Advanced Engineering Dynamics

Second Edition

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To my wife
RONA

Contents

Preface

All who model mechanical systems are aware of the unique demands such activity places on conceptual abilities. We must characterize the manner in which numerous individual components interact, and select the appropriate physical laws applicable to each. No one tells us which variables are important. In complicated situations, a multitude of approaches are likely to be available. Thus, an important aspect of training students in this area is developing a level of experience in identifying the salient aspects of a system. They must learn to identify the pathways by which the basic parameters characterizing the inputs may be connected to the desired information representing the solution. In other words, a basic hallmark of the study of engineering dynamics is problem solving.

Some instructors believe that engineers learn by example. If that statement is true, it is only because an engineer is problem-oriented. One of the most prominent features of this textbook is its wealth of examples and homework problems. I have tried to select systems for this purpose that are recognizable as being relevant to engineering applications, yet sufficiently simplified to enable one to focus on the many facets entailed in implementing the associated theoretical concepts. One example of my approach may be found in the development of the method of Lagrangian multipliers. Some texts employ rather simple systems to illustrate this topic. In contrast, Example 3 in Chapter 7 employes Lagrangian multipliers to obtain the equations of motion for a rolling disk in arbitrary motion. In order to emphasize the continuity of the result with earlier work, this solution is then compared to the case of steady precession of a rolling disk, which is solved in Example 8 of Chapter 5 by using the Newton–Euler equations. The general equations of motion are then used in Examples 4 and 5 of Chapter 7 to illustrate implementation of computational strategies for determining the response of nonlinear systems subject to kinematical constraints. I then use the solution to those equations as a basis for discussing the stabilizing effect of the gyroscopic moment in this situation. This approach is typical, as I often use the same system to highlight comparative features of different principles and approaches. The solutions for most examples are discussed in depth, along with qualitative discussions of the results.

Another feature that distinguishes engineering systems from the problems of classical physics is the importance of a careful characterization of their kinematical features. It is in this area that some texts have their greatest shortcoming. In addition to being important as a self-contained task, a kinematical analysis provides the framework on which kinetics principles are constructed. For example, proficiency in kinematics is a prerequisite to selecting a suitable set of generalized coordinates for a complicated system. A large portion of the present work is devoted to a variety of topics in the kinematics of particles and rigid bodies. Particle motion is treated extensively in Chapter 2, from a variety of viewpoints. The fundamental principles for motion relative to a moving reference frame are developed in Chapter 3. That development begins with a thorough treatment of rotation transformations and their implications for the movements of points in a rotating body. Such study helps the student to break the limits imposed by two-dimensional thinking, and proves to be very useful as a general tool for representing motion variables and forces in three dimensions. The ability to formulate angular velocity and angular acceleration is crucial, yet many texts do not present a consistent methodology for such an evaluation. The development of this topic in Chapter 3 is drawn from the undergraduate text I coauthored with Dr. Joseph Genin.† Our experiences showed this approach to be readily accessible and highly versatile. A large part of the formulation of equations of motion using either Newtonian or Lagrangian concepts must be devoted to constraint equations, so much of Chapter 4 is devoted to applications involving interconnected systems, such as linkages, and to rolling systems.

Most engineering students first learn the Newtonian approach to kinetics principles, and that is as it should be. In that way, the student comes to appreciate the relationship between the external force system, the constraint forces, and the changing linear and angular momenta. It is from such study that most individuals develop physical insight. Chapter 5 is devoted to these topics for bodies in general spatial motion. I present a reasonably complete discussion of the evaluation of moments and products of inertia in this chapter, in order to balance the decreased attention that now seems to be devoted to such tasks in the typical undergraduate course. The emphasis of the kinetics analysis in this chapter is on derivation of the translational and rotational equations of motion for a rigid body, using free-body diagrams as a fundamental modeling tool. I also discuss the application of conservation principles, but only in the context of their use as a simple way of solving differential equations of motion in special circumstances.

A potential difficulty in analytical mechanics is the tendency of some students to view the techniques as rote procedures, which hinders their ability to address new situations. I believe that such difficulties arise because of weaknesses in some presentations of generalized coordinates, virtual displacements, and generalized forces. I have found that discussing these concepts from the viewpoint of the configuration space, as well as the physical space, substantially reinforces understanding of the mutual role of constraint forces and constraint conditions. It also demystifies the concept of a virtual displacement. These matters are treated extensively in the beginning of Chapter 6. Such a development greatly assists the student in appreciating the utility of Hamilton's principle and Lagrange's equations, which are derived in the latter part of that chapter.

I have reserved for Chapter 7 advanced topics that one might consider expendable in a course whose schedule is limited. The first of these pertains to modeling systems that are described by constrained generalized coordinates. The usual reason for such a description is the existence of nonholonomic velocity constraints, but I also give attention to using constrained generalized coordinates for holonomic systems, either as a matter of convenience for the kinematical description or because of the presence of Coulomb friction. Section 2 in this chapter, which discusses computational

† J. H. Ginsberg and J. Genin (1984), *Dynamics*, 2nd ed., Wiley, New York.

methods for solving the differential equations of motion governing unconstrained and constrained generalized coordinates, did not appear in the first edition. This is presently an area of considerable research activity, so my focus here is on the development of techniques that may be implemented directly from the basic equations of motion. Section 7.3 treats Hamilton's canonical equations and Routh's method for cyclic generalized coordinates. Both topics are presented for reasons of completeness and for their insight into the results of the earlier developments.

Study of the Gibbs–Appell equations, which appears in Section 7.4, is important in a number of areas. In my view, the significance of the technique lies in the increased freedom afforded by its kinematical approach based on quasicoordinates, rather than its ability to tailor the formulation to avoid consideration of constraint forces. The conceptual ease with which the Gibbs–Appell equations treat systems subject to many nonholonomic constraints seems to make them very attractive for applications in the area of robotics. Of course, one could implement the equivalent formalism of Kane‡ as an alternative to the Gibbs–Appell equations. I have not done so because I have found the latter to be more accessible to students, primarily because of its continuity with the development of Lagrange's equations.

Chapter 7 closes with a general treatment of linearization of equations of motion. Such a topic is usually considered to be part of a conventional course on the vibration of linear systems. I have included this topic here because foreknowledge that one desires to obtain only linearized equations can influence the basic procedures whereby those equations are obtained. In addition, less careful derivations appearing in some vibrations texts have led to misconceptions. Linearization also provides an important tool for studying the dynamic stability of gyroscopic systems. This is the subject of Chapter 8, which treats free rotation and various simplified gyroscopic systems for inertial guidance and control. In addition to their inherent interest, these studies serve to unify the basic principles and procedures developed in the early chapters.

Adequate coverage of all topics in the first seven chapters can be expected to require at least one semester. The first edition of this text originated from notes I developed for a graduate course in the School of Mechanical Engineering at the Georgia Institute of Technology. It is a three-credit, one-quarter course devoted to the bulk of Chapters 1 to 6, as well as the first section in Chapter 7. Interestingly, a large segment of the enrollment typically consists of students who are performing research in other areas, such as computer-aided design and acoustics. I think in part that this is attributable to the depth of coverage in kinematics, wherein background material – such as the Frenet relations, curvilinear coordinates, and rotation transformations – are developed in a manner that brings out applications in other areas. I also attribute this interest to the fact that the study of dynamics provides an excellent framework for developing an engineering approach to problem solving, in which a variety of concepts must be synthesized in a logical manner. The course at Georgia Tech is a

‡ See for example Kane and Levinson (1985), *Dynamics*, McGraw-Hill, New York. It should be noted that the authors fail to mention the close relationship between their approach and the Gibbs–Appell formulation. A good starting point for discussion of these issues is the paper by E. A. Desloge (1987), "Relationship between Kane's Equations and the Gibbs–Appell Equations," *Journal of Guidance, Control, and Dynamics* 10: 120–122.

prerequisite for the first vibrations course, in order to assure that all students have adequate preparation in the modeling of systems. However, a valid argument can be made that concepts in vibrations provide a useful basis for more general studies in dynamics.

My primary focus in revising the original edition was one of clarification, rather than expansion. Based on my experience as an instructor using the text, I have rewritten several sections, in addition to adding and modifying examples. I also have created some new homework problems highlighting the essential topics. The presentations of curvilinear coordinates in Chapter 2 and rotation transformations in Chapter 3 have been revised to enhance understanding of the basic concepts. Chapter 5 has been reorganized to place the derivations of momentum and energy principles for rigid bodies in closer proximity to their application. Chapter 6 contains an expanded discussion of virtual displacements, generalized forces, and constraint forces from the perspective of the configuration space. As noted earlier, the section in Chapter 7 on computational methods for solving the equations of motion is entirely new.

I hope you enjoy this text as much as I have enjoyed writing it. Whenever I use it in my dynamics course, I find there is something else I would like to say, but I hope you find it covers the issues with which you are most concerned.

Acknowledgments

I am most indebted to my colleague, Dr. Aldo Ferri, for giving me the assurance that writing this edition was a worthwhile endeavor. Being able to discuss complicated issues, and to draw on his experience as a gifted teacher of dynamics, was an immeasurable aid. Many thanks are due to my other colleagues in the Acoustics and Dynamics Group of the G. W. Woodruff School of Mechanical Engineering at Georgia Tech for their understanding when I allowed by concentration on writing this book to detract from my interactions with them. I hope that my graduate students, particularly Drs. Hoang Pham and Kuangchung Wu, did not find such distractions to be too much of an impediment to their Ph.D. studies. Mr. Brian Driessen, who is currently a graduate research assistant, was especially helpful in correcting errors in the printing of the first edition. His assistance with some of the subtleties of the computational techniques was particularly welcome. I would also like to recognize the contribution of the students, in numbers too large to list, who attended my dynamics courses. My interactions with them were my primary guide in selecting material to modify for the second edition. Furthermore, their enthusiasm for the course was contagious. Special thanks are due to Dr. Ward O. Winer, the Director of the School of Mechanical Engineering, whose recognition of the significance of this project was an enormous aid to its completion.

I remain indebted to the individuals who helped me with the first edition. Special thanks are due to Dr. Joseph Genin of New Mexico State University, with whom I co-authored undergraduate texts that led to the basic philosophy on which this book is founded. Dr. Allan D. Pierce, now of Boston University, through his intellectual integrity, spurred me to write the first edition. Being able to share the insights of my colleague, Dr. John G. Papastavridis, was helpful in formulating the treatment of

analytical dynamics in the first edition. I was especially fortunate one day in 1993 when Mrs. Florence Padgett, my editor at Cambridge University Press, visited me and expressed her interest in pursuing the second edition. She has been a delight to work with, and her performance is an exemplar for the world of technical publishing.

My love and appreciation for my wife, Rona A. Ginsberg, is immeasurable. In addition to her editorial assistance, her acceptance of my dedication to this project aided enormously in its completion. She was always there to encourage me when I ran into episodes where writing became difficult. Her understanding and devotion were vital to my effort to balance family and professional concerns. The patience of my sons, Mitchell and Daniel, and my daughter-in-law, Tracie, while I was engrossed in this project was always reassuring. The debt I owe my parents, Rae and David Ginsberg, for the sacrifices they made during my education, is enormous. My mother gave me the motivation and drive that has served me so well, especially in completing this book. My father, who died many years before I began to write the first edition, has been my spiritual guide. Although he never completed high school, he was one of the most intellectually inquisitive people I have ever known. He taught me that learning should be a joyful experience to be pursued for its own sake.

CHAPTER 1

Basic Considerations

1.1 Introduction

The subject of dynamics is concerned with the relationship between the forces acting on a physical object and the motion that is produced by the force system. Our concern in this text shall be situations in which the classical laws of physics (i.e., Newtonian mechanics) are applicable. For our purposes, we may consider this to be the case whenever the object of interest is moving much more slowly than the speed of light. In part, this restriction means that we can use the concept of an absolute (i.e. fixed) frame of reference, which will be discussed shortly.

A study of dynamics consists of two phases: kinematics and kinetics. The objective of a kinematical analysis is to describe the motion of the system. It is important to realize that this type of study does not concern itself with what is causing the motion. A kinematical study might be needed to quantify a nontechnical description of the way a system moves, for example, finding the velocity of points on a mechanical linkage. In addition, some features of a kinematical analysis will always arise in a kinetics study, which analyzes the interplay between forces and motion. A primary objective will be the development of procedures for applying kinematics and kinetics principles in a logical and consistent manner, so that one may successfully analyze systems that have novel features. Particular emphasis will be placed on three-dimensional systems, some of which feature phenomena that you might not have encountered in your studies thus far. This is particularly the case if your prior experiences in the area of dynamics were limited to planar motion problems. As we proceed, you might recognize several topics from your earlier courses, both in engineering and in mathematics. Those topics are treated again here because of their importance, and also in order to gain greater understanding and rigor.

1.2 Newton's Laws

A fundamental aspect of the laws presented by Sir Isaac Newton is the concept of an absolute reference frame, which implies that somewhere in the universe there is an object that is stationary. This concept was discarded in modern physics (relativity theory), but the notion of a fixed reference frame introduces negligible errors for slowly moving objects. The corollary of this concept is the dilemma of what object should be considered to be fixed. Once again, negligible errors are usually produced if one considers the sun to be fixed. However, in most engineering situations it is preferable to use the earth as our reference frame. The primary effect of the earth's motion in most cases is to modify the (in vacuo) free-fall acceleration g resulting from the gravitational attraction between an object and the earth. Other than that effect, it is usually permissible to consider the earth to be an absolute reference frame. (A more careful treatment of the effects of the earth's motion will be part of our study of motion relative to a moving reference frame.)

For the purpose of formulating principles and solving problems, the fixed reference frame will be depicted as a set of coordinate axes, such as xyz. It is important to realize that coordinate axes are also often used to represent the directions for the component description of vectorial quantities. The two uses for a coordinate system are not necessarily related. Indeed, we will frequently describe a kinematical quantity relative to a specified frame of reference in terms of its components along the coordinate axes associated with a different frame of reference.

A remarkable feature of Newton's laws is that they address only objects that can be modeled as a single particle, that is, a body whose mass occupies a single point. Bodies of finite dimension are not formally covered by these laws. The three kinematical quantities for a particle with which we are primarily concerned are position, velocity, and acceleration. By definition, a particle occupies only a single point in space. As time evolves, the point occupies a succession of positions. The locus of all positions occupied by the point is its *path*.

The position of a point, as well as the velocity and acceleration, may be described mathematically by giving three independent coordinate values. Such a description is said to be *extrinsic,* because it does not rely on knowledge of the path. In contrast, an *intrinsic* kinematical description defines position, velocity, and acceleration in terms of the properties of the path.

In either case, the *position* of the point may be depicted by a vector arrow extending from some reference location, such as the origin of the fixed frame of reference, to the point of interest. *We shall always use an overbar to denote a vector quantity.* (A more common notation uses boldface to denote a vector, but the overbar has the advantage of being simpler for handwritten work.) Also, we employ subscripts to denote the point of interest and the reference point. For example, $\bar{r}_{P/O}$ denotes the position of point P with respect to point O (the slash, /, may be translated to mean "with respect to"). A typical position vector is shown in Figure 1.1.

The position changes as time goes by, so $\bar{r}_{P/O}$ is a vector function of time. The rules for differentiation of a vector are the same as those for differentiation of a scalar, except that the order of multiplication cannot be changed in treating cross

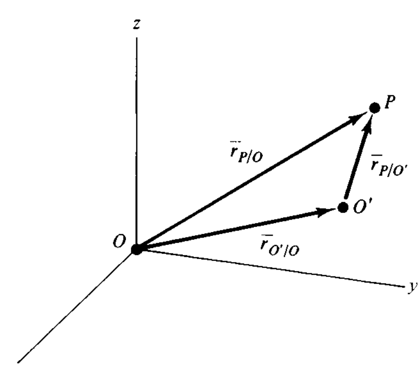

Figure 1.1 Position vectors.

products. The time derivative of the position is called the *velocity.* It is conventional to use one overdot to denote each time derivative. Thus

♦ $$\bar{v} = \frac{d\bar{r}_{P/O}}{dt} = \dot{\bar{r}}_{P/O}. \tag{1.1}$$

Two aspects are notable here. First, no subscripts have been used to denote the velocity vector. If there is any ambiguity as to the point whose velocity is under consideration, the subscript will match that point. It is never necessary to indicate the reference point in the description of velocity, because the velocity is the same as seen from all locations in an absolute frame of reference. This may be proved from Figure 1.1. If points O and O' are both fixed, then the difference in the position of point P relative to these points is constant, that is, $\bar{r}_{O'/O}$ is constant. The derivative of a constant is obviously zero. In Chapter 3 we will treat moving reference frames, in which case we will be interested in the motion relative to that reference frame. Equation (1.1) defines the *absolute* velocity, whereas the velocity seen from a moving reference frame is a relative velocity. The same terminology applies to the description of acceleration, whose definition follows. If it is not specified otherwise, the words velocity and acceleration should be understood to mean the absolute quantities.

Because velocity is a vector, it has an associated magnitude and direction. The magnitude is called the *speed,*

$$v = |\bar{v}|, \tag{1.2}$$

and the direction of $\bar{v}$ tells us the *heading.* Both of these properties are particularly important for formulations using intrinsic (path-related) variables.

Acceleration needs to be considered because it is the only motion parameter that arises in Newton's laws. The basic relation for this quantity is

♦ $$\bar{a} = \dot{\bar{v}} = \ddot{\bar{r}}_{P/O}. \tag{1.3}$$

It might be argued that our senses are accurately attuned to acceleration only when we are experiencing it - it is difficult to judge the acceleration of an object that we are passively observing. Indeed, the time derivative of $\bar{a}$, which is called the *jerk,* occurs primarily in considerations of ride comfort for vehicles.

Newton's laws have been translated in a variety of ways from their original statement in the *Principia* (1687), which was in Latin. We shall use the following version.

First Law
The velocity of a particle can only be changed by the application of a force.

Second Law
The resultant force (that is, the sum of all forces) acting on a particle is proportional to the acceleration of the particle. The factor of proportionality is the mass.

♦ $$\sum \bar{F} = m\bar{a}. \tag{1.4}$$

Third Law
The forces acting on a body result from an interaction with another body such that there is a reactive force (that is, reaction) applied to the other body. The action-

reaction pair consists of forces having the same magnitude, and acting along the same line of action, but having opposite direction.

We realize that the first law is included in the second, but we retain it primarily because it treats systems in static equilibrium without the need to discuss acceleration. The second law is quite familiar, but it must be emphasized that it is a vector relation. Hence, it can be decomposed into as many as three scalar laws, one for each component. The third law is very important to the modeling of systems. The "models" that are created in a kinetics study are free-body diagrams, in which the system is isolated from its surroundings. Careful application of the third law will assist identification of the forces exerted on the body.

The conceptualization of the first and second laws can be traced back to Galileo. Newton's revolutionary idea was the recognition of the third law and its implications for the first and second laws. An interesting aspect of the third law is that it excludes the concept of an inertial force, $-m\bar{a}$, which is usually associated with d'Alembert, because there is no corresponding reactive body. It is for that reason that we shall employ the inertial force concept in Chapter 6 only to develop the principles of analytical mechanics. (D'Alembert employed the artifice of an inertial force as a way of converting dynamic systems into static ones, in order to employ the principle of virtual work. This is the initial step in deriving Lagrange's equations in Chapter 6.)

It is also worth noting that the class of forces described by the third law is limited – any force obeying this law is said to be a *central force.* An example of a noncentral force arises from the interaction between moving electric charges. Such forces have their origin in relativistic effects. Strictly speaking, the study of *classical mechanics* is concerned only with systems that fully satisfy all of Newton's laws. However, many of the principles and techniques are applicable either directly, or with comparatively minor modifications, to relativity theory.

We should note that the acceleration to be employed in Newton's second law is relative to the hypothetical absolute reference frame. However, the same acceleration can also be observed from a variety of moving reference frames, all of which are translating (that is, the reference directions are not rotating) at a constant velocity relative to the fixed reference. Such reference frames are said to be *inertial.* The fact that Newton's laws are valid in any inertial reference frame is the *principle of Galilean invariance,* or the principle of *Newtonian relativity.*

1.3 Systems of Units

Newton's second law brings up the question of the units to be used for describing the force and motion variables. Related to that consideration is the dimensionality of a quantity, which refers to the basic measures that are used to form the quantity. In dynamics, the basic measures are time T, length L, mass M, and force F. The law of dimensional homogeneity requires that these four quantities be consistent with the second law. Thus,

$$F = ML/T^2. \tag{1.5}$$

It is clear from this relation that only three of the four basic measures are independent. Measures for time and length are easily defined, so this leaves the question of whether mass or force is the third independent quantity. When a system of units is defined such that the unit of mass is fundamental, the units are said to be *absolute.* In contrast, when the force unit is fundamental and the mass unit is given by $M = FT^2/L$, the units are said to be *gravitational.* This terminology stems from the relation among the weight w, the mass m, and the free-fall acceleration.

The only system of units to be employed in this text are SI (Standard International), which is a metric MKS (meter-kilogram-second) system, with standardized prefixes for powers of 10 and standard names for derived units. Newton's law of gravitation† states that the magnitude of the attractive force exerted between the earth and a body of mass m is

$$F = \frac{GMm}{r^2}, \tag{1.6}$$

where r is the distance between the centers of mass, G is the universal gravitational constant, and M is the mass of the earth,

$$G = 6.6732(10^{-11}) \text{ m}^3/\text{kg}\cdot\text{s}^2, \qquad M = 5.976(10^{24}) \text{ kg}.$$

The weight w of a body usually refers to the gravitational attraction of the earth when the body is near the earth's surface. When a body near the earth's surface is falling freely in a vacuum, the gravitational attraction is the mass of the body multiplied by the free-fall acceleration, that is,

$$w = mg. \tag{1.7}$$

Matching Eq. (1.6) at the earth's surface to Eq. (1.7) leads to

$$g = \frac{GM}{r_e^2}, \tag{1.8}$$

where r_e is the radius of the earth, $r_e = 6371$ km.

The relationship between g and the gravitational pull of the earth is actually far more complicated than Eq. (1.8). In fact, g depends on the location along the earth's surface. One reason for such variation is the fact that the earth is not perfectly spherical, which means that r_e is not actually constant. Variation in the value of g also arises because the earth is not homogeneous. In addition to these deviations in gravitational force, the value of g is influenced by the motion of the earth, because g is an acceleration measured relative to a noninertial reference frame. (This issue is discussed in Section 3.7.) Consequently, it is not exactly correct to employ Eq. (1.8).

The mass of a particle is constant (assuming no relativistic effects), so defining mass as a fundamental parameter yields an absolute system of units whose definition is not dependent on position; SI units constitute an absolute system. Prior to adoption of the SI standard, many individuals used a gravitational metric system in which grams or kilograms were used to specify the weight of a body. In the SI system, where mass is basic, any body should be described in terms of its mass in kilograms.

† It is implicit to this development that the inertial mass in Newton's second law be the same as the gravitational mass appearing in the law of gravitation. This fundamental assumption, which is known as the *principle of equivalence,* actually is owed to Galileo, who tested the hypothesis with his experiments on various pendulums. Subsequent, more refined, experiments have continued to verify the principle.

Its weight in newtons ($1\ \text{N} \equiv 1\ \text{kg} \times 1\ \text{m/s}^2 = 1\ \text{kg-m/s}^2$) is mg, where g for a standard location on the earth's surface may be taken as

$$g = 9.807\ \text{m/s}^2.$$

The system now known as U.S. customary is another gravitational system. Its basic unit is force, measured in pounds (lb). The body whose weight is defined as a pound must be at a specified location. If that body were moved to a different place then the gravitational force acting on it, and hence the units of force, might be changed. The ambiguity as to a body's weight is one source of confusion in U.S. customary units. Another stems from early usage of the pound as a mass unit. If one also employs a pound-force unit, such that 1 lbf is the weight of a 1-lbm body at the surface of the earth, then $f = ma$ is not satisfied unless the acceleration is measured in multiples of g. This is an unnecessary complication that has been abandoned in most scientific work.

Even when one recognizes that mass is a derived unit in U.S. customary units, the mass unit is complicated by the fact that two length units, feet and inches, are in common use. Practitioners working in U.S. customary units use the standard values

$$g = 32.17\ \text{ft/s}^2 \quad \text{or} \quad g = 386.0\ \text{in./s}^2.$$

Hence, computing the mass as $m = w/g$ will give a value for m that depends on the length unit in use. The *slug* is a standard name for the U.S. customary mass unit, with

$$1\ \text{slug} = 1\ \text{lb}/(1\ \text{ft/s}^2) = 1\ \text{lb-s}^2/\text{ft}.$$

This mass unit is not applicable when inches is the length unit. In order to emphasize this matter, it is preferable for anyone using U.S. customary units to make it a standard practice to give mass in terms of the basic units. For example, a mass might be listed as $5.2\ \text{lb-s}^2/\text{ft}$, or a moment of inertia might be $125\ \text{lb-s}^2\text{-in.}$; SI units avoid all of these ambiguities.

1.4 Vector Calculus

It is assumed here that you are familiar with the basic laws and techniques for the algebra of vectors. Specifically, you should be able to represent a vector in terms of its components and perform calculations such as addition, dot products, and cross products using that component representation. If you feel uncertain about your current proficiency, it is strongly recommended that you take some time to review the appropriate topics. Much of the mathematical software in current use is equipped to carry out these operations.

As mentioned earlier, most of the laws for calculus operations are the same as those for scalar variables. It is only necessary to remember to keep the overbar on vector quantities and to remember that the order in which a cross product is taken is not commutative. In the following, $\bar{A}$ and $\bar{B}$ are time-dependent vector functions, and c and α are scalar functions of time.

Definition of a Derivative

$$\dot{\bar{A}} = \lim_{\Delta t \to 0} \frac{\bar{A}(t+\Delta t) - \bar{A}(t)}{\Delta t}. \tag{1.9}$$

Definite Integration
Let $\bar{B} = \dot{\bar{A}}$. Then

$$\bar{A}(t) = \bar{A}(0) + \int_0^t \bar{B}(\tau)\,d\tau. \tag{1.10}$$

Derivative of a Sum

$$\frac{d}{dt}(\bar{A}+\bar{B}) = \dot{\bar{A}} + \dot{\bar{B}}. \tag{1.11}$$

Derivative of Products

$$\frac{d}{dt}(c\bar{A}) = \dot{c}\bar{A} + c\dot{\bar{A}}, \tag{1.12}$$

$$\frac{d}{dt}(\bar{A}\cdot\bar{B}) = \dot{\bar{A}}\cdot\bar{B} + \bar{A}\cdot\dot{\bar{B}}, \tag{1.13}$$

$$\frac{d}{dt}(\bar{A}\times\bar{B}) = \dot{\bar{A}}\times\bar{B} + \bar{A}\times\dot{\bar{B}}. \tag{1.14}$$

Chain Rule for Differentiation
Let $\bar{A}$ be a function of some parameter α and let α be a function of time. (This obviously means that $\bar{A}$ is an implicit function of time.) Then

$$\dot{\bar{A}} = \frac{d\bar{A}}{d\alpha}\frac{d\alpha}{dt} = \dot{\alpha}\frac{d\bar{A}}{d\alpha}. \tag{1.15}$$

These rules will be used in the next chapter to treat the component representation of vectors with respect to various triads of directions.

1.5 Energy and Momentum

A basic application of the calculus of vectors in dynamics is the derivation of energy and momentum principles, which are integrals of Newton's second law. These integrals represent standard relations between velocity parameters and the properties of the force system. We will derive these laws for particle motion here; the corresponding derivations for a rigid body appear in Chapter 5.

Energy principles are useful when we know how the resultant force varies as a function of the particle's position – in other words, when $\sum \bar{F}(\bar{r})$ is known. The *displacement* of a point is intimately associated with energy principles. The displacement is defined as the change in the position occupied by a point at two instants,

$$\Delta\bar{r} = \bar{r}(t+\Delta t) - \bar{r}(t). \tag{1.16}$$

To obtain a differential displacement $d\bar{r}$, we let Δt become the infinitesimal interval dt. A dot product of Newton's second law with a differential displacement of a particle yields

$$\sum \bar{F}\cdot d\bar{r} = m\bar{a}\cdot d\bar{r}. \tag{1.17}$$

Multiplying and dividing the right side by dt, and then using the definition of velocity and acceleration, leads to

$$\sum \bar{F}\cdot d\bar{r} = m\bar{a}\cdot\frac{d\bar{r}}{dt}\,dt = m\frac{d\bar{v}}{dt}\cdot\bar{v}\,dt = \frac{1}{2}m\frac{d}{dt}(\bar{v}\cdot\bar{v})\,dt$$

$$= \frac{1}{2}m\,d(\bar{v}\cdot\bar{v}) = \frac{1}{2}m\,d(v^2) \tag{1.18}$$

The right side is a perfect differential, and the left side is a function of position only owing to the assumed dependence of the resultant force. Hence, we may integrate the differential relation between the two positions. The evaluation of the integral of the left side must account for the variation of the resultant force as the position changes when the particle moves along its path; this is called a *path integral.* We therefore find that

$$\oint_1^2 \sum \bar{F}\cdot d\bar{r} = \frac{1}{2}m(v_2^2 - v_1^2), \tag{1.19}$$

where the subscripts indicate that the speed should be evaluated at either the initial position 1 or the final position 2. The *kinetic energy* is defined as

♦ $$T = \tfrac{1}{2}mv^2, \tag{1.20}$$

and the path integral is the *work done by the force* in moving the particle,

$$W_{1\to 2} = \oint_1^2 \sum \bar{F}\cdot d\bar{r}. \tag{1.21}$$

The subscript notation for W indicates that the work is done in going from the starting position 1 to the end position 2 along the particle's path. The corresponding form of Eq. (1.19) is

$$T_2 = T_1 + W_{1\to 2}; \tag{1.22}$$

this is known as the *work-energy principle.*

The operation of evaluating the work is depicted in Figure 1.2, where a differential amount of work done by the resultant force in moving the particle may be considered in either of two ways. It is the product of the differential distance the particle moves and the component of the resultant force in the direction of movement, or equivalently, the product of the magnitude of the resultant force and the projection of the

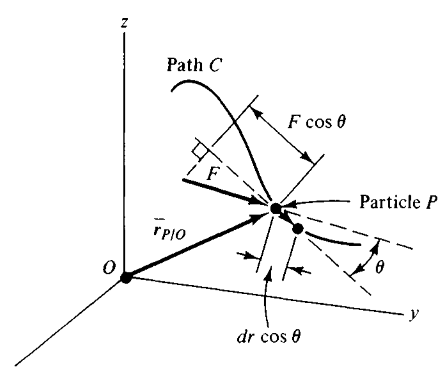

Figure 1.2 Work done by a force.

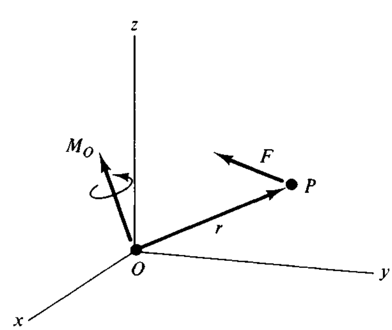

Figure 1.3 Moment of a force.

displacement in the direction of the force. Only in the simple case where the force has a constant component in the direction of the displacement does the work reduce to the simple expression "force multiplied by distance displaced." Evaluation of work is a major part of formulating the work-energy principle. We will find in Chapter 5 that this task is alleviated by introducing the concept of potential energy.

Two momentum principles follow from Newton's second law. The linear impulse-momentum principle is an immediate result when the resultant force is given as a function of time. Because acceleration is the time derivative of velocity, multiplying the second law by dt and integrating over an interval $t_1 \le t \le t_2$ leads to

$$\int_{t_1}^{t_2} \sum \bar{F}\, dt = \int_{t_1}^{t_2} m\bar{a}\, dt = m(\bar{v}_2 - \bar{v}_1). \tag{1.23}$$

The quantity $m\bar{v}$ is the *momentum* of the particle, which we shall denote by the symbol $\bar{P}$. Thus, we have

$$\bar{P} = m\bar{v}, \qquad \bar{P}_2 = \bar{P}_1 + \int_{t_1}^{t_2} \sum \bar{F}\, dt. \tag{1.24}$$

The time integral of the resultant force is the *impulse*. More precise names for the terms appearing in Eq. (1.24) are the linear momentum and linear impulse, because they are associated with the movement of a particle along a (possibly curved) line. The primary utility of the linear impulse-momentum principle is to treat systems excited by impulsive forces - that is, forces that impart a very large acceleration to a body over a very short time interval. Otherwise, the principle is an obvious consequence of knowing the resultant force as a function of time.

The angular momentum principle is associated with the moment the resultant force exerts. Let us evaluate the moment $\bar{M}_O$ about origin O of the fixed reference frame in Figure 1.3. Using the position $\bar{r}$ to form the lever arm leads to

$$\bar{M}_O = \bar{r} \times \sum \bar{F} = \bar{r} \times m\bar{a} = \bar{r} \times m\frac{d\bar{v}}{dt}. \tag{1.25}$$

We now take the time derivative outside the cross product by compensating the equation with an appropriate term to maintain the identity; specifically,

$$\bar{M}_O = \frac{d}{dt}(\bar{r} \times m\bar{v}) - m\frac{d\bar{r}}{dt} \times \bar{v} = \frac{d}{dt}(\bar{r} \times m\bar{v}) - \bar{v} \times m\bar{v}. \tag{1.26}$$

The last term vanishes because the momentum $m\bar{v}$ is parallel to the velocity. The remaining term on the right side of the equation is the time derivative of the moment about origin O of the linear momentum of the particle. We refer to this term as the *angular momentum,* denoted $\bar{H}_O$, because a moment is associated with a rotational tendency. Thus,

◆ $$\bar{H}_O = \bar{r} \times m\bar{v}. \tag{1.27}$$

Substitution of $\bar{H}_O$ onto Eq. (1.26) leads to the derivative form of the *angular impulse-momentum principle,*

◆ $$\bar{M}_O = \frac{d\bar{H}_O}{dt} \equiv \dot{\bar{H}}_O. \tag{1.28}$$

Multiplying the relation by dt and integrating over a time interval $t_1 \le t \le t_2$ leads to

$$(\bar{H}_O)_2 = (\bar{H}_O)_1 + \int_{t_1}^{t_2} \sum \bar{M}_O \, dt, \tag{1.29}$$

where the time integral of the moment is called the *angular impulse* of the resultant force.

Situations where the angular impulse-momentum principle, Eq. (1.29), are needed to study the motion of a particle are few. As is the case for its linear analog, the angular momentum principle might be useful to treat an impulsive force. Also, when the moment of the resultant force about an axis $\bar{e}$ is zero, the principle yields a conservation principle: $\bar{H}_O \cdot \bar{e}$ is constant. The primary utility of the angular momentum principle lies in the application of the derivative form, Eq. (1.27), to a rigid body. We will find in Chapter 5 that the angular momentum of a body is related to the rotation of the body. The study of orbital motion in a gravitational field is another notable application of the principle.

1.6 Brief Biographical Perspective

As we proceed through the various topics, the names of some early scientists and mathematicians will be encountered in a variety of contexts. The magnitude of the contribution of these pioneers cannot be overstated. Indeed, it is a testimonial to their ingenuity that we continue to use so much of their work. A view of the historical relationship between these researchers can greatly enhance our insight. The following is an informal chronological survey of a few individuals who have made key contributions to classical, as opposed to relativistic, physics. More details may be found in the list of references for this chapter.

Galileo, Galilei (1564–1642)
Galileo is best known for experiments on gravity at the leaning tower of Pisa, in his native country, Italy, but there is no conclusive evidence that those experiments actually occurred. From his measurements of the motion of pendulums, which led him to propose the use of a pendulum to provide the time base for a clock, he deduced that gravitational and inertial mass are identical. He refuted Aristotle's ancient statements by observing that the state of motion can only be altered by the presence of other bodies, and that there is no unique inertial reference frame. In astronomy, he developed the astronomical telescope, and used it for many pioneering observations. His last eight years were spent under house arrest for advocating the Copernican view of the solar system, which held that the sun, rather than the earth, is the center of the solar system.

Newton, Sir Isaac (1642–1727)
Newton was a professor of mathematics at Cambridge University whose inspiring work leads many to regard him as one of the two most revolutionary figures in science (Albert Einstein being the other). Newton pursued his studies of physics in England, aware of scientific developments flowering throughout Europe. The foundation for our study of mechanics was laid out by him in *Principia Mathematica Philosophiae Naturalis* (1687). In addition to his basic laws governing the movement

of bodies due to forces, Newton developed the classical law of gravitation, thereby providing a scientific basis for Johannes Kepler's empirical laws for the orbits of the planets. Newton is also generally credited with having developed the mathematical calculus.

Euler, Leonhard (1707-1783)
Euler was Swiss by birth, but his scientific work was done first in Russia, then in Berlin, and finally in Russia when he became a professor of mathematics in St. Petersburg. Some of the greatest contributions to the mechanics of rigid bodies are due to Euler, who was also the most prolific mathematician of his century. He derived many new mathematical principles in order to solve physically meaningful problems. In addition to his contributions to the kinematics and kinetics of rigid bodies, which we shall study in later chapters, he is prominent for his analytical contributions regarding vibration of beams and stability of columns (the Euler buckling load). He also made important contributions to the solution of ordinary differential equations, to geometry and topology, to the theory of functions, and to number theory.

D'Alembert, Jean Le Rond (1717-1783)
As mentioned previously, d'Alembert, who was French, is associated with the notion of an inertial force $-m\bar{a}$. He introduced this artifice as a way of studying dynamic systems by applying principles of statics. We will not use this concept as a technique for formulating problems, but d'Alembert's contribution was crucial to the development by Lagrange of analytical mechanics. Interestingly, d'Alembert and Euler were bitter rivals. In addition to his contributions in the dynamics of particle and rigid body systems, d'Alembert introduced the concept of partial differential equations in order to determine the dynamic response of deformable bodies. D'Alembert was also known as a philosopher and as a music theorist.

Lagrange, Joseph-Louis, Comte de (1736-1813)
The works of Euler and Lagrange are intimately related. Their mutual respect is exemplified by the fact that Lagrange left his native Italy to replace Euler in an important academic position in Berlin upon Euler's recommendation. The fundamental equations of analytical mechanics bear Lagrange's name. These principles, which we will derive in Chapters 6 and 7, employ energy functions and geometrical relations in a viewpoint that emphasizes the way a system behaves as an entity. Lagrange's equations could be considered to be equivalent to Newton's laws, but they are actually more general. Lagrange made important contributions in celestial mechanics and, like Euler, was active in many areas of mathematics, including calculus of variations, the theory of equations, probability theory, and number theory.

Coriolis, Gustave-Gaspard (1792-1843)
In the list we have assembled here, the Frenchman Coriolis is certainly a minor figure. The identification of an acceleration effect attributable to interaction between rotation of a reference frame and movement relative to that reference frame is due to Coriolis. He also made some important contributions to the study of mechanics of materials and collisions of bodies.

Hamilton, Sir William Rowan (1805–1865)
Hamilton, who was Irish, became a professor of astronomy at Trinity College at the age of 22 when he completed his undergraduate work. He is recognized here for development of a unified formulation for classical mechanics. "Hamilton's principle," which draws on concepts from the calculus of variations, contains both the Newtonian and Lagrangian forms of the equations of motion. The generality of his principle is evidenced by the fact that Hamilton reported it in a work on optics. His principle has even been extended to relativistic and quantum mechanics through appropriate redefinitions of the energy functions. Since Newton's laws are axioms, some researchers have argued that Hamilton's principle, rather than Newton's laws, is the foundation for classical mechanics.

Rayleigh, Lord John William Strutt (1842–1919)
Acclaimed as the last of the great British classical physicists, Rayleigh's name appears in the dynamics of particles and rigid bodies in a relatively minor context. He introduced dissipation effects in Lagrange's equations in the same manner that inertia and conservative forces are described, but we now recognize that technique to be of limited validity. Much of Rayleigh's work was in the field of vibrations, which builds on the concepts we develop here. His contributions in acoustics, optics, and electromagnetism are equally significant. He is best known to some as the person who discovered and isolated argon, for which he won the Nobel Prize in 1904.

Perhaps the most remarkable aspect of the foregoing survey is the date of Hamilton's death. Although the basic principles were developed more than a century ago, the versatility of the analytical tools and their level of sophistication have continued to be refined. The subject of mechanics is mature only from a philosophical view; it continues to be an active area for basic and applied research, and many important questions remain to be answered.

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CHAPTER 2

Particle Kinematics

This chapter develops some basic techniques for describing the motion of a particle. Each description is based on a different set of coordinates. The set best suited to a particular situation depends on a variety of factors, but a primary consideration is whether the coordinates naturally fit known aspects of the motion. At the end of this chapter, we will examine situations where more than one of these descriptions may be employed beneficially.

2.1 Path Variables - Intrinsic Coordinates

The idea that the motion of a point should be described in terms of the properties of its path may not seem to be obvious. However, this is precisely how one thinks when using a road map and the speedometer and odometer of an automobile. This type of description is known as *path variables,* or less commonly as *intrinsic coordinates,* because the basic parameters that are considered to change are associated with the properties of the path. The terms *tangent* and *normal components* are also used because those are the primary directions, as we shall see. We assume that the path is known. The most fundamental variable for a specified path is the arclength s along this curve, measured from some starting point to the point of interest. As shown in Figure 2.1, measurement of s requires statement of positive sense along the path. Thus, negative s means that the point has receded, rather than advanced, along its path. It is quite obvious from Figure 2.1 that the position $\bar{r}_{P/O}$ is unambiguously defined by the value of s. Because s changes with time, the position is an implicit function of time, $\bar{r}_{P/O} = \bar{r}(s)$ and $s = s(t)$. It follows that the derivation of formulas for velocity and acceleration will involve chain-rule differentiation. We begin by deriving some basic laws governing the geometry of curves.

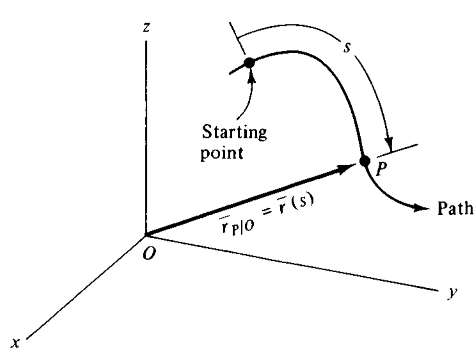


Figure 2.1 Position in path variables.

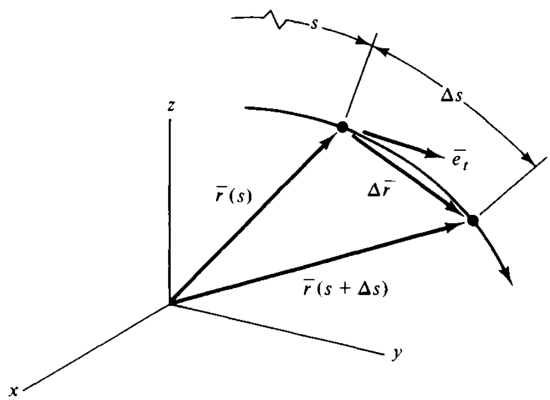


Figure 2.2 Tangent vector.

2.1.1 *Curves in Space - Frenet's Formulas*

Figure 2.2 shows the position vector at two locations that are separated by a small arclength Δs, with Δs becoming ds in the limit of infinitesimal differences. The *displacement* $\Delta\bar{r}$ is the change in the position of the point as it moves from position s to $s+\Delta s$,

$$\Delta\bar{r} = \bar{r}(s+\Delta s) - \bar{r}(s). \tag{2.1}$$

In the limit, the magnitude of $\Delta\bar{r}$ equals ds because a chord progressively approaches the curve. For the same reason, the direction of $\Delta\bar{r}$ approaches tangency to the curve, in the sense of increasing s. This *tangent direction* is defined by the unit tangent vector $\bar{e}_t$, which is the second path-variable parameter. A unit vector has the dimensionless value 1 for magnitude, so

♦ $$\bar{e}_t = \frac{d\bar{r}}{ds}. \tag{2.2}$$

The tangent vector is one of three unit vectors used to describe vectorial quantities in terms of path variables. The second unit vector in the triad is derived by considering the dependence of $\bar{e}_t$ on s. For this evaluation the dot product giving the magnitude of the unit vector $\bar{e}_t$, that is, $\bar{e}_t \cdot \bar{e}_t = 1$, may be differentiated, with the result that

$$\bar{e}_t \cdot \frac{d\bar{e}_t}{ds} = 0. \tag{2.3}$$

In other words, $d\bar{e}_t/ds$ is always perpendicular to $\bar{e}_t$. The *normal direction,* whose unit vector is $\bar{e}_n$, is defined to be parallel to this derivative. Because parallelism of two vectors corresponds to their proportionality, this definition may be written as

♦ $$\bar{e}_n = \rho \frac{d\bar{e}_t}{ds}. \tag{2.4}$$

Because $\bar{e}_n$ is a dimensionless unit vector, the factor of proportionality, ρ, may be found from

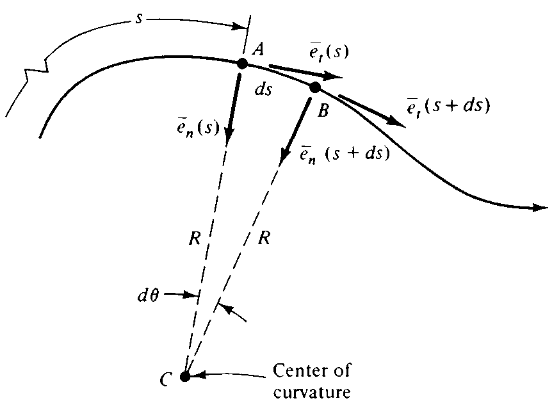
$$\frac{1}{\rho} = \left| \frac{d\bar{e}_t}{ds} \right|. \tag{2.5}$$

Dimensional consistency of Eq. (2.5) requires that ρ be a length parameter; it is the *radius of curvature,* as we will soon demonstrate for a planar path.

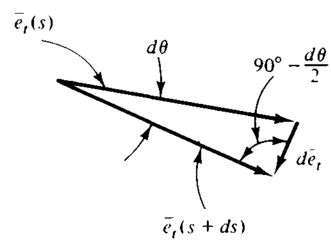
The tangent and normal unit vectors at a selected position form a plane that is tangent to the curve. Although any plane containing $\bar{e}_t$ is tangent to the curve, the plane containing both $\bar{e}_t$ and $\bar{e}_n$ has several interesting features; this plane is referred to as the *osculating plane.*

In order to see why the parameter ρ in Eq. (2.5) is the radius of curvature, let us consider a planar path, in which case the osculating plane is the plane of the curve. Figure 2.3(a) depicts the tangent and normal vectors associated with two points, A and B, that are separated by an infinitesimal distance ds measured along an arbitrary planar path. Point C, which is the intersection of the normal vectors at the two positions along the curve, is the *center of curvature.* Because ds is infinitesimal, the arc AB seems to be circular. The radius R of this arc is the radius of curvature. The formula for the arc of a circle shows that $d\theta = ds/R$.

Now consider the increment $d\bar{e}_t = \bar{e}_t(s+ds) - \bar{e}_t(s)$ in Figure 2.3(b). The angle $d\theta$ between the normal vectors in Figure 2.3(a) is also the angle between the tangent vectors. The vector triangle in Figure 2.3(b) is isosceles because $|\bar{e}_t| = 1$. Hence, the



(a)



(b)

Figure 2.3 Relation between tangent and normal vectors.

angle between $d\bar{e}_t$ and either tangent vector is $90° - d\theta/2$. However, $d\theta$ is infinitesimal, so it must be that $d\bar{e}_t$ is perpendicular to $\bar{e}_t$ in the direction of $\bar{e}_n$. A unit vector has a length of 1, so

$$|d\bar{e}_t| = d\theta |\bar{e}_t| = \frac{ds}{R}.$$

Any vector may be expressed as the product of its magnitude and a unit vector defining the sense of the vector, from which we find that

$$d\bar{e}_t = |d\bar{e}_t| \bar{e}_n = \frac{ds}{R} \bar{e}_n.$$

When this relation is divided by ds, the result agrees with Eq. (2.4) provided that $\rho = R$. Hence, we have proven that the reciprocal of the magnitude of $d\bar{e}_t/ds$ is the radius of curvature of the planar path.

When the path is not planar, the orientation of the osculating plane containing the $\bar{e}_t, \bar{e}_n$ pair will depend on the position along the curve. Nevertheless, ρ is still the radius of curvature. Note that the radius of curvature is generally not a constant, although ρ is obviously the radius of a true circular path.

The development thus far is adequate to obtain formulas for velocity and acceleration. However, additional study of the unit vectors will enhance our understanding of the properties of curves. Because $\bar{e}_t$ and $\bar{e}_n$ are situated in the osculating plane, a third direction is required for the resolution of an arbitrary vector. The direction perpendicular to the osculating plane is called the *binormal;* the corresponding unit vector is $\bar{e}_b$. The cross product of two unit vectors is a unit vector perpendicular to the original two, so we define the binormal direction such that

♦ $$\bar{e}_b = \bar{e}_t \times \bar{e}_n. \tag{2.6}$$

An interesting property arises in the derivative of the $\bar{e}_n$ unit vector, which we may represent in terms of its tangent, normal, and binormal components. The component of an arbitrary vector in a specific direction may be obtained from a dot product with a unit vector in that direction. Multiplying each component by the corresponding unit vector and adding the individual contributions then reproduces the original vector:

$$\frac{d\bar{e}_n}{ds} = \left(\bar{e}_t \cdot \frac{d\bar{e}_n}{ds}\right)\bar{e}_t + \left(\bar{e}_n \cdot \frac{d\bar{e}_n}{ds}\right)\bar{e}_n + \left(\bar{e}_b \cdot \frac{d\bar{e}_n}{ds}\right)\bar{e}_b. \tag{2.7}$$

We obtain the tangential component in Eq. (2.7) from the orthogonality of the unit vectors, which requires that $\bar{e}_t \cdot \bar{e}_n = 0$. Then

$$\bar{e}_t \cdot \frac{d\bar{e}_n}{ds} = -\bar{e}_n \cdot \frac{d\bar{e}_t}{ds} = -\bar{e}_n \cdot \left(\frac{1}{\rho}\bar{e}_n\right) = -\frac{1}{\rho}. \tag{2.8}$$

Similarly, because $\bar{e}_n \cdot \bar{e}_n = 1$, we find that

$$\bar{e}_n \cdot \frac{d\bar{e}_n}{ds} = 0. \tag{2.9}$$

The binormal component of the derivative is generally nonzero; its value is defined as the reciprocal of the *torsion* τ,

$$\frac{1}{\tau} = \bar{e}_b \cdot \frac{d\bar{e}_n}{ds}. \tag{2.10}$$

The reciprocal is used here for consistency with Eq. (2.5), so that τ has the dimension of length. Substitution of Eqs. (2.8)-(2.10) into Eq. (2.7) results in

♦ $$\frac{d\bar{e}_n}{ds} = -\frac{1}{\rho}\bar{e}_t + \frac{1}{\tau}\bar{e}_b. \tag{2.11}$$

The derivative of $\bar{e}_b$ may be obtained by a similar approach. The fact that $\bar{e}_t$, $\bar{e}_n$, and $\bar{e}_b$ are mutually orthogonal, in combination with Eqs. (2.4) and (2.11), yields

$$\bar{e}_t \cdot \bar{e}_b = 0 \;\Rightarrow\; \bar{e}_t \cdot \frac{d\bar{e}_b}{ds} = -\frac{d\bar{e}_t}{ds} \cdot \bar{e}_b = -\frac{1}{\rho}\bar{e}_n \cdot \bar{e}_b = 0, \tag{2.12a}$$

$$\bar{e}_n \cdot \bar{e}_b = 0 \;\Rightarrow\; \bar{e}_n \cdot \frac{d\bar{e}_b}{ds} = -\frac{d\bar{e}_n}{ds} \cdot \bar{e}_b = -\frac{1}{\tau}, \tag{2.12b}$$

$$\bar{e}_b \cdot \bar{e}_b = 1 \;\Rightarrow\; \bar{e}_b \cdot \frac{d\bar{e}_b}{ds} = 0. \tag{2.12c}$$

The result is

♦ $$\frac{d\bar{e}_b}{ds} = -\frac{1}{\tau}\bar{e}_n. \tag{2.13}$$

Because $\bar{e}_n$ is a unit vector, this relation provides the following alternative to Eq. (2.10) for the torsion:

$$\frac{1}{\tau} = \left| \frac{d\bar{e}_b}{ds} \right|. \tag{2.14}$$

Equations (2.4), (2.11), and (2.13) are *Frenet's formulas* for a spatial curve. The first one shows that the change in the tangent vector due to a small increase in s is primarily in the normal direction. The osculating plane is formed from $\bar{e}_t$ and $\bar{e}_n$. We therefore may consider this plane to be the tangent plane that most closely fits the curve at the position of interest. Equation (2.13) shows that the vector normal to the osculating plane primarily changes in the direction of $\bar{e}_n$ with small increments in s. This is equivalent to a rotation of the osculating plane about the tangent direction, which is the source of the terminology "torsion." The osculating plane is constant for a planar curve, which corresponds to an infinite value of τ. The greater the degree to which a curve is twisted in space, the smaller will be the value of τ. In a similar vein, ρ measures the amount by which the curve bends in the osculating plane. A small value of ρ corresponds to a highly bent curve, and ρ is infinite for a straight line.

It is possible to specify a path in a variety of ways. Let us suppose that the path is described in *parametric form*. This means that if α is some parameter with a range of possible values, then the x, y, z coordinates are given in terms of the value of α. In this case, the position vector may be written in component form as

$$\bar{r} = x(\alpha)\bar{i} + y(\alpha)\bar{j} + z(\alpha)\bar{k}. \tag{2.15}$$

When such a description is available, it is possible to apply Frenet's formulas to evaluate the path variables in terms of the parameter α.

We employ the chain rule in order to determine $\bar{e}_t$ according to Eq. (2.2), from which we find that

$$\blacklozenge \quad \bar{e}_t = \frac{d\bar{r}}{d\alpha}\frac{d\alpha}{ds} = \frac{\bar{r}'}{s'},$$
$$\blacklozenge \quad \bar{r}' = x'\bar{i} + y'\bar{j} + z'\bar{k}, \tag{2.16}$$

where a prime denotes differentiation with respect to α. The fact that $\bar{e}_t$ is a unit vector, $|\bar{e}_t| = 1$, then yields a relation by which the arclength may be computed:

$$\blacklozenge \quad s' = (\bar{r}'\cdot\bar{r}')^{1/2} = [(x')^2 + (y')^2 + (z')^2]^{1/2}, \tag{2.17a}$$

$$\blacklozenge \quad s = \int_{\alpha_0}^{\alpha} [(x')^2 + (y')^2 + (z')^2]^{1/2}\, d\alpha, \tag{2.17b}$$

where α_0 is the value at the starting position. The value of s' found from Eq. (2.17a) may be substituted into Eqs. (2.16) in order to evaluate the tangent vector.

The next step is to evaluate $\bar{e}_n$ and ρ, for which the first of Frenet's formulas, Eq. (2.4), is used. From Eqs. (2.4) and (2.16) we have

$$\bar{e}_n = \rho\frac{d\bar{e}_t}{ds} = \rho\frac{d\bar{e}_t}{d\alpha}\frac{d\alpha}{ds} = \frac{\rho}{s'}\left[\frac{\bar{r}''}{s'} - \frac{\bar{r}'s''}{(s')^2}\right]$$
$$= \frac{\rho}{(s')^3}(\bar{r}''s' - \bar{r}'s''). \tag{2.18}$$

The value of s' is given by Eq. (2.17a). Differentiating that relation yields

$$s'' = \frac{\bar{r}'\cdot\bar{r}''}{(\bar{r}'\cdot\bar{r}')^{1/2}} = \frac{\bar{r}'\cdot\bar{r}''}{s'}. \tag{2.19}$$

The desired expression for the normal vector is obtained by substituting Eq. (2.19) into Eq. (2.18), with the result that

$$\blacklozenge \quad \bar{e}_n = \frac{\rho}{(s')^4}[\bar{r}''(s')^2 - \bar{r}'(\bar{r}'\cdot\bar{r}'')]. \tag{2.20}$$

Because $\bar{e}_n$ is a unit vector, using a dot product to form the magnitude of this expression leads to the radius of curvature:

$$\frac{1}{\rho} = \frac{1}{(s')^4}|[\bar{r}''(s')^2 - \bar{r}'(\bar{r}'\cdot\bar{r}'')]|$$
$$= \frac{1}{(s')^4}[\bar{r}''\cdot\bar{r}''(s')^4 - 2(\bar{r}'\cdot\bar{r}'')^2(s')^2 + (\bar{r}'\cdot\bar{r}')(\bar{r}'\cdot\bar{r}'')^2]^{1/2},$$

which simplifies to

$$\blacklozenge \quad \frac{1}{\rho} = \frac{1}{(s')^3}[(\bar{r}''\cdot\bar{r}'')(s')^2 - (\bar{r}'\cdot\bar{r}'')^2]^{1/2}. \tag{2.21}$$

In the case of a planar curve defined in the form $y = y(x)$ and $z = 0$, so that $\alpha = x$, this expression reduces to

$$\rho = \frac{[1 + (y')^2]^{3/2}}{|y''|}, \tag{2.22}$$

which is the same as the formula derived in a course on calculus.

After the result of Eq. (2.21) is substituted back into Eq. (2.20), it is a simple matter to evaluate the binormal direction according to Eq. (2.6):

$$\bar{e}_b = \bar{e}_t \times \bar{e}_n = \frac{\bar{r}'}{s'} \times \frac{\rho}{(s')^4}[\bar{r}''(s')^2 - \bar{r}'(\bar{r}' \cdot \bar{r}'')];$$

♦ $$\bar{e}_b = \frac{\rho}{(s')^3}\bar{r}' \times \bar{r}''. \tag{2.23}$$

An expression for τ may be obtained by applying the third Frenet formula, Eq. (2.13). The result of differentiating Eq. (2.23) may be written as

$$\frac{d\bar{e}_b}{ds} = \frac{1}{s'}\frac{d\bar{e}_b}{d\alpha} = \frac{1}{s'}\frac{d}{d\alpha}\left[\frac{\rho}{(s')^3}\right](\bar{r}' \times \bar{r}'') + \frac{\rho}{(s')^4}(\bar{r}' \times \bar{r}'''). \tag{2.24}$$

Next, we substitute this expression and Eq. (2.20) into Eq. (2.13) to find that

$$\frac{1}{\tau} = -\bar{e}_n \cdot \frac{d\bar{e}_b}{ds}$$

$$= -\frac{\rho}{(s')^4}[\bar{r}''(s')^2 - \bar{r}'(\bar{r}' \cdot \bar{r}'')] \cdot \left[\frac{1}{s'}\frac{d}{d\alpha}\left(\frac{\rho}{(s')^3}\right)(\bar{r}' \times \bar{r}'') + \frac{\rho}{(s')^4}(\bar{r}' \times \bar{r}''')\right].$$

We may simplify this equation by recognizing that a cross product is perpendicular to the individual terms in the product. Hence, carrying out the dot product in the preceding expression, term by term, leads to

♦ $$\frac{1}{\tau} = -\frac{\rho^2}{(s')^6}[\bar{r}'' \cdot (\bar{r}' \times \bar{r}''')]. \tag{2.25}$$

We see from these developments that the parametric description of a curved path enables us to evaluate all the properties of that path. Whether or not the path-variable approach is actually suitable for the description of the motion depends on how the movement along the path is specified, as we shall see in the next section.

Example 2.1 A particle moves along the hyperbolic paraboloidal surface $z = xy/2$ such that $x = 6\sin k\xi$ and $y = -6\cos k\xi$, where x, y, z are in meters and ξ is a parameter. Determine the path-variable unit vectors and the radius of curvature at the position where $\xi = \pi/3k$.

Solution It is possible to obtain the desired results by direct substitution into Eqs. (2.16)–(2.25). For the sake of increased understanding, we shall instead directly carry out the sequential operations indicated by the Frenet formulas. The trigonometric identity for the sine of a double angle yields z as a function of ξ. The corresponding parametric form of the position is

$$\bar{r} = (6\sin k\xi)\bar{i} - (6\cos k\xi)\bar{j} - (9\sin 2k\xi)\bar{k}.$$

Our solution begins by forming $\bar{e}_t$ as a function of ξ according to Eq. (2.16),

$$\bar{e}_t = \frac{d\bar{r}}{ds} = \frac{6k}{s'}[(\cos k\xi)\bar{i} + (\sin k\xi)\bar{j} - (3\cos 2k\xi)\bar{k}].$$

We also will need $s'(\xi)$, which we obtain by setting $|\bar{e}_t| = 1$;

$$s' = 6k(1+9\cos^2 2k\xi)^{1/2}.$$

The first Frenet relation, Eq. (2.4), describes the normal vector as

$$\bar{e}_n = \rho \frac{d\bar{e}_t}{ds} = \frac{\rho}{s'} \frac{d\bar{e}_t}{d\xi}$$

$$= \frac{\rho}{s'}(6k)\left\{\frac{k}{s'}[-(\sin k\xi)\bar{i} + (\cos k\xi)\bar{j} + (6\sin 2k\xi)\bar{k}] - \frac{s''}{(s')^2}[(\cos k\xi)\bar{i} + (\sin k\xi)\bar{j} - (3\cos 2k\xi)\bar{k}]\right\}.$$

We obtain s'' by differentiating s', which yields

$$s'' = -\frac{108\,k^2 \sin 2k\xi \cos 2k\xi}{(1+9\cos^2 2k\xi)^{1/2}}.$$

If the value of the torsion τ were needed, then it would be necessary to retain $\bar{e}_n$ in functional form, in order to form $d\bar{e}_n/ds$ for use in Eq. (2.11) or $\bar{e}_b$ for use in Eq. (2.13). For the present problem we may evaluate all quantities at $\xi = \pi/3k$, with the result that

$$s' = 10.8167k, \qquad s'' = 25.9408k^2,$$

$$\bar{e}_t = 0.27735\bar{i} + 0.48038\bar{j} + 0.83205\bar{k},$$

$$\bar{e}_n = \rho(-0.105905\bar{i} - 0.080869\bar{j} + 0.081990\bar{k}).$$

The requirement that $\bar{e}_n$ be a unit vector yields

$$\rho = 6.392 \text{ m},$$

$$\bar{e}_n = -0.6769\bar{i} - 0.5169\bar{j} + 0.5241\bar{k}.$$

Finally, we compute the binormal unit vector from a cross product according to its definition,

$$\bar{e}_b = \bar{e}_t \times \bar{e}_n = 0.6819\bar{i} - 0.7086\bar{j} + 0.1818\bar{k}.$$

2.1.2 *Kinematical Relations*

Situations in which the path-variable formulation is useful may be recognized by the fact that some aspect of $s(t)$ is given; for example, its rate $\dot{s}$ may be known. We saw in Eq. (2.1) that $\bar{r}_{P/O}$ is a function of time through the corresponding dependence on s. Using the chain rule to evaluate the derivative then yields

$$\bar{v} = \frac{d\bar{r}}{ds}\dot{s}. \tag{2.26}$$

In view of Eq. (2.2), this is equivalent to describing the velocity as

♦ $$\bar{v} = v\bar{e}_t, \qquad v = |\bar{v}| = |\dot{s}|. \tag{2.27}$$

These expressions indicate that the speed v is the rate of change of the arclength to the point. They also show that the velocity is always tangent to the path. Note that we defined $\bar{e}_t$ to point in the direction of increasing s. When the motion is such that the point is returning to the reference position $s = 0$ from positive s, we would set $\dot{s} < 0$ to apply Eqs. (2.27).

