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Rainer Kleber

Dynamic Inventory Management in Reverse Logistics

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With 55 Figures
and 20 Tables

Springer

Dr. Rainer Kleber
Otto-von-Guericke-University Magdeburg
Faculty of Economics and Management
Universitätsplatz 2
39106 Magdeburg, Germany
E-mail: rainer.kleber@ww.uni-magdeburg.de

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Magdeburg, April 2006 *Rainer Kleber*

Preface

Reverse Logistics is an area that has attracted growing attention over the last years both from the industrial as well as from the scientific side. The proper management of reverse flows of products and materials is of considerable importance in many industries because of its influence on economic performance and environmental impact. The respective management tasks, however, are connected with new challenging planning and control problems. This especially holds for product recovery management concerning remanufacturing operations where used products, after being returned to the manufacturer, are reprocessed such that they are as good as new and can be re-integrated into the forward logistics stream.

A major issue in remanufacturing is how to optimally coordinate the potential activities directed at meeting customer demands for serviceable products and to deal with returns of products after end-of-use. The respective decisions refer to finding a proper mix of manufacturing original and remanufacturing used products as well as of stock-keeping and disposing of returned items. Hereby, relevant cost impacts and time patterns of demand and returns have to be taken into consideration.

Up to now, research contributions to this field of Reverse Logistics have addressed only two main aspects that result in high complexity of decision making in product recovery management. One aspect is that of capacity restrictions and fixed costs in manufacturing and remanufacturing systems that makes coordination of lot-sizing a challenging problem. The second aspect refers to uncertainty of demands and returns that leads to complicated stochastic interactions which have to be coped with by appropriate decision rules and safety stock policies. While these issues are highly relevant for operational and tactical decision making, a third aspect with mainly strategic importance has largely been ignored. This is the aspect of time-variability and dynamic change of major input parameters for product recovery decisions. On the one hand, this refers to the variability of product demand and return schemes that can be observed both due to seasonality and the classical life cycle pattern for many product categories. On the other hand, over larger time spans we also face specific cost dynamics caused by experience effects in manufacturing and remanufacturing processes.

It is the commendable contribution of this book that it sheds some light into this complicated field of how to respond most effectively to the dynamically changing environment in product recovery strategy. This response refers to choice of time-varying coordination strategies of manufacturing, remanufacturing and disposal activities as well as to the timing of investment decisions in product recovery technologies. Embedded in these considerations an analysis is developed of how and why to use different kinds of strategic inventories to enable best reactions to dynamic cost, demand and return processes. Based on advanced quantitative modeling and optimization techniques a deep analysis of the addressed complex dynamic decision problems is given.

Summarizing, this book presents major progress in scientifically investigating the field of complex problems of product recovery management induced by several types of dynamics in the planning environment. The underlying dynamic problem aspects are of enormous practical importance, but have not been addressed appropriately in research contributions before. By studying this book the reader will learn novel and interesting findings on how to respond strategically to ongoing changes of a product recovery environment by responsive recovery policies and dynamic inventory management.

Magdeburg, April 2006 *Karl Inderfurth*

Contents

1

Introduction

1.1 Objective and Motivation

The integration of product recovery into regular production processes has developed into a challenge for the manufacturing industry (Guide and Van Wassenhove (2002)). While in the past a firm's main concern was to sell its products leaving the burden of final disposal to society, it is now increasingly assigned responsibility for what happens with the product after use. Consequently, product recovery leads to additional restrictions firms must take into account, but it also enables new opportunities (Stock et al. (2002)).

There are many reasons for this development. An increasing environmental consciousness of the public and limited availability of natural resources to manufacture new products on one side and the necessity to find alternatives to landfilling and incineration of waste led to new regulations that aim at reducing the quantity and environmental impact of waste. Environmental legislation incorporates the prohibition of substances that aggravate material recovery, the enforcement of collection networks, and industry specific take back and recovery obligations. Some of the many examples are the German Recycling and Waste Control Act (Kreislaufwirtschafts- und Abfallgesetz, KrW-/AbfG) enacted in 1996 that extended product responsibility of manufacturers to the end of life phase and the EU Directive on Waste Electric and Electronic Equipment (WEEE) from 2003 which calls for the installment of collection networks. A recent overview on end-of-life legislation issues with examples from the US, Europe, and Japan can be found in Toffel (2003).

In addition, economic motives lead to a voluntary product take back of original equipment manufacturers (OEMs), as detailed and classified in Toffel (2004). First, recovering products allows us to reduce production cost by using recovered material and components in lieu of virgin material and newly produced components. Second, the fact that there is demand for leased products in the marketplace forces us to confront these products again at the end of the lease period. In this example, dealing with returns is part of the 'price we pay' to service the demand for these products. Third, customer behavior seemed to be influenced by the environmental image of firm, and therefore using recovered material in products or merely engaging in product recovery itself increases the demand from this market segment. Fourth, aftermarkets are often lucrative revenue generators, and therefore must be protected against third parties servicing demand for spare parts, etc. Finally, it is also viewed that taking environmentally friendly steps, for instance implementing a product recovery system before take-back laws exist, at times successfully preempts environmental legislation. On the other hand, Reinhardt (1999) also points out that encouraging environmental legislation can lead to an improved position of the firm, forcing competitors into compliance.

An increasing return of used products encourages OEMs to produce more environmental friendly yet recoverable products. A large number of examples for product recovery due to varying incentives are assembled by de Brito et al. (2005) and range from the reprocessing of chemicals in the pharmaceutical industry (see Teunter et al. (2005)) to the remanufacturing of Kodak single-use cameras or of complex products like engines in the car manufacturing industry (the latter two examples will be further detailed below as case studies). Thierry et al. (1995) put forth an overview on strategic product recovery issues and differentiate between product recovery options recycling, repair, cannibalization, refurbishing, and remanufacturing. Out of these options, remanufacturing seems especially appealing to OEMs since large parts of the added value can be recovered (Klausner and Hendrickson (2000)).

Product recovery management is charged with the coordinated planning and control of both production and recovery processes that serve the same demand for materials, parts, or final products. In the context of remanufacturing, both sources are assumed to be perfect substitutes, and recovered products are usually said to be as good as new. It should be noted that although we might choose other recovery options (e.g. repair or refurbishing) to be performed on the returns, we restrict our attention to remanufacturing as we presume substitutability.

When dealing with product returns, logistic processes are more complicated to control since both forward and backward flows must be coordinated. Production planning is more complex since there now exist two possible sources to serve the demand which need to be coordinated, therefore raising new operational questions. These problems receive growing interest from researchers and practitioners alike and are summarized in the field of *Reverse Logistics*. The European Working Group on Reverse Logistics (REVLOG) uses the following definition:

> The research area of Reverse Logistics covers "the process of planning, implementing and controlling backward flows of raw materials, in process inventory, packaging and finished goods, from a manufacturing, distribution or use point, to a point of recovery or point of proper disposal." (de Brito and Dekker (2004))

Quantitative approaches in reverse logistics have been surveyed by Fleischmann et al. (1997) and more recently by Dekker et al. (2004). According to the example of the latter work one can distinguish between three important domains of research within this field, namely: extended supply chain management dealing with relations between different partners inside a reverse logistics system, reverse distribution which includes collection and transportation aspects, and production and inventory management. Here, we focus on the last aspect and assume that an appropriate collection network exists which provides an OEM access to its own used products.

1.2 Inventory Management in Reverse Logistics

There are several reasons to keep stock in traditional production settings, as discussed in Silver et al. (1998), Chapter 3, and inventories can be classified based on their economic motivation. Specifically, safety stock is used to buffer from short term uncertainty of demand and supply, cycle stock is used to account for trade-offs between e.g. fixed setup and holding costs, and anticipation stock is often used to smooth capacity utilization in a dynamic (e.g. seasonal) environment.

Managing inventory in the presence of returns leads to additional complexity. In the case of safety stock, we must now account (in addition to the traditional demand uncertainty) for the uncertainty surrounding the supply of returns, whereas in lot sizing we must coordinate lot sizes and setup times for both production and remanufacturing. Stocks have to be distributed among inventories for serviceables and recoverables (returns). These issues have received attention in research (for reviews on inventory management in reverse logistics see Dekker and van der Laan (2003), de Brito and Dekker (2003) and Fleischmann and Minner (2004)), but they hardly explain the large amount of returned used products held in stock at remanufacturing facilities, as is confirmed by Seitz and Peattie (2004). When adapting our treatment of anticipation stock, we find that the addition of the return stream yields entirely new situations in which we hold stock, which directly result from the dynamic environment firms operate in.

A closer look at the product recovery environment reveals many factors which fluctuate over time. Starting with an obvious one, the demand for the product will vary over time. However, this is no surprise. Frequently, life cycle patterns as well as seasonality will influence demand. In medium-range aggregate production planning (Silver et al. (1998), Chapter 14), we seek to smooth capacity utilization by using seasonal inventory. The resulting solution lies between two extremes of nearly constant production (level) and production which is synchronized with demand (chase). The amounts of returned products may likewise vary over time, as is documented in the following two cases:

Case 1.1. DaimlerChrysler engines (see Kiesmüller et al. (2004))
DaimlerChrysler operates several facilities for recovering parts from used cars, one of which remanufactures used engines for Mercedes Benz cars at the plant Berlin-Marienfelde (MTC). Annually, about 12,000 engines from 28 classes and 800 different model variants are remanufactured. An ABC-classification revealed that 60% of the returns are contributed by 3 classes.
Dynamic issues, i.e. time dependent demands and returns, have to be considered for two reasons. First, demands for an engine class follow the shape of a product life cycle, starting with a phase of increasing sales, followed by the maturing phase and finally declining sales towards the end of a product's life cycle. Returns follow demands in a similar pattern, delayed by the usual life time of an engine and reduced by the number of not returned engines. In the growth phase, demand for remanufactured engines is significantly higher than available returns and all returned cores are remanufactured. Later in the maturing phase, demand decreases and returns can exceed remanufacturing orders. This divides the product life cycle into two main phases, the first with insufficient cores and another one with excessive cores. Similar effects, i.e. dynamic fluctuations of both, demand for remanufactured items and supply of returns, leading to shortage and overage situations have also been reported for car part remanufacturing at Volkswagen (van der Laan et al. (2004)).

Case 1.2. Kodak single-use cameras (see Goldstein (1994) and Guide et al. (2003b))
Introduced by Fuji Photo Film Co. as 'film with lens' and originally designed to be thrown away after use, the single-use camera now is another example of successfully closing the loop on a higher level of product recovery. According to Kodak (2003), about 775 Million cameras have been processed since the start of the product recovery program in 1990 and currently a worldwide return rate of 75% has been achieved. The amount of reusable materials ranges between 77-90% (by weight) of the product. Most recovered parts are plastic bodies, which are reused up to six times and the circuit boards required in flash cameras, which are used up to 10 times.
An important issue that Kodak faces is to deal with dynamic demand and return streams. Goldstein (1994) stated that there is "a lot of seasonality and cyclical behavior" in the market for single-use cameras. This stems from peak selling periods that differ among the various models: Underwater cameras mostly are sold in summer and winter vacation season, while flash cameras sell best around winter holiday season, and peak season for single-use cameras is between March and early September. On the reverse flow side, batch shipments from smaller photofinisher labs to collection facilities can lead to a delay of a couple of months between development of film and shipment to Kodak. On average it took between three and five months for a camera to be returned after being (re)manufactured. Although these numbers have been reduced in recent years, a large number of used products return at the end of a peak season or during off-season.

As the two cases have shown, there will be periods where returns exceed demand (excess returns) and other moments where demand exceeds returns (excess demand). Since available returns can also be seen as a capacity to recover products, an inventory can be used to enlarge the capacity when needed. Fleischmann and Minner (2004) call such stocks 'opportunity stocks', because they enable additional recovery opportunities. From a more strategic point of view, cost parameters themselves can change over time caused not only by external influences (such as ever-increasing disposal fees) but also by internal impacts like learning (or experience) curve effects. Lastly, in the long run, available capacity for product recovery is also not fixed over time but can be changed through capacity expansion or reduction.

Most of product recovery and inventory management models are either restricted to stationary conditions or treat dynamic aspects only numerically, for recent overviews on stochastic inventory control see van der Laan et al. (2004) and for lot sizing issues Minner and Lindner (2004). As a consequence, Dekker et al. (2000) suggest more examination of the effects of non-stationary demand/return conditions on inventory control for joint manufacturing remanufacturing systems.

This thesis concerns itself with the incorporation of dynamic issues in medium and long-term product recovery management. In doing so, we expressly ignore more operative disassembly issues, which would complicate matters. We also restrict ourselves to time-varying deterministic environments, ignoring short term stochastic fluctuations as is common practice in other medium to long term models. We avoid more unnecessary complication by examining the simplest case of a single product or a single part/module. Decisions faced in this realm include (1) when to invest in remanufacturing capabilities (if at all), (2) when to start collecting, hold stock of, and dispose of returns. It specifically deals with use of anticipation inventories for smoothing both capacity supply (e.g. return availability) as well as capacity demand. Strategic implications are expressly considered, especially the decision of whether to engage in a higher level of recovery or not, including aspects such as knowledge acquisition (e.g. experience curve effects) and the additional operational and investment expenditures required to implement product recovery processes (Toffel (2004)). The consideration of more strategic issues in research has been recently demanded by Guide et al. (2003a) since it is seen to be of particular value for practitioners. Long term decisions involve significant sums of money and are often difficult (if not impossible) to change. Examining our problem specifically, we can see that the decision on when to invest if made erroneously could result in opportunity costs rising from either lost recovery cost advantage (if made late) or capital costs (if made early). Investment on product recovery capability is a decision made very carefully by managers, and one that they certainly do not want to get wrong. Likewise, deciding on the correct time to start keeping excess returns is also important and has far reaching effects. If we start too early, we sacrifice capital costs filling our inventory with unusable scrap. This error would be particularly painful if we could have reaped a salvage value by ‘disposing’ of the returns. Waiting too long, on the other hand, results in lost recovery cost advantage.

1.3 Methodology

An appropriate way to examine long term issues is to use a continuous time model, which avoids the discretization of time and the influence of the choice of time units on the model and results. Another advantage of this modeling is that parameters can be given by continuous time functions, eliminating the need to specify them for each time period. Dynamic modeling properties motivates the use of the theory of *Optimal Control* as a solution method. Starting with the pioneering work by Pontryagin et al. (1962) it has reached a wide range of applications in economics and management, see e.g. Seierstad and Sydsæter (1987), Kamien and Schwartz (1991), or Sethi and Thompson (2000). It can be compared with dynamic programming methods developed by Bellman (1957) at about the same time but, according to Feichtinger and Hartl (1986), a main advantage of optimal control is the possibility of gaining insights into the *general structure* of solutions for an *entire problem class*.

Although there are extensions to solve discrete time problems, optimal control literature mainly deals with continuous time systems. A system in this sense is characterized by one or more state variables (e.g. an inventory stock level) which are changed by external influences (demand) or by choosing control variables (production). The development of the states is characterized by a differential equation named *state (transition) equation*. Both, state as well as control variables, can be subject to constraints which have to be considered. An optimal solution is given by optimal trajectories (functions of time) of the state and control variables which maximize (or minimize) a given objective function.

Pontryagin’s Maximum Principle provides a set of necessary conditions for an optimal solution, which is also sufficient under certain conditions. Using the Maximum Principle the dynamic problem is decomposed into an infinite sequence of interrelated static problems, one for each time instant. These are connected by introducing co-state (also called adjoint) variables, which can interpreted as the shadow price of changing the system state. A static objective function, called *Hamiltonian*, is constructed in a way that it measures the total effects of the decisions at a certain time point on the objective. These can be split into a direct and an indirect effect. The direct effect is given for instance by the costs of producing an item. The indirect effect arises since decisions also have an influence on future opportunities by changing the system’s state, e.g. by decreasing or increasing the inventory level. It is measured by the rate of change of the state times the corresponding shadow price (given by the co-state). This yields another advantage of optimal control since the co-states can also be interpreted as the value of e.g. another returned item, a produced, or remanufactured product.

As in dynamic programming, where an optimal decision is taken at each stage (instant of time) assuming that up to that point all decisions have been taken optimally and the same will hold for future decisions, the Hamiltonian has to be maximized at each instant of time by appropriately choosing the controls for given optimal state and co-state values, subject to relevant constraints on control and state variables. This is applied by using standard methods of non-linear programming. Further necessary conditions contain the rate of change of the co-state variables. Thus, a system of differential equations including the state transitions, co-state transitions, and optimal control policies has to be solved.

For an overview on traditional continuous time production and inventory models see e.g. Feichtinger and Hartl (1986), Chapter 9 or Sethi and Thompson (2000), Chapter 6. Well known examples are variants of the HMMS-model (Holt et al. (1960), Thompson and Sethi (1980)) that uses a quadratic objective in order to retain goal levels for both inventory and production. Linear inventory and convex production costs are used in Arrow-Karlin type models (Arrow and Karlin (1958)). More recently, these models have been extended to cope with environmental issues. Wirl (1991) and Hartl (1995) analyzed effects of environmental constraints in the Arrow-Karlin model and Dobos (1998) used the HMMS approach. Product recovery systems including remanufacturing and disposal of returned products under non-linear cost regimes are covered e.g. by Kistner and Dobos (2000). In most practical situations, however, a linear cost regime is present and will be used throughout this work. Recent applications of optimal control in dynamic product recovery are reviewed by Kiesmüller et al. (2004).

1.4 Outline of the Thesis

The road-map followed in the succeeding chapters is given as follows.

In *Chapter 2*, a basic model for product recovery is presented. It aims to explain under which conditions returns should be kept in an anticipation stock. It extends the investigation of a single product/single stage product recovery system by Minner and Kleber (2001) by allowing for discounting. Some attention is paid to the valuation of inventories which in this case can be quite easily accomplished through exploiting the advantages of the solution methodology.

In traditional medium-term aggregate production planning an anticipation stock is used when capacity of the cheaper regular mode becomes binding in order to avoid high costs of overtime. However, the second supply source might also be limited. Therefore, *Chapter 3* discusses the implications of capacity constraints on both the cheaper source (remanufacturing) and the 'overtime' mode (production).

Chapter 4 relaxes the assumption of general preferability of remanufacturing over manufacturing. Knowledge acquisition during repeated remanufacturing operations can itself lead to profitable remanufacturing, even if there is no immediate cost advantage. The influence of learning in the remanufacturing process on stock-keeping decisions is analyzed, revealing that (under certain circumstances) another motive for holding recoverables is to postpone the start of remanufacturing.

Chapter 5 deals with the use of anticipation inventory in controlling remanufacturing capacity over the product life cycle in the most simple case, the choice of the investment time of a remanufacturing facility with unlimited capacity. More specifically, when introducing a new product, two related decisions have to be considered, namely product design and the choice of the recovery mode and a corresponding technology. This is accomplished by considering the influences of such decisions on direct production costs and initial investment expenditures. Taking into account the limited availability of used products in the beginning of a products life cycle and a decreasing time value of the required investment expenditures connected with the set-up of the remanufacturing process, the issues addressed here are when to initiate this process and how the use of a strategic recoverables inventory does affect this decision.

Some concluding remarks are given in *Chapter 6*, along with a recapitulation of the main results of the thesis, as well as a short discussion of related work.

2

A Basic Quantitative Model for Medium and Long Range Product Recovery Planning

2.1 Overview

This chapter deals with a basic dynamic model of product recovery. A single-stage and single item version of a product recovery system is presented and analyzed under a linear cost regime. The remainder of this chapter is organized as follows. The dynamic optimization problem is formulated (Section 2.2) and solved (Section 2.3). Thereby, some general properties of an optimal recovery strategy and a construction method are outlined. In the first instance, this is accomplished by disregarding initial inventories. However, initial recoverable stocks play a role in rolling planning frameworks, and an explicit consideration of positive serviceables will facilitate our discussion in the case of limited capacities (see Chapter 3). We therefore present a method on how to deal with initial inventories in Section 2.4. The findings are illustrated using several numerical examples in Section 2.5. A comparison with the outcome of using an undiscounted (average) cost approach is provided in Section 2.6. The main results are summarized in Section 2.7.

2.2 A Basic Quantitative Model of Dynamic Product Recovery

In the following we analyze a generalized version of the single-product, single-stage product recovery system presented by Minner and Kleber (2001) that allows for discounting. The system under consideration is depicted in Figure 2.1. The system faces an external customer demand rate $d(t)$. Customer return used products with rate $u(t)$. Both rates are deterministic, non-negative, and continuous functions within a finite planning horizon $[0, T]$, and can not be influenced by the decision maker.

As an example consider the following functions for cyclical demand and returns over a planning horizon of $T = 4\pi$:

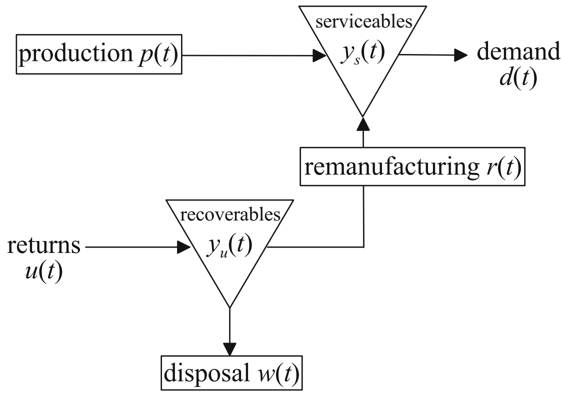

Fig. 2.1. Product recovery system

$$d(t) = 1 + 0.5\sin(t) \text{ and } u(t) = 0.7d(t - \pi)$$

The demand function consists of a time independent base level of 1 which is superimposed by some seasonal influence introduced by a cyclical part $0.5\sin(t)$. A fraction of 70% of previous demand becomes available after staying with the customer for a period of π. Here, it is additionally assumed that there existed demand before the begin of the planning period, i.e. for $t < 0$. This scenario is depicted in Figure 2.2.

Product requirements are satisfied from serviceables inventory which can either be replenished by producing new items with rate $p(t)$ or by remanufacturing of returned products with rate $r(t)$. We assume that remanufactured products have the same quality as produced items and therefore serve as perfect substitutes. Returned products that are not instantaneously remanufactured can either be kept in a recoverables inventory for future remanu-

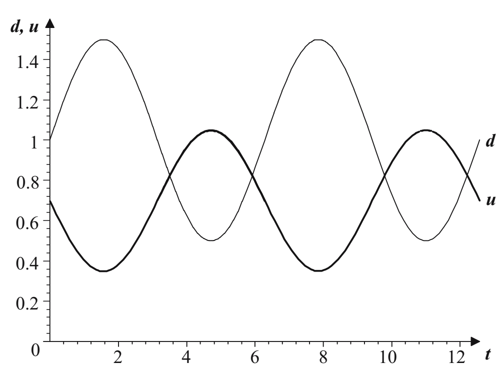

Fig. 2.2. Demands and returns in the example

facturing or disposed of with rate $w(t)$. There are no binding constraints on either process, i.e. it is possible to produce and to remanufacture at any rate. Additionally, the disposal level is assumed not to be limited.

The state of the recovery system is described by a serviceables $y_s(t)$ and a recoverables inventory $y_u(t)$. Initial values for the state variables are given by $y_s(0) = y_s^0$ and $y_u(0) = y_u^0$. The planning horizon T characterizes the end of all obligations for the product. Therefore, both inventories have to be zero at time T, i.e. $y_s(T) = 0,\ y_u(T) = 0$.

Since we assume zero lead times, decisions are implemented instantaneously. Thus, the corresponding state transition equations are represented by differential equations where the rate of change in the serviceables inventory ($\dot{y}_s(t) = dy_s(t)/dt$) equals the sum of production and remanufacturing rate minus demand rate at time t

$$\dot{y}_s(t) = p(t) + r(t) - d(t). \tag{2.1}$$

Analogously, the marginal increase of recoverables inventory is given by the return rate minus the sum of remanufacturing and disposal rates at time t

$$\dot{y}_u(t) = u(t) - r(t) - w(t). \tag{2.2}$$

Since we do not allow for backlogging, the serviceables inventory has to be non-negative

$$y_s(t) \geq 0. \tag{2.3}$$

The availability of recoverable items limits both, remanufacturing and disposal decisions, being expressed by non-negativity of the recoverables inventory

$$y_u(t) \geq 0. \tag{2.4}$$

Further, non-negativity constraints hold for production, remanufacturing, and disposal control variables

$$p(t) \geq 0, \quad r(t) \geq 0, \quad w(t) \geq 0. \tag{2.5}$$

The objective of the model is to satisfy customer demands over the planning horizon with a minimal total discounted cash (out-)flow for production, remanufacturing, disposal and holding serviceables and recoverables inventory. It is assumed that all payments depend linearly on the respective decision, i.e. we assume constant and time independent per unit payments for production $c_p > 0$, remanufacturing $c_r > 0$, and disposing items c_w, as well as constant out-of-pocket inventory holding costs per unit and time unit $h_s > 0$ and $h_u > 0$, respectively. The discount rate $\alpha > 0$ reflects real costs of capital (net of inflation), possibly calculated as weighted average of the companies equity and debt capital (see, e.g., Kaplan and Atkinson (1998)). In contrast to all other parameters, unit payments for disposing of a returned item can also be negative, as would be the case if a salvage revenue can be obtained.

In order to ensure that remanufacturing is an alternative to production, direct costs of remanufacturing must be less than the sum of direct production and disposal costs per unit

$$c_p + c_w > c_r. \tag{2.6}$$

Otherwise, disposal of a returned product and production of a new one is always superior to remanufacturing it, and the model would reduce to a pure production/inventory model with linear direct costs, for which it is optimal to produce the demand at every time point (see Feichtinger and Hartl (1986)) after consuming initial inventories. The difference $(c_p + c_w) - c_r$ is therefore referred to as (direct) recovery cost advantage.

Further we assume, that out-of-pocket holding cost per item and time unit for serviceables is higher than for recoverables

$$h_s > h_u, \tag{2.7}$$

which is reasonable, because serviceables usually are treated more carefully than recoverables. Finally, it is not advantageous to hold unneeded returned products as opposed to disposing of them

$$h_u > \alpha c_w. \tag{2.8}$$

If this would not be the case, disposal would never take place because interests saved when delaying the disposal of an item would always outweigh out-of-pocket holding costs.

Both equations (2.1) and (2.2) imply that production, remanufacturing and disposal are executed with finite rates. But since we assume all controls (processes) to be unrestricted, state variables are allowed to jump, e.g. if there are more recoverables in inventory than needed, one can dispose of a real quantity by an impulse control, which leads to a negative jump in y_u. A (positive) jump in y_s by a production or remanufacturing impulse is more expensive than synchronizing production and remanufacturing with demand because of the additional holding costs incurred. A jump in y_u can only be advantageous at time zero by disposing a quantity w_0 with an impulse control. For later time points, disposal in advance reduces recoverables holding costs. The following proposition summarizes the sketched results on impulse controls.

Proposition 2.1. *A (positive) jump in the serviceables inventory by production or remanufacturing impulse controls will never occur in an optimal policy. A (negative) jump in the recoverables inventory may be advantageous only at time zero.*

A note on Proofs. For all proofs and derivations see the appendices of the respective chapter.

In the following, $x(t^-)$, $x(t^+)$ denote left- and right-side limits of x, respectively. With the introduced assumptions, the following control optimization model with two states (y_s and y_u), three control variables (p, r, w) and a single impulse control variable (w_0) has to be solved subject to the state equations and one jump equation, two pure state constraints, initial and terminal conditions for the state variables as well as non-negativity constraints for all control variables

$$\min NPV = c_w w_0 + \int_0^T e^{-\alpha t} \left(c_p p(t) + c_r r(t) + c_w w(t) + h_s y_s(t) + h_u y_u(t) \right) dt$$

$$\begin{aligned} s.t. \qquad & \dot{y}_s(t) = p(t) + r(t) - d(t), \dot{y}_u(t) = u(t) - r(t) - w(t), \\ & y_s(t) \geq 0,\ y_s(0) = y_s^0,\ y_s(T) = 0, \\ & y_u(t) \geq 0,\ y_u(0^-) = y_u^0,\ y_u(T) = 0, \\ & y_u(0^+) - y_u(0^-) = -w_0, \\ & p(t) \geq 0, u(t) \geq 0, w(t) \geq 0, 0 \leq w_0 \leq y_u(0^-). \end{aligned} \tag{2.9}$$

2.3 Solution of the Model Without Initial Inventories

2.3.1 Solution Methodology

The dynamic optimal control problem (2.9) is solved by applying the direct adjoining approach for Pontryagin's Maximum Principle as shown in Feichtinger and Hartl (1986) for a deterministic, linear, non-autonomous model with pure constraints both in state and control variables and a possible jump in a state. Because of the linear objective function, the necessary conditions given below are also sufficient for an optimal policy.

The solution process as sketched in Section 1.3 requires to solve a system of differential equations which is complicated for several reasons. First, optimal trajectories of control variables are in general not given as closed form expressions and in case of unrestricted controls, there might even be 'jumps' in the slope of the state variables. Second, there might be no predetermined initial or terminal value for the co-state. And finally, co-state variables can be discontinuous under certain conditions. A construction of the optimal trajectory can therefore be arduous. In order to overcome these difficulties, the following general approach is used. Exploiting necessary conditions presented in Section 2.3.2, general properties of an optimal policy can be derived in Section 2.3.3 by defining *cases* with respect to state variables and deriving respective optimal decision structures. Since there can also be other influencing conditions, further sub-cases might be distinguished. Within such defined intervals closed form expressions for the control, state and co-state functions can be stated. Since such a case or policy will in general not be optimal for the whole planning horizon, in Section 2.3.4 it is determined under which conditions a *transition* from one case to another one occurs. These points in time are identified as so called *transition points*. Then, properties of optimal intervals where stock-keeping is present are derived in Section 2.3.5 and finally a solution algorithm is given in Section 2.3.6 that numerically determines the optimal solution.

For ease of the following discussion we first restrict to the case where both initial inventories are equal to zero ($y_s^0 = y_u^0 = 0$), and therefore the jump parameter w_0 must be equal to zero. In Section 2.4 we show what changes when initial stocks have to be considered. For the remainder of this chapter, we omit time indices where appropriate for reasons of simplicity.

2.3.2 Necessary Conditions

There are two equivalent possibilities to formulate necessary conditions. The first way is to use the so-called *present-value formulation* in where the static objective (Hamiltonian) explicitly includes the discount factor $e^{-\alpha t}$. Instead of that, we prefer to state conditions by using the so-called *current-value formulation* which has the advantage that both, the static objective and the static optimality conditions do not change in time.

The current-value Hamiltonian $H(y_s, y_u, p, r, w, t, \lambda_0, \lambda_s, \lambda_u) = H(.)$ for a maximization problem corresponding to (2.9) is given by multiplication of the (negative) objective by a constant λ_0 and the state transitions by adjoint variables $\lambda_s(t)$, $\lambda_u(t)$ respectively

$$\begin{aligned} H(.) = \lambda_0(&-c_p p - c_r r - c_w w - h_s y_s - h_u y_u) \\ &+\lambda_s(p + r - d) + \lambda_u(u - r - w). \end{aligned} \tag{2.10}$$

In the appendix we show that $\lambda_0 = 1$. Then, the adjoints or co-state variables $\lambda_s(t)$ and $\lambda_u(t)$ are unequivocally defined and can be interpreted as shadow price of a change in the corresponding inventory level. Since there are several ways to change the inventory level, the co-states also have the following meaning: $\lambda_s(t)$ is the value of a produced item or the negative value of an additionally demanded item at t. $\lambda_u(t)$ can be seen as value of another returned item or negative value of a disposed item at t. $\lambda_s(t) - \lambda_u(t)$ can be interpreted as value of a remanufactured item. We will also refer to the adjoints as the value of a stored item.

In order to account for the control and state inequality constraints, Lagrange multipliers $\mu_i(t)$, $i = 1, 2, 3$ for the three control variables and $k_j(t)$, $j = 1, 2$ for the two state variables are defined. The corresponding Lagrangian $L(., \mu_1, \mu_2, \mu_3, k_1, k_2) = L(..)$ is given by

$$L(..) = H(.) + \mu_1 \cdot p + \mu_2 \cdot r + \mu_3 \cdot w + k_1 \cdot y_s + k_2 \cdot y_u. \tag{2.11}$$

The derivation of the necessary conditions for the optimal solution is made in accordance with the methodology put forth in Feichtinger and Hartl (1986), pp. 164-169.

Let (y_s^*, y_u^*) represent the optimal trajectory of the state variables and (p^*, r^*, w^*) the (piecewise continuous) trajectory of optimal control policies to problem (2.9). Then, there exist piecewise continuous functions of time λ_s, λ_u, μ_i, $i = 1,2,3$, k_j, $j = 1,2$ and two sets of time points $\theta_s \in \Theta_s$ and $\theta_u \in \Theta_u$ where the co-states λ_s and λ_u jump with corresponding height parameters $\eta_s(\theta_s)$ and $\eta_u(\theta_u)$. Except for points of discontinuity in the controls and junction points, i.e. points where one of the non-negativity constraints for inventories become or leave the state of being binding, the following necessary conditions (2.12)–(2.24) have to hold.

Due to its linearity in all controls, the Hamiltonian is maximized, if

$$p^* = \begin{cases} 0 & \lambda_s < c_p \\ \text{singular} & \lambda_s = c_p \\ \infty & \lambda_s > c_p \end{cases} \tag{2.12}$$

$$r^* = \begin{cases} 0 & \lambda_s - \lambda_u < c_r \\ \text{singular} & \lambda_s - \lambda_u = c_r \\ \infty & \lambda_s - \lambda_u > c_r \end{cases} \tag{2.13}$$

$$w^* = \begin{cases} 0 & \lambda_u > -c_w \\ \text{singular} & \lambda_u = -c_w \\ \infty & \lambda_u < -c_w \end{cases} . \tag{2.14}$$

Note that infinite values for the controls can be excluded as shown below. The so-called bang-bang equations (2.12)–(2.14) can be interpreted as follows. If production costs c_p exceed the value of a stored serviceables item λ_s, the production rate obtains its lower bound of zero. If unit production cost and adjoint variable are equal, the production rate can be positive but it is not to be determined from this condition alone ('singular'). Analogously, the remanufacturing rate is zero if the increase in value of a used product when remanufacturing it (value of a serviceables item minus the value of the recoverable item) is smaller than remanufacturing cost c_r and positive when both are equal. If the value of a returned item is larger than its salvage revenue $(-c_w)$, no items are disposed of.

Further necessary conditions account for inequality conditions for the control variables. Equations (2.15)–(2.17) maximize the Lagrangian. Non-negativity as well as complementary slackness conditions (2.18)–(2.20) must apply:

$$\frac{\partial L}{\partial p} = -c_p + \lambda_s + \mu_1 = 0 \tag{2.15}$$

$$\frac{\partial L}{\partial r} = -c_r + \lambda_s - \lambda_u + \mu_2 = 0 \tag{2.16}$$

$$\frac{\partial L}{\partial w} = -c_w - \lambda_u + \mu_3 = 0 \tag{2.17}$$

$$\mu_1 \geq 0 \quad \mu_1 \cdot p^* = 0 \tag{2.18}$$

$$\mu_2 \geq 0 \quad \mu_2 \cdot r^* = 0 \tag{2.19}$$

$$\mu_3 \geq 0 \quad \mu_3 \cdot w^* = 0 \tag{2.20}$$

From (2.15)–(2.17) together with (2.18)–(2.20) it follows that the infinite upper values for p^*, r^* and w^* in (2.12)–(2.14) cannot occur in an optimal solution because $\lambda_s \leq c_p$ from (2.15) and (2.18), $\lambda_u \geq -c_w$ from (2.16) and (2.19), and $\lambda_s - \lambda_u \leq c_r$ from (2.17) and (2.20).

Equations (2.21)–(2.22) represent optimal co-state transitions, and (2.23)–(2.24) are complementary-slackness conditions for the state variables:

$$\dot{\lambda}_s = \alpha\lambda_s - \frac{\partial L}{\partial y_s} = \alpha\lambda_s + h_s - k_1 \tag{2.21}$$

$$\dot{\lambda}_u = \alpha\lambda_u - \frac{\partial L}{\partial y_u} = \alpha\lambda_u + h_u - k_2 \tag{2.22}$$

$$k_1 \geq 0 \quad k_1 \cdot y_s^* = 0 \tag{2.23}$$

$$k_2 \geq 0 \quad k_2 \cdot y_u^* = 0. \tag{2.24}$$

Conditions (2.21)–(2.22) together with (2.23)–(2.24) imply that if serviceables inventory is positive (therefore $k_1 = 0$), the marginal increase of the value of a serviceables item equals interest on the current value which would be realized by selling the item plus the serviceables out-of-pocket holding cost rate and if recoverables inventory is positive ($k_2 = 0$), the increase of the corresponding value equals the sum of interests to be paid on the current value and the recoverables out-of-pocket holding cost rate.

Now we address times where above conditions might not hold and where adjoints may be discontinuous, i.e. where jumps in λ_s or λ_u and in the Hamiltonian occur. At junction points where a state constraint is binding or it reaches (entry time), leaves (exit time), or just touches (contact time) the boundary, e.g. $y_s^* = 0$, there can be jumps in λ_s at time points $\theta_s \in \Theta_s$ of height $\eta_s(\theta_s)$ such that

$$\lambda_s({\theta_s}^-) = \lambda_s({\theta_s}^+) + \eta_s(\theta_s) \tag{2.25}$$
$$\text{where } \eta_s(\theta_s) \geq 0 \text{ and } \eta_s(\theta_s) \cdot y_s^*(\theta_s) = 0.$$

The same holds for time points $\theta_u \in \Theta_u$ with $y_u^* = 0$

$$\lambda_u({\theta_u}^-) = \lambda_u({\theta_u}^+) + \eta_u(\theta_u) \tag{2.26}$$
$$\text{where } \eta_u(\theta_u) \geq 0 \text{ and } \eta_u(\theta_u) \cdot y_u^*(\theta_u) = 0.$$

Note that only downward jumps are possible for λ_s and λ_u.

Using the necessary conditions described in this section, we can deduce the structure of the optimal solution, which is done in the following subsection.

2.3.3 The Structure of an Optimal Solution

The trajectory of optimal decisions (p^*, r^*, w^*) is determined by the development of the state of the system (consisting of optimal values for state and co-state variables). Especially, the control policy can be given by simple rules which depend on whether a state restriction is binding or not. We therefore distinguish between four different cases with respect to their serviceables and recoverables inventory status. Where necessary, cases are further subdivided, if different control policies are applicable. As a result, we provide optimal decisions within each of the distinguished (sub-)cases and other required properties concerning co-states and demand/return rate developments. Since, in general, none of the cases will be optimal for the entire planning horizon, the question of under which circumstances to enter or to leave a certain case has to be answered as well, which will be discussed later. Thus, the following propositions should provide optimal decisions within an interval of the respective case and given that one does not switch into another case. In the remainder, let $\theta_{e,i}$ and $\theta_{x,i}$ denote the entry and exit time of a Case i interval.

Proposition 2.2 (Optimal decisions in Case 1 intervals).
If both, serviceables and recoverables inventory are positive ($y_s^* > 0$, $y_u^* > 0$), no items are produced ($p^* = 0$), remanufactured ($r^* = 0$), or disposed of ($w^* = 0$).

Under linear cost functions and positive serviceables inventory (to satisfy demand), holding costs can be saved by postponing production to a point in time where inventory is depleted. Remanufacturing cannot be optimal because recoverables holding cost would be substituted by higher holding cost for serviceables. Disposal cannot be optimal because if decisions before t were optimal, items held in the recoverables inventory could have been disposed of in the past realizing recoverables holding cost savings. Applying optimal decisions, the state variables develop as follows

$$\dot{y}_s = -d < 0 \text{ and } \dot{y}_u = u > 0. \tag{2.27}$$

In a Case 1 interval, the adjoint variables increase (from (2.21) and (2.22)) with rates

$$\dot{\lambda}_s = \alpha\lambda_s + h_s \text{ and } \dot{\lambda}_u = \alpha\lambda_u + h_u \tag{2.28}$$

and therefore by assuming the interval to start at $\theta_{e,1}$, solving above first order differential equations (2.28) yields the values of stored serviceables and recoverables

$$\lambda_s(t) = \left(\lambda_s(\theta_{e,1}^+) + \frac{h_s}{\alpha}\right) e^{\alpha(t-\theta_e)} - \frac{h_s}{\alpha} \tag{2.29}$$

$$\lambda_u(t) = \left(\lambda_u(\theta_{e,1}^+) + \frac{h_u}{\alpha}\right) e^{\alpha(t-\theta_e)} - \frac{h_u}{\alpha} \tag{2.30}$$

where $\lambda_s(\theta_{e,1}^+)$, $\lambda_u(\theta_{e,1}^+)$ have to be determined later. Optimal decisions in Case 1 intervals imply the following limitations on the co-states which follow from (2.12)-(2.14)

$$\lambda_s < c_p, \ \lambda_s - \lambda_u < c_r, \text{ and } \lambda_u > -c_w. \tag{2.31}$$

Proposition 2.3 (Optimal decisions in Case 2 intervals). *If serviceables inventory is zero and recoverables inventory is positive ($y_s^* = 0$, $y_u^* > 0$), the optimal policy is not to produce ($p^* = 0$) and not to dispose ($w^* = 0$). The remanufacturing rate equals the demand rate ($r^* = d$).*

Under zero serviceables and positive recoverables inventory, demand is satisfied from remanufacturing only. Disposal is not optimal for the same reason given for Case 1. Thus, co-states are limited by the following conditions

$$\lambda_s = \lambda_u + c_r \text{ and } -c_w < \lambda_u < c_p - c_r. \tag{2.32}$$

In a Case 2 interval starting at $\theta_{e,2}$, both co-states rise at the same rate

$$\dot{\lambda}_s = \dot{\lambda}_u = \alpha \lambda_u + h_u, \tag{2.33}$$

and the adjoint variables (from (2.16) and (2.22)) are given by

$$\lambda_s(t) = \lambda_u(t) + c_r, \tag{2.34}$$

$$\lambda_u(t) = \left(\lambda_u(\theta_{e,2}^+) + \frac{h_u}{\alpha} \right) e^{\alpha(t-\theta_{e,2})} - \frac{h_u}{\alpha}. \tag{2.35}$$

The recoverables inventory develops according to

$$\dot{y}_u = u - d, \tag{2.36}$$

where the direction of this change depends on the relation of demand and return rate.

Proposition 2.4 (Optimal decisions in Case 3 intervals). *If serviceables inventory is positive and recoverables inventory is zero ($y_s^* > 0$, $y_u^* = 0$), the optimal policy is not to produce ($p^* = 0$) and not to remanufacture ($r^* = 0$). All returns are disposed of ($w^* = u$).*

Production cannot be optimal under positive serviceables inventory since holding cost can be saved by postponing production until inventory becomes zero. For the same reason, remanufacturing cannot occur as long as no costs can be saved by shipping recoverables to serviceables inventory. Collecting returns instead of disposing of them implies to terminate Case 3 and to move to Case 1.

In a Case 3 interval starting at $\theta_{e,3}$, the value of a serviceables item is restricted by

$$\lambda_s < c_r - c_w, \tag{2.37}$$

and it increases with rate

$$\dot{\lambda}_s = \alpha\lambda_s + h_s, \tag{2.38}$$

while the value of an item put into the recoverables inventory stays constant at $-c_w$ since an additional return must be disposed of. The adjoint variables (from (2.17) and (2.21)) are

$$\lambda_s(t) = \left(\lambda_s(\theta_e^+) + \frac{h_s}{\alpha}\right) e^{\alpha(t-\theta_e^+)} - \frac{h_s}{\alpha}, \quad \lambda_u(t) = -c_w. \tag{2.39}$$

As in Case 1, the serviceables stock must decrease, since production and remanufacturing do not take place, i.e.

$$\dot{y}_s = -d < 0. \tag{2.40}$$

Proposition 2.5 (Optimal decisions in Case 4 intervals).
If serviceables and recoverables inventories are zero ($y_s^* = 0$, $y_u^* = 0$), optimal decisions depend on how demand relates to the return rate and two subcases can be distinguished:
Subcase 4(1) $\Leftrightarrow d \leq u$
Demand is satisfied completely by remanufacturing returns ($r^* = d$) and excess returns are disposed of ($w^* = u - d$). No items are produced ($p^* = 0$).
Subcase 4(2) $\Leftrightarrow u < d$
All returns are remanufactured ($r^* = u$) and the missing items are produced ($p^* = d - u$). No items are disposed of ($w^* = 0$).

The case of zero inventories is characterized by demand and return synchronized activities. In absence of inventory holding cost, simultaneous production and disposal is suboptimal because of the general recovery advantage assumption. Situations with excess demand require production, while situations with excess returns require disposal.

With zero inventories, the co-states can be interpreted as financial implications of an additional return (λ_u) or additional demand (λ_s). If returns exceed the demand rate ($d(t) \leq u(t)$), the adjoint variables (from (2.16) and (2.17)) are

$$\lambda_s(t) = c_r - c_w, \quad \lambda_u(t) = -c_w, \tag{2.41}$$

i.e. an additional demand unit would be satisfied from remanufacturing a returned unit which would have otherwise been disposed of, whereas another returned item has to be disposed of because it can neither be stored nor remanufactured.

For $u(t) < d(t)$ we find (from (2.15) and (2.16))

$$\lambda_s(t) = c_p, \quad \lambda_u(t) = c_p - c_r. \tag{2.42}$$

Since there are not enough returns available to satisfy demand completely, demand for another item is satisfied by additional production, while an additional return could be used to replace production by remanufacturing.

The main results of the four cases are summarized in Table 2.1.

Table 2.1. Main results of optimal cases in the basic model.

	p^*	r^*	w^*	$\dot{y}_s$	$\dot{y}_u$	λ_s	$\dot{\lambda}_s$	λ_u	$\dot{\lambda}_u$
Case 1: $y_s > 0$, $y_u > 0$									
	0	0	0	$-d$	u	$< c_p$	$\alpha\lambda_s + h_s$	$-c_w <$	$\alpha\lambda_u + h_u$
Case 2: $y_s = 0$, $y_u > 0$									
	0	d	0	0	$u - d$	$\lambda_u + c_r$	$\alpha\lambda_u + h_u$	$-c_w <$ $< c_p - c_r$	$\alpha\lambda_u + h_u$
Case 3: $y_s > 0$, $y_u = 0$									
	0	0	u	$-d$	0	$< c_r - c_w$	$\alpha\lambda_s + h_s$	$-c_w$	0
Case 4: $y_s = 0$, $y_u = 0$									
(1) $(d \leq u)$	0	d	$u - d$	0	0	$c_r - c_w$	0	$-c_w$	0
(2) $(u < d)$	$d - u$	u	0	0	0	c_p	0	$c_p - c_r$	0
Case 4 generalized: $p^* = \max\{d - r^*, 0\}, r^* = \min\{u, d\}, w^* = \max\{u - r^*, 0\}$									

2.3.4 Optimal Transitions Between Cases and Subcases

After determining the optimal production, remanufacturing, and disposal policy for each of the four inventory situations it is necessary to derive conditions for the possible order of different intervals. First, this order is derived from inventory non-negativity constraints which may be binding. Second, a transition from one case to another can be necessary in order to maximize the Hamiltonian, i.e. the change is associated with a higher value of the Hamiltonian. At this point, it is important to distinguish between two kinds of possible transitions, namely *forced* and *automatic*. An automatic transition occurs when applying the optimal control rule for the current case leads to a transition to another case. For instance, the serviceables inventory running empty would necessitate an automatic transition from Case 1 to Case 2. In contrast to this a forced transition is result of a decision. This happens when inventory is build up in anticipation of an impeding use.

There is a relation between the kind of transition and the co-state variables λ_s and λ_u because some transitions can only occur if the co-state variables are not continuous. These transitions are further called *discontinuous* as opposed to *continuous* case changes. For this reason, it is necessary to investigate the continuity of the adjoints.

Proposition 2.6. *λ_s and λ_u are continuous, i.e. jump parameters η_s and η_u vanish everywhere, except at time points $\theta \in \Theta$ where $y_s(\theta) = y_u(\theta) = 0$ and $u(\theta) = d(\theta)$ holds.*

Proposition 2.6 implies that any transition between cases with the exception of transitions from and to (and within) Case 4 intervals must be continuous w.r.t. the co-states. If it is not, from (2.25) and (2.26) we know that the co-states must fall at the jump point. This immediately excludes transitions from Case 4(1) to Case 4(2).

Corollary 2.1. *Within Case 4 Subcase 4(2) is followed by Subcase 4(1). This automatic and discontinuous transition requires demand to equal the return rate. A transition in the opposite direction is not possible.*

From the optimal decisions in Case 1 and Case 3 intervals it follows, that it is not possible to produce or to remanufacture an amount that exceeds current demand. Thus, at any time t where $y_s(t) > 0$ holds, the inventory level must decrease, i.e. $\dot{y}_s(t) = -d(t) \leq 0$.

Corollary 2.2. *It is never optimal to build up a serviceables inventory.*

Corollary 2.2 excludes any transition from Cases 2 or 4 to 1 or 3. Further, the optimal policy in Case 1 requires that this case must terminate in a Case 2 interval. Transitions from Case 1 to Cases 3 or 4 are therefore not possible.

Corollary 2.3. *A Case 1 interval automatically and continuously terminates in another of Case 2.*

A direct transition from a Case 3 ($y_s > 0,\ y_u = 0$) to a Case 2 ($y_s = 0,\ y_u > 0$) interval is not possible because there must be a time of Case 4 ($y_s = 0,\ y_u = 0$) between Case 3 and 2. This leaves two possible transitions at the end of a Case 3 interval, as stated in the following corollary.

Corollary 2.4. *There exist two types of transitions starting at a Case 3 interval. A continuous and automatic transition to Case 4(1) occurs when the serviceables inventory is depleted, and a forced and continuous transition to Case 1 when a decision is made to stop disposing of the returns.*

It remains to be seen which transitions are possible between Case 2 and Case 4 intervals. Here, we have 4 possibilities as stated in the following corollary.

Corollary 2.5. *In order to build up recoverables inventory, a forced transition from Case 4 to Case 2 is necessary. This can happen either continuously (starting in Subcase 4(1)) or discontinuously (starting in Subcase 4(2)).*
When the recoverables inventory is depleted, an automatic transition from Case 2 to Case 4 takes place. This can be either continuous (terminating in Subcase 4(2)) or discontinuous (terminating in Subcase 4(1)).

Figure 2.3 summarizes all possible transitions.

From Corollary 2.2 it follows that only in the model with initial inventories Cases 1 and 3 appear in an optimal time path. Therefore, Cases 2 and 4 and transitions between them (highlighted through a grey shaded area in Figure 2.3) build the focal point of the model and are examined further in the next section.

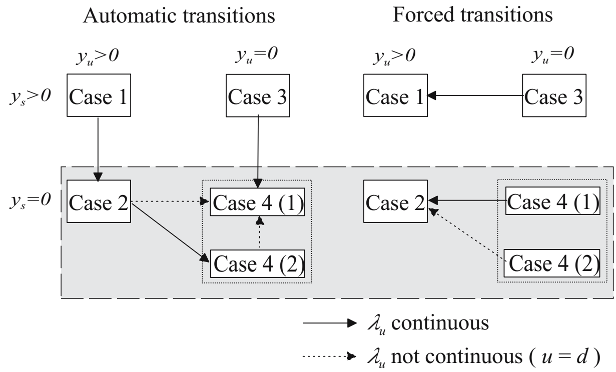

Fig. 2.3. Optimal case transitions in the basic model

2.3.5 Properties of Optimal Return Collection Intervals

After showing which transitions are possible in an optimal solution the question arises at what time these transitions take place. In case of automatic transitions the changes are triggered by following the optimal control policy within a case. But in case of a forced transition the decision maker has to determine the optimal switching time. Therefore, the problem has to be solved to determine the optimal entry point $\theta_{e,2}$ and the optimal exit point $\theta_{x,2}$ of a Case 2 interval. Unfortunately, it is not possible to give an analytical expression for these points, but we can describe the optimal Case 2 intervals and we can provide some properties, especially their optimal entry and exit time. Based on these results an algorithm can be constructed in order to compute these values numerically.

The first property results from the fact that only downward jumps of λ_s and λ_u are permitted inside Case 4 intervals. This leads to the following corollary.

Corollary 2.6 (Location Property).
Let θ denote a time point where $d(\theta) = u(\theta)$. If returns cross demand from above, i.e. there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ with $d(\theta - \varepsilon_1) < u(\theta - \varepsilon_1)$ and $d(\theta + \varepsilon_2) > u(\theta + \varepsilon_2)$ as well as $\varepsilon_1 + \varepsilon_2$ being sufficiently small, then it is always optimal to have a positive recoverables inventory at (and around) time θ ($y_u(\theta) > 0$).

Corollary 2.6 follows immediately from the cost advantage of holding a returned item for an infinitesimal (sufficiently small) time and identifies the time where returns (that otherwise would be disposed of) can be saved before θ and used with higher benefits after θ by remanufacturing it instead of producing a new item. Considering that it might only be beneficial to store returns for a limited time (this aspect is discussed below in more detail), there

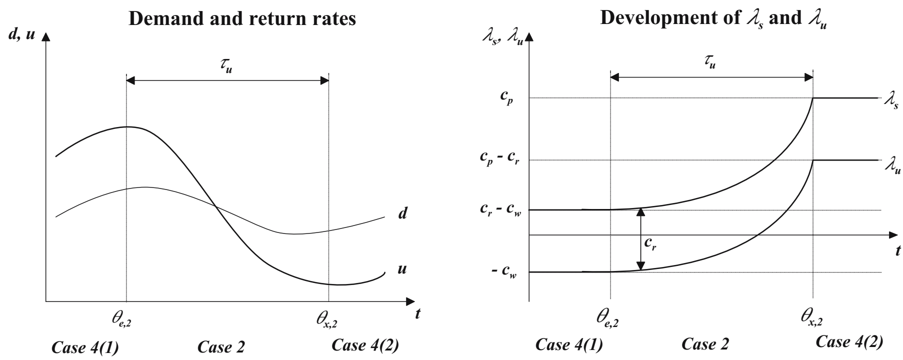

Fig. 2.4. Example of a Case 2 interval which has maximal length.

might be a special case, that returns equal demands for some period. In this case, we have to ensure that this period does not exceed the maximum time for which it is beneficial to store returns.

In the following the maximal length of a Case 2 interval is investigated. Inside this case, a lower bound for the value of a returned item λ_u is given by $-c_w$ and an upper bound is $c_p - c_r$. Due to the increasing co-state λ_u and its continuity, inserting these minimal and maximal values into the co-state development equation within a Case 2 interval (2.35) yields the following condition which is illustrated in Figure 2.4.

Proposition 2.7 (Maximal Length Property).
The maximal length τ_u *of a Case 2 interval* $I = (\theta_{e,2}, \theta_{x,2})$ *is given by*

$$(\theta_{x,2} - \theta_{e,2}) \leq \tau_u := \frac{1}{\alpha} \ln \left(\frac{\alpha(c_p - c_r) + h_u}{-\alpha c_w + h_u} \right).$$

The Maximal Length Property represents a marginal criterion for collecting one returned item which can be interpreted as follows. An additionally collected return at $\theta_{e,2}$ is needed for reuse at time $\theta_{x,2}$. Therefore, it will be stocked for a time interval of length $\theta_{x,2} - \theta_{e,2}$. This can only be optimal if the corresponding holding costs do not exceed the recovery cost advantage and also accounting for discounting and different time values of the respective payments. τ_u will further be referred to as *Maximal Holding Time* (of returns).

