin{Bmatrix} +pq'(\alpha)\, G(\alpha^*)+ \\ +p\left(q(\alpha) + \frac{\lambda(0)}{(r+m)}\right) G'(\alpha^*) \\ +wq'(\alpha)\left(\frac{1}{r} - \prod_P (L_\tau(r))\right) \\ -w\left(\frac{\lambda(0)}{(r+m)} + q(\alpha)\right) \frac{\partial \prod P(L_\tau(r))}{\partial \alpha} \end{Bmatrix} \tag{2.46}$$

For an optimal advertising claim $\frac{\partial J}{\partial \alpha}$=0, leading to:

$$\begin{aligned} \frac{p}{w} &= \frac{A}{B} \\ A = \left(\frac{\lambda(0)}{(r+m)} + q(\alpha)\right) & \frac{\partial \prod_P (L_\tau(r))}{\partial \alpha} \\ & -q'(\alpha)\left(\frac{1}{r} - \prod_P (L_\tau(r))\right) \\ B = q'(\alpha)\, G(\alpha^*) + & \left(q(\alpha) + \frac{\lambda(0)}{(r+m)}\right) G'(\alpha^*) \end{aligned} \tag{2.47}$$

while optimization with the parameter $0 \leq \bar{a} \leq a_{\max}$ yields

$$\bar{a} = \begin{cases} a_{\max} & \text{if } p > w\left(\frac{\prod_P (L_\tau(r)) - \frac{1}{r}}{G(\alpha^*)}\right) \\ 0 & \text{else} \end{cases} \tag{2.48}$$

If we consider a nonlinear advertising cost, while maintaining the constant advertising rate, we have an optimal marginal cost of advertising given by:

$$C'(\bar{a}) = \frac{q(\alpha)\, a^*(r) G(\alpha)}{(r+m)} \left[p + w \frac{1/r - \prod_P (L_\tau(r))}{G(\alpha)}\right] \tag{2.49}$$

Similarly, if we assume that the advertising policy is proportional to the current means of new purchases and the number of clients lost by the firm, then we have:

$$a(t) = \bar{a} + a_1 \lambda(t) + a_2 (\Lambda(t) - \Phi(t)) \tag{2.50}$$

Or,

$$a^*(r) = \frac{\bar{a}}{r} + a_1 \lambda^*(r) + a_2 (\Lambda^*(r) - \Phi^*(r)) \tag{2.51}$$

and (after some elementary manipulations), we have:

$$\begin{aligned} a^*(r) =& \frac{A(r)}{B(r)} \\ A(r) =& \bar{a}\,(r+m) + \\ & + (a_1 r + a_2 - a_2 r \prod_P (L_\tau(r)))\lambda(0) \\ & + a_2\,(r+m)\,\Lambda(0) \\ B(r) =& r\,(r+m) - \\ & - q(\alpha)(a_1 r + a_2 - a_2 \prod_P (L_\tau(r))r) \end{aligned} \tag{2.52}$$

which is optimized with respect to its parameters, a function of the advertising claims.

4. Conclusion and Discussion

Advertising claims once experienced by consumers may determine the propensity to repeat purchase. This paper has focused attention on the design of claims and advertising policies simultaneously based on a stochastic model of advertising efficiency and repeat purchase of experienced clients. First, we have considered a stationary stochastic model on the basis of which a number conclusions were drawn regarding the effects of advertising claims on advertising policies. Subsequently, we have considered an intertemporal for advertising and repeat purchase based on claims verification. The model thus constructed was reduced to a deterministic optimal control model which we have solved under specific assumptions. Explicitly, for infinite and discounted horizon problems, we have shown that the problem can be treated analytically if we assume that the optimal advertising policy is a linear function of the problem's state variables. The results we have obtained have established a clear relationship between optimal claims and optimal advertising policies a s a function of the advertising efficiency, first purchasers response to advertising and the future benefits of derived consumption (such as associated service contracts, products warranties and their like). The simultaneous considerations of these issues in the context of advertising optimization model have not been considered previously.

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ENVIRONMENTAL APPLICATIONS

Chapter 3

CAPITAL RESOURCE SUBSTITUTION, OVERSHOOTING, AND SUSTAINABLE DEVELOPMENT

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Abstract We study an optimal control problem with a man-made capital stock, and a stock of renewable natural resource. They are substitutable inputs in the production of the final good. Starting from low levels of both stocks, the optimal policy consists of three phases. In phase I, the planner builds up the stock of resource above its steady state level, while the man-made capital stock is kept below its steady state level. In phase II, the resource stock declines steadily, while the man-made capital stock continues to grow, until the steady state is reached, and the economy stays thereafter. The model exhibits "overshooting" property.

1. Introduction

Since man-made capital and natural resources are substitutable inputs in the aggregate production function, a natural question that arises is how to optimally accumulate capital and manage the resource stock. The case where the natural resource stock is non-renewable has been studied by Solow under the the maximin criterion, and Dasgupta and Heal (1979) and Pezzy and Withagen

C. Deissenberg and R.F. Hartl (eds.), Optimal Control and Dynamic Games, 41–60.

(1998) under the utilitarian criterion. Solow assumed a Cobb-Douglas production function, and showed that if the share of capital is greater than the share of natural resource, then a constant path of consumption is feasible, and along such a path, the man-made capital stock increases without bound. Dasgupta and Heal (1979) and Pezzy and Withagen (1998) showed that, under the utilitarian criterion, the man-made capital stock will reach a peak, and afterwards both stocks fall to zero asymptotically.

In this paper, we study the optimal path for an economy that produces an output using a stock of capital and a resource input extracted from a stock of renewable natural resource. We retain the Solow-Dasgupta-Heal assumption that capital and resource are substitutable inputs in the production of the final good, but our model differs from theirs because the resource stock is renewable. We wish to find the optimal growth path of the economy under the utilitarian criterion. We show that there exists a unique steady state with positive consumption. We ask the following questions: (i) Can it be optimal to get to the steady state in finite time under the assumption that the utility function is strictly concave? (ii) Can finite-time approach paths to the steady state be smooth, in the sense that there are no jumps in the control variables? (iii) Are there non-smooth paths to the steady state?

The answers to the above questions are as follows.

There exists a set of initial conditions (which forms a one-dimensional manifold, i.e., a curve, in the state space) such that the approach path to the steady state takes a finite time, and is smooth. The path along the manifold toward the steady state involves a steady accumulation of the capital stock, and a steady running down of the resource stock toward its steady state level.

If the initial conditions are not on that one-dimensional manifold, then it may be optimal to get to some point on that manifold first, and then move along the manifold to get to the steady state. The path that gets to a point on the manifold is not smooth at the time it meets the manifold.

We show that starting from low levels of capital stock and resource stock, the optimal policy consists of three phases. In phase I, the planner builds up the stock of resource above its steady state level, while the capital stock is kept below its steady state level. In phase II, the resource stock declines steadily, while the capital stock continues to grow, until the steady state is reached. In phase III, the economy stays at the steady state. Thus, our model exhibits the "overshooting" property.

Before proceeding, we would like to note that there are a number of articles that are somewhat related to our paper, where the authors discussed thr optimal use patterns for renewable resources and the sustainability of economies. Clark et.al. (1979) provided a general formulation with irreversible investment. They focussed on irreversibility, and did not obtain an "overshooting" result. Among the relatively recent papers, Bertratti et.al.(1998) addressed the problem of optimal use of renewable resources under a variety of assumptions about the objective of that economy (with the different types of the utility function.) They constructed a model in which a man-made capital stock and a renewable resource are used for production, and give a very general characterization of the paths which are optimal in various senses. Their basic model is similar to ours, however they focused on different issues. We are not aware of any paper which examines the precise characteristics of steady state and of the approach paths to the steady state in a model with man-made capital and renewable resource.

2. The Model

We consider a continuous-time model. Let K and S denote the stock of man-made capital, and the stock of a renewable natural resource. Let R denote the resource input. The output of the final good is

$$Y = F(K,R) = \sqrt{KR}$$

Output can be consumed, or invested. Let C denote consumption and I denote investment. Then

$$C = F(K,R) - I \tag{3.1}$$

Assume there is no depreciation of capital. Then

$$\dot{K} = I \tag{3.2}$$

Let $\theta(S)$ be the natural growth function of the resource stock. We assume it has the shape of a tent. Specifically, we assume that there exists a stock level $\widehat{S} > 0$ such that $\theta(S) = \omega S$ if $S < \widehat{S}$, and $\theta(S) = \omega\widehat{S} - \delta(S - \widehat{S})$ for $S > \widehat{S}$, where $\omega > 0$, $\delta > 0$. The net rate of growth of the resource stock is

$$\dot{S} = \theta(S) - R \tag{3.3}$$

Remark 1: The function $\theta(S)$ has a kink at $\widehat{S}$, so the derivative $\theta'(S)$ is not defined at $\widehat{S}$. At that point, we define the generalised gradient of $\theta(S)$, denoted by $\partial\theta$, as the real interval $[-\delta, \omega]$, where $-\delta$ is the right-hand derivative, and ω is the left-hand derivative. When applying optimal control theory, we must modify the equation for the shadow price of S when S is at $\widehat{S}$. (This will be discussed in detail later.)

The consumption C yields the utility

$$U(C) = \sqrt{C}$$

The objective of the planner is to maximize the integral of the discounted stream of utility:

$$\max \int_0^\infty \sqrt{C} e^{-\rho t} dt$$

where we assume

$$0 < \rho \leq \omega$$

The maximization is subject to

$$\dot{K} = \sqrt{KR} - C \tag{3.4}$$

$$\dot{S} = \theta(S) - R \tag{3.5}$$

with boundary conditions $K(0) = K_0 > 0$, $S(0) = S_0 > 0$, and

$$\lim_{t\to\infty} K(t) > 0, \lim_{t\to\infty} S(t) \geq 0$$

We focus on the set of initial conditions with $S_0 < \widehat{S}$.

The set of positive stock levels is partitioned into two regions. Region I is the set of points (S, K) such that $0 < S < \widehat{S}$, and $K > 0$. Region II is the set of points (S, K) such that $S \geq \widehat{S}$, and $K > 0$

In what follows we will focus on the optimal paths starting from points in Region I.

CONJECTURES:

There is unique possible steady state.

The set of initial stocks (K_0, S_0) such that the system reaches the steady state smoothly is given by a curve C in the state space (K, S).

The optimal extraction path and the optimal consumption path can then be shown to reach the steady state.

We show that these paths are optimal by showing that they satisfy the necessary conditions of Regions I and II.

3. Region I

3.1 Necessary conditions in Region I

We define the current value Hamiltonian

$$H = \sqrt{C} + \psi_1 \left[\sqrt{KR} - C\right] + \psi_2 [\theta(S) - R]$$

where ψ_1 is the shadow price of man-made capital and ψ_2 is the shadow price of the renewable resource.

The necessary conditions are

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \tag{3.6}$$

$$\frac{\partial H}{\partial R} = \frac{1}{2}\psi_1\sqrt{\frac{K}{R}} - \psi_2 = 0 \tag{3.7}$$

$$\dot{\psi}_1 = \psi_1(\rho - \frac{1}{2}\sqrt{\frac{R}{K}}) \tag{3.8}$$

$$\dot{\psi}_2 = \psi_2 (\rho - \omega) \tag{3.9}$$

Notice that $\psi_1 > 0$ by (3.6). It follows that $\psi_2 > 0$ by (3.7). So, in region I, $\dot{\psi}_2$ can be zero only if $\rho = \omega$.

From equations (3.7), (3.8) and (3.9) we get

$$(\rho - \omega) - (\rho - F_K) = \frac{\dot{\psi}_2}{\psi_2} - \frac{\dot{\psi}_1}{\psi_1} = \frac{1}{F_R}\frac{d(F_R)}{dt}$$

Hence

$$F_K = \omega + \frac{1}{F_R}\frac{d(F_R)}{dt} \tag{3.10}$$

We may call equation (3.10) the *Modified Hotelling Rule:* the rate of capital gain (rate of increase in the price of the extracted resource) plus the biological growth rate must be equated to the rate of interest on the capital good, F_K.

From (3.6) and (3.8), we get

$$\frac{\dot{C}}{2C} = F_K - \rho \tag{3.11}$$

which is the *Ramsey-Euler Rule:* the proportional rate of consumption growth, multiplied by the elasticity of marginal utility, must be equated to the difference between the rate of interest F_K and the utility-discount rate, ρ.

It is convenient to define a new variable x:

$$x(t) = \frac{K(t)}{R(t)}$$

This variable is the capital/resource-input ratio, and is a measure of the *capital intensity* of the production process at time t.

3.2 Steady States in Region I

Can there be a steady state (S_{ss}, K_{ss}) with $0 < S_{ss} < \widehat{S}$? At any steady state, $\dot{C}/C = 0$; thus we must have, from (3.14)

$$x_{ss} = \left(\frac{1}{2\rho}\right)^2$$

Substituting this value into (3.17), and noting that $\psi_2 > 0$ always, we conclude that the steady state requirement $\dot{C}/C = 0$ implies that at the steady state, $\dot{\psi}_2/\psi_2 = 0$. But this is possible only if $\rho = \omega$. Thus we have proved:

FACT 1: There is no steady state with $0 < S_{ss} < \widehat{S}$, unless $\rho = \omega$.

In what follows, we will focus on the case where $\rho < \omega$. Then steady states do not exist in Region I.

3.3 Dynamics in Region I

3.3.1 The time path of capital/resource-input ratio in Region I. Step 1:

We first show that x satisfies the following differential equation

$$-\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}\frac{\dot{x}}{x}=-\omega \tag{3.12}$$

This is shown from the necessary condition (3.6),

$$-\frac{\dot{C}}{2C}=\frac{\dot{\psi}_1}{\psi_1} \tag{3.13}$$

but we know from (3.8)

$$\frac{\dot{\psi}_1}{\psi_1}=\rho-\frac{1}{2}x^{-\frac{1}{2}} \tag{3.14}$$

so we have

$$-\frac{\dot{C}}{2C}=\frac{\dot{\psi}_1}{\psi_1}=\rho-\frac{1}{2}x^{-\frac{1}{2}} \tag{3.15}$$

From (3.7) we get the relationship between ψ_1 and x

$$\frac{\partial H}{\partial R}=\frac{1}{2}\psi_1\sqrt{\frac{K}{R}}-\psi_2=\frac{1}{2}\psi_1\sqrt{x}-\psi_2=0 \tag{3.16}$$

so that

$$\frac{\dot{\psi}_2}{\psi_2}=\frac{\dot{\psi}_1}{\psi_1}+\frac{1}{2}\frac{\dot{x}}{x} \tag{3.17}$$

and using (3.8) yields

$$-\omega=-\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}\frac{\dot{x}}{x} \tag{3.18}$$

Step 2: Solving for $x(t)$:

By multiplying each side of (3.18) by $\sqrt{x}$ gives

$$-\frac{1}{2}+\frac{1}{2}\frac{\dot{x}}{\sqrt{x}}=-\omega x^{\frac{1}{2}} \tag{3.19}$$

Let $y\equiv\sqrt{x}$

$$-\frac{1}{2}+\dot{y}=-\omega y \tag{3.20}$$

the solution can be written in two forms:

$$y(t)=\left(y_0-\frac{1}{2}\right)e^{-\omega t}+\frac{1}{2\omega}$$

or

$$y(t) = \left(y_T - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}$$

where $y_0 = y(0)$ or $y_T = y(T)$ and therefore we have

$$x(t) = \left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2 \tag{3.21}$$

or

$$x(t) = \left(\left(\sqrt{x_0} - \frac{1}{2\omega}\right) e^{-\omega t} + \frac{1}{2\omega}\right)^2$$

This ends Step 2.

Remark 2: If we impose the condition that at some time T the variable $x(T)$ takes the following value (which is its steady state value):

$$x_T = (\frac{1}{2\rho})^2 \tag{3.22}$$

For a given T there is one unique x_0 such that

$$x_0 = x(0) = \left(\left(\sqrt{x_T} - \frac{1}{2}\right) e^{\omega T} + \frac{1}{2}\right)^2$$

and we can use the computed path to note that

$$\dot{x}(t) = -2\omega\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)}\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)$$

since $\sqrt{x_T} - \frac{1}{2\omega} = \frac{1}{2\rho} - \frac{1}{2\omega} > 0$ we have the following lemma.

LEMMA 3.1 *The capital intensity $x(t)$ is decreasing over time.*

It could be useful to compute $\frac{\dot{x}(t)}{x(t)}$

$$\begin{aligned}\frac{\dot{x}(t)}{x(t)} &= -2\omega\frac{\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)} \\ &= -2\omega\left(1 - \frac{1}{\left(\left(\frac{\omega}{\rho} - 1\right) e^{-\omega(t-T)} + 1\right)}\right)\end{aligned}$$

so since $\left(\frac{\omega}{\rho} - 1\right) > 0$ then $\frac{\dot{x}(t)}{x(t)} < 0$

3.3.2 The path of ψ_1. We can solve for ψ_1 from (3.17) and (3.9)

$$\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x} \tag{3.23}$$

with

$$\dot{\psi}_2 = \psi_2 (\rho - \omega) \tag{3.24}$$

so

$$\frac{\dot{\psi}_1}{\psi_1} = \rho - \omega - \frac{1}{2}\frac{\dot{x}}{x}$$

The integration gives

$$\ln \frac{\psi_1 (t)}{\psi_1 (T)} = (\rho - \omega)(t - T) - \ln \sqrt{\frac{x(t)}{x(T)}}$$

or

$$\psi_1 (t) = \psi_{1T} \frac{\sqrt{x_T}}{\sqrt{x(t)}} e^{(\rho-\omega)(t-T)}$$

$$\psi_1 (t) = \psi_{1T} \frac{\sqrt{x_T}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}\right)} \tag{3.25}$$

The denominator is $D(t) = \left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\rho(t-T)} + \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}$ is such that

$$D'(t) = -\rho \left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\rho(t-T)} - (\rho - \omega) \frac{1}{2\omega} e^{-(\rho-\omega)(t-T)}$$

$$D'(t) = \frac{1}{2\omega} (\rho - \omega) e^{-\rho(t-T)} \left(1 - e^{\omega(t-T)}\right) < 0$$

since $\rho < \omega$. So

$$\dot{\psi}_1 (t) > 0 \tag{3.26}$$

3.3.3 The time path of consumption in Region I. The consumption path is

$$\left(\frac{1}{2\psi_1}\right)^2 = C$$

that is

$$C(t) = \frac{1}{\left(2\psi_{1T} \frac{\sqrt{x_T}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)} e^{(\rho-\omega)(t-T)}\right)^2}$$

$$C(t) = C_T \frac{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right) e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2}{x_T} e^{-2(\rho-\omega)(t-T)} \tag{3.27}$$

The evolution of the consumption path is given by

$$\frac{\dot{C}}{C} = -2\frac{\dot{\psi}_1}{\psi_1} < 0 \tag{3.28}$$

LEMMA 3.2 *In region I, $\dot{C}(t)/C(t)$ is negative.*

3.3.4 The time path of extraction in Region I. From the definition of $x = K/R$, we have

$$\dot{K} = \dot{R}x + R\dot{x}$$

and

$$\dot{K} = \sqrt{KR} - C = R\sqrt{x} - C$$

so

$$\dot{R}x + R\dot{x} = R\sqrt{x} - C \tag{3.29}$$

or

$$\dot{R} = R\left(\frac{1}{\sqrt{x}} - \frac{\dot{x}}{x}\right) - \frac{C}{x} \tag{3.30}$$

using (3.12) yields

$$-\omega = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x}$$

so

$$\dot{R} = 2\omega R - \frac{C}{x} \tag{3.31}$$

where $C(t)$ is given by (3.27) and $x(t)$ is given by (3.21) and so

$$\dot{R} = 2\omega R - \frac{C_T \frac{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2}{x_T} e^{-2(\rho-\omega)(t-T)}}{\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2} \tag{3.32}$$

Hence

$$\dot{R} = 2\omega R - \frac{C_T e^{-2(\rho-\omega)(t-T)}}{x_T} \tag{3.33}$$

The exact solution is:

$$R(t) = \frac{1}{2}\frac{C_T}{x_T\rho}\exp(2\omega t - 2\omega T - 2\rho t + 2\rho T) + e^{2\omega t}E$$

and $R(T) = R_T$ so

$$R_T = \frac{C_T}{x_T 2\rho} + e^{2\omega T}E$$

with

$$x_T = \frac{K_{ss}}{R_{ss}} = \left[\frac{1}{2\rho}\right]^2$$
$$C_T = \omega\widehat{S}\left[\frac{1}{2\rho}\right] \tag{3.34}$$

and

$$R_T = \omega\widehat{S} \tag{3.35}$$

so

$$E = \left(R_T - \frac{C_T}{x_T 2\rho}\right)e^{-2\omega T} = 0$$

and

$$R(t) = \omega\widehat{S}e^{2(\omega-\rho)(t-T)} \tag{3.36}$$
$$\dot{R}(t) = 2(\omega-\rho)\,\omega\widehat{S}e^{2(\omega-\rho)(t-T)} > 0$$

3.3.5 The path of capital in region I. We now turn to the capital, we have

$$K = xR$$
$$\frac{\dot{K}}{K} = \frac{\dot{x}}{x} + \frac{\dot{R}}{R} = \frac{\dot{x}}{x} + 2(\omega-\rho)$$

using (3.36) and (3.12) yields

$$\frac{\dot{K}}{K} = -2\omega + x^{-\frac{1}{2}} + 2(\omega-\rho) = x^{-\frac{1}{2}} - 2\rho$$

since $\dot{x} < 0$ we have $\frac{dx^{-\frac{1}{2}}}{dt} > 0$ with $(x(T))^{-\frac{1}{2}} = 2\rho$ and therefore $x^{-\frac{1}{2}} - 2\rho < 0$ for all $t < T$ and thus

$$\frac{\dot{K}}{K} = x^{-\frac{1}{2}} - 2\rho < 0$$

Substituting x and R from (3.36) and (3.21) gives

$$K = xR = \omega\widehat{S}e^{2(\omega-\rho)(t-T)}\left(\left(\sqrt{x_T} - \frac{1}{2\omega}\right)e^{-\omega(t-T)} + \frac{1}{2\omega}\right)^2$$

at time $t = 0$ we have

$$K_0 = K(0) = \omega\widehat{S}e^{-2(\omega-\rho)T}\left(\left(\frac{1}{2\rho}-\frac{1}{2\omega}\right)e^{\omega T}+\frac{1}{2\omega}\right)^2$$

$$\frac{dK_0}{dT} = \omega\widehat{S}e^{-2(\omega-\rho)T}\left(\left(\frac{1}{2\rho}-\frac{1}{2\omega}\right)e^{\omega T}+\frac{1}{2\omega}\right)^2$$

$$\frac{dK_0}{dT} = \omega\widehat{S}\frac{d\left(\left(e^{-(\omega-\rho)T}\left(\left(\frac{1}{2\rho}-\frac{1}{2\omega}\right)e^{\omega T}+\frac{1}{2\omega}\right)\right)^2\right)}{dT}$$

Let $f(T) = e^{-(\omega-\rho)T}\left(\left(\frac{1}{2\rho}-\frac{1}{2\omega}\right)e^{\omega T}+\frac{1}{2\omega}\right)$ we have

$$f(T) = \left(\frac{1}{2\rho}-\frac{1}{2\omega}\right)e^{\rho T}+\frac{1}{2\omega}e^{-(\omega-\rho)T}$$

$$f'(T) = \frac{1}{2\omega}(\omega-\rho)e^{\rho T}\left(1-e^{-\omega T}\right) > 0$$

So

$$\frac{dK_0}{dT} = \omega\widehat{S}2f'(T)f(T) > 0 \tag{3.37}$$

3.3.6 The path of the resource stock. The stock follows

$$\dot{S} = \omega S - R$$

substituting R from (3.36)

$$\dot{S} = \omega S - \omega\widehat{S}e^{2(\omega-\rho)(t-T)}$$
$$S(T) = \widehat{S}$$

Exact solution is:

$$S(t) = \widehat{S}\left(\frac{\omega}{-\omega+2\rho}e^{2(\omega-\rho)(t-T)}+2e^{\omega(t-T)}\frac{-\omega+\rho}{(-\omega+2\rho)}\right)$$

$$\dot{S}(t) = 2\omega(\omega-\rho)e^{\omega(t-T)}\widehat{S}\left(\frac{e^{(\omega-2\rho)(t-T)}-1}{(-\omega+2\rho)}\right) > 0 \tag{3.38}$$

The initial stock must be

$$S_0 = S(0) = \widehat{S}\left(\frac{\omega}{-\omega+2\rho}e^{-2(\omega-\rho)T}+2e^{-\omega T}\frac{-\omega+\rho}{(-\omega+2\rho)}\right) \tag{3.39}$$

So we have

$$\frac{dS_0}{dT} = 2(\omega-\rho)\omega\widehat{S}e^{-\omega T}\left(\frac{1-e^{-(\omega-2\rho)T}}{-\omega+2\rho}\right) < 0 \tag{3.40}$$

and

$$\frac{dK_0}{dT} > 0 \tag{3.41}$$

and therefore in Region I:

$$\frac{dK_0}{dS_0} < 0 \tag{3.42}$$

4. Region II

4.1 The necessary conditions in Region II

The necessary conditions for Region II are a bit more complicated, because at the point $\widehat{S}$ the function $\theta(S)$ is not differentiable. Thus we must deal with a "non-smooth" problem. For a general treatment of non-smooth optimal control problem see Clarke and Winter (1983), or Clarke (1983); here we follow the exposition in Docker et al (2000, pages 74-79).

Since $\theta(S)$ has a kink at $\widehat{S}$, with left-hand derivative equal to $\omega > 0$ and right-hand derivative equal $-\delta$, the generalized gradient of $\theta(.)$ at $\widehat{S}$ is defined as

$$\partial\theta(\widehat{S}) = [-\delta, \omega]$$

The necessary conditions are

$$\frac{\partial H}{\partial C} = \frac{1}{2\sqrt{C}} - \psi_1 = 0 \tag{3.43}$$

$$\frac{\partial H}{\partial R} = \frac{1}{2}\psi_1\sqrt{\frac{K}{R}} - \psi_2 = 0 \tag{3.44}$$

$$\dot{K} = \sqrt{KR} - C \tag{3.45}$$

$$\dot{S} = \omega\widehat{S} - \delta(S - \widehat{S}) - R \text{ if } S > \widehat{S} \tag{3.46}$$

$$\dot{\psi}_1 = \psi_1(\rho - \frac{1}{2}\sqrt{\frac{R}{K}}) \tag{3.47}$$

and, from Docker et al (2000, pages 74-79),

$$-(\dot{\psi}_2 - \rho\psi_2) \in [-\delta\psi_2, \omega\psi_2] \text{ if } S = \widehat{S} \tag{3.48}$$

$$-(\dot{\psi}_2 - \rho\psi_2) = -\delta\psi_2 \text{ if } S > \widehat{S} \tag{3.49}$$

4.2 Steady State in Region II

Consider a possible steady state (S_{ss}, K_{ss}) in Region II. We will show that it is necessary that $S_{ss}{=}\widehat{S}$.

For suppose $S_{ss} > \widehat{S}$. Then, from (3.49)

$$\dot{\psi}_2 = \psi_2(\delta + \rho) > 0 \text{ for } \psi_2 > 0$$

which is in contradiction with

$$\frac{\dot{\psi}_2}{\psi_2} = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x} = 0$$

Thus we have proved:

FACT 2: In Region II, the only possible steady state resource stock level is $S_{ss}=\widehat{S}$.

Let us find the corresponding steady state values of other variables. ¿From (3.46), at the steady state,

$$R_{ss} = \omega\widehat{S}. \tag{3.50}$$

¿From (3.47), at the steady state,

$$\frac{1}{2}\left(\frac{K}{R}\right)^{-\frac{1}{2}} = \rho \tag{3.51}$$

Thus

$$K_{ss} = \omega\widehat{S}\left[\frac{1}{2\rho}\right]^2 \tag{3.52}$$

$$x_{ss} = \frac{K_{ss}}{R_{ss}} = \left[\frac{1}{2\rho}\right]^2$$

Using (3.45), at the steady state

$$C_{ss} = \omega\widehat{S}\left(\frac{1}{2\rho}\right) \tag{3.53}$$

Thus, from (3.43) and (3.53)

$$\psi_{ss1} = \frac{1}{2}\left(\frac{\omega\widehat{S}}{2\rho}\right)^{-\frac{1}{2}}$$

and, from (3.44)

$$\psi_{ss2} = \frac{1}{4}\left(\frac{\omega\widehat{S}}{2\rho}\right)^{-\frac{1}{2}}$$

which is consistent with (3.48) because $\rho \in [-\delta, \omega]$.

5. Convergence to the steady state

We must now solve the following problem: starting from an initial point (S_0, K_0), we must determine if there exists a solution $(S(t), K(t))$ that converges to $(\widehat{S}, \omega\widehat{S}\left[\frac{1}{2\rho}\right]^2)$, possibly at some finite time $\widehat{T}$, where, in general, $\widehat{T} = \widehat{T}(K_0, S_0)$.

The path that converges to the steady state may be smooth (in the sense that there are no jumps in the control variables R and C), or it may exhibit jumps in C and R (say at time T).

5.1 Smooth paths

5.1.1 The time path of capital/resource-input ratio in Region II. **Step 1:**

We first show that x satisfies the following differential equation

$$-\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}\frac{\dot{x}}{x}=\delta \tag{3.54}$$

This is shown from the necessary conditions (3.7), (3.6) and (**??**),

$$\frac{\partial H}{\partial C}=\frac{1}{2\sqrt{C}}-\psi_1=0 \tag{3.55}$$

$$-\frac{\dot{C}}{2C}=\frac{\dot{\psi}_1}{\psi_1} \tag{3.56}$$

but we know from the necessary conditions

$$\frac{\dot{\psi}_1}{\psi_1}=\rho-\frac{1}{2}x^{-\frac{1}{2}} \tag{3.57}$$

so we have

$$-\frac{\dot{C}}{2C}=\frac{\dot{\psi}_1}{\psi_1}=\rho-\frac{1}{2}x^{-\frac{1}{2}} \tag{3.58}$$

The relationship between ψ_1 and x is from

$$\frac{\partial H}{\partial R}=\frac{1}{2}\psi_1\sqrt{\frac{K}{R}}-\psi_2=0 \tag{3.59}$$

so that

$$\frac{\dot{\psi}_2}{\psi_2}=\frac{\dot{\psi}_1}{\psi_1}+\frac{1}{2}\frac{\dot{x}}{x} \tag{3.60}$$

and from the necessary conditions we have

$$\dot{\psi}_2=\psi_2(\delta+\rho) \tag{3.61}$$

so we have

$$\delta+\rho=\rho-\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}\frac{\dot{x}}{x} \tag{3.62}$$

or

$$\delta=-\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}\frac{\dot{x}}{x} \tag{3.63}$$

Step 2: Solving for $x(t)$:
Multiplying each side of by $\sqrt{x}$ gives

$$-\frac{1}{2}+\frac{1}{2}\frac{\dot{x}}{\sqrt{x}}=\delta x^{\frac{1}{2}} \tag{3.64}$$

This solution is given as[1]

$$x(t)=\left(\left(\sqrt{x_T}+\frac{1}{2\delta}\right)e^{\delta(t-T)}-\frac{1}{2\delta}\right)^2$$

or

$$x(t)=\left(\left(\sqrt{x_0}+\frac{1}{2\delta}\right)e^{\delta t}-\frac{1}{2\delta}\right)^2 \tag{3.65}$$

This ends Step 2.

Remark 3: We retrieve the results of Region I if we substitute δ by $-\omega$.

Remark 4: If we impose the condition that at some time T the variable $x(T)$ takes the following value (which is its steady state value):

$$x_T=(\frac{1}{2\rho})^2 \tag{3.66}$$

For a given T there is one unique x_0 such that

$$x(t)=\left(\left(\sqrt{x_0}+\frac{1}{2\delta}\right)e^{\delta t}-\frac{1}{2\delta}\right)^2$$

and we can use the computed path to note that

$$\dot{x}(t)=2\delta\left(\sqrt{x_T}+\frac{1}{2\delta}\right)e^{\delta(t-T)}\left(\left(\sqrt{x_T}+\frac{1}{2\delta}\right)e^{\delta(t-T)}-\frac{1}{2\delta}\right)$$

We have

$$\dot{x}(T)=2\delta\left(\sqrt{x_T}+\frac{1}{2\delta}\right)(\sqrt{x_T})>0$$

but can we have $\dot{x}(t)<0$ for $t\in[0,\tau)$? We must have

$$-\frac{1}{2}+\frac{1}{2}\frac{\dot{x}}{\sqrt{x}}=\delta x^{\frac{1}{2}} \tag{3.67}$$

so $\dot{x}$ must be positive.

LEMMA 3.3 *The capital intensity $x(t)$ is increasing over time.*

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[1]The precise solution will be made available upon request.

5.1.2 The path of ψ_1 in region II. We have

$$\frac{\dot{\psi}_2}{\psi_2} = \delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{1}{2}\frac{\dot{x}}{x}$$

$$\delta + \rho = \frac{\dot{\psi}_1}{\psi_1} + \frac{\delta\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{\delta(t-T)}}{\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{\delta(t-T)} - \frac{1}{2\delta}\right)} \tag{3.68}$$

The solution is given as[2]

$$\psi_1(t) = \frac{\psi_1(T)\,e^{(\delta+\rho)(t-T)}}{\left(\left(1+\frac{\rho}{\delta}\right)e^{\delta(t-T)} - \frac{\rho}{\delta}\right)} \tag{3.69}$$

From

$$C = \left(\frac{1}{2\psi_1}\right)^2$$
$$C = C_T\left(\left(1+\frac{\rho}{\delta}\right)e^{-\rho(t-T)} - \frac{\rho}{\delta}e^{-(\delta+\rho)(t-T)}\right)^2$$
$$\dot{C} = \rho\left(1+\frac{\rho}{\delta}\right)e^{-2\rho(t-T)}C_T\left(-1+e^{-\delta(t-T)}\right)\left(1+\frac{\rho}{\delta}\left(1-e^{-\delta(t-T)}\right)\right)$$
$$\dot{C} < 0$$

since

$$1+\frac{\rho}{\delta}\left(1-e^{-\delta(t-T)}\right) > 0$$

5.1.3 The path of extraction in region II. We now turn to the time path of extraction.

We still have

$$\dot{R} = R\left(\frac{1}{\sqrt{x}} - \frac{\dot{x}}{x}\right) - \frac{C}{x} \tag{3.70}$$

but now

$$\delta = -\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}\frac{\dot{x}}{x} \tag{3.71}$$

so

$$\dot{R} = -2\delta R - \frac{C}{x} < 0 \tag{3.72}$$

$$\dot{R} = -2\delta R - \frac{C_T e^{-2(\delta+\rho)(t-T)}}{x_T} \tag{3.73}$$

[2]The precise solution will be made available upon request.

with

$$x(t) \to x_T \equiv \frac{K_{ss}}{R_{ss}} = \left[\frac{1}{2\rho}\right]^2 \tag{3.74}$$

$$C(t) \to C_T = \omega\widehat{S}\left[\frac{\alpha}{\rho}\right] \tag{3.75}$$

$$\dot{R} = -2\delta R - 2\rho\omega\widehat{S}e^{-2(\delta+\rho)(t-T)} \tag{3.76}$$

The exact solution is given as[3]

$$R(t) = \omega\widehat{S}e^{-2(\delta+\rho)(t-T)}$$

$$\frac{\dot{R}(t)}{R} = -2(\delta+\rho) < 0 \tag{3.77}$$

5.1.4 The path of capital in region II. Concerning the stock of capital we have

$$K = xR$$

so

$$\delta + \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{\dot{x}}{x} \tag{3.78}$$

$$\frac{\dot{K}}{K} = \frac{\dot{x}}{x} + \frac{\dot{R}}{R} = \frac{\dot{x}}{x} - 2(\delta+\rho)$$

$$\frac{\dot{K}}{K} = \frac{1}{\sqrt{x}} - 2\rho \tag{3.79}$$

Since $\frac{\dot{x}}{x} > 0$ then $\frac{d}{dt}\left(\frac{1}{\sqrt{x}}\right) < 0$ so $\frac{1}{\sqrt{x(t)}} > \frac{1}{\sqrt{x(T)}} = 2\rho$ for all $t < T$ and therefore

$$\frac{\dot{K}}{K} > 0. \tag{3.80}$$

Moreover substituting x and R yields

$$K(t) = x(t)R(t)$$

$$K(t) = \left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{\delta(t-T)} - \frac{1}{2\delta}\right)^2 \omega\widehat{S}e^{-2(\delta+\rho)(t-T)} \tag{3.81}$$

there exist a smooth path reaching K_{ss} at T has K_0 that satisfies

$$K_0 = K(0) = \omega\widehat{S}\left(\left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right)e^{-\delta T} - \frac{1}{2\delta}\right)e^{(\delta+\rho)T}\right)^2 \tag{3.82}$$

[3]The precise solution will be made available upon request.

Let $g(T) = \left(\left(\sqrt{x_T} + \frac{1}{2\delta}\right) e^{-\delta T} - \frac{1}{2\delta}\right) e^{(\delta+\rho)T}$

$$g'(T) = \frac{1}{2\delta} e^{\rho T} (\delta+\rho) \left(1 - e^{\delta T}\right) < 0$$

$$\frac{dK_0}{dT} = \omega \widehat{S} g(T) g'(T) < 0 \tag{3.83}$$

5.1.5 The path of the resource stock in region II.

In region II we have

$$\dot{S} = \omega \widehat{S} - \delta(S - \widehat{S}) - R \tag{3.84}$$

Substituting R gives

$$\dot{S} + \delta S = \widehat{S} \left(\omega + \delta - \omega e^{-2(\delta+\rho)(t-T)}\right)$$

Note that

$$\dot{S}(T) + \delta \widehat{S} = \delta \widehat{S}$$

and thus

$$\dot{S}(T) = 0$$

We now solve for the path of the resource stock

$$S' = A\left(\omega + \delta - \omega e^{-2(\delta+\rho)(t-T)}\right) - \delta S$$
$$S(T) = A$$

The exact solution is:

$$S(t) = \omega \frac{\widehat{S}}{\delta} + \widehat{S} + \omega \frac{\widehat{S}}{\delta + 2\rho} e^{-2(\delta+\rho)(t-T)} - 2e^{-\delta(t-T)} \omega \widehat{S} \frac{\delta+\rho}{\delta(\delta+2\rho)} \tag{3.85}$$

we check that $S(T) = \omega \frac{\widehat{S}}{\delta} + \widehat{S} + \omega \frac{\widehat{S}}{\delta+2\rho} - 2\omega \widehat{S} \frac{\delta+\rho}{\delta(\delta+2\rho)} = \widehat{S}$

Moreover

$$\dot{S}(t) = 2\omega \widehat{S} 2 e^{-\delta(t-T)} \frac{\delta+\rho}{(\delta+2\rho)} \left(-e^{-(\delta+2\rho)(t-T)} + 1\right) < 0 \tag{3.86}$$

and there exists a smooth path reaching $\widehat{S}$ at T if S_0 satisfies

$$S_0 = S(0) = \omega \frac{\widehat{S}}{\delta} + \widehat{S} + \omega \frac{\widehat{S}}{\delta+2\rho} e^{2(\delta+\rho)T} - 2e^{\delta T} \omega \widehat{S} \frac{\delta+\rho}{\delta(\delta+2\rho)} \tag{3.87}$$

$$\frac{dS_0}{dT} = \omega \frac{2(\delta+\rho)\widehat{S}}{\delta+2\rho} e^{2(\delta+\rho)T} - 2e^{\delta T} \omega \widehat{S} \frac{\delta+\rho}{(\delta+2\rho)}$$

$$\frac{dS_0}{dT} = 2e^{\delta T} \omega \widehat{S} \frac{\delta+\rho}{(\delta+2\rho)} \left(e^{(\delta+2\rho)T} - 1\right) > 0 \tag{3.88}$$

In Region II we also have

$$\frac{dS_0}{dT} > 0 \tag{3.89}$$

and

$$\frac{dK_0}{dT} < 0 \tag{3.90}$$

so

$$\frac{dK_0}{dS_0} < 0 \tag{3.91}$$

In Both region I and region II we have

$$\frac{dK_0}{dS_0} < 0 \tag{3.92}$$

This implies either overshooting or jump in the control paths.

There exists a set of initial pair of stock levels (K_0, S_0) for which we can find a smooth path that leads to the steady state pair $(\widehat{S}, \widehat{K})$. This set is a one-dimensional manifold which has a negative slope for $0 < S < \widehat{S}$. Or the path converging to the steady state may exhibit jumps in C and R.

This policy implication is that the planner will want to build up the stock of resource before runnig it down to its steady state level. This is an "overshooting" result.

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Chapter 4

HIERARCHICAL AND ASYMPTOTIC OPTIMAL CONTROL MODELS FOR ECONOMIC SUSTAINABLE DEVELOPMENT *

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Abstract In this brief paper one shows the relevance of asymptotic control theory to the study of economic sustainable development. One also proposes a modeling framework where sustainable economic development is represented through a paradigm of optimal stochastic control with two time-scales. This shows that several contributions of rof. Sethi, in the domain of hierarchical and multi-level control models in manufacturing and resource management can also serve to better understand the stakes of sustainability in economic growth and to assess long term environmental policies.

1. Introduction

The theory of economic growth has been using a lot of control theory (and calculus of variations) to attain in the late seventies a remarkable scientific status in the domain of economics and social sciences. In particular the theory of Hamiltonian systems has been used to explain the asymptotic behavior of a growing economy, the key feature being that the economic growth models were likely to exhibit global attractors, both for the state and costate variables. This attractor was called "turnpike", following the work by Cass (1995), McKenzie (1976), Brock & Scheinkman (1976), Rockafellar (1973) and many others. An optimally growing economy in a stationary environment is prone to converge to an

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extremal[1] steady-state which would then characterize a sustainable optimal economic state. The turnpike property has also been an important feature in the study of existence, sufficiency conditions and asymptotic behavior of optimal control problems with zero-discount and overtaking optimality criterion. (see Brock and Haurie (1976) and Carlson, Haurie and Leizarowitz (1994)). In stochastic decision models the turnpike property has been associated with the existence of stationary or invariant state probability measures that are defined by optimal policies (see e.g. Puterman (1994), Veinott (1964)).

An interesting feature of this theory was the result showing that when the pure time preference discount rate $\rho > 0$ was increasing, the sufficient conditions for observing the turnpike property were more stringent. In brief, high discounting could jeopardize sustainability. The economics of the environment and, more precisely, cost-benefit analysis for global climate change mitigation has generated a renewed interest for low discounting or even zero discounting in economic growth models. For example, in Weitzman (1998) Weitzman explained why one should use the lowest rate to discount distant futures. In an interesting monograph edited by Portney and Weyant (1999) several leading authors discuss the proper discount rate to use when dealing with environmental problems that will affect several generations down in the future.

In this note one intends to show that the turnpike theory has an important potential to contribute to a better understanding of the long term economic growth under a global environmental threat and, more generally, of the elusive concept of economic sustainability. Considering the purpose of this essay, most technical developments have deliberately been avoided and the interested reader is referred to the sources given in reference. The paper is organized as follows: Section 4.2 shows the relevance of turnpike theory for sustainable development analysis. One illustrates this property by showing how it would appear in the most popular cost-benefit-analysis (CBA) models, in particular the Nordhaus and Nordhaus-Boyer models DICE-94 or DICE-99 (Nordhaus (1992), Nordhaus (1994), Nordhaus and Boyer (2000)). Section 4.3 shows how the climate change and climate damage uncertainty can be introduced in this class of models, using a two time-scale, hierarchical control scheme. A *limit climate control problem* is formulated in the slow time-scale which serves to determine the optimal long term GHG emissions cap. An auxiliary optimal economic growth problem is then formulated, in the fast time-scale with incentives based on the potential function associated with the solution of the limit climate control problem. The solution of this *transient optimal economic growth* problem will drive the economy toward the long term sustainability goal.

2. Turnpikes for economy-environment models

In this section one considers an archetypal economic growth model with global climate change damage that uses an infinite horizon continuous time optimal control paradigm. The model is directly inspired from the DICE-94 model developed by W. Nordhaus (Nordhaus (1992), Nordhaus (1994), Nordhaus and Boyer (2000)). Using that model one can explore the asymptotic steady-state attractors associated with different possible discount rates and give interpretation of the turnpike property in terms of economic sustainability. Recall first the list of variables used in the DICE model, as given in Table 4.1 The optimal economic

	List of endogenous state variables	
$K(t)$	=	capital stock
$M(t)$	=	mass of GHG in the atmosphere
$T(t)$	=	atmospheric temperature relative to base period
$T^*(t)$	=	deep-ocean temperature relative to base period
	List of control variables	
$I(t)$	=	gross investment
$\mu(t)$	=	rate of GHG emissions reduction
	List of exogenous dynamic variables	
$A(t)$	=	level of technology
$L(t)$	=	labor input (=population)
$O(t)$	=	forcing exogenous GHG
	List of auxiliary variables	
$C(t)$	=	total consumption
$c(t)$	=	per capita consumption
$D(t)$	=	damage from GH warming
$E(t)$	=	emissions of GHGs
$F(t)$	=	radiative forcing from GHGs
$\Omega(t)$	=	output scaling factor due to emissions control and to damages from climate change
$Q(t)$	=	gross world product

Table 4.1. List of variables in the DICE94 model

growth model represents an economy producing a single malleable good which can be consumed or invested in physical production capital. The production process generates emissions of greenhouse gases (GHG) that accumulate in the atmosphere and trigger, through a radiative forcing effect a surface atmospheric temperature (AST) increase. This global temperature change has an impact on the economy measured by the fraction of lost economic output. Emissions abatement can be realised, at a cost also measured in terms of output loss. The formulation of DICE-94 as a control model with infinite time horizon and discounting is given below in Eqs (4.1)-(4.18):

$$\max_{c(\cdot)} \quad \int_0^\infty e^{-\rho t} U(c(t), L(t))\, dt \tag{4.1}$$

s.t.

$$U(c(t), L(t)) = L(t)\frac{c(t)^{1-\alpha} - 1}{1-\alpha} \tag{4.2}$$

$$\dot{L}(t) = g_L(t)L(t) \tag{4.3}$$

$$\dot{g}_L(t) = -\delta_L g_L(t) \tag{4.4}$$

$$Q(t) = \Omega(t)A(t)K(t)^\gamma L(t)^{1-\gamma} \tag{4.5}$$

$$\dot{A}(t) = g_A(t)A(t) \tag{4.6}$$

$$\dot{g}_A(t) = -\delta_A g_A(t) \tag{4.7}$$

$$Q(t) = C(t) + I(t) \tag{4.8}$$

$$c(t) = \frac{C(t)}{L(t)} \tag{4.9}$$

$$\dot{K}(t) = I(t) - \delta K(t) \tag{4.10}$$

$$E(t) = (1-\mu(t))\sigma(t)Q(t) \tag{4.11}$$

$$\dot{M}(t) = \beta_0 E(t) - \delta_M(M(t) - 590) \tag{4.12}$$

$$F(t) = 4.1\frac{\log[M(t)] - \log[590]]}{\log[2]} + O(t) \tag{4.13}$$

$$\dot{T}(t) = \frac{1}{R_1}\{F(t) - \lambda T(t) - \frac{R_2}{\tau_{12}}\{T(t) - T^*(t)\}\} \tag{4.14}$$

$$\dot{T}^*(t) = \frac{1}{\tau_{12}}\{T(t) - T^*(t)\} \tag{4.15}$$

$$D(t) = Q(t)\theta_1(T(t) + \theta_2 T(t)^2) \tag{4.16}$$

$$TC(t) = Q(t)b_1\mu(t)^{b_2} \tag{4.17}$$

$$\Omega(t) = \frac{1 - b_1\mu^{b_2}}{1 + \theta_1(T + \theta_2 T^2)}. \tag{4.18}$$

The principal parameter values are listed in Table (4.2).

For a precise meaning of these parameters, equations and variables, one refers the reader to the two books where DICE-94 and DICE-99 are explained and calibrated Nordhaus (1994), Nordhaus and Boyer (2000). For our purpose it suffices to say that this model captures the fundamental coupling between climate and economics growth that can be also summarized by the diagram shown in Figure 4.1 borrowed from Drouet, Edwards and Hauriex. In this control model the state of the economy evolves through the capital accumulation process, whereas the state of the environment evolves according to the carbon cycle which defines the

α	$=$	1
b_1	$=$	0.045
b_2	$=$	2.15
β_0	$=$	0.64
γ	$=$	0.25
δ_K	$=$	0.10 (per year)
δ_M	$=$	0.0833 (per decade)
λ	$=$	1.41
θ_1	$=$	0.0007
θ_2	$=$	3.57
σ	$=$	0.033

Table 4.2. Parameter values

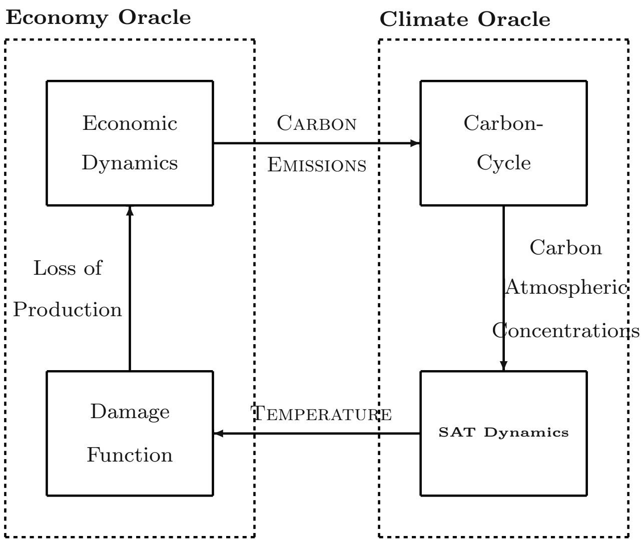

Figure 4.1. GOLDICE framework

concentration of CO_2 in the atmosphere and as a consequence the temperature increase due to the greenhouse effect. The feedback from the climate to the economy is represented by the damage function which determines the loss of economic output due to climate change.

Indeed the first DICE-94 model is very well adapted to the study of an asymptotic behavior[2]. To obtain an asymptotic steady state of the DICE-94 model one pushes the population growth and productivity factor to their asymptotic limits and after equilibrium in the two reservoirs, atmosphere and ocean, one obtains a steady-state path with constant levels of capital, GHG concentration and temperature increase.

It is worthwhile to recall here the definition of turnpikes for optimal control models. For that we use a general formulation of an infinite horizon control model.

$$\max \quad \rho \int_0^\infty e^{-\rho t} L(x(t), u(t))\, dt \tag{4.19}$$

s.t.

$$\dot{x}(t) = f(x(t), u(t)) \tag{4.20}$$

$$u(t) \in \mathcal{U}(x(t)). \tag{4.21}$$

Here x is the state vector ($x = (K, M, T)$ in the DICE-94 model), u is the control vector ($u = (I, C, \mu)$ in the DICE-94 model) and $\mathcal{U}(x)$ is the constraint set linking control and state variables. When the population and the exogenous technological progress have stabilized, the model (4.1)-(4.18) can be put in the form (4.19)-(4.21) of a stationary infinite horizon control problem. When $\rho = 0$ one can define the turnpike as the solution to the steady-state optimization problem

$$\max \quad L(x, u) \tag{4.22}$$

s.t.

$$0 = f(x, u) \tag{4.23}$$

$$u \in \mathcal{U}(x), \tag{4.24}$$

whereas, when $\rho > 0$, the turnpike is solution to the implicit programming problem

$$\max \quad L(x, u) \tag{4.25}$$

s.t.

$$0 = f(x, u) - \rho(x - \bar{x}) \tag{4.26}$$

$$u \in \mathcal{U}(x), \tag{4.27}$$

where $\bar{x}$ is precisely the turnpike value (hence the name *implicit programming* proposed in Feinstein and Luenberger (1981)). These turnpikes are attractors for all the possible optimal trajectories, emanating from all different admissible initial state, when the control system has enough convexity[3].

The solution of (4.22)-(4.24) is an easy mathematical programming problem. The solution of (4.25)-(4.27) involves the computation of a fixed point through a sequence of optimizations where the $\bar{x}$ value is updated. Both problems can be easily solved for the model (4.1)-(4.18), using for example the *solver* tool of *excel*. Table 4.3 below, borrowed from Haurie (2003), gives the asymptotic steady state of any optimal

State variables			State variables		
$\bar{K}$	=	943	$\bar{K}$	=	506
$\bar{M}$	=	911	$\bar{M}$	=	1170
$\bar{T}$	=	2.64	$\bar{T}$	=	3.69
Control variables			**Control variables**		
$\bar{I}$	=	94.3	$\bar{I}$	=	50.6
$\bar{\mu}$	=	0.68	$\bar{\mu}$	=	0.36
Exogenous variables			**Exogenous variables**		
$\bar{A}$	=	0.063	$\bar{A}$	=	0.063
$\bar{L}$	=	12000	$\bar{L}$	=	12000
$\bar{O}$	=	1.15	$\bar{O}$	=	1.15
Auxiliary variables			**Auxiliary variables**		
$\bar{C}$	=	291	$\bar{C}$	=	278
$\bar{E}$	=	41.83	$\bar{E}$	=	75.47
$\bar{F}$	=	3.72	$\bar{F}$	=	5.20
$\bar{Q}$	=	400	$\bar{Q}$	=	342
$\rho = 0\%$			$\rho = 6\%$		

Table 4.3. Turnpike values when ρ=0 % and ρ=6%,respectively

trajectory of the model (4.1)-(4.18) starting from any initial admissible state, for $\rho = 0\%$ and $\rho = 6\%$, respectively.

We notice the important effect of the pure rate of time preference on the asymptotic environmental state, represented by the variables T and M. Also, discounting implies a lower level of asymptotic consumption[4]. We see very well, on this simple numerical example, why Ramsey declared that a zero discount rate is justified on ethical grounds for a multi-generation economic growth model. In the long term a zero discount rate yields a better environment and a higher sustainable utility level. Indeed, using a zero discount rate would impose a high burden and adjustment cost to the present generation whereas climate change will impact future generations more. To cope with this long term cost without imposing too much a toll to the present generations several authors, including Nordhaus and Boyer in their DICE-99 model, have proposed to use a time dependent hyperbolic discount rate $\rho(t) \to 0$. If one introduces such a discounting scheme in the model described above, the asymptotic steady-state will still be the one associated with 0-discounting (see Haurie (2002) for a discussion of the turnpikes for models with uncertain or time varying discount rates).

To conclude this first section one can say that the consideration of sustainable economic development justifies the use of zero discount rates in the assessment of the optimal steady states that characterize economic sustainability. Now, representing the climate dynamics by a couple of differential equations might be considered too simplistic, when one considers the very high level of uncertainty that is still characterizing climate science. In the next section this concept of low or zero-discounting associated with turnpikes is used to build a hierarchical two time-scale model of stochastic interaction between the economy and the environment.

3. Turnpikes for a two time-scale model with stochastic climate change

This section intends to show how the turnpike theory can be used to analyze very long term climate policies when the climate change obeys a stochastic evolution at a time pace which is much slower than the speed of economic adjustment. The new feature of this modeling approach is the introduction of a distinction between the climate variables and a climate change indicator, also called *climate mode* which is the main determinant of the impact and economic damages due to climate change. In this paper one presents uniquely the general idea of the modeling approach, leaving for a more developed paper the task to establish rigorously the conjectures that are introduced here.

3.1 Climate variables and climate modes

The atmospheric concentration of carbon $M(t)$ and the average surface temperature (AST) $T(t)$ are the climate state variables. To represent the uncertainty that concerns the dynamics of these variables, one introduces noise in the carbon cycle and temperature forcing equations that are driving the temperature change. In the DICE models, as well as in most of the integrated assessment models that have been recently proposed to study climate change policies, a damage function is proposed as a function of the AST, for example $D(t) = \beta_1 T^{\beta_2}$, where $D(T)$ measures the loss of output due to global warming. This description of the impact of climate change to the economy is probably inaccurate as the most important consequences will be associated to phenomena described as abrupt climate changes or threshold events. Such events are recognized by climatologists, one knows that many have occurred at the Earth system time scale, including the collapse of large ice shield in the Antarctic, that could trigger a sea-level rise of the order of 6 meters[5]; another such event would be the interruption of the North-Atlantic thermo-haline circulation which could disrupt the Gulf-stream path.

One may therefore introduce the concept of *climate mode* which corresponds to a specific organization of the climate system leading to substantial changes in the distribution of temperature and precipitations. Indeed the switch from the current mode to a different one is a slow paced stochastic process. One can therefore represent the dynamics of the climate-economy systems in two time-scales, a slow one corresponding to the climate mode changes and a fast one corresponding to the economy and GHG concentrations or SAT evolutions.

3.2 Slow dynamics

The slow dynamics of this system represents the evolution of the climate mode which is modeled as a jump process taking values in a finite set of discrete modal states. Let I denote the finite set of possible climate modes and $\{\xi(t) : t \geq 0\}$ a jump process taking values in I, with transition rates

$$\varepsilon q_{k,\ell}(M,T) = \lim_{dt\to 0} \frac{\mathrm{P}[\xi(t+dt) = \ell | \xi(t) = k \text{ and } M(t) = M, T(t) = T]}{dt}. \tag{4.28}$$

The coefficient ε, which will eventually tend to 0, is the ratio between the slow and the fast time scales. One shall also use a *shrinked time-scale* $\tau = \frac{t}{\varepsilon}$. In the shrinked time-scale the transition rates are given by $q_{k,\ell}(M,T)$. Typically, when e.g. $\varepsilon = 0.01$, in the time-scale t one counts in years whereas in the time-scale τ one counts in centuries.

3.3 Fast dynamics

The fast dynamics represents the evolution of economic state variables and the evolution of GHG concentrations and SAT according to the different possible climate modes. One can thus model the climate variable dynamics as a set of controlled diffusion processes indexed over the finite modal set I

$$dM(t) = \varphi^i(M(t), E(t))dt + \sigma^i_M dv_1(t) \tag{4.29}$$
$$dT(t) = \phi^i(M(t), T(t))dt + \sigma^i_M dv_2(t) \tag{4.30}$$
$$i \in I.$$

The economy dynamics is still described by the capital accumulation processes, which is still represented as an ODE:

$$\dot{K}(t) = f^i(K(t), E(t), C(t)) \tag{4.31}$$
$$i \in I.$$

In the shrinked time-scale $\tau = \frac{t}{\varepsilon}$ the equations (4.29-4.34) will write

$$\varepsilon dM(\tau) = \varphi^i(M(\tau), E(\tau))dt + \sqrt{\varepsilon}\sigma^i_M dv_1(\tau) \tag{4.32}$$
$$\varepsilon dT(\tau) = \phi^i(M(\tau), T(\tau))dt + \sqrt{\varepsilon}\sigma^i_M dv_2(\tau) \tag{4.33}$$
$$i \in I.$$

and

$$\varepsilon \dot{K}(\tau) = f^i(K(\tau), E(\tau), C(\tau)) \qquad (4.34)$$
$$i \in I.$$

respectively. In a perspective of centuries or millennia, the economy will be seen as a fast process and the climate variables will tend to be distributed according to some state probability measure, induced by the emission forcing term $E(t)$.

By indexing over I the capital accumulation process in Eq. (4.34) one is able to represent the damage (e.g. the loss of output) due to the change of climate mode.

3.4 The limit climate control problem

One is in the perspective of the *shrinked time-scale* τ (i.e. one adopts the slow dynamics time-scale). The *limit climate control problem* serves to delineate the global long term emission levels that achieve the best compromise between economic consumption and climate change prevention.

Assume the climate is in mode k. Let $\bar{E}^k$, be some constant emission levels and assume that there corresponds to $\bar{E}^k$ a joint steady state (invariant) measures $\bar{m}^k_{M\times T}(\bar{E}^k;\cdot)$, obtained from the diffusion equations (4.32)-(4.33). Then consider the controlled Markov chain with state set I and transition probability rates

$$\tilde{Q}_{k,\ell}(\bar{E}^k) = \int_{M\times T} q_{k,\ell}(\mu,\theta) d\bar{m}^k_{M\times T}(\bar{E}^k;\mu,\theta), \quad k,\ell \in I. \qquad (4.35)$$

While in mode climate k one also considers a reward rate which is defined by

$$\tilde{L}^k(\bar{E}^k) = U^k(\bar{C}^k) \qquad (4.36)$$

where $U^k(\bar{C}^k)$ is the utility of consumption when climate is in mode k, with $\bar{C}^k$ defined by

$$\bar{C}^k = \max\{C : \exists K, 0 = f^k(K, \bar{E}^k, C)\}. \qquad (4.37)$$

The parameter $\bar{C}^k$ is thus defined as the *highest sustainable consumption rate* in the economy, under climate mode k and emissions cap $\bar{E}^k$. Notice that the utility function is indexed over the climate modes ($k \in I$); therefore one can represent the desutility of changing from the current climate mode to a deteriorated one.

The limit climate control problem consists in finding a policy $\varpi^* : I \to \mathcal{E}$ which maximizes the infinite horizon criterion

$$J^i(\varpi) = \mathrm{E}_\varpi[\int_0^\infty e^{-\varrho\tau} \tilde{L}^{\xi(\tau)}(\bar{E}^{\xi(\tau)})\, d\tau | \xi(0) = i] \qquad (4.38)$$

where $\varrho > 0$ is a (possibly very low) discount rate in the *shrinked time-scale*[6] and $(\xi(\tau) : t \geq 0)$ is the Markov chain with transition rates $\tilde{Q}_{k,\ell}(\bar{E}^k)$ while $\bar{E}^i = \varpi(i)$, $i, k, \ell \in I$. The choice of the proper discount rate $\varrho > 0$ is still a delicate problem[7] that one assumes to be solved.

Call J^{*i}, $i \in I$ the optimal potential function associated with the optimal policy ϖ^*, according to (4.38). This potential function satisfies the following dynamic programming equation

$$\varrho J^{*i} = \max_{\bar{E}^i}\{\tilde{L}^i(\bar{E}^i) + \sum_{k\in I}\tilde{Q}_{i,k}(\bar{E}^i)J^{*k}\}; \quad i \in I. \tag{4.39}$$

Remark 1 *The limit climate control problem is thus defined as a controlled Markov chain which serves to determine what should be the long term emissions levels that would keep at their optimal level the probabilities of climate modal changes. This problem is defined in the climate change time-scale, corresponding to the slow modes of the system.*

3.5 Transient control problem

One considers now the problem of defining the optimal economic policy in the time-scale corresponding to the usual economic dynamics. Assume that at time $t = 0$ one is in climate mode i. Consider the economic system, with state equation

$$\begin{aligned} \dot{K}(t) &= f^i(K(t), E(t), C(t)) \\ i &\in I, \end{aligned} \tag{4.40}$$

and the climate dynamics

$$dM(t) = \varphi^i(M(t), E(t))dt + \sigma^i_M d\upsilon_1(t) \tag{4.41}$$

$$dT(t) = \phi^i(M(t), T(t))dt + \sigma^i_M d\upsilon_2(t) \tag{4.42}$$

$$i \in I.$$

Let $x^o = (M^o, T^o, K^o)$ be the initial climate and economic state values.

In the long term perspective, described in the slow (*shrinked*) time-scale $\tau = \frac{t}{\varepsilon}$, the optimal climate-economy policy is defined as the solution of the following problem:

$$\max = \mathrm{E}_{(E(\cdot),C(\cdot))}\left[\int_0^\infty e^{-\rho\tau} U^{\xi(\tau)}(C(\tau))\,d\tau | i, x^o\right] \tag{4.43}$$

s.t.

$$\varepsilon dM(\tau) = \varphi^{\xi(\tau)}(M(\tau), E(\tau))dt + \sqrt{\varepsilon}\sigma_M^{\xi(\tau)} d\upsilon_1(\tau) \tag{4.44}$$

$$\varepsilon dT(\tau) = \phi^{\xi(\tau)}(M(\tau), T(\tau))dt + \sqrt{\varepsilon}\sigma_M^{\xi(\tau)} d\upsilon_2(\tau) \tag{4.45}$$

$$\varepsilon \dot{K}(\tau) = f^{\xi(\tau)}(K(\tau), E(\tau), C(\tau)) \tag{4.46}$$

$$\begin{aligned} q_{k,\ell}(M,T) = \lim_{dt\to 0} \frac{1}{d\tau}\mathrm{P}[\xi(\tau + d\tau) = \ell | \\ \xi(\tau) = k \text{ and } M(\tau) = M, T(\tau) = T]. \end{aligned} \tag{4.47}$$

REMARK 2 *The stochastic control problem (4.43)-(4.47) is defining the climate-economy policy problem where a tradeoff between economic growth and climate risk has to be defined. This problem involves the two time-scales. It falls in the domain of control for singularly perturbed systems. In Filar, Gaitsgory and Haurie the reader will find the theoretical development that justifies the approximation technique proposed in the rest of this section.*

Adapting the results established in Filar, Gaitsgory and Haurie one can show that the solution of the problem can be approximated by considering the following class of deterministic control problems, defined in the fast time scale t (obtained by stretching out the time scale τ): Introduce the auxiliary reward function

$$U(C(t)) + \sum_{k\in I} q_{i,k}(M(t), T(t))J^{*k} \tag{4.48}$$

and consider the average payoff over a long time interval

$$\tilde{g}^i(x^o; \Theta) = \frac{1}{\Theta}\mathrm{E}_{(C(\cdot),\,E(\cdot))}[\int_0^\Theta (U(C(t)) + \sum_{k\in I} q_{i,k}(M(t), T(t))J^{*k})dt | x^o]. \tag{4.49}$$

One looks for the open-loop controls $(C(\cdot),\, E(\cdot))$ that optimize (4.49) subject to the climate and economic dynamics (4.40)-(4.33)[8].

An optimal steady state for the control problem (4.49), (4.40)-(4.33) is a vector $(\bar{K}^i, \bar{C}^i, \bar{E}^i)$ which maximizes

$$U^i(\bar{C}) + \sum_{k\in I} \tilde{Q}_{i,k}(\bar{E})J^{*k} \tag{4.50}$$

where $\tilde{Q}_{i,k}(\bar{E})$ is a transition rate defined according to (4.35), subject to the steady-state constraints

$$0 = f^i(\bar{K}, \bar{E}, \bar{C}) \tag{4.51}$$
$$i \in I.$$

It is not hard to recognize in the solution to this problem the expression of the RHS of the DP equation of the limit climate control problem (4.39), that is

$$\max_{\bar{E}^i}\{\tilde{L}^i(\bar{E}^i) + \sum_{k\in I} \tilde{Q}_{i,k}(\bar{E}^i)J^{*k}\}. \tag{4.52}$$

If the turnpike property holds, when $\Theta \to \infty$ the optimal accumulation path will spend most of the time in the vicinity of the optimal steady state and

$$\lim_{\Theta\to\infty} \tilde{g}^{*i}(x^o; \Theta) = \varrho J^{*i}. \tag{4.53}$$

But, what is important, and the reader is referred to Filar, Gaitsgory and Haurie for a precise proof of this result, is the fact that, when $\varepsilon \to 0$, the control defined by the solution of these auxiliary infinite horizon problems converges to the solution of the original two time-scale stochastic control problem (4.43)-(4.47).

By the way, the auxiliary control problem with reward (4.49) defines an incentive scheme where the climate variables enter into the term $\sum_{k\in I} q_{i,k}(M(t), T(t))J^{*k}$. This corresponds to a valuation of the environmental state variables which is induced by the solution of the limit climate control problem. By imposing a taxing scheme corresponding to this term in the reward function, one drives the economy toward the correct long term sustainable state. The following remark summarizes this decomposition result.

REMARK 3 *By valuing the climate modes in accordance with the potential function that comes out of the solution of the limit climate control problem, one introduces incentives that are asymptotically correct for the economy to be driven, in the fast time scale, toward the appropriate optimal steady state.*

4. Conclusion

In this brief paper one has shown how the concept of turnpike can be used in the study of economic sustainable development, in the presence of long term climate change threats. The focus has also been placed on the difference of time scales between the economic and the climate subsystems that are in interaction.

In summary the following remarks can be drawn from the story told in this short paper:

- The concept of economic sustainability is very close to the concept of turnpike in economic growth models with environmental cost and a stationary context (when typically population and technical progress stabilize)
- Discounting has an important effect on the asymptotic extremal (optimal) steady-state, and hence on the sustainable optimal economic state. The concern for climate change which will impose a toll on future generations speaks for the use of a very low, and ultimately zero, discount-rate in the design of climate/economy policies.
- As the climate change will probably take place in the form of a modal change, including threshold phenomena, with a low probability influenced by the climate variables that are the GHG concentrations and the SAT, it would make sense to design the long term climate policies by solving first a *limit climate control problem* that is posed in the very long term and which serves to define the optimal cap on emissions level that should be ultimately imposed in order to optimally control the climate switches. One may obtain from the solution of this slow paced controlled Markov process problem, a tax scheme (or possibly an emissions trading scheme) that will tend to drive the economic system toward the appropriate long term steady (sustainable) state.

The idea of designing approximate optimal control for multi time-scale systems has been exploited by Prof. Sethi in many important books and papers in the context of manufacturing systems (see Sethi and Zhang (1995a)-Sethi for a short sample). This paper shows that similar ideas could contribute to a better understanding of the coupling between climate and economic dynamics, in the context of economic sustainable development.

Notes

1. It would be an optimal steady-state in the case of zero-discounting.
2. This is less apparent in DICE-99 where a 3-reservoir carbon cycle model, showing no carbon absorption even in the very long term, does not permit to stabilize the concentration of GHG for any emissions level that are higher than the pre-industrial level.
3. The reader is referred to Carlson, Haurie and Leizarowitz (1994) for a more precise formulation of these asymptotic attraction conditions.
4. When assessing sustainable economic development the turnpikes give important information and it is surprising that very little attention has been paid yet to the asymptotic behavior of integrated assessment models.

5. An illustration of the possible (although highly unrealistic) climate modal changes has been given in the movie *The Day after Tomorrow* where a new ice age is triggered by a threshold event.

6. Remind that we positioned ourselves in the slow (or shrinked) time-scale; therefore the nominal discount rate is not yearly but based e.g. on a century or even larger time step.

7. At this slow time-scale the choice of a zero discount rate is still possible, although this would imply a durability of the human species at geological times which contradicts the current knowledge on evolution, paleontology and earth science.

8. Alternatively one may look for an overtaking optimal solution with respect to the criterion

$$\liminf_{\Theta\to\infty} \mathrm{E}_{(C(\cdot),\, E(\cdot))}[\int_0^{\Theta} (U(C(t)) + \sum_{k\in I} q_{i,k}(M(t), T(t))J^{*k})dt|x^o].$$

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Chapter 5

COMMON PROPERTY RESOURCE AND PRIVATE CAPITAL ACCUMULATION WITH RANDOM JUMP

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Abstract We present a model of exploitation of a common property resource when agents can also invest in private and productive capital. The resource extracted from a common pool is non-renewable, but the resource stock is under uncertainty in the sense that the stock might follow jump process. We show that there exists an optimal solution in the model.

1. Introduction

In Long and Katayama (2002) they presented a model of exploitation of a common property resource, when agents can also invest in private and productive capital. The resource extracted from a common pool is non-renewable in the model. We try to extend their result to the case where a common pool is under uncertainty in the sense that it could have a sudden increase or decrease in the process of extraction.

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The extension is quite natural when we see the present state of international crude oil market. Some producing countries encountered the technological difficulties of extraction and/or social hazards. Also the past history showed the unexpected discovery of new oil reserves. Yet the total reserve in the earth planet is limited, and it is expected that the resource is finally exhausted. However, people can accumulate man-made capital for substituting the exhaustible resource and extend the period in which the resource is utilized before it is completely depleted.

Considering these intrinsic aspects of resource economy, we present a model of uncertainty in the process of extraction of the resource and build the capital to substitute for the exhaustible resource. To incorporate it we build a model with a random jump in the stock of the resource.

The main issue is to see whether there is an optimal solution to this model.

2. The model

There are n identical agents having common access to a stock of non-renewable natural resource, denoted by $S(t)$. Each agent i also owns a private capital stock $K_i(t)$. Agent i extracts the amount $R_i(t)$ of the common resource stock ($i = 1, ..., n$). Extraction is costless. Total extraction in the economy at time t is $R(t) = \sum_{i=1}^{n} R_i(t)$, and the reserve depletes according to

$$\dot{S}(t) = -R(t)$$

if it is not subject to any uncertainty.

First assume that each individual extracts equal amount, and so it follows that

$$\dot{S}(t) = -nR_i$$

However, the reserve may be augmented or damaged several times in the finite horizon and the reserve size is affected by those jumps in magnitude.

The jump process takes the form $dJ(t)$, and the resource stock is governed by

$$dS(t) = dJ(t) - nR_i dt \tag{5.1}$$

The stock level at time t is

$$S(t) = S_0 + J(t) - \int_0^t nR_i(s)ds$$

where $J(t)$ is a pure jump process given by

$$J(t) \triangleq \int_0^t \int_{R\setminus\{0\}} S(s-) \cdot zN(ds, dz)$$

$$N(ds, dz) = \lambda ds \cdot N(dz) \quad \text{: Poisson random measure}$$
$$E[N(A)] = \lambda \int_A n(z) dt dz, \; A \in B(R_+ \times R), \text{ a Borel set}$$

Here $n(z)dz$ is Levy measure such that

$$\int_{|z|\leq 1} z^2 n(z) dz < \infty, \quad \int_{|z|>1} n(z) dz < \infty$$

The Levy measure expresses the possible jump size. λ is the average number of jumps to occur during unit time interval.

Assume that the extracted resource cannot be directly consumed. Instead, agent i uses R_i as an input, which, in combination with his labor input and his privately owned capital stock K_i yields an output Y_i of final good. For simplicity we choose a measurement unit by fixing the labor input to unity. Therefore, agent i's production function is

$$Y_i = R_i^{1-\beta} K_i^{\alpha}$$

where $0 < \alpha < 1, \; 0 < \beta < 1$. Agent i consumes $C_i(t)$, and the remaining quantity is invested to accumulate his physical capital. The rate of accumulation of the privately owned capital stock is thus

$$dK_i(t) = \left(R_i^{1-\beta} K_i^{\alpha} - C_i \right) dt \tag{5.2}$$

Each individual utility is increasing in consumption $C_i(t)$:

$$U_i = (1-\gamma)^{-1} C_i^{1-\gamma}$$

where $0 < \gamma < 1$. Each agent wishes to maximize the integral of the stream of discounted utility

$$\max \int_0^{\infty} (1-\gamma)^{-1} C_i^{1-\gamma} e^{-\rho t} dt$$

subject to (5.1) and (5.2), and the initial conditions

$$S(0) = S_0,$$
$$K_i(0) = K_{i0} > 0.$$

γ is the elasticity of marginal utility, and for mathematical simplicity as in Long and Katayama (p. 196, 2002) assume that $\gamma = \alpha$.

3. The cooperative outcome

If the agents cooperate each other, they will collectively seek to maximize the same level of their welfare. They will choose the rate of extraction per agent R_h and consumption per agent C_h to maximize

$$\max \int_0^\infty (1-\gamma)^{-1} C_h^{1-\gamma} e^{-\rho t} dt$$

subject to $\dot{K}_h = R_h^{1-\beta} K_h^\alpha - C$ and $dS(t) = dJ(t) - nR_h(t)$, and the boundary conditions

$$S(0) = S_0,$$
$$K_h(0) = K_0 > 0.$$

Define the value function for this maximization problem by

$$V(S, K_h) = \max_{C_h, R_h > 0} E\left[\int_0^\tau (1-\alpha)^{-1} C_h^{1-\alpha} e^{-\rho t} dt + g\left(S_\tau, K_{h\tau}\right)\right]$$

where $g(\cdot,\cdot)$ is a given function, and

$$\tau = \inf\{t > 0 : (S_t, K_{ht}) \in D\}$$
$$D = \{(S, K) : S > 0, K_h > 0\}$$
$$\partial D = \{(S, K_h) : S = 0\} \cup \{(S, K_h) : K_h = 0\}$$

and $0 < \alpha, \beta < 1$ and $\rho > 0$.