A corresponding formula for acceleration may be obtained by differentiating Eqs. (2.27) with respect to t. When we consider the speed v to be an explicit function of time, we have

$$\bar{a} = \frac{d\bar{v}}{dt} = \dot{v}\bar{e}_t + v\frac{d\bar{e}_t}{ds}\dot{s} = \dot{v}\bar{e}_t + v^2\frac{d\bar{e}_t}{ds}.$$

Equation (2.4) then leads to

$$\blacklozenge \quad \bar{a} = a_t\bar{e}_t + a_n\bar{e}_n,$$
$$\blacklozenge \quad a_t = \dot{v}, \qquad a_n = v^2/\rho. \tag{2.28}$$

Several aspects of this relation are important. The acceleration will have components in the normal and tangent directions. (There is no binormal component because the curve seems locally to lie in the osculating plane.) The normal component of acceleration is always directed toward the center of curvature for that position, because v^2/ρ is never negative. In the case of a circular path, ρ is the radius R. Thus, a_n is the *centripetal acceleration.* It must be emphasized that, even though the speed might be constant, there is always a centripetal acceleration, with two exceptions. If a point comes to rest, even momentarily, then $v = 0$. Alternatively, if the path is a straight line or if the point under consideration is an inflection point, then ρ is infinite. In general, the normal acceleration arises because the velocity direction changes as the point moves along its path. In contrast, the tangential acceleration arises whenever the speed, which is the magnitude of the velocity, changes.

There is a variety of ways in which the arclength s or speed v might be given. Whenever the speed is described in terms of s, rather than time t, the chain rule may be used to find the tangential acceleration. Specifically,

$$\dot{v} = v\frac{dv}{ds}. \tag{2.29}$$

Example 2.2 A particle follows a path defined in parametric form by $x = \frac{1}{2}A\pi\xi^2$, $y = A\xi\sin\pi\xi$, and $z = A\xi\cos\pi\xi$, where A is a constant. The particle gains speed at the constant rate $\dot{v}$. Determine the speed v that the particle should have at the position where $\xi = \frac{1}{2}$ in order for its x component of acceleration to be zero at that location.

Solution Because the value of $\dot{v}$ is given and the path is known in parametric form, we form the acceleration using path variables. The required value of v will be found by setting $\bar{a}\cdot\bar{i} = 0$ at $\xi = \frac{1}{2}$. We begin by using Eqs. (2.16)–(2.21) to obtain the path parameters. Thus

$$\bar{r} = \tfrac{1}{2}A\pi\xi^2\bar{i} + (A\xi\sin\pi\xi)\bar{j} + (A\xi\cos\pi\xi)\bar{k},$$

$$\bar{r}' = A\pi\xi\bar{i} + A(\pi\xi\cos\pi\xi + \sin\pi\xi)\bar{j} + A(-\pi\xi\sin\pi\xi + \cos\pi\xi)\bar{k},$$

$$\bar{r}'' = A\pi\bar{i} + A(-\pi^2\xi\sin\pi\xi + 2\pi\cos\pi\xi)\bar{j} + A(-\pi^2\xi\cos\pi\xi - 2\pi\sin\pi\xi)\bar{k}.$$

The other parameters depend algebraically on $\bar{r}'$ and $\bar{r}''$, so we may evaluate these quantities for $\xi = \frac{1}{2}$, which gives

$$\bar{r}' = A(\tfrac{1}{2}\pi\bar{i} + \bar{j} - \tfrac{1}{2}\pi\bar{k}), \qquad \bar{r}'' = A\pi(\bar{i} - \tfrac{1}{2}\pi\bar{j} - 1\bar{k}).$$

Sequential application of Eqs. (2.17a), (2.16), and (2.20) then yields

$$s' = A(\tfrac{1}{2}\pi^2 + 1)^{1/2} = 2.4361A,$$

$$\bar{e}_t = \frac{\pi\bar{i} + 2\bar{j} - \pi\bar{k}}{2(2.4361)} = 0.64479\bar{i} + 0.41048\bar{j} - 0.64479\bar{k},$$

$$\bar{e}_n = \frac{\rho}{2.4361^4 A}[(2.4361)^2\pi(\bar{i} - \tfrac{1}{2}\pi\bar{j} - 2\bar{k}) - \pi^2(\tfrac{1}{2}\pi\bar{i} + \bar{j} - \tfrac{1}{2}\pi\bar{k})]$$

$$= \frac{\rho}{A}(0.08919\bar{i} - 1.11171\bar{j} - 0.61854\bar{k}).$$

The value of ρ obtained from $|\bar{e}_n| = 1$ is

$$\rho = \frac{A}{(0.08919^2 + 1.11171^2 + 0.61854^2)^{1/2}} = 0.78411A,$$

which corresponds to

$$\bar{e}_n = 0.06994\bar{i} - 0.87171\bar{j} - 0.48501\bar{k}.$$

We may now form the acceleration:

$$\bar{a} = \dot{v}\bar{e}_t + \frac{v^2}{\rho}\bar{e}_n$$

$$= \dot{v}(0.64479\bar{i} + 0.41048\bar{j} - 0.64479\bar{k})$$

$$+ \frac{v^2}{0.78411A}(0.06994\bar{i} - 0.87171\bar{j} - 0.48501\bar{k}).$$

Finally, setting $\bar{a}\cdot\bar{i} = 0$ yields

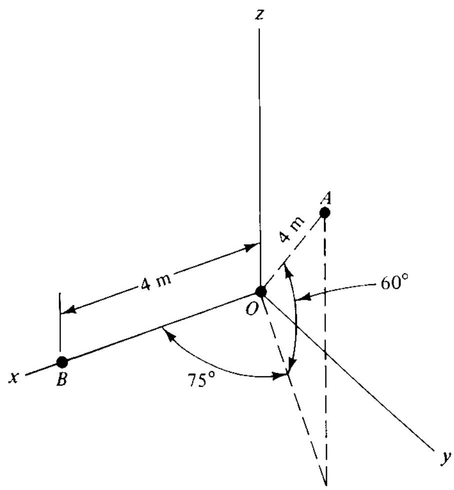
$$0.64479\dot{v} + \frac{0.06994}{0.78411A}v^2 = 0,$$

$$v = 2.6899(-A\dot{v})^{1/2}.$$

The presence of the negative sign means that $\bar{a}\cdot\bar{i} = 0$ is possible only if the speed is decreasing. This situation is a consequence of the fact that the x components of $\bar{e}_n$ and $\bar{e}_t$ are both positive.

Example 2.3 At the instant when a particle is at position A, it has a velocity of 500 m/s directed from point A to point B, and an acceleration of $10g$ directed from point A to point O. Determine the corresponding rate of change of the speed, the radius of curvature of the path, and the location of the center of curvature of the path.

Solution The velocity and acceleration are known, so representing them in terms of tangent and normal components should yield relations for the desired parameters. The given vectors are



Example 2.3

$$\bar{v} = 500\bar{e}_{B/A} \text{ m/s}, \qquad \bar{a} = 10(9.807)\bar{e}_{O/A} \text{ m/s}^2.$$

The unit vectors are defined by the positions of the end points, according to

$$\bar{r}_{A/O} = (4\cos 60°)[(\cos 75°)\bar{i} + (\sin 75°)\bar{j}] + (4\sin 60°)\bar{k}$$
$$= 0.5176\bar{i} + 1.9319\bar{j} + 3.464\bar{k} \text{ m},$$
$$\bar{r}_{B/O} = 4\bar{i},$$
$$\bar{e}_{B/A} = \frac{\bar{r}_{B/A}}{|\bar{r}_{B/A}|} = \frac{\bar{r}_{B/O} - \bar{r}_{A/O}}{|\bar{r}_{B/O} - \bar{r}_{A/O}|} = 0.6598\bar{i} - 0.3660\bar{j} - 0.6563\bar{k},$$
$$\bar{e}_{O/A} = -\frac{r_{A/O}}{|\bar{r}_{A/O}|} = -0.1294\bar{i} - 0.4830\bar{j} - 0.8660\bar{k}.$$

In general, $\bar{v} = v\bar{e}_t$, from which it follows that $\bar{e}_t = \bar{e}_{B/A}$. Then, because $\dot{v}$ is the tangential component of acceleration, we find that

$$\dot{v} = a_t = \bar{a} \cdot \bar{e}_t = 98.07\bar{e}_{O/A} \cdot \bar{e}_{B/A} = 64.70 \text{ m/s}^2.$$

We may evaluate the normal acceleration by forming the difference between $\bar{a}$ and $\dot{v}\bar{e}_t$; specifically,

$$\frac{v^2}{\rho}\bar{e}_n = \bar{a} - \dot{v}\bar{e}_t = 98.07\bar{e}_{O/A} - 64.70\bar{e}_{B/A}$$
$$= -55.38\bar{i} - 23.69\bar{j} - 42.47\bar{k} \text{ m/s}^2.$$

The values of ρ and $\bar{e}_n$ come from the magnitude and direction of this acceleration:

$$\frac{v^2}{\rho} = (55.38^2 + 23.69^2 + 42.47^2)^{1/2} = 73.70 \text{ m/s}^2,$$
$$\rho = \frac{v^2}{73.70} = \frac{500^2}{73.70} = 3{,}392 \text{ m},$$
$$\bar{e}_n = \frac{-55.38\bar{i} - 23.69\bar{j} - 42.47\bar{k}}{73.70} = -0.7514\bar{i} - 0.3214\bar{j} - 0.5763\bar{k}.$$

Finally, we locate the center of curvature C by recalling that it is at ρ units in the $\bar{e}_n$ direction relative to the corresponding point on the path. Hence,

$$\bar{r}_{C/O} = \bar{r}_{A/O} + \rho \bar{e}_n = -2{,}549\bar{i} - 1{,}088\bar{j} - 1{,}951\bar{k} \text{ m}.$$

2.2 Rectangular Cartesian Coordinates

The path-variable description is an intrinsic coordinate formulation because it relies on knowledge of the path for the definition of the unit vectors and of the position. For the remainder of this chapter we shall consider *extrinsic coordinate systems,* in which these properties are defined in a manner that is independent of the path.

The simplest set of extrinsic coordinates is *rectangular Cartesian coordinates.* These are associated with orthogonal xyz axes that are right-handed by convention. Situations where such coordinates might be suitable are recognizable by the fact that vectors (position, velocity, etc.) are described in terms of components with respect to fixed directions, such as left-right and up-down. As shown in Figure 2.4, the components of the position vector are merely the (x, y, z) coordinates projected onto the coordinate axes. These coordinates may all be functions of time, so the position is given by

$$\bar{r}_{P/O} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}. \tag{2.30}$$

Differentiating this expression is a simple matter because the unit vectors are constant. Thus, the velocity is given by

$$\blacklozenge \quad \bar{v} = v_x \bar{i} + v_y \bar{j} + v_z \bar{k},$$
$$\blacklozenge \quad v_x = \dot{x}, \qquad v_y = \dot{y}, \qquad v_z = \dot{z}, \tag{2.31}$$

from which it follows that the acceleration is

$$\blacklozenge \quad \bar{a} = a_x \bar{i} + a_y \bar{j} + a_z \bar{k},$$
$$\blacklozenge \quad a_x = \dot{v}_x, \qquad a_y = \dot{v}_y, \qquad a_z = \dot{v}_z. \tag{2.32}$$

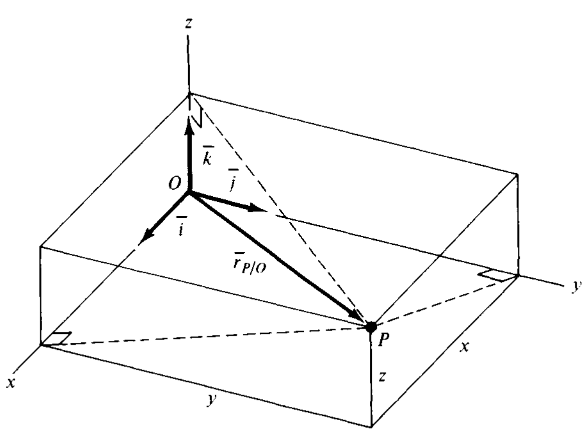
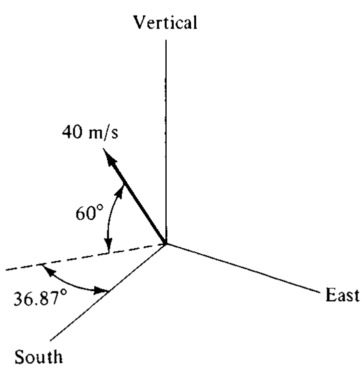


Figure 2.4 Rectangular Cartesian coordinates.

A notable feature of these relations is the uncoupled nature of the motions in the x, y, and z directions. In other words, none of the motion parameters for one direction appear in the other components. One way of regarding this result conceptually is to think of it as a superposition of rectilinear (i.e. straight-line) motions in each of the coordinate directions. (One should not infer from this observation that the motions in the three directions are independent. For example, the acceleration component in one direction might be a function of another coordinate.)

As you might suspect, the simplicity of this formulation limits its usefulness. Practical situations in which the motion is given in terms of fixed directions are not abundant. The most common involves projectile motion near the earth's surface. In that case the force of gravity is considered to be in the downward vertical direction, which means that the acceleration is always downward. Even this case breaks down when one wishes to treat the motion more accurately. For example, should one desire to account for air resistance, the resistance force is always opposite the velocity. Such a force is readily described in path variables as $-f\bar{e}_t$. The description of projectile motion in terms of Cartesian coordinates also encounters difficulty when the motion covers a long range, as with ballistic missiles. Then the gravitational force is always directed toward a fixed point, rather than having a fixed direction. A kinematical description using curvilinear coordinates is more suitable for this type of problem.

Example 2.4 A 200-gram ball is thrown from the ground with the initial velocity $\bar{v}_0$ shown. In addition to its weight, there is a constant wind force of 0.5 newtons acting in the easterly direction. Find the coordinates of the ball at the instant it returns to the elevation from which it was thrown, and the velocity of the ball at that instant.



Example 2.4

Solution The forces acting on the ball are its weight and the wind force, both of which act in fixed directions. We therefore employ Cartesian coordinates, with the z axis aligned vertically and the y axis aligned eastward in order to expedite description of the wind force. Forming $\sum \bar{F} = m\bar{a}$ in terms of components relative to these directions yields

$$0.5\bar{j} - 0.2(9.807)\bar{k} = m\bar{a} = 0.2(\ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}).$$

The initial conditions for the motion are

$$\bar{r}_0 = \bar{0},$$

$$\bar{v}_0 = (40\cos 60°)[(\cos 36.87°)\bar{i} - (\sin 36.87°)\bar{j}] + (40\sin 60°)\bar{k}$$
$$= 16\bar{i} - 12\bar{j} + 34.64\bar{k} \text{ m/s}.$$

We decompose these relations into their individual components as follows:

$$\ddot{x} = 0; \quad \dot{x}_0 = 16 \text{ m/s and } x_0 = 0 \text{ at } t = 0,$$

$$\ddot{y} = 2.5 \text{ m/s}^2; \quad \dot{y}_0 = -12 \text{ m/s and } y_0 = 0 \text{ at } t = 0,$$

$$\ddot{z} = -9.807 \text{ m/s}^2; \quad \dot{z}_0 = 34.64 \text{ m/s and } z_0 = 0 \text{ at } t = 0.$$

The acceleration in each direction is constant, so the individual acceleration equations may each be integrated twice. The first integration yields the velocity components, with the constants of integration selected to match the initial velocity conditions. Thus

$$\dot{x} = 16 \text{ m/s},$$

$$\dot{y} = 2.5t - 12 \text{ m/s},$$

$$\dot{z} = -9.807t + 34.64 \text{ m/s}.$$

The constants of integration for the second integration are used to satisfy the initial positions, with the result that

$$x = 16t, \quad y = 1.25t^2 - 12t, \quad z = -4.9035t^2 + 34.64t \text{ m}.$$

Finally, we evaluate the instant when the ball returns to the x-y plane by setting $z = 0$ for $t > 0$, which occurs when

$$t = 7.064 \text{ s}.$$

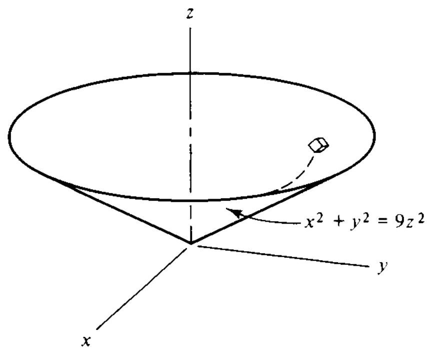
Evaluating the position and velocity components for that instant yields

$$\bar{r} = 113.03\bar{i} - 22.39\bar{j} \text{ m},$$

$$\bar{v} = 16\bar{i} + 5.660\bar{j} - 34.64\bar{k} \text{ m/s}.$$

As a closure, one should note that the problem we solved is unrealistic. The wind force acting on a moving object generally depends on the velocity of that object relative to the flowing air, which is $\bar{v} - \bar{v}_{\text{wind}}$. Such a force will not be constant unless both the velocity of the object and the velocity of the wind are constant. It is reasonable to approximate the wind force as a constant only if the velocity of the object is small in comparison to the wind velocity.

Example 2.5 A right circular cone is defined by $x^2 + y^2 = 9z^2$ (x, y, and z are in millimeters). The vertical position of a block sliding along the interior of such a cone is observed to be $z = 480 - 80t^2$, and $x = y^2/200$. Also, $y > 0$ throughout the motion. Determine the velocity and acceleration of the block when $t = 2$ s.



Example 2.5

Solution Because the intersection of two functions relating x, y, and z is a curve, the first and third of the given functions specify the path of the particle. We elect to use Cartesian coordinates, rather than path variables, because the second of the given relations prescribes the motion along the path in terms of the distance along the z axis. We simplify the functional relationships by using the first and third equations to relate y solely to z. The position equations then become

$$z = 0.480 - 0.080t^2, \quad x = \frac{1}{0.2}y^2, \quad \frac{1}{0.04}y^4 + y^2 = 9z^2 \text{ m.}$$

Differentiation of these expressions yields relations governing $\dot{x}$, $\dot{y}$, and $\dot{z}$:

$$\dot{z} = -0.16t, \qquad \dot{x} = 10y\dot{y}, \qquad (100y^3 + 2y)\dot{y} = 18z\dot{z}.$$

A second differentiation leads to

$$\ddot{z} = -0.16, \qquad \ddot{x} = 10(y\ddot{y} + \dot{y}^2),$$
$$(100y^3 + 2y)\ddot{y} + (300y^2 + 2)\dot{y}^2 = 18(z\ddot{z} + \dot{z}^2).$$

We find values for $t = 2$ s by sequential evaluation of the equations. Because $y > 0$, the position coordinates are

$$x = 0.3903 \quad y = 0.2794 \quad z = 0.160 \text{ m.}$$

These coordinates allow us to evaluate the first derivatives. Thus,

$$\dot{x} = 0.9398 \quad \dot{y} = -0.3364 \quad \dot{z} = -0.320 \text{ m/s.}$$

Next, we use these rate values to solve the equations for the second derivatives; specifically,

$$\ddot{x} = -23.38 \quad \ddot{y} = -8.773 \quad \ddot{z} = -0.160 \text{ m/s}^2.$$

The derivatives are the respective components of the velocity and acceleration, so

$$\bar{v} = 0.9398\bar{i} - 0.3364\bar{j} - 0.320\bar{k} \text{ m/s},$$
$$\bar{a} = -23.38\bar{i} - 8.773\bar{j} - 0.160\bar{k} \text{ m/s}^2.$$

The path may be visualized by noting that it lies on a cone; the projection of the path on the x-y plane is a parabola having x as the axis of symmetry.

2.3 Orthogonal Curvilinear Coordinates

The description we offer here specifies the position of a point by giving the value of a triad of parameters that form an orthogonal mesh in space. Let α, β, and γ be the three parameters, such that a unique transformation exists between the (x, y, z) rectangular Cartesian coordinates and the (α, β, γ) values:

$$\begin{aligned} x &= x(\alpha,\beta,\gamma), \quad & y &= y(\alpha,\beta,\gamma), \quad & z &= z(\alpha,\beta,\gamma); \\ \alpha &= \alpha(x,y,z), \quad & \beta &= \beta(x,y,z), \quad & \gamma &= \gamma(x,y,z). \end{aligned} \tag{2.33}$$

Formulations in terms of cylindrical or spherical coordinates are typical.

When two of the parameters (α, β, γ) are held constant while the third is given a range of values, the first group of Eqs. (2.33) specifies a curve in space in parametric form. When the constant parameter pair is given an assortment of values, the result is a family of curves. Repeating this procedure, with each of the other pairs of parameters held constant, produces two more families of curves; this is the aforementioned mesh. The families of curves are mutually orthogonal in the cases that we shall treat. For this reason, (α, β, γ) are said to be *orthogonal curvilinear coordinates*. The name for each set of coordinates usually corresponds to one of the types of surfaces on which one of the curvilinear coordinates is constant. This is illustrated in Figure 2.5(a) for cylindrical and Figure 2.5(b) for spherical coordinates.

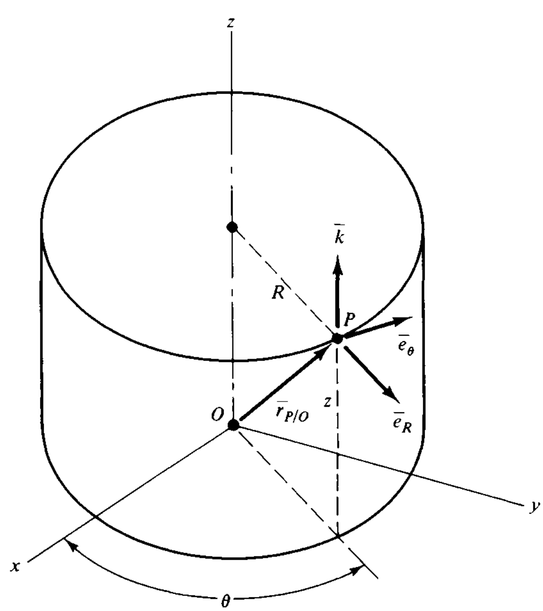


Figure 2.5 (a) Cylindrical coordinates.

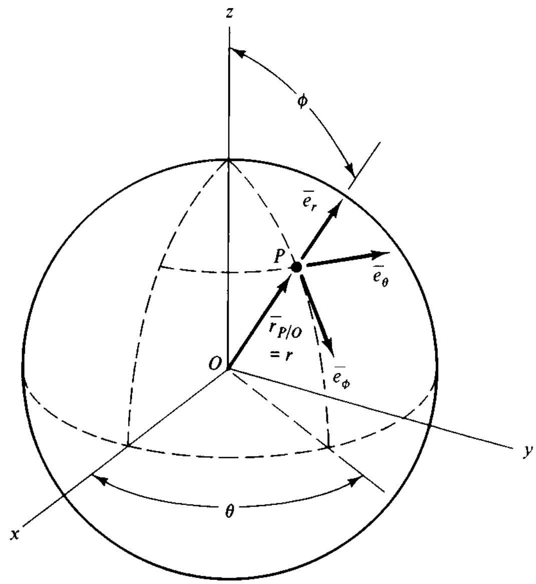


Figure 2.5 (b) Spherical coordinates.

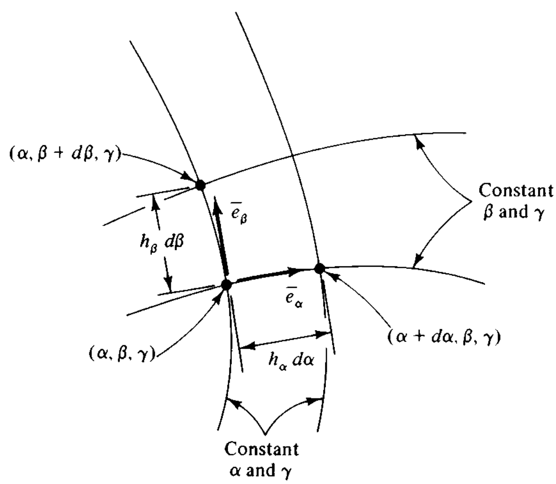


Figure 2.6 Curvilinear coordinate mesh.

It is difficult to depict a three-dimensional situation, so Figure 2.6 shows a two-dimensional grid. Neighboring curves for each family are separated by values of the other coordinate that differ by an infinitesimal value. The distance between intersection points on the grid is not the same as the value of the increment in that coordinate. The ratio of the differential arclength along a coordinate curve between intersections and the increment in the coordinate corresponding to the intersections is the ***stretch ratio*** for that coordinate, denoted h_λ, with $\lambda = \alpha$, β, or γ. The corresponding arclength along a coordinate curve is s_λ, so

$$ds_\lambda = h_\lambda \, d\lambda, \quad \lambda = \alpha, \, \beta, \text{ or } \gamma. \tag{2.34}$$

The relationship between the curvilinear coordinate transformation, Eqs. (2.33), and the stretch ratios will be established shortly.

2.3.1 *Coordinates and Unit Vectors*

Moving along any of the coordinate arcs for a curvilinear coordinate mesh is very much like the situation in path variables. There are three unit vectors $\bar{e}_\lambda$, consisting of the tangent to each of the coordinate curves intersecting at any point. Incrementing one coordinate with the other two fixed is a process of partial differentiation, so the unit vectors may be obtained from

$$\bar{e}_\lambda = \frac{\partial \bar{r}}{\partial s_\lambda};$$

♦ $$\bar{e}_\lambda = \frac{1}{h_\lambda} \frac{\partial \bar{r}}{\partial \lambda}, \quad \lambda = \alpha, \, \beta, \text{ or } \gamma. \tag{2.35}$$

Note that the unit vectors may depend on the value of each curvilinear coordinate. However, in many cases the unit vectors might be independent of one or more of the coordinates. Two such cases are cylindrical and spherical coordinates. The unit vectors obtained for each were depicted in Figure 2.5. In cylindrical coordinates, $\bar{e}_R$ and $\bar{e}_\theta$ depend only on the azimuthal (transverse) angle θ, and $\bar{e}_z = \bar{k}$ is constant. In spherical coordinates, all the unit vectors depend only on the polar angle ϕ and the azimuthal angle θ.

It is conventional to employ a right-handed coordinate system in order to avoid sign errors in the evaluation of cross products. Consistency with this convention is obtained in curvilinear coordinates by ordering (α, β, γ) such that

$$\bar{e}_\alpha \times \bar{e}_\beta = \bar{e}_\gamma. \tag{2.36}$$

Explicit expressions for the stretch ratios may be obtained in such geometrically simple cases as cylindrical and spherical coordinates by drawing diagrams resembling Figure 2.6 for each coordinate line. More difficult cases are treated by using the fact that $\bar{e}_\lambda$ is a unit vector, so that

♦ $$h_\lambda = \left| \frac{\partial \bar{r}}{\partial \lambda} \right|. \tag{2.37}$$

The derivation of the acceleration equation will require differentiation of the unit vectors. Rather than differentiating Eq. (2.35) directly, we shall follow a more circuitous approach that will yield explicit expressions in terms of the stretch ratios. The approach here is similar to the way in which some of the Frenet formulas were derived for path variables. First, the derivative of any unit vector $\bar{e}_\lambda$ with respect to any coordinate μ is resolved into components as

$$\frac{\partial \bar{e}_\lambda}{\partial \mu} = \left(\bar{e}_\alpha \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} \right) \bar{e}_\alpha + \left(\bar{e}_\beta \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} \right) \bar{e}_\beta + \left(\bar{e}_\gamma \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} \right) \bar{e}_\gamma. \tag{2.38}$$

Here, both λ and μ correspond to α, β, or γ, so we must consider permutations of the general term $\bar{e}_\nu \cdot (\partial \bar{e}_\lambda / \partial \mu)$. All cases where the unit vectors $\bar{e}_\lambda$ and $\bar{e}_\nu$ match are covered by

$$\bar{e}_\lambda \cdot \bar{e}_\lambda = 1 \ \Rightarrow \ \bar{e}_\lambda \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} = 0. \tag{2.39}$$

Cases not covered by Eq. (2.39) correspond to $\nu \neq \lambda$. We may evaluate these with the aid of a sequence of identities. It follows from the orthogonality of the unit vectors that

$$\bar{e}_\nu \cdot \bar{e}_\lambda = 0 \ \Rightarrow \ \bar{e}_\nu \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} = -\bar{e}_\lambda \cdot \frac{\partial \bar{e}_\nu}{\partial \mu}, \quad \nu \neq \lambda. \tag{2.40}$$

The following relation originates from Eq. (2.35):

$$\frac{\partial}{\partial \mu}(h_\lambda \bar{e}_\lambda) = \frac{\partial}{\partial \lambda}(h_\mu \bar{e}_\mu).$$

Carrying out the derivatives leads to

$$h_\lambda \frac{\partial \bar{e}_\lambda}{\partial \mu} + \frac{\partial h_\lambda}{\partial \mu} \bar{e}_\lambda = h_\mu \frac{\partial \bar{e}_\mu}{\partial \lambda} + \frac{\partial h_\mu}{\partial \lambda} \bar{e}_\mu. \tag{2.41}$$

We may now consider the various combinations of the general term $\bar{e}_\nu \cdot (\partial \bar{e}_\lambda / \partial \mu)$ when $\nu \neq \lambda$. Because each of the symbols represents one of three possible coordinates, the only combinations fitting the restriction that $\nu \neq \lambda$ are $\mu = \nu \neq \lambda$, $\mu = \lambda \neq \nu$, and $\mu \neq \nu \neq \lambda$. We begin by considering the first case. Such terms are obtained from the dot product of Eq. (2.41) with $\bar{e}_\mu$. Because $\bar{e}_\mu$ and $\bar{e}_\lambda$ are different, it follows from Eq. (2.39) that

$$\bar{e}_\mu \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} = \frac{1}{h_\lambda} \frac{\partial h_\mu}{\partial \lambda}, \quad \mu \neq \lambda. \tag{2.42a}$$

We obtain an expression for the general term $\bar{e}_\nu \cdot (\partial \bar{e}_\lambda / \partial \mu)$ in situations where $\mu = \lambda \neq \nu$ by applying Eq. (2.40) to the foregoing, and then changing the symbol λ to ν. The result is

$$\bar{e}_\nu \cdot \frac{\partial \bar{e}_\mu}{\partial \mu} = -\frac{1}{h_\nu} \frac{\partial h_\mu}{\partial \nu}, \quad \nu \neq \mu. \tag{2.42b}$$

The only remaining case is that for which λ, μ, and ν differ from each other. The dot product of Eq. (2.41) with $\bar{e}_\nu$ in this case yields

$$h_\lambda \bar{e}_\nu \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} = h_\mu \bar{e}_\nu \cdot \frac{\partial \bar{e}_\mu}{\partial \lambda}, \quad \lambda, \nu, \mu \text{ distinct.} \tag{2.43}$$

The next steps involve alternate application of permutations of the properties in Eqs. (2.40) and (2.43) to the *right* side of Eq. (2.43). This gives

$$\begin{aligned} h_\lambda \bar{e}_\nu \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} &= -h_\mu \bar{e}_\mu \cdot \frac{\partial \bar{e}_\nu}{\partial \lambda} = -h_\mu \left(\frac{h_\lambda}{h_\nu} \bar{e}_\mu \cdot \frac{\partial \bar{e}_\lambda}{\partial \nu} \right) \\ &= \frac{h_\mu h_\lambda}{h_\nu} \bar{e}_\lambda \cdot \frac{\partial \bar{e}_\mu}{\partial \nu} = h_\lambda \bar{e}_\lambda \cdot \frac{\partial \bar{e}_\nu}{\partial \mu} = -h_\lambda \bar{e}_\nu \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu}. \end{aligned}$$

The foregoing is a contradiction unless

$$\bar{e}_\nu \cdot \frac{\partial \bar{e}_\lambda}{\partial \mu} = 0, \quad \lambda, \nu, \mu \text{ distinct.} \tag{2.44}$$

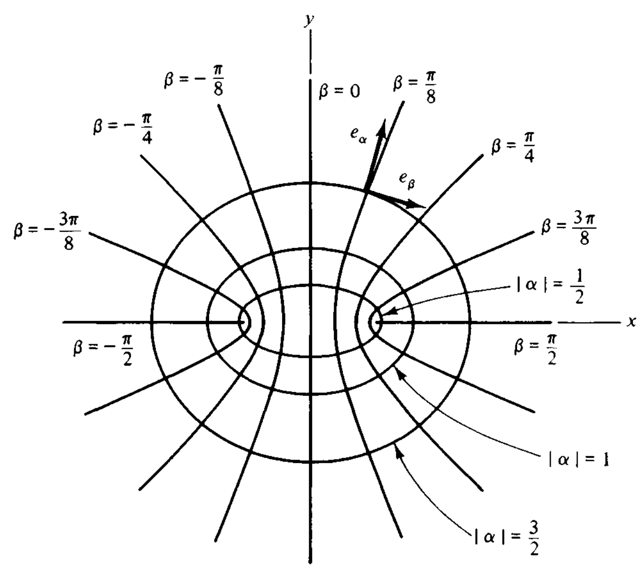
Equations (2.39), (2.42), and (2.44) are the identities we seek.† With these relations, we may express the components of $\partial \bar{e}_\lambda/\partial \mu$ in Eq. (2.38) in terms of the stretch ratios. There are nine combinations of λ and μ values, whose individual components are evaluated by selecting the appropriate case from the identities. We will list only the results for $\lambda = \alpha$. The others follow by permutation of the symbols.

$$\blacklozenge \quad \frac{\partial \bar{e}_\alpha}{\partial \alpha} = -\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\bar{e}_\beta - \frac{1}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma}\bar{e}_\gamma,$$
$$\blacklozenge \quad \frac{\partial \bar{e}_\alpha}{\partial \beta} = \frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\bar{e}_\beta, \qquad \frac{\partial \bar{e}_\alpha}{\partial \gamma} = \frac{1}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha}\bar{e}_\gamma. \tag{2.45}$$

Example 2.6 The two-dimensional hyperbolic-elliptic coordinate system is defined by

$$x = a\cosh\alpha\sin\beta, \qquad y = a\sinh\alpha\cos\beta,$$

where a is a constant. Evaluate the unit vectors of this system in terms of components relative to the x and y axes; then describe the derivatives of the unit vectors.



Example 2.6

Solution The name of this set of coordinates stems, in part, from the fact that lines of constant α are ellipses:

$$\left(\frac{x}{a\cosh\alpha}\right)^2 + \left(\frac{y}{a\sinh\alpha}\right)^2 = 1,$$

† A common operation in tensor analysis is covariant differentiation of a quantity that is defined in terms of basis vectors for an arbitrary curvilinear coordinate system. Such a derivative may be expressed in terms of *Christoffel symbols*. Equations (2.39), (2.42), and (2.44) define some of the Christoffel symbols for the case of an orthogonal coordinate system.

where $2a\cosh\alpha$ and $2a\sinh\alpha$ are the lengths of the major and minor diameters, respectively. Also, lines of constant β are hyperbolas:

$$\left(\frac{x}{a\sin\beta}\right)^2 = \left(\frac{y}{a\cos\beta}\right)^2 + 1,$$

where $x = \pm y\tan\beta$ are the asymptotes and $\pm a\sin\beta$ are the intercepts on the x axis.

The stretch ratios and unit vectors come from the partial derivatives of the position vector:

$$\frac{\partial\bar{r}}{\partial\alpha} = \frac{\partial}{\partial\alpha}(x\bar{i}+y\bar{j}) = (a\sinh\alpha\sin\beta)\bar{i} + (a\cosh\alpha\cos\beta)\bar{j},$$

$$\frac{\partial\bar{r}}{\partial\beta} = \frac{\partial}{\partial\beta}(x\bar{i}+y\bar{j}) = (a\cosh\alpha\cos\beta)\bar{i} - (a\sinh\alpha\sin\beta)\bar{j}.$$

We find from Eq. (2.37) that

$$h_\alpha = \left|\frac{\partial\bar{r}}{\partial\alpha}\right| = a(\sinh^2\alpha\sin^2\beta + \cosh^2\alpha\cos^2\beta)^{1/2}$$

$$= a[(\cosh^2\alpha - 1)\sin^2\beta + \cosh^2\alpha\cos^2\beta]^{1/2} = a(\cosh^2\alpha - \sin^2\beta)^{1/2}.$$

Steps comparable to the foregoing lead to the stretch ratio $h_\beta = |\partial\bar{r}/\partial\beta|$. It is convenient to define

$$h = (\cosh^2\alpha - \sin^2\beta)^{1/2}.$$

The result is that

$$h_\beta = h_\alpha = ah.$$

The corresponding unit vectors are found from Eq. (2.35) to be

$$\bar{e}_\alpha = \frac{1}{h}[(\sinh\alpha\sin\beta)\bar{i} + (\cosh\alpha\cos\beta)\bar{j}],$$

$$\bar{e}_\beta = \frac{1}{h}[(\cosh\alpha\cos\beta)\bar{i} - (\sinh\alpha\sin\beta)\bar{j}].$$

The orthogonality of the mesh is confirmed by these unit vectors, because they show that $\bar{e}_\alpha \cdot \bar{e}_\beta = 0$.

The derivatives of the unit vectors involve partial derivatives of the stretch ratios, which in the present case are obtained from

$$\frac{\partial h}{\partial\alpha} = \frac{1}{h}(\cosh\alpha\sinh\alpha), \qquad \frac{\partial h}{\partial\beta} = -\frac{1}{h}(\sin\beta\cos\beta).$$

The corresponding expressions resulting from Eq. (2.45) are

$$\frac{\partial\bar{e}_\alpha}{\partial\alpha} = -\frac{1}{h}\frac{\partial h}{\partial\beta}\bar{e}_\beta = \frac{\sin\beta\cos\beta}{h^2}\bar{e}_\beta, \qquad \frac{\partial\bar{e}_\beta}{\partial\beta} = -\frac{1}{h}\frac{\partial h}{\partial\alpha}\bar{e}_\alpha = -\frac{\cosh\alpha\sinh\alpha}{h^2}\bar{e}_\alpha,$$

$$\frac{\partial\bar{e}_\alpha}{\partial\beta} = \frac{1}{h}\frac{\partial h}{\partial\alpha}\bar{e}_\beta = \frac{\cosh\alpha\sinh\alpha}{h^2}\bar{e}_\beta, \qquad \frac{\partial\bar{e}_\beta}{\partial\alpha} = \frac{1}{h}\frac{\partial h}{\partial\beta}\bar{e}_\alpha = -\frac{\sin\beta\cos\beta}{h^2}\bar{e}_\alpha.$$

2.3.2 *Kinematical Formulas*

Our task in this section is to express velocity and acceleration in terms of the parameters of a curvilinear coordinate system. For this development we consider the motion to be specified through the dependence of the curvilinear coordinates on time. The velocity is the time derivative of the position vector, which is a function of the curvilinear coordinates. The definition of the unit vectors in Eq. (2.35) then results in

$$\bar{v} = \dot{\alpha}\frac{\partial \bar{r}}{\partial \alpha} + \dot{\beta}\frac{\partial \bar{r}}{\partial \beta} + \dot{\gamma}\frac{\partial \bar{r}}{\partial \gamma} = h_\alpha \dot{\alpha}\bar{e}_\alpha + h_\beta \dot{\beta}\bar{e}_\beta + h_\gamma \dot{\gamma}\bar{e}_\gamma.$$

This expression may be written in summation form as

♦ $$\bar{v} = \sum_{\lambda=\alpha,\beta,\gamma} h_\lambda \dot{\lambda}\bar{e}_\lambda. \tag{2.46}$$

We derive the acceleration by differentiating Eq. (2.46) with respect to time. For this, we consider only the curvilinear coordinates λ to depend explicitly on t, whereas the unit vectors $\bar{e}_\lambda$ and the stretch ratios h_λ depend on t implicitly through their dependence on the coordinates. Application of the chain rule for differentiation then yields

♦ $$\bar{a} = \sum_{\lambda=\alpha,\beta,\gamma}\left[h_\lambda \ddot{\lambda}\bar{e}_\lambda + \sum_{\mu=\alpha,\beta,\gamma}\left(\frac{\partial h_\lambda}{\partial \mu}\bar{e}_\lambda + h_\lambda \frac{\partial \bar{e}_\lambda}{\partial \mu}\right)\dot{\lambda}\dot{\mu}\right]. \tag{2.47}$$

Explicit expressions for a specific set of curvilinear coordinates may be obtained from Eq. (2.47) by evaluating the stretch ratios and the derivatives of the unit vectors according to Eqs. (2.37) and (2.45), respectively.

It is apparent that each acceleration component might consist of several terms, in the most general case. The situation for many common sets of curvilinear coordinates is simplified by the fact that the stretch ratios do not depend on all of the coordinate values. For the coordinates defined in Figure 2.5, we have the following.

Cylindrical Coordinates (R, θ, z)

$$\begin{aligned}
&x = R\cos\theta, \qquad y = R\sin\theta, \qquad z = z;\\
&h_R = 1, \qquad h_\theta = R, \qquad h_z = 1;\\
&\bar{r}_{P/O} = R\bar{e}_R + z\bar{k}, \qquad \bar{e}_z = \bar{k};\\
&\bar{v} = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{k};\\
&\bar{a} = (\ddot{R} - R\dot{\theta}^2)\bar{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\bar{e}_\theta + \ddot{z}\bar{k}.
\end{aligned} \tag{2.48}$$

Spherical Coordinates (r, ϕ, θ)

$$\begin{aligned}
&x = r\sin\phi\cos\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\phi;\\
&h_r = 1, \qquad h_\phi = r, \qquad h_\theta = r\sin\phi;\\
&\bar{r}_{P/O} = r\bar{e}_r;\\
&\bar{v} = \dot{r}\bar{e}_r + r\dot{\phi}\bar{e}_\phi + r\dot{\theta}\sin\phi\,\bar{e}_\theta;\\
&\bar{a} = (\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi)\bar{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi)\bar{e}_\phi\\
&\qquad + (r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi)\bar{e}_\theta.
\end{aligned} \tag{2.49}$$

2.3.3 *Interpretation*

Consideration of Eq. (2.47) shows that in the most general case there are 21 different terms contributing to the acceleration: three from the single summation and nine from each term in the double summation. Because there are only three curvilinear coordinate directions, it is clear that a variety of effects contribute to any acceleration component. Let us examine each type of effect.

The terms in the single summation are intuitively obvious. They express the acceleration tangent to the coordinate curves that arises when the rate of change of the corresponding coordinate is not constant. In order to understand the terms in the double sum we categorize them as to whether or not the indices for each sum are associated with the same curvilinear coordinate. If $\mu = \lambda$, three terms correspond to $\dot{\lambda}^2(\partial h_\lambda/\partial \lambda)\bar{e}_\lambda$. This is another tangent acceleration effect, which arises because constant rates of change of the λ curvilinear coordinate will not lead to a constant rate of movement along that curve if the stretch ratio changes along the curve. The other term corresponding to $\mu = \lambda$ is $\dot{\lambda}^2 h_\lambda(\partial \bar{e}_\lambda/\partial \lambda)$. Because the derivative of a unit vector is always perpendicular to the unit vector, this is an acceleration component perpendicular to the λ coordinate curve that arises from the changing direction of the unit vector as λ changes. Such a change is illustrated in Figure 2.7, associated with $\bar{e}_\lambda$ at two neighboring points on a specific λ curve. The fact that $\dot{\lambda} h_\lambda$ is the velocity component suggests that this acceleration component is analogous to the centripetal acceleration in path variables.

Let us now turn our attention to those terms in Eq. (2.47) that correspond to $\mu \neq \lambda$. There are three combinations fitting this description, corresponding to $\dot{\lambda}\dot{\mu} = \dot{\alpha}\dot{\beta}$, $\dot{\alpha}\dot{\gamma}$, or $\dot{\beta}\dot{\gamma}$. Let us consider the combination of indices $\mu, \lambda = \alpha, \beta$. The first term in the double sum leads to two terms: $[(\partial h_\alpha/\partial \beta)\bar{e}_\alpha + (\partial h_\beta/\partial \alpha)\bar{e}_\beta]\dot{\alpha}\dot{\beta}$. Each of these terms exists if a coordinate transverse to a specified coordinate curve λ changes *and* the stretch ratio for that curve depends on an orthogonal coordinate. Both represent acceleration components tangent to the λ coordinate curve that result because the rate of movement along the curve, $\dot{\lambda} h_\lambda$, changes as a consequence of the nonconstancy of the stretch ratio. The second term in the double summation also leads to two terms for $\lambda, \mu = \alpha, \beta$: $[h_\alpha(\partial \bar{e}_\alpha/\partial \beta) + h_\beta(\partial \bar{e}_\beta/\partial \alpha)]\dot{\alpha}\dot{\beta}$. These are acceleration terms perpendicular to a λ coordinate curve associated with the change in the tangent

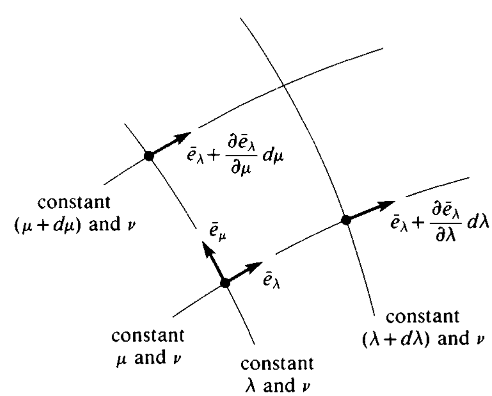


Figure 2.7 Changes in a curvilinear coordinate unit vector.

to that curve, $\bar{e}_\lambda$, resulting from movement to a different λ curve. This is depicted in Figure 2.7, where two λ curves correspond to constant values of the orthogonal coordinate μ that differ by $d\mu$. The tangent vector $\bar{e}_\lambda$ for corresponding points on each curve are not the same.

Consider the terms in the cylindrical and spherical coordinate accelerations, Eqs. (2.48) and (2.49), associated with rate products of different coordinates. In each case there is a factor of 2. From our discussion, it is clear that these arise from two distinctly different effects associated with an interaction of motion along more than one coordinate curve. These mixed-product terms are usually called Coriolis accelerations after G. Coriolis (1792–1843), who successfully explained the phenomenon. Only in the special case of Cartesian coordinates is there a superposition of motion along each of the three coordinate curves resulting from changing the respective coordinate value.

Example 2.7 Derive Eqs. (2.48) for velocity and acceleration in terms of cylindrical coordinates.

Solution The first step is to evaluate the unit vectors and stretch ratios. For the coordinate system in Figure 2.5(a), we have

$$\bar{r} = (R\cos\theta)\bar{i} + (R\sin\theta)\bar{j} + z\bar{k}.$$

Then

$$h_R\bar{e}_R = \frac{\partial\bar{r}}{\partial R} = (\cos\theta)\bar{i} + (\sin\theta)\bar{j},$$

$$h_\theta\bar{e}_\theta = \frac{\partial\bar{r}}{\partial\theta} = -(R\sin\theta)\bar{i} + (R\cos\theta)\bar{j},$$

$$h_z\bar{e}_z = \frac{\partial\bar{r}}{\partial z} = \bar{k}.$$

Setting the magnitude of each unit vector to unity yields the stretch ratios. Thus

$$h_R = \left|\frac{\partial\bar{r}}{\partial R}\right| = 1, \qquad h_\theta = \left|\frac{\partial\bar{r}}{\partial\theta}\right| = R, \qquad h_z = \left|\frac{\partial\bar{r}}{\partial z}\right| = 1,$$

which corresponds to

$$\bar{e}_R = (\cos\theta)\bar{i} + (\sin\theta)\bar{j},$$

$$\bar{e}_\theta = -(\sin\theta)\bar{i} + (\cos\theta)\bar{j}.$$

The derivatives of the unit vectors are required to form Eq. (2.47) for $\bar{a}$. Although Eqs. (2.45) may be applied for this purpose, we may employ direct differentiation with equal ease in the present case. Specifically,

$$\frac{\partial\bar{e}_R}{\partial R} = \bar{0}, \qquad \frac{\partial\bar{e}_R}{\partial\theta} = -(\sin\theta)\bar{i} + (\cos\theta)\bar{j} = \bar{e}_\theta, \qquad \frac{\partial\bar{e}_R}{\partial z} = \bar{0};$$

$$\frac{\partial\bar{e}_\theta}{\partial R} = \bar{0}, \qquad \frac{\partial\bar{e}_\theta}{\partial\theta} = -(\cos\theta)\bar{i} - (\sin\theta)\bar{j} = -\bar{e}_R, \qquad \frac{\partial\bar{e}_\theta}{\partial z} = \bar{0};$$

$$\frac{\partial\bar{e}_z}{\partial R} = \frac{\partial\bar{e}_z}{\partial\theta} = \frac{\partial\bar{e}_z}{\partial z} = \bar{0}.$$

We obtain an expression for the velocity by expanding Eq. (2.46) and substituting the various terms. Thus,

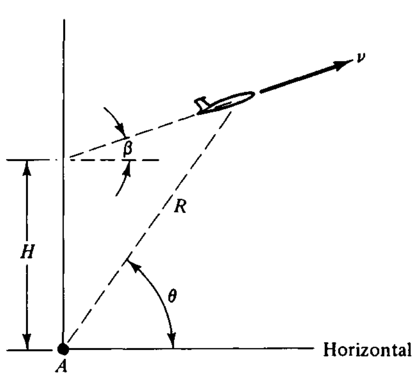
$$\bar{v} = h_R\dot{R}\bar{e}_R + h_\theta\dot{\theta}\bar{e}_\theta + h_z\dot{z}\bar{e}_z = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{k}.$$

Applying the same procedure to Eq. (2.47) yields the acceleration. Specifically,

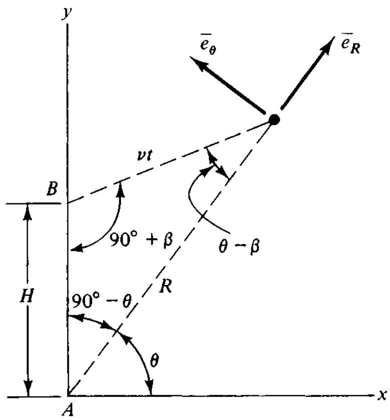
$$\begin{aligned}\bar{a} = {} & h_R\ddot{R}\bar{e}_R + \left(\frac{\partial h_R}{\partial R}\bar{e}_R + h_R\frac{\partial \bar{e}_R}{\partial R}\right)\dot{R}^2 + \left(\frac{\partial h_R}{\partial \theta}\bar{e}_R + h_R\frac{\partial \bar{e}_R}{\partial \theta}\right)\dot{R}\dot{\theta} \\ & + \left(\frac{\partial h_R}{\partial z}\bar{e}_R + h_R\frac{\partial \bar{e}_R}{\partial z}\right)\dot{R}\dot{z} + h_\theta\ddot{\theta}\bar{e}_\theta + \left(\frac{\partial h_\theta}{\partial R}\bar{e}_\theta + h_\theta\frac{\partial \bar{e}_\theta}{\partial R}\right)\dot{R}\dot{\theta} \\ & + \left(\frac{\partial h_\theta}{\partial \theta}\bar{e}_\theta + h_\theta\frac{\partial \bar{e}_\theta}{\partial \theta}\right)\dot{\theta}^2 + \left(\frac{\partial h_\theta}{\partial z}\bar{e}_\theta + h_\theta\frac{\partial \bar{e}_\theta}{\partial z}\right)\dot{\theta}\dot{z} \\ & + h_z\ddot{z}\bar{e}_z + \left(\frac{\partial h_z}{\partial R}\bar{e}_z + h_z\frac{\partial \bar{e}_z}{\partial R}\right)\dot{R}\dot{z} + \left(\frac{\partial h_z}{\partial \theta}\bar{e}_z + h_z\frac{\partial \bar{e}_z}{\partial \theta}\right)\dot{\theta}\dot{z} \\ & + \left(\frac{\partial h_z}{\partial z}\bar{e}_z + h_z\frac{\partial \bar{e}_z}{\partial z}\right)\dot{z}^2, \\ = {} & \ddot{R}\bar{e}_R + \dot{R}\dot{\theta}\bar{e}_\theta + R\ddot{\theta}\bar{e}_\theta + \dot{R}\dot{\theta}\bar{e}_\theta - R\dot{\theta}^2\bar{e}_R + \ddot{z}\bar{k}.\end{aligned}$$

Collecting like components in this expression leads to the same result as Eq. (2.48). Note that the second and fourth terms in the final expression for $\bar{a}$, which combine to form the Coriolis acceleration, originated from different sources. One term corresponds to $h_R(\partial\bar{e}_R/\partial\theta)\dot{R}\dot{\theta}$ and so is a consequence of the dependence of the radial unit vector on the azimuthal angle θ. The other term comes from $(\partial h_\theta/\partial R)\bar{e}_\theta\dot{R}\dot{\theta}$, which arises because the movement in the azimuthal direction accelerates owing to dependence of the azimuthal stretch ratio on the radial distance.

Example 2.8 An airplane climbs at a constant speed v and at a constant climb angle β. The airplane is being tracked by a radar station at point A on the ground. Determine the radial velocity $\dot{R}$ and the angular velocity $\dot{\theta}$ as functions of the tracking angle θ.



Example 2.8



Polar coordinate system.

Solution We shall employ a trigonometric approach here, in which the desired parameters are obtained from differentiation of geometrical relations. (A simpler solution to this problem may be found in Example 2.10, which matches the given velocity to the cylindrical coordinate formulas.) First, we construct the distance vt the airplane has traveled after passing point B above the radar station. This forms one side of a triangle whose other sides are R and H. Then the law of sines yields

$$\frac{R}{\sin(\pi/2+\beta)} = \frac{vt}{\sin(\pi/2-\theta)} = \frac{H}{\sin(\theta-\beta)}.$$

Thus

$$R\sin(\theta-\beta) = H\sin(\pi/2+\beta) \equiv H\cos\beta,$$

$$vt\sin(\theta-\beta) = H\sin(\pi/2-\theta) \equiv H\cos\theta.$$

Differentiating each expression leads to

$$\dot{R}\sin(\theta-\beta) + R\dot{\theta}\cos(\theta-\beta) = 0,$$

$$v\sin(\theta-\beta) + vt\dot{\theta}\cos(\theta-\beta) = -H\dot{\theta}\sin\theta.$$

These are simultaneous equations for $\dot{R}$ and $\dot{\theta}$, whose solution is

$$\dot{\theta} = -\frac{v\sin(\theta-\beta)}{vt\cos(\theta-\beta)+H\sin\theta},$$

$$\dot{R} = \frac{Rv\cos(\theta-\beta)}{vt\cos(\theta-\beta)+H\sin\theta}.$$

These expressions are not in the desired form because they depend on t and R. The equations obtained from the law of sines indicate that

$$R = H\frac{\cos\beta}{\sin(\theta-\beta)}, \qquad vt = H\frac{\cos\theta}{\sin(\theta-\beta)}.$$

We use these relations to eliminate R and vt from the expressions for $\dot{R}$ and $\dot{\theta}$, which leads to

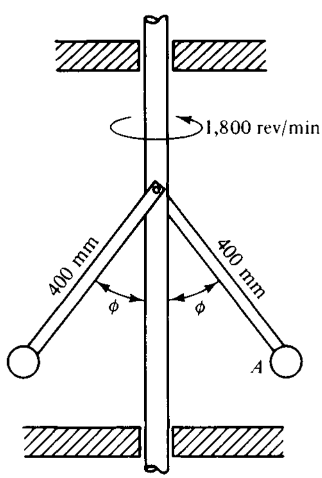
$$\dot{\theta} = -\frac{v}{H}\,\frac{\sin(\theta-\beta)}{\cos\theta\cot(\theta-\beta)+\sin\theta},$$

$$\dot{R} = v\,\frac{\cot(\theta-\beta)\cos\beta}{\cos\theta\cot(\theta-\beta)+\sin\theta}.$$

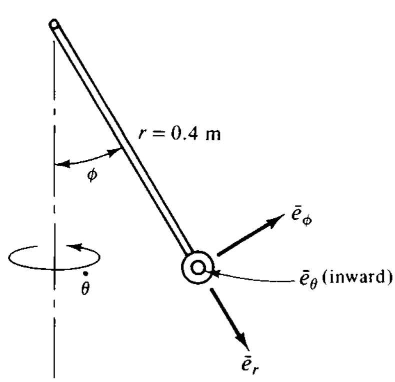
In order to simplify these relations, we multiply numerator and denominator of each by $\sin(\theta-\beta)$, and then employ the identity for the cosine of the difference of angles. This yields

$$\dot{\theta} = -\frac{v}{H}\,\frac{\sin^2(\theta-\beta)}{\cos\beta}, \qquad \dot{R} = v\cos(\theta-\beta).$$

Example 2.9 The flyball governor rotates about the vertical axis at the rate of 1,800 rev/min, and the angle ϕ locating the arms relative to the vertical is known to vary as $\phi = \frac{1}{3}\pi\sin(120\pi t)$ rad, where t is in units of seconds. Determine the velocity and the acceleration of sphere A as a function of time. Then evaluate these expressions for the instants when the elevation of the sphere is a maximum and a minimum.



Example 2.9



Spherical coordinate system.

Solution The motion of each sphere is described in terms of a distance from a fixed point, a rotation about a fixed axis through that point, and an angle relative to the axis, each of which matches a spherical coordinate description. The unit vectors in the sketch are in the sense of increasing coordinate values. (Note that the manner in which the angles are defined leads to a set of unit vectors for which $\bar{e}_r \times \bar{e}_\theta = \bar{e}_\phi$, unlike the definition in Figure 2.5(b).) The radial distance and rotation rate about the fixed axis are both constant, so

$$r = 0.40 \text{ m}, \qquad \dot{r} = \ddot{r} = 0;$$

$$\dot{\theta} = 1{,}800\left(\frac{2\pi}{60}\right) = 60\pi \text{ rad/s}, \qquad \ddot{\theta} = 0;$$

$$\phi = \tfrac{1}{3}\pi \sin(120\pi t) \text{ rad}, \qquad \dot{\phi} = 40\pi^2 \cos(120\pi t) \text{ rad/s};$$

$$\ddot{\phi} = -4{,}800\pi^3 \sin(120\pi t) \text{ rad/s}^2.$$

The spherical coordinate formulas corresponding to these parameters become

$$\bar{v} = 16\pi^2 \cos(120\pi t)\bar{e}_\phi + 24\pi \sin[\tfrac{1}{3}\pi \sin(120\pi t)]\bar{e}_\theta \text{ m/s},$$

$$\begin{aligned}\bar{a} = &-\{640\pi^4 \cos^2(120\pi t) + 1{,}440\pi^2 \sin^2[\tfrac{1}{3}\pi \sin(120\pi t)]\}\bar{e}_r \\ &-\{1{,}920\pi^3 \sin(120\pi t) + 720\pi^2 \sin[\tfrac{2}{3}\pi \sin(120\pi t)]\}\bar{e}_\phi \\ &+ 1{,}920\pi^3 \cos(120\pi t)\cos[\tfrac{1}{3}\pi \sin(120\pi t)]\bar{e}_\theta \text{ m/s}^2.\end{aligned}$$

These general results may now be evaluated at the desired instants. The highest elevation of a sphere corresponds to the maximum value $\phi = \frac{1}{3}\pi$, at which time $\sin(120\pi t) = 1$. Correspondingly, $\cos(120\pi t) = 0$ at this instant, which yields

$$\bar{v} = 65.30\bar{e}_\theta \text{ m/s}, \qquad \bar{a} = -10{,}659\bar{e}_r - 65{,}686\bar{e}_\phi \text{ m/s}^2.$$

The lowest elevation occurs at $\phi = 0$, $\cos(120\pi t) = \pm 1$, which yields

$$\bar{v} = \pm 157.91\bar{e}_\phi \text{ m/s}, \qquad \bar{a} = -62{,}342\bar{e}_r \pm 59{,}532\bar{e}_\theta \text{ m/s}^2.$$

Note that the $\pm$ sign arises because the sphere may be swinging in either direction at $\phi = 0$.

2.4 Joint Kinematical Descriptions

The degree to which the system parameters match those of a coordinate system is a key factor affecting the selection of a kinematical description. A specific concern is whether the quantities that are either given or to be determined are like those for the chosen description. For example, suppose that the path of a particle is known to be as shown in Figure 2.8. If the rate of movement along that path is specified in terms of the speed v, we would certainly want to employ a path-variable description. On the other hand, specification of the rate of motion in terms of the angle θ measured from the x axis would certainly suggest that cylindrical coordinates be employed.

We could consider the kinematical description that best matches the parameters of the actual system to be the "natural" one. We shall investigate here situations in which no one formulation is entirely natural, although more than one has elements

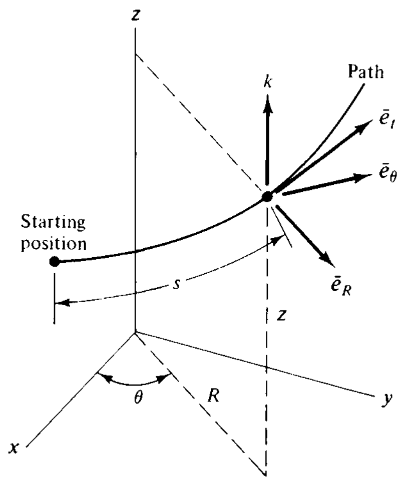


Figure 2.8 Joint usage of path variables and cylindrical coordinates.

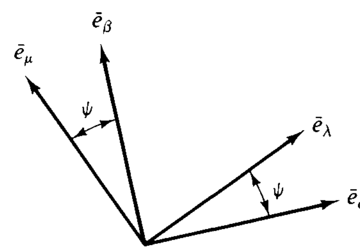


Figure 2.9 Transformation of unit vectors in a plane.

that are suitable. Such a situation arises for the path in Figure 2.8 when the rate of movement is given in terms of the speed, yet we desire to evaluate $\dot{R}$ and $\ddot{R}$. It is almost axiomatic that if one of the kinematical descriptions (such as path variables, Cartesian coordinates, or one of the curvilinear coordinate systems) has some aspect that suits a problem, then it should be employed. Thus, the task that confronts us here is to establish how to implement two different descriptions simultaneously.

The general concept is to match the velocities and accelerations obtained from each of the formulations of interest. This matching depends on the fact that the unit vectors for one formulation may be resolved into components relative to the other. For simplicity, let us begin by considering planar motion. Let $\bar{e}_\alpha, \bar{e}_\beta$ be the unit vectors for one kinematical description (e.g., $\bar{e}_\alpha$ and $\bar{e}_\beta$ are the tangent and normal directions), and let $\bar{e}_\lambda, \bar{e}_\mu$ be the unit vectors for the other description. These unit vectors are depicted in Figure 2.9.

As shown in the figure, the orientation of one set of unit vectors relative to the other is defined by the angle ψ. (The definition of ψ as the angle between $\bar{e}_\alpha$ and $\bar{e}_\lambda$ is arbitrary.) The components of $\bar{e}_\lambda$ and $\bar{e}_\mu$ relative to $\bar{e}_\alpha$ and $\bar{e}_\beta$ are found from this figure to be

$$\begin{aligned} \bar{e}_\lambda &= (\cos\psi)\bar{e}_\alpha + (\sin\psi)\bar{e}_\beta, \\ \bar{e}_\mu &= -(\sin\psi)\bar{e}_\alpha + (\cos\psi)\bar{e}_\beta. \end{aligned} \tag{2.50}$$

The velocity may be expressed in terms of components relative to either set of unit vectors. Thus,

$$\bar{v} = v_\alpha \bar{e}_\alpha + v_\beta \bar{e}_\beta = v_\lambda \bar{e}_\lambda + v_\mu \bar{e}_\mu. \tag{2.51}$$

We are assuming that at this stage, the first set of components v_α, v_β have been related to the parameters associated with (α, β) through the corresponding velocity formula, such as $\bar{v} = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta$ for polar coordinates. A similar operation is also assumed to have been performed for the second set of components. Each velocity component might contain unknown kinematical parameters.