It can easily be shown, that

$$\lim_{\alpha \to 0} \tau_u = \frac{c_p + c_w - c_r}{h_u} \tag{2.43}$$

holds which constitutes the maximal length property as shown by Minner and Kleber (2001) for the undiscounted case. In the absence of time value considerations, an item will be stored until there is equality of holding costs $h_u \cdot \tau_u$ and the direct recovery cost advantage $c_p + c_w - c_r$.

The question arises if all optimal Case 2 intervals have maximal length or when such an interval does not have maximal length. Before we answer this question we provide another property of the optimal Case 2 intervals. Obviously, at the beginning and the end of a Case 2 interval the recoverables inventory is empty. Therefore, cumulative returns and cumulative remanufacturing rate over the entire time interval have to be equal. Further, during the time interval, cumulative returns have to exceed cumulative remanufacturing rates in order to remain in Case 2. This is intuitively clear and leads to the following proposition (presented without proof).

Proposition 2.8 (Inventory Conditions).
Let $I = (\theta_{e,2}, \theta_{x,2})$ be an open time interval where $y_u > 0$ and $y_u(\theta_{e,2}) = y_u(\theta_{x,2}) = 0$. Then,

(i) cumulative demand equals cumulative returns over the whole interval

$$\int_{\theta_{e,2}}^{\theta_{x,2}} (d(t) - u(t))dt = 0, \tag{2.44}$$

(ii) at any point $\theta \in I$, cumulative returns must be larger than cumulative demand

$$\int_{\theta_{e,2}}^{\theta} (u(t) - d(t))dt > 0. \tag{2.45}$$

As an implication of Proposition 2.8 it must hold that at $\theta_{e,2}$, the return rate must not be smaller than the demand rate in order to start collecting returns and at $\theta_{x,2}$, the opposite must hold.

In order for a feasible solution which holds under above inventory conditions, lengths of optimal Case 2 intervals may be smaller than the maximal length given in Proposition 2.7. There are two reasons for this. First, it is possible that there are not enough returns available to save further units for later remanufacturing. In such cases the interval starts at a point in time where returns equal demand rate or at the begin of the planning period (at time zero). Otherwise, it would have been possible to start collecting returns earlier and thereby, increasing the Case 2 interval length. A second reason for non-maximal interval lengths is no excess demands. Then, the Case 2 interval must end at a point where returns equal demands or at the planning horizon T. If this is not the case, additional demand could have been satisfied by extending the Case 2 interval.

Proposition 2.9. *If the interval I does not reach its maximum length, i.e. if $(\theta_{x,2} - \theta_{e,2}) < \tau_u$ holds, and if $\theta_{e,2} > 0$ and $\theta_{x,2} < T$, then*
(i) $u(\theta_{e,2}) = d(\theta_{e,2})$ or
(ii) $u(\theta_{x,2}) = d(\theta_{x,2})$ must hold.

Up to now we have shown how given optimal entry and exit points of Case 2 intervals can be characterized. How these results can be used in order to determine the optimal entry and exit points of optimal Case 2 intervals (without having this information) is shown in the following section where an algorithm for this problem is provided.

2.3.6 Solution Algorithm

The Cases 2 and 4 build the focal point of the model with zero initial inventories, because from Proposition 2.2 it follows that Cases 1 and 3 will never appear in an optimal solution. From the initial and terminal conditions we know that at both points, $t = 0$ and $t = T$, Case 4 must apply. Note that if there exists no time where Corollary 2.6 holds, it is never optimal to have a positive recoverables inventory. For constructing an optimal time path we suggest a forward algorithm that proceeds as follows.

Algorithm 2.1

Step 1
Identify all K time points for which Corollary 2.6 holds.
let θ^k denote the intersection points $k = 1, ..., K$.

Step 2
let $k := 1$.
while $k \leq K$ **do**
  Apply Propositions 2.7, 2.8, and 2.9 to determine the corresponding interval $I^k = (\theta^k_{e,2}, \theta^k_{x,2})$:
    let $\theta^k_{e,2} = \theta^k_{x,2} = \theta^k$. Decrease $\theta^k_{e,2}$ and increase $\theta^k_{x,2}$ so that Proposition 2.8 always holds until one of the terminating conditions given in Propositions 2.7 or 2.9 (i) or (ii) are achieved, $\theta^k_{e,2} = 0$ or $\theta^k_{x,2} = T$.
  let $k := k + 1$.
end while

Step 3
while there exist two consecutive intervals with $\theta^k_{x,2} = \theta^j_{e,2}$, $j > k$ and $\theta^j_{x,2} - \theta^k_{e,2} < \tau_u$ **do**
  let $\theta^k_{x,2} := \theta^j_{x,2}$. Delete point j from the set of construction intervals. Decrease $\theta^k_{e,2}$ and increase $\theta^k_{x,2}$ so that Proposition 2.8 always holds until one of the terminating conditions given in Propositions 2.7 or 2.9 (i) or (ii) are achieved or $\theta^k_{e,2} = 0$ or $\theta^k_{x,2} = T$.
end while

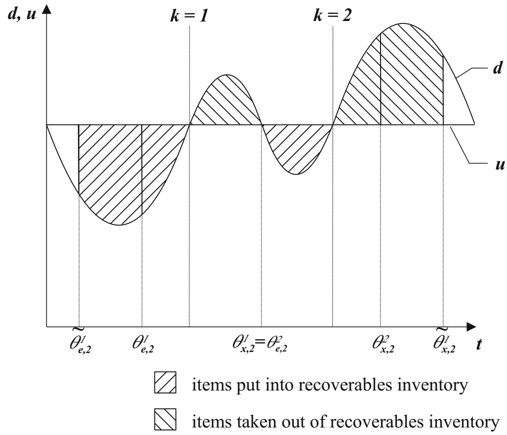

Fig. 2.5. Example for applying the forward algorithm

The following example with a constant return rate and a seasonal demand pattern as depicted in Figure 2.5 is used to illustrate Algorithm 2.1. First, Step 1 identifies two intersection points $k = 1, 2$ where returns intersect demands from above. According to Step 2, the collection intervals $(\theta^1_{e,2}, \theta^1_{x,2})$ for $k = 1$ and $(\theta^2_{e,2}, \theta^2_{x,2})$ for $k = 2$ are extended until one of the stopping criteria holds. In this case, the interval construction ends with criterion (i) and (ii) from Proposition 2.9 because at the time where demands intersect returns from above, the first interval cannot be extended to the right and the second interval cannot be extended to the left without violating the integral condition. Note that (in general) if we assume a rather small value for τ_u, that both interval constructions may not reach this point and the interval construction would terminate because of maximal length reached. Because this example also covers the special situation dealt with by Step 3, we find two intervals that directly follow each other. Let us assume that the cumulative length of both intervals is still smaller than τ_u. Then, a joint interval is constructed by extending the interval to $\tilde{\theta}^1_{e,2}$ on the left and to $\tilde{\theta}^1_{x,2}$ on the right. Here, it is assumed that the construction ends because maximum length of a collection interval is reached, i.e. $\tilde{\theta}^1_{x,2} - \tilde{\theta}^1_{e,2} = \tau_u$.

2.4 Dealing with Initial Inventories

The main difference that arises from allowing for initial stocks is that the jump variable w_0 does not necessarily vanish in problem (2.9). w_0 stands for a disposal quantity (impulse) of unwanted returns which immediately decreases recoverables stock by a quantity (jump). This scenario requires the generalized maximum principle that allows for jumps in state variables, i.e. in our case y_u. Since we know that a jump in y_u is possible only at time zero, the following necessary conditions have to be added (see Feichtinger and Hartl (1986)).

Define the impulse Hamiltonian function H_0 (being valid at time zero)

$$H_0(y_s(0), y_u(0^-), w_0, \lambda_u(0)) = -c_w \cdot w_0 - \lambda_u(0) \cdot w_0. \tag{2.46}$$

Due to its linearity, (2.46) is maximized if (omitting time indices)

$$w_0^* = \begin{cases} 0 & \lambda_u > -c_w \\ singular & \lambda_u \leq -c_w. \end{cases} \tag{2.47}$$

Since the states do not appear in the impulse Hamiltonian the adjoints do not jump at time zero, i.e. $\lambda_u(0^-) = \lambda_u(0^+)$. It is clear that w_0 vanishes if the value of stored returns exceeds (negative) payments ('revenue') for disposal ($\lambda_u > -c_w$). This implies that the disposal rate equals zero. Otherwise, the height of the jump w_0 must be determined by searching for a desired inventory level $\tilde{y}_u^0$ followed by setting the optimal disposal quantity equal to $w_0^* = \max\{0, y_u^0 - \tilde{y}_u^0\}$.

Proposition 2.10. *The desired initial recoverables inventory level $\tilde{y}_u^0$ is given by the net demand during a time interval $[0, \theta^u]$*

$$\tilde{y}_u^0 = \max\left\{0, \int_0^{\theta^u} (d(t) - u(t))dt - y_s^0\right\}.$$

$\theta^u \leq \tau_u$ is the largest value for which holds at any point $\theta < \theta^u$ cumulative demand exceeds cumulative returns

$$\int_\theta^{\theta^u} (d(t) - u(t))dt \geq 0. \tag{2.48}$$

Except for the trivial case $y_s \geq \int_0^T d(t)dt$, a starting interval with $y_s > 0$ will always (at time $\theta^s \in (0, T)$) terminate in an interval with $y_s = 0$ after using all serviceables inventory to satisfy demand, i.e.

$$\theta^s \quad : \quad \int_0^{\theta^s} d(t)dt = y_s^0. \tag{2.49}$$

Let us consider the four cases outlined in Section 2.3.3 as initial conditions (after disposing of quantity w_0^*). Case 4 initial conditions $y_s^0 = \tilde{y}_u^0 = 0$ lead to the results derived before. If the system starts in Case 1 ($y_s^0 > 0$, $\tilde{y}_u^0 > 0$) it will turn to Case 2 ($y_s = 0$, $y_u > 0$) at some time θ^s and finally (at time $\theta^u > \theta^s$) terminate automatically into a Case 4 interval. For a starting Case 2 interval, the same holds with $\theta^s = 0$.

The most interesting case is a starting interval of Case 3 where $y_s^0 > 0$, $\tilde{y}_u^0 = 0$ holds. While the above starting inventory (except Case 4) configuration always terminates with both state variable constraints binding, the end of a Case 3 starting interval can also be driven by a cost advantage of collecting returns. The alternatives are (1) to continue with the Case 3 policy of satisfying demand from serviceables inventory and to dispose of all returns and (2) switch to Case 1 and start collecting excess returns for future recovery within a subsequent Case 2 interval. The conditions for switching from Case 3 to Case 1 can be determined by applying the following modification of Proposition 2.8 (presented without proof).

Proposition 2.11. *Let $I = (\theta_{e,1}, \theta_{x,2})$ be an open time interval (of Case 1 immediately followed by Case 2) where $y_u > 0$ and $y_u(\theta_{e,1}) = y_u(\theta_{x,2}) = 0$. Then,*

(i) cumulative demand equals the sum of cumulative returns over the whole interval plus serviceables inventory at $\theta_{e,1}$

$$\int_{\theta_{e,1}}^{\theta_{x,2}} (d(t) - u(t))dt - y_s(\theta_{e,1}) = 0,$$

(ii) at any point $\theta \in I$ the sum of cumulative returns and serviceables inventory at $\theta_{e,1}$ must be larger than cumulative demand

$$y_s(\theta_{e,1}) + \int_{\theta_{e,1}}^{\theta} (u(t) - d(t))dt > 0.$$

After collecting recoverables in a Case 1 time interval, these stocks are used to remanufacture for demand in a Case 2 interval with positive initial recoverables inventory.

The solution algorithm changes in the following way. First, one determines the desired initial recoverables inventory $\tilde{y}_u^0$ and thus obtains the initial disposal quantity. Considering that until some time point θ^s the demand can be satisfied by using the serviceables inventory, we define the adjusted demand rate

$$\tilde{d}(t) = \begin{cases} 0 & \text{if } t \leq \theta^s \\ d(t) & \text{otherwise.} \end{cases} \tag{2.50}$$

Afterwards, one continues with the proposed solution algorithm using $\tilde{y}_s^0 = 0$, $\tilde{y}_u^0$, i.e. the transformed initial conditions and $\tilde{d}$ are used. Until time θ^u, we always have a time interval with $y_u > 0$. Note, that due to the possible jump in the demand rate at time θ^s, the necessary conditions do not hold at this point, which can lead to a jump in the adjoint variables. At time θ^s it is therefore necessary to check whether $\tilde{d}$ jumps above the return rate which implies that it is optimal to have a recoverables inventory at this time.

2.5 Numerical Examples

In this section the results of the basic model are illustrated by using three examples (Examples 2.1-2.3) based on the demand/return scenario introduced in Section 2.2. According to the Location Property, intervals with positive recoverables inventory will be located around time points $\theta^k \in \{5.92, 12.21\}$. All three examples differ with respect to initial stock levels in both serviceables and recoverables inventories as shown in Table 2.2. Cash flow parameters are set as follows: $c_p = 2$, $c_r = 1$, $c_w = 1$, $h_u = 1$, and $\alpha = 0.1$. Therefore, a Case 2 interval length will not exceed 2.00 units time.

Table 2.2. Initial inventory levels in Examples 2.1-2.3

Example	y_s^0	y_u^0
2.1	0	0
2.2	1	1
2.3	2	1

For now, we consider the case of zero initial inventories (Example 2.1). Using the solution algorithm shown in Section 2.3.6 the following two collection intervals are determined: $[\theta^1_{e,2}, \theta^1_{x,2}] = [4.85, 6.85]$ and $[\theta^2_{e,2}, \theta^2_{x,2}] = [11.82, T]$. The first interval has maximum length, but the second one is shorter because it reaches the end of the planning horizon where no further demand is available. Figure 2.6 shows optimal Case 2 intervals and the optimal development of the value of returns λ_u. The resulting inventory levels are depicted in Figure 2.7.

Outside optimal Case 2 intervals demand is satisfied either from remanufacturing and production if demand exceeds the return rate, or from remanufacturing alone if the opposite applies. In the first case, an additional return would have a value of $c_p - c_r = 1$ since it could be used immediately to forego production. In the second case, the additional return ought to be disposed of, thus having a value of $-c_w = -1$. Inside Case 2 intervals demand is met by remanufacturing currently or previously collected returns, which both are valued equally. The value of an additional (marginal) return increases with time, since the time it spends in inventory until usage decreases and thus, corresponding out-of-pocket holding costs and opportunity costs would reduce.

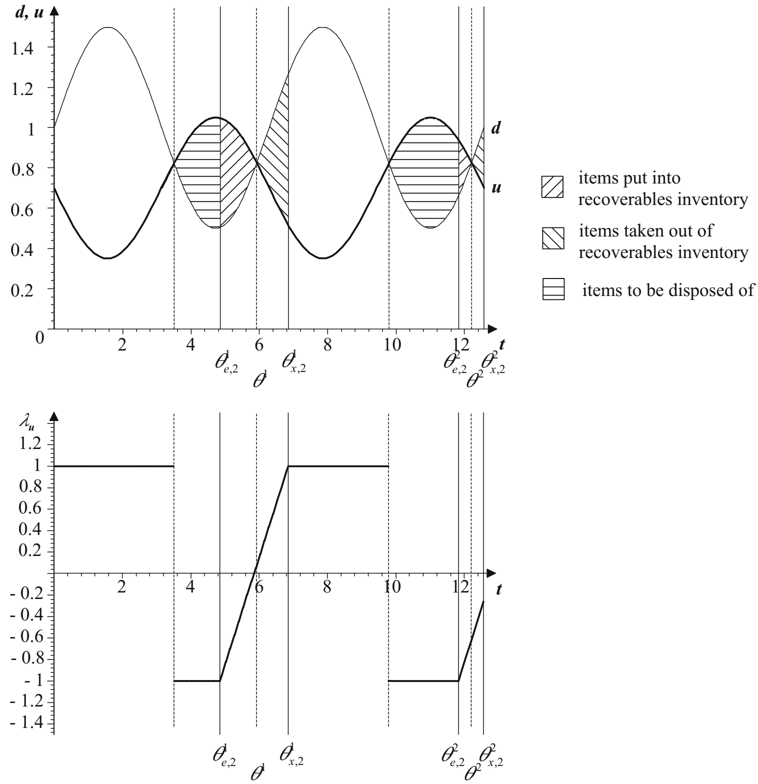

Fig. 2.6. Optimal Case 2 intervals and co-state λ_u development in Example 2.1.

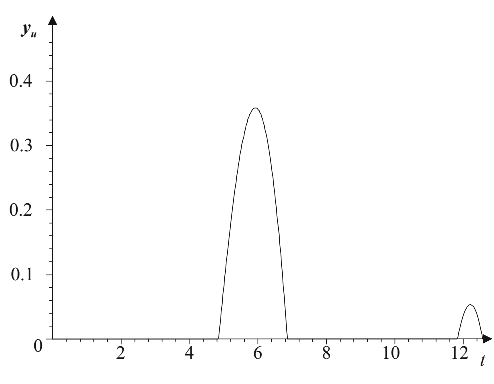

Fig. 2.7. Optimal inventory levels in Example 2.1.

Let us now presume two scenarios with positive initial inventory levels for serviceables and recoverables. Under such situations there are two additional questions to be answered. First, it has to be determined whether to keep all recoverables available at time zero or to dispose of a certain quantity w_0 and secondly, we want to know how operative decisions change.

In Example 2.2 where $y_s^0 = 1, y_u^0 = 1$, the serviceables inventory is depleted first which completes at time $\theta_s = 0.83$. Thus, the desired recoverables inventory level at time zero (which can be used up within economically reasonable time) amounts to $\tilde{y}_u^0 = 0.81$. A quantity of $w_0 = 0.19$ is therefore disposed of at time zero. The remaining returns are kept and further accumulated using incoming returns until after θ_s, upon which recoverables inventory is reduced by filling all demand from remanufacturing. In other words, at θ_s a transition from a Case 1 to a Case 2 interval occurs. The latter finishes at time $\theta_{x,2}^0 = \tau_u = 2$ as the inventory runs empty. Then, the same policy is optimal as described in Example 2.1. Optimal Case 1 and 2 intervals as well as the changed evolution of the value of returns (λ_u) can be seen from Figure 2.8. The corresponding serviceables and recoverables inventory levels are depicted in Figure 2.9.

Since in Example 2.3 initial serviceables stock is higher than in the previous example, it is not used up before $\theta_s = 1.52$. It is possible to acquire sufficient returns during this initial period making it superfluous to have any returns available at time zero ($\tilde{y}_u^0 = 0$). All initial returns are disposed of ($w_0 = 1$). The system therefore starts in a Case 3 situation. Under these circumstances one wants to know whether to have a positive recoverables inventory at θ_s and at what point to start collecting returns, i.e. when to switch from Case 3 to Case 1. Since after θ_s demand surpasses currently available returns, stockkeeping of returns is profitable and it starts at time $\theta_{e,2}^0 = 0.11$. At θ_s a transition from a Case 1 to a Case 2 interval takes place, and since stock-keeping of returns here yields maximal length at $\theta_{x,2}^0 = 2.11$ the recoverables inventory is depleted. As in the previous example, subsequent optimal decisions follow the same pattern as in Example 2.1, being depicted on top of optimal costate developments in Figure 2.10. The changed stock levels are to be found in Figure 2.11.

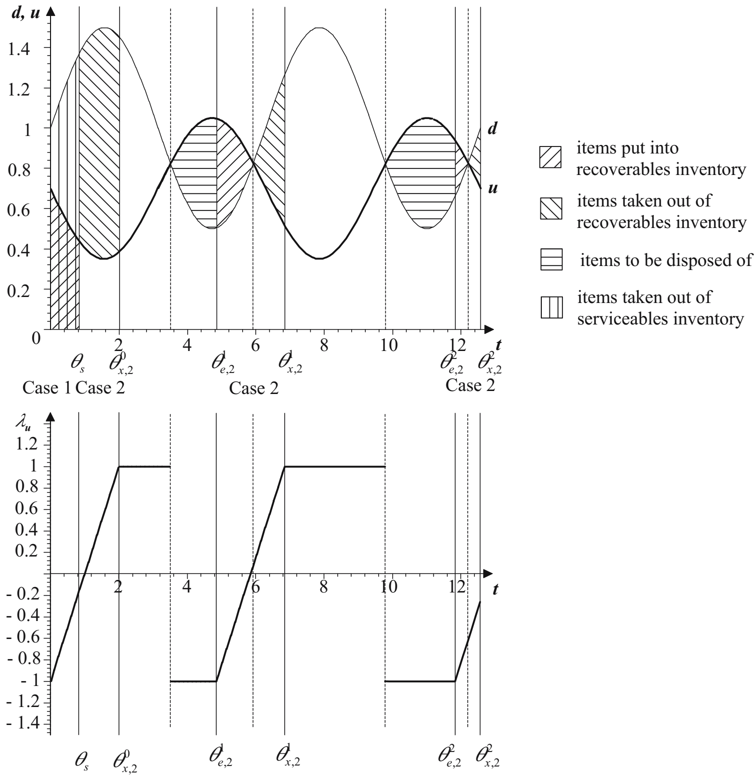

Fig. 2.8. Optimal Case 1/2 intervals and co-state λ_u development in Example 2.2.

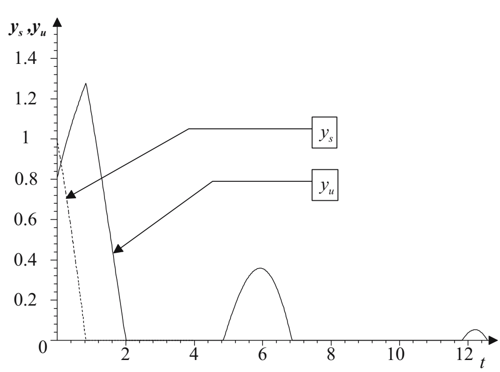

Fig. 2.9. Optimal inventory levels in Example 2.2.

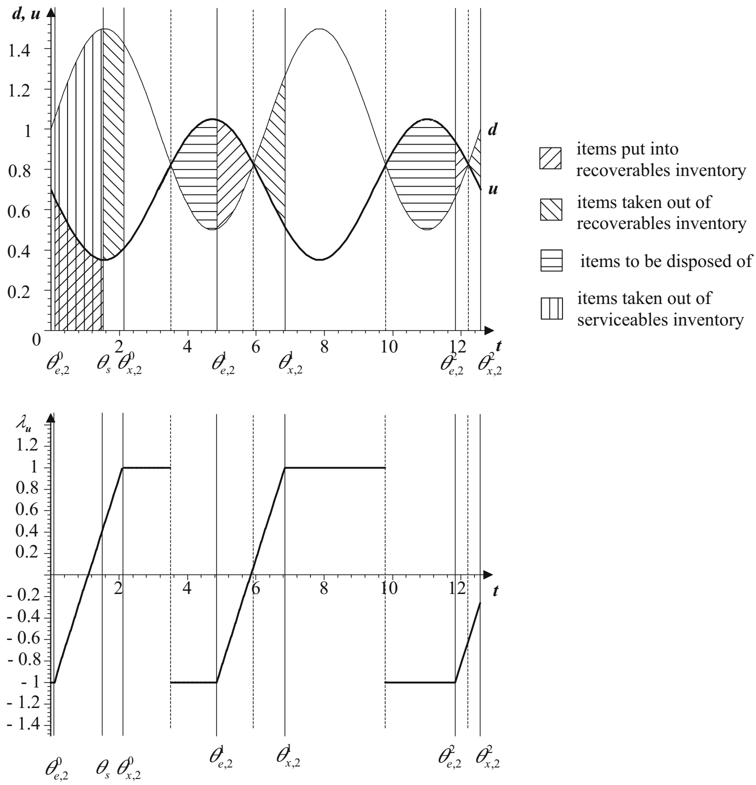

Fig. 2.10. Optimal policies and co-state λ_u development in Example 2.3.

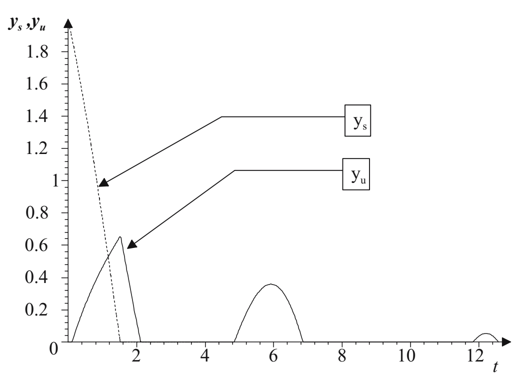

Fig. 2.11. Optimal inventory levels in Example 2.3.

2.6 Comparison of Holding Cost and Discounted Cash Flow

In inventory control Holding Cost (HC, also called Average Cost, AC, for cyclic infinite horizon problems) models often are used as approximation to the as correct recognized Discounted Cash Flow (DCF) approach because they are easier to solve. It has been shown for a number of models (see, e.g., Grubbström (1980), Corbey et al. (1999)) that the HC framework leads to almost the same optimal decision as the DCF approach when choosing the HC holding cost rate appropriately. While in DCF models actual cash flows are considered and holding an item in inventory causes out-of-pocket-costs only, the cost oriented HC approach also includes the opportunity costs of capital tied up in stock. Thus, valuation of inventories becomes crucial in order to find almost correct inventory control decisions.

A lot of effort has been spent in order to determine the 'right' holding cost rate and to correctly value stored items (see, e.g., Teunter et al. (2000), Teunter and van der Laan (2004)). Although there exists general criticism of such undertaking (see, e.g., Fleischmann (2001a)), the deterministic framework presented above can also be used to find a holding cost rate that leads to the same results when comparing above DCF and the corresponding total undiscounted costs solutions provided by Minner and Kleber (2001). Since there is no motivation for having a positive stock in the serviceables inventory in this model, the results are restricted to the holding cost rate for recoverables. In contrast to the usual approach where the HC objective constitutes a first order approximation of the respective DCF objective (Fleischmann (2001a)), in the sequel we deal with properties of the respective optimal solutions.

2.6.1 Holding Cost Results

A main result from solving both discounted and undiscounted models is that the value of stored returns as expressed by the co-state λ_u is not constant over time, and therefore no unequivocal value and with it no 'right' holding cost rate exists. However, given identical parameters for all other operational cash flows and costs a recoverables holding cost rate h_u^{HC} can be determined leading to the same results as are obtained in the DCF approach with out-of-pocket holding cost rate h_u^{DCF} by valuing the returns with some kind of average value.

When solving the HC or the corresponding DCF problem a difference occurs in setting the holding time, i.e. the maximum length an interval with positive recoverables inventory can possibly have. Since α vanishes in (2.22), the co-state λ_u rises linearly with rate $\dot{\lambda}_u = h_u$ in a Case 2 interval where $y_u > 0$. Thus, the Maximal Holding Time in the HC case is given by (see Minner and Kleber (2001), compare with (2.43))

$$\tau_u^{HC} = \frac{c_p + c_w - c_r}{h_u^{HC}}, \tag{2.51}$$

and in the DCF case it is (see Proposition 2.7)

$$\tau_u^{DCF} = \frac{1}{\alpha} \ln\left(\frac{\alpha(c_p - c_r) + h_u^{DCF}}{-\alpha c_w + h_u^{DCF}}\right). \tag{2.52}$$

If $\tau_u^{DCF} = \tau_u^{HC}$, both approaches lead to the same solution.

The difference between both holding cost rates can be modeled as a mark-up $m_h = h_u^{HC} - h_u^{DCF}$. In the following the precise mark-up for the recoverables holding cost rate as well as two approximations for small values of α will be determined. Several numerical examples are used in order to assess the quality of the approximations.

2.6.2 A Comparison of Different Approximations

Now, let us determine the value of the mark-up m_h that assures

$$\tau_u^{HC} = \tau_u^{DCF} \tag{2.53}$$

by assuming identity of all parameters except for the holding cost rates. Further, it is required that out-of-pocket holding costs exceed costs from postponing disposal ($h_u^{DCF} > \alpha c_w$) in order to assure a finite value of τ_u^{DCF}. For instance this is always true in situations where disposal earns a salvage revenue ($c_w < 0$).

Exact solution. For finite values of τ_u^{DCF}, solving equation (2.53) for m_h gives the exact value of the mark-up

$$m_h = \frac{\alpha(c_p + c_w - c_r)}{\ln\left(\frac{\alpha(c_p - c_r) + h_u^{DCF}}{-\alpha c_w + h_u^{DCF}}\right)} - h_u^{DCF}, \tag{2.54}$$

for which two approximations are elaborated.

Approximation 1. Replacing the logarithm term in (2.54) by its first order Taylor series approximation ($\ln x \approx x - 1$) which is valid for small values of α yields

$$\begin{aligned} m_h &\approx \frac{\alpha(c_p + c_w - c_r)}{\frac{\alpha(c_p - c_r) + h_u^{DCF}}{-\alpha c_w + h_u^{DCF}} - 1} - h_u^{DCF} \\ &= \frac{\alpha(c_p + c_w - c_r)}{\frac{\alpha(c_p - c_r) + h_u^{DCF} + \alpha c_w - h_u^{DCF}}{-\alpha c_w + h_u^{DCF}}} - h_u^{DCF} = -\alpha c_w. \end{aligned}$$

A nearly optimal HC recoverables holding cost rate is given by out of pocket costs, reduced by the interests on the disposal cost rate, i.e. $h_u^{DCF} - \alpha c_w$.

This value can be interpreted as follows. Holding costs only play a role during intervals where the recoverables inventory is positive. If an additional return arrives during such a collection interval, there are two options, it can be either be stocked (which will be done in order to shorten the collection interval) or it can be disposed of at cost c_w. Therefore, the value of this item is given by the opportunity cost saved by storing it, i.e. the negative disposal cost rate. Holding an item becomes cheaper by (saved) interests on disposal $-\alpha c_w$ or more expensive by (lost) interests on the salvage value.

Approximation 2. A first order MacLaurin series approach applied to the whole term (2.54) yields the following approximation being valid for small values of α

$$m_h = \alpha \left(\frac{c_p - c_r}{2} + \frac{-c_w}{2} \right) + O(\alpha^2). \tag{2.55}$$

For a complete derivation see page 44. In order to obtain results similar to the DCF solution, the value of a returned item should be set to the unweighted average of the difference of direct production and remanufacturing costs $c_p - c_r$ and the opportunity cost saved by not disposing the item $-c_w$. This makes sense if one considers that m_h has been derived by using a marginal criterion regarding the first item to be put on stock. The value of this (marginal) returned item (represented by the co-state variable) continuously increases during the interval where it is kept on inventory and for small values of α (no discounting) this movement is approximately linear. Since the minimum value of the stored return is given by $-c_w$ (otherwise it would be disposed of) and as a maximum $c_p - c_r$ may be reached (otherwise production would be preferred), above average value results.

Comparison. It remains to be seen whether these approximations perform well or not. In order to answer this question, optimal and approximated holding times are plotted against the discount rate using a number of different scenarios. These were derived by using a factorial design that helps to find out how the parameters interact with each other. Since unit production and remanufacturing costs only appear as a difference in all relevant terms, these parameters have been fixed in such a way that the difference $(c_p - c_r) \in \{-1, 1, 3, 5\}$. The disposal parameter uses the following values in order to account for high, zero, and negative (i.e. positive salvage revenue) per unit cost rates $c_w \in \{-2, 0, 2\}$. Finally, we considered two levels (high and low) for the out-of-pocket holding cost rate $h_u^{DCF} \in \{0.1, 1\}$. By excluding invalid parameter combinations this design leads to eighteen different scenarios. Cost parameters of the examined scenarios as well as the maximal allowed discount rate α_{max} for a finite Maximal Holding Time and average values of the recoverables on stock according to Approximations 1 and 2 can be found in Table 2.3.

In order to gain insight into the quality of the approximations, a maximum permissible discount rate for a given approximation error in the holding time is calculated for maximum relative errors of 1%, 10%, 20% and 50%, see Tables 2.4 and 2.5.

Table 2.3. Cost parameters in examined scenarios.

Scenario	$c_p - c_r$	c_w	h_u^{DCF}	α_{max}	$-c_w$	$\frac{c_p-c_r}{2} - \frac{c_w}{2}$
1	-1	2	0.1	.05	-2	-1.5
2	-1	2	1	.5	-2	-1.5
3	1	0	0.1	∞	0	.5
4	1	0	1	∞	0	.5
5	1	2	0.1	.05	-2	-.5
6	1	2	1	.5	-2	-.5
7	3	-2	0.1	∞	2	2.5
8	3	-2	1	∞	2	2.5
9	3	0	0.1	∞	0	1.5
10	3	0	1	∞	0	1.5
11	3	2	0.1	.05	-2	.5
12	3	2	1	.5	-2	.5
13	5	-2	0.1	∞	2	3.5
14	5	-2	1	∞	2	3.5
15	5	0	0.1	∞	0	2.5
16	5	0	1	∞	0	2.5
17	5	2	0.1	.05	-2	1.5
18	5	2	1	.5	-2	1.5

From comparing values in Tables 2.4 and 2.5 it follows that Approximation 2 performs considerably better than Approximation 1 in all scenarios, which is understandable as it uses a more sophisticated method. Further it can be stated, that the performance of the preferable Approximation 2 depends on the parameters in such a way that it tends to perform better under circumstances that lead to a small (correct) Maximal Holding Time, i.e. for smaller $c_p - c_r$, higher salvage revenues (negative disposal costs) $-c_w$, or higher out-of-pocket holding cost rates h_u^{DCF} (see e.g. Scenario 8). This result is intuitive, since a smaller holding time also implies a smaller impact of compound interest. On the other hand, the approximation should not be used if a high Maximal Holding Time is expected (Scenario 17).

The first approximation tends to perform better as its valuation of returns gets closer to that of the second, i.e. where $-c_w \approx c_p - c_r$ holds (Scenarios 1 to 4, 7 and 8). Although it seems to be (without knowing the technical details) more intuitive than the second, it should generally not be used.

Our results also show that one has to be very careful when designing experiments for determining the 'right' holding cost rate in more complex models, which can not be solved analytically, e.g. when stochastic influences or setup costs are considered. For instance, Teunter et al. (2000) compared five different reasonable methods for setting the holding cost rates in a model with stochastic demand and return rates as well as setup costs for manufacturing and remanufacturing by using simulation methods, but these did not include valuation methods like that one which we derived by using Approximation 2.

Table 2.4. Maximum discount rates α_{max} for different error levels when using Approximation 1.

Scenario	1%	10%	20%	50%
1	0.0019	0.0146	0.0230	0.0348
2	0.0193	0.1461	0.2297	0.3479
3	0.0020	0.0206	0.0425	0.1144
4	0.0201	0.2065	0.4250	1.1440
5	0.0007	0.0060	0.0110	0.0216
6	0.0066	0.0605	0.1104	0.2163
7	0.0021	0.0352	0.2835	> 100
8	0.0209	0.3517	2.8351	> 100
9	0.0007	0.0069	0.0142	0.0381
10	0.0067	0.0688	0.1417	0.3813
11	0.0004	0.0038	0.0073	0.0157
12	0.0040	0.0381	0.0727	0.1570
13	0.0007	0.0080	0.0198	0.1607
14	0.0068	0.0798	0.1977	1.6069
15	0.0004	0.0041	0.0085	0.0229
16	0.0040	0.0413	0.0850	0.2288
17	0.0003	0.0028	0.0054	0.0123
18	0.0029	0.0279	0.0541	0.1232

Table 2.5. Maximum discount rates α_{max} for different error levels when using Approximation 2.

Scenario	1%	10%	20%	50%
1	0.0227	0.0408	0.0454	0.0495
2	0.2274	0.4079	0.4538	0.4945
3	0.0417	0.2214	0.4906	4.5063
4	0.4170	2.2143	4.9064	45.0633
5	0.0109	0.0298	0.0383	0.0484
6	0.1088	0.2981	0.3829	0.4839
7	0.2510	> 100	> 100	> 100
8	2.5103	> 100	> 100	> 100
9	0.0139	0.0738	0.1635	1.5021
10	0.1390	0.7381	1.6355	15.0211
11	0.0071	0.0235	0.0331	0.0474
12	0.0715	0.2349	0.3312	0.4737
13	0.0192	> 100	> 100	> 100
14	0.1925	> 100	> 100	> 100
15	0.0083	0.0443	0.0981	0.9013
16	0.0834	0.4429	0.9813	9.0127
17	0.0053	0.0194	0.0292	0.0464
18	0.0532	0.1938	0.2918	0.4640

2.7 Managerial Insights

In this chapter, we presented an optimization approach that accounts for demand and return dynamics in deterministic recovery management under a linear discounted cost regime. We showed ,that there exist another motivation, besides of cycle stock, safety, and seasonal stocks, for keeping inventory, which arises from a dynamic environment. An *anticipation stock* is used if returns arrive at times when they can not be used immediately, but are stored in order to satisfy demand some time later and therefrom realizing a cost advantage.

The model presented allows the determination of return collection and recovery time intervals. The optimal control framework seems fairly promising to investigate several other aspects of reverse logistics. First, the simultaneous determination of the adjoint trajectories to serviceables and recoverables inventories yields a dynamic economic valuation of returned products. Such value is not simply given by the remanufacturing cost advantage $c_p + c_w - c_r$ but also must account for holding costs from return to reuse period. The first return not disposed of in a collection time interval has a value of $\lambda_u = -c_w$ because of the assumption that undesired returns cannot be rejected. The last collected return, which is immediately recovered and sold, has a value equal to the difference of production and remanufacturing cost rate $\lambda_u = c_p - c_r$. For returns being collected and kept in inventory, the respective inventory holding costs have to be subtracted. This reasoning yields a property regarding a Maximal Holding Time of returns (τ_u) which together with inventory conditions was used to construct a forward solution algorithm. Another interesting implication is that the Maximal Length Property can be used to determine the least required length of the planning horizon within a rolling planning environment (see e.g. Inderfurth et al. (2004) for an overview on production planning for product recovery management). This is due to that fact that only demand and return information for the next τ_u periods are needed in order to answer the question of whether to keep a currently returned item or not.

Further, an optimal way to deal with initial inventories was presented. Since it was shown that it is never optimal to build up a serviceables inventory, after depletion of a given initial stock by filling complete demand for a certain period, production and remanufacturing rates are synchronized with demand rate, which is always possible in the absence of capacity constraints. Regarding an initial recoverables stock, a desired initial recoverables inventory level was determined, which gives the maximum amount of returns that can be used within (economically) reasonable time. If initial stock exceeds desired stock, excess returns have to be disposed of.

In order to investigate commonalities and differences of undiscounted cost (see Minner and Kleber (2001)) and discounted cash flow approach, we compared the results and determined an HC holding cost rate leading to the same optimal solution as our DCF approach. Using an appropriate approximation method, this holding cost rate could be separated into out-of-pocket holding and interest rate based opportunity cost of capital, where the 'value' of returned products is given as an (unweighted) average of the direct remanufacturing advantage $c_p - c_r$ and the opportunity costs saved by not disposing of the item $-c_w$.

The assumption of linear unit costs for production, remanufacturing, and disposal concerns most business practitioners and with appropriate extension it allows for more insights with respect to capacity aspects and economies of scale. When explicitly considering capacity constraints, an optimal production/inventory policy smoothes demand variations. These aspects are considered in more detail in Chapter 3. Economies of scale are often justified by cost degression under learning curve effects, which are discussed in Chapter 4.

2.8 Proofs and Derivations

Proof (Proof of Proposition 2.1).
The proof is divided into three parts. We show that, (i) a production impulse and (ii) a remanufacturing impulse are never optimal, whereas (iii) a disposal impulse can be excluded for all $t > 0$.
(i) Assume that there exists a time point θ where a jump in y_s takes place by an impulse production quantity $v_p > 0$. This quantity is used to satisfy demands from θ until $\theta+\omega$, $\omega > 0$. Therefore, $v_p = \int_\theta^{\theta+\omega} d(t)dt$. The difference $\Delta NPV_{(i)}$ between the impulse control strategy and the financial impact that arise from synchronizing demand with production is given by

$$\Delta NPV_{(i)} = e^{-\alpha\theta} c_p v_p + \int_\theta^{\theta+\omega} e^{-\alpha t} h_s \left(v_p - \int_\theta^t d(s)ds \right) dt - \int_\theta^{\theta+\omega} e^{-\alpha t} c_p d(t)dt\,.$$

Since

$$e^{-\alpha\theta} c_p v_p - \int_\theta^{\theta+\omega} e^{-\alpha t} c_p d(t)\, dt > e^{-\alpha\theta} c_p v_p - e^{-\alpha\theta} c_p \int_\theta^{\theta+\omega} d(t)\, dt = 0$$

it follows

$$\Delta NPV_{(i)} > h_s \int_\theta^{\theta+\omega} e^{-\alpha t} \left(v_p - \int_\theta^t d(s)ds \right) dt > 0.$$

Thus, synchronizing demand with production is always superior to impulse production.
(ii) Assume that there exists a time point θ where an impulse remanufacturing quantity v_r leads to jumps both in y_s and y_u. Then, the difference between this strategy and synchronizing demand and remanufacturing rate is

$$\begin{aligned}
\Delta NPV_{(ii)} = {} & e^{-\alpha\theta} c_r v_r + \int_\theta^{\theta+\omega} e^{-\alpha t} h_s \left(v_r - \int_\theta^t d(s)ds \right) dt \\
& - \int_\theta^{\theta+\omega} e^{-\alpha t} h_u \left(v_r - \int_\theta^t d(s)ds \right) dt - \int_\theta^{\theta+\omega} e^{-\alpha t} c_r d(t)dt \\
& > (h_s - h_u) \int_\theta^{\theta+\omega} e^{-\alpha t} \left(v_r - \int_\theta^t d(s)ds \right) dt,
\end{aligned}$$

which is larger than zero because of $h_u < h_s$.
(iii) Assume that there exists a time point $\theta > 0$ where an impulse disposal quantity $v_w > 0$ leads to a jump in y_u. Because the return rate is finite, there

exists a point $\theta - \omega$, $\omega > 0$ where $y_u(\theta - \omega) > 0$. Then, an earlier disposal $w(\theta-\omega) = \min\{v_w, y_u(\theta-\omega)\}$ saves holding costs of $\int_{\theta-\omega}^{\theta} e^{-\alpha t} h_u w(\theta-\omega)dt > 0$ and the objective value changes by

$$\begin{aligned}
\Delta NPV_{(iii)} &= e^{-\alpha(\theta-\omega)} c_w w(\theta - \omega) - \textstyle\int_{\theta-\omega}^{\theta} e^{-\alpha t} h_u w(\theta - \omega)dt \ - e^{-\alpha\theta} c_w w(\theta) \\
&= \left(e^{-\alpha(\theta-\omega)} - e^{-\alpha\theta}\right) c_w w(\theta - \omega) - \textstyle\int_{\theta-\omega}^{\theta} e^{-\alpha t} h_u w(\theta - \omega)dt \\
&= \left(e^{-\alpha(\theta-\omega)} - e^{-\alpha\theta}\right) c_w w(\theta - \omega) + \tfrac{h_u}{\alpha} \left(e^{-\alpha\theta} - e^{-\alpha(\theta-\omega)}\right) w(\theta-\omega) \\
&= \left(e^{-\alpha(\theta-\omega)} - e^{-\alpha\theta}\right) \left(c_w - \tfrac{h_u}{\alpha}\right) w(\theta - \omega)
\end{aligned}$$

$NPV_{(iii)}$ exceeds zero because of assumption (2.6). Disposal activities will therefore be carried out as soon as possible. Thus, a positive impulse quantity can only appear at time zero, as no earlier disposal is possible.

Proof (Proof of $\lambda_0 = 1$).
We only consider the cases $\lambda_0 = 0$ and $\lambda_0 = 1$. All other cases can be transformed to the case $\lambda_0 = 1$ by rescaling of λ_s and λ_u. First, let us assume $\lambda_0 = 0$. Then, the Lagrangian simplifies to

$$L(..) = (\lambda_s + \mu_1)p + (\lambda_s - \lambda_u + \mu_2)r + (-\lambda_u + \mu_3)w - \lambda_s d + \lambda_u r + k_1 y_s + k_2 y_u, \tag{2.56}$$

and necessary conditions (2.15)–(2.17) change to

$$\frac{\partial L}{\partial p} = \lambda_s + \mu_1 = 0, \tag{2.57}$$

$$\frac{\partial L}{\partial r} = \lambda_s - \lambda_u + \mu_2 = 0, \tag{2.58}$$

$$\frac{\partial L}{\partial w} = -\lambda_u + \mu_3 = 0. \tag{2.59}$$

From non-negativity of multipliers μ_1, μ_2, μ_3 it follows that

$$\lambda_s \leq 0, \lambda_s - \lambda_u \leq 0, \lambda_u \geq 0.$$

Condition $(\lambda_0, \lambda_s, \lambda_u) \neq 0$ for a non-trivial solution requires at least λ_s or λ_u to be different from zero. A positive remanufacturing rate $r > 0$ would require $\lambda_s = \lambda_r = 0$ which contradicts non-triviality (because of $\mu_2 = 0$). Therefore, remanufacturing cannot take place, which leads to an obviously non-optimal solution as $c_p + c_w > c_r$.

Proof (Proof of Proposition 2.2).
From (2.23) and (2.24) it follows that $k_1 = k_2 = 0$. Thus, (2.21) and (2.22) imply

$$\dot{\lambda}_s = \alpha\lambda_s + h_s \text{ and } \dot{\lambda}_u = \alpha\lambda_u + h_u. \tag{2.60}$$

$p > 0$ requires $\mu_1 = 0$ in (2.18) which yields $\lambda_s = c_p$ in (2.15). It follows that $\dot{\lambda}_s = 0$ which contradicts (2.60). $u > 0$ requires $\mu_2 = 0$ in (2.19) which yields $\lambda_s - \lambda_u = c_r$ in (2.16). It follows that $\dot{\lambda}_s = \dot{\lambda}_u$ which contradicts the assumption that $h_s > h_u$. $w > 0$ requires $\mu_3 = 0$ in (2.20) which yields $\lambda_u = -c_w$ in (2.17). It follows that $\dot{\lambda}_u = 0$ which contradicts assumption (2.8).

Proof (Proof of Proposition 2.3).
From (2.24) it follows that $k_2 = 0$. Thus, (2.22) implies

$$\dot{\lambda}_u = \alpha\lambda_u + h_u. \tag{2.61}$$

The proof for $w^* = 0$ is the same as in Case 1. $p > 0$ and (simultaneously) $r > 0$ requires $\mu_1 = \mu_2 = 0$ from (2.18) and (2.19) which yields $\lambda_s = c_p$ in (2.15) and $\lambda_u = c_p - c_r$ in (2.16). It follows that $\dot{\lambda}_u = 0$ which contradicts (2.61). In any interval where $p > 0$ and $r = 0$ hold, the definition of Case 2 ($\dot{y}_s = 0$) requires $p = d$ and $\dot{y}_u = u \geq 0$. From (2.12) and (2.13) it follows $\lambda_u > c_p - c_r$ and $\dot{\lambda}_u = h_u$. Continuity of λ_u within a Case 2 interval (which follows from (2.26)) requires that the optimal control never switches back to $r > 0$, and therefore, y_u will never decrease, which contradicts the final condition $y_u(T) = 0$. The optimal policy in Case 2 is therefore given by $p^* = 0$ and $r^* = d$. (2.13) necessitates $\lambda_s = \lambda_u + c_r$ and $\dot{\lambda}_s = \dot{\lambda}_u$.

Proof (Proof of Proposition 2.4).
From (2.23) it follows that $k_1 = 0$. (2.21) implies

$$\dot{\lambda}_s = \alpha\lambda_s + h_s. \tag{2.62}$$

$p > 0$ requires $\mu_1 = 0$ which yields $\lambda_s = c_p$. It follows that $\dot{\lambda}_s = 0$ which contradicts (2.62). $r = 0, w = 0$ requires $\dot{y}_u = u$ which is not feasible as long as $y_u = 0$. $r > 0, w > 0$ requires $\mu_2 = \mu_3 = 0$ in (2.19) and (2.20) which yields $\lambda_u = -c_w$ in (2.17) and $\lambda_s = c_r - c_w$ in (2.16). It follows that $\dot{\lambda}_s = 0$ which again contradicts (2.62). $r > 0, w = 0$ requires $\mu_2 = 0$ in (2.19) which yields $\lambda_s - \lambda_u = c_r$ in (2.16). It follows that $\dot{\lambda}_s = \dot{\lambda}_u$. Inserting (2.62) and (2.22) yields $k_2 = \alpha(-c_r) + h_u - h_s$ and $k_2 \geq 0$ in (2.24) contradicts the assumptions that $h_s > h_u$ and $c_r > 0$. The optimal policy is therefore given by $r^* = 0$ and $w^* = u$, which is required in order to stay in Case 3 ($\dot{y}_u = 0$ has to hold), and from (2.14), $\lambda_u = -c_w$.

Proof (Proof of Proposition 2.5).
From the definition of Case 4 it follows $\dot{y}_s = \dot{y}_u = 0$ which implies $p + r = d$ and $r + w = u$. When neglecting boundary situations, i.e. where demand or return rate equal zero, this already excludes the alternatives ($p = 0, r = 0$) and ($r = 0, w = 0$). $p > 0, r > 0, w > 0$ requires $\mu_i = 0\ \forall i$ which yields $\lambda_s = c_p$ in (2.15), $\lambda_u = -c_w$ in (2.17). Then, (2.16) becomes $c_p + c_w = c_r$ which contradicts the assumption of a positive recovery advantage (2.6). Similarly, for $p > 0, r = 0, w > 0$ we find $\lambda_s = c_p$, $\lambda_u = -c_w$ and $\mu_2 = c_r - c_p - c_w$ which has to be non-negative (2.19). Again, this contradicts assumption (2.6). For $p > 0, r > 0, w = 0$ we find $r = u$ to ensure $\dot{y}_u = 0$ and then $p = d - u$ to ensure $\dot{y}_s = 0$. This configuration is only feasible if $d \geq u$. For $p = 0, r > 0, w > 0$ we find $r = d$ to ensure $\dot{y}_s = 0$ and then $w = u - d$ to ensure $\dot{y}_u = 0$.

Proof (Proof of Proposition 2.6).
In the following we use the results derived in Feichtinger and Hartl (1986),

especially Corollary 6.3 (p.168). From (2.25) we know that λ_s is always continuous when $y_s > 0$ holds, i.e. inside Cases 1 and 3. The same is true for λ_u if $y_u > 0$ inside Cases 1 and 2. Thus, we will restrict our analysis to examine (i) time points inside intervals where $y_s = 0$ or $y_u = 0$ holds and (ii) entry and exit points of such intervals.

(i) A constraint qualification guarantees the continuity of the adjoint variables λ_s or λ_u inside intervals defined by $y_s = 0$ or $y_u = 0$, respectively. Thus, continuity of the respective adjoint variables is given, if the matrix (with line numbers given on the right hand side)

$$\begin{pmatrix} 1 & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & w & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & y_s & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & y_u \end{pmatrix} \begin{matrix} I \\ II \\ III \\ IV \\ V \end{matrix} \tag{2.63}$$

has full rank of five. First, we note that the first three rows are always independent. Analysis of the above matrix yields that the constraint qualification is not satisfied in three situations.

- If $p = 0$, $r = 0$ and $y_s = 0$ then $I + II = IV$.
  This can only happen in Cases 2 and 4 if $d = 0$. As we already know, in Case 2, λ_u is continuous and $\lambda_s = \lambda_u + c_r$ is therefore also continuous. If one assumes that in Case 4, $u > d = 0$ holds, then $\lambda_s = c_r - c_w$ proves continuity.
- If $r = 0$,$w = 0$ and $y_u = 0$ then $-II - III = V$.
  This can only happen in Cases 3 and 4 if $u = 0$. In Case 3, $\lambda_u = -c_w$ and in Case 4, under the condition $d > u = 0$, $\lambda_u = -c_p - c_r$ holds. Thus, λ_u does not jump.
- If $p = 0$, $w = 0$, $y_s = 0$ and $y_u = 0$ then $I - III = IV + V$.
  This situation occurs in Case 4 ($y_s = 0, y_u = 0$) when demand equals returns and the policy switches from $p > 0, r > 0$ to $r > 0, w > 0$ or vice versa. Only in this case we find a discontinuity of both adjoint variables. The height of both jumps is $\eta_s = \eta_u = c_p + c_w - c_r$. Note, this situation includes the conditions neglected before ($u = d = 0$).

(ii) Let θ_s^1 be the entry time of an interval where $y_s = 0$ holds. Then, λ_s is continuous at this time point if y_s enters this interval in a non-tangential way, i.e. $\dot{y}_s = p + r - d$ jumps. This happens if one of the controls p and r jumps at θ_s^1. An entry point θ_s^1 can only be present at a switch from Cases 1/3 to Cases 2/4. There, r jumps if $d(\theta_s^1) > 0$ which must hold to ensure leaving the first cases. An exit time θ_s^2 does not exist because it will never be optimal to build up a serviceables stock, which can easily be seen from the production and remanufacturing decisions which will in sum never exceed the demand rate.

Let θ_u^1 be the entry time and θ_u^2 the exit time of an interval, where $y_u = 0$ holds. Then, λ_u is continuous at these time points if y_u enters or leaves this interval in a non-tangential way, i.e. $\dot{y}_u = u - r - w$ jumps. That happens again if one of the controls r and w jumps at θ_u^1 or θ_u^2, respectively. This will always be the case except for $u = d$ at one of the time points.

Proof (Proof of Proposition 2.7).
From (2.14) we get a minimal value for λ_u of $\lambda_u^{\min} = -c_w$. For Case 2 we can give an upper bound for $\lambda_u^{\max} = c_p - c_r$ which together with the continuity of λ_u (Proposition 2.6) and (2.35) yields

$$c_p - c_r \geq \left(-c_w + \tfrac{h_u}{\alpha}\right) e^{\alpha\tau_u} - \tfrac{h_u}{\alpha}$$

where $\tau_u = t - \theta_{e,2}$. Solving for τ_u yields

$$\begin{aligned} \alpha(c_p - c_r) + h_u \geq (-\alpha c_w + h_u)\, e^{\alpha\tau_u} &\Leftrightarrow e^{\alpha\tau_u} \leq \tfrac{\alpha(c_p - c_r) + h_u}{-\alpha c_w + h_u} \\ &\Leftrightarrow \tau_u \leq \tfrac{1}{\alpha} \ln\left(\tfrac{\alpha(c_p - c_r) + h_u}{-\alpha c_w + h_u}\right). \end{aligned}$$

Proof (Proof of Proposition 2.9).
Assume that the inequality in Proposition 2.9 holds and that at both time points $u(\theta_{e,2}) > d(\theta_{e,2})$ and $u(\theta_{x,2}) < d(\theta_{x,2})$, which is necessary to build up a stock or to decrease it, respectively. From Proposition 2.6 it follows that λ_u will not jump at these time points and $\lambda_u(\theta_{e,2}) = -c_w$, $\lambda_u(\theta_{x,2}) = c_p - c_r$. Considering that there can be no jump point of λ_u inside the interval I and that λ_u grows at rate $\dot{\lambda}_u = \alpha\lambda + h_u$ yields $\theta_{x,2} - \theta_{e,2} = \tau_u$ which contradicts the assumption.

Proof (Proof of Proposition 2.10). It is optimal to replace production by remanufacturing as long as the cost advantage is larger than the costs that arise from holding recoverables inventory, i.e. for at most τ_u time units. Equation (2.48) ensures, that initial inventories are in fact needed to satisfy demand within the interval.