It is known from Kushner and Dupuis (1992) that this optimization problem is equivalent to the following Hamilton-Jacobi-Bellman equation;

$$\rho V(S, K_h) = \max_{C_h, R_h > 0} \left\{ \begin{array}{l} (1-\alpha)^{-1} C_h^{1-\alpha} + \frac{\partial V}{\partial K}\left(R_h^{1-\beta} K_h^\alpha - C_h\right) \\ \quad +\lambda \int_R \{V(S + Sz, K_h) - V(S, K_h)\}\, n(z) dz \\ \quad +\frac{\partial V}{\partial S}(-nR) \end{array} \right\}$$

$$\text{for } (S, K_h) \in D$$

$$V(S, K_h) = g(S, K_h) \quad \text{for } (S, K_h) \in \partial D$$

The first order conditions for maximization are

$$C_h^{-\alpha} - V_{K_h} = 0$$
$$(1-\beta) R^{-\beta} K_h^\alpha V_{K_h} - nV_S = 0$$

They turn to be

$$C_h = (V_K)^{-1/\alpha} \tag{5.3}$$

and

$$R_h = \left[\frac{nV_S}{(1-\beta)V_{K_h} K_h^{\alpha}} \right]^{-1/\beta}$$

Substituting these conditions into H-B-J equation, we obtain the partial differential equation for $V(S, K_h)$. As Long and Katyayama (2002) have indicated, the solution to the differential equation is not simple. Instead of solving it directly we take the same solution process as the one used by them. As they prove it the partial differential equation has a simple solution

$$V(S, K_h) = AK_h^{1-\alpha} + BS^{1-\beta} \tag{5.4}$$

where A and B are to be determined. Then H-B-J equation becomes

$$\rho V(S, K_h) = \max_{C_h, R_h} \left\{ \begin{array}{l} (1-\alpha)^{-1} C_h^{1-\alpha} + V_{K_h}(R_h^{1-\beta} K_h^{\alpha} - C_h) \\ \quad +\lambda BS^{1-\beta} \int_{R_h} \left\{ (1+z)^{1-\beta} - z^{1-\beta} \right\} \\ \quad -n(z)dz - nR_h V_S \end{array} \right\} \tag{5.5}$$

$$d(\beta, \lambda) \equiv \int_{-\infty}^{\infty} \left\{ (1+z)^{1-\beta} - z^{1-\beta} \right\} n(z)dz \geq 0$$

We first obtain from (5.4) that

$$V_{K_h} = (1-\alpha)AK_h^{-\alpha} \tag{5.6}$$

and

$$V_S = (1-\beta)BS^{-\beta}$$

Substituting these derivatives and (5.4) into (5.5) yields

$$
\begin{aligned}
\rho V &= \rho\left(AK_h^{1-\alpha} + BS^{1-\beta}\right) \\
&= (1-\alpha)^{-1}C_h^{1-\alpha} + (1-\alpha)AK_h^{-\alpha}(R_h^{1-\beta}K_h^{\alpha} - C_h) \\
&\quad + BS^{1-\beta}d\,(\beta) - nR_h(1-\beta)BS^{-\beta} \\
&= (1-\alpha)^{-1}\left\{A\,(1-\alpha)\right\}^{-(1-\alpha)/\alpha} \\
&\quad + A\,(1-\alpha)\,R_h^{1-\beta} - A\,(1-\alpha)\,K_h^{-\alpha}C \\
&\quad + Bd(\beta)S^{1-\beta} - nB(1-\beta)S^{1-\beta}\left(\frac{nB}{A(1-\alpha)}\right)^{-1/\beta} \\
&= (1-\alpha)^{-1}\left\{A\,(1-\alpha)\right\}^{-(1-\alpha)/\alpha}K_h^{1-\alpha} \\
&\quad - A\,(1-\alpha)\left\{A(1-\alpha\right\}^{-1/\alpha}K_h^{1-\alpha} \\
&\quad + A(1-\alpha)\left\{\frac{nB}{A(1-\alpha)}\right\}^{-(1-\beta)/\beta}S^{1-\beta} \\
&\quad + Bd(\beta)S^{1-\beta} - nB(1-\beta)\left\{\frac{nB}{A(1-\alpha)}\right\}^{-/1\beta}S^{1-\beta} \\
&= K_h^{1-\alpha}\left\{A(1-\alpha\right\}^{-1-1/\alpha}\cdot\frac{\alpha}{1-\alpha} \\
&\quad + S^{1-\beta}\left[\frac{(nB)^{-(1-\beta)/\beta}}{\left\{A(1-\alpha\right\}^{-1/\beta}} + Bd(\beta) - (1-\beta)\frac{(nB)^{1-1/\beta}}{\left\{A(1-\alpha\right\}^{-1/\beta}}\right] \\
&= K_h^{1-\alpha}A^{1-1/\alpha}\alpha\,(1-\alpha)^{-1/\alpha} \\
&\quad + S^{1-\beta}\left[\beta\,(nB)^{1-1/\beta}\left\{A(1-\alpha\right\}^{1/\beta} + Bd(\beta)\right]
\end{aligned}
$$

For this equation to hold for any $K_h > 0$ and $S > 0$, it is necessary that the following conditions are satisfied:

$$\rho = A^{-1/\alpha}\alpha(1-\alpha)^{-1/\alpha}$$

and

$$\rho = \left[\beta n^{1-1/\beta}\beta^{-1/\beta}\cdot\left\{A(1-\alpha)\right\}^{1/\beta} + d(\beta)\right] \text{ if } \rho - d(\beta) > 0$$

Therefore,

$$A = \left(\frac{\alpha}{\rho}\right)^{\alpha}(1-\alpha)^{-1} \tag{5.7}$$

From the necessary condition $\delta \equiv \rho - d(\beta, \lambda) > 0$,

$$\delta \triangleq \rho - d(\beta,\lambda) = \beta n^{1-1/\beta}\beta^{-1/\beta}\cdot\{A(1-\alpha)\}^{1/\beta}$$
$$= \beta n^{1-1/\beta}\left(\frac{\alpha}{\rho}\right)^{\alpha/\beta}$$
$$\therefore \delta B^{1/\beta} = \beta n^{1-1/\beta}\left(\frac{\alpha}{\rho}\right)^{\alpha/\beta}$$

or

$$B = \left(\frac{\beta}{\delta}\right) n^{\beta-1}\left(\frac{\alpha}{\rho}\right) \text{ if } \delta > 0 \tag{5.8}$$

Therefore by (5.7) and (5.8)

$$R_h^{-\beta} = \frac{n}{1-\beta}\frac{(1-\beta)S^{-\beta}B}{(1-\alpha)A} = \frac{nB}{(1-\alpha)A}S^{-\beta}$$
$$= n^{\beta}\left(\frac{\beta}{\delta}\right)^{\beta} S^{-\beta}$$
$$\therefore (nR_h)^{-\beta} = \left(\frac{\beta}{\delta}\right)^{\beta} S^{-\beta}\text{or}$$
$$R_h = \left[\frac{\delta}{\beta}\right]\left[\frac{S}{n}\right] \tag{5.9}$$

(5.9) gives the optimal resource extraction rule.

It is natural to see that the optimal extraction by each agent increases as the reserve size per agent becomes larger and the productivity of natural resource in final good production is higher.

Then, using (5.3), (5.5) and (5.7), we obtain the optimal consumption rule:

$$C_h = \left(\frac{\rho}{\alpha}\right) K_h \tag{5.10}$$

The optimal consumption rule (5.10) is independent of the stock of the resource, and as (5.9) shows, the extraction by agent h depends only on the resource stock per head, $\frac{S}{n}$. Notice that Long and Katayama (2002) have derived the optimal extraction function as $R_h = \left[\frac{\rho}{\beta}\right]\left[\frac{S}{n}\right]$ in the absence of jumps in resource size. Our result is obtained by replacing ρ with $\delta = \rho - d(\beta, \lambda)$. Since $d(\beta, \lambda) \geq 0$, the optimal extraction is revealed to be less under jumps than without them for the same level of resource stock. Moreover as $d(\beta, \lambda)$ is increasing in λ, the optimal extraction decreases as the resource jumps more frequently. It is the way for economic agents to react more cautiously to cope with the uncertainty. However, the more cautious behavior is not applied to consumption, since the optimal level C_h is shown to be the same as in Long and Katayama (2002).

Even if agents behave cautiously to cope with uncertainty expressed by Poisson process in this model, it should be noted that the natural resource will eventually be depleted with probability one. It is the intrinsic nature of exhaustible resource.

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Chapter 6

TRANSFER MECHANISMS INDUCING A SUSTAINABLE FOREST EXPLOITATION

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Abstract In this paper our concern is with deforestation as a global environmental issue. Foreign transfers from developed countries to forestry countries have been proposed for this goal. The problem is formulated as a Stackelberg differential game played over an infinite horizon, with the donor community as the leader and the aid recipient country as the follower. We consider different transfer mechanisms through which the donor community subsidizes the forestry country. We compare the results both from the environmental and economic points of view.

1. Introduction

The problem of deforestation in developing countries has received a great attention in the international community due to its important global environmental effect both on biodiversity conservation and on climate change.

The main causes of tropical deforestation seem to be the conversion of forested land to agricultural use and, to a lower level, the forestry activities (see, for example, Southgate (1990), Southgate et al. (1991), Amelung and Diehl (1992), Kaimowitz and Angelsen (1999)). Various models of allocation between forest uses and agricultural uses in devel-

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oping countries have been developed (see, for example, Barbier et al. (1991), Barbier and Burgess (1997)).

Scientists and politicians point out that the problem of deforestation in developing countries requires coordination at an international level. The international externality dimension of the deforestation in developing countries makes of the forest conservation a global environmental issue. Different mechanisms have been proposed in the literature to coordinate the efforts of the developed and developing countries to tackle deforestation (see, for example, Perrings et al. (1995), Pearce and Moran (1994), Panayotou (1994)). In this paper we are interested in the mechanism called financial transfers, which considers aid donation and transfers as a solution for some global environmental issues such as deforestation in developing countries.

The use of financial transfers flowing from developed to (forestry) developing countries to improve forest conservation has received recently some attention in the economic literature. Optimal control theory and differential games are the methodological tools used to show that financial transfers from developed to developing countries may improve both the forest conservation and the welfare of the domestic and/or of the foreign country.

The literature has proposed initially lump-sum aid donations (see, for example, Barbier and Rauscher (1994)). Authors as Stäler (1996) and Mohr (1996) criticized this type of donations as being a passive instrument to prevent deforestation and proposed to make the amount of transfers conditional to the recipient country's effort to improve forest conservation. Later on Van Soest and Lensink (2000) and Fredj et al. (2004) propose a compensation function which makes the amount of transfers also dependent on the deforestation rate. In Martín-Herrán et al. (2004) the authors compare the effect of a compensation function from the developed countries to the developing ones which depends only on the forest stock with one compensation function dependent both on the forest stock and the deforestation rate.

The present paper is also concerned with the design of different aid programs by developed countries to help developing ones keeping their forest. The comparison is made both from the environmental and economic points of view. By environmental point of view we mean the size of the forest both in the short and long runs, which can be viewed as a measure of the conservation of the forest area. By economic point of view we focus on both the aid recipient's and the donor's welfare. As in most of the papers cited in the previous paragraph (except Barbier and Rauscher (1994) and Stäler (1996)) we use differential games as the methodological framework. The game is played à la Stackelberg, where the donor community is the leader and the aid recipient is the follower, since the donor community plays a leadership role in the implementation of the aid program. We use an approach which solves a game with Markovian strategies for the follower and open-loop strategies for the leader. Using this approach the Stackelberg equilibrium has the property of being time-consistent.

The main differences with the papers by Van Soest and Lensink (2000), Fredj et al. (2004) and Martín-Herrán et al. (2004) are as follows:

- All these papers assume a particular specification of the forestry country's revenue function, while in this paper some results are obtained for a general one.

- On one hand, Van Soest and Lensink (2000) and Martín-Herrán et al. (2004) consider utility function for the donor community which are difficult to assess in practice. On the other hand, Fredj et al. (2004) consider a finite time horizon and assume that the objective from the donor's perspective is to maximize the size of the forest at the final date of the aid program. In the present paper we assume an infinite planning period and the donor's objective is to minimize the amount of subsidy flowing to the forestry country but which guarantees the participation constraint of the forestry country in the aid program.

- In this paper we propose specifications for the compensation function which, as far as we know, had not be used before in the literature.

The paper is organized as follows. Section 2 presents the different models we are dealing with. Firstly, we study the basic model where the forestry country exploits the forest without foreign aid. Secondly, we state two different specifications for the compensation function that the donor community can use with the aim of inducing the forestry country to follow a deforestation policy which better preserves the forest. Section 2 also collects the conditions which characterize the optimal paths of the forest stock and the deforestation rate of the different models proposed. Section 3 is devoted to the comparison of the optimal forest stock and deforestation rate paths both in the short and long runs. In Section 4 the donor community's optimization problem is stated and the optimal amount of subsidy flowing to the forestry country is characterized and compared for the different specifications of the subsidy function. Section 5 collects our conclusions.

2. The different models

2.1 The basic model

In this model, the developed country does not provide any aid to the forestry country and the problem becomes an optimal control one, where the forestry country is the unique agent. Let consider that the forestry country has to decide about the optimal allocation of land between forest and agricultural uses. Authors as Ehui et al. (1990), Van Soest and Lensink (2000), Fredj et al. (2004), Martín-Herrán et al. (2004) have previously stated models in which the forestry country's optimization problem can be written as follows:

$$
\begin{aligned}
W^1 = \max_D & \int_0^\infty e^{-rt} R(D(t), F(t))\, dt \\
\text{s.t.} \quad & \dot{F}(t) = -D(t), \quad F(0) = F_0, \\
& D(t) \geq 0.
\end{aligned}
$$

The forestry country choosing the rate of deforestation at time t, $D(t)$, aims at maximizing its stream of discounted revenues, $R(D(t), F(t)) \in C^2(\mathbb{R}, \mathbb{R})$ from agriculture and forest exploitation, e.g. timber production, over an infinite horizon, subject to the time evolution of the forest stock, F. Let us note that the forest is considered as a non-renewable natural resource, and therefore, the dynamics of the forest stock only depends on the rate of deforestation. Parameter F_0 denotes the initial size of the forest.

Next proposition characterizes the optimal paths, solutions of the previous optimal control problem.

Proposition 6.1 *Assuming interior solutions, the optimal forest stock, deforestation rate and shadow-price paths satisfy the following expressions*[1]*:*

$$
\begin{aligned}
\lambda^1 &= \frac{\partial R}{\partial D}(D, F), \\
\dot{F} &= -D, \quad F(0) = F_0, \\
\dot{\lambda}^1 &= r\lambda^1 - \frac{\partial R}{\partial F}(D, F),
\end{aligned}
$$

where λ^1 denotes the costate variable associated with the forest stock.

Proof. We define the current-value Hamiltonian:

$$H^1(F, D, \lambda^1) = R(F, D) - \lambda^1 D.$$

[1] From now on the time argument is omitted when no confusion can arise.

The costate variable λ^1 measures the marginal value of an additional unit of forest stock or equivalently, the costs of deforesting an extra-unit of forested land now rather than in the future, and thus will present a positive sign. For this reason this variable is also called the shadow-price of the forest stock.

Assuming interior solutions, the necessary conditions for optimality derived from Pontryagin's maximum principle are the following:

$$\begin{aligned}\frac{\partial H^1}{\partial D}(F,D,\lambda^1) &= \frac{\partial R}{\partial D}(F,D) - \lambda^1 = 0,\\ \dot{F} &= -D, \quad F(0) = F_0,\\ \dot{\lambda}^1 &= r\lambda^1 - \frac{\partial H^1}{\partial F}(F,D,\lambda^1) = r\lambda^1 - \frac{\partial R}{\partial F}(F,D).\end{aligned}$$

■

The results in the above proposition are intuitive. Indeed, it is readily seen from the optimality conditions that the deforestation policy satisfies the familiar rule of marginal revenue from deforestation ($\frac{\partial R}{\partial D}$) must equal its marginal cost, given here by the costate variable λ^1.

Let denote by $F^1(t), D^1(t)$ the optimal paths of the forest stock and the deforestation rate, respectively.

COROLLARY 6.2 *The steady-state equilibrium of the forest stock denoted by F^{1*} satisfies the following equation:*

$$\frac{\partial R}{\partial D}(0, F^{1*}) = \frac{1}{r}\frac{\partial R}{\partial F}(0, F^{1*}). \tag{6.1}$$

Proof. At the steady-state $\dot{F} = 0, \dot{\lambda}^1 = 0$ and then equation (6.1) can be derived straightforwardly. ■

We consider now the scenario where the donor community participates into the conservation effort of the rainforest by compensating the forestry country for its loss of revenues through a subsidy program. The donor community can be viewed as a group of developed countries, which is more concerned about the sustainable management of the forest. This scenario is formulated as a Stackelberg differential game where the donor community is the leader and the forestry country is the follower. We assume that the players adopt a mixed information structure: the leader plays open-loop strategies and the follower adopts a Markovian response (see, for example, Dockner et al. (2000)). Under this assumption the donor community restricts the space of functions from which it can choose its strategy. The donor community as the leader in the Stackelberg game moves first and proposes to the forestry country a financial transfer given by a subsidy function. The forestry country (follower) optimizes its objective taken into account the leader's announcement and determines its deforestation rate. Once the donor community knows the forestry country's best response to the financial transfer proposed, it chooses the parameter's values which appear in its compensation function to minimize the amount of subsidy flowing to the forestry country. The use of this type of information structure guarantees the time-consistency of the Stackelberg equilibrium if the model is stationary (the leader's objective functional and the system dynamics do not depend explicitly on time) and the time horizon is infinite (see,

Dockner et al. (2000), page 137). Our formulation fits exactly these requirements.

2.2 The second model

We consider, now, that the donor community proposes and the forestry country obtains a subsidy at each time proportional to the size of the forest area as a reward for a better environmental conservation. Let us note that this formulation is equivalent to that appearing in the environmental economics literature which assumes that the forestry country becomes more concerned about the sustainable management of the forest. In the sense it takes into account a preservation value, given by $V(F) = aF$, in its utility function, where a is a positive constant. This kind of hypothesis has been previously considered in problems of forest management (see, for example, Stäler (1996)) and fishing problems (see, for example, Clark (1990)).
The forestry country's optimization problem under this assumption can be rewritten as follows:

$$
\begin{aligned}
W^2 &= \max_D \int_0^\infty e^{-rt}[R(D,F) + aF]\,dt \\
\text{s.t.} \quad & \dot{F} = -D, \quad F(0) = F_0, \\
& D \geq 0.
\end{aligned}
$$

Next proposition characterizes the optimal paths of the forest stock and the deforestation rate:

Proposition 6.3 *Assuming interior solutions, the optimal forest stock, deforestation rate and shadow-price paths satisfy the following expres-*

sions:

$$\lambda^2 = \frac{\partial R}{\partial D}(D,F),$$
$$\dot{F} = -D, \quad F(0) = F_0,$$
$$\dot{\lambda}^2 = r\lambda^2 - \frac{\partial R}{\partial F}(D,F) - a.$$

Proof. It suffices to apply Pontryagin's maximum principle as in Proposition 6.1, taking into account that now the hamiltonian associated with the optimization problem is given by:

$$H^2(F,D,\lambda^2) = R(F,D) + aF - \lambda^2 D,$$

where λ^2 denotes the costate variable associated with the forest stock. ■

Let denote by $F^2(t), D^2(t)$ the optimal paths of the forest stock and the deforestation rate, respectively.

REMARK 6.4 *It is easy to prove that the optimality conditions established in Proposition 6.3 are the same when the follower uses either an open-loop or a Markovian information structure.*

COROLLARY 6.5 *The steady-state equilibrium of the forest stock denoted by F^{2*} satisfies the following equation:*

$$\frac{\partial R}{\partial D}(0,F^{2*}) = \frac{1}{r}\left[\frac{\partial R}{\partial F}(0,F^{2*}) + a\right]. \tag{6.2}$$

REMARK 6.6 *A constant b can be added in the subsidy expression and state $aF+b$ rather than aF with $aF+b>0$. The optimal forest stock and deforestation paths are the same for both specifications and only the optimal payoff functions will differ in the quantity b/r. Later on when the donor community's optimization problem is studied, the subsidy is assumed to be given by $aF+b$.*

2.3 The third model

We focus now on another type of subsidy function. Let consider the following specification:

$$S(t) = \begin{cases} \overline{S} & \text{if} \quad F(t) \geq \overline{F} \quad \forall t \geq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{6.3}$$

where $\overline{F}$ is a threshold for the forest stock. The donor community gives a constant subsidy equal to $\overline{S}/r$ if the deforestation policy applied by the aid recipient country is such that the forest stock is greater than the threshold $\overline{F}$ along the whole infinite planning period. Otherwise, the donor community penalizes the forestry country applying a zero subsidy. Let consider, for example, $\overline{F} = F^{2*}$. That is, the donor community is willing to give a constant subsidy whenever the forest country applies a deforestation policy which gives rise to a forest stock that is greater than the steady-state equilibrium associated with the subsidy which depends linearly on the forest stock.

The forestry country's maximization problem is given by:

$$
\begin{aligned}
W^3 = \max_D & \int_0^\infty e^{-rt}[R(D,F) + S]\, dt \\
\text{s.t.} \quad & \dot{F} = -D, \quad F(0) = F_0, \\
& D \geq 0, \\
& S \quad \text{given in} \quad (6.3).
\end{aligned}
$$

Next proposition characterizes the optimal paths of the forest stock and the deforestation rate when the forest stock remains above the threshold level, F^{2*}, and therefore the forestry country receives the subsidy $\overline{S}/r$.

PROPOSITION 6.7 *Assuming interior solutions, the optimal forest stock, deforestation rate and shadow-price paths satisfy the following expressions:*

$$
\begin{aligned}
\lambda^3 &= \frac{\partial R}{\partial D}(D,F), \\
\dot{F} &= -D, \quad F(0) = F_0, \\
\dot{\lambda}^3 &= r\lambda^3 - \frac{\partial R}{\partial F}(D,F) - \mu, \\
& \mu \geq 0, \; F \geq F^{2*}, \; \mu(F - F^{2*}) = 0,
\end{aligned}
$$

where λ^3 denotes the costate variable associated with the forest stock and μ denotes the Lagrange multiplier associated with the inequality $F(t) \geq F^{2*}$, $\forall t \geq 0$.

Let denote by $F^3(t), D^3(t)$ the optimal paths of the forest stock and the deforestation rate, respectively.

The following three situations can arise:

1. *If $\mu(t) = 0 \ \forall t \geq 0$, then $F^3(t) = F^1(t) \forall t \geq 0$ and $D^3(t) = D^1(t) \ \forall t \geq 0$.*

2. *If at the initial time $t = 0$, $\mu(0) \neq 0$, then $\mu(t) \neq 0 \ \forall t \geq 0$ and $F^3(t) = F^{2*} \ \forall t \geq 0$.*

3 If $\mu(0) = 0$, then there exists a time interval $[0, \tilde{t}]$ in which μ remains null and $F^3(t) = F^1(t)\ \forall t \in [0, \tilde{t}]$. Time $\tilde{t}$ can be infinite (Case 1 above) or finite if there exists $\tilde{t}$ such that $F^1(\tilde{t}) = F^{2*}$.

Therefore, the optimal paths of the forest stock and the deforestation rate can be written as follows:

$$F^3(t) = \begin{cases} F^1(t) & \text{if} \quad t \in [0, \tilde{t}] \quad \text{such that} \quad F^1(\tilde{t}) = F^{2*}, \\ F^{2*} & \text{if} \quad t \geq \tilde{t}, \end{cases} \tag{6.4}$$

$$D^3(t) = \begin{cases} D^1(t) & \text{if} \quad t \in [0, \tilde{t}] \quad \text{such that} \quad F^1(\tilde{t}) = F^{2*}, \\ 0 & \text{if} \quad t \geq \tilde{t}. \end{cases}$$

Proof. The hamiltonian associated with the optimization problem is given by:

$$H^3(F, D, \lambda^3, \mu) = R(F, D) + \overline{S} - \lambda^3 D + \mu(F - F^{2*}).$$

Applying Pontryagin's Maximum Principle, taking into account the inequality $F(t) \geq F^{2*}$, the optimality conditions above can be easily derived.

In the first case of the three situations listed before the optimality conditions in Proposition 6.7 reduce to those in Proposition 6.1 corresponding to the basic model.

Let us note that the second case can only apply if the initial size of the forest, F_0, equals F^{2*}. If $\mu(0) \neq 0$, then $F_0 = F^{2*}$ and since $\dot{F} = -D$ and $D \geq 0$, we can deduce that $F^3(t) \leq F^{2*}$ for all $t \geq 0$. Since the constraint $F^3(t) \geq F^{2*}$ has to be satisfied, the only possibility is to have $F^3(t) = F^{2*}\ \forall t \geq 0$.

In the third case $\mu(0) = 0$, implies $F_0 \geq F^{2*}$. Therefore, by a continuity argument, it can be ensured that the forest stock remains above F^{2*} during a time interval along which $\mu = 0$ and $F^3 = F^1$. Since F^3 is a decreasing function of time we have two possibilities. First, $F^{3*} > F^{2*}$ and therefore, $F^3(t) = F^1(t)$ for all $t \geq 0$. Second, there exists a time $\tilde{t}$ for which $F^3(\tilde{t}) = F^{2*}$. Then, $\mu(t) = 0$, $F^3(t) = F^1(t)\ \forall t \in [0, \tilde{t})$ and $F^3(t) = F^{2*}$, $\forall t \geq \tilde{t}$. ■

Figure 6.1 shows one possible optimal path of the forest stock for model 3. In this case the initial size of the forest $F_0 = 2$ is greater than $F^{2*} = 1$, which at the same time is greater than $F^{1*} = 1/2$ and therefore, $F^3(t)$ attains F^{2*} at time $\tilde{t} = 0.5493$. The optimal path of the forest stock corresponding to the basic model is denoted by a continuous line, while that of model 3 is denoted by stars.

REMARK 6.8 *It is easy to prove that the optimality conditions established in Proposition 6.7 are the same when the follower uses either an open-loop or a Markovian information structure.*

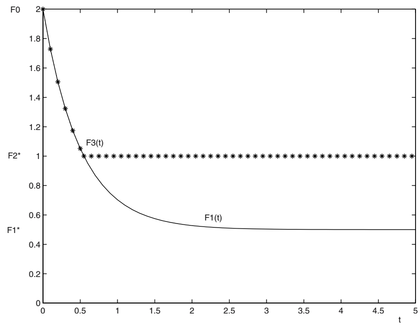

Figure 6.1. Optimal time path of the forest stock: third model

COROLLARY 6.9 *The steady-state equilibrium of the forest stock denoted by F^{3*} satisfies the following equation:*

$$\frac{\partial R}{\partial D}(0, F^{3*}) = \frac{1}{r}\left[\frac{\partial R}{\partial F}(0, F^{3*}) + \mu\right]. \tag{6.5}$$

From the previous proposition we have:

$$F^{3*} = \begin{cases} F^{2*} & \textit{if} \quad F^{2*} \geq F^{1*}, \\ F^{1*} & \textit{if} \quad F^{2*} \leq F^{1*}. \end{cases}$$

Therefore, the solution F^3 coincides with F^1 when $F^{2*} \leq F^{1*}$ or is given by (6.4) if $F^{2*} \geq F^{1*}$.

3. Comparison of forest exploitation in short and long runs

In this section we focus, firstly, on the comparison of the forest stock steady-state equilibria of the different models we are dealing with. Secondly, we compare the optimal paths of the forest stock and the deforestation rate.

Corollary 6.9 shows how the steady-state of the forest stock for the third model compares with the steady-states of the other two models. Next proposition establishes necessary and sufficient conditions to guarantee that the steady-state of the forest stock in the basic model is lower than the forest stock in the second model.

PROPOSITION 6.10 *The steady-state of the forest stock of the basic model and the second model compares as follows:*

$$F^{2*} > F^{1*} \Leftrightarrow \frac{\partial^2 R}{\partial D \partial F}(0, F) - \frac{1}{r}\frac{\partial^2 R}{\partial F^2}(0, F) > 0. \tag{6.6}$$

Proof. Let define the auxiliary function

$$f(F) = \frac{\partial R}{\partial D}(0, F) - \frac{1}{r}\frac{\partial R}{\partial F}(0, F).$$

From the expressions characterizing the steady-state equilibria of the forest stock for the basic and the second models given in Corollaries 6.2 and 6.5, respectively, we have:

$$f(F^{1*}) = 0, \quad f(F^{2*}) = \frac{a}{r} > 0.$$

Therefore, $F^{2*} > F^{1*}$ if and only if f is an increasing function in F, which is equivalent to the right-hand-side in (6.6). ■

Figure 6.2 shows the two possible scenarios. If f is increasing in F (function f_1 in the figure), then $F^{2*} > F^{1*}$. On the contrary, if f is decreasing in F (function f_2 in the figure), then $F^{1*} > F^{2*}$.

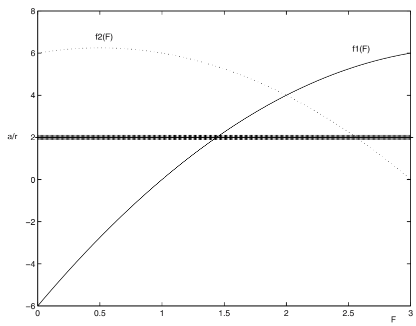

Figure 6.2. Comparison steady-state forest stock: basic and second models

Proposition 6.10 establishes that the donor community attains its objective of a better forest preservation in the long run by subsidizing the forestry country only if condition in (6.6) applies.

REMARK 6.11 *The Hamiltonians H^1, H^2 associated with the basic and second models, respectively are concave in variables (F, D) jointly if and only if the following conditions apply:*

$$\begin{aligned} &\frac{\partial^2 R}{\partial D^2}(F, D) < 0, \quad \frac{\partial^2 R}{\partial F^2}(F, D) < 0, \\ &\frac{\partial^2 R}{\partial D^2}(F, D)\frac{\partial^2 R}{\partial F^2}(F, D) - \left(\frac{\partial^2 R}{\partial D \partial F}(F, D)\right)^2 > 0. \end{aligned} \tag{6.7}$$

If these conditions together with the transversality conditions are satisfied, then the necessary conditions for optimality are also sufficient.

Under conditions in (6.7), we have the following sufficient condition:

$$\text{If} \quad \frac{\partial^2 R}{\partial D \partial F}(F, D) > 0, \quad \text{then} \quad F^{2*} > F^{1*}.$$

We can deduce that if sufficient conditions for the existence of optimal solution hold, complementarity between D and F implies $F^{2*} > F^{1*}$. Let note that this is only a sufficient condition.

In the particular case of the Van Soest & Lensink's model (2000) (analyzed also in Martín-Herrán et al. (2004)) where goods are substitutes, the forestry country's revenue function can be written as:

$$R(F, D) = -a_1 F^2 - a_2 F D - a_3 D^2 + a_4 F + a_5 D + a_6, \tag{6.8}$$

where $a_i > 0$, $i \in \{1, \ldots, 6\}$. With this specification all the second-order partial derivatives are negative. The hamiltonian associated with the forestry country's dynamic optimization problem is concave in (F, D) jointly if and only if $4a_3 a_1 - a_2^2 > 0$.

The forest stock for the basic model with the Van Soest & Lensink's specification can be written as follows:

$$F^{p*} = \frac{a_4 - r a_5}{2a_1 - r a_2},$$

where the superscript p denotes particular specification. Conditions $a_4 - r a_5 > 0, 2a_1 - r a_2 > 0$ guarantee that the optimal trajectory of the forest stock converges towards the positive steady-state F^{p*}.

Let us note that condition (6.6) ensuring $F^{2*} > F^{1*}$ in this case reads $2a_1 - r a_2 > 0$. For this quadratic formulation of the forestry country's revenue function and if goods are substitutes, a discount rate r small enough implies $F^{2*} > F^{1*}$.

We center now on the comparison of the optimal paths of the forest stock for the different models. Unfortunately we are not able to do this comparison for a general specification of the forestry country's revenue function, as we have previously done when ranking the steady-state equilibria. For that reason, we focus on a quadratic formulation. Next proposition collects the result derived from this comparison.

PROPOSITION 6.12 *In the case of a quadratic specification for the forestry country's revenue function the optimal time paths of the forest stock are given by*

$$F^i(t) = (F_0 - F^{i*})e^{-\rho t} + F^{i*}, \quad i = 1, 2,$$

where ρ is the positive solution of the characteristic equation associated with the system of linear ordinary differential equations describing the time evolution of the forest stock and its shadow price, and

$$F^3(t) = \begin{cases} (F_0 - F^{1*})e^{-\rho t} + F^{1*} & \text{if} \quad t \in [0, \tilde{t}] \;\; \text{such that} \;\; F^1(\tilde{t}) = F^{2*}, \\ F^{2*} \quad \text{if} \quad t \geq \tilde{t}. \end{cases}$$

Moreover, the optimal paths of the deforestation rate are

$$D^i(t) = \rho(F_0 - F^{i*})e^{-\rho t}, \quad i = 1, 2,$$

and

$$D^3(t) = \begin{cases} \rho(F_0 - F^{1*})e^{-\rho t} & \text{if} \quad t \in [0, \tilde{t}] \quad \text{such that} \quad F^1(\tilde{t}) = F^{2*}, \\ 0 \quad \text{if} \quad t \geq \tilde{t}. \end{cases}$$

The following equivalence applies:

$$F^{1*} < F^{2*} \;\Leftrightarrow\; F^3(t) \leq F^2(t) \text{ for all } t \geq 0.$$

Proof. The expressions of the optimal forest paths can be easily derived solving the linear system of two ordinary differential equations corresponding to the dynamics of the forest stock and its shadow price. The expressions of the deforestation paths are straightforwardly deduced from the dynamics of the forest stock. The equivalence stated at the end of the proposition can be easily proved taking into account that $F^{1*} < F^{2*}$ is equivalent to $D^1(0) > D^2(0)$. Therefore, $F^3(t)$ is lower or greater than $F^2(t)$ along the whole time horizon depending on whether the slope of function F^2 at zero $(-D^2(0))$ is greater or lower than that of function F^1 $(-D^1(0))$. ■

Figure 6.3 shows the optimal paths of F^1, F^2 and F^3 when $F^{1*} < F^{2*}$. In this figure the optimal paths of F^1, F^2 are denoted by continuous and discontinuous lines, respectively, and F^3 by stars.

4. The donor's optimization problem

Now we turn to the leader's problem and consider the donor community's optimization problem: it aims to minimize the total amount of subsidy flowing to the forestry country but which guarantees the participation constraint of the forestry country in the aid program. In other words, under an open-loop information structure, the donor's objective is to choose the value of a and b in the second model (where the subsidy is given by $S = aF + b$) or $\overline{S}$ in the third model in order to minimize the

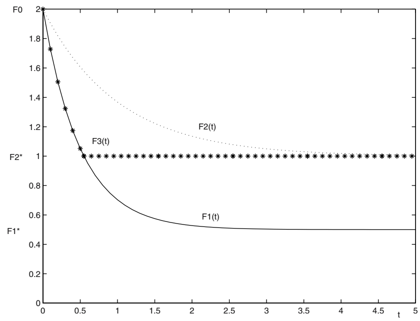

Figure 6.3. Comparison of the optimal time paths of the forest stock in the second and third models when $F^{2*} > F^{1*}$.

total subsidy to be paid to the forestry country, but which ensures at the same time that the aid recipient attains a revenue at least as greater as that it gets in the basic model. As we have previously established the leader make its optimal choices once it knows the optimal paths of forest stock and deforestation rate. Therefore, the donor's optimization problem reads:

$$V^2 = \min_{a,b} \int_0^\infty e^{-rt}(aF^2(t) + b)\, dt \tag{6.9}$$

$$\text{s.t.} \quad W^2 \geq W^1, \tag{6.10}$$

or

$$V^3 = \min_{\overline{S}} \int_0^\infty e^{-rt}\overline{S}\, dt = \min_{\overline{S}} \frac{\overline{S}}{r} \tag{6.11}$$

$$\text{s.t.} \quad W^3 \geq W^1. \tag{6.12}$$

The following two propositions characterize implicitly the optimal expressions of the parameters under the donor's control.

PROPOSITION 6.13 *The donor chooses a, b in problem (6.9)-(6.10) such that the following expression is satisfied:*

$$\int_0^\infty e^{-rt}(aF^2(t)+b)\, dt = \int_0^\infty e^{-rt} \left[R(D^1(t), F^1(t)) - R(D^2(t), F^2(t))\right] dt.$$

Proof. Since the donor in fact wants to minimize the amount of subsidy it gives to the forestry country $\int_0^\infty e^{-rt}(aF^2(t)+b)\, dt$, it is easy to deduce that it is going to choose a, b such that $W^2 = W^1$. That is,

$$W^2 = \int_0^\infty e^{-rt} R(D^2(t), F^2(t))\, dt + \int_0^\infty e^{-rt}(aF^2(t) + b)\, dt =$$
$$W^1 = \int_0^\infty e^{-rt} R(D^1(t), F^1(t))\, dt.$$

Therefore, the total amount of subsidy in this case flowing from the donor community to the forestry country is given by:

$$\int_0^\infty e^{-rt} \left[R(D^1(t), F^1(t)) - R(D^2(t), F^2(t))\right] dt.$$

■

PROPOSITION 6.14 *The optimal expression of $\overline{S}$, solution of the problem (6.11)-(6.12), satisfies the following equation:*

$$\frac{\overline{S}}{r} = \int_0^\infty e^{-rt} \left[R(D^1(t), F^1(t)) - R(D^3(t), F^3(t))\right] dt.$$

Proof. Following the same argument than in the previous proposition the donor is going to choose $\overline{S}$ such that $W^3 = W^1$. In other terms, $\overline{S}$ satisfies the following expression:

$$W^3 = \int_0^\infty e^{-rt} R(D^3(t), F^3(t))\, dt + \frac{\overline{S}}{r} = W^1 = \int_0^\infty e^{-rt} R(D^1(t), F^1(t))\, dt.$$

■

4.1 Comparison of the subsidy schemes from the donor's point of view

From the previous two propositions it can be established which one of the two proposed compensation functions the donor will choose in order to minimize the amount of transfers flowing to the forestry country. Next corollary shows this result.

COROLLARY 6.15 *The donor community offers the subsidy in (6.3) if and only if*

$$\int_0^\infty e^{-rt} \left[R(D^2(t), F^2(t)) - R(D^3(t), F^3(t))\right] dt < 0. \tag{6.13}$$

Proof. It suffices to compare the amount of the total subsidy that the donor gives to the forestry country under the hypotheses of the models 2 and 3 (see Propositions 6.13 and 6.14). ■

Remark 6.16 *From the aid recipient's point of view it is clear that* $W^2 = W^3$. *Moreover, when a quadratic forestry country's revenue function is considered and if* $F^{2*} > F^{1*}$ *we have previously established (see Proposition 6.12) that the affine subsidy in the forest (second model) gives a greater stock of the forest both in the short and long runs than the subsidy scheme proposed in the third model, in fact,* $F^2(t) \geq F^3(t)$ *for all* t.

Unfortunately, it cannot be established whether condition (6.13) in Corollary 6.15 is satisfied or not for a general specification of the forestry country's revenue function. However, for a quadratic specification next proposition collects this result.

Proposition 6.17 *If the forestry country's revenue function is described by the quadratic function (6.8), then the donor community offers always the subsidy given by (6.3).*

Proof. Let denote by a^* the value of a such that $F_0 = F^{2*}$ and define the following function: $H : [0, a^*] \to \mathbb{R}$,

$$H(a) = \int_0^\infty e^{-rt} \left[R(D^2(t,a), F^2(t,a)) - R(D^3(t), F^3(t)) \right] dt.$$

Function H has the following properties:

- H is a continuous function in $[0, a^*]$.
- $H(0) = 0$. This result is obvious since in this case $F^2(t) = F^3(t) = F^1(t)$, $D^2(t) = D^3(t) = D^1(t)$ for all $t \geq 0$ and therefore $R(D^2(t), F^2(t)) - R(D^3(t), F^3(t)) = 0$ for all $t \geq 0$.
- $H(a^*) = 0$. This result is also straightforward because in this case $F^2(t) = F^3(t) = F^{2*}$ for all $t \geq 0$.
- Moreover it can be proved that equation $H(a) = 0$ has two solutions, which are given by $a = 0$ and $a = e^\theta a^*$ with θ satisfying the following equation
$$Ae^{\theta \frac{r+\rho}{\rho}} + Be^\theta = 0,$$
where $A + B = 0$. The only solution of this equation is $\theta = 0$ and then $a = a^*$.

The following properties of $H'(a)$ can also be proved:

- H' is a continuous function in $(0, a^*)$.
- $H'(0) = 0$.

- Since $F_0\, ra_2 - 2\, a_1\, F_0 - ra_5 + a_4 \leq 0$ because $F_0 \geq F^{1*}$ and $ra_2 - 2a_1 < 0$, we have that

$$H'(a^*) = \frac{(F_0\, ra_2 - 2\, a_1\, F_0 - ra_5 + a_4)a_3}{(ra_3 + \sqrt{a_3(r^2 a_3 - 2ra_2 + 4a_1})(ra_2 - 2a_1)} > 0.$$

The properties on H and H' allow us to conclude that $H(a) \leq 0$ if $0 \leq a \leq a^*$. Therefore, the total subsidy given in the third model is lower than the one given in the second model. ■

Propositions 6.12 and 6.17 show that if the forestry country's revenue function is quadratic there exists a trade-off between minimizing the donor's community budget and maximizing the short and long run size of the forest.

5. Conclusions

In this paper we propose a differential game played à la Stackelberg between the donor community (the leader) and the aid recipient (the follower) to analyze the effect of foreign aid donation on forest conservation both in the short and long runs. In the different scenarios analyzed, we assume that the leader plays open-loop strategies while the follower adopts feedback strategies. This information structure leads to time-consistent optimal solutions. The donor community chooses the amount of transfers to send to the aid recipient or forestry country. The latter decides the rate of deforestation.

The Stackelberg game between the donor community and the aid recipient is analyzed under different specifications for the compensation or subsidy function. Our aim is to study the impacts on environmental and economic issues when the players agree on different specifications for the subsidy flowing from the donor community to the forestry country.

We establish necessary and sufficient conditions under which the steady-state of the forest stock is greater when a subsidy mechanism is implemented than when the forestry country does not receive any foreign aid. From the environmental point of view we also compare the optimal paths of the forest stock for the different scenarios, when a quadratic specification for the forestry country's revenue function is considered.

From the economic point of view, we show that for the aid recipient, revenues are the same for both specifications of the subsidy function. We also determine a necessary and sufficient condition that establishes which one of the two proposed compensation functions the donor will choose in order to minimize the amount of transfers flowing to the forestry country. For a quadratic specification of the forestry country's revenue function it is shown that there is a trade-off between minimize the amount of subsidy and maximize the forest stock.

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Chapter 7

CHARACTERIZING DYNAMIC IRRIGATION POLICIES VIA GREEN'S THEOREM

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Abstract We derive irrigation management schemes accounting for the dynamic response of biomass yield to salinity and soil moisture as well as for the cost of irrigation water. The simple turnpike structure of the optimal policy is characterized using Green's Theorem. The analysis applies to systems of arbitrary end conditions. A numerical application of the turnpike solution to sunflower growth under arid conditions reveals that by selecting the proper mix of fresh and saline water for irrigation, significant savings on the use of freshwater can be achieved with negligible loss of income.

1. Introduction

Increasing water scarcity and the alarming deterioration in the quality of many freshwater resources call for improved irrigation efficiency to sustain viable agriculture over vast areas around the globe. Using water of lesser quality and continuously adjusting irrigation rates to the varying needs of the growing

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plants can save a significant fraction of freshwater used in traditional irrigation practices. The trade-offs between the cost of water and the essential contribution of suitable soil moisture to biomass growth give rise to optimization problems that are both theoretically interesting and practically relevant.

An Optimal Control analysis of a dynamic irrigation problem accounting for soil moisture and biomass growth dynamics as well as for the associated cost of irrigation water is presented in Shani, Tsur and Zemel (2004), where the ensuing optimal policy is shown to take a particularly simple form. The policy is defined in terms of two parameters: a turnpike soil moisture $\hat{\theta}$ and a stopping date, such that the optimal moisture process, $\theta(t)$, must be brought from its initial level to the turnpike $\hat{\theta}$ *as rapidly as possible* and maintained at that level until the stopping date, at which time irrigation ceases and the plants are left to grow on the remaining soil moisture until the time of harvest. This simple turnpike behavior is neither unique to the irrigation problem nor is it rare in the dynamic optimization literature. Similar characterizations have been derived for a large variety of economic and management problems (see e.g. Vidale and Wolfe (1957), Sethi (1974), Haruvy, Prasad and Sethi (2003)) and explained by geometrical considerations using Green's Theorem (see Miele (1962), Sethi (1973), Sethi (1977), Sethi and Thompson (2000)). In this method, one eliminates the control, replaces the line integral in the objective functional by an area integral, and compares the values obtained from any two feasible policies by analyzing the sign of the integrand over the area encircled by the trajectories corresponding to these policies. The similar characteristics of the optimal irrigation policy suggest that this problem can also be analyzed in terms of Green's theorem. It turns out that certain features of the irrigation problem render the application of Green's Theorem non trivial in this case, and novel considerations must extend the method to derive the optimal policy.

The purpose of the present work is twofold. First, we adapt the standard Green's Theorem analysis to more complex situations of arbitrary end conditions. Applying the method to the irrigation problem, we explain the simple structure of the optimal policy and provide new links relating the method and Optimal Control theory. Second, we extend the original model of Shani, Tsur and Zemel (2004) by incorporating salinity effects. In many arid regions, brackish water is available to replace scarce freshwater resources for irrigation purposes. The growers, however, must also account for the reduction in yield implied by increased salinity in the root zone. We find that by carefully adjusting the salinity of the irrigation water mix and the parameters of the turnpike policy, the growers can increase the net income from their crop and, more importantly, mitigate freshwater scarcity (manifest in terms of exogenous freshwater quota imposed for each growing season) with only minor income loss.

2. The irrigation management problem

Let $m(t)$ represent the plant biomass at time $t \in [0,T]$, where T denotes the time from emergence to harvest. Marketable yield is derived from the biomass according to the increasing yield function $y(m)$ If yield and biomass are the same, then $y(m) = m$. Often, however, $y(m)$ vanishes for m below some threshold level, but above this level it increases at a rate that exceeds that of the biomass. At each point of time the biomass grows at a rate that depends on the current biomass state as well as on a host of factors including availability of water, salinity, sunlight intensity, day length and ambient temperature. Some of these factors (e.g. soil water content and salinity) can be controlled by the growers who derive irrigation water from two alternative sources: a costly supply of freshwater, and a cheaper supply of saline water from a local aquifer or a wastewater recycling plant. We assume that the mix of water from the two sources is determined at the beginning of the growing season and this fixes the salinity of irrigation water for the entire growing season.

Using the fraction s of saline water as a proxy for the salinity of the irrigation water mix, denoting by $\theta(t)$ the water content in the root zone, and taking all factors that are beyond the growers' control as given, the plant biomass rate of growth depends on $\theta(t)$ and $m(t)$ according to

$$\frac{dm(t)}{dt} \equiv \dot{m}(t) = q(s)g(\theta(t))h(m(t)) \tag{7.1}$$

Implicit in (7.1) is the assumption that the biomass growth rate can be factored to terms depending on s, θ and m separately. For fresh water, the function q is normalized at $q(0) = 1$. Since low salinity bears minor effects while excessive salinity hampers growth, we assume that q is decreasing, with $q'(0) = 0$. The functions g and h are assumed to be strictly concave in their respective arguments, with g vanishing at the wilting point $\theta_{\min}$ and obtaining a maximum at some value $\theta_{\max}$ (too much moisture harms growth). Thus, $g(\theta_{\min}) = 0$, $g'(\theta_{\max}) = 0$, $g'(\theta) > 0$ for $\theta \in (\theta_{\min}, \theta_{\max})$ and $g''(\theta) < 0$ for all θ. Since it is never optimal to increase moisture above the maximum level, we restrict attention to processes with $\theta \leq \theta_{\max}$.

The dynamics of water content in the root zone is determined by mass conservation, implying that the change in $\theta(t)$ at each point of time must equal water input through irrigation, $x(t)$, minus losses due to evapotranspiration, (ET) and drainage (D). (Rainfall can also be incorporated in this framework, but to focus on irrigation management we assume no rainfall.)

Evapotranspiration rate is specified as

$$ET(\theta, m) = \beta q(s)g(\theta)f(m) \tag{7.2}$$

where the coefficient β depends only on climatic conditions and is independent of s, m and θ and $0 \leq f(m) \leq 1$ is a crop scale factor representing the degree of leaves exposure to solar radiation (Hanks (1985)). The use of the same factor $q(s)g(\theta)$ in (7.1) and (7.2) is based on the linear relation between biomass production and evapotranspiration (see deWit (1958)).

The rate of water drainage $D(\theta)$ is assumed to be positive, increasing and convex for the relevant soil moisture range. When all the flow rates are measured in mm day^{-1}and θ is a dimensionless water concentration, the soil water balance can be specified as

$$Z\dot{\theta}(t) = x(t) - \beta q(s)g(\theta(t))f(m(t)) - D(\theta(t)) \tag{7.3}$$

where Z is the root depth and $Z\theta(t)$ measures the total amount of water in the root zone (mm).

Let W_f and W_s denote unit prices of fresh and saline water, respectively, assumed fixed throughout the growing season. For the chosen fraction s of saline water, the growers pay the price $W = (1-s)W_f + sW_s$. At harvest time T they also receive the revenue $py(m(T))$, where p is the output price. For a growing season that lasts a few months we can ignore discounting, and the return to water (excluding expenses on inputs other than water) is $py(m(T)) - W\int_0^T x(t)dt$.

Given the salinity s, we define the relative cost of water $w = W/p$ and formulate the irrigation management problem as finding the irrigation policy $\{x(t), 0 \leq t \leq T\}$ that maximizes

$$V(m_0, \theta_0) = Max_{\{x(t)\}} \left\{ \int_0^T -wx(t)dt + y(m(T)) \right\} \tag{7.4}$$

subject to (7.1), (7.3), $m(0) = m_0$, $\theta(0) = \theta_0$ and $0 \leq x(t) \leq \bar{x}$, where $m_0 > 0$ and $\theta_{\min} \leq \theta_0 \leq \theta_{\max}$ are the initial biomass and soil moisture levels and $\bar{x}$ is an upper bound on the feasible irrigation rate, reflecting physical constraints on irrigation equipment or on soil water absorption capacity. The upper bound $\bar{x}$ exceeds the water loss terms of (7.3) throughout the relevant ranges of m and θ so $\theta(t)$ increases when $x = \bar{x}$ (violations of equivalent assumptions in related contexts are discussed in Sethi (1977)). In the following section we characterize the optimal irrigation policy corresponding to (7.4), with one control variable (the irrigation rate x) and two state variables (the biomass m and the moisture θ). The optimal water mix and the corresponding salinity are considered in a later section.

3. Solution by Green's Theorem

An Optimal Control analysis of the optimization problem (7.4) is presented in Shani, Tsur and Zemel (2004), where the optimal turnpike policy is derived and explained in terms of the linear dependence of the objective and state equations on the control variable x, which gives rise to the typical Most Rapid Approach Path (Spence and Starrett (1975)). The similarity of this policy to the characteristic behavior derived by Sethi and coworkers for a large variety of optimization problems using Green's Theorem suggests that the irrigation problem can also be analyzed by means of this method. We note, however, that certain features of the irrigation problem require proceeding beyond the standard application of Green's Theorem in order to derive the optimal policy.

First, the problem involves two state variables m and θ hence the relevant (state-time) space is three-dimensional. In the standard approach, one would employ Stokes' Theorem instead of Green's, as in Sethi (1976). Here, however, we observe in (7.1) that the biomass process $m(t)$ evolves monotonically in time, hence m can serve as an effective time index, reducing the analysis to the two-dimensional (m, θ) state-space.

Second, the final values of the state variables are free in this problem. Thus, two arbitrary feasible trajectories need not end at the same point and joining them may not give rise to the closed trajectory that the method requires. Moreover, the determination of the turnpike moisture state is further complicated by its dependence on the transversality conditions corresponding to the free endpoints. Finally, fixing the endpoint of the trajectories in the (m, θ) space does not determine the time it takes the processes to get there. Comparing the values derived from trajectories of different durations does not provide the required information, and one must impose the correct duration T to obtain a feasible plan. As we shall see, this constraint introduces an additional term to the effective objective function, which turns out to be instrumental in the determination of the turnpike.

We proceed now to characterize the optimal irrigation policy using Green's Theorem while accounting for the new features listed above. Although the derivation is carried out for the specific irrigation problem presented above, we note that the considerations apply to a large class of optimization problems that are linear in the control or, more generally, satisfy the conditions of Corollary 2.1 of Sethi (1977) and have arbitrary end conditions (free or fixed duration, free or fixed final states, time or state dependent salvage functions etc.). In particular, we study the properties of the family of trajectories which consist of at most three distinct segments: (i) a "nearest approach" segment, leading from the initial state (m_0, θ_0) to some arbitrary turnpike moisture state $\hat{\theta}$ with $x(t) = 0$ if $\theta_0 > \hat{\theta}$ and $x(t) = \bar{x}$ if $\theta_0 < \hat{\theta}$; (ii) a singular segment, which maintains the moisture process fixed at the turnpike state $\hat{\theta}$ by setting

$x(t) = \beta q(s)g(\hat{\theta})f(m(t)) + D(\hat{\theta})$; and (iii) a "nearest exit" segment leading from the turnpike to some final state with $x = 0$ or $x = \bar{x}$. This family, which includes also trajectories in which one or two segments are skipped (e.g. when the turnpike state coincides with the initial or final moisture levels), is termed the family of *turnpike processes*. It turns out that turnpike processes include the optimal policy as well as the processes of extreme duration leading to any given final state. Since the analysis involves processes of arbitrary duration, we shall refer to trajectories consistent with all the constraints of problem (7.4) except for the duration T, as *dynamically feasible*. Evidently, all turnpike processes are dynamically feasible.

We omit, for brevity, the salinity argument from q and the time index from all functions. We use (7.1) and (7.3) to eliminate the control and write the value obtained from any dynamically feasible trajectory Γ initiated at (m_0, θ_0) and ending at some arbitrary final state (m_F, θ_F) with $m_F \geq m_0$ and $\theta_F \leq \theta_{\max}$ as:

$$V_\Gamma = \int_\Gamma -\{[\frac{wD(\theta)}{qg(\theta)h(m)} + \frac{w\beta f(m)}{h(m)}]dm + wZd\theta\} + y(m_F) \tag{7.5}$$

(cf. Hermes and Haynes (1963)). It follows that the difference between the values obtained from any two dynamically feasible trajectories with the same initial and final states (but not necessarily of the same duration) can be evaluated, using Green's Theorem, by

$$\Delta V = \int\int_\sigma \frac{g'(\theta)}{qh(m)g^2(\theta)} w\xi(\theta)d\sigma \tag{7.6}$$

where σ is the area encircled by the graphs of the two trajectories and

$$\xi(\theta) = g(\theta)D'(\theta)/g'(\theta) - D(\theta) \tag{7.7}$$

is an increasing function.

Searching for the roots of the integrand of (7.6) will not yield the correct turnpike state because the trajectories may be, as noted above, of different durations. We can, however, follow the same procedure to obtain the time difference. Recalling (7.1), we write

$$T = \int_0^T dt = \int_\Gamma \frac{dm}{qg(\theta)h(m)} \tag{7.8}$$

Using Green's Theorem again, we find

$$\Delta T = \int\int_\sigma \frac{g'(\theta)}{qh(m)g^2(\theta)} d\sigma \tag{7.9}$$

Hence, for any given constant H

$$\Delta V - H\Delta T = \int\int_{\sigma} \frac{g'(\theta)}{qh(m)g^2(\theta)}[w\xi(\theta) - H]d\sigma \tag{7.10}$$

Since the integrand of (7.9) is positive for all $\theta < \theta_{max}$ it follows that the minimal duration of a dynamically feasible process leading to (m_F, θ_F) must correspond to a turnpike process. To see this, extend the increasing nearest approach segment all the way to θ_{max}. Consider now the decreasing nearest exit segment ending at (m_F, θ_F). If the two segments cross below θ_{max} the trajectory comprising these segments is the dynamically feasible process of shortest duration connecting the initial and final states, since any other dynamically feasible process with the same endpoints must lie below its graph. If the two segments do not cross below θ_{max} the minimum duration is obtained by the three-segment turnpike process comprising them and the singular segment connecting them at θ_{max}.

Similar considerations involving the decreasing nearest approach segment and the increasing nearest exit segment ending at (m_F, θ_F), imply that the maximum dynamically feasible duration is also obtained by a turnpike process. Moreover, the durations of turnpike processes ending at (m_F, θ_F) vary continuously with the turnpike level of their corresponding singular segments. We have, therefore established

PROPOSITION 7.1 *For any dynamically feasible process ending at (m_F, θ_F) there exists a corresponding turnpike process of the same duration and the same final state.*

The values obtained from the two processes of Proposition 7.1 can now be compared:

PROPOSITION 7.2 *The corresponding turnpike process of any dynamically feasible process yields higher (or equal, if the latter process is itself a turnpike process) value than the original process.*

PROOF: Choose the constant H of (7.10) as $H = w\xi(\hat{\theta})$, where $\hat{\theta}$ is the turnpike state associated with the singular segment of the turnpike process. With $\Delta T = 0$, (7.10) reduces to $\Delta V = \int\int_{\sigma} \frac{g'(\theta)}{qh(m)g^2(\theta)}[w\xi(\theta) - H]d\sigma$. Recalling that ξ is increasing, we see that the integrand is positive for all states above the singular segment and negative below it (Figure 7.1). With the counterclockwise convention for closed line integrals, we see that for each of the closed sub-areas encircled by the two trajectories, the turnpike process yields a larger value. Thus, the result follows for the entire trajectories as well. □

The characterization of the optimal policy follows immediately from Proposition 7.2:

PROPOSITION 7.3 *The optimal policy must be a turnpike process.*

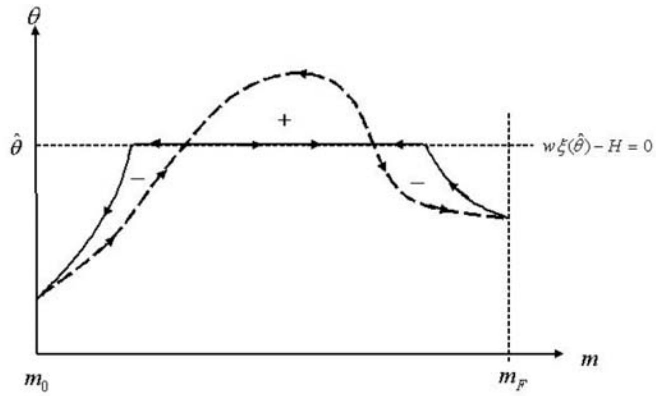

Figure 7.1. Comparing the values obtained from an arbitrary dynamically feasible process (dashed line) and the corresponding turnpike process (solid line). The "+" and "−" symbols indicate the sign of the integrand, and the arrows indicate the direction of the line integration in each of the closed sub-areas. The constant H is adjusted so that the integrand vanishes on the singular segment.

The role of the constant H ought to be explained. In Shani, Tsur and Zemel (2004) we show that the value of H corresponding to the optimal policy equals the constant value of the Hamiltonian under this policy. From Optimal Control theory we know that the Hamiltonian can be regarded as the shadow price associated with a marginal increase in T. In the context of (7.10), this shadow price assumes the role of the Lagrange multiplier associated with the constraint that the duration must be fixed at the given time T. Indeed, with $\Delta T = 0$ the term including H in the right-hand-side of (7.10) gives a vanishing contribution for any value of this constant (see (7.9)). Nevertheless, the particular choice of the Hamiltonian value for H ensures that the integrand has a definite sign for each sub-area, as required by Green's Theorem method.

While Proposition 7.3 characterizes the optimal policy as a turnpike process, it does not provide specific information on the nearest exit segment. In Shani, Tsur and Zemel (2004) we establish that this segment must be decreasing, so it is optimal to cease irrigation prior to T and reduce the moisture level towards the end of the growing season. This is indeed a common practice among growers. The property is most easily demonstrated via the transversality condition associated with the free final moisture state; it will not be further considered here.

Unlike previous work based on Green's Theorem, the turnpike state is not explicitly specified in this problem, because neither the final state nor the Hamiltonian are a priori given. In fact, the full dynamics of the state and costate variables, as well as the relevant transversality conditions must be invoked to derive the turnpike and the final states. Nevertheless, the power of the method to derive and explain the simple structure of the optimal policy is evident also for the more complicated problem considered here.

4. Salinity and scarcity

We turn now to study the effects of salinity on the optimal policy. Since $q'(0) = 0$, mixing a small amount of saline water must have a negligible effect on plant growth, yet it helps to reduce the cost of irrigation water. Higher salinity levels hamper growth and reduce yields. These trade-offs suggest an internal solution for the optimal salinity. We illustrate these trade-offs by applying the model to the growth of Ornamental sunflower (*Helianthus annuus* var dwarf yellow) in the Arava Valley in Israel. Lack of precipitation throughout the growing period and deep groundwater (120 m below soil surface) ensure that irrigation is the only source of water. The biomass and moisture dynamics are modeled using the functional specifications of Shani, Tsur and Zemel (2004):

$$\dot{m} = q(s)(1.21\Theta - 1.71\Theta^2)m(1 - m/491) \tag{7.11}$$

and

$$\dot{\theta} = [x - 0.19q(s)(1.21\Theta - 1.71\Theta^2)m(1 - m/785.6) - 3600\Theta_d^{5.73}]/600 \tag{7.12}$$

where $\Theta = (\theta - 0.09)/0.31, \Theta_d = (\theta - 0.04)/0.36$ and $q(s) = 1/\left[1 + (4s/3)^3\right]$ (Dudley and Shani (2003)).

Marketable yield for sunflowers is obtained only at biomass levels above 350 $g \cdot m^{-2}$. At the maximal biomass ($m = 491$ $g \cdot m^{-2}$) the yield comprises 80% of the biomass. Assuming a linear increase gives rise to the following yield function:

$$y(m) = \begin{cases} 0 & \text{if } m < 350\ g \cdot m^{-2} \\ 2.79(m - 350) & \text{if } m \geq 350\ g \cdot m^{-2} \end{cases} \tag{7.13}$$

The initial soil water and biomass levels were taken at $\theta_0 = 0.1$ (just above water content at the wilting point $\theta_{\min} = 0.09$, where Θ and the growth rate vanish) and $m_0 = 10$ $g \cdot m^{-2}$ (about 2% of the maximal obtainable biomass). The maximal feasible irrigation rate is $\bar{x} = 41.8$ $mm \cdot day^{-1}$ and the growing period lasts 45 days. Sunflower seeds are sold at about \$1 kg^{-1}, yielding the relative water prices of $w_f = W_f/p = 0.3$ $kg \cdot m^{-3}$ (freshwater) and $w_s = W_s/p = 0.1$ $kg \cdot m^{-3}$ (saline water).

Using fresh water only, a numerical implementation of the optimal policy based on the above specifications gave rise to the turnpike level $\hat{\theta} = 0.148$. Irrigating at the maximal rate brings soil moisture to the turnpike at $t_1 = 0.7$ day, at which time irrigation rate is tuned so as to maintain the soil water content fixed at $\hat{\theta}$ for the major part of the growing period of $T = 45$ days. However, during the last 2.8 days irrigation is avoided because the gain in yield due to continued irrigation is not sufficient to cover the cost of the water needed to maintain the high soil water content. The corresponding harvested yield is 350 $g \cdot m^{-2}$ for the optimal policy—about 10% below the maximal attainable yield. With irrigation costs of \$1020 ha^{-1} (about half of which is due to drainage), the net income (excluding labor and other inputs) from the optimal policy amounts to \$2480 ha^{-1}.

With the option to use saline water, we find that the optimal water mix is $s = 0.21$. The lower unit cost of water allows increasing slightly the turnpike moisture (to $\hat{\theta} = 0.15$) and the total amount of irrigation water so that the harvested yield increases marginally (to 351.3 $g \cdot m^{-2}$). However, this yield is obtained with a smaller irrigation bill of \$934 ha^{-1}, leaving a net income of \$2579 ha^{-1} to the growers.

The saving on freshwater may be even more important. Under the optimal mix, the saving on this precious resource amounts to 16%. Indeed, in arid regions such as the Arava valley, freshwater scarcity might dominate its nominal cost in determining the total quantity of applied irrigation. Scarcity turns into a binding constraint when growers are allocated an exogenous quota of freshwater below their use under the optimal (nonbinding) policy. The effect of the binding freshwater quota takes the form of a fixed shadow price to be added to the relative cost of water. Shani, Tsur and Zemel (2004) show that the shadow price should be adjusted so that the effective unit cost implies irrigation using exactly the allocated freshwater quota. As an example, assume that the quota amounts to only 75% of freshwater used for irrigation under the mixed-water policy discussed above. The constraint corresponds to adding a shadow price of \$0.42 m^{-3} to the nominal freshwater cost $W_f = \$0.3\ m^{-3}$. The increased effective cost implies an increase in salinity to $s = 0.282$ and an 18% decrease in the total amount of irrigation water, reducing the yield by 9% (to 318.6 $g \cdot m^{-2}$). Accounting for the smaller water bill, however, we find that the net income loss is a mere 5% (to \$2458 ha^{-1}).

It is instructive to compare this quota-bound policy with the outcome of the unbound policy based on freshwater only. Although the net incomes differ by less than 1%, the bound, mixed-water policy uses only 63% of freshwater required by the unbound policy. Indeed, if the same freshwater quota were imposed on growers without access to the saline resource, the net income would drop to \$1998 ha^{-1}, representing an income loss of nearly 20%. We see, therefore, that by carefully adjusting the turnpike policy, the growers can exploit the saline water resource to mitigate the significant losses implied by freshwater scarcity.

Finally, we remark that the same methodology can incorporate other effects of salinity. Assume, for example, that saline irrigation water is eventually drained to an underlying freshwater aquifer, or that it increases soil salinity for the following seasons, and the environmental damage is proportional to the cumulative amount of salt applied. The damage in such cases can be modeled as an additional fixed component of the unit cost of saline water. Except for changes in the numerical values of the optimal parameters, the characteristic turnpike policy and the solution methodology will not be affected.

5. Concluding comments

In a recent publication, Shani, Tsur and Zemel (2004) used Optimal Control theory to derive dynamic irrigation schemes that account for soil moisture and biomass growth dynamics as well as for the associated cost of irrigation water. The results reveal two important features: (i) Although the biomass and soil moisture dynamics are quite complex, the optimal policy displays an extremely simple turnpike behavior, and (ii) The turnpike policy is robust to a wide range of variations and extensions which, in spite of adding significant new considerations to the optimization tradeoffs, can modify the numerical values of the optimal parameters but not the characteristic behavior of the optimal policy.

The analysis applied here, based on Green's Theorem, provides a simple and elegant explanation to both features. The turnpike behavior follows from simple geometric considerations with little recourse to the complexities of the dynamic system. Evidently, the original method requires some modifications to cope with the new properties of the optimization problem considered here. In particular, we exploit the observation that turnpike trajectories not only provide the maximum for the objective, but also give rise to the minimal and maximal dynamically feasible process durations, as well as to the continuum of durations between these extremes. In fact, this observation follows from the same geometric considerations used to compare the objectives of competing trajectories.

We also note the simple relation between the Hamiltonian and the optimal turnpike state. The Hamiltonian function, which is the key element of Optimal Control theory, is typically absent in Green's Theorem analysis. Its role in the present formulation, thus, provides an interesting link between these two complementary methodologies of deriving optimal dynamic solutions.

It has been suggested (Haynes (1966), Sethi (1976)) that simple turnpike behavior can characterize the solutions of a variety of complex multi-dimensional dynamic optimization problems. Indeed, recent economic studies of optimal R&D strategies in the context of resource scarcity and economic growth (see Tsur and Zemel (2000), Tsur and Zemel (2002), Tsur and Zemel (2003), Tsur and Zemel (2004)) reveal interesting examples of such behavior for multi-dimensional infinite horizon problems with discounted utilities. Analyzing such problems in terms of Green or Stokes' Theorems (as in Haynes (1966) or Sethi (1976)) remains a challenge for future research.

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APPLICATIONS IN ECONOMICS AND FINANCE

Chapter 8

VOLATILITY FORECASTS AND THE PROFITABILITY OF AUTOMATED TRADING STRATEGIES

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Abstract Traditional approaches to forecast option prices and implement trading strategies make use of implied volatilities. Noh, Engle, and Kane (1994) propose a different approach. Based on conditional variance models of the GARCH type they forecast volatility and use these forecasts to predict future option prices. In combination with simple trading rules Noh et al. evaluate the profitability of these forecasts for the S&P 500 index. In this paper we take up their approach and apply it to Bund future options. We show that volatility forecasts together with simple option trading strategies create value. The profits can be significant even when transaction costs are taken into account.

1. Introduction

Traditional models of option price forecasts use implied volatilities to predict future prices. The theoretical basis for this approach is rooted in option pricing theory. In efficient capital markets volatility must be reflected in option prices. While there is a large body of literature to test this proposition empirically (see e.g. Schmalensee and Trippi (1978)) Noh, Engle, and Kane (1994) take a different approach to forecast option prices. They make use of a GARCH model to predict the volatility of

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asset returns and use the forecasts to predict option prices. ARCH and GARCH models as proposed by Engle (1982) and Bollerslev (1986) are asset return models which capture the dynamic behavior of conditional volatility without presupposing any option price formula. ARCH and GARCH volatilities can, however, be used to predict future option prices. This is done by taking a GARCH volatility forecast together with an option pricing formula to calculate future option prices.[1] Applying a GARCH model in such a way is equivalent to an efficiency test of the option market.

Noh, Engle, and Kane (1994) analyze the efficiency of the market for the S&P 500 index option. Their analysis is set up as follows. First, historical asset returns of the S&P 500 are used to estimate a GARCH(1,1) model which in turn is used to predict daily volatilities. Second, the Black and Scholes (1973) option price formula is used together with the volatility forecasts to predict future option prices. The option price forecasts, and hence the efficiency of the option market, are analyzed on the basis of simple trading rules. Noh, Engle, and Kane (1994) take *at-the-money straddles* to benefit from the accuracy of the volatility forecasts. Since a straddle is a delta-neutral strategy, mainly changes in volatility cause a shift in the straddle price and therefore they make use of the following strategy. If the forecast of the future straddle price is above today's closing price, the straddle is bought (long position) and if it is below, it is sold (short position). Rates of returns for this strategy are calculated on a daily basis. It turns out that GARCH volatility forecasts and corresponding price forecasts for *near-the-money straddles* can result in significant profits even after transaction costs are taken into account. As a consequence the market for the S&P 500 index option is inefficient in such a way that historical information as exploited in a GARCH model generates value.

In this paper we take up the approach proposed by Noh, Engle, and Kane (1994) and apply it to Bund future options as traded at LIFFE.[2] Hence, our objective is twofold. Firstly, we need to analyze whether or not GARCH models are appropriate to capture the dynamics of conditional variances for interest rate futures markets. Secondly, we are interested in the efficiency of the option market on the Bund future. The trading rules applied in this paper are the same as proposed by Noh, Engle, and Kane (1994). We use *near-the-money straddles* and

[1] It should be noted, however, that it is not theoretically sound to use any possible option pricing model together with GARCH volatilities. A theoretically sound approach must guarantee that the option pricing formula is consistent with the returns generating process.

[2] For a similar analysis with DAX data see Schmitt and Kaehler (1996).

take a long position whenever price forecasts are above the latest available option (straddle) price and take a short positions otherwise. As mentioned above, trading straddles as compared to trading single calls or puts results from the property that *at-the-money straddles* are delta neutral, and that straddle prices are insensitive with respect to dividend payments (see Hull (2003)). Delta neutrality, however, can only work if the distortions caused by positive or negative gamma is small. Given the number of contracts and the limited time span (one day) over which the position is held this assumption seems to be justified.

It turns out that a GARCH(1,1) model of the Glosten, Jagannathan, and Runkle (1993) (GJR) type is capable of capturing all the dynamic structure of conditional variances. In particular, volatility clustering and the leverage effect are identified for the returns of the Bund future and are properly taken care by the GJR model. Implementing volatility forecasts together with *at-the-money straddles*, we find that continuous daily trading does not result in abnormal returns since empirically observable bid-ask spreads impose too high transaction costs.[3] Hence we optimize our trading strategies with respect to a simple filter rule. We concentrate on the empirical fact of volatility clustering and use the GARCH model to identify such periods with high clustering. Applying such a filter we find that our trading strategy results in significant profits even when transaction costs (bid-ask spreads) are taken into account. Whenever a trading rule shows a profit potential actual profits can arbitrarily be increased by the number of contracts bought (sold). This approach neglects any sensible risk measure. Therefore we introduce an additional trading rule for which we determine the number of contracts based on a Value at Risk (VaR) measure. We then compare the performance of our trading rule by means of the Sharpe ratio to that of the DAX which we identify as a well diversified portfolio. We find that the performance of our option strategy clearly dominates that of the market portfolio (measured by the DAX). Finally we calculate the efficient frontier based on a bond futures portfolio and the straddle strategy derived in the paper and show that significant risk/return improvements are possible.

Our paper is organized as follows. In Section 8.2 we present different volatility models, discuss their theoretical structure together with the empirical estimation and show how these estimates can be used in option trading. As for the volatility models, we will focus on several GARCH specifications and the exponentially weighted moving average (EWMA) model. Section 8.3 presents the estimation results of the dif-

[3]During the period that comprises our sample actual transaction costs at LIFFE were small so that the profitability of the trading strategy largely depended on the bid ask spread.

ferent return models and their corresponding volatilities for the Bund future. In Section 8.4 we quantify the performance of our trading strategy. Finally, Section 8.5 summarizes our main findings and concludes the paper.

2. Measuring Volatility of Asset Returns

The literature on the econometric modelling of financial time series does not contain a standard and well accepted definition for volatility. Geyer (1992) surveys the literature on volatility measures and distinguishes nine different approaches for computing volatility. Their differences are based on the choice of frequency (i.e. whether daily or monthly data are used), the treatment of the mean, and the use of overlapping or non overlapping observations. Bollerslev, Chou, and Kroner (1992) provide an overview of empirical applications of GARCH models to financial time series, Bollerslev, Engle, and Nelson (1994) focus on theoretical aspects of GARCH models, and Gourieroux (1997) discusses in detail how GARCH models can be incorporated in financial decision problems such as asset pricing and portfolio management. Here we do not want to give a literature survey on volatility models. We rather concentrate on those specifications that are employed in the present study. Hence, our selection of volatility specifications is driven by their forecasting abilities.

The basic idea behind many parametric volatility models is the assumption that asset returns can be broken up into a predictable part, $c_t = E_t(r_{t+1})$ (i.e. the conditional mean E_t reflecting the information available at time t) and an unexplained portion, ϵ_{t+1},

$$r_{t+1} = c_t + \epsilon_{t+1}. \tag{8.1}$$

Based on this specification, the conditional variance,σ^2_{t+1}, can be calculated as

$$\sigma^2_{t+1} = E_t(r^2_{t+1}) - c^2_t \equiv E_t(\epsilon^2_{t+1}). \tag{8.2}$$

In case the conditional mean is close to zero, $c_t = 0$, the most common approach to estimate the conditional variance, σ^2_{t+1}, is to make use of a weighted sum of past squared returns,

$$\sigma^2_{t+1} = \sum_{i=0}^{k-1} \omega_i(t) r^2_{t-i} \tag{8.3}$$

where $\omega_i(t)$ are the weights on past squared returns, which can depend on the information available at time t. Different specifications of the weights lead to different volatility models.

2.1 Naive Model

The updated sample variance puts constant weights, $\omega_i(t) = 1/k$, on past observations, and hence ignores the dynamic structure of volatilities. It simply calculates the estimate on historical volatilities over some pre-specified time window.

$$\sigma_{t+1}^2 = \frac{1}{k}\sum_{i=0}^{k-1} r_{t-i}^2 \tag{8.4}$$

with r_t as compounded return on day t, i.e., $r_t = \ln(S_t/S_{t-1})$ and k as the length of the time window.

2.2 Exponentially Weighted Moving Average Model

The main disadvantage of the naive model is that it gives equal weight to all observations in the sample, thus neglecting the stronger impact of recent innovations. This is the reason why the naive model is not capable of mimicking volatility clustering present in financial time series. This has led to the introduction of alternative moving average models for computing stock return volatilities (see e. g.Taylor (1994)). Comparing several specifications, Taylor found that the Exponentially Weighted Moving Average (EWMA) did best in terms of empirical performance. The idea behind the EWMA moel is the following. The conditional volatility of the current period $t+1$ is calculated as an MA(∞) process of weighted squared returns where the weights decay exponentially, i.e.,

$$\sigma_{t+1}^2 = (1-\lambda)\sum_{i=0}^{\infty} \lambda^i r_{t-i}^2 = \lambda\sigma_t^2 + (1-\lambda) r_t^2, \tag{8.5}$$

with λ as a constant decay factor.

The definition of the EWMA model shows that it is a generalization of the standard naive variance estimator with decaying weights. While the definition of the EWMA is rather straight forward, its use depends on the estimation of the weight λ. It can be estimated by minimizing an appropriate error measure. Moreover, with specification (8.5) an initial value for σ_0^2 is needed in empirical estimation. Usually the unconditional sample variance is chosen.