The next step is to convert the (λ, μ) components to (α, β) components. This is achieved by substituting Eqs. (2.50) into Eq. (2.51), with the result that

$$\begin{aligned}\bar{v} &= v_\alpha \bar{e}_\alpha + v_\beta \bar{e}_\beta \\ &= v_\lambda \{[(\cos\psi)\bar{e}_\alpha] + [(\sin\psi)\bar{e}_\beta]\} + v_\mu \{[-(\sin\psi)\bar{e}_\alpha] + [(\cos\psi)\bar{e}_\beta]\}.\end{aligned}$$

These are two descriptions of the velocity in terms of the same set of components. Equality of vectors requires that their corresponding components be equal, which leads to the following algebraic relations:

$$\begin{aligned}v_\alpha &= v_\lambda \cos\psi - v_\mu \sin\psi, \\ v_\beta &= v_\lambda \sin\psi + v_\mu \cos\psi.\end{aligned} \tag{2.52}$$

These relations may be used to solve for two unknown parameters in the velocity components.

As an illustration of this procedure, suppose that (α, β) represents path variables and (λ, μ) represents polar coordinates. Substitution of the respective velocity components into Eq. (2.52) then yields

$$\begin{aligned}v &= \dot{R}\cos\psi - R\dot{\theta}\sin\psi, \\ 0 &= \dot{R}\sin\psi + R\dot{\theta}\cos\psi.\end{aligned}$$

The values of the radial distance R and the angle of orientation θ are known if the position is specified. Thus, the previous display shows two relations among the three rate variables v, $\dot{R}$, and $\dot{\theta}$.

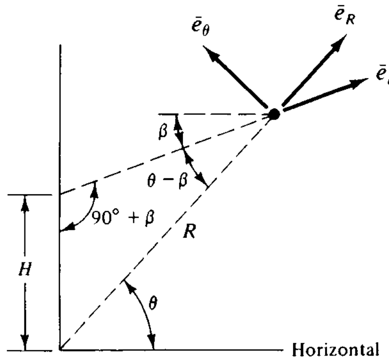
Relations such as those listed in Eqs. (2.52) could be used in either of two general situations. It might be that the velocity is already known in terms of components in the (λ, μ) system. In that case, Eqs. (2.52) provide the conversion to components and parameters associated with the (α, β) system. The more interesting situation is that of a mixed description, that is, one in which the velocity is only partially known in terms of either of the two descriptions. In that case, Eqs. (2.52) provide the means of ascertaining the unknown parameters in each system, and thereby the velocity itself.

The same approach may be applied to treat acceleration. Specifically, the individual formulas for acceleration may be matched by employing the unit vector transformation, Eq. (2.50). However, doing so requires that the velocity parameters, such as $|\bar{v}|$ or $\dot{R}$, be evaluated first because they occur in the acceleration components. In other words, the velocity relations must be solved before accelerations can be addressed.

This discussion has treated the case of planar motion, but the same procedure also applies to three-dimensional motion. The kinematical formulas in that case have three components, so matching corresponding components will lead to three simultaneous equations. The primary difficulty that arises in this extension is the evaluation of the transformation of the unit vectors. The component representation in Eq. (2.50) was achieved by visual projections of a unit vector onto the other directions. The same procedure may be performed in a three-dimensional case if the geometry is not too complicated. An alternative approach for determining the unit vector components when the configuration is difficult to visualize uses rotation transformation properties established in the next chapter.

Example 2.10 Use the concept of a joint kinematical description to determine $\dot{R}$ and $\dot{\theta}$ for the airplane in Example 2.8.

Solution The path and speed of the airplane are given, both of which are path-variable parameters. We must determine the rates of change of polar coordinates, which are cylindrical coordinates with $z = 0$. Thus, we draw a sketch that depicts the unit vectors for both formulations at an arbitrary θ.



Unit vectors.

The velocity in terms of each set of unit vectors is

$$\bar{v} = v\bar{e}_t = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta.$$

Resolving $\bar{e}_t$ into polar coordinate components yields

$$\bar{e}_t = \cos(\theta - \beta)\bar{e}_R - \sin(\theta - \beta)\bar{e}_\theta,$$

so that

$$\bar{v} = v\cos(\theta - \beta)\bar{e}_R - v\sin(\theta - \beta)\bar{e}_\theta = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta.$$

The result of matching like components is

$$\dot{R} = v\cos(\theta - \beta), \qquad \dot{\theta} = -\frac{v}{R}\sin(\theta - \beta).$$

All that remains is to express R in terms of θ, which we find from the law of sines as follows:

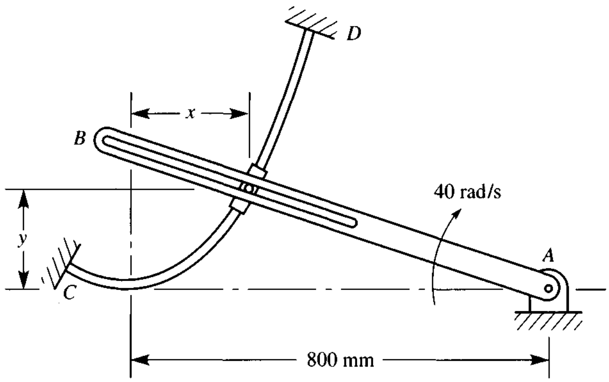
$$\frac{R}{\sin(\pi/2 + \beta)} = \frac{H}{\sin(\theta - \beta)} \Rightarrow R = H\frac{\cos\beta}{\sin(\theta - \beta)}.$$

Thus

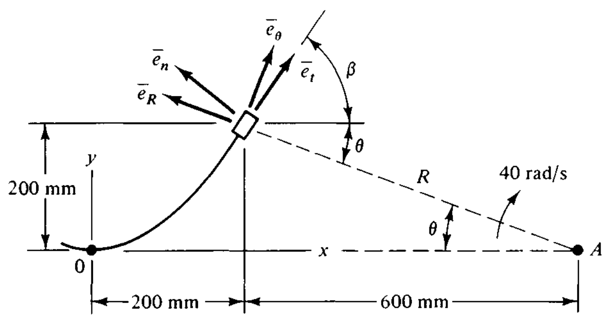
$$\dot{R} = v\cos(\theta - \beta), \qquad \dot{\theta} = -\frac{v}{H}\frac{\sin^2(\theta - \beta)}{\cos\beta}.$$

There is no doubt that this solution is easier than the one in Example 2.8. In essence, the joint kinematical description avoids the need to differentiate functions, because the various kinematical formulas are themselves derivatives of the position.

Example 2.11 Arm AB rotates clockwise at the constant rate of 40 rad/s as it pushes the slider along guide CD, which is described by $y = x^2/200$ (x and y are in



Example 2.11



Coordinate systems.

millimeters). Determine the velocity and acceleration of the collar when it is at the position $x = 200$ mm.

Solution The planar motion is specified by a rotation rate, but the path is not described in terms of polar coordinates. Hence, we shall follow an approach that employs path variables and polar coordinates. A sketch shows both sets of unit vectors at $x = 200$ mm, which corresponds to $y = x^2/200 = 200$ mm.

The polar coordinates are found from a right triangle to be

$$R = [(600)^2 + (200)^2]^{1/2} = 632.5 \text{ mm}, \qquad \theta = \tan^{-1}\left(\frac{200}{600}\right) = 18.435°.$$

The slope of the guide bar at this location yields the angle of orientation of the tangent vector:

$$\beta = \tan^{-1}\left(\frac{dy}{dx}\right) = \tan^{-1}\left(\frac{x}{100}\right) = 63.435°.$$

Matching like velocity components in each formulation is the next step. We find that

$$\bar{v} = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta = v\bar{e}_t = v[-\cos(\theta+\beta)\bar{e}_R + \sin(\theta+\beta)\bar{e}_\theta];$$

$$\bar{v}\cdot\bar{e}_R = \dot{R} = -v\cos(\theta+\beta), \qquad \bar{v}\cdot\bar{e}_\theta = R\dot{\theta} = v\sin(\theta+\beta).$$

The value of $\dot{\theta}$ is given to be 40 rad/s, and R, θ, and β have been evaluated. The corresponding results obtained from these relations are

$$v = 25{,}557 \text{ mm/s} = 25.56 \text{ m/s},$$

$$\dot{R} = -3{,}614 \text{ mm/s} = -3.614 \text{ m/s}.$$

Because we have evaluated all velocity parameters, we may now follow a similar procedure for acceleration. In order to resolve $\bar{e}_n$, we note that it is oriented toward the center of curvature, which is up and to the left for the parabolic curve;

$$\bar{a} = (\ddot{R} - R\dot{\theta}^2)\bar{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\bar{e}_\theta = \dot{v}\bar{e}_t + \frac{v^2}{\rho}\bar{e}_n$$

$$= \dot{v}[-\cos(\theta+\beta)\bar{e}_R + \sin(\theta+\beta)\bar{e}_\theta] + \frac{v^2}{\rho}[\sin(\theta+\beta)\bar{e}_R + \cos(\theta+\beta)\bar{e}_\theta].$$

The unknowns in these equations are $\dot{v}$, $\ddot{R}$, and ρ. We compute the latter from Eq. (2.22), which gives, for $x = 200$ mm,

$$\rho = \frac{[1 + (x/100)^2]^{3/2}}{(1/100)} = 223.6 \text{ mm}.$$

The result of matching like acceleration components is

$$\bar{a}\cdot\bar{e}_R = \ddot{R} - R\dot{\theta}^2 = -\dot{v}\cos(\theta+\beta) + \frac{v^2}{\rho}\sin(\theta+\beta),$$

$$\bar{a}\cdot\bar{e}_\theta = R\ddot{\theta} + 2\dot{R}\dot{\theta} = \dot{v}\sin(\theta+\beta) + \frac{v^2}{\rho}\cos(\theta+\beta).$$

The value of $\dot{\theta}$ is constant, and we found $\dot{R}$ and v earlier. Hence, we solve the a_θ equation for $\dot{v}$, then substitute that result into the equation for a_R:

$$\dot{v} = \frac{2\dot{R}\dot{\theta} - (v^2/\rho)\cos(\theta+\beta)}{\sin(\theta+\beta)} = -3.755(10^5) \text{ mm/s}^2,$$

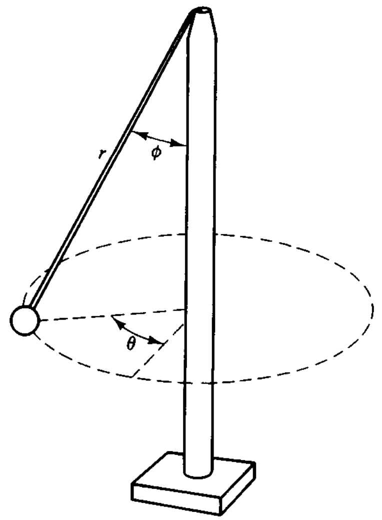
$$\ddot{R} = R\dot{\theta}^2 - \dot{v}\cos(\theta+\beta) + \frac{v^2}{\rho}\sin(\theta+\beta) = 1.6435(10^6) \text{ mm/s}^2.$$

The parameters that were requested are the velocity and acceleration, which we may form in terms of either kinematical formulation. Thus

$$\bar{v} = 25.56\bar{e}_t \text{ m/s} = -3.614\bar{e}_R + 25.30\bar{e}_\theta \text{ m/s},$$

$$\bar{a} = -375.5\bar{e}_t + 584.2\bar{e}_n \text{ m/s}^2 = 631.4\bar{e}_R - 289.1\bar{e}_\theta \text{ m/s}^2.$$

Example 2.12 The cord suspending a spherical pendulum is pulled in at a constant rate of 5 m/s. At the instant when the pendulum is 2 m long, the azimuth angle $\theta = 0°$ and the angle of inclination of the cable is $\phi = 30°$. At this instant, $\dot{\theta} = 2$ rad/s, $\ddot{\theta} = 0$, $\dot{\phi} = -5$ rad/s, and $\ddot{\phi} = -10$ rad/s^2. Determine the speed and the rate of change of the speed of the small body at the end of the cable at this instant. Also, determine the corresponding radius of curvature of the body's path at this position.



Example 2.12

Solution In this situation the motion is fully specified in terms of spherical coordinates centered at the top of the post, whereas the desired parameters are path variables. Hence, the procedure is to construct the velocity and acceleration using spherical coordinates, and then to relate those expressions to the path-variable formulas.

Because the cable is being pulled in at a constant rate, we set $\dot{r} = -5$ m/s and $\ddot{r} = 0$ when $r = 2$ m. Substitution of these values and the stated rotation rates into Eqs. (2.49) gives

$$\bar{v} = -5\bar{e}_r - 10\bar{e}_\phi + 2\bar{e}_\theta \text{ m/s},$$

$$\bar{a} = -52\bar{e}_r + 26.54\bar{e}_\phi - 44.64\bar{e}_\theta \text{ m/s}^2.$$

The speed is the magnitude of the velocity, so

$$v = (v_r^2 + v_\phi^2 + v_\theta^2)^{1/2} = 11.358 \text{ m/s}.$$

Because $\bar{v} = v\bar{e}_t$, we find that the tangent vector is

$$\bar{e}_t = \bar{v}/v = -0.4402\bar{e}_r - 0.8805\bar{e}_\phi + 0.17609\bar{e}_\theta.$$

We know that $\dot{v}$ is the tangential component of acceleration, which we find by using a dot product:

$$\dot{v} = \bar{a} \cdot \bar{e}_t = -8.336 \text{ m/s}^2.$$

Finally, in order to evaluate ρ, we form the difference between the total acceleration and the tangential acceleration. Specifically,

$$\frac{v^2}{\rho}\bar{e}_n = \bar{a} - \dot{v}\bar{e}_t = -55.67\bar{e}_r + 19.20\bar{e}_\phi - 43.17\bar{e}_\theta \text{ m/s}^2.$$

We evaluate the magnitude of this acceleration and use the value of v to find

$$\rho = \frac{v^2}{(55.67^2+19.20^2+43.17^2)^{1/2}} = 1.7668 \text{ m}.$$

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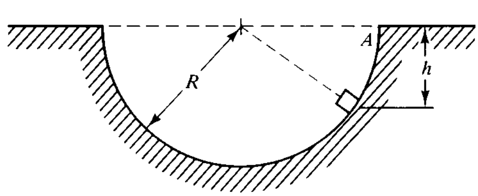
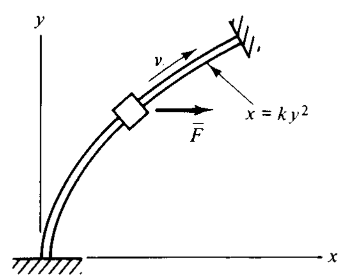
Problems

2.1 A particle follows a planar path defined by $x = k\zeta^2$ and $y = 2k[1-\exp(\zeta)]$, such that its speed is $v = \beta\zeta$, where k and β are constants. Determine the velocity and acceleration at $\zeta = 0.5$.

2.2 A small block slides in the interior of a smooth semicircular cylinder. Because friction is negligible, the speed of the block is given by $v^2 = 2gh$, where h is the vertical distance the block has fallen. Determine the velocity and acceleration of the block as a function of the distance the block travels in a case where the block is released at position A.



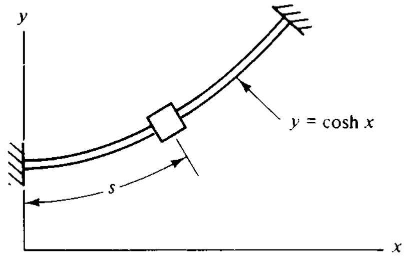
Problem 2.2



Problem 2.3

2.3 The collar slides over the stationary guide defined by $x = ky^2$ in the vertical plane. The speed of the collar is the constant value v. This motion is implemented by applying a force $\bar{F}$ of variable magnitude parallel to the x axis. Derive an expression for the magnitude of $\bar{F}$ and of the normal reaction as functions of the y coordinate of the collar.

2.4 A slider moves over a curved guide whose shape in the vertical plane is given by $y = \cosh x$. Starting from $x = 0$, the speed is observed to vary as $v = v_0(1-ks)$, where s



Problem 2.4

is the distance traveled and k is a constant. Derive expressions for the velocity and acceleration of the slider as a function of x.

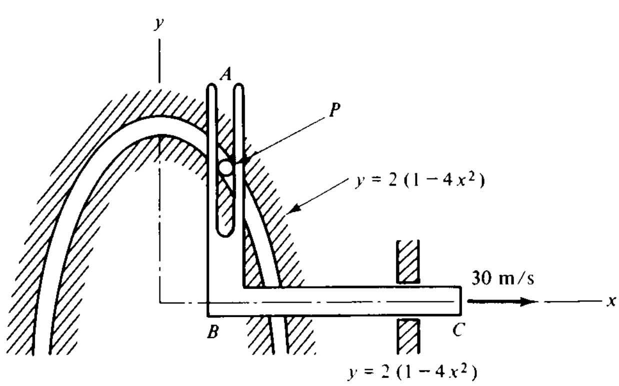
2.5 A helix is defined by $x = ec\psi$, $y = e\sin(c\psi)$, $z = -e\cos(c\psi)$. Determine the path variable unit vectors, the radius of curvature, and the torsion of this curve as a function of ψ.

2.6 A particle moves along the paraboloid of revolution $y = (x^2 + z^2)/\alpha$, such that $x = -\alpha \sinh k\zeta$ and $z = \alpha \cosh k\zeta$, where x, y, and z are in meters, ζ is a parameter, and α and k are constants. At the position where $\zeta = 1/k$, the particle's speed is $5\alpha k$ and its speed is decreasing at the rate $2\alpha k^2$. Determine the velocity and acceleration at this position.

2.7 Determine the radius of curvature and the torsion of the path in Problem 2.6 at the given position.

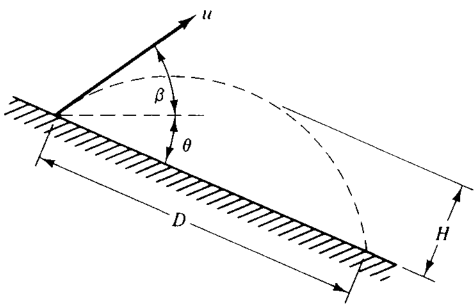
2.8 A particle moves along the paraboloid of revolution $z = (x^2 + y^2)/k$ such that $x = k\omega\zeta \sin \omega\zeta$ and $y = k\omega^2\zeta^2$, where k and ω are constants and ζ is a parameter. Consider the case where the parameter $\zeta = t^2$, where t is measured in seconds. Derive expressions for the velocity and acceleration.

2.9 Pin P is pushed by arm ABC through the groove, $y = 2(1 - 4x^2)$, where x and y are in meters. The velocity of arm ABC is constant at 30 m/s to the right. Determine the velocity and acceleration of the pin at the position $x = 0.25$ m.

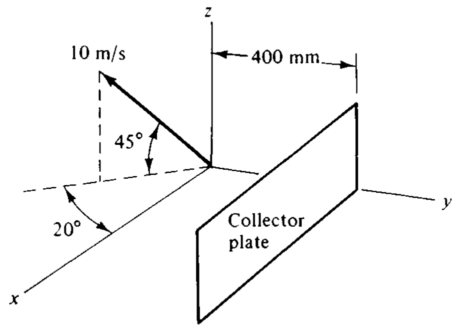


Problem 2.9

2.10 A ball is thrown down an incline whose angle of elevation is θ. The initial velocity is u at an angle of elevation β. Derive an expression for the distance D measured along the incline at which the ball will return to the incline. Also determine the maximum



Problem 2.10



Problems 2.11 and 2.12

height H (measured perpendicularly to the incline) of the ball, and the corresponding velocity of the ball at that position.

2.11 A 10-mg dust particle is injected into an electrostatic precipitator with an initial velocity of 10 m/s, as shown. The z axis is vertical and the electrostatic force is 2 mN, acting in the positive y direction. Determine the location at which the dust particle will strike a collector plate that is situated in the vertical plane, $y = 400$ mm.

2.12 A 10-mg dust particle is injected into an electrostatic precipitator with an initial velocity of 10 m/s, as shown. The z axis is vertical and the attractive force on the particle is $0.4 - y$ mN acting in the positive y direction, where y is measured in meters. Determine the location at which the dust particle will strike a collector plate that is situated in the vertical plane, $y = 400$ mm.

2.13 For laminar flow at low Reynolds number, the air resistance on an object is $-c\bar{v}$, where c is a constant and $\bar{v}$ is the velocity of the object. A sphere of mass m is thrown from the ground with an initial speed v_0 at an angle of elevation β in the (vertical) x-y plane. Determine the position and velocity of the sphere as a function of time.

2.14 Current flowing through a coiled wire sets up a magnetic field $\bar{B}$ that is essentially constant in magnitude and parallel to the axis of the coil, so $\bar{B} = B\bar{k}$. The force acting on a charged particle moving through this field at velocity $\bar{v}$ is given by $\bar{F} = \beta\bar{v} \times \bar{B}$, where β is a constant. Suppose such a particle is injected into this field at the origin, with an arbitrary initial velocity. Derive an expression for the position of this particle as a function of time, and identify the corresponding path. Assume that gravity is negligible.

2.15 Derive the expressions for velocity and acceleration in terms of spherical coordinates.

2.16 A ball rolls on the interior of a paraboloid of revolution given by $x^2 + y^2 = cz$. The angle of rotation about the z axis is $\theta = 4\pi \sin^2(\omega t)$, and the elevation of the ball is $z = b\theta$, where b, c, and ω are constants. Determine the velocity and acceleration when $t = 4\pi/3\omega$.

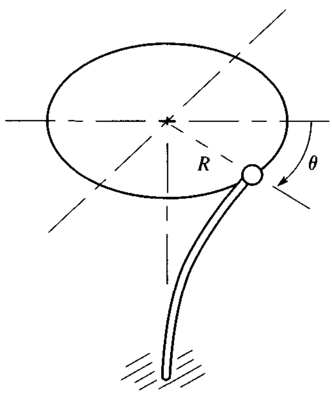
2.17 In a Eulerian description of fluid flow, particle velocity components are described as functions of the current position of a particle. The polar velocity components of fluid particles in a certain flow are known to be $v_R = (A \cos \phi)/R^2$ and $v_\phi = (A \sin \phi)/R^2$, where R and ϕ are the polar coordinates of the particle. Determine the corresponding expressions for the acceleration.

2.18 Observation of a small mass attached to the end of the flexible bar reveals that the path of the particle is essentially an ellipse in the horizontal plane. The polar coordinates for this motion are

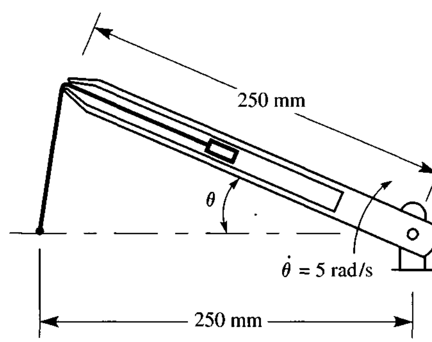
$$R = \frac{ab}{[b^2 + (a^2 - b^2)(\cos\theta)^2]^{1/2}}, \qquad \dot{\theta} = \frac{K}{R^2},$$

where $2a$ and $2b$ are the diameters of the ellipse. Determine the acceleration of the particle in an arbitrary position.

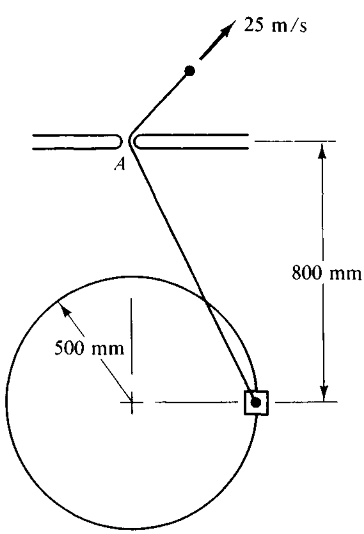
2.19 The cable, whose length is 250 mm, is fastened to the 500-g block. Clockwise rotation of the arm at a constant angular speed of 5 rad/s causes the block to slide outward. The motion occurs in the vertical plane, and frictional resistance is negligible. Determine the tensile force in the cable and the force exerted by the block on the walls of the groove when $\theta = 53.1301°$.



Problem 2.18



Problem 2.19

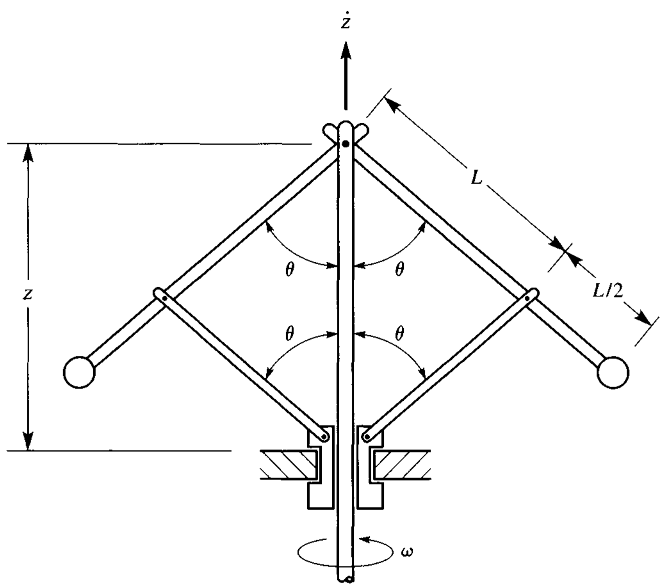


Problem 2.21

2.20 A particle follows a planar path defined in polar coordinates by $R = R(\theta)$ such that $\dot{\theta}$ is constant. Derive expressions for the velocity and acceleration of the particle. Then use those results to derive an expression for the radius of curvature of a path in polar coordinates.

2.21 A cable that passes through a hole at point A is pulled inward at the constant rate of 25 m/s, thereby causing the 0.2-kg collar to move along the circular guide bar. The system is situated in the vertical plane. Determine the speed and the rate of change of the speed of the slider at the instant shown in the sketch. Also evaluate the corresponding tension in the cable.

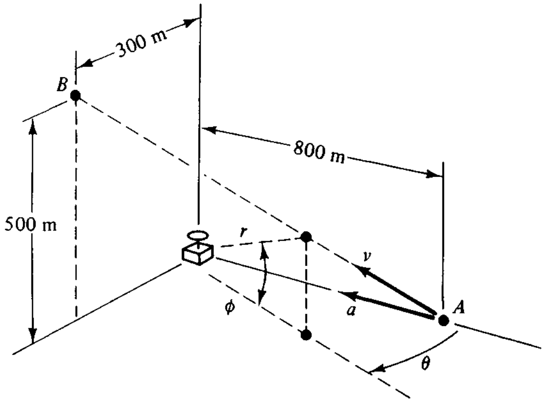
2.22 In the flyball governor shown, changes in the radial distance to the balls results in changing the length z, which may be measured magnetically. Consider a situation where the angular speed ω for rotation about the vertical axis is constant, and the length is changing at the constant rate $\dot{z}$. Derive an expression for the acceleration of a ball in terms of the angle θ for the links.



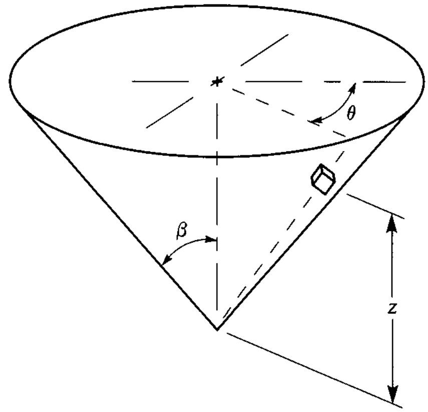
Problem 2.22

2.23 (See figure, next page.) A radar station at the origin measures the azimuth angle θ, the elevation angle ψ, and the radial distance r to a target. At the instant when a missile passes position A, its velocity is 500 m/s directed from point A to point B. Its acceleration at this instant is $10g$, directed toward the origin. Determine the first and second derivatives of these position parameters at this instant.

2.24 (See figure, next page.) A small block is pushed along the interior of the stationary cone such that the azimuth angle to the block is $\theta = \frac{1}{2}\alpha t^2$, where α is a constant. The vertical distance measured from the apex of the cone is a specified function of time $z(t)$. Derive expressions for the velocity and acceleration of the block using (a) cylindrical coordinates and (b) spherical coordinates.

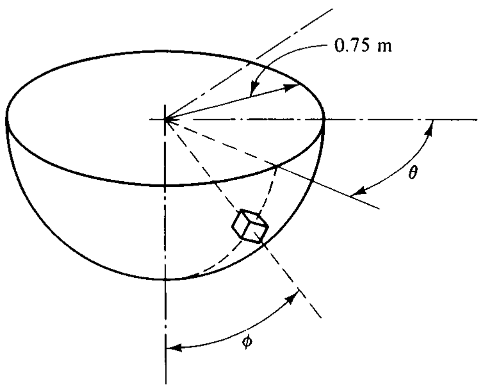


Problem 2.23



Problem 2.24

2.25 A 250-gram block is pushed over the smooth interior of a hemispherical shell whose interior radius is 0.75 meters. The force $\bar{F}$ causing this motion acts tangentially to the hemispherical surface. The block rotates about the vertical centerline at the constant rate of one revolution in 2 seconds, and it moves inward such that the polar angle is $\phi = \pi/2 - 0.5t^2$. Determine the force $\bar{F}$ at the instant when $\phi = \pi/6$.

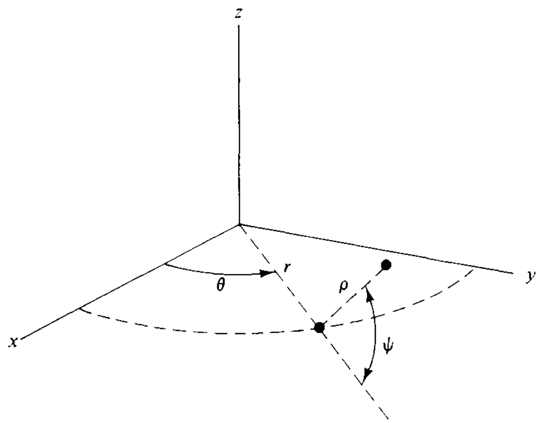


Problem 2.25

2.26 Toroidal coordinates (ρ, θ, ψ) are useful for magnetohydrodynamic studies in the fusion reactor known as a *tokamak*. These coordinates reference position to a circle of radius r such that the transformation to Cartesian coordinates is $x = (r + \rho \cos \psi) \cos \theta$, $y = (r + \rho \cos \psi) \sin \theta$, $z = \rho \sin \psi$. Derive expressions for the unit vectors for this coordinate system, and then describe the derivatives of the unit vectors with respect to the toroidal coordinates.

2.27 Obtain expressions for velocity and acceleration in terms of the toroidal coordinates in Problem 2.26.

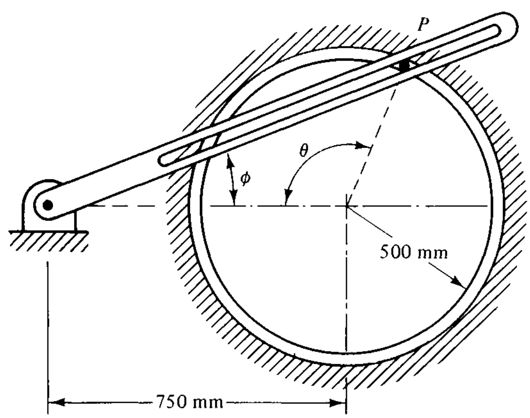
2.28 The instantaneous velocity of a point is $\bar{v} = 10\bar{i} - 4\bar{j} + 6\bar{k}$ m/s, and the acceleration is $\bar{a} = -30\bar{i} - 25\bar{j} + 15\bar{k}$ m/s^2. Determine the corresponding speed, rate of change of the speed, and radius of curvature of the path.



Problems 2.26 and 2.27

2.29 An airplane heading eastward is observed to be in a 20° climb at a speed of 2,400 km/hr. At this instant its acceleration components are $2g$ eastward, $5g$ northward, and $1.5g$ downward. Determine the rate of change of the speed, as well as the radius of curvature and the location relative to the airplane of the center of curvature of the path.

2.30 Pin P slides inside the 500-mm radius groove at a constant speed of 1 m/s. Determine the values of $\dot{\phi}$ and $\ddot{\phi}$ at the instant when $\theta = 90°$.

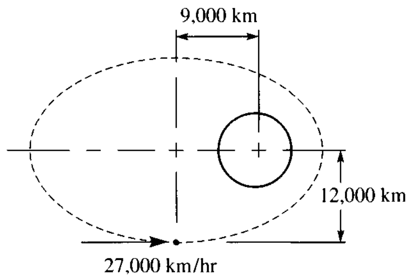


Problems 2.30 and 2.31

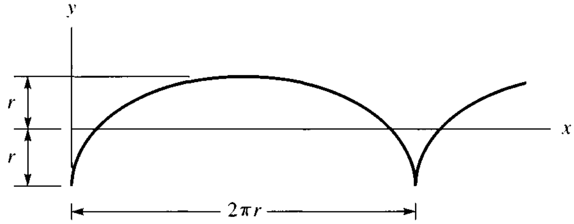
2.31 Solve Problem 2.30 for the instant when $\theta = 135°$.

2.32 The elevation of the center of mass of an automobile following an extremely bumpy road is observed to be $y = 0.1 \sin(\pi x/3)$, with x and y in meters. Its speed at $x = 1$ m is measured as 20 m/s, and the speed at that position is decreasing at 5 m/s^2. Determine the horizontal and vertical components of the acceleration at that instant.

2.33 A body is in an elliptical orbit about the earth. The magnitude of the acceleration of this body is $g(R_e/R)^2$, where R is the distance from the body to the center of the



Problem 2.33



Problem 2.34

earth, $R_e = 6{,}370$ km, and $g = 9.807$ m/s^2. At the position shown, the speed of the body is 27,000 km/hr. Determine the rate of change of the speed and the radius of curvature of the orbit at this position.

2.34 A wheel, whose radius is r, rolls without slipping. A point on the perimeter of the wheel follows a cycloidal path, described in parametric form by

$$x = r(\xi - \sin\xi), \qquad y = -r\cos\xi.$$

The parameter ξ is observed to depend on time according to $\xi = ct$. Derive expressions for the speed and the rate of change of speed of this point as a function of ξ. Also determine the radius of curvature of the cycloid as a function of ξ.

2.35 A particle moves in a helical path defined in terms of cylindrical coordinates by $R = b$ and $z = \frac{3}{4}R\theta$. The normal component of acceleration is known to vary with time according to $a_n = c(\cos\omega t)^2$.

(a) Derive an expression for the earliest time for which the speed is a maximum. Determine the speed and acceleration at that instant.

(b) Derive an expression for the earliest time at which the tangential acceleration is a maximum. Determine the speed and the acceleration at that instant.

CHAPTER 3

Relative Motion

A moving body, such as an automobile, frequently provides a useful reference frame for our observations of motion. Even when we are not moving, it is often easier to describe the motion of a point by reference to a moving object. This is the case for many common machines, such as linkages. In this chapter we shall develop the ability to correlate observations of position, velocity, and acceleration from fixed and moving reference frames.

Figure 3.1 depicts a general situation in which point P is being observed from a moving reference frame xyz, whereas XYZ is a fixed reference frame. In order to make use of xyz, we must know the absolute position of its origin, $\bar{r}_{O'/O}$. It is apparent from Figure 3.1 that the absolute position $\bar{r}_{P/O}$ is related to the relative position $\bar{r}_{P/O'}$ by

$$\blacklozenge \qquad \bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}. \tag{3.1}$$

We must represent each of these vectors with respect to components in the same direction. Let us introduce for this purpose the $\hat{x}\hat{y}\hat{z}$ coordinate system, whose origin always coincides with point O', but whose axes always remain parallel to the respective fixed axes of XYZ. Such a reference frame executes a *translational motion*. The coordinates of point P with respect to this reference frame are $(\hat{x}_P, \hat{y}_P, \hat{z}_P)$. Because of the parallelism of the axes we may decompose Eq. (1) into its components, so that

$$X_P = X_O + \hat{x}_P, \qquad Y_P = Y_O + \hat{y}_P, \qquad Z_P = Z_O + \hat{z}_P. \tag{3.2}$$

This conversion between coordinates is referred to as a *translation transformation*.

Now consider describing $\bar{r}_{P/O'}$ by giving the coordinates (x_P, y_P, z_P) measured with respect to xyz. This complicates the task of adding $\bar{r}_{O'/O}$ and $\bar{r}_{P/O'}$, because the axes used to represent the vectors are not parallel. Relating the components of $\bar{r}_{P/O'}$ with respect to xyz to those measured relative to $\hat{x}\hat{y}\hat{z}$ requires a rotation transformation.

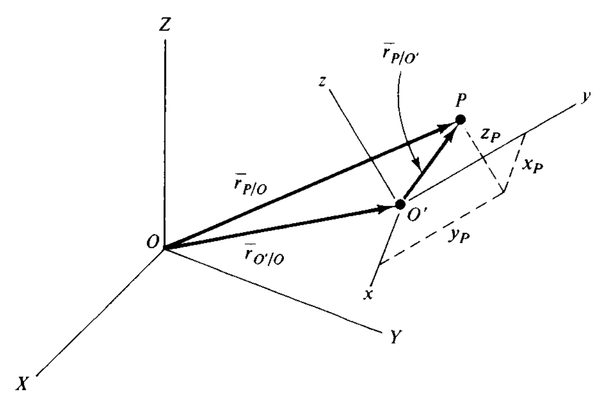


Figure 3.1 Absolute and relative position.

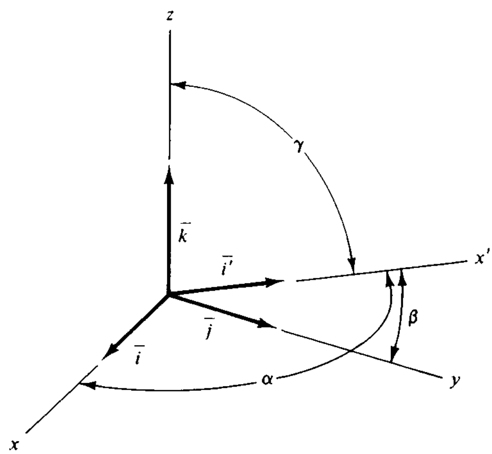


Figure 3.2 Direction angles.

3.1 Rotation Transformations

Let us consider a general situation in which two coordinate systems, xyz and $x'y'z'$, are employed to represent the components of a vector. Only the orientation of the axes is of interest here, so the origins of the coordinate systems coincide. Figure 3.2 depicts the *direction angles* α, β, γ between the x' axis and each of the xyz axes. An examination of Figure 3.2 shows that the values of the direction angles should be limited to the range $0 \leq \alpha, \beta, \gamma \leq \pi$ in order to avoid ambiguity. The components of $\bar{i}'$ are the projections of the vector onto the axes of xyz, which, in turn, are determined from the direction angles according to

$$\begin{aligned} \bar{i}' &= (\bar{i}'\cdot\bar{i})\bar{i} + (\bar{i}'\cdot\bar{j})\bar{j} + (\bar{i}'\cdot\bar{k})\bar{k} \\ &= (\cos\alpha)\bar{i} + (\cos\beta)\bar{j} + (\cos\gamma)\bar{k}. \end{aligned} \tag{3.3}$$

This expression indicates that the cosines of the direction angles are more significant to our investigation: they are the *direction cosines*. We obviously are equally interested in all the unit vectors. Thus:

♦ *Define $l_{p'q} = l_{qp'}$ to be the cosine of the angle between axis p' and axis q, with p and q representing x, y, or z.*

Extending Eq. (3.3) to the other unit vectors then yields

$$\begin{aligned} \bar{i}' &= l_{x'x}\bar{i} + l_{x'y}\bar{j} + l_{x'z}\bar{k}, \\ \bar{j}' &= l_{y'x}\bar{i} + l_{y'y}\bar{j} + l_{y'z}\bar{k}, \\ \bar{k}' &= l_{z'x}\bar{i} + l_{z'y}\bar{j} + l_{z'z}\bar{k}. \end{aligned} \tag{3.4}$$

It is convenient to rewrite these equations in matrix form as

♦ $$\begin{Bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{Bmatrix} = [R] \begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix}, \tag{3.5}$$

where

$$\blacklozenge \qquad [R] = \begin{bmatrix} l_{x'x} & l_{x'y} & l_{x'z} \\ l_{y'x} & l_{y'y} & l_{y'z} \\ l_{z'x} & l_{z'y} & l_{z'z} \end{bmatrix}. \tag{3.6}$$

The matrix $[R]$ is the *rotation transformation matrix.* It is a generalization of the conversion between coplanar pairs of unit vectors that we employed to discuss joint kinematical descriptions in Section 2.4.

Several important properties of $[R]$ follow from the fact that $\bar{i}, \bar{j}, \bar{k}$ constitute an orthogonal set of unit vectors, as do $\bar{i}', \bar{j}', \bar{k}'$. Suppose we were to follow steps parallel to the preceding in order to establish the transformation $[R']$, describing the unit vectors $\bar{i}, \bar{j}, \bar{k}$ in terms of their components with respect to $x'y'z'$. By direct analogy with Eqs. (3.5) and (3.6), we have

$$\begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix} = [R'] \begin{Bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{Bmatrix}, \tag{3.7}$$

where the elements of $[R']$ are the corresponding direction cosines between an axis of xyz and an axis of $x'y'z'$. For example, $R'_{12} = l_{xy'}$. However, because $l_{xy'} \equiv l_{y'x}$, it must be that $R'_{12} = R_{21}$. More generally, it follows from the definition of the direction cosines that $R'_{nm} = R_{mn}$, so $[R'] = [R]^{\mathrm{T}}$.

A different relation for $[R']$ results from solving Eq. (3.5) for the unit vectors xyz, which gives

$$\begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix} = [R]^{-1} \begin{Bmatrix} \bar{i}' \\ \bar{j}' \\ \bar{k}' \end{Bmatrix}. \tag{3.8}$$

A comparison of Eqs. (3.6) and (3.7) shows that $[R'] = [R]^{-1}$. Thus, we find that:

♦ *The matrix $[R']$ representing the inverse transformation is the inverse of the original transformation matrix $[R]$, which is identical to the transpose of $[R]$;*

$$\blacklozenge \qquad [R'] = [R]^{-1} = [R]^{\mathrm{T}}. \tag{3.9}$$

Matrices satisfying Eq. (3.9) are said to be *orthonormal.* The terminology arises from consideration of the identity that results from Eq. (3.9), $[R][R]^{\mathrm{T}} = [U]$, where $[U]$ is the unit identity matrix. To obtain an element in this product we observe that a column of $[R]^{\mathrm{T}}$ is the same as a row of $[R]$. Thus, the elements of the product are

$$l_{p'x} l_{q'x} + l_{p'y} l_{q'y} + l_{p'z} l_{q'z} = \delta_{pq}, \quad p, q = x, y, z, \tag{3.10}$$

where δ_{pq} denotes the Kronecker delta; $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ otherwise. Because the left side of Eq. (3.10) is the dot product of $\bar{e}_{p'}$ and $\bar{e}_{q'}$, we see that the identity $[R][R]^{\mathrm{T}} = [U]$ is a statement that the unit vectors of $x'y'z'$ are mutually orthogonal.

The equality $[R][R]^{\mathrm{T}} = [U]$ gives rise to an important property. Recall from matrix algebra that the determinant of a product of matrices is identical to the product of the individual determinants. Furthermore, the determinant of $[R]^{\mathrm{T}}$ is identical to the determinant of $[R]$. Simultaneous satisfaction of both properties leads to the conclusion that $|[R]| = 1$, which is a useful check for computations.

Equation (3.10) is symmetric in p and q. Hence, it represents six equations (one for each p, q pair) relating the nine direction cosines. It follows that there are only three independent direction angles. The selection of which angles are arbitrary is not entirely free. For example, the values of α, β, and γ in Figure 3.2 are not independent because $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. This restriction arises because these three angles locate only one axis.

The importance of the transformation matrix stems from the fact that it relates the components of arbitrary vectors with respect to two coordinate systems, not just the unit vectors. In order to demonstrate this feature, we recall that any vector $\bar{A}$ is independent of the coordinate system used to describe its components, so

$$\bar{A}=A_x\bar{i}+A_y\bar{j}+A_z\bar{k}=A_{x'}\bar{i}'+A_{y'}\bar{j}'+A_{z'}\bar{k}'. \tag{3.11}$$

This expression may be written in matrix form as

$$[\bar{i}'\ \bar{j}'\ \bar{k}']\begin{Bmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{Bmatrix}=[\bar{i}\ \bar{j}\ \bar{k}]\begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}. \tag{3.12}$$

In order to eliminate the unit vectors, substitute the transpose of Eq. (3.7) into (3.12). The transpose of a product is the product of the transposes, so

$$[\bar{i}'\ \bar{j}'\ \bar{k}']\begin{Bmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{Bmatrix}=[\bar{i}'\ \bar{j}'\ \bar{k}'][R']^{\mathrm{T}}\begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}. \tag{3.13}$$

In view of the inverse property in Eq. (3.9), the foregoing reduces to

♦ $$\begin{Bmatrix}A_{x'}\\A_{y'}\\A_{z'}\end{Bmatrix}=[R]\begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}. \tag{3.14}$$

A particularly simple type of transformation arises when the $x'y'z'$ system may be pictured as being the result of a rotation about one of the axes of the xyz coordinate system. The three possibilities, involving rotation about either the x, y, or z axis, are depicted in Figure 3.3. Denote the corresponding transformation matrices by a subscript that corresponds to the rotation axis. Then Eq. (3.6) leads to

$$[R_x]=\begin{bmatrix}1&0&0\\0&\cos\theta_x&\sin\theta_x\\0&-\sin\theta_x&\cos\theta_x\end{bmatrix},\quad [R_y]=\begin{bmatrix}\cos\theta_y&0&-\sin\theta_y\\0&1&0\\\sin\theta_y&0&\cos\theta_y\end{bmatrix}, \tag{3.15a,b}$$

$$[R_z]=\begin{bmatrix}\cos\theta_z&\sin\theta_z&0\\-\sin\theta_z&\cos\theta_z&0\\0&0&1\end{bmatrix}. \tag{3.15c}$$

Note that in each of the above, the rotation angle is positive in the sense of the right-hand rule. Specifically, if the extended thumb of the right hand points in the positive sense of the axis when the fingers curl in the sense of the rotation, then the angle is positive.

One of the most common types of vectors to be involved in a rotation transformation is the position of a point with respect to the origin. In such cases the vector

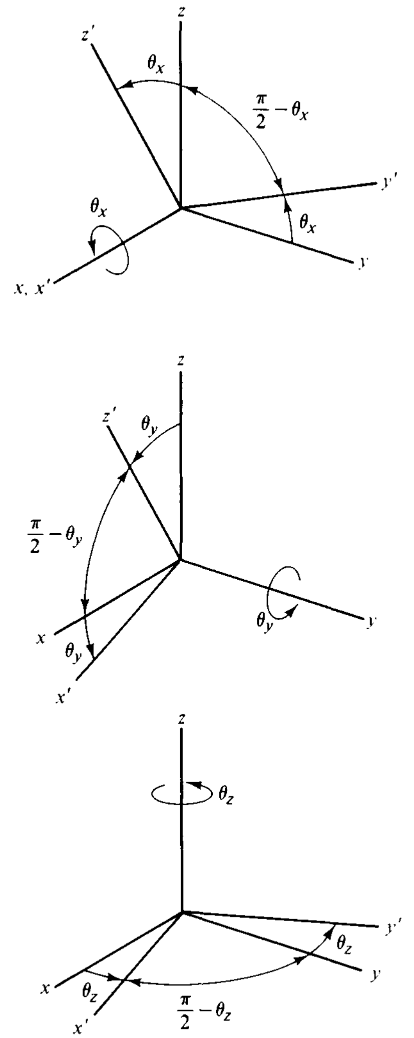


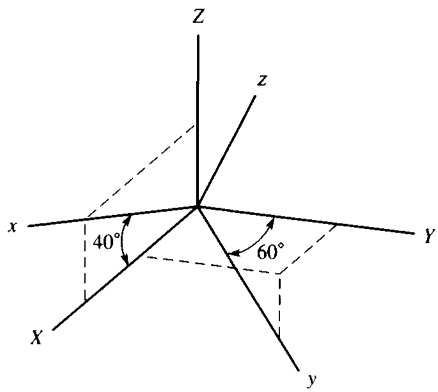
Figure 3.3 Rotations about coordinate axes.

components appearing in Eq. (3.14) describe the position coordinates of the point relative to $x'y'z'$ in terms of its components relative to xyz. The transformation of position coordinates may be used in two ways. Sometimes the position of the point with respect to a fixed reference frame XYZ is known. The position coordinates relative to a reference frame that has moved away from XYZ may be found directly from Eq. (3.14). A less obvious situation arises when we follow a point that remains fixed relative to a moving reference frame. In that case, the coordinates of the point relative to the *rotated* reference frame do not change from the values they had prior to any rotation. The subsequent coordinates of the point relative to the *fixed* reference frame may then be found by inverting Eq. (3.14), for which the orthonormal property, Eq. (3.9), is suitable.

Example 3.1 A force $\bar{F}$ may be described in terms of its components with respect to either the XYZ or xyz reference frames shown in the sketch.

(a) If $\bar{F} = 100\bar{I} - 50\bar{J} + 150\bar{K}$ N, determine the components of the force relative to the xyz coordinate system.

(b) If $\bar{F} = 100\bar{i} - 50\bar{j} + 150\bar{k}$ N, determine the components of the force relative to the XYZ coordinate system.



Example 3.1

Solution We will find the transformation matrix by first resolving the unit vectors of xyz into components relative to XYZ and then imposing the orthogonality condition. (A simpler solution may be obtained by following the methods in the next section.) Information regarding orientation of the x and y axes is given, so we write the associated unit vectors in terms of their direction angles relative to XYZ:

$$\bar{i} = l_{xX}\bar{I} + l_{xY}\bar{J} + l_{xZ}\bar{K},$$
$$\bar{j} = l_{yX}\bar{I} + l_{yY}\bar{J} + l_{yZ}\bar{K}.$$

Because the x axis lies in the X–Z plane at an angle of 40° from the X axis, and the direction angle between the Y and y axes is 60°, we have

$$l_{xX} = \cos 40^\circ, \qquad l_{xY} = 0, \qquad l_{xZ} = \sin 40^\circ, \qquad l_{yY} = \cos 60^\circ.$$

These expressions indicate that

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = \begin{bmatrix} \cos 40^\circ & 0 & \sin 40^\circ \\ l_{yX} & \cos 60^\circ & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{bmatrix}.$$

Because $[R]$ is an orthogonal transformation, we set $[R][R]^{\mathrm{T}} = [U]$:

$$\begin{bmatrix} 0.7660 & 0 & 0.6428 \\ l_{yX} & 0.50 & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{bmatrix}\begin{bmatrix} 0.7660 & l_{yX} & l_{zX} \\ 0 & 0.50 & l_{zY} \\ 0.6428 & l_{yZ} & l_{zZ} \end{bmatrix} = [U].$$

The corresponding equations derived from each element of the product are

(1, 2) $0.7760 l_{yX} + 0.6428 l_{yZ} = 0,$

(1, 3) $0.7660 l_{zX} + 0.6428 l_{zZ} = 0,$

(2, 2) $l_{yX}^2 + 0.25 + l_{yZ}^2 = 1,$

(2, 3) $l_{yX} l_{zX} + 0.5 l_{zY} + l_{yZ} l_{zZ} = 0,$

(3, 3) $l_{zX}^2 + l_{zY}^2 + l_{zZ}^2 = 1.$

We solve eqs. (1, 2) and (2, 2) first, and then use that result to determine the other direction cosines:

(1, 2) $l_{yZ} = -1.1918 l_{yX};$

(2, 2) $(1.1918^2 + 1) l_{yX}^2 = 0.75,$
$l_{yX} = 0.5567,\ l_{yZ} = -0.6634;$

(1, 3) $l_{zZ} = -1.1918 l_{zX};$

(2, 3) $0.5567 l_{zX} + 0.50 l_{zY} - 0.6634(-1.1918 l_{zX}) = 0,$
$l_{zY} = -2.695 l_{zX};$

(3, 3) $(1 + 2.695^2 + 1.1918^2) l_{zX}^2 = 1,$
$l_{zX} = -0.3218,\ l_{zY} = 0.8660,\ l_{zZ} = 0.3830.$

Note that we selected the signs of l_{yX} and l_{zX} according to whether the given sketch indicated that the angle between the respective axes was acute or obtuse.

Now that $[R]$ is known, we may transform the vectors. In case (a), we know the XYZ components, so

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = [R] \begin{Bmatrix} 100 \\ -50 \\ 150 \end{Bmatrix} = \begin{Bmatrix} 173.02 \\ -68.84 \\ -18.03 \end{Bmatrix};$$

$$\bar{F} = 173.02\bar{i} - 68.84\bar{j} - 18.03\bar{k} \text{ N}.$$

In case (b), we use the inverse transformation because we know the xyz components. Specifically,

$$\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = [R]^{\mathrm{T}} \begin{Bmatrix} 100 \\ -50 \\ 150 \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 104.90 \\ 154.90 \end{Bmatrix};$$

$$\bar{F} = 0.50\bar{I} + 104.90\bar{J} + 154.90\bar{K} \text{ N}.$$

3.2 Finite Rotations

A *spatial rotation* features rotation about two or more nonparallel axes. Kinematics and kinetics studies of rigid bodies in three dimensions require describing and analyzing such motion. In addition, the general task of evaluating the rotation transformation $[R]$ between two coordinate systems often is more readily achieved by picturing one coordinate system as having moved away from the other in a sequence of simple rotations.

The ultimate orientation of a reference frame that undergoes a spatial rotation clearly will depend upon both the orientation of each axis of rotation and the amount of rotation about each axis. It is less apparent that the final alignment of the reference frame is dependent also on the sequence in which the individual rotations occur.

Two situations commonly arise. The conceptually simpler case involves *space-fixed axes,* by which we mean that the rotation axes have fixed orientations in space. The contrasting situation is that of *body-fixed axes.* In a body-fixed rotation sequence, each rotation is about one of the axes of the coordinate system at the preceding step in the sequence. For example, a body-fixed sequence $\theta_y, \theta_z, \theta_x$ occurs first about the initial position of the y axis, then about the orientation of the z axis after the first rotation, then finally about the x axis after the second rotation. We shall see that although a body-fixed rotation is more difficult to describe in words than is a space-fixed rotation, the transformation matrix for a body-fixed rotation is easier to derive.

3.2.1 *Body-Fixed Rotations*

We begin by following a specific sequence of body-fixed rotations. The first rotation θ_x occurs about the original orientation of the x axis and the second rotation θ_y occurs about the new orientation of the y axis. After we have derived the transformation for this case, we will generalize the result to an arbitrary sequence of rotations. Note that we use the right-hand rule to define the sense of the rotation. Specifically: curl the fingers of your right hand in the direction of the rotation; if the extended thumb of that hand points in the positive direction of that axis, then the rotation angle is positive.

As shown in Figure 3.4(a), we choose the fixed XYZ system such that it coincides with the initial orientation of xyz. We mark the orientation of xyz after the θ_x rotation as $x_1 y_1 z_1$. The transformation matrix for a single-axis rotation about the x axis was given in Eq. (3.15a). Adapting that matrix to the current notation leads to

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = [R_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_1] = [R_x], \tag{3.16}$$

where we use $[R_1]$ to denote the transformation, rather than $[R_x]$, in order to emphasize that it corresponds to the first rotation.

The result of the second rotation is depicted in Figure 3.4(b). The θ_y rotation moves xyz from $x_1 y_1 z_1$ to its final orientation. Since this corresponds to a single-axis rotation about the y_1 axis, we may apply Eq. (3.15b) directly. Thus,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_2] \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = [R_2][R_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_2] = [R_y]. \tag{3.17}$$

As an alternative to Eq. (3.17), the (x, y, z) values could have been expressed directly in terms of the (X, Y, Z) values by using the overall transformation matrix $[R]$, so that

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_2][R_1]. \tag{3.18}$$

The virtue of the notation we have employed comes from the recognition that each transformation $[R_i]$ is the result of the ith rotation. We may conclude that Eq. (3.18) is valid for any set of body-fixed rotations. In other words:

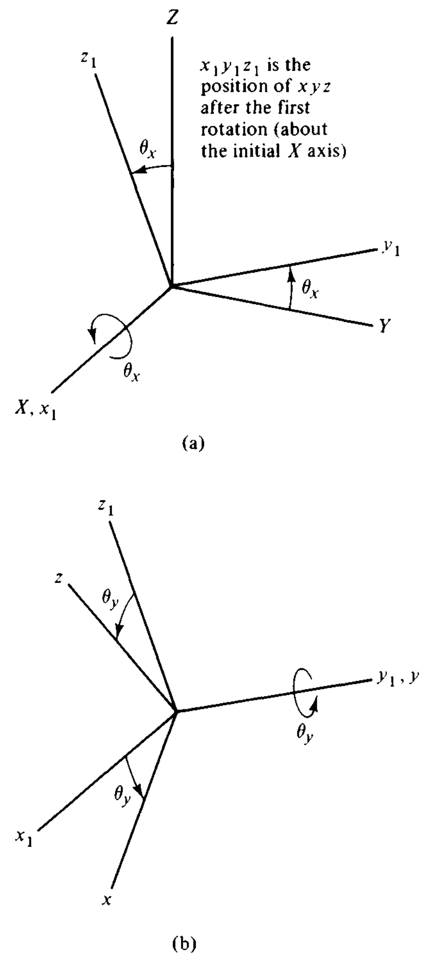


Figure 3.4 Body-fixed rotations.

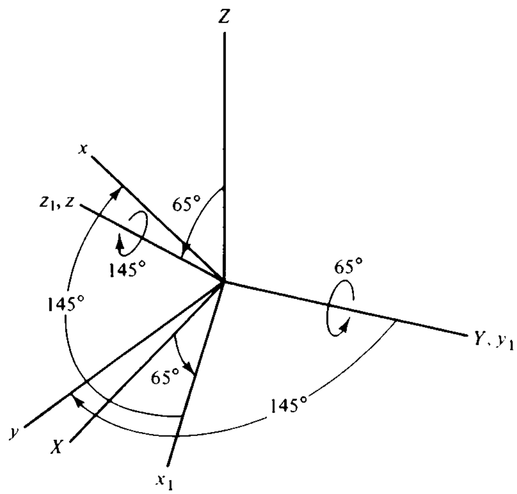
♦ *Let xyz be a reference frame that undergoes a sequence of rotations about its own axes, and let XYZ mark the initial orientation of xyz. The transformation from XYZ to the final xyz components is obtained by premultiplying (from right to left) the sequence of transformation matrices for the individual single-axis rotations. For n rotations,*

♦ $$[R] = [R_n]\cdots[R_2][R_1]. \tag{3.19}$$

Example 3.2 An xyz coordinate system, which initially coincided with the XYZ coordinate system, first undergoes a rotation $\theta_1 = 65°$ about its y axis, followed by $\theta_2 = -145°$ about its z axis. For this rotation determine:

(a) the coordinates relative to xyz in its final orientation of a stationary point at $X = 2$, $Y = -3$, $Z = 4$ meters; and

(b) the displacement components relative to XYZ of a point that remains situated at $x = 2$, $y = -3$, $z = 4$ meters relative to the moving reference frame.



Coordinate systems.

Solution We begin by sketching the orientation of xyz after the first rotation, which we mark as $x_1y_1z_1$. The rotation $\theta_1 = 65°$ is about the y axis, so the first transformation is

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = [R_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix},$$

$$[R_1] = \begin{bmatrix} \cos\theta_1 & 0 & -\cos(\theta_1 + 90°) \\ 0 & 1 & 0 \\ \cos(90° - \theta_1) & 0 & \cos\theta_1 \end{bmatrix} = \begin{bmatrix} 0.4226 & 0 & -0.9063 \\ 0 & 1 & 0 \\ 0.9063 & 0 & 0.4226 \end{bmatrix}.$$

The second rotation is about the axis marked z_1. The second transformation, for $\theta_2 = -145°$, is

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_2] \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix},$$

$$[R_2] = \begin{bmatrix} \cos\theta_2 & -\cos(90° - \theta_2) & 0 \\ \cos(90° - \theta_2) & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.8192 & -0.5736 & 0 \\ 0.5736 & -0.8192 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The combination of the two transformations is

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_2][R_1] = \begin{bmatrix} -0.3462 & -0.5736 & 0.7424 \\ 0.2424 & -0.8192 & -0.5199 \\ 0.9063 & 0 & 0.4226 \end{bmatrix}.$$

In case (a), the coordinates relative to XYZ are known, so we employ $[R]$ directly to obtain

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} 2 \\ -3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 3.998 \\ 0.863 \\ 3.503 \end{Bmatrix} \text{m}.$$

In case (b) it is the final xyz coordinates that are known. The inverse of $[R]$ then returns the coordinates to XYZ, so

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = [R]^{\mathrm{T}} \begin{Bmatrix} 2 \\ -3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 2.206 \\ 1.310 \\ 4.735 \end{Bmatrix} \text{m}.$$

The displacement is the difference between the initial and final position of the point, so

$$\{\Delta r\} = \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix} = \begin{Bmatrix} 2.206 \\ 1.310 \\ 4.735 \end{Bmatrix} - \begin{Bmatrix} 2 \\ -3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 0.206 \\ 4.310 \\ 0.735 \end{Bmatrix} \text{m}.$$

3.2.2 *Space-Fixed Rotations*

We shall develop here the transformation matrix for a sequence of rotations about axes that are fixed in space. The method will follow a course that parallels the development in the previous section. Thus, we shall begin by considering a set of rotations θ_X about the fixed X axis, followed by θ_Y about the fixed Y axis.

In addition to xyz, which undergoes both rotations, and XYZ, which remains fixed, we introduce two other reference frames. Figure 3.5(a) shows $x_1y_1z_1$, which is the orientation of xyz after the first rotation, θ_X. This reference frame does not undergo the second rotation. The other reference frame we need is $x_2y_2z_2$, which is defined in Figure 3.5(b) to be the reference frame that coincided with XYZ before the second rotation.