Proof (Derivation of Equation (2.55)).
For a first order MacLaurin approximation of (2.54) **(A)** the value of m_h as well as **(B)** its first derivative at $\alpha = 0$ have to be determined. Thereby, the following relationship will be used

$$\frac{\partial}{\partial\alpha} \ln\left(\frac{\alpha(c_p - c_r) + h_u^{DCF}}{-\alpha c_w + h_u^{DCF}}\right) = \frac{h_u^{DCF}(c_p + c_w - c_r)}{(\alpha(c_p - c_r) + h_u^{DCF})(-\alpha c_w + h_u^{DCF})}.$$

(A) Since in the first term of (2.54) both numerator and denominator tend to 0 as α tends to 0, l'Hôpital's rule must be applied

$$\begin{aligned} \lim_{\alpha\to+0} m_h = \frac{"0"}{0} - h_u^{DCF} &= \lim_{\alpha\to+0} \frac{c_p + c_w - c_r}{\frac{h_u^{DCF}(c_p + c_w - c_r)}{(\alpha(c_p - c_r) + h_u^{DCF})(-\alpha c_w + h_u^{DCF})}} - h_u^{DCF} \\ &= \frac{(h_u^{DCF})^2}{h_u^{DCF}} - h_u^{DCF} = 0. \end{aligned}$$

(B) The first derivative of (2.54) is given by

$$\frac{\partial m_h}{\partial \alpha} = \frac{(c_p + c_w - c_r)\ln X - \frac{\alpha h_u^{DCF}(c_p+c_w-c_r)^2}{(\alpha(c_p-c_r)+h_u^{DCF})(-\alpha c_w+h_u^{DCF})}}{[\ln X]^2} \tag{2.64}$$

where $X = \left(\frac{\alpha(c_p-c_r)+h_u^{DCF}}{-\alpha c_w+h_u^{DCF}}\right)$.

Again, l'Hôpital's rule is used in order to find the right hand side limit of (2.64) as α tends to 0, i.e.

$$\begin{aligned}
&\lim_{\alpha\to+0} \frac{\partial m_h}{\partial \alpha} = \frac{\text{“0”}}{0} \\
&= \lim_{\alpha\to+0} \frac{(c_p + c_w - c_r)\left(1 - \frac{(h_u^{DCF})^2+\alpha^2 c_w(c_p-c_r)}{(\alpha(c_p-c_r)+h_u^{DCF})(-\alpha c_w+h_u^{DCF})}\right)}{2\ln\left(\frac{\alpha(c_p-c_r)+h_u^{DCF}}{-\alpha c_w+h_u^{DCF}}\right)} = \frac{\text{“0”}}{0} \\
&= \lim_{\alpha\to+0} \frac{-2\alpha c_w(c_p-c_r) + \frac{\left((h_u^{DCF})^2+\alpha^2 c_w(c_p-c_r)\right)\left(h_u^{DCF}(c_p-c_r)-hc_w-2\alpha c_w(c_p-c_r)\right)}{(\alpha(c_p-c_r)+h_u^{DCF})(-\alpha c_w+h_u^{DCF})}}{2h_u^{DCF}} \\
&= \frac{\left(\frac{(h_u^{DCF})^2\left(h_u^{DCF}(c_p-c_r)-hc_w\right)}{(h_u^{DCF})^2}\right)}{2h_u^{DCF}} = \frac{c_p - c_r}{2} + \frac{-c_w}{2}.
\end{aligned}$$

Thus, the first order MacLaurin approximation of m_h is given by

$$m_h = \left(\frac{c_p - c_r}{2} + \frac{-c_w}{2}\right)\alpha + O(\alpha^2).$$

3

On the Effects of Capacity Constraints in Product Recovery

3.1 Motivation

In the previous chapter, a product recovery system with dynamic demand and return streams was investigated. It was found that a recoverables inventory is built up if there are some excess returns which can later be used to replace production with remanufacturing and thereby realizing a cost advantage of recovering returned items, which is for a certain time period larger than the resulting out-of-pocket holding costs and the decrease of the recovery cost advantage due to discounting. When reconsidering the optimal policies and their dynamics, there are a number of reasons which necessitate highly flexible processes. Firstly, since the serviceables inventory is not used, production and remanufacturing rates are synchronized with demand and undergo (in sum) the same variations. Secondly, in situations where it is optimal to have a positive recoverables inventory, demand is served from remanufacturing alone, which also means that the production rate is zero. Finally, at time points where stock-keeping starts or where the recoverables inventory is depleted, both production and remanufacturing rates undergo sudden and considerable changes, e.g. the production rate jumps from zero up to the difference of demand and return rates.

In many practical situations, processes are not that flexible and therefore, a smoothing of manufacturing and remanufacturing volumes is required. In medium-range aggregate production planning of a pure production system, if it is not possible to influence the productive capacity ('hard' constraint), a level strategy (see Silver et al. (1998), Chapter 14) is employed where a serviceables inventory is built up at times when the constraint is not binding and used up when it is. An application of optimal control theory to a production/inventory model is presented e.g. by Gaimon (1988). There, in addition to production quantities, optimal prices and capacity expansion are determined in order to maximize total profits. Another modeling option for inflexible processes is to use convex cost functions, which represent 'soft' constraints where regular capacity can be extended at increasing (overtime) unit costs. An application of convex cost functions has been presented by Kistner and Dobos (2000) for a product recovery system without disposal option. But there it is assumed that the number of returns at t directly depends on the current demand rate at t, i.e. it is modeled as a fraction of demand. Finally, process levels are smoothed if there are costs of changing e.g. the production rate or deviating from an optimal level. A production/inventory model as presented by Thompson and Sethi (1980) for instance penalizes the deviation from a target production rate and inventory level by using quadratic cost functions.

In this chapter, we deal with 'hard' capacity constraints that cannot be influenced. These restrictions can be present in different ways. If the processes take place in different facilities, independent capacity constraints, i.e. maximum production $\bar{p} > 0$ or remanufacturing rate $\bar{r} > 0$, are given. A similar approach can be applied if, due to legislative regulations, also the disposal rate is limited. If production and remanufacturing take place in a common facility, the joint capacity usage can be limited. Given a total capacity $\bar{a} > 0$ and capacity requirements coefficients $a_p > 0$ and $a_r > 0$ for both processes this can be modeled as

$$a_p p(t) + a_r r(t) \leq \bar{a}. \tag{3.1}$$

The analysis of bottleneck situations, i.e. situations where a constraint becomes binding, adds another motive for holding inventory because in situations where e.g. demand exceeds the current total capacity of the system, it is necessary to have a positive (serviceables or recoverables) inventory. An optimal solution must therefore answer the question which inventory to use. Going one step further, an issue of operations strategy would be to choose from different capacity levels or expansion paths. In such a case, capacity constraints would be functions of time. A discussion of the connection of classical inventory management and capacity expansion strategies is for instance to be found in Slack and Lewis (2003). The optimal control framework used to solve dynamic problems of product recovery is able to deal with capacity dynamics because there is no general reliance on static parameters. Although not explicitly stated below, the following analysis is therefore not only restricted to stationary constraints but these can also be assumed to be (given) continuous functions of time.

In order to account for the medium and long term horizon of capacity expansion, the application of a discounted cash flow approach as introduced in Chapter 2 is continued in the following. Changes that are relevant when assuming a zero discount rate are stated explicitly. A simple application to product recovery is presented in Chapter 5, where the questions are answered on when to make an (unrestricted) remanufacturing process available, if at all.

The remainder of this chapter is organized as follows. Single capacity constraints for production and remanufacturing are considered and pure effects (i.e. under exclusion of situations where stock-keeping already took place in the basic model) of introducing the respective restriction are discussed. The investigation is supported by several numerical examples. For technical reasons and in order to simplify the discussion, demand and return rates are assumed to be strictly positive, i.e.

$$d(t) > 0 \text{ and } u(t) > 0. \tag{3.2}$$

3.2 Limited Production Capacity

3.2.1 Changes to the Basic Model

In this section, the effects of a single capacity constraint are shown. Thereby, it is assumed that production and remanufacturing take place in different facilities and the latter process is only limited by available returns. Thus, solely the production process is restricted

$$p(t) \leq \bar{p}(t). \tag{3.3}$$

An optimization problem is given by (2.9) and an additional constraint (3.3).

Since both sources to satisfy demand are limited, a feasible solution exists if for every point in time $0 \leq t \leq T$ the cumulative demand up to this point reduced by initial serviceables does not exceed the sum of maximal production and cumulative returns, i.e.

$$\int_0^t d(s)ds - y_s^0 \leq y_u^0 + \int_0^t (\bar{p}(s) + u(s))ds \quad \forall t \in [0, T]. \tag{3.4}$$

In order to simplify our discussion it is assumed that there exist no 'points of contact' of demand, return and constraint functions, e.g. if demand and constraint rates meet ($d(t) = \bar{p}(t)$) then they truly intersect ($\dot{d}(t) \neq \dot{\bar{p}}(t)$).

3.2.2 Properties of an Optimal Solution

Necessary Conditions

As only a pure control constraint was added to control optimization model (2.9), the Hamiltonian (2.10) does not change. By defining a Langrange multiplier μ_4 in order to account for the additional restriction (3.3), the Lagrangian is

$$L(..) = H(.) + \mu_1 \cdot p + \mu_2 \cdot r + \mu_3 \cdot w + \mu_4 \cdot (\bar{p} - p) + k_1 \cdot y_s + k_2 \cdot y_u. \tag{3.5}$$

The proof of $\lambda_0 = 1$ can be accomplished in the same way as in the basic model and is therefore not treated here. Proceeding as in Section 2.3.2, necessary conditions (2.12)–(2.24) change as follows. The Hamiltonian maximizing condition (2.12) now reads

$$p^* = \begin{cases} 0 & \lambda_s < c_p \\ \text{singular} & \lambda_s = c_p \\ \bar{p} & \lambda_s > c_p \end{cases}, \tag{3.6}$$

equation (2.15) is replaced by

$$\frac{\partial L}{\partial p} = -c_p + \lambda_s + \mu_1 - \mu_4 = 0, \tag{3.7}$$

and because of constraint (3.3) we get an additional non-negativity as well as a complementary slackness condition

$$\mu_4 \geq 0 \quad \mu_4 \cdot (\bar{p} - p^*) = 0. \tag{3.8}$$

All other conditions remain unaltered.

In contrast to the basic model, production at the upper bound $\bar{p}$ takes place, if the value of an item to be added to the serviceables inventory (λ_s) exceeds production unit costs. Consequently, also the value of a used product can be higher than $c_p - c_r$, as it would (arriving at a time where $\lambda_s > c_p$) allow for a higher cost advantage when replacing production.

The Structure of an Optimal Solution

Although we still distinguish between four cases with respect to the serviceables and recoverables inventory status, Cases 1 and 2 may show different optimal policies than provided in Propositions 2.2 and 2.3. Thus, the following propositions provide further sub-cases within intervals of the respective case.

Proposition 3.1 (Optimal decisions in Case 1 intervals).
If both serviceables and recoverables inventory are positive ($y_s^* > 0$, $y_u^* > 0$), no items are remanufactured ($r^* = 0$) or disposed of ($w^* = 0$). The optimal decision on whether to produce new items or not depends on the co-state $\lambda_s < \lambda_u + c_r$ and two subcases can be distinguished:
Subcase 1(1) $\Leftrightarrow \lambda_s < c_p$ *and* $\lambda_u > -c_w$
No items are produced ($p^* = 0$).
Subcase 1(2) $\Leftrightarrow \lambda_s > c_p$ *and* $\lambda_u > c_p - c_r$
Production takes place at its upper bound ($p^* = \bar{p}$).

In contrast to the model without a capacity constraint, Subcase 1(2) shows that it is possible to produce even if there are serviceables available and if there exists a positive recoverables stock from which current demand could be satisfied completely. Such a decision only makes sense if one considers that there might be a later bottleneck situation for which both stocks are held for. It requires that the value of a serviceables item is higher than actual unit production costs and that the value of a recoverables item exceeds the difference of production and remanufacturing costs. Production takes place at its upper limit.

The co-states development takes place as shown in the basic model, i.e.

$$\dot{\lambda}_s = \alpha\lambda_s + h_s \text{ and } \dot{\lambda}_u = \alpha\lambda_u + h_u. \quad (3.9)$$

Because of the different decision structure in Subcase 1(2) and in contrast to the first subcase, the serviceables inventory level changes with rate $\bar{p} - d$, and it increases if demand rate is smaller than the manufacturing constraint.

Proposition 3.2 (Optimal decisions in Case 2 intervals).
If serviceables inventory is zero and recoverables inventory is positive ($y_s^* = 0$, $y_u^* > 0$), no items are disposed of. The optimal decisions on production and remanufacturing depend on the co-state λ_u and on the relationship between demand and return rates. Three subcases are distinguished:
Subcase 2(1) $\Leftrightarrow -c_w < \lambda_u < c_p - c_r$, $\lambda_s < c_p$
No items are produced ($p^* = 0$) and the remanufacturing rate equals the demand rate ($r^* = d$).
Subcase 2(2) $\Leftrightarrow \lambda_u > c_p - c_r$, $\lambda_s = c_p$, *and* $d \leq \bar{p}$
Production equals demand rate ($p^* = d$) and no items are remanufactured ($r^* = 0$).
Subcase 2(3) $\Leftrightarrow \lambda_u > c_p - c_r$, $\lambda_s > c_p$, *and* $\bar{p} < d$
Production takes place at its upper bound ($p^* = \bar{p}$) and remanufacturing is used to fill remaining demand ($r^* = d - \bar{p}$).

Under zero serviceables and positive recoverables inventory, there are two new subcases when considering a restricted production process. Both have in common that production becomes favorable, since the value of a serviceables item at least equals production unit costs ($\lambda_s \geq c_p$). Therefore, as many items as possible are manufactured without building up a serviceables stock ($p^* = \min\{d, \bar{p}\}$) and thus, more returns can be saved for later use as would be the case in Subcase 2(1). While in Subcase 2(2) no items are remanufactured, because their increase in value would be smaller than remanufacturing costs. In Subcase 2(3) one is indifferent between satisfying demand from remanufacturing or production of new items ($\lambda_s = \lambda_u + c_r$).

Since there is no change in optimal decisions in Cases 3 and 4, Propositions 2.4 and 2.5 are still applicable. Table 3.1 summarizes the main results of the four cases.

Optimal Transitions Between Cases and Subcases

From the optimal decisions inside Case 4 intervals it can be seen that the maximal possible demand rate that can be satisfied immediately without having inventory available is $u + \bar{p}$. If this *current capacity* of the product recovery system is exceeded by the demand rate, either serviceables have to be available in stock (as it would be the case in a pure production system) or recoverables must have been collected before in order to satisfy excess demand. Intervals where the current capacity of the system is smaller than demand are further

Table 3.1. Main results of optimal cases when considering a manufacturing constraint.

	p^*	r^*	w^*	$\dot{y}_s$	$\dot{y}_u$	λ_s	$\dot{\lambda}_s$	λ_u	$\dot{\lambda}_u$
Case 1: $y_s > 0$, $y_u > 0$									
(1)	0	0	0	$-d$	u	$< c_p$ $(\lambda_s < \lambda_u + c_r)$	$\alpha\lambda_s + h_s$	$-c_w <$	$\alpha\lambda_u + h_u$
(2)	$\bar{p}$	0	0	$\bar{p} - d$	u	$c_p <$ $(\lambda_s < \lambda_u + c_r)$	$\alpha\lambda_s + h_s$	$c_p - c_r <$	$\alpha\lambda_u + h_u$
Case 2: $y_s = 0$, $y_u > 0$									
(1)	0	d	0	0	$u - d$	$< c_p$ $(\lambda_s = \lambda_u + c_r)$	$\alpha\lambda_u + h_u$	$-c_w < \lambda_u < c_p - c_r$	$\alpha\lambda_u + h_u$
(2) $(d \leq \bar{p})$	d	0	0	0	u	c_p	0	$c_p - c_r <$	$\alpha\lambda_u + h_u$
(3) $(\bar{p} < d)$	$\bar{p}$	$d - \bar{p}$	0	0	$u - d + \bar{p}$	$c_p <$ $(\lambda_s = \lambda_u + c_r)$	$\alpha\lambda_u + h_u$	$c_p - c_r <$	$\alpha\lambda_u + h_u$
Subcases 2(2) and (3) generalized: $p^* = \min\{d, \bar{p}\}, r^* = \max\{0, d - \bar{p}\}, w^* = 0$									
Case 3: $y_s > 0$, $y_u = 0$									
	0	0	u	$-d$	0	$< c_r - c_w$	$\alpha\lambda_s + h_s$	$-c_w$	0
Case 4: $y_s = 0$, $y_u = 0$									
(1) $(d \leq u)$	0	d	$u - d$	0	0	$c_r - c_w$	0	$-c_w$	0
(2) $(u < d)$	$d - u$	u	0	0	0	c_p	0	$c_p - c_r$	0
Case 4 generalized: $p^* = \max\{d - r^*, 0\}, r^* = \min\{u, d\}, w^* = \max\{u - r^*, 0\}$									

called *bottleneck situations*. The required serviceables and recoverables to deal with such situations are referred to as *bottleneck stock*.

In Section 2.3.4, we distinguished between two types of case transitions, namely *forced* and *automatic*. Here, a third type is introduced, which helps to differentiate between forced transitions that are made to build up anticipation stock (those already known from the basic model) and others which are required to deal with bottleneck situations. These are called *constraint-forced* transitions. Constraint-forced transitions require co-states to have a higher value than was allowed in the basic model. This becomes possible because the necessity to fill demand may require us to keep stock for a longer time period than was possible in the basic model where production always was able to satisfy demand immediately. As before, *continuous* and *discontinuous* transitions are distinguished. Subsequently, $A \rightarrow B$ reads as a transition from a Case A to a Case B interval.

In order to determine dynamic properties of the system, especially on possible case and subcase transitions, it is necessary to know under which conditions the co-states, λ_s and λ_u, are allowed to jump. Proposition 3.3 collects all situations, in which one or both co-state variables may be discontinuous.

Proposition 3.3 (Continuity of λ_s and λ_u).
λ_s and λ_u are continuous, i.e. jump parameters η_s and η_u vanish everywhere, except at time points θ where one of the following conditions holds:

(i) $2(3) \rightarrow 2(2)$ *:*
  λ_s jumps if $d(\theta) = \bar{p}(\theta)$ and $\dot{d}(\theta) < \dot{\bar{p}}(\theta)$.

(ii) $4(2) \rightarrow 4(1)$ *:*
  λ_s and λ_u jump if $u(\theta) = d(\theta)$ and $\dot{d}(\theta) < \dot{u}(\theta)$.

(iii) $1(2) \rightarrow 2(2)$ *as well as* $2(3) \rightarrow 1(2)$ *:*
  λ_s jumps if $d(\theta) = \bar{p}(\theta)$ and $\dot{d}(\theta) < \dot{\bar{p}}(\theta)$.

(iv) $2(1) \rightarrow 4(1)$ *as well as* $4(2) \rightarrow 2(1)$ *:*
  λ_s and λ_u jump if $d(\theta) = u(\theta)$ and $\dot{d}(\theta) < \dot{u}(\theta)$.

(v) $2(3) \rightarrow 4(2)$ *:*
  λ_s and λ_u jump if $d(\theta) = u(\theta) + \bar{p}(\theta)$ and $\dot{d}(\theta) < \dot{u}(\theta) + \dot{\bar{p}}(\theta)$.

The following Corollaries 3.1–3.7 have been developed by using results regarding optimal decisions within the subcases as well as co-states properties, especially by using Proposition 3.3. A transition from one case to another is excluded if (1) optimal decisions and requirements on parameter relation together do not allow for accumulation or depletion of a certain inventory, (2) an upward jump in one or both co-states is required, or (3) downward jumps are necessitated for other switches than those stated above.

Corollary 3.1. *Within Case 1 Subcase 1(1) can be followed by Subcase 1(2). This constraint-forced and continuous transition requires the value of a serviceables item to equal production unit costs ($\lambda_s = c_p$). At switching time, the value of recoverables must exceed the difference of unit production and remanufacturing costs ($\lambda_u > c_p - c_u$).*

Case 1(1) only allows for a maximum value of the serviceables held in stock. When surpassing c_p without emptying the serviceables stock, a switch to Subcase 1(2) must take place. A reverse transition is not possible because it would require a jump in the serviceables co-state λ_s which is forbidden in an interval with positive inventory level.

Corollary 3.2. *Within Case 2 the following transitions are possible:*

2(1) → 2(2). *This constraint-forced and continuous transition requires* $\lambda_s = c_p$, $\lambda_u = c_p - c_r$, *and* $d \leq \bar{p}$.

2(1) → 2(3). *This constraint-forced and continuous transition requires* $\lambda_s = c_p$, $\lambda_u = c_p - c_r$, *and* $d > \bar{p}$.

2(3) → 2(2). *This automatic and discontinuous transition requires demand to intersect the production constraint from above (*$d = \bar{p}$ *and* $\dot{d} < \dot{\bar{p}}$*).*

As the value of a recoverables item reaches the difference of unit production and remanufacturing cost, a transition from Subcase 2(1) to one of the other subcases takes place. Whether Subcase 2(2) or 2(3) is reached depends on the relation between demand rate and constraint. If this relation changes, i.e. demand gets smaller than the constraint, a transition from Subcase 2(3) to 2(2) takes place.

Corollary 3.3. *Within Case 4 Subcase 4(2) is followed by Subcase 4(1). This automatic and discontinuous transition requires demand to intersect the return rate from above (*$d = u$ *and* $\dot{d} < \dot{u}$*).*

Other transitions between subcases within a case are not optimal. Now, we turn to transitions between different cases.

Corollary 3.4. *Starting at a Case 1 interval where* $y_s > 0$ *and* $y_u > 0$ *the following transitions are possible:*

1(1) → 2(1). *This automatic and continuous transition requires* $\lambda_s = \lambda_u + c_r$ *and* $\lambda_u < c_p - c_u$.

1(1) → 2(2). *This constraint-forced and continuous transition requires* $\lambda_s = c_p$, $\lambda_u > c_p - c_u$, *and* $d \leq \bar{p}$.

1(1) → 2(3). *This constraint-forced and continuous transition requires* $\lambda_s = c_p$, $\lambda_u > c_p - c_u$, *and* $d > \bar{p}$.

1(2) → 2(2). *This automatic and discontinuous transition requires* $d = \bar{p}$ *and* $\dot{d} < \dot{\bar{p}}$.

1(2) → 2(3). *This automatic and continuous transition requires* $\lambda_s = \lambda_u + c_r$ *and* $d > \bar{p}$.

Since returns are neither remanufactured nor disposed of, a Case 1 interval terminates into a Case 2 subcase when depleting the serviceables inventory. Depending on the value of recoverables λ_u, Case 1(1) switches either to Case 2(1) (if $\lambda_u < c_p - c_u$) or to Cases 2(2)/2(3). Case 1(2) can be left if demand exceeds the manufacturing constraint at the end if this interval. Since $\lambda_u > c_p - c_u$ also holds there, only transitions to Cases 2(2)/2(3) are possible.

Corollary 3.5. *Starting at a Case 2 interval where* $y_s = 0$ *and* $y_u > 0$ *the following transitions are possible:*

2(1) → 4(1). *This automatic and discontinuous transition requires* $d = u$ *and* $\dot{d} \leq \dot{u}$.

2(1) → 4(2). *This automatic and continuous transition requires* $u \leq d$, $\lambda_s = c_p$, $\lambda_u = c_p - c_r$.

2(2) → 1(2). *This constraint-forced and continuous transition requires* $d < \bar{p}$.

2(3) → 1(2). *This constraint-forced and discontinuous transition requires* $d = \bar{p}$, $\dot{d} < \dot{\bar{p}}$.

2(3) → 4(2). *This automatic and discontinuous transition requires* $d = \bar{p} + u$, $\dot{d} \leq \dot{\bar{p}} + \dot{u}$.

In addition to the case transitions known from the basic model, a switch from Case 2(2)/(3) to Case 1(2) is possible, requiring demand not to be larger than the remanufacturing constraint ($d \leq \bar{p}$). Further, a depleting recoverables inventory allows for a transition from Case 2(3) to 4(2). This requires demand to be larger than the current capacity of the system at the end of the Case 2(3) interval as well as the opposite inside the Case 4(2) interval in order to be able to satisfy demand without having a positive stock.

Corollary 3.6. *Starting at a Case 3 interval where* $y_s > 0$ *and* $y_u = 0$ *the following transitions are possible:*

3 → 1(1). *This forced and continuous transition requires* $d \leq u$.

3 → 4(1). *This automatic and continuous transition requires* $\lambda_s = c_r - c_w$ *and* $d \leq u$.

Transitions starting at a Case 3 interval are the same as discussed for the basic model.

Corollary 3.7. *Starting at a Case 4 interval where* $y_s = y_u = 0$ *the following transitions are possible:*

4(1) → 2(1). *This forced and continuous transition requires* $d \leq u$.

4(2) → 2(1). *This forced and discontinuous transition requires* $d = u$ *and* $\dot{d} < \dot{u}$.

4(2) → 2(2). *This constraint-forced and continuous transition requires* $d \leq \bar{p}$.

4(2) → 2(3). *This constraint-forced and continuous transition requires* $d > \bar{p}$.

Adding to the already known switches, Case 4(2) can be left for a transition into a Case 2(2)/2(3) interval in order to build up a bottleneck stock.

As in the basic model, an accumulation of serviceables through a transition from any subcase of Cases 2/4 to 1(1) or 3 is excluded because building up stock would require $\dot{y}_s > 0$ which contradicts optimal decisions within the mentioned subcases with positive serviceables inventory level (1(1) and 3). Since also 1(2) does not terminate into those cases, they are only possible during an initial period. We therefore distinguish between transitions that can only occur at the beginning of the planning horizon when having a positive initial inventory level (depicted in Figure 3.1) and others that occur during

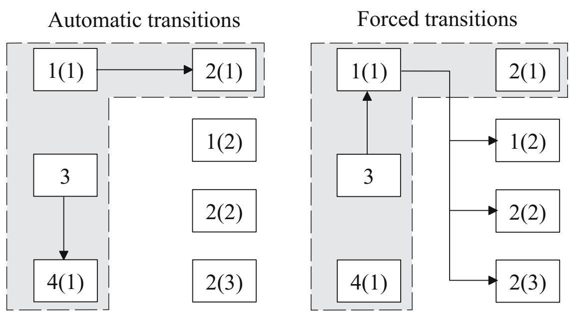

Fig. 3.1. Optimal case transitions at the beginning of the planning horizon when considering a manufacturing constraint

the planning period after using up initial stock (see Figure 3.2).

In what follows, we concentrate on the latter kind of case changes by assuming $y_s^0 = y_u^0 = 0$. A discussion on how to deal with initial stock would proceed in a similar manner as been applied for the basic model (see Section 2.4). Additional transitions have to be considered if parts of initial stock are used during a subsequent bottleneck situation.

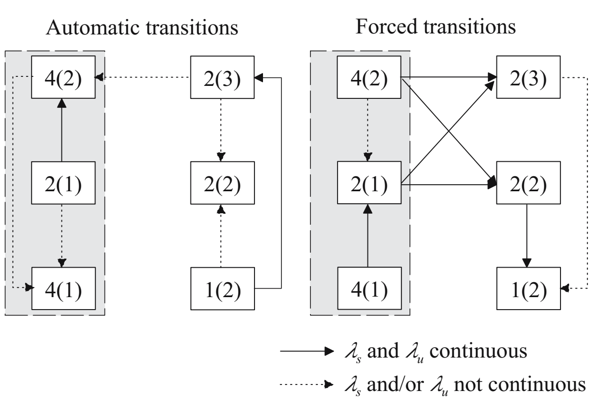

Fig. 3.2. Optimal case transitions when considering a manufacturing constraint

3.2.3 Pure Effects of a Manufacturing Constraint

Figure 3.2 distinguishes between two reasons for keeping stock. On the one hand, as in the basic model, the anticipation of a later change in the demand/return relationship leads to an accumulation of returns in a Case 2(1) interval (shaded area in Figure 3.2). This motive only allows for using the recoverables inventory. Another cause that may also require a serviceables stock is to prepare for a bottleneck situation using Cases 2(2), 2(3) and 1(2). The required constraint-forced transitions are shown on the right hand side of Figure 3.2. A general solution to our problem requires the simultaneous consideration of both motivations, because a bottleneck situation for instance may not become binding if there is enough anticipation stock available to satisfy demand from remanufacturing returns.

To begin with, we deal with *pure effects* of a manufacturing constraint. Therefore, it is assumed that returns do not exceed demand during the planning horizon, i.e.

$$d(t) > u(t)\ \forall t \in [0, T] \tag{3.10}$$

If the opposite holds, demand can always be satisfied from (unrestricted) remanufacturing alone. Later, *combined effects* are discussed, that occur when allowing demand and return rate to intersect.

Limitation (3.10) immediately excludes Case 4(1) and Case 2(1) intervals, where the cost advantage of remanufacturing motivates stock keeping. Therefore, the number of case transitions reduces to those depicted in Figure 3.3. Two further implications result. Firstly, since recoverables can immediately

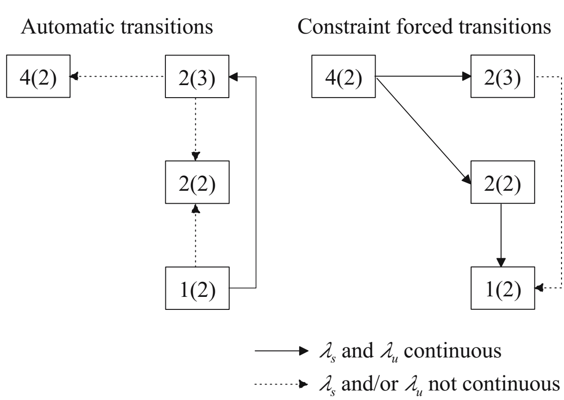

Fig. 3.3. Optimal case transitions when considering a manufacturing constraint assuming $u < d$

be used to serve demand, they obtain a minimal value of $\lambda_u \geq c_p - c_r$. Likewise, since production is necessary to fill part of the demand, serviceables are not valued less than production costs, i.e. $\lambda_s \geq c_p$. Secondly, production capacity always becomes binding in intervals where demand exceeds the current production and remanufacturing capacity. This leads to the following proposition.

Proposition 3.4 (Location of a bottleneck interval).
If demand exceeds current capacity of the system, i.e.

$$d(\theta) > \bar{p}(\theta) + u_r(\theta), \tag{3.11}$$

it is necessary to have a positive (serviceables or recoverables) inventory at θ.

Proof. Proposition 3.4 directly follows from the definition of Case 4 intervals.

In order to simplify the discussion we consider a situation with a single bottleneck interval as well as no planning horizon effects, i.e. we do not deal with situations where the treatment of a bottleneck interval requires to take the borders of the planning period into account. Furthermore, it is assumed that a valid solution exists, i.e. feasibility condition (3.4) holds.

Let $\theta_{e,b}$ the begin and $\theta_{x,b}$ the end of a bottleneck situation as described in the preceding proposition with

$$\theta_{e,b} : d(\theta_{e,b}) = \bar{p}(\theta_{e,b}) + u(\theta_{e,b}) \text{ and } \dot{d}(\theta_{e,b}) > \dot{\bar{p}}(\theta_{e,b}) + \dot{u}(\theta_{e,b}),$$
$$\theta_{x,b} : d(\theta_{x,b}) = \bar{p}(\theta_{x,b}) + u(\theta_{x,b}) \text{ and } \dot{d}(\theta_{x,b}) < \dot{\bar{p}}(\theta_{x,b}) + \dot{u}(\theta_{x,b}).$$

Reconsidering the limited current capacity of the system, at $\theta_{e,b}$ a certain number of serviceables or recoverables have to be stocked in advance to satisfy *bottleneck demand* D_B. It is defined as that part of demand during the bottleneck interval $(\theta_{e,b}, \theta_{x,b})$ which can not be satisfied by using current capacity of the system, i.e.

$$D_B = \int_{\theta_{e,b}}^{\theta_{x,b}} d(t) - (u(t) + \bar{p}(t))\, dt. \tag{3.12}$$

Now, the question will be answered how to collect stock and under which circumstances to use each of the inventories.

In an optimal solution Case 4(2) cannot be followed by Case 1(2) or vice versa, which means that before starting an interval where both inventories are positive (Case 1(2)), there must be an interval where only returns are collected (Case 2). In a Case 1(2) interval returns are collected and only a transition back to Case 2 is possible. Therefore, the following sequences can occur in an optimal solution

$$\text{Case 2} \to \text{Case 4(2)} \to \text{Case 2 and Case 2} \to \text{Case 1(2)} \to \text{Case 2.}$$

As no further stock is required at the end of the bottleneck interval $(\theta_{x,b})$, Case 4(2) is present afterwards. Since $d > \bar{p}$ holds at this point, a transition from Subcase 2(3) to Case 4(2) must take place.

Let θ_c denote the start time of a corresponding *collection interval* $(\theta_c, \theta_{e,b})$. This time point depends on how many items have to be stored to cover bottleneck demand and how long it takes to collect them. Case 4(2) is present before θ_c. Depending on the relation between demand and maximum production rate, at θ_c either a transition from Case 4(2) to Case 2(2) or to Case 2(3) takes place. Within the Case 2 interval, recoverables inventory increases with rate

$$\dot{y}_u = \min\{u, u - (d - \bar{p})\} \text{ (inside Case 2(2)/2(3))} \tag{3.13}$$

and since both switches are continuous transitions, co-states develop starting with

$$\lambda_s(\theta_c) = c_p \text{ and } \lambda_u(\theta_c) = c_p - c_r. \tag{3.14}$$

During the collection interval, building up a serviceables stock and thus switching from Case 2 to Case 1(2) becomes preferable in a period where demand falls below the capacity constraint followed by another where the opposite holds. This situation technically reflects the impossibility of a transition from Subcase 2(2) to 2(3) without passing a Case 1(2) interval, and leads to the following corollary.

Corollary 3.8 (Location property of a Case 1(2) interval). *Let θ denote a time point inside a collection interval where $d(\theta) = \bar{p}(\theta)$. If demand crosses the manufacturing capacity from below, i.e. $\dot{d}(\theta) > \dot{\bar{p}}(\theta)$, then it is always optimal to have a positive serviceables inventory at time θ.*

In a Case 1(2) interval, the recoverables inventory increases with rate

$$\dot{y}_u = u \text{ (inside Case 1(2))} \tag{3.15}$$

which is larger than it would have been when staying in Case 2 where $d(\theta) > \bar{p}(\theta)$. Therefore, using the serviceables inventory allows to save additional returns for later use during the bottleneck interval and likewise decreases the length of the collection interval thus saving holding costs in the recoverables inventory. This reasoning will later be used to develop a maximal length criterion.

Since there might be several points meeting Corollary 3.8 within the collection period, we assume that there are $n \geq 0$ Case 1(2) intervals $J^i = (\theta^i_{e,1(2)}, \theta^i_{x,1(2)})$, $i = 1, 2, ..., n$. Analogously to Proposition 2.8, the following inventory conditions must hold (presented without proof).

Proposition 3.5 (Inventory Conditions of Case 1(2) intervals). *Let $J^i = (\theta^i_{e,1(2)}, \theta^i_{x,1(2)})$ be an open time interval where $y_s > 0$, $y_u > 0$ and $y_s(\theta^i_{e,1(2)}) = y_s(\theta^i_{x,1(2)}) = 0$. Then,*

(i) cumulative production equals cumulative demand over the whole interval

$$\int_{\theta^i_{e,1(2)}}^{\theta^i_{x,1(2)}} (\bar{p}(t) - d(t))dt = 0, \tag{3.16}$$

(ii) at any point $\theta \in J^i$, cumulative production must be larger than cumulative demand

$$\int_{\theta^i_{e,1(2)}}^{\theta} (\bar{p}(t) - d(t))dt > 0. \tag{3.17}$$

Comparing the optimal return collection between Case 2 and a Case 1(2) intervals to be seen in (3.13) and (3.15), in the latter case additional returns are stored with a total quantity of $\int_{\theta^i_{e,1(2)}}^{\theta^i_{x,1(2)}} (\max\{\bar{p}(t) - d(t), 0\})\, dt$. Using this result, inventory conditions regarding the use of a recoverables inventory can be given (presented without proof and omitting the non-negativity condition).

Proposition 3.6 (Inventory Condition for the joint collection and bottleneck interval).
Let $I = (\theta_c, \theta_{x,b})$ be an open time interval where $y_u > 0$ with $y_s(\theta_c) = y_u(\theta_c) = y_s(\theta_{x,b}) = y_u(\theta_{x,b}) = 0$ and $J^i = (\theta^i_{e,1(2)}, \theta^i_{x,1(2)}) \subset I$, $i = 1, 2, ..., n$ be open time intervals where $y_s > 0$, $y_u > 0$ and for which Proposition 3.5 applies. Then, cumulative stored returns must equal bottleneck demand

$$\int_{\theta_c}^{\theta_{x,b}} (u(t) - \max\{d(t) - \bar{p}(t), 0\})\, dt + \sum_{i=1}^{n} \int_{\theta^i_{e,1(2)}}^{\theta^i_{x,1(2)}} (\max\{\bar{p}(t) - d(t), 0\})\, dt = 0. \tag{3.18}$$

As already mentioned, increasing the length of a Case 1(2) interval leads to a decreasing length of the collection period. But this is only profitable as long as the induced serviceables holding costs lead to a higher reduction of recoverables holding costs. Using this trade-off a condition for maximal Case 1(2) interval lengths can be given.

Proposition 3.7 (Maximal Length of a Case 1(2) interval).
The maximal length of a Case 1(2) interval $J^i = (\theta^i_{e,1(2)}, \theta^i_{x,1(2)})$ is time dependent and it is given by

$$\theta^i_{x,1(2)} - \theta^i_{e,1(2)} \leq \frac{1}{\alpha} \ln \left(\frac{\alpha c_r + h_s - h_u}{\alpha \left(c_p - \lambda_u(\theta^i_{e,1(2)}) \right) + h_s - h_u} \right). \tag{3.19}$$

The reasoning behind marginal criterion (3.19) is as follows. Assume there are two possibilities to satisfy the last demand unit of a bottleneck interval. This can be done either by reducing θ_c (increase the length of the collection period) or by decreasing $\theta^i_{e,1(2)}$ (increase length of Case 1(2) interval). Independent of this (marginal) decision, the same amount of returns will be on stock at $\theta^i_{x,1(2)}$, i.e. at the end of the Case 1(2) interval. In an optimal solution, one has to be indifferent between both options and holding costs up to this time need to be balanced. A Case 1(2) interval with maximal length, however, implies the following sequence

$$\text{Case } 2(2) \rightarrow \text{Case } 1(2) \rightarrow \text{Case } 2(3).$$

There are two interesting implications of Proposition 3.7. On the one hand, maximal Case 1(2) interval lengths increase during a collection period. And on the other, there might be a time after which the maximal length of the Case 1(2) interval becomes infinity, because a long collection period leads to such a high value of recoverables that the serviceables co-state is not able to catch up. This is the case if the denominator in the logarithm term of 3.19 becomes negative.

Corollary 3.9. *There exists no maximal holding time for a Case 1(2) interval* $J^i = (\theta^i_{e,1(2)}, \theta^i_{x,1(2)})$ *if it starts at a time point* $\theta^i_{e,1(2)}$ *which is larger than a critical value* $\theta^{crit}_{e,1(2)}$

$$\theta^{crit}_{e,1(2)} : \lambda_u(\theta^{crit}_{e,1(2)}) > c_p + \frac{h_s - h_u}{\alpha} \tag{3.20}$$

There exist two other types of situations where (3.19) does not hold with equality. First, the collection interval cannot start earlier because there is no excess capacity available (demand would exceed capacity constraint) and thus, inventory condition (Proposition 3.5) would be violated. This situation requires $d(\theta^i_{e,1(2)}) = \bar{p}(\theta^i_{e,1(2)})$ and $\dot{d}(\theta^i_{e,1(2)}) < \dot{\bar{p}}(\theta^i_{e,1(2)})$ and Case 1(2) is reached originating in a Case 2(3) interval. Second, a Case 1(2) interval may not be extended because of absent excess demand. This would require a switch from Case 1(2) to Case 2(2).

In rather short term problems, discounting can be neglected and co-state developments inside intervals with positive inventory simplify to linear functions, i.e. $\dot{\lambda}_u = h_u$ in intervals where $y_u > 0$ and $\dot{\lambda}_s = h_s$ if $y_s > 0$. The results with respect to cases and case transitions remain unaffected but the maximal holding time criterion given in Proposition 3.7 changes as follows

$$\theta^i_{x,1(2)} - \theta^i_{e,1(2)} \leq \frac{\lambda_u(\theta^i_{e,1(2)}) - (c_p - c_r)}{h_s - h_u}. \tag{3.21}$$

Maximum holding time is reached if equality holds. As in the equivalent criterion under consideration of discounting, there is indifference between (1) increasing the Case 1(2) interval and accepting additional holding costs in the serviceables inventory or (2) enlarging the length of collection interval.

3.2.4 Mixed Effects of a Manufacturing Constraint

When allowing for intersections of demand and return functions, both the anticipation as well as the capacity motive for keeping stock play a role, if for instance during an interval where returns are collected for use during a later bottleneck interval the demand and return rates intersect. Thus, it might happen that returns exceed demand before the start of the collection interval θ_c. These excess returns can be used to replace production which would take place right after θ_c by remanufacturing of returns that have been stored before this time. Then, the collection interval is preceded by a Case 2(1) interval, where anticipation stock is built up. The same trade-off has to be struck which is already known from the basic model and the maximal length of a Case 2(1) interval is given by Proposition 2.7.

Since part of the anticipation stock is used to fill bottleneck demand, inventory condition (3.18) must be adapted as follows.

Proposition 3.8 (Inventory Condition for the joint collection and bottleneck interval, adapted for mixed effects).
Let $I = (\theta_{e,2(1)}, \theta_{x,b})$ be an open time interval where $y_u > 0$ with $y_s(\theta_c) = y_u(\theta_c) = y_s(\theta_{x,b}) = y_u(\theta_{x,b})$ and $J^i = (\theta^i_{e,1(2)}, \theta^i_{x,1(2)}) \subset I,\ i = 1, 2, ..., n$ be open time intervals where $y_s > 0$, $y_u > 0$ and for which Proposition 3.5 applies. Let θ_c denote the time of leaving Case 2(1). Then, cumulative stored returns must equal bottleneck demand

$$\int_{\theta_{e,2(1)}}^{\theta_c} (u(t) - d(t))\, dt + \int_{\theta_c}^{\theta_{x,b}} (r(t) - \max\{d(t) - \bar{p}(t), 0\})\, dt + \sum_{i=1}^{n} \int_{\theta^i_{e,1(2)}}^{\theta^i_{x,1(2)}} (\max\{\bar{p}(t) - d(t), 0\})\, dt = 0. \tag{3.22}$$

Note that it might happen that a bottleneck situation will not become binding when demand is completely satisfied by remanufacturing anticipation stock.

3.2.5 Numerical Examples

Three examples are used to show the essential properties of optimal solutions in the presence of a limited production rate. Let demand, return, capacity constraint over a planning horizon of $T = 2\pi$ be given as stated in Table 3.2 (approximately 6 months). The discount rate is set to zero ($\alpha = 0$) and holding cost parameters are $h_s = 1.5$ and $h_u = 1$. The remaining relevant cash flow parameters as well as the beginning and end of the respective bottleneck intervals can also be found in Table 3.2.

In Example 3.1 (see Figure 3.4) demand d is generally larger than (constant) return rate u and capacity constraint $\bar{p}$, and for some positive time it even becomes larger than the current production and remanufacturing capacity of the system $\bar{p} + u$. Therefore, an interval with positive recoverables

Table 3.2. Demands, returns and capacity constraint functions as well as cash flow parameters in three considered scenarios

Example	$d(t)$	$u(t)$	$\bar{p}(t)$	$\theta_{e,b}$	$\theta_{x,b}$	c_p	c_r	c_w
3.1	$3.1 + 1.5\sin(t^2/12)$	3	1	2.8	5.5	4	2	1
3.2	$4 + 2\sin(6 - t)$	0.5	4	3.1	5.7	4	2	1
3.3	$1.8 + 3\sin(t^2/12)$	3	1	3.1	5.3	3	2	1

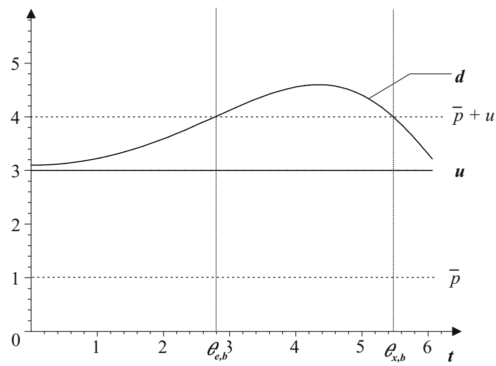

Fig. 3.4. Demands, returns and capacity constraint in Example 3.1

inventory must start and terminate in Subcase 4(2) and optimal decisions in the collection interval follow Subcase 2(3). For determining the start time θ_c of the collection interval, inventory condition (3.18) can be used which simplifies to

$$\int_{\theta_c}^{\theta_{x,b}} (r(t) - \max\{d(t) - \bar{p}(t), 0\})\, dt = 0 \tag{3.23}$$

since there is no Case 1(2) interval to be considered ($n = 0$). Solving (3.23) yields $\theta_c = 0.7$. Optimal decisions in Example 3.1 can be found in Figure 3.5, and the optimal co-state movements are depicted in Figure 3.6.

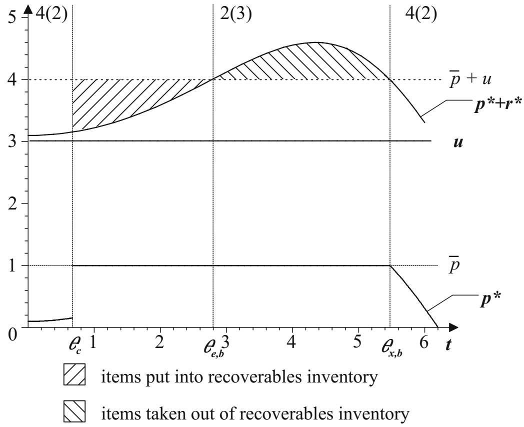

Fig. 3.5. Optimal solution of Example 3.1

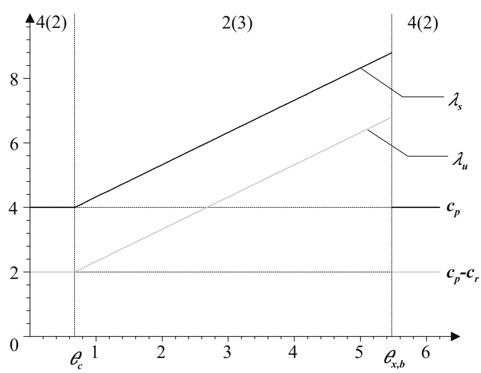

Fig. 3.6. Optimal co-state developments in Example 3.1

In Example 3.2, which is depicted in Figure 3.7, there is a point satisfying Corollary 3.8 inside the collection interval (at 2.9) and thus, it is optimal to have a Case 1(2) interval at (and around) this time. The optimal solution must satisfy inventory conditions (3.16), (3.17), (3.18), as well as Maximal Length Condition for a Case 1(2) interval (3.21), and it is depicted in Figure 3.8. The collection interval starts at $\theta_c = 0.45$, a switch to Case 1(2) takes place at $\theta_{e,1(2)} = 1.65$ and back to Case 2(3) is located at $\theta_{x,1(2)} = 4.06$.

Since the extension of both collection and Case 1(2) intervals is not limited and because of a zero discount rate, the recoverables co-state at the begin of the Case 1(2) interval is given by $\lambda_u(\theta_{e,1(2)} = 1.65) = c_p - c_r + h_u(\theta_{e,1(2)} - t_c) = 3.2$ and (3.21) holds with equality, yielding maximal length of the Case 1(2) interval of $\theta_{x,1(2)} - \theta_{e,1(2)} \leq 1.2/.5 = 2.4$. Equality holds except for a rounding error. The optimal co-state developments can be seen from Figure 3.9.

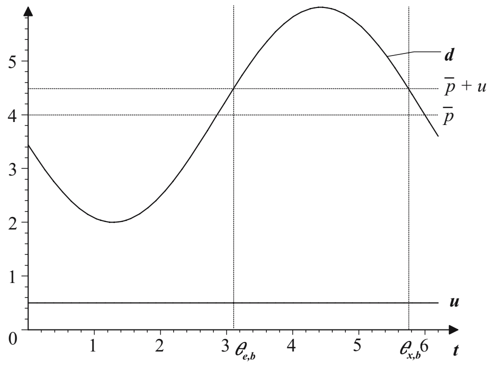

Fig. 3.7. Demands, returns and capacity constraint in Example 3.2

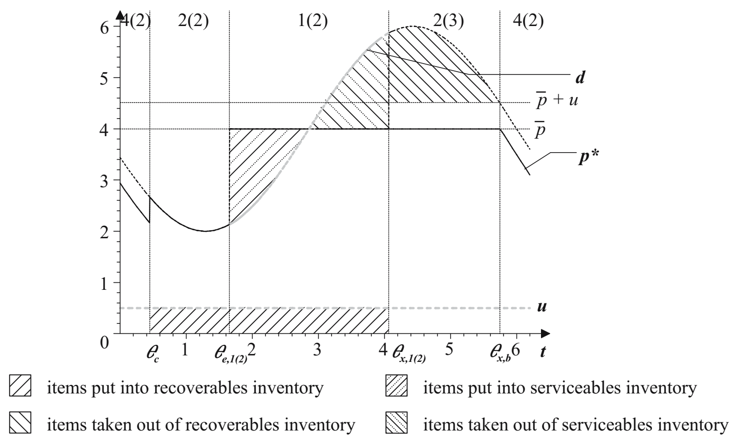

Fig. 3.8. Optimal solution of Example 3.2

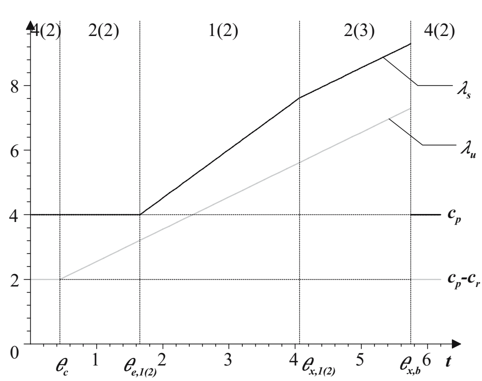

Fig. 3.9. Optimal co-state developments in Example 3.2

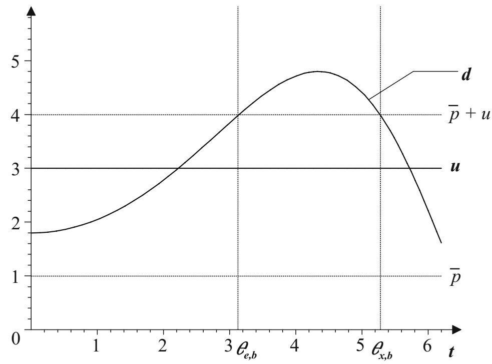

Fig. 3.10. Demands, returns and capacity constraint in Example 3.3

Now, we will examine Example 3.3 (see Figure 3.10), where it is optimal to have a positive inventory even in the unrestricted case because demand and return functions intersect. There the maximal holding time in the recoverables inventory $\tau_u = 2$. The optimal unrestricted solution is depicted in Figure 3.11, showing a period where the capacity constraint is violated (grey shaded area).

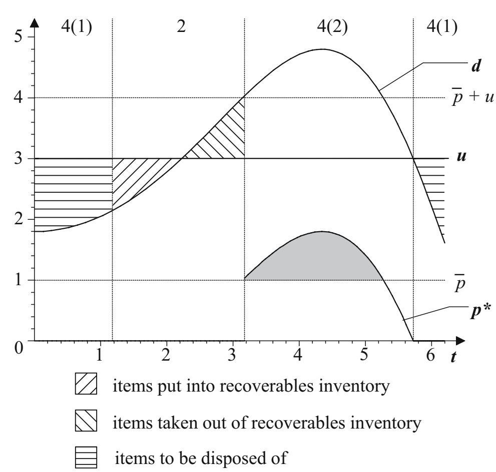

Fig. 3.11. Optimal solution of the unrestricted problem for Example 3.3

The optimal solution under consideration of the manufacturing constraint consists of a sequence 4(1) → 2(1) → 2(3) → 4(2). Using inventory condition (3.22) (again with $n = 0$, see Example 3.1) as well as maximum length property of Case 2(1) intervals both optimal time point when to switch from Case

4(1) to Case 2(1) $\theta_{e,2(1)}$ as well as time θ_c after which manufacturing takes place are determined: $\theta_{e,2(1)} = 0.6$ and $\theta_c = 2.6$. The solution of Example 3.3 is depicted in Figure 3.12, and the optimal co-state movements are to be found in Figure 3.13.

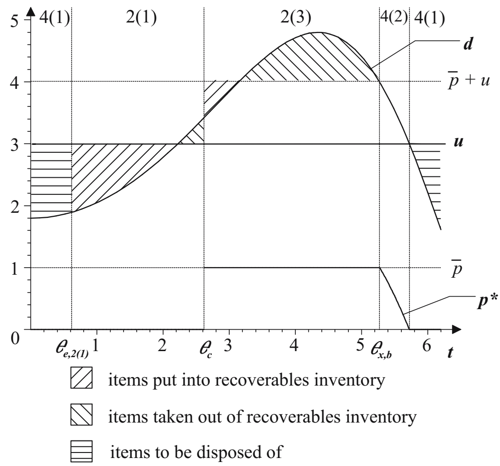

Fig. 3.12. Optimal solution of Example 3.3

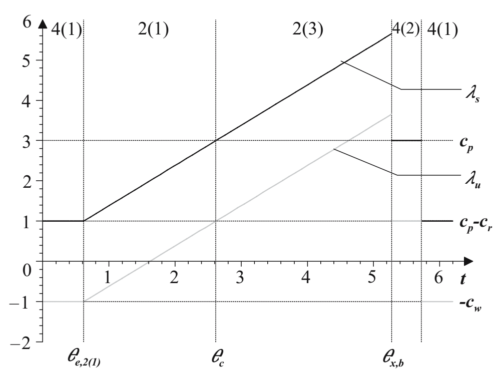

Fig. 3.13. Optimal co-state developments in Example 3.3

A number of more complex examples, also including several successive bottleneck situations within the planning horizon, can be found in Baranczyk (2001). In this work, a discrete time approximation of the constraint model was solved using linear programming methods.

3.2.6 Managerial Insights

Adding a production capacity constraint to the basic model leads to an additional motivation for keeping stock, namely to prevent against bottleneck situations where current capacity of the system does not suffice to satisfy demand. In contrast to the results presented in Chapter 2, generally both inventories can be used. Since there exists a holding cost advantage, bottlenecks are primarily served using previously collected returns. The size of the bottleneck determines how many returns to save and when to start collecting returns. Building up a serviceables stock during the collection interval is beneficial in periods where excess production capacity is available. It is used in order to shorten the collection period, because more returns are accumulated. The optimal policy must balance a trade-off regarding the holding costs which can be expressed by an interval length property.