2.3 GARCH Model

The most successful model for describing nonlinear dynamics and non-normality of stock returns is the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model, introduced by Bollerslev

(1986) based on the ARCH model of Engle (1982). It is a model that builds on time-varying second order moments. Applications of GARCH models in finance are surveyed in Gourieroux (1997) and the theoretical foundation in Bollerslev, Engle, and Nelson (1994). While the class of GARCH models is rather flexible and admits a large variety of different specifications, we concentrate on the GARCH(1,1) model. In the GARCH(p,q) specification the estimate of the volatility is not only given by a function of past returns but also by a function of past volatilities. The latter capture volatility clustering present in many asset return data. A full specification of the GARCH(1,1) model with a nonzero mean is given by

$$\begin{aligned} r_t &= c + \epsilon_t \text{ with } \epsilon_t = u_t \sigma_t \text{ and } u_t \sim \text{ N(0,1)} \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned} \tag{8.6}$$

This specification implies that the conditional variance follows an autoregressive process for which stationarity is satisfied when the sum of α and β is less than unity. If this condition is violated there is a unit root in the conditional variance process and the corresponding model is referred to as an Integrated GARCH (IGARCH) model. The introduction of conditional time-dependent second moments is capable of generating volatility clustering as well as leptokurtic unconditional returns. This holds true even when a Gaussian distribution is specified for the standardized residuals u_t.

Comparing the GARCH model for quantifying market volatilities with the EWMA specification introduced above, we observe that the two are closely related. The major differences, however, stem from the way the model parameters are estimated and from their stationarity assumptions.

The most popular extensions to the standard GARCH model are the Exponential-GARCH (E-GARCH) model, proposed by Nelson (1991), and the model proposed by Glosten, Jagannathan, and Runkle (1993) (GJR-model). These modifications offer improved fits relative to the standard GARCH-model. Since we will make use of both the EGARCH and GJR-model, we present their analytical structures.

While the traditional GARCH model can successfully be applied to capture fat tailed returns and volatility clustering, the EGARCH model is capable of capturing the leverage effect present in stock returns. The specification of the EGARCH(1,1) model is given by

$$\begin{aligned} r_t &= c + \epsilon_t \text{ with } \epsilon_t = u_t \sigma_t \text{ and } u_t \sim \text{ N(0,1)} \\ \log \sigma_t^2 &= \omega + \alpha \left(\frac{\mid \epsilon_{t-1}^2 \mid}{\sigma_{t-1}^2} - \sqrt{2/\pi} \right) + \beta \log \sigma_{t-1}^2 + \theta \frac{\epsilon_{t-1}}{\sigma_{t-1}} \end{aligned} \tag{8.7}$$

Finally, the GJR-model allows for a quadratic response of volatility to news with different coefficients for good and bad news, and hence incorporates asymmetries in the model. Its specification is given by

$$r_t = c + \epsilon_t \text{ with } \epsilon_t = u_t \sigma_t \text{ and } u_t \sim \text{N(0,1)}$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 + \theta S_{t-1}^- \epsilon_{t-1}^2, \tag{8.8}$$

with $S_t^- = 1$ if $\epsilon < 0$ and $S_t^- = 0$ otherwise. Each of the above models can be used to derive volatility forecasts and hence can be applied to evaluate the profitability of trading strategies. As was pointed out in the introduction, we will take a long position in a straddle whenever the forecasted volatility is larger than the present one and take a short position otherwise. This strategy is based on the relationship that an increase in the GARCH volatility forecast corresponds to an increase in the option price forecast, so that the forecasted price is above the latest available closing price. In the next sections we apply these techniques to Bund future options traded at LIFFE during the period of 1988 and 1995. First, we present the volatility forecasts and then evaluate the profitability of option trading strategies based on these forecasts.

3. Estimation of Alternative Volatility Models

As pointed out above, we will analyze the Bund future options market. Our data set of daily closing prices for the Bund future starts at the beginning of 1988 and runs through August 1995. We follow standard procedures and take futures prices for those series only for which the time to maturity is larger than one month. This gives us a consistent series, that is, however, not stationary. Hence, instead of using the prices we employ returns series.

The daily returns are calculated as the differences in the logarithms of the prices and hence constitute continuously compounded rates. They exhibit much the same characteristics than stock returns. They are characterized by volatility clustering and fat tailed distributions as can be seen from the summary statistics reported in Table 8.1. In this table we present descriptive statistics not only for the returns based on closing prices but also for returns calculated from opening to closing and closing to opening prices. These statistics show the following results: the mean is very close to zero so that it can be neglected if necessary and all series have excess kurtosis.

As a final first step we analyze the return series for possible autocorrelations. Looking at the autocorrelation functions of the returns and the squared returns suggests that there are no significant linearities in the data but strong nonlinearities that can be the result of heteroscedastic-

Table 8.1. Descriptive statistics of daily returns

	Close to Close	Opening to Close (Active Period)	Close to Opening (Passive Period)
Mean	-4.7E-5 %	-0.014 %	0.014 %
Standard Dev.	0.360 %	0.326 %	0.196 %
Skewness	-0.163	0.066	0.034
Kurtosis	7.137	7.081	12.734

ity. This again points to the use of a GARCH model. In what follows we estimate three alternative GARCH models. A standard GARCH(1,1), an EGARCH(1,1) and a GJR(1,1) specification. Tables 8.2, 8.3 and 8.4 report the estimation results.

Table 8.2. GARCH (1,1) estimates

Variable	Coefficient	t-Statistic	Probability
c	8.29E-5	1.17	0.241
ω	1.4E-7	5.15	0.000
α	0.0646	8.74	0.000
β	0.9249	136.13	0.000

Table 8.3. EGARCH (1,1) estimates

Variable	Coefficient	t-Statistic	Probability
c	4.62E-5	0.66	0.509
ω	-0.1707	-5.42	0.000
α	0.1314	9.10	0.000
β	0.9847	359.00	0.000
θ	-0.0348	-4.63	0.000

These results imply the following. First, there is a strong persistence in conditional variances as expressed by values for β above 0.9 for all three specifications. Second, for both the GARCH(1,1) and for the GJR(1,1) model the sum of the estimated parameters is close to unity suggesting an integrated process for the conditional variance. Finally, returns show significant leverage effects. In order to select one of the models as the appropriate one, we performed several diagnostic checks which are reported in the following tables. Table 8.5 reports some de-

Table 8.4. GJR(1,1) estimates

Variable	Coefficient	t-Statistic	Probability
c	4.59E-5	0.64	0.524
ω	1.56E-7	5.54	0.000
α	0.0318	4.09	0.000
β	0.9330	139.39	0.000
θ	0.0441	4.42	0.000

scriptive statistics of the normalized residuals for the three models. From these we see that the kurtosis is substantially reduced which is in accord with international evidence on stock returns. Table 8.6 reports the test statistic of the Box-Ljung autocorrelation test, which indicates that both the residuals and the squared residuals are not autocorrelated. Finally, the BDS test reported in Table 8.7 supports consistency with i.i.d. residuals for all three specifications. Hence, to decide on one of the alternative models we make use of the estimated likelihood as shown in Table 8.8. On the basis of these results we will mainly use the GJR(1,1) model for the following analysis.

Table 8.5. Descriptive statistics of normalized residuals

Model	Mean	Std. Dev.	Skewness	Kurtosis
GARCH(1,1)	-0.0165	0.9981	-0.1435	4.653
EGARCH(1,1)	-0.0016	0.9984	-0.0341	4.801
GJR(1,1)	-0.0027	0.9980	-0.0906	4.667

Table 8.6. Autocorrelation test (Box-Ljung test with lag = 20)

	Normalized Residuals		Squared Normalized Residuals	
Model	χ^2-Statistic	Probability	χ^2-Statistic	Probability
GARCH(1,1)	21.35	0.377	14.00	0.830
EGARCH(1,1)	23.79	0.252	15.83	0.727
GJR(1,1)	21.27	0.381	12.84	0.884

As pointed out in Section 8.2 the exponentially weighted moving average exhibits many characteristics of a GARCH model. The two important differences are that the EWMA does violate the strict stationarity

Table 8.7. BDS test of daily returns and normalized returns

	Embedding Dimension = 2		Embedding Dimension = 5	
Model	BDS-Statistic	Probability	BDS-Statistic	Probability
Daily Returns	8.08	0.000	13.12	0.000
GARCH(1,1)	-1.23	0.220	-1.41	0.160
EGARCH(1,1)	-1.36	0.173	-1.17	0.241
GJR(1,1)	-1.19	0.235	-1.01	0.312

Table 8.8. Estimated likelihood of GARCH models

Model	Number of Parameters	Log-Likelihood
GARCH(1,1)	4	9276.29
EGARCH(1,1)	5	9273.31
GJR(1,1)	5	9282.18

assumption and that it does not include any constant. Since the GARCH estimates demonstrate that the data are consistent with stationary variances the EWMA model does not seem to be a correct specification. Nevertheless we make use of this volatility model and estimate its parameters via least squares methods. The corresponding estimated dynamic equation is given by

$$\hat{\sigma}_{t+1}^2 = 0,0704 m_t + 0,9296 \hat{\sigma}_t^2 \tag{8.9}$$

with $m_t = (r_t - c)$ and c as the sample mean and σ_0 as the sample variance. Since our primary objective is to use the five different models for volatility forecasts, we need to evaluate their relative forecasting performance. This is not a trivial exercise, since volatility is not directly observable. Hence, we need to identify some benchmark. Here we choose the sample standard deviation as a reference. Based on this measure, Table 8.9 presents the in-sample and out-of-sample forecasting performance evaluated along the lines of two different measures, the mean squared error (MSE) and the mean absolute error (MAE). Based on these results, again the GJR model dominates the others. Hence, for the remainder of the paper we mainly use this specification.

4. Profitability of Option Trading Strategies

Our objective in this paper is to study the efficiency of the option market. For that reason, we analyze whether or not volatility forecasts based

Table 8.9. Forecasting performance in-sample and out-of-sample

	In-Sample		Out-of-Sample	
Model	MSE (10^{-6})	MAE (10^{-3})	MSE (10^{-6})	MAE (10^{-3})
Sample Std. Dev.	6.289	1.733	6.903	1.903
Naive Model	5.470	1.625	6.566	1.929
EWMA	5.341	1.607	6.511	1.924
GARCH(1,1)	5.307	1.640	6.416	1.902
EGARCH(1,1)	5.272	1.634	6.408	1.911
GJR(1,1)	5.238	1.631	6.378	1.884

on historical returns can be used to derive profitable trading strategies. We proceed as follows: From the volatility forecast we get information as to whether volatility in the next period is going to increase or decrease. We use this information as buying or selling signal for a straddle. If the forecasted volatility is lower than the current one (i.e. volatility decreases) we go short and if volatility increases we take a long position. Figure 8.1 below depicts the profits resulting from this trading rule over a three years trading period starting in August 1992 and ending in August 1995. In this experiment the sample period is divided into an in-sample and an out-of-sample period. The in-sample period is used to estimate the volatility model and the out-of-sample period is used to evaluate the return performance of the trading strategy.

Based on the volatility forecasts a single straddle is bought or sold every day during the out-of-sample period starting on August 1, 1992. These profits are accumulated. The results allow for the following interpretation: First, using historical information in returns series and predicting future volatilities on the basis of a GARCH-model generates substantial gross-profits. They are highest for the GJR model and lowest for the GARCH(1,1) specification. It is also interesting to note that the EWMA model does very well relative to the more flexible GARCH approach. Second, these profits vary significantly over different years. While during the period of August 1992 to August 1993 losses occurred, for the remaining two years substantial profits can be observed. These results, however, do not imply that the strategies generate value since transaction costs are not taken into account.

As a second experiment we look at a strategy that we call *strategy with constant trades*. In this case we buy or sell straddles with a constant value equal to 100. Figures 8.2 and 8.3 show the cumulative profits for two alternative scenarios: the case where only long positions are feasible and the case where both long and short positions are possible.

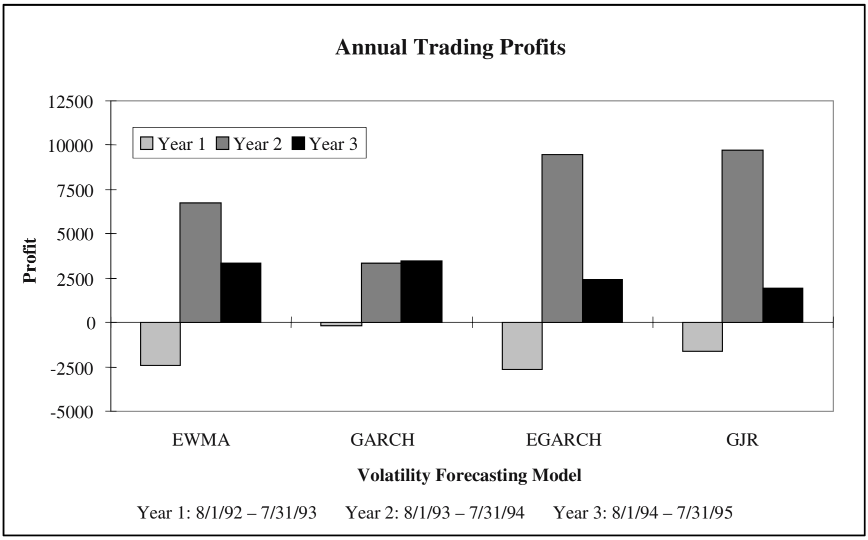

Figure 8.1. Annual trading profits.

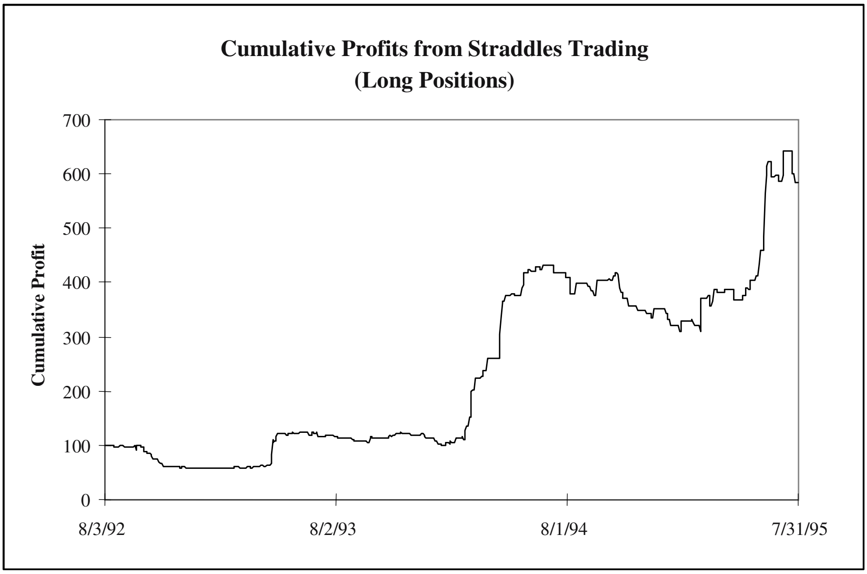

Figure 8.2. Cumulative profits from straddles trading (long positions).

But again we ignore transaction costs, and assume a risk free rate of zero. Again, the results are very impressive and show the enormous potential for excess profits. Although the performance of the automated

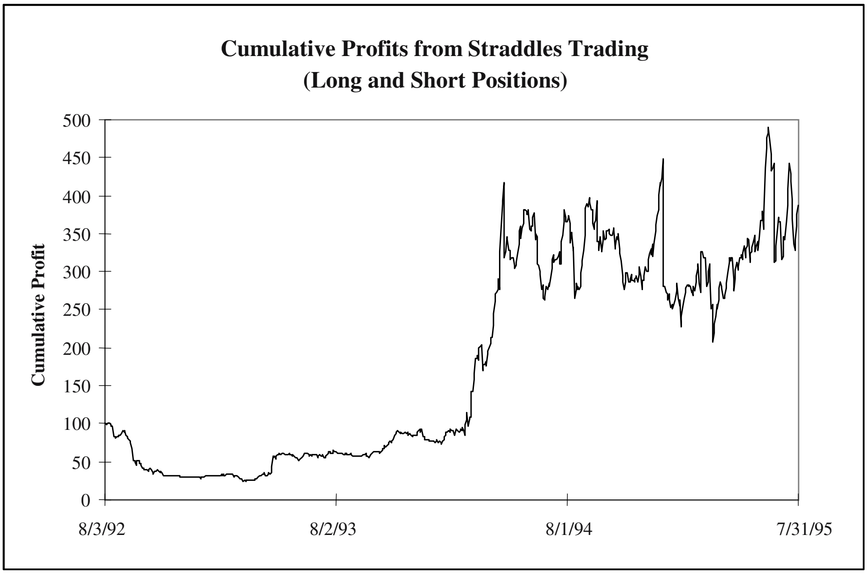

Figure 8.3. Cumulative profits from straddles trading (long and short positions).

trading strategy is rather poor during the period of August 1992 to the end of 1993, there are substantial gains during early 1994 and smooth increases over the rest of the out-of-sample-period. To understand this development we plot in Figure 8.4 the implied volatilities based on an appropriate option pricing model and observe that during the initial period until the beginning of 1994 the market has been very stable, whereas volatilities increased substantially during 1994 up until 1995 with strong volatility clustering effects. During this period the predictive power of the GARCH model is very good and results in large increases in profits.

Although the above results are quite impressive, they suffer from two shortcomings. Transaction costs are not taken into account and every trading (forecasting) signal is used to execute a trade. But as Figure 8.4 suggests, big increases in profits can be earned when there are substantial changes in volatilities combined with strong volatility clustering. Led by this empirical observation, we introduce a filter and take into account transaction costs to arrive at more realistic scenarios. For a filter rule we initiate trades only during that periods for which volatility clustering is to be expected. These are the periods following a large change in volatility. When applying an ex-post optimized filter and taking into account a bid-ask spread of 0.04 DEM and a flat rate of 25 DEM as transaction costs, we arrive at profits shown in Figures 8.5 and 8.6.

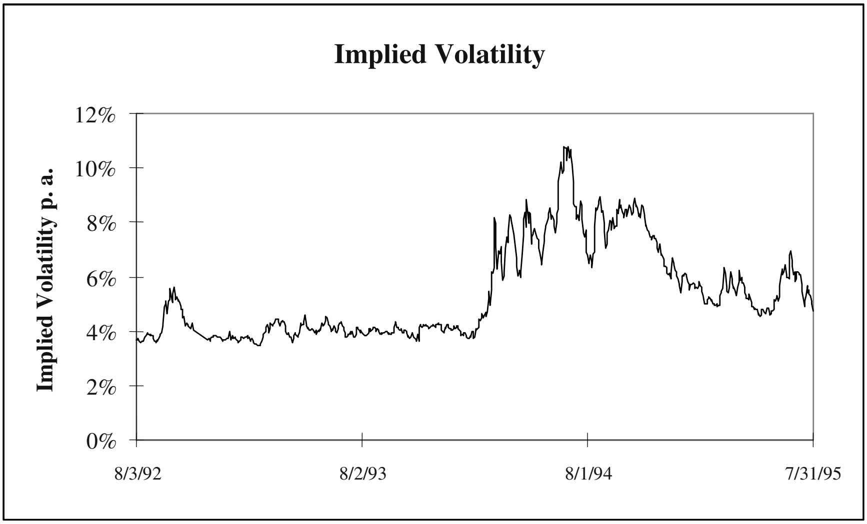

Figure 8.4. Implied volatility.

Again we distinguish two scenarios. One in which only long positions, and one in which both long and short positions are possible. Moreover we relax the assumption of constant trades and calculate the number of contracts simply by investing all the money available in every period in straddles. Under this regime it turns out that an initial value of 100 (August 1992) can be turned into a terminal value at the end of July 1995 of more than 300.

Although these strategies already take into account transaction costs, some criticism of the approach is still valid. Our analysis ignores any risk considerations. To overcome this problem we make use of a Value at Risk (VaR) approach to calculate the number of contracts traded. In particular we use a historical simulation to calculate the maximum possible loss due to a straddle price change that occurs in 5 % of all cases. We then calculate the number of straddle contracts in such a way that this maximum possible loss (given the pre-specified confidence level of 5 %) is covered by our existing capital. Figures 8.7 and 8.8 present the corresponding cumulative profits for the case of long positions only and the case where long and short positions are possible, respectively.

To underline the profits suggested in the diagrams we additionally report different performance measures for the trading strategies using the VaR approach in Table 8.10. It turns out that our strategies together with the refinements generate substantial value. As Figures 8.7 and 8.8, however, show the automated trading strategies only generate value in

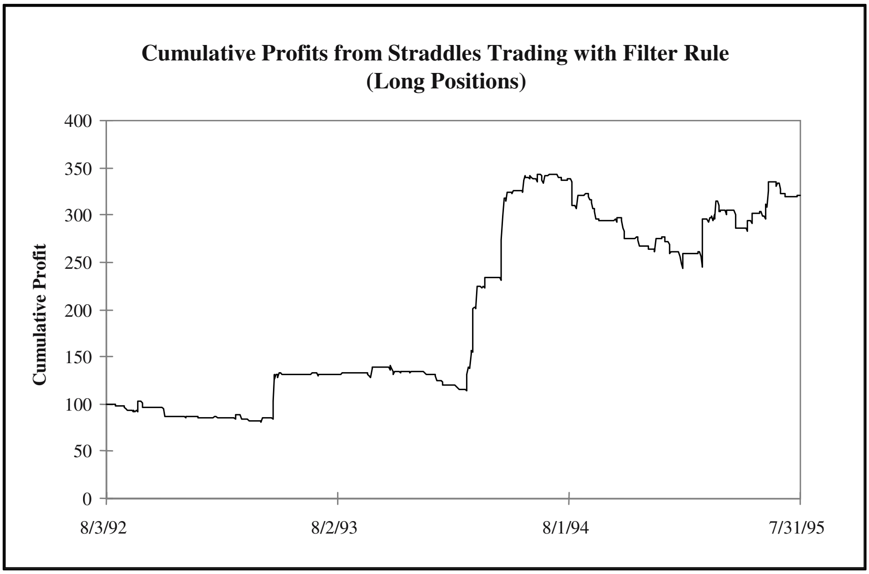

Figure 8.5. Cumulative profits from straddles trading with filter rule (long positions).

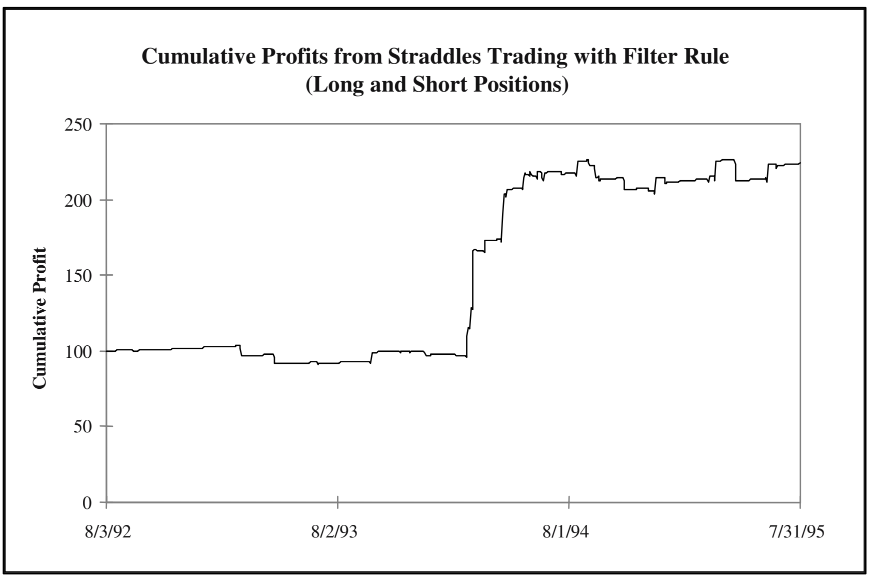

Figure 8.6. Cumulative profits from straddles trading with filter rule (long and short positions).

periods with large volatility clustering, i.e., during turbulent times. For our out-of-sample-period this only occurred during 1994 when in Feb-

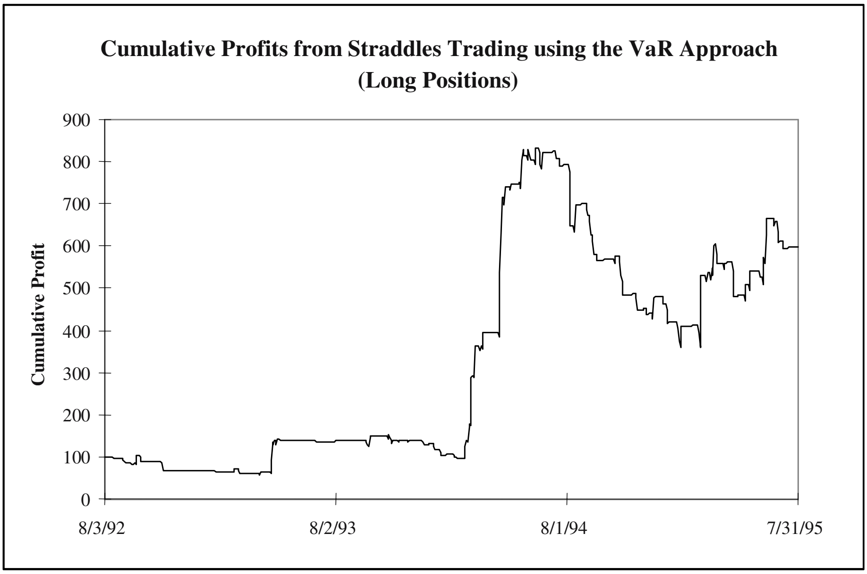

Figure 8.7. Cumulative profits from straddles trading using the VaR approach (long positions).

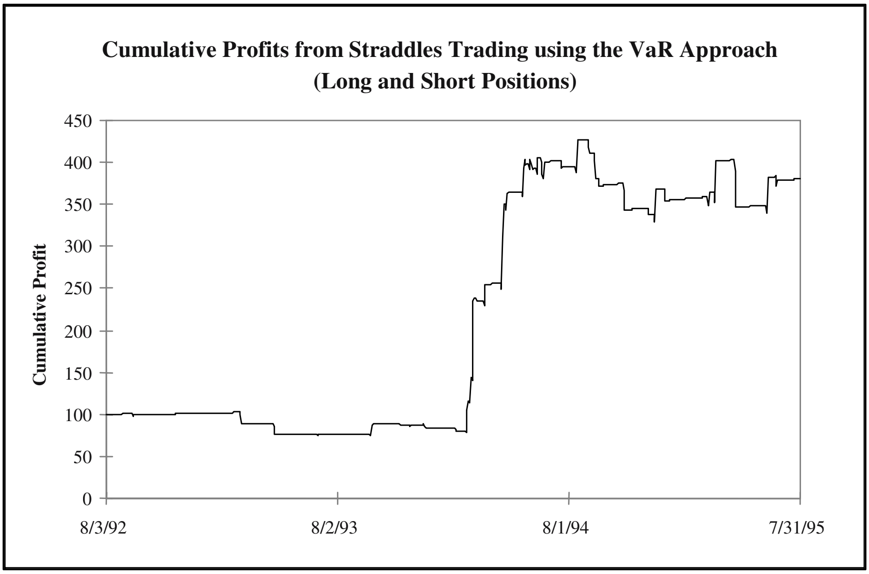

Figure 8.8. Cumulative profits from straddles trading using the VaR approach (long and short positions).

ruary an unexpected increase in the Fed-rate triggered a very turbulent year for interest rate securities.

This observation suggests that an automated trading strategy like the one analyzed here should not be applied in isolation but should be part of a general portfolio strategy. Therefore we now look at the portfolio (diversification) effects of our strategy when used jointly with a portfolio of Bund futures. In Figure 8.9 we present a risk and expected return diagram for a portfolio consisting of the underlying Bund future and the straddle portfolio derived through our trading strategy. The resulting efficient frontier suggests that the automated trading strategy not only generates value but significantly improves the diversification effects in a portfolio that holds the straddle strategy as an additional asset.

Table 8.10. Strategy characteristics

	Long Positions Only	Long and Short Positions
Average Return p.a.	59.5 %	44.5 %
Volatility (SD) p.a.	73.4 %	46.7 %
Avg. Number of Trades p.a.	26.67	12.33
Sharpe Ratio	0.76	0.87

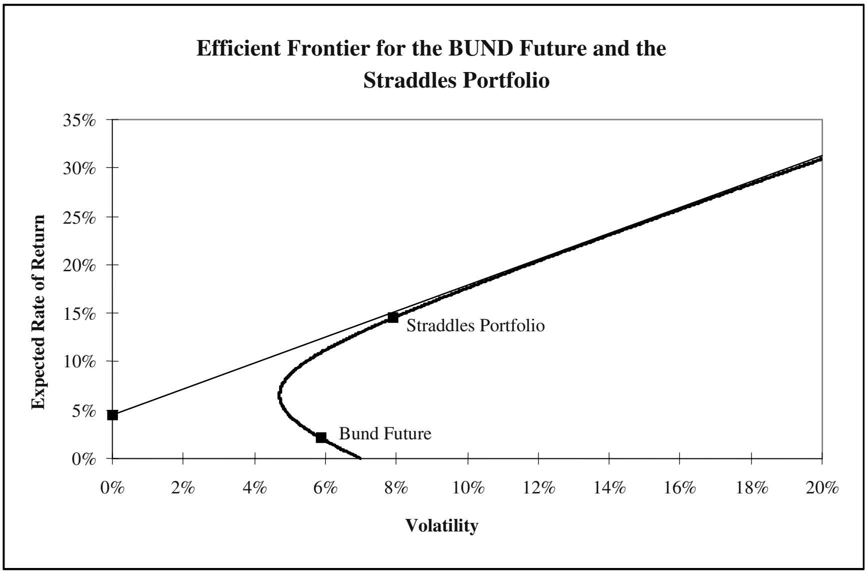

Figure 8.9. Efficient frontier for Bund future and straddles portfolio.

5. Conclusions

In this paper we employ volatility forecasts to evaluate the profitability of option trading strategies. We first present several volatility models theoretically and then use these specifications to empirically evaluate the efficiency of the Bund future options market. It turns out that volatility forecasts based on historical returns are capable of adding value when used together with simple trading rules. We derive profits for several different variations of the trading rule and find in all cases abnormal returns.

Our results need some qualifications, however. Firstly, our analysis is based on closing prices, at which actual trades cannot be executed. Therefore the profitability of the simple trading rules critically depends on how far the closing prices come to those price at which the last trades of a day are carried out. Secondly, our results clearly indicate that during periods of low volatility the forecasting model does not perform very well. Only if there is sufficient volatility clustering, so that a GARCH model is very accurate in predicting conditional variances, our trading rule can outperform the market.

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Chapter 9

TWO-PART TARIFF PRICING IN A DYNAMIC ENVIRONMENT

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Abstract A two-part tariff is a non-linear pricing technique in which the price of a product or service is composed of two parts: an entrance fee and a charge per unit of consumption. Compared to linear pricing, this methodology leads to higher profits by allowing a firm more freedom in extracting the consumer surplus. It is widely used in telecommunication services.

This chapter documents recent developments on non-linear pricing in a dynamic and competitive environment. The developments can also be viewed as extensions of the linear dynamic pricing literature by allowing a two-part tariff scheme.

1. Introduction

Two-part tariffs are widely practiced in the Internet and telecommunication networks. With the growth of these industries, the pricing of services that take into account the growth of subscribers, as well as the demand for services by members of the service, poses a challenge to managers. There are several key issues associated with this. The first has to do with the fact that services typically offer a two-part price that consists of a membership fee and a usage price. The membership fee is a fee to join the network while the usage price is variable. This raises an immediate question of what should be the optimal two-part tariff for a firm serving a growing network of subscribers. Appealing to Optimal Control, Fruchter and Rao (2001) offer an answer to this question. The paper deals with a situation, which is similar to that of a durable product in the sense that a customer becomes a member of the network

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only once, just as a customer buys only one unit of a durable product. In other words, the customer adopts the service. However, the situation described in the paper is also similar to a non-durable product in the sense that the customer pays an ongoing fee and a usage price that could be thought of as a repeat purchase price. Durable goods pricing has been analyzed by Dolan and Jeuland (1981) and Kalish (1983) for a monopolist, and by Bass and Rao (1985) and Dockner and Jorgensen (1988) under competition. However, past work has not explicitly considered a two-part pricing policy that changes over time with the growth of a network. In a recent paper on the pricing of cellular phones, Jain, Muller and Vilcassim (1999) examined the question of how the pricing of a complementary product, such as the handset, influences the pricing of the metered service, phone calls. They conclude that under certain cost conditions and competition in a two-period world, the price of the telephones decreases over time while that of the calls is non-decreasing. In developing the model, they assume that the average demand per customer, in minutes of phone calls, decreases over time as the network size grows. The assumption is consistent with data and has also been observed by Manova et al. (1998). Fruchter and Rao (2001) make a similar assumption. However, instead of focusing on complementary products, they focus on network membership and usage by network members.

Another issue I want to address is that in reality the consumers, while belonging to the network of one company, communicate with some who belong to the same network and others, who belong to competing networks. In the latter case, consumers are using both networks. From the point of view of one of the networks, it incurs a greater cost to connect "its" customer with a user of the competing network than what it would incur if both users were on the same network. We can expect that this difference in the marginal cost of serving a customer (depending on whether they connect to a user on one's own network or a rival network) would have pricing implications. Laffont, Rey and Tirole (1998b) examined pricing in such situations and showed that this could lead to network-based price discrimination. Network-based price discrimination in their model has a user connecting to another customer on the same network paying a lower price than she would if she were connecting to a customer of a rival network. There are several examples of network-based pricing strategies. One is MCI's Friends-and-Family Program that offers better discounts to calls by MCI's customers made to other MCI customers than to calls across other networks. Similarly, Orange—a wireless service operator in Hong Kong—provides a basic service plan that charges HK$1 for each minute of calls between Orange and other networks but only HK$0.2 for each minute of calls within its own network. A discount of 80% is rather large relative to cost differentials. Also, while charging for interconnection costs has been a long-standing norm in the telecommunication industry, the use of network-based price discounts is a relatively new practice and its prevalence varies across situations. Naturally, the question arises whether network-based pricing strategies would arise from other strategic reasons that may exist in addition to differences in cost. To address this question, Fruchter, Shi and Rao (2004) depart from Laffont, Rey and Tirole (1998b), by extending the two-part pricing policy of Fruchter and Rao (2001) to a competitive network situation. Like Laffont, Rey and Tirole (1998b), they study a duopoly in which each competing firm follows a two-part pricing scheme consisting of an access (or membership) fee and a usage fee. They differentiate between the usage fee for communications within a network, and across networks. The difference between the two usage fees, if any, they call a *network-based price discount*. Thus, Fruchter, Shi and Rao (2004) innovate by studying pricing strategies in a dynamic duopoly with each firm being concerned not only with current revenues but also with future revenues that depend, in turn, on the growth of network size. It is a well-established proposition in marketing that the adoption of a service follows a diffusion process similar to the one captured by the Bass model (Bass (1969)). Since the diffusion process is affected by price, it becomes necessary to capture the dynamic competition in the duopoly. Therefore, unlike Laffont, Rey and Tirole (1998b), who cast the pricing problem as a static game, they investigate the firms' optimal network-based pricing strategies using a differential game framework to accommodate dynamic pricing. They derive the optimal pricing policy as a Nash equilibrium feedback strategy with prices depending on the network sizes, i.e., on the number of subscribers. The equilibrium prices that they obtain are characterized by a network-based price discount, even if there are no interconnection costs. The solution offers a strategic rationale based on customer acquisition for network-based discounts, especially in situations where the observed magnitude of the discount is significantly at variance with cost differences due to interconnection costs. Fruchter, Shi and Rao (2004) is firmly rooted in the literature on dynamic pricing strategies, and may therefore be seen as a generalization of some previous models. Like Dockner and Jorgensen (1988) and Bass and Rao (1985), it uses a differential game to study the optimal dynamic pricing strategies for oligopoly. Like Dolan and Jeuland (1981), Kalish (1983), Dockner and Jorgensen (1988), and Fruchter and Rao (2001), it takes into account the diffusion effect on market demand. Unlike the previous work, it takes into account the interdependence of demand across networks, a phenomenon that has assumed importance in recent years.

Next we present the dynamic two-part tariff-pricing problem in a competitive network.

2. The two-part tariff dynamic and competitive pricing problem

We consider a market with two competing networks, each of which offers network telecommunication service, either through a fixed line or wireless technology. A consumer has to be a subscriber of one of the two networks in order to use the service. The two networks are interconnected so that users at one network can communicate with users at the other network. Each firm adopts a pricing policy that consists of an ongoing membership fee and usage-based fees for the metered service. We distinguish between two communication services based on the location of users: (1) communication within the same network and (2) communication across the two competing networks. Following Laffont, Rey and Tirole (1998a), we will refer to communications that are initiated and terminated within the same network as "on-net" calls; we use the term "off-net" calls to refer to communications initiated at one network but terminated at the other network. Networks are allowed to adopt network-based price discrimination by providing discounts to the on-net communications. Let $k_i(t)$ denote the ongoing membership fee charged by firm i at time t. Denote $p_{ii}(t)$ and $p_{ij}(t)$ to be firm i's unit price for on- and off-net calls, respectively, where the first subscript denotes the originating network and the second subscript denotes the ending network. We denote $N_i(t)$ to be the number of subscribers to firm i's service at time t. Consumers of network i have an average demand for communication of $D_i(t)$ at time t, which consists of both on-net communication $D^{ii}(t)$ and off-net communication $D^{ij}(t)$, where $D^i(t) = D^{ii}(t) + D^{ij}(t)$. We assume that average on-net communications $D^{ii}(t)$ depends on the size of one's own network $N_i(t)$ and on-net price $p_{ii}(t)$, i.e. $D^{ii}(t) = D^{ii}(N_i, p_{ii})$ and $D^{ij}(t) = D^{ij}(N_i, N_j, p_{ij})$. As standard, we assume that consumer communication demand decreases with price. However, the effect of network size $N_i(t)$ on average demand $D^{ii}(t)$ is not as straightforward. Average demand $D^{ii}(t)$ may increase with the growth of network size. For the growth of the network we assume the following general specification, $\dot{N}_i(t) = f^i(N_i, N_j, k_i, k_j, p_{ii}, p_{ij}, p_{jj}, p_{ji})$. The network size increases with competing network's prices but decreases with one's own network's prices. Also, as network size increases, the growth increases, thus having more value as more customers are in the network. However, with time, there are less people left to join the network, so there are saturation effects.

The duopolist's problem of determining a two-part pricing policy, $\{k_i, [p_{ii}, p_{ij}]\}$, to maximize the present value of its profit stream over the planning period, can be formulated as a differential game

$$\begin{cases} \max\limits_{k_i, p_{ii}, p_{ij}} \prod_i \\ s.t. \quad \dot{N}_1 = f^1(N_1, N_2, k_1, k_2, p_{11}, p_{22}, p_{12}, p_{21}), \quad N_1(0) = 0 \\ \text{and} \quad \dot{N}_2 = f^2(N_1, N_2, k_1, k_2, p_{11}, p_{22}, p_{12}, p_{21}), \quad N_2(0) = 0 \end{cases} \tag{9.1}$$

where

$$\prod_i = \int_0^\infty N_i[(k_i - c_i^k) + (p_{ii} - c_{ii})D^{ii}(N_i, p_{ii}) + (p_{ij} - c_{ij}) \\ D^{ij}(N_1, N_2, p_{ij})]e^{-rt}dt$$

denotes the sum of the discounted profits of firm i over an infinite horizon. There are three cost parameters related to $\prod_i$. The first, c_i^k, is the cost per subscriber that would correspond, for example, to the billing and customer service provided by firm i to each customer. The second, c_{ii} is the cost of providing on-net metered service that would correspond, for example, to the minutes of telephone calls between two of the firm i's customers. The third factor, c_{ij}, is the cost of providing off-net metered service of telephone calls originated from firm i and ended at firm j. When networks are connected for off-net communication, the originating network often has to incur the interconnection costs. As a result, variable cost for off-net communication is typically higher than that for on-net communication. In other words, $c_{ii} \leq c_{ij}$. We are looking for a feedback Nash equilibrium strategy that solves this differential game.

3. The equilibrium pricing policies

Using dynamic optimization techniques (for example, Kamien and Schwartz (1991)), the two-part feedback Nash equilibrium $\{k_i^*, [p_{ii}^*, p_{ij}^*]\}$, satisfies the following conditions (for proof see Fruchter, Shi and Rao (2004))

$$p_{ii}^* = \frac{\eta_{D^{ii}}^{p_{ii}}}{\eta_{D^{ii}}^{p_{ii}} - 1}\left(c_{ii} - \lambda_i^i \frac{f_{p_{ii}}^i}{D_{p_{ii}}^{ii} N_i} - \lambda_j^i \frac{f_{p_{ii}}^j}{D_{p_{ii}}^{ii} N_i}\right) \tag{9.2}$$

$$p_{ij}^* = \frac{\eta_{D^{ij}}^{p_{ij}}}{\eta_{D^{ij}}^{p_{ij}} - 1}\left[c_{ij} - \lambda_i^i \frac{f_{p_{ij}}^i}{D_{p_{ij}}^{ij} N_i}\left(1 - \frac{f_{p_{ii}}^i}{f_{p_{ij}}^i}\right) - \lambda_j^i \frac{f_{p_{ij}}^j}{D_{p_{ij}}^{ij} N_i}\left(1 - \frac{f_{p_{ii}}^j}{f_{p_{ij}}^j}\right)\right] \tag{9.3}$$

$$N_i + \lambda_i^i f_{k_i}^i + \lambda_j^i f_{k_i}^j = 0\,. \tag{9.4}$$

In (9.2)–(9.4), λ_i^i and λ_j^i represent the on- and off-net shadow prices; they take into account the dynamic effects, in our context, the value of subscription to the firm's network and rival's network, respectively. The behavior of the $\{k_i^*, [p_{ii}^*, p_{ij}^*]\}$, over time depends on these shadow prices and the additional factors as it is reflected by these relationships. A more extensive analysis is presented in the sequel. We now present special cases.

3.1 Special cases

The results in (9.2)–(9.4) contain a number of special cases that have already been studied in the literature.

- By setting, $\lambda_i^i = \lambda_j^i = 0$ we get the static-pricing rule.
- By setting $f^i = D^{ii}N_i$ means that firm i's profit is only derived from new demand f^i instead of $D^{ii}N_i$. As in the case of durable goods, we can replicate the studies of Dockner and Jorgensen (1988) for duopoly dynamic pricing for durable goods.
- A more special case is on monopoly durable good pricing policy. In this case, the off-net shadow price is zero and is obtained by using the well-known monopoly durable goods pricing rule derived in Kalish (1983).
- By setting the off-net shadow price equal to zero, we obtain the solution for the monopoly case (Fruchter and Rao (2001)). Next we want to further analyse this special case.

3.2 Analysing the monopoly policy

By setting $\lambda_j^i = 0$ in (9.2) and (9.4) and removing the indices, we replicate the results by Fruchter and Rao (2001) for the membership fee[1] and usage price of the monopoly network. Thus,

$$p^* = \frac{\eta_D^p}{\eta_D^p - 1}\left(c - \lambda \frac{f_p}{D_p N}\right) \tag{9.5}$$

and

$$N + \lambda f_k = 0\,. \tag{9.6}$$

Considering (9.6), since $N(0) = 0$ and $f_k < 0$ we have $\lambda(0) = 0$ and $\lambda(t) > 0$, for $t > 0$. This is a formalization of the intuition that an

[1] For a positive membership fee.

additional subscriber to a network results in an increase in the firm's future profit. It is easy to show that the sufficient condition for optimality implies that the membership fee k increases with network size N. Thus, the optimal membership fee constitutes a penetration strategy: At the beginning it is low, and when the number of customers increases, the membership fee increases. To understand this, first note that a low membership fee implies that if the firm could, it would even pay to acquire customers. The reason for this is that once a person becomes a customer, i.e., a member of the network, they continue to buy from the service provider. It therefore pays to "acquire" these customers by offering them a low membership fee. However, at some point, when N is sufficiently large, the value to adding a customer is less than it was in the beginning. At this point, it no longer makes sense to pay to acquire customers. Indeed, it may become optimal to make the customers pay for being a part of the network. And so k increases.

Of course, we could imagine that customer acquisition could also be accomplished by setting the usage price low. Considering (9.5), it turns out (cf. Fruchter and Rao (2001)) that this is not necessarily part of the optimal policy. Indeed, if demand is not too sensitive to usage price, i.e., for sufficient low D_N, the optimal usage price could constitute a skimming strategy. The policy on the usage price is driven by the fact that early adopters are also heavy users in this model. This, in turn, means that customer acquisition is better accomplished through a lower membership fee rather than a low usage price. And so, the usage price can be held relatively high.

3.3 Analysing the competitive policy

¿From (9.4) and considering the sufficient conditions for optimality (cf. Fruchter, Shi and Rao (2004)), it is easy to see under some additional assumptions that $f^i_{k_i} < 0$, $f^i_{k_j} > 0$, and $f^i_{k_i k_i} < 0$ and $f^j_{k_i k_i} \leq 0$, that the on-net shadow price should be positive for any positive time, i.e. $\lambda^i_i(t) > 0$, for $t > 0$; thus the value of subscription to the firm's network results in an increase in the firm's future profit. This result is consistent with the intuition that a firm's long-run profit increases with its customer base. Furthermore, there is a neighbourhood of $t = 0$, $0 < t < t_1$, where λ^i_j should be positive too. This shows that at least for some of the time, increase in a competitor's network size is good for business! Of course, this makes sense because consumers in a firm's network can make calls to more users (those in a competitor's network), and so revenue increases. In particular, it shows that in the early stages of the game, a network's future profit may increase with additional subscribers to the other network. When networks provide network-based discounts, the extra subscribers to network j increases a subscriber's chance to enjoy the on-net discounts within network j. Consequently, the subscription to network i becomes less attractive. However, with a larger rival network (j), a firm can increase its profit as its subscribers demand more off-net communications. Such a benefit is likely to be bigger in the early stage of the game ($\boldsymbol{t} \leq \boldsymbol{t}_1$) because in the beginning, the total network size is small and hence a new subscriber is expected to further increase the communication demand.

Considering again (9.4) and the sufficient condition for optimality, it is easy to show that the optimal membership fee increases with the size of the network. Thus, k_i increases with N_i (for a formal proof see Fruchter, Shi and Rao (2004)). This implies that in competition, as in monopoly, the optimal membership fee follows a penetration strategy.

Rewriting (9.2), it is easy to show that if

$$\lambda_i^i f_{p_{ii}}^i + \lambda_j^i f_{p_{ii}}^j < 0 \tag{9.7}$$

then the dynamic on-net price is lower than the static (when $\lambda_i^i = \lambda_j^i = 0$) on-net price. The condition (9.7) is necessary for a unit decrease of a firm's on-net price p_{ii} to have a positive effect on the firm's future profit, in addition to the increase of the firm's immediate profit. Specifically, a unit decrease of p_{ii} can increase the firm's network size by $(-f_{p_{ii}}^i)$. The increased network size will lead to an increase of the firm's profits in the subsequent periods by $(-\lambda_i^i f_{p_{ii}}^i)$. In addition, a unit decrease of p_{ii} can decrease the other firm's network size by $f_{p_{ii}}^j$, which will also lead to a gain of the firm's profits from the subsequent periods by $(-\lambda_i^j f_{p_{ii}}^j)$. Overall, a unit decrease of a firm's on-net price would increase the firm's future profits by $-(\lambda_i^i f_{p_{ii}}^i + \lambda_j^i f_{p_{ii}}^j)$. Thus, the condition $\lambda_i^i f_{p_{ii}}^i + \lambda_j^i f_{p_{ii}}^j < 0$ captures the long-term effect of a firm's current price change. Since $f_{p_{ii}}^i < 0$, $f_{p_{ii}}^j > 0$, $\lambda_i^i(t) > 0$ and $\lambda_j^i(t) > 0$ only for some $t < t_1$, the condition $\lambda_i^i f_{p_{ii}}^i + \lambda_j^i f_{p_{ii}}^j < 0$ should hold as we move away from $t = t_1$. Therefore, we can offer the managerial guideline that after some time the equilibrium dynamic on-net price should be below the static (myopic) on-net price.

A similar result is true for the off-net price (see Fruchter, Shi and Rao (2004)). Furthermore, if we compare the dynamic network-based discount with the static, we conclude (see Fruchter, Shi and Rao (2004)) that if

$$\left(-\frac{1}{D_{p_{ii}}^{ii}}\right) \sum_{m=i,j} \lambda_m^i f_{p_{ii}}^m > \left(-\frac{1}{D_{p_{ij}}^{ii}}\right) \sum_{m=i,j} \lambda_m^i f_{p_{ij}}^m \tag{9.8}$$

then the dynamic discount is lower than the static. This result shows that the network discount should not be as high in the dynamic case as in the static case. This points to why the network-based price discount would depend on strategic reasons in addition to the cost reasons that occur in static models as well. In other words, managers can gain by an understanding of the effect of their decisions on network dynamics and following the optimal network-based pricing strategy as obtained from an analysis of the appropriate dynamic game. Since $\lambda_i^i > 0$, condition (9.8) is more likely to hold with a decreased ratio of $f_{p_{ii}}^i / f_{p_{ij}}^i$. A decreased ratio of $f_{p_{ii}}^i / f_{p_{ij}}^i$ implies a smaller effect of a firm's on-net price (relative to its off-net price) on consumers' network choice decisions. Therefore, with a smaller $f_{p_{ii}}^i / f_{p_{ij}}^i$, firms have more incentive to lower the off-net price, leading to a smaller network-based price discount. This is the key intuition here.

A special case—when customers are equally likely to call any person on their network and the persons on a competing network (termed as uniform calling pattern) show that the firm's dynamic network-based discount increases with the membership fee thus with the network size. The combination of the penetration strategy for the membership fee and the "penetration strategy for the discount" uncovers an interesting property of the dynamic two-part tariff. Unlike the static setting where the network-based discount is a result of interconnection costs (Laffont, Rey and Tirole (1998b)), in the dynamic setting the firms focus on customer acquisition at the beginning through a low membership fee, and then, when the customer base is large enough, the firm can effectively retain customers by offering a higher discount for on-net usage.

4. Symmetric competition: The feedback Nash equilibrium strategy

To be able to get an explicit solution and a further characterization, in this section we assume a symmetric competition. More precisely, we suppose that both firms are symmetric, and so have the same cost parameters and the same parameters in the demand functions and growth of their networks. We look for a symmetric equilibrium, that is,

$$k_i^* = k_j^* = k^*, \quad p_{ii}^* = p_{jj}^* = p_{on}^*, \quad p_{ij}^* = p_{ji}^* = p_{off}^*. \tag{9.9}$$

This implies

$$N_i = N_j = N \text{ and } \lambda_i^i = \lambda_j^j = \lambda_{on} \text{ and } \lambda_i^j = \lambda_j^i = \lambda_{off}\,. \tag{9.10}$$

Also

$$f^i = f^j = f \text{ and } D^{ii} = D^{jj} = D^{on}, \quad D^{ij} = D^{ji} = D^{off}. \tag{9.11}$$

The following theorem on feedback equilibrium will prove useful in providing a solution for the optimal policy.

THE MAIN THEOREM: *Consider the differential game stated in (9.1) and assume that the sufficient condition*[2] *holds in some neighborhood of* $(\boldsymbol{k}^*, \boldsymbol{p}^*_{on}, \boldsymbol{p}^*_{off})$. *Then the necessary conditions (3.2)–(3.4) define a unique local time-invariant feedback Nash equilibrium of the form,*

$$\left.\begin{array}{lcl} k^* & = & k^*(N, \Phi(N), \varphi(N)) \\ p^*_{on} & = & p^*_{on}(N, \Phi(N), \varphi(N)) \\ p^*_{off} & = & p^*_{off}(N, \Phi(N), \varphi(N))\,, \end{array}\right\} \tag{9.12}$$

where $\Phi(N)$ *and* $\varphi(N)$ *satisfy the following system of backward differential equations,*

$$\begin{aligned} &\Phi'(N) f(N, k^*, p^*_{on}, p^*_{off}) = r\Phi(N) - (k^* - c^k) \\ &-(p^*_{on} - c_{on})\left[D^{on}(p^*_{on}, N) + N D^{on}_N(p^*_{on} N)\right] \\ &-(p^*_{off} - c_{off})\left[D^{off}(p^*_{on}, N) + N D^{off}_N(p^*_{off} N)\right] \\ &-[\Phi(N) + \varphi(N)] f_N(N, k^*, p^*_{on}, p^*_{off}) \\ &-[\Phi(N) f^i_{k_j}(N, k^*, p^*_{on}, p^*_{off}) + \varphi(N) f^j_{k_j}(N, k^*, p^*_{on}, p^*_{off})] \\ &\qquad \frac{\partial k_j}{\partial N_i}(N, \Phi(N), \varphi(N), k^*, p^*_{on}, p^*_{off}) \\ &-[\Phi(N) f^i_{p_{jj}}(N, k^*, p^*_{on}, p^*_{off}) + \varphi(N) f^j_{p_{jj}}(N, k^*, p^*_{on}, p^*_{off})] \\ &\qquad \frac{\partial p_{jj}}{\partial N_i}(N, \Phi(N), \varphi(N), k^*, p^*_{on}, p^*_{off}) \\ &-[\Phi(N) f^i_{p_{ji}}(N, k^*, p^*_{on}, p^*_{off}) + \varphi(N) f^j_{p_{ji}}(N, k^*, p^*_{on}, p^*_{off})] \\ &\qquad \frac{\partial p_{ji}}{\partial N_i}(N, \Phi(N), \varphi(N), k^*, p^*_{on}, p^*_{off}) \\ &\lim_{t\to\infty} \Phi(N(t)) e^{-rt} = 0 \end{aligned} \tag{9.13}$$

[2]See condition (B1) in Appendix B of Fruchter, Shi and Rao (2004).

and

$$\varphi'(N)f(N,k^*,p^*_{on}f^*_{off}) = r\varphi(N)$$
$$-(p^*_{off}-c_{off})ND^{off}_N - [\Phi(N)+\varphi(N)]f_N(N,k^*,p^*_{on}p^*_{off})$$
$$-[\Phi(N)f^i_{k_j}(N,k^*,p^*_{on},p^*_{off})+\varphi(N)f^j_{k_j}(N,k^*,p^*_{on},p^*_{off})]$$
$$\frac{\partial k_j}{\partial N_j}(N,\Phi(N),\varphi(N),k^*,p^*_{on},p^*_{off})$$
$$-[\Phi(N)f^i_{p_{jj}}(N,k^*,p^*_{on},p^*_{off})+\varphi(N)f^j_{p_{jj}}(N,k^*,p^*_{on},p^*_{off})]$$
$$\frac{\partial p_{jj}}{\partial N_j}(N,\Phi(N),\varphi(N),k^*,p^*_{on},p^*_{off})$$
$$-[\Phi(N)f^i_{p_{ji}}(N,k^*,p^*_{on},p^*_{off})+\varphi(N)f^j_{p_{ji}}(N,k^*,p^*_{on},p^*_{off})]$$
$$\frac{\partial p_{ji}}{\partial N_j}(N,\Phi(N),\varphi(N),k^*,p^*_{on},p^*_{off})$$
$$\lim_{t\to\infty}\varphi(N(t))e^{-rt}=0 \tag{9.14}$$

where $i,j=1,2,\ i\neq j$

Proof: See Fruchter, Shi and Rao (2004).

A numerical analysis of the symmetric equilibrium (see Fruchter, Shi and Rao (2004)) offers insights into the role of competition in dynamic network-based pricing. In the early stage of the diffusion process, when competition for new customers is high, price discrimination through the use of a network-based discount is not critical. But as the network size grows, and the diffusion process enters a low growth phase, focus turns to price discrimination and so the network-based discount is higher. Indeed, equilibrium dynamic prices in later stages approach the static, or myopic, prices characterized by Laffont, Rey and Tirole (1998b).

5. Conclusions

The dynamic two-part tariff competitive pricing policy presented here extends the existing dynamic pricing research, e.g. Kalish (1983), Dockner and Jorgensen (1988), and Fruchter and Rao (2001). For the dynamic network-based pricing, we can recommend the following managerial guidelines. As in the case of monopoly, the membership fee follows a penetration strategy being low at the beginning and increasing as the customer base increases. The network-based discount, under certain conditions, for instance, when customers are equally likely to call any person on their network and the persons on a competing network, also follows a penetration strategy. The combination of the penetration strategy for the membership fee and the "penetration strategy for the network-based discount" leads to the following managerial guideline: when a firm has a small network, the firm should focus on acquiring new customers through a low membership fee. As its network grows, the firm should turn more attention to customer retention by offering a higher network-based price discount.

The results from the numerical example of symmetric equilibrium show that the dynamic network-based prices are lower than the static counterparts. Consequently, the network-based price discount is also smaller in the dynamic game than in the static game. An interesting pattern emerges that a firm's dynamic network-based price discount increases with the firm's network size. Since the firms' networks expand over time, the result suggests a small (or none at all) network-based price discount in the early stage of the diffusion process, but a large network-based price discount in the late stage of the diffusion process. From the numerical analysis of symmetric equilibrium, we find that firms compete intensively for valuable new customers in the early stage of the diffusion process. As a result, despite the interconnection cost, the difference between the on-net and off-net prices is very small. In contrast, since the new adopters arriving in the later stage of the diffusion process are less valuable, we observe much softer price competition and the equilibrium prices approach the static prices that Laffont, Rey and Tirole (1998b) characterized. A valuable direction for future research would be to empirically investigate the network-based price discounts over the diffusion process and test the predictions of this model.

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Chapter 10

NUMERICAL SOLUTIONS TO LUMP-SUM PENSION FUND PROBLEMS THAT CAN YIELD LEFT-SKEWED FUND RETURN DISTRIBUTIONS

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Abstract The paper is about pension fund problems where an agent pays an amount x_0 to the fund manager and is repaid, after time T, a lump sum $x(T)$. Such problems admit an analytical solution for specific, rather unrealistic formulations. Several practical pension fund problems are converted in the paper into Markov decision chains solvable through approximations. In particular, a couple of problems with a non-differentiable asymmetric (with respect to risk) utility function are solved, for which left-skewed fund-return distributions are reported. Such distributions ascribe more probability to higher payoffs than the right-skewed ones that are common among analytical solutions.

1. Introduction

This paper[1] is about lump-sum pension fund problems *i.e.*, such where an amount x_0 is paid to the fund manager by an agent who is repaid, after time T, a lump sum $x(T)$. The latter is called here the pension.

The purpose of this paper is to propose approximately optimal solutions to several such pension fund problems where performance measures are asymmetric with respect to risk. We will show that the fund return distributions obtained for those measures can be left skewed (negatively). Such distributions ascribe *more probability to higher payoffs* than the right-skewed ones that are common among analytical solutions to prob-

[1]This paper draws from Krawczyk (2003).

C. Deissenberg and R.F. Hartl (eds.), Optimal Control and Dynamic Games, 155–176.

lems characterised by the HARA (Hyperbolic Absolute Risk-Aversion) family of utility functions. If most utility realisations (mode) are low then a policy which leads to that result might not be acceptable for a realistic fund manager.

The problem of how to produce an acceptable portfolio-performance strategy is well recognised in the *static* context. Markowitz (1952) is credited with pioneering the classical mean-variance portfolio selection problem whose solution balances a good average yield with a certainty of achieving it. Since Markowitz (1959) seminal book, many authors have worked on extensions and other methods for portfolio optimisation that would allow for hedging against uncertainties and/or assure an acceptable level of payoff; see Bertsimas *et al.* (2004), de Athayde & Flôres Jr. (2004), Rockafellar & Uryasev (2000) or Krawczyk (1990).

On the other hand, the mainstream research onto *dynamic* portfolio management has been following the seminal works by Samuelson (1969) and Merton (1969) (also Merton (1971)) and concentrated on solutions to the HARA problems. Such solutions typically provide an optimal strategy in closed form. The solutions are generically risk-sensitive but leave aside the utility distribution issues; see *e.g.*, Brennan *et al.* (1997), Fleming & Sheu (2000), Morck *et al.* (1989).

Recently, problems of hedging and/or assurance of attainment of a desired payoff (thus "realistic") have been investigated in the context of dynamic portfolio management. For example, Howe *et al.* (1996) discuss multiperiod minimax hedging strategies, Gülpinal *et al.* (2004) apply mean-variance analysis to multistage portfolio management and Frey & Runggaldier (1999) apply risk-minimising hedging approach to dynamic strategy determination. Most importantly for "certainty" of payoff achievement, Value-at-Risk and Conditional Value-at-Risk constraints have been used in Yiu (2004) and Bogentoft *et al.* (2001), respectively, to produce acceptable management policies in the dynamic context.

Basak & Shapiro (2001) consider two types of "realistic" portfolio managers: "VaR-RM" — that incorporate a Value-at-Risk constraint in their optimisation problem formulation and "LEL-RM" — that limit their expected losses, and who appear very much like the Conditional Value-at-Risk managers from Bogentoft *et al.* (2001). Basak & Shapiro (2001) obtain closed form solutions for a CRRA utility function for either manager and compare performance of their portfolios. The problem with a Value-at-Risk constraint yields a solution in which while (as expected) the Value-at-Risk is limited, larger losses will be incurred "in the most adverse states of the world" than under no VaR constraint. The authors show that this is because a VaR risk manager minimises the probability of the loss regardless of the magnitude. A limit on *expected losses* (LEL) can produce strategies that lead to much smaller expected losses than under the VaR constraint. Overall, the provided distribution plots indicate that an introduction of a VaR constraint and of a LEL limit "improve" the wealth distribution in that the difference between the mean and median reduces vis-a-vis an unconstrained solution. However, the plots are still such that the mean dominates the median, which signifies a low mode.

With the exception of Basak & Shapiro (2001) who added constraints to a CRRA utility function, explicit solutions to "realistic" dynamic portfolio management problems are usually beyond the simple quadratures. Consequently, the authors have used stochastic programming and/or discretisation techniques to solve their problems.

The pension fund problems solved in this paper will be with "realistic" preferences and they will be solved numerically. Attention will be paid to the utility realisations' distributions rather than to moments and parameters which were of prime interest in most of the above papers. Notwithstanding, a Value-at-Risk will be gauged for each solution.

Numerical solutions will be obtained in this paper through a discretisation scheme inspired by the Kushner (1990) approach[2]. His approach consists of a discretisation method, capable of producing solutions to stochastic optimal control problems. There have been successful implementations of this approach to *infinite*-horizon decision problems *e.g.*, Munk (2000) and stochastic differential games Haurie *et al.* (1994). However, in Krawczyk (2001) (also see Krawczyk & Windsor (1997) and Krawczyk (1999)) a similar but simpler approach that produced numerical solutions to a few *finite*-horizon stochastic optimal control problems was used. Instead of looking for a solution to the Hamilton-Jacobi-Bellman (*HJB*) equation, as in the Kushner approach, a Markov decision chain, discrete in time and space, was solved. This is a more elementary exercise: instead of looking for a numerical solution to a second-order partial differential equation (*HJB*), a first order difference equation (Bellman's) needs to be solved. This approach will be used in this paper to the pension fund problems' solution.

Following is a brief outline of what the paper contains. A pension fund problem is outlined in Section 10.2 as one of stochastic optimal control. A Markovian approximation method suitable for the solution of such a problem is applied in Section 10.3 to several optimal investment problems with asymmetric utility functions. Concluding remarks are

[2]For an up-to-date and complete treatment of Kushner's Markov chain approximation method in stochastic control, see Kushner & Dupuis (2001).

contained in Section 10.4. The approximation method appears original and is explained in Appendix 10.A.

2. Pension fund problems

2.1 General considerations

A plausible situation in financial management is one, in which an agent pays an amount x_0 to a fund manager at time 0, to be repaid, at time T, a *lump sum* $x(T)$ - called here the pension. The latter is a result of an investment policy[3] $\mu(x_t, t), t \in [0, T]$ adopted by the fund manager, and also of the market conditions. The latter are deterministically non predictable and usually modelled with the help of stochastic processes. This causes the pension problem to be also stochastic.

There are several practical questions both agent and manager need to answer before x_0 can be accepted. Among them:

- what lump sum $\bar{x}_T$ can the manager "promise" for a given x_0 or, alternatively, what x_0 should be paid in for a promised $\bar{x}_T$?

- how should the "promise" be formulated: deterministically or in some probabilistic terms?

seem to be crucial for the fund management. We shall try to answer them in the sequel.

The manager's policy of administering a fund depends obviously on their objective function. The latter can be maximisation of the fund expected value, maximisation of probability to obtain a target amount, shortfall minimisation *etc.* Once the objective function is revealed, the manager's policy can be computed as a solution to a stochastic optimal control problem associated with the objective function. The problem solution should routinely comprise an optimal decision rule $\mu(x_t, t)$ and also a distribution of x_T. Knowing the former is crucial for the manager to control the portfolio. The latter is "practical" in that it tells the pension buyer what they can, or should, expect as x_T.

Knowing the distribution of x_T also helps the manager. It gives them an idea of what probabilities, or risks, are associated with obtaining a particular realisation of the objective function. For example, the distribution may suggest that, for a given x_0 there is a "probable" terminal value $\bar{x}_T$, which the manager may choose to advertise as the pension target (subject to some legislative constraints on financial advertising).

[3] If appropriate, the rule will be multidimensional, comprising a consumption rule, administration fee *etc.*

2.2 A stochastic optimal control problem

We can model a pension fund problem as above using a simplified version of Merton (1971)'s model of optimal portfolio selection (*cf.* Fleming & Rishel (1975), pp. 160-161; also Tapiero (1998)).

As usual, the stock portfolio consists of two assets, one "risky" and the other "risk free". If the price per share of the risky asset $p(t)$ changes according to

$$dp(t) = \alpha\, p(t)dt + \sigma\, p(t)dw \tag{10.1}$$

where w is a one-dimensional standard Brownian motion, while the price $q(t)$ per share for the risk free asset changes according to

$$dq(t) = rq(t)dt$$

then the wealth $x(t)$ at time $t \in [0, T]$ changes according to the following stochastic differential equation

$$dx(t) = (1 - u(t))rx(t)dt + u(t)x(t)(\alpha dt + \sigma dw) - U(t)dt. \tag{10.2}$$

Here, r, α, σ are constants with $r < \alpha$ and $\sigma > 0$. The symbol $u(t)$ (respectively, $1 - u(t)$) denotes the fraction of the wealth invested in the risky (respectively, risk free) asset at t and $U(t)$ is the "consumption" rate (which could also model a management fee intensity).

In Fleming & Rishel (1975), $U(t)$ is the consumption rate and the agent's objective is to find an optimal two-dimensional strategy $\mu = [u(x,t), U(x,t)]$, such that[4]

$$0 \leq u(t) \leq 1, \quad \text{and} \quad U(t) \geq 0, \tag{10.3}$$

and which maximises the expected discounted total utility

$$J(0, x(0); \mu) = \mathbb{E}\left(\int_0^T e^{-\varrho t} g(U(t))dt \,\middle|\, x(0) = x_0 \right) \tag{10.4}$$

given the discount rate $\varrho > 0$ and assuming that $g(U(t))$ is the manager's utility function (*e.g.*, $g(U(t)) = [U(t)]^\gamma$, with $0 < \gamma < 1$). In Fleming & Rishel (1975) there is no value assigned to wealth at T while x_0 is the wealth at the initial time 0. However, utility (10.4) can be augmented to

$$J(0, x(0); \mu) = \mathbb{E}\left(\int_0^T e^{-\varrho t} g(U(t))dt \, + e^{-\varrho T} s(x_T) \,\middle|\, x(0) = x_0 \right) \tag{10.5}$$

[4]Constraints (10.3) mean no borrowing or short selling. These restrictions have been weakened in the literature; however, they may be reasonable in some situations and they make the problem "strictly" constrained, which is a feature that the optimisation algorithm needs to be tested on.

where $s(x_T)$ is a function of final wealth. The problem "maximise (10.5) subject to (10.2) and other relevant constraints" (like *e.g.*, $x(t) \geq 0$) can now model a pension fund problem. Indeed, making $U = const$ and maximising (10.5) in u only (with $g(\cdot) \equiv 0$ or not), defines a problem that captures the task of a pension fund manager. On the other hand, max (10.5) s.t. (10.2) (*plus* constraints) defines a stochastic optimal control problem. It is this problem that lacks a closed from solution for some "realistic" utility functions and therefore needs to be solved numerically.

3. Pension fund models and solutions

3.1 The yield's expected utility maximisation

Maximisation of concave utility functions is a rather popular method to establish an optimal policy for portfolio management (see *e.g.*, Morck *et al.* (1989), the classical Merton (1971) and *many* more). Perhaps the functions are so popular because they frequently produce analytical solutions. Sadly, some of them deliver policies that generate rather unacceptable utility realisation distributions.

Suppose an initial outlay x_0 is be managed for T years. An academically learned fund director may choose to manage this outlay according to a policy that maximises a concave, risk-averse utility function. Let us investigate what policy maximises a terminal fund *utility* defined as

$$s(x_T) = \big[x(T)\big]^{\delta}, \quad 0 < \delta < 1. \tag{10.6}$$

The optimal policy is analytically obtainable. Set $g\big(U(t)\big) = 0$ and follow one of classical Merton's references[5] to obtain

$$u(t) = \frac{(\alpha - r)}{(1-\delta)\sigma^2}\bigg|_{\delta=.5} = \frac{2(\alpha - r)}{\sigma^2}\,. \tag{10.7}$$

The optimal strategies are risk sensitive. However, for a given value of parameter δ, which reflects an agent's degree of risk aversion, they are constant in time and independent of state. (To focus attention we will assume $\delta = .5$.)

Let us look at Figure 10.1 to see what the *utility* realisation distribution for this policy is. The distribution was obtained using optimal strategy (10.7) for wealth equation (10.2) and the following model para-

[5] Also, Fleming & Rishel (1975), Tapiero (1998), Korn & Korn (2001) and many more.

meters[6]:

$$x_0 = 40\,000\,,\ T = 10,\ r = .05,\ \alpha = .11,\ \sigma = .4,\ \varrho = .11\,.$$

These parameters seem rather plausible (*e.g.*, 5% *p.a.* return on the secure asset, $\alpha = \varrho$, ten years until retirement, *etc.*) hence, the numerical experiments conducted below should be readily interpretable. What we can see first, is that the distribution is rather strongly skewed to the right. This means that most utility realisation for the optimal policy (10.7) are low.

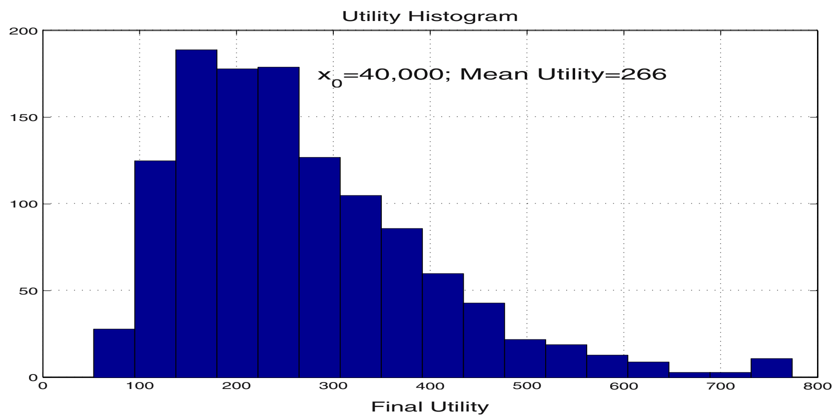

Figure 10.1. Final utility distribution.

Observe the final fund yield distribution in Figure 10.2. The figure informs the fund manager about the actual fund payoffs and is probably more relevant for their decision of whether policy (10.7) is *acceptable*.

[6]As only some distribution shapes could be computed analytically, all distributions in this paper have been obtained experimentally to ensure some comparability of the results. This and the following distributions were obtained as histograms of 1200 fund realisations.

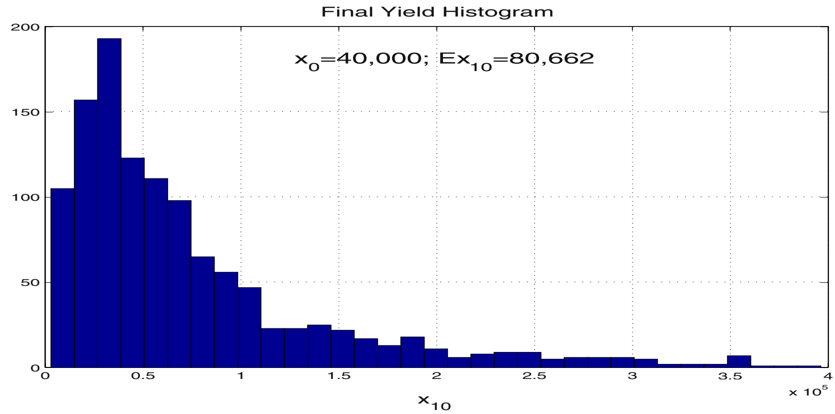

Figure 10.2. Fund yield distribution for the yield's expected utility maximisation.

This figure[7] also helps to (approximately) compute a value at risk, which will be useful to compare performance of various strategies. Two risk measures, Value-at-Risk (**VaR**) and Conditional Value-at-Risk (**CVaR**) will be calculated. The measures explanation can be found in *e.g.*, Rockafellar & Uryasev (2000) or Basak & Shapiro (2001); here, we notice briefly that for a specified probability level β, the ${}_{\boldsymbol{\beta}}$**VaR** of a policy is the lowest amount of loss α whose probability does not exceed β. For the same probability level β, the ${}_{\boldsymbol{\beta}}$**CVaR** is the mean loss above the amount α. We well define any final fund yield below \$ 40 000 as *loss*.

For the policy maximising the risk averse utility function (10.6) with an initial outlay $x_0 = 40\,000$, $\beta = .9$ and the loss defined as above we can see that the mean payoff is as large as 80 662; however, there is also a lot to be lost: ${}_{\mathbf{.9}}\mathbf{VaR} = \bar{\alpha}_{.9} \approx 28\,000$ and ${}_{\mathbf{.9}}\mathbf{CVaR} \approx 34\,000$. Moreover, we can see that the probability of earning less than the "secure" revenue, which one would receive by investing in the risk-free asset only:

$$40\,000 \exp\{(r - \text{"management fee"})10\} = 53{,}994 \ ,$$

for a 2% management fee, is more than .5.[8] With probability >.4, the final payoff will be less than the initial outlay $x_0 = 40\,000$. Evidently, using a policy that maximises the expected final yield utility (10.6) is a very risky strategy of managing a portfolio. We conclude that the

[7] Acutually, this histogram looks almost indistinguishable from one that shows a fund yield distribution obtained for a risk neutral utility function, see Krawczyk (2003).

[8] To see this and also prove the next claim normalise the area under the histogram to 1 and integrate from zero to 53,994 and 40 000 , respectively.

maximisation of the final payoff utility cannot be a *realistic* pension fund target.

3.2 Shortfall risk minimisation

Given a fairly disperse distribution of $x(T)$ when a policy of yield's expected utility maximisation is applied, a realistic portfolio manager might target a specific amount $\bar{x}_T$ as one to be repaid to an agent who has deposited x_0. (Should $x(T)$ be deliverable with a high probability, the manager could used it *e.g.* for advertising and thus beat the vague promises by the competition.) Hence, $x(T)$ might be viewed as the manager's liability H_T at a given future time T. (See *e.g.* Frey & Runggaldier (1999) for an optimisation model that computes risk-minimizing hedging strategies.)

We will use a numerical method[9] to compute a policy that minimises the risk of not delivering $\bar{x}_T$. We will examine the distribution of $x(T)$ to assess how likely receiving $\bar{x}_T$ is.

In "real life", $\bar{x}_T = H_T$ may be measurable with respect to the $\sigma-$algebra generated by a stochastic process related, or not, to the wealth dynamics (10.2). For example, the price process $p(t)$ -see (10.1)- of the risky asset could be the process, which generates the algebra. This would be a realistic feature of the model, which would help adjusting $x(T)$ to different market situations. There may be situations in which an investment policy *realisation* might appear profitable but in fact be a result of high inflation. In that case, the "pension" $\bar{x}_T$ the agent would like to secure, would need to be higher than if inflation was low. Conversely, with a less expanding process $p(t)$, the pension or liability H_T could be lower.

Thus H_T may be a random variable, *contingent* upon the prices of the risky asset. We say that a liability is (perfectly) hedged, respectively super-hedged, if

$$x_u(T) = H_T \qquad \text{almost surely} \tag{10.8}$$

or

$$x_u(T) \geq H_T \qquad \text{almost surely,} \tag{10.9}$$

respectively, where u is an investment strategy. There may not exist a strategy that achieves a perfect hedge and, to achieve a superhedging, it may be necessary to start from a rather large initial outlay x_0.

[9]Explained in Appendix 10.A. Briefly, the policy will be determined as an optimal solution to a Markov decision chain approximating the underlying stochastic optimal control problem.