The transformation from XYZ to $x_1y_1z_1$ is straightforward to obtain, because the X and x_1 axes are coincident. The corresponding transformation is $[R_x]$ in Eq. (3.15a). We therefore have

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = [R_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_1] = [R_x]. \tag{3.20}$$

Similarly, the transformation from XYZ to $x_2y_2z_2$ is given by $[R_y]$ in Eq. (3.15b), because the Y and y_2 axes remain coincident in the rotation about the Y axis. Thus, we have

$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = [R_2] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_2] = [R_y]. \tag{3.21}$$

It still remains to relate the final xyz reference frame to any of the others. This is the reason for the introduction of $x_2y_2z_2$. In the second rotation, xyz goes from $x_1y_1z_1$ to its final orientation, while $x_2y_2z_2$ goes from XYZ to its final position. Because both reference frames undergo the same rotation, their relative orientation is not altered. Hence, the relation between xyz and $x_2y_2z_2$ is the same as that between $x_1y_1z_1$ and XYZ. It follows from Eq. (3.20) that

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_1] \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}. \tag{3.22}$$

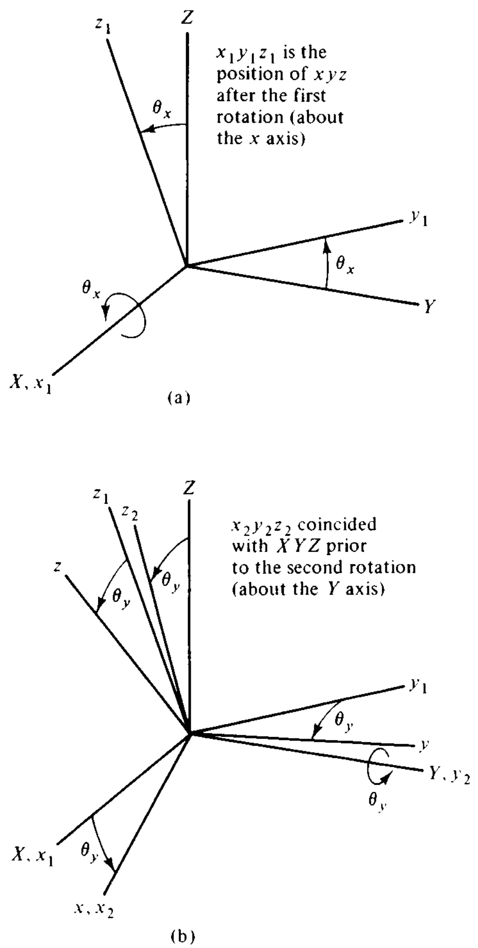


Figure 3.5 Space-fixed rotations.

It is a simple matter to eliminate the intermediate coordinate values by substituting Eq. (3.21) into Eq. (3.22). The result is that

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_1][R_2]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = [R]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_1][R_2]. \tag{3.23}$$

As we did for body-fixed axes, we may conclude that Eqs. (3.23) are generally valid. Specifically:

♦ *Let xyz be a reference frame that undergoes a sequence of rotations about the space-fixed axes XYZ with which it initially coincided. The transformation from XYZ to the final xyz components is obtained by postmultiplying (from left to right) the sequence of transformation matrices for the individual single-axis rotations. For n rotations,*

♦ $$[R] = [R_1][R_2]\cdots[R_n]. \tag{3.24}$$

The similarity of Eqs. (3.19) and (3.24) is significant. We see that the result of a sequence of body-fixed rotations matches that obtained from the reverse sequence of space-fixed rotations, and vice versa. This similarity is also a source of errors. The single-axis transformations in each case might be identical. However, for a given sequence of rotations, the order of multiplication of the individual matrices must be consistent with the type of rotation: premultiplication (right to left) for body-fixed rotations and postmultiplication (left to right) for space-fixed rotations. In a situation where the overall rotation involves both types of rotations, we may follow Eqs. (3.19) and (3.24) by premultiplying for the body-fixed rotations and postmultiplying for the space-fixed rotations. For example, a sequence described by $[R_1]$ and $[R_2]$ about body-fixed axes, followed by $[R_3]$ about a space-fixed axis and then $[R_4]$ about a body-fixed axis, would lead to $[R] = [R_4][R_2][R_1][R_3]$.

We have seen, in both this section and the preceding one, that the sequence in which individual rotations occur must be considered when the overall transformation is formed. Suppose we were to reverse the sequence in which two space-fixed rotations occur; this would reverse the order of multiplication in Eq. (3.23), so that the transformation in this case would be $[\hat{R}] = [R_2][R_1]$. Because $[R_2][R_1]$ does not, in general, equal $[R_1][R_2]$, $[\hat{R}]$ will differ from the transformation $[R]$ in Eq. (3.23). The same observation would arise if we were to reverse the sequence in which the individual body-fixed rotations forming Eq. (3.19) occur. We must conclude that, in either type of rotation:

♦ *The final orientation of a coordinate system depends on the sequence in which rotations occur, as well as the magnitude of the individual rotations and the orientation of their respective axes.*

An important corollary is that finite spatial rotations cannot be represented as vectors, because vector addition is independent of the order of addition.

Example 3.3 Consider Example 3.2 in the case where the rotations of an xyz coordinate system are $\theta_1 = 65°$ about the Y axis followed by $\theta_2 = -145°$ about the fixed Z axis, where xyz coincides with XYZ prior to any rotation. As was requested in that problem, determine for this set of rotations:

(a) the coordinates relative to xyz in its final orientation of a stationary point at $X = 2$, $Y = -3$, $Z = 4$ meters; and
(b) the final coordinates relative to XYZ of a point that is situated at $x = 2$, $y = -3$, $z = 4$ meters relative to the moving reference frame.

Solution The transformation matrix for the individual rotations are the same as in the previous example, but they combine differently because the rotations are about space-fixed axes. Here, the overall transformation results from the sequential product from left to right, that is,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_1][R_2].$$

The values of $[R_1]$ and $[R_2]$ are detailed in the solution to Example 3.2; the corresponding product just specified is

$$[R]=\begin{bmatrix}-0.3462 & -0.2424 & -0.9063\\ 0.5736 & -0.8192 & 0\\ -0.7424 & -0.5199 & 0.4226\end{bmatrix}.$$

Direct application of $[R]$ for case (a) yields

$$\begin{Bmatrix}x\\y\\z\end{Bmatrix}=[R]\begin{Bmatrix}2\\-3\\4\end{Bmatrix}=\begin{Bmatrix}-3.590\\3.605\\1.765\end{Bmatrix},$$

whereas using $[R]^{-1}$ in case (b) gives

$$\begin{Bmatrix}X\\Y\\Z\end{Bmatrix}=[R]^{\mathrm{T}}\begin{Bmatrix}2\\-3\\4\end{Bmatrix}=\begin{Bmatrix}-5.383\\-0.107\\-0.122\end{Bmatrix}.$$

3.2.3 *Rotation about an Arbitrary Axis*

We have seen that a general rotation transformation can be obtained from a sequence of simple rotations about various coordinate axes. The approach we employed to treat space-fixed rotations is also useful to describe a rotation that occurs about an arbitrary axis. Such a situation appears in Figure 3.6, where we have defined two fixed coordinate systems: XYZ, which is the one of interest, and $X'Y'Z'$, which is defined to have its Z' axis align with the rotation axis, but otherwise is arbitrary. The transformation from XYZ to $X'Y'Z'$ is $[R']$. Let xyz and $x'y'z'$ be the moving reference frames whose axes initially aligned with the respective fixed coordinate systems. Because xyz and $x'y'z'$ experience the same rotation, and therefore maintain their relative orientation, $[R']$ also describes the relation between these coordinate systems. Thus we have

$$\begin{Bmatrix}X'\\Y'\\Z'\end{Bmatrix}=[R']\begin{Bmatrix}X\\Y\\Z\end{Bmatrix},\qquad \begin{Bmatrix}x'\\y'\\z'\end{Bmatrix}=[R']\begin{Bmatrix}x\\y\\z\end{Bmatrix}. \tag{3.25}$$

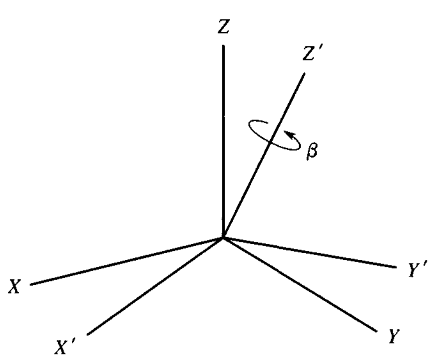


Figure 3.6 Rotation about an arbitrary axis.

We let β denote the angle of rotation about the Z' axis. The transformation from $X'Y'Z'$ to $x'y'z'$ is therefore

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [R_\beta] \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix}, \quad [R_\beta] = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.26}$$

Eliminating both sets of primed coordinates from these relations yields a product of transformations like that obtained from a sequence of rotations. This leads to the following conclusion:

◆ *The effect of a rotation about an arbitrary axis is equivalent to a sequence of body-fixed rotations in which the coordinate system is first rotated to bring one of its axes into coincidence with the rotation axis, followed by a rotation about that axis, then the reverse of the first rotation;*

◆ $$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R']^{\mathrm{T}}[R_\beta][R']. \tag{3.27}$$

The converse of the foregoing development is *Euler's theorem,* which states that any rotation is equivalent to a single rotation β about an axis. A theorem of matrix algebra makes it a simple matter to evaluate the equivalent angle, given $[R]$. The relation between $[R]$ and $[R_\beta]$ is an *orthogonal similarity transformation.* (We will encounter this in greater detail in Chapter 5 when we discuss the inertial properties of rigid bodies.) An important property of such a transformation is constancy of the trace of the matrix, which is the sum of the diagonal terms. Thus, $\mathrm{tr}[R] = \mathrm{tr}[R_\beta]$, from which it follows that the angle of rotation must satisfy

$$1 + 2\cos\beta = \mathrm{tr}[R]. \tag{3.28}$$

To determine the orientation of the equivalent axis Z', we note that the orientation of the rotation axis relative to xyz remains constant. Hence, the direction cosines of Z' with respect to xyz after the rotation are the same as its direction cosines with respect to XYZ. However, the direction cosines of any vector with respect to xyz and XYZ are related by $[R]$. Consequently, we have

$$\begin{Bmatrix} l_{Z'X} \\ l_{Z'Y} \\ l_{Z'Z} \end{Bmatrix} = [R] \begin{Bmatrix} l_{Z'X} \\ l_{Z'Y} \\ l_{Z'Z} \end{Bmatrix} \Rightarrow [[R] - [U]] \begin{Bmatrix} l_{Z'X} \\ l_{Z'Y} \\ l_{Z'Z} \end{Bmatrix} = \{0\}. \tag{3.29}$$

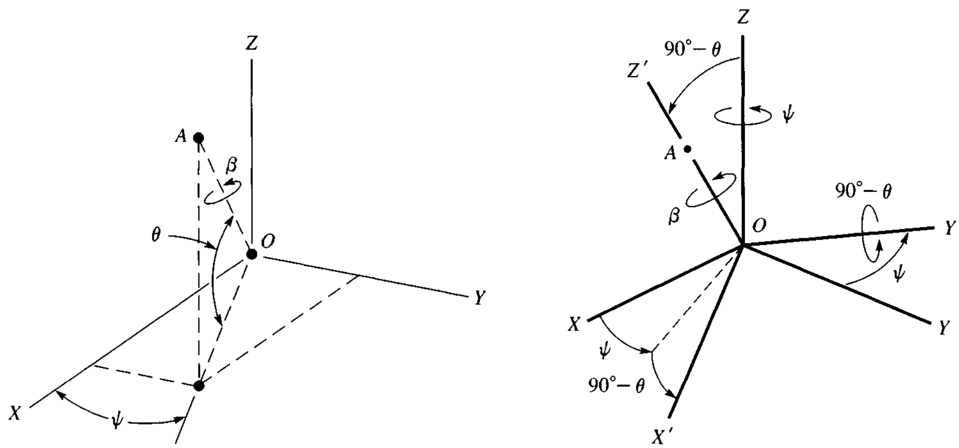
It follows from the second expression that $[R]$ has an eigenvalue of unity. (Indeed, all three of its eigenvalues are unity, which is a corollary of Euler's theorem; see Goldstein 1980.) Two of the three component equations represented by Eq. (3.29) yield two of the direction cosines to Z' in terms of the third. The individual values may then be determined by satisfying the requirement that $l_{Z'X}^2 + l_{Z'Y}^2 + l_{Z'Z}^2 = 1$.

Example 3.4 A reference frame xyz, initially coincident with fixed reference frame XYZ, is rotated through an angle β about axis OA, counterclockwise when viewed from A to O. The orientation of this fixed rotation axis is specified by the azimuthal angle ψ and angle of elevation θ relative to the X–Y plane. Point P has constant coordinates of $(0.5, -0.2, 0.4)$ meters with respect to xyz.

(a) Describe the transformation from XYZ to xyz as a sequence of single-axis rotations.

(b) For the case where $\psi = 30°$, $\theta = 75°$, and $\beta = 53.1301°$, confirm that Eq. (3.28) is satisfied.

(c) Determine the coordinates of point P relative to XYZ corresponding to the rotation angles in part (b).



Example 3.4 Coordinate systems.

Solution Application of Eq. (3.27) requires that we first define a fixed coordinate system whose Z' axis is aligned with the rotation axis OA. Such a coordinate system may be obtained by considering $X'Y'Z'$ to initially coincide with XYZ, and then imparting a pair of body-fixed rotations, first by ψ about the Z' axis and then by $90° - \theta$ about the Y' axis. The corresponding transformation is

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = [R'] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R'] = [R_\theta][R_\psi],$$

where

$$[R_\theta] = \begin{bmatrix} \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix}, \qquad [R_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transformation $[R_\beta]$ is as given in Eq. (3.15c). Substitution of $[R']$ derived here into Eq. (3.27) shows that the rotation transformation matrix from XYZ to xyz is

$$[R] = [R_\psi]^{\mathrm{T}}[R_\theta]^{\mathrm{T}}[R_\beta][R_\theta][R_\psi].$$

Recall that the inverse of a rotation about a coordinate axis is a rotation by the same amount in the opposite sense. Therefore, this expression for $[R]$ suggests that the rotation about axis OA is equivalent to a sequence of five body-fixed rotations about the axes of xyz: (1) ψ about the z axis, (2) $90°-\theta$ about the y axis, (3) β about the z axis, (4) $90°-\theta$ about the negative y axis, and (5) ψ about the negative z axis.

The specific transformation corresponding to the given values of ψ, θ, and β is

$$[R]=\begin{bmatrix} 0.62010 & 0.78434 & -0.01693 \\ -0.76114 & 0.60670 & 0.22932 \\ 0.19013 & -0.12932 & 0.97321 \end{bmatrix}.$$

The trace of $[R]$ is 2.20001. According to Eq. (3.28), the rotation angle β is

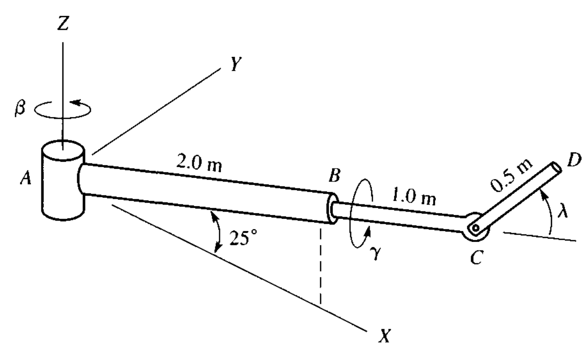
$$\beta=\cos^{-1}\left(\frac{\text{tr}[R]-1}{2}\right)=53.1297°,$$

which (aside from numerical round-off error) is identical to the given value.

With the transformation $[R]$ known, it is straightforward to determine the coordinates of point P after the rotation. The given coordinates are constant with respect to xyz. We use the inverse property to solve the transformation, with the result that

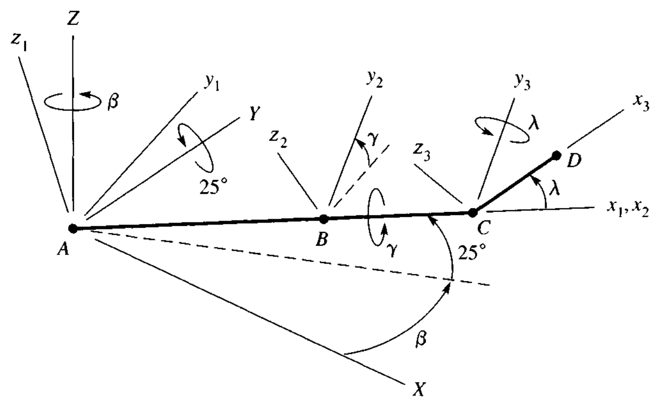
$$\begin{Bmatrix} X_P \\ Y_P \\ X_P \end{Bmatrix}=[R]^{\text{T}}\begin{Bmatrix} 0.5 \\ -0.2 \\ 0.4 \end{Bmatrix}=\begin{Bmatrix} 0.53833 \\ 0.21911 \\ 0.33496 \end{Bmatrix}\text{ m.}$$

Example 3.5 The angles β, γ, and λ in the robotic linkage are individually controllable. Initially, $\beta=\gamma=\lambda=0$, in which position the axis of the pin for rotation λ is horizontal and the linkage lies in the X-Z plane. The system is given a set of rotations consisting of $\beta=50°$, $\gamma=30°$, $\lambda=60°$. Determine the coordinates of end D with respect to the fixed XYZ coordinate after these rotations.



Example 3.5

Solution Rather than pursuing any shortcuts that depend on the particular configuration of this system, we shall develop a general approach. Furthermore, note that the sequence in which the rotations are applied is not specified. It will become apparent in the course of the solution that such information is irrelevant for the present problem. We describe the position of end B as the sum of the position vectors through each link,



Coordinate systems.

$$\bar{r}_{D/A} = \bar{r}_{B/A} + \bar{r}_{C/B} + \bar{r}_{D/C}.$$

We define reference frames $x_1y_1z_1$ fixed to arm AB, $x_2y_2z_2$ fixed to arm BC, and $x_3y_3z_3$ fixed to arm CD, with the x axis for each aligned with the corresponding arm. The transformations for the last two reference frames relative to the preceding one in the linkage are defined by a single-axis rotation. The rotation from $x_2y_2z_2$ to $x_3y_3z_3$ is λ about the negative y_2 (or y_3) axis, while the rotation from $x_1y_1z_1$ to $x_2y_2z_2$ is γ about the positive x_1 (or x_2) axis:

$$[R_3] = \begin{bmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{bmatrix}, \qquad [R_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}.$$

The corresponding transformation from the fixed XYZ reference frame to $x_1y_1z_1$ may be considered to consist of a $\theta = 25°$ rotation about the negative Y axis, followed by rotation β about the Z axis. Because both of these rotations are about space-fixed axes, we have

$$[R_1] = [R_Y][R_Z];$$

$$[R_Y] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \qquad [R_Z] = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that this is the only place throughout the solution where the sequence of rotations becomes an issue. The alternative to the foregoing is to consider $x_1y_1z_1$ to rotate first by β about the z_1 axis and then by θ about the y_1 axis. This leads to the same $[R_1]$ as before because the rotations in this case are about body-fixed axes.

We employ the rotation transformations to describe the relative position vectors. For simplicity of notation, we define (x_B, y_B, z_B), (x_C, y_C, z_C), and (x_D, y_D, z_D) as the coordinates of the respective points relative to the body-fixed reference frames. Specifically,

$$\bar{r}_{B/A} = [x_B \ y_B \ z_B] \begin{Bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{Bmatrix}, \qquad \bar{r}_{C/B} = [x_C \ y_C \ z_C] \begin{Bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{Bmatrix},$$

$$\bar{r}_{D/C} = [x_D y_D z_D] \begin{Bmatrix} \bar{i}_3 \\ \bar{j}_3 \\ \bar{k}_3 \end{Bmatrix}.$$

The rotation transformations enable us to express each set of unit vectors in terms of components relative to XYZ, according to

$$\begin{Bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{Bmatrix} = [R_1] \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix}, \quad \begin{Bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{Bmatrix} = [R_2] \begin{Bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{Bmatrix} = [R_2][R_1] \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix},$$

$$\begin{Bmatrix} \bar{i}_3 \\ \bar{j}_3 \\ \bar{k}_3 \end{Bmatrix} = [R_3] \begin{Bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{Bmatrix} = [R_3][R_2][R_1] \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix}.$$

Now let (X_D, Y_D, Z_D) denote the coordinates of point D with respect to the fixed reference frame. We substitute the unit vector transformations into the respective position vectors $\bar{r}_{B/A}$, $\bar{r}_{C/B}$, and $\bar{r}_{D/C}$, and then substitute those expressions into the first equation for $r_{D/A}$. Cancelling the common factor formed by the column of XYZ unit vectors leads to

$$\begin{aligned} [X_D \;\; Y_D \;\; Z_D] = {} & [x_B \;\; y_B \;\; z_B][R_1] \\ & + [x_C \;\; y_C \;\; z_C][R_2][R_1] + [x_D \;\; y_D \;\; z_D][R_3][R_2][R_1]. \end{aligned}$$

Taking the transpose of this equation leads to the more familiar columnar form,

$$\begin{Bmatrix} X_D \\ Y_D \\ Z_D \end{Bmatrix} = [R_1]^{\mathrm{T}} \begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} + [R_1]^{\mathrm{T}}[R_2]^{\mathrm{T}} \begin{Bmatrix} x_C \\ y_C \\ z_C \end{Bmatrix} + [R_1]^{\mathrm{T}}[R_2]^{\mathrm{T}}[R_3]^{\mathrm{T}} \begin{Bmatrix} x_D \\ y_D \\ z_D \end{Bmatrix}.$$

This is the general relation for points having arbitrary coordinates. To obtain the specific values, we note that the only nonzero coordinate values for the points of interest are $x_B = 2$ m, $x_C = 1$ m, and $x_D = 0.5$ m. We substitute the given angles $\theta = 25°$, $\beta = 50°$, $\gamma = 30°$, and $\lambda = 60°$ into the transformations $[R_j]$, then use the coordinate values just given. This yields

$$\begin{Bmatrix} X_D \\ Y_D \\ Z_D \end{Bmatrix} = \begin{Bmatrix} 1.9269 \\ 2.0117 \\ 1.7292 \end{Bmatrix} \text{m}.$$

It is interesting to observe that the procedure we have employed here leads to a combination of translation and rotation transformations. In essence, each sequence of products in the general expression for (X_D, Y_D, Z_D) represents a rotation transformation between coordinate systems having concurrent origins. Summing these individual products is a process of translating the origin from C to B to A.

3.3 Angular Velocity and Derivatives of Rotating Vectors

Discrepancies between spatial rotations that differ only in their sequence become less significant as the magnitude of each rotation decreases. The limiting case of infinitesimal rotations yields a result that is independent of the sequence. In order to demonstrate this fact, let us evaluate the transformation matrix associated with a set of space-fixed rotations $d\theta_X, d\theta_Y, d\theta_Z$, in that order, about the fixed reference frame axes.

The transformation matrices for the individual rotations are given by Eqs. (3.15). Second-order differential quantities are negligible compared with first-order terms, so we set $\cos d\theta = 1$ and $\sin d\theta = d\theta$. The limiting forms of the individual transformation matrices are therefore

$$[R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d\theta_X \\ 0 & -d\theta_X & 1 \end{bmatrix}, \qquad [R_2] = \begin{bmatrix} 1 & 0 & -d\theta_Y \\ 0 & 1 & 0 \\ d\theta_Y & 0 & 1 \end{bmatrix},$$
$$[R_3] = \begin{bmatrix} 1 & d\theta_Z & 0 \\ -d\theta_Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.30}$$

Second-order differentials are also negligible when these matrices are multiplied. The resulting transformation is found from Eqs. (3.24) to be

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_1][R_2][R_3]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} 1 & d\theta_Z & -d\theta_Y \\ -d\theta_Z & 1 & d\theta_X \\ d\theta_Y & -d\theta_X & 1 \end{bmatrix}\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}. \tag{3.31}$$

The foregoing is the general transformation between the fixed and moving coordinate systems. Let us apply it to the situation where we wish to follow an arbitrary point P whose position relative to the moving coordinate system remains constant. If the coordinates of point P with respect to the fixed coordinate system were (X_0, Y_0, Z_0) prior to any rotation, then these will also be the constant coordinates of point P with respect to the moving system. Let (X_f, Y_f, Z_f) be the fixed reference frame coordinates after the rotation. Solving Eq. (3.31) with the aid of the orthonormal property yields

$$\begin{Bmatrix} X_f \\ Y_f \\ Z_f \end{Bmatrix} = \begin{bmatrix} 1 & -d\theta_Z & d\theta_Y \\ d\theta_Z & 1 & -d\theta_X \\ -d\theta_Y & d\theta_X & 1 \end{bmatrix}\begin{Bmatrix} X_0 \\ Y_0 \\ Z_0 \end{Bmatrix}. \tag{3.32}$$

Consider changing the order in which $[R_1]^T$, $[R_2]^T$, and $[R_3]^T$ are multiplied to form Eq. (3.32). The final result will be the same because second-order differentials are negligible with respect to first-order differentials. Since the right-to-left order of multiplication in Eq. (3.24) matches the rotation sequence, we may conclude that:

♦ *The final orientation of a coordinate system is unaffected by the sequence in which a set of infinitesimal rotations are performed.*

An important corollary follows from the fact that the transformations for space-fixed and body-fixed axes differ only by the sequence in which the individual rotation transformations are multiplied. Consequently, we observe from the foregoing statement that the same transformation is obtained if a set of infinitesimal rotations are imparted about body-fixed or space-fixed axes.

Equation (3.32) relates the final coordinates of a point to the initial values. The differential change in the (fixed-frame) coordinates is found to be

$$\begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} = \begin{Bmatrix} X_f - X_0 \\ Y_f - Y_0 \\ Z_f - Z_0 \end{Bmatrix} = \begin{bmatrix} 0 & -d\theta_Z & d\theta_Y \\ d\theta_Z & 0 & -d\theta_X \\ -d\theta_Y & d\theta_X & 0 \end{bmatrix} \begin{Bmatrix} X_0 \\ Y_0 \\ Z_0 \end{Bmatrix}. \tag{3.33}$$

The increments described by Eq. (3.33) are the components of the infinitesimal displacement $d\bar{r}$. The vector form of $d\bar{r}$ is

$$\begin{aligned} d\bar{r} &= dX\bar{I} + dY\bar{J} + dz\bar{K} \\ &= (d\theta_Y Z_0 - d\theta_Z Y_0)\bar{I} + (d\theta_Z X_0 - d\theta_X Z_0)\bar{J} + (d\theta_X Y_0 - d\theta_Y X_0)\bar{K}. \end{aligned} \tag{3.34}$$

A simpler representation of the foregoing is obtained by using a cross product. Let $\overline{d\theta}$ represent a differential rotation vector,

$$\overline{d\theta} = d\theta_X \bar{I} + d\theta_Y \bar{J} + d\theta_Z \bar{K}. \tag{3.35}$$

We describe the relative position $\bar{r}_{P/O'}$ in terms of components with respect to the fixed reference frame, so that

$$\bar{r}_{P/O'} = X_0 \bar{I} + Y_0 \bar{J} + Z_0 \bar{K}. \tag{3.36}$$

Then the resulting expression for the infinitesimal displacement is

$$d\bar{r} = \overline{d\theta} \times \bar{r}_{P/O'}. \tag{3.37}$$

Several aspects of Eqs. (3.35) and (3.37) are noteworthy. The individual rotations were defined in Figures 3.5(a) and (b) to be positive according to the right-hand rule. Thus, the component representation of $\overline{d\theta}$ in Eq. (3.35) is equivalent to a vectorial superposition of the infinitesimal rotations, as was anticipated earlier. The overbar is placed above the entire symbol $\overline{d\theta}$ in order to emphasize that there is no finite rotation vector from which the differential is formed. A principal advantage of Eq. (3.37) over Eq. (3.33) is that the vector form does not rely on a coordinate system to represent components. Specifically, Eq. (3.37) remains valid if its vectors are represented in terms of components relative to the axes of the moving reference frame. (We often shall use such a description.)

Finally, observe that the position vector and its differential occur in Eq. (3.37) only because the derivation began by considering the position coordinates. Suppose we had considered an arbitrary vector $\bar{A}$ that is attached to a moving reference frame, so that its components relative to that frame are constant. The result would have been an expression for the infinitesimal change in $\bar{A}$, that is,

$$d\bar{A} = \overline{d\theta} \times \bar{A}. \tag{3.38}$$

There is a simple explanation for this relation. Figure 3.7(a) shows a typical vector $\bar{A}$ before and after an infinitesimal rotation $\overline{d\theta}$. The direction of the rotation vector is parallel to the axis of rotation, and the angle of rotation is $|\overline{d\theta}|$. The change in $\bar{A}$ is found as the difference between the new and previous vectors. This difference is depicted in Figure 3.7(b), where the tails of the vectors have been brought to the axis represented by $\overline{d\theta}$. The sketch shows that only the component of $\bar{A}$ perpendicular to the axis changes; call this component $\bar{A}_\perp$. The line in Figure 3.7(b) representing $\bar{A}_\perp$

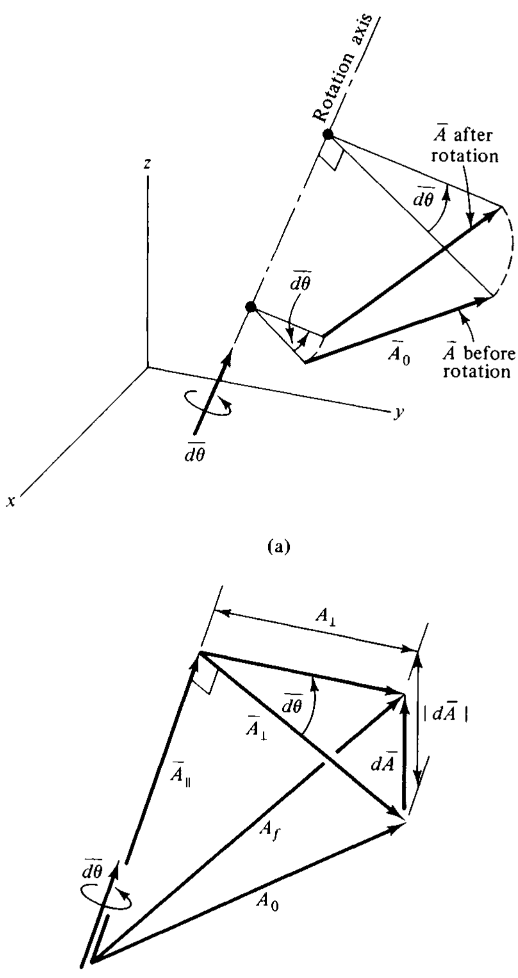


Figure 3.7 Change in a vector due to rotation.

rotates through the angle $|\overline{d\theta}|$. Hence, the arc that represents $d\bar{A}$ has a length $A_\perp |\overline{d\theta}|$, and the direction of $d\bar{A}$ is perpendicular to both $\bar{A}$ and the $\overline{d\theta}$ vector. The magnitude of a cross product is defined to be the product of the magnitude of one vector and the perpendicular component of the other vector, and the direction of the product is perpendicular to the individual vectors. It follows that the pictorial analysis fully agrees with Eq. (3.38). In other words, the change in $\bar{A}$ results from the movement of its tip perpendicular to the plane formed by $\overline{d\theta}$ and $\bar{A}$, in the sense of the rotation. We will often recall this interpretation when we treat the change of a vector due to a rotation.

The properties of a differential change in a variable are much like those for the rate of change of that variable. Hence, we define the angular velocity to be

$$\bar{\omega} = \frac{\overline{d\theta}}{dt} = \dot{\theta}_X \bar{I} + \dot{\theta}_Y \bar{J} + \dot{\theta}_Z \bar{K}. \tag{3.39}$$

Similarly, dividing Eq. (3.38) by dt yields the rate of change of any vector having constant components relative to a moving reference frame. We have:

♦ $$\dot{\bar{A}} = \bar{\omega} \times \bar{A}. \quad (3.40)$$

This theorem may be interpreted in words as:

♦ *Let $\bar{A}$ be a vector whose components relative to a moving reference frame are constant, and let $\bar{\omega}$ be the angular velocity of that reference frame. Then the rate of change of $\bar{A}$ is the cross product of $\bar{\omega}$ and $\bar{A}$.*

Our primary application for this theorem will be to differentiate the unit vectors of a moving reference frame.

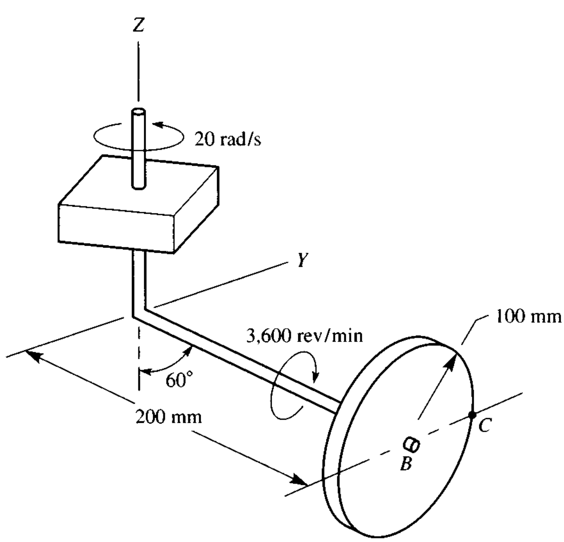
Another generalization results from the fact that $\bar{\omega}$, like $\overline{d\theta}$, is a vector quantity. This permits us to represent the angular velocity as the vector sum of rotational speeds about arbitrary axes, rather than only the axes of the fixed reference frame. Specifically:

♦ *Let xyz be a coordinate system that is undergoing a spatial rotation. Let $\bar{e}_i$ be a unit vector parallel to the* ith *axis of rotation in the sense of the rotation according to the right-hand rule, and let ω_i be the corresponding rate of rotation in radians per second. Then the angular velocity of xyz is given by*

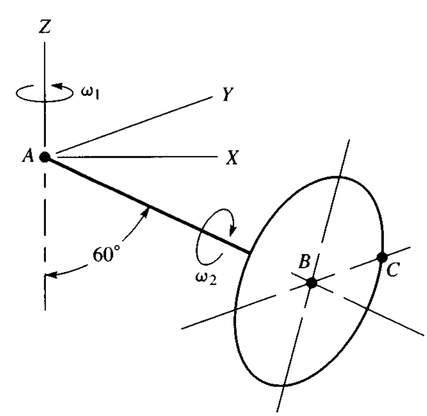
♦ $$\bar{\omega} = \sum_i \omega_i \bar{e}_i. \quad (3.41)$$

Equations (3.40) and (3.41) are powerful tools that we shall employ frequently in our study of kinematics, and they will play a vital role for the kinetics of rigid bodies.

Example 3.6 The disk is rotating about shaft AB at 3,600 rev/min as the system rotates about the vertical axis at 20 rad/s. Determine the angular velocity of the disk.



Example 3.6



Coordinate systems.

Use this angular velocity to determine the approximate displacement of point C on the perimeter of the disk 2 μs after the instant depicted in the sketch. Compare this result to the precise one obtained from the rotation transformation.

Solution The given time interval is very small, so we may approximate increments over the interval as differentials. This leads to

$$\Delta\bar{r}_C \approx (\bar{\omega}\times\bar{r}_{C/A})\Delta t,$$

where the reference point for the position is selected as point A because that point is the fixed one in the rotation.

We obtain the angular velocity by vectorially adding the rotation rates. Thus,

$$\bar{\omega} = \omega_1\bar{K} + \omega_2\bar{e}_{A/B},$$

where, according to the right-hand rule, the sense of the rotation ω_2 about shaft AB is directed from point B to point A. We shall describe the displacement in terms of components relative to the fixed reference frame. We resolve $\bar{e}_{A/B}$ into components relative to XYZ, which yields

$$\bar{e}_{A/B} = -(\sin 60°)\bar{I} + (\cos 60°)\bar{K}.$$

Substituting ω_1, ω_2, and $\bar{e}_{A/B}$ into the expression for $\bar{\omega}$ yields

$$\begin{aligned}\bar{\omega} &= 20\bar{K} + 3{,}600\left(\frac{2\pi}{60}\right)(-0.8660\bar{I} + 0.50\bar{K})\\ &= -326.5\bar{I} + 208.5\bar{K} \text{ rad/s.}\end{aligned}$$

At this instant, the position is

$$\begin{aligned}\bar{r}_{C/A} &= 0.2[(\sin 60°)\bar{I} - (\cos 60°)\bar{K}] + 0.10\bar{J}\\ &= 0.17321\bar{I} + 0.10\bar{J} - 0.10\bar{K} \text{ m.}\end{aligned}$$

Thus, for $\Delta t = 2(10^{-6})$ s, we find

$$\begin{aligned}\Delta\bar{r}_{C/A} &\approx (-326.5\bar{I} + 208.5\bar{K})\times(0.17321\bar{I} + 0.10\bar{J} - 0.10\bar{K})[2(10^{-6})]\\ &= (-41.70\bar{I} + 6.929\bar{J} - 65.30\bar{K})(10^{-6}) \text{ m.}\end{aligned}$$

To perform the corresponding computation using rotation transformations, we first evaluate the angles of rotation corresponding to constant rates ω_1 and ω_2 over a 2-μs interval: $\theta_1 = \omega_1\Delta t = 4(10^{-5})$ rad, $\theta_2 = \omega_2\Delta t = 7.540(10^{-4})$ rad. Let xyz be a body-fixed reference frame, having origin A, whose x axis aligns with shaft AB and whose y axis coincided with the Y axis prior to any rotations. We may picture the final orientation of this reference frame as being obtained from a sequence of rotations that moves xyz away from initial coincidence with XYZ. The sequence consists of a rotation θ_1 about the positive fixed Z axis, followed by a 30° rotation about the body-fixed y axis, followed by a rotation θ_2 about the negative body-fixed x axis. Hence, the rotation transformation from XYZ to xyz is

$$\begin{aligned}[R] &= [R_x][R_y][R_Z]\\ &= \begin{bmatrix}1 & 0 & 1\\ 0 & \cos\theta_2 & -\sin\theta_2\\ 0 & \sin\theta_2 & \cos\theta_2\end{bmatrix}\begin{bmatrix}\cos 30° & 0 & -\sin 30°\\ 0 & 1 & 0\\ \sin 30° & 0 & \cos 30°\end{bmatrix}\begin{bmatrix}\cos\theta_1 & \sin\theta_1 & 0\\ -\sin\theta_1 & \cos\theta_1 & 0\\ 0 & 0 & 1\end{bmatrix}.\end{aligned}$$

Note that, because the computation of displacement will require subtracting numbers of nearly equal magnitude, at least eight significant figures in $[R]$ should be retained.

The coordinates of point C with respect to xyz remain constant at (0.20, 0.10, 0), so the coordinates of this point with respect to XYZ subsequent to the rotations are

$$\begin{Bmatrix} X_C \\ Y_C \\ Z_C \end{Bmatrix}_f = [R]^{\mathrm{T}} \begin{Bmatrix} 0.20 \\ 0.10 \\ 0 \end{Bmatrix}.$$

The initial coordinates of point C with respect to XYZ are the components of $\bar{r}_{C/A}$ given previously. The corresponding displacement is

$$\begin{Bmatrix} \Delta X_C \\ \Delta Y_C \\ \Delta Z_C \end{Bmatrix} = \begin{Bmatrix} X_C \\ Y_C \\ Z_C \end{Bmatrix}_f - \begin{Bmatrix} 0.2\cos 30° \\ 0.10 \\ -0.2\sin 30° \end{Bmatrix} = \begin{Bmatrix} -41.699 \\ 6.898 \\ -65.297 \end{Bmatrix} (10^{-6})\text{ m}.$$

The closeness of these displacement components to those computed from $(\bar{\omega} \times \bar{r}_{C/A})\Delta t$ is to be expected, because the angles θ_1 and θ_2 are very small. Indeed, nearly comparable values would be obtained by merely superposing the displacements from the initial position of point C associated with rotations θ_1 and θ_2.

Example 3.7 The Frenet formulas give the derivatives of the path-variable unit vectors with respect to the arclength s along an arbitrary curve. Because $\dot{s} = v$, these derivatives may be converted to time rates of change of the unit vectors. Furthermore, the orthonormal directions represented by these unit vectors form a moving reference frame. Use Eq. (3.40) to determine the angular velocity of the $\bar{e}_t \bar{e}_n \bar{e}_b$ reference frame in terms of the path-variable parameters.

Solution It is useful to begin by recalling the Frenet formulas:

$$\frac{d\bar{e}_t}{ds} = \frac{1}{\rho}\bar{e}_n, \qquad \frac{d\bar{e}_n}{ds} = -\frac{1}{\rho}\bar{e}_t + \frac{1}{\tau}\bar{e}_b, \qquad \frac{d\bar{e}_b}{ds} = -\frac{1}{\tau}\bar{e}_n.$$

In order to convert these to time derivatives, we observe that if $\bar{e}$ is a unit vector that depends on the arclength s locating a point, and if $s = s(t)$, then the derivative of $\bar{e}$ may be obtained from

$$\dot{\bar{e}} = \frac{d\bar{e}}{ds}\dot{s} = v\,\frac{d\bar{e}}{ds}.$$

Hence, we have

$$\dot{\bar{e}}_t = v\,\frac{d\bar{e}_t}{ds} = \frac{v}{\rho}\bar{e}_n, \qquad \dot{\bar{e}}_n = v\,\frac{d\bar{e}_n}{ds} = -\frac{v}{\rho}\bar{e}_t + \frac{v}{\tau}\bar{e}_b, \qquad \dot{\bar{e}}_b = v\,\frac{d\bar{e}_b}{ds} = -\frac{v}{\tau}\bar{e}_n.$$

Now let $\bar{\omega}$ be the angular velocity of $\bar{e}_t \bar{e}_n \bar{e}_b$, which may be written in component form as

$$\bar{\omega} = \omega_t \bar{e}_t + \omega_n \bar{e}_n + \omega_b \bar{e}_b.$$

Each unit vector has constant components relative to the reference frame, so Eq. (3.40) applies. Thus,

$$\dot{\bar{e}}_t = \bar{\omega} \times \bar{e}_t = \omega_b \bar{e}_n - \omega_n \bar{e}_b = \frac{v}{\rho}\bar{e}_n,$$

$$\dot{\bar{e}}_n = \bar{\omega} \times \bar{e}_n = -\omega_b \bar{e}_t + \omega_t \bar{e}_b = -\frac{v}{\rho}\bar{e}_t + \frac{v}{\tau}\bar{e}_b,$$

$$\dot{\bar{e}}_b = \bar{\omega} \times \bar{e}_b = \omega_n \bar{e}_t - \omega_t \bar{e}_n = -\frac{v}{\tau}\bar{e}_n.$$

Matching like components in each equation leads to

$$\omega_t = \frac{v}{\tau},\ \omega_n = 0,\ \omega_b = \frac{v}{\rho} \quad \Rightarrow \quad \bar{\omega} = \frac{v}{\tau}\bar{e}_t + \frac{v}{\rho}\bar{e}_b.$$

We see from this result that a sharp bend in the curve (small ρ) causes a rapid rotation about the binormal direction, which is perpendicular to the osculating plane. Similarly, a sharp twist (small τ) causes a rapid rotation about the tangent direction. There is no rotation about the normal direction because the curve locally lies in the osculating plane.

3.4 Angular Acceleration

One of the primary properties of the motion of a reference frame is its angular acceleration, which is defined as the time derivative of the angular velocity. It is conventional to denote the angular acceleration as $\bar{\alpha}$, so that

♦ $$\bar{\alpha} = \dot{\bar{\omega}}. \tag{3.42}$$

We saw in the preceding section that $\bar{\omega}$ is the sum of rotation rates about various axes. Even if the rotation rates are constants, there will be an angular acceleration whenever any of the axes do not have a fixed orientation.

Let us consider the manner in which $\bar{\omega}$ may be differentiated. The only general statement regarding $\bar{\omega}$ is Eq. (3.41), which calls for a summation of rotation rate vectors that are formed according to the right-hand rule. Each unit vector $\bar{e}_i$ has been defined to be aligned with an axis of rotation. In order to expedite the description of each $\bar{e}_i$, we proceed as follows:

♦ *Define a moving reference frame $x_i y_i z_i$ for each rotation, such that one of the axes of $x_i y_i z_i$ always coincides with that rotation axis. Hence, one of the axes of $x_i y_i z_i$ coincides with $\bar{e}_i$. Let $\bar{\Omega}_i$ be the angular velocity of $x_i y_i z_i$.*

Note that each of these reference frames may be, but is not necessarily, the same as the xyz frame whose angular acceleration is being evaluated.

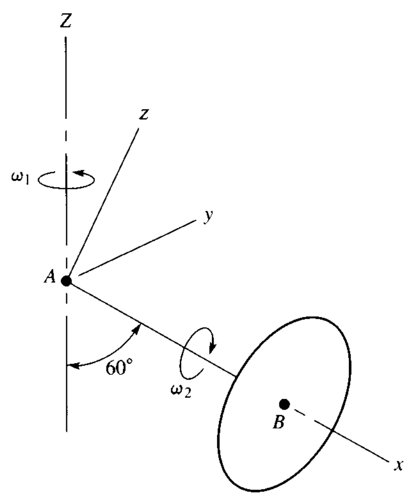
By definition, the unit vector $\bar{e}_i$ is fixed relative to $x_i y_i z_i$. Hence, the derivative of $\bar{e}_i$ is known from Eq. (3.40) to be $\bar{\Omega}_i \times \bar{e}_i$. The rules for differentiating a sum and a product then lead to

♦ $$\bar{\alpha} = \sum_i [\dot{\omega}_i \bar{e}_i + \omega_i (\bar{\Omega}_i \times \bar{e}_i)]. \tag{3.43}$$

After Eq. (3.43) has been formed using the unit vectors of the various reference frames, all terms in $\bar{\omega}$ and $\bar{\alpha}$ should be transformed to a common coordinate system. The principles derived in Chapter 5 governing the kinetics of a rigid body require that the components of $\bar{\omega}$ and $\bar{\alpha}$ be expressed relative to a body-fixed reference frame. Hence, we usually describe the angular motion of xyz in terms of components relative to its own axes.

The task of finding the angular acceleration is a key aspect of evaluations based on the concepts of relative motion. Hence, it is imperative to understand the meaning of the terms in Eqs. (3.41) and (3.43). In particular, each $\bar{\Omega}_i$ corresponds to the angular velocity of a reference frame in which the ith axis of rotation seems to be fixed. The term in Eq. (3.43) containing $\bar{\Omega}_i \times \bar{e}_i$ represents the effect of changing the direction of the ith axis of rotation; the effect is perpendicular to that axis. In contrast, the term $\dot{\omega}_i \bar{e}_i$, which is parallel to the ith axis, arises whenever the rotation rate is changed from its current value. Note that a rotation about a fixed axis, such as that in a planar motion, produces only the latter effect. It is for this reason that intuitive judgments obtained from experience with planar motion are often incorrect.

Example 3.8 Determine the angular acceleration of the disk in Example 3.6.



Coordinate systems.

Solution The procedure here differs from the solution to Example 3.6, owing to the need to define reference frames associated with each rotation. As shown in the sketch, the Z axis of the fixed XYZ frame coincides with the vertical axis of rotation, so $\bar{e}_1 = \bar{K}$. For the other rotation, we attach the xyz reference frame to the flywheel. (The location of the origin of xyz is unimportant for these operations.) We align the x axis with shaft AB, so that $\bar{e}_2 = \bar{i}$ throughout the motion.

The corresponding general description of the angular velocity is

$$\bar{\omega} = \omega_1 \bar{K} - \omega_2 \bar{i},$$

where the sign for the ω_2 rotation is a consequence of the right-hand rule. Both rotation rates are constant. We set $\dot{\bar{K}} = \bar{0}$ because $\bar{K}$ is constant. In contrast, the angular velocity of xyz is $\bar{\omega}$, so $\dot{\bar{i}} = \bar{\omega} \times \bar{i}$. We may assume that the rotation rates are constant, because it is not stated otherwise. The derivative of the general expression for $\bar{\omega}$ is therefore

$$\bar{\alpha} = -\omega_2(\bar{\omega} \times \bar{i}).$$

Employing $\bar{\omega}$ and $\bar{\alpha}$ in subsequent computations would require their resolution into components relative to a single coordinate system. As mentioned in the discussion following Eq. (3.43), it is standard practice to employ the xyz axes for this purpose. We could consider xyz to have undergone an arbitrary rotation about shaft AB. This would complicate the task of evaluating the components of $\bar{K}$ because it would not lie in any of the coordinate planes. The axisymmetry of the disk makes it equally valid - and much more convenient - to define xyz such that, at the instant of interest, its z axis lies in the vertical plane formed by the two rotation axes. This is the orientation depicted in the sketch. Thus,

$$\bar{K} = -(\cos 60°)\bar{i} + (\sin 60°)\bar{k}.$$

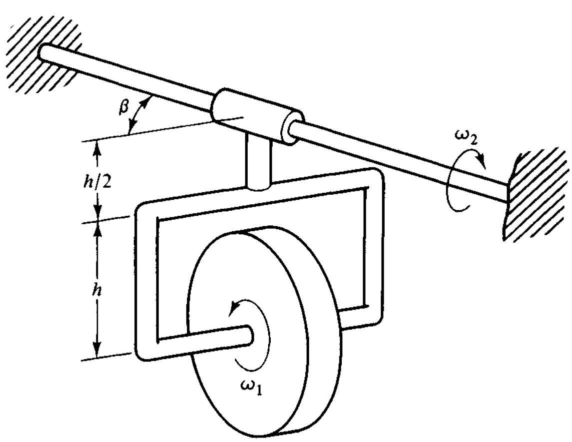
An interesting corollary of the foregoing procedure is that the orientations of the fixed reference frame axes are unimportant if they are not rotation axes. Substituting for $\bar{K}$, ω_1, and ω_2 in the general expressions yields

$$\bar{\omega} = 20(-0.50\bar{i} + 0.866\bar{k}) - 120\pi\bar{i} = -387.0\bar{i} + 17.32\bar{k} \text{ rad/s},$$

$$\bar{\alpha} = -(120\pi)(-387.0\bar{i} + 17.32\bar{k}) \times \bar{i} = -6{,}530\bar{j} \text{ rad/s}^2.$$

A simple verification of this result for $\bar{\alpha}$ is the observation that, as the system rotates about the fixed vertical axis, the tip of the angular velocity term $-\omega_2\bar{i}$ moves in the negative y direction. This agrees with the direction of the computed value of $\bar{\alpha}$. The rotation about the vertical axis does not contribute to the angular acceleration, because its rate is constant and its axis retains a constant orientation.

Example 3.9 The gyroscopic turn indicator consists of a flywheel that spins about its axis of symmetry at the constant rate ω_1, as the assembly rotates about the fixed horizontal shaft at the variable rate ω_2. The angle β locating the axis of the flywheel relative to the horizontal shaft is an arbitrary function of time. Determine the angular acceleration of the flywheel at an arbitrary instant.



Example 3.9

Solution We define xyz to be attached to the flywheel, so that the angular motion of xyz is identical to that of the flywheel. The ω_2 rotation is about the fixed horizontal shaft. Correspondingly, we define the fixed reference frame such that the unit vector parallel to that rotation axis is $\bar{e}_2 = -\bar{I}$. (There is no need to specify the other fixed axes, because we will resolve all terms into components relative to the moving reference frame.) The ω_1 rotation is about the axis of the flywheel, and this direction is fixed relative to the flywheel. We therefore define xyz such that the unit vector along this rotation axis is $\bar{e}_1 = \bar{i}$. The third rotation is about an axis that is always perpendicular to the horizontal shaft. In order to describe it, we attach an $x'y'z'$ reference frame to the gimbal supporting the flywheel. We define the z' axis to coincide with the $\dot{\beta}$ rotation, so $\bar{e}_3 = \bar{k}'$. (Note that we have defined the x' axis in the sketch to coincide with the axis of the flywheel. Consequently, we could have alternatively defined $\bar{e}_1 = \bar{i}'$. However, it would be incorrect to assign $\bar{e}_3$ to $\bar{k}$, because the z axis rotates relative to the gimbal.)

The angular velocity of the flywheel is the sum of the individual effects, so

$$\bar{\omega} = -\omega_2 \bar{I} + \omega_1 \bar{i} + \dot{\beta}\bar{k}'.$$

We associated $\bar{e}_1$ with xyz and $\bar{e}_2$ with XYZ. The corresponding angular velocities to be used in differentiating the unit vectors of those reference frames are

$$\bar{\Omega}_1 = \bar{\omega} \quad \text{and} \quad \bar{\Omega}_2 = \bar{0}.$$

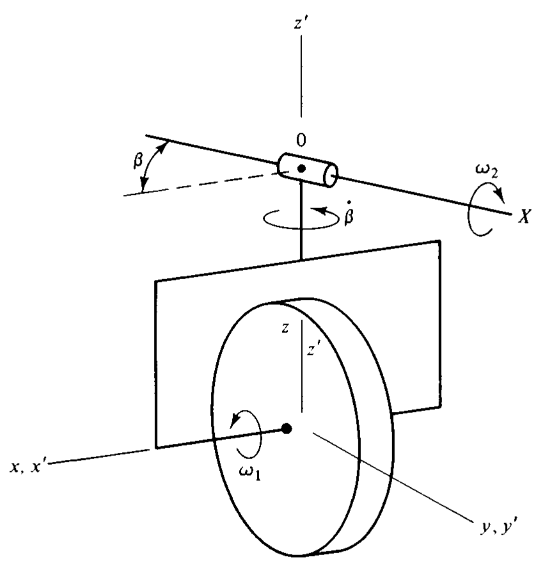
The angular motion of $x'y'z'$ is like that of xyz, except that it does not include the spinning rotation ω_1. Hence, the angular velocity of $x'y'z'$ is

$$\bar{\Omega}_3 = -\omega_2 \bar{I} + \dot{\beta}\bar{k}'.$$

Using these angular velocities to differentiate the general expression for $\bar{\omega}$ yields

$$\bar{\alpha} = -\dot{\omega}_2 \bar{I} + \ddot{\beta}\bar{k}' + \omega_1(\bar{\omega} \times \bar{i}) + \dot{\beta}(\bar{\Omega}_3 \times \bar{k}'),$$

where we have set $\dot{\omega}_1 = 0$, as specified.



Coordinate systems.

Both $\bar{\omega}$ and $\bar{\alpha}$ must be represented in terms of a single set of directions, for which we select xyz. The resolution is simplified significantly if we define xyz so that the z and z' axes coincide at the instant of interest; this is the configuration in the sketch. Then,

$$\bar{I} = -(\cos\beta)\bar{i} + (\sin\beta)\bar{j}, \qquad \bar{k}' = \bar{k}.$$

Substitution of these unit vectors into the previous relations yields

$$\bar{\omega} = (\omega_1 + \omega_2\cos\beta)\bar{i} - (\omega_2\sin\beta)\bar{j} + \dot{\beta}\bar{k},$$

$$\bar{\Omega}_3 = (\omega_2\cos\beta)\bar{i} - (\omega_2\sin\beta)\bar{j} + \dot{\beta}\bar{k},$$

$$\begin{aligned}\bar{\alpha} &= -\dot{\omega}_2[-(\cos\beta)\bar{i} + (\sin\beta)\bar{j}] + \ddot{\beta}\bar{k} \\ &\quad + \omega_1[(\omega_1 + \omega_2\cos\beta)\bar{i} - (\omega_2\sin\beta)\bar{j} + \dot{\beta}\bar{k}] \times \bar{i} \\ &\quad + \dot{\beta}[(\omega_2\cos\beta)\bar{i} - (\omega_2\sin\beta)\bar{j} + \dot{\beta}\bar{k}] \times \bar{k} \\ &= (\dot{\omega}_2\cos\beta - \omega_2\dot{\beta}\sin\beta)\bar{i} + (-\dot{\omega}_2\sin\beta + \omega_1\dot{\beta} - \omega_2\dot{\beta}\cos\beta)\bar{j} \\ &\quad + (\ddot{\beta} + \omega_1\omega_2\sin\beta)\bar{k}.\end{aligned}$$

It is important to recognize that, in Examples 3.8 and 3.9, one axis of the body-fixed xyz reference frame was selected to always align with the body-fixed rotation axis. This defines uniquely the orientation of that axis at any instant. The axial symmetry of the body to which xyz was attached enabled us to select the instantaneous orientations of the other axes of xyz as we wished; this selection was made to facilitate the description of the other axes of rotation. Following such a procedure in other systems may not be possible. One such case may be found in Example 3.14.

3.5 Derivative of an Arbitrary Vector

We saw in Section 3.3 that the derivative of a vector having constant components relative to a moving reference frame is determined by the angular velocity of that reference frame. In Section 3.4 we treated a much more general situation in which the vector to be differentiated (i.e. $\bar{\omega}$) had variable components. In that case, the individual contributions were described in terms of unit vectors $\bar{e}_i$ that were oriented arbitrarily. These unit vectors were not necessarily associated with the same reference frame. Here, we shall address the conventional representation of a vector quantity, in which all unit vectors are associated with the axes of the (moving) xyz reference frame.

Let $\bar{\omega}$ denote the angular velocity of xyz, and let $\bar{A}$ be a vector whose components A_x, A_y, A_z relative to the moving reference frame are functions of time. The component representation of $\bar{A}$ is

$$\bar{A} = A_x\bar{i} + A_y\bar{j} + A_z\bar{k}. \tag{3.44}$$

The unit vectors, as well as the components, in this expression are functions of time. Equation (3.40) describes the derivatives of $\bar{i}$, $\bar{j}$, and $\bar{k}$. The angular velocity to be used for the differentiation in each case is $\bar{\omega}$, because the unit vectors are those for the moving axes. The rules for the derivative of a sum and a product therefore yield

$$\dot{\bar{A}} = \dot{A}_x\bar{i} + \dot{A}_y\bar{j} + \dot{A}_z\bar{k} + A_x(\bar{\omega}\times\bar{i}) + A_y(\bar{\omega}\times\bar{j}) + A_z(\bar{\omega}\times\bar{k});$$

♦ $$\dot{\bar{A}} = \frac{\delta \bar{A}}{\delta t} + \bar{\omega} \times \bar{A}, \tag{3.45}$$

where $\delta\bar{A}/\delta t$ describes the rate of change of $\bar{A}$ due to the time dependency of the components:

♦ $$\frac{\delta \bar{A}}{\delta t} = \dot{A}_x \bar{i} + \dot{A}_y \bar{j} + \dot{A}_z \bar{k}. \tag{3.46}$$

These relations have a simple explanation. Suppose you were to observe any vector quantity, such as a position $\bar{r}$, while situated on a moving reference frame. In describing the velocity in terms of coordinates measured relative to the axes of the moving frame, the term $\delta\bar{r}/\delta t$ is the only effect you would observe. In general, $\delta\bar{A}/\delta t$ is a time derivative as seen from a moving reference frame. It may be considered to be the partial derivative of $\bar{A}$, based on holding the orientation of xyz constant. We know from the previous section that the term $\bar{\omega} \times \bar{A}$ gives the portion of the change of $\bar{A}$ that is attributable to rotation of the reference frame. Hence, Eq. (3.45) is merely a statement that these two effects superpose, as you might have expected.

3.6 Velocity and Acceleration Using a Moving Reference Frame

Equations (3.45) and (3.46) will be employed in a variety of ways. Here we shall use those expressions to derive formulas that relate observations of velocity and acceleration relative to moving and fixed reference frames. The general situation confronting us is depicted in Figure 3.8, which is the same as the diagram that introduced this chapter. We have already seen that the observations of the position of point P from the fixed and moving reference frame are related by

$$\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}. \tag{3.47}$$

By definition, the absolute velocity is the time derivative of the position with respect to the fixed reference frame. Differentiating Eq. (3.47) yields

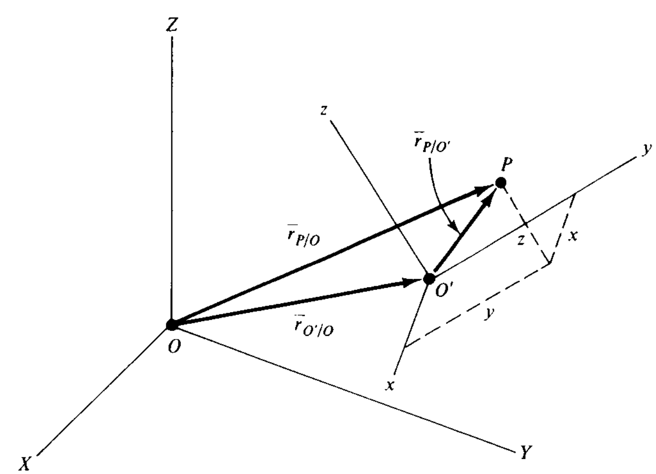


Figure 3.8 Position relative to a moving reference frame.

$$\bar{v}_P = \bar{v}_{O'} + \frac{d}{dt}\bar{r}_{P/O'}. \tag{3.48}$$

Presumably the (moving) xyz reference frame has been chosen for its convenience in describing the position of point P. We therefore describe $\bar{r}_{P/O'}$ in terms of the coordinates of point P with respect to the axes of this reference frame. Thus,

♦ $$\bar{r}_{P/O'} = x\bar{i} + y\bar{j} + z\bar{k}. \tag{3.49}$$

Differentiating Eq. (3.49) is not difficult, because it matches the situation addressed in the previous section. We apply Eq. (3.45), which superposes the effects of the changing coordinate values and the rotation of the vectors. This gives

$$\frac{d}{dt}\bar{r}_{P/O'} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + \bar{\omega} \times \bar{r}_{P/O'}. \tag{3.50}$$

Suppose you were an observer moving with the xyz reference frame; you would see only the (x, y, z) coordinates change. Thus, the first three terms on the right side of Eq. (3.50) describe the velocity of point P as seen from the moving reference frame. Let $(\bar{v}_P)_{xyz}$ denote this *relative velocity,* where the subscript P details the point under consideration and the trailing group of subscripts describes the reference frame from which the motion is viewed. In this notation, combining Eqs. (3.48) and (3.50) leads to

♦ $$\bar{v}_P = \bar{v}_{O'} + (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}, \tag{3.51}$$

where the relative velocity is

♦ $$(\bar{v}_P)_{xyz} = \frac{\delta}{\delta t}\bar{r}_{P/O'} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k}. \tag{3.52}$$

Note that it is a matter of convenience to omit the reference-frame specification when indicating an absolute velocity (and acceleration).

Equation (3.51) confirms a superposition of effects that we could have anticipated. The rotation of the reference frame contributes a transverse velocity $\bar{\omega} \times \bar{r}_{P/O'}$ that is perpendicular to the plane formed by $\bar{\omega}$ and $\bar{r}_{P/O'}$. This term combines with the velocity of the origin O', and the relative velocity, to form $\bar{v}_P$.

Our intuition is not so correct when applied to acceleration. A relation for the acceleration of a point is obtained by differentiating Eq. (3.51). The derivative of the velocity of origin O' is its acceleration. Because Eq. (3.52) gives the relative velocity in terms of its components relative to the moving frame, we employ Eq. (3.45) to differentiate $(\bar{v}_P)_{xyz}$:

$$\frac{d}{dt}(\bar{v}_P)_{xyz} = (\bar{a}_P)_{xyz} + \bar{\omega} \times (\bar{v}_P)_{xyz}, \tag{3.53}$$

where the *relative acceleration* is

♦ $$(\bar{a}_P)_{xyz} = \frac{\delta}{\delta t}(\bar{v}_P)_{xyz} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}. \tag{3.54}$$

The third term in Eq. (3.51) is the product $\bar{\omega} \times \bar{\rho}_{P/O'}$, and Eq. (3.50) gives the derivative of $\bar{r}_{P/O'}$. The angular acceleration is the derivative of the angular velocity, so the total derivative of the third term is

$$\frac{d}{dt}(\bar{\omega} \times \bar{r}_{P/O'}) = \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times [(\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}]. \tag{3.55}$$

The resulting acceleration relation is thereby found to be

♦ $$\bar{a}_P = \bar{a}_{O'} + (\bar{a}_P)_{xyz} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) + 2\bar{\omega} \times (\bar{v}_P)_{xyz}. \tag{3.56}$$

The final term in Eq. (3.56), that preceded by the factor 2, is called the *Coriolis acceleration*. In order to understand this term, consider the second derivative of a product; specifically,

$$\frac{d^2}{dt^2}(uv) = \ddot{u}v + 2\dot{u}\dot{v} + u\ddot{v}.$$

Because acceleration is the second derivative of position, the occurrence of a factor 2 should not be surprising. For further insight, recall the analysis in Chapter 2 of curvilinear coordinates for particle motion. We saw there that the Coriolis acceleration originates from two distinct effects. The same is true here, since Eqs. (3.53) and (3.55) both contribute to the overall effect. The term in Eq. (3.53) is associated with the change in the direction of the relative velocity resulting from rotation of the reference frame. In contrast, the term in Eq. (3.55) corresponds to the change in the components of $\bar{\omega} \times \bar{r}_{P/O'}$ that results from changing the components of $\bar{r}_{P/O'}$. Comparable effects were found for curvilinear coordinates. Experienced dynamicists recognize Coriolis acceleration as a combination of two explainable effects resulting from movement relative to a rotating reference frame. Thus it is, to a certain extent, a misnomer to use a single name to describe the corresponding term in Eq. (3.56).