Depending on the size of the bottleneck situation and the time that is required to collect returns, holding costs might achieve substantial dimensions and it is questionable whether the assumption to satisfy all demand should be kept. When allowing for a backlogging of demands, improvements are possible because in such an environment parts of the demand during a bottleneck situation are collected and satisfied after its end. See Kiesmüller et al. (2000) for a discussion of the profitability of having backorders in a product recovery system without capacity restrictions. One step further, the question has to be answered which parts of the demand to satisfy at all. This can be answered by using a profit maximizing approach.

3.3 Limited Remanufacturing Capacity

3.3.1 Changes to the Basic Model

This section deals with the influence of a remanufacturing constraint $\bar{r}(t) > 0$ on optimal decisions in the product recovery system introduced in Chapter 2. The corresponding optimization problem is given by (2.9) and an additional constraint

$$r(t) \leq \bar{r}(t). \tag{3.24}$$

Since production is unrestricted, demand can always be satisfied without producing in advance. Therefore, a condition ensuring the existence of a feasible solution as provided in Section 3.2 is not needed.

In what follows, since we are not dealing with boundary effects, starting inventory levels are set to zero, i.e. $y_s^0 = y_u^0 = 0$. As an additional simplifying condition, it is assumed that there exists no joint intersection of demand, return and maximal remanufacturing rate within the planning horizon, i.e.

$$\nexists\, t \in [0, T] \text{ with } d(t) = u(t) = \bar{r}. \tag{3.25}$$

Without restricting generality too much, (3.25) together with the assumption of strict positive demand and return rates leads to a considerable reduction of possible case transitions.

3.3.2 Properties of an Optimal Solution

Necessary Conditions

Proceeding in the same way as in the model with a remanufacturing constraint (see Section 3.2), the Lagrangian now reads

$$L(..) = H(.) + \mu_1 \cdot p + \mu_2 \cdot r + \mu_3 \cdot w + \mu_4 \cdot (\bar{r} - r) + k_1 \cdot y_s + k_2 \cdot y_u. \quad (3.26)$$

Necessary conditions (2.12)–(2.24) known from the basic model are adapted as follows. The Hamiltonian maximizing condition (2.13) changes to

$$r^* = \begin{cases} 0 & \lambda_s - \lambda_u < c_r \\ \text{singular} & \lambda_s - \lambda_u = c_r \\ \bar{r} & \lambda_s - \lambda_u > c_r \end{cases}, \quad (3.27)$$

equation (2.16) is replaced by

$$\frac{\partial L}{\partial r} = -c_r + \lambda_s - \lambda_u + \mu_2 - \mu_4 = 0, \quad (3.28)$$

and constraint (3.24) leads to an additional non-negativity as well as complementary slackness condition

$$\mu_4 \geq 0 \quad \mu_4 \cdot (\bar{r} - r^*) = 0. \quad (3.29)$$

The remaining conditions remain unchanged.

When introducing a remanufacturing constraint, the difference of the values of a serviceables item λ_s and a recoverables item λ_u can be higher than remanufacturing unit costs c_r because the latter might not be usable due to the capacity restriction. Since production and disposal rate are unbounded, the maximal value of serviceables is given by unit production costs ($\lambda_s \leq c_p$), and the value of returns cannot be lower than disposal revenue ($\lambda_u \geq -c_w$).

The Structure of an Optimal Solution

The introduction of a remanufacturing constraint allows for additional subcases in each of the four cases being distinguished with respect to their serviceables and recoverables status.

Proposition 3.9 (Optimal decisions in Case 1 intervals).
If both, serviceables and recoverables inventory are positive ($y_s^* > 0$, $y_u^* > 0$), no items are produced ($p^* = 0$) or disposed of ($w^* = 0$). The optimal decision

on whether to remanufacture or not depends on the relation of both co-states and two subcases can be distinguished:
Subcase 1(1) $\Leftrightarrow \lambda_s - \lambda_u < c_r$
No items are remanufactured ($r^* = 0$).
Subcase 1(2) $\Leftrightarrow \lambda_s - \lambda_u > c_r$
Remanufacturing takes place at its upper bound ($r^* = \bar{r}$).

In analogy to the model with a restricted production rate, it is now possible to remanufacture even if demand could be served from serviceables stock. In addition to Subcase 1(1) which is already known from the basic model there is another Subcase 1(2). There the remanufacturing rate obtains its upper bound if the change of the value of a recoverables item when remanufacturing it to serviceables stock ($\lambda_s - \lambda_u$) exceeds unit remanufacturing costs. It requires that the value of a serviceables λ_s is located in the interval $(c_r - c_w, c_p)$ and for the value of a returned item $\lambda_u \in (-c_w, c_p - c_r)$ must hold. The inventory levels change with rates $\dot{y}_s = \bar{r} - d$ and $\dot{y}_u = u - \bar{r}$.

Proposition 3.10 (Optimal decisions in Case 2 intervals).
If serviceables inventory is zero and recoverables inventory is positive ($y_s^* = 0$, $y_u^* > 0$), no items are disposed of. The optimal decisions on production and remanufacturing depend on the co-states and on the relationship between demand and return rates. Three subcases are distinguished:
Subcase 2(1) $\Leftrightarrow c_r - c_w < \lambda_s < c_p$, $\lambda_u < c_p - c_r$, $d < \bar{r}$
No items are produced ($p^* = 0$) and the remanufacturing rate equals the demand rate ($r^* = d$).
Subcase 2(2) $\Leftrightarrow \lambda_s = c_p$, $\lambda_u > c_p - c_r$
Production equals demand rate ($p^* = d$) and no items are remanufactured ($r^* = 0$).
Subcase 2(3) $\Leftrightarrow \lambda_s = c_p$, $\lambda_u < c_p - c_r$ *and* $\bar{r} < d$
Remanufacturing takes place at its upper bound ($r^* = \bar{r}$) and production is used to fill remaining demand ($p^* = d - \bar{r}$).

Similar to the previous section, two additional cases have to be considered in which production takes place. In Subcase 2(2) demand is filled completely from producing new items and thus, the recoverables inventory level must rise ($\dot{y}_u = u > 0$). As will be shown below, this subcase is not relevant for determining the optimal solution. Subcase 2(3) shows both remanufacturing with maximal rate as well as production. This makes sense in situations where demand is so high that it cannot be satisfied using the preferred mode.

Proposition 3.11 (Optimal decisions in Case 3 intervals).
If serviceables inventory is positive and recoverables inventory is zero ($y_s^* > 0$, $y_u^* = 0$), the optimal policy is not to produce ($p^* = 0$). The value of a returned item equals the disposal revenue ($\lambda_u = -c_w$), and the optimal decision on whether to remanufacture or not depends on the co-state λ_s. The following subcases are possible:

Subcase 3(1) $\Leftrightarrow \lambda_s < c_r - c_w$
No items are remanufactured ($r^* = 0$) *and all returns are disposed of* ($w^* = u$).
Subcase 3(2) $\Leftrightarrow c_r - c_w < \lambda_s < c_p$ *and* $u > \bar{r}$
Remanufacturing takes place at its upper bound ($r^* = \bar{r}$) *and remaining returns are disposed of* ($w^* = u - \bar{r}$).

Although it is not optimal to produce when having a positive serviceables inventory, in a new Subcase 3(2) remanufacturing takes place with maximal possible rate. The serviceables stock develops inside this case with rate $\dot{y}_s = \bar{r} - d$ and it increases if the constraint exceeds demand and decreases if the opposite holds. Such a decision is reasonable if one considers that serviceables are accumulated by remanufacturing more items than currently required in order to satisfy demand in a situation where the remanufacturing constraint is binding. Returns that exceed the capacity are disposed of if they can not be used within a reasonable period.

Proposition 3.12 (Optimal decisions in Case 4 intervals).
If serviceables and recoverables inventories are zero ($y_s^* = 0$, $y_u^* = 0$), *optimal decisions depend on how demand rate, return rate, and remanufacturing constraint relate to each other and three subcases can be distinguished:*
Subcase 4(1) $\Leftrightarrow d \leq u$ *and* $d < \bar{r}$
Demand is satisfied completely by remanufacturing returns ($r^* = d$) *and excess returns are disposed of* ($w^* = u - d$). *No items are produced* ($p^* = 0$).
Subcase 4(2) $\Leftrightarrow u < d$ *and* $u < \bar{r}$
All returns are remanufactured ($r^* = u$) *and the missing items are produced* ($p^* = d - u$). *No items are disposed of* ($w^* = 0$).
Subcase 4(3) $\Leftrightarrow \bar{r} < u$ *and* $\bar{r} < d$
Remanufacturing takes place at its upper bound ($r^* = \bar{r}$) *and the missing items are produced* ($p^* = d - \bar{r}$). *Remaining returned items are disposed of* ($w^* = u - \bar{r}$).

In the absence of both recoverables and serviceables stock, the decision on remanufacturing depends on the relation of demand, return and maximum remanufacturing rate. It is given by the minimum of those three values ($r^* = \min\{p, u, \bar{r}\}$) and the determination of how many items to produce and/or to dispose of directly follows this decision.

The value of an additional return λ_u is either given by $c_p - c_r$ if it can be used immediately (Subcase 4(2)), or it is $-c_w$ if the returned item cannot be used due to insufficient demand or capacity. This happens in Subcases 4(1) and 4(3). The value of a serviceables item λ_s is $c_r - c_w$ if demand can be satisfied from remanufacturing alone (Subcase 4(1)), requiring both enough returns and remanufacturing capacity, or it is given by c_p if the last demanded unit was satisfied from production. This is relevant in Subcases 4(2) and 4(3).

Table 3.3 summarizes the results of the four cases.

Table 3.3. Main results of optimal cases when considering a remanufacturing constraint.

	p^*	r^*	w^*	$\dot{y}_s$	$\dot{y}_u$	λ_s	$\dot{\lambda}_s$	λ_u	$\dot{\lambda}_u$
Case 1: $y_s > 0$, $y_u > 0$									
(1)	0	0	0	$-d$	$u-d$	$< c_p$ ($\lambda_s < \lambda_u + c_r$)	$\alpha\lambda_s + h_s$	$-c_w <$	$\alpha\lambda_u + h_u$
(2)	0	$\bar{r}$	0	$\bar{r}-d$	$u-\bar{r}$	$c_r - c_w < \lambda_s < c_p$ ($\lambda_u + c_r < \lambda_s$)	$\alpha\lambda_s + h_s$	$-c_w < \lambda_u < c_p - c_r$	$\alpha\lambda_u + h_u$
Case 2: $y_s = 0$, $y_u > 0$									
(1) ($d < \bar{r}$)	0	d	0	0	$u-d$	$c_r - c_w < \lambda_s < c_p$ ($\lambda_s = \lambda_u + c_r$)	$\alpha\lambda_u + h_u$	$-c_w < \lambda_u < c_p - c_r$	$\alpha\lambda_u + h_u$
(2)	d	0	0	0	u	c_p	0	$c_p - c_r <$	$\alpha\lambda_u + h_u$
(3) ($\bar{r} < d$)	$d-\bar{r}$	$\bar{r}$	0	0	$u-\bar{r}$	c_p	0	$-c_w < \lambda_u < c_p - c_r$	$\alpha\lambda_u + h_u$
Subcases 2(1) and 2(3) generalized: $p^* = \max\{0, d-\bar{r}\}, r^* = \min\{d, \bar{r}\}, w^* = 0$									
Case 3: $y_s > 0$, $y_u = 0$									
(1)	0	0	u	$-d$	0	$< c_r - c_w$	$\alpha\lambda_s + h_s$	$-c_w$	0
(2) ($\bar{r} < u$)	0	$\bar{r}$	$u-\bar{r}$	$\bar{r}-d$	0	$c_r - c_w < \lambda_s < c_p$	$\alpha\lambda_s + h_s$	$-c_w$	0
Case 4: $y_s = 0$, $y_u = 0$									
(1) ($d < u, d < \bar{r}$)	0	d	$u-d$	0	0	$c_r - c_w$	0	$-c_w$	0
(2) ($u < d, u < \bar{r}$)	$d-u$	u	0	0	0	c_p	0	$c_p - c_r$	0
(3) ($\bar{r} < u, \bar{r} < d$)	$d-\bar{r}$	$\bar{r}$	$u-\bar{r}$	0	0	c_p	0	$-c_w$	0
Case 4 generalized: $p^* = \max\{d - r^*, 0\}, r^* = \min\{u, d, \bar{r}\}, w^* = \max\{u - r^*, 0\}$									

Optimal Transitions Between Cases and Subcases

Because of the additional subcases, there is a substantial increase of the number of possible case transitions. Therefore, cases are excluded that never occur in an optimal solution (especially when neglecting initial stocks) before describing relevant transitions. Similar to the case of a manufacturing constraint, Subcases 1(1) and 3(1) can only be present at the beginning of the planning period. This is because from the optimal serviceables state developments inside Case 1(1) and Case 3(1) intervals ($\dot{y}_s < 0$) it follows that any transition from a case where $y_s = 0$ to one of these two cases is forbidden. Moreover, because of co-state continuity if the respective inventory is non-empty, there is no possibility for transitions from Case 3(2) to Case 1(1)/3(1) or from Case 1(2) to Case 1(1), because a downward jump in λ_s would be required. Thus, the occurrence of both cases requires positive initial inventory conditions, leading to similar effects as discussed for the basic model.

In any interval where Case 2(2) applies, the recoverables stock must increase ($\dot{y}_u > 0$). Therefore, a switch to another case where $y_u = 0$ is excluded. Since the value of a recoverables item exceeds the difference of unit production and remanufacturing cost rate ($\lambda_u > c_p - c_r$), a transition to another case where $y_u > 0$ (Cases 1(2), 2(1), and 2(3)) cannot take place. There the co-state requires a lower value and a downward jump in the co-state is excluded when having positive recoverables inventory. Case 2(2) will therefore never be left contradicting final condition $y_u(T) = 0$. The following corollary collects results regarding the exclusion of cases.

Corollary 3.10 (Cases that do not occur in an optimal solution). *It is never optimal to have intervals of Cases 1(1), 3(1), and 2(2) in a solution to a problem with zero initial and terminal conditions.*

Following our usual procedure, Proposition 3.13 collects all situations, under which one or both co-states are allowed to jump.

Proposition 3.13 (Continuity of λ_s and λ_u).
λ_s and λ_u are continuous, i.e. jump parameters η_s and η_u vanish everywhere, except at time points θ where one of the following conditions hold:

(i) *λ_s jumps if $d(\theta) = \bar{r}(\theta)$ and $\dot{d}(\theta) < \dot{\bar{r}}(\theta)$ hold in situations where $y_s = 0$, allowing for the following transitions:*
$2(3) \rightarrow 2(1)$, $4(3) \rightarrow 4(1)$, $4(3) \rightarrow 2(1)$, $1(2) \rightarrow 2(1)$, $3(2) \rightarrow 4(1)$, $2(3) \rightarrow 1(2)$, $4(3) \rightarrow 1(2)$, *and* $4(3) \rightarrow 3(2)$.

(ii) *λ_s and λ_u jump if $u(\theta) = d(\theta)$and $\dot{d}(\theta) < \dot{u}(\theta)$ hold in situations where $y_s = y_u = 0$, allowing for the following transitions:*
$4(2) \rightarrow 4(1)$, $2(1) \rightarrow 4(1)$, *and* $4(2) \rightarrow 2(1)$.

(iii) *λ_u jumps if $u(\theta) = \bar{r}(\theta)$ and $\dot{u}(\theta) > \dot{\bar{r}}(\theta)$ hold in situations where $y_u = 0$, allowing for the following transitions:*
$4(2) \rightarrow 4(3)$, $1(2) \rightarrow 3(2)$, $1(2) \rightarrow 4(3)$, $2(3) \rightarrow 4(3)$, *and* $4(2) \rightarrow 2(3)$.

The following Corollaries 3.11–3.16 state possible case and subcase transitions and are derived by using the same methods described in the previous section.

Corollary 3.11. *Within Case 2 Subcase 2(3) can be followed by Subcase 2(1). This automatic and discontinuous transition requires* $d = \bar{r}$ *and* $\dot{d} < \dot{\bar{r}}$.

In Case 2(3) as many items as possible are remanufactured ($r = \bar{r}$). This is only feasible without accumulating a serviceables stock as long as demand exceeds the remanufacturing constraint. As the demand rates falls below the constraint, decisions must change and an automatic transition to Case 2(1) takes place.

Corollary 3.12. *Within Case 4 the following transitions are possible:*

4(2) → 4(1). *This automatic and discontinuous transition requires* $d = u < \bar{r}$ *and* $\dot{d} < \dot{u}$.

4(2) → 4(3). *This automatic and discontinuous transition requires* $u = \bar{r} < d$ *and* $\dot{u} > \dot{\bar{r}}$.

4(3) → 4(1). *This automatic and discontinuous transition requires* $d = \bar{r} < u$ *and* $\dot{d} < \dot{\bar{r}}$.

Since there is no stock available in Case 4 intervals, the minimum of demand, return rate and constraint determines the remanufacturing rate. Therefore, as another component begins to decide about this rate, also the subcase of Case 4 must change.

Now, possible transitions between different cases are presented.

Corollary 3.13. *Starting at a Case 1(2) interval where* $y_s > 0$ *and* $y_u > 0$ *the following transitions are possible:*

1(2) → 2(1). *This automatic and discontinuous transition requires* $d = \bar{r}$ *and* $\dot{d} < \dot{\bar{r}}$.

1(2) → 2(3). *This automatic and continuous transition requires* $\lambda_s = c_p$ *and* $\bar{r} < d$.

1(2) → 3(2). *This automatic and discontinuous transition requires* $u = \bar{r}$ *and* $\dot{u} > \dot{\bar{r}}$.

1(2) → 4(3). *This automatic and discontinuous transition requires* $\lambda_s = c_p$, $u = \bar{r} < d$ *and* $\dot{u} > \dot{\bar{r}}$.

In Case 1(2) remanufacturing takes place at its upper bound. The serviceables inventory decreases if demand exceeds the constraint and recoverables inventory depletes if more recoverables are remanufactured than currently returned. Depending on the development of demand, return, and constraint rates, one of both inventories depletes first, and a switch to Case 2 or Case 3 occurs. As a special case, a transition to Case 4(3) takes place if both inventories are emptied simultaneously.

Corollary 3.14. *Starting at a Case 2 interval where* $y_s = 0$ *and* $y_u > 0$ *the following transitions are possible:*

2(1) → 1(2). *This constraint-forced and continuous transition requires* $d < \bar{r}$.

2(1) → 4(1). *This automatic and discontinuous transition requires* $d = u < \bar{r}$, $\dot{d} \leq \dot{u}$

2(1) → 4(2). *This automatic and continuous transitions requires* $\lambda_s = c_p$, $\lambda_u = c_p - c_r$, *and* $u < d < \bar{r}$.

2(3) → 1(2). *This constraint-forced and discontinuous transition requires* $d = \bar{r}$, $\dot{d} < \dot{\bar{r}}$.

2(3) → 4(2). *This automatic and continuous transition requires* $\lambda_u = c_p - c_r$ *and* $u < \bar{r} < d$.

2(3) → 4(3). *This automatic and discontinuous transition requires* $u = \bar{r} < d$ *and* $\dot{u} > \dot{\bar{r}}$.

Similar to the model with a limited production rate, a Case 2 interval may terminate into Case 1(2), requiring demand not to exceed the constraint ($d \leq \bar{r}$). A switch to Case 4 either takes place under conditions known from the basic model (2(1) → 4(1)/4(2), $u \leq d < \bar{r}$), or it occurs starting from a Case 2(3) interval, which necessitates $u \leq \bar{r} < d$.

Corollary 3.15. *Starting at a Case 3(2) interval where* $y_s > 0$ *and* $y_u = 0$ *the following transitions are possible:*

3(2) → 1(2). *This automatic and continuous transition requires* $u > \bar{r}$.

3(2) → 4(1). *This automatic and discontinuous transition requires* $d = \bar{r} < u$ *and* $\dot{d} < \dot{\bar{r}}$.

3(2) → 4(3). *This automatic and continuous transition requires* $\lambda_s = c_p$, $\bar{r} < u$, *and* $\bar{r} < d$.

As before, after completing a Case 3(2) interval, either (1) a constraint-forced switch to Case 1(2) is possible, if the return rate is larger than the remanufacturing constraint, or (2) a depleting serviceables inventory leads to a transition to Case 4 requiring demand to exceed the remanufacturing constraint.

Corollary 3.16. *Starting at a Case 4 interval where* $y_s = y_u = 0$ *the following transitions are possible:*

4(1) → 1(2). *This constraint-forced and continuous transition requires* $d < \bar{r} < u$.

4(1) → 2(1). *This forced and continuous transition requires* $d < u$ *and* $d < \bar{r}$.

4(1) → 3(2). *This constraint-forced and continuous transition requires* $d < \bar{r} < u$.

4(2) → 2(1). *This forced and discontinuous transition requires* $d = u < \bar{r}$, $\dot{d} < \dot{u}$

4(2) → 2(3). *This constraint-forced and discontinuous transition requires* $u = \bar{r} < d$ *and* $\dot{u} > \dot{\bar{r}}$.

4(3) → 1(2). *This constraint-forced and discontinuous transition requires* $d = \bar{r} < u$ *and* $\dot{d} < \dot{\bar{r}}$.

4(3) → 2(1). *This forced and discontinuous transition requires* $d = \bar{r} < u$ *and* $\dot{d} < \dot{\bar{r}}$.

4(3) → 2(3). *This constraint-forced and continuous transition requires* $d > \bar{r}$ *and* $u > \bar{r}$.

4(3) → 3(2). *This constraint-forced and discontinuous transition requires* $d = \bar{r} < u$ *and* $\dot{d} < \dot{\bar{r}}$.

Starting in a Case 4 interval, a broad range of forced transitions is possible. These can be organized into four groups. The first is already known from the basic model and uses the anticipation of a change in the relation between demands and returns motive. It terminates in a Case 2(1) interval. A second group, culminating in a Case 2(3) interval uses a related motivation. Here, in a situation with high demand ($d > u$ and $d > \bar{r}$), a recoverables stock is collected in a period where $u > \bar{r}$ in anticipation of a return rate falling below the constraint. In the third group, where a switch to a Case 3(2) interval takes place, situations with a high return rate ($d < u$,$u > \bar{r}$) require to use a serviceables inventory in anticipation of a demand increasing above the constraint. A combination of the last two motivations leads to a transition to Case 1(2), necessitating that both collecting returns and building serviceables stock can be done simultaneously. This requires $d \leq \bar{p} < u$ at switching time.

All possible transitions are depicted in Figure 3.14.

3.3.3 Pure Effects of a Remanufacturing Constraint

When considering a remanufacturing constraint, the current capacity of the product recovery system always suffices to immediately fill demand. Therefore, there is no such inherent reason that requires us to build up stock in order to obtain feasibility as it has been when assuming a limited manufacturing rate. Production in advance never takes place as can be seen from optimal decisions in cases where the serviceables inventory is positive. On the other side, the recovery cost advantage suggests to efficiently use available capacity and thus there are situations where it might be preferable to keep stock to 'smooth'

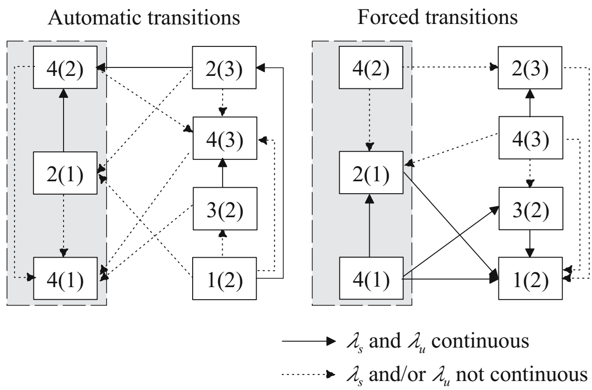

Fig. 3.14. Optimal case transitions when considering a remanufacturing constraint

its usage. The corresponding *constraint-forced* transitions are discussed in the following. As a result, additional motivations for keeping stock are developed.

Since we are concentrating an pure effects of a limited remanufacturing rate, and in order to reduce the potential complexity of the solution, we restrict ourselves to two scenarios with respect to the relation of demand and return rates. Thus not all transitions present in Figure 3.14 are considered. First, pure effects of a remanufacturing constraint are discussed by assuming a situation with *high demand*, i.e. the return rate is smaller than the demand rate during the whole planning period. Afterwards, we deal with a situation where the opposite holds (*low demand*). In doing so, we neglect situations where the anticipation motive leads to keeping stock. Some of the results are applied in a model for product recovery strategy and presented in Section 5.5.

Pure Effects of a Remanufacturing Constraint when u < d

In a planning situation where demand exceeds the return rate, Case 4(1) plays no role. Exclusion of all switches between cases that require $d \leq u$ leaves transitions as depicted in Figure 3.15.

Here, available returns can immediately be used to fill demand as long as these do not exceed the remanufacturing capacity (Case 4(2)). If they do, all returns exceeding the constraint cannot immediately be used, and are therefore called *bottleneck returns*. An optimal solution must answer the question on how to deal with these items. For simplifying the discussion, let us assume there exists a single interval where returns do not exceed maximal remanufacturing rate amid of two adjacent *bottleneck intervals* during the considered planning period. This situation is sufficient to explain all of the relevant effects. Let $\theta^u_{x,b}$ represent the end of the preceding and $\theta^u_{e,b}$ the begin of the subsequent bottleneck interval with

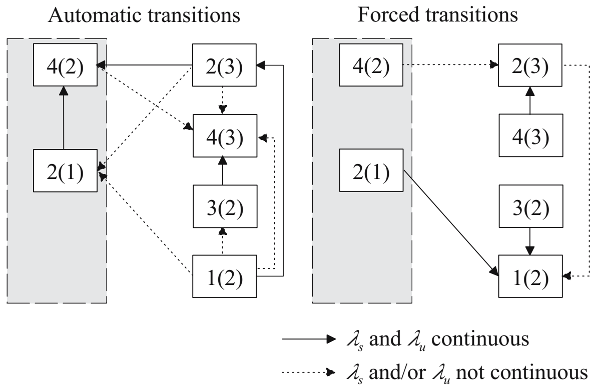

Fig. 3.15. Optimal case transitions when considering a remanufacturing constraint assuming high demand

$$\theta^u_{x,b} : u(\theta^u_{x,b}) = \bar{r}(\theta^u_{x,b}) \text{ and } \dot{u}(\theta^u_{x,b}) < \dot{\bar{r}}(\theta^u_{x,b}),$$
$$\theta^u_{e,b} : u(\theta^u_{e,b}) = \bar{r}(\theta^u_{e,b})) \text{ and } \dot{u}(\theta^u_{e,b}) > \dot{\bar{r}}(\theta^u_{e,b}).$$

At the end of the preceding bottleneck period it might be preferable to keep returns and use them at a time where the return rate has fallen below the constraint. There, the stored returns can be used to maximize remanufacturing capacity usage and replace an otherwise necessary production of new items. The following corollary results from the fact that Case 4(3) can only terminate into a Case 2(3) interval.

Corollary 3.17 (Location of an interval with positive recoverables inventory when demand is high).
Let θ denote a time point where $u(\theta) = \bar{r}(\theta) < d(\theta)$. If returns cross the constraint from above, i.e. $\dot{u}(\theta) < \dot{\bar{r}}(\theta)$, then it is always optimal to have a positive recoverables inventory at (and around) time θ ($y_u(\theta) > 0$).

Let $\theta_{e,u}$ denote the start of a bottleneck *collection period*, i.e. the time of a transition from Case 4(3) to Case 2(3) at which accumulation of returns starts. Then, at $\theta_{e,u}$ the return rate must not be smaller than the remanufacturing constraint ($u(\theta_{e,u}) \geq \bar{r}(\theta_{e,u})$) in order to start building up stock. The end of a bottleneck interval $\theta^u_{x,b}$ (which satisfies Corollary 3.17) separates the collection from a *consumption period* where stored recoverables are used up, being completed at time $\theta_{x,u}$.

Figure 3.16 shows a situation where demand exceeds both the return rate as well as the remanufacturing constraint. There exists a point satisfying Corollary 3.17 which therefore separates a bottleneck return collection period from a return consumption period. Before $\theta_{e,u}$ part of the returns, i.e. those exceeding the constraint, are disposed of and after $\theta_{x,u}$ the production rate gets higher than would be required when remanufacturing with highest

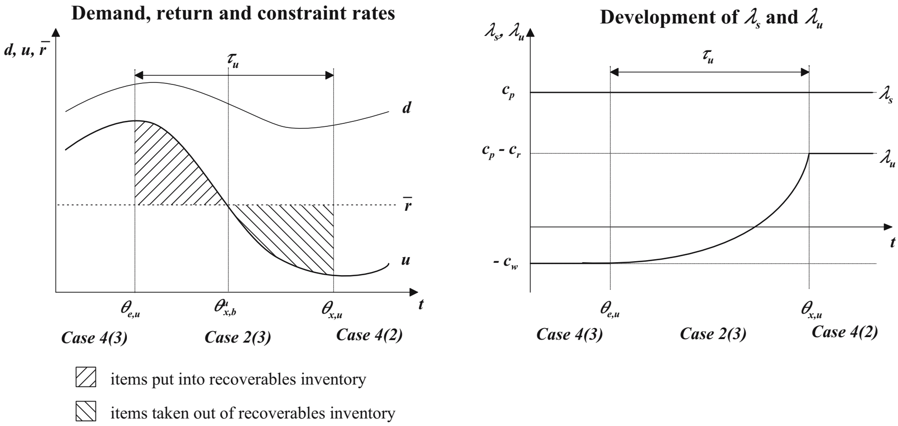

Fig. 3.16. An interval where returns are kept during a bottleneck collection interval for later use within a consumption period with maximal length.

possible rate. Therefore, a trade-off is balanced which is similar to that one being known from the basic model for collecting a returned item and later using it for replacing production by remanufacturing, instead of disposing of it. A corresponding marginal criterion leads to a maximal holding time condition for recoverables as presented in the following proposition.

Proposition 3.14 (Maximal Length of an interval where $y_u > 0$). *The maximal length of an interval $I = (\theta_{e,u}, \theta_{x,u})$ where $y_u > 0$ holds is given by*

$$\theta_{x,u} - \theta_{e,u} \leq \tau_u \tag{3.30}$$

where τ_u is determined in Proposition 2.7.

A bottleneck collection and consumption period having maximal length is exemplified in Figure 3.16 showing a transition sequence $4(3) \to 2(3) \to 4(2)$. Here also optimal co-state movements can be seen. Since production takes place in all cases, the serviceables value is c_p. The value of a returned item rises during the Case 2(3) interval starting with $-c_w$ at $\theta_{e,u}$ and reaching $c_p - c_r$ at $\theta_{x,u}$.

Intervals with positive recoverables inventory not having maximal length require intersections of return rate and constraint at either $\theta_{e,u}$ or $\theta_{x,u}$. In the first case, the collection period cannot start earlier, since there are not sufficient returns that exceed the remanufacturing capacity. A corresponding case sequence would be $4(2) \to 2(3) \to 4(2)$. Secondly, the consumption period must end, because there is no remaining excess capacity available to remanufacture stored returns resulting in a sequence $4(3) \to 2(3) \to 4(3)$.

Another interesting question would be what happens if demand intersects the constraint. By definition this is only possible inside a consumption period because otherwise the assumption that demands exceed the return rate would be contradicted. As demand falls below the constraint, it is no longer necessary to remanufacture with maximum rate and a transition from Case 2(3) to 2(1) takes place at a time θ with $d = \bar{r}$ and $\dot{d} < \dot{\bar{r}}$. An example where the joint interval (where $y_u > 0$) has maximal length is depicted in Figure 3.17. It shows a sequence 4(3) $\rightarrow$ 2(3) $\rightarrow$ 2(1) $\rightarrow$ 4(2). Since production is not

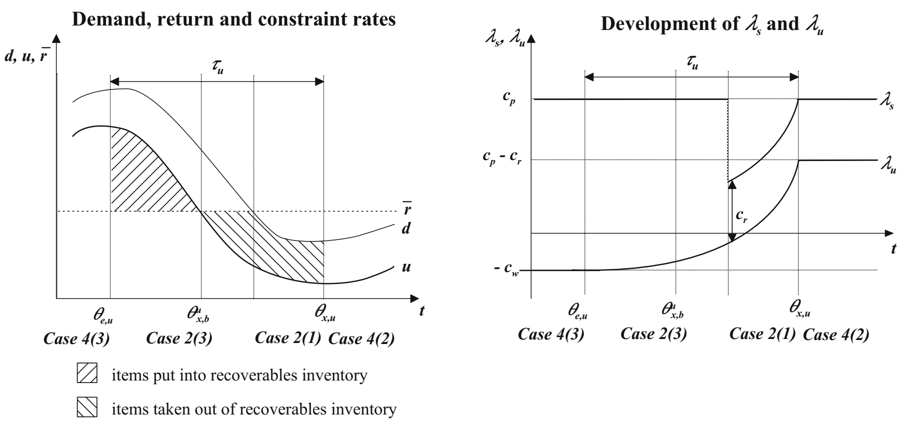

Fig. 3.17. A consumption period showing a Case 2(1) interval

required inside Case 2(1) intervals, the value of serviceables drops down to the sum of the value of recoverables and the remanufacturing cost rate at the transition time. The same reasons as explained above allow also in this case for Case 2 intervals of less than maximal length corresponding to sequences 4(2) $\rightarrow$ 2(3) $\rightarrow$ 2(1) $\rightarrow$ 4(2) and 4(3) $\rightarrow$ 2(3) $\rightarrow$ 2(1) $\rightarrow$ 4(3).

If inside a consumption period demand crosses the constraint from below, it is preferable to remanufacture with maximum rate before the intersection time and to keep serviceables in stock, since it now allows to fill additional demand from remanufactured and kept items (see Figure 3.18). The following corollary results from the impossibility of a transition from Case 2(1) to Case 2(3).

Corollary 3.18 (Location of an interval with positive serviceables inventory when demand is high).
Let θ denote a time point where $u(\theta) < d(\theta) = \bar{r}(\theta)$ and $y_u(\theta) > 0$. If demand crosses the constraint from below, i.e. $\dot{d}(\theta) > \dot{\bar{r}}(\theta)$, then it is always optimal to have a positive serviceables inventory at (and around) time θ ($y_s(\theta) > 0$).

There might be several time points meeting Corollary 3.18 within a consumption period. Therefore, it is assumed that there are $n > 0$ intervals $J^i = (\theta^i_{e,s}, \theta^i_{x,s})$, $i = 1, 2, ..., n$ where it is optimal to keep serviceables ($y_s > 0$).

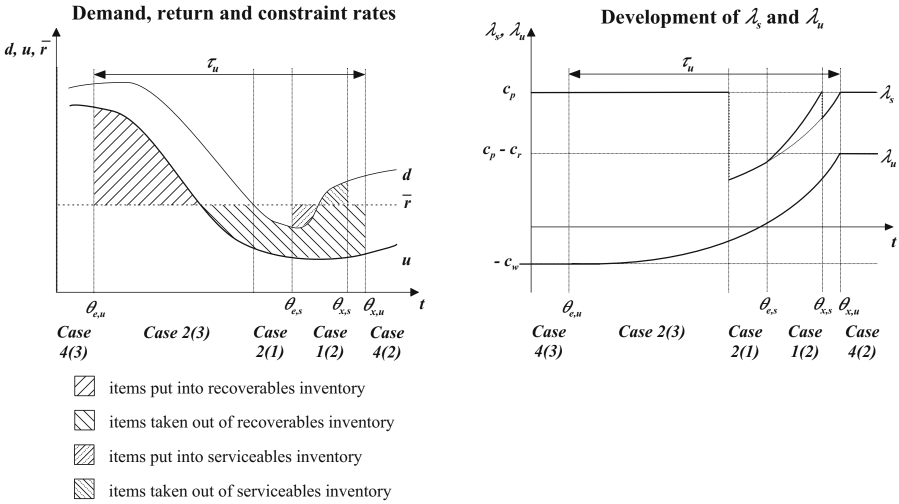

Fig. 3.18. A consumption period showing a Case 1(2) interval.

For each one of these, the following inventory conditions must hold (presented without proof).

Proposition 3.15 (Inventory Conditions of intervals with positive serviceables inventory).
Let $J^i = (\theta^i_{e,s}, \theta^i_{x,s})$ *be an open time interval where* $y_s > 0$ *and* $y_s(\theta^i_{e,s}) = y_s(\theta^i_{x,s}) = 0$. *Then,*

(i) cumulative remanufacturing equals cumulative demand over the whole interval

$$\int_{\theta^i_{e,s}}^{\theta^i_{x,s}} (\bar{r}(t) - d(t))dt = 0, \tag{3.31}$$

(ii) at any point $\theta \in J^i$, *cumulative remanufacturing must be larger than cumulative demand*

$$\int_{\theta^i_{e,s}}^{\theta} (\bar{r}(t) - d(t))dt > 0. \tag{3.32}$$

Inside intervals with serviceables in stock, more demand is satisfied from remanufacturing than inside Case 2 and thus, additional returns are used adding up to $\int_{\theta^i_{e,s}}^{\theta^i_{x,s}} (\max\{\bar{r}(t) - d(t), 0\})dt$. This result can be used in order to derive inventory conditions for the recoverables inventory (presented without proof and omitting the non-negativity condition).

Proposition 3.16 (Inventory Condition for the joint collection and consumption interval).
Let $I = (\theta_{e,u}, \theta_{x,u})$ be an open time interval where $y_u > 0$ with $y_s(\theta_{e,u}) = y_u(\theta_{e,u}) = y_u(\theta_{x,b}) = 0$ and $J^i = (\theta^i_{e,s}, \theta^i_{x,s}),\ i = 1, 2, ..., n$ be open time intervals where $y_s > 0$, $y_u > 0$ and for which Proposition 3.15 applies. Then, cumulative stored returns during bottleneck periods must equal consumed returns

$$\int_{\theta_{e,u}}^{\theta_{x,u}} (u(t) - \min\{d(t), \bar{r}(t)\})\, dt - \sum_{i=1}^{n} \int_{\theta^i_{e,s}}^{\theta^i_{x,s}} (\max\{\bar{r}(t) - d(t), 0\})\, dt = 0. \quad (3.33)$$

As an implication from consuming more recoverables, also the consumption period reduces when increasing the length of an interval with a positive serviceables stock. This lowers costs as long as the induced serviceables holding costs are accompanied by a higher reduction of recoverables holding costs. This trade-off yields the following proposition.

Proposition 3.17 (Maximal Length of a Case 1(2) interval).
The maximal length of a Case 1(2) interval $J^i = (\theta^i_{e,s}, \theta^i_{x,s})$ is time dependent and it is given by

$$\theta^i_{x,s} - \theta^i_{e,s} \leq \frac{1}{\alpha} \ln \left(\frac{\alpha c_p + h_s}{\alpha \left(\lambda_u(\theta^i_{e,s}) + c_r \right) + h_s} \right). \quad (3.34)$$

In short term problems when assuming $\alpha = 0$, condition (3.34) changes to

$$\theta^i_{x,s} - \theta^i_{e,s} \leq \frac{c_p - \lambda_u(\theta^i_{e,s}) - c_r}{h_s}. \quad (3.35)$$

Marginal criterion (3.34) (respectively (3.35)) can be interpreted as follows. Assume there are two possibilities to use a collected returned unit available at $\theta^i_{e,s}$. This can be either (1) keeping it until $\theta_{x,u}$ where it is remanufactured to fill demand at that point (increase the consumption period) or (2) remanufacturing it immediately to stock at $\theta^i_{e,s}$ for satisfying demand at $\theta^i_{x,s}$ and thus, increasing the length of the interval where $y_s > 0$. In an optimal solution, one needs to be indifferent between both options except for situations explained later. As another implication of Proposition 3.17, the maximal length of an interval with positive serviceables inventory must decrease during a consumption period, because the later $\theta^i_{e,s}$ the higher the recoverables co-state and thus, the right hand side of (3.34) must fall. This is because remanufacturing cost advantage is balanced against total holding costs for a marginal item. The later a stored return is remanufactured and transferred to serviceables stock the higher are already accumulated recoverables holding costs and thus less serviceables holding cost are allowed to be added.

Since the serviceables co-state increases faster than the recoverables value inside a Case 1(2) value, the serviceables inventory must be emptied before

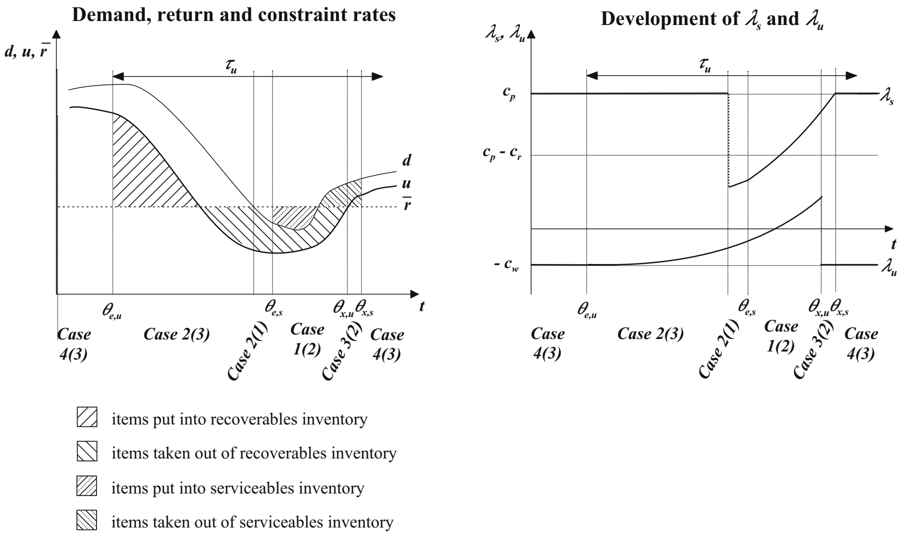

Fig. 3.19. A consumption period terminating into a Case 3(2) interval.

depleting the recoverables inventory except for one reason. Namely that if the time period, where the recoverables inventory is positive is limited because the return rate rises above the remanufacturing constraint at the end of this interval $\theta_{x,u}$, then the consumption period does not reach maximal length. There, it might be preferable to hold serviceables even longer than until $\theta_{x,u}$. This allows for sequences (at the end of a consumption period) $1(2) \to 4(3)$, if both inventories are depleted simultaneously and $1(2) \to 3(2) \to 4(3)$ if recoverables inventory is emptied first. An example is depicted in Figure 3.19.

Pure Effects of a Remanufacturing Constraint when u > d

Planning situations under which the return rate always exceeds the demand rate never show Case 4(2) intervals, and excluding all transitions requiring $d \geq u$ leaves switches as depicted in Figure 3.20. In what follows, we concentrate on a main effect of a remanufacturing constraint when demand is low by additionally assuming $u \geq \bar{r}$. Thus, there are always sufficient returns available for remanufacturing and stockkeeping is not performed under use of a recoverables inventory. Under this assumption therefore only sequences

$$\text{Case 4} \;\to\; \text{Case 3(2)} \;\to\; \text{Case 4}$$

occur in an optimal solution. Further results could be obtained when allowing for more complex interactions as has been accomplished in the case of high demand.

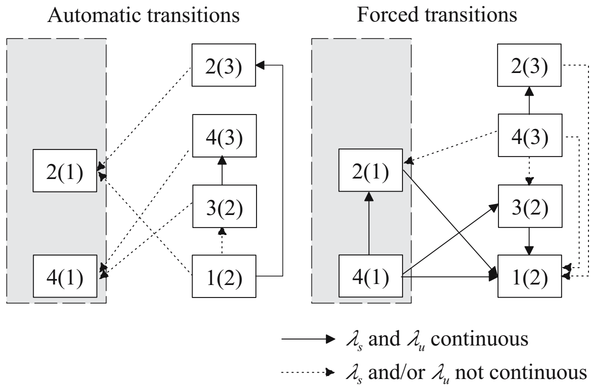

Fig. 3.20. Optimal case transitions when considering a remanufacturing constraint assuming low demand.

As long as demand does not exceed the remanufacturing constraint, it is satisfied from current remanufacturing, and excess returns are disposed of (Case 4(1)). An optimal solution must answer the question from which source to satisfy demand that exceeds the remanufacturing capacity, which is therefore called *bottleneck demand*. Analogously to the previously considered situation, our discussion is restricted to a single interval where $d < \bar{r}$. Let $\theta^d_{x,b}$ be the end of the preceding and $\theta^d_{e,b}$ the beginning of a subsequent *bottleneck interval* with

$$\theta^d_{x,b} : d(\theta^d_{x,b}) = \bar{r}(\theta^d_{x,b}) \text{ and } \dot{d}(\theta^d_{x,b}) < \dot{\bar{r}}(\theta^d_{x,b}),$$
$$\theta^d_{e,b} : d(\theta^d_{e,b}) = \bar{r}(\theta^d_{e,b})) \text{ and } \dot{d}(\theta^d_{e,b}) > \dot{\bar{r}}(\theta^d_{e,b}).$$

Before $\theta^d_{e,b}$ it is favorable to remanufacture as many returns as possible and to build up a serviceables stock. These items can later be used to replace otherwise necessary production. Since Case 4(3) can only be preceded by a Case 3(2) interval, Corollary 3.19 follows.

Corollary 3.19 (Location of an interval with positive serviceables inventory when demand is low).
Let θ denote a time point where $d(\theta) = \bar{r}(\theta) < u(\theta)$. If demand crosses the constraint from below, i.e. $\dot{d}(\theta) > \dot{\bar{r}}(\theta)$, then it is always optimal to have a positive serviceables inventory at (and around) time θ ($y_s(\theta) > 0$).

Let $\theta_{x,s}$ denote the end of a *serviceables consumption period*, i.e. the time of a transition from Case 3(2) to Case 4. At $\theta_{x,s}$, demand must not be smaller than the remanufacturing constraint in order to deplete the serviceables inventory. The beginning of a bottleneck interval $\theta^d_{e,b}$ separates a *serviceables collection* from the respective consumption period. The first is assumed to

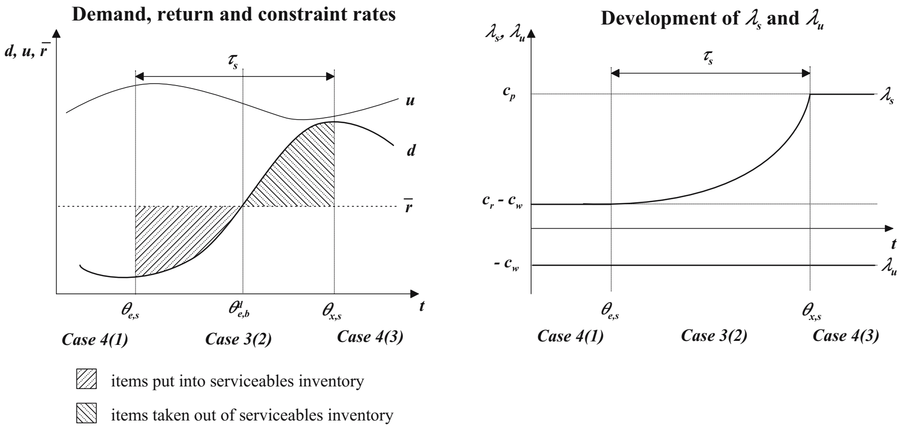

Fig. 3.21. An interval where returns are remanufactured with maximal rate during a serviceables accumulation interval for later use within a demand bottleneck with maximal length.

start at $\theta_{e,s}$. For the interval $(\theta_{e,s}, \theta_{x,s})$ an inventory condition as stated in Proposition 3.15 must hold.

Figure 3.21 shows a situation where the return rate exceeds both demand and remanufacturing constraint. A point satisfying Corollary 3.19 separates a serviceables accumulation period from a consumption period. Before $\theta_{e,s}$, all returns exceeding current demand are disposed of. After $\theta_{x,s}$, the part of demand which is higher than the remanufacturing capacity is served from production. Therefore, a marginal criterion applies for remanufacturing a returned item to stock and later using it for serving demand from inventory and replacing production instead of disposing of it. A maximal holding time condition for serviceables results as presented in the following proposition.

Proposition 3.18 (Maximal Length of an interval where $\mathbf{y_s > 0}$). *The maximal length of an interval $I = (\theta_{e,s}, \theta_{x,s})$ where $y_s > 0$ holds is given by*

$$\theta_{x,s} - \theta_{e,s} \leq \tau_s = \frac{1}{\alpha} \ln \left(\frac{\alpha c_p + h_s}{\alpha(c_r - c_w) + h_s} \right). \tag{3.36}$$

When assuming a short planning period and setting α to zero, (3.36) simplifies to

$$\theta_{x,s} - \theta_{e,s} \leq \tau_s = \frac{c_p + c_w - c_r}{h_s} \tag{3.37}$$

Figure 3.21 illustrates an interval where $y_s > 0$ with maximal length showing a sequence $4(1) \to 3(2) \to 4(3)$ as well as optimal co-state movements. While the serviceables value steadily increases, the value of returns is given by $-c_w$ because in all cases returns must partly be disposed of. There are two reasons for not having a Case 3(2) interval with maximal length. Either there is no remaining capacity available at $\theta_{e,s}$ and thus, the serviceables accumulation period cannot be extended, or consumption of serviceables ends because there exists no further bottleneck demand at $\theta_{x,s}$. The first case leads to a sequence $4(3) \rightarrow 3(2) \rightarrow 4(3)$ and the second to $4(1) \rightarrow 3(2) \rightarrow 4(1)$.

3.3.4 Managerial Insights

Although the current capacity of the system always suffices to satisfy demand without using inventories, a limitation of the cheaper mode, namely remanufacturing, motivates stock-keeping in order to improve the usage of available capacity. Even when restricting to pure effects, i.e. neglecting interactions with the anticipation stock motive, this motivation allows for a number of different situations where stock-keeping takes place, but all of these jointly require that there are periods where the remanufacturing rate is restricted through another circumstance which can be either insufficient returns or demand.

In situations with high demand ($d > u$), returns are kept in intervals where they cannot be immediately used for later remanufacturing if an interval with returns exceeding the constraint is followed by another where the opposite holds. The maximum length of these intervals is given by the Maximal Holding Time τ_u which is already known from the basic model. If during a return consumption period demand falls below the remanufacturing capacity it might be preferable to additionally build up a serviceables inventory through remanufacturing with maximum rate. This stock is used up in a period where demand rises above the constraint. A maximal length criterion holds which also takes into account the period where returns have been held in the recoverables stock and therefore, the maximal length of an interval with positive serviceables stock decreases in time.

If the return rate is larger than the demand rate ($d < u$), primarily a serviceables inventory is used in situations where an interval with demand being lower than the remanufacturing capacity is followed by another where demand exceeds the constraint. As before, a maximal length of the interval can be given.

3.4 Proofs

Proof (Proof of Proposition 3.1). From (2.23) and (2.24) it follows that $k_1 = k_2 = 0$. Thus, (2.21) and (2.22) imply

$$\dot{\lambda}_s = \alpha\lambda_s + h_s > 0 \text{ and } \dot{\lambda}_u = \alpha\lambda_u + h_u > 0. \tag{3.38}$$

The proofs for $r = 0$ and $w = 0$ are the same as in Proposition 2.2. Thus,

$$\lambda_s < \lambda_u + c_r \text{ and } \lambda_u > -c_w \tag{3.39}$$

$0 < p < \bar{p}$ requires $\mu_1 = 0$ in (2.18) and $\mu_4 = 0$ in (3.8) yielding $\lambda_s = c_p$ in (3.7) which contradicts (3.38).

$p = 0$ requires $\lambda_s < c_p$ and $p = \bar{p}$ necessitates $\lambda_s > c_p$ from (3.6). Together with (3.39), in the latter case it follows $\lambda_u > c_p - c_r$.

Proof (Proof of Proposition 3.2). From (2.24) it follows that $k_2 = 0$. Thus, (2.22) implies

$$\dot{\lambda}_u = \alpha \lambda_u + h_u > 0. \tag{3.40}$$

The proof for $w = 0$ is the same as in Case 1 (see proof of Proposition 2.2). Thus,

$$\lambda_u > -c_w \tag{3.41}$$

follows from (2.14).
$0 < p < \bar{p}$ requires $\mu_1 = \mu_4 = 0$ from (2.18) and (3.8) which yields $\lambda_s = c_p$ in (3.7). If simultaneously $r > 0$ than from (2.19) it follows $\mu_2 = 0$, leading to $\lambda_u = c_p - c_r$ in (2.16). It follows that $\dot{\lambda}_u = 0$ which contradicts (3.40).
In any interval where $0 < p < \bar{p}$ and $r = 0$ hold, the definition of Case 2 ($\dot{y}_s = 0$) requires $p = d < \bar{p}$ and $\dot{y}_u = u \geq 0$. From (3.6) and (2.13) it follows $\lambda_u > c_p - c_r$.
$r > 0$ from (2.13) necessitates $\lambda_s = \lambda_u + c_r$ and $\dot{\lambda}_s = \dot{\lambda}_u$. $p = 0$ requires $r = d$ in order to fill the demand and $\lambda_s < c_p$ from (3.6) from which $\lambda_u < c_p - c_r$ follows. $p = \bar{p}$ requires (using the definition of the case) $d > \bar{p}$ and thus $r = d - \bar{p}$ as well as $\lambda_s > c_p$ from (3.6) from which $\lambda_u > c_p - c_r$ follows.

Proof (Proof of Proposition 3.3). This proof proceeds in the same way as the proof to Proposition 2.6. (*i*) time points inside intervals where $y_s = 0$ or $y_u = 0$ holds are examined and (*ii*) we deal with entry and exit points of such intervals.
(i) First we look for circumstances under which the matrix

$$\begin{pmatrix} 1 & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & w & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & \bar{p}-p & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & y_s & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & y_u \end{pmatrix} \begin{matrix} I \\ II \\ III \\ IV \\ V \\ VI \end{matrix} \tag{3.42}$$

has not full rank of six. Reconsidering $\bar{p} > 0$, $d > 0$, and $u > 0$, this is not the case in the following five situations:

- (*i-i*) If $p = 0$, $r = 0$ and $y_s = 0$ then $I + II = V$.
  This requires $d = 0$ and can therefore be excluded.
- (*i-ii*) If $p = \bar{p}$, $r = 0$ and $y_s = 0$ then $-IV + II = V$.
  This requires $d = \bar{p}$ and a jump in λ_s may occur
  (a) inside Case 2 intervals when switching from Subcase 2(3) to 2(2). From the definition of both cases it follows $\dot{d} < \dot{\bar{p}}$. This completes the proof of Proposition 3.3(i).
  (b) inside Case 4 intervals. But here $r = \min\{u, d\} > 0$ and therefore the co-state is continuous.

(c) at entry and exit times of Case 4 intervals when switching from/to Case 2. Since only downward jumps are allowed for λ_s and because $d > 0$, this would allow for a transition from Case 2(3) to Case 4(2). Additionally, a jump in λ_u is required which can only occur under conditions put forth in part (*ii*) of this proof.