A reasonable criterion of a portfolio selection is then that of *shortfall risk minimisation* where, given an admissible set A of strategies u (*e.g.*, A contains u constrained as in Section 10.2.2) and an initial outlay x_0, one looks for a strategy $u^* \in A$ for which

$$\mathbb{E}\{\max(0, H_T - x_{u^*}(T))\} = \inf_{u \in A} \mathbb{E}\{\max(0, H_T - x_u(T))\} \qquad (10.10)$$

where $\max(0, H_T - x_u(T))$ represents the *shortfall i.e.*, the amount by which the final value $x_u(T)$ of the portfolio *falls short* of the goal H_T.

Now, we will solve numerically[10] the above shortfall minimisation problems. Let T=10, x_0=40 000 , *etc.* as before and let $H_{10} = \pi\, p(10) x_0$ where $\pi = 83\%$ is some inflation "relaxation" coefficient. Bearing in mind that such $\mathbb{E}p(10) \approx 3$, H_{10} will *on average* equal \$100 000 (so, these results are somehow comparable across the many examples solved in this paper).

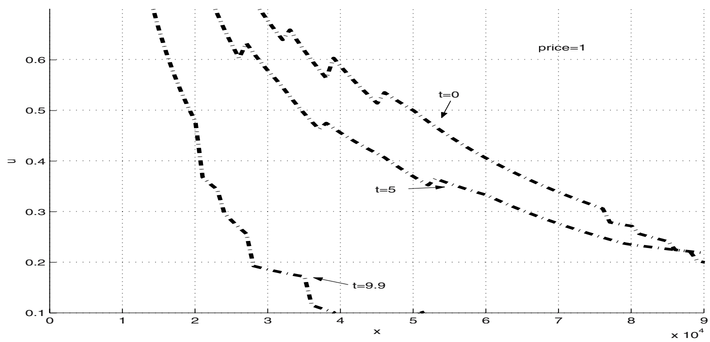

Figure 10.3. Shortfall minimising policies.

Figure 10.3 shows the policy[11] graphs $u(x,p)|_{p=1}$. The horizontal axis is wealth (x_t) and the vertical axis is the corresponding investment u_t. Each curve in the figure corresponds to a strategy for a different time. The times represented are: the beginning of the investment period ($t = 0$), the middle time ($t = 5$) and a final time ($t = 9.9$).

The shortfall minimising policies are state *dependent.* This is in sharp contrast to *flat* policies (10.7). Here, when you are far from the "target" you invest a lot. However, as times goes by you become less aggressive. For example, assuming the price level $p = 1\ \forall t$, the same level of wealth

[10] For the ease of computations $\max(0, H_T - x_u(T))^2$ was minimised.
[11] An updated version of Windsor & Krawczyk (1997) was used for numerical solutions obtained here and in the rest of this paper.

of 40 000 implies an investment of (approximately) 0.6, 0.45 and 0.1 at times 0, 5 and 9.9 respectively.

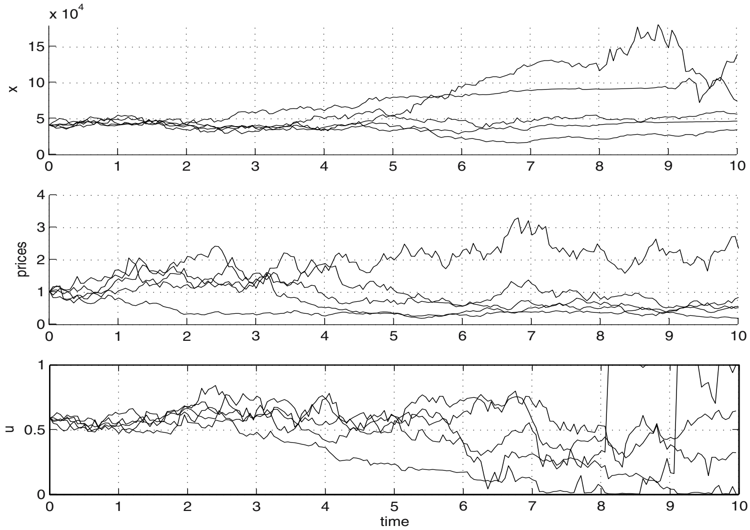

Figure 10.4. A sample of time profiles for the shortfall minimisation problem.

Figure 10.4 presents a sample of policy realisations (bottom panel) and the corresponding wealth and prices time profiles. The profiles comprise probably quite extreme cases of superhedging (one) and of substantial underhedging (three). We can see that "anything" is possible with policy u^* that minimises shortfall (10.10). However, much more useful for the policy assessment is to examine Figure 10.5, which shows the fund yield distribution.

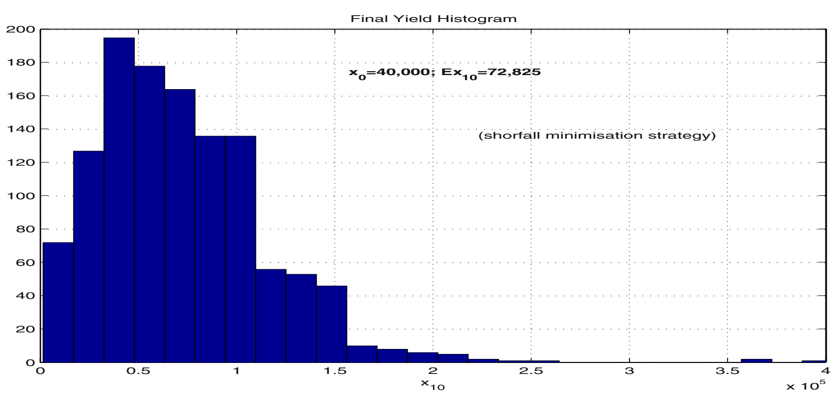

Figure 10.5. Fund yield distribution for the shortfall minimisation problem.

Here, both the median and mean appear on the right hand side of the initial outlay value of 40 000. Also, $_{.9}\mathbf{VaR} \approx 16\,000$ (for loss=40 000) is smaller than before. Clearly, the shortfall minimising policies generate a more desirable type of skewness than that in Figure 10.2.

3.3 Cautious policies

Notwithstanding a better yield distribution for a shortfall minimisation policy than for yield's utility maximisation, the spread of possible realisations of $x_{u^*}(T)$ might still appear too wide for the manager to implement u^*. We would argue that this might be because of no "reward" for superhedging x_T in problem (10.10). In real life, pension fund managers will want to strongly avoid any yield below x_T (as in (10.10)) but they will also enjoy yields above x_T. We will want to capture this "realistic" feature of portfolio management though a non-symmetric (with respect to risk) utility function. Consider

$$\begin{aligned} J(0, x_0; u^*) &= \max_{u \in A} \mathbb{E}\left(s(x_u(T)) \middle| x(0) = x_0 \right) && (10.11) \\ \text{where} \quad A & \ \text{contains } u \text{ constrained as in Section 10.2.2 and} \\ s(x_u(T)) &= \begin{cases} (x_u(T) - \bar{x}_T)^\kappa & \text{if} \quad x_u(T) \geq \bar{x}_T, \\ -(\bar{x}_T - x_u(T))^a & \text{otherwise} \end{cases} && (10.12) \\ & \quad 0 < \kappa < 1,\ a > 1\,. \end{aligned}$$

This criterion[12] reflects the client's (and manager's) wish to dispose of sufficient funds to meet $\bar{x}_T$, which could be advertised as a likely payoff. The reward for exceeding $\bar{x}_T$ is moderate ($0 < \kappa < 1$) while the punishment for not reaching it might be made substantial ($a > 1$). The policies will be computed for two different combinations of κ and a. We will call these policies *cautious* because the manager is "cautious" in enjoying wealth above $\bar{x}_T$.

Suppose that the desired value of the pension is $\bar{x}_{10} = 100\,000$. The rest of the problem parameters are as before *i.e.*, $x_0 = 40\,000$, $T = 10$, $r = .05$, *etc.* Also notice that $\bar{x}_{10} = 100\,000$ will be reached with *certainty* if x_0 =74,082 is invested at time 0 in the secure asset (management fee = 2%) and also if $100\,000\exp(-.03t)$ is invested at time t.

Optimal investment policies resulting from the solution to (10.11), (10.12) are shown in Figure 10.6 for two different pairs of κ and a.

In the top panel, the three curves represent a cautious policy obtained for a client whose preferences are reflected by $\kappa = \frac{1}{2}$ and $a = 2$. Here,

[12] The expectation will be well defined under the implicit integrability assumption.

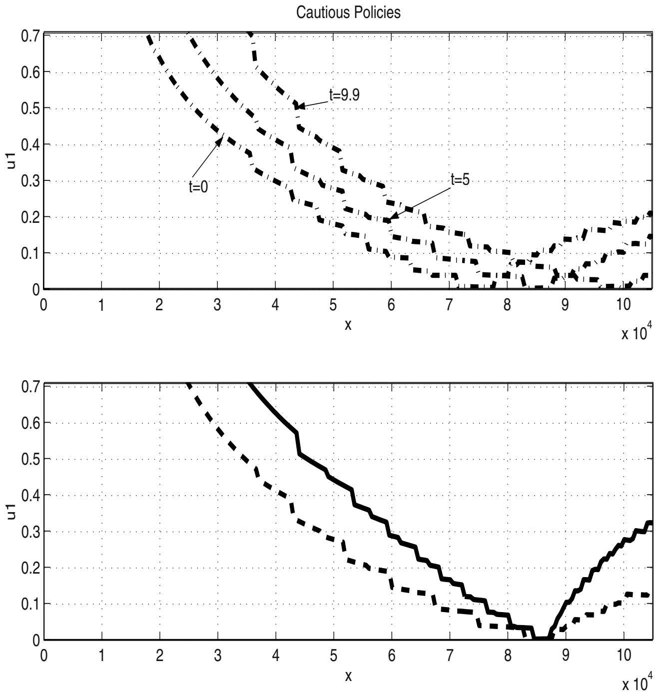

Figure 10.6. Cautious policies.

the preferences tell us about a client's "quadratic" fear of not reaching $\bar{x}_{10}$ and a "square-root" enjoyment from exceeding it.

The lower panel shows two strategy lines for $t = 5$. The dash-dotted line is as in the top panel. The solid line corresponds to a client's preferences reflected by $\kappa = \frac{9}{10}$ and $a = \frac{3}{2}$. Here, the client's attitude is also cautious but more relaxed about not meeting $\bar{x}_{10} = 100\,000$; moreover he (or she) enjoys exceeding it more than the previous client does.

The top-panel policy lines get higher as the time-to-maturity shortens and steeper as wealth x_t decreases. This is an interesting solution: if wealth is far from desired or there is not enough time for the pension fund to grow, the manager invests in the risky investment more than if he (or she) has higher wealth and/or longer time to grow it. In other words, the further the investor is from the state, from which meeting $\bar{x}_{10}$ is certain (*e.g.*, x_0=74,082 at $t = 0$), the higher the investment in the risky asset is.

The lower panel of Figure 10.6 shows that the optimal policies of the "cautiously relaxed" customer are bolder than those of their "cautious" counterpart. This means that at a given state (x_t, t) the relaxed will be happy to commit more funds to the risky investment than the cautious.

Figure 10.7 shows a sample of (five, as in Figure 10.4) state and control realisations corresponding to the cautious policy. A difference between the large spread of the final states in Figure 10.4 and this figure's rather concentrated distribution of the state variable is clearly visible.

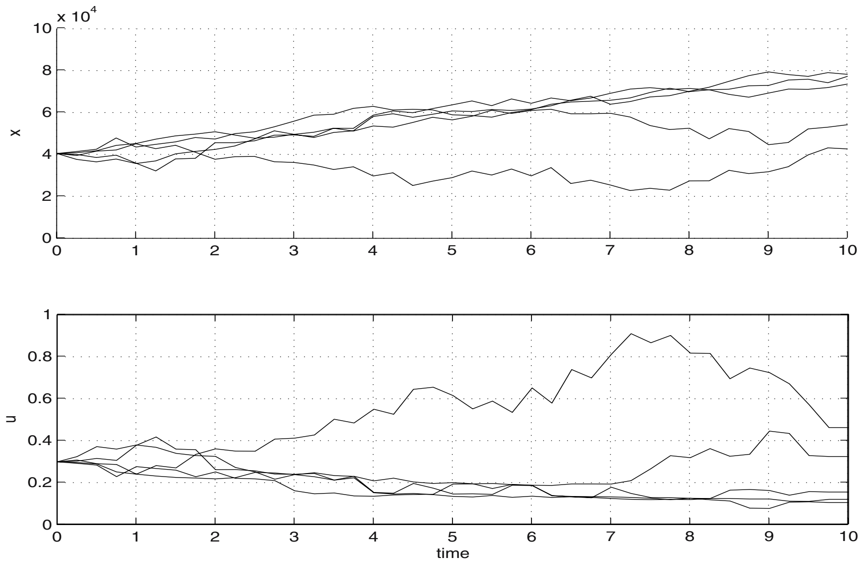

Figure 10.7. Time profiles for the cautious policy.

The usefulness of the strategies obtained as solutions to the utility maximisation problem (10.11), (10.12) for pension fund management can be assessed with the help of the histograms presented in Figure 10.8.

The two strategies (cautious and relaxed-cautious) were used to manage an initial outlay of $x_0 = \$40\,000$. See Figure 10.8 top panel for the yeld distributions. The lower panel of this figure shows the yield distribution of when the expected yield maximisation policy was used (in Krawczyk (2003)) to generate $Ex_{10} \approx 100\,000 = \bar{x}_{10}$. One can see that the type of skewness, which the cautious policies generate, helps the manager to form a strong expectation of a satisfactory final payoff. For example, for the current set of data, we can say that $x_{10} \geq 70\,000$ has the probability of about .75 , for the cautiously relaxed policy. Under the expected yield maximisation policy, the probability of achieving this result is less than .3 and less .4 for the yield's utility maximisation (Figure 10.2) and shorfall minimisation (Figure 10.5). It is also easy to

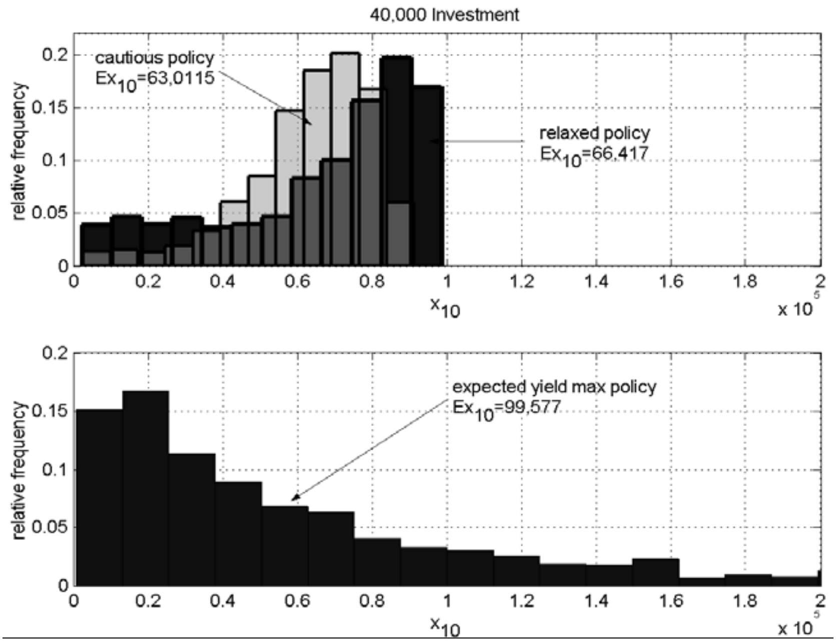

Figure 10.8. Policy comparisons.

see that the risk measures VaR and CVaR for the cautious policies are very small.

A further analysis of the final yield distributions, calculated in Krawczyk (2001) for several initial deposits $x_0 \in [40\,000\,, 74\,082]$, would help the manager to make up their mind which particular $\underline{x}_0$ to accept, in return for a ten year "bond" or pension $\bar{x}_{10}$.

Notice that the portfolio problems with a HARA utility function considered in the Merton problem ((10.5) or (10.4) subject to (10.2) and other relevant constraints) is homogeneous and its "classical" optimal investment solution does not depend on the initial outlay x_0, see (10.7). Our solution does *not* have this homogeneity property. It advocates *different* strategies for different clients depending on their wealth. These strategies appear satisfactory as judged by the distribution skewness and the risk measures VaR and CVaR.

4. Conclusion

We have analysed some fund yield distributions generated by policies that optimise classical risk averse utility functions and concluded that the distributions ascribe high probability to low payoffs. This makes them not applicable for pension fund management. Their high probability of low payoffs might be a result of the optimised utility functions being *insufficiently* risk averse to guarantee an acceptable fund performance. As an alternative to these utility functions an asymmetric and non differentiable utility function was proposed in this paper. That function captures the investor's different feelings about exceeding versus failing the target (see (10.12)). Policy rules that optimise this function can generate left-skewed utility distributions that promise pension fund high yields realised with high probability.

As to the bullet point questions asked in Section 10.2 (page 158), the following answers may be suggested.

1 A relationship between $\bar{x}_T$ and x_0 can be read from the fund yield histograms. The manager can advertise $\bar{x}_T$ such that its realisation is 90% probable (or 95%, *etc.*) and/or whose VaR/CVaR is acceptable.

2 Any "promised" $\bar{x}_T$ is stochastic in nature; however, it can be formulated deterministically if the manager is comfortable with its distribution.

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Appendix: A Simple Markovian Approximation

The stochastic optimal control problem from Section 10.2.2 can be formulated in more general terms. The state process is

$$\mathbf{X} = \{\mathbf{X}(t) \in X \quad \mathbb{R}^n,\, t \geq 0,\, \mathbf{X}(0) = \mathbf{x}_0 \, - \text{ given}\}$$

that satisfies the following stochastic differential equation

$$d\mathbf{X}(t) = f\big(\mathbf{X}(t), \mathbf{u}(t), t\big)dt + b\big(\mathbf{X}(t), \mathbf{u}(t), t\big)d\mathbf{W}(t) \tag{10.A.1}$$

where $\mathbf{u}(t) \in U \subset \mathbb{R}^m$ is the control, $\mathbf{W}(t)$ is a Wiener process, $f\big(\mathbf{X}(t), \mathbf{u}(t), t\big)$ is a drift, and $b\big(\mathbf{X}(t), \mathbf{u}(t), t\big)$ is diffusion[13]. The *optimal* control rule μ that determines the control $\mathbf{u}$ is admissible[14] and Markovian

$$\mathbf{u}(t) = \mu(t, \mathbf{X}(t)) \tag{10.A.2}$$

and chosen so as to maximise a functional J

$$\max_{\mathbf{u}} J(0, x_0; \mathbf{u}) \tag{10.A.3}$$

where furthermore

$$J(\tau, x; \mathbf{u}) = \mathbb{E}\left(\int_\tau^T g\big(\mathbf{X}(t), \mathbf{u}(t), t\big)\mathrm{d}t + s\big(\mathbf{X}(T)\big) \,\Big|\, \mathbf{X}(\tau) = x\right) \tag{10.A.4}$$

is the profit-to-go function. We have observed in Section 10.3 that for a "realistic" choice of g and s in (10.A.4) a closed-form solution to the stochastic optimal control problem thus defined becomes difficult if not impossible. On the other hand, finite Markov decision chains are solvable through the dynamic programming technique albeit the solution time can rise exponentially in the number of the chain states. We will now discretise (10.A.4), (10.A.1) in three steps to produce a solvable Markov decision chain[15]. First, the state equation (10.A.1) is discretised in time using the Euler-Maruyama approximation (*cf.* Kloeden & Platen (1992)). Then, the state space is restricted to a finite dimensional discrete state grid and, finally, the transition probabilities and rewards for these discrete states are specified.

Euler-Maruyama Approximation. An Euler-Maruyama approximation[16] of a process $\mathbf{X} \subset \mathbb{R}^1$ that satisfies equation (10.A.1) is a stochastic process

$$\mathbf{Y} = \{\mathbf{Y}_\ell \in X, \quad \ell = 0, 1, \ldots, N\}$$

satisfying the equation (called the iterative scheme)

$$\mathbf{Y}_{\ell+1} = \mathbf{Y}_\ell + f\big(\mathbf{Y}_\ell, \mathbf{u}_\ell, \tau_\ell\big)\big(\tau_{\ell+1} - \tau_\ell\big) + b\big(\mathbf{Y}_\ell, \mathbf{u}_\ell, \tau_\ell\big)\big(\mathbf{W}(\tau_{\ell+1}) - \mathbf{W}(\tau_\ell)\big) \tag{10.A.5}$$

---

[13] For the formal treatment of the optimally controlled diffusion process refer to Fleming & Rishel (1975).
[14] *E.g.*, constrained as in Section 10.2.2.
[15] See Krawczyk (2001) (also see Krawczyk & Windsor (1997) and Krawczyk (1999)) for the discretisation details.
[16] The approximation scheme is introduced here for a one dimensional process. The extension of the scheme to $\mathbb{R}^n$ is obvious.

$\tau \in \{0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$ where $\{\tau_0, \tau_1, \ldots \tau_N\}$ is a partition of the the time interval $[0, T]$.

The the initial and subsequent values are, respectively

$$\mathbf{Y}_0 = \mathbf{X}(0) = x_0, \qquad \mathbf{Y}_\ell = \mathbf{Y}(\tau_\ell). \tag{10.A.6}$$

The discretisation does not have to be equidistant. However, for a N-step time discretisation using a constant time step δ

$$\tau_\ell = \ell\,\delta \;\text{ where }\; \delta = \tau_{\ell+1} - \tau_\ell = \frac{T}{N}. \tag{10.A.7}$$

Discrete State Space. The discrete state space for stage ℓ is denoted by $\overline{X}_\ell \subset \mathbb{R}^1$. Let the upper and lower bounds of the state grid be

$$\overline{U}_\ell = \max \overline{X}_\ell \quad \text{and} \quad \overline{L}_\ell = \min \overline{X}_\ell\,,$$

respectively. A point $x \in X$ is defined to be within the grid $\overline{X}_\ell$ if $\overline{L}_\ell \le x \le \overline{U}_\ell$. The collection of the discrete state spaces for all the stages, $\{\overline{X}_\ell\}_{\ell=0}^N$, is denoted $\overline{X}$ and called the discrete state space. We can say the scheme approximates a point of X at stage ℓ by the points of $\overline{X}_\ell$ which are "adjacent" to it[17].

Transition Probabilities. Consider the stochastic process $\mathbf{Y} = \{\mathbf{Y}_\ell,\ \ell = 0, 1, 2, \ldots N\}$ where $\mathbf{Y}_\ell$ is defined through (10.A.5). For a given control sequence $\mathbf{u}_\ell$ and equidistant discretisation times, the iterative scheme (10.A.5) can be replaced by a *weak* Taylor approximation of an Itô diffusion process (10.A.1) (see Kloeden & Platen (1992)) as follows

$$\mathbf{Y}_{\ell+1} = \mathbf{Y}_\ell + \delta f_\ell + b_\ell \widetilde{\Delta \mathbf{W}}_\ell. \tag{10.A.8}$$

where $\widetilde{\Delta \mathbf{W}}_\ell$ is a *convenient approximation* of the random increments

$$\Delta \mathbf{W}_\ell = \mathbf{W}(\tau_{\ell+1}) - \mathbf{W}(\tau_\ell), \quad \text{for} \quad \ell = 0, 1, 2, \ldots N-1$$

of the Wiener process $\mathbf{W} = \{\mathbf{W}(t), t \ge 0\}$ where $\Delta \mathbf{W}_\ell$ are known (*ibid.*) to be independent Gaussian random variables with mean $\mathbb{E}(\Delta \mathbf{W}_\ell) = 0$ and variance $\mathbb{E}((\Delta \mathbf{W}_\ell)^2) = \tau_{\ell+1} - \tau_\ell$. Process $\widetilde{\Delta \mathbf{W}}_\ell$ needs to have similar moment properties to those of $\Delta \mathbf{W}_\ell$. In the pension fund model, we will use an easily generated two-point random variable taking values $\pm\sqrt{\delta}$ *i.e.*,

$$P\left(\widetilde{\Delta \mathbf{W}}_\ell = \pm\sqrt{\delta}\right) = \frac{1}{2}. \tag{10.A.9}$$

This approximation of the continuously distributed perturbation $\Delta \mathbf{W}_\ell$ by a two-value noise is of course arbitrary[18]

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[17] In $\mathbb{R}^n$ two states are **adjacent** if their projections onto each of the n coordinate axes are adjacent *cf.* Krawczyk & Windsor (1997) or Krawczyk (2001).

[18] One can obviously use other more realistic discrete representations of $\widetilde{\Delta \mathbf{W}}_\ell$ *e.g.*, it can be modelled as a three-point distributed random variable T_ℓ with

$$P\left(T_\ell = \pm\sqrt{3\delta}\right) = \frac{1}{6} \qquad P\left(T_\ell = 0\right) = \frac{2}{3}.$$

However, (10.A.9) appears sufficient for the approximating solutions' convergence. No matter how simple or complex the approximations are, they should preserve the original distribution's first and second moments and depend on the partition interval's length. The latter feature guarantees that, for all such approximations, the smaller δ the less diffuse the states become, to which the process transits.

Now, suppose that at some time τ_ℓ, $\mathbf{Y}_\ell = \overline{\mathbf{Y}}_\ell \in \overline{X}_\ell$. The noise discretisation method implies that for $\delta > 0$ the process reaches (in $\mathcal{R}^1$), at $\ell + 1$:

$$\mathbf{Y}^-_{\ell+1} = \overline{\mathbf{Y}}_\ell + \delta f_\ell - b_\ell\sqrt{\delta} \qquad \text{with probability} \quad \frac{1}{2} \tag{10.A.10}$$

$$\mathbf{Y}^+_{\ell+1} = \overline{\mathbf{Y}}_\ell + \delta f_\ell + b_\ell\sqrt{\delta} \qquad \text{with probability} \quad \frac{1}{2}. \tag{10.A.11}$$

Suppose there are two adjacent states to each $\mathbf{Y}^-_{\ell+1}$ and $\mathbf{Y}^+_{\ell+1}$: the "lower" adjacent states $\overline{\mathbf{Y}}^{-\ominus}_{\ell+1}, \overline{\mathbf{Y}}^{+\ominus}_{\ell+1}$ and the "upper" adjacent states $\overline{\mathbf{Y}}^{-\oplus}_{\ell+1}, \overline{\mathbf{Y}}^{+\oplus}_{\ell+1}$. We assign the probabilities for the process transition from $\overline{\mathbf{Y}}_\ell$ to each of the adjacent states as follows:

$$p(\overline{\mathbf{Y}}_\ell, \overline{\mathbf{Y}}^{-\oplus}_{\ell+1}|\mathbf{u}_\ell) = \frac{1}{2}\,\frac{\mathbf{Y}^-_{\ell+1} - \overline{\mathbf{Y}}^{-\ominus}_{\ell+1}}{h^-_\ell}, \qquad p(\overline{\mathbf{Y}}_\ell, \overline{\mathbf{Y}}^{-\ominus}_{\ell+1}|\mathbf{u}_\ell) = \frac{1}{2}\,\frac{\overline{\mathbf{Y}}^{-\oplus}_{\ell+1} - \mathbf{Y}^-_{\ell+1}}{h^-_\ell}, \tag{10.A.12}$$

$$p(\overline{\mathbf{Y}}_\ell, \overline{\mathbf{Y}}^{+\oplus}_{\ell+1}|\mathbf{u}_\ell) = \frac{1}{2}\,\frac{\mathbf{Y}^+_{\ell+1} - \overline{\mathbf{Y}}^{+\ominus}_{\ell+1}}{h^+_\ell} \qquad p(\overline{\mathbf{Y}}_\ell, \overline{\mathbf{Y}}^{+\ominus}_{\ell+1}|\mathbf{u}_\ell) = \frac{1}{2}\,\frac{\overline{\mathbf{Y}}^{+\oplus}_{\ell+1} - \mathbf{Y}^+_{\ell+1}}{h^+_\ell} \tag{10.A.13}$$

where $h^-_\ell = \overline{\mathbf{Y}}^{-\oplus}_{\ell+1} - \overline{\mathbf{Y}}^{-\ominus}_{\ell+1}$ and $h^+_\ell = \overline{\mathbf{Y}}^{+\oplus}_{\ell+1} - \overline{\mathbf{Y}}^{+\ominus}_{\ell+1}$. So, the probabilities are calculated using an *inverse distance method*, see Krawczyk (2001), and multiplying (or "weighting") the (relative) distances by $\frac{1}{2}$. If any of the states $\mathbf{Y}^{-\ominus}, \mathbf{Y}^{+\ominus}$, *etc.* overlap one another, the respective probabilities have to be summed up.

The above discretisation method is very simple and intuitive (despite complex notation) yet, as noted, preserves the first two moments of the original distribution so that the overall discretisation scheme is *weakly consistent.*[19]

Constraints. It has to be borne in mind that discretisation of a constraint is sensitive to the choice of the discretisation steps and has to be dealt with "carefully". *E.g.*, a state constraint $x(t) > 0$ for $t \in [t_1, t_2]$ cannot be automatically translated to $x_\ell > 0$ and needs to allow for the values of δ and h. [20]

Transition Rewards. Let the control strategy be Markovian (10.A.2) and action at state ℓ computed as

$$\mathbf{u}_\ell = \mu(\ell, \overline{\mathbf{Y}}_\ell),\ \overline{\mathbf{Y}}_\ell \in \overline{X}_\ell,\ \ell = 0, 1, ..N-1. \tag{10.A.15}$$

---

[19]Kushner lists in Kushner (1990), page 1002, three conditions for consistency of an approximation scheme. Conditions "1" and "2" (about continuity of the Markov chain expected value and variance) are easily satisfied:

1 $\mathbb{E}(y^h_{\ell+1} - y^h_\ell) = \delta f_\ell$;

2 $\mathbb{E}[(y^h_{\ell+1} - y^h_\ell) - \mathbb{E}(y^h_{\ell+1} - y^h_\ell)]^2 = \delta\, b^2_\ell$;

condition "3" can be satisfied if the time discretisation interval is allowed to become "shorter" *i.e.*,

3 $|y^h_{\ell+1} - y^h_\ell| = \mathcal{O}(\sqrt{\delta})$.

Moreover, consistency fails along the boundary of the discrete state space so the scheme is *locally* weakly consistent. This is not surprising since it would be impossible for a system constrained to lie within a finite space to follow the behaviour of a system which is not similarly constrained at the points where the constraints become active. However, this feature is common to all approximation schemes of this kind.

[20]It appears (from Merton (1971)) that the best discrete-time counterpart of continuous time constraint $x(t) \geq 0$ is

$$x_\ell\big(1 + \delta(r + u_{1,\ell}(\alpha - r))\big) - \delta U_{2,\ell} \geq 0 \tag{10.A.14}$$

where δ is the time discretisation step.

Recalling (10.A.4), note that for the approximating problem, the decision maker receives a reward that depends on the state at stage ℓ and on the action $\mathbf{u}_\ell$

$$\gamma(\overline{\mathbf{Y}}_\ell, \mathbf{u}_\ell, \ell) = \delta\, g(\overline{\mathbf{Y}}_\ell, \mathbf{u}_\ell, \ell), \tag{10.A.16}$$

$\ell = 0, 1, 2, \dots N-1$. The overall reward for the Markov decision chain $\mathbf{Y}$, starting from $\overline{\mathbf{Y}}_0 = x_0 \in \overline{X}_0$ and controlled by $\mathbf{u} = \{\mathbf{u}_0, \mathbf{u}_1, \dots \mathbf{u}_{N-1}\}$ can be determined as

$$J(0, \overline{x}_0; \mathbf{u}) = \mathbb{E}\left(\textstyle\sum_{\ell=0}^{N-1} \gamma(\overline{\mathbf{Y}}_\ell, \mathbf{u}_\ell, \ell) + s(\overline{\mathbf{Y}}_N) \,\middle|\, \overline{\mathbf{Y}}_0 = x_0 \right) \tag{10.A.17}$$

Finally, the problem:

$$\begin{cases} \max_{\mathbf{u}} & J(0, \overline{x}_0; \mathbf{u}) \\ & \text{subject to} \\ \mathbf{Y}_{\ell+1} & = \mathbf{Y}_\ell + \delta f_\ell + b_\ell \widetilde{\Delta \mathbf{W}}_\ell, \end{cases} \tag{10.A.18}$$

with the transition probabilities defined as above is the *Markov decision chain* approximating the original continuous-time optimisation problem (10.A.1)-(10.A.4). Now we can use value iteration for the solution of the above Markov decision chain.

There are two crucial parameters for the solution method outlined above: the number of states and the number of time steps. One expects that increasing these numbers would improve the solution's accuracy. However, the value-iteration computation times also increase. Relatively recent papers Rust (1997), Rust (1997)a report results mitigating the curse of dimensionality for a certain subclass of Markov decision chains through use of randomisation. The Markovian approximation described above leads to a similar conclusion. Notice that the time required to compute the optimal decision for the current state is *largely* independent of both the number of time steps and the number of states. Its independence of the number of states is a consequence of the approximation scheme scanning only the adjacent states in the next stage. Doubling the number of states means that twice the time is taken for each stage and the computation time doubles. Doubling the number of time steps leaves the computation time for each stage fixed but doubles the number of stages and hence the computation time doubles (see Krawczyk (2001)).

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Chapter 11

DIFFERENTIATED CAPITAL AND THE DISTRIBUTION OF WEALTH

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Abstract We present a one-sector growth model with finitely many households who differ from each other with respect to their endowments, their preferences, and the type of capital supplied to firms. There is monopolistic competition on the capital market and perfect competition on all other markets. We show that there exists a unique stationary equilibrium and that all households have strictly positive wealth in this equilibrium. We study how the stationary equilibrium depends on the time-preference rates of the households and on the elasticity of substitution between different types of capital. We also analyze the stability of the stationary equilibrium.

1. Introduction

In his seminal contribution, Ramsey (1928) conjectured that, in a stationary equilibrium of what is now considered as the standard neoclassical growth model, only the most patient household(s) would own capital while all other households would consume their entire income without possessing any capital at all. A formal proof of this so-called 'Ramsey conjecture' was first given by Becker (1980).[1]

If the number of households who share the smallest time-preference rate is large, the Ramsey conjecture makes sense. However, if there are only a few most patient households (or even a single most patient household), then the result is conceptually inconsistent with one of the most fundamental assumptions under which it is derived, namely with

[1]Becker (1980) assumes that households cannot borrow. Bewley (1982) analyzes the model without the no-borrowing constraint.

C. Deissenberg and R.F. Hartl (eds.), Optimal Control and Dynamic Games, 177–196.

the assumption that the households act as price takers. More specifically, if all the capital is owned by only a few households, as predicted by the Ramsey conjecture, then these households must realize that they have market power on the capital market. Consequently, they will not take the interest rate as exogenously given. To address this issue, Sorger (2002) has studied a model in which the households take the inverse capital demand function rather than the interest rate as given. The capital market is therefore an oligopoly, on which the households interact strategically. Using this framework, Sorger has shown by means of examples that the Ramsey conjecture may fail to hold. Becker (2003) has extended Sorger's examples to the case of a general Cobb-Douglas production function and has derived a necessary and sufficient condition for the Ramsey conjecture to be true. He has also derived a number of comparative statics results for stationary strategic Ramsey equilibria.

The analysis of the dynamic oligopoly model from Sorger (2002) and Becker (2003) becomes cumbersome when the number of households is large. For this reason, both Sorger and Becker have restricted their studies to the case of two households and a Cobb-Douglas technology. In the present paper we extend the analysis to the case of an arbitrary number of households and a general production technology. In order to avoid the messy algebra that arises in the oligopolistic framework, we assume that each household supplies a different type of capital and that there is monopolistic competition on the capital market. As in the oligopolistic model from Sorger (2002) and Becker (2003), households face elastic capital demand functions and can therefore influence the rate of return on their capital holdings. At the same time, however, the monopolistically competitive model retains a convenient feature from the perfectly competitive one in Becker (1980), namely that households take the evolution of aggregate variables other than the interest rate (e.g., aggregate output) as given.

We model monopolistic competition using the formal approach pioneered by Dixit and Stiglitz (1977) and Ethier (1982). The Dixit/Stiglitz model has figured prominently in many modern endogenous growth theories, notably in Aghion and Howitt (1992), Grossman and Helpman (1991), and Romer (1990). Contrary to most applications of this approach, the model of the present paper does not feature a symmetry property with respect to the differentiated goods. This is due to the heterogeneity of the households, especially to the heterogeneity with respect to their time-preference rates.

The paper is organized as follows. The model is formulated in section 11.2, where we describe the behavior of households, the behavior of firms, and the market clearing conditions. In section 11.3 we prove the existence and uniqueness of a stationary equilibrium and characterize some of its properties. Our most important observation is that the Ramsey conjecture does not hold, because all households own strictly positive amounts of capital. We also show that the perfectly competitive model (in which the Ramsey conjecture is true) emerges as a limiting case of our model when the elasticity of substitution between different types of capital approaches $+\infty$. Finally, we show that a mean preserving spread of the distribution of time-preference rates across households leads to a reduction of the marginal productivity of capital and to an increase of aggregate output and the real wage.

The most important parameter in the Dixit/Stiglitz model of monopolistic competition is the elasticity of substitution between different types of capital. Section 11.4 presents a detailed analysis of the effects caused by changes of this parameter. We first show that an increase of the elasticity of substitution has positive effects on output, the capital holdings of the most patient household(s), and the mean capital stock.[2] Individual capital holdings of all households other than the most patient one(s), however, depend non-monotonically on the elasticity of substitution: their steady state capital holdings increase for small values of this parameter and decrease for large values. Similar monotonicity properties hold also for the capital income of individual households. Finally, we show that an increase of the elasticity of substitution increases wealth inequality as well as the inequality of capital income. In other words, whereas higher competition leads to higher aggregate output it reduces equality.

The paper concludes with section 11.5 in which we analyze the stability of the stationary equilibrium. In the case where the heterogeneity across households is sufficiently weak we can prove that the stationary equilibrium is saddlepoint stable and that equilibria converging to the stationary one are locally monotonic. For the general case we have to resort to numerical calculations. They seem to confirm that saddlepoint stability holds true also for strongly heterogeneous households but that equilibria can exhibit damped oscillations when they converge to the stationary equilibrium.

2. Model formulation

We consider a dynamic general equilibrium model in continuous time. There exists a finite number of infinitely-lived households who own the

[2]The mean capital stock will be formally defined in subsection 11.2.2 below. In general, it is different from the aggregate capital stock or the per-capita capital stock.

production inputs labor and capital. Labor supply is homogeneous across households but capital supply is not: each household provides a different type of capital. This implies that there is monopolistic competition on the capital market and that households face elastic capital demand functions. Taking these demand functions and the evolution of the wage rate as given, the households maximize their life-time utility of consumption subject to non-negativity constraints on capital and consumption.

Firms produce a single homogeneous output which the households can either consume or convert one-for-one into their individual type of capital.[3] We choose output as the numeraire. Firms rent the factor inputs from the households and take the wage rate and the rental rate of capital as given. All markets are assumed to clear at all times.

2.1 Households

There exist $H \geq 2$ infinitely-lived households which will be indexed by h. Households are heterogeneous with respect to their preferences and their endowments. Household h is characterized by its time-preference rate ρ_h, its labor endowment of ℓ_h units per time-period, its initial capital endowment k_{h0}, and its instantaneous utility function u_h.

The household perfectly anticipates the wage rate $w(t)$ at all future dates $t \geq 0$ and supplies labor inelastically. Labor is homogeneous across households. Capital, on the other hand, is heterogeneous. Each household supplies a different type of capital and has therefore monopoly power on the capital market. We denote the inverse demand function for capital of type h at time t by $R_h(\cdot, t)$.[4] The household takes this function as given for all future dates $t \geq 0$. For simplicity we assume that capital does not depreciate.[5]

The household chooses a consumption path $c_h(\cdot)$ and a capital path $k_h(\cdot)$ in order to maximize lifetime utility. Instantaneous utility depends only on consumption. The economic life of household h is therefore

[3]The assumption that the rate at which the output good can be converted into capital is the same for all types of capital could be relaxed at the expense of a more complicated exposition. More specifically, if one unit of output could be converted in τ_h units of type-h capital, then one would have to replace the left-hand side of the differential equation (11.1) below by $\dot{k}_h(t)/\tau_h$. The transformation rates τ_h would then of course also show up in the equilibrium conditions. We assume $\tau_h = 1$ for all h.

[4]The function $R_h(\cdot, t)$ will be derived from the firms' optimization problem in subsection 11.2.3 below. It will turn out that the inverse demand function is the same for all types of capital so that we could drop the household index h from $R_h(\cdot, t)$. For clarity of exposition, however, we will keep the notation $R_h(\cdot, t)$.

[5]It would be straightforward to introduce capital depreciation into the model. The rates of capital depreciation could also be household-specific.

described by the solution of the following optimal control problem P_h:

$$\begin{aligned} \text{maximize} \quad & \int_0^{+\infty} e^{-\rho_h t} u_h(c_h(t))\,\mathrm{d}t \\ \text{subject to} \quad & \dot{k}_h(t) = R_h(k_h(t),t)k_h(t) + w(t)\ell_h - c_h(t) \\ \text{and} \quad & c_h(t) \geq 0\,,\ k_h(t) \geq 0\,,\ k_h(0) = k_{h0}. \end{aligned} \tag{11.1}$$

The following standard assumptions are imposed on the preferences and endowments of households.

H1: For all h it holds that $\rho_h > 0$, $\ell_h > 0$, and $k_{h0} > 0$. The function $u_h : \mathbb{R}_+ \mapsto \mathbb{R}$ is continuous, strictly increasing, and strictly concave. Moreover, it is twice continuously differentiable on the interior of its domain and satisfies $\lim_{c_h \to 0} u_h'(c_h) = +\infty$ and $\lim_{c_h \to +\infty} u_h'(c_h) = 0$.

It will be shown below that the return on capital, $R_h(k_h,t)k_h$, is a strictly increasing function of k_h which is infinitely steep at $k_h = 0$. This property ensures that it is always optimal for a household to maintain a strictly positive capital stock at all times. In other words, the non-negativity constraint $k_h(t) \geq 0$ will never be binding. From the Inada-type conditions in H1 it follows furthermore that the non-negativity constraint $c_h(t) \geq 0$ cannot be binding. An optimal solution to problem P_h must therefore be an interior one and, consequently, it must satisfy the Euler equation

$$\dot{c}_h(t) = \frac{u_h'(c_h(t))}{u_h''(c_h(t))}\left[\rho_h - R_h(k_h(t),t) - R_h'(k_h(t),t)k_h(t)\right] \tag{11.2}$$

where $R_h'(k_h(t),t) = \partial R_h(k_h(t),t)/\partial k_h$. Conversely, any pair of functions $(c_h(\cdot), k_h(\cdot))$ which is feasible for P_h and which satisfies equation (11.2) plus the transversality condition

$$\lim_{t \to +\infty} e^{-\rho_h t} u_h'(c_h(t))k_h(t) \leq 0 \tag{11.3}$$

qualifies as an optimal solution of problem P_h provided that $R_h(k_h,t)k_h$ is concave with respect to k_h for all $t \geq 0$. The latter property will also be verified below.

Having described the behavior of every single household, we now make an assumption on the entire set of households.

H2: It holds that $\rho_h \leq \rho_{h'}$ whenever $h \leq h'$. Furthermore, it holds that $\sum_{h=1}^H \ell_h = 1$.

The first part of H2 simply says that households are ordered according to increasing time-preference. If we define $h^* = \max\{h \mid \rho_h = \rho_1\}$, then it follows that the set of most patient households is given by $\{1, 2, \ldots, h^*\}$. The second part of H2 normalizes the aggregate labor supply to 1. Assumption H2 does not present any loss of generality.

2.2 Firms

There is a single, perfectly competitive production sector which transforms capital and labor into output. Labor is homogeneous but capital comes in H different types corresponding to the different households. At each point in time the firms rent labor and capital from the households and maximize profits. Firms solve a static profit maximization problem at each point in time and act as price takers on the output market and on all factor markets.

Let us denote the labor input at time t by $L(t)$ and the input of type-h capital by $K_h(t)$. It is assumed that output is given by

$$Y(t) = F(K(t), L(t)),$$

where $K(t)$ is an ordinary mean value of the H individual capital input levels $K_h(t)$. More specifically, we assume that

$$K(t) = \mathcal{M}_\sigma(K_1(t), K_2(t), \ldots, K_H(t))$$

where

$$\mathcal{M}_\sigma(K_1, K_2, \ldots, K_H) = \left(H^{-1} \sum_{h=1}^{H} K_h^\sigma\right)^{1/\sigma}.$$

The mathematical properties of ordinary means are well understood; see, e.g., Hardy, Littlewood, and Polya (1952), chapter 2. For example, it holds that $\mathcal{M}_\sigma(K_1, K_2, \ldots, K_H)$ is homogeneous of degree 1 with respect to $K_1, K_2, \ldots, K_H$ and non-decreasing with respect to σ.[6] Furthermore, it holds that

$$\lim_{\sigma\to 0} \mathcal{M}_\sigma(K_1, K_2, \ldots, K_H) = \left(\prod_{h=1}^{H} K_h\right)^{1/H},$$

$$\lim_{\sigma\to -\infty} \mathcal{M}_\sigma(K_1, K_2, \ldots, K_H) = \min\{K_h \,|\, h = 1, 2, \ldots, H\},$$

$$\lim_{\sigma\to +\infty} \mathcal{M}_\sigma(K_1, K_2, \ldots, K_H) = \max\{K_h \,|\, h = 1, 2, \ldots, H\}.$$

The parameter σ measures the substitution possibilities between different types of capital, with $\sigma = 1$ corresponding to perfect substitutability. Formally, the elasticity of substitution between two different types of capital is given by $1/(1-\sigma)$. We shall refer to $K(t)$ as the mean capital stock.

[6]The ordinary mean $\mathcal{M}_\sigma(K_1, K_2, \ldots, K_H)$ is in fact strictly increasing with respect to σ if $\sigma > 0$ and if not all arguments K_h are equal.

We impose the following assumptions on the production technology.

F1: The production function $F : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is homogeneous of degree 1. The intensive production function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by $f(K) = F(K, 1)$ is continuous, strictly increasing, and strictly concave. Moreover, f is twice continuously differentiable on the interior of its domain and satisfies $f(0) = 0$, $\lim_{K \to 0} f'(K) = +\infty$, and $\lim_{K \to +\infty} f'(K) = 0$.

F2: It holds that $\sigma \in (0, 1)$.

Assumption F1 says that F is a neoclassical production function satisfying the usual properties. Assumption F2 restricts attention to the case of substitution elasticity greater than 1. This is a standard assumption in models of monopolistic competition which ensures that the capital demand functions (which will be derived below) have rental rate elasticities greater than 1 in absolute value.

As before, we denote the wage rate at time t by $w(t)$. Furthermore, the rental rate of type-h capital is denoted by $r_h(t)$. The representative firm takes these factor prices as given and seeks to maximize its profit

$$Y(t) - \sum_{h=1}^{H} r_h(t) K_h(t) - w(t) L(t)$$

subject to the technological constraints and to non-negativity constraints for all input factors. The first-order conditions for an interior profit maximum are

$$r_h(t) = H^{-1} F_1(K(t), L(t)) [K(t)/K_h(t)]^{1-\sigma}, \tag{11.4}$$

$$w(t) = F_2(K(t), L(t)), \tag{11.5}$$

where F_1 and F_2 denote the partial derivatives of F with respect to the first and second argument, respectively.

Note that, for any fixed mean capital stock $K(t) > 0$, the right-hand side of (11.4) is a strictly decreasing function of $K_h(t)$ with limits $+\infty$ and 0 as $K_h(t)$ approaches 0 and $+\infty$, respectively. This shows that output $Y(t)$ as a function of the individual capital stock $K_h(t)$ satisfies the Inada conditions. This implies that the non-negativity constraints on capital inputs will never be binding. As a consequence of the Inada conditions on the production function F stated in assumption F1, it follows furthermore that the non-negativity constraint on labor input cannot be binding. Together with the convexity assumptions stated in F1 these properties imply that conditions (11.4)-(11.5) provide a complete characterization of the firms' behavior.

2.3 Equilibrium

We are now ready to close the model by imposing clearing conditions for all markets. According to assumption H2, the aggregate labor supply equals 1 such that labor market clearing requires $L(t) = 1$. Substituting this into equation (11.5) and using the well-known fact that $F_2(K, L) = f(K/L) - (K/L)f'(K/L)$ we obtain

$$w(t) = f(K(t)) - K(t)f'(K(t)). \tag{11.6}$$

Equilibrium on the market for type-h capital requires $k_h(t) = K_h(t)$. Substituting this along with $L(t) = 1$ and $F_1(K, L) = f'(K/L)$ into (11.4) we get

$$r_h(t) = H^{-1}f'(K(t))[K(t)/k_h(t)]^{1-\sigma}.$$

We assume that the households perfectly anticipate aggregate output $f(K(t))$ and the mean capital stock $K(t)$ at all dates $t \geq 0$. The above equation represents therefore the inverse demand function for type-h capital. In other words, the function R_h appearing in the capital accumulation equation (11.1) and the corresponding Euler equation (11.2) is given by

$$R_h(k_h, t) = H^{-1}f'(K(t))[K(t)/k_h]^{1-\sigma}. \tag{11.7}$$

Note that this implies that the return on type-h capital is given by $R_h(k_h, t)k_h = H^{-1}f'(K(t))K(t)^{1-\sigma}k_h^{\sigma}$, which is a strictly increasing and strictly concave function of k_h and that this function is infinitely steep at $k_h = 0$. These properties have been used in subsection 11.2.1 to show that the non-negativity constraints on capital holdings are not binding and that the Euler equation and the transversality condition form a set of sufficient optimality conditions for household h's optimization problem P_h.

We summarize the equilibrium conditions in the following proposition.

Proposition 11.1 *A family of functions* $\left\{K(\cdot), (c_h(\cdot), k_h(\cdot))_{h=1}^{H}\right\}$ *is an equilibrium if and only if the following conditions hold for all* h *and all*

$t \geq 0$:

$$\dot{k}_h(t) = H^{-1} f'(K(t)) K(t)^{1-\sigma} k_h(t)^\sigma + [f(K(t)) - K(t) f'(K(t))]\ell_h - c_h(t), \tag{11.8}$$

$$\dot{c}_h(t) = \frac{u_h'(c_h(t))}{u_h''(c_h(t))} \left(\rho_h - \sigma H^{-1} f'(K(t))[K(t)/k_h(t)]^{1-\sigma}\right), \tag{11.9}$$

$$K(t) = \left[H^{-1} \sum_{h=1}^{H} k_h(t)^\sigma\right]^{1/\sigma}, \tag{11.10}$$

$$\lim_{t \to +\infty} e^{-\rho_h t} u_h'(c_h(t)) k_h(t) \leq 0, \tag{11.11}$$

$$c_h(t) \geq 0\,,\ k_h(t) \geq 0. \tag{11.12}$$

PROOF: The proposition is easily proved by substituting the expressions for $w(t)$ and $R_h(k_h(t), t)$ from (11.6) and (11.7), respectively, into the households' feasibility and optimality conditions (11.1), (11.2), and (11.3). □

3. The stationary equilibrium

An equilibrium is called stationary if it consists of constant functions. We start the present section by proving the existence and uniqueness of a stationary equilibrium. To this end let us define $\bar{K}$ as the unique positive number satisfying

$$f'(\bar{K}) = (H/\sigma) \left(\frac{1}{H} \sum_{h=1}^{H} \rho_h^{-\sigma/(1-\sigma)}\right)^{-(1-\sigma)/\sigma}. \tag{11.13}$$

Existence and uniqueness of $\bar{K}$ follows from assumption F1.

THEOREM 11.2 *There exists a unique stationary equilibrium given by*

$$K(t) = \bar{K} \tag{11.14}$$

$$k_h(t) = \bar{k}_h := \bar{K} \left[\sigma f'(\bar{K})/(H\rho_h)\right]^{1/(1-\sigma)}, \tag{11.15}$$

$$c_h(t) = \bar{c}_h := [f(\bar{K}) - \bar{K} f'(\bar{K})]\ell_h + \rho_h \bar{k}_h/\sigma. \tag{11.16}$$

PROOF: Imposing the stationarity condition $\dot{c}_h(t) = 0$ in (11.9) implies that

$$k_h(t) = K(t) \left[\sigma f'(K(t))/(H\rho_h)\right]^{1/(1-\sigma)}.$$

This proves (11.15) and, after substituting the latter into (11.10), one obtains (11.13) and (11.14) by straightforward algebra. Equation (11.16) is now easily obtained by substituting the results into (11.8) and using $\dot{k}_h(t) = 0$. This shows that (11.13)-(11.16) must hold in a stationary equilibrium. Conversely, it is easy to see that the constant solution specified by (11.13)-(11.16) satisfies the equilibrium conditions (11.8)-(11.12). This completes the proof of the theorem. □

Note that the stationary equilibrium values $\bar{K}$, $\bar{k}_h$, and $\bar{c}_h$ do not depend on the utility functions of the households. The only preference parameters that affect these values are the time-preference rates. This is a property that holds also in the competitive setting from Becker (1980) and the oligopolistic setting from Sorger (2002) and Becker (2003). Note furthermore that (11.13) can be rewritten in the form

$$f'(\bar{K}) = (H/\sigma)\mathcal{M}_{-\sigma/(1-\sigma)}(\rho_1, \rho_2, \ldots, \rho_H), \tag{11.17}$$

which shows that the marginal productivity of the mean capital stock in the stationary equilibrium is proportional to an ordinary mean of the time-preference rates. It is therefore possible to use results about ordinary means to study how the time-preference rates of the households and the elasticity of substitution between different types of capital affect the stationary equilibrium. As an example, let us characterize the limits of the stationary equilibrium as σ approaches its extreme values 0 and 1, respectively.

It follows from (11.17) and the properties of ordinary means that

$$\lim_{\sigma \to 0} [\sigma f'(\bar{K})] = H \left(\prod_{h=1}^{H} \rho_h \right)^{1/H}.$$

Since the right-hand side of this equation is a finite positive number, we obtain $\lim_{\sigma \to 0} f'(\bar{K}) = +\infty$. Together with assumption F1 this implies that $\lim_{\sigma \to 0} \bar{K} = 0$. Substituting these results back into (11.15) and (11.16), we see that $\lim_{\sigma \to 0} \bar{k}_h = \lim_{\sigma \to 0} \bar{c}_h = 0$ for all h. The interpretation of this result is as follows. As σ approaches 0, the inverse capital demand functions become unit-elastic and the profit that each household earns from holding capital, $R_h(k_h(t), t)k_h(t)$, becomes flat. Thus, in the extreme case $\sigma = 0$, the households do not have any incentive to hold capital and, therefore, $\bar{k}_h = 0$ must be true for all h in the stationary equilibrium. As a consequence, the mean capital stock, aggregate output, and aggregate consumption must be equal to 0 as well.

Now let us turn to the other extreme case. As σ converges to 1, the elasticity of substitution between different types of capital becomes infinitely large. This means that different types of capital are hardly considered to be different by final goods producers and the capital market approaches a perfectly competitive one. We therefore call the limiting case $\sigma = 1$ the competitive limit. The stationary equilibrium in the competitive limit is characterized in the following lemma.

LEMMA 11.3 *Consider the stationary equilibrium from theorem 11.2. It holds that*[7]

$$\lim_{\sigma \to 1} \bar{k}_h = \begin{cases} \bar{\bar{K}} H/h^* & \text{if } h \le h^*, \\ 0 & \text{if } h > h^*, \end{cases} \tag{11.18}$$

$$\lim_{\sigma \to 1} \bar{c}_h = \begin{cases} [f(\bar{\bar{K}}) - \bar{\bar{K}} f'(\bar{\bar{K}})]\ell_h + \rho_h \bar{\bar{K}} H/h^* & \text{if } h \le h^*, \\ [f(\bar{\bar{K}}) - \bar{\bar{K}} f'(\bar{\bar{K}})]\ell_h & \text{if } h > h^*, \end{cases} \tag{11.19}$$

where $\bar{\bar{K}}$ is the unique positive number satisfying $f'(\bar{\bar{K}}) = H\rho_1$.

PROOF: It follows from (11.17), from assumption H2, and from the properties of ordinary means that $\lim_{\sigma \to 1} f'(\bar{K}) = H\rho_1$. This, in turn, implies that $\bar{K}$ approaches the finite and positive value $\bar{\bar{K}}$ as σ converges to 1. Using these properties, it follows from equation (11.15) that $\lim_{\sigma \to 1} \bar{k}_h = 0$ for all $h > h^*$. Equation (11.15) also implies that $\bar{k}_h = \bar{k}_{h'}$ whenever both h and h' are elements of $\{1, 2, \ldots, h^*\}$. Using these results together with (11.10), it follows that $\lim_{\sigma \to 1} \bar{k}_h = \bar{\bar{K}} H/h^*$ for all $h \le h^*$. The remaining statements of the lemma are now simple consequences of theorem 11.2. □

The above lemma shows that the boundary case $\sigma = 1$ corresponds to the competitive Ramsey equilibrium studied in Becker (1980).

A simple consequence of theorem 11.2 is that, in the stationary equilibrium, all households own positive amounts of capital; see (11.15). It follows that the Ramsey conjecture, according to which in a stationary equilibrium only the most patient household(s) own capital, does not hold in the case of a monopolistically competitive capital market. Only in the competitive limit $\sigma = 1$ does the Ramsey conjecture hold, as can be seen from lemma 11.3. However, even for values of σ strictly smaller than 1, it is true that more patient households own more capital. This can be easily seen from the following equation, which is an implication of (11.15):

$$\bar{k}_h / \bar{k}_{h'} = (\rho_{h'}/\rho_h)^{1/(1-\sigma)}. \tag{11.20}$$

As in the case of an oligopolistic capital market, which is studied in Sorger (2002) and Becker (2003), it holds that the ranking of households

[7] Recall that $h^* = \max\{h \,|\, \rho_h = \rho_1\}$ such that $\{1, 2, \ldots, h^*\}$ is the set of most patient households.

according to their wealth coincides with the ranking according to their patience.

To conclude the present section, let us study the dependence of the stationary equilibrium allocation on the distribution of the time-preference rates. Because $\rho^{-\sigma/(1-\sigma)}$ is a strictly convex function of ρ, it follows from (11.13) that a mean-reserving spread of the distribution of the time-preference rates across households would lead to a reduction of $f'(\bar{K})$ and, hence, to an increase of the mean capital stock $\bar{K}$, aggregate output $f(\bar{K})$, and the real wage $\bar{w} = f(\bar{K}) - \bar{K}f'(\bar{K})$. A similar result has already been found by Becker (2003) in the case of two households acting as duopolists on the market for (non-differentiated) capital.

4. Changes in substitutability

In the previous section we have characterized the stationary equilibrium for any given value of the elasticity of substitution between different types of capital. In particular, we have discussed the two limiting cases $\sigma \to 0$ and $\sigma \to 1$. The present section discusses in more detail how the stationary equilibrium is affected by changes of the parameter σ.

Let us begin by analyzing the effects of σ on the mean capital stock and aggregate output. Since $-\sigma/(1-\sigma)$ is a decreasing function of σ, it follows from the properties of ordinary means mentioned in subsection 11.2.2 that $\mathcal{M}_{-\sigma/(1-\sigma)}(\rho_1, \rho_2, \ldots, \rho_H)$ is a decreasing function of σ. Because of (11.17) this implies that $f'(\bar{K})$ is strictly decreasing with respect to σ. Together with assumption F1 this shows that the mean capital stock $\bar{K}$ and aggregate output $f(\bar{K})$ are strictly increasing with respect to σ. We conclude that stronger competition has a positive level effect on mean capital and aggregate output.

Individual capital holdings, however, are in general non-monotonic functions of σ. To see this, recall from the previous section that, for $\sigma \to 0$, all individual capital stocks $\bar{k}_h$ converge to 0. For $\sigma \to 1$, on the other hand, these capital stocks approach positive values if $h \leq h^*$ and they approach 0 for $h > h^*$. In the latter case we would therefore expect that $\bar{k}_h$ is increasing for small values of σ and decreasing for large σ. Instead of trying to verify these properties analytically for general production functions f, we restrict ourselves to illustrating them by means of numerical examples.

Example 11.4 Our first model economy consists of 3 households only. We assume that differences in time-preference rates are the only source of heterogeneity. More specifically, we set $\ell_1 = \ell_2 = \ell_3 = 1/3$, $\rho_1 = 3\%$, $\rho_2 = 4\%$, and $\rho_3 = 5\%$. As for the production technology we assume $f(K) = K^{1/3}$.

Using these specifications, figure 11.1 shows the capital holdings $\bar{k}_1$ of the (unique) most patient household 1 and the mean capital stock $\bar{K}$ in the stationary equilibrium. We see that both of these variables are increasing functions of σ. The capital holdings of the two other households, $\bar{k}_2$ and $\bar{k}_3$, are shown in figure 11.2. As expected, they are hump-shaped with the more patient household 2 attaining its maximal wealth level at a higher value of σ than the most impatient household 3.

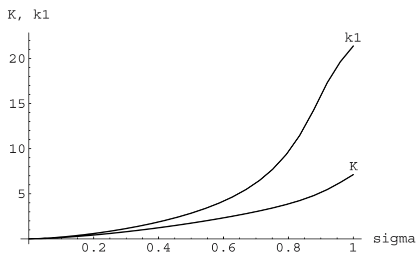

Figure 11.1. Type-1 capital and mean capital as functions of σ.

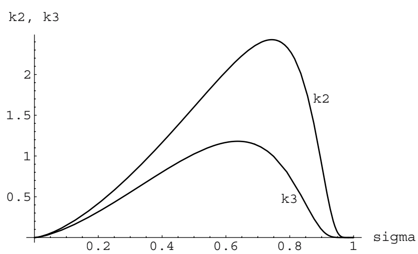

Figure 11.2. Type-2 capital and type-3 capital as functions of σ.

Let us now turn to income. Since there is no capital depreciation, consumption and income must coincide in a stationary equilibrium. The first term on the right-hand side of (11.16) is labor income, whereas the second one describes capital income. Let us denote capital income of household h in the stationary equilibrium by $\bar{y}_h$. Using (11.20), it is easily seen that

$$\bar{y}_h/\bar{y}_{h'} = (\rho_{h'}/\rho_h)^{\sigma/(1-\sigma)}. \tag{11.21}$$

This shows that more patient households receive higher capital income than less patient ones. The capital income of the three households from example 11.4 are shown in figure 11.3.

Since all households face a common competitive wage rate, it follows that households with higher labor endowment have also higher labor

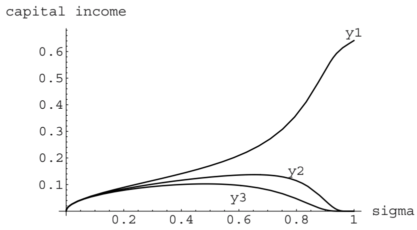

Figure 11.3. Capital income as a function of σ.

income. The ranking of households according to their total income (or, equivalently, according to their consumption levels) depends therefore both on the labor endowments and on the time-preference rates of the households.

We conclude this section by a brief analysis of the effect of σ on wealth and income inequality. It is evident from (11.20) that $\bar{k}_h/\bar{k}_{h'}$ is a strictly increasing function of σ whenever $\rho_h < \rho_{h'}$. As the elasticity of substitution increases, the wealth of a more patient household relative to that of a more impatient one increases. It is easy to see that this property implies also that the Gini coefficient of the wealth distribution in the unique stationary equilibrium is a strictly increasing function of σ. A completely analogous argument (using (11.21) instead of (11.20)) shows that the Gini coefficient of the distribution of capital income is also a strictly increasing function of σ. Stronger competition leads therefore to higher wealth inequality and capital income inequality. We illustrate this finding again by means of a numerical example.

Example 11.5 Consider an economy consisting of $H = 26$ households. As in example 11.4 we assume that differences in time-preference rates are the only source of heterogeneity. Each household has a labor endowment of $1/26$ units per time period. The time-preference rate of household h is assumed to be $0.03 + 0.0008(h - 1)$. In other words, the time-preference rates depend linearly on the household number h with the most patient household having $\rho_1 = 3\%$ and the most impatient one having $\rho_{26} = 5\%$. As for the production technology we assume again $f(K) = K^{1/3}$.

The Gini coefficient of the stationary wealth distribution of this economy is shown in figure 11.4. It starts at a value of roughly 10% when $\sigma = 0$ and increases monotonically towards 100% as σ approaches the competitive limit, where all capital is held by a single household. The Gini coefficient of the distribution of capital income $\bar{y}_h = \rho_h \bar{k}_h/\sigma$ is depicted in figure 11.5. It looks similar to the Gini coefficient of the wealth distribution, except that it starts out at 0 when $\sigma = 0$. The Gini coefficients of total income will obviously depend on the distribution of labor endowments and are not shown here.

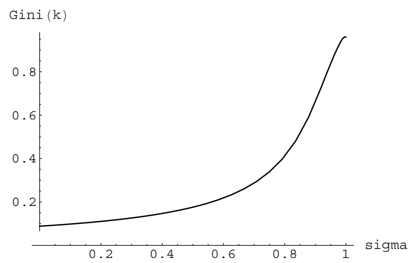

Figure 11.4. Gini coefficient of the wealth distribution as a function of σ.

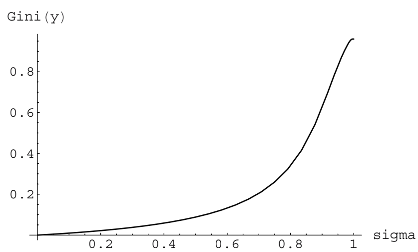

Figure 11.5. Gini coefficient of the distribution of capital income as a function of σ.

5. Stability analysis

So far our analysis was restricted to the unique stationary equilibrium. The dynamics of the economy are described by the differential equations (11.8)-(11.9). In the present section we present a few results regarding the stability of the stationary equilibrium and the transition dynamics towards it.

It will be convenient to introduce the following notation:

$$\begin{aligned}
\phi &= (1-\sigma)f'(\bar{K}) + \bar{K}f''(\bar{K}), \\
\theta_h &= -\bar{c}_h u_h''(\bar{c}_h)/u_h'(\bar{c}_h), \\
z_h &= \rho_h/[\sigma f'(\bar{K})], \\
\psi_h &= -\bar{K}f''(\bar{K})\ell_h.
\end{aligned}$$

Note that θ_h is the relative risk aversion of the utility function u_h evaluated at the consumption level of household h in the stationary equilibrium. Straightforward algebra shows that the elements of the Jacobian matrix of the $(2H)$-dimensional system (11.8)-(11.9) evaluated at the stationary equilibrium are given by

$$\begin{aligned}
\frac{\partial \dot{k}_h(t)}{\partial k_h(t)} &= \rho_h + \left[\phi H^{-1/(1-\sigma)} z_h^{-\sigma/(1-\sigma)} + \psi_h\right] z_h, \\
\frac{\partial \dot{k}_h(t)}{\partial k_{h'}(t)} &= \left[\phi H^{-1/(1-\sigma)} z_h^{-\sigma/(1-\sigma)} + \psi_h\right] z_{h'}, \\
\frac{\partial \dot{k}_h(t)}{\partial c_h(t)} &= -1, \\
\frac{\partial \dot{c}_h(t)}{\partial k_h(t)} &= \frac{\bar{c}_h z_h^2}{\theta_h \bar{K}} \left[\sigma\phi - (1-\sigma)\rho_h H^{1/(1-\sigma)} z_h^{-(1-2\sigma)/(1-\sigma)}\right], \\
\frac{\partial \dot{c}_h(t)}{\partial k_{h'}(t)} &= \frac{\sigma \bar{c}_h z_h z_{h'} \phi}{\theta_h \bar{K}}, \\
\frac{\partial \dot{k}_h(t)}{\partial c_{h'}(t)} &= \frac{\partial \dot{c}_h(t)}{\partial c_h(t)} = \frac{\partial \dot{c}_h(t)}{\partial c_{h'}(t)} = 0,
\end{aligned}$$

where h and h' are arbitrary but mutually different households.

Let us start the stability analysis by assuming that the households are homogeneous. In this case, there exist real numbers ρ, $\bar{c}$, and θ such that $\rho_h = \rho$, $\bar{c}_h = \bar{c}$, and $\theta_h = \theta$ hold for all h. Furthermore, assumption H2 implies $\ell_h = 1/H$ and equation (11.13) implies $f'(\bar{K}) = H\rho/\sigma$. It follows that $z_h = 1/H$ and $\psi_h = -Kf''(\bar{K})/H$ for all h. If we define

$$\begin{aligned}
\alpha &= (1-\sigma) f'(\bar{K})/H^2, \\
\beta &= \frac{\bar{c}}{\theta \bar{K}} \left[\sigma\phi/H^2 - (1-\sigma)\rho\right], \\
\gamma &= \frac{\sigma \bar{c} \phi}{H^2 \theta \bar{K}},
\end{aligned}$$

then it follows that the Jacobian matrix of (11.8)-(11.9) evaluated at the stationary equilibrium has the form

$$\begin{pmatrix}
\rho+\alpha & \alpha & \dots & \alpha & -1 & 0 & \dots & 0 \\
\alpha & \rho+\alpha & \dots & \alpha & 0 & -1 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha & \alpha & \dots & \rho+\alpha & 0 & 0 & \dots & -1 \\
\beta & \gamma & \dots & \gamma & 0 & 0 & \dots & 0 \\
\gamma & \beta & \dots & \gamma & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma & \gamma & \dots & \beta & 0 & 0 & \dots & 0
\end{pmatrix}. \tag{11.22}$$

We can now prove the following result.

THEOREM 11.6 *The eigenvalues of the Jacobian matrix (11.22) are given by*

$$\lambda_{1,2} = \rho/2 \pm \sqrt{\rho^2/4 - \beta + \gamma},$$
$$\lambda_{3,4} = (\rho + \alpha H)/2 \pm \sqrt{(\rho + \alpha H)^2/4 - \beta - \gamma(H-1)},$$

whereby both λ_1 and λ_2 have multiplicity $H-1$ and both λ_3 and λ_4 have multiplicity 1. All eigenvalues are real. Exactly H eigenvalues are negative (namely λ_1 and λ_3) and exactly H are positive (namely λ_2 and λ_4).

PROOF: See appendix. □

Theorem 11.6 demonstrates that, in the case of homogeneous households, the stationary equilibrium is locally saddlepoint stable. This implies that, for every vector of initial capital endowments that is sufficiently close to the vector $(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_H)$, there exists a unique equilibrium that converges towards the stationary equilibrium. Note that theorem 11.6 allows us to conclude that the same property remains true also for the case of heterogeneous households as long as the heterogeneity is not too strong. Formally, this follows from the fact that the eigenvalues of a matrix are continuous functions of the matrix. Another implication of theorem 11.6 is that, in the case of homogeneous households, convergence towards the stationary equilibrium is locally monotonic. This is implied by the fact that the eigenvalues are real. It is worth emphasizing that these results hold true for arbitrary utility functions and production functions satisfying assumptions H1, F1, and F2.

The computation of the eigenvalues of the Jacobian matrix in the general case of heterogeneous households is analytically intractable. We have therefore computed them numerically for a number of examples. In all these examples we found saddlepoint stability, i.e, there were exactly H stable eigenvalues and H unstable eigenvalues. However, it was easy to come up with examples in which the stable eigenvalues were complex numbers. One of these examples is reported below. If the stable eigenvalues are complex it follows that the equilibria which converge towards the stationary equilibrium exhibit damped oscillations.

EXAMPLE 11.7 Consider an economy consisting of $H = 2$ households. The time-preference rates are 1.2% for household 1 and 4.8% for household 2. We assume that $\sigma = 4/5$ and that the intensive production function satisfies $f(1) = 1$ and $f'(1) = 3Q/25$, where $Q = (2/257)^{1/4}$.

Both households are endowed with half a unit of labor per period. It follows from theorem 11.2 that the unique stationary equilibrium is given by $\bar{K} = 1$, $\bar{k}_1 = 2048Q/257$, $\bar{k}_2 = 2Q/257$, $\bar{c}_1 = 1/2 + 153Q/2570$, and $\bar{c}_2 = 1/2 - 153Q/2570$. Now assume that the utility functions are such that $\theta_1 = 1$ and $\theta_2 = 2$ and that the production function is such that $f''(1) = -3$. Using these specifications, we find that the eigenvalues of the Jacobian matrix evaluated at the stationary equilibrium are $-0.604686 \pm 0.179266i$, 0.187691, and 2.96384. The two stable eigenvalues are conjugate complex numbers and the equilibria that converge to the stationary equilibrium exhibit damped oscillations.

Appendix

This appendix presents the proof of theorem 11.6. We need two technical lemmas. Let I be the $H \times H$ unit matrix and let E the $H \times H$ matrix in which all entries are equal to 1.

LEMMA A.1 *Let A, B, C, and D be real $H \times H$ matrices and assume that B is non-singular. Then it follows that $\lambda \in \mathbb{C}$ is an eigenvalue of*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

if and only if λ is a solution of the equation

$$\text{Det}\left[C - (D - \lambda I)B^{-1}(A - \lambda I)\right] = 0.$$

PROOF: The number λ is an eigenvalue if and only if there exists a non-zero vector $(x\,y)$ with $x \in \mathbb{C}^H$ and $y \in \mathbb{C}^H$ such that the equations

$$Ax + By = \lambda x,$$
$$Cx + Dy = \lambda y$$

hold. Because B is non-singular, the first equation implies $y = -B^{-1}(A - \lambda I)x$. Substituting this result into the second equation we get

$$\left[C - (D - \lambda I)B^{-1}(A - \lambda I)\right] x = 0.$$

If x were equal to 0 then it would follow that y is also equal to 0, which is a contradiction to the requirement that $(x\,y)$ is non-zero. Thus x must be non-zero and the above equation can only hold if the determinant of $C - (D - \lambda I)B^{-1}(A - \lambda I)$ is equal to 0. This completes the proof of the lemma. □

LEMMA A.2 *Let p and q be arbitrary complex numbers and define the $H \times H$ matrix M by $M = pI + qE$. Then it follows that the eigenvalues of M are given by p (with multiplicity $H - 1$) and $p + Hq$ (with multiplicity 1).*

PROOF: It is easily seen that the vector $x_1 = (1, 1, \ldots, 1)$ is an eigenvector of M with eigenvalue $p + Hq$ and that the vectors

$$\begin{aligned} x_2 &= (1, -1, 0, \ldots, 0), \\ x_3 &= (1, 0, -1, \ldots, 0), \\ &\vdots \\ x_H &= (1, 0, 0, \ldots, -1) \end{aligned}$$

are eigenvectors of M corresponding to the eigenvalue p. Since the vectors x_1, x_2, ..., x_H are linearly independent, the result follows. □

PROOF OF THEOREM 11.6: We can write the Jacobian matrix in (11.22) as

$$J = \begin{pmatrix} \rho I + \alpha E & -I \\ (\beta - \gamma) I + \gamma E & 0 \end{pmatrix}.$$

According to lemma A.1 it follows that λ is an eigenvalue of J if and only if it satisfies the equation

$$\mathrm{Det}\left[(\beta - \gamma - \rho\lambda + \lambda^2) I + (\gamma - \alpha\lambda) E\right] = 0.$$

The determinant of a matrix is equal to the product of its eigenvalues. Using lemma A.2 we can therefore rewrite the above equation as

$$(\beta - \gamma - \rho\lambda + \lambda^2)^{H-1}[\beta + \gamma(H-1) - (\rho + \alpha H)\lambda + \lambda^2] = 0.$$

Solving this equation yields the stated values for the eigenvalues $\lambda_{1,2,3,4}$. The eigenvalues $\lambda_{1,2}$ satisfy $g(\lambda) := \beta - \gamma - \rho\lambda + \lambda^2 = 0$. Because of

$$g(0) = \beta - \gamma = -\frac{(1-\sigma)\rho\bar{c}}{\theta\bar{K}} < 0$$

it follows that the solutions of $g(\lambda) = 0$ must be real and of opposite sign. The eigenvalues $\lambda_{3,4}$ satisfy $G(\lambda) := \beta + \gamma(H-1) - (\rho + \alpha H)\lambda + \lambda^2 = 0$. It holds that

$$G(0) = \beta + \gamma(H-1) = \frac{\bar{c}}{\theta\bar{K}}\left[\frac{\sigma\phi}{H} - (1-\sigma)\rho\right] = \frac{\sigma\bar{c}f''(\bar{K})}{\theta H},$$

where we have used $\rho = \sigma f'(\bar{K})/H$ for the last step. Therefore, it holds that $G(0) < 0$ and it follows that the solutions of $G(\lambda) = 0$ must be real and of opposite sign. This completes the proof of the theorem. □

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Chapter 12

OPTIMAL FIRM CONTRIBUTIONS TO OPEN SOURCE SOFTWARE

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Abstract This paper examines open source software development in a competitive environment. The quality of open source software improves over time based upon contributions by firms and users. A firm's decision to contribute is interesting because it also augments competitors' software quality in future periods subject to compatibility considerations with their existing software. A differential game model is developed to understand why firms are increasingly involved in open source software development by determining the optimal contributions and software quality over time. We obtain a closed-loop Nash equilibrium solution. Examples are given to derive insights from this model.