The other terms in Eq. (3.56) could have been predicted in advance. The additive nature of the accelerations of the origin O' of the xyz reference frame and of point P relative to xyz requires no explanation. The term $\bar{\alpha} \times \bar{r}_{P/O'}$ is the angular acceleration contribution, analogous to the velocity term $\bar{\omega} \times \bar{r}_{P/O'}$ resulting from an angular velocity. In spatial motion, the angular acceleration is usually not parallel to the angular velocity. As a result, the direction of the corresponding acceleration might occasionally differ from expectations.

Now consider the term $\bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'})$. Figure 3.9 shows the construction of this acceleration. The magnitude of $(\bar{\omega} \times \bar{r}_{P/O'})$ is $r_\perp |\bar{\omega}|$, and the corresponding direction is perpendicular to the plane formed by $\bar{\omega}$ and $\bar{r}_{P/O'}$. Then $\bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'})$ is perpendicular to the rotation axis, pointing inward; its magnitude is $|\bar{\omega}|^2 r_\perp$. In other words, this term describes the centripetal acceleration that would be found if the reference frame were rotating at rate $|\bar{\omega}|$ about an axis that intersects the origin O' and is parallel to $\bar{\omega}$.

One aspect of both relative velocity and relative acceleration greatly facilitates their evaluation. These terms may be regarded as the effects that would be present if the reference frame were held stationary yet the relative motion remained unchanged. These velocity and acceleration effects were described in Eqs. (3.52) and (3.54), respectively, in terms of a Cartesian coordinate description. However, other kinematical descriptions, such as those employing path variables and curvilinear coordinates, might be more appropriate in some situations. When such an approach is taken, it becomes necessary to convert the corresponding unit vectors to the set of components used to represent the other vectors in the velocity and acceleration relations.

It is instructive to close this discussion by considering two special cases. The situation of a translating xyz frame corresponds to $\bar{\omega}$ being identically equal to zero; hence, $\bar{\alpha}$ also is zero. The relative motion equations then reduce to the following.

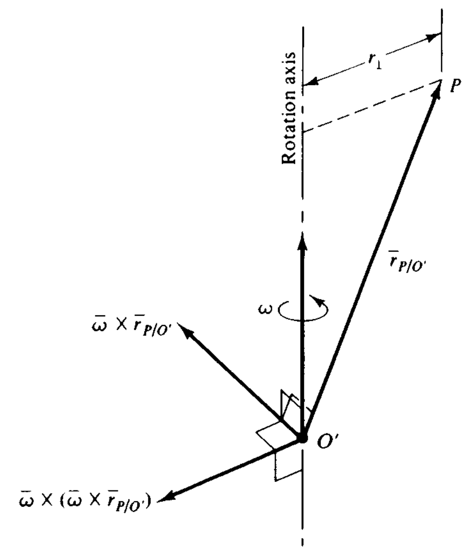


Figure 3.9 Centripetal acceleration.

Translating Reference Frame xyz

$$\bar{v}_P = \bar{v}_{O'} + (\bar{v}_P)_{xyz},$$
$$\bar{a}_P = \bar{a}_{O'} + (\bar{a}_P)_{xyz}. \tag{3.57}$$

The motion of the origin and of the point relative to the moving reference frame are additive - there are no corrections for direction changes due to rotation.

Let us use Eqs. (3.57) to reexamine fixed reference frames. Suppose that xyz, as well as XYZ, is fixed. Then Eqs. (3.57) show that the velocity and acceleration are the same, regardless of which reference frame is selected. This verifies our earlier statement that there is no need to indicate the origin of the reference frame in the notation for velocity and acceleration.

Another interesting situation arises if the reference frame is translating at a constant velocity. This means that $\bar{a}_{O'} = \bar{0}$. Note that the origin O' must be following a straight path in order for it to have no acceleration. The second of Eqs. (3.57) shows that acceleration viewed from the fixed and moving references is identical. A reference frame with this type of motion is said to be an *inertial* or *Galilean* reference frame. The terminology arises because the absolute acceleration is observable from the reference frame, so the frame may be employed to formulate Newton's laws.

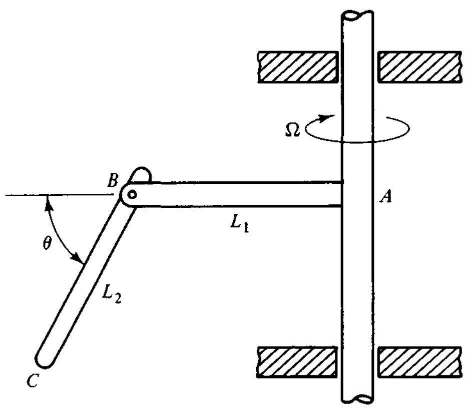
The second special case arises when point P is fixed with respect to the moving reference frame. This is the situation treated in Section 3.2 to study position changes in finite rotation. The velocity and acceleration relations simplify substantially, because $(\bar{v}_P)_{xyz}$ and $(\bar{a}_P)_{xyz}$ are both identically zero.

Fixed Position Relative to xyz

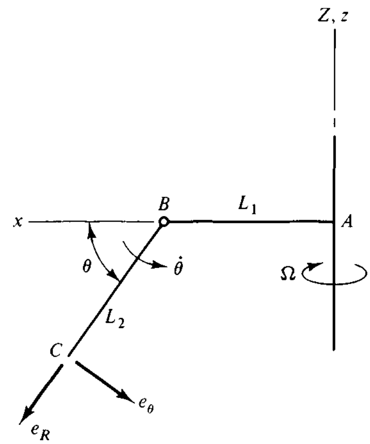
$$\bar{v}_P = \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{P/O'},$$
$$\bar{a}_P = \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}). \tag{3.58}$$

A primary reason for highlighting this situation is that it is descriptive of the motion of a rigid body. If the reference frame is attached to the body, then the position vectors between points in the body have constant components relative to the moving reference frame. Also, the angular motions of the body and of the reference frame are synonymous in this case. The motion of rigid bodies is the focus of the next chapter.

Example 3.10 Bar BC is pinned to the T-bar, which is rotating about the vertical axis at the constant rate Ω. The angle θ is an arbitrary function of time. Determine the velocity and acceleration of point C using an xyz reference frame that is attached to the T-bar with its x axis aligned with segment AB.



Example 3.10



Coordinate systems.

Solution Placing the origin of xyz at point A makes it a trivial matter to describe the motion of the reference point. The specified xyz reference frame only rotates about the vertical axis, which is fixed, so we designate $\bar{e}_1 = -\bar{K} = -\bar{k}$. Hence, the motion of the reference frame is

$$\bar{v}_A = \bar{a}_A = \bar{0}, \qquad \bar{\omega} = -\Omega\bar{K} = -\Omega\bar{k}, \qquad \bar{\alpha} = \bar{0}.$$

Relative to xyz, point C follows a circular parth centered at point B. Using the polar coordinates defined in the sketch to represent the relative motion yields

$$(\bar{v}_C)_{xyz} = L_2\dot{\theta}\bar{e}_\theta, \qquad (\bar{a}_C)_{xyz} = -L_2\dot{\theta}^2\bar{e}_R + L_2\ddot{\theta}\bar{e}_\theta.$$

We transform these expressions to xyz components in order to employ them in the relative motion equation. First, we form

$$\bar{e}_R = (\cos\theta)\bar{i} - (\sin\theta)\bar{k}, \qquad \bar{e}_\theta = -(\sin\theta)\bar{i} - (\cos\theta)\bar{k}.$$

Substitution of these unit vectors into the relative velocity and acceleration yields

$$(\bar{v}_C)_{xyz} = L_2\dot{\theta}[-(\sin\theta)\bar{i} - (\cos\theta)\bar{k}],$$

$$(\bar{a}_C)_{xyz} = L_2[(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta)\bar{i} + (\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)\bar{k}].$$

Also, the vector from the origin of xyz to point C is

$$\bar{r}_{C/A} = (L_1 + L_2 \cos\theta)\bar{i} - (L_2 \sin\theta)\bar{k}.$$

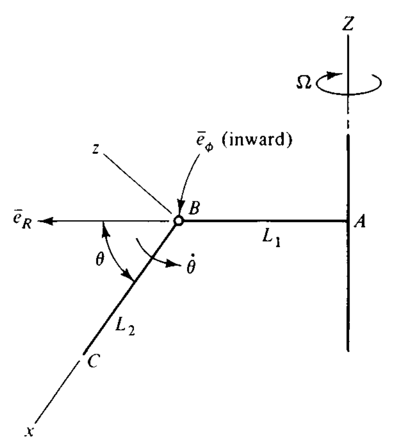
We are now ready to form the absolute motion. The result of substituting the individual terms into Eq. (3.51) is

$$\begin{aligned}\bar{v}_C &= \bar{v}_A + (\bar{v}_C)_{xyz} + \bar{\omega}\times\bar{r}_{C/A}\\ &= -(L_2\dot{\theta}\sin\theta)\bar{i} - (L_1 + L_2\cos\theta)\Omega\bar{j} - (L_2\dot{\theta}\cos\theta)\bar{k}.\end{aligned}$$

Similar steps for Eq. (3.56) yield

$$\begin{aligned}\bar{a}_C &= \bar{a}_A + (\bar{a}_C)_{xyz} + \bar{\alpha}\times\bar{r}_{C/A} + \bar{\omega}\times(\bar{\omega}\times\bar{r}_{C/A}) + 2\bar{\omega}\times(\bar{v}_C)_{xyz}\\ &= -[L_2\ddot{\theta}\sin\theta + L_2\dot{\theta}^2\cos\theta + (L_1 + L_2\cos\theta)\Omega^2]\bar{i}\\ &\quad + (2L_2\Omega\dot{\theta}\sin\theta)\bar{j} - L_2(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\bar{k}.\end{aligned}$$

Example 3.11 Determine the velocity and acceleration of point C in Example 3.10 using an xyz reference frame that is attached to bar BC.



Coordinate systems.

Solution Because xyz must be fixed to bar BC, the origin of xyz must have a stationary position as viewed from that body. For that reason, we cannot place the origin of xyz at the stationary point A. Point B is suitable as this origin. We know that this point follows a circular path centered at point A. The radial direction for this motion is $\bar{e}_R$ in the sketch, and we define the azimuthal direction to be inward relative to the plane of the sketch. Hence, we have

$$\bar{v}_B = L_1\Omega\bar{e}_\phi, \qquad \bar{a}_B = -L_1\Omega^2\bar{e}_R.$$

We align the axes of xyz to expedite the description of $\bar{r}_{C/B}$, $\bar{\omega}$, and $\bar{\alpha}$. For this reason, we let the x axis coincide with bar BC and select the z axis to be situated in the vertical plane, as shown. One rotation of xyz occurs about the vertical axis at rate Ω, so $\bar{e}_1 = -\bar{K}$. The other rotation is $\dot{\theta}$ about the axis of the pin, outward as viewed in the sketch according to the right-hand rule. The y axis always coincides with this axis of rotation, and y must be outward in the sketch because xyz must be a right-handed set of coordinates. Hence, we set $\bar{e}_2 = \bar{j}$, which leads to

$$\bar{\omega} = -\Omega \bar{K} + \dot{\theta}\bar{j}.$$

Differentiating this expression in accordance with the general relation, Eq. (3.43), gives, for constant Ω,

$$\bar{\alpha} = \ddot{\theta}\bar{j} + \dot{\theta}(\bar{\omega} \times \bar{j}).$$

One benefit of fixing xyz to bar BC is that point C is stationary relative to xyz, so

$$(\bar{v}_C)_{xyz} = (\bar{a}_C)_{xyz} = \bar{0}.$$

As the last step prior to forming Eqs. (3.51) and (3.56), we transform all vectors to components relative to xyz. The various unit vectors are found by referring to our sketch, which gives

$$\bar{e}_R = (\cos\theta)\bar{i} + (\sin\theta)\bar{k}, \qquad \bar{e}_\phi = -\bar{j}, \qquad \bar{K} = -(\sin\theta)\bar{i} + (\cos\theta)\bar{k},$$

which then leads to

$$\bar{v}_B = -L_1\Omega\bar{j}, \qquad \bar{a}_B = -L_1\Omega^2[(\cos\theta)\bar{i} + (\sin\theta)\bar{k}];$$

$$\bar{\omega} = (\Omega\sin\theta)\bar{i} + \dot{\theta}\bar{j} - (\Omega\cos\theta)\bar{k},$$

$$\bar{\alpha} = (\Omega\dot{\theta}\cos\theta)\bar{i} + \ddot{\theta}\bar{j} + (\Omega\dot{\theta}\sin\theta)\bar{k}.$$

A verification of the correctness of this expression is that the two components containing the product $\Omega\dot{\theta}$ form a vector in the direction of $\bar{e}_R$. This is the direction in which the tip of the angular velocity $\dot{\theta}\bar{j}$ moves owing to the rotation about the vertical axis.

The velocity and acceleration relations, Eqs. (3.51) and (3.56), now give, for $\bar{r}_{C/B} = L_2\bar{i}$,

$$\bar{v}_C = \bar{v}_B + \bar{\omega} \times \bar{r}_{C/B} = -(L_1 + L_2\cos\theta)\Omega\bar{j} - L_2\dot{\theta}\bar{k},$$

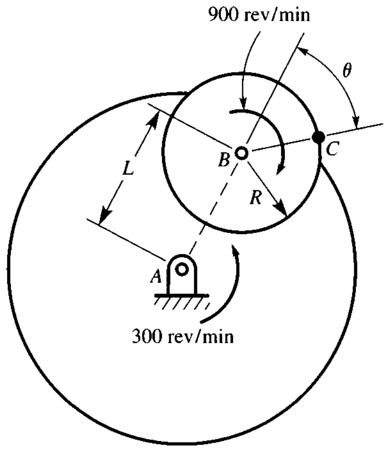
$$\begin{aligned}\bar{a}_C &= \bar{a}_B + \bar{\alpha} \times \bar{r}_{C/B} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{C/B}) \\ &= -[L_2\dot{\theta}^2 + (L_1 + L_2\cos\theta)\Omega^2\cos\theta]\bar{i} \\ &\quad + (2L_2\Omega\dot{\theta}\sin\theta)\bar{j} - [L_2\ddot{\theta} + (L_1 + L_2\cos\theta)\Omega^2\sin\theta]\bar{k}.\end{aligned}$$

If we were to transform the present components to those used in Example 3.10, we would find that the results represent identical vectors.

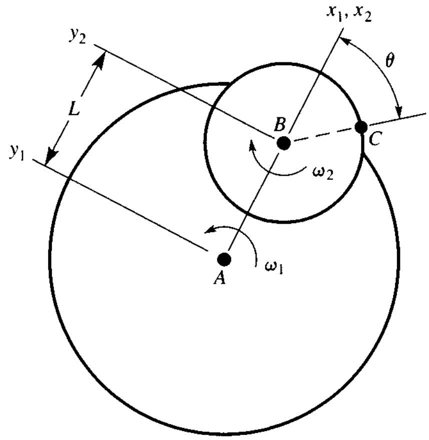
Example 3.12 Disk B rotates at 900 rev/min relative to the turntable, which is rotating about a fixed axis at a constant rate of 300 rev/min. Determine the acceleration of point C on the perimeter of the disk at the instant shown using:

(a) a moving coordinate system that is attached to the turntable; and
(b) a moving coordinate system that is attached to the disk.

Solution This is a case of planar motion, so we orient the z axis normal to the plane in both formulations. The reference frame $x_1y_1z_1$ in the first case is attached to the turntable, so center point A is a convenient origin. Placing the x axis along line AB leads to the following description of the motion of $x_1y_1z_1$:



Example 3.12



Coordinate systems.

$$\bar{v}_A = \bar{a}_A = \bar{0}; \qquad \bar{\omega} = 300\left(\frac{2\pi}{60}\right)\bar{k}, \qquad \bar{\alpha} = \bar{0}.$$

We could use polar coordinates to formulate the relative velocity and acceleration; but, for the sake of variety, we shall employ relative motion equations for this task. We visualize the motion that would remain if the turntable were stationary. From this viewpoint, the angular motion of the disk relative to the turntable is

$$\bar{\omega}_{\text{rel}} = -900\left(\frac{2\pi}{60}\right)\bar{k} \text{ rad/s}, \qquad \bar{\alpha}_{\text{rel}} = \bar{0}.$$

Points B and C have fixed positions when viewed from the disk, so we may employ Eqs. (3.58) with the angular motion being that of the disk. Correspondingly, we have

$$\begin{aligned}(\bar{v}_C)_{xyz} &= \bar{\omega}_{\text{rel}} \times \bar{r}_{C/B} = (-30\pi\bar{k}) \times [(R\cos\theta)\bar{i} - (R\sin\theta)\bar{j}] \\ &= -30\pi R[(\sin\theta)\bar{i} + (\cos\theta)\bar{j}],\end{aligned}$$

$$(\bar{a}_C)_{xyz} = \bar{\omega}_{\text{rel}} \times (\bar{\omega}_{\text{rel}} \times \bar{r}_{C/B}) = -900\pi^2[(R\cos\theta)\bar{i} - (R\sin\theta)\bar{j}].$$

We substitute these quantities into Eqs. (3.51) and (3.56) to find

$$\begin{aligned}\bar{v}_C &= \bar{v}_A + (\bar{v}_C)_{xyz} + \bar{\omega} \times \bar{r}_{C/A} \\ &= -30\pi R[(\sin\theta)\bar{i} + (\cos\theta)\bar{j}] + 10\pi\bar{k} \times [(L + R\cos\theta)\bar{i} - (R\sin\theta)\bar{j}] \\ &= -20\pi R(\sin\theta)\bar{i} + 10\pi(L - 2R\cos\theta)\bar{j},\end{aligned}$$

$$\begin{aligned}\bar{a}_C &= \bar{a}_A + (\bar{a}_C)_{xyz} + \bar{\alpha} \times \bar{r}_{C/A} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{C/A}) + 2\bar{\omega} \times (\bar{v}_C)_{xyz} \\ &= -100\pi^2(L + 4R\cos\theta)\bar{i} + 400\pi^2(R\sin\theta)\bar{j}.\end{aligned}$$

A formulation based on fixing reference frame $x_2y_2z_2$ to the disk is considerably easier. We place the origin of $x_2y_2z_2$ at center point B, because it is the only point on the disk that follows a simple path. The rotation of $x_2y_2z_2$ consists of a superposition of $\omega_1 = 10\pi$ rad/s in the sense of $\bar{e}_1 = \bar{k}$, and $\omega_2 = 30\pi$ rad/s in the sense of $\bar{e}_2 = -\bar{k}$. Thus,

$$\bar{\omega} = \omega_1\bar{k} - \omega_2\bar{k} = -20\pi\bar{k} \text{ rad/s}, \qquad \bar{\alpha} = \bar{0};$$

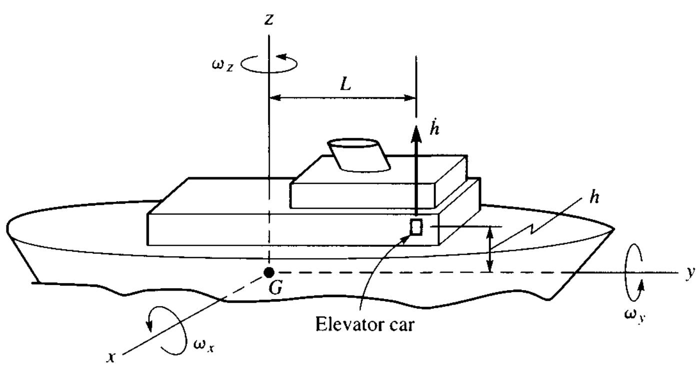
$$\bar{v}_B = 10\pi L\bar{j}, \qquad \bar{a}_B = -100\pi^2 L\bar{i}.$$

There is no relative motion to evaluate in this case, so we have

$$\begin{aligned}\bar{v}_C &= \bar{v}_B + \bar{\omega}\times\bar{r}_{C/B} = 10\pi L\bar{j} + (-20\pi\bar{k})\times[(R\cos\theta)\bar{i} - (R\sin\theta)\bar{j}]\\ &= -20\pi R(\sin\theta)\bar{i} + 10\pi(L - 2R\cos\theta)\bar{j},\end{aligned}$$

$$\begin{aligned}\bar{a}_C &= \bar{a}_B + \bar{\omega}\times(\bar{\omega}\times\bar{r}_{C/B})\\ &= -100\pi^2 L\bar{i} - 400\pi^2[(R\cos\theta)\bar{i} - (R\sin\theta)\bar{j}]\\ &= -100\pi^2(L + 4R\cos\theta)\bar{i} + 400\pi^2(R\sin\theta)\bar{j}.\end{aligned}$$

Example 3.13 Let ω_x, ω_y, and ω_z denote the pitch, roll, and yaw rates (respectively) of a ship about xyz axes that are attached to the ship with the orientations shown. All of these rotation rates are variable quantities. The origin of xyz coincides with the center of mass G of the ship. Consider an elevator car whose path perpendicularly intersects the centerline at a distance L forward from the center of mass. Let $h(t)$ denote the height of the elevator above the centerline. The velocity and acceleration of the center of mass at this instant are $\bar{v}_G$ and $\bar{a}_G$. Determine the corresponding velocity and acceleration of the car.



Example 3.13

Solution The elevator follows a straight path relative to the ship, so it is convenient to attach xyz to the ship. The given rotations are about body-fixed axes, so we have $\bar{e}_1 = \bar{i}$, $\bar{e}_2 = \bar{j}$, and $\bar{e}_3 = \bar{k}$, corresponding to the rates ω_x, ω_y, and ω_z, respectively. Thus, the rotation of xyz is

$$\bar{\omega} = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k},$$

$$\begin{aligned}\bar{\alpha} &= \dot{\omega}_x\bar{i} + \dot{\omega}_y\bar{j} + \dot{\omega}_z\bar{k} + \omega_x(\bar{\omega}\times\bar{i}) + \omega_y(\bar{\omega}\times\bar{j}) + \omega_z(\bar{\omega}\times\bar{k})\\ &= \dot{\omega}_x\bar{i} + \dot{\omega}_y\bar{j} + \dot{\omega}_z\bar{k} + \bar{\omega}\times(\omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}) = \dot{\omega}_x\bar{i} + \dot{\omega}_y\bar{j} + \dot{\omega}_z\bar{k}.\end{aligned}$$

It should be noted that these expressions for $\bar{\omega}$ and $\bar{\alpha}$ are generally true. They indicate that the angular acceleration components are always the derivatives of the angular velocity components, provided that all components are relative to body-fixed axes. This observation is a key aspect of the development of kinetics principles in Chapter 5.

The position of the elevator relative to the center of mass is

$$\bar{r}_{E/G} = L\bar{j} + h\bar{k}.$$

Relative to xyz, the elevator executes a rectilinear motion at speed $\dot{h}$ parallel to the z axis, so

$$(\bar{v}_E)_{xyz} = \dot{h}\bar{k}, \qquad (\bar{a}_E)_{xyz} = \ddot{h}\bar{k}.$$

Correspondingly, we find

$$\begin{aligned}\bar{v}_E &= \bar{v}_G + (\bar{v}_E)_{xyz} + \bar{\omega} \times \bar{r}_{E/G} \\ &= \bar{v}_G + (\omega_y h - \omega_z L)\bar{i} - \omega_x h\bar{j} + (\dot{h} + \omega_x L)\bar{k},\end{aligned}$$

$$\begin{aligned}\bar{a}_E &= \bar{a}_G + (\bar{a}_E)_{xyz} + \bar{\alpha} \times \bar{r}_{E/G} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{E/G}) + 2\bar{\omega} \times (\bar{v}_E)_{xyz} \\ &= \bar{a}_G + (\dot{\omega}_y h - \dot{\omega}_z L + \omega_x \omega_y L + \omega_x \omega_z h + 2\omega_y \dot{h})\bar{i} \\ &\quad + (-\dot{\omega}_x h + \omega_y \omega_z h - \omega_z^2 L - \omega_x^2 L - 2\omega_x \dot{h})\bar{j} \\ &\quad + (\ddot{h} + \dot{\omega}_x L - \omega_x^2 h - \omega_y^2 h + \omega_y \omega_z L)\bar{k}.\end{aligned}$$

Some of the acceleration terms were foreseeable. The acceleration of the elevator relative to the ship is represented by the $\ddot{h}$ term, and the angular acceleration effects are contained in the $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ terms. In the same vein, the ω_x^2, ω_y^2, and ω_z^2 terms represent centripetal accelerations about the respective axes. The terms that are not intuitive are those containing products of rotation rates about different axes, as well as the Coriolis acceleration terms, which feature a product of $\dot{h}$ and a rotation rate.

Example 3.14 The cooling fan consists of a shaft that rotates about the vertical axis at angular speed ω_1 while the blades rotate around the shaft at angular rate $\dot{\theta}$, where θ is the angle of rotation of one of the blades from the top-center position. Both rotation rates are constant. Derive expressions for the velocity and acceleration of the blade tip P in terms of components relative to the body-fixed xyz reference system shown in View C–C.

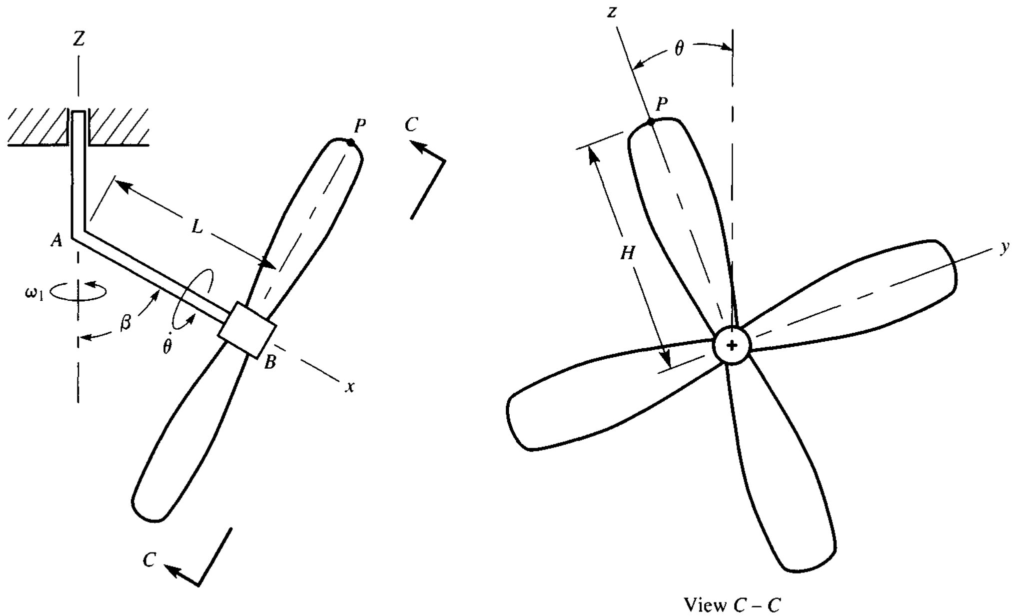
Solution Had it not been specified otherwise, it would be preferable to employ a body-fixed $x'y'z'$ reference frame, parallel to the given xyz but with origin at point A. However, by employing the given system we will gain greater versatility in treating the variety of situations that may occur. The rotation of xyz is the sum of the rotation ω_1 about the fixed Z axis and the rotation $\dot{\theta}$ about the x axis. The general expressions for $\bar{\omega}$ and $\bar{\alpha}$ are therefore

$$\bar{\omega} = \omega_1 \bar{K} + \dot{\theta}\bar{i}, \qquad \bar{\alpha} = \dot{\theta}(\bar{\omega} \times \bar{i}).$$

Several approaches for expressing $\bar{K}$ in terms of xyz components are available; we shall evaluate the rotation transformation from XYZ to xyz. If the X axis is defined such that shaft AB lies in the XZ plane at the instant depicted in the diagram, then

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_\theta][R_\beta],$$

where $[R_\beta]$ describes a rotation of $90° - \beta$ about the Y axis and $[R_\theta]$ describes a rotation about the x axis:



Example 3.14

$$[R_\beta]=\begin{bmatrix}\sin\beta & 0 & -\cos\beta\\ 0 & 1 & 0\\ \cos\beta & 0 & \sin\beta\end{bmatrix},\qquad [R_\theta]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}.$$

Thus, the overall transformation is

$$[R]=\begin{bmatrix}\sin\beta & 0 & -\cos\beta\\ (\cos\beta)(\sin\theta) & \cos\theta & (\sin\beta)(\sin\theta)\\ (\cos\beta)(\cos\theta) & -\sin\theta & (\sin\beta)(\cos\theta)\end{bmatrix}.$$

The last column of $[R]$ consists of the direction cosines between the Z axis and the xyz axes, from which we find that

$$\bar{K}=-(\cos\beta)\bar{i}+(\sin\beta)(\sin\theta)\bar{j}+(\sin\beta)(\cos\theta)\bar{k},$$

$$\bar{\omega}=[-\omega_1\cos\beta+\dot{\theta}]\bar{i}+\omega_1(\sin\beta)(\sin\theta)\bar{j}+\omega_1(\sin\beta)(\cos\theta)\bar{k},$$

$$\bar{\alpha}=\dot{\theta}\omega_1(\sin\beta)[(\cos\theta)\bar{j}-(\sin\theta)\bar{k}].$$

To describe the motion of the origin of xyz, we observe that point B is on shaft AB, which is rotating without angular acceleration about the vertical shaft. Because $\bar{r}_{B/A}=L\bar{i}$, this leads to

$$\bar{v}_B=\omega_1\bar{K}\times\bar{r}_{B/A}=L\omega_1(\sin\beta)[(\cos\theta)\bar{j}-(\sin\theta)\bar{k}],$$

$$\bar{a}_B=\omega_1\bar{K}\times(\omega_1\bar{K}\times\bar{r}_{B/A})$$
$$=L\omega_1^2(\sin\beta)[-(\sin\beta)\bar{i}-(\cos\beta)(\sin\theta)\bar{j}-(\cos\beta)(\cos\theta)\bar{k}].$$

As a check on the correctness of these expressions, note that $|\bar{v}_B|=L\omega_1(\sin\beta)$ and $|\bar{a}_B|=L\omega_1^2(\sin\beta)$, as they should because point B is following a circular path of radius $L\sin\beta$ at angular speed ω_1.

Because point P belongs to the blade and xyz is fixed to the blade, there is no relative motion:

$$\bar{r}_{P/B} = H\bar{k}; \qquad (\bar{v}_P)_{xyz} = \bar{0}, \qquad (\bar{a}_P)_{xyz} = \bar{0}.$$

Substituting the individual terms into the relative motion equations (3.51) and (3.56) gives

$$\begin{aligned}\bar{v}_P &= H\omega_1(\sin\beta)(\sin\theta)\bar{i} + [L\omega_1(\sin\beta)(\cos\theta) + H\omega_1(\cos\beta) - H\dot{\theta}]\bar{j}\\ &\quad - L\omega_1(\sin\beta)(\sin\theta)\bar{k},\\ \bar{a}_P &= [-L\omega_1^2(\sin\beta)^2 - H\omega_1^2(\sin\beta)(\cos\beta)(\cos\theta) + 2H\omega_1\dot{\theta}(\sin\beta)(\cos\theta)]\bar{i}\\ &\quad + [-L\omega_1^2(\sin\beta)(\cos\beta)(\sin\theta) + H\omega_1^2(\sin\beta)^2(\cos\theta)(\sin\theta)]\bar{j}\\ &\quad + [-L\omega_1^2(\sin\beta)(\cos\beta)(\cos\theta) - H(-\omega_1\cos\beta + \dot{\theta})^2\\ &\qquad - H\omega_1^2(\sin\beta)^2(\sin\theta)^2]\bar{k}.\end{aligned}$$

It is interesting to observe that each term in $\bar{a}_P$ containing a product of ω_1 and $\dot{\theta}$ also contains a factor of 2. These are Coriolis acceleration effects associated with the interaction of the two rotation rates. These terms arise even though the Coriolis acceleration term $2\bar{\omega} \times (\bar{v}_P)_{xyz}$ in Eq. (3.56) vanishes in our solution.

3.7 Observations from a Moving Reference System

The treatment in the previous section implicitly assumed that the motion of some point could be more readily described in terms of a moving reference frame, rather than a fixed one. However, this is not always the case. In some situations, the absolute motion is known and the relative motion must be evaluated. For example, it might be necessary to ensure that one part of a machine merges with another part in a smooth manner, as in the case of gears. The influence of the earth's motion on the dynamic behavior of a system is another important case where aspects of the absolute motion are known.

One approach is to interchange the absolute and relative reference frames, based on the fact that the kinematical relationships do not actually require that one of the reference frames be stationary. Thus, in this viewpoint, if the angular velocity of xyz relative to XYZ is $\bar{\omega}$, then the angular velocity of XYZ as viewed from xyz is $-\bar{\omega}$. The difficulty with this approach is that it is prone to errors, particularly in signs, because of the need to change the observer's viewpoint for the formulation. The simpler approach, which does not require redefinitions of the basic quantities, manipulates the previous relations.

The concept is quite straightforward. When the absolute velocity $\bar{v}_P$ and absolute acceleration $\bar{a}_P$ are known, Eqs. (3.51) and (3.56) may be solved for the relative motion parameters. Specifically,

$$(\bar{v}_P)_{xyz} = \bar{v}_P - \bar{v}_{O'} - \bar{\omega} \times \bar{r}_{P/O'}, \tag{3.59}$$

$$(\bar{a}_P)_{xyz} = \bar{a}_P - \bar{a}_{O'} - \bar{\alpha} \times \bar{r}_{P/O'} - \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) - 2\bar{\omega} \times (\bar{v}_P)_{xyz}. \tag{3.60}$$

If it is appropriate to do so, the relative velocity may be removed from the acceleration relation by substitution of Eq. (3.59). The result is

$$(\bar{a}_P)_{xyz} = \bar{a}_P - \bar{a}_{O'} - \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) - 2\bar{\omega} \times (\bar{v}_P - \bar{v}_{O'}). \tag{3.61}$$

The steps required to apply these relations are like those already established, because the angular velocity and angular acceleration still describe the rotation of the moving reference frame as seen from the fixed one.

One of the most common applications of these relations is to cases where the rotation of the earth must be considered. Our observations are usually in terms of earth-based instruments. However, Newton's second law relates the forces acting on a body to the motion relative to some hypothetical inertial reference frame, that is, $\bar{a}_P = \sum \bar{F}/m$ for a particle.

Consider an observer at point O' on the earth's surface. A natural definition for the reference frame employed by this observer is east-west and north-south for position along the surface, and vertical for measurements off the surface. Such a reference frame is depicted in Figure 3.10, where the $\bar{i}$ vector is northward and the $\bar{j}$ vector is westward. The observation point O' in the figure is located by the latitude angle λ measured from the equator and the longitude angle ϕ measured from some reference location, such as the prime meridian (the longitude of Greenwich, England).

For our present purposes, it is adequate to employ an approximate model of the earth. The earth rotates about its polar axis at $\omega_e = 2\pi$ rad/(23.934 hr) = $7.292(10^{-5})$ rad/s. The orbital rate of rotation of the earth about the sun, ω_0, is smaller by an approximate factor of 365, because one such revolution requires a full year. The centripetal acceleration of a point at the equator due to the spin about the polar axis is $\omega_e^2 R_e$, where R_e is the earth's diameter, $R_e \approx 6{,}370$ km. For comparison, the centripetal acceleration due to the orbital motion is $\omega_0^2 R_0$, where the mean orbital radius is $R_0 \approx 149.6(10^6)$ km. We note that $\omega_0^2 R_0 \approx 0.176\omega_e^2 R_e$, and $\omega_e^2 R_e$ is itself quite feeble (≈ 0.034 m/s^2). Furthermore, the centripetal acceleration associated with the earth's orbital motion is essentially balanced by the effect of the sun's gravitational attraction, since that balance produces the orbit. For these reasons, it is reasonable to

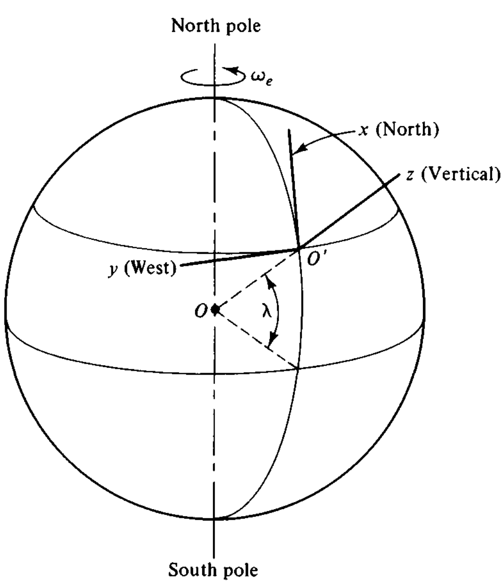


Figure 3.10 Reference frame fixed to the earth.

consider the center of the earth to be stationary. If we also ignore the relatively minor wobble of the polar axis, our model of the earth reduces to a sphere that rotates about the (fixed) polar axis at the constant rate ω_e. The corresponding expression for the acceleration relative to the earth is

$$(\bar{a}_P)_{xyz} = \frac{\sum \bar{F}_a}{m} + \frac{\bar{F}_g}{m} - \bar{a}_{O'} - \bar{\omega}_e \times (\bar{\omega}_e \times \bar{r}_{P/O'}) - 2\bar{\omega}_e \times (\bar{v}_P)_{xyz}, \tag{3.62}$$

where the term $\bar{F}_g$ represents the gravitational force acting on the body and $\sum \bar{F}_a$ represents the applied loads and reactions.

Now consider a particle near the earth's surface at the instant after it has begun to fall freely. Let point O' be close to the particle, so that $\bar{r}_{P/O'} \approx \bar{0}$. If air resistance is negligible then there are no applied forces, and $\sum \bar{F}_a = \bar{0}$. By definition, the acceleration observed from the earth is g vertically downward. Recall that the z axis in Figure 3.10 was defined as the upward vertical, which now means that the observed free-fall acceleration is $-g\bar{k}$. Then Eq. (3.62) gives

$$-g\bar{k} = \frac{\bar{F}_g}{m} - \bar{a}_{O'}. \tag{3.63}$$

This relation may be quantified by using the inverse square law for gravity, as well as the relative motion equation to describe $\bar{a}_{O'}$. The result is

$$g\bar{k} = \frac{GM_e}{R_e^3}\bar{r}_{O'/O} + \bar{\omega}_e \times (\bar{\omega}_e \times \bar{r}_{O'/O}). \tag{3.64}$$

There are two primary aspects of interest in this relation. The centripetal acceleration term is not parallel to the position $\bar{r}_{O'/O}$ except at the poles, where $\bar{\omega}_e$ is parallel to $\bar{r}_{O'/O}$, and at the equator, where $\bar{\omega}_e$ is perpendicular to $\bar{r}_{O'/O}$. Hence, the $\bar{k}$ direction, which people perceive as vertical, generally does not intersect the center of the earth. (It should be noted that $\bar{k}$ does coincide with a meridional plane, which is any plane formed by $\bar{\omega}_e$ and $\bar{r}_{O'/O}$.) At a latitude of 45°, $\bar{k}$ deviates from $\bar{r}_{O/O'}$ by approximately 0.1°. Equally significant is the effect of the centripetal acceleration on the weight mg that is measured at the earth's surface. This effect is most noticeable at the equator, where the centripetal acceleration is $\omega_e^2 R_e$ parallel to $\bar{r}_{O'/O}$. The value $g = 9.807$ m/s^2 represents a reasonable average value when the latitude is not specified.

In view of Eq. (3.63), the acceleration relative to the earth given by Eq. (3.62) becomes

$$(\bar{a}_P)_{xyz} = \frac{\sum \bar{F}_a}{m} - g\bar{k} - \bar{\omega}_e \times (\bar{\omega}_e \times \bar{r}_{P/O'}) - 2\bar{\omega}_e \times (\bar{v}_P)_{xyz}. \tag{3.65}$$

Usually, the term $\bar{\omega}_e \times (\bar{\omega}_e \times \bar{r}_{P/O'})$ may be neglected, unless $\bar{r}_{P/O'}$ is a large fraction of the earth's radius. Therefore, the primary difference between Eq. (3.65) and the form of Newton's second law that ignores the motion of the earth is the Coriolis term.

It is a straightforward matter to describe each of the terms in Eq. (3.65) in terms of components relative to the earth-based xyz reference frame. The position, velocity, and acceleration relative to this system may be described by the Cartesian coordinates (x, y, z). Because the angular velocity of the earth is parallel to the polar axis and the deviation of the z axis from the line to the center of the earth is small, the angular velocity is essentially

$$\bar{\omega}_e = \omega_e[(\cos\lambda)\bar{i} + (\sin\lambda)\bar{k}]. \tag{3.66}$$

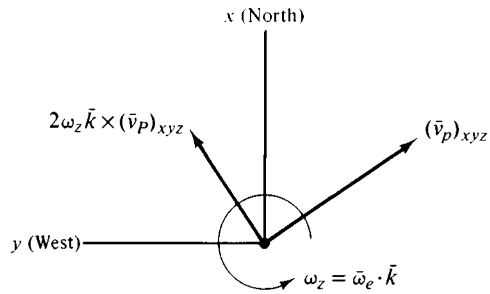


Figure 3.11 Coriolis acceleration due to the earth's rotation.

Correspondingly, Eq. (3.65) becomes

$$\begin{aligned}\ddot{x}-2\omega_e\dot{y}\sin\lambda &= \frac{F_x}{m},\\ \ddot{y}+2\omega_e(\dot{x}\sin\lambda-\dot{z}\cos\lambda) &= \frac{F_y}{m},\\ \ddot{z}+2\omega_e\dot{y}\cos\lambda &= \frac{F_z}{m}-g,\end{aligned} \tag{3.67}$$

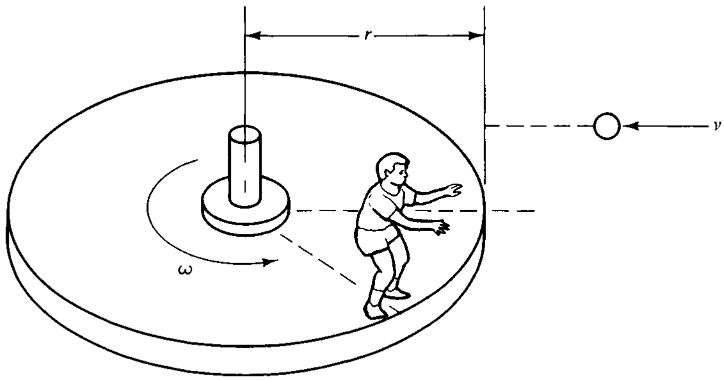
where (F_x, F_y, F_z) are the components of the applied forces. These equations may be solved for the forces required to have a specified motion relative to the earth. Alternatively, they may be regarded as a set of coupled differential equations for the relative position in situations where the forces are specified.

The fact that the Coriolis term is perpendicular to the velocity as seen by an observer on the earth leads to some interesting anomalies. In the Northern Hemisphere, the component of $\bar{\omega}_e$ perpendicular to the earth's surface is outward. If a particle is constrained to follow a horizontal path relative to the earth in the Northern Hemisphere, the Coriolis term $2\bar{\omega}_e\times(\bar{v}_P)_{xyz}$ is as shown in Figure 3.11. It follows that a horizontal force acting to the left of the direction of motion is required if that direction is to be maintained.

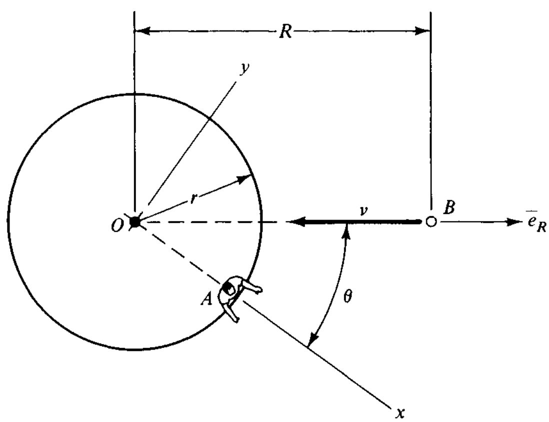
A story that has been passed down from professor to student over the years, without substantiation, states that a railroad line had two sets of north-south tracks along which trains ran in only one direction. For the track along which trains ran northward, the inner surface of the east rail was supposedly more shiny, because of the westward Coriolis force it had to exert on the flange of the wheels. Correspondingly, the track for trains running south was shinier on the inner surface of the west rail. The veracity of this story is questionable, owing to the smallness of the force in comparison with such other effects as wind and elevation changes.

If a transverse force is not present to maintain a particle in a straight relative path, as required by Eq. (3.65), then the particle will deviate to the right. This observation leads to a qualitative explanation of the fact that a liquid being drained through the center of a perfectly symmetrical cylindrical tank will exhibit a counterclockwise spiraling flow. (The flow will be clockwise in the Southern Hemisphere.) The same phenomenon acts on a much larger scale to set up the flow patterns in hurricanes and typhoons. Goldstein (1980) offers an excellent discussion of these effects. Meteorological models used to predict general weather patterns must account for the Coriolis acceleration effect.

Example 3.15 A child standing on a merry-go-round rotating about the vertical axis at the constant rate ω attempts to catch a ball traveling in the radial direction horizontally at speed v. Determine the velocity and acceleration of the ball as seen by the child.



Example 3.15



Coordinate systems.

Solution We will evaluate the relative motion by using kinematical formulas. However, it is useful first to develop a contrasting solution that differentiates the relative position directly. Let $t = 0$ be the instant when the child catches the ball, so $t < 0$ characterizes an arbitrary instant before the ball is caught. Correspondingly, the angle θ locating the child is $\theta = \omega(-t)$ and the distance R to the ball is $R = r + v(-t)$.

We assume that the child is stationary with respect to the turntable, so we attach xyz to that body. In order to expedite the construction of $\bar{r}_{B/A}$, we place the origin of xyz at the center of the turntable and align the x axis with the radial line to the child. Correspondingly, we find

$$\begin{aligned}\bar{r}_{B/A} &= (R\cos\theta - r)\bar{i} + (R\sin\theta)\bar{j} \\ &= [(r - vt)\cos\omega t - r]\bar{i} - (r - vt)(\sin\omega t)\bar{j}.\end{aligned}$$

By definition, the relative velocity is obtained by differentiating the components of the relative position; that is,

$$\begin{aligned}(\bar{v}_B)_{xyz} &= \frac{\delta}{\delta t}(\bar{r}_{B/A}) \\ &= [-v\cos\omega t-(r-vt)\omega\sin\omega t]\bar{i}+[v\sin\omega t-(r-vt)\omega\cos\omega t]\bar{j}.\end{aligned}$$

Similarly, we obtain the relative acceleration by differentiating the relative velocity components:

$$\begin{aligned}(\bar{a}_B)_{xyz} &= \frac{\delta}{\delta t}(\bar{v}_B)_{xyz} \\ &= [2v\omega\sin\omega t-(r-vt)\omega^2\cos\omega t]\bar{i}+[2v\omega\cos\omega t+(r-vt)\omega^2\sin\omega t]\bar{j}.\end{aligned}$$

The solution using the kinematical formulas barely resembles the operations in the previous solution. The motion of the reference frame is defined by

$$\bar{v}_A = r\omega\bar{j}, \qquad \bar{a}_A = -r\omega^2\bar{i}; \qquad \bar{\omega} = \omega\bar{k}, \qquad \bar{\alpha} = \bar{0}.$$

We also know the absolute motion of the ball to be

$$\bar{v}_B = -v\bar{e}_R = -v[(\cos\theta)\bar{i}+(\sin\theta)\bar{j}], \qquad \bar{a}_B = \bar{0}.$$

We write the position as

$$\bar{r}_{B/A} = (R\cos\theta - r)\bar{i} + (R\sin\theta)\bar{j},$$

so the relative velocity equation yields

$$\begin{aligned}(\bar{v}_B)_{xyz} &= v_B - v_A - \bar{\omega}\times\bar{r}_{B/A} \\ &= -v[(\cos\theta)\bar{i}+(\sin\theta)\bar{j}] = r\omega\bar{j} - (\omega\bar{k})\times[(R\cos\theta-r)\bar{i}+(R\sin\theta)\bar{j}] \\ &= (-v\cos\theta+\omega R\sin\theta)\bar{i}+(-v\sin\theta-\omega R\cos\theta)\bar{j}.\end{aligned}$$

Substitution of $R = r - vt$ and $\theta = -\omega t$ into this expression would yield the same result as the previous one.

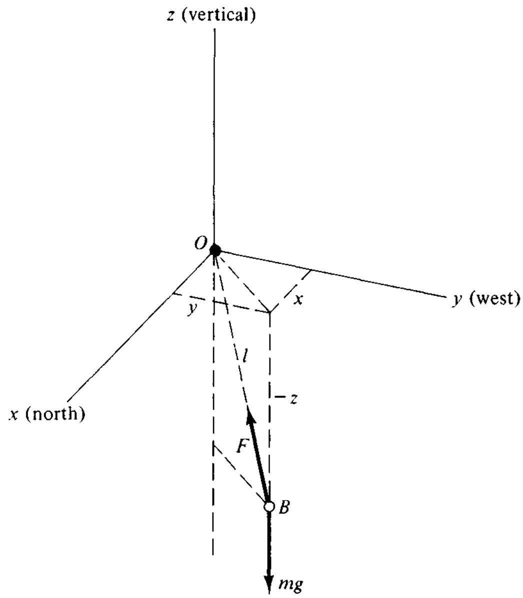
The same approach leads to the relative acceleration, except that the result for relative velocity is required to form the Coriolis acceleration. Hence,

$$\begin{aligned}(\bar{a}_B)_{xyz} &= \bar{a}_B - \bar{a}_A - \bar{\alpha}\times\bar{r}_{B/A} - \bar{\omega}\times(\bar{\omega}\times\bar{r}_{B/A}) - 2\bar{\omega}\times(\bar{v}_B)_{xyz} \\ &= r\omega^2\bar{i} + \omega^2[(R\cos\theta-r)\bar{i}+(R\sin\theta)\bar{j}] \\ &\quad -2\omega[(-v\cos\theta+\omega R\sin\theta)\bar{j}-(-v\sin\theta-\omega R\cos\theta)\bar{i}] \\ &= (-\omega^2 R\cos\theta - 2\omega v\sin\theta)\bar{i}+(-\omega^2 R\sin\theta+2\omega v\cos\theta)\bar{j}.\end{aligned}$$

This expression is equivalent to the acceleration in the first solution to this problem.

We could employ either approach with equal ease because this problem treated a case of planar motion. The relative motion equations become increasingly advantageous as the rotation of the reference frame becomes more complicated.

Example 3.16 When a small ball is suspended by a cable from an ideal swivel joint that permits three-dimensional motion, the system is called a *spherical pendulum*. Suppose such a pendulum, whose cable length is l, is released from rest relative to the earth with the ball at a distance $b \ll l$ north of the point below the pivot. Analyze



Free-body diagram.

the effect of the earth's rotation on the motion. It may be assumed that the angle between the suspending cable and the vertical is always very small.

Solution We may apply Eqs. (3.67) directly to this system. A free-body diagram of the ball shows the weight mg and the tensile force F exerted by the cable, which may be described in terms of xyz components as

$$\bar{F} = F\bar{e}_{O/B} = F\frac{\bar{r}_{O/B}}{|\bar{r}_{O/B}|} = F\left(\frac{-x\bar{i} - y\bar{j} + z\bar{k}}{l}\right).$$

Because the cable length l is constant, the z coordinate (which is negative) must satisfy

$$z = -[l^2 - x^2 - y^2]^{1/2}.$$

We obtain the corresponding velocity and acceleration components by successive differentiation:

$$\dot{z} = (x\dot{x} + y\dot{y})(l^2 - x^2 - y^2)^{-1/2},$$
$$\ddot{z} = (x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2)(l^2 - x^2 - y^2)^{-1/2} + (x\dot{x} + y\dot{y})^2(l^2 - x^2 - y^2)^{-3/2}.$$

Specifying that the angle with the vertical line be small means that $x \ll l$ and $y \ll l$. Incorporating this approximation into the previous equations yields

$$\dot{z} \approx \frac{1}{l}(x\dot{x} + y\dot{y}), \qquad \ddot{z} \approx \frac{1}{l}(\dot{x}^2 + \dot{y}^2).$$

It follows that for motions in which the angle from the vertical remains small, we may use the approximations $\dot{z} \approx 0$ and $\ddot{z} \approx 0$. In other words, we may consider the ball to move in the horizontal plane.

The corresponding form of Eqs. (3.67) is

$$\ddot{x} - 2\omega_e \dot{y} \sin\lambda = -\frac{F}{m}\frac{x}{l},$$

$$\ddot{y} + 2\omega_e \dot{x} \sin\lambda = -\frac{F}{m}\frac{y}{l},$$

$$2\omega_e \dot{y} \cos\lambda = \frac{F}{m}\frac{z}{l} - g.$$

Now observe that $z \approx -l$ because x and y are small, and that the Coriolis acceleration in motion relative to the earth is much weaker than the free-fall acceleration. The last of the preceding equations of motion therefore leads to the approximation that $F/m \approx g$. We introduce this approximation in the first two equations of motion, which become

$$\ddot{x} + \Omega^2 x - 2p\dot{y} = 0, \qquad \ddot{y} + \Omega^2 y + 2p\dot{x} = 0, \tag{1}$$

where the coefficients are

$$\Omega^2 = g/l, \qquad p = \omega_e \sin\lambda.$$

Evaluating the motion requires that we solve this pair of linear, coupled, ordinary differential equations. We could solve these equations by using the method of characteristic exponents, but an examination of the equations leads to a much briefer solution. We observe that if the Coriolis effect were not present, $p = 0$, then the equations for x and y would be uncoupled and the fundamental solutions for both variables would be $\sin(\Omega t)$ and $\cos(\Omega t)$. In either equation, the order of the derivatives of y is one different from the order of the derivatives of x. The combination of these two features suggest that both x and y vary sinusoidally, with a 90° phase difference between them. We therefore consider the trial solution

$$x = A\cos(\mu t + \phi), \qquad y = B\sin(\mu t + \phi), \tag{2}$$

where the amplitudes A and B, frequency μ, and phase angle ϕ are to be determined.

Substituting the trial forms into the equations of motion (1) leads to

$$(\Omega^2 - \mu^2)A - 2p\mu B = 0, \qquad -2p\mu A + (\Omega^2 - \mu^2)B = 0. \tag{3}$$

In order for A and B to be nonzero, the determinant of this pair of homogeneous equations for A and B must vanish. This leads to the characteristic equation

$$(\Omega^2 - \mu^2)^2 - 4p^2\mu^2 = 0,$$

which has roots $\mu = \pm p \pm (p^2 + \Omega^2)^{1/2}$. We need only the positive roots, which are

$$\mu_1 = (p^2 + \Omega^2)^{1/2} - p, \qquad \mu_2 = (p^2 + \Omega^2)^{1/2} + p. \tag{4}$$

Thus, there are two general solutions of eqs. (1). Because both eqs. (3) have the same solution for B in terms of A when the characteristic equation is satisfied, the two roots μ_j lead to two solutions for B_j in terms of A_j for each general solution. The second of eqs. (3) indicates that

$$B_j = \frac{2p\mu_j}{\Omega^2 - \mu_j^2} A_j. \tag{5}$$

This expression may be simplified further by substituting for the frequencies μ_j. Equations (4) show that $\Omega^2-\mu_1^2=2p\mu_1$ and $\Omega^2-\mu_2^2=-2p\mu_1$, from which it follows that $B_1=A_1$ and $B_2=-A_2$. The corresponding general solution of the equations of motion is therefore

$$\begin{aligned} x &= A_1\cos(\mu_1 t+\phi_1)+A_2\cos(\mu_2 t+\phi_2), \\ y &= A_1\sin(\mu_1 t+\phi_1)-A_2\sin(\mu_2 t+\phi_2). \end{aligned} \tag{6}$$

The coefficients A_1, A_2, ϕ_1, and ϕ_2 must satisfy initial conditions. In the statement of the problem, the ball was released from rest relative to xyz at a distance b to the north of the pivot. Thus, the initial conditions are

$$x=b, \quad y=0, \quad \dot{x}=\dot{y}=0 \quad \text{when } t=0.$$

The corresponding solution obtained from eqs. (6) is

$$\begin{aligned} x &= \frac{b}{\mu_1+\mu_2}[\mu_2\cos(\mu_1 t)+\mu_1\cos(\mu_2 t)], \\ y &= \frac{b}{\mu_1+\mu_2}[\mu_2\sin(\mu_1 t)-\mu_1\sin(\mu_2 t)]. \end{aligned} \tag{7}$$

It is convenient at this stage to simplify the characteristic roots by making use of the fact that $p \ll \Omega$ owing to the smallness of ω_e. We therefore expand each characteristic exponent in powers, and drop higher-order terms. This leads to

$$\mu_1=\Omega-p, \qquad \mu_2=\Omega+p.$$

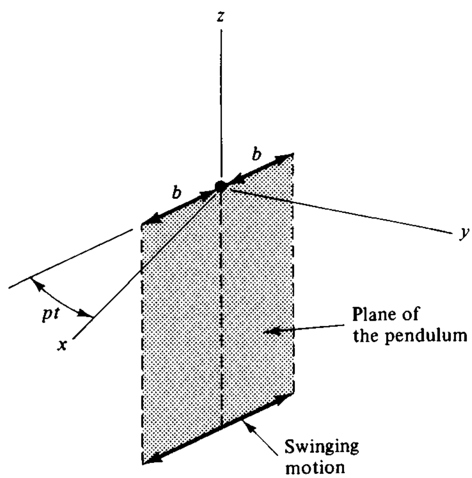
The values of μ_1 and μ_2 are very close, $\mu_1\approx\mu_2\approx\Omega$. Minor errors will be introduced if we use this approximation in the coefficients, but not the frequencies, of the sinusoidal terms. Combining this approximation with the trigonometric identities for the sum of sines or cosines leads to

$$\begin{aligned} x &\approx \frac{1}{2}b[\cos(\mu_1 t)+\cos(\mu_2 t)] = b\cos\left(\frac{\mu_1-\mu_2}{2}t\right)\cos\left(\frac{\mu_1+\mu_2}{2}t\right) \\ &\approx b\cos(pt)\cos(\Omega t), \\ y &\approx \frac{1}{2}b[\sin(\mu_1 t)-\sin(\mu_2 t)] = b\sin\left(\frac{\mu_1-\mu_2}{2}t\right)\cos\left(\frac{\mu_2+\mu_1}{2}t\right) \\ &\approx -b\sin(pt)\cos(\Omega t). \end{aligned}$$

The nature of the path becomes obvious when we observe that $\sin(pt)$ and $\cos(pt)$ vary much more slowly than $\sin(\Omega t)$ because $p \ll \Omega$. The preceding solutions satisfy

$$y=-x\tan(pt),$$

which is the equation of a straight line whose slope is $-\tan(pt)$ if we neglect the variation in the value of pt. As shown in the diagram, the path at each instant seems to be at an angle pt relative to the north (i.e., the x axis), measured clockwise when viewed downward. To an observer on the earth, the vertical plane in which the cable lies therefore seems to rotate about the vertical axis at $\omega_z=-p=-\omega_e\sin\lambda$. This is exactly opposite the angular velocity component of the earth in the vertical direction –



Motion of the spherical pendulum.

an observer viewing the pendulum from a fixed reference frame in outer space sees the plane of the pendulum as being fixed.

The movement of the plane of a spherical pendulum relative to the earth was used in 1851 by the French physicist Jean Louis Foucault (1819–1869) to demonstrate the earth's rotation. What is perhaps the most famous Foucault pendulum in current use may be found in the General Assembly building at United Nations headquarters in New York City.

In closing, we should note that the spherical pendulum for arbitrary, small initial values would seem to follow an elliptical path. The major and minor axes of the ellipse would rotate relative to the earth at angular speed $\omega_z = -\omega_e \sin \lambda$.

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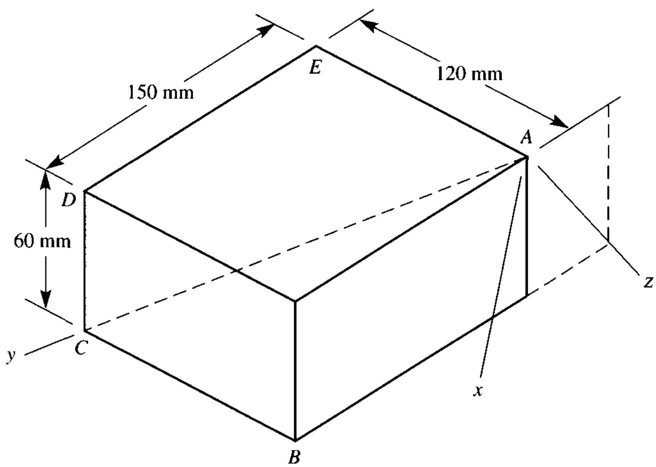
Problems

3.1 (See figure, next page.) The y axis coincides with the main diagonal for the box, and the z axis coincides with the plane of the right face. Determine the coordinates relative to xyz of corners D and E.

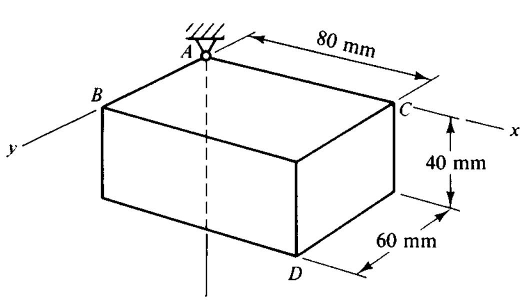


Problem 3.1

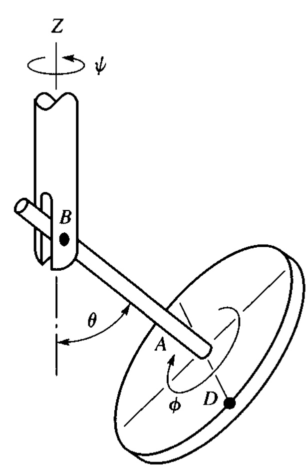
3.2 The rectangular box is supported from the ceiling by a ball-and-socket joint at corner A. Commencing from the position shown, the box is rotated about edge AB by 40° clockwise as viewed from A to B. Then the box is rotated about edge AC by 80° counterclockwise as viewed from A to C. Determine the displacement of corner D due to these rotations.

3.3 The rectangular box is supported from the ceiling by a ball-and-socket joint. Commencing from the position shown, the box is given a rotation about the fixed X axis of 60°, followed by a rotation about the fixed Z axis of −120°. Determine the displacement of corner D due to these rotations.

3.4 It is desired to impart to the box in Problem 3.3 a rotation about a single axis that is equivalent to the rotations specified there. Determine the orientation of that axis and the angle of rotation about that axis.