- (*i-iii*) If $r = 0$, $w = 0$ and $y_u = 0$ then $II + III = -VI$.
  This requires $u = 0$ and can therefore be excluded.
- (*i-iv*) If $p = 0$, $w = 0$, $y_s = y_u = 0$ then $I - III = V + VI$.
  This situation occurs when demand equals returns ($d = u$) and is already known from the basic model. See proof of Proposition 2.6.
  (a) Inside Case 4 intervals a jump in both λ_s and λ_u occurs when switching from Case 4(2) to Case 4(1). The definition of both cases requires $\dot{d} < \dot{u}$. This completes the proof of Proposition 3.3(ii).
  (b) At entry and exit times of Case 4 intervals in the case of a transition from/to Case 2, a jump in λ_s may occur when switching *(b1)* from Case 4(2) to Case 2(1), *(b2)* from Case 2(1) to Case 4(1) as well as *(b3)* from Case 2(3) to Case 4(2). As before, all transitions *(b1-3)* also require a jump in λ_u. Therefore, *(b3)* would require $d - \bar{p} = u < d$ (necessary for a non-tangential entry into the $y_u = 0$ interval) which contradicts $d = u$ because of $\bar{p} > 0$.
- (*i-v*) If $p = \bar{p}$, $w = 0$, $y_s = y_u = 0$ then $-IV - III = V + VI$.
  This requires $d = u + \bar{p} > u$ and may appear
  (a) inside Case 4 intervals. Since this inequality shows the presence of Subcase 4(2), the co-states are fixed and therefore continuous.
  (b) at entry and exit times of Case 4 intervals when switching from/to Case 2, λ_s may jump when switching from Case 2(3) to Case 4(2). As before, this transition also requires for a jump in λ_u.

(ii) Let θ_s^1 be the entry time of an interval where $y_s = 0$ holds, i.e. the begin of a Case 2 or Case 4 interval. Then, λ_s is continuous at this time point if y_s enters this interval in a non-tangential way, i.e. $\dot{y}_s = p + r - d$ jumps. This happens if the sum of the controls p and r jumps at θ_s^1. An entry point θ_s^1 can only be present at a switch *(a)* from Case 1 to Case 2 or *(b)* from Case 3 to Case 4 (other transitions are not possible when applying the optimal decisions in Case 1 and Case 3 intervals). In the first case *(a)*, $p + r$ does not jump if $d = \bar{p}$ holds at the end of Subcase 1(2) interval, where $\lambda_u(\theta_s^1) > c_p - c_r$. Since $y_u > 0$ holds for both intervals, λ_u must be continuous, thus only transitions to Cases 2(2) or (3) are possible. Since $\lambda_s < \lambda_u + c_r$ holds inside Subcase 1(2), $\lambda_s = \lambda_u + c_r$ inside Subcase 2(3), and λ_u is continuous during this transition ($y_u > 0$), λ_s must be continuous as well when switching from Subcase 1(2) to 2(3). In the second case *(b)*, $p + r = 0$ holds at the end of a Case 3 interval, thus the Case 4 interval is entered non-tangentially, because by definition of this case $p + r = d > 0$ must hold. Thus the only not continuous entry into an interval where $y_s = 0$ holds (i.e. where λ_s jumps) is a transition from Case 1(2) to 2(2) if $d = \bar{p}$. In order to leave Case 1(2), $d > \bar{p}$ is required at the end of such an interval. This leads to $\dot{d} < \dot{\bar{p}}$ at switching time. This completes the first part of the proof of Proposition 3.3(iii).

An exit time θ_s^2 is present at a transition from Case 2 to Case 1 and from Case 4 to Case 1 or Case 3. A switch from Case 4 to Case 1(1) or Case 3 is not possible because there no serviceables stock can be accumulated by applying optimal case decisions. Since upward jumps in the co-states are not allowed, a transition from Case 4 to Case 1(2) must take place continuously. The same holds for transitions from Case 2 (1) to Case 1(2). Proceeding similarly as for the entry time shows that only the transition from Case 2(3) to Case 1(2) allows for a jump in λ_s, if $d = \bar{p}$. In order to enter Case 1(2), $d > \bar{p}$ is required at the begin of such an interval. Again, this leads to $\dot{d} < \dot{\bar{p}}$ at switching time. This completes the second part of the proof of Proposition 3.3(iii).

Let θ_u^1 be the entry time and θ_u^2 the exit time of an interval, where $y_u = 0$ holds (Case 3 or Case 4). Then, λ_u is continuous at these time points if y_u enters or leaves this interval in a non-tangential way, i.e. $\dot{y}_u = u - r - w$ jumps. That occurs if $r + w$ jumps at (c) θ_u^1 or (d) θ_u^2, respectively. (c) Case 3 can not be reached, neither from a Case 1 nor a Case 2 interval by following optimal policies. An entry point of Case 4 is only present after a Case 2 interval. Thus, λ_u will always be continuous except for transitions from Case 2(1) to Case 4(1) where $d = u$ and $\dot{d} < \dot{u}$, and for transitions from Case 2(3) to Case 4(2) where $d = u + \bar{p}$ and $\dot{d} < \dot{u} + \dot{\bar{p}}$. In both cases, also λ_s may jump, because the system is inside a boundary situation w.r.t. the serviceables inventory and situations as put forth in (*i-iv*) and (*i-v*), respectively, apply. This completes the proof of the first part of (iv) as well as part (v) of Proposition 3.3. In case (d), all transitions starting at Case 3 or Case 4 intervals can be excluded or must proceed continuously in λ_u with the exception of a transition from Case 4(2) to Case 2(1) requiring $u = d$. From (*i-iv*), also λ_s jumps under such circumstance. This completes the second part of the proof of Proposition 3.3(iv).

Proof (Proof of Proposition 3.7). From Proposition 3.1 we get a minimal value for λ_s of $\lambda_s^{\min} = c_p$. For Case 1(2) we can give an upper bound for $\lambda_s^{\max} = \lambda_u + c_r$. Given $\lambda_u(\theta^i_{e,1(2)})$, the recoverables co-state is

$$\lambda_u(t) = \left(\lambda_u(\theta^i_{e,1(2)}) + \tfrac{h_u}{\alpha}\right) e^{\alpha(t-\theta^i_{e,1(2)})} - \tfrac{h_u}{\alpha}.$$

Together with continuity of λ_s and λ_u inside the Case 1(2) interval and the solution of co-state development (3.9) yields

$$\left(c_p + \tfrac{h_s}{\alpha}\right) e^{\alpha\left(\theta^i_{x,1(2)} - \theta^i_{e,1(2)}\right)} - \tfrac{h_s}{\alpha} \leq \lambda_s(\theta^i_{x,1(2)}) \text{ and}$$
$$\lambda_s(\theta^i_{x,1(2)}) \leq \left(\lambda_u(\theta^i_{e,1(2)}) + \tfrac{h_u}{\alpha}\right) e^{\alpha\left(\theta^i_{x,1(2)} - \theta^i_{e,1(2)}\right)} - \tfrac{h_u}{\alpha} + c_r$$

Solving the combined inequality for $\theta^i_{x,1(2)} - \theta^i_{e,1(2)}$ finally yields

$$\theta^i_{x,1(2)} - \theta^i_{e,1(2)} \le \frac{1}{\alpha} \ln \left(\frac{\alpha c_r + h_s - h_u}{\alpha \left(c_p - \lambda_u(\theta^i_{e,1(2)}) \right) + h_s - h_u} \right).$$

Proof (Proof of Proposition 3.9). From (2.23) and (2.24) it follows that $k_1 = k_2 = 0$. Thus, (2.21) and (2.22) imply

$$\dot{\lambda}_s = \alpha\lambda_s + h_s > 0 \text{ and } \dot{\lambda}_u = \alpha\lambda_u + h_u > 0. \tag{3.43}$$

The proofs for $p = 0$ and $w = 0$ are the same as in Proposition 2.2. Thus,

$$\lambda_s < c_p \text{ and } \lambda_u > -c_w. \tag{3.44}$$

$0 < r < \bar{r}$ requires $\mu_2 = 0$ in (2.19) and $\mu_4 = 0$ in (3.29) yielding $\lambda_s - \lambda_u = c_r$ in (3.28). This requires $\dot{\lambda}_s = \dot{\lambda}_u$ which contradicts (3.43) because $h_s > h_u$.
$r = 0$ requires $\lambda_s - \lambda_u < c_r$ and $r = \bar{r}$ necessitates $\lambda_s - \lambda_u > c_r$ from (3.27). Together with (3.44), in the latter case it follows $\lambda_u < c_p - c_r$ and $\lambda_s > c_r - c_w$.

Proof (Proof of Proposition 3.10). From (2.24) it follows that $k_2 = 0$. Thus, (2.22) implies

$$\dot{\lambda}_u = \alpha\lambda_u + h_u > 0. \tag{3.45}$$

The proof for $w = 0$ is the same as in Case 1 (see proof of Proposition 2.2). Thus,

$$\lambda_u > -c_w \tag{3.46}$$

from (2.14).
$0 < r < \bar{r}$ requires $\mu_2 = \mu_4 = 0$ from (2.19) and (3.29) which yields $\lambda_s - \lambda_u = c_r$ in (3.28). If simultaneously $p > 0$, then from (2.18) it follows $\mu_1 = 0$, leading to $\lambda_s = c_p$ in (2.15) as well as $\lambda_u = c_p - c_r$ in (3.28). It follows $\dot{\lambda}_u = 0$ which contradicts (3.45).
In any interval where $0 < r < \bar{r}$ and $p = 0$ hold, the definition of Case 2 ($\dot{y}_s = 0$) requires $r = d < \bar{r}$ and $\dot{y}_u = u - d$. From (2.12) and (3.27) it follows $\lambda_s = \lambda_u + c_r$, $\lambda_u < c_p - c_r$, and $\dot{\lambda}_s = \dot{\lambda}_u$.
$p > 0$ from (2.12) necessitates $\lambda_s = c_p$ and $\dot{\lambda}_s = 0$. $r = 0$ requires $p = d$ in order to fill the demand and $\lambda_s - \lambda_u < c_r$ from (3.27) which also requires $\lambda_u > c_p - c_r$. $p > 0$ and $r = \bar{r}$ requires (using the definition of the case) $d > \bar{r}$ and thus $p = d - \bar{r}$ as well as $\lambda_u < c_p - c_r$ from (3.27).

Proof (Proof of Proposition 3.11).
From (2.23) it follows that $k_1 = 0$. (2.21) implies

$$\dot{\lambda}_s = \alpha\lambda_s + h_s > 0. \tag{3.47}$$

$p > 0$ requires $\mu_1 = 0$ from (2.18) which yields $\lambda_s = c_p$. It follows that $\dot{\lambda}_s = 0$ which contradicts (3.47). Thus, $p^* = 0$ and

$$\lambda_s < c_p. \tag{3.48}$$

$r=0, w=0$ requires $\dot{y}_u = u$ which is not feasible as long as $y_u = 0$.
$0 < r < \bar{r}$ requires $\mu_2 = \mu_4 = 0$ in (2.19) and (3.29) which yields $\lambda_s - \lambda_u = c_r$ in (3.28). It follows that $\dot{\lambda}_s = \dot{\lambda}_u$. Applying to (3.47) and (2.22) and solving for k_2 yields $k_2 = \alpha(-c_r) + h_u - h_s$ and $k_2 \geq 0$ in (2.24) contradicts the assumptions that $h_s > h_u$ and $c_r > 0$.
$r = 0$ and $w > 0$ necessitates $w = u$ in order to stay in Case 3 ($\dot{y}_u = 0$ has to hold). It further requires $\mu_3 = \mu_4 = 0$ in (2.20) and (3.29), yielding $\lambda_u = -c_w$ in (2.17) and finally $\lambda_s < c_r - c_w$ in (3.28).
$r = \bar{r}$ and $w > 0$ requires $w = u - \bar{r}$ as well as $u > \bar{r}$. As before, $\lambda_u = -c_w$ holds and together with $\mu_2 = 0$ from (2.19) this yields $\lambda_s > c_r - c_w$ in (3.28).

Proof (Proof of Proposition 3.12).
From the definition of Case 4 it follows $\dot{y}_s = \dot{y}_u = 0$ which implies $p + r = d$ and $r + w = u$. When neglecting boundary situations, i.e. where demand or return rate equal zero, or where demand or return rate exactly match the remanufacturing constraint, this already excludes the alternatives ($p = 0, r = 0$), ($r = 0, w = 0$), ($p = 0, r = \bar{r}$), and ($r = \bar{r}, w = 0$).
For $p > 0, 0 \leq r < \bar{r}, w > 0$ we find $\lambda_s = c_p$, $\lambda_u = -c_w$ and $\mu_2 = c_r - c_p - c_w$ which has to be non-negative (2.19). This contradicts the assumption of a positive recovery advantage (2.6).
For $p = 0, 0 < r < \bar{r}, w > 0$ we find $r = d$ to ensure $\dot{y}_s = 0$ and then $w = u - d$ to ensure $\dot{y}_u = 0$. This decision requires $d < \bar{r}$ and $d < u$. The co-states are given by $\lambda_s = c_r - c_w$ and $\lambda_u = -c_w$.
For $p > 0, 0 < r < \bar{r}, w = 0$ we find $r = u$ to ensure $\dot{y}_u = 0$ and then $p = d - u$ to ensure $\dot{y}_s = 0$. This configuration is only feasible if $d > u$ and $u < \bar{r}$. The co-states are given by $\lambda_s = c_p$ and $\lambda_u = c_p - c_r$.
$p > 0, r = \bar{r}, w > 0$ requires $p = d - \bar{r}$ and $w = u - \bar{r}$ leading to $d > \bar{r}$ as well as $u > \bar{r}$. The co-states are given by $\lambda_s = c_p$ and $\lambda_u = -c_w$.

Proof (Proof of Proposition 3.13). This proof proceeds in the same way as the proof to Proposition 2.6. (i) time points inside intervals where $y_s = 0$ or $y_u = 0$ holds are examined and (ii) we deal with entry and exit points of such intervals. Thereby, boundary Cases 1(1), 3(1) and 2(2) are neglected.
(*i*) First we look for circumstances under which the matrix

$$\begin{pmatrix} 1 & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & w & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & \bar{r}-r & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & y_s & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & y_u \end{pmatrix} \begin{matrix} I \\ II \\ III \\ IV \\ V \\ VI \end{matrix} \tag{3.49}$$

has not full rank of six. Reconsidering $\bar{r} > 0$, $d > 0$, $u > 0$, as well as assumption (3.25), this is not the case in the following five situations:

- (*i-i*) If $p = 0$, $r = 0$ and $y_s = 0$ then $I + II = V$.
  This requires $d = 0$ and can therefore be excluded.

- (i-ii) If $p = 0$, $r = \bar{r}$ and $y_s = 0$ then $I - IV = V$.
  This requires $d = \bar{r}$ and a jump in λ_s may occur
  (a) inside Case 2 intervals when switching from Subcase 2(3) to 2(1).
  (b) inside Case 4 intervals when switching from Subcase 4(3) to 4(1).
  (c) at entry and exit times of Case 4 intervals when switching from/to Case 2. Since only downward jumps are allowed for λ_s, this would allow for transitions from Case 2(3) to Case 4(1) as well as from Case 4(3) to Case 2(1). From the definition of Case 2(3) and 4(1), the first transitions would require $d = u = \bar{r}$ which contradicts (3.25).
  From the definition of the cases or in order to clear the recoverables inventory, $\dot{d} < \dot{\bar{r}}$ is required in all relevant situations *(a-c)*. This completes the first part of the proof of Proposition 3.13(i).
- (i-iii) If $r = 0$, $w = 0$ and $y_u = 0$ then $II + III = -VI$.
  This requires $u = 0$ and can therefore be excluded.
- (i-iv) If $p = 0$, $w = 0$, $y_s = y_u = 0$ then $I - III = V + VI$.
  This situation occurs when demand intersects returns from above ($d = u$, $\dot{d} < \dot{u}$) and is already known from the basic model. See proof of Proposition 2.6.
  (a) Inside Case 4 intervals a jump in both λ_s and λ_u occurs when switching from Case 4(2) to Case 4(1).
  (b) At entry and exit times of Case 4 intervals in the case of a transition from/to Case 2, a jump in λ_s may occur when switching (b1) from Case 4(2) to Case 2(1) and (b2) from Case 2(1) to Case 4(1). All transitions (b1-2) also require for a jump in λ_u. In anticipation of part (ii) of this proof, from optimal decisions within the respective cases it can be seen that intervals where $y_u = 0$ holds are entered or exited in a non-tangential way and thus, also λ_u jumps. This completes the proof of Proposition 3.13(ii).
- (i-v) If $r = \bar{r}$, $w = 0$, $y_u = 0$ then $-IV + III = -VI$.
  This requires $u = \bar{r}$ and may appear inside Case 4 intervals when switching from Case 4(2) to Case 4(3). Here also $\dot{u} > \dot{\bar{r}}$ holds. This completes the first part of the proof of Proposition 3.13(iii).

(ii) Let θ_s^1 be the entry time and θ_s^2 the exit time of an interval where $y_s = 0$ holds, i.e. the beginning of a Case 2 or Case 4 interval. Then, λ_s is continuous at these time points if y_s enters or leaves this interval in a non-tangential way, i.e. $\dot{y}_s = p + r - d$ jumps. This happens if the sum of the controls p and r jumps at θ_s^1.
(ii-i) An entry point θ_s^1 can be present at a switch *(a)* from Case 1(2) to Case 2 or *(b)* to Case 4 and *(c)* from Case 3 to Case 4 (other transitions are not possible when applying the optimal decisions in Case 1 and Case 3 intervals).

(a) In the first case, the transition is non-tangential ($p + r$ does not jump) if $d = \bar{r}$ holds at the end of Subcase 1(2). Since $\lambda_s > \lambda_u + c_r$ holds inside Subcase 1(2) and $\lambda_s = \lambda_u + c_r$ inside Subcase 2(1), λ_s jumps at transition time. In order to leave Case 1(2), $d > \bar{r}$ must hold. Together

with the definition of Subcase 2(1), $\dot{d} < \dot{\bar{r}}$ must hold at θ_s^1. Since $\lambda_s < c_p$ holds inside Subcase 1(2) and $\lambda_s = c_p$ in Subcase 2(3), a transition from Subcase 1(2) to Subcase 2(3) would be continuous.

(b) In situation *(b)*, $d > \bar{r} > u$ must hold in order to clear both inventories simultaneously. The only transition requiring a jump in λ_s would be to Case 4(1). But from the definition of a Case 4 interval $d = \bar{r} = u$ must hold which is excluded by assumption (3.25).

(c) In the third case, $d > \bar{r}$ must hold to empty the serviceables inventory. The only transition requiring a jump in λ_s is from Case 3(2) to Case 4(1). This necessitates $d = \bar{r}$ as well as $\dot{d} < \dot{\bar{r}}$ at θ_s^1.

(*ii-ii*) An exit time θ_s^2 is present at a transition from *(a)* Case 2 to Case 1 and from Case 4 *(b)* to Case 1 or *(c)* Case 3.

(a) A jump in λ_s only is required at a switch from Case 2(3) to Case 1(2) and it requires $d = \bar{r}$. Since $d < \bar{r}$ is required to build up serviceables stock during a Case 1(2) interval, $\dot{d} < \dot{\bar{r}}$ is required.

(b) A switch from Case 4 to Case 1(2) necessitates $d < \bar{r} < u$ at the beginning of the Case 1(2) interval. This already excludes a transition starting in Case 4(2) since this would require $d = \bar{r} = u$. A transition starting in Case 4(3) needs $d = \bar{r}$ as well as $\dot{d} < \dot{\bar{r}}$.

(c) $d < \bar{r}$ at the beginning of the Case 3(2) interval is required for a switch from Case 4 to Case 3(2). As before only a transition starting in Case 4(3) is possible, and $d = \bar{r}$ as well as $\dot{d} < \dot{\bar{r}}$ must hold at θ_s^2.

This completes the last part of the proof of Proposition 3.13(i).

Let θ_u^1 be the entry time and θ_u^2 the exit time of an interval, where $y_u = 0$ holds (Case 3 or Case 4). Then, λ_u is continuous at these time points if y_u enters or leaves this interval in a non-tangential way, i.e. $\dot{y}_u = u - r - w$ jumps. That occurs if $r + w$ jumps at (c) θ_u^1 or (d) θ_u^2, respectively.
(*ii-iii*) An entry time θ_u^1 is present when switching from Case 1(2) *(a)* to Case 3(2) or *(b)* to Case 4, and *(c)* from Case 2 to Case 4.

(a) A non-tangential transition from Case 1(2) to Case 3(2) requires $u = \bar{r}$. In order to leave Case 1(2), $u < \bar{r}$ must hold. Together with the definition of Subcase 3(2), $\dot{u} > \dot{\bar{r}}$ must hold at θ_u^1.

(b) When switching from Case 1(2) to Case 4, $d > \bar{r} > u$ must hold in order to clear both inventories simultaneously. The only transition requiring a jump in λ_u terminates in Case 4(3), requiring $u = \bar{r}$. From the definition of a Case 4 interval $\dot{u} > \dot{\bar{r}}$ must hold.

(c) Besides the transition already discussed in (*i-iv*)*(b)* requiring a simultaneous jump in λ_s and λ_u, only a switch from Case 2(3) to Case 4(3) is possible. Depleting the recoverables inventory requires $u < \bar{r}$. Together with the definition of a Case 4(3) interval, besides $u = \bar{r}$ as well as $\dot{u} > \dot{\bar{r}}$ hold.

(*ii-iv*) An exit time θ_u^2 is present when switching *(a)* from Case 3(2) to Case 1(2) and from Case 4 *(b)* to Case 1, or *(c)* to Case 2.

(a) A transition from Case 3(2) to Case 1(2) proceeds continuously.
(b) A discontinuous switch w.r.t. λ_u from Case 4 to Case 1(2) would only be possible starting with a Case 4(2) interval. $d < \bar{r} < u$ must hold in order to build up both inventories simultaneously. From the definition of a Case 4(2) interval $d = \bar{r} = u$ must hold which is excluded by assumption (3.25).
(c) As before, a transition of situation *(c)* has been discussed in *(i-iv)(b)*. Aside from this, a switch from Case 4(2) to Case 2(3) is possible. A start of collecting returns in the recoverables inventory requires $u > \bar{r}$ at the beginning of a Case 2(3) interval. Together with the definition of a Case 4(2) interval, $u = \bar{r}$ and $\dot{u} > \dot{\bar{r}}$ hold at θ_u^2.

This completes the proof of Proposition 3.13(iii).

Proof (Proof of Proposition 3.14).
In all cases where $y_u > 0$ it must hold that $-c_w < \lambda_u < c_p - c_r$ and $\dot{\lambda}_u = \alpha\lambda_u + h_u$. Proceeding in the same way as done in the proof of Proposition 2.7 finally yields $\theta_{x,c} - \theta_{e,c} \leq \tau_u$.

Proof (Proof of Proposition 3.17). From Proposition 3.9 we get a minimal value for λ_s of $\lambda_s^{\min} = \lambda_u(\theta_{e,s}^i) + c_r$. The upper bound is given by $\lambda_s^{\max} = c_p$. Together with continuity of λ_s inside the interval with positive serviceables stock and the solution of co-state development (3.9) yields

$$\left(\lambda_u(\theta_{e,s}^i) + c_r + \tfrac{h_s}{\alpha}\right) e^{\alpha\left(\theta_{x,s}^i - \theta_{e,s}^i\right)} - \tfrac{h_s}{\alpha} \leq \lambda_s(\theta_{x,s}^i) \leq c_p$$

Solving the combined inequality for $\theta_{x,s}^i - \theta_{e,s}^i$ finally yields

$$\theta_{x,s}^i - \theta_{e,s}^i \leq \frac{1}{\alpha} \ln\left(\frac{\alpha c_p + h_s}{\alpha\left(\lambda_u(\theta_{e,s}^i) + c_r\right) + h_s}\right).$$

Proof (Proof of Proposition 3.18). From Proposition 3.12 we get a minimal value for λ_s of $\lambda_s^{\min} = c_r - c_w$. The upper bound is given by $\lambda_s^{\max} = c_p$. Together with continuity of λ_s inside the interval with positive serviceables stock and the solution of co-state development (3.9) yields

$$\left(c_r - c_w + \tfrac{h_s}{\alpha}\right) e^{\alpha(\theta_{x,s} - \theta_{e,s})} - \tfrac{h_s}{\alpha} \leq \lambda_s(\theta_{x,s}^i) \leq c_p$$

Solving the combined inequality for $\theta_{x,s} - \theta_{e,s}$ finally yields

$$\theta_{x,s} - \theta_{e,s} \leq \frac{1}{\alpha} \ln\left(\frac{\alpha c_p + h_s}{\alpha\left(c_r - c_w\right)) + h_s}\right).$$

4

Knowledge Acquisition and Product Recovery

4.1 Motivation

So far, our analysis based on the assumption that all processes maintained an initial performance level and constant cost rates. However, ongoing competition and the search for profit maximization provide incentives for productivity improvements. This chapter relates to an empirical phenomenon found quite often in practical applications dating back to the 1920's and intends to explore effects of acquiring knowledge in product recovery on strategic recoverables inventory management. The learning curve, introduced by Wright (1936), shows a (potential) relationship between cumulative output and labor hours per unit produced. It's empirical evidence has been proven in a large number of studies, for comprehensive literature surveys see Yelle (1979) or Dutton and Thomas (1984). Traditionally, learning is seen as autonomous acquisition of tacit knowledge which is more prominent in labor intensive processes, but it also has been proven to occur in highly mechanized industries.

Because of the shortening of manufacturing times or more generally, reduction of input quantities, learning leads to an increase of the capacity of a manufacturing system and a decrease of direct production costs. This relationship has led to two main representations of the experience curve, as the relationship between cumulative output and unit costs is called. Let c^0 be the unit costs of producing the first item, c the unit costs of the X-th one, and $b > 0$ a learning parameter. According to Zangwill and Kantor (1998), the most widespread used functional forms of learning curves are

(a) the so-called *Power Law* form

$$c(X) = c^0 \cdot X^{-b} \tag{4.1}$$

(b) and the *Mixed Exponential Learning* form

$$c(X) = c^0 \cdot e^{-bX}. \tag{4.2}$$

The first type (a) represents the classical type first introduced by Wright (1936). This form predicts a reduction of unit production costs by a constant percentage each time the cumulative output doubles. The second function (b) is well-known from psychological learning theory, but also used for economic analyses (see Kantor and Zangwill (1991)). Both functional forms have in common that unit costs persistently decrease at a decreasing rate, i.e. further efficiency improvements become increasingly difficult to achieve. It should be noted, however, that since these functions only represent results of a system's inherent learning process, efforts have been taken to explain the underlying causes, resulting in refinements of the functional forms. Adler and Clark (1991) for instance investigate the impact of forced engineering changes and training on productivity improvement. Further factors that influence the learning rate have been elaborated by Argote and Epple (1990). These factors include organizational forgetting and employee turnover, both having a negative effect, and different ways to transfer knowledge between products or different departments of a company, which in turn increase the learning rate.

Quantitative approaches integrating learning effects into production planning decisions have been surveyed by Gulledge and Khoshnevis (1987). They include the following issues

- *break-even analysis*
  Pegels (1976) aims to predict at which level of production volume total costs equal total revenue or at which point marginal costs equal marginal revenue.
- *aggregate production planning*
  Ebert (1976) extends the well known HMMS model (Holt et al. (1960)) by introducing a dependence between productivity of workforce and cumulative production. Liao (1979) and Reeves and Sweigart (1981) consider this issue on a more detailed level aiming to determine the optimal *product-mix*. In this work, a number of different resource requirements depend on the number of manufactured items of a product.
- *economic lot sizing*
  The influence of learning on economic manufacturing quantity has been reviewed by Smunt and Morton (1985) and, more recently, by Jaber and Bonney (2001). In this context a number of different learning and 'forgetting' effects are investigated. For instance learning in the set-up of batch production leads to smaller lot sizes and a reduction of cycle stocks.
- *dynamic pricing*
  Li and Rajagopalan (1998) analyze a dynamic profit maximization model, where a monopolistic firm sets price and production quantities by assuming a generic learning function and additionally considering knowledge depreciation.

Cost reductions due to learning are not restricted to production alone. It can also be presumed to occur for remanufacturing processes, if there are repeated operations performed on a large number of similar items. In such a case, and since remanufacturing operations usually include labor intensive tasks (Guide et al. (2000)), there exists a considerable potential for cost-reductions from acquiring e.g. tacit disassembly know how or due to specialized tools developed during the remanufacturing process. It must also be stated, that if there is a large diversity of remanufactured products with only a few information on how to deal with them correctly, sufficient experience might be more difficult to obtain. Under such circumstances remanufacturing usually would be dominated by lower levels of product recovery. This especially holds for independent recyclers whereas OEM have a competitive advantage due to an easier access to required information. See Toffel (2004) for a more detailed discussion.

In addition to the traditional volume learning effects, there is a second level of using the knowledge that was acquired during recovery operations. As indicated by Toffel (2004), a transfer of product recovery knowledge back to the production stage may lead to improvements in product design and manufacturing processes such that overall profitability increases.

The single-use camera case (see Chapter 1) provides a good example of large scale product recovery. Figure 4.1 exemplifies the increase in yearly volume indicating that considerable productivity improvements have been realized. With respect to the above-mentioned second level learning, there is an ongoing process of design changes to increase the reusability level. This is facilitated by the fact that the products are designed for ‘reverse compatibility’, which means that a new product generation can by design use some parts recovered from a previous generation. This helps to serve customer demand while still benefiting from a high product recovery level. For a more detailed discussion of this issue see Goldstein (1994).

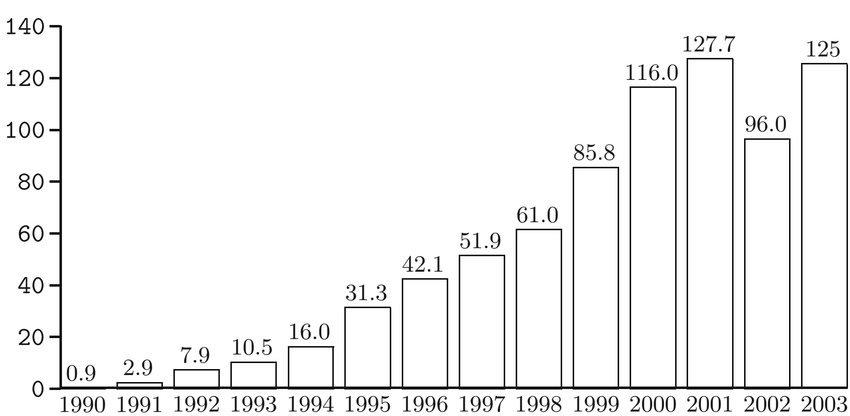

Fig. 4.1. Single-use cameras recycled by Kodak between 1990 and 2003 (in Millions). Sources: Annual Reports 1999-2003, The Kodak Corporation.

Quantitative approaches devoted to effects of knowledge accumulation in product recovery can rarely be found. Exceptions are offered in the context of Total Quality Management. For instance, Lapré et al. (2000) present a model where learning leads to a reduction of unwanted by-products or waste inside the production process. The scarcity of these approaches is intuitive, because similar effects are expected when dealing with the pure remanufacturing process as encountered in production planning. When dealing with an *integrated* product recovery system new results are likely to occur. This chapter starts to fill this gap by analyzing the impact of acquiring remanufacturing knowledge on optimal decisions in such an environment. Possible research questions are how strategic stock-keeping changes and what would be the effect on optimal remanufacturing and production policies. For instance, as a strategic implication of productivity improvements, remanufacturing can be profitable in the long run, even if there is no immediate cost advantage over the production of new items, because subsequent unit remanufacturing costs are lowered. Here, benefits are derived from using an optimal control framework because, as explained in Chapter 2, the indirect effect of current decisions on future expenses is also valued.

The remainder of this chapter is organized as follows. In Section 4.2 an optimal control model is introduced which is further analyzed in Section 4.3. Special attention is drawn on the impact of zero and nonzero interest rates and by further distinguishing between specific cost/cash flow conditions, the main qualitative additions to the results of the basic formulation known from Chapter 2 are identified in Section 4.4. Numerical examples are used in Section 4.5 to illustrate the findings and final conclusions are presented in Section 4.6.

4.2 A Model with Remanufacturing Knowledge Acquisition

In this section a model is presented where current decisions on remanufacturing have an additional impact on future cash flow, because they change a knowledge stock which in turn lowers unit remanufacturing costs. The model uses the basic formulation introduced in Chapter 2 which has been adapted in order to account for knowledge acquisition. The following changes and assumptions are made.

Since only learning effects in the remanufacturing process are considered, it is assumed that the manufacturing process uses a 'mature' or old technology with negligible learning rate. Further, a re-transfer of remanufacturing knowledge back to the production stage (second level learning) is disregarded here, because this would destroy homogeneity of produced and later returned used products. Rate effects and knowledge decay are also not considered. A possible method for including knowledge depreciation into a dynamic framework has been presented by Li and Rajagopalan (1998).

Learning in the remanufacturing shop is based on cumulative remanufacturing volume R which is derived, given an initial stock of knowledge R_0, by using

$$\dot{R}(t) = r(t) \geq 0 \text{ and } R(0) = R_0 > 0. \tag{4.3}$$

Constant out-of-pocket remanufacturing unit costs are replaced by a generic exogenous and deterministic function $c_r(R)$ measuring the learning curve effect realized so far. For this function we assume

$$c_r'(R) < 0,\ c_r''(R) > 0, \text{ as well as } \lim_{R\to\infty} c_r(R) \geq 0, \tag{4.4}$$

i.e. unit remanufacturing costs decrease at a decreasing rate. Initial remanufacturing costs $c_r(R_0)$ can be so high that assumption (2.6) is violated and thus, a situation with a negative initial remanufacturing cost advantage $(c_p + c_w - c_r(R_0) < 0)$ in general is allowed.

All processes are supposed to be unrestricted. A serviceables inventory is therefore not considered here ($y_s(t) = 0\ \forall t$). Production quantity $p(t)$ immediately follows, when setting a corresponding remanufacturing rate $r(t)$, from the necessity to immediately satisfy all demand. Therefore, we have

$$p(t) = d(t) - r(t). \tag{4.5}$$

Non-negativity of the production rate requires the return rate not to exceed the demand rate $(r(t) < d(t))$.

Besides an unchanged recoverables inventory transition equation

$$\dot{y}_u(t) = u(t) - r(t) - w(t) \tag{4.6}$$

initial and final recoverables inventory levels are set to zero ($y_u(0) = y_u(T) = 0$). As usual, it should not be advantageous to hold unneeded returned products

$$h_u > \alpha c_w. \tag{4.7}$$

For analytical convenience but without loss of generality we assume strictly positive demand and return rates during the whole planning horizon

$$d(t) > 0 \text{ and } u(t) > 0. \tag{4.8}$$

Thus, a constraint qualification required for the results presented in the next section is always satisfied.

Now, an optimal control problem with two states (R and y_u) and two control variables (r and w) has to be solved such that the total discounted cash outflow during a finite planning horizon is minimized. The problem is subject to the state equations, a pure state constraint, initial and (partly) terminal conditions for the state variables as well as non-negativity constraints for control variables and an upper limit for the remanufacturing rate

$$
\begin{aligned}
\min NPV = & \int_0^T e^{-\alpha t} \left(c_p(d(t) - r(t)) + c_r(R(t))r(t) + c_w w(t) + h_u y_u(t) \right) dt \\
s.t. \quad & (4.3), (4.6), \\
& y_u(t) \geq 0 \;\; y_u(0) = 0, \, y_u(T) = 0, \\
& d(t) - r(t) \geq 0, r(t) \geq 0, w(t) \geq 0.
\end{aligned}
\tag{4.9}
$$

4.3 Optimality Conditions and General Results

Necessary Conditions

There is no fundamental difference in optimal control theory when dealing with dynamic instead of static parameters and thus, the general solution procedure known from Chapter 2 can also be applied to the extended problem. Since the remanufacturing costs depend on a state, the objective loses the property of linearity in both controls and states. Necessary conditions as presented below are therefore no longer sufficient for optimality.

After introducing a new co-state variable λ_R, which corresponds to the stock of accumulated experience R, the Hamiltonian reads as

$$
H(.) = \lambda_0(-c_p(d - r) - c_r(R)r - c_w w - h_u y_u) + \lambda_u(u - r - w) + \lambda_R r \tag{4.10}
$$

Using Kuhn-Tucker multipliers μ_1, μ_2, μ_3, and k_2 the Lagrangian is given by

$$
L(..) = H(.) + \mu_1(d - r) + \mu_2 r + \mu_3 w + k_2 y_u. \tag{4.11}
$$

In the appendix we show that $\lambda_0 = 1$. The adjoint λ_R rates the impact of the current remanufacturing decision on future costs and can therefore be interpreted as the shadow 'price' or *value of acquiring knowledge* and thereby changing the stock of experience R. It incorporates the discounted value of all improvements in future remanufacturing costs.

Now, let (y_u^*, R^*) represent the optimal trajectory of the state variables and (r^*, w^*) be a piecewise continuous trajectory of optimal control policies to problem (4.9). Then, there exists a continuous function of time λ_R as well as piecewise continuous functions of time λ_u, μ_i, $i = 1, 2, 3$, k_2 and a set of points $\theta_u \in \Theta_u$ where the co-state λ_u jumps with corresponding height parameters $\eta_u(\theta_u)$. Except for points of discontinuity in the controls and junction points, the following necessary conditions (4.12)–(4.24) must hold.

The Hamiltonian is maximized, if the following bang-bang equations (4.12) and (4.13) hold

$$
r^* = \begin{cases} 0 & \lambda_u > c_p - (c_r(R^*) - \lambda_R) \\ singular & \lambda_u = c_p - (c_r(R^*) - \lambda_R) \\ d & \lambda_u < c_p - (c_r(R^*) - \lambda_R) \end{cases} \tag{4.12}
$$

$$
w^* = \begin{cases} 0 & \lambda_u > -c_w \\ singular & \lambda_u = -c_w. \end{cases} \tag{4.13}
$$

The remanufacturing rate is zero and demand is filled from production, if the value of a recoverables item exceeds the difference of current production expenses and the *net expenses of remanufacturing* (current direct costs less the future cost reductions valued by λ_R). r^* equals demand if the opposite holds and it can take any value between 0 and the demand rate, if there is indifference (equality of both sides). Similarly, a returned item is kept only, if its value exceeds a salvage revenue ($-c_w$).

In order to maximize the Lagrangian, (4.14) and (4.15) must hold. Non-negativity of controls as well as complementary slackness conditions apply as stated in (4.16)-(4.18).

$$\frac{\partial L}{\partial r} = c_p - c_r(R^*) - \lambda_u + \lambda_R - \mu_1 + \mu_2 = 0 \tag{4.14}$$

$$\frac{\partial L}{\partial w} = -c_w - \lambda_u + \mu_3 = 0 \tag{4.15}$$

$$\mu_1 \geq 0 \quad \mu_1 \cdot (d - r^*) = 0 \tag{4.16}$$

$$\mu_2 \geq 0 \quad \mu_2 \cdot r^* = 0 \tag{4.17}$$

$$\mu_3 \geq 0 \quad \mu_3 \cdot w^* = 0 \tag{4.18}$$

Optimal co-state transitions are given in (4.19) and (4.20), and complementary slackness condition for the recoverables inventory is represented by (4.21).

$$\dot{\lambda}_u = \alpha\lambda_u - \frac{\partial L}{\partial y_u} = \alpha\lambda_u + h_u - k_2 \tag{4.19}$$

$$\dot{\lambda}_R = \alpha\lambda_R - \frac{\partial L}{\partial R} = \alpha\lambda_R + c_r'(R^*) \cdot r^* \tag{4.20}$$

$$k_2 \geq 0 \quad k_2 \cdot y_u^* = 0. \tag{4.21}$$

New condition (4.20) can be interpreted as follows. The value of knowledge (i) increases with the interest to be paid on it ($\alpha\lambda_R$), i.e. a later acquisition of experience through remanufacturing would have been cheaper in terms of the discounted value of required remanufacturing expenses, and it (ii) decreases with the rate at which remanufacturing unit costs decrease ($\dot{c}_r^* = c_r'(R^*) \cdot r^*$). This is because a current cost reduction lowers the remaining reduction potential.

Transversality conditions for the co-state variables are given in (4.22) and (4.23)

$$\lambda_u(T) \text{ is free,} \tag{4.22}$$

$$\lambda_R(T) = 0. \tag{4.23}$$

The value of stored recoverables at the end of the planning period can not be predetermined because there the inventory is forced to zero. Instead of this,

$\lambda_u(T)$ depends on the relation of demand and return functions. At the same time, remanufacturing knowledge becomes useless. Therefore, the corresponding co-state takes on a value of zero.

As in the basic model, there are points $\theta_u \in \Theta_U$ with $y_u^* = 0$ where downward jumps in λ_u and in the Hamiltonian occur.

$$\lambda_u(\theta_u^-) = \lambda_u(\theta_u^+) + \eta_u(\theta_u) \tag{4.24}$$
$$\text{where } \eta_u(\theta_u) \geq 0 \text{ and } \eta_u(\theta_u) \cdot y_u^*(\theta_u) = 0.$$

Downward jumps in λ_R are not permitted.

Properties of an Optimal Solution

In this section particular features are derived that must hold for any optimal solution. We start with general results regarding the co-states, which are followed by a derivation of optimal policies in different cases in the state space. Lastly, conditions for case transitions are deduced. These findings are later used in Section 4.4 to examine optimal policies in more specific situations with respect to cash flow parameters.

General Properties of the Co-states λ_R and λ_u

Reconsidering co-state movement (4.20), bounds for the value of acquiring knowledge λ_R as well as for its rate of change can be derived. Except for the special case where discounting can be neglected ($\alpha = 0$), the direction of the co-state development is not predetermined in (4.20), but it depends on the relation of the two terms on the right hand side of (4.20). Since c_r' is strictly negative, the second term is negative. If additionally the first term is less than zero, requiring the co-state to be negative, then the co-state is not able to rise again which contradicts transversality condition (4.23). Thus, the value of acquiring knowledge is limited to non-negative values, i.e.

$$0 \leq \lambda_R(t). \tag{4.25}$$

The largest possible decrease of λ_R is given by the rate of change of remanufacturing costs (obtained when assuming a zero discount rate) and the maximal rise is determined by the opportunity costs of a later learning. Thus, in an optimal solution the following must hold

$$\dot{c}_r(t) \leq \dot{\lambda}_R(t) \leq \alpha\lambda_R(t), \tag{4.26}$$

i.e. the shadow price of acquiring knowledge must not decrease faster than current remanufacturing costs and it does not increase more rapidly than the interest to be paid on it.

The maximal possible decrease of λ_R in (4.26) determines an upper bound

$$\lambda_R(t) \leq c_r\left(R^*(t)\right) - c_r(R^*(T)) . \tag{4.27}$$

Otherwise, transversality condition (4.23) again would be infeasible. Inequality (4.27) implies that the value of knowledge acquisition at time t lies below the remaining remanufacturing cost reduction. As can easily be seen when inserting $\alpha = 0$ into (4.20), equality must hold in (4.27) if $\alpha = 0$. The following lemma summarizes the results regarding λ_R (presented without proof).

Lemma 4.1. *The value of acquiring knowledge λ_R must lie between zero and the remaining remanufacturing cost reduction. If discounting can be neglected ($\alpha = 0$), then λ_R exactly equals the remaining remanufacturing cost reduction.*

Another important result concerns the upper bound of the value of returns (λ_u). In the basic model, this bound was given by the difference of manufacturing and remanufacturing costs, $\lambda_u^{\max} = c_p - c_r$. In our case, a similar but time dependent boundary applies if remanufacturing takes place, i.e. $r^* > 0$ requires $\lambda_u \leq c_p - (c_r(R^*) - \lambda_R)$ from (4.12). The first derivative of the right hand side of this inequality w.r.t. time yields

$$\frac{d}{dt}\left(c_p - \left(c_r(R^*(t)) - \lambda_R(t)\right)\right) = \alpha\lambda_R(t). \tag{4.28}$$

That means that the difference between production expenses and net expenses of remanufacturing increases with non-negative rate $\alpha\lambda_R$. Consequently, if $\alpha = 0$ it is time independent and equals the projected difference of production and remanufacturing expenses at the end of the planning period. This leads us to Lemma 4.2 (presented without proof).

Lemma 4.2. *The maximal level which the value of returns is allowed to achieve if remanufacturing takes place ($r^* > 0$) rises with rate $\alpha\lambda_R$. In case of zero discounting ($\alpha = 0$), this level is constant and given by $\lambda_u^{\max} = c_p - c_r(R^*(T))$.*

Cases in State Space and Optimal Policies

Since only a single stocking point (the recoverables inventory) is considered, there are only two cases, one with a positive ($y_u > 0$) and another with a zero inventory level ($y_u = 0$). In order to remain consistent with Chapter 2, these cases are named Case 2 and 4, respectively. The basic model can be seen as a special case with a zero learning rate, all policies (cases and subcases) that apply are also relevant in the present situation. Further policies are available and conditions for the already known subcases change as shown in the following two propositions.

Proposition 4.1 (Optimal decisions in Case 2 intervals).
If recoverables inventory is positive ($y_u^* > 0$), the optimal policy is not to dispose of returns ($w^* = 0$). The decision on whether to remanufacture or to

produce depends on a relationship based on co-states and current remanufacturing costs. Two subcases can be distinguished:
Subcase 2(1) $\Leftrightarrow \lambda_u < c_p - (c_r(R^*) - \lambda_R)$
Demand is satisfied from remanufacturing returns ($r^* = d$) and no items are produced ($p^* = 0$).
Subcase 2(2) $\Leftrightarrow \lambda_u > c_p - (c_r(R^*) - \lambda_R)$
Remanufacturing does not take place ($r^* = 0$) and demand is filled from production alone ($p^* = d$).

Under positive recoverables inventory conditions, two different policies might apply. Both have in common that disposal is not optimal, because otherwise items could have been disposed of earlier (from stock or directly when arriving) which would have saved holding costs. Thus $\lambda_u > -c_w$. The question on how to fill demand, either completely from remanufacturing out of stock or by producing new items, depends on whether or not the value of returns exceeds the difference of production expenses c_p and net remanufacturing costs (including indirect learning effects) $c_r(R^*) - \lambda_R$.

The already known Subcase 2(1) was used in the basic model for building an anticipation stock. Subcase 2(2), however, might be used in order to postpone remanufacturing decisions, if holding an item and remanufacturing it later would be cheaper than immediately processing it.

Proposition 4.2 (Optimal decisions in Case 4 intervals).
If recoverables inventory is empty ($y_u^* = 0$), optimal decisions depend on the net recovery cost advantage and on how demand relates to the return rate. Here three subcases can be distinguished:
Subcase 4(1) $\Leftrightarrow d \leq u$, $\lambda_u = -c_w$, *and* $c_p + c_w > c_r(R^*) - \lambda_R$
Demand is satisfied completely by remanufacturing returns ($r^* = d$) and excess returns are disposed of ($w^* = u - d$). No items are produced ($p^* = 0$).
Subcase 4(2) $\Leftrightarrow u < d$, $\lambda_u = c_p - c_r(R^*) + \lambda_R$, *and* $c_p + c_w > c_r(R^*) - \lambda_R$
All returns are remanufactured ($r^* = u$) and the missing items are produced ($p^* = d - u$). No items are disposed of ($w^* = 0$).
Subcase 4(3) $\Leftrightarrow \lambda_u = -c_w$, $c_p + c_w < c_r(R^*) - \lambda_R$
Remanufacturing does not take place ($r^* = 0$) and demand is filled from production alone ($p^* = d$). All returns are disposed of ($w^* = u$).

In the case of zero inventories, besides those two policies already known from the basic model, a third subcase applies. While the first two aim to maximize current remanufacturing volume, Subcase 4(3) is characterized by a zero remanufacturing rate and disposal of all returns. It is preferred to the other two if net remanufacturing costs exceed the costs of producing a new item and disposing of the old one, i.e. in situations where remanufacturing is not preferable. Depending on optimal decisions λ_u either takes on its lower bound $-c_w$ if disposal takes place (Subcases 4(1) and 4(3)) or its upper bound $\lambda_u = c_p - c_r(R^*) + \lambda_R$ in case it does not (Subcase 4(2)).

Main results of both cases including state and co-state developments are summarized in Table 4.1.

Table 4.1. Main results of optimal cases in a product recovery system with learning.

	r^* ($\dot{R}$)	w^*	$\dot{y}_u$	λ_u	$\dot{\lambda}_u$	λ_R	$\dot{\lambda}_R$	k_2
Case 2: $y_u > 0$								
(1)	d	0	$u-d$	$-c_w <$ $< A$	$\alpha\lambda_u + h_u$ (> 0)	$B <$	$\alpha\lambda_R + c_r'(R^*)d$	0
(2)	0	0	u	$-c_w <$ $A <$	$\alpha\lambda_u + h_u$ (> 0)		$\alpha\lambda_R$	0
Case 4: $y_u = 0$								
(1) $(d \leq u)$	d	$u-d$	0	$-c_w$ $< A$	0	$B <$	$\alpha\lambda_R + c_r'(R^*)d$	$h_u - \alpha c_w$
(2) $(u < d)$	u	0	0	A $-c_w <$	$\alpha\lambda_R$	$B <$	$\alpha\lambda_R + c_r'(R^*)u$	$\alpha(\lambda_u - \lambda_R) + h_u$
(3)	0	u	0	$-c_w$ $A <$	0	$< B$	$\alpha\lambda_R$	$h_u - \alpha c_w$

Abbreviations:
$A = c_p - (c_r(R^*) - \lambda_R)$ ... cost advantage of remanufacturing including learning
$B = c_r(R^*) - c_p - c_w$ ... current (direct) cost **dis**advantage of remanufacturing

Optimal Transitions Between Cases and Subcases

As in the previous chapters, we classify transitions with respect to their dependence on an actual decision to build up stock, and differentiate between *forced* and *automatic* case transitions. A new type is named a *learning forced* transition which happens, if the optimal policy switches between non-performance and start of the remanufacturing process. As usual, $A \rightarrow B$ reads as a transition from a Case A to a Case B interval, and *continuous* and *discontinuous* transitions are distinguished.

An exploration of the continuity properties of the adjoints leads to Proposition 4.3.

Proposition 4.3 (Continuity of the co-states). *λ_R is always continuous. λ_u is continuous, i.e. jump parameter η_u vanishes everywhere, except at time points $\theta \in \Theta$ where $y_u(\theta) = 0$ and $u(\theta) = d(\theta)$ holds.*

Corollaries 4.1-4.3 provide results regarding possible case transitions. The proofs are not completely straightforward and therefore explicitly given in the proof section. Since not all transitions are possible under all cash flow parameter conditions, we distinguish between low (**L**), moderate (**M**), and high (**H**) values of $c_r(R^*(t))$ being defined as follows

- **L**ow: $c_r(R^*(t)) \leq c_p + c_w$ (and consequently, $\alpha(c_r(R^*(t)) - c_p) < h_u$), There currently exists a positive direct recovery cost advantage.
- **M**oderate: $c_r(R^*(t)) > c_p + c_w$ and $\alpha(c_r(R^*(t)) - c_p) \leq h_u$, Remanufacturing does not immediately pay off. Out of pocket holding costs are higher than savings from deferring to remanufacture an item.

- **H**igh: $h_u < \alpha(c_r(R^*(t)) - c_p)$ (including $c_r(R^*(t)) > c_p + c_w$), Remanufacturing is so expensive, that holding an item is cheaper than the interests that are saved when postponing remanufacturing of that item.

The following discussion concentrates on transitions that are not known from the basic model. Each transition incorporating one of the new cases (Subcase 2(2) and 4(3)) will be briefly discussed and it is indicated which cost condition is required.

Corollary 4.1. *Within Case 2 Subcase 2(2) is followed by a Subcase 2(1) interval. This continuous and automatic transition requires $\lambda_u = c_p - (c_r(R^*) - \lambda_R)$ and is only possible under high remanufacturing cost conditions (**H**).*

Subcase 2(2) accumulates returns. In order to deplete the stock, a switch to Subcase 2(1) is required. This happens at a time at which the value of returns equals the difference of production and net remanufacturing costs. The required cost scenario (**H**) indicates that Subcase 2(2) is used to postpone the start of remanufacturing in Subcase 2(1) while still able to recover value from collected returns.

Corollary 4.2. *Within Case 4 the following transitions are possible:*
4(2) → 4(1). *This automatic and discontinuous transition requires $d = u$ and $\dot{d} < \dot{u}$.*
4(3) → 4(1). *This learning forced and continuous transition requires $d < u$, $\lambda_R = c_r(R^*) - c_p - c_w$, and moderate or high remanufacturing costs (**M,H**).*
4(3) → 4(2). *This learning forced and continuous transition requires $d > u$, $\lambda_R = c_r(R^*) - c_p - c_w$, and moderate remanufacturing costs (**M**).*

As the value of acquiring knowledge reaches a critical value, i.e. the current recovery cost disadvantage, a switch from Subcase 4(3) to one of the other two subcases takes place. Transition 4(3) → 4(2) additionally requires a moderate cost situation. Otherwise, stock-keeping of returns and postponing the start of remanufacturing would be preferable. This is feasible since there is excess demand available after a transition time which could be filled from stored returns.

Corollary 4.3. *Between a Case 2 and a Case 4 interval the following transitions are possible:*
2(1) → 4(1). *This automatic and discontinuous transition requires $d = u$ and $\dot{d} < \dot{u}$.*
2(1) → 4(2). *This automatic and continuous transition requires $d > u$.*
4(1) → 2(1). *This forced and continuous transition requires $d < u$.*
4(2) → 2(1). *This forced and discontinuous transition requires $d = u$ and $\dot{d} < \dot{u}$.*
4(3) → 2(2). *This learning forced and continuous transition requires high remanufacturing costs (**H**).*

A transition from Case 4(3) to 2(2) takes place as disposal of returns ceases and accumulation for later remanufacturing during a Subcase 2(1) interval starts. Since Subcase 2(2) is left terminating in a Subcase 2(1) interval, the same remanufacturing cost condition must hold.

All possible transitions and relevant initial cost conditions are depicted in Figure 4.2. The grey shaded area highlights transitions that are known from the basic model.

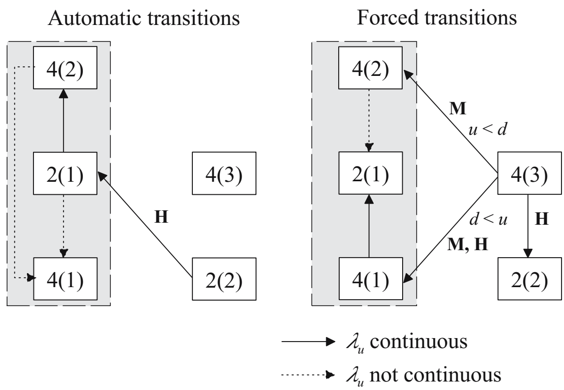

Fig. 4.2. Optimal case transitions in a product recovery system with learning.

Corollaries 4.1-4.3 showed that there are no optimal transitions from the already known (2(1),4(1),4(2)) to new subcases (2(2), 4(3)). The latter might therefore only occur during an initial period and the whole planning period can be divided into two parts, the first where no remanufacturing takes place and another where it does. Let $\theta_l \in [0, T]$ be the time which divides the two parts, i.e. after which remanufacturing is used. Corollary 4.4 summarizes the results regarding case transitions (presented without proof).

Corollary 4.4. *After remanufacturing a first returned item at time θ_l this process is continued until the end of the planning horizon, i.e. $r^*(t) \geq 0 \ \forall t \geq \theta_l$. Cases 4(3) and 2(2) are only present during an initial period and in both cases it holds that $R^* = R_0$.*

Therefore, for any transition involving Subcases 2(2) and 4(3) it holds $R^*(t) = R_0$ and the ($\mathbf{L}/\mathbf{M}/\mathbf{H}$) distinction introduced before bases on *initial unit remanufacturing costs* $c_r(R_0)$. Corollaries 4.1-4.3 have to be adapted accordingly.