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1. Introduction

Open source (OS) software is software that is licensed for free use, modification and redistribution (Raymond (2001)). Together with the compiled software, the source code is made available to users, so that they can examine and modify the code if they should choose to do so. In particular, a volunteer group may be set up to oversee the collection of code modifications and improvements, and to bring out new versions of the software that incorporate these changes. The users thus become partners in the development of the software.

While OS software development in the past has been characterized as unstructured and temporary collaborations by individual hackers, the new reality is that successful open source development has become structured, and oftentimes funded by interested commercial firms. Some commercial firms have kept open source programmers on their payroll, while others have contributed source code. A case in point is the billion dollars that IBM, HP, Novell, Intel and others have invested in Linux and OS Development Labs. [1]

One of the benefits obtained from revealing the source code is that many programmers are able to examine it for defects, or bugs. Even highly internally tested software is prone to bugs that might trigger under particular settings that have not been tested. In a method akin to the peer review system and open publications of academic research, OS software tends to be more reliable and less buggy. Thus, user testing and contributions explains why firms might consider opening their code in monopoly markets.

What is surprising, however, is that even in competitive markets, firms should contribute to open source. For example, Red Hat and Novell compete for the same business clients, and IBM and HP both offer servers running Linux software. In such a context, the competitors are also able to examine and incorporate the contributed software into their own bundle of software programs. The contributing firm thus helps its

[1]Thus, IBM hosts, coordinates, and provides support for several open source projects. It has several dedicated teams of in-house developers in charge of major projects. For example, IBM's journaled file system technology, currently used in IBM enterprise servers, is an in-house development project managed by a small, core group of contributors known as the JFS core team. HP is hosting a number of open source software projects that run on various HP systems, including Handhelds.org, HP OfficeJet Linux Driver, and OpenSSI Clusters for Linux. Motorola's Metrowerks subsidiary acquired the assets of Linux tools and solutions vendor Embedix. Metrowerks draws on the Embedix assets to provide Linux-based app development tools and platforms for PDAs, smart handheld devices, residential gateways, and digital TVs. Nokia recently released the Nokia Developers Suite for J2ME which runs over Linux. Open Office is the best known open source project of Sun Microsystems. Darwin is the best known open source project by Apple Computer.

competitors. In this paper, we shall examine why this is optimal. One reason may be that the user contributions are high. Thus, by participating in open source, the firm benefits from the high externality from users. Second, the firm may be able to contribute in a manner that its competitor cannot take full advantage of its contributions due to compatibility issues. For example, software firms may provide an OS interface to access their proprietary software, which helps them tap into the OS software market, but only indirectly benefits others. Finally, the competition may not be too intense.

Past research into open source has focused on the behavioral and economic incentives of open source programmers to freely contribute their work to the public (e.g., Hertel, Niedner and Hermann (2003); Kogut and Metiu (2001); Lakhani and von Hippel (2003)). A few works have focused on profit maximizing firms, optimizing investment and prices. Among those works, Haruvy, Prasad and Sethi (2003) evaluate a model where the open source code is sold as part of a commercial product. Haruvy, Sethi and Zhou (2004) examine a monopolist developing open source with a complementary commercial product and Haruvy, Prasad, Sethi and Zhang (2004) study open source as a public good. The present paper complements the existing literature by examining three components hitherto understudied-competition, compatibility and user contributions.

2. Model and Analysis

There are two firms indexed $i = 1, 2$. Each firm i has a commercial product, H_i, which it sells for a profit. The quality of the commercial product of firm i is determined by its investment, u_i. Each firm i can also contribute code, a_i, to an open source. The open source product, by virtue of being open, cannot be sold and is distributed freely. However, the commercial and open source products are complementary, meaning that their sales will go hand in hand.

Each firm derives quality S_i from the open source. Note that even though the open source code is public and freely available, the two firms may derive different qualities from it. That is, S_1 and S_2, are not necessarily identical. That is because the firms' complementary commercial product may not be 100% compatible. As such, each firm's open source contributions are geared towards its own complementary product's unique needs and so its open source contributions may not be as useful to the competitor. If the two firms' complementary products were exactly compatible, then the two open source qualities would be the same and $S_1 = S_2$. The open source and commercial product qualities are determined over time according to the two firms' investment levels according to the equations below.

$$\dot{S}_i = \eta_1 a_i + \eta_2 a_{3-i} + \eta_3(D_1 + D_2), \quad S_i(0) = S_{i0} \tag{12.1}$$

$$\dot{H}_1 = u_1, \quad H_1(0) = H_{10} \tag{12.2}$$

$$\dot{H}_2 = u_2, \quad H_2(0) = H_{20} \tag{12.3}$$

Consumer demand for each commercial product depends on the quality and price of that commercial product relative to the competition, as well as the quality of the complementary software (which is open source). Accordingly, the demand functions are given by

$$D_1 = \gamma_1 S_1 + \gamma_2 H_1 - \gamma_3 H_2 - \gamma_4 p_1 + \gamma_5 p_2 \tag{12.4}$$

$$D_2 = \gamma_1 S_2 + \gamma_2 H_2 - \gamma_3 H_1 - \gamma_4 p_2 + \gamma_5 p_1 \tag{12.5}$$

Each firm maximizes its profit with respect to commercial product and open source investments and price, taking costs of investment into consideration. Their objective functions are:

$$J_1(a_1, u_1, p_1) = \int_0^T (p_1 D_1 - c_{s1} a_1^2 - c_{h1} u_1^2) dt \tag{12.6}$$

$$J_2(a_2, u_2, p_2) = \int_0^T (p_2 D_2 - c_{s2} a_2^2 - c_{h2} u_2^2) dt \tag{12.7}$$

We summarize the notation and delineate the state and control variables for convenience.

S_i	Quality of open source software. State variable
H_i	Quality of proprietary software for firm i. State variable
$a_i(t)$	Investment in open source software by firm i. Control var.
$u_i(t)$	Investment in proprietary software by firm i. Control var.
$p_i(t)$	Price of software charged by firm i. Control var.
$\eta_1,\ \eta_2$	Compatibility parameters
η_3	Parameters for user contribution
γ_j	Demand parameters, $j = 1$ to 5
$c_{S1},\ c_{S2}$	Cost parameters for the common component software
$c_{H1},\ c_{H2}$	Cost parameter for the private component software
T	Duration of the game
S_{i0}, H_{i0}	Initial values

Then our problem becomes a standard differential game problem which is given by (12.1)-(12.7). For differential games, there are different equilibrium solution concepts depending on different information assumptions (see, e.g. Basar and Olsder (1982); Mehlmann (1988); Fudenberg and Tirole (1991)). Among them, open-loop and closed-loop Nash equilibrium solutions are used most frequently. When the players adopt the open-loop Nash equilibrium solution concept, they design the time path concerning the control variable(s) at the initial time and then stick to it forever. When players adopt the closed-loop Nash equilibrium solution, they do not precommit control variable(s) and their strategies at any time may depend on the history of the game up to that time, and, in particular, on the states. Further, closed-loop solutions can be divided into some more detailed classes according to different information structures (see, e.g., Dockner, Jorgensen, Long and Sorger (2000)).

As is well known, the closed-loop solution concept is more reasonable than the open-loop solution concept for general differential game problems, but it often needs to solve partial differential equations if, further, the Hamilton-Jacobi-Bellman (HJB) equation is introduced, which has to recur to numerical approaches in most cases. However, due to the special structure of our problem, the necessary conditions for the closed-loop Nash equilibrium solution will be given by a linear two-point boundary value problem (TPBVP) with constant coefficient matrix. It has a unique analytical solution under certain conditions. The characteristics implied in our model are helpful to identify a more general class of differential game models which admit analytical closed-loop Nash equilibrium solution.

Now, we use Pontryagin's maximum principle to analyze possible closed-loop solutions. The Hamiltonians for the two firms are:

$$\begin{aligned}
L_1 &= p_1 D_1 - c_{s1} a_1^2 - c_{h1} u_1^2 + \lambda_0 \dot{S}_1 + \lambda_1 \dot{S}_2 + \lambda_2 \dot{H}_1 + \lambda_3 \dot{H}_2 \\
&= p_1(\gamma_1 S_1 + \gamma_2 H_1 - \gamma_3 H_2 - \gamma_4 p_1 + \gamma_5 p_2) - c_{s1} a_1^2 - c_{h1} u_1^2 \\
&\quad + \lambda_0 \{\eta_1 a_1 + \eta_2 a_2 + \eta_3 [(\gamma_1 S_1 + \gamma_2 H_1 - \gamma_3 H_2 - \gamma_4 p_1 + \gamma_5 p_2) \\
&\quad + (\gamma_1 S_2 + \gamma_2 H_2 - \gamma_3 H_1 - \gamma_4 p_2 + \gamma_5 p_1)]\} \\
&\quad + \lambda_1 \{\eta_1 a_2 + \eta_2 a_1 + \eta_3 [(\gamma_1 S_1 + \gamma_2 H_1 - \gamma_3 H_2 - \gamma_4 p_1 + \gamma_5 p_2) \\
&\quad + (\gamma_1 S_2 + \gamma_2 H_2 - \gamma_3 H_1 - \gamma_4 p_2 + \gamma_5 p_1)]\} + \lambda_2 u_1 + \lambda_3 u_2
\end{aligned} \tag{12.8}$$

$$\begin{aligned}
L_2 &= p_2 D_2 - c_{s2} a_2^2 - c_{h2} u_2^2 + \mu_0 \dot{S}_1 + \mu_1 \dot{S}_2 + \mu_2 \dot{H}_1 + \mu_3 \dot{H}_2 \\
&= p_2(\gamma_1 S_2 + \gamma_2 H_2 - \gamma_3 H_1 - \gamma_4 p_2 + \gamma_5 p_1) - c_{s2} a_2^2 - c_{h2} u_2^2 \\
&\quad + \mu_0 \{\eta_1 a_1 + \eta_2 a_2 + \eta_3 [(\gamma_1 S_1 + \gamma_2 H_1 - \gamma_3 H_2 - \gamma_4 p_1 + \gamma_5 p_2) \\
&\quad + (\gamma_1 S_2 + \gamma_2 H_2 - \gamma_3 H_1 - \gamma_4 p_2 + \gamma_5 p_1)]\} \\
&\quad + \mu_1 \{\eta_1 a_2 + \eta_2 a_1 + \eta_3 [(\gamma_1 S_1 + \gamma_2 H_1 - \gamma_3 H_2 - \gamma_4 p_1 + \gamma_5 p_2) \\
&\quad + (\gamma_1 S_2 + \gamma_2 H_2 - \gamma_3 H_1 - \gamma_4 p_2 + \gamma_5 p_1)]\} + \mu_2 u_1 + \mu_3 u_2
\end{aligned} \tag{12.9}$$

The first order conditions for optimal control are:

$$\begin{cases} \frac{\partial L_1}{\partial a_1} = -2c_{s1}a_1 + \eta_1\lambda_0 + \eta_2\lambda_1 = 0 \\ \frac{\partial L_1}{\partial u_1} = -2c_{h1}u_1 + \lambda_2 = 0 \\ \frac{\partial L_1}{\partial p_1} = \gamma_1 S_1 + \gamma_2 H_1 - \gamma_3 H_2 - 2\gamma_4 p_1 + \gamma_5 p_2 + \eta_3(\gamma_5 - \gamma_4)\lambda_0 \\ \quad +\eta_3(\gamma_5 - \gamma_4)\lambda_1 = 0 \\ \frac{\partial L_2}{\partial a_2} = -2c_{s2}a_2 + \eta_2\mu_0 + \eta_1\mu_1 = 0 \\ \frac{\partial L_2}{\partial u_2} = -2c_{h2}u_2 + \mu_3 = 0 \\ \frac{\partial L_2}{\partial p_2} = \gamma_1 S_1 + \gamma_2 H_2 - \gamma_3 H_1 - 2\gamma_4 p_2 + \gamma_5 p_1 + \eta_3(\gamma_5 - \gamma_4)\mu_0 \\ \quad +\eta_3(\gamma_5 - \gamma_4)\mu_1 = 0 \end{cases} \tag{12.10}$$

As can be seen from the equations, (12.10) is a system which is linear in controls. Under the condition $\gamma_4 > \gamma_5$, we have:

$$\begin{cases} a_1 = \frac{\eta_1}{2c_{s1}}\lambda_0 + \frac{\eta_2}{2c_{s1}}\lambda_1 \\ u_1 = \frac{\lambda_2}{2c_{h1}} \\ p_1 = \frac{1}{4\gamma_4^2 - \gamma_5^2}[2\gamma_1\gamma_4 S_1 + \gamma_1\gamma_5 S_2 + (2\gamma_4\gamma_2 - \gamma_3\gamma_5)H_1 \\ \quad +(\gamma_2\gamma_5 - 2\gamma_3\gamma_4)H_2 + (2\eta_3\gamma_4\gamma_5 - 2\gamma_3\gamma_4^2)\lambda_0 + (2\eta_3\gamma_4\gamma_5 - 2\gamma_3\gamma_4^2)\lambda_1 \\ \quad +(\eta_3\gamma_5^2 - \eta_3\gamma_4\gamma_5)\mu_0 + (\eta_3\gamma_5^2 - \eta_3\gamma_4\gamma_5)\mu_1] \\ a_2 = \frac{\eta_2}{2c_{s2}}\mu_0 + \frac{\eta_1}{2c_{s2}}\mu_1 \\ u_2 = \frac{\mu_3}{2c_{h2}} \\ p_2 = \frac{1}{4\gamma_4^2 - \gamma_5^2}[\gamma_1\gamma_5 S_1 + 2\gamma_1\gamma_4 S_2 + (\gamma_2\gamma_5 - 2\gamma_3\gamma_4)H_1 \\ \quad +(2\gamma_2\gamma_4 - \gamma_3\gamma_5)H_2 + (\eta_3\gamma_5^2 - \eta_3\gamma_4\gamma_5)\lambda_0 + (\eta_3\gamma_5^2 - \eta_3\gamma_4\gamma_5)\lambda_1 \\ \quad +(2\eta_3\gamma_4\gamma_5 - 2\eta_3\gamma_4^2)\mu_0 + (2\eta_3\gamma_4\gamma_5 - 2\eta_3\gamma_4^2)\mu_1] \end{cases} \tag{12.11}$$

In fact, $\gamma_4 > \gamma_5$ is a very weak condition. It says that the consumer demand is more sensitive to the price of a commercial product itself than to the other related commercial products' prices.

From (12.11), all controls are linear combinations of states and co-states. They can be rewritten as

$$U = AX \tag{12.12}$$

where

$$U = [a_1, u_1, p_1, a_2, u_2, p_2]^T,$$

$$X = [S_1, S_2, H_1, H_2, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \mu_0, \mu_1, \mu_2, \mu_3]^T$$

and A is a 6 by 12 coefficient matrix whose elements are functions of parameters $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \eta_1, \eta_2, \eta_3, c_{s1}, c_{h1}, c_{s2}, c_{h2}$.

From the necessary condition for closed-loop Nash optimality, the costate variables should satisfy the following ordinary differential equations with all terminal values equal to zero:

$$\begin{cases} \dot{\lambda}_0 = -\frac{\partial L_1}{\partial S_1} - (\frac{\partial L_1}{\partial a_2}\frac{\partial a_2}{\partial S_1} + \frac{\partial L_1}{\partial u_2}\frac{\partial u_2}{\partial S_1} + \frac{\partial L_1}{\partial p_2}\frac{\partial p_2}{\partial S_1}) \\ \dot{\lambda}_1 = -\frac{\partial L_1}{\partial S_2} - (\frac{\partial L_1}{\partial a_2}\frac{\partial a_2}{\partial S_2} + \frac{\partial L_1}{\partial u_2}\frac{\partial u_2}{\partial S_2} + \frac{\partial L_1}{\partial p_2}\frac{\partial p_2}{\partial S_2}) \\ \dot{\lambda}_2 = -\frac{\partial L_1}{\partial H_1} - (\frac{\partial L_1}{\partial a_2}\frac{\partial a_2}{\partial H_1} + \frac{\partial L_1}{\partial u_2}\frac{\partial u_2}{\partial H_1} + \frac{\partial L_1}{\partial p_2}\frac{\partial p_2}{\partial H_1}) \\ \dot{\lambda}_3 = -\frac{\partial L_1}{\partial H_2} - (\frac{\partial L_1}{\partial a_2}\frac{\partial a_2}{\partial H_2} + \frac{\partial L_1}{\partial u_2}\frac{\partial u_2}{\partial H_2} + \frac{\partial L_1}{\partial p_2}\frac{\partial p_2}{\partial H_2}) \\ \dot{\mu}_0 = -\frac{\partial L_2}{\partial S_1} - (\frac{\partial L_2}{\partial a_1}\frac{\partial a_1}{\partial S_1} + \frac{\partial L_2}{\partial u_1}\frac{\partial u_1}{\partial S_1} + \frac{\partial L_2}{\partial p_1}\frac{\partial p_1}{\partial S_1}) \\ \dot{\mu}_1 = -\frac{\partial L_2}{\partial S_2} - (\frac{\partial L_2}{\partial a_1}\frac{\partial a_1}{\partial S_2} + \frac{\partial L_2}{\partial u_1}\frac{\partial u_1}{\partial S_2} + \frac{\partial L_2}{\partial p_1}\frac{\partial p_1}{\partial S_2}) \\ \dot{\mu}_2 = -\frac{\partial L_2}{\partial H_1} - (\frac{\partial L_2}{\partial a_1}\frac{\partial a_1}{\partial H_1} + \frac{\partial L_2}{\partial u_1}\frac{\partial u_1}{\partial H_1} + \frac{\partial L_2}{\partial p_1}\frac{\partial p_1}{\partial H_1}) \\ \dot{\mu}_3 = -\frac{\partial L_2}{\partial H_2} - (\frac{\partial L_2}{\partial a_1}\frac{\partial a_1}{\partial H_2} + \frac{\partial L_2}{\partial u_1}\frac{\partial u_1}{\partial H_2} + \frac{\partial L_2}{\partial p_1}\frac{\partial p_1}{\partial H_2}) \end{cases} \tag{12.13}$$

where the terms included in the brackets describe the feedback effects from the rival's control variables. It is easy to see from (12.8) and (12.9) that

$$\begin{cases} \frac{\partial L_1}{\partial S_1} = \gamma_1 p_1 + \eta_3\gamma_1\lambda_0 + \eta_3\gamma_1\lambda_1 \\ \frac{\partial L_1}{\partial S_2} = \eta_3\gamma_1\lambda_0 + \eta_3\gamma_1\lambda_1 \\ \frac{\partial L_1}{\partial H_1} = \gamma_2 p_1 + \eta_3(\gamma_2-\gamma_3)\lambda_0 + \eta_3(\gamma_2-\gamma_3)\lambda_1 \\ \frac{\partial L_1}{\partial H_2} = -\gamma_3 p_1 + \eta_3(\gamma_2-\gamma_3)\lambda_0 + \eta_3(\gamma_2-\gamma_3)\lambda_1 \\ \frac{\partial L_2}{\partial S_1} = \eta_3\gamma_1\mu_0 + \eta_3\gamma_1\mu_1 \\ \frac{\partial L_2}{\partial S_2} = \gamma_1 p_2 + \eta_3\gamma_1\mu_0 + \eta_3\gamma_1\mu_1 \\ \frac{\partial L_2}{\partial H_1} = -\gamma_3 p_2 + \eta_3(\gamma_2-\gamma_3)\mu_0 + \eta_3(\gamma_2-\gamma_3)\mu_1 \\ \frac{\partial L_2}{\partial H_2} = \gamma_2 p_2 + \eta_3(\gamma_2-\gamma_3)\mu_0 + \eta_3(\gamma_2-\gamma_3)\mu_1 \end{cases} \tag{12.14}$$

By substituting p_1 and p_2 from (12.11) into (12.14), it becomes a system which is linear in the states and costates. In other words, all the first terms outside the brackets of the RHS of (12.13) are linear with states and costates. Second, note that all $\frac{\partial a_i}{\partial Y}$ and $\frac{\partial u_i}{\partial Y}$ ($i = 1, 2$; $Y \in \{S_1, S_2, H_1, H_2\}$) are zero by (12.11). Then all the first two terms included in the brackets must be zero. This means that the terms left at work in the brackets are only $\frac{\partial L_i}{\partial p_j}\frac{\partial p_j}{\partial Y}$. Using (12.11), we have that all $\frac{\partial p_j}{\partial Y_i}$ are constants. By (12.8) and (12.9), we get

$$\begin{cases} \frac{\partial L_1}{\partial p_2} = \gamma_5 p_1 + \eta_3(\gamma_5-\gamma_4)\lambda_0 + \eta_3(\gamma_5-\gamma_4)\lambda_1 \\ \frac{\partial L_2}{\partial p_1} = \gamma_5 p_2 + \eta_3(\gamma_5-\gamma_4)\mu_0 + \eta_3(\gamma_5-\gamma_4)\mu_1 \end{cases} \tag{12.15}$$

Substituting again p_1 and p_2 from (12.11) into (12.15), it becomes a system which is linear in states and costates. Then all the third terms included in the brackets are linear in states and costates. Hence, the entire RHS of (12.13) is linear in states and costates. Namely, (12.13) is a system of linear ordinary differential equations with terminal values equal to zero.

Therefore the necessary conditions for closed-loop Nash equilibrium are given by the four linear system equations (12.1)-(12.3) with four initial boundary conditions, and the eight linear differential equations (12.13) with eight terminal conditions. In other words, it becomes a linear TPBVP problem with constant coefficients. For convenience in later description, we denote this TPBVP problem by

$$\dot{X} = BX \tag{12.16}$$

where $X = [S_1, S_2, H_1, H_2, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \mu_0, \mu_1, \mu_2, \mu_3]^T$, B is a constant 12×12 coefficient matrix. The boundary conditions are given by

$$\begin{cases} X(1)_{t=0} = S_{10} \\ X(2)_{t=0} = S_{20} \\ X(3)_{t=0} = H_{10} \\ X(4)_{t=0} = H_{20} \\ X(5)_{t=T} = 0 \\ X(6)_{t=T} = 0 \\ X(7)_{t=T} = 0 \\ X(8)_{t=T} = 0 \\ X(9)_{t=T} = 0 \\ X(10)_{t=T} = 0 \\ X(11)_{t=T} = 0 \\ X(12)_{t=T} = 0 \end{cases} \tag{12.17}$$

While the necessary conditions are given by a linear TPBVP problem with constant coefficient matrix which is mathematically tractable, and well-known algorithms are available for it (see, e.g., Agrawal and Fabien (1999)), it is hard to derive sufficient conditions for the optimality, since there are too many parameters and the relationships between parameters involved are extremely complicated. However, a large number of simulations for a very wide range of parameters have demonstrated the optimality of solutions obtained in this way.

3. Solution

As mentioned above, there are many numerical methods available for general TPBVP problems, some of which are fixed in the toolboxes of mathematical software. For example, one can immediately give a numerical solution to such problems by writing a short program including bvp4c.m in Matlab. However, for the above linear TPBVP problem with constant coefficients, we are able to get its analytical solution under certain conditions (e.g., Dockner, Jorgensen, Long and Sorger (2000); Sethi and Thompson (2000)), which can yield more theoretical insight. We shall adopt such a method in the following examples.

It is well known that there are two major steps in solving the TPBVP problem consisting of (12.16) and (12.17). The first is to find the general solution by computing the matrix exponential function of $\exp(tB)$; the second is to determine a particular solution by using the boundary conditions (12.17). The second step is relatively simple. It only needs to substitute the boundary conditions into the general solution obtained by the first step. Then a system of 12 algebraic equations with 12 unknown variables is given. Under appropriate conditions, it has a unique solution and the particular solution to the TPBVP problem is determined accordingly. However, one may encounter some practical problems in the first step. At the first glance, the idea of calculating the matrix exponential function of $\exp(tB)$ might be to see if it can be obtained directly by $V\mathrm{diag}(\exp(\mathrm{diag}(\lambda_i)))V^{-1}$, where V is a full set of eigenvectors V with corresponding eigenvalues $\lambda_i(\ i = 1, \ldots 12)$. But eigenvalue multiplicity of coefficient matrix B makes this approach invalid for our problem. As we know, a great deal of numerical methods have been developed to calculate $\exp(tB)$ (e.g. Moler and Loan (1978); Moler and Loan (2003)). However, high-accuracy algorithms are often at the price of a longer computation time and a larger storage memory.

Our following examples are based on a very straightforward method-Taylor series expansion. One important reason for using such a method is that it enables us to see quite clearly which and how factors affect the results by an oversimplified method, and it allows us not to spend additional time studying the numerical errors introduced by a particular algorithm. Theoretically, $e^{tB} = \sum\limits_{k=0}^{\infty} \frac{(tB)^k}{k!}$. We may use the sum of the first $N+1$ terms for the approximation of the matrix exponential function. Then the theoretical error is the sum of the later left infinite terms $\sum\limits_{k=N+1}^{\infty} \frac{(tB)^k}{k!}$. Of course, the larger the number N is, the more precise the approximation is. But for numerical computation, N is always limited. If N is given, then the error is mainly affected by two factors–t and B. Their effects on the convergence rate of Taylor series are directly reflected in the final results. Recognizing the potential effect of these two factors on the solution is very necessary, since it is helpful to understand why we may observe some counter-intuitive even strange results sometimes in numerical examples. They are, as expected here, most likely either from a longer time horizon leading to a slower convergence rate, relationship between parameters leading to a slow convergence rate of $\frac{B^k}{k!}$, or relationship between the length of time horizon and coefficient matrix B leading to a slow convergence rate of $\frac{(tB)^k}{k!}$. With all these in mind, one can identify the precise reason for the possible "bad" results immediately. From the special structure of coefficient matrix B and expressions of the elements in B, [2] we can see that $\frac{(tB)^k}{k!}$ converges very quickly for a given t if the cost parameters are large enough.

In the following examples, we set $N = 10$. This N is sufficient to give a quite precise result for a very wide range of parameters. Comparison between the results obtained by this way and those by bvp4c.m or Laplace transformation in Matlab indicates that differences are usually less than 10^{-3}. Yet, one thing we should pay more attention to is that, if we adopt the approach of Laplace and inverse Laplace transformation functions in Matlab, we should include some phrases in the program to ignore the imaginary parts of the matrix exponential function, since they are introduced by numerical error and should strictly be zero in theory. In fact, we can observe that they are often less than 10^{-15} in our examples. So they can be ignored safely.

4. Simulations

To derive insights from this model, we examined the sensitivity response to the parameters. Whereas several parameter values were considered, only some representative ones are presented here.

The first parameter we examine is user contribution. In many instances, users are the main contributors to the software. This is particularly true with statistical open source software such as R (www.r-project.org) where the users are uniquely qualified to contribute their own routines. On the other end of the spectrum, server applications are supported mainly by the companies that manufacture the servers (IBM, HP) and these companies have full-time dedicated staff to these projects. Users contribute mostly bug reports and patches, which are useful but make less than 3% of all contribution. We look at the extreme end of the spectrum where user contribution is non-existent ($\eta_3 = 0$) as well as a more reasonable user contribution level at $\eta_3 = 0.02$.

The second parameter we examine is compatibility. As explained in the introduction, the two firms derive different qualities from the software due to potential incompatibility. The two extreme points of this spectrum are $\eta_2 = 0$ and $\eta_2 = 1$. In the first, the firms have zero

[2] Here, we omit the general expression for the coefficient matrix B because of space limitation.

compatibility, whereas in the latter, they enjoy full compatibility and $S_1 = S_2$. The four cases are shown in the table below:

Table 12.1. The four cases.

	User Contribution	*No User Contribution*
Full compatibility	Case 1 ($\eta_2 = 1$, $\eta_3 > 0$)	Case 2 ($\eta_2 = 1$, $\eta_3 = 0$)
No compatibility	Case 3 ($\eta_2 = 0$, $\eta_3 > 0$)	Case 4 ($\eta_2 = 0$, $\eta_3 = 0$)

Case1: Compatible Open Source with User Contribution: $\eta_1 = \eta_2$ and $\eta_3 > 0$

Case2: Compatible Open Source without user contribution: $\eta_1 = \eta_2$ and $\eta_3 = 0$

Case3: Incompatible Open Source with user contribution: $\eta_2 = 0$ and $\eta_3 > 0$

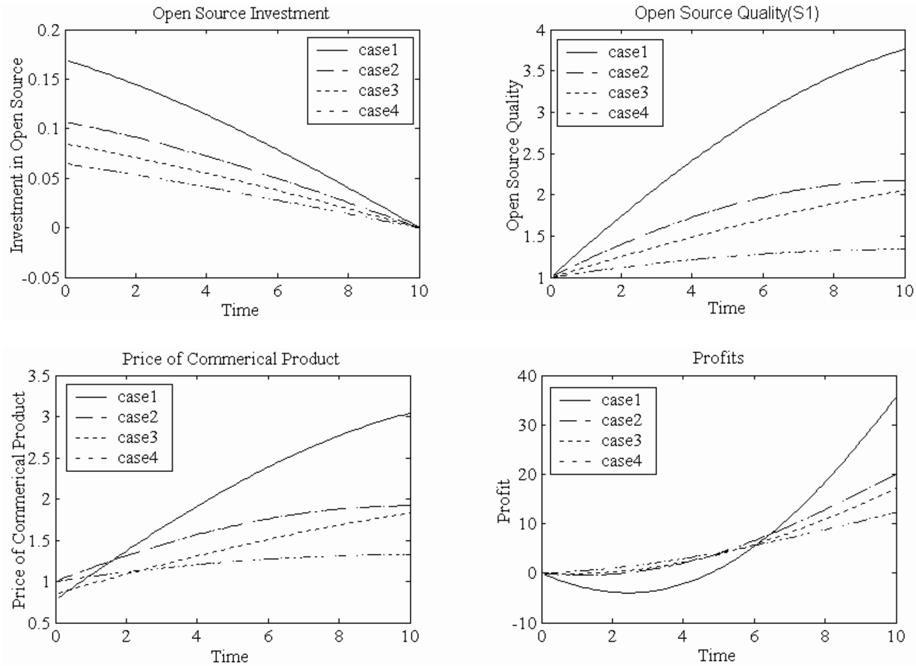
Case4: Closed source: $\eta_2 = 0$ and $\eta_3 = 0$.

Our benchmark model is case 1 with $\eta_1 = 1.0$; $\eta_2 = 1.0$; $\eta_3 = 0.02$; $\gamma_1 = 1.0$; $\gamma_2 = 1.0$; $\gamma_3 = 0.5$; $\gamma_4 = 1.0$; $\gamma_5 = 0.5$; $c_{s1} = 100$; $c_{h1} = 100$; $c_{s2} = 100$; $c_{h2} = 100$; $T = 10$. Case 2 has the same parameters as case 1, except $\eta_3 = 0$. Case 3 is the same as case 1 except $\eta_2 = 0$. Case 4 is the same as case 1 except $\eta_2 = 0$ and $\eta_3 = 0$. Appendix 1 gives a particular solution of case 1.

As expected and as shown by figure 1, open source investment and quality, as well as the commercial product's price, are the highest in case 1, where both compatibility and externality from user contribution are positive. They are lowest in case 4, where both are zero. Profits closely follow the rankings of investment quality.

Next, we examine whether compatibility and externality from user contributions can result in free-riding. Public goods (to which open source belongs) are susceptible to free-riding, where firms would enjoy the benefits of open source without contributing. To isolate the two effects, we examine cases 2 and 3 for asymmetric firms.

As an illustration for case 2 with asymmetric firms, consider $\eta_1 = \eta_2 = 1$, $\eta_3 = 0$, $\gamma_1 = 1.0$, $\gamma_2 = 1.0$, $\gamma_3 = 0.5$, $\gamma_4 = 1.0$, $\gamma_5 = 0.5$, $c_{s1} = c_{h1} = c_{h2} = 50$, $c_{s2} = 500$, $T = 10$. The initial values are $S_1 = S_2 = 2.5$, $H_1 = 5$, and $H_2 = 5$. In this parameterization, the cost of investment in open source for firm 2 is ten times the cost for firm 1. As it turns out, firm 2 invests less and reaps a higher profit. In the above example, total profit (the integral of profit over t) to firm 1 is 209.8, whereas total profit to firm 2 is 336.7. The combined total profit is 546.5. A similar picture emerges when the firms have asymmetric



Figur 1.1. Comparison of the four cases– An illustrativ example

productivity in open source such that $\eta_1 > \eta_2$. Figure 2 shows open source investment and profits for the above parameterization.

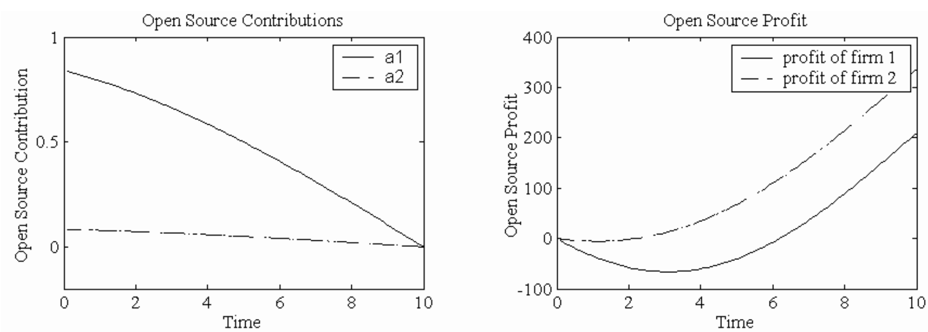


Figure 1.2. An Asymmetric Scenario for Case 2

Observation 1. *When the main open source externality comes from compatibility, the firm with the higher cost of investment in the open source will invest less in open source and make a higher profit than the firm with the lower cost of investment.*

The above observation is a classical free-riding scenario. The seemingly advantaged firm, which has the lower cost, ends up at a disadvantage as the weaker firm free-rides its investment in the public good. This finding is due to the fact that the marginal benefit of investment in the public good exceeds marginal cost for the firm with the lower cost / higher impact. As such it will invest more. The benefit, however, accrues equally to both firms.

With case 3, however, no such parallel exists and no free-riding occurs. Consider the following parameterization: $\eta_1 = 1, \eta_2 = 0$, $\eta_3 = 0.02$, $\gamma_1 = 1.0$, $\gamma_2 = 1.0$, $\gamma_3 = 0.5$, $\gamma_4 = 1.0$, $\gamma_5 = 0.5$, $c_{h1} = c_{h2} = 50$, $c_{s1} = 5$, $c_{s2} = 500$, $T = 10$. As we see below, firm 1 invests more than firm 2 and outperforms firm 2 in terms of profit.

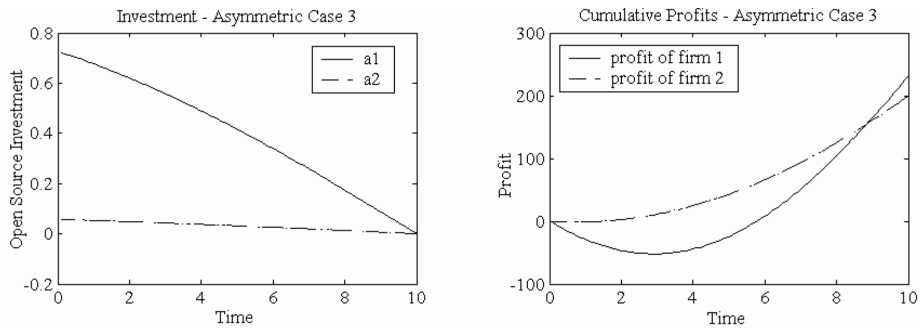


Figure 1.3. An Asymmetric Scenario for Case 3

This result leads us to the following observation.

Observation 2. *When the main open source externality comes from user ontributions, the firm with the higher cost of investment in the open source will invest less in open source and make a lower profit than the firm with the lower cost of investment.*

5. Conclusion

Open source has long been an enigma in the academic and business communities. To succeed, a firm pursuing open source development must rely on seemingly selfless contributions by a community of users as well as on contributions by competitors, whose interests are seemingly opposite the firm's.

Each open source project is unique. Some generate a great deal of excitement and goodwill in the user community. Others, like specialized server modules, are drier and less exciting to the community at large but are nevertheless important to the firm. The firm must be able to design its software development strategy to accommodate the degree of public interest and subsequent public contribution. However, even in cases where the public is not actively involved in development, the firm may nevertheless engage in co-opetition with its rivals. That is, the two may find themselves competing in one market and yet cooperatively contributing to the development of the open source product. This will happen when the two open source applications share a great deal of overlap. Since the open source is a public good, a degree of free riding may be inevitable but the outcome is still beneficial to both firms.

Our aim in the present work is to tackle both of these aspects of open source development– namely, user and competitor involvement– within a competitive environment. Since development is a dynamic rather than static process, we apply the theory of differential games to gain insight into this environment. Due to the special structure of this problem, the closed-loop Nash equilibrium solution is given by a linear two-point boundary problem with constant coefficients, which admits analytical solution under certain conditions.

We find that both the degree of user involvement and compatibility with a rival's product positively affect profits. However, only the former does so without the caveat of free-riding. Free-riding is not necessarily a problem for the firm that ends up bearing the bulk of the investment burden. However, free-riding may result in reduced incentives for smaller firms to invest and in reluctance by larger firms to share their technologies.

Note that in the present work, community goodwill towards the company, as measured by the user contribution parameter η_3 was assumed exogenous and proportional to demand. Increasingly, there is evidence that company actions may influence the degree of user contributions. Haruvy, Prasad and Sethi (2003) study a model where open source contributors respond negatively to excessive profit taking on the part of the company. As survey evidence accumulates on the motives of open source contributors, future extensions should enhance the user contribution function to include such motives. Another shortcoming of the present model, and a task for future extensions, is that it deals with only two firms. Open source in many cases benefits from the contributions of many firms. We believe the insights gained here are generalizable to the many-firm scenario, but the mathematical derivation of that proof is at this stage intractable. Generalizing this conclusion for n firms is needed for a more complete characterization of open source development.

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Appendix:

This Appendix lists the solution of case 1 corresponding to the parameters: $\eta_1 = 1.0$, $\eta_2 = 1.0$, $\eta_3 = 0.02$, $\gamma_1 = 1.0$, $\gamma_2 = 1.0$, $\gamma_3 = 0.5$, $\gamma_4 = 1.0$, $\gamma_5 = 0.5$, $c_{s1} = c_{h1} = c_{s2} = c_{h2} = 100$, $S_1 = S_2 = H_1 = H_2 = 1$, $T = 10$.

$$S_1 = 1. + .3829t - .5126 \times 10^{-2}t^2 - .6241 \times 10^{-3}t^3 + .3461 \times 10^{-5}t^4 + .3131 \times 10^{-6}t^5 - .8888 \times 10^{-9}t^6 - .7460 \times 10^{-10}t^7 + .1117 \times 10^{-12}t^8 + .1034 \times 10^{-13}t^9 - .7243 \times 10^{-17}t^{10}$$

$$S_2 = 1. + .3829t - .5126 \times 10^{-2}t^2 - .6241 \times 10^{-3}t^3 + .3461 \times 10^{-5}t^4 + .3131 \times 10^{-6}t^5 - .8888 \times 10^{-9}t^6 - .7460 \times 10^{-10}t^7 + .1117 \times 10^{-12}t^8 + .1034 \times 10^{-13}t^9 - .7243 \times 10^{-17}t^{10}$$

$$H_1 = 1. + .1073t - .2766 \times 10^{-2}t^2 - .2172 \times 10^{-3}t^3 + .2044 \times 10^{-5}t^4 + .1098 \times 10^{-6}t^5 - .5865 \times 10^{-9}t^6 - .2635 \times 10^{-10}t^7 + .8777 \times 10^{-13}t^8 + .3680 \times 10^{-14}t^9 - .7872 \times 10^{-17}t^{10}$$

$$H_2 = 1. + .1073t - .2766 \times 10^{-2}t^2 - .2172 \times 10^{-3}t^3 + .2044 \times 10^{-5}t^4 + .1098 \times 10^{-6}t^5 - .5865 \times 10^{-9}t^6 - .2635 \times 10^{-10}t^7 + .8777 \times 10^{-13}t^8 + .3680 \times 10^{-14}t^9 - .7872 \times 10^{-17}t^{10}$$

$$\lambda_0 = 25.59 - 1.458t - .1428t^2 + .2189 \times 10^{-2}t^3 + .1207 \times 10^{-3}t^4 - .9674 \times 10^{-6}t^5 - .4073 \times 10^{-7}t^6 + .2000 \times 10^{-9}t^7 + .7341 \times 10^{-11}t^8 - .2357 \times 10^{-13}t^9 - .8211 \times 10^{-15}t^{10}$$

$$\lambda_1 = +8.251 - .7029t - .2489 \times 10^{-1}t^2 + .1106 \times 10^{-2}t^3 + .2195 \times 10^{-4}t^4 - .5272 \times 10^{-6}t^5 - .7692 \times 10^{-8}t^6 + .1192 \times 10^{-9}t^7 + .1436 \times 10^{-11}t^8 - .1565 \times 10^{-13}t^9 - .1661 \times 10^{-15}t^{10}$$

$$\lambda_2 = 21.47 - 1.106t - .1303t^2 + .1636 \times 10^{-2}t^3 + .1098 \times 10^{-3}t^4 - .7038 \times 10^{-6}t^5 - .3689 \times 10^{-7}t^6 + .1404 \times 10^{-9}t^7 + .6623 \times 10^{-11}t^8 - .1574 \times 10^{-13}t^9 - .7381 \times 10^{-15}t^{10}$$

$$\lambda_3 = -4.545 + .2601 \times 10^{-1}t + .4649 \times 10^{-1}t^2 + .1167 \times 10^{-4}t^3 - .3842 \times 10^{-4}t^4 - .4352 \times 10^{-7}t^5 + .1267 \times 10^{-7}t^6 + .1917 \times 10^{-10}t^7 - .2234 \times 10^{-11}t^8 - .3862 \times 10^{-14}t^9 + .2445 \times 10^{-15}t^{10}$$

$$\mu_0 = 8.251 - .7029t - .2489 \times 10^{-1}t^2 + .1106 \times 10^{-2}t^3 + .2195 \times 10^{-4}t^4 - .5272 \times 10^{-6}t^5 - .7692 \times 10^{-8}t^6 + .1192 \times 10^{-9}t^7 + .1436 \times 10^{-11}t^8 - .1565 \times 10^{-13}t^9 - .1661 \times 10^{-15}t^{10}$$

$$\mu_1 = 25.59 - 1.458t - .1428t^2 + .2189 \times 10^{-2}t^3 + .1207 \times 10^{-3}t^4 - .9674 \times 10^{-6}t^5 - .4073 \times 10^{-7}t^6 + .2000 \times 10^{-9}t^7 + .7341 \times 10^{-11}t^8 - .2357 \times 10^{-13}t^9 - .8211 \times 10^{-15}t^{10}$$

$$\mu_2 = -4.545 + .2601 \times 10^{-1}t + .4649 \times 10^{-1}t^2 + .1167 \times 10^{-4}t^3 - .3842 \times 10^{-4}t^4 - .4352 \times 10^{-7}t^5 + .1267 \times 10^{-7}t^6 + .1917 \times 10^{-10}t^7 - .2234 \times 10^{-11}t^8 - .3862 \times 10^{-14}t^9 + .2445 \times 10^{-15}t^{10}$$

$$\mu_3 = 21.47 - 1.106t - .1303t^2 + .1636 \times 10^{-2}t^3 + .1098 \times 10^{-3}t^4 - .7038 \times 10^{-6}t^5 - .3689 \times 10^{-7}t^6 + .1404 \times 10^{-9}t^7 + .6623 \times 10^{-11}t^8 - .1574 \times 10^{-13}t^9 - .7381 \times 10^{-15}t^{10}$$

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PRODUCTION, MAINTENANCE AND TRANSPORTATION

Chapter 13

THE IMPACT OF DYNAMIC DEMAND AND DYNAMIC NET REVENUES ON FIRM CLOCKSPEED

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Abstract A firm's new product development clockspeed is determined by the frequency of new product introductions to the marketplace. Using a simple analytic model, we derive an optimal firm NPD clockspeed that is driven by several external market and internal organizational related factors. Specifically, we analyze the impact of dynamic sales/demand curves and dynamic net revenues on the optimal pace of new product introductions.

1. Introduction

Recent empirical literature defines an industry's clockspeed as a measure of the evolutionary life cycle capturing the dynamic nature of the industry. Among other factors, the rate of new product development is one of the primary drivers of clockspeed. For example, Fine (1998) suggests that one metric which can be used to measure an industry clockspeed is the rate of new product introduction or intervals between new product generations. In their study of the electronics industry, Mendelson and Pillai (1999) also show that higher industry clockspeed is associated with faster execution in product development activities. Furthermore, these authors find that firms operating in faster-moving business environments tend to accelerate their own internal operations such that their own individual clockspeed is synchronized with the corresponding industry clockspeed.

A variety of other mechanisms driving the speed of new product development for individual firms are also discussed in the literature. Eisen-

C. Deissenberg and R.F. Hartl (eds.), Optimal Control and Dynamic Games, 215–229.

hardt and Brown (1998) recommend that in rapidly shifting industries, “time pacing” new product development efforts relative to the calendar can help managers to better manage transitions between new product development projects, and to build organizational momentum towards achieving ambitious new product development goals. Moreover, these authors advocate that firms should synchronize the pace of change with their own marketplace and internal capabilities. In contrast, they posit that “event pacing”, (i.e. timing your internal new product development efforts relative to competitors, shifts in technology, etc.) may be an erratic and ineffective strategy in fast paced industries. For example, Bayus (1998) also names key factors that are expected to significantly influence a firm’s decision regarding new product introductions, including competitive pressure, market opportunity, and internal pressure such as market share and time since last product introduction.

An issue related to new product introduction is the importance of “product rollovers,” or the simultaneous management of new product introduction and displacement of old products. Billington, Lee and Tang (1998) discuss problems associated with product rollovers. In particular, they analyze two primary strategies: the solo product roll (where one product completely replaces the previous generation) and the dual product roll (where both products remain in the market). These authors find that a firm’s choice of an appropriate rollover strategy depends on factors such as the inventory related rollover costs, customer service, and the firm’s market position.

We introduce a simple analytic model which identifies an optimal firm level clockspeed. First, we determine the optimal number of generations of new products that should be introduced for a given planning horizon. The optimal firm clockspeed is driven by the following forces: (i) average demand forecasts, (ii) dynamic profits earned over time, (iii) cannibalization of older products, and (iv) organizational constraints limiting the pace of new product development. Thus, this model offers managerial insights concerning the dynamics of new product development activities on the firm level.

A key factor influencing firm clockspeed is the anticipated shape of the demand/sales curve for each generation of a new product. For example, the product life cycle curve is often associated with the introduction, growth and decline of a product in the marketplace via some kind of diffusion process. Conversely, a common assumption in the literature addressing the optimal time-to-market for new product introductions is that sales are constant for both old and new generations of products. Finally, certain types of products experience the highest levels of sales immediately following their introduction to the market, with declining sales throughout the remainder of the planning horizon.

In addition, we characterize the impact of dynamic net revenues on firm clockspeed. For instance, consider the PC industry that exhibits extreme price competition exemplified by prices that decrease rapidly throughout the product life cycle. For this type of industry, we derive analytical results showing that rapidly decreasing prices are associated with an accelerating firm clockspeed, (i.e. new products are introduced at a faster pace.) In contrast, consider industries where learning curves effectively decrease unit costs such that net revenues are increasing over the life cycle. Indeed, we find that firm clockspeed is actually decelerated (i.e. new products are introduced at a slower pace) due to the decreasing unit costs associated with the learning effects. Therefore, the dynamic nature of the net revenue stream has a significant impact on the pace of new product development and introduction at the firm level.

The primary focus of this paper is to develop analytic insights into factors driving firm level clockspeed. Specifically, structural results concerning the impact of dynamic net revenues and dynamic demand are developed. Carrillo (2004) also addresses firm level clockspeed and its relationship to industry level trends utilizing numerical examples. However, because the model in Carrillo (2004) is a discrete model with complex development cost, demand and marketshare functions, no analytic results are derived. For a comprehensive review of the literature in this area, see Carrillo (2004).

The remainder of the paper is organized as follows. In Section 2, a firm level model of clockspeed is introduced, while Section 3 contains the subsequent analysis for this model. Finally, Section 4 contains a summary of conclusions and future directions for research.

2. Model

In this section, we introduce a simple analytic model which explains key factors influencing the optimal pacing of new product introductions. First, we begin this section by introducing appropriate notation and motivating key assumptions necessary for further analysis. After introducing the model, we analyze optimal solutions and discuss the corresponding managerial insights.

2.1 Notation and Assumptions

Each of the generations of new products introduced into the market has the following characteristics: (a) a net profit earned for each unit produced and sold, (b) a corresponding sales/demand rate, and (c) development costs incurred to bring the new generation to the market. The following paragraphs describe these particular parameters for a firm.

First, let n represent the primary decision variable denoting the number of generations of new products to be introduced during a given planning horizon of length T for a particular firm. Because this decision variable effectively determines the pace of new product development for a given planning horizon, we refer to it simply as the firm's NPD clockspeed. An alternate measure of firm clockspeed can be captured by dividing the number of generations of new products introduced by the length of the planning horizon, (i.e. n/T). However, to simplify notation, we simply refer to n as the firm NPD clockspeed and t' (= T/n) as the firm's average life cycle length per generation. Also, to gain analytic insights into the various factors driving firm clockspeed, we assume that n is a continuous variable.

Second, let $\dot{Q}(t)$denote the anticipated sales rate at time t for each generation of new products introduced into the market at time t=0. Therefore, the cumulative demand for a product through time t of the product life cycle is denoted by Q(t). Initially, we make no specific assumptions concerning the shape of the sales/demand curve. Later, we investigate the impact of specific types of demand curves on the firm's pacing of new product introduction.

Third, another key assumption concerns the issue of "product rollovers," or the simultaneous management of new product introduction and displacement of old products. Furthermore, we assume that the introduction of a new product completely replaces all sales of the older product currently in the market. Billington, Lee and Tang (1998) refer to this strategy as the solo product rollover (whereby one product completely replaces the previous generation).

Fourth, the firm's internal development costs are a crucial factor. As such, we now consider the impact on overall profit explicitly as a separate function which is independent of the firm's average net revenues. For simplicity, we assume that the development costs are fixed for each new generation of products introduced to the market. Specifically, let c_D denote the firm development costs per each new generation new products introduced to the market.

Fifth, a key piece of the model describes the net revenue earned from sales of the new product throughout its lifetime. Let π(t) represent the net revenue earned from sale of a unit of for each generation of new products at time t during a product life cycle, (i.e. revenue - unit cost). Note that time t=0 reflects the time at which each generation is introduced to the market. Factors influencing the net revenue include both demand side influences (i.e. revenues earned) and supply side influences (i.e. unit costs).

Furthermore, we assume that net revenue derived during each product life cycle is dynamic and can be described by a time varying pattern following the introduction of a new product. To capture the effects of dynamic revenue, π(t) can be expressed as a continuous function of a simple average multiplied by a time varying agent, as shown in Equation (13.1). By definition, π_0 represents the average net revenue earned during the planning horizon, while α(t) represents the time-varying influence due to the dynamic net revenue changes over the product's life cycle.

$$\pi(t) \;=\; \pi_0\, \alpha(t); \quad \int_0^T \alpha(t)\, dt \;=\; T. \tag{13.1}$$

Next, we assume that the firm's potential to introduce new generations of products into the marketplace is limited by its own internal and/or technological capabilities. Specifically, let n_o reflect the maximal number of new product generations that the organization can introduce during a planning horizon of length T. Furthermore, n_o should be set to reflect the lesser of the firm's production, design, or supply capabilities. For example, the firm may not have adequate production flexibility to meet demand for the multiple generations of new products, (see Gaimon and Morton (2004)). The firm's internal design capacity may be limited either in the magnitude of manpower needed, or in the requisite skill mix. Alternatively, the constraint may reflect design limitations relating to the actual physical technology of the components embedded in the new product. Lastly, suppliers may not have the necessary capacity to meet demand for the multiple generations of new products.

2.2 The Objective

The objective is to maximize profit associated with sales of new products over the total planning horizon. Because we assume that (a) average net revenues earned during a product life cycle are similar throughout the planning horizon, and (b) each new product completely cannibalizes sales of older products, then the average profit earned during each product life cycle is essentially the same. The problem reduces to one of determining how many generations of new products (n) to introduce during a planning horizon of length T, given that each new generation earns revenues for an average life cycle length t' = T/n. The objective function and the constraint limiting the maximal pace of new product introductions are shown below. To simplify subsequent analysis, the dynamic sales and net revenue effects have both been combined. Specifically, let Q_a(t) represent the adjusted cumulative sales which reflect the cumulative sales at time t as adjusted for the time varying effects from the dynamic net revenues.

Max

$$n \int_0^{T/n} \pi_0 \alpha(t)\dot{Q}(t)\, dt - n\, c_D \quad = \quad n\, \pi_0\, Q_a(T/n) - n\, c_D \tag{13.2}$$

where

$$Q_a(T/n) = \int_0^{T/n} \alpha(t)\, \dot{Q}_a(t) dt \tag{13.3}$$

St:

$$0 \leq n \leq n_o. \tag{13.4}$$

3. Analysis

To obtain a solution for the constrained optimization problem, we formulate a Lagrangian as shown in Equation (13.5). Let λ denote the Lagrange multiplier associated with the organizational barrier, (i.e., λ represents the marginal value to the firm's profit derived from an additional new product generation made available through the firm's organizational capabilities). In Equations (13.6) and (13.7) we state the Karush-Kuhn-Tucker (KKT) necessary conditions for optimality of n and λ. Note that in general, strict concavity of the objective function with the linear constraint ensures that the KKT conditions are both necessary and sufficient. However, if the objective function is indeed convex, then a boundary solution occurs. We consider both possibilities later in the analysis.

$$L \ = \ n\, \pi_0\, Q_a(T/n) \ - \ nc_D + \lambda \ (n_o \ - \ n) \tag{13.5}$$

$$n* : \ \pi_0\, Q_a(T/n) \quad = \quad T/n\, \pi_0\, \dot{Q}_a(T/n) \ + c_D + \ \lambda \tag{13.6}$$

$$\lambda* : \quad \lambda \ (n_o \ - \ n) \quad = \quad 0 \tag{13.7}$$

The first order condition in Equation (13.6) states that the optimal number of new product introductions is that value for which the marginal gain in profit equals the marginal costs. Specifically, the marginal gain in profit is derived from the sales of an additional new product life cycle introduced during the planning horizon. The marginal cost is due to (i) the truncation of the length of time during which all products introduced during the planning horizon remain in the market, (ii) the additional development costs, and (iii) the marginal value of the organizational pace constraint. Therefore, the optimal number of generations of the new product for an industry must balance the extra revenues earned from an additional product life cycle with the truncation of the sales of its older product lines during the current planning horizon.

Equation (13.7) is simply the complementary slackness constraint, which assures that the pace of new product introduction does not exceed the organizational barrier. From these optimality constraints, we can derive two different cases depending on whether or not the constraint is tight. Specifically, Case 1 occurs when the constraint is not tight (i.e. $\lambda = 0$), whereas Case 2 occurs when the constraint is tight, (i.e. $\lambda > 0$). In the following paragraphs, we discuss the managerial implications for each situation.

First, we consider the situation where the organizational constraint is not tight, (i.e. Case 1 occurs). In this case, the optimal clockspeed which maximizes firm profits actually lags the maximal pace allowed by the firm's organizational capabilities. In other words, it is more profitable to slow the pace of new product introductions relative to the maximal pace dictated by firm's organizational capabilities. Furthermore, by speeding the industry clockspeed beyond the optimal value n*, the costs of limiting the length of all of the new product life cycles during a give planning horizon outweigh the benefits of one additional new product introduction.

Next, we consider the situation where the organizational constraint is tight, (i.e. Case 2 occurs). In this case, the speed of new product introduction is dictated by the organizational and/or technological barriers. Moreover, an incentive exists for firms to invest in organizational capabilities to speed new product introduction. Furthermore, we can calculate the additional profits for the firm from breaking the organizational barrier, which can be weighed against the costs of achieving that goal. In particular, this information is captured by λ^*, which represents the marginal value to the firm's profit derived from an additional product introduction made available through enhancing the firms organizational capabilities.

To investigate the existence of both Case 1 and Case 2 solutions, we first analyze the shape of the diffusion curve. In particular, the objective function must be a concave function of n for a non-boundary point solution to exist (i.e. Case 1). Furthermore, concavity of the objective function ensures that second order conditions of optimality are fulfilled. The following sections investigate the impact of alternative forms of the adjusted diffusion curve on the optimal firm clockspeed.

3.1 Constant Adjusted Sales Curve

Consider first the situation where the adjusted sales curve remains flat for the total length of the planning horizon, due to a constant sales level and constant net revenues earned for each unit. Specifically, let $\dot{Q}(t) = K$ and $\alpha(\mathrm{t}) = 1$ where K represents a constant. Theorem 13.1 summarizes the consequences of this combination of sales and net revenues on the optimal firm clockspeed.

THEOREM 13.1 *Suppose the adjusted diffusion curve is concave over the planning horizon. Then, the optimal clockspeed n*=0.*

Proof: From Equation (13.6), $\pi_0 KT/n \quad = \quad T/n\, \pi_0 K \ + c_D + \ \lambda.$

Essentially, the marginal benefits from an additional generation are always exactly equal to the marginal costs from truncating all other generations of new products. However, because the additional development costs, the total marginal costs will always exceed the marginal benefits in this scenario. To interpret this situation, consider a firm that already has a generation of products on the market. If the sales curve and net revenues are flat, then the firm can always earn that consistent revenue stream without introducing any new products to the market. Basically, the firm foregoes any development costs and continues offering the original product on the market.

The assumption of constant sales is a common one used in the time-to-market literature. In reality, there are limits to the assumption of constant sales and net revenues. For most products, it's likely that sales and net revenues will likely decline at some point in the product life cycle, necessitating the introduction of new products. Similarly, firms may introduce new generations of enhanced products to the market in an attempt to gain overall marketshare. Therefore, the sales curve may actually be increasing for each new generation of products introduced to the market.

3.2 Peaking Adjusted Sales Curve

Next, we consider the situation where the firm's demand curve for each new generation of products reflects a classic life cycle pattern which peaks at some time during the planning horizon. As the basis for our analysis, we use a variation of the simple Bass diffusion model (1969). The Bass diffusion model usually describes the pattern of demand for a new product at the industry level including the influence of both innovators and imitators buying the product. Specifically, innovators will buy the new product upon its introduction to the market, while imitators are more cautious and wait until the product has been on the market before making an initial purchase. This model has been shown to be fairly robust, and has been empirically validated for a wide variety of industries. In addition, empirical evidence exists which supports the utilization of such diffusion models at the individual firm and/or brand level of analysis.

Here, we consider a version of the basic Bass diffusion model which explicitly incorporates a time varying influence agent. From the "time adjusted" Bass model shown in Bass et al (1994), commonly known as the "Generalized Bass Model (GBM)," we have Equation (13.8)-(13.10). The cumulative demand for a product through time t of the product life cycle is denoted by $Q_a(t)$, while the number of adopters at time t is denoted by $\dot{Q}_a(t)$. Also, m is the scale parameter corresponding to the ultimate number of adopters for a particular type of product. The shape parameters denoted by p and q represent the coefficients of innovation and imitation respectively. Finally, $\alpha(t)$ represents the time-varying influence from the dynamic net revenues.

$$\dot{Q}_a(t) = (p + qQ_a(t)/m)(m - Q_a(t))\alpha(t) \tag{13.8}$$

$$Q_a(t) = m\left(1 - e^{-(p+q)u(t)}\right)\Big/\left(1 - (q/p)e^{-(p+q)u(t)}\right) \tag{13.9}$$

$$u(t) \;=\; \int_0^t \alpha(\tau)\, d\tau \tag{13.10}$$

Next, we analyze the impact of a peaking adjusted diffusion curve on the optimal firm clockspeed. Several basic relationships necessary to complete our analysis are given in Equations (13.11) - (13.13).

$$\partial\, Q_a(T/n)/\partial n = -T/n^2 \dot{Q}_a(T/n) \tag{13.11}$$

$$\partial\, \dot{Q}_a(T/n)\Big/\partial n = -T/n^2 \ddot{Q}_a(T/n) \tag{13.12}$$

$$\ddot{Q}_a(T/n) = \dot{Q}_a(T/n)\ \left[(q\ -\ p\ -\ 2\,q/mQ_a(T/n))\ \alpha(T/n) + 1/\alpha(T/n)\dot{\alpha}(T/n)\right] \tag{13.13}$$

THEOREM 13.2 *Suppose the adjusted diffusion curve is concave and peaks at time t_P such that the following holds:*

$$\begin{aligned} \ddot{Q}_a(t) &> 0 for\ t < t_P \\ \ddot{Q}_a(t) &= 0 for\ t = t_P \\ \ddot{Q}_a(t) &< 0 for\ t > t_P \end{aligned}$$

Also, define n_P such that $n_P = T/\ t_P$. Then, $n^* < n_P$, (i.e. $t'^* > t_P$.)

Proof: The proof can be seen graphically in Figure 1 at the end of the chapter by considering the area under the adjusted diffusion curve. Note that if $\mathrm{n}_O \geq \mathrm{n}_P$ ($\mathrm{t}_O \leq \mathrm{t}_P$), then $\mathrm{n}^* < \mathrm{n}_P$ ($\mathrm{t'}^* > \mathrm{t}_P$). Else if $\mathrm{n}_O < \mathrm{n}_P$ ($\mathrm{t}_O > \mathrm{t}_P$), then $\mathrm{n}^* \leq \mathrm{n}_O < \mathrm{n}_P$, ($\mathrm{t'}^* \geq \mathrm{t}_O > \mathrm{t}_P$).

The implications of Theorem 13.2 are illustrated in Figure 1. Note that the marginal benefits (i.e. $Q_a(T/n)$) are represented by the area under the diffusion curve to the left of t', whereas the marginal costs (i.e. $T/n\dot{Q}_a(T/n)$) are represented by the area inside the square region. Hence, the optimal clockspeed (n*) for the unconstrained problem corresponds to the average life cycle length (t') where the difference between the area representing the marginal benefits and the area representing the marginal costs are exactly equal to the development costs. Furthermore, from Figure 1, it is apparent that the optimal life cycle length (t') occurs after the peak diffusion time (t_P). Now we consider the impact of the organizational constraint on this result.

Suppose the adjusted diffusion curve of a particular firm peaks at some point in time, and the organizational barrier for this firm exceeds the pace of peak sales (i.e. $\mathrm{n}_O > \mathrm{n}_P$ and $\mathrm{t}_O < \mathrm{t}_P$). Then the optimal industry clockspeed is such that all products introduced will remain in the market longer than the time necessary to achieve peak sales. Furthermore, a Case 1 solution is optimal and the industry clockspeed lags the organizational capabilities. However, if the pace afforded by the firm's capabilities lags the pace of peak sales (i.e. $\mathrm{n}_O < \mathrm{n}_P$ and $\mathrm{t}_O > \mathrm{t}_P$), then a Case 2 solution may be optimal and the industry clockspeed will be determined by the organizational capabilities. In either situation, the optimal industry clockspeed lags that value which allows each new product generation to remain in the market long enough to meet peak demand. Moreover, the optimal firm clockspeed allows new products to reach the decline stage of the life cycle prior to replacing them in the market.

To illustrate, consider the following example. Suppose a firm has a constant net revenue stream, (i.e. $\dot{\alpha}(t) = 0$) and a classic growth curve is appropriate such that the rate of change in the number of adopters for the new products increases then decreases over time. Specifically, the coefficient of innovation is less than the coefficient of imitation for the

Bass model, (i.e. p<q). Furthermore, suppose that the organizational constraint for this type of industry is such that it allows new products to be introduced to the market at a rate which exceeds the peak diffusion for each new generation, (i.e. $n_O > n_P$). Then, the optimal clockspeed for this type of firm is slower than that which allows all new product to reach the peak level of sales prior to the introduction of the next

3.3 Declining Adjusted Sales Curve

In contrast to Theorem 13.2 which assumes that the diffusion curve is concave, Theorem 13.3 addresses the situation where the adjusted diffusion curve is actually decreasing over time. In addition, to derive the results in Theorem 13.3, we first consider the case where development costs are negligible. Under these conditions, the second order conditions of optimality are violated, and a boundary solution will always be optimal.

THEOREM 13.3 *If the rate of change in the number of adopters for the new products is decreasing over time, (i.e. $\ddot{Q}_a(t) \leq 0 \ \forall t$) and if the development costs are negligible (i.e. c_D=0), then a Case 2 solution will be optimal and $n^* = n_O$.*

Proof: The proof can be seen graphically in Figure 2 by considering the area under the adjusted diffusion curve. Note that for $\ddot{Q}_a(t) \leq 0$, the total area under the diffusion curve (i.e. $Q_a(t)$) is always greater than the rectangular area representing the average diffusion rate (i.e. $t\dot{Q}_a(t)$) at any point in time. Therefore, $Q_a(t) \geq t\dot{Q}_a(t) \, \forall t$.

To illustrate, consider the following example. Suppose an industry is fiercely competitive and characterized by decreasing net revenue stream, (i.e. $\dot{\alpha}(T/n) \leq 0$). Such an industry may offer the greatest premium revenue to the industry "pioneers" that are first to market. For this industry, the coefficient of innovation exceeds the coefficient of imitation for the Bass model, (i.e. p > q). From Equation (13.6), we have $\ddot{Q}_a(T/n) < 0$, since $\dot{Q}_a(T/n) \geq 0$. That is, if the rate of change in the number of adopters for the new products is decreasing, and the development costs are very small, the firm's clockspeed will always match the pace of the organizational capabilities.

While the assumption of negligible development costs may be unreasonable, the results of Theorem 13.3 can be interpreted as the limit of the optimal clockspeed as development costs become smaller. When development costs are significant, however, a Case 1 solution may be optimal even when the adjusted sales curve is decreasing.

3.4 Declining Net Revenues

In Theorem 13.4, we characterize the impact of the change in net revenue over time on the peak diffusion time, assuming that the demand for the firm follows a classic product life cycle which peaks then declines.

THEOREM 13.4 *Suppose the diffusion curve is represented by the Bass model in Equation (13.8) , with $p < q$. Compare three alternate scenarios each with similar diffusion parameters but differing dynamic revenue streams earned over time. Specifically, in scenario 1, the net revenue is increasing over time (i.e. $\dot{\alpha}(t) > 0$) and the diffusion curve peaks at t_{P1}. In scenario 2, the net revenue is constant over time (i.e. $\dot{\alpha}(t) = 0$) and the diffusion curve peaks at t_{P2}. In scenario 3, net revenue is decreasing over time (i.e. $\dot{\alpha}(t) < 0$) and the diffusion curve peaks at t_{P2}. Then, $t_{P1} > t_{P2} \quad > t_{P3}$.*

Proof: To find t$_P$, let $\ddot{Q}_a(t_P) = 0$ such that $2q/m\, Q_a(t) - \dot{\alpha}(t)/[\alpha(t)]^2 = q - p$. The proof follows from the fact that $Q_a(t) \geq 0$ and q - p >0.

¿From Theorem 13.4, the dynamic nature of the net revenue stream over the product life cycle has a significant impact on the adjusted diffusion curve. In particular, for firms encountering extreme price competition such that net revenue is decreasing throughout the product life cycle, the peak demand occurs earlier in the planning horizon. Furthermore, it's more likely that the organizational barrier will be tight, and the industry will exhibit characteristics of a Case 2 solution. For a Case 1 industry which is unconstrained by organizational barriers, the effect of decreasing net revenue is to increase the firm clockspeed. Moreover, as price competition in the industry increases (i.e. the net revenue exhibits a faster rate of decline), the firm's clockspeed necessarily increases. To illustrate, consider the personal computer (PC) industry that is characterized by rapidly decreasing prices ($\sim$ 50% per year). The effect of the decreasing net revenue is that PC makers are introducing new models at a faster rate every year, confirming the empirical evidence found by Mendelson and Pillai (1999).

In contrast, when significant learning effects associated with decreasing cost structures dominate any price reductions, net revenue is actually increasing over time, (i.e. $\dot{\alpha}(t) > 0$). In this type of industry, the impact of learning effects on the firm's internal costs actually causes the peak demand to occur later in the planning horizon. Intuitively, firms in this type of industry can earn more profit per unit later in the planning horizon, then the new products should remain in the market longer to reap these benefits prior to replacement by newer generations. Therefore, strong learning curve effects in an industry are associated with a slower pace of new product introduction. To illustrate, consider the aircraft industry which is traditionally associated with strong learning effects. In his recent book, Fine (1998) reports that the product clockspeed for this industry is between ten and twenty years.

4. Conclusions

Using a simple analytic model, an optimal firm "clockspeed" can easily be derived. The derived clockspeed is mostly dependent on marketing, technology, and operations related factors such as (i) average demand forecasts, (ii) dynamic profits earned over time, (iii) cannibalization of older products, and (iv) technology and/or production constraints limiting the pace of new product development. Implicitly, there exists some optimal firm related rate of new product introductions, whereby exceeding this optimal level decreases firm profits.

Two different cases exist and may be appropriate for different types of firms. In the first type, the optimal clockspeed which maximizes profits for the firm actually lags the maximal pace that the firm could achieve. This situation results when dynamic demand follows a traditional life cycle growth curve with both growth and decline stages. Furthermore, the optimal firm clockspeed in this case advocates introducing new generations of products such that each generation is allowed to reach the decline stage before replacing it with a new generation. Therefore, market related demand factors are crucial drivers of the firm's clockspeed in this situation.

In contrast, the second case depicts the situation where the speed of new product introduction is dictated by the firm's organizational capabilities. Factors which contribute directly to this phenomena include a competitive environment with declining sales/demand curves along with declining net revenues for the firm's new product. In this case, an incentive exists for firms to invest in enhancing its production, design, and/or supply related capabilities to increase the frequency of new product introductions. Therefore, firm's operating in this type of environment should carefully assess the trade-off between the additional revenues that could be earned by speeding up its pace of new product introduction and the costs of enhancing these internal capabilities.

A key factor influencing firm clockspeed is the anticipated shape of the demand/sales curve for each generation of a new product. In general, when demand curves are relatively flat, there is little incentive for the firm to introduce multiple generations of new products, particularly if development costs are formidable. In contrast, when demand curves are declining and development costs are low, the firm should introduce new products at the maximal pace possible with its current organizational capabilities. Finally, when the demand curves follow a traditional growth and decline pattern typically associated with the classical diffusion process, the firm optimally introduces each new generation of products after the old product has reached the declining phase of the product life cycle.

While this paper addresses the impact of dynamic net revenues and demand on firm clockspeed via an analytic framework, a more careful examination of some of the model assumptions is warranted. The assumption of a "single product rollover" whereby the introduction of a new product completely replaces the sales of the older product currently in the market is fairly common in the literature. However, the relaxation of this assumption to allow for the sales of multiple generations of products simultaneously can be further investigated. In addition, a more sophisticated linkage of the potential marketshare between multiple generations of a product may yield further insights into firm level NPD clockspeed. Finally, a more detailed analysis of alternate development cost structures is appropriate.

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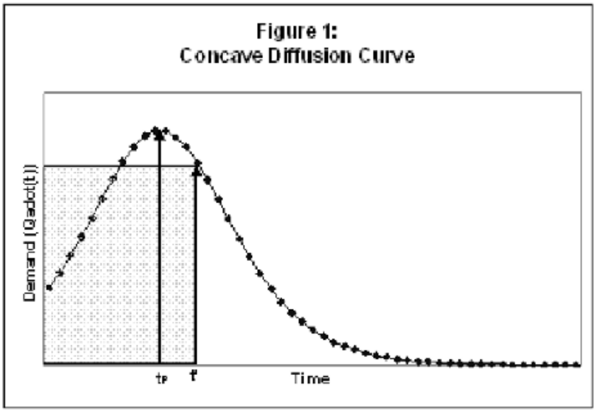

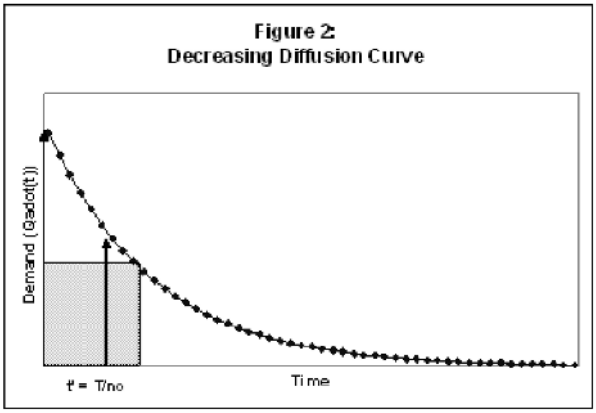

Chapter 14

HIBERNATION DURATIONS FOR CHAIN OF MACHINES WITH MAINTENANCE UNDER UNCERTAINTY

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Abstract Maintenance of a machine and its replacements by newer ones in the course of a predetermined planning horizon with fixed intermediate dates for potential replacement opportunities is considered. Using the Kamien-Schwartz optimal control model for maintenance, allowance for ceasing of production until installation of a new machine is studied with respect to regeneration points.

1. Introduction

We consider a single machine and its possible replacements (allowed on a calendar of potential regeneration points) over time. The probability distribution of machine failure can be improved by predictive or preventive maintenance. The natural hazard rate for which the machine was designed for, can thus be reduced to a more favorable effective hazard rate.

If the retirement date of a machine is not required to be equal to the installment date of its successor, then the length of the hibernation duration for the production operations need to be determined. When capital expenditures of an organization are made at fixed points on a calendar (such as release of funds in first week of each quarter, or semi annually on first weeks of March and September), then new machine purchases may have to wait for these dates for the availability of the acquisition funds. In the meantime it is possible that the machine waiting for replacement may operate under potentially unprofitable circumstances.

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Selling the machine on hand and waiting idle may be more attractive than suffering unfavorable production costs, or a rapid deterioration in its resale/salvage value. In addition to such factors, constraints on delivery dates of the machine supplier can possibly prevent installation of a replacement at the retirement time of its predecessor. Hibernation can also be considered when buying the currently available machine yields negative expected net present value of cash flow, making it preferable to wait idle until the availability of profitable technologies.

We use the term hibernation to indicate such deliberate non-production periods where the system waits for the arrival of a new and profitable machine. If hibernation is allowed, when should they be scheduled? Answers to such questions may also put pressure for realignment of the calendar for the regeneration points, as well as company policies on borrowing versus use of internal funds. These in turn may raise considerations for the modification of machine replacement time windows.

2. The Model

The main model to be used is that of Kamien and Schwartz (1971) which was recently imbedded into a dynamic programming model by Dogramaci and Fraiman (2004) (in short D-F), for potential machine replacements at fixed intermediate dates over the planning horizon.

Notation:

T : Length of planning horizon consisting of T equal length periods. Starting point of each period constitutes a potential for the acquisition of a machine (a replacement opportunity), i.e. a regeneration point. Generalization of the model for periods of unequal lengths is straightforward and will not be addressed here.

j : Integer indicating a specific regeneration point in the planning horizon. Chronologically the one at the start of the terminal period of the planning horizon is set as $j = 1$, and earlier ones have higher values (in order to serve as index for computational backsweep operations.)

$F_j(t)$: Probability that a machine of vintage j (bought when there were j periods to go until the end of the planning horizon) fails at or before t units of time from its purchase date.

$h_j(t) = [dF_j(t)/dt]/[1 - F_j(t)]$: Natural hazard rate of a machine (of vintage j).

$u(t)$: intensity of maintenance effort at time t. $u(t) \in [\underline{U_j}, \overline{U_j}]$, $0 \leq \underline{U_j} < \overline{U_j} \leq 1$ where $\underline{U_j}$ and $\overline{U_j}$ denote minimum and maximum allowable intensities on a machine of vintage j.

$h_j(t)[1 - u(t)]$: Effective hazard rate of the machine.

$M_j(u(t))h_j(t)$: Cost of maintenance effort at time t. $M_j(u(t))$ is continuously differentiable with respect to $u(t)$, with $M_j^{'} > 0$, $M_j^{''} > 0$, and $M_j(0) = 0$.

r: Discount rate indicating time value of money.

D_j: Cost of acquiring and installing a machine of vintage j.

R_j: Revenue net of all costs except maintenance $u(t)$ generated by a machine of vintage j.

$S_j(t)$: Resale value at time t, of a working machine of vintage j. $0 \leq S_j(t) \leq R_j/r$.

L_j: Junk value of a failed machine costs due to in-service failure . $L_j < S_j(t)$.