Problems 3.2 to 3.4

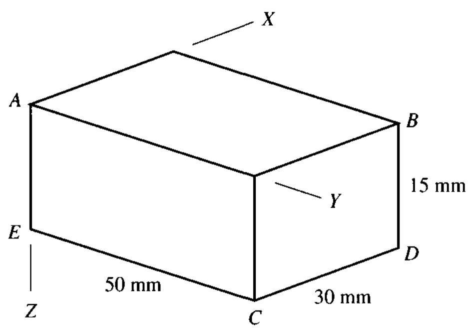


Problem 3.5

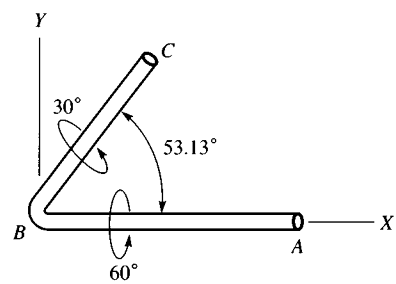
3.5 The rotation of the disk is specified by the rotation angles ψ about the fixed vertical axis, θ about the horizontal axis perpendicular to the plane of shaft AB and the vertical, and ϕ by which the disk rotates about shaft AB. (These are, respectively, the Eulerian angles of precession, nutation, and spin, which will be discussed in Chapter 4.) There are six possible sequences in which the rotations may take place. Let xyz be a coordinate system that is fixed to the disk with an orientation that coincides with the XYZ system when $\psi = \theta = \phi = 0$. Prove that the transformation from XYZ to xyz components is the same, regardless of the sequence in which the rotations occur.

3.6 Starting from the position shown, the box is rotated by 40° about face diagonal AB, clockwise as viewed from corner B toward corner A. Determine the coordinates of corner C relative to the fixed reference frame XYZ after this rotation.

3.7 Starting from the position shown, the box is rotated by angle θ about main diagonal AD, counterclockwise as viewed from corner D toward corner A. The angle between the fixed Y axis and the unit vector $\bar{e}_{E/A}$ after the rotation is 110°. Determine θ.



Problems 3.6 and 3.7



Problems 3.8 and 3.9

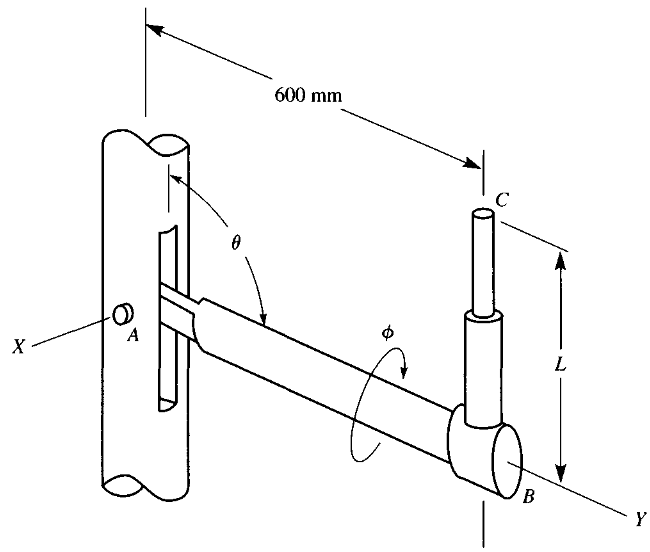
3.8 The bent rod is given a pair of rotations, first by 60° about line AB, and then 30° about line BC, with the sense of each rotation as shown in the sketch. Let xyz be a coordinate system fixed to the rod that initially aligned with the fixed XYZ system shown. Determine the transformation by which vector components with respect to XYZ may be converted to components with respect to xyz.

3.9 Consider the rotation of the bent rod in Problem 3.8. Determine the orientation of the axis and the angle of rotation of the single rotation that would be equivalent to the pair of rotations specified there.

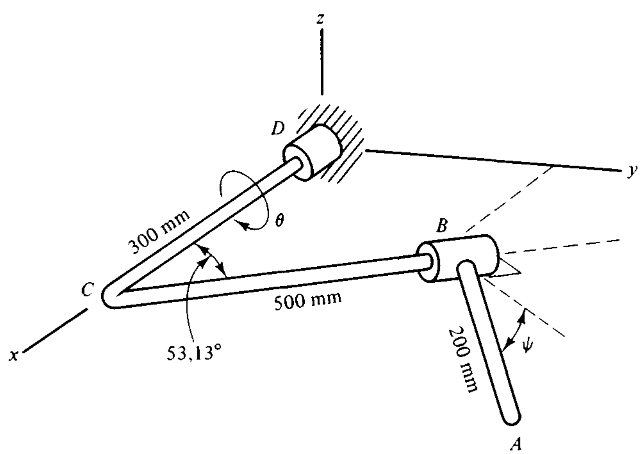
3.10 (See figure, next page.) In the initial position shown, shaft AB is aligned with the fixed Y axis, and hydraulic cylinder BC (which may pivot about shaft AB) is vertical. In this position the length of BC is $L = 400$ mm. From the initial position, the system rotates by 36.87° about the vertical axis, the angle θ for shaft AB remains 90°, and the hydraulic cylinder rotates about shaft AB by $\phi = 120°$. In the final position, $L = 800$ mm. Determine the corresponding displacement of end C.

3.11 (See figure, next page.) Solve Problem 3.10 for the case where the angle $\theta = 50°$ in the final position and all other motions are as specified there.

3.12 (See figure, next page.) Bar AB is welded to collar B, which pivots about segment BC of the triangular arm BCD. The rotation of arm BCD about the horizontal x axis is the angle θ, and the rotation of bar AB relative to the triangular bar is the angle ψ.



Problems 3.10 and 3.11

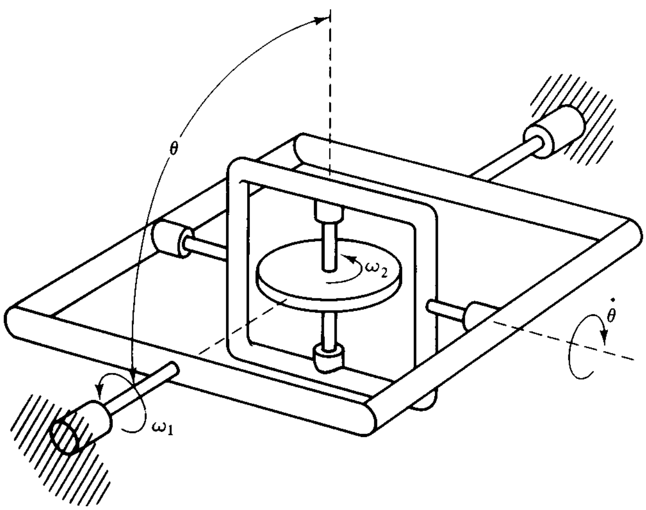


Problem 3.12

Consider the case where $\theta = 110°$ and $\psi = 20°$. Determine the displacement of end A from its position when $\theta = \psi = 0°$.

3.13 The radiator fan of an automobile engine whose crankshaft is aligned with the longitudinal axis is rotating at 1,000 rev/min, clockwise as viewed from the front of the vehicle. The automobile is following a 60-m radius left turn at a constant speed of 75 km/hr. Determine the angular velocity and angular acceleration of the fan.

3.14 The flywheel of the gyroscope rotates about its own axis at $\omega_2 = 6{,}000$ rev/min, and the outer gimbal support is rotating about the horizontal axis at the rate $\omega_1 = 10$ rad/s, $\dot{\omega}_1 = 100$ rad/s^2. Determine the angular velocity and angular acceleration of the flywheel if θ is held constant at 75°.



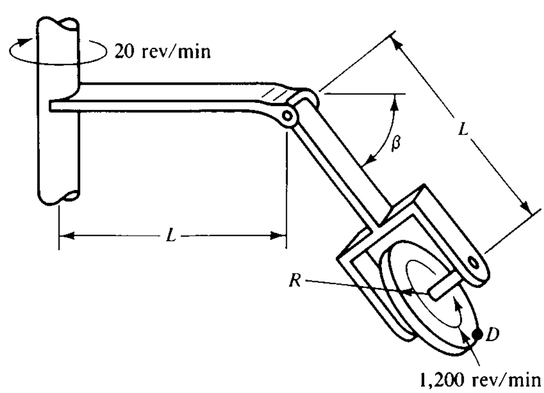
Problems 3.14 and 3.15

3.15 The flywheel of the gyroscope rotates about its own axis at $\omega_2 = 6{,}000$ rev/min. At the instant when $\theta = 120°$, the inner gimbal support is rotating relative to the outer gimbal at $\dot{\theta} = 6$ rad/s and $\ddot{\theta} = -90$ rad/s^2. The corresponding rotation of the outer gimbal about the horizontal axis is $\omega_1 = 10$ rad/s, $\dot{\omega}_1 = 100$ rad/s^2. Determine the angular velocity and angular acceleration of the flywheel at this instant.

3.16 The disk spins about its own axis at 1,200 rev/min as the system rotates about the vertical axis at 20 rev/min. Determine the angular velocity and angular acceleration of the disk if β is constant at 30°.

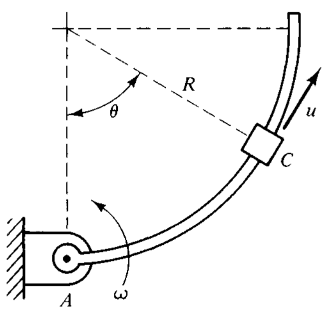
3.17 Solve Problem 3.16 for the case where $\dot{\beta} = 10$ rad/s and $\ddot{\beta} = -50$ rad/s^2 when $\beta = 36.87°$.

3.18 The disk spins about its own axis at 1,200 rev/min as the system rotates about the vertical axis at 20 rev/min. The angle of inclination is constant at $\beta = 30°$. Determine the velocity and acceleration of point D on the perimeter of the disk when it is at its lowest position, as shown.

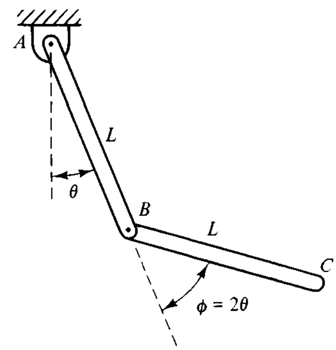


Problems 3.16 to 3.18

3.19 Collar C moves at the constant speed u relative to the curved bar, which rotates in the horizontal plane at the constant rate ω. Derive expressions for the velocity and acceleration of the collar as a function of the angle θ.



Problem 3.19



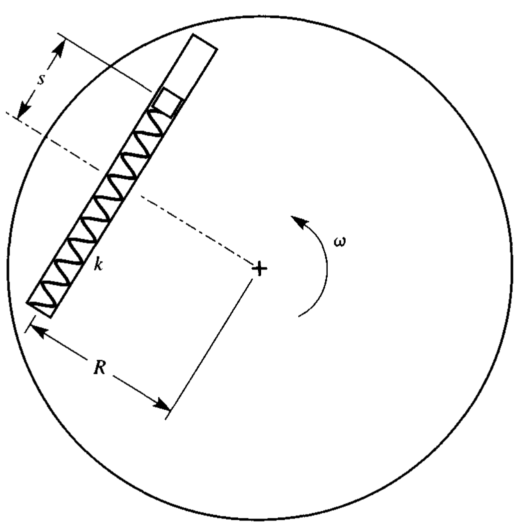
Problem 3.20

3.20 A servomotor maintains the angle ϕ of bar BC relative to bar AB at $\phi = 2\theta$, where θ is the angle of inclination of bar AB. Determine the acceleration of end C corresponding to $\theta = 15°$, $\dot{\theta} = 50$ rad/s, and $\ddot{\theta} = 0$.

3.21 A speed governor consists of a block of mass m that slides within a smooth groove in a housing. The unstretched length of the spring, whose stiffness is k, is selected such that the block is situated at $s = 0$ when there is no rotation. The system rotates about the vertical axis at angular speed ω.

(a) Derive a differential equation governing s as a function of time in the case where ω is an arbitrary function of time.

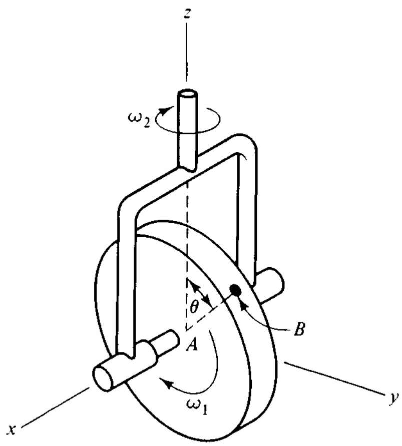
(b) Derive an expression for the normal force exerted by the groove wall on the block in terms of ω and s.



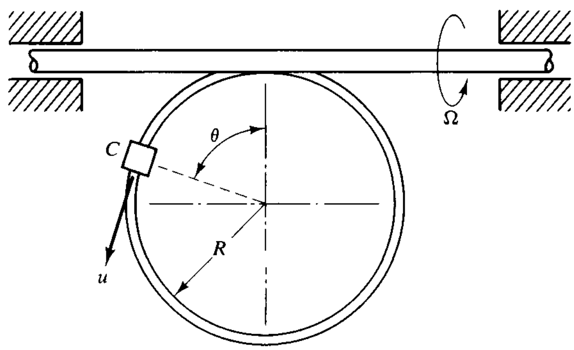
Problem 3.21

(c) Determine the natural frequency of oscillation when ω is constant, and explain how that result can be used to monitor when ω exceeds a critical value.

3.22 The disk rotates at ω_1 about its axis, and the rotation rate of the forked shaft is ω_2. Both rates are constant. Use two different approaches to determine the velocity and acceleration of an arbitrarily selected point B on the perimeter. Describe the results in terms of components relative to the xyz axes in the sketch.



Problem 3.22



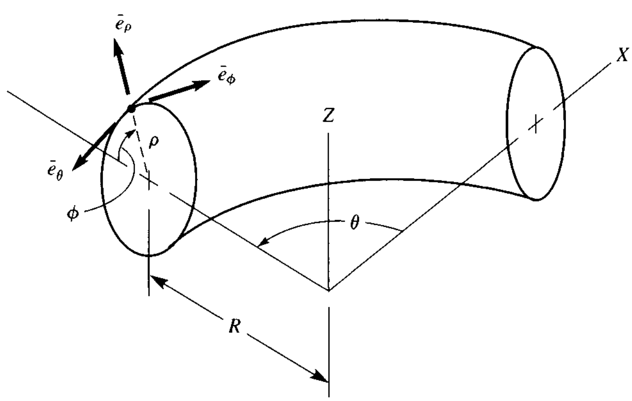
Problem 3.23

3.23 Collar C slides relative to the curved rod at a constant speed u, while the rod rotates about the horizontal axis at the constant rate Ω. Determine the acceleration of the collar in terms of θ.

3.24 The rotation rates of the bars in Problem 3.12 are constant at $\dot{\theta} = 20$ rad/s, $\dot{\psi} = 10$ rad/s. For the instant when $\theta = 120°$ and $\psi = 50°$, determine the velocity and acceleration of end A of the bar.

3.25 Use the concepts of relative motion to derive the formulas for velocity and acceleration of a point in terms of a set of spherical coordinates.

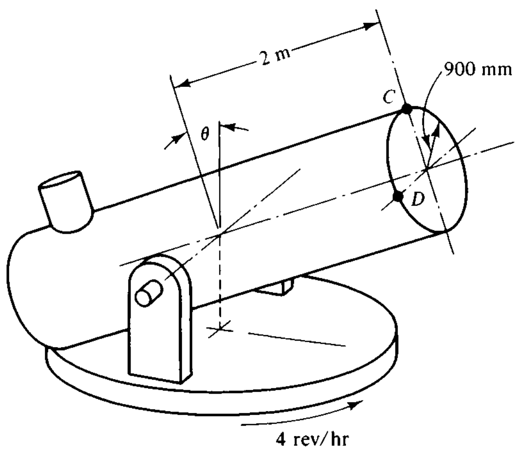
3.26 The sketch defines an orthogonal curvilinear coordinate system (ρ, θ, ϕ), known as *toroidal* coordinates. The radius R is constant. Use the concepts of relative motion



Problem 3.26

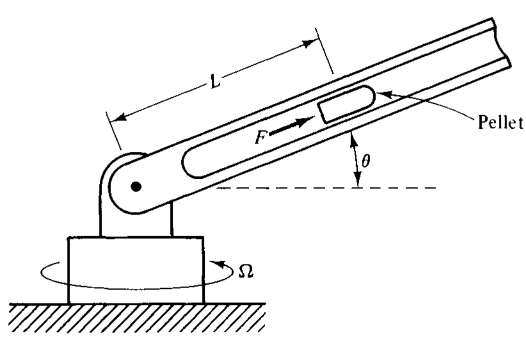
to derive the corresponding formulas for velocity and acceleration of a point in terms of the unit vectors of this coordinate system.

3.27 The telescope rotates about the fixed vertical axis at 4 rev/hr as the angle θ oscillates at $\theta = (\pi/3)\sin(\pi t/7{,}200)$ radians, where t has units of seconds. Determine the velocity and acceleration of points C and D as functions of time.



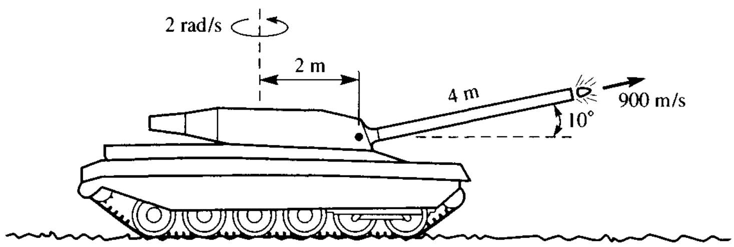
Problem 3.27

3.28 A pellet of mass m moves through the smooth barrel, which rotates about the vertical axis at angular speed Ω as the angle of elevation of the barrel is increased at the rate $\dot{\theta}$. Both rates are constant. At the instant before the pellet emerges, its speed relative to the barrel is u. At that instant, the magnitude of the propulsive force $\bar{F}$, which acts parallel to the barrel, is a factor of 50 times greater than the weight of the pellet. Derive expressions for the acceleration term $\dot{u}$ and for the force the pellet exerts on the walls of the barrel at this instant.



Problem 3.28

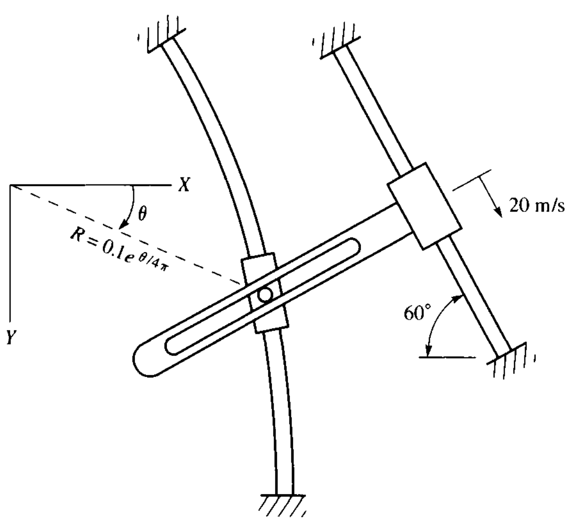
3.29 In the position shown, the turret is rotating about the vertical axis at the constant rate of 2 rad/s. At this instant the barrel is being raised at the rate of 0.8 rad/s, which is decreasing at 200 rad/s^2. The tank is at rest. Immediately preceding its emergence, a 20-kg cannon shell is traveling at a speed of 900 m/s relative to the barrel, and the



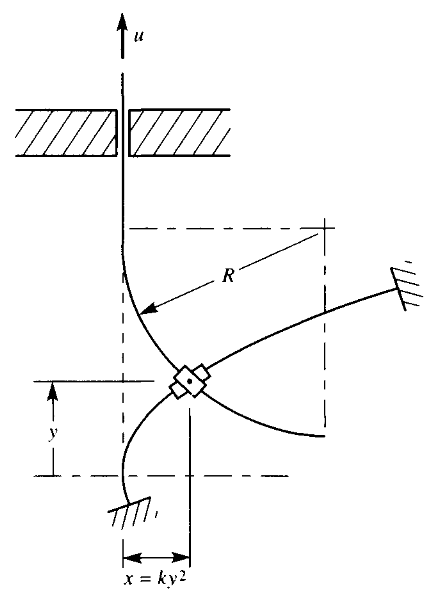
Problem 3.29

internal propulsive pressure within the barrel has been dissipated. Determine the acceleration of the cannon shell at this instant, and the corresponding forces exerted by the cannon shell on the barrel.

3.30 A collar slides in the horizontal plane over a spirally curved rod defined in polar coordinates by $R = 0.1\exp[\theta/(4\pi)]$ m. The motion is actuated by the translating arm, which contains a groove that pushes a pin in the collar. The speed of the arm is constant at 20 m/s. Determine the velocity and acceleration of the collar in the position where $\theta = 0.8$ rad.



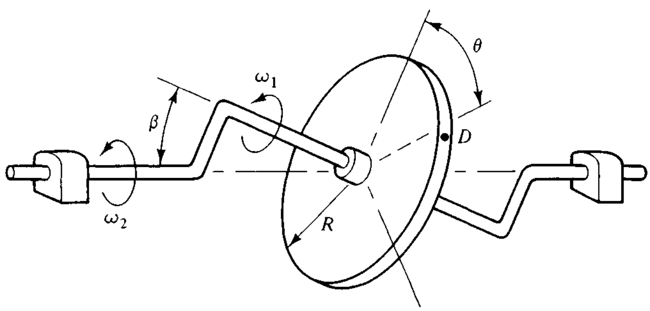
Problem 3.30



Problem 3.31

3.31 Two collars are pinned together, such that they simultaneously slide over a fixed rod and a rod that translates upward at a constant speed u. Derive expressions for the velocity and acceleration of the collar as a function of the vertical distance y from the axis of the fixed rod to the collars.

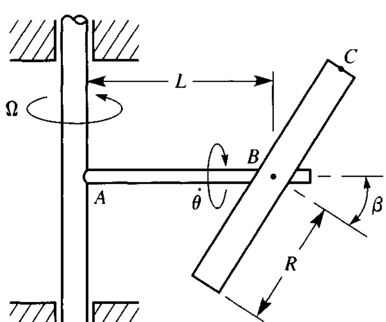
3.32 The disk is mounted on a bent shaft, about which it rotates at variable rate ω_1 while the shaft rotates about the horizontal axis at constant rate ω_2. In the position $\theta = 0$,



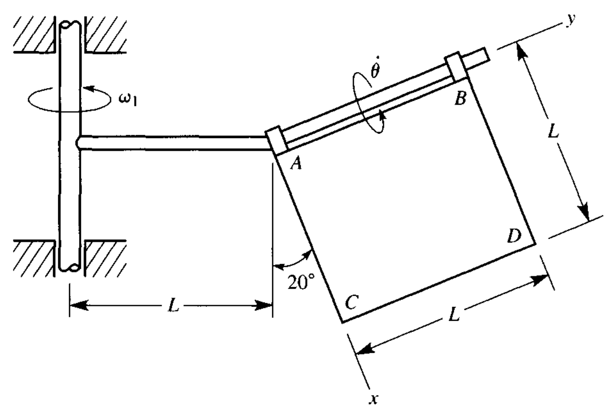
Problem 3.32

the diametral line through point D is coplanar with the bent shaft. Determine the velocity and acceleration of point D as a function of θ.

3.33 The disk has been misaligned in its mounting onto shaft AB of the T-bar assembly. As a result, the axis of the disk forms a constant angle β relative to AB. The angle of rotation θ of the disk about the shaft is defined such that point C is at its highest elevation when $\theta = 0$, which is the position depicted in the sketch. The rotation rates Ω about the vertical axis and $\dot{\theta}$ about shaft AB are constant. Derive expressions for the velocity and acceleration of point C at arbitrary θ. Describe the results in terms of components relative to a body-fixed xyz system whose x axis always coincides with the diametral line from B to C.



Problem 3.33



Problem 3.34

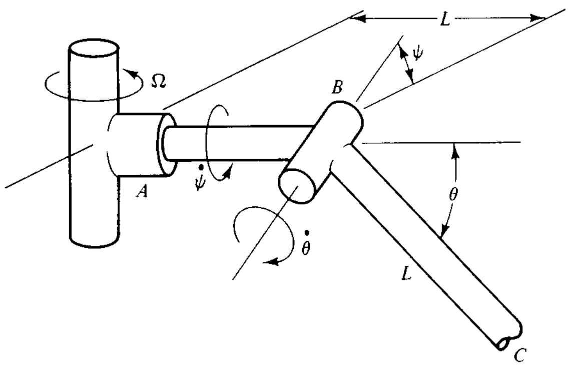
3.34 The square plate is welded to a bent shaft that rotates about the fixed vertical axis at constant angular speed ω_1. The angle of rotation of the plate about its shaft is θ, which is an arbitrary function of time. The position depicted in the sketch corresponds to $\theta = 0$, where the plate is situated in the vertical plane.

(a) Derive expressions for the angular velocity and angular acceleration of the plate, valid for arbitrary θ, in terms of components with respect to the body-fixed xyz system.

(b) Derive corresponding expressions for the velocity and acceleration of corner D.

3.35 A servomotor at joint A rotates arm AB about its axis through an angle ψ, and another servomotor at joint B controls the angle θ of arm BC relative to arm AB.

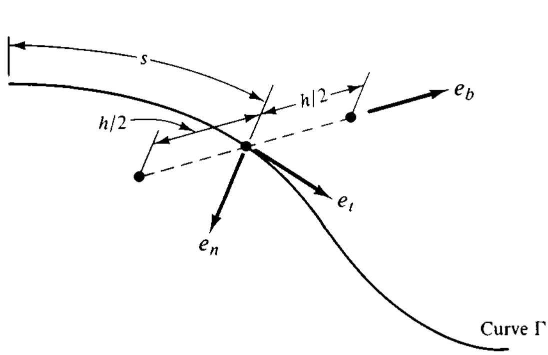
When $\psi = 0°$, arm BC lies in the vertical plane. Consider a situation where $\dot{\theta}$ is constant at 2 rad/s, while $\dot{\psi} = 4$ rad/s and $\ddot{\psi} = 12$ rad/s^2. Concurrently with these rotations, the entire assembly is rotating about the vertical axis at the constant rate $\Omega = 8$ rad/s. Determine the angular velocity and angular acceleration of link BC if $\psi = 0°$ and $\theta = 30°$ at this instant.



Problems 3.35 and 3.36

3.36 Consider the system in Problem 3.35 in a situation where $\dot{\theta}$ is constant at 2 rad/s, while $\dot{\psi}$ is constant at 4 rad/s. The entire assembly is rotating about the vertical axis at the constant rate $\Omega = 8$ rad/s. Determine the velocity and acceleration of end C at the instant when $\psi = 53.13°$ and $\theta = 20°$.

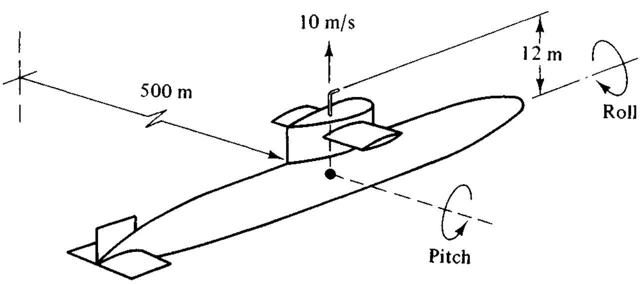
3.37 Consider a roller coaster that is constructed such that a car follows a specified curve Γ while the axles of the car are always parallel to the binormal to Γ. In order to achieve this, each point on the tracks is located by measuring equal distances $h/2$ (where h is the distance between wheels) in the binormal direction relative to the point on Γ. As a result, the longitudinal axis of a car is always parallel to the tangential direction. Derive an expression for the acceleration of an arbitrary point P in the car in terms of the speed v of a car, the rate of increase $\dot{v}$, and the properties of curve Γ. The coordinates of point P are (x, y, z) relative to a body-fixed coordinate system whose origin follows curve Γ and whose orientation is selected such that $\bar{i} = \bar{e}_t$ and $\bar{j} = \bar{e}_n$.



Problem 3.37

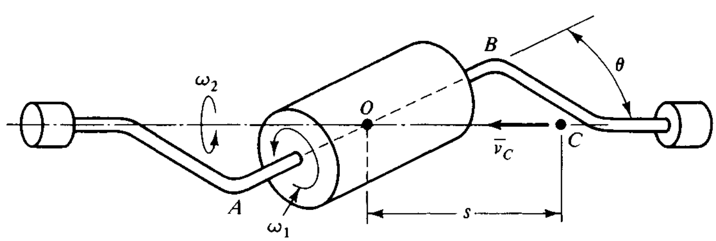
3.38 An airplane is executing a manuever in which its roll, pitch, and yaw rates are 2 rad/s, 0.5 rad/s, and −0.2 rad/s, respectively. The yaw rate is decreasing at 10 rad/s^2, and the other rates are constant. Determine the velocity and acceleration of the nose relative to the center of mass G, as seen by an observer on the ground. The point of interest at the nose is 5 m forward and 0.2 m below point G.

3.39 A submarine is following a horizontal 500-m circle at 20 knots (1 knot = 1.852 km/hr). At the instant when it is aligned with the horizontal and vertical directions, it is rolling at 0.5 rad/s and pitching at 0.1 rad/s, as shown. Both rates are maxima at this instant. Determine the velocity and acceleration of the tip of the periscope, which is being extended at the constant rate of 10 m/s.



Problem 3.39

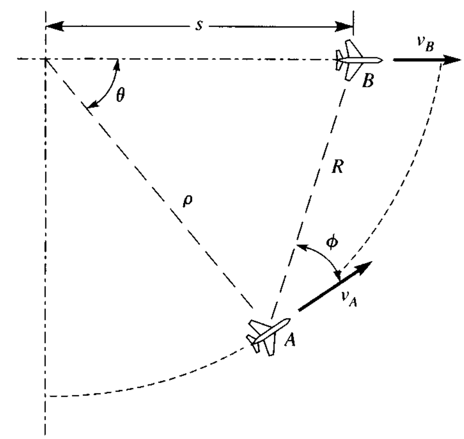
3.40 A test chamber for astronauts rotates about axis AB at a constant angular speed ω_1 as the entire assembly rotates about the horizontal axis at angular speed ω_2, which also is constant. An astronaut is seated securely in the chamber at center point O, which is collinear with both axes of rotation. Object C has a constant absolute velocity $\bar{v}_C$ parallel to the horizontal axis. Determine the acceleration of this object as seen by the astronaut.



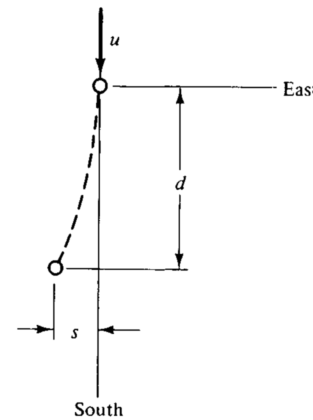
Problem 3.40

3.41 Airplane B travels eastward at constant speed v_B, while airplane A executes a constant radius turn at constant speed v_A. At an arbitrary instant, the angle θ and distance s locating the airplanes are known. Radar equipment on aircraft A can measure the separation distance R and the angle ϕ relative to the longitudinal axis, as well as the rates of change of these parameters. Derive expressions for $\dot{R}$, $\ddot{R}$, $\dot{\phi}$, and $\ddot{\phi}$.

3.42 A small disk slides with negligible friction on a horizontal sheet of ice. The initial velocity of the disk was u in the southerly direction. Determine the distance and sense of the shift s in the position after the disk has traveled distance d southward. How would this result have changed if the initial velocity were northward or eastward?



Problem 3.41

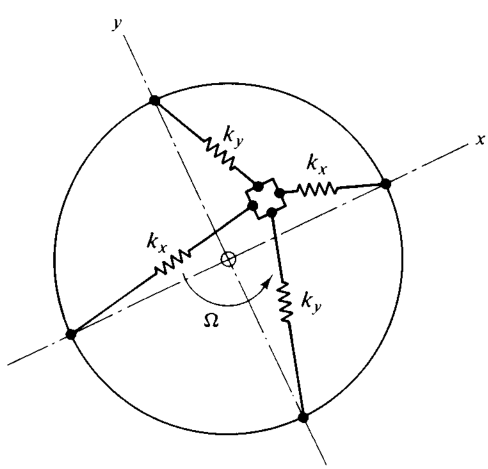


Problem 3.42

3.43 A ball is thrown vertically from the ground at speed v. Assuming that air resistance is negligible, derive an expression for the shift due to the Coriolis effect in the position where it returns to the ground. Evaluate the result for $v = 40$ m/s at a latitude of 45°.

3.44 An object falls in a vacuum after being released at a distance H above the surface of the earth. The line extending from the center of the earth to this object is at latitude λ, and point O' on the earth's surface is concurrent with this line. Determine the location (east-west and north-south relative to point O') at which the object strikes the ground in each of the following cases:
(a) the block is initially at rest relative to the earth; and
(b) the block was initially at rest relative to a reference frame that translates with the center of the earth but does not execute the earth's spinning rotation.
For the sake of simplicity, the gravitational attraction may be considered to be constant at mg. Explain the difference between the results in cases (a) and (b).

3.45 A small block of mass m is attached to a horizontal turntable by two pairs of opposing springs having stiffness k_x and k_y. The springs are unstretched when the block



Problem 3.45

coincides with the axis of the turntable, and the (x, y) coordinates of the block relative to the turntable are much less than the radius of the turntable. Derive linearized differential equations for x and y for the case where the turntable rotates at the constant rate Ω. Then solve those equations for initial conditions in which the block is released from rest relative to the turntable at $x = b$, $y = 0$. Discuss how this system may be used as an analog for the Foucault pendulum.

CHAPTER 4

Kinematics of Rigid Bodies

The concept of a rigid body is an artificial one, in that all materials deform when forces are applied to them. Nevertheless, this artifice is very useful when we are concerned with an object whose movement due to deformation is only a minor part of its motion. In addition, formulations of the motion of deformable bodies often find it convenient to decompose the overall motion into rigid body and deformational contributions.

4.1 General Equations

A rigid body is defined to be a collection of particles whose distance of separation is invariant. In this circumstance, any set of coordinate axes xyz that is scribed in the body will maintain its orientation relative to the body. Such a coordinate system forms a *body-fixed reference frame.* The orientation of xyz relative to the body and the location of its origin are arbitrary. A typical situation is depicted in Figure 4.1.

Because all points in the rigid body maintain their relative position, their velocity and acceleration relative to xyz is zero. Thus the velocity and acceleration of point A in Figure 4.1 are given by

$$\begin{aligned} \bar{v}_A &= \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{A/O'}, \\ \bar{a}_A &= \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{A/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{A/O'}). \end{aligned} \tag{4.1}$$

Similar relations apply for another point B:

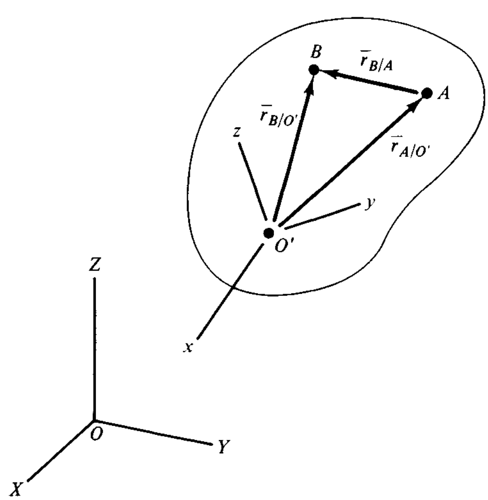


Figure 4.1 Position of points in a rigid body.

$$\bar{v}_B = \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{B/O'},$$
$$\bar{a}_B = \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{B/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{B/O'}). \tag{4.2}$$

The foregoing yield relations between the motion of A and B that do not require that either point be the origin. Subtracting each of Eqs. (4.1) from the corresponding Eqs. (4.2) yields

$$\bar{v}_B - \bar{v}_A = \omega \times (\bar{r}_{B/O'} - \bar{r}_{A/O'}),$$
$$\bar{a}_B - \bar{a}_A = \bar{\alpha} \times (\bar{r}_{B/O'} - \bar{r}_{A/O'}) + \bar{\omega} \times [\bar{\omega} \times (\bar{r}_{B/O'} - \bar{r}_{A/O'})]. \tag{4.3}$$

Because $\bar{r}_{B/A} = \bar{r}_{B/O'} - \bar{r}_{A/O'}$, we find that

♦ $$\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A},$$
♦ $$\bar{a}_B = \bar{a}_A + \bar{\alpha} \times \bar{r}_{B/A} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{B/A}). \tag{4.4}$$

The arbitrariness of the origin O' could have been used as an argument for deriving Eqs. (4.4) directly. However, the approach used here leads to an important observation.

♦ *Given a set of n points in a rigid body, there are n − 1 independent equations in the form of Eqs. (4.4) between their velocities or accelerations. These equations may be obtained by relating one point to each of the other n − 1.*

As demonstrated in Eqs. (4.1), (4.2), and (4.4), other relations we might write are linear combinations of the independent ones. This interdependence between the motion of points in a body is a consequence of the rigidity, which maintains points at fixed relative positions.

Equations (4.4) describe the velocity and acceleration of point in a rigid body as the combination of the movement of point A and a rotational effect about point A. A comparable description of position may be obtained by combining the translational transformation of coordinates, Eq. (3.2), with Euler's theorem, Eqs. (3.28) and (3.29), which represent an arbitrary rotation as a single rotation about an axis. These observations are manifestations of *Chasle's theorem,* which states:

♦ *The* general motion *of a rigid body is a superposition of a* translation *and a* pure rotation. *In the translation, all points follow the movement of an arbitrary point A in the body and the orientation remains constant. The rotational portion of the motion is such that the arbitrary point A remains at rest.*

Note the arbitrariness of the point selected for the translation. This means that the only unique property of the kinematics of a body is the rotation – as described by its current orientation, its angular velocity, and its angular acceleration. Various methods for locating a point via intrinsic and extrinsic coordinates were discussed in Chapter 2. The next section will present a standardized way for describing orientation.

A basic tool in the analysis of velocity for a body in planar motion is the "instant center" method. In essence, this technique is based on considering a body in general motion (translation plus rotation) to be rotating about a rest point, which is called the *instantaneous center of zero velocity* or, more briefly, the *instant center.* In general, the instant center has an acceleration, so the point on the body that has no velocity changes as the motion evolves. For this reason the method is not suitable for the evaluation of acceleration. The instant center method also is not useful for evaluating the velocity of bodies in arbitrary spatial motion, for reasons that will become apparent.

In order to explore the instant center method, consider the velocities of two points A and B in a rigid body. If point A is at rest then $\bar{v}_B = \bar{\omega} \times \bar{r}_{B/A}$, where $\bar{\omega}$ is the angular velocity of the body. According to this relationship, the speed of point B is proportional to the distance from that point to the axis parallel to $\bar{\omega}$ that intersects point A. Also, the direction of $\bar{v}_B$ will be perpendicular to the radial vector $\bar{r}_{B/A}$ in the sense of the rotation according to the right-hand rule. The instant center may be located by using these properties for two points whose velocities are in known directions. The velocities of other points may then be computed by using the relation for circular motion, $v = \omega r$, where r is the distance from the instant center to the point of interest.

The difficulty is that, in general, there is no point for which $\bar{v}_A = \bar{0}$. This is readily proven by taking a dot product of $\bar{v}_B$ in Eq. (4.4) with $\bar{\omega}$, which yields

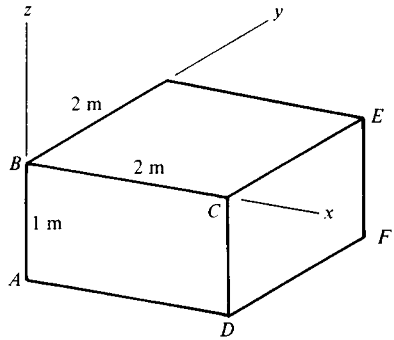
$$\bar{\omega} \cdot \bar{v}_B = \bar{\omega} \cdot \bar{v}_A.$$

This relation states that all points in a body have the same velocity component parallel to $\bar{\omega}$. Therefore, if there is a situation where $\bar{v}_A = \bar{0}$, all points in the body must have velocities that are perpendicular to $\bar{\omega}$. The conditions imposed on many systems do not satisfy such a restriction, although it is identically satisfied for planar motion.

Pure spatial rotation is another exception, because it is, by definition, the case where some point in the body actually is fixed. General motion occurs when a rotating body has no fixed point. Chasle's theorem could be used to represent a general motion as the superposition of a translation in the direction of $\bar{\omega}$ that follows a selected point A, and a rotation about an axis parallel to $\bar{\omega}$ through point A. This is a *screw motion*. The terminology stems from an analogy with the movement of a screw with a right-handed thread, which is to advance in the direction of the outstretched thumb of the right hand when the fingers of that hand are curled in the direction that the screw turns. We shall not pursue such a representation because it does little to improve our ability to formulate problems. However, some people do find it to be a useful way to visualize spatial motion.

Example 4.1 Observation of the motion of the block reveals that at a certain instant the velocity of corner A is parallel to the diagonal AE. At this instant components relative to the body-fixed xyz coordinate system of the velocities of the other corners are known to be $(v_B)_x = 10$, $(v_C)_z = 20$, $(v_D)_x = 10$, and $(v_E)_y = 5$, where all values are in units of meters/second. Determine whether these values are possible, and if so, evaluate the velocity of corner F.

Solution We know that $\bar{v}_A = v_A \bar{e}_{E/A}$, with the sign of v_A unspecified, and also that $\bar{v}_B \cdot \bar{i} = 10$, $\bar{v}_C \cdot \bar{k} = 20$, $\bar{v}_D \cdot \bar{i} = 10$, and $\bar{v}_E \cdot \bar{j} = 5$ (units are m/s). If these values are possible, there will be values of $\bar{\omega}$ and the velocity of any point that satisfy the relative velocity equations. We select point A for this representation, because the only unknown aspect of its velocity is the speed, that is,



Example 4.1

$$\bar{v}_A = \bar{v}_A \bar{e}_{E/A} = v_A \frac{\bar{r}_{E/A}}{|\bar{r}_{E/A}|} = v_A \frac{2\bar{i}+2\bar{j}+\bar{k}}{[2^2+2^2+1]^{1/2}} = \frac{1}{3} v_A (2\bar{i}+2\bar{j}+\bar{k}).$$

The angular velocity is unknown, so

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}.$$

The velocity equations relating point A to the other points are

$$\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A}, \qquad \bar{r}_{B/A} = \bar{k};$$

$$\bar{v}_C = \bar{v}_A + \bar{\omega} \times \bar{r}_{C/A}, \qquad \bar{r}_{C/A} = 2\bar{i} + \bar{k};$$

$$\bar{v}_D = \bar{v}_A + \bar{\omega} \times \bar{r}_{D/A}, \qquad \bar{r}_{D/A} = 2\bar{i};$$

$$\bar{v}_E = \bar{v}_A + \bar{\omega} \times \bar{r}_{E/A}, \qquad \bar{r}_{E/A} = 2\bar{i} + 2\bar{j} + \bar{k}.$$

We substitute the expressions for $\bar{v}_A$ and $\bar{\omega}$ into these equations, and then evaluate the given velocity components by taking the appropriate dot products. This leads to

$$\bar{v}_B \cdot \bar{i} = \tfrac{2}{3} v_A + \omega_y = 10,$$

$$\bar{v}_C \cdot \bar{k} = \tfrac{1}{3} v_A - 2\omega_y = 20,$$

$$\bar{v}_D \cdot \bar{i} = \tfrac{2}{3} v_A = 10,$$

$$\bar{v}_E \cdot \bar{j} = \tfrac{2}{3} v_A - \omega_x + 2\omega_z = 5.$$

Although we have four equations for the four unknown parameters, the equations are not solvable. The third equation yields $v_A = 15$ m/s, from which the first equation gives $\omega_y = 0$, while the second gives $\omega_y = -7.5$ rad/s. Therefore, the motion is not possible. Note that if the first two equations were compatible, we still would not be able to determine ω_z because the given velocity conditions do not properly constrain the motion.

4.2 Eulerian Angles

Three independent direction angles define the orientation of a set of xyz axes. Because there are a total of nine direction angles locating xyz with respect to a

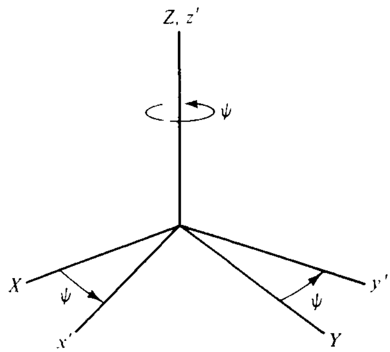


Figure 4.2 Precession.

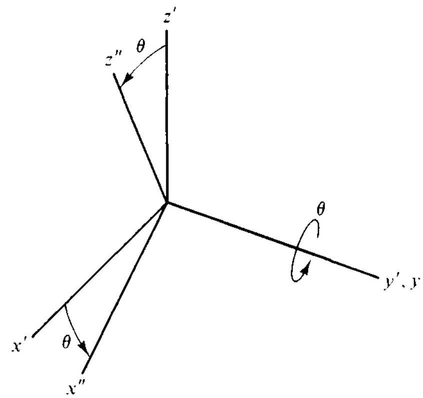


Figure 4.3 Nutation.

fixed reference frame XYZ, an independent set of angles may be selected in a variety of ways. Eulerian angles treat this matter as a specific sequence of rotations.

Let us follow the intermediate orientations of a moving reference frame as it is rotated away from its initial alignment with XYZ. The first rotation, called the *precession,* is about the fixed Z axis. The angle of rotation in the precession is denoted ψ, as depicted in Figure 4.2. The orientation of the moving reference frame after it has undergone only the precession is denoted as $x'y'z'$. The transformation from XYZ to $x'y'z'$ may be found from Figure 4.2 to be

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [R_\psi] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \tag{4.5}$$

where

$$[R_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.6}$$

The second rotation is about the y' axis. The orientation of the moving reference frame after this rotation is denoted $x''y''z''$ in Figure 4.3. This is the *nutation,* and θ is the angle of nutation. The second transformation is given by

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = [R_\theta] \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}, \tag{4.7}$$

where

$$[R_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}. \tag{4.8}$$

The last rotation is the *spin,* in which the reference frame moves from $x''y''z''$ to its final orientation. The z'' axis is the axis for the spin, and the angle of spin is denoted as ϕ. The transformation from $x''y''z''$ to the final xyz system is found from Figure 4.4 to be

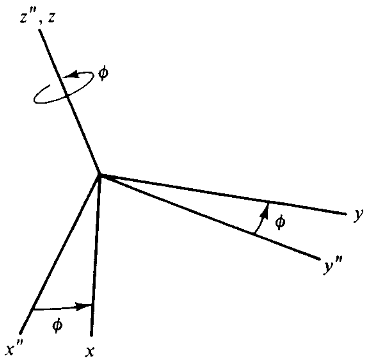


Figure 4.4 Spin.

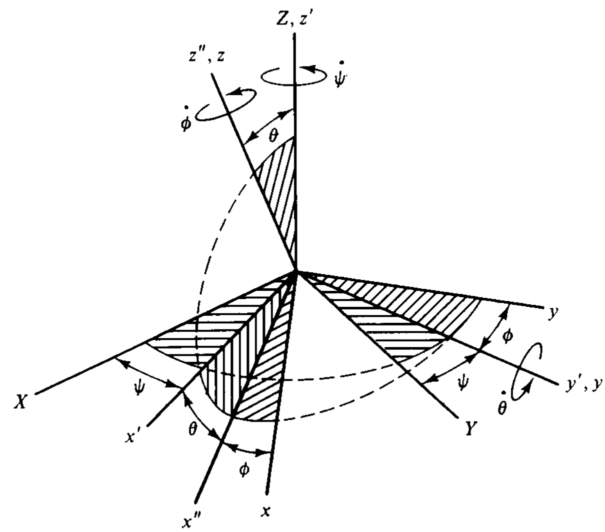


Figure 4.5 Eulerian angles and reference frames.

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_\phi] \begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix}, \tag{4.9}$$

where

$$[R_\phi] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.10}$$

We obtain the overall transformation by combining Eqs. (4.5), (4.7), and (4.9), with the result that

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R] = [R_\phi][R_\theta][R_\psi]. \tag{4.11}$$

The full sequence of rotations is depicted in Figure 4.5 by following the tips of the axes.

The angular velocity and angular acceleration are readily expressed in terms of the angles of precession, nutation, and spin by adding the rotation rates about the respective axes. For this task, we note that the precession axis is defined to be the (fixed) Z axis, so the precessional portion of the angular velocity is always $\dot{\psi}\bar{K}$. The nutation occurs about the y' axis. (We sometimes shall use the term *line of nodes* to refer to the y' axis, because points on this axis do not move in the nutation.) A general description of the nutational angular velocity is therefore $\dot{\theta}\bar{j}'$. Finally, the spin is about the z'' axis. Because the z'' and z axes remain coincident, the spin angular velocity may be written as $\dot{\phi}\bar{k}$. The angular velocity is the (vector) sum of the individual rotation rates, so

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k}. \tag{4.12}$$

We obtain a general expression for angular acceleration from the foregoing by noting that $\bar{K}$ is fixed in space, whereas $\bar{j}'$ and $\bar{k}$ are unit vectors for $x'y'z'$ and xyz, respectively. Thus, $\dot{\bar{k}} = \bar{\omega} \times \bar{k}$ and $\dot{\bar{j}}' = \bar{\omega}' \times \bar{j}'$, where $\bar{\omega}'$ is the angular velocity of $x'y'z'$. Because $x'y'z'$ undergoes only precession, this term is

$$\bar{\omega}' = \dot{\psi}\bar{K}. \tag{4.13}$$

Using $\bar{\omega}$ and $\bar{\omega}'$ to differentiate the unit vectors then leads to

$$\begin{aligned}\bar{\alpha} &= \ddot{\psi}\bar{K} + \ddot{\theta}\bar{j}' + \dot{\theta}\dot{\bar{j}}' + \ddot{\phi}\bar{k} + \dot{\phi}\dot{\bar{k}} \\ &= \ddot{\psi}\bar{K} + \ddot{\theta}\bar{j}' + \dot{\theta}(\bar{\omega}' \times \bar{j}') + \ddot{\phi}\bar{k} + \dot{\phi}(\bar{\omega} \times \bar{k}).\end{aligned} \tag{4.14}$$

In order to use these expressions in computations, they must be transformed to a common set of components. Many situations require xyz components. From Figure 4.5, we find that the unit vectors $\bar{K}$ and $\bar{j}'$ are

$$\begin{aligned}\bar{K} &= \sin\theta[-(\cos\phi)\bar{i} + (\sin\phi)\bar{j}] + (\cos\theta)\bar{k}, \\ \bar{j}' &= (\sin\phi)\bar{i} + (\cos\phi)\bar{j}.\end{aligned} \tag{4.15}$$

Thus, the angular velocity and angular acceleration are

$$\begin{aligned}\bar{\omega} &= (-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi)\bar{i} \\ &\quad + (\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)\bar{j} + (\dot{\psi}\cos\theta + \dot{\phi})\bar{k},\end{aligned} \tag{4.16}$$

$$\begin{aligned}\bar{\alpha} &= (-\ddot{\psi}\sin\theta\cos\phi + \ddot{\theta}\sin\phi - \dot{\psi}\dot{\theta}\cos\theta\cos\phi + \dot{\psi}\dot{\phi}\sin\theta\sin\phi + \dot{\phi}\dot{\theta}\cos\phi)\bar{i} \\ &\quad + (\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi + \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi)\bar{j} \\ &\quad + (\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta)\bar{k}.\end{aligned} \tag{4.17}$$

These expressions, particularly the one for $\bar{\alpha}$, are quite complicated. For that reason, the $x''y''z''$ axes, which do not undergo the spin, are sometimes selected for the representation. Then

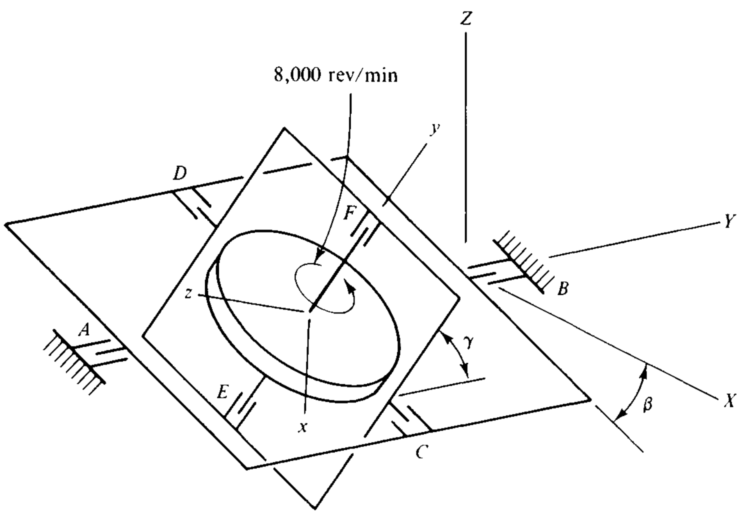
$$\bar{K} = -(\sin\theta)\bar{i}'' + (\cos\theta)\bar{k}''; \qquad \bar{j}' = \bar{j}'', \qquad \bar{k} = \bar{k}''. \tag{4.18}$$

Substitution of Eqs. (4.18) into Eqs. (4.12) and (4.14) results in substantially simpler expressions. They will be equivalent to Eqs. (4.16) and (4.17) for the instant when $\phi = 0$, corresponding to $\bar{i}'' = \bar{i}$ and $\bar{j}'' = \bar{j}$.

Utilization of Eulerian angles requires recognition of the appropriate axes of rotation. This involves identifying a fixed axis of rotation as the precession axis. Then the nutation axis precesses orthogonally to the precession axis. Finally, the spin axis precesses and nutates, while it remains perpendicular to the nutation axis. In many cases, the nutation or spin rates may be zero, in which case either of the respective angles is constant. This results in a degree of arbitrariness in the selection of the axes. Indeed, the case of rotation about a single axis can be considered to be solely precession, nutation, or spin, as one wishes.

One should note that there is no accepted standard as to how the coordinate axes should be assigned to the rotation axes. Some texts define the nutation such that the x' axis is the line of nodes. Also, it is quite possible that the formulas based on Eulerian angles may not be directly applicable. For example, because the representation addresses a motion featuring no more than three rotations, one set of Eulerian angles cannot describe a motion that features more than three rotations. Such situations could, however, be treated by defining multiple sets of transformations. Another case where the Eulerian angle formulation is inadequate arises when a motion consists of three rotations in which no two rotational axes are orthogonal. No line of nodes is evident in that case. Here again, more than one set of Eulerian angles would be required.

Example 4.2 A free gyroscope consists of a flywheel that rotates relative to the inner gimbal at the constant angular speed of 8,000 rev/min, while the rotation of the inner gimbal relative to the outer gimbal is $\gamma = 0.2\sin(100\pi t)$ rad. The rotation of the outer gimbal is $\beta = 0.5\sin(50\pi t)$ rad. Use the Eulerian angle formulas to determine the angular velocity and angular acceleration of the flywheel at $t = 4$ ms. Express the results in terms of components relative to the body-fixed xyz and space-fixed XYZ reference frames, where the z axis is parallel to the Z axis at $t = 0$.



Example 4.2

Solution The primary task in applying the Eulerian angle formulas is identification of precession, nutation, and spin in terms of the given rotations. The angle β is the rotation about the fixed Y axis, so we shall replace ψ in the formulas by β, and $\bar{K}$ by $\bar{J}$. The line of nodes, which is the nutation axis, must be perpendicular to the precession axis, and the spin axis must be perpendicular to the line of nodes. Therefore, we identify the y axis as the spin axis and axis CD as the line of nodes, which we designate as x'. Correspondingly, θ in the formulas becomes γ, the angle between the precession and spin axes, and $\bar{k}$ will be replaced by $\bar{j}$. Finally, we identify the spin angle by noting that the z and Z axes were coincident at $t = 0$, at which time $\beta = \gamma = 0$ also. Because y' was the line of nodes in the derivation, in the formulas we replace $\bar{j}'$ by $\bar{i}'$, $\bar{j}$ by $\bar{i}$, and $\bar{i}$ by $\bar{k}$. Also, the given constant spin rate leads to the spin angle $\phi = 8{,}000(2\pi/60)t$ rad.

In terms of the present notation, Eqs. (4.16) and (4.17) become

$$\bar{\omega} = (-\dot{\beta}\sin\gamma\cos\phi + \dot{\gamma}\sin\phi)\bar{k} + (\dot{\beta}\sin\gamma\sin\phi + \dot{\gamma}\cos\phi)\bar{i} + (\dot{\beta}\cos\gamma + \dot{\phi})\bar{j},$$

$$\begin{aligned}\bar{\alpha} &= (-\ddot{\beta}\sin\gamma\cos\phi + \ddot{\gamma}\sin\phi - \dot{\beta}\dot{\gamma}\cos\gamma\cos\phi + \dot{\beta}\dot{\phi}\sin\gamma\sin\phi + \dot{\phi}\dot{\gamma}\cos\phi)\bar{k} \\ &\quad + (\ddot{\beta}\sin\gamma\sin\phi + \ddot{\gamma}\cos\phi + \dot{\beta}\dot{\gamma}\cos\gamma\sin\phi + \dot{\beta}\dot{\phi}\sin\gamma\cos\phi - \dot{\gamma}\dot{\phi}\sin\phi)\bar{i} \\ &\quad + (\ddot{\beta}\cos\gamma + \ddot{\phi} - \dot{\beta}\dot{\gamma}\sin\gamma)\bar{j}.\end{aligned}$$

The Eulerian angles and their derivatives at $t = 4$ ms are

$$\beta = 0.2939 \text{ rad}, \quad \dot{\beta} = 63.54 \text{ rad/s}, \quad \ddot{\beta} = -7{,}252 \text{ rad/s}^2;$$

$$\gamma = 0.19021 \text{ rad}, \quad \dot{\gamma} = 19.416 \text{ rad/s}, \quad \ddot{\gamma} = -18{,}773 \text{ rad/s}^2;$$

$$\phi = 3.351 \text{ rad}, \quad \dot{\phi} = 837.8 \text{ rad/s}, \quad \ddot{\phi} = 0.$$

The corresponding angular velocity and acceleration are

$$\bar{\omega} = -21.49\bar{i} + 900.2\bar{j} + 7.71\bar{k} \text{ rad/s},$$

$$\bar{\alpha} = 11{,}934\bar{i} - 7{,}354\bar{j} - 14{,}256\bar{k} \text{ rad/s}^2.$$

These are the component representations in terms of the body-fixed xyz axes. We may employ the rotation transformation in Eq. (4.11) to convert the result to the space-fixed XYZ axes, provided that we preserve the permutation of the axis labels in the present application. Thus, we write

$$\begin{Bmatrix} z \\ x \\ y \end{Bmatrix} = [R]\begin{Bmatrix} Z \\ X \\ Y \end{Bmatrix},$$

where

$$\begin{aligned}[R] &= [R_\phi][R_\gamma][R_\beta] \\ &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.85909 & -0.47723 & 0.18494 \\ 0.47876 & -0.87707 & -0.03931 \\ 0.18096 & 0.05477 & 0.98196 \end{bmatrix}.\end{aligned}$$

Applying $[R]^{-1} = [R]^{\mathrm{T}}$ to evaluate the XYZ components of $\bar{\omega}$ and $\bar{\alpha}$ yields

$$\begin{Bmatrix} \omega_Z \\ \omega_X \\ \omega_Y \end{Bmatrix} = [R]^{\mathrm{T}}\begin{Bmatrix} 7.71 \\ -21.49 \\ 900.2 \end{Bmatrix} = \begin{Bmatrix} 145.98 \\ 64.47 \\ 886.19 \end{Bmatrix},$$

$$\begin{Bmatrix} \alpha_Z \\ \alpha_X \\ \alpha_Y \end{Bmatrix} = [R]^{\mathrm{T}}\begin{Bmatrix} -14{,}256 \\ 11{,}934 \\ -7{,}354 \end{Bmatrix} = \begin{Bmatrix} 16{,}630 \\ -4{,}066 \\ -10{,}327 \end{Bmatrix}.$$

Thus

$$\bar{\omega} = 64.5\bar{I} + 886.2\bar{J} + 146.0\bar{K} \text{ rad/s},$$

$$\bar{\alpha} = -4{,}066\bar{I} - 10{,}327\bar{J} + 16{,}630\bar{K} \text{ rad/s}^2.$$

4.3 Interconnections

Interesting kinematical questions arise when the motion of a body is restricted by other objects. Such conditions are associated with pin or slider connections between bodies, as well as with a variety of other methods for constructing mechanical systems. The kinematical manifestation of these connections are *constraint equations,* which are mathematical statements of conditions that the connection imposes on the motion of a point, or on the angular motion of a body. The kinematical constraints are imposed by *constraint forces* (and couples), which are more commonly known as *reactions.* The role of constraint forces will be treated in the chapters on kinetics.

A simple, though common, constraint condition arises when a body is only permitted to execute a *planar motion.* By definition, planar motion means that all points in the body follow parallel planes, which can only happen if the angular velocity is always perpendicular to these planes. Let the X–Y plane of the fixed reference frame and the x–y plane of the body-fixed reference frame be coincident planes of motion. Points that differ only in their z coordinate execute the same motion in this case, so they may be considered to be situated in the x–y plane. Hence, the kinematical equations for planar motion are

$$\begin{aligned}
&\bar{\omega} = \omega\bar{K} = \omega\bar{k}, \qquad \bar{\alpha} = \dot{\omega}\bar{K} = \dot{\omega}\bar{k};\\
&\bar{r}_{B/A} = X\bar{I} + Y\bar{J} = x\bar{i} + y\bar{j};\\
&\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A},\\
&\bar{a}_B = \bar{a}_A + \bar{\alpha} \times \bar{r}_{B/A} - \omega^2 \bar{r}_{B/A}.
\end{aligned} \tag{4.19}$$

Note that the centripetal acceleration term here is simplified from $\bar{\omega} \times (\bar{\omega} \times \bar{r}_{B/A})$ to $-\omega^2 \bar{r}_{B/A}$ by an identity that is valid only when $\bar{r}_{B/A}$ is perpendicular to $\bar{\omega}$. These relations are depicted in Figure 4.6.

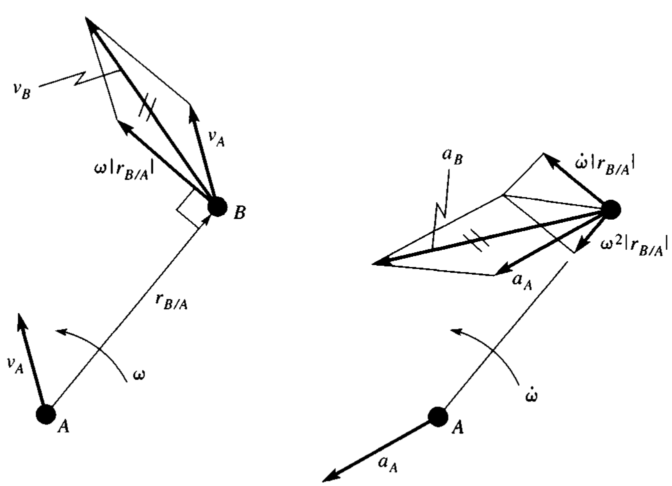


Figure 4.6 Velocity and acceleration in planar rigid-body motion.

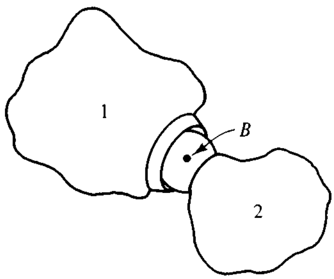


Figure 4.7 Ball-and-socket joint.