As another interesting implication of the split planning period the development of the value of acquiring knowledge λ_R in general proceeds as follows

$$\dot{\lambda}_R(t) = \begin{cases} \alpha\lambda_R & t < \theta_l \\ \alpha\lambda_R + c_r'(R^*) \cdot r^* & t \geq \theta_l \end{cases}. \tag{4.29}$$

Using boundary condition (4.23) the solution of this differential equation is

$$\lambda_R(t) = \begin{cases} e^{\alpha(t-\theta_l)} \left(-\int\limits_{\theta_l}^{T} e^{\alpha(\theta_l - s)} \left(c_r'(R^*(s)) \cdot r^*\right) ds \right) & t < \theta_l \\ -\int\limits_{t}^{T} e^{\alpha(t-s)} \left(c_r'(R^*(s)) \cdot r^*\right) ds & t \geq \theta_l \end{cases} . \tag{4.30}$$

Equation (4.30) can be interpreted as follows. At any point later than θ_l, the value of knowledge acquisition is given by the discounted value (in terms of that time point) of all later cost reductions $\dot{c}_r = c_r'(R^*) \cdot r^*$. For all time points $t < \theta_l$, $\lambda_R(\theta_l)$ is only adapted to account for different time value, i.e. it is discounted from θ_l down to t.

4.4 Optimal Policies in Specific Situations

This section deals with qualitative additions of remanufacturing knowledge accumulation on the strategic stock-keeping known from the basic model. Since the effects differ when having a zero or non-zero interest rate, in the next two subsections optimal policies for each of these two cases are developed.

4.4.1 Optimal Policy with a Zero Interest Rate

The case of negligible discounting is characterized by an equal valuation of all payments independent of their timing. A postponement of remanufacturing decisions does therefore not make sense and the solution takes on a simple structure as provided in Proposition 4.4.

Proposition 4.4. *If the interest rate is zero, then either remanufacturing takes place right from start of the planning period or it does not take place at all, i.e.* $\theta_l \in \{0, T\}$.

Thus, the optimal solution either is characterized by a sequence of intervals of Cases 2(1), 4(1) and 4(2) (*Type 1*), or alternatively Case 4(3) is present throughout the planning period (*Type 2*). Both types are now examined in detail.

A *Type 1* solution exhibits the same structure as the basic model without initial inventories. It requires the value of acquiring knowledge (1) to exceed a possible direct recovery cost disadvantage ($\lambda_R(t) > c_r(R^*(t)) - c_p - c_w \; \forall t$) and (2) to continuously decrease with the same rate at which remanufacturing costs fall ($\dot{\lambda}_R = \dot{c}_r$). The value of returns depend on the relation of demand and return functions and on whether it is optimal to build up an anticipation stock of recoverables. Optimal Case 2(1) intervals require attributes like a Location Property (Corollary 2.6) and Inventory Conditions (Proposition 2.8), which both remain unchanged. A Maximal Length Property similar to Proposition 2.7 can be derived by using results provided by Lemma 4.2. Subsequently, let $\theta_{e,i}$ and $\theta_{x,i}$ denote the entry and exit time of a Case i interval.

Proposition 4.5 (Maximal Length Property where $\alpha = 0$). *If the interest rate is zero, then the maximal length of a Case 2(1) interval $I = (\theta_{e,2(1)}, \theta_{x,2(1)})$ is given by*

$$(\theta_{x,2(1)} - \theta_{e,2(1)}) \leq \tau_u := \max\left\{0, \frac{c_p + c_w - c_r(R^*(T))}{h_u}\right\}.$$

The Maximal Holding Time τ_u is constant over time, and it immediately anticipates all later-acquired experience right from the beginning. It now balances a trade-off between incurred holding costs and the *net* recovery cost advantage.

A *Type 2* solution is characterized by disposal of all returns which is only optimal if $\lambda_u(t) = -c_w \; \forall t$. Since there is no learning, the value of acquiring knowledge is zero and it never exceeds the recovery cost disadvantage, i.e. $0 = \lambda_R(t) < c_r(R^*(t)) - c_p - c_w \; \forall t$.

It remains to be seen which type of solution applies. In the case of low initial remanufacturing costs (**L**) where $c_r(R_0) \leq c_p + c_w$, Case 4(3) is invalid and remanufacturing takes place. But if we have a situation with moderate initial remanufacturing costs where $c_r(R_0) > c_p + c_w$ (**M**, while **H** conditions are excluded by definition when interest rate is zero) it is questionable whether the investments spent for 'riding down the experience curve' (i.e. into the initial recovery cost disadvantage) can later be recaptured. Since independent of their timing all payments are valued equally, this question can be answered by using a break-even type of analysis. Let $\tilde{R}$ be a return quantity at which total remanufacturing costs equal total costs of disposal and production of new items, i.e.

$$\tilde{R} \; : \; \int_{R_0}^{R_0+\tilde{R}} c_r(x)\, dx = (c_p + c_w)\tilde{R}. \tag{4.31}$$

Then, the following proposition is used to decide upon the solution type.

Proposition 4.6. *If the interest rate is zero, then remanufacturing takes place if the total remanufacturing quantity surpasses a break-even total remanufacturing quantity, i.e. $R^*(T) \geq \tilde{R}$.*

Since the total remanufacturing quantity $R^*(T)$ plays a role both in determining τ_u and in the question whether to remanufacture or not, the choice of the planning horizon T becomes a critical decisive factor. A longer planning period would lead to an increased Maximal Holding Time and thus to more stock-keeping. But also remanufacturing would more likely pay off because total remanufacturing rises. This result complies with the strategic focus of the learning curve approach.

The main problem when constructing the optimal solution, is to find the right total remanufacturing quantity $R^*(T)$ which on one side depends on the

Maximal Holding Time τ_u and on the other side is required to determine just it. In order to overcome these difficulties we propose the following solution algorithm.

Algorithm 4.1

Step 1
Start with an initial total remanufacturing quantity $\hat{R}(T)$. Determine a preliminary solution by using Algorithm 2.1 (see Section 2.3.6) assuming a Maximal Holding Time of $\tau_u = \max\left\{0, \frac{c_p+c_w-c_r(\hat{R}(T))}{h_u}\right\}$. Evaluate for this solution the total remanufacturing quantity $R^0(T)$.

Step 2
let $i := 0$.
repeat
  let $i := i + 1$
  Determine a preliminary solution under the assumption of a Maximal Holding Time $\tau_u = \max\left\{0, \frac{c_p+c_w-c_r(R^{i-1}(T))}{h_u}\right\}$. Evaluate for this solution the total remanufacturing quantity $R^i(T)$.
until $|R^i(T) - R^{i-1}(T)| < \varepsilon$.

Step 3
The solution determined in Step 2 is (approximately) optimal if $R^i(T) \geq \tilde{R}$. Otherwise remanufacturing does not take place.

Algorithm 4.1 starts with an initial guess of the total remanufacturing quantity (Step 1) and it iteratively improves this value in Step 2 until a sufficient precision (measured by a parameter ε) is reached. In Step 3, it is checked whether remanufacturing is performed at all. A possible way to initialize $\hat{R}(T)$ would be to use break-even quantity $\tilde{R}$ or the maximal potential total experience given by the sum of initial experience and accumulated return rate, i.e. $R_0 + \int_0^T u(t)dt$. The algorithm converges, because if $R^{i-1}(T)$ was chosen too large, then terminal remanufacturing costs are under- and Maximal Holding Time is overestimated. Thus, still too many items are remanufactured and therefore $R^i(T)$ is also located in between $R^*(T)$ and $R^{i-1}(T)$. A similar argument holds for too low values of $R^{i-1}(T)$.

4.4.2 Optimal Policy with a Positive Interest Rate

When discounting matters, three main modifications to the previous case occur. First of all, even if remanufacturing does take place it is not necessarily useful to start it at the beginning of the planning horizon. Therefore, the question must be answered when to start this process. A trade-off is struck between early remanufacturing that leads to higher but later direct cost savings, and a lower discounted value of the initial expenses if it is started later. Secondly, under (**H**) initial remanufacturing cost conditions, a strategic inventory might be used to postpone remanufacturing. Finally, the Maximal Holding Time is no longer constant. The following proposition uses results derived in Lemma 4.2.

Proposition 4.7 (Maximal Length Property where $\alpha > 0$).
If the interest rate is positive, then the maximal length of a Case 2 interval $I = (\theta_{e,2}, \theta_{x,2})$ is time dependent, and it increases over time. It is given as a function of the exit time of the respective Case 2(1) interval

$$(\theta_{x,2} - \theta_{e,2}) \leq \tau_u(\theta_{x,2}) := \frac{1}{\alpha} \ln \left(\frac{\alpha \left(c_p - (c_r(R^*(\theta_{x,2})) - \lambda_R(\theta_{x,2}))\right) + h_u}{-\alpha c_w + h_u} \right).$$

The Maximal Length Property corresponds to that known from the basic model (Proposition 2.7), but here remanufacturing expenses (c_r) are replaced by net remanufacturing costs ($c_r(R^*(\theta_{x,2})) - \lambda_R(\theta_{x,2})$). In the case of positive discounting, not all later acquired experience is anticipated because the respective cost savings are valued less than current expenses. Since this difference in time value decreases as time advances, the Maximal Holding Time rises with time, i.e. a later Case 2 interval is allowed to be longer than an earlier one.

Next, we consider the different initial remanufacturing cost situations.

Low Initial Remanufacturing Costs: $\mathbf{c_r(R_0) \leq c_p + c_w}$

Under (**L**) conditions there is an immediate advantage of remanufacturing over producing new items and disposing of the old ones. The optimal policy in such a situation therefore is to start remanufacturing as early as possible, i.e. $\theta_l = 0$ and to follow a *Type 1* policy. An anticipation stock of returns is held under circumstances as described before, but in contrast to the zero discounting case, the Maximal Holding Time τ_u utilizes Proposition 4.7 and must therefore be determined individually for each Case 2(1) interval. This is accomplished by applying a simple procedure as sketched below.

Algorithm 4.2

Start with the solution of Algorithm 4.1. Let n be the number of Case 2 intervals and **for all** $i = 1$ **to** n **let** $\tau_u^i = (\theta_{x,2}^i - \theta_{e,2}^i)$.
repeat
  for all $i = 1$ **to** n.
    if $\tau_u(\theta_{x,2}^i) < \tau_u^i$ **then** decrease τ_u^i
    if $\tau_u(\theta_{x,2}^i) > \tau_u^i$ **then** increase τ_u^i
  end for
  Determine a new solution by using Algorithm 2.1 under the assumption of a sequence of individual Maximal Holding Times $\{\tau_u^i\}$.
until $\max\{\tau_u(\theta_{x,2}^i) - \tau_u^i\} < \varepsilon$.

Algorithm 4.2 iteratively improves an initial guess for the sequence of Maximal Holding Times $\{\tau_u^i\}$. Note that the algorithm neglects the possibility of a Case 2 interval that has been joined during interval construction (see Algorithm 2.1 in Section 2.3.6). This means that when decreasing the Maximal Holding Time during Step 2 it might occur that this interval is split into two succeeding independent intervals, and for each one a different Maximal Holding Time must be determined.

Moderate Initial Remanufacturing Costs: $\mathbf{c_r(R_0) > c_p + c_w}$ and $\boldsymbol{\alpha}\mathbf{(c_r(R_0) - c_p) \leq h_u}$

If the initial remanufacturing expenses exceed the sum of direct cost of producing a new item and disposing of the old one (**M**) it is possible to start remanufacturing later than at time zero, which leads to a third solution type (*Type 3*) being attributed by a period where Case 4(3) is present followed by another sequence including Cases 2(1)/4(1)/4(2). Depending on the current demand/return situation both periods are connected by a transition 4(3) → 4(1) or 4(3) → 4(2) at time θ_l. For these transitions, a break-even like condition for the net recovery cost advantage at the time where remanufacturing starts is provided by Proposition 4.8.

Proposition 4.8. *At the optimal start time of remanufacturing* $\theta_l^* > 0$ *it must hold that*

$$\lambda_R(\theta_l^*) = c_r(R_0) - c_p - c_w. \tag{4.32}$$

Thus, at θ_l^* the value of acquiring knowledge must exactly outweigh the initial recovery cost disadvantage. If it would be lower, then the net recovery cost advantage would be negative and production would be preferable. In the opposite case, remanufacturing could have been started earlier since there also was a positive cost advantage.

Similarly to the zero interest rate case, having a point satisfying Proposition 4.8 does not necessarily mean that investments in knowledge acquisition pay off. But in contrast to that case it is not possible to formulate a simple condition as presented in Proposition 4.6. But when considering the differences in the time values it needs to be said that the total remanufacturing quantity must be higher than break-even quantity $\tilde{R}$ derived before in order to amortize initial high costs.

Since only first order necessary optimality conditions were considered, local and global optimality as well as existence and uniqueness of a candidate satisfying condition (4.32) is not assured. A possible solution method would therefore be to numerically search for all points for which break-even condition (4.32) holds and to compare the respective objective value of each candidate with the non-performance of remanufacturing ($\theta_l = T$) and with start of remanufacturing at time zero ($\theta_l = 0$).

High Initial Remanufacturing Costs: $\mathbf{h_u < \alpha(c_r(R_0) - c_p)}$ (and $\mathbf{c_r(R_0) > c_p + c_w}$)

Under high remanufacturing cost conditions (**H**), a strategic inventory might be used in order to further postpone the start of remanufacturing. This occurs as a sequence 4(3) $\rightarrow$ 2(2) $\rightarrow$ 2(1) $\rightarrow$ 4(1)/4(2), i.e. at time $\theta_{e,2(2)}$ disposal of returns stops which then are accumulated during a Subcase 2(2) interval. After switching to Subcase 2(1) at time θ_l, the returns are used up. This requires that at the end of the respective interval demand exceeds the return rate in order to deplete the recoverables stock.

The Case 2 interval is built around a time point θ_l where

$$\lambda_u(\theta_l) = c_p - c_r(R^*(\theta_l)) + \lambda_R(\theta_l) \tag{4.33}$$

which can be interpreted as another Location Property. Aside a Maximal Length Property as provided with Proposition 4.7, Inventory Conditions must hold which assure a positive stock-level during the whole Case 2 interval and a zero level at exit time. These considerations lead to Proposition 4.9 (presented without proof).

Proposition 4.9 (Inventory Conditions).
Let $I = (\theta_{e,2(2)}, \theta_l) \cup [\theta_l, \theta_{x,2(1)})$ *be the open time interval of a sequence* 2(2) $\rightarrow$ 2(1) *where* $y_u > 0$ *and* $y_u(\theta_{e,2}) = y_u(\theta_{x,2}) = 0$. *Then,*

(i) cumulative demand equals cumulative returns over the whole interval

$$\int_{\theta_{e,2(2)}}^{\theta_l} u(t)\,dt + \int_{\theta_l}^{\theta_{x,2(1)}} (u(t) - d(t))\,dt = 0, \tag{4.34}$$

(ii) at any point $\theta \in [\theta_l, \theta_{x,2(1)})$, *cumulative returns must be larger than cumulative demand, especially*

$$\int_{\theta_{e,2(2)}}^{\theta_l} u(t)\,dt + \int_{\theta_l}^{\theta} (u(t) - d(t))\,dt > 0. \tag{4.35}$$

This result has been adapted from Proposition 2.8 by additionally taking into account the two different policies inside the interval where recoverables are kept.

4.5 Numerical Examples

In this section four numerical examples based on two different demand/return scenarios are used to exemplify the main results of the model with learning. All examples use a learning function according to the Power Law, i.e.

$$c_r(R) = c_r^0 R^{-b} \tag{4.36}$$

holds with an 80% progress ratio (b=0.32) and $R_0 = 1$. Common cash flow parameters are $c_p = 2$, $c_w = 1$.

Example 4.1 The first example aims to show the effects of learning when having a zero interest rate. It uses a scenario which remains basically the same as Example 2.1 in Chapter 2 except for remanufacturing cost rate. Demand and return functions are given as follows

$$d(t) = 1 + 0.5\sin(t) \text{ and } u(t) = 0.7d(t - \pi). \tag{4.37}$$

The planning horizon has been extended to $T = 6\pi$ in order to increase the number of possible Case 2 intervals. This cyclical scenario is depicted in Figure 4.3.

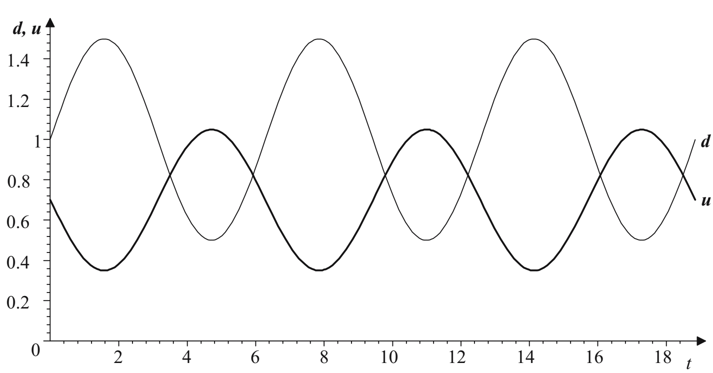

Fig. 4.3. Demands and returns in Examples 4.1 and 4.2.

Further parameters are $c_r^0 = 3$ and $h_u = 1$. Thus, there is no initial recovery cost disadvantage leading to a break-even total remanufacturing quantity $\tilde{R} = 0$. The solution is naturally of *Type 1* ($\theta_l = 0$).

Algorithm 4.1 was used to solve this example yielding a total optimal remanufacturing quantity $R^*(T) = 11.1371$, $c_r(R^*(T)) = 1.3872$ and thus, a Maximal Holding Time $\tau_u = 1.6128$. The optimal solution is depicted in Figure 4.4 showing three collection intervals where the first two have maximal length and the last one is shorter because the planning horizon is reached. Optimal Case 2(1) intervals are shown in Table 4.2.

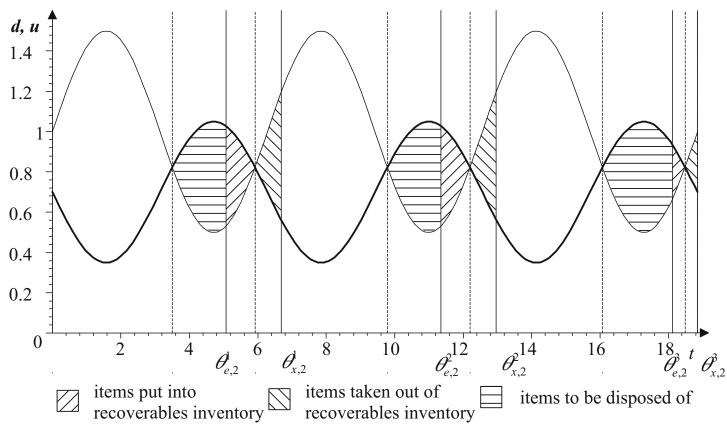

Fig. 4.4. Optimal solution of Example 4.1.

Table 4.2. Optimal Case 2(1) intervals in Examples 4.1 and 4.2.

	$\theta^1_{e,2}$	$\theta^1_{x,2}$	$\theta^2_{e,2}$	$\theta^2_{x,2}$	$\theta^3_{e,2}$	$\theta^3_{x,2}$
Example 4.1	5.0715	6.6842	11.3547	12.9674	18.1107	T
Example 4.2	5.0752	6.6814	11.3141	12.9990	18.1107	T

The optimal development of the value of returns λ_u can be found in Figure 4.5 and Figure 4.6 exhibits the decreasing slope of the value of acquiring knowledge λ_R.

The sensitivity of the Maximal Holding Time on the length of the planning horizon can be seen when modifying it, $T = 4\pi$ would lead to $\tau_u = 1.4126$ whereas $T = 8\pi$ yields $\tau_u = 1.7394$.

Example 4.2 For the second example the discount rate was modified to $\alpha = 0.2$. This leads to modified optimal Case 2(1) intervals which are determined by using Algorithm 4.2 and also shown in Table 4.2. The first two intervals

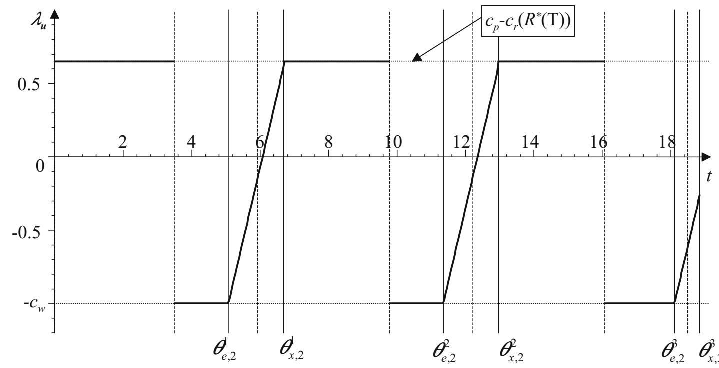

Fig. 4.5. Optimal development of co-state λ_u in Example 4.1.

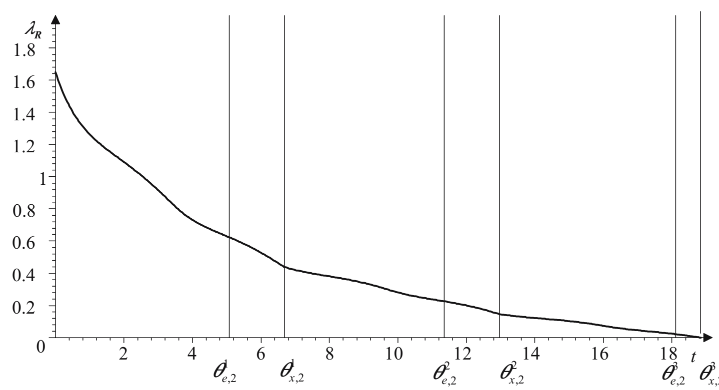

Fig. 4.6. Optimal development of co-state λ_R in Example 4.1.

each having full length show an increasing holding time of $\tau_u(\theta^1_{x,2(1)}) = 1.6062$ and $\tau_u(\theta^2_{x,2(1)}) = 1.6849$. As before, the length of the third collection interval is limited by the end of the planning period, but at this point a maximal length of $\tau_u(\theta^3_{x,2(1)}) = 1.7277$ be allowed. Figure 4.7 shows the optimal solution.

The optimal recoverables co-state development is depicted in Figure 4.8. The upper dotted line shows how the maximal allowed value of returns rises with time and therefore, the Maximal Holding Time increases. The evolution of the value of acquiring knowledge can be seen from Figure 4.9.

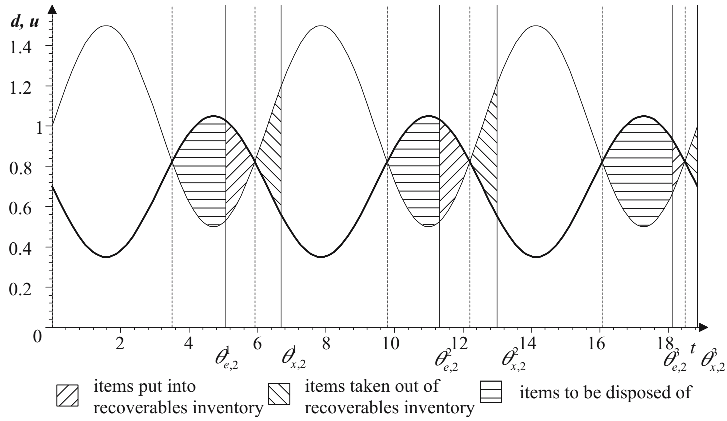

Fig. 4.7. Optimal solution of Example 4.2.

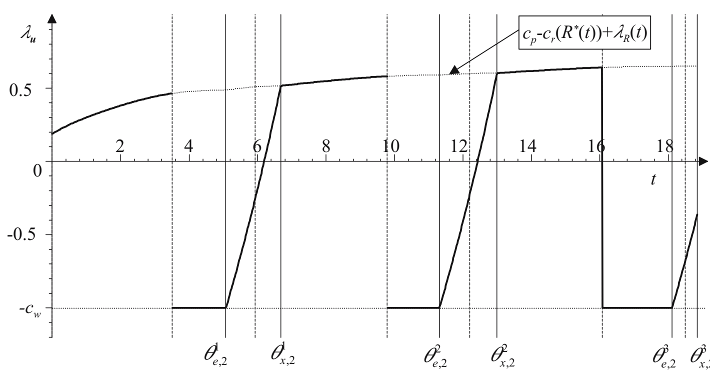

Fig. 4.8. Optimal development of co-state λ_u in Example 4.2.

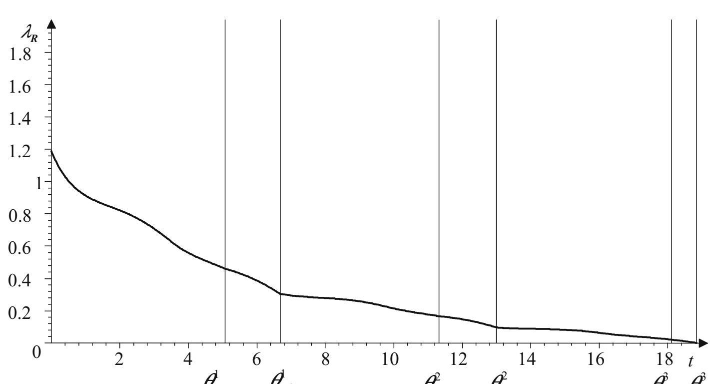

Fig. 4.9. Optimal development of co-state λ_R in Example 4.2.

Example 4.3 The third and fourth examples are used to exemplify the problem of choosing the time at which remanufacturing starts (θ_l). In both examples a scenario is used where the planning horizon $T = 3\pi$. Demand and return functions are

$$d = 1 + \sin(t/2) \text{ and } u = 0.75\, d(t - 1.5\pi). \tag{4.38}$$

While the demand function approximates a product life cycle, the return function is set in a way that it shows two peaks. This could for instance be motivated by assuming that each peak corresponds to one out of two classes of returns: (i) commercial returns, e.g. defective items that are assumed to occur more often in the beginning of the product life cycle and (ii) used products. Both types are assumed to be remanufactured using the same process and at the same costs. This scenario is depicted in Figure 4.10.

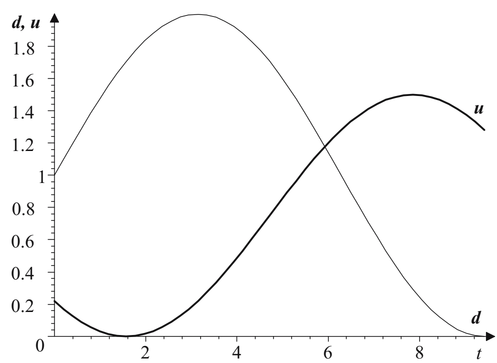

Fig. 4.10. Demands and returns in Examples 4.3 and 4.4.

Cash flow parameters are as in Example 4.2 except for initial remanufacturing costs, which now are given by a moderate level of $c_r^0 = 3.8$. Therefore, a situation with a positive discount rate and parameter condition (**M**) is considered.

The optimal time where remanufacturing starts is determined using Proposition 4.8. Since there is no reason for keeping stock in this example, $\lambda_R(\theta_l)$ follows from inserting the optimal remanufacturing decisions in Cases 4(1)/(2) into (4.30)

$$\lambda_R(\theta_l) = -\int_{\theta_l}^{T} e^{\alpha(\theta_l - s)} \left(c_r' \left(R_0 + \int_{\theta_l}^{s} \min\{d(t), u(t)\}\, dt \right) \min\{d(s), u(s)\} \right) ds \tag{4.39}$$

Inserting (4.39) into (4.32) and solving for θ_l yields 1.7176 and 5.9718. Comparing the respective objective values with the non-performance of remanufacturing and with starting at time zero yields $\theta_l = 1.7176$ as the optimal time to start remanufacturing. The optimal policy is therefore characterized by disposal of commercial returns up to θ_l because later cost savings that are due to the acquired knowledge do not suffice to compensate the higher time value of remanufacturing expenses when starting early.

Figure 4.11 shows the optimal development of λ_R.

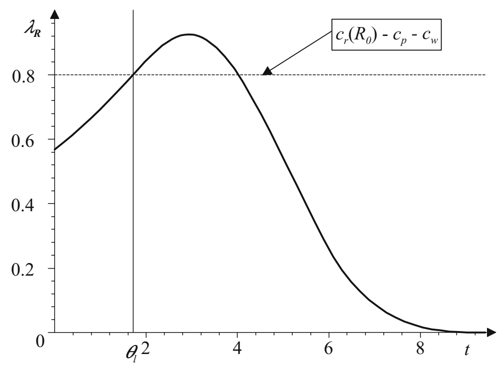

Fig. 4.11. Optimal development of co-state λ_R in Example 4.3.

Example 4.4 For the fourth example out-of-pocket holding costs are reduced to $h_u = 0.3$. Therefore (**H**) conditions apply and it is beneficial to have a positive recoverables stock under conditions that characterize the optimal start time of remanufacturing in the previous example. Maximal Length Property (Proposition 4.7), Inventory Conditions (Proposition 4.9) and the Location Property like condition (4.33) are used to determine a sequence $4(3) \rightarrow 2(2) \rightarrow 2(1) \rightarrow 4(2)$. The optimal solution is represented by $\theta_{e,2(2)} = 1.2304$, $\theta_l = 3.9514$, and $\theta_{x,2(1)} = 4.3646$, and is illustrated in Figure 4.12. Optimal co-state developments are depicted in Figures 4.13 and 4.14.

Compared with Example 4.3 a smaller holding cost rate (or relatively higher initial remanufacturing costs) leads to the possibility to use a strategic recoverables stock in order to postpone the start time of remanufacturing θ_l. High initial remanufacturing expenses are pooled at and after that time when they are valued less than at the time when returns would have been remanufactured without stock-keeping. Moreover, a higher total number of returns is recovered because return collection starts earlier than remanufacturing in the case without stock-keeping.

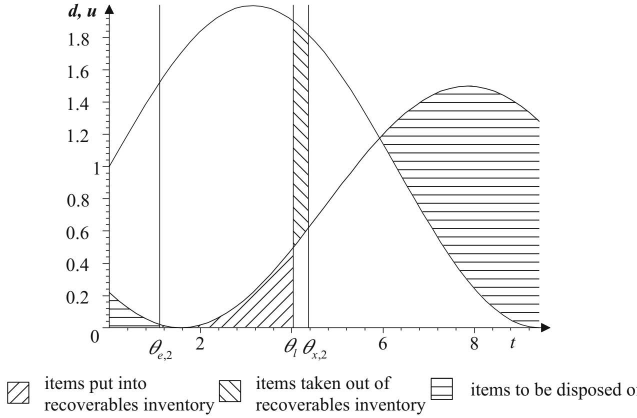

Fig. 4.12. Optimal solution of Example 4.4.

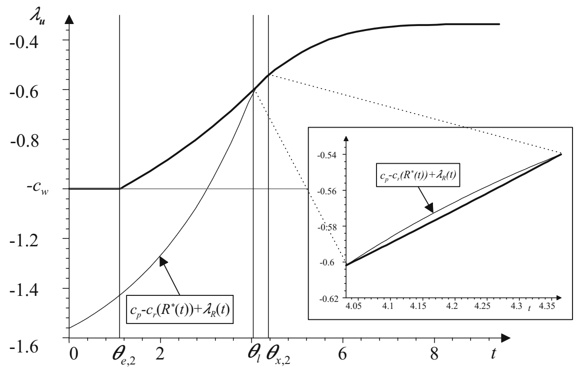

Fig. 4.13. Optimal development of co-state λ_u in Example 4.4.

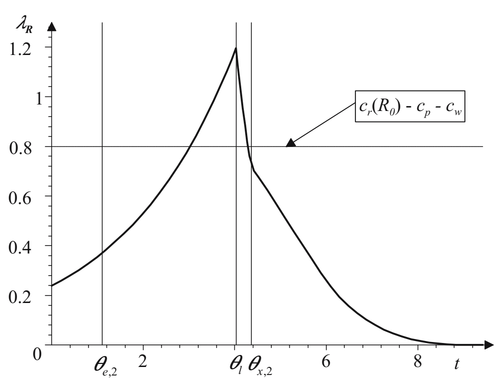

Fig. 4.14. Optimal development of co-state λ_R in Example 4.4.

4.6 Conclusions and Managerial Insights

In this chapter the effects of introducing a 'learning' remanufacturing process into the optimal control framework from Chapter 2 are investigated. The anticipation of later knowledge acquisition led to the possibility to remanufacture used products even if there exists no immediate cost advantage. More specifically when neglecting discounting, the Maximal Holding Time only depends on remanufacturing costs at the end of the planning horizon after all cost reductions due to learning have occurred. The decision on whether to remanufacture starting at time zero or not at all is made based on a simple break-even like condition for the total remanufacturing quantity during the planning period. In case of positive interest rate this process might start later than the begin of the planning period. The case with discounting is further characterized by two main additions to strategic stock-keeping. First, the maximal length of collection intervals increases with time and secondly, we discover another motivation for stock-keeping. Strategic inventory can be used to postpone the beginning of remanufacturing under conditions where initial remanufacturing costs are high enough that more interest is saved when delaying it compared to incurred holding costs.

Results are sensitive to planning horizon changes. This especially holds for small planning periods with considerable potential for cost-improvements after the end of the planning period. Since the learning curve concept is a strategic one, the planning horizon should be chosen sufficiently large. But used together with the product life cycle concept, this analysis can be a helpful tool for deciding whether to engage in remanufacturing at all.

Besides learning curve effects, there is a number of other reasons why cost parameters change over time. For instance, this might occur when varying prices for expensive raw materials lead to significant fluctuations of direct production costs. In addition to a reason for stock-keeping of these materials in anticipation of a price increase such a situation also leads to price dependent holding times for recoverables. Another example might be that due to the growing regulation in many industrialized countries intended to reduce the outcome of waste, using the disposal option becomes more and more expensive, influencing both stock-keeping and the decision when to start the remanufacturing process.

4.7 Proofs

Proof (Proof of $\lambda_0 = 1$).
This proof proceeds in a similar manner as its equivalent in Chapter 2. Let us assume $\lambda_0 = 0$. Then, the Lagrangian reduces to

$$L(..) = \lambda_u(u - r - w) + \lambda_R r + \mu_1(d - r) + \mu_2 r + \mu_3 w + k_2 y_u, \quad (4.40)$$

and necessary conditions (4.14), (4.15), (4.19), and (4.20) change to

$$\frac{\partial L}{\partial r} = -\lambda_u + \lambda_R - \mu_1 + \mu_2 = 0 \tag{4.41}$$

$$\frac{\partial L}{\partial w} = -\lambda_u + \mu_3 = 0 \tag{4.42}$$

$$\dot{\lambda}_u = \alpha\lambda_u - \frac{\partial L}{\partial y_u} = \alpha\lambda_u - k_2 \tag{4.43}$$

$$\dot{\lambda}_R = \alpha\lambda_R - \frac{\partial L}{\partial R} = \alpha\lambda_R. \tag{4.44}$$

From (4.44) together with transversality condition (4.23) it follows that $\lambda_R = 0$ and $\dot{\lambda}_R = 0$. Non-triviality condition $(\lambda_0, \lambda_u, \lambda_R) \neq 0$ thus requires $\lambda_u \neq 0 \; \forall t$. Non-negativity of μ_3 requires $\lambda_u > 0$ and $\mu_3 > 0$ in (4.42). Thus, $w = 0$ in (4.18). In order to achieve equality with zero in (4.41), $\mu_2 > 0$ must hold, which in turn requires $r = 0$. Thus $\dot{y}_u = u > 0$ which contradicts final inventory condition $(y_u(T) = 0)$.

Proof (Proof of Proposition 4.1).
$y_u > 0$ requires $k_2 = 0$ from (4.21); (4.19) thus reduces to

$$\dot{\lambda}_u = \alpha\lambda_u + h_u > 0. \tag{4.45}$$

$w > 0$ necessitates $\mu_3 = 0$ in (4.18) yielding $\lambda_u = -c_w$ in (4.15). It follows $\dot{\lambda}_u = 0$, which contradicts (4.45). Thus, $\lambda_u > -c_w$ and $w^* = 0$.

Three cases are possible for remanufacturing rate r:

- **2(1)** $r = d$ $(p = 0)$ requires $\mu_2 = 0$ in (4.17) yielding $\lambda_u < c_p - c_r(R) + \lambda_R$ in (4.14). States and co-states develop as follows: $\dot{y}_u = u - d$, $\dot{R} = d$, and $\dot{\lambda}_R = \alpha\lambda_R + c_r'(R) \cdot d$.
- $0 < r < d$ $(p = d - r > 0)$ requires $\mu_1 = \mu_2 = 0$ in (4.16) and (4.17). Inserting into (4.14) yields

  $$\lambda_u = c_p - c_r(R) + \lambda_R. \tag{4.46}$$

  Differentiating both sides with respect to time leads to $\dot{\lambda}_u = \alpha\lambda_R$. Together with (4.45) it follows $\lambda_R = \lambda_u + \frac{h_u}{\alpha}$. Re-inserting into (4.46) leads to

  $$c_r(R) = c_p + \frac{h_U}{\alpha}. \tag{4.47}$$

  Since the RHS of (4.47) is a constant, R must not change which contradicts the definition of this case $(\dot{R} = r > 0)$.
- **2(2)** $r = 0$ $(p = d)$ requires $\mu_1 = 0$ in (4.16) yielding $\lambda_u > c_p - c_r(R) + \lambda_R$ in (4.14). States and co-states develop as follows: $\dot{y}_u = u > 0$, $\dot{R} = 0$, and $\dot{\lambda}_R = \alpha\lambda_R$.

Proof (Proof of Proposition 4.2).
Case 2 conditions $(y_u = 0)$ require $p = d - r$ and $w = u - r$. Therefore, four situations with respect to the remanufacturing rate r are to be distinguished.

- $0 < r < \min\{d, u\}$ $(p > 0, w > 0)$ requires $\mu_1 = \mu_2 = \mu_3 = 0$ in (4.16)–(4.18), leading to $\lambda_u = -c_w$ in (4.15). Therefore, $0 = c_p - c_r(R) + c_w + \lambda_R$ in (4.14). Differentiating RHS with respect to time yields $0 = \alpha\lambda_R$ which necessitates $\lambda_R = 0$ as well as $\dot{\lambda}_R = 0$. Inserting both values into (4.20) leads to $0 = c_r'(R) \cdot r$ which contradicts $r > 0$.
- **4(1)** $r = d < u$ $(p = 0, w > 0)$ requires $\mu_2 = \mu_3 = 0$ in (4.17) and (4.18). This yields $\lambda_u = -c_w$ in (4.15), $\dot{\lambda}_u = 0$, as well as $k_2 = h_u - \alpha c_w > 0$ in (4.19). Inserting μ_2 and λ_u into (4.14) leads to $\lambda_R > c_r(R) - c_p - c_w$. Remaining optimal state and co-state movements are given by $\dot{\lambda}_R = \alpha\lambda_R + c_r'(R) \cdot d$ and $\dot{R} = d$.
- **4(2)** $0 < r = u < d$ $(p > 0, w = 0)$ requires $\mu_1 = \mu_2 = 0$ in (4.16) and (4.17). Inserting into (4.14) leads to

$$\lambda_u = c_p - c_r(R) + \lambda_R > -c_w. \tag{4.48}$$

  Differentiating with respect to time leads to $\dot{\lambda}_u = \alpha\lambda_R$. Together with (4.19) it follows $k_2 = \alpha(\lambda_u - \lambda_R) + h_u$. Solving inequality (4.48) for λ_R yields $\lambda_R > c_r(R) - c_p - c_w$. State and co-states develop as follows: $\dot{\lambda}_R = \alpha\lambda_R + c_r'(R) \cdot u$ and $\dot{R} = u$.
- **4(3)** $r = 0$ $(p > 0, w > 0)$ requires $\mu_1 = \mu_3 = 0$ in (4.16) and (4.18). Proceeding as in the first subcase yields $\lambda_u = -c_w$ in (4.15), $\dot{\lambda}_u = 0$, $k_2 = h_u - \alpha c_w > 0$, as well as $\lambda_R < c_r(R) - c_p - c_w$. Optimal state and co-state movements are $\dot{\lambda}_R = \alpha\lambda_R$ and $\dot{R} = 0$.

Proof (Proof of Proposition 4.3).
Continuity of λ_R has already been established when rendering necessary optimality conditions (see Section 4.3). The proof of continuity of λ_u proceeds in two steps. In the first step (i) points inside intervals where $y_u = 0$ holds are examined. Afterwards, step (ii) deals with entry and exit points of such an interval.
(i) A constraint qualification guarantees the continuity of adjoint variable λ_u inside intervals where $y_u = 0$. Thus, continuity is given, if the matrix (with line numbers given on the right hand side)

$$\begin{pmatrix} -1 & 0 & d-r & 0 & 0 & 0 \\ 1 & 0 & 0 & r & 0 & 0 \\ 0 & 1 & 0 & 0 & w & 0 \\ -1 & -1 & 0 & 0 & 0 & y_u \end{pmatrix} \begin{matrix} I \\ II \\ III \\ IV \end{matrix}$$

has full rank of four. Analysis of the above matrix yields that the constraint qualification is not satisfied in three situations.

- If $p = d - r = 0$ and $r = 0$ then $I = -II$.
  This can only happen in Case 4 $(y_u = 0)$ in situations where $d = 0$ which is excluded by assumption (4.8) $(d > 0)$.

- If $r = 0$,$w = 0$ and $y_u = 0$ then $-II - III = IV$.
  This can only happen in Case 4 if $u = 0$ which again contradicts assumption (4.8) ($u > 0$).
- If $p = d - r = 0$, $w = 0$, and $y_u = 0$ then $I - III = IV$.
  This situation occurs in Case 4 ($y_u = 0$) when demand equals returns and the policy switches from $p > 0, r > 0$ to $r > 0, w > 0$ or vice versa. Only in this case we find a discontinuity λ_u. The height of this jump is $\eta_u = c_p + c_w - c_r(R) + \lambda_R$.

(ii) Let θ_u^1 be the entry time and θ_u^2 be the exit time of an interval, where $y_u = 0$ holds. Then, λ_u is continuous at this point if y_u enters this interval in a non-tangential way, i.e. $\dot{y}_u = u - r - w$ jumps. This requires a jump in $r + w$ which might occur if $d = u$. Then, a tangential transition between cases 2(1) and 4(1) or 4(2) happens. The height of this jump is $\eta_u < c_p + c_w - c_r(R) + \lambda_R$. Any other tangential transition (between 2(1) and 4(3)) would require $d = u = 0$ which is excluded by assumption (4.8).

Proof (Proof of Corollary 4.1).
Since returns are accumulated but not used in Subcase 2(2) the only transition starting in such an interval is one that terminates into Subcase 2(1), where recoverables inventory can be depleted. Thus, any sequence of transitions between Subcases of Case 2 must terminate in a Subcase 2(1) interval. Both types of transition must take place continuously, and at transition time it must hold that

$$-c_w < \lambda_u = c_p - (c_r(R^*) - \lambda_R) \tag{4.49}$$

as well as $\dot{\lambda}_u = \alpha\lambda_u + h_u > \frac{d}{dt}(c_p - (c_r(R^*) - \lambda_R)) = \alpha\lambda_R$ for $2(1) \to 2(2)$ or $\dot{\lambda}_u = \alpha\lambda_u + h_u < \frac{d}{dt}(c_p - c_r(R^*) + \lambda_R) = \alpha\lambda_R$ for $2(2) \to 2(1)$, respectively. Inserting the value for λ_u as determined in (4.49) finally leads us to the following cost condition at transition time

$$h_u \begin{cases} \leq \alpha\,(c_p - c_r(R^*)) & \text{for } 2(1) \to 2(2) \\ > \alpha\,(c_p - c_r(R^*)) & \text{for } 2(2) \to 2(1) \end{cases} . \tag{4.50}$$

It can easily be seen that since c_r is a decreasing function, a transition $2(1) \to 2(2)$ prevents from a later transition in the opposite direction, and that a transition therefore is not possible.

Transition $2(2) \to 2(1)$ takes place requiring condition 4.49 as well as high remanufacturing costs (**H**).

Proof (Proof of Corollary 4.2).
$4(1) \to 4(2)$ would require an upward jump in λ_u except for a situation where $-c_w = \lambda_u = c_p - c_r(R^*) + \lambda_R$ implying $\lambda_R = c_r(R^*) - c_p - c_w$ which contradicts the definition of both cases in Proposition 4.2.
$4(1) \to 4(3)$ and $4(2) \to 4(3)$ necessitate $\lambda_R = c_r(R^*) - c_p - c_w$ and

$\dot{\lambda}_R < \dot{c}_r(R^*) \Leftrightarrow \frac{d}{dt}(\lambda_R - \dot{c}_r(R^*)) < 0$ at transition time. This contradicts (4.28).
4(2) $\rightarrow$ 4(1) proceeds discontinuous and thus, it requires $d = u$.
4(3) $\rightarrow$ 4(1) and 4(3) $\rightarrow$ 4(2) are continuous and need $-c_w = \lambda_u = c_p - c_r(R^*) + \lambda_R \Leftrightarrow c_r(R^*) > c_r(R^*) - \lambda_R = c_p + c_w$. Moderate or high remanufacturing costs are necessitated (**M,H**). In the second case, at and after entering Subcase 4(2) it must hold that $k_2 = \alpha(\lambda_u - \lambda_R) + h_u \geq 0$. Thus, $h_u \geq \alpha(c_r(R^*) - c_p)$ only allowing for moderate remanufacturing costs (**M**).

Proof (Proof of Corollary 4.3).
Case 2 can only be left starting from a Subcase 2(1) interval. Transitions to Subcases 4(1) and 4(2) proceed under the same conditions as in the basic model without learning. 2(1) $\rightarrow$ 4(3) would require a downward jump in λ_u. For a non-tangential transition, $d = u = 0$ are required, which contradicts assumption (4.8).
Transitions starting in Subcases 4(1) and 4(2) to 2(1) proceed under the same conditions as in the basic model without learning. 4(1) $\rightarrow$ 2(2) requires $-c_w = \lambda_u = c_p - c_r(R^*) + \lambda_R \Leftrightarrow \lambda_R - c_r(R^*) = -c_w - c_p$. Since in Subcase 4(1) it holds that $\lambda_R - c_r(R^*) > -c_w - c_p$ and RHS of this inequality increases with rate $\alpha\lambda_R > 0$ (see (4.28)), the transition can be excluded.
4(2) $\rightarrow$ 2(2) necessitates $-c_w < \lambda_u = c_p - c_r(R^*) + \lambda_R \Leftrightarrow \lambda_u - \lambda_R = c_p - c_r(R^*)$. At the end of the Subcase 4(2) interval it must hold that $k_2 = \alpha(\lambda_u - \lambda_R) + h_u \geq 0$. Inserting previously determined value for the co-state difference $\lambda_u - \lambda_R$ yields $h_u \geq \alpha(c_r(R^*) - c_p)$. In analogy to the proof of Corollary 4.1 this would exclude a later transition 2(2) $\rightarrow$ 2(1) and thus contradict the zero final inventory level condition.
4(3) $\rightarrow$ 2(2) does not require any special condition, but since Subcase 2(2) can only be left under high remanufacturing cost (**H**), it must also hold here. For a transition 4(3) $\rightarrow$ 2(1), switching time is (a) determined by demand/return developments and at the same time (b) the value of learning must exactly equal the initial cost disadvantage of remanufacturing which also depends on demand and return development. Thus, the transition is a rare event which is not of practical interest and can be neglected in our discussion. In order to keep generality, it is further treated as a special case of a 4(3) $\rightarrow$ 2(2) $\rightarrow$ 2(1) with zero Case 2(2) length.

Proof (Proof of Proposition 4.4).
A zero interest rate excludes the case of high initial remanufacturing costs (**H**). All transitions requiring such circumstances are impossible. Therefore, Subcase 2(2) will not be present in an optimal solution.

Subcase 4(3) can not be left, because from Lemma 4.1 together with the definition of the case in Proposition 4.2 inside this case it holds that $\lambda_R = c_r(R_0) - c_r(R^*(T)) < c_r(R_0) - c_p - c_w$, and therefore $c_r(R^*(T)) > c_p + c_w$. Throughout the planning period it is not possible to obtain a positive recovery cost advantage and remanufacturing does not occur.

Proof (Proof of Proposition 4.5).
From (4.13), a minimal value for λ_u is given ($\lambda_u^{\min} = -c_w$). For a Case 2(1) interval the upper bound is provided by Lemma 4.2 as follows $\lambda_u^{\max} = c_p - c_r(R^*(T)$. In the undiscounted case the development of λ_u is linear within a Case 2 interval which together with its continuity yields

$$-c_w \leq \lambda_u(t) = \lambda_u(\theta_{e,2}) + h_u \cdot (t - \theta_{e,2}) \leq c_p - c_r(R^*(T) \qquad (4.51)$$

Let $t = \theta_{x,2}$ and $\lambda_u(\theta_{e,2}) = -c_w$. Solving (4.51) for $\theta_{x,2} - \theta_{e,2}$ yields

$$\begin{aligned} -c_w + h_u \cdot (\theta_{x,2} - \theta_{e,2}) &\leq c_p - c_r(R^*(T) \\ \Leftrightarrow (\theta_{x,2} - \theta_{e,2}) &\leq \frac{c_p + c_w - c_r(R^*(T))}{h_u} \end{aligned}$$

Proof (Proof of Proposition 4.7).
The proof proceeds in the same way as the proof of Proposition 4.5 with $\lambda_u^{\min} = -c_w$, but now we have $\lambda_u^{\max}(t) = c_p - c_r(R^*(t) + \lambda_R(t)$. Solving differential equation (4.19) assuming $\lambda_u(\theta_{e,2}) = \lambda_u^{\min}$ leads to

$$\lambda_u(\theta_{x,2}) = \left(-c_w + \frac{h_u}{\alpha}\right) e^{\alpha(\theta_{x,2} - \theta_{e,2})} - \frac{h_u}{\alpha} \leq c_p - c_r(R^*(\theta_{x,2})) + \lambda_R(\theta_{x,2}), \qquad (4.52)$$

i.e. it must be smaller than $\lambda_u^{\max}(\theta_{x,2})$. Solving for $(\theta_{x,2} - \theta_{e,2})$ finally yields

$$(\theta_{x,2} - \theta_{e,2}) \leq \frac{1}{\alpha} \ln \left(\frac{\alpha\,(c_p - c_r(R^*(\theta_{x,2})) + \lambda_R(\theta_{x,2})) + h_u}{-\alpha c_w + h_u} \right). \qquad (4.53)$$

From (4.28) we know $\dot{\lambda}_u^{\max}(t) = \alpha \lambda_R(t) \geq 0$. Thus the numerator in the logarithm expression in (4.53) increases and with it the whole term.

Proof (Proof of Proposition 4.8).
Proposition 4.8 directly follows from Corollaries 4.2 and 4.4.

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5

Technology Selection in the Context of Reverse Logistics

5.1 Motivation

When developing new products and setting up production facilities, firms often have the choice between different technologies in order to manufacture the product. Besides the quality and service aspects, this decision has a major impact on direct (variable) production costs and necessary capital expenditures in building and maintaining new facilities or modifying existing ones. In the context of reverse logistics, an additional issue has to be considered: tightened recycling and reuse legislation and environmental awareness of customers forces firms to take back their products from customers after use. At this point, the selected production technology also affects the ways on how to deal with returned/used products. That raises the question of whether to design and produce a product for single use only, or in a way that allows for reuse after some recovery process (e.g. rework, upgrading or remanufacturing, see de Brito and Dekker (2004) for an overview on available options). This can yield additional profit, as some of the added value will not be lost under certain recovery options, as it would be the case with material-recycling or disposing of the returned item. On the other side, there can be higher expenses for setting up production facilities, as well as higher direct production unit costs, that are caused by the necessity to add properties to the product in order to make it usable. This is illustrated by the following real-life example.

Case 5.1. CopyMagic (See Thierry et al. (1995) and Thierry (1997)) As a multinational copier manufacturing company, CopyMagic sells its products in all segments of the copier market, mainly by using leasing contracts. This creates a continuous flow of returned used products namely off-lease copiers. Depending on the returned product one or more of the following product recovery options is used: repair, cannibalization, remanufacturing and recycling. In particular the last two options require a special product design which is different from a 'classical' single use product which can only be disposed of after use. Design for recycling requires a reduction in the number of used materials, a replacement of non-recyclable with recyclable materials and easy separability of materials. Design requirements for remanufacturing go even further. Here it must be ensured that a product or its components are (in principle) capable to be used for more than one life time for the product to be sold 'as-good-as-new'. This causes higher production expenses and together with outlays for product recovery the question is which recovery option is preferable.

Although there exists a wealth of literature on operative issues in reverse logistics (for literature surveys see Fleischmann et al. (1997), Guide et al. (2000), or Fleischmann (2001b)), aspects of financial justification of product recovery, which are highly influential to investment decisions, have widely been neglected so far. Because they require both, a dynamic consideration as well as the application of discounted cash flow (DCF) techniques, which substantially complicates the matter, models usually deal with stationary situations and focus on average cost/profit. Debo et al. (2005) for instance consider the problem of technology selection in connection with market segmentation. Thereby, they assume a situation where remanufactured products are valued less than newly produced items, but both compete on the same market where customers have a heterogeneous willingness-to-pay. The chosen technology is characterized by a level of remanufacturability influencing variable production costs, but investment expenditures are not considered. The objective is to select a remanufacturability level which maximizes average profit under equilibrium conditions.

Durable products tend to remain with the customer for a considerable amount of time compared to the time period where they are sold. Demand may be subject to dynamic processes like the product life cycle and thus, a static (equilibrium) analysis based on average costs is often not appropriate. A dynamic DCF framework is required which also takes into account the time value of money and, especially, of investments. In the previous chapters it was assumed that both, production and remanufacturing facilities already existed, i.e. the choice upon remanufacturability and the selection of an appropriate technology has already been made. Since introducing a technology for remanufacturing may require additional investments, it remains to be seen if these investments will pay off or not. Furthermore, when considering more than one technology with different investments and variable unit costs, the optimal product recovery technology has to be determined. Consequently, a technology in our sense is described by four key characteristics – investment expenditure for setting up the production and remanufacturing facilities and variable costs for producing new items as well as remanufacturing returns.

Regarding the remanufacturing activities, it is seldom preferable to start it immediately when production starts, as this often requires a considerable capital commitment while not yet yielding a large number of returns. For instance in the case of engine remanufacturing in car industry (see, e.g., Seitz and Peattie (2004)) the usual life time of an engine must be considered while specialized equipment is required that sometimes must meet even higher standards than the one used for producing new engines. New production technologies like the use of aluminum instead of steel for engine components necessitate high investment in the remanufacturing process as well. Therefore, a decision has to be made when to introduce the process. This decision has to balance a trade-off which is given by the fact that postponing the investment reduces its time value whereas an earlier process introduction increases the potential benefit from remanufacturing.

Another question is whether to hold returned items in strategic inventory for later use or just to keep items that are necessary to satisfy current demand and therefore dispose excess returns. Practitioners often apply simplistic rules (e.g. collecting all returns and disposing of none, see Kiesmüller et al. (2004)). Both issues are related to the problem of timing a capacity expansion known from production/inventory theory (see, e.g., Slack and Lewis (2003)), where a (serviceables) inventory is used to postpone capacity expansion compared to a strategy without stockkeeping where capacity expansion must lead demand in order to avoid backorders. It is questionable whether recoverables inventory would influence the remanufacturing investment time in the same way.