$f_{(j)}$: Optimal dynamic programming value function at stage j of backward sweep. This is the net present value (with respect to node j) of an optimal regeneration and maintenance policy when there are j periods to go until the end of the planning horizon. It will be computed for $j = 1, 2, ..., T$ in that order. Subscripts in parentheses indicate stage number of dynamic programming calculations, rather than equipment vintage. $f_{(0)} = 0$.

$V(j, K)$: Optimal expected net present value for a vintage j machine acquired at time $T - j$, in other words at node j, at cost of D_j dollars with the intention of keeping it for K periods ($K \leq j$) and subsequent replacements (if any). Present value is computed with respect to the time when the machine is introduced to the production system ($T - j$. Maximum value of K is j. However, managerial considerations can dictate it to be shorter.

Z_j: Hibernation time (measured in terms of machine age): Planned retirement age of machine of chosen at node j. If hibernation is not allowed, $Z_j = K$. Otherwise, $0 \leq Z_j \leq K$.

K_{Z_j}: Closest regeneration point downstream of Z_j. ($0 \leq Z_j \leq K_{Z_j} \leq K$). K_{Z_j} is the smallest integer larger than or equal to Z_j.

$V(j, K)$ shall be determined after $f_{(j-1)}$,,$f_{(0)}$ are obtained, and will in turn feed into the computation of $f_{(j)}$ as follows:

$$f_{(j)} = \max_{K=1,..,j_K} [V(j, K)] , \quad j = 1, 2, ..., T; \; j_K \leq j. \tag{14.1}$$

j_K is the upper bound on intended machine life for vintage j, as dictated by technical, safety, and managerial considerations. If there is no such limit, then one can set $j_K = j$. At node j different types of machines may be available, (and hibernation times of each of these alternatives may be different.) If there are alternative models, i.e. a variety of technologies available at time $T - j$, then $V(j, K)$ can be solved for each and the alternative with largest expected net present value may be chosen.

Consider any point in time t, during the time span addressed by any $V(j, K)$. With probability $1 - F_j(t)$ the machine has not yet failed implying a cash flow rate of $R_j - M_j(u(t))h_j(t)$. On the other hand failure of the machine at time t is associated with probability density $dF_j(t)/dt = [1 - u(t)]h_j(t)[1 - F_j(t)]$ and cash flow of L_j right away, as well as $f_{(j-\tau-1)}$ which with respect to time t, is the nearest downstream optimal dynamic programming value function. The index number of the nearest downstream regeneration point is $j-\tau-1$. In case machine fails at time t a new one is bought at this node. (Values of τ, $\tau + 1$, $\cdots$ are chosen to target such nodes.) Thus $V(j, K)$ is obtained by solving the following problem.

$$
\begin{aligned}
&V(j, K) = \\
&= \max_{u(t), Z_j} \sum_{\tau=0}^{K_{Z_j}-1} \int_{\tau}^{\min[(\tau+1), Z_j]} \{e^{-rt}\{[R_j - M_j(u(t))h_j(t)][1 - F_j(t)] \\
&\quad + L_j[1 - u(t)]h_j(t)[1 - F_j(t)]\} \\
&\quad + e^{-r(\tau+1)} f_{(j-\tau-1)}[1 - u(t)]h_j(t)[1 - F_j(t)]\}dt \\
&\quad + [1 - F_j(Z_j)]\, [e^{-rZ_j} S_j(Z_j) + e^{-rK_{Z_j}} f_{(j-K_{Z_j})}] - D_j
\end{aligned} \tag{14.2}
$$

subject to

$$\frac{dF_j(t)}{dt} = [1 - u(t)]h_j(t)[1 - F_j(t)] \tag{14.3}$$

with

$$0 \leq \underline{U}_j \leq u(t) \leq \overline{U}_j \leq 1, \;\; F_j(0) = 0, \;\; t \in [0, Z_j] \text{ and } 0 \leq Z_j \leq K_{Z_j} \leq K.$$

If solution of (14.2)-(14.3) above yields $V(j, K) < 0$, then managerial policies allowing, we can set $V(j, K) = 0$ (implying that an imaginary machine of zero costs and revenues) and stay idle from time $T - j$ until $T - j + K$.

In the objective function (14.2), jumps from $f_{(j-1)}$ to $f_{(j-2)}$ to $f_{(j-3)} \cdots$ are addressed by breaking the problem into K unit period segments and imbedding each into the adjacent upstream one.

3. A Solution Procedure

The procedure proposed here builds upon the D-F approach with the added complexity of checking for hibernation possibilities. We first investigate the (potentially) last period of usage to check whether $K_{Z_j} =$

K. Hence, the machine of vintage j , to be used for K periods is studied from $t = K - 1$ to K.

$$
\begin{aligned}
& J_{j,K-1,F_j(K-1)} = \\
& = \max_{u(t)} \int\limits_{t=K-1}^{K} \{e^{-rt}\{[R_j - M_j(u(t))h_j(t)][1 - F_j(t)] \\
& \quad + L_j[1 - u(t)]h_j(t)[1 - F_j(t)] \\
& \quad + e^{-rK} f_{(j-K)}[1 - u(t)]h_j(t)[1 - F_j(t)]\}dt \\
& \quad + e^{-rK}[S_j(K) + f_{(j-K)}][1 - F_j(K)]
\end{aligned}
\tag{14.4}
$$

subject to

$$
\frac{dF_j(t)}{dt} = [1 - u(t)]h_j(t)[1 - F_j(t)] \tag{14.5}
$$

with $0 \leq \underline{U}_j \leq u(t) \leq \overline{U}_j \leq 1$, $F_j(K-1)$ given, and $F_j(K)$ free.

The probability that the machine would still be up and running is reflected in the value of the state variable at time (in this context, time=age) $K - 1 : F_j(K - 1)$. The optimal value of this problem, $J^*_{j,K-1,F_j(K-1)}$, feeds in as a salvage value to the adjacent optimal control problem from $K - 2$ to $K - 1$. D-F showed that for $\tau = 1, ..., K$, $J^*_{j,\tau-1,F_j(\tau-1)}$ is a linear function of the starting value of the state variable $F_j(\tau - 1)$. Thus the problem starting at $\tau - 1$ needs only to be solved for a starting state variable value of $F_j(\tau - 1) = 0$. Its optimal value will be imbedded into the adjacent earlier problem on the left (i.e. into the model that starts at time $\tau - 2$) as salvage value term, in the form: $[1 - F_j(\tau - 1)\ ]\ J^*_{j,\tau-1,0}$. Thus the objective function in (14.4) can be stated for $F_j(K - 1) = 0$ as:

$$
\begin{aligned}
&J_{j,K-1,0} = \\
&= \max_{u(t)} \int_{t=K-1}^{K} \{e^{-rt}\{[R_j - M_j(u(t))h_j(t)][1 - F_j(t)] \\
&\quad + L_j[1 - u(t)]h_j(t)[1 - F_j(t)]\} \\
&\quad + e^{-rK} f_{(j-K)}[1 - u(t)]h_j(t)[1 - F_j(t)]\}dt \\
&\quad + e^{-rK}[S_j(K) + f_{(j-K)}][1 - F_j(K)] \\
&= \max_{u(t)} \int_{t=K-1}^{K} \{e^{-rt}\{[R_j - M_j(u(t))h_j(t)][1 - F_j(t)] \\
&\quad + L_j[1 - u(t)]h_j(t)[1 - F_j(t)]\}dt \\
&\quad + \int_{t=K-1}^{K} \{e^{-rK} f_{(j-K)} \frac{dF_j(t)}{dt}\}dt \\
&\quad + e^{-rK}[S_j(K) + f_{(j-K)}][1 - F_j(K)] \\
&= \max_{u(t)} \int_{t=K-1}^{K} \{e^{-rt}\{[R_j - M_j(u(t))h_j(t)][1 - F_j(t)] \\
&\quad + L_j[1 - u(t)]h_j(t)[1 - F_j(t)]\}\}dt \\
&\quad + e^{-rK} f_{(j-K)}[F_j(K) - F_j(K-1)] \\
&\quad + e^{-rK}[S_j(K) + f_{(j-K)}][1 - F_j(K)]
\end{aligned}
$$

Since $F_j(K-1) = 0$, the objective function of the problem becomes,

$$
\begin{aligned}
&J_{j,K-1,0} = \\
&= \max_{u(t)} \int_{t=K-1}^{K} \{e^{-rt}\{[R_j - M_j(u(t))h_j(t)][1 - F_j(t)] \\
&\quad + L_j[1 - u(t)]h_j(t)[1 - F_j(t)]\}\}dt \\
&\quad + e^{-rK}[S_j(K)][1 - F_j(K)] + e^{-rK} f_{(j-K)}
\end{aligned} \tag{14.6}
$$

Since 14.6 subject to 14.5 is structurally a standard K-S model, any hibernation possibility in this period can be studied in the context of a free terminal time problem. Keeping j and $K-1$ fixed, and calling the terminal time Z_j , the terminal condition for

$$\begin{aligned}
J_{j,K,0}(Z_j) &= \\
&= \max_{u(t)} \int\limits_{t=K-1}^{Z_j} \{e^{-rt}\{[R_j - M_j(u(t))h_j(t)][1 - F_j(t)] \\
&\quad + L_j[1-u(t)]h_j(t)[1-F_j(t)]\}\}dt \\
&\quad + e^{-rZ_j}[S_j(Z_j)][1-F_j(Z_j)] + e^{-rK}f_{(j-K)} \qquad (14.7)
\end{aligned}$$

involves the evaluation of

$$\begin{aligned}
&e^{-rZ_j}\left(1-F_j(Z_j)\right)\left[R_j - M_j\left(u^*(Z_j)\right)h_j(Z_j) + L_j\left(1-u^*(Z_j)\right)h_j(Z_j)\right. \\
&\left.\quad -\left(r + \left(1-u^*(Z_j)\right)h(Z_j)\right)S_j(Z_j) + dS_j(Z_j)/dZ_j\right] \qquad (14.8)
\end{aligned}$$

where $u^*(Z_j)$ denotes the optimal value of the control at the optimal hibernation time. (See for example Kamien and Schwartz (1971) or Sethi and Thompson (2000) ch. 9.)

$u^*(Z_j)$ is chosen so as to maximize the following:

$$\max_{0 \leq u(Z_j) \leq 1} \quad \{(S_j(Z_j) - L_j)\, u(Z_j) - M_j[u(Z_j)]\} \qquad (14.9)$$

The expression in square brackets in (14.8) determines sign of the marginal benefit (negative if cost) of an infinitesimal increase in terminal time and shall be denoted by $B(Z_j)$.

$$\begin{aligned}
B\left(Z_j\right) &= R_j - M_j\left(u^*(Z_j)\right)h_j(Z_j) + \; L_j\left(1-u^*(Z_j)\right)h_j(Z_j) \\
&\qquad - \left(r + \left(1-u^*(Z_j)\right)h(Z_j)\right)S_j(Z_j) + \frac{dS_j(Z_j)}{dZ_j} \qquad (14.10)
\end{aligned}$$

and can be numerically evaluated for any candidate terminal time. It is clear that at optimal Z_j, we must have $B\left(Z_j\right) \geq 0$. Otherwise for some $\varepsilon > 0$, $Z_j - \varepsilon$ (which may be less than $K-1$) may be more profitable.

Since all the expressions can now be numerically evaluated, the procedure involves the following:

1 If $B(Z_j) \geq 0$ for all $Z_j \in [K-1, K]$ then we can set $Z_j := K$, implying no hibernation.

2 If $B(Z_j) \leq 0$ for all $Z_j \in [K-1, K]$ then one can set $K := K-1$ and if the new $K \geq 1$, solve this one-period-shorter problem for hibernation possibility.

3 Otherwise, using numerical search, find the values of Z_j for which $B(Z_j) = 0$ and compute the corresponding values of $J_{j,Z_j,0}$ as well

as for $Z_j = K - 1$, and $Z_j = K$. Pick the Z_j for which $J_{j,Z_j,0}$ is largest. (If this $J_{j,Z_j,0} \leq 0$ then set $K := K - 1$ and if the new $K \geq 1$, solve this one-period-shorter problem for hibernation possibility.)

4. Implications for Realigning the Calendar for Regeneration Points.

Allowance for hibernation relaxes the D-F model to ensure non- negative expected net present values for a machine and in particular, for the cash flow towards the end of its life.

If optimal value of hibernation time does not turn out to be an integer, the management may be advised to evaluate the allowance of shorter periods between regeneration points. Numerical experiments of D-F had indicated that reduction of such granularity increases the computational time as a polynomial function of the number of regeneration points. This evaluation also needs to take into account other considerations including whether acquisitions (or deliveries) of machines at the newly proposed times are feasible. While the optimal control model cannot comprise the non-quantifiable factors of managerial decisions, it nevertheless can serve as a useful tool for providing some of the basic building blocks that feed into the final decision.

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Chapter 15

SELF-ORGANIZED CONTROL OF IRREGULAR OR PERTURBED NETWORK TRAFFIC

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Abstract We present a fluid-dynamic model for the simulation of urban traffic networks with road sections of different lengths and capacities. The model allows one to efficiently simulate the transitions between free and congested traffic, taking into account congestion-responsive traffic assignment and adaptive traffic control. We observe dynamic traffic patterns which significantly depend on the respective network topology. Synchronization is only one interesting example and implies the emergence of green waves. In this connection, we will discuss adaptive strategies of traffic light control which can considerably improve throughputs and travel times, using self-organization principles based on local interactions between vehicles and traffic lights. Similar adaptive control principles can be applied to other queueing networks such as production systems. In fact, we suggest to turn push operation of traffic systems into pull operation: By removing vehicles as fast as possible from the network, queuing effects can be most efficiently avoided. The proposed control concept can utilize the cheap sensor technologies avail-

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able in the future and leads to reasonable operation modes. It is flexible, adaptive, robust, and decentralized rather than based on precalculated signal plans and a vulnerable traffic control center.

1. Introduction

Traffic control in networks has a long history. Early efforts have aimed at synchronizing traffic signals along a one-way, then a two-way arterial. There is still potential for improvement in this direction, as is attested by some recent research efforts [Stamatiadis and Gartner (1999)] or prompted by the development of new theoretical tools [Lotito *et al.* (2002), Mancinelli *et al.* (2001)]. Synchronization of traffic along arterials results in so-called green-waves, the aim of which is simply to ensure that traffic flows smoothly along main streets. Expected benefits of green waves are reduced fuel consumption and travel times.

The green-wave approach can be generalized to networks, yielding pre-calculated signal control schemes, such as TRANSYT [Robertson (1997)]. In principle such schemes are completely coercive: they force the traffic flow to comply with pre-calculated patterns, optimizing such criteria as the total travel time spent. Since traffic demand varies, the need for some responsiveness of the signal control was felt very soon. The SCOOT system [Robertson and Bretherton (1991)], an outgrowth of TRANSYT, allows for smooth change in the signal settings in response to changes in the traffic demand.

Among the strategies making use of precalculated controls, let us mention SCATS [Sims and Dobinson (1979), Lin and Chen (2004)], which relies on a library of controls (green durations, offsets, ...) according to traffic conditions. Even the optimization criterion depends on the traffic state. The system might, at night, minimize the number of stops, maximize throughput at day time under normal conditions, and aim at postponing the onset of congestion under heavy traffic conditions.

More recent developments stress greater adaptability. For instance UTOPIA [Mauro and Di Taranto (1989)] combines a regional control based on prediction of traffic flow through the main network arteries with the action of local intersection controllers. The regional control simply serves as a reference for local control.

OPAC [Gartner (1990)] optimizes queues in accordance with the "store-and-forward" concept [Papageorgiou (1991)], based on dynamic programming, with a rolling horizon. OPAC is fundamentally designed to manage intersections but extends to networks.

Even more decentralized and demand-responsive at a very local level, PRODYN [Henry and Farges (1989)] optimizes traffic at intersections by switching traffic lights on a traffic-actuated basis. Optimality is achieved through the dynamic programming technique. PRODYN also tries to coordinate neighboring intersections.

A further development includes dynamic assignment into the calculation of optimal traffic light settings as well as non-mandatory management schemes (user information). METACOR [Elloumi *et al.* (1994)], based on an optimal control strategy with a rolling horizon, is a good example of this approach. In the same line of approach, TUC [Diakaki *et al.* (2003)] displays two innovative features:

1. a reference strategy is calculated for the network (for a given situation),
2. a filter is included into the algorithm which calculates the commands. The aim of the filter is to detect and adjust deviations from the nominal traffic situation, and also to detect in real time deviations in parameter values.

A notable trend in recent research on demand-responsive traffic management systems is greater reliance on artificial intelligence (AI) methods, prompted by an ever growing complexity of algorithms, models and data. Let us cite some examples of this trend: [Li *et al.* (2004), Sayers *et al.* (1998), Niittymäki (2002)] and CLAIRE [Scémama (1994)].

Overall, no matter how sophisticated these classical approaches,

- either their responsiveness is limited and they appear as tools both coercive and normative (imposing a traffic situation rather than responding to it),
- or they are completely demand-responsive (CLAIRE or PRODYN for instance) and lack a global coordination. The TUC strategy might be viewed as a nice compromise.

All classical approaches require vast amounts of data collection and processing, as well as huge processing power. Further, global coordination notoriously requires data difficult to obtain or elaborate such as dynamic origin-destination matrices or dynamic assignment data. Finally, the systems described so far have a difficult time responding to exceptional events, accidents, temporary building sites or other changes in the road network, natural or industrial disasters, catastrophes, terrorist attacks etc.

Hence the usefulness of the decentralized and self-organized approach advocated in this paper is its greater degree of flexibility, its independence of a central traffic control center, and its greater robustness with respect to local perturbations or failures. As shown in Sec. 15.4 and summarized in Sec. 15.5, our autonomous adaptive control based on a traffic-responsive self-organization of traffic lights leads to reasonable operations, including synchronization patterns such as green waves. In particular, our principle of self-control is suited for irregular (i.e. non-Manhattan type) road networks with counterflows, with main roads (arterials) and side roads, with varying inflows, and with changing turning or assignment fractions. This distinguishes our approach from simplified scenarios investigated elsewhere [Brockfeld *et al.* (2001), Fouladvand and Nematollahi (2001), Huang and Huang (2003)]. Another interesting feature is that our approach considers not only "pressures" on the traffic lights related to delay times. It also takes into account "counter-pressures" when subsequent road sections are full, i.e. when green times cannot be effectively used.

2. Modeling traffic flow in urban road networks

In our model of urban road traffic, road networks are composed of nodes (intersections, plazas, dead ends, or cross sections of the road), which are connected by directed links i, representing homogeneous road sections without changes in capacity.

2.1 Traffic flow on network links

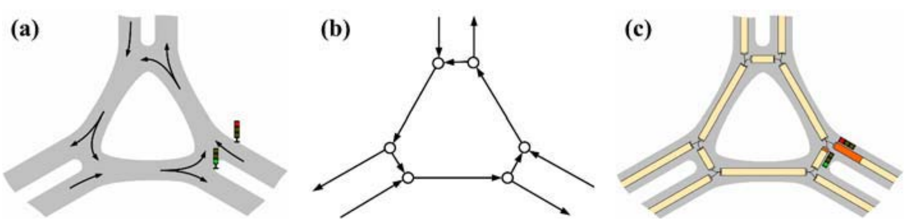

Figure 15.1. A road network (a) can be considered as a directed graph (b). The directed links represent homogeneous road sections, while the nodes correspond to junctions. (c) The road sections may or may not be controlled by traffic lights.

2.1.1 Homogeneous road sections. Our road sections i are characterized by a constant number I_i of lanes, over which traffic is assumed to be equally distributed. Different lanes turning into different directions may be treated as separate road sections, depending on the respective design of the infrastructure. Road sections can have a very large length L_i, which is in favor of numerical efficiency. The dynamics within a link of the road network is described by the section-based queueing-theoretical traffic model by Helbing (2003b). It is directly related to the equation of vehicle conservation [Lighthill and Whitham (1955)] and briefly introduced, here. The average velocity of vehicles on link i around place x at time t is denoted by $V_i(x,t)$, the spatial density per lane by $\rho_i(x,t)$, and the flow per lane by $Q_i(x,t) = \rho_i(x,t)V_i(x,t)$. The flow is approximated by a triangular flow-density relationship

$$Q_i(x,t) = \begin{cases} \rho_i(x,t)V_i^0 & \text{if } 1/\rho_i(x,t) > \left(1/\rho^{\text{jam}} + TV_i^0\right) \\ \frac{1}{T}\left[1 - \rho_i(x,t)/\rho^{\text{jam}}\right] & \text{otherwise (in congested traffic).} \end{cases} \tag{15.1}$$

While the increasing line $\rho_i V_i^0$ describes free traffic moving with speed V_i^0, the falling "jam line" describes congested traffic, in which the average vehicle distance $1/\rho_i$ is given by an effective vehicle length $l^{\text{eff}} = 1/\rho^{\text{jam}}$ (= vehicle length plus minimum front-bumper-to-back-bumper distance) plus a safety distance TV_i which grows linearly with the speed V_i. The proportionality factor is the (safe) time gap T kept in congested traffic. Therefore, our model is based on only three intuitive parameters: the maximum jam density ρ^{jam}, the free velocity V_i^0 (speed limit) on link i, and the time gap in congested traffic T. In our paper, we have chosen $V_i^0 = 14$ m/s $= 50$ km/h, $\rho^{\text{jam}} = 150$ vehicles per kilometer and lane, and $T = 1.8$ s.

We should note that there are other macroscopic traffic models such as the non-local, gas-kinetic-based traffic (GKT) model [Treiber *at al.* (1999)], which can describe the aggregate dynamics of traffic flows more accurately than this model. The "GKT model" has even been successfully implemented to simulate traffic flows on all German freeways, taking into account information by local detectors and floating car data. However, the dynamics of urban traffic is dominated by the dynamics of the traffic lights, which justifies simplifications in favor of numerical efficiency and analytical treatment. The section-based traffic model covers the most essential features of traffic flow in urban road networks, e.g. the transition between free and congested traffic, the spreading and interaction of vehicle queues, etc. Its particular strengths are its transparency, numerical stability, and computational efficiency. Compared to microsimulation models of urban traffic such as cellular automata models [Cremer and Ludwig (1986), Esser and Schreckenberg (1997), Nagel *et al.* (2000)], the treatment of lane changes, intersections, and turning operations is much easier, and analytical investigations are possible.

2.1.2 Propagation of perturbations. The particular simplicity of the section-based traffic model results from its two constant characteristic velocities: While perturbations of free traffic propagate together with the cars at the speed V_i^0, in congested traffic perturbations travel upstream with the constant velocity

$$c = -1/(T\rho^{\text{jam}})\,, \tag{15.2}$$

which has the typical value of −3.7 m/s or −13.3 km/h.

A favorable property of the section-based traffic model is that all relevant quantities can be determined from the boundary flows, which makes the model very efficient. For example, the dynamics inside a road section i can be easily derived from the arrival flow $Q_i^{\text{arr}}(t)$ and the departure flow $Q_i^{\text{dep}}(t)$ per lane with the two characteristic velocities V_i^0 and c, see Fig. 15.2.

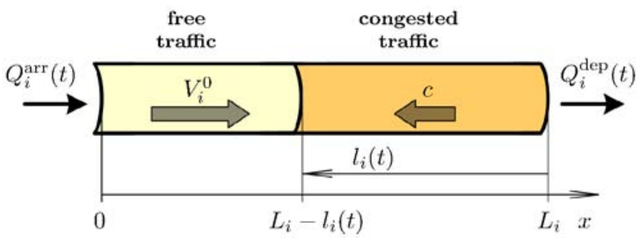

Figure 15.2. A road section i of length L_i with an area l_i of congested traffic at the downstream end (right). Due to the constant propagation speeds V_i^0 and c of perturbations in free and congested traffic, respectively (see big arrows), the internal dynamics can be easily calculated based on the boundary flows $Q_i^{\text{arr}}(t)$ and $Q_i^{\text{dep}}(t)$ only.

The interior flow per lane is given by

$$Q_i(x,t) = \begin{cases} Q_i^{\text{arr}}\left(t - \frac{x}{V_i^0}\right) & \text{if } x < L_i - l_i(t) \text{ (in free traffic)}\,, \\ Q_i^{\text{dep}}\left(t - \frac{L_i - x}{|c|}\right) & \text{if } L_i - l_i(t) \le x \le L_i\,. \end{cases} \tag{15.3}$$

That is, the flow is determined by the downstream boundary in the area of congested traffic of length $l_i(t) \ge 0$, while it is given by the arrival flow in the area $x < L_i - l_i(t)$ of free traffic. The density can be obtained via

$$\rho_i(x,t) = \begin{cases} Q_i(x,t)/V_i^0 & \text{if } x < L_i - l_i(t) \text{ (in free traffic)}\,, \\ [1 - TQ_i(x,t)]\rho^{\text{jam}} & \text{if } L_i - l_i(t) \le x \le L_i\,. \end{cases} \tag{15.4}$$

The average velocity is calculated via the formula $V_i(x,t) = Q_i(x,t)/\rho_i(x,t)$, if $\rho_i(x,t) > 0$.

The temporal change of the number $N_i(t)$ of vehicles per lane on road section i can be also determined from the arrival and departure flows:

$$\frac{dN_i}{dt} = Q_i^{\text{arr}}(t) - Q_i^{\text{dep}}(t)\,. \tag{15.5}$$

The time-dependent change of the congested area of length $l_i(t)$ will be discussed in the next paragraph.

2.1.3 Movement of congestion fronts. Since our road sections are homogeneous by definition, congestion can only be triggered at their downstream ends. While the congested area might eventually expand over the entire road section, the downstream end remains at $x = L_i$. The upstream end lies at $x = L_i - l_i(t)$, where jumps $\Delta\rho_i$ and ΔQ_i occur in the density and in the flow, respectively. In order to ensure the conservation of vehicles, the condition $\Delta Q_i = -\Delta\rho_i \cdot dl_i/dt$ must be fulfilled. Therefore, the border line between free and congested traffic moves with the following velocity [Helbing (2003b)]:

$$\frac{dl_i}{dt} = -\frac{Q_i^{\text{arr}}\big(t - [L_i - l_i(t)]/V_i^0\big) - Q_i^{\text{dep}}\big(t - l_i(t)/|c|\big)}{\rho_i^{\text{arr}}\big(t - [L_i - l_i(t)]/V_i^0\big) - \rho_i^{\text{dep}}\big(t - l_i(t)/|c|\big)} \,. \tag{15.6}$$

Note that, within the congested area of length $l_i(t)$, one might find areas of quasi-free traffic, where the vehicles reach the maximum free velocity V_i^0 and the maximum flow $Q_i^{\max}$ per lane that is possible according to the flow-density relationship (15.1):

$$Q_i^{\max} = \left(T + \frac{1}{V_i^0 \rho^{\text{jam}}}\right)^{-1} \,. \tag{15.7}$$

This value corresponds to vehicles accelerating out of a traffic jam every $T = 1.8$ seconds. Nevertheless, the value $1/T$ is not completely reached, as each subsequent vehicle has to drive an additional distance $l^{\text{eff}} = 1/\rho^{\text{jam}}$ in order to reach the respective measurement cross section. This requires an additional time interval of l^{eff}/V_i^0 as in the formula above (see Fig. 15.3).

Let us shortly discuss two special cases of formula (15.6): If the departure flow is stopped due to a red traffic light, we obtain the simplified relationship

$$\frac{dl_i}{dt} = \left[\frac{\rho^{\text{jam}}}{Q_i^{\text{arr}}\big(t - [L_i - l_i(t)]/V_i^0\big)} - \frac{1}{V_i^0}\right]^{-1} \approx \frac{Q_i^{\text{arr}}\big(t - [L_i - l_i(t)]/V_i^0\big)}{\rho^{\text{jam}}} \,. \tag{15.8}$$

If the traffic light turns green at time t_0', the end of the traffic jam still propagates upstream at the speed (15.8) with new arriving vehicles. However, at the same time, an area of quasi-free traffic with maximum flow $Q_i^{\max}$ propagates upstream with velocity c from the downstream boundary. Therefore, the effective length $l_i^{\text{eff}}(t)$ of the vehicle queue is

$$l_i^{\text{eff}}(t) = l_i(t) - |c|(t - t_0') \,. \tag{15.9}$$

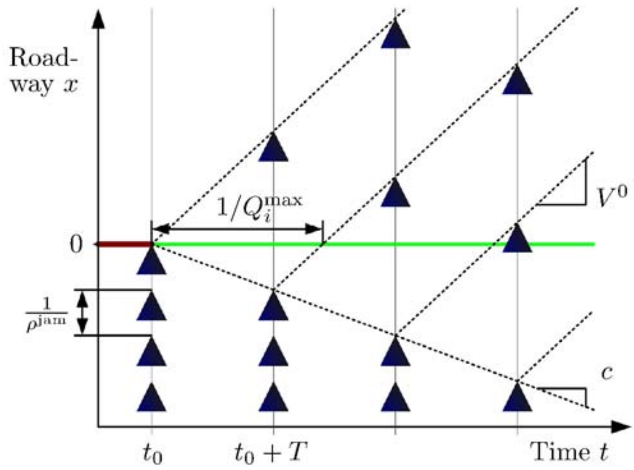

Figure 15.3. Illustration of queued vehicles (triangles in the lower left corner) and freely moving vehicles after a traffic light turns green (triangles in the upper right part). The characteristic speeds V^0 and c are indicated by diagonal lines.

If this effective queue has been fully resolved at time t^*, i.e. $l_i^{\text{eff}}(t^*) = 0$, it takes an additional time $l_i(t^*)/V_i^0$ until the last vehicle of that queue has left the road section i. Therefore, we reach $l_i(t) = 0$ and, thereby, free traffic on the whole road section i, at time $t^* + l_i(t^*)/V_i^0$. Before this point in time, vehicles that have moved out of the queue may still be trapped again by a red traffic light at the end of road section i.

2.1.4 Travel time. Let the travel time $T_i(t)$ be the time a vehicle needs to pass through the road section i when entering it at time t. Then, the actual number $N_i(t)$ of vehicles inside the road section is given by

$$N_i(t) = \int_t^{t+T_i(t)} dt'\, Q_i^{\text{dep}}(t') \,. \tag{15.10}$$

This formula implies the following delay-differential equation describing how the travel time T_i depends on the boundary flows [Helbing (2003b)]:

$$\frac{dT_i}{dt} = \frac{Q_i^{\text{arr}}(t)}{Q_i^{\text{dep}}\big(t + T_i(t)\big)} - 1 \,. \tag{15.11}$$

According to this, the travel time can be predicted based on the anticipated departure flow, e.g. when a certain traffic light control is assumed (see Secs. 15.2.1.5 and 15.4.3).

2.1.5 Delay time. Since the travel time would exactly be L_i/V_i^0 without congestion, any deviation from that can be understood as the time a vehicle has been delayed due to congestion. Therefore, we may introduce the delay time

$$T_i^{\text{del}}(t) = T_i - \frac{L_i}{V_i^0} \,. \tag{15.12}$$

Since L_i/V_i^0 is time-independent, the right hand side of equation (15.11) applies to dT_i^{del}/dt as well.

Consider a road section with a constant arrival flow $Q_i^{\text{arr}}(t)$ and a departure flow $Q_i^{\text{dep}}(t) = \gamma_i(t)Q_i^{\max}$ being controlled by a traffic light. As the buffer size is given by the maximum number $L_i\rho^{\text{jam}}$ of vehicles per lane on road section i, from Eq. (15.5) we can derive

$$\begin{aligned}\frac{1}{t}\int_0^t dt'\, Q_i^{\text{arr}}(t') &\le \frac{L_i\rho^{\text{jam}}}{t} + \frac{1}{t}\int_0^t dt'\, Q_i^{\text{dep}}(t') \\ &\le \frac{L_i\rho^{\text{jam}}}{t} + \frac{Q_i^{\max}}{t}\int_0^t dt'\, \gamma_i(t') \\ &= \frac{L_i\rho^{\text{jam}}}{t} + u_i Q_i^{\max} \end{aligned} \tag{15.13}$$

with the average green time fraction

$$u_i = \frac{1}{t}\int_0^t dt'\, \gamma_i(t') \,. \tag{15.14}$$

For $t \to \infty$ we can see that the average arrival rate per lane on road section i should not exceed the maximum flow times the green time fraction u_i. Otherwise, we will have a growing queue, until the maximum storage capacity $I_i L_i \rho^{\text{jam}}$ for vehicles on road section i has been reached.

The throughput is reduced if a downstream road section j is sometimes fully congested, as this limits the departure flow. Moreover, the delay time can temporarily increase, if the arrival of vehicles at the upstream boundary of road section i is not synchronized with the green phase of the traffic light at the downstream end. Such a synchronization of arrivals in i with the desired departure times is hard to reach in an irregular road network. As a consequence, vehicles tend to queue up at a red light before they can leave a road section i (see Fig. 15.4). Note, however, that a green light reaches maximum efficiency when it serves vehicles which have queued up before.

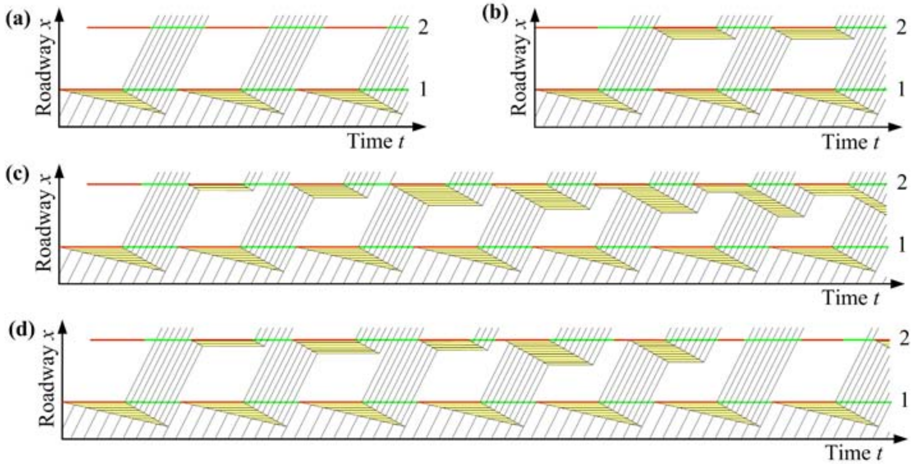

Figure 15.4. Trajectories of freely moving vehicles (diagonal lines) and queued vehicles (horizontal lines) in dependence of the traffic light control at two subsequent intersections 1 and 2. In all four displayed scenarios, vehicles arrive with identical time headways (i.e. constant arrival rate) at traffic light 1, which operates periodically. Traffic light 2 is operated in different modes: (a) The frequency and time offset are adapted to the first traffic light, as required by a green wave. (b) The frequency is the same as for the first traffic light, but has a non-optimal time offset. (c) The frequency (and cycle time) differs from the one of the first traffic light. (d) The green time varies stochastically, but the average green time fraction is the same. When the frequencies are the same, but the time offset is not properly adjusted, a certain fraction of vehicles is stopped, see (b). If the frequencies are different, it is most likely that vehicles will be stopped by a red light, potentially even for several times, see (c). In such cases, a stochastic variation of green time periods can be favorable, see (d).

Let us now study the case where the waiting queues cannot be cleared completely within one green phase. How long is a vehicle delayed, if it joins a queue of length $l_i(t_0)$ at time t_0? The totally required green time needed until the vehicle can leave the road section i is given by

$$T_i^{\text{req}}(t_0) = \frac{l_i(t_0)\rho^{\text{jam}}}{Q_i^{\max}}\,, \tag{15.15}$$

since $l_i(t_0)\rho^{\text{jam}}$ is the number of vehicles per lane to be served and $Q_i^{\max}$ the service rate. Let us now estimate the overall time passed until the downstream boundary of road section i is reached. It is given by the formula

$$T_i^{\text{pass}}(t_0) = T_i^{\text{req}}(t_0) + \text{overall red and yellow times in between.} \tag{15.16}$$

The time delay of vehicle i by queuing, red and yellow times is the overall time passed minus the travel time $l_i(t_0)/V_i^0$ in free traffic:

$$\begin{aligned} T_i^{\text{del}}(t_0) &= T_i^{\text{pass}} - \frac{l_i(t_0)}{V_i^0} \qquad (15.17)\\ &= l_i(t_0)\left(\frac{\rho^{\text{jam}}}{Q_i^{\max}} - \frac{1}{V_i^0}\right) + \text{overall red and yellow times.} \end{aligned}$$

Generally, this formula is difficult to express, as its result depends sensitively on the respective red and green phases. However, the formula for the average delay time becomes quite simple. Just remember that the average green time fraction is u_i and the average fraction of red and yellow times must be $1 - u_i$. Therefore, the average delay $\overline{T_i^{\text{del}}}$ as a function of the average queue length $\overline{l_i}$ and the green time fraction u_i is estimated by the formula

$$\begin{aligned} \overline{T_i^{\text{del}}} &\approx \overline{l_i}\left(\frac{\rho^{\text{jam}}}{Q_i^{\max}} - \frac{1}{V_i^0}\right) + \frac{1-u_i}{u_i} \times \text{totally required green time } T_i^{\text{req}} \\ &= \overline{l_i}\left(\frac{\rho^{\text{jam}}}{u_i Q_i^{\max}} - \frac{1}{V_i^0}\right). \qquad (15.18) \end{aligned}$$

According to this, the average delay time $\overline{T_i^{\text{del}}}$ is proportional to the average queue length $\overline{l_i}$, but a large green time fraction u_i is helpful. Note that the formulas of this section are not only applicable to situations with fixed cycle times and signal programs. They are also applicable to situations where the red and green phases are varying.

2.1.6 Potential flows and traffic states. The in- and outflow of a road section is not only limited by capacity constraints such as $Q_i^{\max}$, but also by the actual state of traffic. We will, therefore, denote the potential arrival and departure flows per lane by $Q_i^{\text{arr,pot}}(t)$ and $Q_i^{\text{dep,pot}}(t)$, respectively. Congestion is triggered if $Q_i^{\text{dep}}(t) > Q_i^{\text{dep,pot}}(t)$, and resolved if $l_i(t) = 0$. In the case where the road section is entirely congested, i.e. $l_i(t) = L_i$, this state remains until $Q_i^{\text{arr}}(t) < Q_i^{\text{arr,pot}}(t)$. The potential flows are determined as follows: As long as there is no congestion, the potential departure flow is given by the former arrival flow $Q_i^{\text{arr}}(t - L_i/V_i^0)$. When the downstream end of road section i is congested, vehicles are queued up and can depart with the maximum possible flow $Q_i^{\max}$. Altogether, we have

$$Q_i^{\text{dep,pot}}(t) = \begin{cases} Q_i^{\text{arr}}(t - L_i/V_i^0) & \text{if } l_i(t) = 0\,, \\ Q_i^{\max} & \text{if } l_i(t) > 0\,. \end{cases} \tag{15.19}$$

At the upstream end, the maximum possible flow $Q_i^{\max}$ can enter road section i as long as it is not entirely congested. Otherwise, the arrival flow is limited by the former departure flow $Q_i^{\text{dep}}(t - L_i/|c|)$. This implies

$$Q_i^{\text{arr,pot}}(t) = \begin{cases} Q_i^{\max} & \text{if } l_i(t) < L_i\,, \\ Q_i^{\text{dep}}(t - L_i/|c|) & \text{if } l_i(t) = L_i\,. \end{cases} \tag{15.20}$$

In cases, where the outflow of the road section is to be controlled by a traffic light, the potential departure flow $Q_i^{\text{dep,pot}}(t)$ must be multiplied with a prefactor $\gamma_i(t)$. A green light corresponds to $\gamma_i(t) = 1$, a red light to $\gamma_i(t) = 0$. Note that it is also possible to vary $\gamma_i(t)$ gradually to account for drivers passing the signal during yellow phases.

2.2 Traffic flows through network nodes

A node of the road network connects one or several incoming road sections i with one or several outgoing road sections j, see figure 15.5(a). It may represent a junction or a link of two subsequent homogeneous road sections i and $i+1$ with different speed limits V_i^0, V_{i+1}^0 or numbers I_i, I_{i+1} of lanes. Since nodes are assumed to have no storage capacity, the total in- and outflow have to be the same (Kirchhoff's law):

$$\underbrace{\sum\nolimits_i Q_i^{\text{dep}}(t)}_{\text{inflow}} = \underbrace{\sum\nolimits_j Q_j^{\text{arr}}(t)}_{\text{outflow}}\,. \tag{15.21}$$

Furthermore, the flows have to be non-negative and must not exceed the potential flows specified in Sec. 15.2.1.6.

$$0 \le Q_i^{\text{dep}}(t) \le Q_i^{\text{dep,pot}}(t)\,, \qquad 0 \le Q_j^{\text{arr}}(t) \le Q_j^{\text{arr,pot}}(t)\,. \tag{15.22}$$

The fraction of the inflow Q_i^{dep} that diverges from road section i to road section j is denoted by $\alpha_{ij}(t)$. Due to normalization we have

$$\sum_j \alpha_{ij}(t) = 1\,. \tag{15.23}$$

The turning or assignment coefficients α_{ij} may depend on the driver destinations d as well as on the actual traffic situation, see Daganzo (1995) and Sec. 15.3. Finally, note that the arrival flow $Q_j^{\text{arr}}(t)$ is composed of all turning flows $Q_i^{\text{dep}}(t)\alpha_{ij}(t)$ entering road section j:

$$Q_j^{\text{arr}}(t) = \sum_i Q_i^{\text{dep}}(t)\alpha_{ij}(t)\,. \tag{15.24}$$

For a more detailed treatment of network nodes see Lebacque (2005).

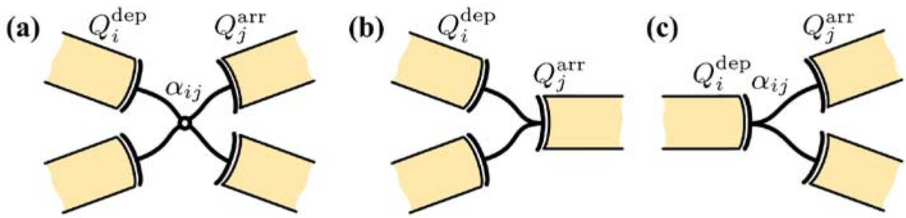

Figure 15.5. (a) A node of the road network distributes the vehicular flows between the road sections that are connected to it. It makes sense to distinguish two special cases: (b) merges into a single road section and (c) diverges from one road section into several others.

2.2.1 Merges. In the case where traffic flows from several incoming road sections i merge into one outgoing road section j, as shown in Fig. 15.5(b), two cases can be distinguished: As long as the subsequent road section j has sufficient capacity to admit the potential flows of all incoming road sections i, i.e. $Q_j^{\text{arr,pot}}(t) \geq \sum_i Q_i^{\text{dep,pot}}(t)$, the flow through the node is given by the upstream traffic conditions in the road sections i. Otherwise, some of the upstream departure flows $Q_i^{\text{dep}}(t)$ have to be restricted. But which ones? According to practical experience, small traffic flows $Q_i^{\text{dep}}(t)$ can almost always squeeze in, while flows from equivalent roads tend to share the capacity $Q_j^{\text{arr,pot}}$ equally. Note that in scenarios with main roads having a right of way, the corresponding flow is to be served first. The remaining capacity is subsequently distributed among the side roads.

2.2.2 Diverges. Figure 15.5(c) shows the case where traffic diverges from one road section into several others. This is, for example, the case when a road splits up into lanes for turning left, continuing straight ahead, or turning right. For diverges, the throughput is determined by a cascaded minimum-function:

$$Q_i^{\text{dep}}(t) = \min\left\{ Q_i^{\text{dep,pot}}(t), \ \min_j \frac{Q_j^{\text{arr,pot}}(t)}{\alpha_{ij}(t)} \right\}. \tag{15.25}$$

The first term on the right-hand side is obvious, as any restriction of the potential departure flow $Q_i^{\text{dep,pot}}(t)$ of road section i limits the flows to all outgoing road sections j. The second term on the right-hand side follows from the fact that the fraction α_{ij} of the departure flow $Q_i^{\text{dep}}(t)$ to any subsequent road section j is limited by its potential arrival flow $Q_j^{\text{arr,pot}}(t)$, i.e.

$$Q_i^{\text{dep}}(t)\alpha_{ij} \le Q_j^{\text{arr,pot}}(t) \quad \forall j\,. \tag{15.26}$$

In the special case of a node connecting only two subsequent road sections i and $j = i+1$, we have $\alpha_{ij} = 1$ and the throughput is just limited by the minimum of both potential flows:

$$Q_i^{\text{dep}}(t) = \min\left\{ Q_i^{\text{dep,pot}}(t), \ Q_{i+1}^{\text{arr,pot}}(t) \right\} = Q_{i+1}^{\text{arr}}(t)\,. \tag{15.27}$$

The last equality follows from Eq. (15.24).

3. Traffic assignment

The simplest way to model turning at intersections is by turning coefficients $\alpha_{ij}(t)$, which assume that a certain fraction $\alpha_{ij}(t)$ of the departure flow $Q_i^{\text{dep}}(t)$ turns into road section j. In many theoretical studies, the coefficients α_{ij} are kept constant. However, it is well-known that the turning fractions vary in the course of the day, which is often taken into account by using historical, time-dependent turning coefficients $\alpha_{ij}(t)$ from a database [Chrobok *et al.* (2000)]. Moreover, even if the same origin-destination flows would repeat each week, delays due to perturbations in the traffic flow (e.g. due to an accident) would cause different time-dependent turning fractions. Therefore, a better treatment is based on dynamic traffic assignment.

In order to integrate dynamic traffic assignment in our model, let us denote the destination node of vehicles by d. Moreover, let $N_{id}(t)$ represent the number of driver-vehicle units on the directed link i, which finally want to arrive at d. This implies

$$N_i(t) = \sum_d N_{id}(t)\,. \tag{15.28}$$

The quantity $Q_{id}^{\text{arr}}(t)$ shall denote the flow of vehicles with destination d entering the link i, and $Q_{id}^{\text{dep}}(t)$ the flow of vehicles leaving it. We have

$$Q_i^{\text{arr}}(t) = \sum_d Q_{id}^{\text{arr}}(t) \quad \text{and} \quad Q_i^{\text{dep}}(t) = \sum_d Q_{id}^{\text{dep}}(t)\,. \tag{15.29}$$

Finally, let $\overline{j}$ be the starting node of link j and $\underline{j} = k$ its ending node. Moreover, let $T_{\overline{j}k}(t)$ be the travel time on link j and $\widehat{T}_{kd}(t)$ the minimum travel time between two nodes k and d (as can, for example, be determined by the Dijkstra algorithm). Then, the minimum travel time to note d via link j (i.e. node k) is given by $T_{\overline{j}k}(t) + \widehat{T}_{kd}(t)$, and the minimum travel time $\widehat{T}_{\overline{j}d}(t)$ from node $\overline{j}$ to destination d at time t is determined via

$$\widehat{T}_{\overline{j}d}(t) = \min_k [T_{\overline{j}k}(t) + \widehat{T}_{kd}(t)]\,, \tag{15.30}$$

where the minimum function extends over all successors k of node $\overline{j}$. Instead of this, we may use the following approximate relationship:

$$\widehat{T}_{\overline{j}d}(t) = \min_k [T_{\overline{j}k}(t) + \widehat{T}_{kd}(t - \Delta t)]\,. \tag{15.31}$$

The advantage of (15.31) over (15.30) is that the information about travel times gradually propagates to the present location of the car (namely by one link each time step Δt). A delayed evaluation of Dijkstra's shortest path algorithm saves computer time and models this information flow, the speed of which is controlled by Δt. Another advantage is the determination of travel times based on a local algorithm.

Based on this travel time information, we may distribute the departure flows $Q_i^{d,\text{dep}}(t)$ over neighboring links according to a multinomial logit model [Ben-Akiva, McFadden *et al.* (1999)]. Accordingly, we specify the turning probabilities of cars with destination d at node $\overline{j} = \underline{i}$ as

$$p_{\overline{j}k}^d(t) = \frac{\exp\{-\beta[T_{\overline{j}k}(t) + \widehat{T}_{kd}(t - \Delta t)]/\widehat{T}_{\overline{j}d}^0\}}{\sum_{k'} \exp\{-\beta[T_{\overline{j}k'}(t) + \widehat{T}_{k'd}(t - \Delta t)]/\widehat{T}_{\overline{j}d}^0\}}\,, \tag{15.32}$$

where $\widehat{T}_{\overline{j}d}^0$ is the minimum travel time from $\overline{j}$ to d during free traffic (at three o'clock during the night). The coefficient β describes the sensitivity with respect to changes in the relative travel time and is also a measure for the reliability of travel time estimates. Finally, the time-dependent assignment coefficients can be calculated as

$$\alpha_{ij}(t) = \sum_d \frac{Q_{id}^{\text{dep}}(t)}{Q_i^{\text{dep}}(t)}\, p_{\underline{i}j}^d(t)\,, \tag{15.33}$$

where $\underline{i} = \overline{j}$ and $\underline{j} = k$. This assumes individual route choice decisions without central coordination, i.e. selfish routing.

We must still decide how to determine travel times. On the one hand, one may use the expected travel times $T_{\overline{j}k}(t) = T_{\overline{j}\underline{j}}(t) = T_j(t)$ according to Eq. (15.11) (or, as a second best alternative, the instantanous link travel times). On the other hand, one may use travel time information $T^*_{\overline{j}k}(t)$ of comparable days from a database [Chrobok *et al.* (2000)]. While for close links, the expected travel time may be a good (and the instantaneous travel time a reasonable) estimate of the actual travel time, it becomes less reliable the more remote the respective link is. For remote links, a travel time estimate based on measurements of similar previous days may be more reliable. Therefore, we propose to use a weighted mean value generalizing formula (15.31):

$$\widehat{T}_{\overline{j}d}(t) = \min_k [T_{\overline{j}k}(t) + \mathrm{e}^{-\lambda T_{\overline{j}k}(t)} \widehat{T}_{kd}(t - \Delta t) + (1 - \mathrm{e}^{-\lambda T_{\overline{j}k}(t)}) T^*_{kd}(t)] \, . \tag{15.34}$$

In this formula, the travel time $T^*_{kd}(t)$ from node k to d is taken from a database, the weights are exponentially decaying with increasing travel times, and $\lambda > 0$ is a suitably chosen calibration parameter.

Right now it is not clear what happens if traffic lights adapt to the traffic situation and drivers try to adjust to the traffic lights at the same time. Driver adaptation is a reasonable strategy for signal plans that are fixed or determined by the time of the day. However, it may perturb attempts to optimize traffic by self-organized control. Therefore, the study of route choice behavior in the context of adaptive traffic light control requires careful study. A method to stabilize the system dynamics, if needed, would be road pricing (see Sec. 15.5.1.1).

4. Self-organized traffic light control

4.1 Why traffic lights?

For the illustration of the advantages of oscillatory traffic control, let us assume a conventional four-armed intersection with identical capacities $Q_i^{\max} = Q^{\max}$. The arrival time of vehicles shall be stochastic. Vehicles are assumed to obstruct the intersection area (i.e. the node) for a time period of $1/Q^{\max}$ in case of compatible flow directions. For incompatible, e.g. crossing flows, the blockage time shall be $\tau = sT$ with $s > 1$. The maximum average throughput Q^{cap} of the intersection is, therefore, bounded by the following inequality:

$$\frac{1}{T} > Q^{\max} \geq Q^{\text{cap}} \geq \frac{1}{\tau} = \frac{1}{sT} \, . \tag{15.35}$$

The exact value of Q^{cap} depends on the fractions of compatible and incompatible flows. For compatible flows only, we have $Q^{\text{cap}} = Q^{\max}$. If the vehicle flows were always incompatible, one would have $Q^{\text{cap}} = 1/\tau = 1/(sT)$.

Let us now cluster vehicles into platoons of n vehicles by the use of suitable adaptive traffic lights. Moreover, let the green phases last for the time periods $\Delta\tau_i$. Between the green periods, we will need yellow lights for a time period of τ to prevent accidents. An estimate of the capacity Q^{cap} of the signalized intersection is then

$$Q^{\text{cap}} = \frac{\sum_{i=1}^{k} Q_i^{\max} \Delta\tau_i}{\sum_{i=1}^{k}(\Delta\tau_i + \tau)} = Q^{\max} \frac{\sum_{i=1}^{k} \Delta\tau_i}{T^{\text{cyc}}} , \tag{15.36}$$

where $T^{\text{cyc}} = k\tau + \sum_i \Delta\tau_i$ is the average cycle time. Of course, there are different possible schemes to control the intersection, but we can show that for n-vehicle platoons with $\Delta\tau_i = n/Q^{\max}$, the capacity of the signalized intersection is

$$Q^{\text{cap}}_{(n)} = \frac{kn/Q^{\max}}{kn/Q^{\max} + ksT} = \left(\frac{1}{Q^{\max}} + \frac{sT}{n} \right)^{-1} . \tag{15.37}$$

This is greater than the capacity $1/(sT)$ of an uncontrolled intersection with incompatible flows, if

$$sT \left(1 - \frac{1}{n} \right) > \frac{1}{Q^{\max}} \ge T , \tag{15.38}$$

i.e. if s or n are large enough. In other words: Forming vehicle platoons (clusters) by oscillatory traffic lights can increase the intersection capacity. This, however, requires that the green times are fully used. Otherwise, at small arrival rates, traffic lights would potentially delay vehicles.

Despite of the simplifications made in the above considerations, the following conclusions are quite general: It is most efficient if vehicles can pass the intersection immediately one by one, if the arrival rates are small. Above a certain threshold, however, it is more efficient to form vehicle platoons by means of traffic lights. This is certainly the case, if the sum of arrival flows exceeds the capacity of an unsignalized intersection with incompatible flows. According to formula (15.36), the capacity of a signalized intersection can be increased by increasing the green time fractions $\Delta\tau_i/T^{\text{cyc}}$. This can be done by increasing the cycle time T^{cyc} in cases of high arrival flows Q_i^{arr}. Thereby, the relative blockage time by yellow lights is reduced.

4.2 Self-induced oscillations

In pedestrian counterflows at bottlenecks, one can often observe oscillatory changes of the passing direction, as if the pedestrian flows were controlled by a traffic light. Inspired by this, we have suggested to generalize this principle to the self-organized control of intersecting vehicle flows [see the newspaper article by Stirn (2003)]. This idea was described in 2003 in the DFG proposal He 2789/5-1 entitled "Self-organized traffic signal control based on synchronization phenomena in driven many-particle systems and supply networks". The control concept elaborated in the meantime has been submitted for a patent. For visualizations of some traffic scenarios see the videos available at `www.trafficforum.org trafficlights/`.

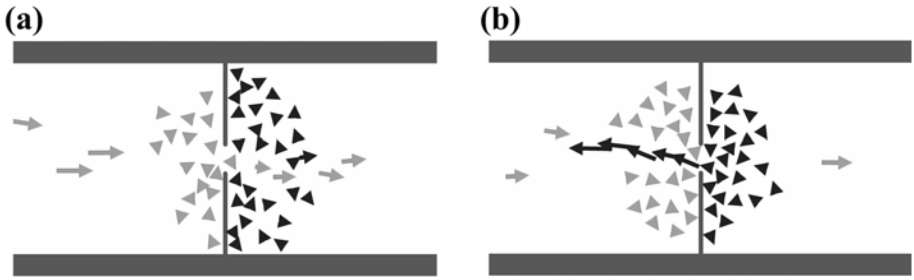

Figure 15.6. Alternating pedestrian flows at a bottleneck. These oscillations are self-organized and occur due to a pressure difference between the waiting crowd on one side and the crowd on the other side passing the bottleneck [after Helbing and Molnár (1995), Helbing (1997)].

Oscillations are a organization pattern of conflicting flows which allows to optimize the overall throughput under certain conditions (see Sec. 15.4.1). In pedestrian flows (see Fig. 15.6), the mechanism behind the self-induced oscillations is as follows: Pressure builds up on that side of the bottleneck where more and more pedestrians have to wait, while it is reduced on the side where pedestrians can move ahead and pass the bottleneck. If the pressure on one side exceeds the pressure on the other side by a certain amount, the passing direction is changed.

Transferring this self-organization principle to urban vehicle traffic, we define red and green phases in a way that considers "pressures" on the traffic light by road sections waiting to be served and "counter-pressures" from the subsequent road sections depending on the degree of congestion on them. Generally speaking, these pressures depend on delay times, queue lengths, or potentially other quantities as well. The proposed control principle is self-organized, autonomous, and adaptive to the respective local traffic situation, as will be shown below.

4.3 Basic switching rules for traffic lights

Our switching rules for traffic lights will have to solve the following control problems:

- The number of vehicles on a road section served by a green time period should be proportional to the average arrival flows $\overline{Q_i^{\text{arr}}}$, at least if these are small.
- In order to avoid time losses due to yellow lights, switching of traffic lights should be minimized under saturated traffic conditions. However, single vehicles and small queues need to be served as well after some maximum cycle time T^{max}.
- Despite of the desire to maintain green lights as long as possible, signal control should be able to react to changing traffic conditions in a flexible way. Unfortunately, the change of traffic conditions depends on traffic light control itself, so that a reliable forecast is only possible over short time periods.
- Under suitable conditions, traffic lights should synchronize themselves to establish green waves.

The synchronization of traffic lights is not only a matter of the adjustment of green and red time periods, i.e. of the frequency of control cycles: The adaptation of the time offset is also crucial for the establishment of green waves. While the adaptation problem is easily solvable for Manhattan-like road networks, the situation for irregular road networks is much more complex. Green waves may, in fact, cause major obstructions of crossing flows. Therefore, it is a great difficulty to find suitable rules which flows to prioritize. While addressing these points in the next paragraphs, we will develop a suitable control approach step by step. The resulting control principles may be also used to resolve conflicts between competing flows in other complex systems like production networks [Helbing (2003a, 2004, 2005), Helbing *et al.* (2004)], see Sec. 15.5.1.2.

The philosophy of our traffic light control is the minimization of the cumulative or average travel time and, therefore, of the cumulative delay time. Minimizing the overall delay time means to serve as many vehicles by the traffic lights as possible, i.e. to maximize the average departure rate (the average throughput). Let us explain this principle in more detail: If the traffic light is red or yellow, we have $\gamma_i(t) = 0$ and the overall departure rate is $I_i Q_i^{\mathrm{dep}}(t) = 0$. Otherwise, if the traffic light is green ($\gamma_i(t) = 1$), we find

$$I_i Q_i^{\mathrm{dep}}(t) = \begin{cases} I_i Q_i^{\mathrm{arr}}(t - L_i/V_i^0) & \text{if } l_i(t) = 0\,, \\ \min_j [I_j Q_j^{\mathrm{dep}}(t - L_i/|c|)/\alpha_{ij}] & \text{if } l_j(t) = L_j\,, \\ I_i Q_i^{\max} & \text{otherwise.} \end{cases} \tag{15.39}$$

A green light should be provided for the road section whose vehicle flow during a certain future time period is expected to be highest, taking into account any yellow-light related time losses. This principle tends to serve the road with the largest outflow, i.e. the largest number I_i of lanes (see the third condition). However, it matters how long the maximum flow can be maintained, i.e. how large the number number $I_i l_i \rho^{\mathrm{jam}}$ of queued vehicles is. Moreover, vehicles in road section i will be hardly able to depart (see the second condition), if one of the subsequent road sections j is completely congested by the expected number $I_i Q_i^{\max} \alpha_{ij}(t - t_0')$ of vehicles arriving between time t_0' and t. That is, a green light starting at time t_0' would usually end when the condition

$$I_i Q_i^{\max}(t - t_0')\alpha_{ij} = I_j[L_j - l_j(t_0')]\rho^{\mathrm{jam}} \tag{15.40}$$

is valid for the first time. Freely moving vehicles (see the first conditions) will have an impact comparable to the reduction of a queue (third condition) only, if

$$\frac{1}{t - t_0} \int_{t_0}^{t} dt'\, Q_i^{\mathrm{arr}}(t - L_i/V_i^0) = \frac{N_i^{\mathrm{arr}}(t - L_i/V_i^0) - N_i^{\mathrm{arr}}(t_0 - L_i/V_i^0)}{t - t_0} \tag{15.41}$$

is of the order $Q_i^{\max}$, where

$$N_i^{\mathrm{arr}}(t - L_i/V_i^0) = \int_0^t dt'\, Q_i^{\mathrm{arr}}(t' - L_i/V_i^0)\,. \tag{15.42}$$

Summarizing this, the expected number $\Delta N_i^{\exp}$ of vehicles served before interruption by a red light at time t_1 can be often estimated by the cascaded minimum function

$$\begin{aligned} \Delta N_i^{\exp} &= I_i Q_i^{\max}\,(t_1 - t_0') \\ &= \rho^{\mathrm{jam}} \min \Bigg[\underbrace{I_i l_i(t_0')}_{\text{pressure}}, \underbrace{\min_j \left(\frac{I_j[L_j - l_j(t_0')]}{\alpha_{ij}} \right)}_{\text{counter-pressure}} \Bigg]\,, \end{aligned} \tag{15.43}$$

where $t_1 - t'_0$ denotes the expected green time. However, generalizations of this formula are needed for the treatment of low traffic (see Sec. 15.4.5) and green waves (see Sec. 15.4.6).

As our control philosophy requires to reduce queues as fast as possible, the decision to serve a certain road section i should be based on the greatest value of $\sum \Delta N_i^{\text{exp}}/(t_1 - t'_0)$, where the sum extends over all flows compatible with Q_i^{dep}. If a switching time τ is necessary, the relevant formula is $\sum \Delta N_i^{\text{exp}}/(t_1 - t'_0 + \tau)$, instead. The switching decision should be regularly revised (e.g. every time period τ), as the traffic situation may change.

Note that formula (15.43) implies that, given an equal number of lanes, green times are more likely for long queues, which could be said to exert some "pressure" on the traffic light. However, if road sections j demanded by turning flows are congested, this exerts some "counter-pressure". This will suppress green lights in cases where they would not allow to serve vehicles, i.e. where they would not make sense. As a consequence, while cycle times increase with growing arrival rates as long as these can be served, they may go down again when the road network is too congested.

4.4 Oscillations at a merge bottleneck

For the purpose of illustration, let us discuss a merge bottleneck (see Fig. 15.8). The two merging road sections $i \in \{1, 2\}$ shall have the overall capacities $I_i Q^{\max}$ with $I_1 \geq I_2$, while the subsequent section j shall have the capacity $I_j Q^{\max} \geq I_1 Q^{\max}$, so that no congestion will occur in the subsequent road section. Let us assume that the arrival flows Q_i^{arr} are constant in time. Furthermore, let us assume that the traffic light for road section 2 turns red at times t_0, t_2, etc., while the red lights for road section 1 start at t_1, t_3, etc. The green times for road section 1 begin after an yellow time period of τ, i.e. at times $t'_{2k} = t_{2k} + \tau$ and last for the time periods $t_{2k+1} - t'_{2k}$.

We can distinguish the following cases:

1. Equivalent road sections: If $I_1 = I_2$, the queues on both road sections will be completely cleared in an alternating way, see Fig. 15.7(a). In case of growing vehicle queues, the green times grow accordingly.

2. One main and one side road ($I_1 > I_2$):

   (i) If the arrival flow Q_2^{arr} of road section 2 (the side road) is low, both roads are completely cleared.

(ii) In many cases, however, the queue length in the side road grows in the course of time, while the queue in the main road (road section 1) is completely cleared, see Fig. 15.7(b). As a consequence, road section 2 will be fully congested after some time period, which limits a further growth of the queue and discourages drivers to use this road section according to our traffic assignment rule. In extreme cases, when no maximum cycle time is implemented (see Sec. 15.4.4.3), the main road may have a green light all the time, while road section 2 (the side road) is never served, see Fig. 15.7(c).

(iii) If the sum $\sum_i I_i Q_i^{\rm arr}$ of overall arrival flows exceeds the capacity $I_j Q^{\rm max}$ of the subsequent road section j, the queue on both road sections will grow, see Fig. 15.7(d).

We will now discuss these cases in more detail.

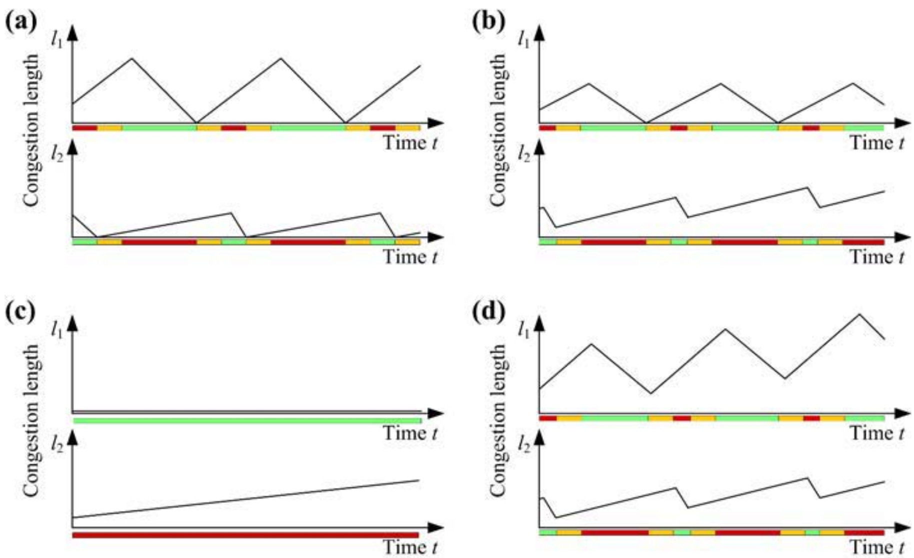

Figure 15.7. Different cases of the self-organized control of a merge bottleneck: (a) The vehicle queue in each road section is completely cleared, before the traffic light turns red. (b) The traffic light in the side road turns red, before the vehicle queue has fully disappeared, but the main road is fully cleared. (c) In extreme cases, if a maximum cycle time is not enforced, the side road would never get a green light and the main road would always be served. (d) When the sum of arrival rates is higher than the capacity of the subsequent road section, the vehicle queues in both road sections may grow under certain conditions (see text).

4.4.1 Equivalent road sections. Let us assume the queue length on road section 2 is zero at time t_0 and the traffic light switches to red in order to offer a green light to road section 1 at time $t'_0 = t_0 + \tau$. The queue length at time t is given by

$$l_1(t) = l_1(t'_0) + C_1(t - t'_0) \, , \qquad (15.44)$$

where

$$C_i = \left(\frac{\rho^{\text{jam}}}{Q_i^{\text{arr}}} - \frac{1}{V_i^0} \right)^{-1} = \frac{Q_i^{\text{arr}}}{\rho^{\text{jam}} - Q_i^{\text{arr}}/V_i^0} \qquad (15.45)$$

according to Eq. (15.8). Note that, in the limit of small arrival rates Q_i^{arr}, this queue expansion velocity is proportional to Q_i^{arr}. The reduction of the queue starts with the green phase and is proportional to c. We, therefore, have the following equation for the length of the effective queue (= queue length minus area of quasi-free traffic):

$$l_1^{\text{eff}}(t) = l_1(t) + c(t - t'_0) = l_1(t'_0) + C_1(t - t'_0) - |c|(t - t'_0) \, . \qquad (15.46)$$

The effective queue length disappears at time

$$t_0^* = t'_0 + \frac{l_1(t'_0)}{|c| - C_1} \, . \qquad (15.47)$$

However, the last vehicle of the queue needs an additional time period of $l_1(t_0^*)/V_1^0$ to leave the road section, so that the queue length $l_1(t)$ in road section 1 becomes zero at time $t = t_1$ with

$$t_1 = t_0^* + \frac{l_1(t_0^*)}{V_1^0} = \cdots = t'_0 + l_1(t'_0) \frac{1 + |c|/V_1^0}{|c| - C_1} \, . \qquad (15.48)$$

At that time, the traffic light for road section 1 switches to red and road section 2 is served by a green light starting at $t'_1 = t_1 + \tau$. Analogous considerations show that the queue in road section 2 is cleared at time

$$t_2 = t'_1 + l_2(t'_1) \frac{1 + |c|/V_2^0}{|c| - C_2} \, . \qquad (15.49)$$

The next green time for road section 1 starts at time $t'_2 = t_2 + \tau$ and ends at

$$t_3 = t'_2 + l_1(t'_2) \frac{1 + |c|/V_1^0}{|c| - C_1} \, . \qquad (15.50)$$

We can determine the queue length $l_1(t'_2)$ at the beginning of the green phase as the queue length that has built up during the previous red phase of length $t_2 - t'_1$ and two yellow phases of duration τ each. As a consequence, we find $l_1(t_2') = C_1(t_2 - t_1' + 2\tau)$. In the stationary case we have $l_1(t_2') = l_1(t_0')$ and $l_1(t_1) = 0$, as the queue on road section 1 is completely cleared at time t_1. This eventually leads to a rather complicated formula for $t_2 - t_1'$, which is proportional to the respective queue length. For small values of the arrival rates Q_i^{arr}, one can show that the green times are proportional to C_i and Q_i^{arr}. That is, the duration of the green phases is proportional to the arrival rates, as expected, if the arrival rates are small enough. The cycle time grows linearly with $Q_1^{\text{arr}} + Q_2^{\text{arr}}$.

4.4.2 One main and one side road. If both road sections are completely cleared as in case (i) above, the mathematical treatment is analogous to the previous section. More interesting is case (ii), in which the traffic light for road section 2 switches to red already before the queue is cleared completely, see Fig. 15.7(b). While Eqs. (15.48) and (15.50) are still valid, we have to find other expressions for t_2 and $l_1(t_2') = l_1(t_0')$. Let t_2^+ be the time point in which the queue of length $l_1(t_2)$ in road section 1 at time t_2 would be completely resolved, if the traffic light would turn green for road section 1 at time t_2. Road section 1 could for sure deliver an overall flow of $I_1 Q^{\max}$ between $t_2' = t_2 + \tau$ and t_2^+, while the departure flow from road section 1 could be much smaller than $I_1 Q^{\max}$ afterwards. In order to switch to green in favor of road section 1, it is, therefore, reasonable to demand

$$I_1 Q^{\max}[t_2^+ - (t_2 + \tau)] \geq I_2 Q^{\max}(t_2^+ - t_2)\,. \tag{15.51}$$

This formula considers the time loss τ by switching due to the intermediate yellow period, and it presupposes that $Q^{\max}(t_2^+ - t_2) \geq l_2(t_2)\rho^{\text{jam}}$, i.e. road section 2 can maintain the maximum flow $Q^{\max}$ until t_2^+. Our philosophy is to give a green light to the road section which can serve most vehicles during the next time period $t_2^+ - t_2$. The equation to determine $t_2^+ = t_2^- + l_1(t_2)/V_1^0$ is $l_1(t_2) = |c|[t_2^- - (t_2 + \tau)]$ with $l_1(t_2) = C_1(t_2 - t_1)$. This leads to $t_2^- = t_2 + \tau + C_1(t_2 - t_1)/|c|$ and

$$t_2^+ = (t_2 + \tau) + \left(\frac{C_1}{|c|} + \frac{C_1}{V_1^0}\right)(t_2 - t_1)\,, \tag{15.52}$$

while Eq. (15.51) implies

$$t_2^+ - t_2 \geq \frac{\tau}{1 - I_2/I_1}\,. \tag{15.53}$$

Together with Eq. (15.52) we find

$$t_2 - t_1 = \frac{\tau}{I_1/I_2 - 1} \Bigg/ \left(\frac{C_1}{|c|} + \frac{C_1}{V_1^0}\right)\,. \tag{15.54}$$

For $I_1 = I_2$, one can immediately see that the traffic light would never switch before the queue in road section 2 is fully resolved. However, early switching could occur for $I_1 > I_2$.

Once the traffic light is turned green at time t_2, the vehicles which have queued up until time t_2^+ will be served with the overall rate $I_1 Q^{\max}$ as well, until the departure flow is given by the lower arrival flow Q_1^{arr} at time t_3 and later. The time point t_2^* at which the effective queue resolves is given by $l_1(t_2^*) = |c|[t_2^* - (t_2 + \tau)]$, which results in

$$t_2^* - t_2 = \frac{t_2^- - t_2}{1 - C_1/|c|} = \frac{\tau + C_1(t_2 - t_1)/|c|}{1 - C_1/|c|} . \tag{15.55}$$

The last vehicle of the queue has left road section 1 at time t_3 with

$$t_3 - t_2 = \frac{t_2^+ - t_2}{1 - C_1/|c|} = \frac{\tau}{(1 - I_2/I_1)(1 - C_1/|c|)} . \tag{15.56}$$

Afterwards, the overall departure flow drops indeed to $I_1 Q_1^{\text{arr}}$, and the traffic light tends to turn red if $I_1 Q_1^{\text{arr}} < I_2 Q^{\max}$. Otherwise, it will continue to stay green during the whole rush hour. Considering $l_1(t_3) = 0 = l_1(t_1)$ and $l_1(t_2') = C_1(t_2' - t_1)$, one can determine all quantities. One can show that the green time fraction for road section 1 grows proportionally to Q_1^{arr}, if τ is small. Moreover, one can derive that the green time fractions of both road sections and the cycle time $T^{\text{cyc}} = t_3 - t_1$ are proportional to C_1, i.e. the main road dominates the dynamics. The queue length on road section 2 tends to grow, as it is never fully cleared.

If $I_1 Q_1^{\text{arr}} + I_2 Q_2^{\text{arr}} > I_j Q^{\max}$, it can also happen that the queues grow in both road sections. This is actually the case, if $I_1 Q_1^{\text{arr}} > I_j Q^{\max}$, see Fig. 15.7(d). Moreover, in the case $I_2 Q^{\max} < I_1 Q_1^{\text{arr}}$, road section 2 would never be served, see Fig. 15.7(c). This calls for one of several possible solutions: 1. Allow turning on red. 2. Decide to transform the side road into a dead end. 3. Build a bridge or tunnel. 4. Use roundabouts or other road network designs which do not require traffic lights. 5. Treat main and side roads equivalently, i.e. set $I_1 = I_2 = 1$ in the above formulas, or specify suitable parameter values for I_i, although it will increase the overall delay times. 6. Restrict the red times to a maximum value at the cost of increased overall delay times and reduced intersection throughput.

4.4.3 Restricting red times. In order to avoid excessive cycle times, one has to set upper bounds. This may be done as follows: Let

$T^{\max}$ be the maximum allowed cycle time,

$$\overline{\gamma_i} = \frac{1}{T^{\max}} \int\limits_{t-T^{\max}}^{t} dt' \; \gamma_i(t') \tag{15.57}$$

the green time fraction within this time interval, and

$$\overline{Q_i^{\rm arr}} = \frac{1}{T^{\max}} \int\limits_{t-T^{\max}}^{t} dt' \; Q_i^{\rm arr}(t') \tag{15.58}$$

the average arrival rate. If $\overline{\gamma_i}$ exceeds a specified green time fraction u_i^0, the green light will be switched to red. This approach also solves the problem that even small vehicle queues or single vehicles must be served within some maximum time period.

The green time fractions u_i^0 may slowly vary in time and could be specified proportionally to the relative arrival rate $\overline{Q_i^{\rm arr}}/\sum_{i'} \overline{Q_{i'}^{\rm arr}}$, with some correction for the yellow time periods. However, it is better to determine the green time fractions u_i^0 in a way that helps to optimize the system performance (see Sec. 15.5.1.1).

4.4.4 Intersection capacity and throughput. Let us finally calculate the average throughput $Q^{\rm all}$ of the signalized intersection. When the traffic volume is low, it is determined by the sum $\sum_i I_i \overline{Q_i^{\rm arr}}$ of average arrival flows, while at high traffic volumes, it is given by the intersection capacity

$$Q^{\rm cap} = Q^{\max} \frac{I_1(t_3 - t_2') + I_2(t_2 - t_1')}{t_3 - t_1} = Q^{\max} \frac{I_1(t_3 - t_2') + I_2(t_2 - t_1')}{T^{\rm cyc}} \, . \tag{15.59}$$

This implies

$$Q^{\rm all} = \min \left(\sum_i I_i \overline{Q_i^{\rm arr}}, Q^{\rm cap} \right) . \tag{15.60}$$

According to these formulas, the losses in throughput and capacity by the yellow times 2τ are reduced by longer green times $t_3 - t_2'$ and $t_2 - t_1'$. Our calculations indicate that our switching rule automatically increases the cycle time $T^{\rm cyc} = t_3 - t_1$ and the intersection capacity $Q^{\rm cap}$, when the arrival rates $Q_i^{\rm arr}$ of equivalent roads with $I_1 = I_2$ or the arrival rate $Q_1^{\rm arr}$ of a main road are increased. Figure 15.8 shows the cycle time $T^{\rm cyc}$, throughput $Q^{\rm all}$, and green time fraction u_1 as a function of $Q_i^{\rm arr} = Q^{\rm arr}$ for different values of I_1/I_2.