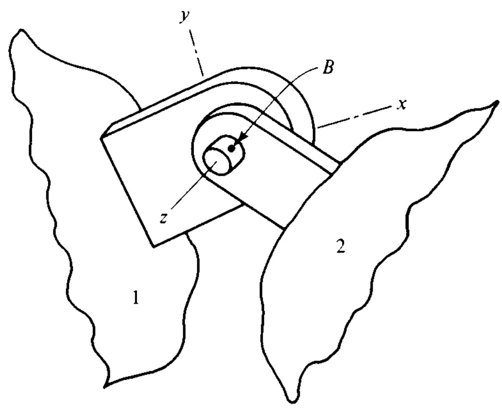


Figure 4.8 Pin connection.

Constraint conditions arising from a connection may be determined by examining its characteristics. For example, the *ball-and-socket joint* connecting bodies 1 and 2 in Figure 4.7 allows the bodies to have arbitrary orientations. However, point B_1 at the center of the ball moves in unison with point B_2 at the center of the socket. Thus the constraint conditions are

$$\bar{v}_{B1} = \bar{v}_{B2}, \qquad \bar{a}_{B1} = \bar{a}_{B2}. \tag{4.20}$$

These equations allow us to consider the joint as occupying a single point B that belongs to either rigid body.

The case of a *pin connection* between bodies has some elements in common with a ball-and-socket joint. Figure 4.8 depicts such a connection, with the z axis aligned along the axis of the pin. As with the ball-and-socket joint, both bodies have the same motion at their point of commonality. Consequently, Eqs. (4.20) must be satisfied. However, the pin also introduces restrictions on the rotations. In order to develop the constraint equations for angular motion, let us define xyz in Figure 4.8 to be fixed to body 1 with the z axis coincident with the axis of the pin. The only rotation of body 2 relative to body 1 permitted by the pin connection is spinning about the z axis. We let $\dot{\phi}$ denote the rate of this spin, so the angular velocities are related by

$$\bar{\omega}_2 = \bar{\omega}_1 + \dot{\phi}\bar{k}. \tag{4.21}$$

We find the corresponding constraint condition on angular acceleration by differentiating this relation. Because $\bar{k}$ is a unit vector that is fixed relative to body 1, we have

$$\bar{\alpha}_2 = \bar{\alpha}_1 + \ddot{\phi}\bar{k} + \dot{\phi}(\bar{\omega}_1 \times \bar{k}). \tag{4.22}$$

Equations (4.21) and (4.22) are constraint equations on the angular motion that must be satisfied in addition to Eqs. (4.20) for the connecting points. When it is more convenient to employ a coordinate system that does not have an axis aligned with the pin, $\bar{k}$ may be transformed to components relative to the desired coordinate system.

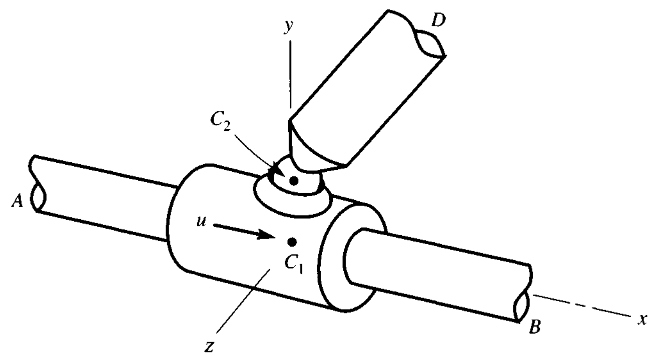


Figure 4.9 Collar connection.

Another common method for connecting bodies consists of a *collar* that slides over a bar, as depicted in Figure 4.9. (This connection is also known as a *slider.*) As with the ball-and-socket joint, point C_2 in the figure denotes the center of the ball. The collar is free to slide over bar AB. We characterize the constraint condition in this case by attaching a reference frame to bar AB. With respect to this reference frame, the collar only seems to be moving in or out along line AB. (Either sense may be assumed if it is not specified.) The collar is usually small in comparison to the bodies it joins, in which case we may neglect the distance from point C_2 to bar AB. When we let u denote the speed of the collar relative to bar AB, we find that the motion of point C_2 with respect to bar AB is given by

$$(\bar{v}_{C2})_{AB} = u\bar{e}_{B/A}, \qquad (\bar{a}_{C2})_{AB} = \dot{u}\bar{e}_{B/A}. \tag{4.23}$$

These relations assume that bar AB is straight. If this is not the case, we must add a centripetal acceleration term $(u^2/\rho)\bar{e}_n$, where ρ is the radius of curvature of bar AB at the current location of the collar and $\bar{e}_n$ is the instantaneous normal direction at that point.

To further characterize the motion of point C_2, we let C_1 denote the point on bar AB that coincides at this instant with the projection of point C_2 onto bar AB. Setting $\bar{r}_{C2/C1} = \bar{0}$ reduces the rigid-body motion equations (4.4) relating these points to:

$$\bar{v}_{C2} = \bar{v}_{C1} + u\bar{e}_{B/A}, \qquad \bar{a}_{C2} = \bar{a}_{C1} + \dot{u}\bar{e}_{B/A} + 2\bar{\omega}_{AB} \times u\bar{e}_{B/A}. \tag{4.24}$$

The collar introduces no restriction on the angular motion of the bodies it joins if bar CD is fastened to it by a ball-and-socket joint. However, a common connection method is a pin, Figure 4.10(a), or a fork-and-clevis joint, Figure 4.10(b). If the cross-section of bar AB is not circular, interference prevents the collar from spinning about bar AB. Then the constraints on angular motion are the same as Eqs. (4.21) and (4.22) for a pin connection. This is contrasted by the situation where the collar is free to rotate about the axis of bar AB because the cross-section of bar AB is circular. Bar AB then acts like another pin. We treat the angular motion constraints for this connection by fixing $x_1y_1z_1$ to bar AB with the x_1 axis aligned with bar AB, while $x_2y_2z_2$ is fixed to bar CD and the z_2 axis is aligned with the pin, as shown in the figure. The connection permits bar CD to spin at rate $\dot{\psi}$ about bar AB and at rate $\dot{\phi}$ about the pin, so the angular velocity-constraint condition is

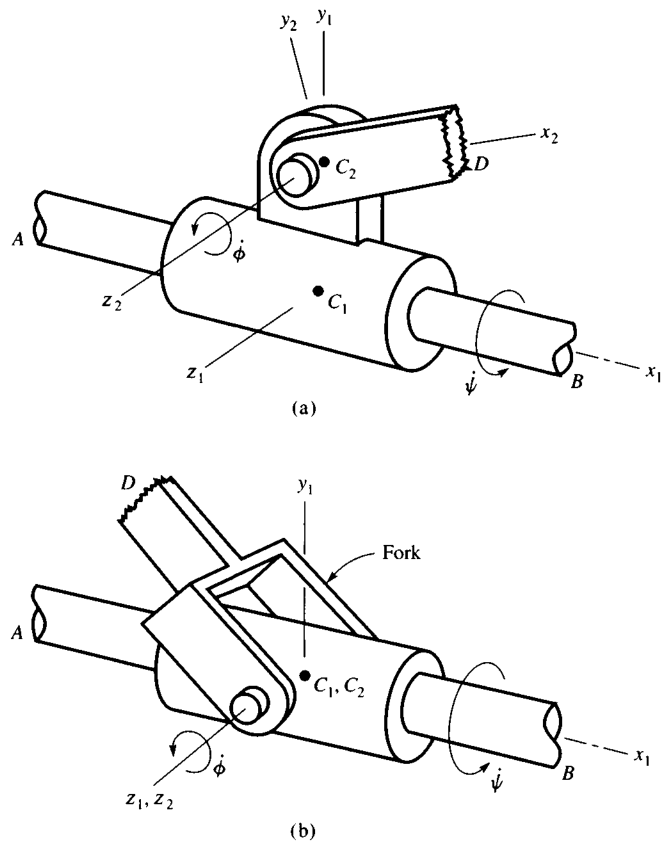


Figure 4.10 Collar connections that restrict rotation. (a) Collar with a pin. (b) Collar with a clevis joint.

$$\bar{\omega}_2 = \bar{\omega}_1 + \dot{\psi}\bar{i}_1 + \dot{\phi}\bar{k}_2. \tag{4.25}$$

We use the angular velocity of the respective bodies to differentiate the unit vectors. The corresponding angular acceleration constraint is therefore

$$\bar{\alpha}_2 = \bar{\alpha}_1 + \ddot{\psi}\bar{i}_1 + \dot{\psi}(\bar{\omega}_1 \times \bar{i}_1) + \ddot{\phi}\bar{k}_2 + \dot{\phi}(\bar{\omega}_2 \times \bar{k}_2). \tag{4.26}$$

To employ Eqs. (4.25) and (4.26) in the context of a problem solution, we would need to transform all vector quantities to a common coordinate system.

It should be obvious from the discussion thus far that the connections need to be examined in detail to identify all constraints on the motion. If all of the permutations and novel features of various types of connections were to be tabulated, it would not aid our understanding. It is preferable to consider each unfamiliar connection on a case-by-case basis, and then to employ the type of reasoning developed thus far to identify the constraint equations.

Describing a system's constraint conditions in mathematical terms is one aspect of an overall kinematical study. It also is necessary to relate the velocity and acceleration of constrained points in each body through Eqs. (4.4). When such relations are broken down into components, one obtains algebraic equations for the kinematical

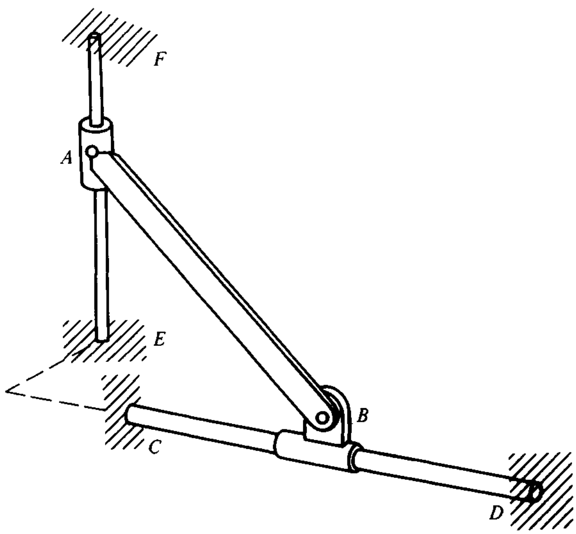


Figure 4.11 Spatial motion of a bar.

variables describing the motion of each body. In combination with the constraint equations, the result will be a system of simultaneous equations.

If the motion of the system is *fully constrained,* then this system of equations will be solvable such that, for each body, the linear motion of a point (velocity or acceleration) and the angular motion may be evaluated. If the system is only *partially constrained,* then the number of kinematical variables exceeds the number of kinematical equations. The simultaneous equations may then be solved for a set of excess variables in terms of the other variables. The excess variables in this case depend on the nature of the force system, so their evaluation requires a kinetics study. Another possibility is that the kinematical equations are not solvable. In that case, there are too many constraints on the motion of the system. This means that no motion is possible - such a system is *rigid.*

In order to demonstrate these matters, consider bar AB in Figure 4.11, which is constrained by collars that follow non-coplanar guide bars. The connection at collar A is a ball-and-socket joint, whereas the one at collar B is a pin. Because the guide bars are fixed, the velocity constraint equations (4.24) reduce to

$$\bar{v}_A = u_A \bar{e}_{F/E}, \qquad \bar{v}_B = u_B \bar{e}_{D/C}, \tag{4.27}$$

where the sense of these velocities will be determined by the sign of u_A and u_B. Let us ignore for now the constraint on the rotation of bar AB that is introduced by the pin on collar B. Then the angular velocity of the bar is an unknown $\bar{\omega}_{AB}$ having three components. Because points A and B are in the same rigid body, their velocities are related by

$$\bar{v}_A = \bar{v}_B + \bar{\omega}_{AB} \times \bar{r}_{A/B}. \tag{4.28}$$

This kinematical relation reduces to three scalar equations for five unknowns: u_A, u_B, and the three components of $\bar{\omega}_{AB}$. In order to further characterize the system, we need to account for the constraint on $\bar{\omega}_{AB}$ introduced by the pin on collar B. The round cross-sectional shape of guide CD permits collar B and bar AB to execute an arbitrary rotation about the guide bar. This condition corresponds to the motion constraints described by Eqs. (4.25) and (4.26). The guide bar is fixed, so it serves as the precession axis for the rotation of bar AB. Thus, one component of the angular velocity of bar AB is $\dot{\psi}\bar{e}_{D/C}$. In addition, bar AB may spin about the axis of the pin in the collar. The corresponding contribution to the angular velocity is $\dot{\phi}\bar{e}_{\phi}$, where $\bar{e}_{\phi}$ corresponds to $\bar{k}_2$ in the earlier derivation. We obtain a description of $\bar{e}_{\phi}$ from the pin's perpendicular orientation to the plane formed by guide CD and bar AB, so that

$$\bar{e}_{\phi} = \frac{\bar{r}_{B/A} \times \bar{r}_{D/C}}{|\bar{r}_{B/A} \times \bar{r}_{D/C}|}. \tag{4.29}$$

Precession and spin are the only rotations permitted by the sliding collar connection, so the angular velocity of bar AB is given by

$$\bar{\omega}_{AB} = \dot{\psi}\bar{e}_{D/C} + \dot{\phi}\frac{\bar{r}_{B/A} \times \bar{r}_{D/C}}{|\bar{r}_{B/A} \times \bar{r}_{D/C}|}. \tag{4.30}$$

This construction reduces the number of velocity unknowns in the system to four: u_A, u_B, $\dot{\psi}$, and $\dot{\phi}$. Two possibilities now arise. In a fully constrained situation, the overall motion will be defined through some kinematical input such as a specified motion for either collar. This removes the corresponding velocity parameter from the list of unknowns, thereby making Eqs. (4.27), (4.28), and (4.30) into a solvable set. In contrast, if the motion is induced by a given set of forces, so that none of the kinematical parameters are specified, then the system is partially constrained. In that case, the foregoing kinematical equations lead to characterization of the motion in terms of a single kinematical unknown. Kinetics principles would relate this unknown to the force system.

Let us consider other possibilities. Suppose that bar AB were connected to both collars by pins. That would introduce another constraint on $\bar{\omega}_{AB}$ like Eq. (4.30). In most cases, it would not be possible to satisfy both angular motion constraints simultaneously unless $\bar{\omega}_{AB} = \bar{0}$. This is the case of a rigid system. One notable exception where two pin connections would be acceptable is planar motion. This occurs when the guide bars are coplanar, and the axis of each pin is perpendicular to the plane containing the guide bars. Then $\bar{\omega}_{AB}$ has only a single component perpendicular to the plane, and the velocity relation, Eq. (4.28), has components in the plane only. In other words, the planar system with pin connections to the collars is not rigid.

Another situation of partial constraint is obtained when bar AB is connected to both collars by ball-and-socket joints, because Eq. (4.30) then does not apply. The lack of constraint in such a case is associated with the ability of bar AB to spin about its own axis. Such a rotation does not affect the motion of either collar. The kinematical equations can therefore be solved for a relation between u_A and u_B, although there will be no unique solution for $\bar{\omega}_{AB}$. If it is desirable for the number of equations and unknowns to match, a solution could be found by considering one of the joints to be a pin. Another approach in this case would be to obtain an additional relation by setting $\bar{\omega}_{AB} = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}$ and then requiring that $\bar{\omega}_{AB}\cdot\bar{e}_{B/A} = 0$; this corresponds to removing the rotation of the bar about its own axis.

Thus far, the discussion has addressed the analysis of velocities only. The treatment of acceleration follows a parallel development. However, it is necessary to evaluate the velocities first. One reason for this is that the angular velocities occur in the acceleration relation between two points in a body, Eq. (4.4). They also arise when we differentiate an angular velocity to derive a corresponding constraint on angular acceleration, as we did to derive Eqs. (4.22) and (4.26). A third place where velocity parameters arise in an acceleration analysis is in the characterization of the acceleration of a collar sliding along a curved bar, as was noted following Eq. (4.23). In all other respects, the analysis of acceleration is essentially a retracing of the procedure by which velocities are obtained.

An area of special interest in kinematics is concerned with *linkages,* where bars are interconnected in order to convert an input motion to a different output motion. From the standpoint of our general approach to rigid-body motion, the treatment of linkages presents no special problems. The constrained points in the system are the ends of the linkage, the connection points, and any point whose motion is specified. The constraint conditions on velocity (and then acceleration) of the constrained points are expressed according to the type of connection, for example, a pin or collar. All constraint conditions on the rotation of each member of the linkage resulting from the method of interconnection are used to express the angular velocity (and then the angular acceleration).

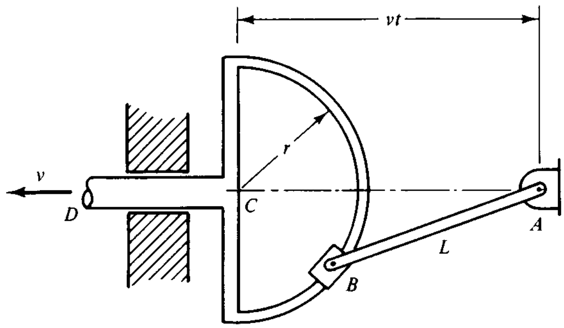
After the constraint equations have been formulated, the basic approach is to relate the velocity (and then acceleration) of the constrained points using the kinematical relation between the motion of two points in a rigid body. This may be achieved by starting at each end and working inward toward a selected connection in a progressive manner, from one link to its neighbor. The constraint equations are used to describe the various terms that arise in the basic kinematical equations for each member.

The ultimate result of the procedure will be two vector expressions for the velocity (or acceleration) of the selected connection, with each relation corresponding to different paths through the linkage. The two expressions must be equal because they describe the same point. Equating the corresponding components of each expression leads to a set of algebraic equations that should be solvable for all unknown parameters. (This assumes that the linkage is fully constrained, so that its motion is defined by the input.)

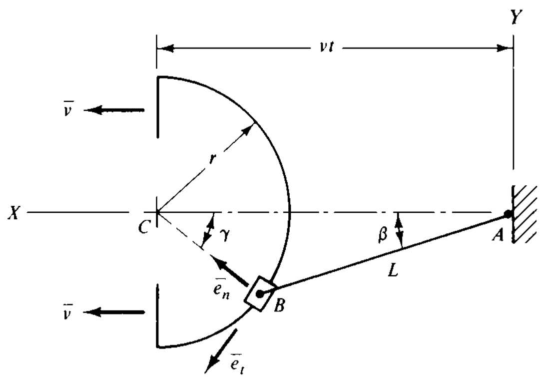
There is one complication that arises in treating linkages. The position of all constrained points in the linkage at any given instant must be described. This is not an overwhelming difficulty for planar linkages, but the geometrical relations for an arbitrary spatial linkage require careful consideration in their formulation. Rotation transformation matrices sometimes prove to be useful for this task.

Example 4.3 Collar B is pinned to arm AB as it slides over a circular guide bar. The guide bar translates to the left at a constant speed v, such that the distance from pivot A to the center C is vt. Derive expressions for the angular velocity and angular acceleration of arm AB.

Solution In general, the task of treating linkages is expedited by defining a space-fixed set of axes XYZ. This coordinate system, which is depicted in the figure, is sometimes referred to as a *global coordinate system*. It assists us in relating vectors that are associated with different bodies.



Example 4.3



Coordinate system and unit vectors.

We use the law of cosines to express the angles locating collar B:

$$\cos\beta = \frac{v^2t^2+L^2-r^2}{2Lvt}, \qquad \cos\gamma = \frac{v^2t^2+r^2-L^2}{2rvt}.$$

Because the foregoing given β and γ at any instant t, we could differentiate these expressions in order to determine the time derivatives of these angles. However, the derivatives are cumbersome, particularly for the second derivative. Also, an approach based on differentiation of analytical expressions is inherently limited to situations where the geometrical relations are relatively uncomplicated.

We shall therefore employ kinematical relations to solve this problem. Our approach is based on the recognition that guide bar CD forms a translating reference frame for the collar's motion. The vectors $\bar{e}_t$ and $\bar{e}_n$ in the sketch are the path-variable unit vectors for the motion of the collar relative to guide bar CD. We also know that the collar is pinned to bar AB. Since body CD is translating at speed v to the left, we have

$$\bar{v}_B = (\bar{v}_B)_{CD} + u\bar{e}_t = v\bar{I} + u\bar{e}_t = \bar{\omega}_{AB} \times \bar{r}_{B/A},$$

where u is the relative speed of the collar. Resolving the vectors into components relative to XYZ yields

$$\bar{v}_B = v\bar{I} + u[(\sin\gamma)\bar{I} - (\cos\gamma)\bar{J}] = (-\dot{\beta}\bar{K}) \times [(L\cos\beta)\bar{I} - (L\sin\beta)\bar{J}];$$

$$\bar{v}_B \cdot \bar{I} = v + u\sin\gamma = -\dot{\beta}L\sin\beta,$$

$$\bar{v}_B \cdot \bar{J} = -u\cos\gamma = -\dot{\beta}L\cos\beta.$$

The solution of the component equations is

$$u = -\frac{v}{\sin\gamma + \cos\gamma\tan\beta} \equiv -\frac{v\cos\beta}{\sin(\gamma+\beta)},$$

$$\dot{\beta}L = -\frac{v}{\cos\beta\tan\gamma + \sin\beta} \equiv -\frac{v\cos\gamma}{\sin(\gamma+\beta)}.$$

The corresponding velocity of collar B is

$$\bar{v}_B = (\bar{v}_B \cdot \bar{I})\bar{I} + (\bar{v}_B \cdot \bar{J})\bar{J} = -\dot{\beta}L[(\sin\beta)\bar{I} + (\cos\beta)\bar{J}]$$

$$= \frac{v\cos\gamma}{\sin(\gamma+\beta)}[(\sin\beta)\bar{I} + (\cos\beta)\bar{J}].$$

The same approach is valid for acceleration. The translational velocity of guide bar CD is constant, so we have

$$\bar{a}_B = \dot{u}\bar{e}_t + \frac{u^2}{r}\bar{e}_n = \bar{\alpha}_{AB} \times \bar{r}_{B/A} - \omega_{AB}^2\bar{r}_{B/A}.$$

In component form, this yields

$$\bar{a}_B = \dot{u}[(\sin\gamma)\bar{I} - (\cos\gamma)\bar{J}] + \frac{u^2}{r}[(\cos\gamma)\bar{I} + (\sin\gamma)\bar{J}]$$

$$= (-\ddot{\beta}\bar{K}) \times [(L\cos\beta)\bar{I} - (L\sin\beta)\bar{J}] - \dot{\beta}^2[(L\cos\beta)\bar{I} - (L\sin\beta)\bar{J}];$$

$$\bar{a}_B \cdot \bar{I} = \dot{u}\sin\gamma + \frac{u^2}{r}\cos\gamma = -\ddot{\beta}L\sin\beta - \dot{\beta}^2L\cos\beta,$$

$$\bar{a}_B \cdot \bar{J} = -\dot{u}\cos\gamma + \frac{u^2}{r}\sin\gamma = -\ddot{\beta}L\cos\beta + \dot{\beta}^2L\sin\beta.$$

The identities for the sine and cosine of the sum of two angles leads to the following solutions of the component equations:

$$\dot{u} = -\frac{\dot{\beta}^2L}{\sin(\gamma+\beta)} - \frac{u^2}{r}\cot(\gamma+\beta),$$

$$\ddot{\beta}L = -\dot{\beta}^2L\cot(\beta+\gamma) - \frac{u^2}{r\sin(\gamma+\beta)}.$$

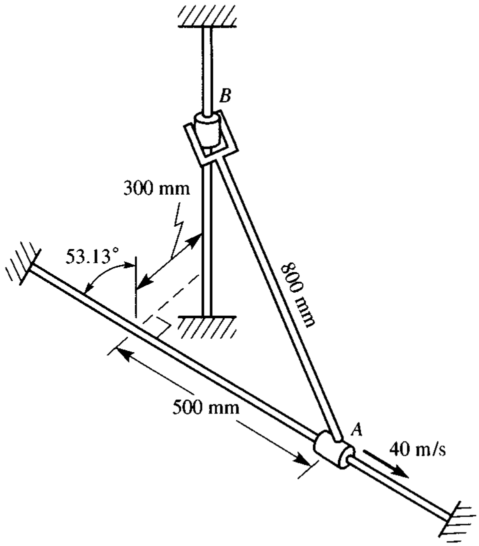
The corresponding expression for the acceleration in component form is

$$\bar{a}_B = (\bar{a}_B \cdot \bar{I})\bar{I} + (\bar{a}_B \cdot \bar{J})\bar{J}$$

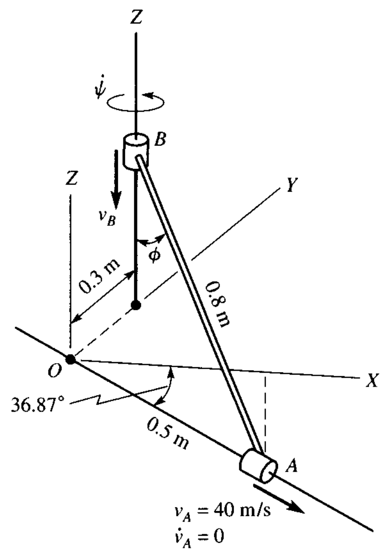
$$= -\frac{\dot{\beta}^2Lr\sin\gamma - u^2\sin\beta}{r\sin(\gamma+\beta)}\bar{I} + \frac{\dot{\beta}^2Lr\cos\gamma + u^2\cos\beta}{r\sin(\gamma+\beta)}\bar{J}.$$

Our solutions for $\bar{v}_B$ and $\bar{a}_B$ are implicit functions of time, because we first expressed β and γ in terms of t, and then found $\dot{\beta}$ and u in terms of β and γ.

Example 4.4 Collar A moves downward and to the right at a constant speed of 40 m/s. The connection of link AB to collar A is a ball-and-socket joint, while that at collar B is a pin. Determine the velocity and acceleration of collar B, and the angular velocity and angular acceleration of bar AB, for the position shown.



Example 4.4



Coordinate system and kinematical parameters.

Solution We begin by expressing the constraints on the motion of each collar in terms of components relative to the fixed XYZ axes. Because each follows a straight path, we have

$$\bar{v}_A = 40[(\cos 36.87°)\bar{I} - (\sin 36.87°)\bar{K}] = 32\bar{I} - 24\bar{K}\text{ m/s}, \qquad \bar{a}_A = \bar{0};$$

$$\bar{v}_B = v_B(-\bar{K}), \qquad \bar{a}_B = \dot{v}_B(-\bar{K}).$$

Next, we describe the constraint on the angular motion of bar AB imposed by the pin at end B. The angular velocity of bar AB consists of a precession about the guide bar at end B and a spin about the axis of the pin, which is orthogonal to the vertical guide bar and bar AB. Thus

$$\bar{\omega}_{AB} = \dot{\psi}\bar{K} + \dot{\phi}\bar{e}_\phi, \quad \bar{e}_\phi = \frac{\bar{r}_{AB} \times \bar{K}}{|\bar{r}_{A/B} \times \bar{K}|}.$$

The pin, which defines the orientation of $\bar{e}_\phi$, executes the precessional rotation, so the angular acceleration is

$$\bar{\alpha}_{AB} = \ddot{\psi}\bar{K} + \ddot{\phi}\bar{e}_\phi + \dot{\phi}(\dot{\psi}\bar{K} \times \bar{e}_\phi).$$

In order to express these vectors in component form, we use the Pythagorean theorem to evaluate the vertical distance between the collars at this instant,

$$H = [0.8^2 - 0.3^2 - (0.5\cos 36.87°)^2]^{1/2} = 0.6245\text{ m}.$$

Then

$$\bar{r}_{A/B} = (0.5\cos 36.87°)\bar{I} - 0.3\bar{J} - H\bar{K}$$
$$= 0.4\bar{I} - 0.3\bar{J} - 0.6245\bar{K} \text{ m};$$
$$\bar{e}_\phi = \frac{-0.3\bar{I} - 0.4\bar{J}}{[0.3^2 + 0.4^2]^{1/2}} = -0.6\bar{I} - 0.8\bar{J};$$
$$\bar{\omega}_{AB} = \dot{\phi}(-0.6\bar{I} - 0.8\bar{J}) + \dot{\psi}\bar{K},$$
$$\bar{\alpha}_{AB} = \ddot{\psi}\bar{K} + \ddot{\phi}(-0.6\bar{I} - 0.8\bar{J}) + \dot{\psi}\dot{\phi}(0.8\bar{I} - 0.6\bar{J}).$$

Now that the effects of the constraints have been characterized, it remains only to relate the motion of ends A and B. The velocity relation gives

$$\bar{v}_A = \bar{v}_B + \bar{\omega}_{AB} \times \bar{r}_{AB};$$
$$32\bar{I} - 24\bar{K} = -v_B\bar{K} + (-0.6\dot{\phi}\bar{I} - 0.8\dot{\phi}\bar{J} + \dot{\psi}\bar{K}) \times (0.4\bar{I} - 0.3\bar{J} - 0.6245\bar{K});$$
$$\bar{v}_A \cdot \bar{I} = 32 = 0.4996\dot{\phi} + 0.3\dot{\psi},$$
$$\bar{v}_A \cdot \bar{J} = 0 = -0.3747\dot{\phi} + 0.4\dot{\psi},$$
$$\bar{v}_A \cdot \bar{K} = -24 = -v_B + 0.5\dot{\phi};$$
$$\dot{\phi} = 40.99 \text{ rad/s}, \qquad \dot{\psi} = 38.40 \text{ rad/s}; \qquad v_B = 44.50 \text{ m/s}.$$

Substitution of these results into the earlier expressions yields

$$\bar{\omega}_{AB} = -24.60\bar{I} - 32.79\bar{J} + 38.4\bar{K} \text{ rad/s};$$
$$\bar{v}_B = -44.50\bar{K} \text{ m/s}.$$

We may now relate the acceleration of the collars using

$$\bar{a}_A = \bar{a}_B + \bar{\alpha}_{AB} \times \bar{r}_{A/B} + \bar{\omega}_{AB} \times (\bar{\omega}_{AB} \times \bar{r}_{AB}).$$

When we substitute the results for $\dot{\phi}$ and $\dot{\psi}$ into the general expression for $\bar{\alpha}$, we obtain

$$\bar{\alpha}_{AB} = (-0.6\ddot{\phi} + 12{,}593.0)\bar{I} + (-0.8\ddot{\phi} - 944.5)\bar{J} + \ddot{\psi}\bar{K}.$$

The constraints on the accelerations of the collars are that $\bar{a}_A = \bar{0}$ and $\bar{a}_B = -\dot{v}_B\bar{K}$, which leads to

$$\bar{a}_A = \bar{0} = -\dot{v}_B\bar{K} + [(-0.6\ddot{\phi} + 12{,}593.0)\bar{I} + (-0.8\ddot{\phi} - 944.5)\bar{J} + \ddot{\psi}\bar{K}]$$
$$\times (0.4\bar{I} - 0.3\bar{J} - 0.6245\bar{K}) + (-24.60\bar{I} - 32.79\bar{J} + 38.40\bar{K})$$
$$\times [(-24.60\bar{I} - 32.79\bar{J} + 38.40\bar{K}) \times (0.4\bar{I} - 0.3\bar{J} - 0.6245\bar{K})];$$
$$\bar{a}_A \cdot \bar{I} = 0 = (0.4996\ddot{\phi} + 589.8 + 0.3\ddot{\psi}) - 672.2,$$
$$\bar{a}_A \cdot \bar{J} = 0 = (-0.3747\ddot{\phi} + 786.4 + 0.4\ddot{\psi}) + 1{,}732.9,$$
$$\bar{a}_A \cdot \bar{K} = 0 = -\dot{v}_B + 0.5\ddot{\phi} + 1{,}049.4.$$

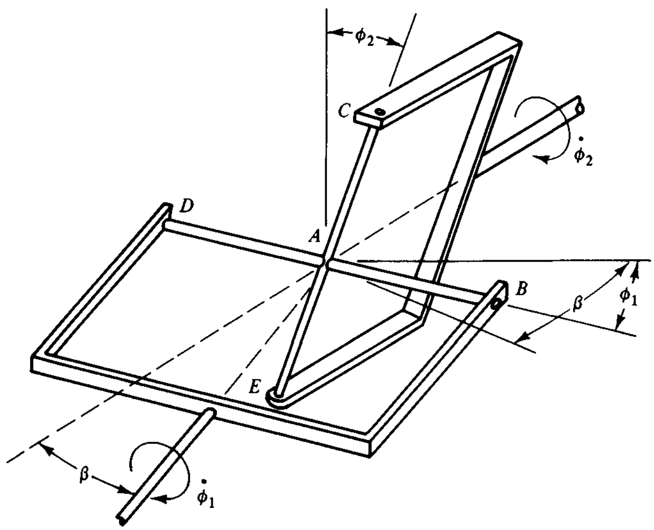
The solution of these simultaneous equations is

$$\ddot{\phi} = 2{,}526 \text{ rad/s}^2, \qquad \ddot{\psi} = -3{,}932 \text{ rad/s}^2, \qquad \dot{v}_B = 2{,}312 \text{ m/s}^2,$$

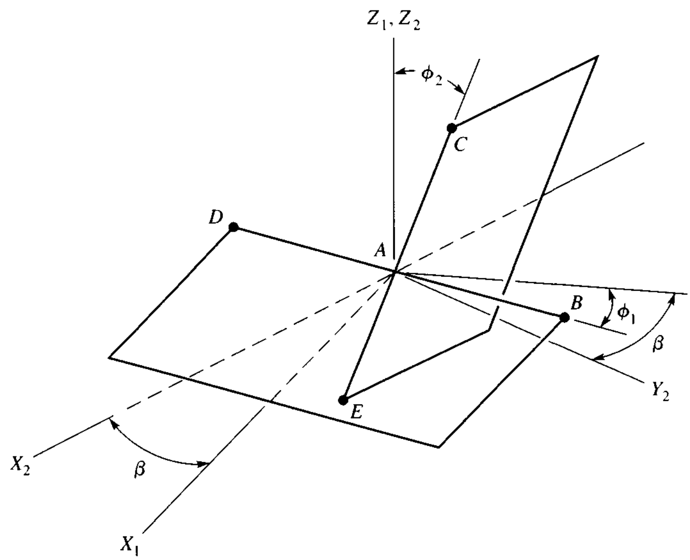
from which we obtain

$$\bar{\alpha}_{AB} = -256\bar{I} - 2{,}965\bar{J} - 3{,}932\bar{K} \text{ rad/s}^2, \qquad \bar{a}_B = -2{,}312\bar{K} \text{ ft/s}^2.$$

Example 4.5 Two shafts lying in a common horizontal plane at a skew angle β are connected by a cross-link universal joint that is called a *cardan joint*. Derive an expression for the rotation rate ω_2 in terms of ω_1 and the instantaneous angle of rotation ϕ_1, where cross-link AB is horizontal when $\phi_1 = 0$.



Example 4.5



Coordinate systems.

Solution There are a variety of approaches to this problem, but they all require that the angles of rotation ϕ_1 and ϕ_2 of the respective shafts be related. Once such a relation is developed, it is a simple matter to obtain the angular rates by differentiation.

Arm AC is perpendicular to arm AB, so $\bar{e}_{C/A}\cdot\bar{e}_{B/A}=0$. In order to describe these unit vectors, we define coordinate axes $X_1Y_1Z_1$ and $X_2Y_2Z_2$ such that each X axis is parallel to the corresponding axis of rotation and each Z axis is perpendicular to the plane formed by the two rotation axes. Because arm AB rotates about the X_1 axis while arm AC rotates about the X_2 axis, we have

$$\bar{e}_{B/A}=(\cos\phi_1)\bar{J}_1-(\sin\phi_1)\bar{K}_1, \qquad \bar{e}_{C/A}=(\sin\phi_2)\bar{J}_2+(\cos\phi_2)\bar{K}_2.$$

Because the Z axes coincide, and the angle between the Y axes is β, the orthogonality condition becomes

$$\bar{e}_{B/A}\cdot\bar{e}_{C/A}=\cos\phi_1\sin\phi_2\cos\beta-\sin\phi_1\cos\phi_2=0,$$

$$\tan\phi_2=\frac{1}{\cos\beta}\tan\phi_1.$$

This is a general relation between the angles, so it may be differentiated. Thus

$$\frac{1}{\cos^2\phi_2}\dot{\phi}_2=\frac{1}{\cos\beta\cos^2\phi_1}\dot{\phi}_1.$$

In order to remove the dependence on ϕ_2, we employ a trigonometric identity to find

$$\cos^2\phi_2\equiv\frac{1}{1+\tan^2\phi_2}=\frac{1}{1+\tan^2\phi_1/\cos^2\beta},$$

which leads to

$$\dot{\phi}_2=\frac{\cos\beta}{\sin^2\phi_1+\cos^2\beta\cos^2\phi_1}\dot{\phi}_1.$$

It is interesting to observe that the maximum and minimum values of $\dot{\phi}_2$ are $\dot{\phi}_1/\cos\beta$ at $\phi_1=0$ and π, and $\dot{\phi}_1\cos\beta$ at $\phi_1=\pi/2$ and $3\pi/2$, respectively. This oscillation relative to the input speed $\dot{\phi}_1$ makes the cardan joint by itself unsuitable as a constant-velocity joint. In front-engine, rear-wheel-drive automobiles, two opposed cardan joints are employed in the drive train; the reciprocal arrangement produces a final speed that matches the input.

4.4 Rolling

A common constraint condition arises when bodies rotate as they move over each other. The fact that the contacting surfaces cannot penetrate each other imposes a restriction on the velocity components perpendicular to the plane of contact (that is, the tangent plane). Figure 4.12 shows two surfaces in contact, as viewed edgewise along their plane of contact. The z axis in the figure is defined to be normal to the plane of contact. Because the surface of each body is impenetrable, the velocity components normal to the contact plane must match. Let C_1 and C_2 be contacting points on each body. Then

$$\bar{v}_{C1}\cdot\bar{k}=\bar{v}_{C2}\cdot\bar{k}. \qquad (4.31)$$

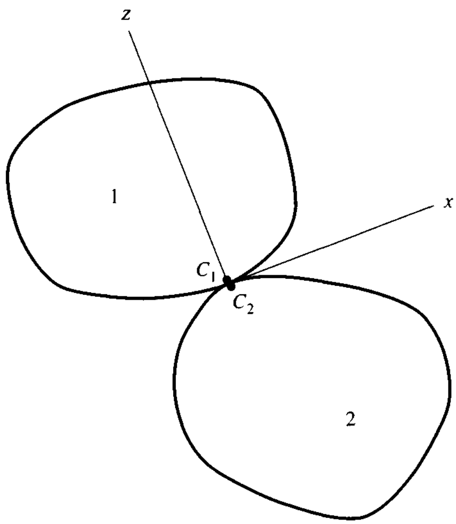


Figure 4.12 Rolling contact.

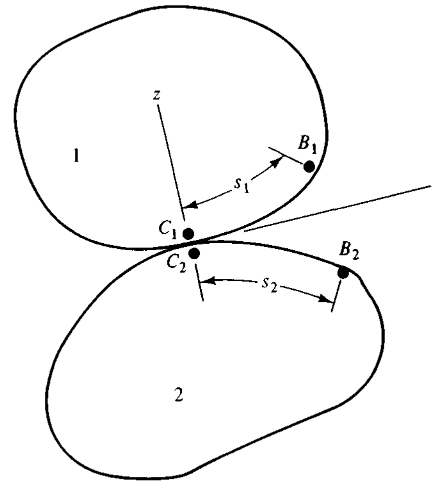


Figure 4.13 No-slip condition.

The special case of *rolling without slipping* imposes an additional constraint. A variety of viewpoints are available to treat this type of motion. Consider two pairs of points on the perimeter of bodies that roll over each other without slipping, such as B_i and C_i in Figure 4.13. These points are selected such that B_1 and B_2 were the points of contact at an earlier instant, with C_1 and C_2 the current points of contact. (The figure considers a planar situation, in order to readily depict all points of contact.) The elimination of slipping means that the arclength s_1 along the perimeter of body 1 between the points B_1 and C_2 is the same as the arclength s_2 along body 2 between points B_2 and C_2. The restriction on arclengths is one way of describing the constraint imposed by the condition of no slipping.

The most common application of such a description is for a wheel rolling along the ground. The path of a point on the circumference of the wheel is a *cycloid.* The geometrical parameters needed to characterize this path are depicted in Figure 4.14. The origin of the fixed reference frame has been placed at the starting position of the center of the wheel. Point A on the cylinder contacted the ground initially, whereas point B is the current contact point. When there is no slippage, the arclength between

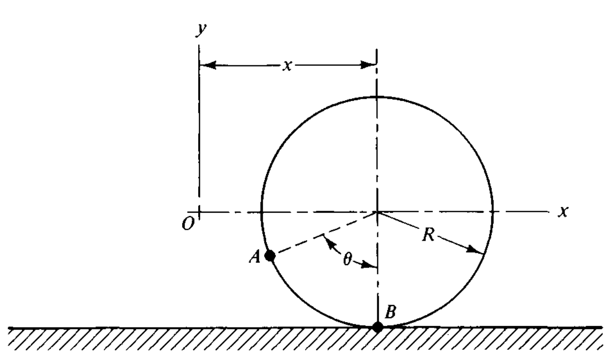


Figure 4.14 Rolling wheel.

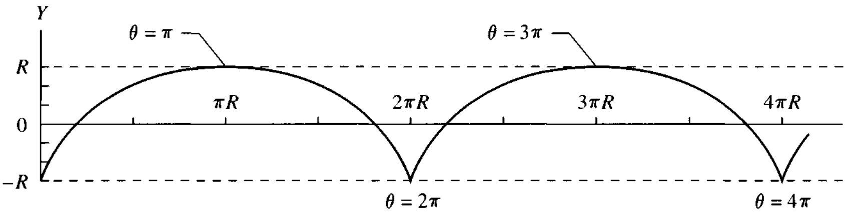


Figure 4.15 Cycloidal path.

points A and B is the same as the distance x that the center of the wheel has displaced. Thus,

$$x = R\theta. \tag{4.32}$$

From this relation the position of point A may be described, in parametric form as a function of θ, as

$$\bar{r}_{A/O} = R(\theta - \sin\theta)\bar{i} - (R\cos\theta)\bar{j}. \tag{4.33}$$

The cycloidal path is depicted in Figure 4.15.

Expressions for the velocity and acceleration of point A may be found by differentiation of the foregoing relationship, in conjunction with the center's speed being $v = \dot{x} = R\dot{\theta}$. The results are

$$\begin{aligned} \bar{v}_A &= v(1-\cos\theta)\bar{i} + v(\sin\theta)\bar{j}, \\ \bar{a}_A &= \left[\dot{v}(1-\cos\theta) + \frac{v^2}{R}\sin\theta\right]\bar{i} + \left[\dot{v}\sin\theta + \frac{v^2}{R}\cos\theta\right]\bar{j}. \end{aligned} \tag{4.34}$$

An aspect of the velocity and acceleration of particular relevance to further developments arises at $\theta = 0$, where $\bar{v}_A = \bar{0}$ and $\bar{a}_A = (v^2/R)\bar{j}$. In other words, the contact point comes to rest, and the acceleration is upward. This corresponds to the cusp in the cycloidal path.

One difficulty with a formulation in terms of arclengths is that it becomes increasingly difficult to use as the complexity of the motion increases. This is particularly true for spatial three-dimensional motion. The alternative method that we shall develop uses constraint conditions on velocity and acceleration that are derived from the "equal arclength" rule.

Consider the limiting situation where the points of contact B_i and C_i in Figure 4.13 correspond to instants that are very close. The points of contact on each body then seem to have the same motion along the contact plane at the instant they contact. Hence, the velocities of contacting points of bodies that roll over each other without slipping must match in all directions. The constraint condition is

♦ $$\bar{v}_{C1} = \bar{v}_{C2} \quad \text{for no slipping.} \tag{4.35}$$

Acceleration is more complicated, because the contacting points on each body come together and then separate. This means that they have different accelerations in the normal direction. A common misconception arises from the case of the rolling wheel in Figure 4.14, as well as other planar situations. The second of Eqs. (4.34) indicates that the acceleration of point A on the circumference of the wheel is upward

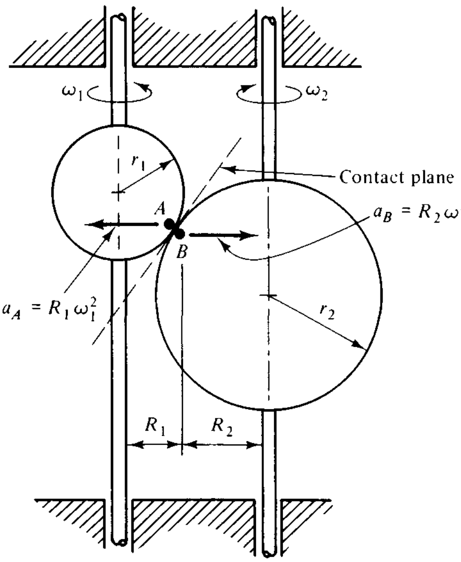


Figure 4.16 Contact between rotating spheres.

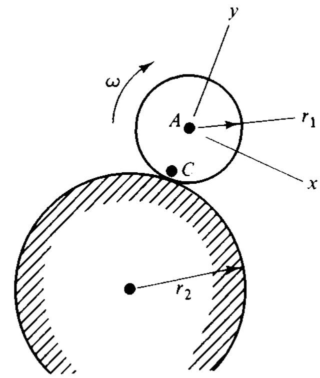


Figure 4.17 Rolling of circular shapes.

when the point contacts the ground. This is often taken (incorrectly) to be generally valid. Specifically, the result is interpreted to mean that, in the absence of slipping, the contact points may only accelerate relative to each other perpendicularly to the contact plane. However, the statement is not true in many cases of spatial motion, where the normal to the plane of contact does not necessarily coincide with the osculating plane of the path of a contacting point.

As a way of identifying the difficulty, consider the two spheres in Figure 4.16 that rotate at constant rates ω_1 and ω_2 about fixed parallel axes, such that there is no slipping between the contacting points A and B. The perpendicular distances R_1 and R_2 from the points of contact to the respective axes of rotation may be found from similar triangles. The no-slip velocity condition is $v_A = v_B$, which is satisfied when the ratio of the rotation rates is $\omega_2/\omega_1 = R_1/R_2$. Because points on each sphere follow circular paths, their (centripetal) acceleration is perpendicular to the rotation axes, as shown in the figure. As a result, each acceleration has a component parallel to the plane of contact.

The lack of a direct constraint condition for acceleration presents a dilemma whose resolution lies in the existence of another constraint. The shape of each rolling body is constant. That shape is usually expressed by a functional relationship for the distance from a reference point on the body to points on the perimeter. Round objects are of primary concern for engineering applications; the point of reference in that case is the center. The constraint that there is a constant distance from the contact point to the center of a round body must be satisfied. In effect, this means that the center is subject to a kinematical constraint.

In order to explore this idea, consider the planar situation of a planetary gear rolling over a fixed inner gear, Figure 4.17. Because the distance from the center A of the planetary gear to the point of contact C is constant, point A follows a circular path of radius r_1+r_2. Thus, for the xyz coordinate system depicted in the figure, the velocity and acceleration of point A are described according to path variables as

$$\bar{v}_A = v_A\bar{i}, \qquad \bar{a}_A = \dot{v}_A\bar{i} - \frac{v_A^2}{r_1+r_2}\bar{j}. \tag{4.36}$$

Because there is no slipping at point C, the velocity constraint $\bar{v}_C = \bar{0}$ yields

$$\bar{v}_A = \bar{\omega}\times\bar{r}_{A/C} = (-\omega\bar{k})\times r_1\bar{j} = \omega r_1\bar{i}. \tag{4.37}$$

Thus, the constraint that the center A move tangent to its circular path, expressed by the first of Eqs. (4.36), is satisfied by $\bar{v}_A$ in Eq. (4.37), which we obtained independently. Comparing the two expressions for $\bar{v}_A$ yields the familiar relation

$$v_A = \omega r_1. \tag{4.38}$$

Now consider acceleration. Because of the round shape and the restriction to planar motion, the algebraic relation between the speed of each geometric center and the angular speeds of the contacting bodies will not change. We may differentiate such an expression with respect to time, so that

$$\dot{v}_A = \dot{\omega} r_1. \tag{4.39}$$

Substitution of Eqs. (4.38) and (4.39) into Eq. (4.36) yields a complete description of $\bar{a}_A$ without consideration of $\bar{a}_C$.

The same approach may be extended directly to treat cases of spatial motion in which the orientation of the contacting bodies relative to each other does not change as the bodies roll. An example of such a situation is a cone rolling over a plane. Example 4.8 illustrates another situation of this type.

In some spatial motions, the relationships for rotation rates might be position-dependent. Nevertheless, the basic concept of differentiating a general expression remains unchanged. A common system in which the geometrical properties are not constant is a rolling coin. Figure 4.18 depicts a disk that is rolling in a wobbly manner

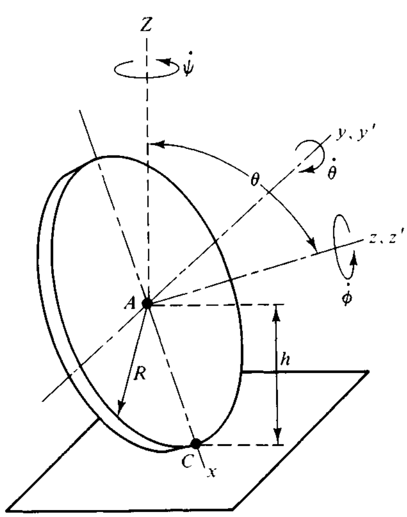


Figure 4.18 Rolling disk in arbitrary motion.

but without slipping over a flat surface. Let $\dot{\psi}$ be the precession rate (about the fixed Z axis), and let $\dot{\theta}$ be the nutation rate (about the horizontal axis in the plane of the disk). Let $x'y'z'$ be a reference frame whose origin is at the center of the disk, with the z' axis coincident with the axis of the disk and the y' axis coincident with the line of nodes for the Eulerian angles. The body-fixed xyz axes spin about the z' axis at $\dot{\phi}$ relative to $x'y'z'$.

The importance of the round shape in this case is that, if all other quantities are held fixed, the motion of the system will not be altered by changing the value of the spin angle ϕ. As a consequence of this invariance, the $x'y'z'$ axes may be used to obtain a general relation for the velocity of point A. The expressions for the angular velocity $\bar{\omega}'$ of the $x'y'z'$ reference frame and for $\bar{\omega}$ of the disk in terms of the Eulerian angles are

$$\begin{aligned} \bar{\omega}' &= -(\dot{\psi}\sin\theta)\bar{i}' + \dot{\theta}\bar{j}' + (\dot{\psi}\cos\theta)\bar{k}', \\ \bar{\omega} &= -(\dot{\psi}\sin\theta)\bar{i}' + \dot{\theta}\bar{j}' + (\dot{\phi} + \dot{\psi}\cos\theta)\bar{k}'. \end{aligned} \tag{4.40}$$

The condition that point C not slip relative to the ground leads to relations between the motion of the center A and the Eulerian angles. It follows from $\bar{v}_C = \bar{0}$ that

$$\bar{v}_A = \bar{\omega} \times \bar{r}_{A/C} = -R(\dot{\phi} + \dot{\psi}\cos\theta)\bar{j}' + R\dot{\theta}\bar{k}'. \tag{4.41}$$

This is a general relation for $\bar{v}_A$ in terms of the Eulerian angles. It therefore may be differentiated to determine $\bar{a}_A$. Toward this end, remember that $\bar{\omega}'$ must be employed to express the derivatives of the unit vectors in Eq. (4.41). Thus,

$$\begin{aligned} \bar{a}_A &= \frac{\delta \bar{v}_A}{\delta t} + \bar{\omega}' \times \bar{v}_A \\ &= R[\dot{\theta}^2 + (\dot{\phi} + \dot{\psi}\cos\theta)\dot{\psi}\cos\theta]\bar{i}' + R(-\ddot{\phi} - \ddot{\psi}\cos\theta + 2\dot{\psi}\dot{\theta}\sin\phi)\bar{j}' \\ &\quad + R[\ddot{\theta} + (\dot{\phi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta]\bar{k}'. \end{aligned} \tag{4.42}$$

Equations (4.41) and (4.42) provide general relations between the motion of the center and the Eulerian angles when there is no slipping. Although they contain a variety of effects, one in particular is readily identifiable. The components of $\bar{v}_A$ and $\bar{a}_A$ perpendicular to the contact plane - that is, in the direction of the Z axis - are found with the aid of Figure 4.18 to be

$$\bar{v}_A \cdot \bar{K} = R\dot{\theta}\cos\theta, \qquad \bar{a}_A \cdot \bar{K} = R(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta).$$

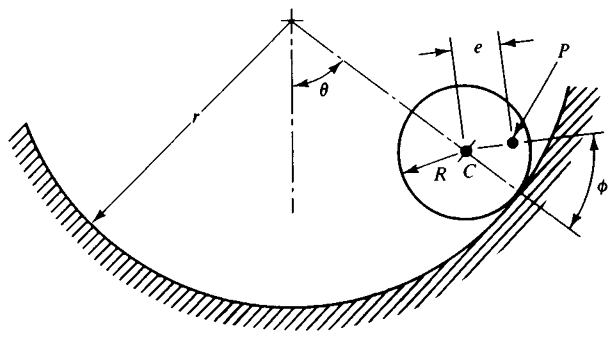
We now observe from Figure 4.18 that the elevation of point A above the ground is

$$h = R\sin\theta.$$

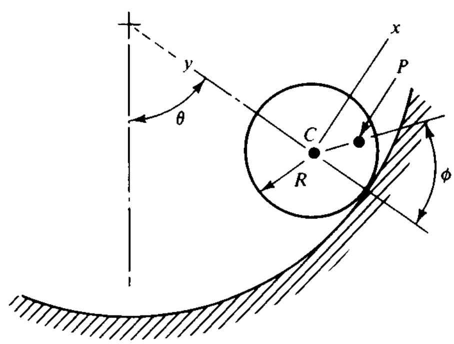
Successive differentiation of h shows that $\dot{h} = \bar{v}_A \cdot \bar{K}$ and $\ddot{h} = \bar{a}_A \cdot \bar{K}$.

Although the roundness of the disk played a less obvious role in this motion, it was crucial. If the disk were elliptical, it would have been necessary to describe the velocity of the center point as a function of the spin angle and the properties of the ellipse. Differentiating such a representation would have been substantially more difficult than the corresponding tasks in the case of a circular disk.

Example 4.6 The cylinder of radius R rolls without slipping inside a semicylindrical cavity. Point P is collinear with the vertical centerline when the vertical angle θ locating the cylinder's center C is zero. Derive expressions for the velocity and acceleration



Example 4.6



Coordinate system.

of point P in terms of ϕ and the speed v of the center C, valid when $\dot{v} \neq 0$. Then convert those expressions to express them in terms of θ.

Solution It is convenient to orient the body-fixed reference frame such that, at the instant of interest, the x axis is tangent to the path of the center and the y axis intersects the center of curvature. Movement of the wheel to the right then corresponds to

$$\bar{v}_C = v\bar{i}, \qquad \bar{a}_C = \dot{v}\bar{i} + \frac{v^2}{r-R}\bar{j}; \qquad \bar{\omega} = -\omega\bar{k}, \qquad \bar{\alpha} = -\dot{\omega}\bar{k}.$$

Because there is no slipping at the contact point A, it also must be that

$$\bar{v}_C = \bar{\omega} \times \bar{r}_{C/A}.$$

We match the two expressions for $\bar{v}_C$ to obtain a relation for ω, and then differentiate it, with the result that

$$\omega = \frac{v}{R} \Rightarrow \dot{\omega} = \frac{\dot{v}}{R}.$$

Now that we have characterized $\bar{\omega}$, $\bar{\alpha}$, $\bar{v}_C$, and $\bar{a}_C$, we may evaluate the velocity and acceleration of point P:

$$\bar{v}_P = \bar{v}_C + \bar{\omega} \times \bar{r}_{P/C} = v\bar{i} + \left(-\frac{v}{R}\bar{k}\right) \times [(e\sin\phi)\bar{i} - (e\cos\phi)\bar{j}]$$
$$= v\left(1 - \frac{e}{R}\cos\phi\right)\bar{i} - v\left(\frac{e}{R}\sin\phi\right)\bar{j},$$
$$\bar{a}_P = \bar{a}_C + \bar{\alpha} \times \bar{r}_{P/C} - \omega^2 \bar{r}_{P/C}$$
$$= \dot{v}\bar{i} + \frac{v^2}{r-R}\bar{j} + \left(-\frac{\dot{v}}{R}\bar{k}\right) \times [(e\sin\phi)\bar{i} - (e\cos\phi)\bar{j}]$$
$$- \left(\frac{v}{R}\right)^2 [(e\sin\phi)\bar{i} - (e\cos\phi)\bar{j}]$$
$$= \left[\dot{v}\left(1 - \frac{e}{R}\cos\phi\right) - \frac{v^2 e}{R^2}\sin\phi\right]\bar{i} + \left[-\dot{v}\frac{e}{R}\sin\phi + v^2\left(\frac{1}{r-R} + \frac{e}{R^2}\cos\phi\right)\right]\bar{j}.$$

To express the velocity and acceleration in terms of the vertical angle θ, we must establish the relation between ϕ and θ. For this, we return to the no-slip condition. It was stated that point P is on the vertical centerline when $\theta = 0$. The equal-arclength form of the no-slip constraint requires the arclengths measured along the cylinder and along the cavity from the current points of contact to the initial points be the same, which leads to

$$r\theta = R(2\pi - \phi) \;\Rightarrow\; \phi = 2\pi - \frac{r}{R}\theta.$$

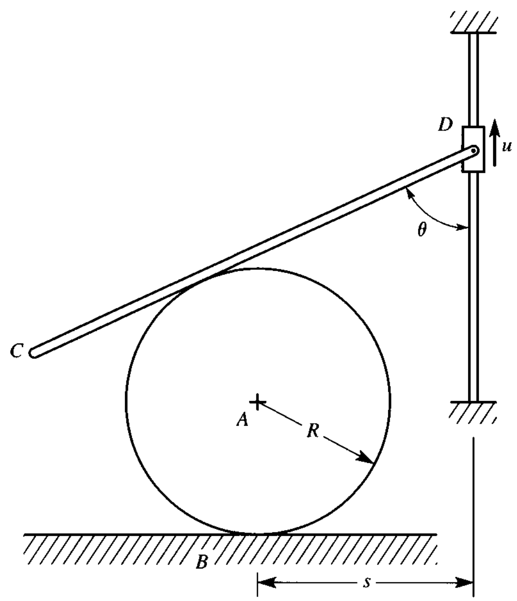
According to this relation, the condition $\theta = 0$ corresponds to $\phi = 2\pi$, which represents one revolution. We could also have obtained this relation by using the velocity constraint equation, as follows. We have established that the angular velocity of the cylinder is $\omega = v/R$. Considering point C to follow a circular path of radius $r - R$, with the radial line rotating at rate $\dot{\theta}$, yields $v = (r-R)\dot{\theta}$. The *total angle* of rotation of the cylinder is $\theta + \phi$ in the counterclockwise sense, so its angular velocity is $\bar{\omega} = (\dot{\theta} + \dot{\phi})(-\bar{k})$. We therefore have

$$-(\dot{\theta} + \dot{\phi}) = \frac{(r-R)\dot{\theta}}{R} \;\Rightarrow\; \dot{\phi} = -\frac{r}{R}\dot{\theta} \;\Rightarrow\; \phi = -\frac{r}{R}\theta.$$

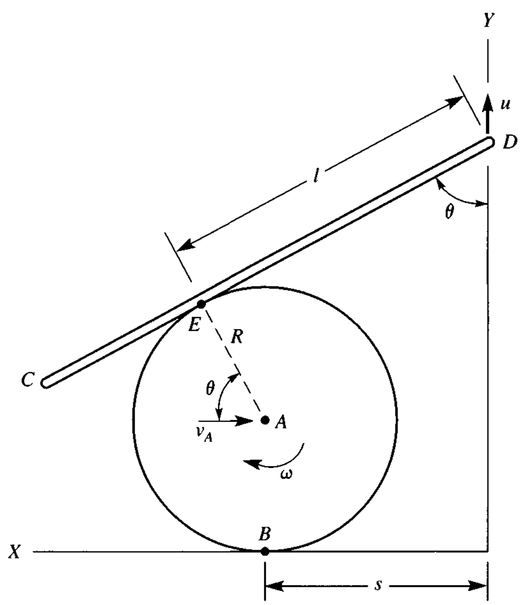
Note that we set $\phi = 0$ when $\theta = 0$ to integrate this relation, which is the reason for the 2π difference between this relation and the one obtained from the equal-arclength rule. Because the trigonometric functions are periodic in 2π, either relation yields the same result when it is used to eliminate ϕ in favor of θ in the solutions for $\bar{v}_P$ and $\bar{a}_P$.

Example 4.7 Rack CD, which meshes with gear A, is actuated by moving collar D upward at the constant speed u. Rack B, over which gear A rolls, is stationary. Derive expressions for the velocity and acceleration of the center of gear A in terms of the angle θ and the distance s.

Solution Our approach here is based on treating the system as a linkage in which some of the constrained points are subject to the no-slip condition. We begin with the velocity analysis. The contact points E on the gear and on the rack must



Example 4.7



Constrained points and geometry.

have the same velocity if there is no slippage between the bodies. Our approach for rolling bodies considers the center to be a constrained point, so we write

$$\bar{v}_E = \bar{v}_A + \bar{\omega}_A \times \bar{r}_{E/A} = \bar{v}_D + \bar{\omega}_{CD} \times \bar{r}_{E/D}.$$

We describe these vectors in terms of the global XYZ coordinate system in the sketch, which leads to

$$\bar{v}_A = -v_A \bar{I}, \qquad \bar{v}_D = u\bar{J}; \qquad \bar{\omega}_A = \frac{v_A}{R}\bar{K}, \qquad \bar{\omega}_{CD} = \dot{\theta}\bar{K};$$

$$\bar{r}_{E/A} = (R\cos\theta)\bar{I} + (R\sin\theta)\bar{J}, \qquad \bar{r}_{E/D} = (l\sin\theta)\bar{I} - (l\cos\theta)\bar{J},$$

where the expression for $\bar{\omega}_A$ stems from the requirement that there be no slippage at the stationary point B on the lower rack. We wish to express the results in terms of s and θ, so we use the horizontal distance from the origin to point E in order to eliminate l:

$$l = \frac{s + R\cos\theta}{\sin\theta} \quad\Rightarrow\quad \bar{r}_{E/D} = (s + R\cos\theta)[\bar{I} - (\cot\theta)\bar{J}].$$

We substitute $\bar{r}_{E/A}$ and $\bar{r}_{E/D}$ into the equation for $\bar{v}_E$, and equate like components, from which we obtain

$$\bar{v}_E \cdot \bar{I} = -v_A(1 + \sin\theta) = \dot{\theta}(s + R\cos\theta)\cot\theta,$$

$$\bar{v}_E \cdot \bar{J} = v_A \cos\theta = u + \dot{\theta}(s + R\cos\theta).$$

Solving these equations yields

$$v_A = u\,\frac{\cos\theta}{1+\sin\theta}, \; \dot{\theta} = -u\,\frac{\sin\theta}{s + R\cos\theta} \quad\Rightarrow\quad \bar{v}_A = -u\,\frac{\cos\theta}{1+\sin\theta}\bar{I}.$$

We lack an effective kinematical constraint relating the acceleration of point E on each body. We therefore determine the acceleration of the center of gear A by noting that its path is straight, with $\bar{e}_t = -\bar{I}$, and that the preceding result for its speed is generally valid. We differentiate v_A to determine the tangential acceleration, and then use the previous expression for $\dot{\theta}$ to obtain a result in terms of s and θ. We thereby obtain

$$\dot{v}_A = u\dot{\theta}\left[\frac{-(\sin\theta)(1+\sin\theta) - \cos^2\theta}{(1+\sin\theta)^2}\right] \equiv -u\dot{\theta}\frac{1}{1+\sin\theta},$$

$$\bar{a}_A = \dot{v}_A \bar{e}_t = -u^2\,\frac{\sin\theta}{(s + R\cos\theta)(1+\sin\theta)}\bar{I}.$$

Example 4.8 The shaft of disk A rotates about the vertical axis at the constant rate Ω as the disk rolls without slipping over the inner surface of the cylinder. Determine the angular velocity and angular acceleration of the disk and the acceleration of the point on the disk that contacts the cylinder.