This chapter is organized as follows. In Section 5.2, we propose a dynamic environment for strategic decision making in the context of reverse logistics based on simple assumptions regarding the product life cycle and an availability cycle for returns. Three investment projects representing different environmental policies are introduced in Section 5.3, and for each one the optimal policy parameters are determined. Main results of a numerical study, in which effects of strategic inventory are examined together with a comparison of simple heuristic rules with optimal strategies are presented in Section 5.4. Section 5.5 discusses the effects of introducing a remanufacturing constraint and the last section provides conclusions and further research possibilities.

5.2 A Dynamic Modeling Environment for Strategic Decision Making

In this section, we propose a simple generic environment for investment decisions for product recovery which integrates used product returns into a product life cycle development. First, we present the required assumptions on such a dynamic situation which are used in the optimization procedures of the next section. Later, we show that these structural properties hold for a specific modeling environment.

A Generic Dynamic Environment

In the following, we consider a demand/return scenario as illustrated in Figure 5.1 which complies with the following assumptions:

A.1 Demand $d(t)$ is assumed to be a deterministic continuously differentiable function of time showing the typically unimodal shape of a product life cycle with its maximum located at $t_d^{max} > 0$. At the end demand must vanish, i.e. $\lim_{t\to\infty} d(t) = 0$.

A.2 Returns $u(t)$ are not available prior to time point $\Delta > 0$ ($u(t) = 0 \, \forall \, t < \Delta$) and otherwise given by an unimodal function of time with $u(t) > 0 \, \forall \, t \geq \Delta$ representing the availability cycle of returns. Further, the return function has a maximum at t_u^{max} and is continuously differentiable for $t > \Delta$. As demand, returns finally vanish, i.e. $\lim_{t\to\infty} u(t) = 0$.

A.3 There exists at most a single intersection of demand and return functions $t_I \geq \max\{t_d^{max}, \Delta\}$ for which it holds $u(t) < d(t)$ if $t < t_I$ and $u(t) > d(t)$ if $t > t_I$. We thus presume that returns do not exceed demand during the growth phase of the product life cycle. If demand always exceeds the return rate there is no intersection, and t_I is set to infinity.

It is intuitively clear that at t_I the product life cycle has already entered its decline phase, i.e. demand is decreasing and intersects the return rate from above. Using Δ and t_I, one can distinguish between three different regions. In Region I, which ends at Δ, there are practically no returns accessible. Demand has to be filled completely by producing new items. Region II shows

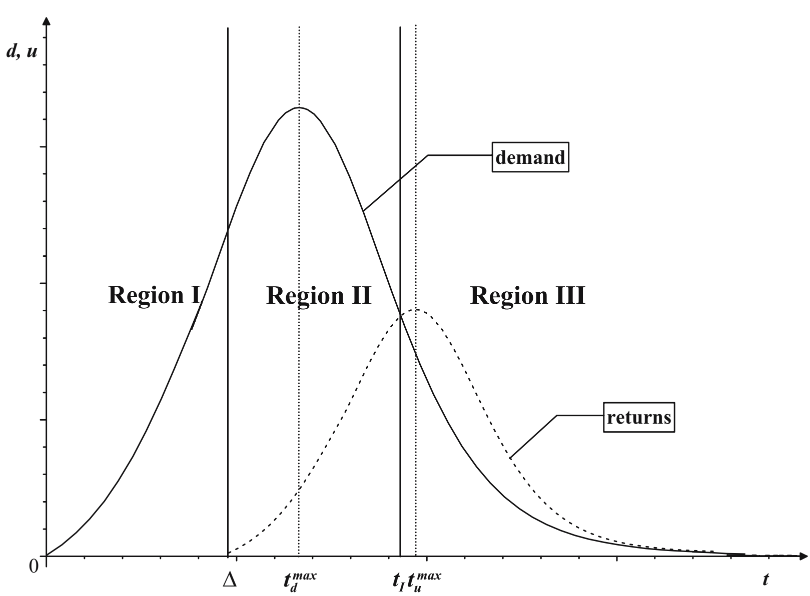

Fig. 5.1. A generic environment complying with assumptions A.1-A.3.

less returns than demand (excess demand) and Region III is characterized by returns exceeding demand (excess returns) and decreasing demand. If the remanufacturing option is available, product returns can be used in both regions (II and III) to satisfy part or all of the demand. Moreover, in the last region it makes no sense to keep more than currently needed returns, because returns remain larger than demand until the end of the product's life cycle. Therefore, excess returns are to be disposed of.

A Specific Model

The demand process can be modeled by using one of the many different new-product diffusion approaches which are quite common in marketing theory (for an overview see Mahajan et al. (1993)), but have also been introduced to operations management, for instance in combined forecasting and inventory models (see Kurawarwala and Matsuo (1996)). The most prominent example being applicable to durable goods has been developed by Bass (1969). According to Mahajan et al. (1993), this model and its revised forms have been proven to have a good predictive capabilities and have been successfully applied in retail service, industrial technology, pharmaceutical and consumer-durable markets. The Bass model uses three parameters, $M > 0$ is the number of potential adopters, representing the cumulative demand as time reaches infinity. $P > 0$ denotes the so-called coefficient of innovation and $Q \geq 0$ is the coefficient of immitation. All parameters are assumed to be known in advance. In the continuous variant of the model, demand at time t is given by

$$d(t) = \frac{MP(P+Q)^2 \cdot e^{-(P+Q)t}}{(P + Qe^{-(P+Q)t})^2} \tag{5.1}$$

This function comprises the demand of two kinds of adopters, namely innovators and immitators. These two groups differ in the reason why they are buying the product. An innovator's decision to buy the product is independent of the decision taken by others and it is purely induced externally by marketing techniques. Thus, a constant fraction P of the remaining potential customers' demand at time t is demanded by this group. This relation leads to a typically positive initial demand rate $d(0) = MP > 0$. In contrast to the first kind of adopters, immitators are more likely to buy the product the higher the proportion of potential customers who have already bought the product, i.e. they are said to be internally influenced. Demand rate $d(t)$ reaches its maximum at the Point of Inflection of the typically S-shaped cumulative adoptions curve located at $t_d^{max} = -\frac{1}{P+Q} \ln(P/Q)$ (for $P < Q$, otherwise it is zero).

Forecasting of product returns becomes complicated because there is uncertainty with respect to time, quantity and quality of returns. Nevertheless, there exists a dependence of the return flow with historic demand data. De Brito and van der Laan (2002) give an overview on recent publications on the forecasting of product returns that exploit this connection by additionally using past return flow properties. Uncertainty with respect to time and quantity of returns can be reduced considerably when using leasing contracts that oblige the customer to give back the product after a fixed period $\Delta > 0$. But it is likely that not all products can be sold in this manner and some returns may not be in a condition to be recovered, only a fraction $F \in (0, 1]$ of previous demands becomes available for remanufacturing. In the simplest case, both parameters can be assumed to be deterministic and known in advance. Thus, the return rate $u(t)$ is given by

$$u(t) = \begin{cases} 0 & \text{for } t < \Delta \\ F \cdot d(t - \Delta) & \text{otherwise} \end{cases} . \tag{5.2}$$

The return rate reaches its maximum at $t_u^{max} = t_d^{max} + \Delta$.

A typical demand/return situation with parameters $P = 0.028$, $Q = 0.25$, and $M = 100000$, $F = 0.4$, and $\Delta = 5$ is depicted in Figure 5.2, both showing growth, maturity and decline phases of the respective life cycle. We now consider properties of the planning situation, exemplified by using demand and return functions as defined above. Proposition 5.1 discusses the relation between them.

Proposition 5.1. *Under demand and return conditions (5.1) and (5.2), there exists at most one intersection point t_I of demand and returns. There is no intersection point, if*

(i) $F < e^{-(P+Q)\Delta}$, *or*

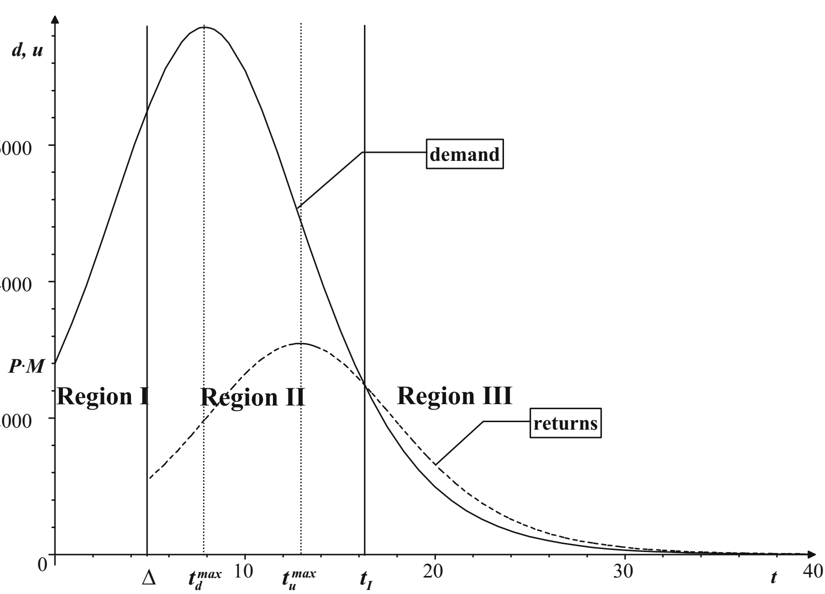

Fig. 5.2. Demands and returns in a specific model.

(ii) $F \cdot P \cdot M > d(\Delta)$.

Proposition 5.1 states that there exist two settings where product life cycle and availability cycle of returns do not intersect. If *(i)* either the time lag Δ and/or the return fraction F are sufficiently small, returns will never exceed demand. In this case, t_I is defined to be infinity. If *(ii)* the time lag Δ is large and/or return fraction F is high enough, then returns at Δ, i.e. $u(\Delta) = Fd(0) = F \cdot P \cdot M$, immediately exceed demand rate $d(\Delta)$. Here, t_I is defined to equal Δ.

5.3 Three Investment Projects

In this section we present three basic investment projects differing with respect to design and technology decisions. After introducing the respective optimal dynamic policies that describe how production, remanufacturing, and disposal decisions evolve over time, individual policy parameters are derived for each investment project. Finally, rules on how to determine the optimal parameters are deduced from minimizing the Net Present Value (NPV) of the respective series of payments within the planning horizon which for analytical convenience is set to infinity.

The following types of operational cash outflows are considered: investment expenditures for production and remanufacturing processes, constant production, remanufacturing and disposal per unit payments as well as out-of-pocket inventory holding costs. Revenues, as well as payments connected with the take back of used products are not taken into account, since both demand as well as returns are assumed to be given and thus not subject of our considerations. Investment expenses include all discounted outlays for acquiring, maintaining, extending, and (possibly) salvage revenues for selling facilities with sufficient capacity. Therefore, restrictions on operative processes are supposed to never become binding.

Regarding the environmental policy of the firm and the existence of a strategic inventory that keeps returns for a later use, the following capital investment projects are considered:

(a) Design for single use

Products are designed in such a way that they can not be remanufactured. Thus, all returns have to be disposed of at costs c_w, which can be positive, if actual payments are necessary, or negative, if there is a positive salvage value. The corresponding investment into the production process at $t = 0$ is $K_p^s > 0$. Direct unit production costs are given by $c_p^s > 0$.

(b) Design for reuse

Products are designed such, that all returned products can serve as perfect substitutes to newly produced items after remanufacturing. This also implies that products might be remanufactured several times, i.e. we assume unlimited (or at least sufficient) durability. See Geyer and Van Wassenhove (2005) for a discussion on the impacts of limited remanufacturability. Investment expenditures for setting up production at $t = 0$ amount to $K_p^r > K_p^s$, producing each item would cost $c_p^r > c_p^s$. Both values exceed the above expenses for single use design because of the additional requirements which are needed for a later remanufacturing. Returning used products can be remanufactured at unit costs $c_r > 0$ after introducing a remanufacturing process at time $t_r \geq \Delta$. This leads to a cash outlay of $K_r > 0$. In order to assure that this investment project constitutes a viable option a benefit must be realized if, instead of simultaneous production of a new item and disposal of the returned one, the latter is merely remanufactured, i.e. there is a positive *direct recovery cost advantage*

$$c_p^r + c_w - c_r > 0. \tag{5.3}$$

Otherwise, no remanufacturing would take place. A possibility to store returns is not considered; all returns arriving before t_r must be disposed of.

(c) Design for reuse with strategic inventory

In addition to (b), it is now possible to keep returns for later use in inventory, e.g. in order to store items at a time where remanufacturing is not yet possible. The inventory level at t is denoted by $y_u(t)$. Out-of-pocket holding costs are assumed to be proportional to the time and quantity of used products on stock. The respective holding cost parameter is denoted by $h_u > 0$. A meaningful solution is assured if it is not advantageous to hold unneeded returned products as opposed to disposing of them, i.e.

$$h_u > \alpha c_w. \tag{5.4}$$

If this were not the case, disposal would not take place, because delaying the disposal of an item saves interests on the expense which would be higher than out-of-pocket costs incurred by holding the item.

5.3.1 Valuation of Investment Project (a) - Design for Single Use

When assuming a single use product, the optimal dynamic policy is obviously to dispose of all returns immediately upon receipt, i.e. $w^*(t) = u(t)$. Product requirements are satisfied by producing new items ($p^*(t) = d(t)$) as depicted in Figure 5.3 for the demand/return scenario known from Figure 5.2. This leads to the following expression for the Net Present Value

$$NPV_a = K_p^s + \int_0^\infty e^{-\alpha t} \left[c_p^s d(t) + c_w u(t) \right] dt. \tag{5.5}$$

NPV_a can be considered as a benchmark against which the financial benefit of the other investment projects have to be compared.

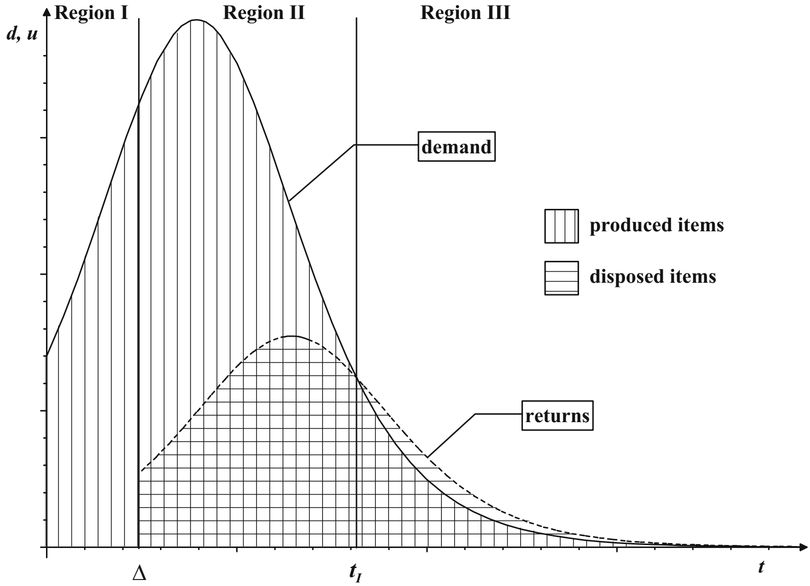

Fig. 5.3. Optimal decisions in investment project (a)

5.3.2 Investment Project (b) - Design for Reuse

Dynamic policy. The remanufacturing option is available after a capital expenditure of K_r at time t_r, subdividing the planning horizon into two parts. Before t_r, only production can be used to satisfy demand. Since returned items cannot be stored, the same policy applies as for investment project (a), i.e.

$$p^*(t) = d(t),\ r^*(t) = 0,\ w^*(t) = u(t) \quad \forall t < t_r.$$

The optimal policy for $t \geq t_r$ is to remanufacture as many units as possible, to produce excess demand (if necessary), and to dispose of remaining returns, yielding

$$p^*(t){=}\max\{d(t) - u(t), 0\},\ r^*(t){=}\min\{d(t), u(t)\},\ w^*(t){=}\max\{u(t) - d(t), 0\} \quad \forall t \geq t_r.$$

The dynamic policy is shown in Figure 5.4.

Optimization of policy parameter. The Net Present Value of investment project (b) NPV_b depends on the investment time t_r, and minimizing it leads to the following non-linear optimization problem

$$\min_{t_r}\ NPV_b = K_p^r + \int_0^{t_r} e^{-\alpha t}\left[c_p^r d(t) + c_w u(t)\right] dt + e^{-\alpha t_r} K_r \tag{5.6}$$

$$+\int_{t_r}^{\infty} e^{-\alpha t}\left[c_p^r \max\{d(t) - u(t), 0\} + c_r \min\{d(t), u(t)\} + c_w \max\{u(t) - d(t), 0\}\right] dt$$

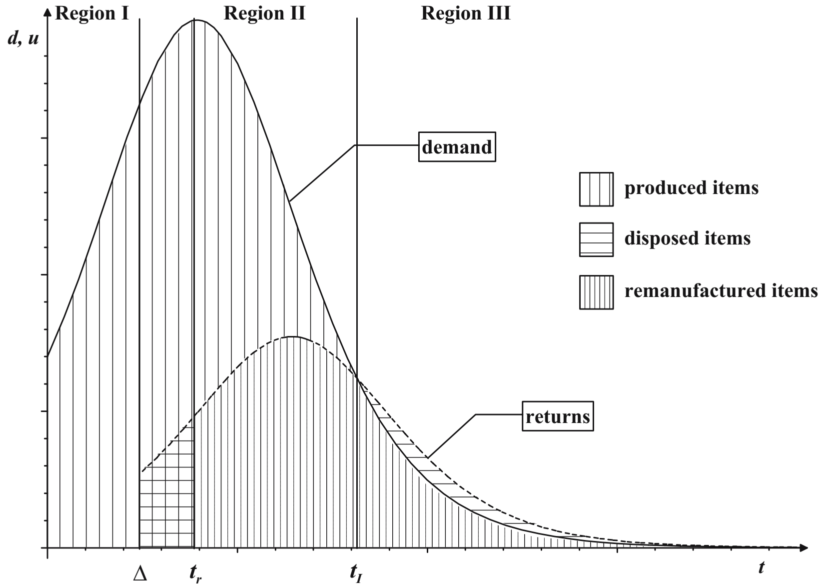

Fig. 5.4. Optimal decisions in investment project (b)

Although the properties of this function depend to a certain extent on the underlying demand and return functions, it can be shown that based on the assumptions A.1-A.3 from Section 5.2, $NPV_b(t_r)$ is strictly decreasing and convex for $t_r < \Delta$, and there exists a point t after which it finally becomes a strictly decreasing and convex function. Depending on the parameters, there is either a single local minimum between Δ and t_I which is followed by a local maximum or the function is decreasing during the whole planning period $[0, \infty)$. The first derivative of $NPV_b(t_r)$ is a continuous function except for $t_r = \Delta$, given a jump discontinuity of the return function $u(t)$ at Δ. Typical shapes of the objective and its first derivative for the demand/return scenario used in Figure 5.2 are depicted in Figure 5.5 for relatively high (i) and low (ii) expenditures for the remanufacturing facility.

When choosing the investment time t_r, a trade-off has to be made between the lower discounted value of investment expenses K_r if the introduction of the remanufacturing process is postponed and a larger realized recovery cost advantage if it is placed earlier. Reconsidering the dynamic environment introduced in Section 5.2 and exemplified in Figure 5.4, it is easy to see that in Region I, where no returns are yet available, changing t_r only affects the discounted value of expenses required for the recovery investment. Therefore, postponing it would always be preferable. Analogously for Region III, as time goes to infinity, the demand rate approaches zero, and with it the potential

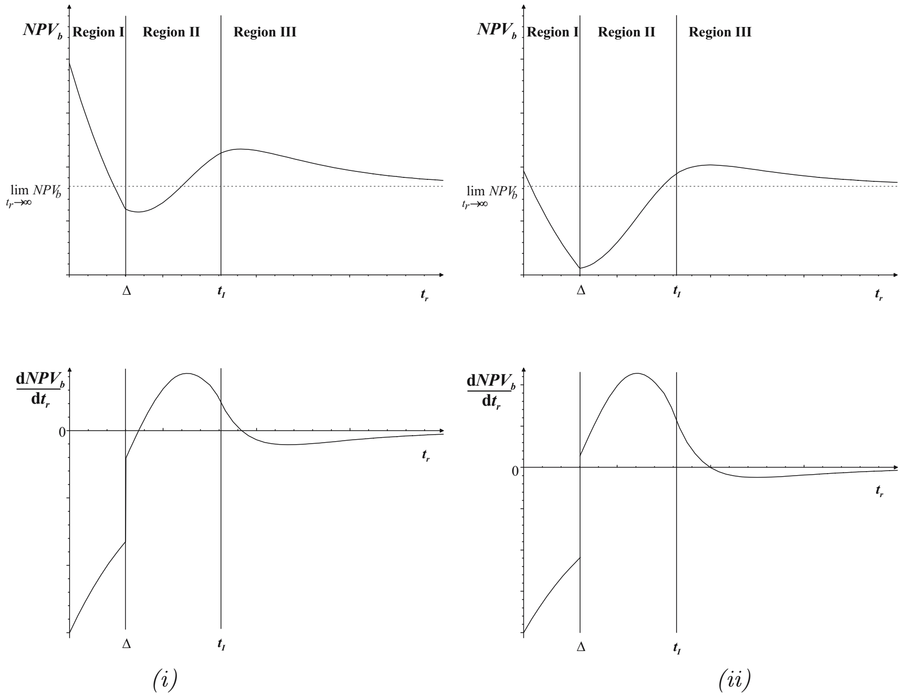

Fig. 5.5. Typical shape of the objective function and its first derivative in a demand/return scenario with *(i)* high and *(ii)* low investment expenditures.

current benefit of remanufacturing diminishes and again, a delay (this time until infinity) of the recovery investment is favorable. This consideration leads to local properties of the optimal investment time t_r^* (given it is finite and thus, located in Region II) as expressed by Proposition 5.2 which is derived by exploiting the first and second order derivatives of the objective (5.6).

Proposition 5.2. *If it exists, a finite investment time* t_r^* *must be located within the half open interval* $[\Delta, \min\{t_u^{max}, t_I\})$*, and one of the following situations must apply*

$$\begin{array}{lll} (i)\ u(t_r^*)(c_p^r + c_w - c_r) = \alpha K_r & \text{and}\ \ \dot{u}(t_r^*) > 0 & \text{if}\ \ t_r^* > \Delta, \\ (ii)\ u(t_r^*)(c_p^r + c_w - c_r) \geq \alpha K_r & & \text{if}\ \ t_r^* = \Delta. \end{array}$$

Proposition 5.2 states that at t_r^*, the current cost advantage of remanufacturing $u(t_r^*)(c_p^r + c_w - c_r)$ must at least earn interests on the investment, i.e. αK_r. If *(i)* equality holds, the remanufacturing rate must increase at t_r^* to start recouping the investment expenses. This is not possible at or later than either the time where returns reach their maximum t_u^{max} or the intersection time of demand and return rate t_I. Because in both cases the remanufacturing rate no longer increases. Thus, investment time t_r^* must lie in an interval given by $[\Delta, \min\{t_u^{max}, t_I\})$. As a special case *(ii)*, at time Δ there may be a jump in the return rate (and consequently in the potential remanufacturing rate) from $u(\Delta^-) = 0$ up to a point $u(\Delta^+) > 0$ where a higher recovery cost advantage may be realized than interests on investment expenses require. This leads to a different condition in Proposition 5.2. Both situations are sketched in Figure 5.5.

Resorting the first order condition in Proposition 5.2 leads to a critical value for the return rate u^{crit}

$$u^{crit} = \frac{\alpha K_r}{c_p^r + c_w - c_r} \tag{5.7}$$

which can be interpreted as a dynamic break-even point. If the investment into a remanufacturing facility takes place at all, then it is placed at a point in time where an increasing return rate surpasses u^{crit} as seen in Figure 5.6.

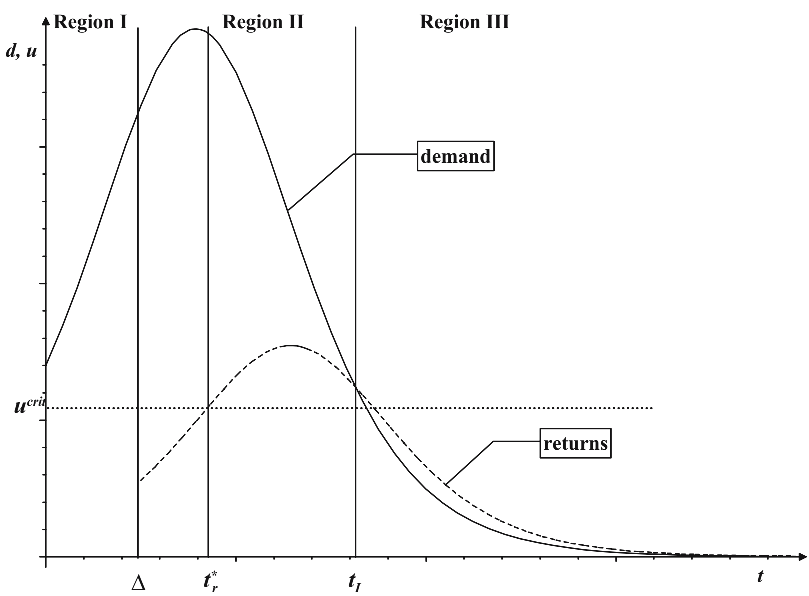

Fig. 5.6. Return rate surpassing u^{crit} at optimal final investment time t_r^*.

Since the remanufacturing rate is limited to the available returns, a finite investment time t_r^* does not exist if the return rate never exceeds the critical return rate u^{crit} in Region II. In such a case, the initial interest rate on the investment exceeds the maximum possible current cost advantage of remanufacturing (to be earned at $\min\{t_u^{max}, t_I\}$) as stated in the following corollary.

Corollary 5.1. *There exists no finite optimal investment time* t_r^*, *if*

$$u\left(\min\{t_u^{max}, t_I\}\right) \leq u^{crit} \;\Leftrightarrow\; u\left(\min\{t_u^{max}, t_I\}\right)\left(c_p^r + c_w - c_r\right) \leq \alpha K_r. \quad (5.8)$$

So far, only local conditions ensuring the possibility to *start* recouping the investment expenditures required for the remanufacturing facility have been considered. Of course, these investments have to be paid off *completely*. If the expenses are rather high, even in the optimal case, the cumulative cost advantage may not be sufficient for amortization. In this case, the optimal investment time is infinity. By comparing the values of the objective function of the finite candidate satisfying Proposition 5.2 with its limit as time approaching infinity, the following sufficient (global) condition for optimality of a finite investment time t_r^* must hold.

Proposition 5.3. *For the optimal investment time* t_r^* *it must hold that the total realized advantage of remanufacturing discounted to* t_r^* *at least equals the expenses needed for setting up the remanufacturing facility*

$$K_r \leq \int_{t_r^*}^{\infty} e^{-\alpha(t-t_r^*)}\left[c_p^r + c_w - c_r\right]\min\{d(t), u(t)\}dt\,. \quad (5.9)$$

Comparison with Investment project (a). There exists no simple rule for determining the best of the two investment projects, but by comparing (optimal) Net Present Values it can be stated that investment project (b) is preferable to (a) if the total discounted net advantage of remanufacturing $A_r^b \geq 0$

$$A_r^b = \int_{t_r^*}^{\infty} e^{-\alpha t}(c_p^r + c_w - c_r)\min\{d(t), u(t)\}dt \;-\; e^{-\alpha t_r^*}K_r \quad (5.10)$$

exceeds the increase of the total discounted expenditures for the production process $D_p^b > 0$

$$D_p^b = \left(K_p^r - K_p^s\right) + \left(c_p^r - c_p^s\right)\int_0^{\infty} e^{-\alpha t}d(t)dt. \quad (5.11)$$

It is easy to see that if there is no finite optimal investment time for investment project (b), then $A_r^b = 0$ holds and (a) should be chosen. If A_r^b is positive, then preferability of the design for reuse depends on the increase of initial investments in the production process compared with a single use production as well as on the increase of direct production costs.

5.3.3 Investment project (c) - Design for Reuse with Strategic Inventory

Dynamic policy. As an extension to investment project (b), returns can be stored in a recoverables inventory. As such, the problem becomes truly dynamic, because the decision to store a returned item influences future possibilities of remanufacturing. In analogy to (b) and given a value for t_r and for the system's state at this time $y_u(t_r)$, the planning horizon can be subdivided into two parts. Prior to t_r, the question arises when to start collecting returns in order to achieve the desired stock. Obviously, it is not useful to dispose of returns during the collection period, because otherwise one could have started gathering later and thus, saved holding costs. Therefore, to each value $y_u(t_r)$ a corresponding time point $t_e \leq t_r$ can be given where disposal stops and all returns are put to stock, being defined by the following equation

$$\int_{t_e}^{t_r} u(s)ds = y_u(t_r). \tag{5.12}$$

Since stock-keeping can start earliest at Δ, i.e. $t_e \geq \Delta$, the maximum possible quantity on stock at t_r is given by

$$y_u(t_r) \leq \int_{\Delta}^{t_r} u(s)ds. \tag{5.13}$$

Thus, optimal decisions in the first part are given by

$$\begin{aligned} &p^*(t) = d(t),\, r^*(t) = 0,\, w^*(t) = u(t) \quad && \forall\, t < t_e, \\ &p^*(t) = d(t),\, r^*(t) = 0,\, w^*(t) = 0 && \forall\, t_e \leq t < t_r. \end{aligned}$$

The optimal solution of the second part is derived by using results of the basic model presented in Chapter 2. First, the recoverables inventory is depleted by filling excess demand $d(t) - u(t)$ from remanufacturing stored returns. This is completed at a time $t_x \geq t_r$, given by

$$\int_{t_r}^{t_x} \left(d(s) - u(s)\right) ds = y_u(t_r). \tag{5.14}$$

Completion time t_x must not be larger than t_I because afterwards returns always exceed the demand rate and carrying an inventory is no longer necessary. Beside (5.13), this gives another condition for $y_u(t_r)$

$$y_u(t_r) \leq \int_{t_r}^{t_I} \left(d(s) - u(s)\right) ds. \tag{5.15}$$

After t_x, the same policy is used as in investment project (b) because it is not useful to build up stock again, yielding the following optimal decisions in the second part

$$p^*(t) = 0, r^*(t) = d(t), w^*(t) = 0 \qquad \forall\, t_r \leq t < t_x,$$
$$p^*(t) = \max\{d(t) - u(t), 0\}, r^*(t) = \min\{d(t), u(t)\}, w^*(t) = \max\{u(t) - d(t), 0\} \qquad \forall t \geq t_x.$$

The dynamic policy in investment project (c) is depicted in Figure 5.7. Of course, this policy requires sufficient capacity and a high flexibility both in the production as well as in remanufacturing process. This is for instance the case if workers that normally are employed to produce new items, can easily be switched to remanufacture used products. In the presence of capacity constraints, a more complex model would be necessary for which only numerical results can be obtained. See Section 5.5.

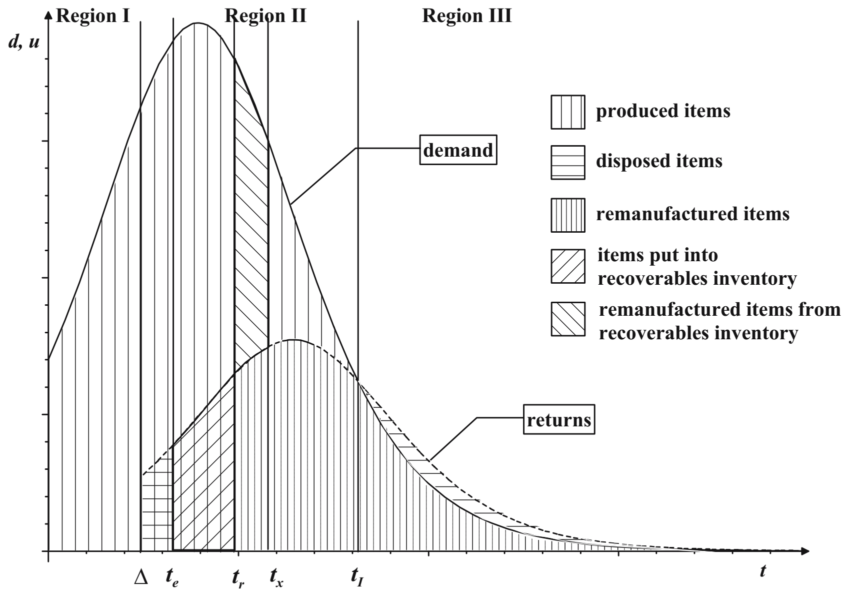

Fig. 5.7. Optimal decisions in investment project (c)

Optimization of policy parameters. In contrast to investment project (b), the optimal solution to policy class (c) not only consists of investment time t_r^*, but additionally the corresponding recoverables stock $y_u^*(t_r^*)$ has to be determined. Equivalently, and probably even more interesting than the actual stock value, the starting time of collecting returns t_e^* will be determined. In the following, we restrict ourselves to find the best finite solution, i.e. $\Delta \leq t_e < t_r < t_I$. Since the option of investing never, i.e. $t_e = t_r = \infty$, still belongs to the set of solution candidates, it has to be considered in order to find the global optimum. Note that we disregard demand and return constraints as given by Proposition 5.1 *(ii)* (see Section 5.2) in the following, because under such circumstances (returns exceed demands immediately after Δ) it is obviously not optimal to hold recoverables for later use.

Restricting ourselves to finite solution candidates, we get the following optimization problem (5.16)-(5.21).

$$\min_{t_e,\ t_r} \ NPV_c = K_p^r + \int_0^{t_e} e^{-\alpha t}\big[c_p^r d(t) + c_w u(t)\big]\ dt \tag{5.16}$$

$$+ \int_{t_e}^{t_r} e^{-\alpha t}\big[c_p^r d(t) + h_u y_u(t)\big]\ dt + e^{-\alpha t_r} K_r + \int_{t_r}^{t_x} e^{-\alpha t}\left[c_r d(t) + h_u y_u(t)\right]\ dt$$

$$+ \int_{t_x}^{\infty} e^{-\alpha t}\big[c_p^r \max\{d(t) - u(t), 0\} + c_r \min\{d(t), u(t)\} + c_w \max\{u(t) - d(t), 0\}\big]\ dt$$

with

$$y_u(t;t_e,t_r,t_x) = \begin{cases} \int_{t_e}^{t} u(s)ds & \text{for } t \in [t_e,t_r] \\ \int_{t_e}^{t_r} u(s)ds - \int_{t_r}^{t} (d(s)-u(s))ds & \text{for } t \in (t_r,t_x] \\ 0 & \text{otherwise} \end{cases} \quad (5.17)$$

and t_x being implicitly defined by a function $f(t_e,t_r,t_x)$

$$t_x : f(t_e,t_r,t_x) = \int_{t_e}^{t_r} u(s)ds - \int_{t_r}^{t_x} (d(s)-u(s))ds = 0 \quad (5.18)$$

subject to the restrictions

$$\Delta \leq t_e, \quad (5.19)$$

$$t_e \leq t_r, \quad (5.20)$$

$$\int_{t_x}^{t_I} (d(s)-u(s))\, ds \geq 0. \quad (5.21)$$

The objective function (5.16) incorporates all payments connected with the optimal policies in each of the above distinguished regions. Function (5.17) is used to determine the inventory level and (5.18) gives an implicit definition of point t_x where the inventory is depleted. Constraint (5.19) ensures continuity of the objective by limiting the admissible set and (5.20) is needed in order to assure a meaningful solution. Restriction (5.21) is equivalent to $t_x \leq t_I$ but technically it is easier to handle. This inequality represents remaining excess demand between t_x and t_I, which must be non-negative.

Due to the quite general assumptions on demand and return functions, objective (5.16) is neither a (quasi-)convex function in each nor in both decision parameters. Moreover, the admissible region is not convex because of our general assumptions on demand and return developments in (5.21). Hence, conditions derived below by using standard methods of non-linear programming are only necessary for optimality. As a consequence, a solution candidate can represent a local minimum, maximum, or a saddle point. Further, there may be several solution candidates for a single problem instance, in order to find the optimal solution the respective objective values need to be compared.

In the following, four different types of solution candidates are distinguished. For ease of representation, a candidate is given by a triplet (t_e,t_r,t_x), bearing in mind that t_x is a function of the other two points.

Proposition 5.4 (Solution Candidates). *If (t_e^*,t_r^*,t_x^*) is an optimal solution to problem (5.16)-(5.21), $t_e^* < t_r^* (< t_x^*)$ must hold, and one of the following four cases applies*

(i) $\Delta < t_e^*$, $t_x^* < t_I$ (interior solution)
(ii) $\Delta = t_e^*$, $t_x^* = t_I$ (complete use of interval $[\Delta, t_I]$)
(iii) $\Delta = t_e^*$, $t_x^* < t_I$ (availability of returns is binding restriction)
(iv) $\Delta < t_e^*$, $t_x^* = t_I$ (availability of excess demand is binding restriction)

Figure 5.8 shows all cases, for which candidates for the optimal solution can be determined. Now, by exploiting first order necessary conditions, properties

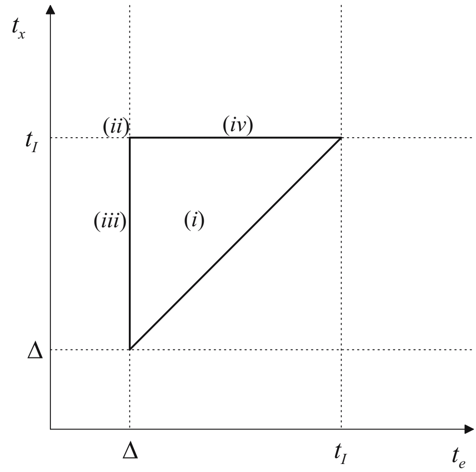

Fig. 5.8. Cases representing candidates for the optimal solution.

of these cases are discussed. Firstly, a general condition regarding the optimal holding time $t_x^* - t_e^*$ can be given.

Proposition 5.5 (Maximal Holding Time). *If (t_e^*, t_r^*, t_x^*) is an optimal solution to problem (5.16)-(5.21), it must hold that $t_x^* - t_e^*$ does not exceed a maximal holding time τ_u, i.e.*

$$t_x^* - t_e^* \leq \frac{1}{\alpha} \ln \left(\frac{\alpha(c_p^r - c_r) + h_u}{-\alpha c_w + h_u} \right) =: \tau_u. \tag{5.22}$$

Maximal holding time τ_u as defined in Proposition 5.5 comprises the same marginal criterion known from the basic model (see Section 2.3) which balances the cost advantage of storing an otherwise disposed item between t_e^* and t_x^* in order to replace production by remanufacturing at t_x^* and the required holding costs.

The following Propositions 5.6-5.9 present results for each of the cases.

Proposition 5.6 (Case (i) - interior solution). *A triplet (t_e, t_r, t_x) with $\Delta < t_e < t_r < t_x < t_I$ is a solution candidate to problem (5.16)-(5.21) of Case (i), if it satisfies the following equations*

$$t_x - t_e = \tau_u, \tag{5.23}$$
$$-e^{-\alpha t_r}[c_p^r - c_r]d(t_r) = h_u \int_{t_r}^{t_x} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_x}[c_p^r - c_r]d(t_r) - \alpha e^{-\alpha t_r} K_r. \tag{5.24}$$

Equations (5.23) and (5.24) follow from setting the first derivatives of the objective (5.16) to zero and can be interpreted as follows. Since $t_x - t_e$ equals the maximal holding time τ_u, the decision maker must be indifferent between (1) disposing of a (marginal) return unit arriving at t_e, and producing a new one to meet demand at t_x or (2) holding this item until t_x when it is remanufactured to serve demand. Next, at t_r one needs to be indifferent between starting the remanufacturing process and thereby realizing the direct cost advantage of remanufacturing immediately or to postpone it which saves interests on the investment expenses. Then, a (marginal) demand $d(t_r)$ is served from producing new items and the thus saved (marginal) return is kept until t_x which results in holding costs and lowers the discounted value of the direct remanufacturing cost advantage.

Using (5.23) together with the definition of the case requires

$$t_I - \Delta > \tau_u, \tag{5.25}$$

which has to be assured first in order to find a Case (i) solution candidate. Then, simultaneously solving (5.18), (5.23) and (5.24) for (t_e, t_r, t_x) yields the candidate, given it exists.

Proposition 5.7 (Case (ii) - complete use of interval $[\Delta, t_I]$). *A triplet (t_e, t_r, t_x) with $\Delta = t_e < t_r < t_x = t_I$ is a solution candidate to problem (5.16)-(5.21) of Case (ii), if the following conditions are satisfied*

$$e^{-\alpha\Delta} c_w d(t_r) \geq h_u \int_{\Delta}^{t_r} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_r}[c_p^r - c_r] d(t_r) + \alpha e^{-\alpha t_r} K_r, \tag{5.26}$$

$$-e^{-\alpha t_r}[c_p^r - c_r] d(t_r) \geq h_u \int_{t_r}^{t_I} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_I}[c_p^r - c_r] d(t_r) - \alpha e^{-\alpha t_r} K_r. \tag{5.27}$$

Inequality (5.26) implies that it would be preferable to put additional returns in stock at t_e for use at t_r by simultaneously lowering t_e and t_r, even at the cost of an earlier investment. But this is not possible because $\Delta = t_e$. Likewise, using (5.27), the value of the objective could be lowered by postponing investment time t_r. This is also forbidden because we would need to increase $t_x = t_I$ which again is not possible.

A Case (ii) candidate may only exist, if

$$t_I - \Delta \leq \tau_u. \tag{5.28}$$

Thus, it is not possible to have a planning situation where we could obtain solution candidates in both Case (i) and Case (ii) simultaneously. If the Case (ii) pre-requirement is fulfilled, from (5.18) one gets a value for t_r which is verified if (5.26) and (5.27) hold.

Proposition 5.8 (Case (iii) - availability of returns is binding restriction). *A triplet (t_e, t_r, t_x) with $\Delta = t_e < t_r < t_x < t_I$ is a solution candidate to problem (5.16)-(5.21) of Case (iii), if the following conditions are satisfied*

$$t_x - \Delta \leq \tau_u, \tag{5.29}$$

$$-e^{-\alpha t_r}[c_p^r - c_r] d(t_r) = h_u \int_{t_r}^{t_x} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_x}[c_p^r - c_r] d(t_r) - \alpha e^{-\alpha t_r} K_r. \tag{5.30}$$

From (5.29) we know that maximal holding time is not yet reached. But, in contrast to Case (ii), from (5.30) we are indifferent regarding the postponement of t_r. Placing it earlier is also not possible, because t_e is fixed to Δ. A Case (iii) candidate is to be found by simultaneously solving equations (5.18) and (5.30) for t_r and t_x by assuming $t_e = \Delta$. The result is a solution candidate if inequality (5.29) is satisfied.

Proposition 5.9 (Case (*iv*) - availability of excess demand is binding restriction). *A triplet* (t_e, t_r, t_x) *with* $\Delta < t_e < t_r < t_x = t_I$ *is a solution candidate to problem (5.16)-(5.21) of Case (iv), if the following conditions are satisfied*

$$t_I - t_e \leq \tau_u, \tag{5.31}$$

$$e^{-\alpha t_e} c_w d(t_r) = h_u \int_{t_e}^{t_r} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_r}[c_p^r - c_r] d(t_r) + \alpha e^{-\alpha t_r} K_r. \tag{5.32}$$

As before, but with t_x fixed to t_I, (5.31) implies to decrease t_e which is not possible without changing t_r. Choosing t_r requires indifference between disposing a (marginal) returned item at t_e or using it to lower t_r which in turn causes an increase in associated holding and interest expenses due to sooner investment but it also replaces production by remanufacturing at t_r. The determination of a Case (*iv*) candidate requires to simultaneously solve equations (5.18) and (5.32) for t_e and t_r assuming $t_x = t_I$. The result is a solution candidate if inequality (5.31) is satisfied.

Algorithm 5.1

Step 1
if $t_I - \Delta > \tau_u$
  then Simultaneously solve (5.23),(5.24), and (5.18) for (t_e, t_r, t_x).
    add result to set of candidates.
  else Set $t_e = \Delta$, $t_x = t_I$ and solve (5.18) for t_r.
    add result to set of candidates, if (5.26) and (5.27) are satisfied.
set $t_e = \Delta$, and simultaneously solve (5.30) and (5.18) for t_r and t_x and
    add result to set of candidates, if (5.29) is satisfied.
set $t_x = t_I$, and simultaneously solve (5.32) and (5.18) for t_e and t_r and
    add result to set of candidates, if (5.31) is satisfied.
add $(t_e, t_r, t_x) = (\infty, \infty, \infty)$ to set of candidates.

Step 2
For all candidates, evaluate the objective value (5.16). Smallest value gives the (global) optimum.

Comparison with Investment project (b). Investment project (c) generally leads to a lower Net Present Value than (b) because it is a generalization of (b) which uses a strategic inventory to maximize the benefit from replacing production by remanufacturing. Another interesting question is how the possibility to hold returns for later use affects investment time t_r. Unfortunately, there is no general answer. A scenario that allows for postponing the investment time is that it no longer has a direct effect on the remanufacturability of returns since these also can be put in stock and remanufactured later. Other aspects make it possible to start remanufacturing earlier, e.g. a higher direct cost advantage of remanufacturing can be realized at t_r because demand is sourced completely from remanufacturing returns. In order to gain more insight into this question we conducted a numerical investigation presented in the next section.

5.4 A Numerical Investigation

The purpose of this study is threefold. First, a pre-test should show that all types of (finite) solution candidates of investment project (c) as presented in Proposition 5.4 are relevant for determining the optimal solution. Further, an assessment of the potential benefit derived from permitting stock-keeping had to be performed and finally, the influence of a strategic recoverables inventory on the investment time t_r was assessed. A second test was used to clarify, under which conditions simple heuristic rules relevant for a practical application perform sufficiently well. Since the effects of changes in the interacting parameters are manifold, we decided to perform the study based on a large number of randomly generated examples.

In this pre-test study we used a demand function according to the Bass model (see Section 5.2) with parameters $P = 0.01$, $Q = 0.3$, $M = 100\,000$. Since we did not have real-live data, the parameters for each one of 30 000 instances were generated from uniform probability distributions over each of the following ranges, partly including extreme values: $Range(F) = [0; 1]$, $Range(\Delta) = [0; 12]$, $Range(\alpha) = [0.05; 0.15]$, $Range(c_p^r) = [4; 5]$, $Range(c_r) = [1; 3]$, $Range(c_w) = [-1; 1]$, $Range(K_r) = [20\,000; 100\,000]$ and $Range(h_u) = [0; 1]$. Since the cash outlays for setting up the production facility were not relevant for our comparison, we set K_p^r equal to zero.

In total, 10 658 examples showed finite optimal solutions for investment project (c) according to Cases (*i*)-(*iv*). Of these, 4 685 (44.0%) belonged to Case (*i*), 90 (0.8%) to Case (*ii*), 5 872 (55.1%) to Case (*iii*), and only 11 (0.1%) examples were Case (*iv*) solutions. Although all cases are relevant for determining the optimal solution these numbers show that in more than half of all considered instances it was optimal to immediately start storing returns. Less than half of the instances showed an interior solution.

Next, by comparing the optimal objective values for investment projects (b) and (c), we found that the benefit from keeping stock averaged to about 2% but the maximum difference was more than 11%, found in a scenario with the following parameters: $F = 0.63$, $\Delta = 0.60$, $\alpha = 0.14$, $c_p^r = 4.23$, $c_r = 1.07$, $c_w = 0.93$, $K_r = 91\,628$, $h_u = 0.16$. Thus, taking into account that aside of operational expenses also investment expenditures where savings amount to a remarkable amount of money.

Regarding the investment time t_r, the results indicate that it is usually (i.e. in 81.7% of all considered examples) postponed due to the strategic inventory, except for cases where the optimal investment time is infinite in investment project (b) but finite in (c) because of the additional benefit from storing returns. This happened for 15.7% of all examples. But 280 instances (2.6%)

including all Cases (i)-(iv) exhibited the opposite behavior. Particularly noteworthy, all examples where the availability of excess demand was a binding restriction (Case (iv)) exhibited an earlier investment time when allowing for stockkeeping. This result was confirmed by another 2 000 instances, which were generated in order to increase the number of Case (iv) solutions, where we changed the ranges for the following parameters: $Range(F) = [0.8; 1]$, $Range(\Delta) = [0; 2]$, $Range(\alpha) = [0.1; 0.15]$, $Range(K_r) = [60\,000; 100\,000]$, $Range(h_u) = [0.1; 0.3]$. Thereby, 161 (11.5%) out of 1396 finite optimal solutions were of Case (iv).

In a second test we compared the performance of four simple heuristic rules, which are described in the following. While the first two neglect the possibility of keeping stock and just try to select an appropriate investment time, H3 and H4 use more or less sophisticated methods to control storing of returns for later use by following the most common investment project (c) cases identified before. Since the change of the optimal investment time when allowing for stock-keeping averaged to just about 1.1 in the pre-test, all heuristics except of the first use as investment time t_r the value which is optimal for investment project (b). The heuristics are now explained in more detail.

H1 The remanufacturing facility is set up as soon as the first returns arrive, i.e. at time $t_r = \Delta$. Thus, there is no need for stock-keeping. This heuristic neglects the decreasing time value of investment expenses due to discounting and is expected to perform well if investment expenditures for the remanufacturing facility are low or discount rate is small.

H2 The optimal solution of investment project (b) is used as a second heuristic. It should lead to good results in circumstances where the Maximal Holding Time is relatively small, e.g. where out-of-pocket holding costs are large.

H3 The third heuristic combines H2 with a simple rule with respect to stock keeping. Returns are kept starting at time Δ and used up after t_r. If there are any items left on stock at t_I, these are disposed of. This heuristic approximately corresponds to a Case (iii) solution candidate and might perform well if out-of-pocket holding costs are low because it disregards a possible limitation of stock-keeping in time.

H4 In contrast to H3, this last heuristic only keeps stock that can be used up before t_I and stored no longer than the Maximal Holding Time. Thus, building up the anticipation inventory may start later than Δ leading to a Case (i) candidate like solution. Since this heuristic considers both, holding time and the time value of investment expenses, its performance is expected to be superior to that of the other heuristics.

The heuristic rules are tested for a collection of classes of randomly generated instances. The variety is based on a single demand and a number of return scenarios, as well as on levels of key parameters, namely direct recovery cost advantage, discount rate, out-of-pocket holding cost rate, and investment expenditures for setting up the remanufacturing facility. For each of these parameters two ranges representing comparably high and low values have been defined. More precisely, we used the following experimental design.

- Demand function is fixed as in the pre-test ($P = 0.01$, $Q = 0.3$, $M = 100\,000$).
- Four return scenarios are used with a low/high return fraction F and small/large duration of use Δ as depicted in Table 5.1. Since after t_I not

Table 5.1. Four considered return scenarios

Scenario	F	*Delta*	t_I	Total returns	Usable returns
I	0.4	3	28.4	40 000	39 982
II	0.4	6	17.7	40 000	33 059
III	0.7	3	15.1	70 000	62 562
IV	0.7	6	15.3	70 000	46 063

  all returns can be used, the last column in Table 5.1 shows the maximal usable number of returns which better expresses the potential benefit from remanufacturing.
- As in the pre-test, initial investment expenditures into the remanufacturing facilities were set to zero ($K_p^r = 0$). The difference $c_p^r - c_r$ is normalized to 1. Objective values are calculated by using $c_p^r = 1$ and $c_r = 0$.
- Since the other parameters (c_p^r and c_r) are fixed, the recovery cost advantage only depends on the disposal cost rate c_w. Two intervals are considered, one with a relative low disposal costs, i.e. $c_w \in (-1, 0)$, and another one with a comparably high cost rate $c_w \in (0, 1)$. In the first case, the direct recovery cost advantage ranges between 0 and 1, and in the second it is in an interval between 1 and 2.
- In order to find a possible impact of discounting, α is assumed to belong to one out of two intervals, being either low ($\alpha \in (0.05, 0.1)$) or high ($\alpha \in (0.1, 0.15)$).
- Holding cost rate h_u was assumed to be taken either out of an interval with a relative low level, i.e. $h_u \in (0, 0.25)$, or from another with comparably high level $h_u \in (0.25, 0.5)$. Hereby it is ensured that only values are used which satisfy assumption (5.4).
- For the investment expenditures K_r we chose the following two ranges: $K_r \in (0, 40\,000)$ and $K_r \in (40\,000, 80\,000)$. The upper border is motivated by the fact that because of discounting, in case of a highest possible recovery cost advantage a and lowest possible discount rate, this number

represents maximal investment expenditures that can be earned from remanufacturing in Scenario III where most usable returns are present.

In total, there have been 64 combinations of scenario and parameter intervals which correspond to a certain setting (4x2x2x2x2 factorial design). Since we fixed some of the parameters, we could suffice with only 200 examples for each setting (12 800 in total) yielding enough material for statistical tests. For each example the relative errors of the heuristic (H1-H4) objective values were calculated. In order to do a fair comparison, only those examples were considered, under which remanufacturing actually would be useful, i.e. where a finite investment project (c) solution is optimal. With other words, it is assumed that the decision maker is able to decide whether remanufacturing actually makes sense or not, and the only concern is ascertaining at which time to invest and whether and when to start collecting returns. By appropriately grouping the examples, a sensitivity analysis of the average performance of the heuristics with respect to return scenarios as well as the examined key parameters was performed.

This analysis was complemented by statistical tests which ensured the comparison of average performance of the heuristics one against each other, where the results originated from the same experiments (matched pairs), but the change of the heuristics performance due to different settings (independent group means), two different types of tests had to be performed. In the first case, a paired t-test was carried out which, because of the large sample sizes, was approximated by a Normal z Test. For comparing independent group means w.r.t. the same heuristic in different settings, a single-sided version of the approximative two groups Normal z Test was performed. Because of the large sample size, the significance was tested on a 99% level. For a detailed treatment of the tests performed see Section 5.7. All comparisons of average errors are significant except where stated otherwise. In spite of this procedure, since it is not possible to generate a general setting which integrates all possible demand/return situations and cash flow parameter combinations, all following statements should rather be seen to express tendencies, which should be verified before applying to an actual situation.

The main results of the study are presented in Tables 5.2-5.6 showing average and maximal relative errors of the heuristic solutions with respect to the objective. Here also the fraction of finite optimal investment project (c) solutions in the respective subset of all experiments can be found, represented by the sign #. This number expresses a relationship between the setting and an average profitability of remanufacturing. The variability of the performances of the heuristics is presented in Section 5.7.

Overall results and scenario comparison. Considering all examples (see last row in Table 5.2), H4 performs best in terms of the average error as previously expected. It also shows the smallest maximal deviation from the optimal solution. Also not unexpectedly, H1 performs worse than all other heuristics in both criteria. Especially when comparing the performance with H2, a large benefit can be obtained only by postponing the investment time. Introducing a simple rule for stock-keeping (H3 instead of H2) yields in average an additional benefit, but under circumstances described below it can also lead to a substantial performance loss.

Table 5.2. Maximal and average NPV deviations of heuristics from optimal solution (in percent) within the considered return scenarios.

		H1		H2		H3		H4	
Scenario	#	avg.	max.	avg.	max.	avg.	max.	avg.	max.
I	27.8	8.9	49.2	3.1	13.6	1.5	13.7	1.2	13.6
II	25.1	5.3	27.7	2.1	8.1	1.0	6.5	0.8	6.5
III	44.4	11.7	59.0	4.2	15.9	2.1	33.3	1.4	14.5
IV	35.5	6.2	33.4	2.8	10.2	1.3	11.9	1.2	8.0
Overall	33.2	8.4	59.0	3.2	15.9	1.6	33.3	1.2	14.5

By comparing the results in the different scenarios and reconsidering the corresponding number of usable returns, it can be seen that the higher this number the higher the potential benefit from remanufacturing. In such cases, specifically the number of instances where remanufacturing takes place increases (e.g. in Scenario III it is higher than in Scenario II). The performance of the heuristic approaches decreases if either the return fraction F increases or returns arrive (relatively) early. Since both cases allow for higher investment expenses, an erroneous determination of investment time has a higher impact on the performance.