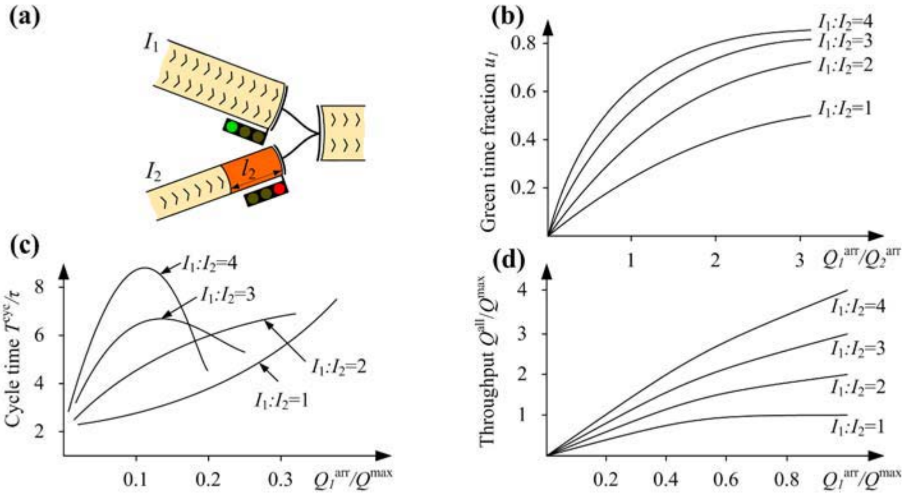

Figure 15.8. (a) Illustration of the traffic control of a merge bottleneck for constant arrival rates and a non-congested outflow. The characteristic behavior of the proposed self-organized traffic light control depends on the number I_i of lanes of the entering road sections i and on the arrival rates Q_i^{arr}: (b) Actual green time fraction u_1 for $Q_2^{\text{arr}} = \text{const.}$ and variable Q_1^{arr}, (c) cycle time T^{cyc} as compared to the yellow time period τ for $Q_2^{\text{arr}} = Q_1^{\text{arr}}$, and (d) actual throughput Q^{all} of the signalized intersection in comparison with the maximum uninterrupted flow $Q^{\max}$ per lane for $Q_2^{\text{arr}} = Q_1^{\text{arr}}$.

4.5 Serving single vehicles at low traffic volumes

While traffic lights have been invented to efficiently coordinate and serve vehicle flows at high traffic volumes, they should ideally provide a green light for every arriving vehicle at low average arrival rates $\overline{Q}_i^{\text{arr}}$. According to formula (15.39), the departure flow $Q_i^{\text{dep}}(t)$ will, in fact, be 0 most of the time on all road sections. Only during short time periods, single vehicles will randomly cause positive values of $Q_i^{\text{arr}}(t - L_i/V_i^0)$ on one of the road sections i. The traffic light should be turned green shortly before the arrival of the vehicle at the downstream boundary of this road section. If switching requires a time period of τ, the arrival flow $Q_i^{\text{arr}}(t - L_i/V_i^0 + \tau)$ would need to trigger a switching of the traffic light in favor of road section i. Considering this and formula (15.39), it is essential to take a switching decision based on the departure flow $Q_i^{\text{dep}}(t + \tau)$ expected at time $t + \tau$. The departure flow $Q_i^{\text{dep}}(t)$ can, in fact, be forecasted for a certain time period based on available flow data and assumed states of neighboring signals. In order to minimize the time period τ, it makes sense to switch any traffic light to red, if no other vehicle is following. That is, at low traffic volumes, all traffic lights would be red most of the time. However, any single vehicle would trigger an anticipative green light upon arrival, so that vehicles would basically never have to wait at a red light.

4.6 Emergence of green waves through self-organized synchronization

In order to let green waves emerge in a self-organized way, the control strategy must show a tendency to form vehicle groups, i.e. convoys, and to serve them just as they approach an intersection. For this to happen, small vehicle clusters must potentially be delayed, which gives them a chance to grow. When they are released, the corresponding "convoys" may themselves trigger a green wave.

In fact, the ideal situation would be that traffic flow from road section i arrives at location $L_j - l_j(t)$ in a subsequent road section j just when the effective queue $l_j^{\text{eff}}(t)$ has resolved. This is equivalent with the need to arrive at location L_j just at the moment when the queue length $l_j(t)$ becomes zero. Under such conditions, free arrival flows $Q_j^{\text{arr}}(t - L_j/V_j^0)$ with values around $Q_i^{\max}$ would immediately follow the high outflow $Q_j^{\text{dep}} = Q_j^{\max}$ from the (resolving) congested area in road section j (here, we assume $I_i = I_j$). As a consequence, the green light at the end of road section j would be likely to continue. This mechanism could establish a synchronization among traffic lights, i.e. a green wave by suitable adjustment of the time offsets, triggered by vehicle flows. As it requires a time period $\Delta t_j = [L_j - l_j(t)]/V_j^0$ to reach the upstream congestion front in section j, it will be required to turn the signal of the previous road section i green a time period Δt_j before the effective queue is expected to resolve. This time period defines the necessary forecast time interval.

When the effective queue of length $l_j^{\text{eff}}(t)$ is resolved, the related sudden increase in $L_j - l_j(t)$ can cause a sudden increase in ΔN_i^{pot} and, thereby, possibly trigger a switching of the traffic light. The emergence of green waves obviously requires that the green light at the end of road section j should stay long enough to resolve the queue. This is likely, if road section j is a main road (arterial), see our considerations in Sec. 15.4.4.2.

In a more abstract sense, the intersections in the road network can be understood as self-sustained oscillators which are coupled by the vehicle flows between them. Therefore, one might expect them to synchronize like many natural systems do [Pikovsky *et al.* (2001)]. Interestingly, even if the intersections are not coupled artificially with some communication feedback, the weak coupling via vehicle flows is sufficient to let larger areas of the road network synchronize. The serving direction percolates through the network, stabilizes itself for a while and is then taken over by another serving direction. In other words, neighboring intersections affect each other by interactions via vehicle flows, which favors a mutual adjustment of their rhythms. This intrinsic mechanism introduces order, so that vehicle flows are coordinated.

5. Summary and outlook

In this contribution, we have presented a section-based traffic model for the simulation and analysis of network traffic. Moreover, we have proposed a decentralized control strategy for traffic flows, which has certain interesting features: Single arriving vehicles always get a green light. When the intersection is busy, vehicles are clustered, resulting in an oscillatory and efficient service (even of intersecting main flows). If possible, vehicles are kept going in order to avoid capacity losses produced by stopped vehicles. This principle bundles flows, thereby generating main flows (arterials) and subordinate flows (side roads and residential areas). If a road section cannot be used due to a building site or an accident, traffic flexibly re-organizes itself. The same applies to different demand patterns in cases of mass events, evacuation scenarios, etc. Finally, a local dysfunction of sensors or control elements can be handled and does not affect the overall system. A large-scale harmonization of traffic lights is reached by a feedback between neighboring traffic lights based on the vehicle flows themselves, which can synchronize traffic signals and organize green waves. In summary, the system is self-organized based on local information, local interactions, and local processing, i.e. decentralized control. However, a multi-hierarchical feedback may further enhance system performance by increasing the speed of large-scale information exchange and the speed of synchronization in the system.

We should point out some interesting differences compared to conventional traffic control:

- The green phases of a traffic light depend on the respective traffic situation on the previous *and* the subsequent road sections. They are basically determined by actual and expected queue lengths and delay times. If no more vehicles need to be served or one of the subsequent road sections is full, green times for one direction will be terminated in favor of green times for other directions. The default setting corresponds to red lights, as this enables one to respond quickly to approaching traffic. Therefore, during light traffic conditions, single vehicles can trigger a green light upon arrival at the traffic signal.

- Our approach does not use precalculated or predetermined signal plans. It is rather based on self-organized red and green phases. In particularly, there is no fixed cycle time or a given order of green phases. Some roads may be even served more frequently than others. For example, at very low traffic volumes it can make sense to serve the same road again before all other road sections have been served. In other words, traffic optimization is not just a matter of green times and their permutation.

- Instead of a traffic control center, we suggest a distributed, local control in favor of greater flexibility and robustness. The required information can be gathered by optical or infrared sensors, which will be cheaply available in the future. Complementary information can be obtained by a coupling with simulation models. Apart from the section-based model proposed in this paper, one can also use other (e.g. microsimulation) models with or without stochasticity, as our control approach does not depend on the traffic model. Travel time information to enhance route choice decisions may be transmitted by mobile communication.

- Pedestrians could be detected by modern sensors as well and handled as additional traffic streams. Alternatively, they may get green times during compatible green phases for vehicles or after the maximum cycle time $T^{\max}$. Public transport (e.g. busses or trams) may be treated as vehicles with a higher weight. A natural choice for the weight would be the average number of passengers. This would tend to prioritize public transport and to give it a green light upon arrival at an intersection. In fact, a prioritization of public transport harmonizes much better with our self-organized traffic control concept than with precalculated signal plans.

5.1 Future research directions

5.1.1 Towards the system optimum. Traffic flow optimization in networks is not just a matter of durations, frequencies, time offsets and the order of green times, which may be adjusted in the way described above. Conflicts of flows and related inefficiencies can also be a result of the following problems:

- Space which is urgently required for certain origin-destination flows may be blocked by other flows, causing a spill-over and blockage of upstream road sections. One of the reasons for this is the cascaded minimum function (15.25). It may, therefore, be helpful to restrict turning only to subsequent road sections that are normally not fully congested (i.e. wide and/or long road sections).

- Giving green times to compatible vehicle flows may cause the over-proportional service of certain road sections. These over-proportional flows may be called parasitic. They may cause the blockage of space in subsequent road sections which would be needed for other flow directions. In order to avoid parasitic flows, it may be useful to restrict the green times of compatible flow directions.

- Due to the selfish route choice behavior, drivers tend to distribute over alternative routes in a way that establishes a Wardrop equilibrium (also called a Nash or user equilibrium) [Papageorgiou (1991)]. This reflects the tendency of humans to balance travel times [Helbing *et al.* (2002)]. That is, all subsequent road sections j of i used to reach a destination d are characterized by (more or less) equal travel times. If the travel time on one path was less than on alternative ones, more vehicles would choose it, which would cause more congestion and a corresponding increase in travel times.

In order to reach the *system optimum*, which is typically defined by the minimum of the overall travel times, the drivers have to be coordinated. This would be able to further enhance the capacity of the traffic network, but it would require the local adaptation of signal control parameters. For example, the enforcement of *optimal* green time fractions u_i^0 based on the method described in Sec. 15.4.4.3 would be one step into this direction, as it is not necessarily the best, when green time fractions are specified proportionally to the arrival rates Q_i^{arr}.

Unfortunately, green time fractions u_i^0 do not allow to differentiate between different origin-destination flows using the same road section. Such a differentiation would allow one to reserve certain capacities (i.e. certain fractions of road sections) for specific flows. This could be reached by advanced traveller information systems (ATIS) [Hu and Mahmassani (1997), Mahmassani and Jou (2000), Schreckenberg and Selten (2004)] together with suitable pricing schemes, which would increase the attractiveness of some routes compared to others.

Different road pricing schemes have been proposed, each of which has its own advantages and disadvantages or side effects. Congestion charges, for example, could discourage to take congested routes required to reach minimum *average* travel times, while conventional tolls and road pricing may reduce the trip frequency due to budget constraints (which potentially interferes with economic growth and fair chances for everyone's mobility).

In order to activate capacity reserves, we therefore propose an automated route guidance system based on the following principles: After specification of their destination, drivers should get individual route choice recommendations in agreement with the traffic situation and the route choice proportions required to reach the system optimum. If an individual selects a faster route instead of the recommeded route it should, on the one hand, have to pay an amount proportional to the increase in the overall travel time compared to the system optimum. On the other hand, drivers not in a hurry should be encouraged to take the slower route i by receiving the amount of money corresponding to the related decrease in travel times. Altogether, such an ATIS could support the system optimum while allowing for some flexibility in route choice. Moreover, the fair usage pattern would be cost-neutral for everyone, i.e. traffic flows of potential economic relevance would not be suppressed by extra costs.

5.1.2 On-line production scheduling. Our approach to self-organized traffic light control could be also transfered to a flexible production scheduling, in order to cope with problems of multi-goal optimization, with machine breakdowns, and variations in the consumption rate. This could, for example, help to optimize the difficult problem of re-entrant production in the semiconductor industry [Beaumariage and Kempf (1994), Diaz-Rivera *et al.* (2000), Helbing (2005)].

In fact, the control of network traffic flows shares many features with the optimization of production processes. For example, travel times correspond to cycle times, cars with different origins and destinations to different products, traffic lights to production machines, road sections to buffers. Moreover, variations in traffic flows correspond to variations in the consumption rate, congested roads to full buffers, accidents to machine breakdowns, and conflicting flows at intersections to conflicting goals in production management. Finally, the cascaded minimum function (15.25) reflects the fact that the scarcest resource governs the maximum production speed: If a specific required part is missing, a product cannot be completed. All of this underlines the large degree of similarity between traffic and production networks [Helbing (2005)]. As a consequence, one can apply similar methods of description and similar control approaches.

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Chapter 16

A STOCHASTIC OPTIMAL CONTROL POLICY FOR A MANUFACTURING SYSTEM ON A FINITE TIME HORIZON

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Abstract We consider a problem of optimal production control of a single reliable machine. Demand is described as a discrete-time stochastic process. The objective is to minimize linear inventory/backlog costs over a finite time horizon. Using the necessary conditions of optimality, which are expressed in terms of co-state dynamics, we develop an optimal control policy. The policy is parameterized and its parameters are calculated from a computational procedure. Numerical examples show the convergence or divergence of the policy when the expected demand is greater or smaller than the production capacity. A non-stationary case is also presented.

1. Introduction

Uncertainty of demand is one of the major factors affecting decision-making in production planning and control (Nahmias, 2001). The problem of finding optimal control policies even for simple manufacturing systems under uncertainty has proved to be challenging both at the modeling stage and in analysis. In continuous time, hedging point policies have proved to be optimal for a class of stochastic systems. For such policies, a machine produces: a) at full capacity if the inventory level is

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lower than the hedging point; b) nothing if the inventory level is higher than the hedging point; and c) as much as the demand if the inventory level is equal to the hedging point.

The value of the optimum hedging point for systems with one unreliable machine and one part-type was first obtained by Akella and Kumar (1986), for the discounted cost problem and by Bielecki and Kumar (1988) for the average cost problem. These works assumed exponential distribution of the machine's work and repair time, constant deterministic demand rate, and linear surplus/backlog cost structure. Other researchers such as El-Ferik et al. (1998), Feng and Yan (2000), and Gershwin (1994) considered more general production environments and achieve only partial characterization of optimal policies. A comprehensive survey of research in optimal control of stochastic manufacturing systems can be found in Sethi et al. (2002). Perkins and Srikant (1997), extended the problem of Akella and Kumar to the case of two-part type and obtained an optimal policy. Perkins and Srikant (1998), subsequently enlarging upon their previous research into the problem of multiple part type, presented new results about the structure of the optimal policy and provided bounds on the optimal hedging points.

In this paper, we formulate a continuous-time optimal control policy when demand is discrete-time. The policy is not stationary, since the system is considered on a finite horizon. Within the intervals between demand realizations, the policy takes a specific form, different from hedging. It is proved to be optimal with the aid of the optimality conditions. We focus on developing a computation procedure for finding the control policy at each interval between demand realizations and on implementing the procedure for both the cases when demand distribution is stationary and with changes in time.

The paper is organized as follows: Section 16.2 introduces the system model and notation. An optimality condition is also provided at the end of the section. In Section 16.3 an optimal solution is characterized and parameterized. An algorithm for calculating the exact characteristics and parameters of the solution is presented. Section 16.4 provides numerical examples. Finally, we conclude the paper in Section 16.5.

2. Problem formulation and optimality conditions

Consider a machine whose production is intended to track an uncertain demand over a finite time horizon, $\tau \in [0, T]$. The machine is assumed reliable and no other source of uncertainty (except for demand realizations) is relevant. Let ω be a scenario of demand realizations,

$\omega = (\xi_0, \xi_1, ..., \xi_{T-1})$, where ξ_t is a realization of random demand d_t at t. Without loss of generality, we assume that the time horizon, T, is integer and the demand period is 1, t=0,1,...,T-1. The cumulative demand process, $D_\tau(\omega) = \sum_{t=0}^{T-1} \xi_t \theta(\tau - t)$ is assumed right-continuous; $\theta(\tau)$ is the unit step function, $\theta(\tau) = 1$ if $\tau \geq 0$ and $\theta(\tau) = 0$ if $\tau < 0$. Such a process can be defined by a joint distribution of d_t. As usual, we assume that d_t are independent random variables with bounded distribution density $\pi_t(s)$. Note that d_t are allowed to be distributed differently. An admissible control $U = \{u_\tau(\omega)\}$ defines the intensity of production at time τ so that

$$0 \leq u_\tau(\omega) \leq V, \tag{16.1}$$

where V is the maximal production rate. We assume that U is a predictable process over $\Im_\tau$, the σ-algebra generated by the values of the demand process on the interval $[0, \tau)$. The process $X = \{x_\tau(\omega)\}$ describes the inventory or surplus/backlog level at time τ. If control U is given, then the process X satisfies the equation

$$x_\tau(\omega) = x_0 + \int_0^\tau u_s(\omega) ds - D_\tau(\omega). \tag{16.2}$$

The total expected cost is defined as

$$J(x, U) = E\left[\int_0^T Cost(x_\tau(\omega)) d\tau | x_0 = x\right], \tag{16.3}$$

where E denotes the expectation, $Cost(x)$ is a continuous and piecewise continuously differentiable convex cost function. Our goal is to find an admissible control U, which satisfies (16.1) and minimizes the performance measure (16.3). The derivation of the necessary optimality conditions for the problem stated here is done similarly to that of the problem of optimal control of an unreliable machine with constant deterministic demand, as presented in Khmelnitsky et al. (2004), and can be written as

$$u_\tau = \begin{cases} V, \ E[\psi_\tau(\omega) | \Im_\tau] > 0 \\ 0, \ E[\psi_\tau(\omega) | \Im_\tau] < 0 \\ \in [0, V], \ E[\psi_\tau(\omega) | \Im_\tau] = 0 \end{cases} \tag{16.4}$$

where the co-state variable $\psi_t(\omega)$ is defined as

$$\psi_\tau(\omega) = -\int_\tau^T \frac{\partial Cost(x_s(\omega))}{\partial x} ds. \tag{16.5}$$

In what follows, we omit dependence of $u_\tau(\omega)$ and $x_\tau(\omega)$ on ω.

3. Solution method

In this section we assume linear cost function $Cost(x)$, i.e., $Cost(x) = xsig(x)$, where $sig(x) = \begin{cases} c^+, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -c^-, \text{ if } x < 0 \end{cases}$, and $\frac{\partial Cost(x)}{\partial x} = sig(x)$. From (16.5) it follows that $\psi_T = 0$ for all ω. Therefore, $E\left[\psi_T | x_T = x\right] = 0$ for every x and the costate dynamics can be determined recursively, starting from $t = T$, as described below.

Suppose $E\left[\psi_{t+1} | x_{t+1} = x\right]$ is known as a function of x for some t=0,...,T-1. Then, just before the demand realization at t+1

$$E[\psi_{t+1-} | \, x_{t+1-} = x] = \int_0^\infty \pi_{t+1}(s) E[\psi_{t+1} | \, x_{t+1} = x - s] ds. \qquad (16.6)$$

We denote this function as

$$A_{t+1}(x) = E[\psi_{t+1-} | \, x_{t+1-} = x].$$

Since no demand occurs within the interval $\tau \in (t, t+1)$, the function $E\left[\psi_t | x_t = x\right]$ unambiguously depends on the function $A_{t+1}(x)$. Therefore, by taking into account (16.6), it also depends on the function $E\left[\psi_{t+1} | x_{t+1} = x\right]$. That is, the costate variable moves from right to left. At the same time, the state variable moves from left to right and the dependence of x_{t+1-} on x_t will be denoted as $B_t(x)$, $x_{t+1-} = B_t(x_t)$. Section 16.3.1 below further clarifies these dependencies.

LEMMA 16.1 *The function* $A_t(x) = \int_0^\infty \pi_t(s) E\left[\psi_t | x_t = x - s\right] ds$ *is non-increasing.*

Proof: Consider a specific scenario starting at t, $\omega = (..., \xi_t, \xi_{t+1}, ..., \xi_{T-1})$ and two trajectories that correspond to this scenario, $x_\tau^{(1)}$ and $x_\tau^{(2)}$, $t \leq \tau \leq T$. The first trajectory starts with a lower inventory level than the other, $x_{t-}^{(1)} = x_1 < x_{t-}^{(2)} = x_2$. Therefore, $x_t^{(1)} = x_1 - \xi_t < x_t^{(2)} = x_2 - \xi_t$. In the sequel, the two trajectories retain the order, i.e., $x_\tau^{(1)} < x_\tau^{(2)}$ for $t \leq \tau \leq T$. Now, from (16.5) and the convexity of the $Cost(x)$ function it follows that $\psi_t^{(1)} \geq \psi_t^{(2)}$. Since the last inequality is true for each scenario, it is true also for the expected values, $E\left[\psi_t | x_t = x_1 - s\right] \geq E\left[\psi_t | x_t = x_2 - s\right]$ for all s. This proves the lemma.

3.1 State-costate dynamics

Consider a time interval $\tau \in [t, t+1)$. Let $x_t = x$ be given at the left boundary of the interval and the function $A_{t+1}(\cdot)$ be given at the right limit of the interval. That is, the state variable is given from left and the costate variable is given from right. Then, the state-costate dynamics within the interval can be calculated. In particular, the state value at the end of the interval, $B_t(x)$, and the costate variable at the beginning of the interval, $E[\psi_t|x_t = x]$, can also be calculated. We will show that $E[\psi_t|x_t = x]$ depends solely on x and on $A_{t+1}(B_t(x))$, i.e., $E[\psi_t|x_t = x] = C(x, A_{t+1}(B_t(x)))$.

The calculation of the functions $C(\cdot,\cdot)$ and $B_t(\cdot)$ given $A_{t+1}(\cdot)$ differs for six cases, $n = 1, ..., 6$ as presented below. Consider the first case, n=1, which corresponds to large values of x. For sufficiently large x, the costate variable must be negative in order not to produce waste inventory. This follows from the optimality conditions (16.4). From the costate dynamic equation (16.5), we have $\dot{\psi}_\tau = c^+$, that is, within the interval $\tau \in [t, t+1), \psi_\tau$ increases with the constant rate c^+, as shown in Figure 16.1a). During the interval, the state variable does not change, $B_t(x) = x$, and the costate value increases by c^+, $C(x, z) = z - c^+$. The first case works as far as the costate value at the right limit of the interval is negative. Therefore, the marginal x for this case is such that $A_{t+1}(B(x)) = A_{t+1}(x) = 0$. Thus, the first case is true for $A^{-1}_{t+1}(0) < x < \infty$. The other five cases, n=2,. . . ,6 are analyzed similarly.

The six cases are:

n=1, if $A^{-1}_{t+1}(0) < x < \infty$, then $B_t(x) = x$ and $C(x, z) = z - c^+$. See Figure 1a).

n=2, if $\max\{0, A^{-1}_{t+1}(c^+) - V\} < x \le A^{-1}_{t+1}(0)$, then $B_t(x) = y$, where y is the root of the equation $\frac{y-x}{V} = \frac{A_{t+1}(y)}{c^+}$, and $C(x, z) = z - c^+$. See Figure 1b).

n=3, if $0 < x \le \max\{0, A^{-1}_{t+1}(c^+) - V\}$, then $B_t(x) = x + V$ and $C(x, z) = z - c^+$. See Figure 1c).

n=4, if $\min\{0, y - V\} < x \le 0$, then $B_t(x) = y$, where y is the root of the equation $\frac{y}{V} = \frac{A_{t+1}(y)}{c^+}$, and $C(x, z) = -c^- \frac{x}{V}$. See Figure 1d).

n=5, if $-V < x \le \min\{0, y - V\}$, where y is the root of the equation $\frac{y}{V} = \frac{A_{t+1}(y)}{c^+}$, then $B_t(x) = x + V$, and $C(x, z) = z + c^+ + (c^+ - c^-)\frac{x}{V}$. See Figure 1e).

n=6, if $-\infty < x \le -V$, then $B_t(x) = x + V$, and $C(x, z) = z + c^-$. See Figure 1f).

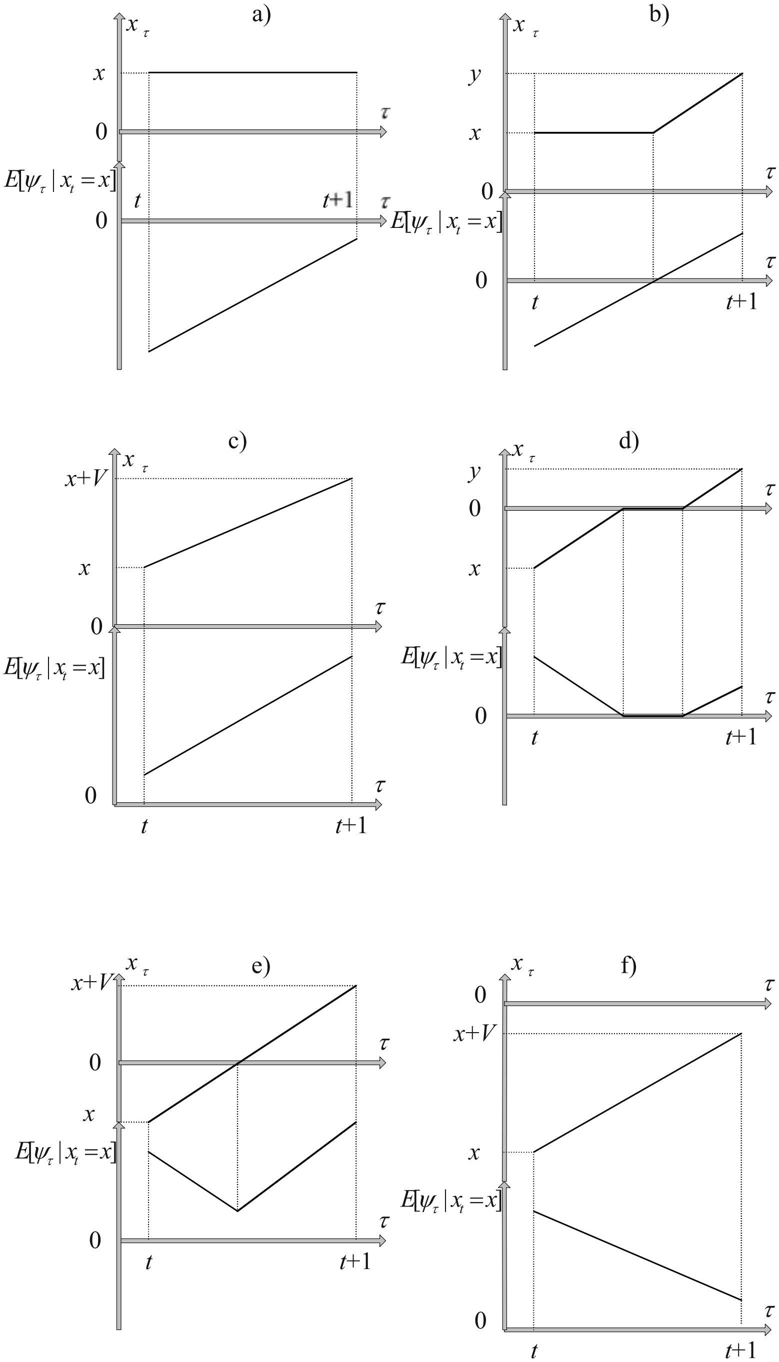

Figure 16.1. State-costate behavior for the six control regimes

3.2 Feedback control rule

In the previous section we showed that over the time interval $\tau \in [t, t+1)$, t=0,...,T-1, the state-costate dynamics takes one of the six possible types, depending on x_t, t, and the parameters of the problem. Therefore, the control u_τ, $\tau \in [t, t+1)$ is a function of x_t and t, too, as follows: For n=1, $u_\tau = 0$, $\tau \in [t, t+1)$.
For $n = 2$,

$$u_\tau = \begin{cases} 0, & \tau \in [t, t+1-\frac{B_t(x_t)-x_t}{V}) \\ V, & \tau \in [t+1-\frac{B_t(x_t)-x_t}{V}, t+1) \end{cases} \qquad (16.7)$$

For n=3, 5 and 6, $u_\tau = V$, $\tau \in [t, t+1)$.

For n=4, $u_\tau = \begin{cases} V, & \tau \in [t, t+\frac{-x_t}{V}) \\ 0, & \tau \in [t+\frac{-x_t}{V}, t+1-\frac{B_t(x_t)}{V}) \\ V, & \tau \in [t+1-\frac{B_t(x_t)}{V}, t+1) \end{cases}$.

The presented control rule satisfies the optimality condition 16.4. Therefore, it is optimal, since the problem 16.1-16.3 is convex.

3.3 Algorithm for calculating the optimal feedback control rule

Step 1. Set $t = T$-1 and $A_{t+1}(x) = E\left[\psi_{t+1-} | x_{t+1-} = x\right] = E\left[\psi_{T-} | x_{T-} = x\right] = 0$ for all x.

Step 2. Calculate $C(x, 0)$ and $B_t(x)$ for all x as discussed in Section 16.3.1

Step 3. Set $t = t-1$ and determine the optimal feedback control rule from (1.7), $u_\tau = u_\tau(t, x_t, \tau)$, $\tau \in [t, t+1)$.

Step 4. If t=0, stop, otherwise go to step 5.

Step 5. Calculate the function $A_{t+1}(x)$, as

$$A_{t+1}(x) = \int_0^\infty \pi_t(s) C(x-s, A_{t+2}(B_t(x-s)))ds. \qquad (16.8)$$

Step 6. Calculate $B_t(x)$ and $C(x, A_{t+1}(B_t(x)))$ as discussed in Section 16.3.1 Go to step 3.

4. Numerical results

The complexity of the algorithm in Section 16.3.3 is defined by the accuracy with which the function $A_{t+1}(x)$ is calculated from 16.8. If for each t, $A_{t+1}(x)$ is calculated at N points and the integral in 16.8)is

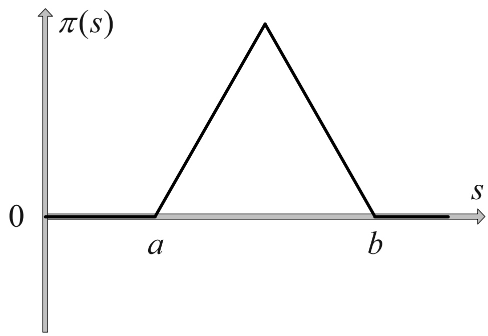

Figure 16.2. Probability density function of demand

calculated by the trapeze method with K mesh intervals, ($\pi_t(s)$ is pertinent in K+1 mesh points), the complexity of each loop of the algorithm is $O(KN)$. The total complexity is $O(KNT)$.

We implemented the algorithm of the previous section for the following parameters of the problem: T=40, V=1, $c^+ = 0.4$, $c^- = 1$.

4.1 Convergence and divergence

For a stationary demand, $\pi_t(\cdot) = \pi(\cdot)$ for all t, with the production capacity being greater than the expected demand, $E[d_t] = E[d] < V$, the control policy converges. That is, there exists a limit function $B(x)$, such that

$$\lim_{T\to\infty} B_t(x) = B(x)$$

for each t and x.

We used the probability density function of demand, $\pi(s)$, as follows (see Figure 16.2)

$$\pi(s) = \begin{cases} 0, \text{ if } s < a \text{ or } s \geq b \\ \frac{4(s-a)}{(b-a)^2}, \text{ if } a \leq s < \frac{a+b}{2} \\ \frac{4(b-s)}{(b-a)^2}, \text{ if } \frac{a+b}{2} \leq s < b \end{cases}$$

with the parameters a=0.1 and b=1.7.

Figures 16.3 and 16.4 present the function $A_t(x)$ calculated by the developed algorithm. The function does not converge over the whole axis. For a very large positive x, the costate function increases by c^+ at each time period (see case n=1 in Section 16.3.1), and for a very large negative x, the costate function decreases by c^- at each time period (see case n=6 in Section 16.3.1). Therefore,

$$\lim_{x\to\infty} A_t(x) = -c^+(T-t), \quad \lim_{x\to-\infty} A_t(x) = c^-(T-t).$$

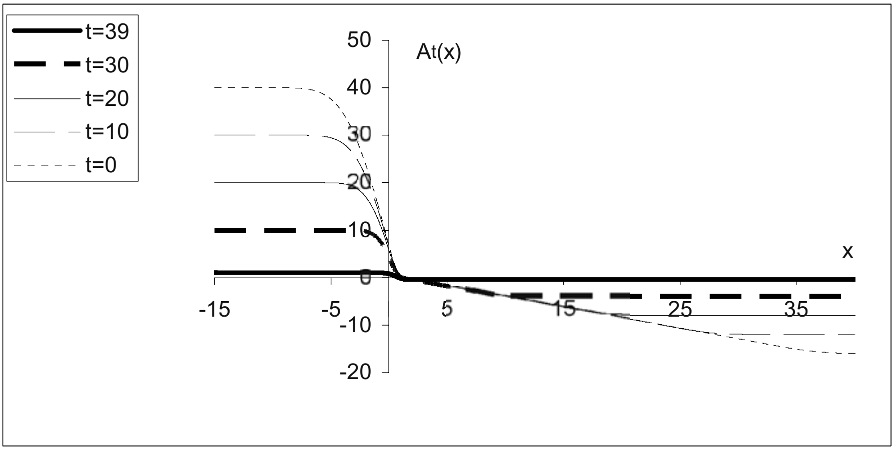

Figure 16.3. Function $A_t(x)$ for t=39, 30, 20, 10, 0: general scale.

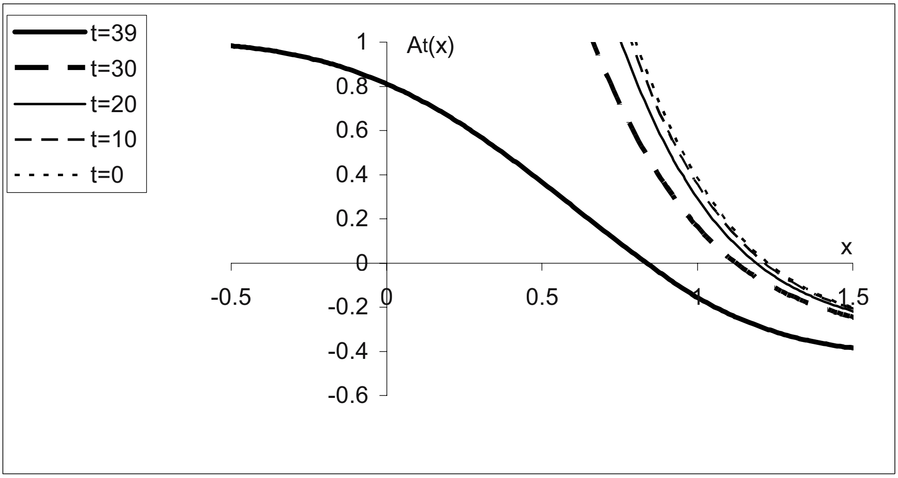

Figure 16.4. Function $A_t(x)$ for t=39, 30, 20, 10, 0: the scale of $A_t^{-1}(0)$ and $A_t^{-1}(c^+)$.

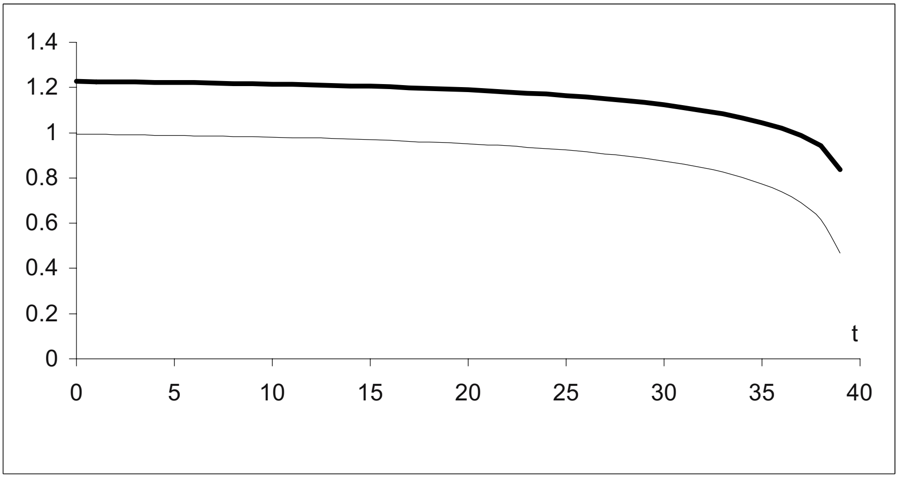

Figure 16.5. Convergence of $A_t^{-1}(0)$ (bold line) and $A_t^{-1}(c^+)$ (thin line)

.

However, the control strategy depends on the function $A_t(x)$ only within the area between $A_t^{-1}(c^+)$ and $A_t^{-1}(0)$. Therefore, if the function $A_t(x)$ converges within the area between $A_t^{-1}(c^+)$ and $A_t^{-1}(0)$, the control strategy converges as well. Figure 16.5 shows the convergence of the last two values as t approaches zero. Figure 16.6 shows how much inventory is to be stored at the end of period t, $B_t(x)$, if inventory at the beginning of period t is x. The value of production within the period is $B_t(x) - x$.

The next experiment shows the divergence of the control strategy. Here we used the parameters a=0.5 and b=1.9 , i.e. $E[d] > V$. The rest of the parameters are same. Figures 16.7-16.10 are similar to Figures 16.3-16.6 of the previous experiment.

4.2 Changing in time demand

The developed algorithm works for a non-stationary demand distribution as well. We used the demand distribution from the previous experiments with the parameters a=0.1 and b=0.8 for t=1,3,5, ... ,39, and a=1.1 and b=1.8 for t=0,2,4, ... ,38. The control policy converges to two different forms, one for even time periods and the other for the odd periods. Figure 16.11 shows the limit policy approximated at t=0 and at t=1.

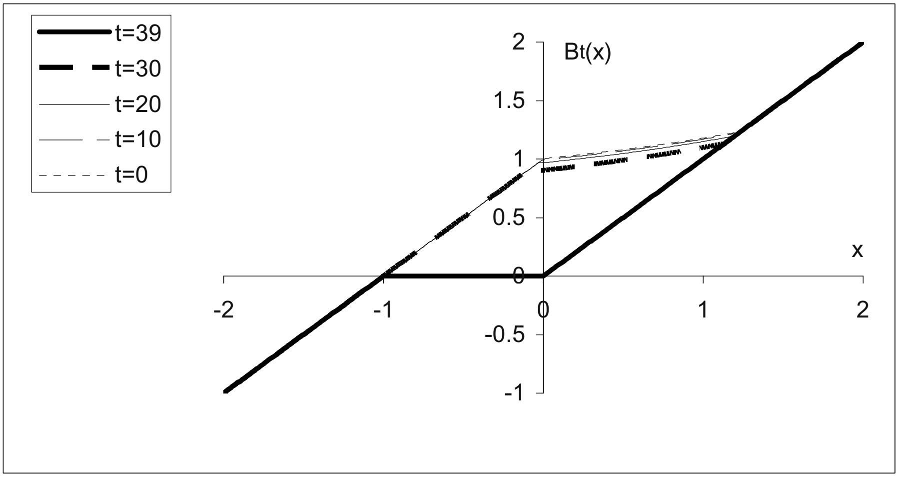

Figure 16.6. Convergence of $B_t(x)$.

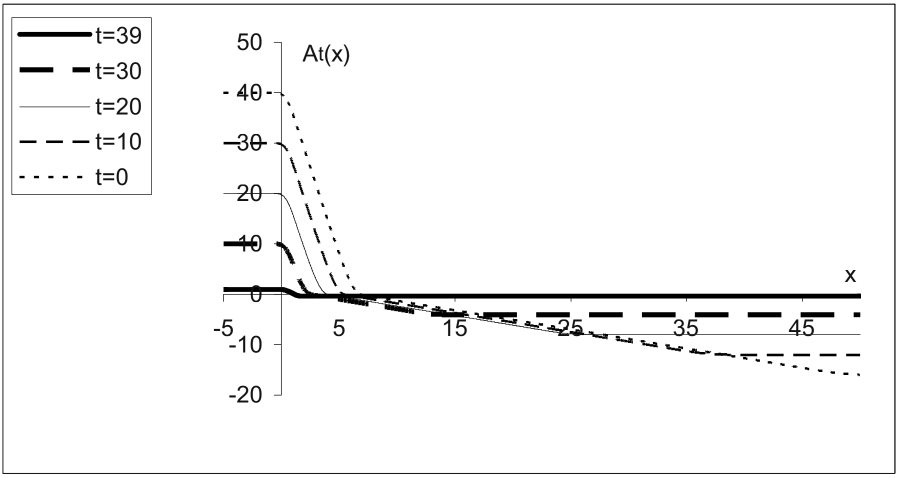

Figure 16.7. Function $A_t(y)$ for t=39, 30, 20, 10, 0: general scale.

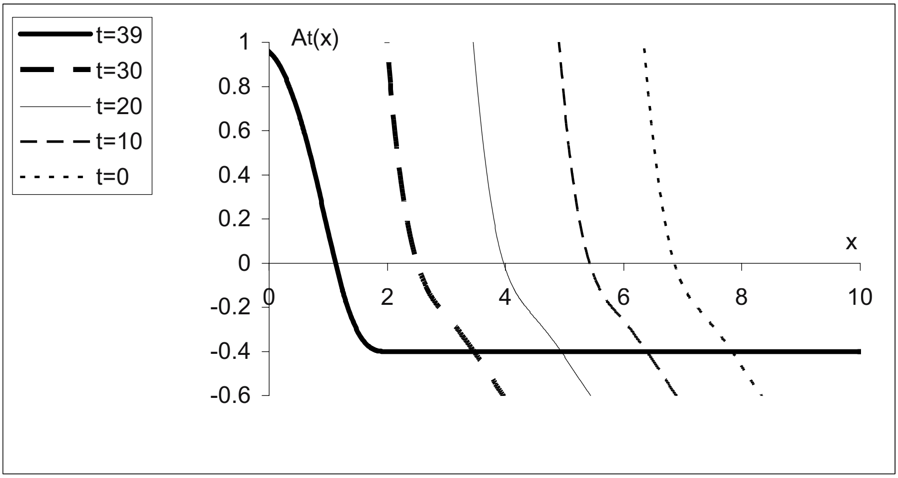

Figure 16.8. Function $A_t(y)$ for t=39, 30, 20, 10, 0: general scale.

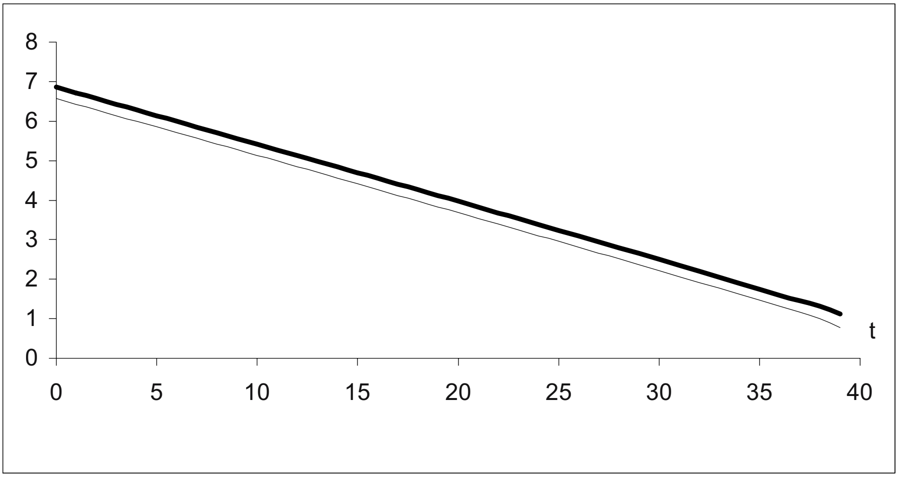

Figure 16.9. Divergence of $A_t^{-1}(0)$ (bold line) and $A_t^{-1}(c^+)$ (thin line).

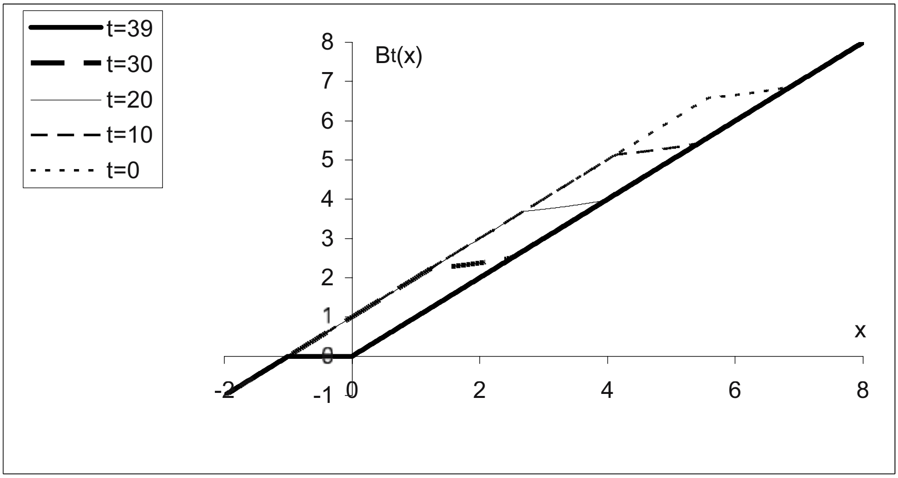

Figure 16.10. Divergence of $B_t(x)$.

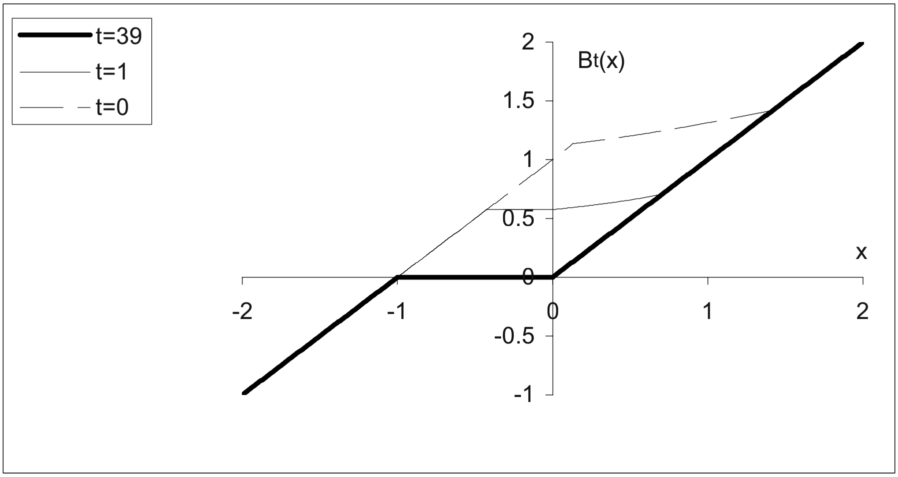

Figure 16.11. Two limit levels of $B_t(x)$

5. Conclusions

In this paper we have shown how the optimality conditions of a stochastic optimal control problem can be used to construct a solution method. The constructed method uses both the state and costate dynamics of the problem and converges (when possible) to a limit strategy optimal far from the end of the planning horizon. In particular, the method can be used to approximate the limit strategy. The approach and the method can be generalized to more complex cases that would include stochastic and non-stationary machine capacity, as well as nonlinear cost structure.

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Chapter 17

ON A STATE-CONSTRAINED CONTROL PROBLEM IN OPTIMAL PRODUCTION AND MAINTENANCE

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Abstract We consider a control problem introduced by Cho, Abad and Parlar (1993) which "incorporates a dynamic maintenance problem into a production control model". For a quadratic production cost function we present a detailed numerical study of optimal control policies for different final times. The maintenance control is either composed by bang-bang and singular arcs or is purely bang-bang. In the case of a linear production cost, we show that both production and maintenance control are purely bang-bang. A recently developed second order sufficiency test is applied to prove optimality of the computed controls. This test enables us to calculate sensitivity derivatives of switching times with respect to perturbation parameters in the system. Furthermore, numerical results are presented in the case where a state constraint on the number of good items is added to the control problem.

C. Deissenberg and R.F. Hartl (eds.), Optimal Control and Dynamic Games, 289–308.

1. Introduction

Cho, Abad and Parlar (1993) have considered a production process whose performance declines over time in the absence of maintenance. Preventive maintenance may be applied to the process to slow down the rate of decline of (or improve) the process performance which is measured in terms of good (non-defective) units of items produced. The authors restrict the analysis of optimal control solutions to a quadratic production cost function while the maintenance cost function is assumed to be linear. Using an algorithm that is tailored to the specific control problem, they compute a maintenance control which is composed of bang-bang and singular arcs while the production control is a continuous function.

The present paper pursues *three objectives* which are discussed in Sections 17.3-17.5. The *first objective* (Section 17.3) is to present numerical results that improve on those in Cho, Abad and Parlar (1993). We apply recently developed optimization techniques to obtain feasible controls that satisfy the necessary optimality conditions in Pontryagin's maximum principle with high accuracy. For a variety of finite time horizons, different control structures are elaborated. Surprisingly, no sufficient conditions are available in the literature which would bear upon the type of controls encountered in the model.

The *second objective* (Section 17.4) is the computation of optimal controls in the case where both the maintenance and production cost functions are linear. It is shown that the production and maintenance controls are purely bang-bang. In addition, we are able to verify numerically that the computed bang-bang controls provide a strict strong minimum. This is achieved by applying a new sufficiency test developed in Agrachev, Stefani and Zezza (2002), Maurer and Osmolovskii (2003,2004), Maurer, Büskens, Kim and Kaya (2004) and Osmolovskii and Maurer (2005).

The *third goal* is discussed in Section 17.5 where an additional state constraint, a lower bound for the number of good items, is imposed. We show that the optimal control solution contains one boundary arc and is bang-bang on interior arcs. To prove sufficient optimality conditions, we extend the second order test in Section 17.4 to handle a boundary arc.

Table 17.1. Notation

variable	*description*
$x(t)$	inventory level at time $t \in [0, T]$ (*state*), fixed final time $T > 0$
$y(t)$	proportion of 'good' units of end items produced at time t: process performance (*state*)
$u(t)$	scheduled production rate (*control*)
$m(t)$	preventive maintenance rate to reduce the proportion of defective units produced (*control*)
$\alpha(t)$	obsolescence rate of the process performance in the absence of maintenance
$s(t)$	demand rate
$\rho > 0$	discount rate

2. The control model for optimal production and maintenance

The notations for the optimal control problem are displayed in Table 17.1. The dynamics of the process is described as

$$\begin{aligned} \dot{x}(t) &= y(t)u(t) - s\,, & x(0) &= x_0 > 0\,, \\ \dot{y}(t) &= -(\alpha + m(t))y(t) + m(t)\,, & y(0) &= y_0 > 0\,. \end{aligned} \tag{17.1}$$

Bounds on the control variables are given by

$$0 \le u(t) \le U, \quad 0 \le m(t) \le M \quad \text{for } 0 \le t \le T\,. \tag{17.2}$$

Since all demands must be satisfied, the following state constraint is imposed:

$$0 \le x(t) \quad \text{for } 0 \le t \le T\,. \tag{17.3}$$

Later, we shall consider an additional state constraint, a lower bound on the number of good items,

$$S(y(t)) := y(t) - y_{\min} \ge 0 \quad \text{for } 0 \le t \le T\,. \tag{17.4}$$

The optimal control problem then is to maximize the total discounted profit plus the salvage value of $y(T)$,

$$\begin{aligned} J(x, y, u, m) = \; & \int_0^T [ws - hx(t) - \Phi(u(t)) - cm(t)]e^{-\rho t}\,dt \\ & + by(T)e^{-\rho T}, \end{aligned} \tag{17.5}$$

under the constraints (17.1)–(17.3), resp., (17.1)–(17.4). Here, w, h, c, ρ, b are positive constants and $\Phi(u)$ denotes the production cost function that will be chosen either as the *quadratic* function

$$\Phi(u) = ru^2, \quad r > 0\,, \tag{17.6}$$

or *linear* function

$$\Phi(u) = qu, \quad q > 0\,. \tag{17.7}$$

Note that the maintenance cost in (17.5) is taken as the *linear* function $c \cdot m$. Computations will reveal that the optimal inventory $x(t)$ always satisfies $x(t) > 0$ for $0 \le t < T$ and $x(T) = 0$ for both types of production cost functions. Hence, we shall replace the state constraint (17.3) by the terminal condition

$$x(T) = 0\,. \tag{17.8}$$

We choose constants that are identical to those in Cho, Abad and Parlar (1993) and specify the linear function in (17.7):

Table 17.2. Control problem data

in eq. (17.1)	:	$s \equiv 4$,	$\alpha \equiv 2$,	$x_0 = 3$,	$y_0 = 1$	
in eq. (17.2)	:	$U = 3$,	$M = 4$			
in eq. (17.5)	:	$\rho = 0.1$,	$w = 8$,	$h = 1$,	$c = 2.5$,	$b = 10$
in eq. (17.6), (17.7)	:	$r = 2$,	$q = 4$			

First, we discuss the basic set of necessary optimality conditions in Pontryagin's maximum principle; cf., Feichtinger and Hartl (1986), Sethi and Thompson (2004). The current value Hamiltonian in normal form is given by

$$\begin{aligned} H(x, y, u, m, \lambda_x, \lambda_y) = \; & (ws - hx - \Phi(u) - cm) \\ & + \lambda_x(yu - s) + \lambda_y(-(\alpha + m)y + m), \end{aligned} \tag{17.9}$$

where λ_x, λ_y are the adjoint variables. The adjoint equations and transversality conditions yield in view of $x(T) = 0$ and the salvage term in the cost functional:

$$\begin{aligned} \dot{\lambda}_x &= \rho\lambda_x - \frac{\partial H}{\partial x} = \rho\lambda_x + h, & \lambda_x(T) &= \nu, \\ \dot{\lambda}_y &= \rho\lambda_y - \frac{\partial H}{\partial y} = \lambda_y(\rho + \alpha + m) - \lambda_x u, & \lambda_y(T) &= b. \end{aligned} \tag{17.10}$$

The multiplier ν is not known a priori and will be determined later. The maximization of the Hamiltonian with respect to the control variable u is equivalent to the maximization problem

$$\max_{0\leq u\leq U}\left(-\Phi(u)+\lambda_x(t)y(t)u\right). \tag{17.11}$$

In the next two sections, this maximization will be evaluated separately for the quadratic, resp., linear production cost function (17.6), resp., (17.7). Since the maintenance control m enters the Hamiltonian linearly, the control m is determined by the sign of the switching function

$$\sigma^m(t)=\frac{\partial H}{\partial m}=-c+\lambda_y(t)(1-y(t)) \tag{17.12}$$

as the policy

$$m(t)=\left\{\begin{array}{lccl} M & , & \text{if} & \sigma^m(t)>0 \\ 0 & , & \text{if} & \sigma^m(t)<0 \\ \text{singular} & , & \text{if} & \sigma^m(t)\equiv 0 \quad \text{for } \ t\in I_{\text{sing}}\subset[0,T] \end{array}\right\}. \tag{17.13}$$

Cho, Abad and Parlar (1993) have shown that a singular arc occurs indeed for a quadratic production cost function. In the next section, we shall give a more detailed account of the singular case.

3. Quadratic production cost function: bang-bang and singular maintenance control

In this section, we omit the state constraint (17.4). Assuming the quadratic production cost $\Phi(u)=ru^2$ with $r=2$, our computations for time horizons $T>0.75$ indicate that the control constraint $0\leq u(t)\leq U=3$ never becomes active. Upon deleting this constraint, the maximization in (17.11) with respect to the control u immediately yields the relation

$$u(t)=\frac{\lambda_x(t)y(t)}{2r}. \tag{17.14}$$

A singular maintenance control $m_{\text{sing}}(t)$ in (17.13) is characterized by the equations $d^k\sigma^m/dt^k=0$ for $t\in I_{\text{sing}}\subset[0,T]$, $k=0,1,2$. Using the control policy (17.14) we find after some lengthy computations that the second derivative of the switching function is given by

$$\begin{aligned} \frac{d^2\sigma^m}{dt^2}&=A+B\cdot m_{\text{sing}}=0,\\ A&=-(h+\rho\lambda_x)u(1-y)-\alpha\lambda_x u(1+y)+\alpha^2\lambda_y\\ &\quad+\alpha\rho\lambda_y-\lambda_x(1-y)/(2r)(y(h+\rho\lambda_x)-\alpha y\lambda_x),\\ B&=\lambda_x u(1-y)+\alpha\lambda_y-\lambda_x^2(1-y)^2/(2r). \end{aligned} \tag{17.15}$$

Hence, the singular arc is of order one (cf. Bell and Jacobson (1975), Krener (1977)) and the singular maintenance control is determined by the formula

$$m_{\text{sing}}(t) = -A(t)/B(t)\,. \tag{17.16}$$

We only mention that it is even possible to derive a feedback expression $m_{\text{sing}} = m_{\text{sing}}(x, y)$ for the singular control. Namely, one can use the relations $\sigma = \dot{\sigma} = 0$ to obtain the adjoints λ_x and λ_y as functions of x and y. The evaluation of formula (17.16) requires that the generalized Legendre-Clebsch condition

$$B(t) = \frac{\partial}{\partial m}\left(\frac{d^2}{dt^2}\frac{\partial H}{\partial m}\right)(t) > 0$$

holds along the singular arc, cf. McDanell and Powers (1971), Krener (1977). We shall verify this condition a posteriori.

The numerical approach in Cho, Abad and Parlar (1993) is tailored to the specific problem under consideration. It is based on a decentralized approach in which the process is divided into two subproblems which are linked via an interaction variable. Moreover, it is assumed that the maintenance control has the following three stages for the final time $T = 1$:

$$m(t) = \left\{ \begin{array}{lcll} 0 & , & \text{for} & 0 \le t \le t_1 \\ \text{singular} & , & \text{for} & t_1 < t \le t_2 \\ M = 4 & , & \text{for} & t_2 < t \le T \end{array} \right\}. \tag{17.17}$$

We can confirm this control structure but choose an entirely different approach. We optimize a discretized version of the control problem for a high number N of grid points $\tau_i = i \cdot T/N$, $i = 0, 1, ..., N$; cf. Betts (2001), Büskens (1998), Büskens and Maurer (2000). The integration method for the ODE system (17.1) and (17.5) is Heun's method. Furthermore, we implement the discretized control problem in the programming language AMPL of Fourer, Gay and Kernighan (1993) and use the interior point optimization code LOQO of Vanderbei and Shanno (1999). For $N = 5000$ grid points, the computed state, control and adjoint functions are displayed in Figs. 17.1-17.3. Table 17.3 shows the solution data for different values of T which illustrate different control policies (17.17)–(17.19).

The optimization approach AMPL/LOQO determines the switching times t_i only with a precision of up to 3 decimals. The precise values of the switching times in Table 17.3 are obtained with a direct optimization approach which is similar to that in Maurer, Büskens, Kim and Kaya (2004) used for bang-bang controls.

Table 17.3. Solution data for the quadratic cost function, control policies (17.17)–(17.19).

Values for	$T = 1$	$T = 0.99$	$T = 0.8$
J	25.36401	25.54044	26.41073
$\lambda_x(0)$	10.72807	10.59144	2.689152
$\lambda_y(0)$	7.910491	7.743679	1.439564
$\lambda_x(T)$	12.90783	12.73409	3.745814
t_1	0.517947	0.567540	0.621544
t_2	0.701577	0.591157	
t_3		0.684511	

Fig. 17.2 indicates that the maintenance control is *discontinuous* at the junction points: the computed values for $T = 1$ are $m_{\text{sing}}(t_1) = 3.1324$ and $m_{\text{sing}}(t_2) = 2.0998$. Hence, the necessary junction conditions in Theorem 1 in McDanell and Powers (1971) are fulfilled. The switching function $\sigma^m(t)$ matches precisely the control policy (17.13) and (17.17). It is worth noting that the singular control obtained via discretization and optimization methods coincides precisely with the singular control (17.15) and (17.16) when the computed adjoints are inserted. Thus, we have computed a solution candidate that satisfies the first order *necessary* conditions in Pontryagin's maximum principle with high accuracy.

Note that the singular maintenance rate differs considerably from that depicted in Fig. 17.3 in Cho, Abad and Parlar (1993). The adjoint function $\lambda_x(t)$ corresponds to the adjoint function $\lambda_1(t) + \eta(t)$, $\lambda_1(T) = 0$, for which Cho, Abad and Parlar (1993) determine the value $\eta \equiv 12.5$. Both adjoint functions agree only qualitatively.

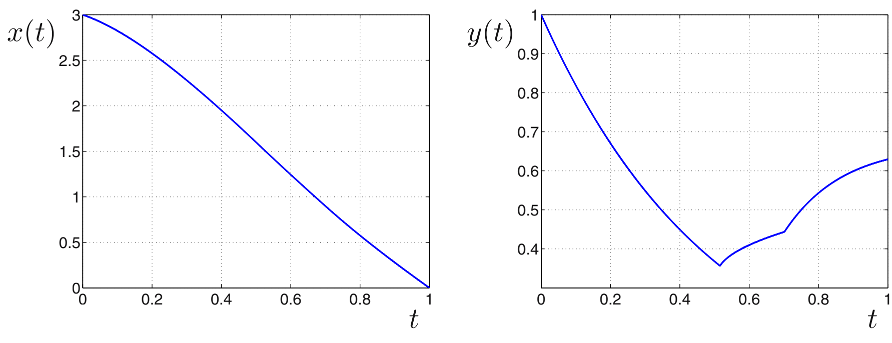

Figure 17.1. State variables $x(t)$ and $y(t)$.

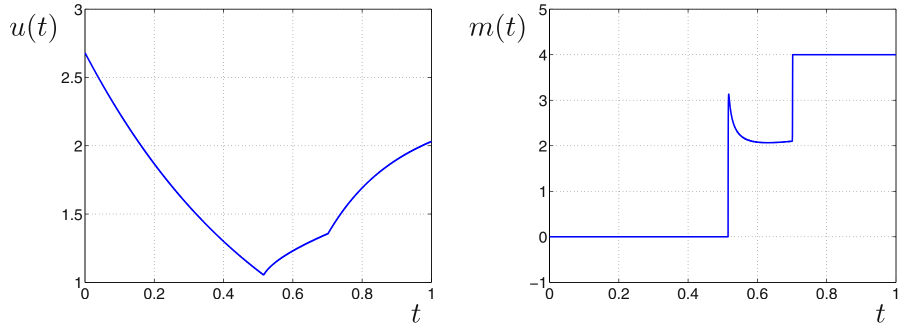

Figure 17.2. Control variables $u(t)$ and $m(t)$.

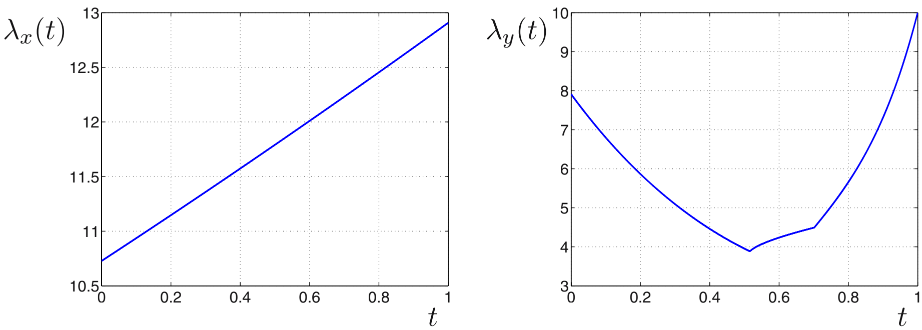

Figure 17.3. Adjoint variables $\lambda_x(t)$ and $\lambda_y(t)$.

Now the question arises whether *sufficient conditions* exist that bear upon the computed candidate solution. The well known sufficiency theorem in Feichtinger and Hartl (1986), p. 36, Satz 2.2, requires the maximized Hamiltonian $H^{\max}$ to be *concave* in the variables (x, y) when the adjoint functions $\lambda_x(t), \lambda_y(t)$ are inserted. However, using the maximizing control $u(t) = \lambda_x(t)y(t)/2r$, we obtain

$$H^{\max}(x, y, \lambda_x(t), \lambda_y(t)) = \frac{y^2 \lambda_x(t)^2}{4r} + lin(x, y),$$

where $lin(x, y)$ denotes a linear function in the variables (x, y). Since $\lambda_x(t) \neq 0$ holds in $[0, T]$, the maximized Hamiltonian is *strictly convex* in the variable y. This means that the above mentioned sufficiency theorem is not applicable here.

Other types of sufficient conditions that could be eventually applied to the present problem belong to the class of second-order sufficient conditions (SSC) for local optimality. SSC for optimal control problems

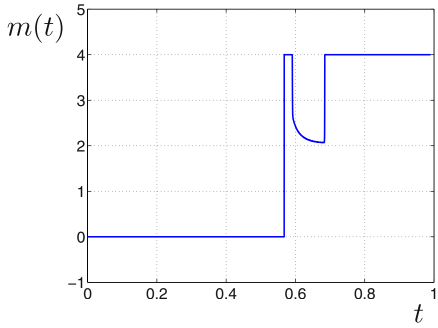

Figure 17.4. Optimal maintenance control for $T = 0.99$.

where the strict Legendre condition holds with respect to all control components (regular control) may be found in Malanowski and Maurer (1996), Maurer and Pickenhain (1995), Milyutin and Osmolovskii (1998) and Zeidan (1994). SSC for purely bang-bang controls have recently been derived in Agrachev, Stefani and Zezza (2002), Maurer and Osmolovskii (2003,2004) and Osmolovskii and Maurer (2005). However, the control problem studied in this paper exhibits a mixture of a regular control and a bang-singular control. Thus we are faced with the surprising fact that the literature does not provide SSC which are applicable to the control problem considered here. This poses a theoretical challenge which is currently under investigation.

Now we briefly study optimal controls for time horizons $T \neq 1$. It turns out that the structure (17.17) of the maintenance control does not prevail for all $T \neq 1$. E.g., for the final time $T = 0.99$, the maintenance control has an additional bang-bang arc preceding the singular arc:

$$m(t) = \left\{ \begin{array}{lll} 0 & , \text{ for } & 0 \le t \le t_1 \\ M = 4 & , \text{ for } & t_1 < t \le t_2 \\ \text{singular} & , \text{ for } & t_2 < t \le t_3 \\ M = 4 & , \text{ for } & t_3 < t \le T \end{array} \right\}. \tag{17.18}$$

The control is displayed in Fig. 17.4 while the solution data are given in Table 17.3.

For final times $T \in [0.15, 0.9819]$, the singular arc amidst the two upper bang-bang arcs disappears and the maintenance control takes the simple bang-bang form with one switching time

$$m(t) = \left\{ \begin{array}{lll} 0 & , \text{ for } & 0 \le t \le t_1 \\ M = 4 & , \text{ for } & t_1 < t \le T \end{array} \right\}. \tag{17.19}$$

Table 17.3 contains the solution data for $T = 0.8$. For time horizons $T \leq 0.75$ the production rate $u(t)$ is zero for all $t \in [0, T]$.

For values $T > 1$ one observes that the singular arc in (17.17) remains part of the control in the range $T \in [0.9975, 1.4695]$. For values of $T \in [1.4696, 1.583]$, the simple bang-bang policy (17.19) is optimal.

4. Linear production and maintenance cost functions: bang-bang controls

In this section, we assume that the production cost function is linear,

$$\Phi(u) = q \cdot u, \quad q = 4\,.$$

Hence, we can expect that the control constraint $0 \leq u(t) \leq U$, $U = 3$, in (17.2) becomes active. The production control u is determined by the sign of the associated switching function

$$\sigma^u(t) = \frac{\partial H}{\partial u} = -q + \lambda_x(t) y(t) \tag{17.20}$$

according to

$$u(t) = \left\{ \begin{array}{lll} 0 & , & \text{if} \quad \sigma^u(t) < 0 \\ U = 3 & , & \text{if} \quad \sigma^u(t) > 0 \end{array} \right\}. \tag{17.21}$$

We refrain from giving a theoretical discussion of singular production and maintenance controls, since numerical computations indicate that both control components are purely bang-bang for a certain range of time horizons T.

To determine the optimal solution for the final time $T = 1$, we implement again the optimization package AMPL/LOQO in Fourer, Gay and Kernighan (1993), Vanderbei and Shanno (1999). Using Heun's integration method and $N = 5000$ grid points, we obtain the following bang-bang production and maintenance control:

$$(u(t), m(t)) = \left\{ \begin{array}{lll} (U, 0) & , & \text{for} \quad 0 \leq t \leq t_1 \\ (0, 0) & , & \text{for} \quad t_1 < t \leq t_2 \\ (0, M) & , & \text{for} \quad t_2 < t \leq t_3 \\ (U, M) & , & \text{for} \quad t_3 < t \leq T = 1 \end{array} \right\}. \tag{17.22}$$

The computed state, control and switching functions are shown in Figs. 17.5-17.7. The solution data are given in Table 17.4.

The switching of the maintenance rate from zero to the upper bound $M = 3$ is plausible. The production policy where one switches from full to zero production and back to full production was not clear from the

Table 17.4. Solution data for linear cost function $\Phi(u) = q \cdot u$.

Values for	$T = 1$
J	25.79803
$\lambda_x(0)$	7.386222
$\lambda_y(0)$	6.358150
$\lambda_x(T)$	9.214515
t_1	0.346523
t_2	0.727053
t_3	0.841542

onset. This policy is not in agreement with the statement in Cho, Abad and Parlar (1993), Conclusion, Finding (3): "For a linear production cost function it is always optimal not to produce units until the on-hand inventory is completely depleted".

In contrast to the results for quadratic production cost, we can actually prove that the purely bang-bang control (17.22) provides a strict strong minimum. We use the recently derived second-order sufficient conditions (SSC) in Agrachev, Stefani and Zezza (2002), Maurer and Osmolovskii (2003,2004), Osmolovskii and Maurer (2005) and apply the numerical verification technique in Maurer, Büskens, Kim and Kaya (2004). Observe first that the production and maintenance controls do not switch simultaneously. Furthermore, one reads off Fig. 17.7 and can check it numerically that the following *strict bang-bang property* holds:

$$\begin{array}{lll} \dot{\sigma}^u(t_1) < 0, \quad \dot{\sigma}^m(t_2) > 0, & \dot{\sigma}^u(t_3) > 0, & \\ \sigma^u(t) \neq 0 \;\text{ for } t \neq t_1, t_3, & \sigma^m(t) \neq 0 \;\text{ for } t \neq t_2\,. \end{array} \tag{17.23}$$

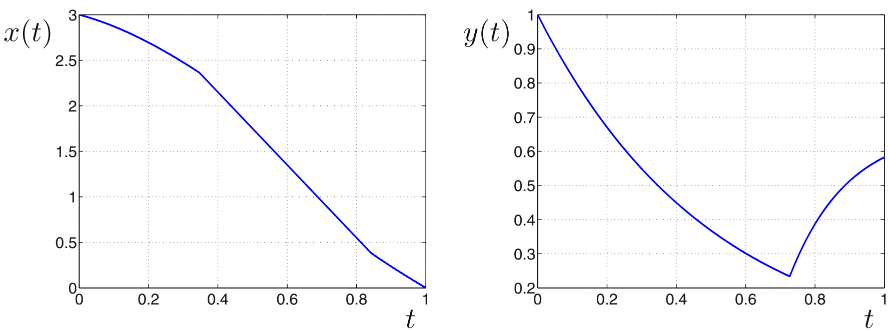

Figure 17.5. State variables $x(t)$ and $y(t)$.

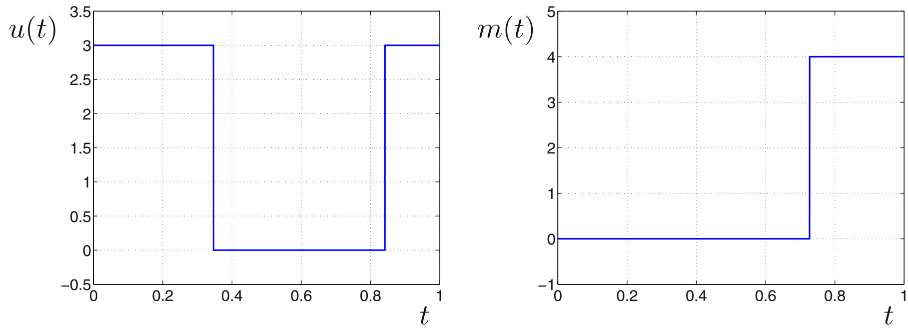

Figure 17.6. Control variables $u(t)$ and $m(t)$.

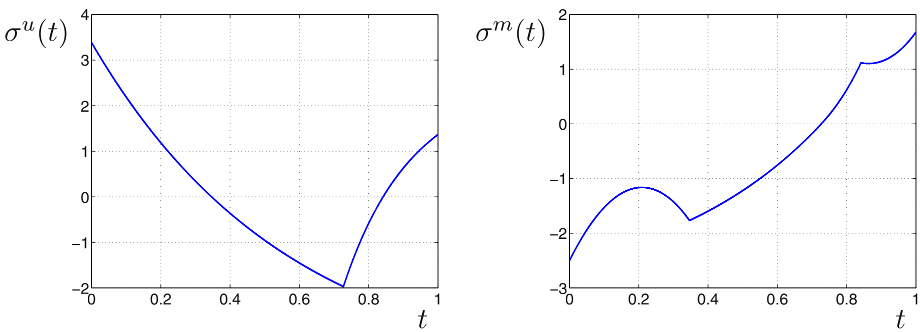

Figure 17.7. Switching functions $\sigma^u(t)$ and $\sigma^m(t)$.

Let us sketch the crucial part of the SSC test in Agrachev, Stefani and Zezza (2002), Maurer, Büskens, Kim and Kaya (2004). The bang-bang control problem is reformulated as an optimization problem where the *optimization variables* are the *arc durations* $\xi_1 = t_1$, $\xi_2 = t_2 - t_1$ and $\xi_3 = t_3 - t_2$ of the bang-bang arcs. Therefore, the switching times are related to the arc durations by $t_k = \sum_{i=1}^{k} \xi_k$ for $k = 1, 2, 3$. Since the final time T is fixed, the duration of the last bang-bang arc is given by $T - t_3 = T - (\xi_1 + \xi_2 + \xi_3)$. Hence, we consider the optimization variable $z := (\xi_1, \xi_2, \xi_3)$. The control policy (17.22) then defines functions $u(t; z)$, $m(t; z)$ which determine the solution of the ODE system (17.1) as functions $x(t; z)$ and $y(t; z)$. Therefore, the cost functional (17.5) can be rewritten as the function

$$F(z) = F(\xi_1, \xi_2, \xi_3) := J(x(t; z), y(t; z), u(t; z), m(t; z)) \qquad (17.24)$$

which is to be maximized subject to the equality constraint

$$G(z) = G(\xi_1, \xi_2, \xi_3) := x(T; z) = 0\,. \qquad (17.25)$$

The Lagrangian function for the optimization problem (17.24) and (17.25) is defined by

$$\mathcal{L}(z,\nu) = F(z) + \nu G(z), \tag{17.26}$$

where ν is a multiplier which coincides with that in the transversality condition $\lambda_x(T) = \nu$ in (17.10). The second order test in Agrachev, Stefani and Zezza (2002), Maurer and Osmolovskii (2003,2004) is equivalent to the following three conditions where partial derivatives are denoted by subscripts and the asterisk denotes the transpose:

$$\begin{array}{ll} (a) & \mathcal{L}_z(z,\nu) = F_z(z) + \nu G_z(z) = 0, \quad G_z(z) \neq (0,0,0), \\ (b) & h^* \mathcal{L}_{zz}(z,\nu) h > 0 \quad \forall\, h \neq (0,0,0),\ G_z(z)h = 0. \end{array} \tag{17.27}$$

Note that condition (a) is equivalent to the adjoint equations and the switching conditions; cf. Maurer, Büskens, Kim and Kaya (2004). The SSC test (b) can be performed with the routine NUDOCCCS of Büskens (1998). We get the following results : $\xi_1 = t_1 = 0.346523$, $\xi_2 = t_2 - t_1 = 0.380530$, $\xi_3 = t_3 - t_2 = 0.114489$ and the multiplier $\nu = \lambda_x(T) = 9.214515$. The Jacobian of $G(z)$ and the Hessian of $\mathcal{L}(z,\nu)$ are computed as

$$\mathcal{L}_{zz}(z,\nu) = \begin{pmatrix} 41.62 & 21.04 & -3.44 \\ 21.04 & 21.05 & -3.46 \\ -3.44 & -3.46 & 34.53 \end{pmatrix}, \ G_z(z) = -(0.32, 1.82, 1.35). \tag{17.28}$$

Hence, condition (a) in (17.27) holds. Condition (b) in (17.28) requires that the Hessian $\mathcal{L}_{zz}(z,\nu)$ given in (17.28) be positive definite on the kernel of $G_z(z)$. This property can be easily checked numerically. Thus, in view of the strict bang-bang property (17.23) we may conclude that the control policy (17.22) yields a strict strong minimum.

The test of SSC has another important consequence for stability and sensitivity of the optimal bang-bang control when the data of the control problem are perturbed. It follows from the sensitivity results in Fiacco (1983), Büskens and Maurer (2001), and Kim and Maurer (2003) that, for small perturbations of the data, the optimal control has the same number of switching times as the unperturbed optimal control. Moreover, it can be shown that the switching times are differentiable functions of any parameter in the system.

To illustrate this result, we choose $\alpha = 2$ as the nominal reference parameter. Using the code NUDOCCCS in Büskens (1998) we obtain the following parametric sensitivity derivatives of the optimal switching times:

$$\frac{dt_1}{d\alpha} = -0.0109557, \quad \frac{dt_2}{d\alpha} = -0.0816061, \quad \frac{dt_3}{d\alpha} = -0.1059373\,.$$

The negative signs of $dt_k/d\alpha$ are quite reasonable since a higher obsolescence rate α would urge an economist to stop full production and to apply full maintenance at an earlier time. The advantage of the numerical approach then lies in the fact that one is able to pass from qualitative decisions to more precise quantitative decisions. Another example is the influence of the cost parameter q in the production cost $q \cdot u$. The sensitivity derivatives at the nominal parameter $q = 4$ are evaluated as

$$\frac{dt_1}{dq} = -0.0134343, \quad \frac{dt_2}{dq} = -0.01841493, \quad \frac{dt_3}{dq} = -0.00849733\,.$$

5. Linear production and maintenance cost and state constraint $y(t) \geq y_{\min}$

As in the preceding section, we assume *linear* production and maintenance cost functions and impose the additional state constraint (17.4),

$$S(y(t)) := y(t) - y_{\min} \geq 0 \quad \forall\, t \in [0, T]. \tag{17.29}$$

For a review of the maximum principles for state-constrained control problems, we refer to Hartl, Sethi and Vickson (1995), Sethi and Thompson (2004). The state constraint (17.29) has *order one*, since the first derivative $\dot{y} = -(\alpha + m)y + m$ contains the control variable m explicitly. We directly adjoin the state constraint to the Hamiltonian H in (17.9) and obtain the augmented Hamiltonian

$$\tilde{H}(x, y, u, m, \lambda_x, \lambda_y, \mu) = H(x, y, u, m, \lambda_x, \lambda_y) + \mu(y - y_{\min})\,, \tag{17.30}$$

with a non-negative multiplier function $\mu \geq 0$. On a *boundary arc* with

$$y(t) = y_{\min} \quad \text{for } t \in I_b \subset [0, T],$$

the *boundary control* $m_b(t)$ is determined via the relation $\dot{y}(t) = 0$ as

$$m_b(t) = \frac{\alpha y_{\min}}{1 - y_{\min}}\,. \tag{17.31}$$

We are looking for feasible boundary controls which lie in the interior of the control set,

$$0 < m_b(t) < M = 4\,, \tag{17.32}$$

which implies $y_{\min} < M/(\alpha + M)$. The adjoint equations are modified on the boundary as follows:

$$\begin{aligned} \dot{\lambda}_x &= \rho\lambda_x - \frac{\partial \tilde{H}}{\partial x} = \rho\lambda_x + h, \\ \dot{\lambda}_y &= \rho\lambda_y - \frac{\partial \tilde{H}}{\partial y} = \lambda_y(\rho + \alpha + m) - \lambda_x u - \mu\,. \end{aligned} \tag{17.33}$$

On interior arcs with $y(t) > y_{\min}$ the controls are shown to be bang-bang. The assumption (17.32) on the boundary control implies that the optimal control m is *discontinuous* at the entry and exit time of a boundary arc. Since the order is equal to one, we infer from Corollary 5.2 (ii) in Maurer (1977) that *the adjoint variables do not have jumps at junctions with the state boundary* and thus are *continuous* across junctions. The multiplier μ is implicitly determined by the fact that the switching function vanishes,

$$\sigma^m(t) = \lambda_y(t)(1 - y(t)) - c \equiv 0 \quad \forall \; t \in I_b \subset [0, T]$$

which follows from assumption (17.32) and the maximization of the Hamiltonian. In view of $y(t) \equiv y_{\min}$, the second adjoint variable is determined by $\lambda_y(t) \equiv c/(1 - y_{\min}) = 25/6$. Hence, we have $\dot{\lambda}_y = 0$ and the second adjoint equation in (17.33) yields the formula

$$\mu = \lambda_y(\rho + \alpha + m_b) - \lambda_x u\,. \tag{17.34}$$

Let us choose the data $T = 1, \alpha = 2$ and $y_{\min} = 0.4$. Then the state inequality constraint (17.29) will become active, since the unconstrained solution $y(t)$ achieves its minimum at $y(t_2) = 0.2366$, cf., Fig. 17.5. The boundary control (17.31) is $m_b(t) \equiv 4/3$. With $N = 5000$ grid points, the optimization package AMPL/LOQO in Fourer, Gay and Kernighan (1993), Vanderbei and Shanno (1999) yields the following control policy with one boundary arc:

$$(u(t), m(t)) = \left\{ \begin{array}{lcll} (U,0) & , & \text{for} & 0 \le t \le t_1 \\ (0,0) & , & \text{for} & t_1 < t \le t_2 \\ (0,m_b) & , & \text{for} & t_2 < t \le t_3 \\ (0,M) & , & \text{for} & t_3 < t \le t_4 \\ (U,M) & , & \text{for} & t_4 < t \le T = 1 \end{array} \right\}. \tag{17.35}$$

The solution data are given in Table 17.5. The state, control and adjoint variables are displayed in Figs. 17.8-17.10. The production control u is zero along the boundary arc. Hence, the multiplier μ in (17.34) is given by $\mu(t) = \lambda_y(t)(\rho + \alpha + m_b) \equiv 515/36 = 14.3056 > 0$.

To show local optimality of this solution, we apply a second order test that generalizes the one for bang-bang controls which was briefly explained in the last section. The bang-bang control problem is transcribed into an optimization problem where the optimization variables are the arc durations $\xi_1 = t_1$, $\xi_2 = t_2 - t_1$, $\xi_3 = t_3 - t_2$, $\xi_4 = t_4 - t_3$ of the bang-bang arcs and the boundary arc. The duration of the last bang-bang arc is given by $T - t_4 = T - (\xi_1 + \xi_2 + \xi_3 + \xi_4)$. We consider the optimization variable $z := (\xi_1, \xi_2, \xi_3, \xi_4)$. The control policy (17.35) defines functions $u(t; z)$, $m(t; z)$ which determine the solution of the ODE

Table 17.5. Solution data for linear cost functions and state constraint $y_{\min} = 0.4$.

Values for	$T = 1$ and $y(t) \geq 0.4$
J	25.49798
$\lambda_x(0)$	6.879734
$\lambda_y(0)$	6.430174
$\lambda_x(T)$	8.654760
t_1	0.308045
t_2	0.458145
t_3	0.753127
t_4	0.813690

system (17.1) as functions $x(t;z)$ and $y(t;z)$. The cost function (17.5) is determined as the function

$$F(z) = F(\xi_1, \xi_2, \xi_3, \xi_4) := J(x(t;z), y(t;z), u(t;z), m(t;z)) \qquad (17.36)$$

which has to be maximized subject to equality constraints that represent the terminal condition and the entry condition for the boundary arc

$$G(z) = G(\xi_1, \xi_2, \xi_3, \xi_4) := (x(T;z), y(t_2;z) - y_{\min}) = (0,0)\,. \qquad (17.37)$$

Computations show that second order sufficient conditions of the form (17.27) hold when we replace the second condition in (17.27) (a) by the regularity condition $rank(G_z(z)) = 2$. Moreover, the switching functions satisfy sign conditions which generalize the strict bang-bang property in (17.23):

$$\begin{aligned} &\dot\sigma^u(t_1) < 0, \quad \dot\sigma^m(t_2-) > 0, \quad \dot\sigma^m(t_3+) > 0, \quad \dot\sigma^u(t_4) > 0, \\ &\sigma^u(t) \neq 0 \quad \text{for} \quad t \neq t_1, t_4, \qquad \sigma^m(t) \neq 0 \quad \text{for} \quad 0 \leq t < t_3,\ t_4 < t \leq T. \end{aligned}$$

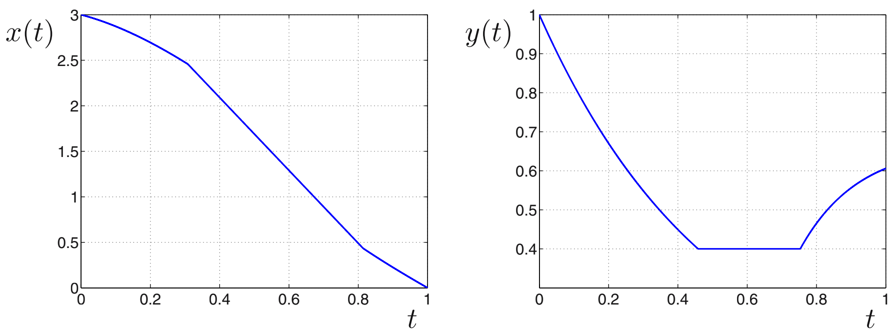

Figure 17.8. State variables $x(t)$ and $y(t)$.

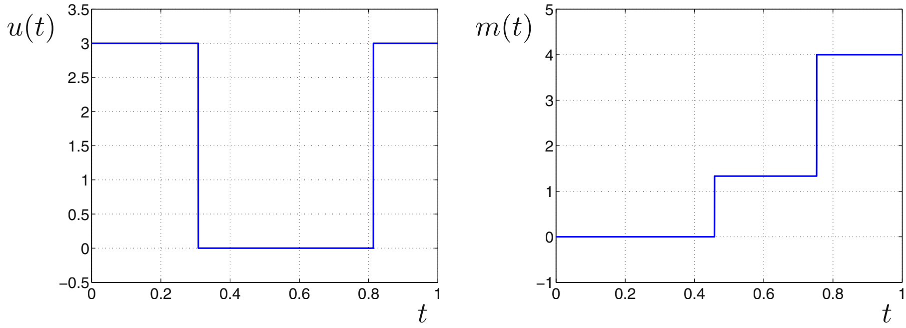

Figure 17.9. Control variables $u(t)$ and $m(t)$.

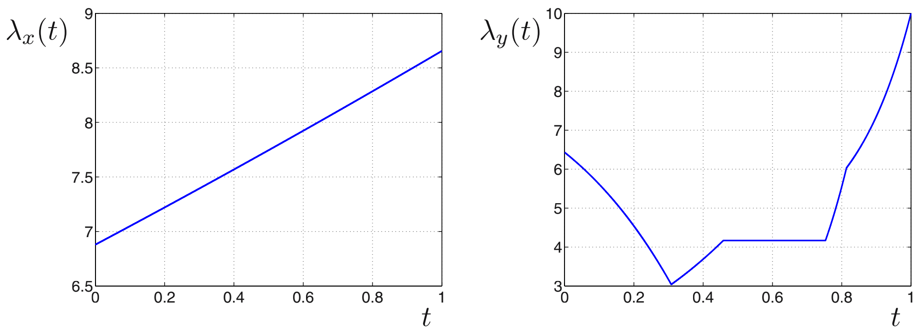

Figure 17.10. Adjoint variables $\lambda_x(t)$ and $\lambda_y(t)$.

6. Conclusion

We have considered a control problem of optimal production and maintenance which was discussed in Cho, Abad and Parlar (1993) for a quadratic production cost and linear maintenance cost function. Two recently developed optimization methods were applied to compute candidate solutions that satisfy the first order necessary conditions in Pontryagin's maximum principle with high accuracy. A surprising fact in this numerical study was that no sufficient conditions could be found that would bear upon the control problem. This is due to the fact that two different types of control functions occur in the problem, the production control being a continuous function while the maintenance control is bang-bang and singular.

In the case where both the production and maintenance cost function are linear, we have found that production and maintenance control are purely bang-bang. We could apply a recent second order test to verify numerically that the computed bang-bang control provides a strong minimum. The second order test is based on an associated optimization problem where the unknown switching times are optimized directly.

Finally, we have added to the control problem a state constraint of practical interest, namely, a lower bound on the number of good items. It is shown that the optimal solution contains one boundary arc while on interior arcs both controls are bang-bang. The second test for bang-bang controls was extended to include boundary arcs. Thus we could again verify numerically that the computed solution provides a strong minimum.

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METHODOLOGICAL ADVANCES

Chapter 18

RELIABILITY INDEX

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Abstract The reliability index is a useful indicator to compute the failure probability. If J is the performance of interest and if J is a Normal random variable, the failure probability is computed by

$$P_f = N(-\beta)$$

and β is the reliability index. When J is a nonlinear function of n normal random variables $(X_1, \ldots, X_n)$, then the preceding formula can be generalized, with some approximation. One uses a nice property of the reliability index, to be the shortest distance of the origin to the failure region. This method introduced by B.M. Ayyub, provides an analytic alternative to the Monte Carlo method.

1. Introduction

The objective of this article is to discuss a method to compute the failure probability (or its opposite the reliability) of an element, subject to several random inputs. It is an analytical approach, which can be compared to the Monte Carlo approach, common in this type of problem. This work relies on the presentation of B M. Ayyub (2003), where the method is introduced. We give a rigorous treatment of the main results.

2. Reliability Assessment

The reliability of an element of a system can be determined based on a performance function. Call J this performance function. It is assumed to be a function $J(X_1, \cdots, X_n)$, where the X_i are random variables. The limit state is when $J = 0$. When $J < 0$, the element is in a failure state, whereas when $J > 0$, it is in the survival state. The failure probability

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is defined as

$$P_f = P(J < 0)$$

Of course,we can write

$$P_f = \int \cdots \int \{J < 0\} f_{X_1,\cdots,X_n}(x_1, \cdots, x_n) dx_1 \cdots dx_n$$

where

$$f_{X_1,\cdots,X_n}(x_1, \cdots, x_n)$$

represents the joint probability density of the random variables. This formula is hardly computable, and one relies naturally of the Monte Carlo approach to estimate it. The reliability index method that we are going to develop is an alternative, which can be less costly than the Monte Carlo approach, especially when the variables are gaussian, or close to gaussian.

3. The case of a linear performance with two inputs

This case permits to understand the essence of the method. We assume that

$$J = S - L$$

If we are in the domain of materials, S represents the structural strength of the material, and L the load effect. If we are in the economic field, then S can represent the supply and L the demand. S, L are random variables. Let μ, σ denote the mean and standard deviation of J. One defines

$$\beta = \frac{\mu}{\sigma}$$

as the reliability index. If J is normally distributed and $N(x)$ represents the cumulative distribution of the standard normal variable, then one has

$$P_f = N(-\beta) = 1 - N(\beta)$$

If S, L are normally distributed, with means μ_S, μ_L, standard deviations, σ_S, σ_L and if they are not correlated, then one has the formula

$$\beta = \frac{\mu_S - \mu_L}{\sqrt{\sigma_S^2 + \sigma_L^2}}$$

We assume $\mu_S - \mu_L > 0$, so the reliability index is positive. Define the reduced variables

$$Y_S = \frac{S - \mu_S}{\sigma_S}, Y_L = \frac{L - \mu_L}{\sigma_L}$$

In a plane of coordinates Y_S, Y_L, the line

$$Y_L = \frac{\sigma_S}{\sigma_L} Y_S + \frac{\mu_S - \mu_L}{\sigma_L}$$

represents the limit state. To the right hand side of this line, we have the survival region, and to the left hand side, we have the failure region. The origin is in the survival region. As easily seen, the reliability index β represents the shortest distance to the failure region.

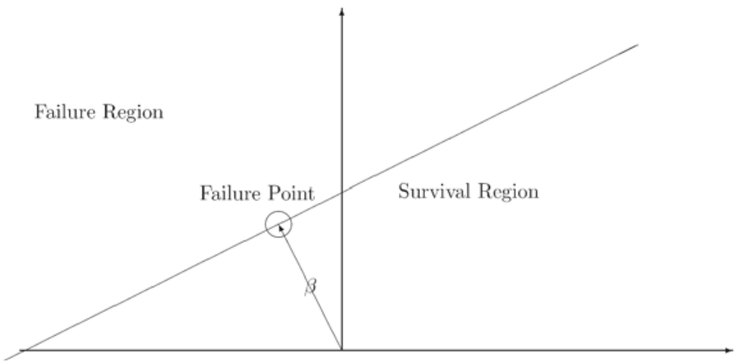

Figure 18.1. Performance Function for a Linear,Two-Random Variable Case in Normalized Coordinates

The point on the limit state line that corresponds to the shortest distance β is called the Failure Point. It is the most likely failure point. This is explained as follows: Since Y_L, Y_S are standard, normal and independent, the probability density

$$P(Y_S = x, Y_L = y | \text{ System has failed}) =$$
$$= \frac{1}{2\pi P_f}\left(\exp -\frac{1}{2}(x^2+y^2)\right) 1\!\!\mathrm{I}_{\{y \geq \frac{x\sigma_S+\mu_S-\mu_L}{\sigma_L}\}}$$

This expression is maximum in x, y when

$$y = \frac{x\sigma_S + \mu_S - \mu_L}{\sigma_L}$$
$$x \text{ minimizes } x^2 + \frac{(x\sigma_S + \mu_S - \mu_L)^2}{\sigma_L^2}$$

The point which is obtained in this way is the failure point.

4. Generalization to Nonlinear Performance Functions

Suppose now that the performance function is given by

$$J(X_1, \cdots, X_n)$$

where the X_i are normal, with mean μ_{X_i} and standard deviation σ_{X_i}. We assume that they are independent. Approximations are possible where they are not gaussian, and when they are gaussian, but correlated. Failure occurs when $J < 0$, and survival when $J \geq 0$. If we consider the reduced variables

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}$$

then we can consider the Y_i as coordinates in an Euclidean space R^n. The region

$$J(\mu_{X_1} + \sigma_{X_1} Y_1, \cdots, \mu_{X_n} + \sigma_{X_n} Y_n) \geq 0$$

is the survival region, and the limit state corresponds to $J = 0$.

The probability density of $Y_1, \cdots, Y_n$ conditional to $J < 0$ is given by

$$\frac{1}{(2\pi)^{\frac{n}{2}} P_f} \left(\exp -\frac{1}{2} \sum_{i=1}^{n} y_i^2 \right) \mathbb{1}_{\{J(\mu_{X_1} + \sigma_{X_1} y_1, \cdots, \mu_{X_n} + \sigma_{X_n} y_n) < 0\}}$$

where P_f is the probability of failure, $P_f = P(J < 0)$. The most likely failure point is obtained by solving the problem

$$\min \sum_{i=1}^{n} y_i^2, \text{ subject to} J(\cdots) \leq 0$$

If J is a convex function, then the failure domain is convex. We assume

$$\mu_J = J(\mu_{X_1}, \cdots, \mu_{X_n}) > 0$$

so that the origin is not in the failure domain. Note that μ_J is not the mathematical expectation of J, but the certainty equivalent. The assumption means that the system is operating correctly, when the random factors are replaced by their means. With this assumption the distance is strictly positive. We denote it by β. Under convexity assumptions,the minimum $\hat{y}_i$ is uniquely defined and is located on the boundary of the failure domain.

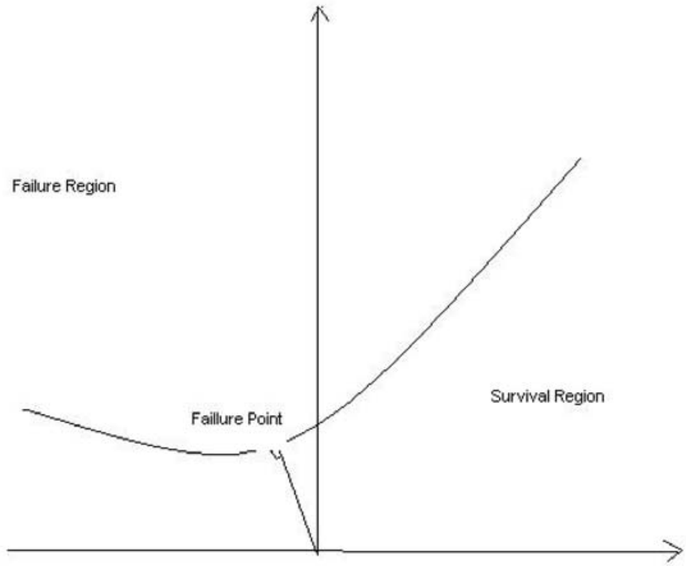

Figure 18.2. Performance function for a Nonlinear Two-Random Variable Case

By Lagrange multiplier theory,the optimal y_i must satisfy the system of equations

$$y_i + \lambda \frac{\partial J}{\partial X_i} \sigma_{X_i} = 0$$

where λ is the Lagrange multiplier. Recalling the definition of β, one has

$$\lambda = \frac{\beta}{\sqrt{\sum_i (\sigma_{X_i} \frac{\partial J}{\partial X_i})^2}}$$

Set

$$\alpha_i = \frac{\frac{\partial J}{\partial X_i} \sigma_{X_i}}{\sqrt{\sum_i (\sigma_{X_i} \frac{\partial J}{\partial X_i})^2}}$$

then

$$y_i = -\beta \alpha_i$$

Considering β as a parameter, the preceding relations form a system of nonlinear equations in $y_1, \cdots, y_n$. Now β is obtained by writing the condition

$$J(\mu_{X_1} - \sigma_{X_1} \beta \alpha_1, \cdots, \mu_{X_n} - \sigma_{X_n} \beta \alpha_n) = 0$$

The next question is to compute $P_f = P(J \leq 0)$. Here we proceed with an approximation. We shall expand J around the point

$$\hat{X}_i = \mu_{X_i} - \sigma_{X_i}\beta\alpha_i$$

instead of expanding it around the mean μ_{X_i}. Writing

$$J(\cdots, \hat{X}_i + X_i - \hat{X}_i, \cdots) \sim \sum_i \frac{\partial J}{\partial X_i}(X_i - \hat{X}_i)$$

We approximate J by a normal random variable with mean $\mu_{X_i} - \hat{X}_i$, and variance

$$(\sigma_J)^2 = \sum_i (\sigma_{X_i}\frac{\partial J}{\partial X_i})^2$$

Writing

$$\mu_J = \mu_{X_i} - \hat{X}_i = \beta\sigma_J$$

we obtain that the reliability index is the same ratio as in the linear case, provided the real mean and standard deviation of J are replaced by the formulas μ_J, σ_J above. The nice thing is that the formula

$$P_f = N(-\beta) = 1 - N(\beta)$$

remains a valid approximation, with the reliability index computed as explained.

If the X_i are not normally distributed, we can apply the preceding procedure, by making first a new approximation, which is to find the equivalent normal distribution. This amounts to finding, for each i two numbers $\mu_{X_i}^N, \sigma_{X_i}^N$, such that

$$N(\frac{\hat{X}_i - \mu_{X_i}^N}{\sigma_{X_i}^N}) = F_i(\hat{X}_i)$$
$$N'(\frac{\hat{X}_i - \mu_{X_i}^N}{\sigma_{X_i}^N}) = F_i'(\hat{X}_i)\sigma_{X_i}^N$$

where $F_i(x)$ denotes the cumulative probability distribution of the variable X_i.

We deduce easily

$$\sigma_{X_i}^N = \frac{N'\left(N^{-1}[F_i(\hat{X}_i)]\right)}{F_i'(\hat{X}_i)}$$

$$\mu_{X_i}^N = \hat{X}_i - N^{-1}[F_i(\hat{X}_i)]\sigma_{X_i}^N$$

Having determined $\mu_{X_i}^N, \sigma_{X_i}^N$ for each variable, one can compute the reliability index as explained above.

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Chapter 19

THE DIRECT METHOD FOR A CLASS OF INFINITE HORIZON DYNAMIC GAMES

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Abstract In this paper we present an extension of a direct solution method, originally due to Leitmann (1967) for single-player games on a finite time interval, to a class of infinite horizon N-player games in which the state equation is affine in the strategies of the players. Our method, based on a coordinate transformation method, gives sufficient conditions for an open-loop Nash equilibrium. An example is presented to illustrate the utility of our results.

1. Introduction

Recently, there has been a series of papers by the authors in which a direct solution method has been used to obtain solutions to open-loop finite-horizon differential games with prescribed two-point boundary conditions. The purpose of this paper is to extend the direct method, alluded to above, to address problems defined on the infinite time horizon. Problems of this type have many important applications in economics and consequently the extension of the direct method to these types of models will significantly enlarge the class of applied problems to which this method can be utilized.

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2. The general model considered

We consider an N-player game in which the dynamics of the j-th player, $j = 1, 2, \ldots, N$, at time $t \geq t_0$ is denoted by $\mathbf{x}_j(t)$, generated by control $u_j(t)$,and satisfies an affine control system of the form

$$\dot{\mathbf{x}}_j(t) = F_j(t, \mathbf{x}(t)) + G_j(t, \mathbf{x}(t))u_j(t), \quad \text{a.e.} \quad t_0 \leq t \tag{19.1}$$

with fixed initial condition,

$$\mathbf{x}_j(t_0) = \mathbf{x}_{0j}, \tag{19.2}$$

control constraints

$$u_j(t) \in U_j(t) \subset \mathbb{R}^{m_j}, \quad \text{a.e.} \quad t_0 \leq t \tag{19.3}$$

and state constraints

$$\mathbf{x}_j(t) \in X_j(t) \subset \mathbb{R}^{n_j}, \quad \text{for} \quad t_0 \leq t \tag{19.4}$$

We assume here that for each $j = 1, 2, \ldots, N$ $\mathbf{x}_j(\cdot) : [t_0, +\infty) \to \mathbb{R}^{n_j}$, $u_j(\cdot) : [t_0, +\infty) \to \mathbb{R}^{m_j}$, and $\mathbf{x}(\cdot) \doteq (\mathbf{x}_1(\cdot), \mathbf{x}_2(\cdot), \ldots, \mathbf{x}_N(\cdot)) : [t_0, +\infty) \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_N} = \mathbb{R}^{\mathbf{n}}$. The functions $F_j(\cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{n_j}$ and $G_j(\cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{n_j \times m_j}$ are continuous for each $j = 1, 2, \ldots, N$ and also for each $t \in [t_0, +\infty)$ sets $X_j(t)$ and $U_j(t)$ are convex subsets of $\mathbb{R}^{n_j}$ and $\mathbb{R}^{m_j}$. Additionally, we assume that each matrix, $G_j(t, \mathbf{x})$, for $(t, \mathbf{x}) \in [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}}$, has a left inverse $G_j(t, \mathbf{x})^{-1}$ that is also continuous.

The objective of each player is to minimize a performance criterion of the form,

$$J_j(\mathbf{x}(\cdot), u_j(\cdot)) = \int_{t_0}^{+\infty} F_j^0(t, \mathbf{x}(t), u_j(t))\, dt \tag{19.5}$$

in which $F_j^0(\cdot, \cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{m_j} \to \mathbb{R}$ is assumed to be continuous. Clearly, it is unreasonable to expect that each player will be able to minimize his/her performance and consequently we seek a Nash equilibrium. To define this we have the following definitions and notation.

Definition 1 We say a pair of functions $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\} : [t_0, +\infty) \to \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{m}}$ is an admissible trajectory-strategy pair if and only if $t \to \mathbf{x}(t)$ is locally absolutely continuous on $[t_0, +\infty)$ (i.e., it is absolutely continuous on each compact subinterval of $[t_0, +\infty)$), $t \to \mathbf{u}(t)$ is Lebesgue measurable on $[t_0, +\infty)$, for each $j = 1, 2, \ldots, N$, the relations (19.1)–(19.4) are satisfied, and for each $j = 1, 2, \ldots, N$, the functionals (19.5) are finite Lebesgue integrals.

REMARK 1 For brevity we will refer to an admissible trajectory-strategy pair as an admissible pair. Also, for a given admissible pair, $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$, we will follow the traditional convention and refer to $\mathbf{x}(\cdot)$ as an admissible trajectory and $\mathbf{u}(\cdot)$ as an admissible strategy.

For a fixed $j = 1, 2, \dots, N$, $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ and $y_j \in \mathbb{R}^{n_j}$, we use the notation $[\mathbf{x}^j, y_j]$ to denote a new vector in $\mathbb{R}^{\mathbf{n}}$ in which $\mathbf{x}_j \in \mathbb{R}^{n_j}$ is replaced by $y_j \in \mathbb{R}^{n_j}$. That is,

$$[\mathbf{x}^j, y_j] \doteq (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{j-1}, y_j, \mathbf{x}_{j+1}, \dots, \mathbf{x}_N).$$

Analogously $[\mathbf{u}^j, v_j] \doteq (u_1, u_2, \dots, u_{j-1}, v_j, u_{j+1}, \dots, u_N)$ for all $\mathbf{u} \in \mathbb{R}^{\mathbf{m}}$, $v_j \in \mathbb{R}^{m_j}$. With this notation we now have the following two definitions.

DEFINITION 2 Let $j = 1, 2, \dots, N$ be fixed and let $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ be an admissible pair. We say that a pair of functions $\{y_j(\cdot), v_j(\cdot)\} : [t_0, +\infty) \to \mathbb{R}^{n_j} \times \mathbb{R}^{m_j}$ is an admissible trajectory-strategy pair for player j relative to $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ if and only if the pair

$$\{[\mathbf{x}(\cdot)^j, y_j(\cdot)], [\mathbf{u}(\cdot)^j, v_j(\cdot)]\}$$

is an admissible pair.

DEFINITION 3 An admissible pair $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$ is a Nash equilibrium if and only if for each $j = 1, 2, \dots, N$ and each pair $\{y_j(\cdot), v_j(\cdot)\}$ that is admissible for player j relative to $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$,

$$\begin{aligned} J_j(\mathbf{x}^*(\cdot), u_j^*(\cdot)) &= \int_{t_0}^{+\infty} F_j^0(t, \mathbf{x}^*(t), u_j^*(t))\, dt \\ &\leq \int_{t_0}^{+\infty} F_j^0(t, [\mathbf{x}^*(t)^j, y_j(t)], v_j(t))\, dt \\ &= J_j([\mathbf{x}^*(\cdot)^j, y_j(\cdot)], v_j(\cdot)). \end{aligned}$$

Our goal in this paper is to provide a "direct method" which in some cases will enable us to determine a Nash equilibrium. We point out that relative to a fixed Nash equilibrium $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$ each of the players in the above game solves an optimization problem taking the form of a standard problem of optimal control. Thus, under suitable additional assumptions, it is relatively easy to derive a set of necessary conditions (in the form of a Pontryagin-type maximum principle) that must be satisfied by all Nash equilibria. Unfortunately these conditions are only necessary and not sufficient. Further, it is well known that non-uniqueness is always a source of difficulty in dynamic games so that in general the necessary conditions are not uniquely solvable (as is often the case in optimal control theory, when sufficient convexity is imposed). Therefore it is important to be able to find usable sufficient conditions for Nash equilibria.

2.1 The associated variational game.

We observe that, under our assumptions, the algebraic equations,

$$z_j = F_j(t, \mathbf{x}) + G_j(t, \mathbf{x})u_j \quad j = 1, 2, \ldots N, \tag{19.6}$$

can be solved for u_j in terms of t, z_j, and $\mathbf{x}$ to obtain

$$u_j = G_j(t, \mathbf{x})^{-1} \left(z_j - F_j(t, \mathbf{x}) \right), \quad j = 1, 2, \ldots N. \tag{19.7}$$

As a consequence we can define the extended real-valued functions $L_j(\cdot, \cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{n_j} \to \mathbb{R} \cup +\infty$ as

$$L_j(t, \mathbf{x}, z_j) = \begin{cases} F_j^0(t, \mathbf{x}, G_j(t, \mathbf{x})^{-1}(z_j - F_j(t, \mathbf{x}))) & \text{if } \mathbf{x} \in \mathbf{X}(t) \\ & \text{and } G_j(t, \mathbf{x})^{-1} \left(z_j - F_j(t, \mathbf{x}) \right) \in U_j(t) \\ +\infty & \text{otherwise.} \end{cases} \tag{19.8}$$

With these functions we can consider the N-player variational game in which the objective functional for the jth player is defined by

$$I_j(\mathbf{x}(\cdot)) = \int_{t_0}^{t_f} L_j(t, \mathbf{x}(t), \dot{x}_j(t))\, dt. \tag{19.9}$$

With this notation we have the following additional definitions.

Definition 4 A locally absolutely continuous n-vector valued function $\mathbf{x}(\cdot) : [t_0, +\infty) \to \mathbb{R}^{\mathbf{n}}$ is said to be admissible for the variational game if and only if it satisfies the initial conditions given in equation (19.2) and such that the map $t \to L_j(t, \mathbf{x}(t), \dot{\mathbf{x}}_j(t))$ is finitely Lebesgue integrable on $[t_0, +\infty)$ for each $j = 1, 2, \ldots, N$.

Definition 5 Let $\mathbf{x}(\cdot) : [t_0, +\infty) \to \mathbb{R}^{\mathbf{n}}$ be admissible for the variational game and let $j \in \{1, 2, \ldots, N\}$ be fixed. We say that $y_j(\cdot) : [t_0, +\infty) \to \mathbb{R}^{n_j}$ is admissible for player j relative to $\mathbf{x}(\cdot)$ if and only if $[\mathbf{x}^j(\cdot), y_j(\cdot)]$ is admissible for the variational game.

Definition 6 We say that $\mathbf{x}^*(\cdot) : [t_0, +\infty) \to \mathbb{R}^{\mathbf{n}}$ is a Nash equilibrium for the variational game if and only if for each $j = 1, 2, \ldots, N$

$$I_j(\mathbf{x}^*(\cdot)) \leq I_j([\mathbf{x}^{*j}(\cdot), y_j(\cdot)])$$

for all functions $y_j(\cdot) : [t_0, +\infty) \to \mathbb{R}^{n_j}$ that are admissible for player j relative to $\mathbf{x}^*(\cdot)$.

Clearly the variational game and our original game are related. In particular we have the following theorem given in Carlson and Leitmann (2004).

THEOREM 1 Let $\mathbf{x}^*(\cdot)$ be a Nash equilibrium for the variational game defined above. Then there exists a function $\mathbf{u}^*(\cdot) : [t_0, +\infty) \to \mathbb{R}^{\mathbf{m}}$ such that the pair $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$ is an admissible trajectory-strategy pair for the original dynamic game. Moreover, it is a Nash equilibrium for the original game as well.

Proof:. See Carlson and Leitmann (2004), Theorem 7.1. □

REMARK 2 The above result holds in a much more general setting than indicated above; see Carlson and Leitmann (2005). We chose the restricted setting since it is sufficient for our needs in the analysis of the model we will consider in the next section.

With the above result we now focus our attention on the variational game. In 1967, for the case of one player variational games (i.e., the calculus of variations), Leitmann (1967) and later Leitmann (2001) presented a technique (the "direct method") for determining solutions of these games by comparing their solutions to that of an equivalent problem whose solution is more easily determined than that of the original. This equivalence was obtained through a coordinate transformation. Since then this method has been used successfully to solve a variety of problems. Recently, Carlson (2002) and in Leitmann (2004) an extension of this method was presented that expands the utility of the approach and also made a useful comparison with a technique originally given by Carathéodory (1982) in the early twentieth century. Also, Dockner and Leitmann (2001) extended the original direct method to include the case of open-loop dynamic games. Finally, the extension of Carlson to the method was also modified in Leitmann (2004) to the include the case of open-loop differential games in Carlson and Leitmann (2004). All of these results are applied to problems over a finite linterval with both fixed initial and fixed terminal conditions. We now present a modified version of the basic lemma for open-loop dynamic games which enables us to consider the problem addressed here, based on an infinite horizon result of Leitmann (2001a).

LEMMA 1 For $j = 1, 2, \ldots, N$ let $\mathbf{x}_j = z_j(t, \tilde{\mathbf{x}}_j)$ be a transformation of class C^1 having a unique inverse $\tilde{\mathbf{x}}_j = \tilde{z}_j(t, \mathbf{x}_j)$ for all $t \in [t_0, +\infty)$ such that there is a one-to-one correspondence $\mathbf{x}(t) \Leftrightarrow \tilde{\mathbf{x}}(t)$, for all admissible trajectories $\mathbf{x}(\cdot)$ satisfying the initial conditions (19.2) and for all $\tilde{\mathbf{x}}(\cdot)$ satisfying

$$\tilde{\mathbf{x}}_j(t_0) = \tilde{z}_j(t_0, \mathbf{x}_{0j})$$

for all $j = 1, 2, \ldots, N$. Further for each $j = 1, 2, \ldots, N$ let $\tilde{L}_j(\cdot, \cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{n_j} \to \mathbb{R}$ be a given integrand. For a given admissible

$\mathbf{x}^*(\cdot) : [t_0, +\infty) \to \mathbb{R}^{\hat{n}}$ suppose the transformations $\mathbf{x}_j = z_j(t, \tilde{\mathbf{x}}_j)$ are such that there exists a C^1 function $H_j(\cdot,\cdot) : [t_0, +\infty) \times \mathbb{R}^{n_j} \to \mathbb{R}$ so that the functional identity

$$L_j(t, [\mathbf{x}^{*j}(t), \mathbf{x}_j(t)], \dot{\mathbf{x}}_j(t)) - \tilde{L}_j(t, [\mathbf{x}^{*j}(t), \tilde{\mathbf{x}}_j(t)], \dot{\tilde{\mathbf{x}}}_j(t)) = \frac{d}{dt} H_j(t, \tilde{x}_j(t)) \qquad (19.10)$$

holds on $[t_0, +\infty)$ and such that $\lim_{t_f \to +\infty} H_j(t_f, \tilde{x}_j(t_f)) = 0$ for $j = 1, 2, \ldots, N$. If $\tilde{\mathbf{x}}_j^*(\cdot)$ yields an extremum of $\tilde{I}_j([\mathbf{x}^{*j}(\cdot), \cdot])$ with $\tilde{\mathbf{x}}_j^*(\cdot)$ satisfying the transformed boundary conditions, then $\mathbf{x}_j^*(\cdot)$ with $\mathbf{x}_j^*(t) = z_j(t, \tilde{\mathbf{x}}^*(t))$ yields an extremum for $I_j([\mathbf{x}^{*j}(\cdot), \cdot])$ with the initial conditions (19.2).

Moreover, the function $\mathbf{x}^*(\cdot)$ is an open-loop Nash equilibrium for the variational game.

Proof.. To prove this result we fix $j = 1, 2, \ldots, N$ and observe that if $\tilde{\mathbf{x}}_j^*(\cdot)$ is a minimizer of $\tilde{I}_j([\mathbf{x}^{*j}(\cdot), \cdot])$ then for any admissible $\tilde{y}(\cdot) : [t_0, +\infty) \to \mathbb{R}^{n_j}$ that satisfies the initial condition

$$\tilde{y}(0) = z_j(0, \mathbf{x}_{0j})$$

we have as a consequence of the functional identity and the optimality of $\mathbf{x}_j^*(\cdot)$ that

$$\begin{aligned}
\tilde{I}_j([\mathbf{x}^{*j}(\cdot), \tilde{\mathbf{x}}_j^*(\cdot)]) &= \lim_{t_f \to +\infty} \int_{t_0}^{t_f} \tilde{L}_j(t, [\mathbf{x}^{*j}(t), \tilde{\mathbf{x}}_j^*(t)], \dot{\tilde{\mathbf{x}}}_j^*(t))\, dt \\
&\leq \tilde{I}_j([\mathbf{x}^{*j}(\cdot), \tilde{y}(\cdot)]) \\
&= \lim_{t_f \to +\infty} \int_{t_0}^{t_f} \tilde{L}_j(t, [\mathbf{x}^{*j}(t), \tilde{y}(t)], \dot{\tilde{y}}(t))\, dt \\
&= \lim_{t_f \to +\infty} \int_{t_0}^{t_f} L_j(t, [\mathbf{x}^*(t), z_j(t, \tilde{y}(t))], \frac{d}{dt} z_j(t, \tilde{y}(t)))\, dt \\
&\quad - \lim_{t_f \to +\infty} [H_j(t_f, \tilde{y}(t_f))] + H_j(t_0, \tilde{y}(t_0)) \\
&= I_j([\mathbf{x}^*(\cdot), z_j(\cdot, \tilde{y}(\cdot))] + H_j(t_0, z_j(t_0, \mathbf{x}_{0j})),
\end{aligned}$$

where we have used the fact that $\lim_{t_f \to +\infty} H_j(t_f, \tilde{y}(t_f)) = 0$. As a result of the functional identity we also have

$$I_j([\mathbf{x}^{*j}(\cdot), z_j(\cdot, \tilde{\mathbf{x}}_j^*(\cdot))]) + H_j(t_0, z_j(t_0, \mathbf{x}_{0j})) = \tilde{I}_j([\mathbf{x}^{*j}(\cdot), \tilde{\mathbf{x}}_j^*(\cdot)])$$

implying that

$$I_j([\mathbf{x}^{*j}(\cdot), z_j(\cdot, \tilde{\mathbf{x}}_j^*(\cdot))]) \leq I_j([\mathbf{x}^*(\cdot), z_j(\cdot, \tilde{y}(\cdot))]).$$

From this, as a consequence of the one-to-one correspondence between the two sets of admissible trajectories, it follows that $\mathbf{x}_j^*(\cdot) = z_j(\cdot, \tilde{\mathbf{x}}_j^*(\cdot))$ is a minimizer for $I_j([\mathbf{x}^{*j}(\cdot), \cdot])$ over all trajectories meeting the prescribed initial conditions. Further, since this holds for each j it is easy to see that $\mathbf{x}^*(\cdot)$ is an open-loop Nash equilibrium for the variational game. □

Under assumptions of sufficient smoothness for the functions $H_j(\cdot, \cdot)$ we have the following useful and immediate corollaries. For more details we refer the reader to Carlson and Leitmann (2004).

Corollary 1 The existence of $H_j(\cdot, \cdot)$ in (19.10) implies that the following identities hold for $(t, \tilde{x}_j, \tilde{p}_j) \in (t_0, +\infty) \times \mathbb{R}^{n_j} \times R^{n_j}$ and for $j = 1, 2, \ldots, N$:

$$\begin{aligned} L_j(t, [\mathbf{x}^{*j}(t), z_j(t, \tilde{x}_j)], \frac{\partial z_j(t, \tilde{x}_j))}{\partial t} &+ \langle \nabla_{\tilde{x}_j} z_j(t, \tilde{x}_j), \tilde{p}_j \rangle) \\ &- \tilde{L}_j(t, [\mathbf{x}^{*j}(t), \tilde{x}_j], \tilde{p}_j) \\ \equiv \frac{\partial H_j(t, \tilde{x}_j)}{\partial t} &+ \langle \nabla_{\tilde{x}_j} H_j(t, \tilde{x}_j), \tilde{p}_j \rangle, \end{aligned} \tag{19.11}$$

in which $\nabla_{\tilde{x}_j} H_j(\cdot, \cdot)$ denotes the gradient of $H_j(\cdot, \cdot)$ with respect to the variables x_j and $\langle \cdot, \cdot \rangle$ denotes the usual scalar or inner product.

Corollary 2 For each $j = 1, 2, \ldots, N$ the left-hand side of the identity, (19.11) is linear in $\tilde{p}_j$, that is, it is of the form,

$$\theta_j(t, \tilde{x}_j) + \langle \psi_j(t, \tilde{x}_j), \tilde{p}_j \rangle$$

and,

$$\frac{\partial H_j(t, \tilde{x}_j)}{\partial t} = \theta_j(t, \tilde{x}_j) \quad \text{and} \quad \nabla_{\tilde{x}_j} H_j(t, \tilde{x}_j) = \psi(t, \tilde{x}_j)$$

on $[t_0, +\infty) \times \mathbb{R}^{n_j}$.

Corollary 3 For integrands $L_j(\cdot, \cdot, \cdot)$ of the form,

$$\begin{aligned} L_j(t, [\mathbf{x}^{*j}(t), x_j(t)], \dot{x}_j(t)) =& \dot{x}_j'(t) a_j(t, [\mathbf{x}^{*j}(t), x_j(t)]) \dot{x}_j(t) \\ &+ b_j(t, [\mathbf{x}^{*j}(t), x_j(t)])' \dot{x}_j(t) + c_j(t, [\mathbf{x}^{*j}(t), x_j(t)]), \end{aligned}$$

and

$$\begin{aligned} \tilde{L}_j(t, [\mathbf{x}^{*j}(t), x_j(t)], \dot{x}_j(t)) =& \dot{x}_j'(t) \alpha_j(t, [\mathbf{x}^{*j}(t), x_j(t)]) \dot{x}_j(t) \\ &+ \beta_j(t, [\mathbf{x}^{*j}(t), x_j(t)])' \dot{x}_j(t) + \gamma_j(t, [\mathbf{x}^{*j}(t), x_j(t)]), \end{aligned}$$

with $a_j(t, [\mathbf{x}^{*j}(t), x_j(t)]) \neq 0$ and $\alpha_j(t, [\mathbf{x}^{*j}(t), x_j(t)]) \neq 0$, the class of transformations that permit us to obtain equation (19.11) must satisfy,

$$\left[\frac{\partial z_j(t, \tilde{x}_j)}{\partial \tilde{x}_j}\right]' a_j(t, [\mathbf{x}^*(t)^j, z_j(t, \tilde{x}_j)]) \left[\frac{\partial z_j(t, \tilde{x}_j)}{\partial \tilde{x}_j}\right] = \alpha_j(t, [\mathbf{x}^*(t)^j, \tilde{x}_j])$$

for $(t, x_j) \in [t_0, +\infty) \times \mathbb{R}^{n_j}$.

We conclude this section with a simple example for which the solution can be obtained using the above theory.

Example 1. We consider two firms which produce an identical product. The production cost for each firm is given by the total cost function,

$$C(u_j) = \frac{1}{2}u_j^2, \quad j = 1, 2,$$

in which u_j refers to a jth firm's production level. Each firm supplies all that it produces to the market at all times. The amount supplied at each time effects the price, $P(t)$ and the total inventory at the market determines the price according to the ordinary control system,

$$\dot{P}(t) = s[a - u_1(t) - u_2(t) - P(t)] \quad \text{a.e.} \quad t \in [t_0, +\infty). \tag{19.12}$$

Here $s > 0$ refers to the speed at which the price adjusts to the price corresponding to the total quantity (i.e., $u_1(t)+u_2(t)$)and $a > 0$ is a fixed constant related to the linear demand function. The model assumes a linear demand rate given by $\Pi = a - X$ where X denotes total supply related to a price Π. Thus the dynamics above says that the rate of change of price at time t is proportional to the difference between the actual price $P(t)$ and the idealized price $\Pi(t) = a - u_1(t) - u_2(t)$. We assume that the initial price is given. This leads to the initial condition,

$$P(t_0) = P_0. \tag{19.13}$$

Additionally we also impose the constraints

$$u_j(t) \geq 0 \quad \text{for almost all} \quad t \in [t_0, +\infty). \tag{19.14}$$

and

$$P(t) \geq 0 \quad \text{for} \quad t \in [t_0, +\infty). \tag{19.15}$$

The goal of each firm is to maximize its accumulated profit, assuming that it sells all that it produces, over the interval, $[t_0, +\infty)$ given by the integral functional,

$$J_j(P(\cdot), u_j(\cdot)) = \int_{t_0}^{+\infty} e^{-\delta t} \left[P(t)u_j(t) - \frac{1}{2}u_j^2(t)\right] dt, \tag{19.16}$$

in which $\delta > 0$ is a fixed positive discount rate. To put the above dynamic game into the framework to use the direct method let $\alpha, \beta > 0$ satisfy $\alpha+\beta = 1$ and consider the ordinary 2-dimensional control system,

$$\dot{x}(t) = -\, s(\alpha x(t) + \beta y(t) - a) - \frac{s}{\alpha} u_1(t), \quad \text{a.e. } t_0 \leq t \tag{19.17}$$

$$\dot{y}(t) = -\, s(\alpha x(t) + \beta y(t) - a) - \frac{s}{\beta} u_2(t), \quad \text{a.e. } t_0 \leq t \tag{19.18}$$

with the initial conditions,

$$x(t_0) = y(t_0) = P_0 \tag{19.19}$$

For the remainder of our discussion we focus on the first player as the computation of the second player is the same. We begin by observing that the integrand for player 1 is

$$\begin{aligned} L_1(x, y, p) = e^{-\delta t} \Bigg\{ & \frac{\alpha^2}{2s^2} p^2 + \frac{\alpha^2 a^2}{2} + \left(\frac{\alpha^2}{2} + \alpha \right) (\alpha x + \beta y)^2 \\ & + \left[\frac{\alpha}{s} (\alpha x + \beta y) - \frac{\alpha^2}{s} (a - (\alpha x + \beta y)) \right] p \\ & - a(\alpha^2 + \alpha)(\alpha x + \beta y) \Bigg\}. \end{aligned} \tag{19.20}$$

Inspecting this integrand we choose $\tilde{L}(\cdot, \cdot, \cdot)$ to be,

$$\tilde{L}(\tilde{x}, \tilde{y}, \tilde{p}) = e^{-\delta t} \left[\frac{\alpha^2}{2s^2} \tilde{p}^2 + \frac{\alpha^2 a^2}{2} \right]$$

from which we immediately deduce, applying Corollary 3, that the appropriate transformation, $z_1(\cdot, \cdot)$, must satisfy the partial differential equation

$$\left(\frac{\partial z_1}{\partial \tilde{x}} \right)^2 = 1$$

giving us that $z_1(t, \tilde{x}) = f(t) \pm \tilde{x}$ and that

$$\frac{\partial z_1}{\partial t} + \frac{\partial z_1}{\partial \tilde{x}} \tilde{p} = \dot{f}(t) \pm \tilde{p}.$$

From this we now compute,

$$
\begin{aligned}
&L_1(f(t)\pm\tilde{x}, y^*(t), \dot{f}(t)\pm\tilde{p}) - \tilde{L}(\tilde{x}, y^*(t), \tilde{p})\\
&\quad = e^{-\delta t}\left\{\frac{\alpha^2}{2s^2}(\dot{f}(t)\pm\tilde{p})^2 + \frac{\alpha^2 a^2}{2}\right.\\
&\qquad + \left(\frac{\alpha^2}{2}+\alpha\right)(\alpha(f(t)\pm\tilde{x}) + \beta y^*(t))^2\\
&\qquad + \left[\frac{\alpha}{s}(\alpha(f(t)\pm\tilde{x}) + \beta y^*(t))\right.\\
&\qquad - \left.\frac{\alpha^2}{s}(a - (\alpha(f(t)\pm\tilde{x}) + \beta y^*(t)))\right](\dot{f}(t)\pm\tilde{p})\\
&\qquad \left. -a(\alpha^2+\alpha)(\alpha(f(t)\pm\tilde{x}) + \beta y^*(t))\right\}\\
&\qquad - e^{-\delta t}\left\{\frac{\alpha^2}{2s^2}\tilde{p}^2 + \frac{\alpha^2 a^2}{2}\right\}\\
&\quad = e^{-\delta t}\left\{\frac{\alpha^2}{2s^2}\dot{f}(t)^2 + \left(\frac{\alpha^2}{2}+\alpha\right)[\alpha(f(t)\pm\tilde{x}) + \beta y^*(t)]^2\right.\\
&\qquad - a\left(\alpha^2+\alpha\right)[\alpha(f(t)\pm\tilde{x}) + \beta y^*(t)]\\
&\qquad \left. + \left[\left(\frac{\alpha^2}{s}+\frac{\alpha}{s}\right)[\alpha(f(t)\pm\tilde{x}) + \beta y^*(t)] - \frac{\alpha^2 a}{s}\right]\dot{f}(t)\right\}\\
&\qquad \pm e^{-\delta t}\left\{\frac{\alpha^2}{s^2}\dot{f}(t) + \left(\frac{\alpha^2}{s}+\frac{\alpha}{s}\right)[\alpha(f(t)\pm\tilde{x}) + \beta y^*(t)] - \frac{\alpha^2 a}{s}\right\}\tilde{p}\\
&\quad \doteq \frac{\partial H_1(t,\tilde{x})}{\partial t} + \frac{\partial H_1(t,\tilde{x})}{\partial \tilde{x}}\tilde{p}.
\end{aligned}
$$

From this we compute the mixed partial derivatives to obtain,

$$
\begin{aligned}
\frac{\partial^2 H_1}{\partial\tilde{x}\partial t}(t,\tilde{x}) &= e^{-\delta t}\left[\pm 2\left(\frac{\alpha^2}{2}+\alpha\right)[\alpha(f(t)\pm\tilde{x}) + \beta y^*(t)]\,\alpha\right.\\
&\qquad\qquad \left.\mp a\alpha(\alpha^2+\alpha) \pm \alpha\left(\frac{\alpha^2}{s}+\frac{\alpha}{s}\right)\dot{f}(t)\right]\\
&= \pm e^{-\delta t}\left[\alpha^3(\alpha+2)(f(t)\pm\tilde{x}) + \alpha^2\beta(\alpha+2)y^*(t)\right.\\
&\qquad\qquad \left. -\alpha^2(\alpha+1)a + \frac{\alpha^2}{s}(\alpha+1)\dot{f}(t)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 H_1}{\partial t \partial \tilde{x}}(t, \tilde{x}) = & \pm e^{-\delta t} \left\{ \frac{\alpha^2}{s^2} \ddot{f}(t) + \left(\frac{\alpha^2}{s} + \frac{\alpha}{s} \right) \left[\alpha \dot{f}(t) + \beta \dot{y}^*(t) \right] \right\} \\
& \mp \delta e^{-\delta t} \left\{ \frac{\alpha^2}{s^2} \dot{f}(t) + \frac{\alpha(\alpha+1)}{s} [\alpha(f(t) \pm \tilde{x}) \right. \\
& \left. + \beta y^*(t)] - \frac{\alpha^2 a}{s} \right\} \\
= & \pm e^{-\delta t} \left\{ \frac{\alpha^2}{s^2} \ddot{f}(t) + \frac{\alpha^2}{s}(\alpha+1)\dot{f}(t) + \frac{\alpha\beta}{s}(\alpha+1)\dot{y}^*(t) \right. \\
& - \frac{\alpha^2 \delta}{s} \dot{f}(t) - \frac{\delta \alpha^2}{s}(\alpha+1) f(t) \\
& \left. \mp \frac{\delta \alpha^2}{s}(\alpha+1)\tilde{x} - \frac{\delta \alpha \beta}{s}(\alpha+1) y^*(t) + \frac{\alpha^2 \delta}{s} a \right\}.
\end{aligned}
$$

Assuming sufficient smoothness and equating the mixed partial derivatives we obtain the following equation:

$$
\ddot{f}(t) - \delta \dot{f}(t) - (\alpha s^2(\alpha+2) + \delta s(\alpha+1)) f(t) = h_1(t, \tilde{x})
$$

where

$$
\begin{aligned}
h_1(t, \tilde{x}) = & (\beta s^2(\alpha+2) + \frac{\delta \beta s}{\alpha}) y^*(t) - \frac{\beta s}{\alpha}(\alpha+1)\dot{y}^*(t) \\
& \pm (\alpha s^2(\alpha+2) - \delta s(\alpha+1))\tilde{x} - a(s^2(\alpha+1) + \delta s).
\end{aligned}
$$

A similar analysis for player 2 yields:

$$
\begin{aligned}
L_2(x, y, q) = & e^{-\delta t} \left\{ \frac{\beta^2}{2s^2} q^2 + \frac{\beta^2 a^2}{2} + \left(\frac{\beta^2}{2} + \beta \right) (\alpha x + \beta y)^2 \right. \\
& \left[\frac{\beta}{s}(\alpha x + \beta y) - \frac{\beta^2}{s}(a - (\alpha x + \beta y) \right] q \qquad (19.21) \\
& \left. - a(\beta^2 + \beta)(\alpha x + \beta y) \right\},
\end{aligned}
$$

and so choosing

$$
\tilde{L}_2(\tilde{x}, \tilde{y}, \tilde{q}) = e^{-\delta t} \left\{ \frac{\beta^2}{2s^2} q^2 + \frac{\beta^2 a^2}{2} \right\}
$$

gives us that the transformation $z_2(\cdot, \cdot)$ is obtained by solving the partial differential equation

$$
\left(\frac{\partial z_2}{\partial \tilde{y}} \right)^2 = 1,
$$

which of course gives us, $z_2(t, \tilde{y}) = g(t) \pm \tilde{y}$. Proceeding as above we arrive at the following differential equation for $g(\cdot)$,

$$\ddot{g}(t) - \delta\dot{g}(t) - (\beta s^2(\beta + 2) + \delta s(\beta + 1))g(t) = h_2(t, \tilde{y})$$

where

$$\begin{aligned} h_2(t, \tilde{y}) =&(\alpha s^2(\beta + 2) + \frac{\delta\alpha s}{\beta})x^*(t) - \frac{\alpha s}{\beta}(1 + \beta)\dot{x}^*(t) \\ &\pm (\beta s^2(\beta + 2) - \delta s(\beta + 1))\tilde{y} - a(s^2(\beta + 1) + \delta s). \end{aligned}$$

Now the auxiliary variational problem we must solve consists of minimizing the two functionals,

$$\int_{t_0}^{+\infty} e^{-\delta t}\left(\frac{\alpha^2}{2s^2}\dot{\tilde{x}}^2(t) + \frac{\alpha a^2}{2}\right) dt \text{ and } \int_{t_0}^{+\infty} e^{-\delta t}\left(\frac{\beta^2}{2s^2}\dot{\tilde{y}}^2(t) + \frac{\beta a^2}{2}\right) dt$$

over some appropriately chosen initial conditions. We observe that these two minimization problems are easily solved if these conditions take the form,

$$\tilde{x}(t_0) = c_1 \quad \text{and} \quad \tilde{y}(t_0) = c_2$$

for arbitrary but fixed constants c_1 and c_2. The solutions are in fact,

$$\tilde{x}^*(t) \equiv c_1 \quad \text{and} \quad \tilde{y}^*(t) \equiv c_2$$

According to our theory we then have that the solution to our variational game is,

$$x^*(t) = f(t) \pm c_1 \quad \text{and} \quad y^*(t) = g(t) \pm c_2.$$

In particular, using this information in the equations for $f(\cdot)$ and $g(\cdot)$ with $\tilde{x} = c_1$ and with $\tilde{y} = c_2$ we obtain the following equations for $x^*(\cdot)$ and $y^*(\cdot)$,

$$\begin{aligned} \ddot{x}^*(t) - \delta\dot{x}^*(t) - [\alpha s^2(\alpha + 2) + \delta s(\alpha + 1)]x^*(t) =&h_1(t, 0) \\ \ddot{y}^*(t) - \delta\dot{y}^*(t) - [\beta s^2(\beta + 2) + \delta s(\beta + 1)]y^*(t) =&h_2(t, 0). \end{aligned}$$

with the initial conditions,

$$x^*(t_0) = P_0 \quad \text{and} \quad y^*(t_0) = P_0.$$

These equations coincide exactly with the Euler-Lagrange equations, as derived by the Maximum Principle for the open-loop variational game without constraints. Additionally we note that as these equations are derived here via the direct method we see that they become sufficient conditions for a Nash equilibrium of the unconstrained system, and hence for the constrained system for solutions which satisfy the constraints Moreover, we also observe that we can recover the functions $H_j(\cdot,\cdot)$, for $j = 1, 2$, since we can recover both $f(\cdot)$ and $g(\cdot)$ by the formulas

$$f(t) = x^*(t) \mp c_1 \quad \text{and} \quad g(t) = y^*(t) \mp c_2.$$

The required functions are now recovered by integrating the partial derivatives of $H_1(\cdot,\cdot)$ and $H_2(\cdot,\cdot)$ which can be computed. Moreover, if the functions $x^*(\cdot)$ and $y^*(\cdot)$ are bounded, which implies that $f(\cdot)$ and $g(\cdot)$ are also bounded, we further have that,

$$\lim_{t\to+\infty} H_1(t, \tilde{x}^*(t)) = 0 \quad \text{and} \quad \lim_{t\to+\infty} H_2(t, \tilde{y}^*(t)) = 0.$$

Consequently, we see that in this instance the solution to our variational game is given by the solutions of the above Euler-Lagrange system, provided the resulting strategies and the price satisfy the requisite constraints. Finally, we can obtain the solution to the original problem by taking,

$$P^*(t) = \alpha x^*(t) + \beta y^*(t),$$

$$u_1^*(t) = \alpha\left(a - P^*(t) - \frac{1}{s}\dot{x}^*(t)\right),$$

and

$$u_2^*(t) = \beta\left(a - P^*(t) - \frac{1}{s}\dot{y}^*(t)\right).$$

Of course, we still must check that these functions meet whatever constraints are required (i.e., $u_i(t) \geq 0$ and $P(t) \geq 0$).

There is one special case of the above analysis in which the solution can be obtained easily. This is the case when $\alpha = \beta = \frac{1}{2}$. In this case the above Euler-Lagrange system becomes,

$$\begin{aligned}
\ddot{x}^*(t) - \delta\dot{x}^*(t) - \left(\frac{5}{4}s^2 + \frac{3\delta}{2}s\right)x^*(t) &= \\
&\left(\frac{5}{4}s^2 + \delta s\right)y^*(t) - \frac{3}{2}s\dot{y}^*(t) - a\left(\frac{3}{2}s^2 + \delta s\right) \\
\ddot{y}^*(t) - \delta\dot{y}^*(t) - \left(\frac{5}{4}s^2 + \frac{3\delta}{2}s\right)y^*(t) &= \\
&\left(\frac{5}{4}s^2 + \delta s\right)x^*(t) - \frac{3}{2}s\dot{x}^*(t) - a\left(\frac{3}{2}s^2 + \delta s\right).
\end{aligned}$$

Using the fact that $P^*(t) = \frac{1}{2}(x^*(t) + y^*(t))$ for all $t \in [t_0, +\infty)$ we can multiply each of these equations by $\frac{1}{2}$ an add them together to obtain the following equation for $P^*(\cdot)$,

$$\ddot{P}^*(t) + \left(\frac{3}{2}s - \delta\right)\dot{P}^*(t) - \frac{5}{2}(s+\delta)\,sP^*(t) = -\left(\frac{3}{2}s + \delta\right)as,$$

for $t_0 \leq t$. This equation is an elementary non-homogeneous second order linear equation with constant coefficients whose general solution is given by

$$P^*(t) = Ae^{r_1(t-t_0)} + Be^{r_2(t-t_0)} + \left(\frac{3s+2\delta}{5(s+\delta)}\right)a$$

in which r_1 and r_2 are the characteristic roots of the equation and A and B are arbitrary constants. More specifically, the characteristic roots are roots of the polynomial

$$r^2 + \left(\frac{3}{2}s - \delta\right)sr - \frac{5}{2}(s+\delta)\,s = 0$$

and are given by

$$r_1 = s \quad \text{and} \quad r_2 = \delta - \frac{5}{2}s.$$

Thus, to solve the dynamic game in this case we select A and B so that $P^*(\cdot)$ satisfies the fixed initial condition and remains bounded. That is, we require $0 \leq \delta < \frac{5}{2}s$, we put $A = 0$, and choose B to be,

$$B = P_0 - \left(\frac{(3s+2\delta)a}{5(\delta+s)}\right).$$

Further, we note that we can also take

$$x^*(t) = y^*(t) = P^*(t)$$

and so obtain the optimal strategies as

$$u_1^*(t) = u_2^*(t) = \frac{1}{2}\left(a - P^*(t) - \frac{1}{s}\dot{P}^*(t)\right).$$

and of course subject to the requirement that the control constraints given by (19.14) and state constraints (19.15) are met. Regarding these conditions it is an easy matter to see that the optimal price (as chosen above) satisfies $P^*(t) \geq 0$ for all $t \geq 0$ since $P_0 \geq 0$. To satisfy the control constraints, we observe that

$$\begin{aligned} u_i^*(t) &= \frac{1}{2}\left[a - \left(P_0 - \left(\frac{(3s+2\delta)a}{5(\delta+s)}\right)\right)e^{(\delta-\frac{5s}{2})(t-t_0)} - \left(\frac{(3s+2\delta)a}{5(\delta+s)}\right)\right. \\ &\quad \left. -\frac{1}{s}\left(\delta - \frac{5s}{2}\right)\left(\frac{(3s+2\delta)a}{5(\delta+s)}\right)e^{(\delta-\frac{5s}{2})(t-t_0)}\right] \\ &\to \frac{1}{2}\left[1 - \frac{(3s+2\delta)}{5(\delta+s)}\right]a \end{aligned}$$

as $t \to +\infty$ so that a necessary condition for the optimal stratiegies to remain positive would be that

$$\frac{1}{2}\left[1 - \frac{(3s+2\delta)}{5(\delta+s)}\right] \geq 0,$$

or that $s \leq 3\delta$. In addition, we observe that since the optimal strategies have the form $E + De^{r(t-t_0)}$ with E and D cosntants it follows that they must be strictly monotonic. This implies that for the control constraints to be satisfied all we need check is that their initial values, $u_i(t_0)$, be positive. Thus we must choose the parameters so that,

$$\begin{aligned} 2u_i^*(t_0) &= \left[a - \left(P_0 - \left(\frac{(3s+2\delta)a}{5(\delta+s)}\right)\right) - \left(\frac{(3s+2\delta)a}{5(\delta+s)}\right)\right. \\ &\quad \left. -\frac{1}{s}\left(\delta - \frac{5s}{2}\right)\left(\frac{(3s+2\delta)a}{5(\delta+s)}\right)\right] \\ &= a - P_0 - \frac{1}{s}\left(\delta - \frac{5s}{2}\right)\left(\frac{(3s+2\delta)a}{5(\delta+s)}\right) \\ &= \left[1 - \frac{1}{s}\left(\delta - \frac{5s}{2}\right)\left(\frac{(3s+2\delta)}{5(\delta+s)}\right)\right]a - P_0 \\ &= \left(\frac{25s^2 + 14s\delta - 4\delta^2}{10s(s+\delta)}\right)a - P_0 \\ &= \left(\frac{25(s+\frac{7}{25})^2 + 51\delta^2}{10s(s+\delta)}\right)a - P_0 \\ &\geq 0. \end{aligned}$$

Summarizing, the direct method provides an open-loop Nash equilibrium for this example whenever the parameters, s, a, P_0, and δ, are chosen so that $0 < \frac{s}{3} < \delta < \frac{5s}{2}$ and

$$P_0 < \left(\frac{25(s+\frac{7}{25})^2 + 51\delta^2}{10s(s+\delta)}\right)a.$$

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Suresh P. Sethi

Suresh Sethi (www.utdallas.edu/∼sethi), Ashbel Smith Professor and Director of Center for Intelligent Supply Networks at University of Texas at Dallas, has made fundamental contributions in a number of disciplines including operations management, finance, marketing, operations research (applied mathematics), industrial engineering, and optimal control.

Suresh Sethi was born in Ladnun, India in 1945. He received his B.Tech. in Mechanical Engineering from the Indian Institute of Technology, Bombay, in 1967. He graduated with an M.B.A. from Washington State University, Pullman, WA in 1969. In that same year, he joined the Graduate School of Industrial Administration at Carnegie Mellon University and there he received his M.S.I.S. in 1971 and his Ph.D. in 1972.

He became a Full Professor at University of Toronto at the age of 33, where he served from 1973-1997 in various positions including General Motors Research Professor and Director of Laboratory for Manufacturing Research. There, he founded the Doctoral Program in Operations Management. He joined the University of Texas at Dallas in 1997 to build the Operations Management area including a doctoral program. Currently, the area has about 10 faculty and 20 doctoral students.

Sethi's doctoral thesis at Carnegie Mellon explored the applications of optimal control theory to functional areas of management. Sethi extended the theory to deal with the peculiarities of management problems, such as the nonnegativity constraints and time lags. He has made pioneering applications in the areas of operations management, marketing and finance. Few individuals have contributed more toward the application of optimal control theory to managerial problems than Sethi. His work has been extensive in coverage and penetrating in analysis. His thesis and the subsequent work eventually led to the 1981 Sethi-Thompson book (481 pages) that brought the theory of optimal control to management schools. The second edition (505 pages) of this classic text became available in Fall 2000.

In the areas of operations management, Sethi has applied optimal control theory to HMMS-type production planning problems, machine maintenance and replacement problems, simultaneous production and pricing problems, etc. Sethi has surveyed this area in a 1978 paper.

Sethi has made numerous applications of optimal control in the area of marketing. His application of Green's Theorem to solve for optimal advertising expenditures in the Vidale-Wolfe model is now a classic. In 1983, Sethi also introduced stochastic optimal control to the marketing area. This paper has found a number of interesting extensions. Prasad and Sethi develops a competitive extension of the Vidale-Wolfe advertising model, and solves explicitly the resulting stochastic differential game. Bass et al. introduces generic advertising in the model and solve the resulting differential game explicitly to obtain the Nash equilibrium levels of both brand and generic advertising expenditures by the competitors. Sethi and Feichtinger, Hartl, and Sethi have provided extensive reviews of the literature of optimal control of advertising models in 1977 and 1994, respectively.

In 1978, Sethi began to look into the fundamental problem of how long-term planning influences immediate decisions. His work on decision, forecast and rolling horizons has provided a logical foundation for the practice of finite horizon assumptions and the choice of horizon. Sethi has published extensively on the topic, and he is considered to be one of the foremost scholars in the area. Recently, Chand, Hsu, and Sethi have surveyed the field in an article commissioned by Manufacturing Services & Operations Management (2002), a leading journal in the area.

Among his contributions in the finance/economics area, the most important is his work on the classical consumption-portfolio problem. He is responsible for bringing the realistic features of subsistence consumption and bankruptcy into the classical problem. His 1986 paper with Karatzas, Lechozky and Shreve published in Mathematics of Operations Research broadens the scope of the classical problem and provides an explicit solution of the problem. It represents a landmark paper in the area. It gave new life to the classical problem, which lay dormant for 15 years, and it inspired mathematicians and mathematical economists to study the problem with new perspectives. Sethi's further work on the problem includes risk-averse behaviour of agents subject to bankruptcy. The work of Sethi and co-authors on the problem has appeared in a 428-page book by Sethi, published by Kluwer. In its Fellow citation, the New York Academy of Sciences mentions that Harry Markowitz, a 1990 Nobel Laureate in Economics, places Dr. Sethi "among the leaders in financial theory."

Sethi has also made significant contributions to the area of sequencing and scheduling. In a seminal co-authored 1992 paper, a new paradigm for scheduling jobs and sequencing of robot moves simultaneously in a robotic cell is introduced and analyzed. A deep conjecture regarding the optimality of the solution, referred to as the Sethi conjecture among the co-authors, generated considerable research interest including at least five doctoral theses and over 40 papers. The conjecture is now partially resolved. A book co-authored by Dawande, M., Geismar, H., and Sriskandarajah, C. on the topic is forthcoming in 2005.

In the area of flexible manufacturing systems, two surveys by Sethi has defined and surveyed the various concepts of manufacturing flexibility (Browne et al. and Sethi and Sethi). These are very well-known and highly cited papers in the area. In addition, Sethi has published research on flexible transfer lines and flexible robotic cells.

Over the last fifteen years, Sethi has been looking into the complex problem of production planning in stochastic manufacturing systems. The work has resulted in a new theory of hierarchical decision making in stochastic manufacturing systems. While the work is still continuing, a significant plateau reached by Sethi and co-authors resulted in a 1994 book by Sethi and Zhang (419 pages). In reviews, the book is variously described as impressive, pathbreaking, and profound. The late Herbert A. Simon, the 1978 Nobel Laureate in Economics, stated, "Suresh Sethi has clearly made a series of important extensions to the treatment of hierarchical systems and applications to management science problems, and the book with Zhang is an impressive piece of work." A review in Discrete Event Dynamic Systems (July 1996) states: "This is a truly remarkable book, in which Sethi and Zhang have contributed enormously to the area of hierarchical controls in manufacturing." The theory leads to a reduction of the intractable stochastic optimization problem into simpler problems, which could then be solved to obtain a provably near-optimal solution of the original problem. More specifically, the theory gives rise to a relatively simpler model for higher-level management decisions and a reduced model for lower level decisions on the shop floor. The importance of the scheme lies in the facts that the simpler higher-level model is realistic enough so that it captures the essential features of the manufacturing process, and the lower level model can be reduced since it is guided by higher-level decisions. In his 1994 text Manufacturing Systems Engineering, Gershwin mentions: "There have been many hierarchical scheduling and planning algorithms, some quite practical and successful....However, outside of the work of Sethi and his colleagues, there had been little systematic justification of this structure." A follow-up of the 1994 book dealing with the average cost criteria, co-authored with Zhang and Zhang, is soon to appear.

More recently, Sethi has been studying inventory problems with Markovian demands or world-driven demands with discounted as well as average cost criteria. In addition, he has generalized the assumption on cost functions to include those having polynomial growth. Moreover; he has analyzed the case when the demand depends on a Markov process, which in turn depends on other decisions such as promotion. A book titled Markovian Demand Inventory Models is currently in progress to appear in the International Series on Operations Research and Management Science published by Springer.

Beginning with his 2001 paper, Sethi started studying the optimality of base-stock and (s,S) type policies in cases of forecast updates and multiple delivery modes. In this paper, Sethi, Yan and Zhang introduce a general forecast updating scheme, termed peeling layers of an onion, and show the optimality of a forecast-dependent base-stock policy with two delivery modes. Fix cost was introduced in a subsequent paper. Finally, it is shown that the base-stock policy is no longer optimal for other than the two fastest modes when there are three or more consecutive delivery modes. Sethi had studied a variety of supply chain contracts with demand forecast updates. Gan, Sethi and Yan look at the issue of coordination in a supply chain consisting of risk-averse agents. They develop a definition of coordination in this case, and obtain coordinating contracts in a variety of supply chains with agents observing different risk-averse objectives. Bensoussan, Feng and Sethi generalize the standard newsvendor problem to include two ordering stages, a forecast update at the second stage, and an overall service constraint. A book titled Inventory and Supply Chain Management with Forecast Updates is to appear in the International Series on Operations Research and Management Science published by Springer in 2005.

Sethi has published over 300 papers in the areas of operations research, operations management, optimal control, mathematical finance and economics, industrial engineering, and semiconductor manufacturing. He has presented his work at many scholarly conferences, universities and research institutions. He serves on several editorial boards of journals in the areas of operations research, operations management, applied mathematics, and optimal control. Over the years, Sethi's research has been supported by a number of sponsors including NSERC, SSHRC, Manufacturing Research Corporation of Ontario, and Research Grant Council (Hong Kong).

In recognition of his contributions, Sethi has received many honors. The Canadian OR Society recognized his work on operations research by bestowing on him the 1996 CORS Award of Merit. In 1997 he gave a distinguished Bartlett Memorial lecture in mathematics at the University of Tennessee. In 1999 he was elected a Fellow of New York Academy of Sciences for his outstanding contributions in a variety of research areas. In 2001 the Institute of Electrical and Electronics Engineers named him IEEE Fellow for his extraordinary accomplishments in optimal control. Suresh Sethi was awarded an INFORMS Fellow in 2003. The American Association for the Advancement of Science elected him an AAAS Fellow in that same year. Other honors include C.Y. O'Connor Fellow, Curtin University, Perth, Australia (1998); Honorary Professor at Zhejiang University of Technology, Hangzhou, China (appointed in 1996); Fellow of the Canadian Academy of Sciences and Humanities or The Royal Society of Canada (1994); Visiting Erskine Fellow at the University of Canterbury, Christchurch, New Zealand (1991); Connaught Senior Research Fellow at the University of Toronto (1984-85). He is listed in Canadian Who's Who, Marquis Who's Who in the World and Marquis Who's Who in America (2001).

Sethi is a member of INFORMS, MSOM, SIAM, IEEE, CORS, POMS, ORSI, DSI, AAAS, NYAS, Royal Society of Canada, Phi Kappa Phi, and Beta Gamma Sigma.

Sethi has been very successful at mentoring post-doctoral fellows and PhD students. These researchers have gone on to make important contributions to both teaching and research in Operations Management. This list includes Dr. Sita Bhaskaran (General Motors), Professor Richard Hartl (University of Vienna), Professor Qing Zhang (University of Georgia), Professor Steef van de Velde (Erasmus University), Dr. Dirk Beyer (Hewlett Packard), Dr. Feng Cheng (IBM), Dr. Hanqin Zhang (Chinese Academy of Sciences), Professor Gerhard Sorger (University of Vienna), Professor Abel Cadenillas (University of Alberta), Dr. Wulin Suo (Queens University), Professor Houmin Yan and Dr. Xun Yu Zhou (Chinese University of Hong Kong), Professor Suresh Chand and Dr. Arnab Bisi (Purdue University), Professor Chelliah Sriskandarajah (University of Texas at Dallas), Dr. Ruihua Liu (University of Dayton).

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