Low versus high recovery cost advantage (disposal cost rate). Having high disposal costs or a high recovery cost advantage noticeably increases the number of instances where remanufacturing makes sense as shown in Table 5.3. Performance of the considered heuristics decreases except for H3 where the difference lacks significance, because of its generally large variability of relative deviations from the optimal solution. Another reason why this heuristic does not perform much worse is the positive effect of the recovery cost advantage on the Maximal Holding Time τ_u. Thus, the profitability of using a recoverables inventory increases, but also the possible error when neglecting τ_u decreases. This reasoning also explains why H3 performs poorly compared with H2 if c_w is low.

Low versus high discount rate. Although the effect can hardly be termed large (see Table 5.4), a higher discount rate leads to a decreasing profitability of remanufacturing, but it also lowers the precision of H1-H3. This especially holds for H1, which does not consider time value. The potential benefit of keeping stock increases (H2 performs worse), because it would allow for a further postponement of the investment time which has a stronger effect than would be with a lower discount rate. In contrast to H3 which performs worse when increasing the discount rate, there is probably (although it lacks significance)

Table 5.3. Maximal and average NPV deviations of heuristics from optimal solution (in percent) with relative high and low recovery cost advantages.

		H1		H2		H3*		H4	
c_w	#	avg.	max.	avg.	max.	avg.	max.	avg.	max.
low	15.3	3.8	38.3	1.2	10.1	1.4	33.3	0.3	7.4
high	51.1	9.8	59.0	3.8	15.9	1.6	14.5	1.5	14.5

* Averages are not significantly different.

an improvement for H4, which is due to H4 reacting on a modification in discounting both by changing the investment time and by correctly adapting the modified Maximal Holding Time.

Table 5.4. Maximal and average NPV deviations of heuristics from optimal solution (in percent) with relative high and low discount rates.

		H1		H2		H3		H4*	
α	#	avg.	max.	avg.	max.	avg.	max.	avg.	max.
low	36.4	6.0	40.3	2.9	15.3	1.4	15.0	1.3	13.6
high	30.1	11.4	59.0	3.5	15.9	1.8	33.3	1.1	14.5

* Averages are not significantly different.

Low versus high out-of-pocket holding cost rate. Similarly to a high discount rate, a high out-of-pocket holding cost rate decreases the profitability of remanufacturing, especially of items that have been kept in stock for later use (see Table 5.5). As intuition suggests, it improves the performance of those heuristics which do not keep stock, but surprisingly H3 and H4 also perform better. For H3 this at first glance appears counter-intuitive, but it becomes reasonable because less stock-keeping also results in a smaller deviation from the approximated investment time from the optimum and an improvement of the investment time estimation overcompensating higher holding costs.

Table 5.5. Maximal and average NPV deviations of heuristics from optimal solution (in percent) with relative high and low out-of-pocket holding cost rate.

		H1		H2		H3		H4	
h_u	#	avg.	max.	avg.	max.	avg.	max.	avg.	max.
low	34.9	9.9	59.0	4.0	15.9	1.9	20.3	1.8	14.5
high	31.6	6.8	44.2	2.3	9.0	1.3	33.3	0.6	7.7

Low versus high investment expenditures for remanufacturing facility. High investment expenditures clearly lead to a strong decrease of the number of instances where remanufacturing makes sense (see Table 5.6). All heuristics perform worse, as they do not correctly reflect the potential of changing the investment time due to stock-keeping, which has a larger effect if investment expenditures increase. This can especially be seen from the average error H4 exhibited.

Table 5.6. Maximal and average NPV deviations of heuristics from optimal solution (in percent) with relative high and low investment expenditures.

		H1		H2		H3		H4	
K_r	#	avg.	max.	avg.	max.	avg.	max.	avg.	max.
low	59.0	6.9	46.8	2.9	15.9	1.5	33.3	1.1	13.6
high	7.4	20.7	59.0	5.4	15.4	2.6	14.5	2.4	14.5

Summary of results. Summarizing the results it can be seen that 'dumb' investment time rule H1 should not be used, because there exists a considerable amount of savings to be realized by applying H2. It is also clear that the use of an anticipation stock not only becomes reasonable because of the additional remanufacturing but it also can be used to change (mostly postponing, as seen in the pre-test) the investment time t_r. This yields the biggest effect with low out-of-pocket holding costs or a high recovery cost advantage. The question on whether to apply one of the heuristics (H2-H4) depends (a) on the situation and (b) on the error which the decision maker is willing to accept. Especially, the knowledge of data and computational requirements which H4 necessitates are comparable to those needed for finding the optimal solution.

5.5 Effects of a Limited Remanufacturing Capacity

So far, capacity aspects did not play a role in our discussion. The optimal solution, however, shows a large variability in both production and remanufacturing rates, leading to an optimal dynamic strategy which requires demand to be completely satisfied from remanufacturing between t_r and t_x and therefore to stop production completely. In this section we deal with the changes caused by a time independent capacity constraint for the remanufacturing process $\bar{r}$ when applying the results derived in Chapter 3.

Investment Project (b). In case of investment project (b), the optimal policy for $t \geq t_r$ changes to $r^*(t) = \min\{d(t), u(t), \bar{r}\}$, $p^*(t) = d(t) - r^*(t)$, $w^*(t) = u(t) - r^*(t)$ and thus, remanufacturing would be limited if $\bar{r} < u(\min\{t_u^{\max}, t_I\})$. The results with respect to the optimal investment time as stated in Proposition 5.2 remain valid but existence and pay back conditions have to be adapted. A finite investment time does not exist if the critical return rate u^{crit} as defined in Corollary 5.1 exceeds the constraint, i.e.

$u^{crit} > \bar{r}$. In this case, the interest rate on the investment always exceeds the maximal feasible cost advantage of remanufacturing. The pay-off condition presented in Proposition 5.3 now must take into account the changed optimal policy and thus, the total discounted net advantage of remanufacturing A_r^b from (5.10) changes to

$$\bar{A}_r^b = \int_{t_r^*}^{\infty} e^{-\alpha t}(c_p^r + c_w - c_r) \min\{d(t), u(t), \bar{r}\} dt - e^{-\alpha t_r^*} K_r. \qquad (5.33)$$

Since returns are not stored, profitability of remanufacturing reduces because at times when the constraint is binding, part of returns can not be used and have to be disposed of.

Investment Project (c). A (binding) remanufacturing capacity may change the dynamic policy allowing for stock-keeping in two ways (see Figure 5.9). Regarding the already known stock-keeping motive around investment time

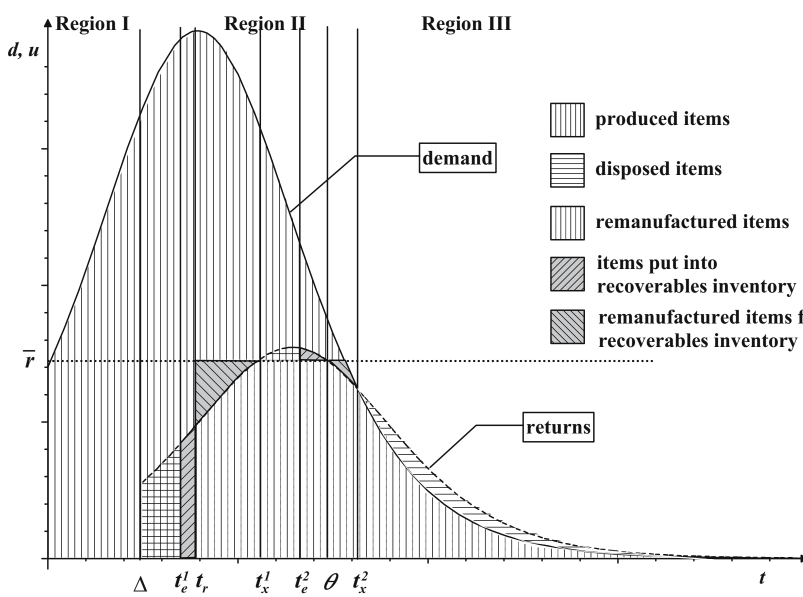

Fig. 5.9. Optimal decisions in investment project (c) when having a limited remanufacturing rate.

t_r (leading to interval $[t_e^1, t_x^1]$), since only part of demand can be satisfied from remanufacturing there is a tendency towards having less returns in inventory at t_r. In an extreme case it might no longer be necessary to have a positive stock at all if the return rate already exceeds the constraint, i.e. $u(t_r) > \bar{r}$. Secondly, by reconsidering the pure effects of a remanufacturing constraint derived in Corollary 3.18, having a positive recoverables stock is beneficial, if there is a time θ in between $t_u^{\max}$ and t_I (where demand is still higher than return rate but return rate is already falling) with $u(\theta) = \bar{r}$. Of course, this requires a situation where $t_u^{\max} < t_I$. Starting at $t_e^2 < \theta$, returns which exceed the remanufacturing constraint can be saved and used up after this time by remanufacturing with maximal remanufacturing rate. This is completed at a time $t_x^2 \leq t_I$. Since depending on the present situation both effects can interact with each other, i.e. returns that are collected before t_r might be used up after θ, a general solution procedure for the constrained problem is considerably more complicated.

5.6 Conclusions and Possible Extensions

In this chapter we used properties of a dynamic situation consisting of a product life cycle and an availability cycle for returns in order to find optimal dynamic policies for three investment projects that differ with respect to the environmental policy. For investment projects that incorporate remanufacturing, the time of the remanufacturing investment proved to be a crucial decision variable because it influences both the time value of the expenses accompanied with it, but the advantage that can be obtained by replacing production of new products by remanufacturing of returns. We have shown that improvements exist if returns can be kept in a strategic inventory. From our experiments it looks that this option should not be implemented as a rule of thumb e.g. by generally keeping all returns. That being said, further research using real life data is required to quantify the performance losses of heuristic approaches.

A number of possibilities exist for further research. It would be interesting to see how robust our results behave when assuming imperfect knowledge on future demand and return developments. A more complex demand/return situation (for instance with a relaunch of the product life cycle) can be solved by using general results for controlling the product recovery system as shown in Chapter 2. But in contrast to the simplified situation introduced above, only numerical results can be derived.

Although capacity aspects have been neglected while performing the analysis they have an impact on profitability of the considered investment projects. In cases, where the remanufacturing process is limited by a capacity constraint the potential advantage of remanufacturing generally decreases but as a side effect, an additional motivation for stock-keeping must be taken into consideration, namely to deal with a situation where the constraint leaves the state of being binding. As an extension, for instance the optimal level of remanufacturing capacity may be obtained. Furthermore, in a strategic model optimal production/remanufacturing capacity expansion and contraction paths can be determined. As in pure production models, building up capacity might be postponed or avoided by using a strategic serviceables inventory. Next, since there are two options to fill the demand, the choice of capacity in the remanufacturing shop influences the required production capacity and vice versa.

We assumed that demand development does not depend on the chosen technology. Marketing aspects like consumer awareness towards environmental conscious products are neglected. Further, competition both on demand as well on the return side are not considered. For instance, the easier products are recoverable, the higher the possibility that other firms will want to participate and to carry out remanufacturing in competition against the OEM. This can be seen for example in the case of refilling toner cartridges for laser printers utilized by Majumder and Groenevelt (2001) to motivate a two-period model which aims to explain how the level of remanufacturability of a product influences competition. An active returns acquisition management as has been described by Guide and Van Wassenhove (2001) can therefore lead to further cost reductions. See Minner and Kiesmüller (2002) for a detailed analysis of return acquisition in a dynamic framework.

5.7 Proofs and Statistical Tests

Proofs

Proof (Proof of Proposition 5.1).
An intersection point $t \geq \Delta$ of demand and return functions requires

$$d(t) = \frac{MP(P+Q)^2 \cdot e^{-(P+Q)t}}{(P+Qe^{-(P+Q)t})^2} = \frac{FMP(P+Q)^2 \cdot e^{-(P+Q)(t-\Delta)}}{(P+Qe^{-(P+Q)(t-\Delta)})^2} = r(t). \quad (5.34)$$

Let $X(t) := e^{-(P+Q)t}$ be a strictly decreasing function of time with co-domain $(0, 1]$ and $Y := e^{(P+Q)\Delta} > 1$ be a constant, then (5.34) can (omitting time indices) be rewritten as

$$\frac{MP(P+Q)^2X}{(P+QX)^2} = \frac{FMP(P+Q)^2XY}{(P+QXY)^2}$$
$$\Leftrightarrow P^2 + 2PQXY + Q^2X^2Y^2 = FYP^2 + 2FPQXY + FQ^2X^2Y.$$

Rearranging terms leads to the following quadratic equation of form $ax^2 + bx + c = 0$ in $X(t)$

$$Q^2Y(Y-F) \cdot X^2 + 2PQY(1-F) \cdot X + P^2(1-FY) = 0. \quad (5.35)$$

A solution to (5.35) is given by $\left(x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)$

$$\begin{aligned} X &= \frac{-2PQY(1-F) \pm \sqrt{(2PQY(1-F))^2 - 4(Q^2Y(Y-F))(P^2(1-FY))}}{2Q^2Y(Y-F)} \\ &= -\frac{P(1-F)}{Q(Y-F)} \pm \frac{\sqrt{4FP^2Q^2Y(1-Y)^2}}{2Q^2Y(Y-F)} \\ &= -\frac{P(1-F)}{Q(Y-F)} \pm \frac{P|1-Y|\sqrt{FY}}{QY(Y-F)} \end{aligned} \quad (5.36)$$

Two positive real roots may exist if the first term in (5.36) is positive, which is not true because $1-F$ as well as $Y-F$ never take on negative values. Since $1-Y<0$ and given X exists, it is

$$X = \frac{P/Y(Y-1)\sqrt{FY}-P(1-F)}{Q(Y-F)}, \tag{5.37}$$

and intersection t_I is uniquely given $t_I = \frac{-\ln(X)}{P+Q}$.
From (5.35) one can see, that a positive root X requires $1-FY<0$. Therefore, if

$$F \leq 1/Y = e^{-(P+Q)\Delta} \tag{5.38}$$

no intersection between the demand and return function can exist and we can deduce using $F<1$, that $r(t) < d(t)\,\forall t$. Note that this condition is sufficient only.
In order to derive a necessary and sufficient condition for the existence of intersection point t_I, it additionally has to be assured that X (as derived from using (5.37)) is located in the co-domain $(0,1]$. Further, for the inverse of X it must hold that the derived time point $t_I = X^{-1}$ exceeds Δ, which is equivalent to $r(\Delta) = Fd(0) < d(\Delta)$.

Proof (Proof of Proposition 5.2).
When inserting $u(t)=0$ for $t<\Delta$ and replacing the max / min operators in (5.6) the objective is (omitting time indices) given by

$$NPV_b = \begin{cases} K_p^r + \int_0^{\Delta} e^{-\alpha t} c_p^r d\,dt + e^{-\alpha t_r} K_r \\ + \int_{\Delta}^{t_I} e^{-\alpha t}[c_p^r(d-u)+c_r u]dt + \int_{t_I}^{\infty} e^{-\alpha t}[c_r d + c_w(u-d)]dt & t_r < \Delta \\ K_p^r + \int_0^{\Delta} e^{-\alpha t} c_p^r d\,dt + \int_{\Delta}^{t_r} e^{-\alpha t}[c_p^r d + c_w u]dt + e^{-\alpha t_r} K_r \\ + \int_{t_r}^{t_I} e^{-\alpha t}[c_p^r(d-u)+c_r u]dt + \int_{t_I}^{\infty} e^{-\alpha t}[c_r d + c_w(u-d)]dt & \Delta \leq t_r < t_I \\ K_p^r + \int_0^{\Delta} e^{-\alpha t} c_p^r d\,dt + \int_{\Delta}^{t_r} e^{-\alpha t}[c_p^r d + c_w u]dt + e^{-\alpha t_r} K_r \\ + \int_{t_r}^{\infty} e^{-\alpha t}[c_r d + c_w(u-d)]dt & t_I \leq t_r \end{cases} \tag{5.39}$$

The first derivative of the objective (5.39) is different for each of the three regions as defined in Section 5.2 and given by

$$\frac{\partial NPV_b}{\partial t_r} = \begin{cases} -\alpha e^{-\alpha t_r} K_r & \text{for } t_r < \Delta & \text{(Region 1)} \\ \text{undefined} & \text{for } t_r = \Delta \\ e^{-\alpha t_r}[(c_p^r + c_w - c_r)u(t_r) - \alpha K_r] & \text{for } \Delta < t_r < t_I & \text{(Region 2)} \\ e^{-\alpha t_r}[(c_p^r + c_w - c_r)d(t_r) - \alpha K_r] & \text{for } t_I \leq t_r & \text{(Region 3).} \end{cases} \tag{5.40}$$

Note that except for $t_r = \Delta$, (5.40) is continuous. Candidates for t_r^* are given by points where (5.40) changes its sign from negative to positive. This is not possible in *R*egion 1, where the derivative is negative.
The time point $t_{r,1} = \Delta$ is a candidate for t_r^* if $\lim_{t_r \to \Delta+0} \frac{\partial NPV_b}{\partial t_r} \geq 0$. This leads to

$$u(\Delta)(c_p^r + c_w - c_r) \geq \alpha K_r. \tag{5.41}$$

Further candidates, $t_{r,2}$ and $t_{r,3}$, are given by setting the first derivative in the two remaining regions to zero, which gives the following conditions

$$u(t_{r,2})(c_p^r + c_w - c_r) = \alpha K_r \text{ for } \Delta < t_{r,2} < t_I \text{ or} \tag{5.42}$$

$$d(t_{r,3})(c_p^r + c_w - c_r) = \alpha K_r \text{ for } t_I \leq t_{r,3}. \tag{5.43}$$

The second derivative of NPV_b is given by

$$\frac{\partial^2 NPV_b}{\partial t_r^2} = \begin{cases} \alpha^2 e^{-\alpha t_r} K_r & \text{for } t_r < \Delta \\ \text{undefined} & \text{for } t_r = \Delta \\ e^{-\alpha t_r}[(c_p^r + c_w - c_r)(\dot{u}(t_r) - \alpha u(t_r)) + \alpha^2 K_r] & \text{for } \Delta < t_r < t_I \\ \text{undefined} & \text{for } t_r = t_I \\ e^{-\alpha t_r}[(c_p^r + c_w - c_r)(\dot{d}(t_r) - \alpha d(t_r)) + \alpha^2 K_r] & \text{for } t_I < t_r. \end{cases} \tag{5.44}$$

Inserting (5.42) and (5.43) for αK_r in the respective part of (5.44), conditions for a local minimum can be derived. $t_{r,2}$ is a local minimum if

$$\left.\frac{\partial^2 NPV_b}{\partial t_r^2}\right|_{t_r = t_{r,2}} = e^{-\alpha t_r}(c_p^r + c_w - c_r)\dot{u}(t_{r,2}) > 0 \Leftrightarrow \dot{u}(t_{r,2}) > 0 \tag{5.45}$$

and thus, $t_{r,2} < t_u^{max}$. Analogously, $t_{r,3}$ is a local minimum if $\dot{d}(t_{r,3}) > 0$. Since demand must decrease for any point $t_{r,3} \geq t_I$ (from assumption), candidate $t_{r,3}$ fails the second order necessary conditions.
From our assumptions about the return function (unimodal) it follows that if inequality (5.41) holds, i.e. $t_{r,1} = \Delta$ is a candidate for an optimal solution, there will be no candidate $t_{r,2}$ and vice versa. Hence, there exists at most a single finite solution, located in an half open interval $[\Delta, \min\{t_u^{max}, t_I\})$.

Note, that this proof must be adapted in order to apply to a situation as defined in Proposition 5.1 *(ii)* and as a result, only $t_{r,1} = \Delta = t_I$ is a solution candidate, because immediately afterwards the remanufacturing rate must decrease.

Proof (Proof of Proposition 5.3).
Since NPV_b decreases for sufficient high t_r, i.e. there exists a time $t_r > t_I$ for which it holds $\forall t > t_r \ : \ d(t) < \frac{\alpha K_r}{c_p + c_w - c_r}$, solution candidate $t_r^\infty = \infty$ (invest never) has to be considered.
In order to find the best alternative, the Net Present Value of the payment stream arising by assuming a relevant finite candidate $\tilde{t}_r \in [\Delta, \min\{t_u^{max}, t_I\})$, i.e. $NPV_b(\tilde{t}_r)$, has to be compared with $NPV_b(t_r^\infty)$, which is given by

$$NPV_b(t_r^\infty) = K_p^r + \int_0^\infty e^{-\alpha t}\left[c_p^r d(t) + c_w u(t)\right] dt. \tag{5.46}$$

This gives an expression of the total discounted advantage of remanufacturing A_r^b

$$\begin{aligned} A_r^b &= NPV_b(t_r^\infty) - NPV_b(\tilde{t}_r) \\ &= -e^{-\alpha\tilde{t}_r} K_r + \int_0^\infty e^{-\alpha t}\left[c_p^r d(t) + c_w u(t)\right] dt \end{aligned}$$

$$-\int_{\tilde{t}_r}^{\infty} e^{-\alpha t}\left[c_p^r \max\{d(t)-u(t),0\} + c_r \min\{d(t),u(t)\} + c_w \max\{u(t)-d(t),0\}\right] dt$$

$$= -e^{-\alpha\tilde{t}_r} K_r + \int_{\tilde{t}_r}^\infty e^{-\alpha t}\left[(c_p^r + c_w - c_r)\min\{d(t),u(t)\}\right] dt \tag{5.47}$$

Therefore, $t_r^* = \tilde{t}_r$ if $A_r^b > 0$, i.e.

$$K_r \leq \int_{\tilde{t}_r}^\infty e^{-\alpha(t-\tilde{t}_r)}\left[c_p^r + c_w - c_r\right]\min\{d(t),u(t)\}dt. \tag{5.48}$$

Otherwise $t_r^* = \infty$.

Proof (Proof of Propositions 5.4 to 5.9).
Propositions 5.4 to 5.9 are results of an optimization approach which is carried through in the following. Since both the objective function NPV_c and constraint (5.21) are in general not convex, in accordance with Sydsæter and Hammond (1995), p 608 the following solution method is used:

(1) Determination of the partial derivatives of the objective function.
(2) Identification of possible solution candidates (Steps 1 and 2 in Sydsæter and Hammond (1995)) using standard methods of Nonlinear Programming. This proofs Proposition 5.4. Exploring a joint property of all valid cases proofs Proposition 5.5 while individual properties confirm results stated in Propositions 5.6-5.9.
(3) Comparison of values of NPV_c at candidate points against each other (Step 3) and with the Net Present Value of investing never $NPV_b(t_r^\infty)$ as given in (5.46). Smallest value is the (global) minimal value of NPV_c (Step 5). As this requires actual data, we will omit this part.

(1) Partial derivatives of the objective function
Subsequently, the partials of t_x with respect to t_e and t_r are needed. This is applied by using the implicit differentiation rules

$$\frac{\partial t_x}{\partial t_e} = -\frac{\frac{\partial f}{\partial t_e}}{\frac{\partial f}{\partial t_x}} = -\frac{u(t_e)}{d(t_x)-u(t_x)} \tag{5.49}$$

$$\frac{\partial t_x}{\partial t_r} = -\frac{\frac{\partial f}{\partial t_r}}{\frac{\partial f}{\partial t_x}} = \frac{d(t_r)}{d(t_x)-u(t_x)} \tag{5.50}$$

The first partial derivative of $NPV_c(t_e, t_r)$ with respect to t_e is given by

$$\frac{\partial NPV_c}{\partial t_e} = e^{-\alpha t_e}\left[c_p^r d(t_e) + c_w u(t_e)\right] - e^{-\alpha t_e}\left[c_p^r d(t_e) + h_u y_u(t_e)\right]$$
$$+ \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_e} dt$$
$$+ \begin{pmatrix} e^{-\alpha t_x}\left[c_r d(t_x) + h_u y_u(t_x)\right] + \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_x} dt \\ -e^{-\alpha t_x}\left[c_p^r(d(t_x) - u(t_x)) + c_r u(t_x)\right] \end{pmatrix} \frac{\partial t_x}{\partial t_e}.$$

Collecting terms and inserting $y_u(t_e) = y_u(t_x) = 0$ yields

$$\frac{\partial NPV_c}{\partial t_e} = e^{-\alpha t_e} c_w u(t_e) + \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_e} dt$$
$$+ \left(-e^{-\alpha t_x}\left[c_p^r - c_r\right](d(t_x) - u(t_x)) + \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_x} dt\right) \frac{\partial t_x}{\partial t_e}.$$

Since the partial of $y_u(t)$ with respect to t_e equals

$$\frac{\partial y_u(t)}{\partial t_e} = \begin{cases} \text{undefined} & \text{for } t = t_e \\ -u(t_e) & \text{for } t \in (t_e, t_x) \\ \text{undefined} & \text{for } t = t_x \\ 0 & \text{otherwise} \end{cases} \tag{5.51}$$

and because of $\frac{\partial y_u(t)}{\partial t_x} = 0$ it follows that

$$\frac{\partial NPV_c}{\partial t_e} = e^{-\alpha t_e} c_w u(t_e) - \int_{t_e}^{t_x} e^{-\alpha t} h_u u(t_e) dt \; - e^{-\alpha t_x}\left[c_p^r - c_r\right](d(t_x) - u(t_x)) \frac{\partial t_x}{\partial t_e}.$$

Replacing $\frac{\partial t_x}{\partial t_e}$ by $-\frac{u(t_e)}{d(t_x) - u(t_x)}$ using (5.49) gives

$$\frac{\partial NPV_c}{\partial t_e} = \left(e^{-\alpha t_e} c_w - h_u \int_{t_e}^{t_x} e^{-\alpha t} dt \; + e^{-\alpha t_x}\left[c_p^r - c_r\right]\right) u(t_e). \tag{5.52}$$

Equation (5.52) can be interpreted as follows. If t_e increases, a marginal return $u(t_e)$ arriving at this time is no longer stored but disposed of, leading to additional unit costs of $e^{-\alpha t_e} c_w u(t_e)$. Since this (marginal) return is not available for later remanufacturing, t_x decreases and thus, costs for producing a (marginal) new product at t_x are caused, given by $e^{-\alpha t_x}\left(c_p^r - c_r\right) u(t_e)$. On the other hand, storing less returns reduces inventory holding costs per unit by $h_u \int_{t_e}^{t_x} e^{-\alpha t} dt \, u(t_e)$.

Evaluating the integral in (5.52) finally yields

$$\frac{\partial NPV_c}{\partial t_e} = \left(e^{-\alpha t_e}\left[c_w - \frac{h_u}{\alpha}\right] + e^{-\alpha t_x}\left[c_p^r - c_r + \frac{h_u}{\alpha}\right]\right) u(t_e). \tag{5.53}$$

The first partial derivative of $NPV_c(t_e, t_r)$ with respect to t_r is given by

$$\frac{\partial NPV_c}{\partial t_r} = e^{-\alpha t_r}\left[c_p^r d(t_r) + h_u y_u(t_r)\right] - \alpha e^{-\alpha t_r} K_r - e^{-\alpha t_r}\left[c_r d(t_r) + h_u y_u(t_r)\right]$$
$$+ \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_r} dt$$
$$+ \begin{pmatrix} e^{-\alpha t_x}\left[c_r d(t_x) + h_u y_u(t_x)\right] + \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_x} dt \\ -e^{-\alpha t_x}\left[c_p^r(d(t_x) - u(t_x)) + c_r u(t_x)\right] \end{pmatrix} \frac{\partial t_x}{\partial t_r}.$$

Collecting terms and inserting $y_u(t_x) = 0$ yields

$$\frac{\partial NPV_c}{\partial t_r} = e^{-\alpha t_r}\left[c_p^r - c_r\right] d(t_r) - \alpha e^{-\alpha t_r} K_r + \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_r}\, dt$$
$$+ \left(-e^{-\alpha t_x}\left[c_p^r - c_r\right](d(t_x) - u(t_x)) + \int_{t_e}^{t_x} e^{-\alpha t} h_u \frac{\partial y_u(t)}{\partial t_x} dt \right) \frac{\partial t_x}{\partial t_r}.$$

Using

$$\frac{\partial y_u(t)}{\partial t_r} = \begin{cases} \text{undefined} & \text{for } t = t_r \\ d(t_r) & \text{for } t \in (t_r, t_x) \\ \text{undefined} & \text{for } t = t_x \\ 0 & \text{otherwise} \end{cases} \tag{5.54}$$

it follows that

$$\frac{\partial NPV_c}{\partial t_r} = e^{-\alpha t_r}\left[c_p^r - c_r\right] d(t_r) - \alpha e^{-\alpha t_r} K_r + \int_{t_r}^{t_x} e^{-\alpha t} h_u d(t_r)\, dt$$
$$-e^{-\alpha t_x}\left[c_p^r - c_r\right](d(t_x) - u(t_x)) \tfrac{\partial t_x}{\partial t_r}.$$

Replacing $\frac{\partial t_x}{\partial t_r}$ by $\frac{d(t_r)}{d(t_x) - u(t_x)}$ from (5.50) yields

$$\frac{\partial NPV_c}{\partial t_r} = \left(\left(e^{-\alpha t_r} - e^{-\alpha t_x}\right)\left[c_p^r - c_r\right] + h_u \int_{t_r}^{t_x} e^{-\alpha t} dt \right) d(t_r) - \alpha e^{-\alpha t_r} K_r. \tag{5.55}$$

A later investment time t_r decreases the Net Present Value of the investment expenses by $\alpha e^{-\alpha t_r} K_r$. A (marginal) demand $d(t_r)$ is no longer satisfied by remanufacturing returns at t_r, which instead are stored for a later use at t_x. Therefore, a cost reduction at t_x by remanufacturing instead of producing faces an increase in costs at t_r. Additional holding costs are caused by storing the (marginal) return.
Continuing in the same manner as above gives

$$\frac{\partial NPV_c}{\partial t_r} = \left(e^{-\alpha t_r} - e^{-\alpha t_x}\right)\left[c_p^r - c_r + \tfrac{h_u}{\alpha}\right] d(t_r) - \alpha e^{-\alpha t_r} K_r. \tag{5.56}$$

(2) Identification of solution candidates
By introducing Lagrange multipliers $\mu_i,\ i = 1, 2, 3$ which are associated with constraints (5.19)-(5.20) the Lagrangian $\mathcal{L}(t_e, t_r, \mu_1, \mu_2, \mu_3)$ is defined as

$$\mathcal{L}(t_e, t_r, \mu_1, \mu_2, \mu_3) = NPV_c(t_e, t_r) - \mu_1(t_e - \Delta) - \mu_2(t_r - t_e)$$
$$-\mu_3 \left(\int_{t_x}^{t_I} (d(s) - u(s)) ds \right). \tag{5.57}$$

The partials of $\mathcal{L}(t_e, t_r, \mu_1, \mu_2, \mu_3)$ have to equal zero:

$$\frac{\partial \mathcal{L}}{\partial t_e} = \frac{\partial NPV_c}{\partial t_e} - \mu_1 + \mu_2 + \mu_3(d(t_x) - u(t_x))\frac{\partial t_x}{\partial t_e}$$
$$= \frac{\partial NPV_c}{\partial t_e} - \mu_1 + \mu_2 - \mu_3 u(t_e) = 0 \quad (5.58)$$
$$\frac{\partial \mathcal{L}}{\partial t_r} = \frac{\partial NPV_c}{\partial t_r} - \mu_2 + \mu_3(d(t_x) - u(t_x))\frac{\partial t_x}{\partial t_r}$$
$$= \frac{\partial NPV_c}{\partial t_r} - \mu_2 + \mu_3 d(t_r) = 0 \quad (5.59)$$

The complementary slackness conditions are given by

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = \Delta - t_e \leq 0,\ \mu_1 \geq 0,\ \mu_1(\Delta - t_e) = 0 \quad (5.60)$$
$$\frac{\partial \mathcal{L}}{\partial \mu_2} = t_e - t_r \leq 0,\ \mu_2 \geq 0,\ \mu_2(t_e - t_r) = 0 \quad (5.61)$$
$$\frac{\partial \mathcal{L}}{\partial \mu_3} = -\int_{t_x}^{t_I} (d(s) - u(s))ds \leq 0,\ \mu_3 \geq 0,\ \mu_3 \left(\int_{t_x}^{t_I} (d(s) - u(s))ds\right) = 0 \quad (5.62)$$

Having three constraints, either active or inactive, in total eight cases have to be distinguished. Cases 1-4 are excluded, cases (i)-(iv) make up solution candidates in Proposition 5.4.

Case 1 $t_e = t_r$, $\Delta = t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds = 0 \Leftrightarrow \Delta = t_e = t_r = t_x = t_I$
This case is only feasible if $\Delta = t_I$ and can therefore be excluded immediately.

Case 2 $t_e = t_r$, $\Delta = t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds > 0 \Leftrightarrow \Delta = t_e = t_r = t_x < t_I$
From $t_x < t_I$ using (5.62), $\mu_3 = 0$ follows. Inserting this value into (5.59) gives

$$\mu_2 = \left.\frac{\partial NPV_c}{\partial t_r}\right|_{t_r = t_x = \Delta} = -\alpha e^{-\alpha\Delta} K_r < 0.$$

This contradicts $\mu_2 \geq 0$.

Case 3 $t_e = t_r$, $\Delta < t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds = 0 \Leftrightarrow \Delta < t_e = t_r = t_x = t_I$
Case 3 does not include an interval with a positive recoverables stock and has already been excluded from the set of solution candidates to investment project (b). See proof to Proposition 5.2.

Case 4 $t_e = t_r$, $\Delta < t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds > 0 \Leftrightarrow \Delta < t_e = t_r = t_x < t_I$
Both, first and third conditions are inactive. Then, $\mu_1 = 0$ and $\mu_3 = 0$ by (5.60) and (5.62), respectively. From (5.58) it follows

$$\mu_2 = -\left.\frac{\partial NPV_c}{\partial t_e}\right|_{t_e = t_r = t_x} = -e^{-\alpha t_r}\left[c_p^r + c_w - c_r\right] u(t_r).$$

Because $c_p^r + c_w - c_r > 0$ from (5.3) and $u(t) > 0\ \forall t \geq \Delta$, μ_2 is negative and thus contradicts $\mu_2 \geq 0$.

Since all cases (1-4) with $t_e = t_r$ are excluded, for the optimal solution (t_e^*, t_r^*) it holds $t_e^* < t_r^*$. The remaining cases constitute the different types of solution candidates as stated in Proposition 5.4. This completes the proof to Proposition 5.4.

For all remaining cases (i)-(iv) holds $t_e^* < t_r^*$ which requires from (5.61) $\mu_2 = 0$. Inserting this value into (5.58) (reconsidering non-negativity of μ_1, μ_3, and $u(t_e^*)$) necessitates $\frac{\partial NPV_c}{\partial t_e} \geq 0$, yielding

$$\tfrac{\partial NPV_c}{\partial t_e} = \left(e^{-\alpha t_e^*}\left[c_w - \tfrac{h_u}{\alpha}\right] + e^{-\alpha t_x^*}\left[c_p^r - c_r + \tfrac{h_u}{\alpha}\right]\right) u(t_e^*) \geq 0 \quad (5.63)$$

Because of $u(t) > 0\ \forall t \geq \Delta$ this is equivalent to

$$e^{-\alpha t_e^*}\left[c_w - \tfrac{h_u}{\alpha}\right] + e^{-\alpha t_x^*}\left[c_p^r - c_r + \tfrac{h_u}{\alpha}\right] \geq 0$$

and solving for $t_x^* - t_e^*$ finally yields

$$t_x^* - t_e^* \leq \frac{1}{\alpha} \ln\left(\frac{\alpha\,(c_p - c_r) + h_u}{-\alpha c_w + h_u}\right) =: \tau_u, \quad (5.64)$$

i.e. the maximum length condition known from Kleber et al. (2002) for the special case of a single product. This completes the proof to Proposition 5.5.

Case (i) $t_e < t_r$, $\Delta < t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds > 0 \Leftrightarrow \Delta < t_e < t_x < t_I$
None of the conditions is active. Thus, $\mu_1 = 0$, $\mu_2 = 0$ as well as $\mu_3 = 0$. Inserted into (5.58), this yields

$$\tfrac{\partial NPV_c}{\partial t_e} = \left(e^{-\alpha t_e} c_w - h_u \int_{t_e}^{t_x} e^{-\alpha t} dt\, + e^{-\alpha t_x}\left[c_p^r - c_r\right]\right) u(t_e) = 0.$$

Proceeding as above (5.63)-(5.64) leads to

$$t_x - t_e = \tau_u. \quad (5.65)$$

Further, inserting Lagrange Multipliers into (5.59) requires

$$\tfrac{\partial NPV_c}{\partial t_r} = \left((e^{-\alpha t_r} - e^{-\alpha t_x})\,[c_p^r - c_r] + h_u \int_{t_r}^{t_x} e^{-\alpha t} dt\right) d(t_r) - \alpha e^{-\alpha t_r} K_r = 0.$$

$$\Leftrightarrow -e^{-\alpha t_r}[c_p^r - c_r]d(t_r) = h_u \int_{t_r}^{t_x} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_x}[c_p^r - c_r]d(t_r) - \alpha e^{-\alpha t_r} K_r \quad (5.66)$$

This completes the proof to Proposition 5.6.

Case (ii) $t_e < t_r$, $\Delta = t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds = 0 \Leftrightarrow \Delta = t_e < t_r < t_x = t_I$
Inserting $\mu_2 = 0$ into (5.59) yields

$$\mu_3 = -\left.\frac{\partial NPV_c}{\partial t_r}\right|_{t_x=t_I} \frac{1}{d(t_r)}$$
$$= \quad -\left(e^{-\alpha t_r} - e^{-\alpha t_I}\right)\left[c_p^r - c_r\right] - h_u \int_{t_r}^{t_I} e^{-\alpha t} dt \, + \alpha e^{-\alpha t_r} K_r \frac{1}{d(t_r)}.$$

$\mu_3 \geq 0$ requires

$$\left(\left(e^{-\alpha t_r} - e^{-\alpha t_I}\right)\left[c_p^r - c_r\right] + h_u \int_{t_r}^{t_I} e^{-\alpha t} dt\right) d(t_r) - \alpha e^{-\alpha t_r} K_r \leq 0$$

$$\Leftrightarrow -e^{-\alpha t_r}[c_p^r - c_r]d(t_r) \geq h_u \int_{t_r}^{t_I} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_I}[c_p^r - c_r]d(t_r) - \alpha e^{-\alpha t_r} K_r. \tag{5.67}$$

Further, inserting μ_3 into (5.58) leads to

$$\mu_1 = \left.\frac{\partial NPV_c}{\partial t_e}\right|_{t_e=\Delta,\ t_x=t_I} + \left.\frac{\partial NPV_c}{\partial t_r}\right|_{t_x=t_I} \frac{u(\Delta)}{d(t_r)}$$
$$= \left(e^{-\alpha\Delta} c_w - h_u \int_{\Delta}^{t_I} e^{-\alpha t} dt \, + e^{-\alpha t_I}\left[c_p^r - c_r\right]\right) u(\Delta)$$
$$+ \left(\left(e^{-\alpha t_r} - e^{-\alpha t_I}\right)\left[c_p^r - c_r\right] + h_u \int_{t_r}^{t_I} e^{-\alpha t} dt\right) u(\Delta) - \alpha e^{-\alpha t_r} K_r \frac{u(\Delta)}{d(t_r)}$$
$$= \quad \left(e^{-\alpha\Delta} c_w - h_u \int_{\Delta}^{t_r} e^{-\alpha t} dt \, + e^{-\alpha t_r}[c_p^r - c_r]\right) u(\Delta) - \alpha e^{-\alpha t_r} K_r \frac{u(\Delta)}{d(t_r)}. \tag{5.68}$$

Since $\mu_1 \geq 0$, (5.68) implies

$$\left(e^{-\alpha\Delta} c_w - h_u \int_{\Delta}^{t_r} e^{-\alpha t} dt \, + e^{-\alpha t_r}[c_p^r - c_r]\right) d(t_r) - \alpha e^{-\alpha t_r} K_r \geq 0$$
$$\Leftrightarrow e^{-\alpha\Delta} c_w d(t_r) \geq h_u \int_{\Delta}^{t_r} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_r}[c_p^r - c_r]d(t_r) + \alpha e^{-\alpha t_r} K_r. \tag{5.69}$$

This completes the proof to Proposition 5.7.

Case (iii) $t_e < t_r$, $\Delta = t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds > 0 \Leftrightarrow \Delta = t_e < t_r < t_x < t_I$
Both second and third conditions are inactive. Then, $\mu_2 = 0$ and $\mu_3 = 0$ from (5.61) and (5.62), respectively. Inserting both values into (5.59) yields

$$\frac{\partial NPV_c}{\partial t_r} = \left(\left(e^{-\alpha t_r} - e^{-\alpha t_x}\right)\left[c_p^r - c_r\right] + h_u \int_{t_r}^{t_x} e^{-\alpha t} dt\right) d(t_r) - \alpha e^{-\alpha t_r} K_r = 0.$$

$$\Leftrightarrow -e^{-\alpha t_r}[c_p^r - c_r]d(t_r) = h_u \int_{t_r}^{t_x} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_x}[c_p^r - c_r]d(t_r) - \alpha e^{-\alpha t_r} K_r. \tag{5.70}$$

This completes the proof to Proposition 5.8.

Case (iv) $t_e < t_r$, $\Delta < t_e$, $\int_{t_x}^{t_I}(d(s) - u(s))ds = 0 \Leftrightarrow \Delta < t_e < t_x = t_I$
Both, first and second conditions are inactive. Then, $\mu_1 = 0$ and $\mu_2 = 0$ from (5.60) and (5.61), respectively. Both values inserted into (5.58) yields the value for μ_3

$$\mu_3 = \frac{1}{u(t_e)} \left.\frac{\partial NPV_c}{\partial t_e}\right|_{t_x=t_I} = e^{-\alpha t_e} c_w - h_u \int_{t_e}^{t_I} e^{-\alpha t} dt + e^{-\alpha t_I} \left[c_p^r - c_r\right]$$

Inserting μ_3 into (5.59) gives

$$\left.\frac{\partial NPV_c}{\partial t_r}\right|_{t_x=t_I} + \frac{d(t_r)}{u(t_e)} \left.\frac{\partial NPV_c}{\partial t_e}\right|_{t_x=t_I} = 0$$
$$\Leftrightarrow e^{-\alpha t_e} c_w d(t_r) = h_u \int_{t_e}^{t_r} e^{-\alpha t} dt\, d(t_r) - e^{-\alpha t_r}[c_p^r - c_r] d(t_r) + \alpha e^{-\alpha t_r} K_r. \tag{5.71}$$

This completes the proof to Proposition 5.9.

Statistical Tests

The objective of the statistical analysis was to test the average performance of the heuristics **(A)** one against each other, where the results originated from the same experiments (matched pairs), but also **(B)** the change of the heuristics performance due to different settings (independent group means), two different types of tests had to be performed. In the first case, a paired t-test was carried out which, because of the large sample sizes, was approximated by a Normal z Test. For comparing independent group means w.r.t. the same heuristic in different settings, a single-sided version of the approximately two groups Normal z Test was performed. Because of the large sample size, the significance level α was set to 1% resulting in a quantile value of about 2.33. The difference of two average values can in both tests be assumed statistically significant if the corresponding absolute test value is larger than 2.33.

(A) The results of the first test are shown in Tables 5.7 and 5.8. As can be seen, the equality of average performances of two different heuristics can be rejected under all considered scenarios.

(B) For use during the second test, the standard deviation of the relative error when applying heuristic solutions instead of the optimal solution was required, which is depicted in Figure 5.9. Using these values, test values of the two groups Normal z Test have been calculated as shown in Figure 5.10. Only in two cases were differences of the compared means not significantly large. These have been highlighted by bold letters.

Table 5.7. Average difference, sample standard deviation and z test values for comparison of the average performance of different heuristics.

	H1 vs. H2			H1 vs. H3			H1 vs. H4		
	mean	std.dev.	z test	mean	std.dev.	z test	mean	std.dev.	z test
Overall	0.052	0.077	44.62	0.069	0.085	52.81	0.072	0.084	55.88
Scenario I	0.058	0.080	21.73	0.074	0.084	26.10	0.077	0.083	27.51
Scenario II	0.033	0.046	20.37	0.044	0.049	25.46	0.045	0.048	26.54
Scenario III	0.075	0.097	28.95	0.096	0.109	33.20	0.103	0.107	36.11
Scenario IV	0.034	0.049	23.40	0.048	0.055	29.48	0.050	0.055	30.86
c_w low	0.027	0.040	20.75	0.024	0.046	16.27	0.035	0.044	24.79
c_w high	0.060	0.083	41.44	0.082	0.089	52.70	0.083	0.090	52.97
α low	0.031	0.047	31.61	0.046	0.056	39.57	0.047	0.056	41.16
α high	0.079	0.095	36.37	0.096	0.103	40.84	0.102	0.102	44.18
h_u low	0.059	0.087	32.11	0.081	0.095	40.35	0.082	0.094	41.04
h_u high	0.045	0.063	32.30	0.055	0.070	35.48	0.062	0.071	39.37
K_r low	0.040	0.060	40.81	0.054	0.070	48.05	0.058	0.069	51.63
K_r high	0.153	0.112	29.66	0.181	0.107	36.96	0.183	0.107	37.27

Table 5.8. Average difference, sample standard deviation and z test values for comparison of the average performance of different heuristics (continued).

	H2 vs. H3			H2 vs. H4			H3 vs. H4		
	mean	std.dev.	z test	mean	std.dev.	z test	mean	std.dev.	z test
Overall	0.016	0.030	35.11	0.020	0.025	52.24	0.004	0.016	14.92
Scenario I	0.016	0.025	18.49	0.019	0.022	25.54	0.003	0.011	8.12
Scenario II	0.011	0.016	19.79	0.012	0.014	24.41	0.001	0.005	7.52
Scenario III	0.021	0.042	18.91	0.028	0.032	33.21	0.007	0.025	10.59
Scenario IV	0.014	0.022	22.23	0.016	0.020	27.40	0.002	0.008	7.36
c_w low	-0.003	0.032	-2.77	0.008	0.011	23.50	0.011	0.030	11.57
c_w high	0.022	0.027	46.33	0.023	0.027	50.10	0.001	0.007	12.10
α low	0.015	0.023	31.83	0.017	0.021	38.01	0.002	0.008	10.51
α high	0.017	0.037	20.71	0.023	0.028	36.87	0.006	0.022	12.08
h_u low	0.022	0.030	34.14	0.023	0.029	37.35	0.001	0.008	6.25
h_u high	0.010	0.028	15.56	0.016	0.019	39.94	0.007	0.022	13.85
K_r low	0.015	0.029	31.09	0.019	0.023	50.03	0.004	0.017	14.12
K_r high	0.028	0.035	17.49	0.030	0.035	18.60	0.002	0.007	6.73

Table 5.9. Standard deviation of the relative error of heuristic solutions from optimal solution.

	H1 std.dev.	H2 std.dev.	H3 std.dev.	H4 std.dev.
Overall	0.089	0.029	0.021	0.016
Scenario I	0.090	0.026	0.018	0.016
Scenario II	0.052	0.017	0.010	0.010
Scenario III	0.111	0.035	0.028	0.019
Scenario IV	0.059	0.023	0.015	0.014
c_w low	0.046	0.013	0.029	0.007
c_w high	0.094	0.029	0.017	0.017
α low	0.061	0.027	0.016	0.015
α high	0.107	0.030	0.025	0.017
h_u low	0.100	0.033	0.020	0.019
h_u high	0.072	0.018	0.022	0.009
K_r low	0.072	0.026	0.020	0.014
K_r high	0.112	0.034	0.025	0.026

Table 5.10. Z test values for comparison of the average performance of the same heuristic between different scenarios/settings.

	H1	H2	H3	H4
Scenario I vs. II	10.09	9.67	8.13	6.26
Scenario I vs. III	-6.75	-9.03	-6.52	-3.02
Scenario I vs. IV	-8.18	-10.39	-8.43	-3.30
Scenario II vs. III	-18.40	-19.75	-14.36	-10.18
Scenario II vs. IV	-3.27	-7.70	-6.74	-6.48
Scenario III vs. IV	-16.26	-12.95	-9.37	-4.29
c_w low vs. high	-26.98	-40.15	**-1.70**	-30.15
α low vs. high	-19.61	-6.16	-5.06	**2.09**
h_u low vs. high	11.85	22.09	9.18	25.69
K_r low vs. high	-26.15	-15.14	-9.28	-10.53

6

Conclusions

We have addressed dynamic issues in product recovery management. A continuous time framework was used which enables to account for external dynamic aspects such as demand seasonality and product life cycle effects but it also enables us to deal with internal cost dynamics. Since we dealt with fairly simple models, optimal policies could be determined that lead to general insights into the optimal behavior of product recovery systems.

In a basic model, there is a direct cost advantage of remanufacturing over production, an anticipation stock is used in order to enhance product recovery opportunities if a period with excess returns is followed by another with excess demand. As a main result, a maximal holding time for returns was derived which balances the direct cost advantage of remanufacturing and holding costs. This period limits the time interval which can be influenced by a current decision and it therefore can be used in order to specify the minimal length of the planning horizon within a rolling planning scheme for an integrated production planning. This approach was also useful in valuing product returns which is of importance for accounting issues.

An extended version of the basic model was used to investigate the smoothing effect of anticipation stocks in capacitated product recovery systems. In the case of a manufacturing constraint the current capacity of the system can be insufficient to service demand and thus, stock-keeping is required for bottleneck situations. Exploiting the holding cost advantage of recoverables, such demands are primarily filled from remanufacturing of additionally collected returns. Since the length of a corresponding collection interval depends on the total 'bottleneck size', it might exceed a time period which is motivated through a recovery cost advantage. In such cases, a serviceables inventory is used at times where excess production capacity is available during the collection interval in order to reduce its length. For determining the respective maximal interval lengths, a trade-off between serviceables and recoverables holding costs is made.

Beside the availability of returns there might also exist other limitations for remanufacturing, e.g. a maximal possible rate. Although in contrast to the previously considered case, the current capacity of the system always suffices to immediately satisfy demand, cost reductions due to exploiting the recovery cost advantage led to a diversity of situations where inventory is used. In a situation where demand is higher than both return rate and remanufacturing constraint, for instance before the constraint leaves the binding state recoverables are collected for later use. Under other circumstances even pre-remanufacturing to a serviceables stock can be favorable.

We also considered two strategic applications of dynamic product recovery. The integration of recovery knowledge acquisition into the basic framework allowed for remanufacturing even under circumstances where there is no immediate advantage over producing new items. Besides the usual results of fully or partly anticipating learning effects when having zero or positive interest rates, another outcome affects the decision when to start the remanufacturing process. This is only possible because there exists another way to fill demand. This decision is accompanied by another motivation to keep stock, namely to postpone the start of remanufacturing. A similar effect results mainly from the combined problem of choosing upon the product's design, a corresponding recovery mode, and the investment time into a remanufacturing facility under product life cycle conditions.

There exist several other recent applications of using the optimal control approach for dynamic product recovery than those already presented in the introductory chapter. Under the assumption of different variants of the same basic product with differing profitability of remanufacturing (e.g. spare parts for the original and an upgraded version of the product), it must be determined for which demand class returns should be used. In extending the basic model presented in Chapter 2, Kleber et al. (2002) showed that in contrast to that case it can be optimal to satisfy a certain demand from production although there still are recoverables available on stock from which demand could be serviced. The underlying trade-off balances differences in the profitability between the available recovery options and holding costs.

Another option that could be considered is not to immediately satisfy all demand but to backlog parts of it. This is preferable at the end of time periods with excess demand which are satisfied when excess returns become available. Here a trade-off between the associated costs (like price cuts required to keep a customer) which also may depend on the time a customer is required to wait and the remanufacturing cost advantage is balanced. An optimal control model analyzing this issue has been presented by Kiesmüller et al. (2000).

Throughout this work it was assumed that demand and returns could not be influenced by the decision maker. An active return acquisition management, however, can considerably enhance profitability by increasing or reducing the availability of returns when needed. This can be implemented through incentives like advertising or buy back prices for returns. An application of optimal control for dynamically setting a buy back price is provided by Minner and Kiesmüller (2002).

List of Symbols

Basic Notation Introduced in Chapters 2 and 3

Indices and general

t	time index
$(.)(t)$	time dependence
$.^*$	optimal values
x^-	left-side limit of x
x^+	right-side limit of x

Parameters of the dynamic environment

$d(t)$	demand rate at time t
$u(t)$	return rate at time t
T	planning horizon

Processes (states and variables)

$y_s(t)$	physical stock serviceables
$y_u(t)$	physical stock recoverables
y_s^0	initial stock serviceables
y_u^0	initial stock recoverables
$p(t)$	production rate
$\bar{p}$	production capacity
$r(t)$	remanufacturing rate
$\bar{r}$	remanufacturing capacity
$w(t)$	disposal rate
w_0	initial disposal quantity

Cost and cash flow parameters

α	discount rate or continuous interest rate
c_p	variable unit production cost

c_r	variable unit remanufacturing cost
c_w	variable unit disposal cost or negative salvage revenue
h_s	holding cost rate serviceables
h_u	holding cost rate recoverables

Important optimization variables

H	Hamiltonian
L	Lagrangian
$\lambda_.$	co-state / adjoint variable
$\mu_.$	Kuhn-Tucker-Multiplier for pure control conditions
$k_.$	Kuhn-Tucker-Multiplier for pure state conditions
$\theta_. \in \Theta_.$	time points where co-state jumps
$\eta_.$	jump heights at time points where co-state is not continuous
$\theta_{e,i}$	entry time of a Case i interval
$\theta_{x,i}$	exit time of a Case i interval

Miscellaneous

Case A $\rightarrow$ Case B Transition from Case A to Case B

Additional Notation in Chapter 4

Learning related parameters

b	learning rate parameter
$c^0_{(r)}$	unit costs of producing (remanufacturing) a first item
$R(t)$	cumulative remanufacturing up to t as a proxy to remanufacturing knowledge
R_0	initial stock of remanufacturing knowledge
$\tilde{R}$	break-even total remanufacturing quantity under zero discount rate conditions

Important optimization variables

$\lambda_R(t)$	value of acquiring knowledge at time t
$\lambda_u^{\max}$	maximum level of λ_u if remanufacturing takes place
θ_l	time after which remanufacturing takes place

Additional Notation in Chapter 5

Parameters of the dynamic environment

M	number of potential adopters
P	coefficient of innovation

Q	coefficient of immitation
Δ	time delay of returns
F	fraction of demanded products being available for remanufacturing
t_d^{max}	time point of maximum demand rate
t_u^{max}	time point of maximum return rate
t_I	intersection point of demand and return functions

Cash flow parameters

K_p^s	initial investment expenditures for single use production
K_p^r	initial investment expenditures for reuse production
K_r	investment expenditures for remanufacturing facility
c_p^s	variable unit production cost at single use production
c_p^r	variable unit production cost at reuse production

Policy parameters

t_r	time of remanufacturing investment
t_e	start time of storing collected returns
t_x	time where all stored returns are used up

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