Solution Let $\dot{\phi}$ denote the spin rate about the axis of the disk. The z axis of the body-fixed reference frame is defined in the sketch to coincide with this axis, so the angular velocity of the disk is

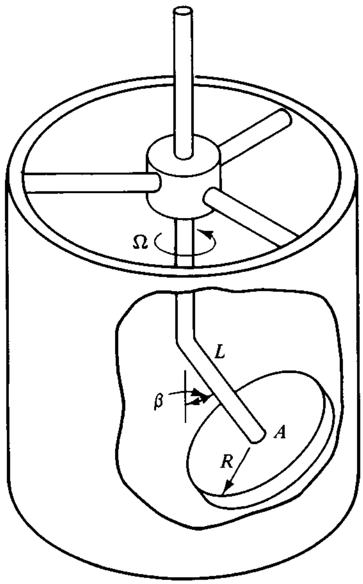
$$\bar{\omega} = -\Omega\bar{K} - \dot{\phi}\bar{k}.$$

Constancy of Ω leads to $\ddot{\phi} = 0$. Thus, the angular acceleration of the disk is

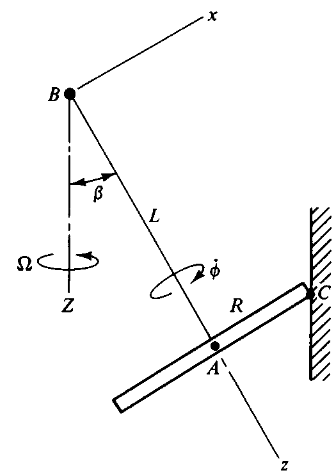
$$\bar{\alpha} = -\dot{\phi}(\bar{\omega} \times \bar{k}).$$

Resolving all vectors into components relative to xyz leads to

$$\bar{\omega} = (\Omega\sin\beta)\bar{i} - (\Omega\cos\beta + \dot{\phi})\bar{k}, \qquad \bar{\alpha} = (\Omega\dot{\phi}\sin\beta)\bar{j}.$$



Example 4.8



Coordinate system and constrained points.

It remains to relate $\dot{\phi}$ to Ω, for which we use the condition that there is no slipping at contact point C. Because point B occupies a fixed position relative to the disk, we could base the formulation on the fact that points B and C are two points on the disk that are at rest at this instant. For the sake of greater generality, we shall instead relate the velocity of point C to the center A, which follows a horizontal circle of radius $L \sin\beta$ with the radial line rotating at angular rate Ω. Hence,

$$\bar{v}_A = -(\Omega L \sin\beta)\bar{j}.$$

Before presenting the correct analysis, we should address an error that is extremely common for novices. It is tempting to argue that, because $\bar{v}_C = \bar{0}$, the speed of point A must be $R\dot{\phi}$. This is incorrect because it ignores the fact that the precession also is responsible for rotation about the z axis.

The correct procedure relates the velocity of points A and C, which are on the same body. It follows that

$$\bar{v}_A = \bar{v}_C + \bar{\omega} \times \bar{r}_{A/C} = \bar{0} + \bar{\omega} \times (-R\bar{i}),$$

$$-(\Omega L \sin\beta)\bar{j} = (\Omega \cos\beta + \dot{\phi})R\bar{j},$$

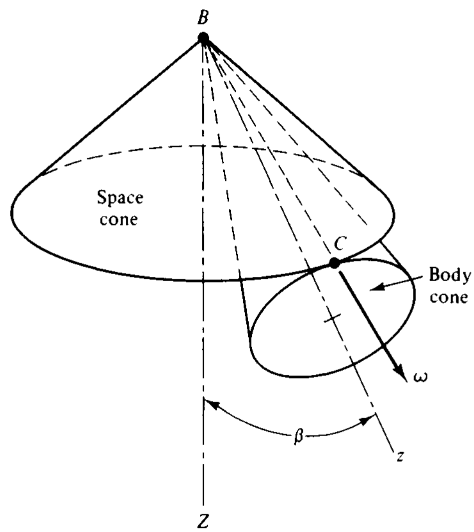
from which we obtain

$$\dot{\phi} = -\Omega\left(\frac{L}{R}\sin\beta + \cos\beta\right).$$

When we use this rotation rate to form the angular motion vectors, we find

$$\bar{\omega} = (\Omega \sin\beta)\left(\bar{i} + \frac{L}{R}\bar{k}\right)$$

$$\bar{\alpha} = -\Omega^2\left(\frac{L}{R}\sin^2\beta + \sin\beta\cos\beta\right)\bar{j}.$$



Space and body cones.

We already identified $\bar{v}_C = \bar{0}$. The acceleration of the contact point is obtained from the relative motion relation that refers this point to the center A. The motion of point A was noted previously to be a circular path in the horizontal plane, with Ω being the rate at which the radial line rotates;

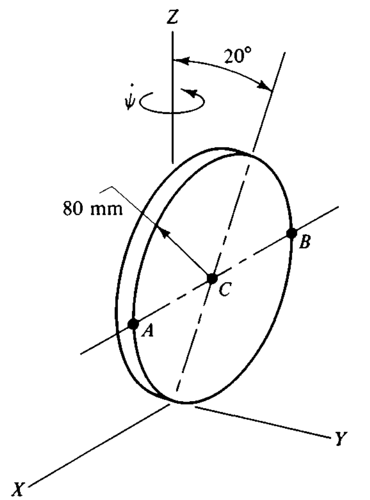
$$\bar{a}_A = \Omega^2 (L \sin\beta)\bar{e}_n = (L\Omega^2 \sin\beta)[-(\cos\beta)\bar{i} - (\sin\beta)\bar{j}],$$

from which we obtain

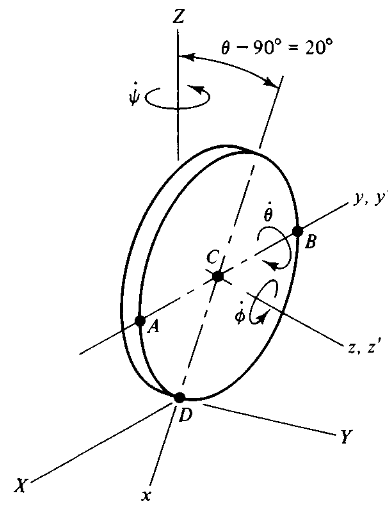
$$\bar{a}_C = \bar{a}_A + \bar{\alpha} \times \bar{r}_{C/A} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{C/A})$$
$$= -\Omega^2 \left(\frac{L}{R} \sin^2\beta + \sin\beta \cos\beta \right)(L\bar{i} - R\bar{k}).$$

In order to interpret these results, we recall that $\bar{r}_{C/B} = R\bar{i} + L\bar{k}$, which leads to the observation that $\bar{\omega} \times \bar{r}_{C/B} = \bar{0}$ and $\bar{a}_C \cdot \bar{r}_{C/B} = 0$. In other words, $\bar{\omega}$ is parallel to $\bar{r}_{C/B}$, while $\bar{a}_C$ is perpendicular to that direction. Both features have a straightforward explanation. First, note that the locus of lines connecting pivot B and contact point C is a cone; this is called the *space cone*. On the other hand, the line connecting point B to a specific point on the rim of the disk lies on a cone relative to the xyz reference frame, which is fixed to the disk; this is the *body cone*. The last sketch depicts both cones. The motion of the disk is equivalent to the body cone rolling without slipping over the space cone. The instantaneous axis of rotation is the line of contact between the cones. The acceleration of any point on the body cone that is on this line of contact is normal to the rotation axis. The concept of space and body cones is particularly useful for the treatment of the rotation of bodies in free flight, which we examine in Chapter 8.

Example 4.9 A disk rolls without slipping on the X-Y plane. At the instant shown, the horizontal diameter ACB is parallel to the X axis. Also at this instant, the horizontal components of the velocity of the center C are known to be 5 m/s in the X



Example 4.9



Coordinate systems and kinematical parameters.

direction and 3 m/s in the Y direction, while the Y component of the velocity of point B is 6 m/s. Determine the precession, nutation, and spin rates for the Eulerian angles in Figure 4.18.

Solution For convenience in describing the given components of $\bar{v}_B$ and $\bar{v}_C$, we define the reference frame $x'y'z'$, whose y' axis is the line of nodes and whose z' axis is the axis of the disk. The body-fixed xyz reference frame spins relative to $x'y'z'$, so the coincidence of the reference frames is an instantaneous occurrence.

Because we have labeled the axes for the Eulerian angles consistently with Figure 4.18, we may employ Eqs. (4.40) directly. Note that those expressions use components relative to $x'y'z'$, which is especially suitable here because the y' axis is parallel to the X-Y plane. We defined θ to be the angle between the Z and z axes, so we set $\theta = 110°$ for the instant of interest, which yields

$$\bar{\omega} = -0.9397\dot{\psi}\bar{i}' + \dot{\theta}\bar{j}' + (\dot{\phi} - 0.3420\dot{\psi})\bar{k}'.$$

We set $\bar{v}_D = \bar{0}$ because there is no slipping at the contact point. We refer the velocities of points B and C to this point, which leads to

$$\bar{v}_C = \bar{\omega} \times (-0.08i') = -0.08(\dot{\phi} - 0.3420\dot{\psi})\bar{j}' + 0.08\dot{\theta}\bar{k}',$$

$$\begin{aligned}\bar{v}_B &= \bar{\omega} \times (-0.08\bar{i}' + 0.08\bar{j}') \\ &= 0.08(\dot{\phi} - 0.3420\dot{\psi})(-\bar{i}' - \bar{j}') + 0.08(\dot{\theta} - 0.9397\dot{\psi})\bar{k}'.\end{aligned}$$

These velocities must match the given components. The fact that $\bar{j}' = -\bar{I}$ at this instant substantially expedites the evaluation of dot products, which we find to be

$$\begin{aligned}\bar{v}_C \cdot \bar{I} = 5 &= 0.08[-(\dot{\phi} - 0.3420\dot{\psi})\bar{j}' \cdot \bar{I} + \dot{\theta}\bar{k}' \cdot \bar{I}] \\ &= 0.08(\dot{\phi} - 0.3420\dot{\psi}),\end{aligned}$$

$$\bar{v}_C \cdot \bar{J} = 3 = 0.08[-(\dot{\phi} - 0.3420\dot{\psi})\bar{j}' \cdot \bar{J} + \dot{\theta}\bar{k}' \cdot \bar{J}] = 0.08\dot{\theta}\cos 20°,$$

$$\bar{v}_B \cdot \bar{J} = 6 = 0.08[-(\dot{\phi} - 0.3420\dot{\psi})(\bar{i}' \cdot \bar{J} + \bar{j}' \cdot \bar{J}) + (\dot{\theta} - 0.9397\dot{\psi})\bar{k}' \cdot \bar{J}]$$
$$= 0.08[-(\dot{\phi} - 0.3420\dot{\psi})(-\sin 20°) + (\dot{\theta} - 0.9397\dot{\psi})(\cos 20°)].$$

The solution of these simultaneous equations is

$$\dot{\psi} = -18.260, \ \dot{\theta} = 39.907, \ \dot{\phi} = 56.255 \text{ rad/s}.$$

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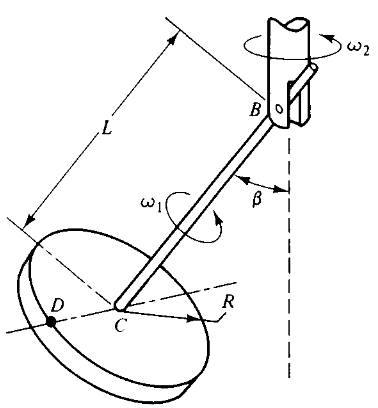
Problems

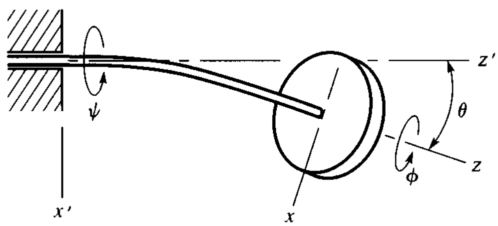
4.1 A gyropendulum consists of a flywheel that rotates at constant angular speed ω_1 relative to shaft BC. This shaft is pinned to the vertical shaft, which rotates at constant angular speed ω_2. The angle β measuring the inclination of shaft BC is an arbitrary function of time. Use the Eulerian angle formulas for angular velocity and angular acceleration, Eqs. (4.16) and (4.17), to derive expressions for the velocity and acceleration of point D, which coincides with the horizontal diameter at the instant of interest.



Problem 4.1

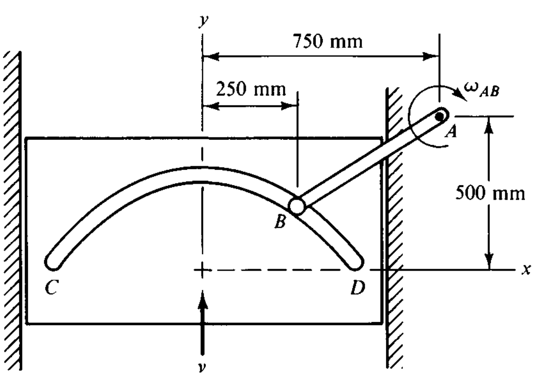
4.2 Consider a body whose orientation is described by Eulerian angles. Derive the transformation from space-fixed to body-fixed axes for a sequence beginning with precession $\psi = 20°$, followed by nutation $\theta = -60°$, then spin $\phi = 140°$. Is it possible to obtain the same transformation with a different sequence beginning with nutation θ', followed by spin ϕ', then precession ψ'? If so, determine the values of θ', ϕ', and ψ'.

4.3 A rigid disk is welded to the end of a flexible shaft that rotates about bearing A. The bending deformation of the shaft is such that its centerline forms a curve in a plane that always contains the bearing's axis. The rotation of this plane about the bearing's axis is the precession ψ. The tangent to this curve at end B is the axis of symmetry of the disk, and the angle between the bearing's axis and the disk's axis is the nutation angle θ. Torsional deformation of the shaft produces a spin ϕ about the disk's axis. Let xyz be a set of axes attached to the disk, and let $x'y'z'$ be a set of axes that undergo only the precessional motion. The orientation of $x'y'z'$ is such that the curved centerline of the shaft is always situated in the $x'z'$ plane, with z' being the bearing axis. It is observed that at some instant $\theta = 10°$, $\phi = -5°$, and the angular velocity of the disk is $\bar{\omega} = 17\bar{i}' - 20\bar{j}' + 48\bar{k}'$ rad/s. Determine the corresponding precession, nutation, and spin rates. Then express the angular velocity in terms of components relative to xyz.

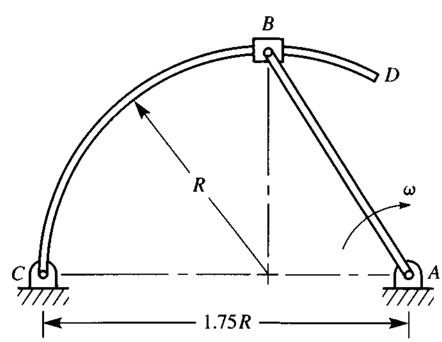


Problem 4.3

4.4 Pin B slides through groove CD in a plate that translates upward at speed v. The groove forms the parabolic curve $y = 300 - x^2/400$, where x and y have units of millimeters. In the position shown, bar AB is rotating clockwise at 40 rad/s, and that rate is decreasing at 160 rad/s^2. Determine the corresponding values of v and $\dot{v}$.



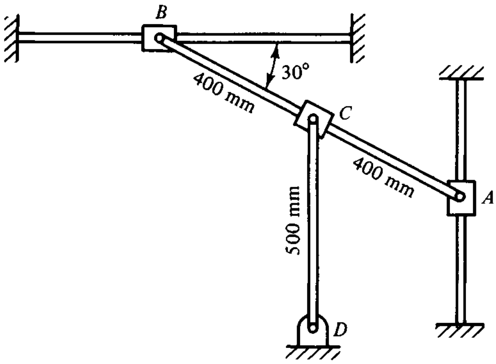
Problem 4.4



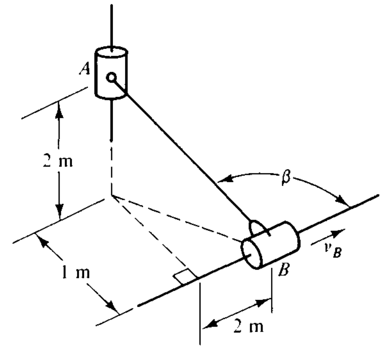
Problem 4.5

4.5 Bar AB rotates at the constant rate ω, causing collar B to slide over curved bar CD. Determine the angular velocity and angular acceleration of CD in the position shown.

4.6 Collar C slides over bar AB. When the system is in the position shown, slider A is moving downward at 600 mm/s and its speed is decreasing at 15 m/s^2. Determine the corresponding angular velocity and angular acceleration of each bar.



Problem 4.6

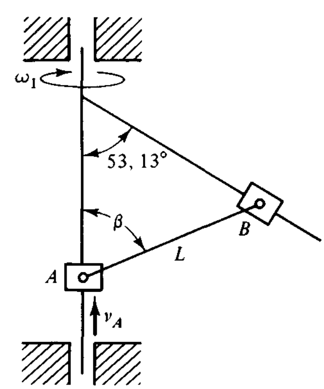


Problems 4.7 and 4.8

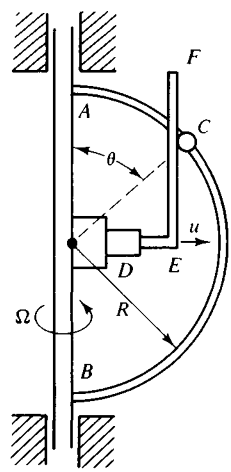
4.7 Collar A is connected to bar AB by a ball-and-socket joint, whereas the connection between collar B and bar AB is a forked pin. For the position shown, $v_B = 3$ m/s. Determine the velocity of slider A and the value of $\dot{\beta}$, where β is the angle between AB and the horizontal guide.

4.8 Determine the angular acceleration of bar AB in the linkage in Problem 4.7 for the case where $v_B = 5$ m/s and $\dot{v}_B = -20$ m/s^2 in the given position.

4.9 Collar A is pushed upward at $v_A = 30$ m/s, while the entire system precesses about the vertical axis at 2,400 rev/min. Determine the velocity of the midpoint of bar AB in the position where $\beta = 53.13°$. The length of the bar is $L = 600$ mm.



Problems 4.9 and 4.10

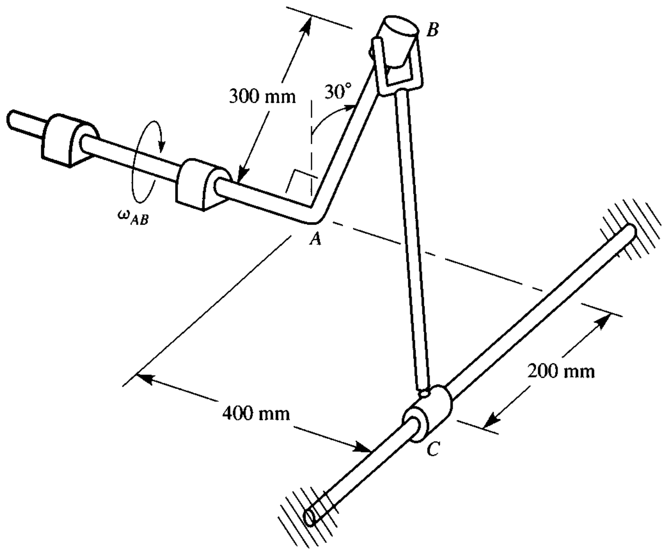


Problem 4.11

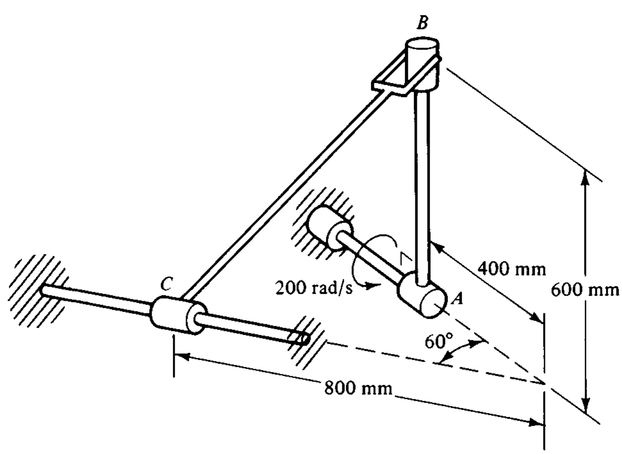
4.10 Collar A is pushed upward at constant speed v_A, while the entire system precesses about the vertical axis at ω_1. Determine the angular velocity and angular acceleration of bar AB in the position where $\beta = 90°$.

4.11 Bead C slides relative to the curved guide bar AB, which rotates about the vertical axis at the constant rate Ω. The movement of the slider is actuated by arm DEF, which pushes the collar outward from the vertical axis at a constant rate u. Determine the velocity and acceleration of the slider as a function of θ.

4.12 and 4.13 Bar BC is attached by a fork-and-clevis joint to cap B, which is free to rotate about the axis of bar AB. The connection between bar BC and collar C is a ball-and-socket joint. The guide bar for collar C is horizontal, as is the fixed shaft about which bar AB rotates. The rotation rate is $\omega_{AB} = 200$ rad/s, which is constant. Determine the angular velocity and angular acceleration of bar BC, and the velocity and acceleration of collar C in the position shown.

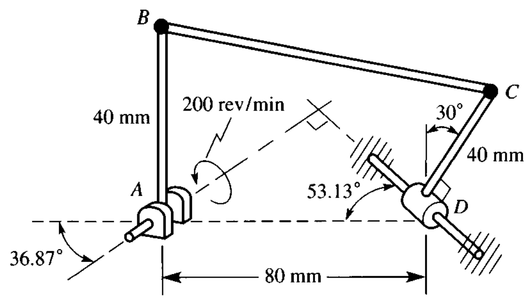


Problem 4.12



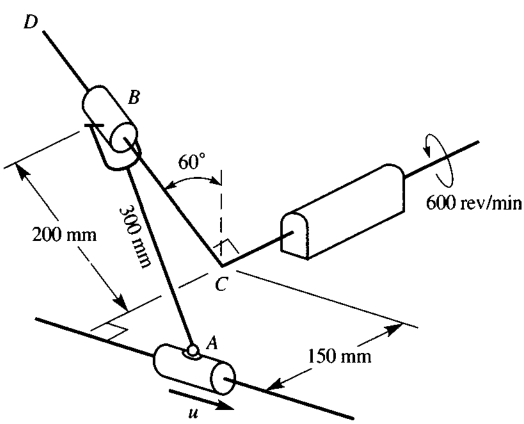
Problem 4.13

4.14 The axes of bearings A and D lie in the same horizontal plane, and intersect orthogonally. Bar AB rotates at the constant rate of 200 rev/min. Connections B and C are ball-and-socket joints. Determine the velocity and acceleration of joint C at the instant shown.

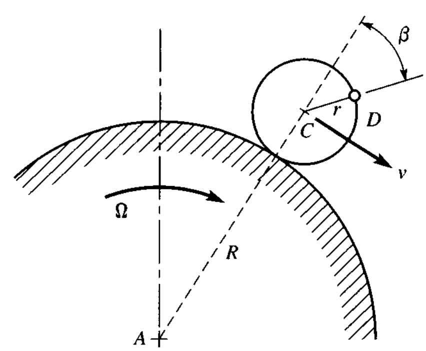


Problem 4.14

4.15 Bar AB is attached to collar A by a ball-and-socket joint, while end B is fastened to the collar by a fork-and-clevis joint. The cross-section of bar CD is circular. Bar CD rotates about the horizontal axis at a constant rate of 600 rpm. Let u denote the speed of collar A in the position shown, which is increasing at the rate $\dot{u}$. Determine in terms of u and $\dot{u}$ the angular velocity and angular acceleration of bar AB in this position.



Problem 4.15

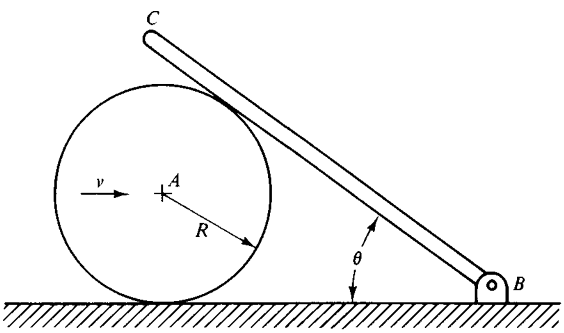


Problem 4.16

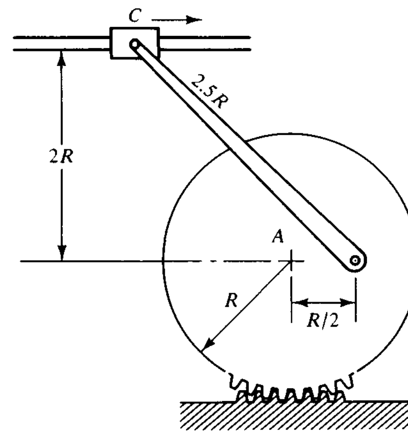
4.16 A disk rolls without slipping over the exterior of a large drum. The drum rotates clockwise at constant angular speed Ω. In the position shown, the center of the disk has a speed v, which is increasing at the rate $\dot{v}$. Derive expressions for the velocity and acceleration of point D, which is situated at an arbitrary angle β relative to the line of centers.

4.17 (See figure, next page.) In the position shown, cylinder A is moving to the right such that its center has a speed v. There is no slipping between the cylinder and bar BC, but there is slipping between the cylinder and the ground. Determine the angular velocity and angular acceleration of bar BC, and the velocity and acceleration of the cylinder at the point where it contacts the ground.

4.18 (See figure, next page.) Collar C has a constant speed v to the right, and the rack is stationary. Determine the angular velocity and angular acceleration of gear A at the instant shown.

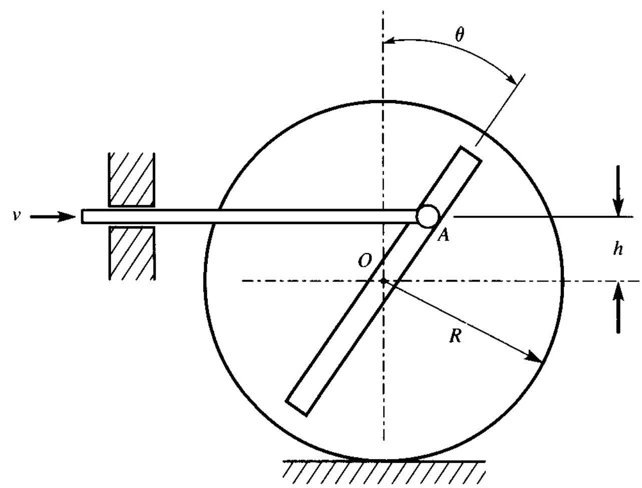


Problem 4.17



Problem 4.18

4.19 Movement of the actuating rod at constant speed v pushes the connecting pin A through the groove in the gear, thereby causing the gear to roll over the rack. Determine the angular velocity and angular acceleration of the gear as a function of θ.

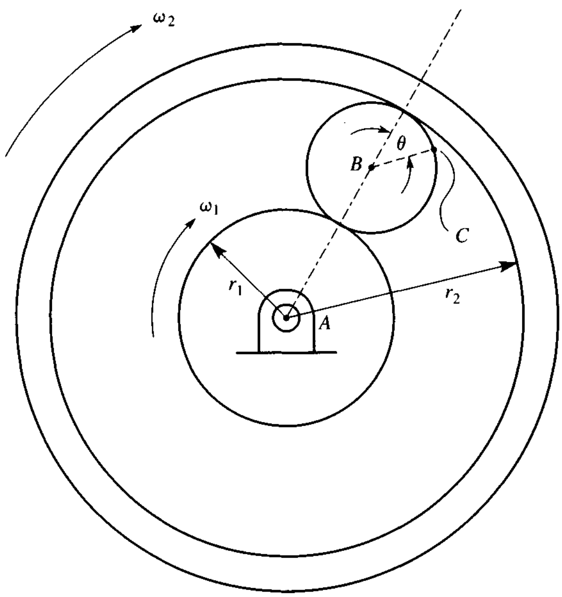


Problem 4.19

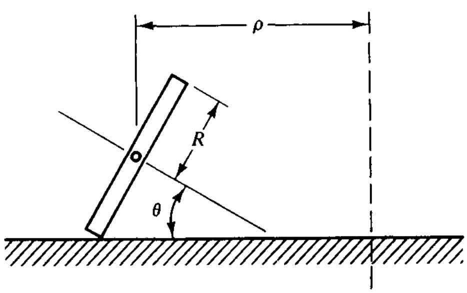
4.20 The angular velocities of the inner and outer gears are counterclockwise at the constant values ω_1 and ω_2, respectively. Determine the velocity and acceleration of point C on the perimeter of the planetary gear as a function of the angle θ locating the instantaneous position of point C relative to the radial line.

4.21 A disk rolls without slipping over the ground such that the angle of tilt θ is constant. The center follows a horizontal circular path of radius ρ at constant speed v. Derive an expression for the angular velocity and angular acceleration of the disk.

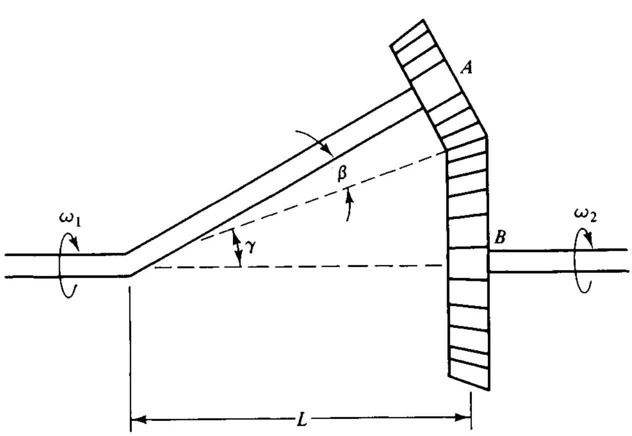
4.22 Gear A rotates freely about its shaft, which rotates at constant rate ω_1 about the horizontal axis. Gear B is stationary ($\omega_2 = 0$). Determine the angular velocity and angular acceleration of gear A.



Problem 4.20



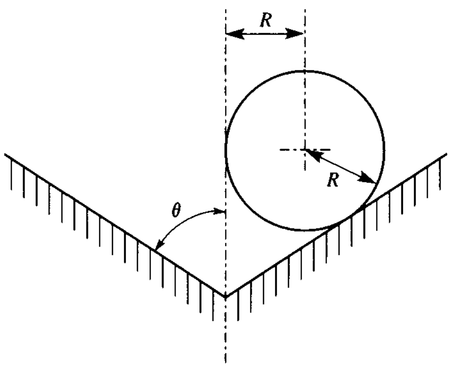
Problem 4.21



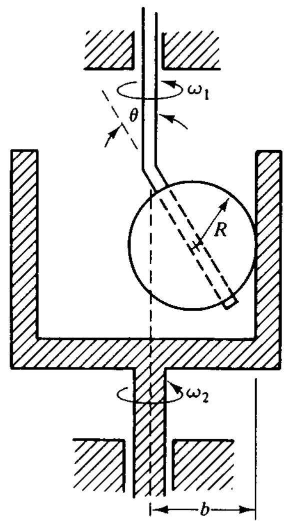
Problems 4.22 and 4.23

4.23 Gear A is free to spin about its shaft, which rotates at variable rate ω_1 about the horizontal axis. The angular speed of gear B is the variable rate ω_2. Determine the angular velocity and angular acceleration of gear A.

4.24 A sphere of radius R rolls without slipping in the interior of a cone such that the distance from the axis of the cone to the center of the sphere is constant at R. The speed of the center of the sphere is the constant value v. Derive expressions for the angular velocity and angular acceleration of the sphere in terms of v, R, and the apex angle θ.



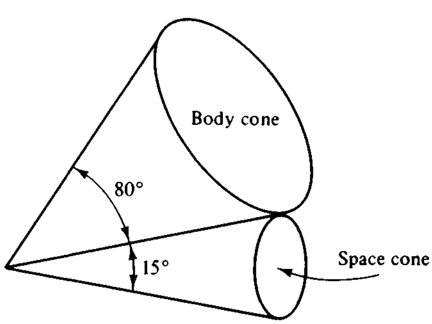
Problem 4.24



Problem 4.25

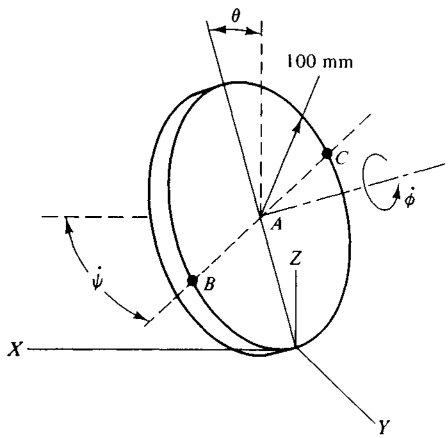
4.25 The sphere rolls without slipping over the interior wall of a hollow cylinder that rotates about its axis at ω_2. The angular speed of the vertical shaft driving the sphere is ω_1. Both rotation rates are constant. Determine the angular velocity and angular acceleration of the sphere.

4.26 The body cone executes three revolutions about the stationary space cone in a 1-s interval. Determine the angular velocity and angular acceleration of the body cone.



Problem 4.26

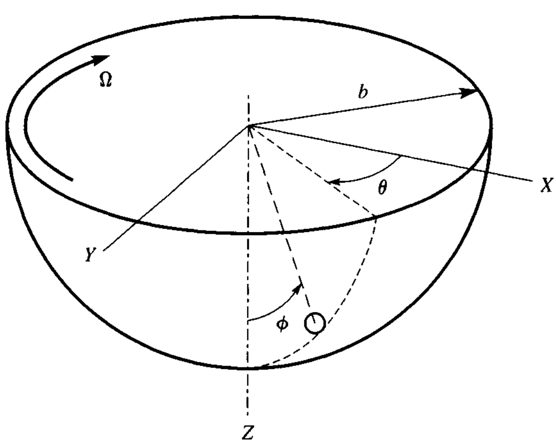
4.27 The disk rolls without slipping over the horizontal X-Y plane. At the instant when $\theta = 36.87°$, the X and Y components of the velocity of point B on the horizontal diameter of the disk are 8 m/s and −4 m/s, respectively. The X and Y components of the velocity of center A at this instant are 4 m/s and 2 m/s. Determine the precession angle ψ between the horizontal diameter BAC and the X axis, and also evaluate the precession, nutation, and spin rates.



Problems 4.27 and 4.28

4.28 The disk is rolling without slipping. At the instant when the angle of inclination $\theta = 30°$, the disk is observed to be spinning at $\dot{\phi} = 5$ rad/s. At this instant, the speed of points B and C on its horizontal diameter are 1 and 2 m/s, respectively. Determine the corresponding precession and nutation rates.

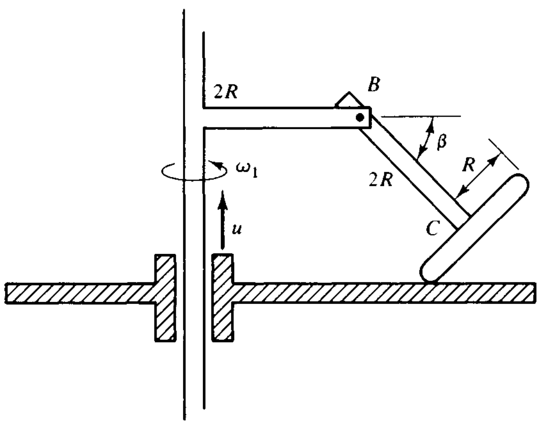
4.29 A sphere of radius R rolls without slipping over the interior of a hemispherical shell of radius b that rotates about the vertical axis at constant rate Ω. (The system is analogous to a ball in a roulette wheel.) The polar and azimuth angles locating the center of the sphere are (respectively) ϕ and θ, defined with respect to the fixed XYZ



Problem 4.29

coordinate system. Both angles are arbitrary functions of time. Derive expressions for the angular velocity and angular acceleration of the sphere.

4.30 Shaft BC is pinned to the T-bar, which rotates at the constant angular speed ω_1. Wheel C rotates freely relative to shaft BC. The platform, over which wheel C rolls, is raised at the constant speed u, causing angle β to decrease. The wheel does not slip relative to the platform in the direction transverse to the diagram, but slipping in the radial direction is observed to occur. Derive expressions for the angular velocity and the angular acceleration of the wheel.



Problem 4.30

CHAPTER 5

Newtonian Kinetics of a Rigid Body

Chasle's theorem states that the general motion of a rigid body can be represented as a superposition of a translation following any point in the body and a pure rotation about that point. Kinematics studies are concerned only with the description of that motion. The developments in this chapter will disclose how the motion is related to the force system acting on the body. The resultant force may be regarded intuitively as the net tendency of the force system to push a body, so it may be expected to be related to the translational effect. Similarly, the resultant moment may be considered to be the net rotational effect. We shall confirm and quantify these expectations in the following presentation for general spatial motion, and then specialize the derived principles for the case of planar motion.

5.1 Fundamental Principles

Newton's laws govern the motion of a particle. A rigid body may be treated as a collection of particles whose motions are not independent. In the first part of this chapter, we shall derive the basic kinetics principles for rigid-body motion. The foundation for these derivations is Newton's second law, which describes inertial effects, and the third law, which describes the nature of the force system.

5.1.1 *Basic Model*

From a philosophical perspective, we initially recognize the atomic nature of matter by considering a body to consist of N particles having mass m_i. However, the enormous value of N and the correspondingly small value of m_i associated with an atomic representation ultimately will lead us to quantify the general relations by employing a continuum model, in which the mass m_i becomes an infinitesimal mass dm. In the most general situation, there are no kinematical relations between the motion of the particles forming the system of interest. (Such would be the case if we were to consider a gaseous medium.) We begin by developing kinetics principles for such a system. Then we account for the kinematical relations associated with a rigid-body model, in which the particles are constrained to maintain a fixed relative position. If one wished to model deformation effects, this kinematical specification would be replaced by a constitutive law, such as Hooke's law for elasticity, that relates the stresses to the strains. In the rigid-body model, the stress resultants are constraint forces exerted internally within the system.

Figure 5.1 depicts three particles out of the full set, and two types of force. The internal force exerted on particle i by particle j is denoted $\bar{f}_{ij}$. Such forces are associated with contact with neighboring particles, as well as with gravitational and other body forces exerted between the various particles in the system. The force $\bar{F}_i$ arises from the action of other objects on the ith particle. Contributors to $\bar{F}_i$ may be body

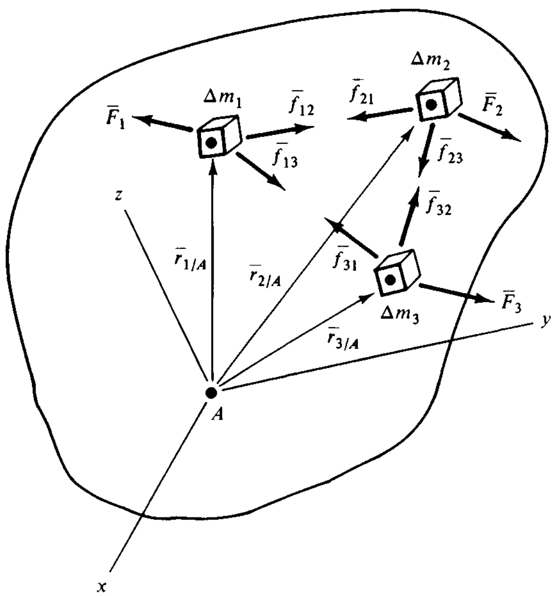


Figure 5.1 Infinitesimal mass elements.

forces, applied loads, or constraint forces (that is, reactions) that result from the manner in which the system is supported.

A key aspect of the internal forces arises from Newton's third law (action and reaction). According to that law, each pair of forces $\bar{f}_{ij}$ and $\bar{f}_{ji}$ $(i \neq j)$ is equal in magnitude but oppositely directed, so

$$\bar{f}_{ij} + \bar{f}_{ji} = \bar{0}. \tag{5.1}$$

In addition, the two forces forming each interaction share a common line of action, which is the line connecting particles i and j. When either force is moved along its line of action to the other particle for the purpose of computing the moment about an arbitrary point A, it becomes apparent that the two forces cancel each other in the moment sum. Thus

$$\bar{r}_{i/A} \times \bar{f}_{ij} + \bar{r}_{j/A} \times \bar{f}_{ji} = \bar{0}. \tag{5.2}$$

As a result of Eqs. (5.1) and (5.2), the internal forces will not appear explicitly in the equations of motion. Nevertheless, they are important because of their role in enforcing the rigidity of the body.

5.1.2 *Resultant Force and Point Motion*

The resultant of all forces acting on any mass particle must satisfy Newton's second law, which for particle i is

$$\bar{F}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{f}_{ij} = m_i \bar{a}_i, \tag{5.3}$$

where N is the number of particles constituting the system. Now add Eq. (5.3) for each particle by summing over i:

$$\sum_{i=1}^{N} \bar{F}_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{f}_{ij} = \sum_{i=1}^{N} m_i \bar{a}_i. \tag{5.4}$$

Consider a specific set of values for each i, j pair in the double summation above, such as $i, j = 1, 2$. The corresponding contribution is $\bar{f}_{12} + \bar{f}_{21}$. By virtue of Eq. (5.1), these two terms cancel. The same result applies to every other pair, so the force sum may be rewritten as

$$\sum_{i=1}^{N} \bar{F}_i = \sum_{i=1}^{N} m_i \bar{a}_i. \tag{5.5}$$

The left side is identifiable as the sum of the external forces exerted on the system by other objects. We shall now express the term on the right side in a more meaningful form.

First, we write the acceleration of each particle as the second derivative of the position. That each particle has constant mass makes it possible to sum prior to differentiation, with the result that

$$\sum_{i=1}^{N} \bar{F}_i = \frac{d^2}{dt^2}\left(\sum_{i=1}^{N} m_i \bar{r}_{i/O}\right). \tag{5.6}$$

The key observation about this form is that the right side is related to the position of the center of mass. Let the coordinates of each particle relative to the center of mass be (X_i, Y_i, Z_i), and let the coordinates of the center of mass G be (X_G, Y_G, Z_G). The mass is the sum of the individual masses,

$$m = \sum_{i=1}^{N} m_i. \tag{5.7}$$

In order to locate the center of mass G, the moment of the distributed gravitational force system is equated to the moment of the total weight acting at point G. That evaluation leads to the *first moments of mass* with respect to each coordinate, which are

$$mX_G = \sum_{i=1}^{N} m_i X_i, \qquad mY_G = \sum_{i=1}^{N} m_i Y_i, \qquad mZ_G = \sum_{i=1}^{N} m_i Z_i. \tag{5.8}$$

When we multiply these equations by $\bar{i}$, $\bar{j}$, and $\bar{k}$ (respectively) and add them, we obtain the first moment of mass about the origin:

♦ $$m\bar{r}_{G/O} = \sum_{i=1}^{N} m_i \bar{r}_{i/O}. \tag{5.9}$$

This expression allows us to replace the summation on the right side of Eq. (5.6) with a single term involving the center of mass. The mass is constant, so the specified time derivatives in that equation lead to

♦ $$\sum \bar{F} = m\bar{a}_G, \tag{5.10}$$

where $\sum \bar{F}$ denotes the resultant (i.e. sum) of the external forces.

Note that Eq. (5.10) is valid for any collection of particles. If the particles move independently, it is one of many equations of motion for the various particles. The primary value of this relation lies in its application for the collection of particles forming a rigid body. Chasle's theorem indicates that a complete description of the motion of a rigid body entails specification of the motion of one point in the body, and of the rotation of the body. For a kinematical study, the choice for a point is arbitrary, but it will always be the center of mass when the resultant force is to be considered.

The similarity of Eq. (5.10) to Newton's second law, which treats only a particle, is important. It shows that modeling an object as a particle is equivalent to focusing attention on the motion of the center of mass of that object. In contrast, a rigid-body model is needed to evaluate the rotational effects. The next task is to identify a relation between the moment exerted by the force system and the corresponding rotational motion.

5.1.3 *Resultant Moment and Rotation*

As we did for the evaluation of the resultant force acting on a rigid body, we begin the treatment of rotational effects by considering a general collection of particles. The resultant moment of the forces exerted on a group of particles is obtained by summing the contribution from each particle. A typical situation, in which moments about arbitrary point A will be evaluated, is depicted in Figure 5.2. In view of Newton's second law, the moment of the forces acting on particle i about this point must equal the corresponding moment of the $m_i \bar{a}_i$ vector; that is,

$$(\bar{M}_A)_i = \bar{r}_{i/A} \times \left[\bar{F}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{f}_{ij} \right] = \bar{r}_{i/A} \times m_i \bar{a}_i. \tag{5.11}$$

It is logical to employ point A as the reference point for the kinematical description because the moment is computed about that point. The most general description of acceleration for a collection of particles moving independently is $\bar{a}_i = \bar{a}_A + \bar{a}_{i/A}$, where the last term represents the acceleration of particle i relative to point A, as seen from the inertial reference frame:

$$\bar{a}_{i/A} = \frac{d}{dt} \bar{v}_{i/A} = \frac{d^2}{dt^2} \bar{r}_{i/A}. \tag{5.12}$$

We substitute $\bar{a}_i = \bar{a}_A + (d/dt)\bar{v}_{i/A}$ into Eq. (5.11) and invoke the identity for a derivative of a product; the result is

$$\begin{aligned}(\bar{M}_A)_i &= \bar{r}_{i/A} \times m_i \bar{a}_A + \bar{r}_{i/A} \times m_i \frac{d}{dt} \bar{v}_{i/A} \\ &= m_i \bar{r}_{i/A} \times \bar{a}_A + \frac{d}{dt} (\bar{r}_{i/A} \times m_i \bar{v}_{i/A}),\end{aligned} \tag{5.13}$$

where the last form results because $[(d/dt)\bar{r}_{i/A} \times m_i \bar{v}_{i/A}] \equiv \bar{v}_{i/A} \times m_i \bar{v}_{i/A} = \bar{0}$.

The term $m_i \bar{v}_{i/A}$ represents the momentum of particle i relative to point A. Hence, the term in Eq. (5.13) contained in parentheses is the moment of the relative momentum or, more simply, the *angular momentum* about point A. We now sum this equation for each particle, and recall from Eq. (5.2) that the moments of the internal forces cancel in such a process. This leads to

$$\bar{M}_A = \sum_{i=1}^{N} \bar{r}_{i/A} \times \bar{F}_i = \sum_{i=1}^{N} m_i \bar{r}_{i/A} \times \bar{a}_A + \sum_{i=1}^{N} \frac{d}{dt} (\bar{r}_{i/A} \times m_i \bar{v}_{i/A}). \tag{5.14}$$

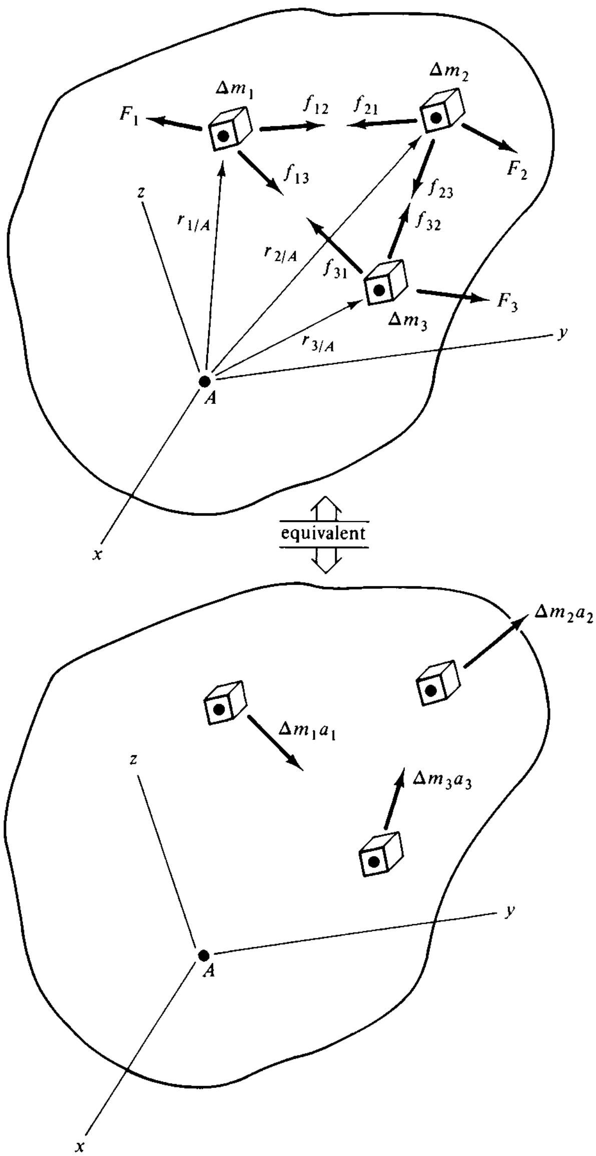


Figure 5.2 Forces acting on the mass elements of a rigid body.

The last sum is the total angular momentum of the system about point A, and the second sum is the first moment of mass about point A. Consequently, the moment equation reduces to

♦ $$\bar{M}_A = \sum_{i=1}^{N} \bar{r}_{i/A} \times \bar{F}_i = m\bar{r}_{G/A} \times \bar{a}_A + \dot{\bar{H}}_A, \tag{5.15}$$

where $\bar{H}_A$ denotes the angular momentum of the system about point A,

♦ $$\bar{H}_A = \sum_{i=1}^{N} (\bar{r}_{i/A} \times m_i \bar{v}_{i/A}). \tag{5.16}$$

In the special case where point A is fixed in the inertial reference frame, these equations reduce to the sum of the angular impulse-momentum equations (1.26) and (1.27) for each particle relative to an inertial reference point.

As was true for the resultant-force equation, the resultant moment exerted on a system of particles yields one of many equations required to describe the motion of a group of independently moving particles. By contrast, for the special case where the particles constitute a rigid body, Eq. (5.15) is sufficient by itself to describe the rotation. We begin the specialization by introducing the first restriction on the choice of point A. Requiring this point to be a point in the body† enables us to employ the kinematical relationship between the velocity of two points in a rigid body. Because $\bar{v}_{i/A} = \bar{v}_i - \bar{v}_A = \bar{\omega} \times \bar{r}_{i/A}$, the angular momentum becomes

$$\blacklozenge \qquad \bar{H}_A = \sum_{i=1}^{N} m_i [\bar{r}_{i/A} \times (\bar{\omega} \times \bar{r}_{i/A})]. \tag{5.17}$$

It is clear from this expression that the resultant moment affects the rotation of a body. However, Eq. (5.15) also contains the linear acceleration of point A. This represents a coupling of the linear and angular motion of the body. We may obtain a moment equation that depends solely on the rotation of the body by further restricting the choice of point A. If point A is selected such that $\bar{r}_{G/A} \times \bar{a}_A = \bar{0}$, then

$$\blacklozenge \qquad \sum \bar{M}_A = \dot{\bar{H}}_A. \tag{5.18}$$

The condition $\bar{r}_{G/A} \times m\bar{a}_A = \bar{0}$, required for validity of Eq. (5.18), is obtained when point A satisfies one of the following criteria.

(1) Point A is the center of mass G: $\bar{r}_{G/A} = \bar{0}$. The center of mass is always an admissible point for summing moments.
(2) Point A has no acceleration: $\bar{a}_A = \bar{0}$. This situation could arise if there is a point on the body that is constrained to follow a straight path at a constant speed, but such cases are comparatively rare. The more usual situation arises when a body is in pure rotation about point A.
(3) Point A is accelerating directly toward or away from the center of mass. In this case, $\bar{a}_A$ is parallel to $\bar{r}_{G/A}$, so their cross product vanishes.

Very few types of motion fit the third criterion. One exception arises in planar motion when a disk or sphere rolls without slipping. Even then, the body must be balanced - that is, the center of mass and the geometric centroid must coincide. The acceleration of a disk rolling in a plane is depicted in Figure 5.3. The center C is accelerating, but the no-slip condition requires that the acceleration of the contact point A be normal to the surface. Because $\bar{a}_A$ is directed toward the center C, it would be permissible to formulate a moment sum about the contact point, provided that point C is the center of mass. However, if the wheel is unbalanced, so that the center of mass G is eccentric, then $\bar{a}_A$ will usually not be parallel to $\bar{r}_{G/A}$.

Because of its lack of generality, we will not consider the third criterion in selecting a point for summing moments. In contrast, there are strong reasons for selecting

† We shall develop principles governing angular momentum only relative to a reference point on the body. Related principles - governing angular momentum of a system of particles relative to a point that is fixed in the inertial reference frame - can also be developed. However, they are less suitable for the study of rigid-body motion.

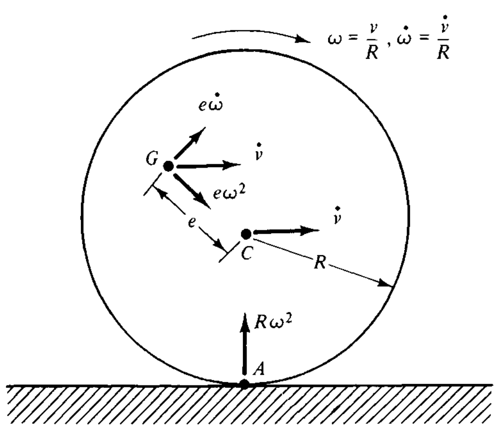


Figure 5.3 Accelerations in an eccentric disk.

the fixed point for a body in pure rotation, in accord with the second criterion. In order for a point to be held stationary, reaction forces must be exerted at that point. This is exemplified by a ball-and-socket joint, which exerts an arbitrary reaction force having three components. These unknown reactions do not contribute to a moment sum about the stationary point. The ability to eliminate unknown reactions from the moment equation for a body in pure rotation makes it worthwhile to formulate moments about the stationary point. Note that this is the only case where the point for the moment sum is selected on the basis of eliminating reactions. In static systems, of course, all points are stationary, so any point is permissible for the moment sum.

Because the angular momentum $\bar{H}_A$ in Eq. (5.17) is a function of $\bar{\omega}$, the moment equation (5.18) is the principle we need for studying the angular motion of a rigid body. Its form is analogous to Eq. (5.10) for the motion of the center of mass. The total linear momentum of the system is $\bar{P} = m\bar{v}_G$, as may be seen by differentiating Eq. (5.9) with respect to time. Then the equation governing the motion of the center of mass may be rewritten as

$$\blacklozenge \qquad \sum \bar{F} = \dot{\bar{P}}, \qquad \bar{P} = m\bar{v}_G = \sum_{i=1}^{N} m_i \bar{v}_i. \tag{5.19}$$

In other words, the linear or angular effect of the external force system equals the rate of change of the corresponding type of momentum for the body.

5.1.4 *Kinetic Energy*

Linear and angular momentum are fundamental kinetic properties associated with the motion of a body. Another such quantity is the kinetic energy. In addition to its appearance in the work-energy principle, kinetic energy will play a prominent role for describing inertia properties when we quantify the equation of rotational motion. It also is the primary quantity for the analytical mechanics developed in Chapter 6.

We begin by describing the kinetic energy of a system of independently moving particles. Because kinetic energy is a scalar, we obtain the total energy T of the system by adding the values for each particle:

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2} \sum_{i=1}^{N} m_i \bar{v}_i \cdot \bar{v}_i. \tag{5.20}$$

Let B denote an arbitrary reference point to which the velocity of all particles is referred, so that $\bar{v}_i = \bar{v}_B + \bar{v}_{i/B}$. The corresponding form for the system's kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^{N} m_i (\bar{v}_B + \bar{v}_{i/B}) \cdot (\bar{v}_B + \bar{v}_{i/B}) \\ &= \frac{1}{2} \sum_{i=1}^{N} m_i (\bar{v}_B \cdot \bar{v}_B) + \bar{v}_B \cdot \sum_{i=1}^{N} m_i \bar{v}_{i/B} + \frac{1}{2} \sum_{i=1}^{N} m_i \bar{v}_{i/B} \cdot \bar{v}_{i/B}. \end{aligned} \tag{5.21}$$

We factor out of each sum those terms that are independent of the particle number. The first sum in Eq. (5.21) then yields the total mass. The second sum is the first moment of mass relative to point B, which locates the center of mass C relative to point B. Correspondingly, the kinetic energy of the system becomes

$$T = \frac{1}{2} m (\bar{v}_B \cdot \bar{v}_B) + m \bar{v}_B \cdot \bar{v}_{G/B} + \frac{1}{2} \sum_{i=1}^{N} m_i \bar{v}_{i/B} \cdot \bar{v}_{i/B}. \tag{5.22}$$

One viewpoint of this expression is that the kinetic energy of a system of particles is associated with three effects: translation of all particles following the reference point (the first term in Eq. (5.22)), motion of the particles relative to the reference point (the third term), and an interaction of the motions of the reference point and of the center of mass relative to the reference point. Substantial simplification results when we select the reference point such that the second term vanishes. This may be achieved in a variety of ways, some of which are not useful for a rigid body. For this reason, let us now specialize Eq. (5.22) to the collection of particles that form a rigid body. In that case $\bar{v}_{i/B} = \bar{\omega} \times \bar{r}_{i/B}$, so the kinetic energy becomes

$$T = \frac{1}{2} m (\bar{v}_B \cdot \bar{v}_B) + m \bar{v}_B \cdot (\bar{\omega} \times \bar{r}_{G/B}) + \frac{1}{2} \sum_{i=1}^{N} m_i (\bar{\omega} \times \bar{r}_{i/B}) \cdot (\bar{\omega} \times \bar{r}_{i/B}). \tag{5.23}$$

The sum may be written in a more recognizable form by using an identity for the scalar triple product,

$$(\bar{a} \times \bar{b}) \cdot \bar{c} = \bar{a} \cdot (\bar{b} \times \bar{c}).$$

We employ this identity with $\bar{a} = \bar{\omega}$, $\bar{b} = \bar{r}_{i/B}$, and $\bar{c} = \bar{\omega} \times \bar{r}_{i/B}$, which yields

$$\begin{aligned} T &= \frac{1}{2} m \bar{v}_B \cdot \bar{v}_B + m \bar{v}_B \cdot (\bar{\omega} \times \bar{r}_{G/B}) + \frac{1}{2} \sum_{i=1}^{N} m_i \bar{\omega} \cdot [\bar{r}_{i/B} \times (\bar{\omega} \times \bar{r}_{i/B})] \\ &= \frac{1}{2} m \bar{v}_B \cdot \bar{v}_B + m \bar{v}_B \cdot (\bar{\omega} \times \bar{r}_{G/B}) + \frac{1}{2} \bar{\omega} \cdot \bar{H}_B, \end{aligned} \tag{5.24}$$

where $\bar{H}_B$ is the angular momentum relative to point B, as defined in Eq. (5.17).

In order to avoid evaluating of the second term in Eq. (5.24), we shall restrict our selection for point B. Placing this point at the center of mass of the body gives $\bar{r}_{G/B} = \bar{0}$. Alternatively, if point B is the stationary point for a body in pure rotation, then $\bar{v}_B = \bar{0}$. The latter condition also holds whenever point B is situated on the instantaneous axis of rotation for a body in general planar motion. However, such a point is not fixed relative to the body, so the inertia properties (discussed in the next section) required to form $\bar{H}_B$ will not be constant. Another possibility is to assign point B to anywhere on an axis that is parallel to $\bar{\omega}$ and intersects the center of mass, for then $\bar{\omega} \times \bar{r}_{G/B} = \bar{0}$. Once again, such a point gives rise to nonconstant inertia properties. For these reasons, the alternatives we shall employ to evaluate the kinetic energy of a rigid body are:

♦ $$T = \tfrac{1}{2} m \bar{v}_G \cdot \bar{v}_G + \tfrac{1}{2} \bar{\omega} \cdot \bar{H}_G \quad \text{for all motions;} \tag{5.25a}$$

♦ $$T = \tfrac{1}{2} \bar{\omega} \cdot \bar{H}_O \quad \text{for pure rotation about point } O. \tag{5.25b}$$

5.2 Evaluation of Angular Momentum and Inertia Properties

Our next task is to develop an effective way in which to compute the angular momentum associated with a specified rotation of the body. Clearly, it is not conceivable to do this by adding the contribution of every atomic particle forming the body. Our approach is consistent with the usual ways that finite sums extending over numerous small elements are treated in engineering-oriented calculus courses.

5.2.1 *Moments and Products of Inertia*

In a continuum model of a rigid body, the particles become differential elements of mass dm having infinitesimal dimensions in all directions. These elements dm fill the region occupied by the body. In this viewpoint, any summation over the particles forming the body becomes an integral over this region.

In Figure 5.4, *the xyz coordinate system is fixed to the body with its origin at point A*. The position vector $\bar{r}_{i/A}$ and angular velocity $\bar{\omega}$ may then be described in terms of coordinates relative to xyz as

$$\begin{aligned} \bar{r}_{i/A} &= x\bar{i} + y\bar{j} + z\bar{k}, \\ \bar{\omega} &= \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}. \end{aligned} \tag{5.26}$$

We substitute these expressions into Eq. (5.17) for the angular momentum, and convert the summation to an integral. This transforms the general relation to

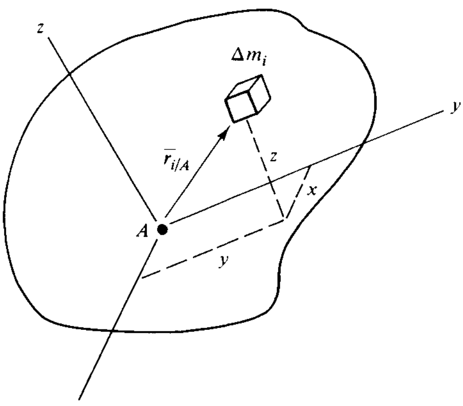


Figure 5.4 Position of a mass element.

$$\bar{H}_A = \iiint (x\bar{i}+y\bar{j}+z\bar{k}) \times [(\omega_x \bar{i}+\omega_y \bar{j}+\omega_z \bar{k}) \times (x\bar{i}+y\bar{j}+z\bar{k})]\, dm. \tag{5.27}$$

The rotation rates are overall properties of the motion - they are independent of the position relative to the body. Hence, ω_x, ω_y, and ω_z may be factored out of the integral after the cross products have been evaluated. The result is

♦ $$\bar{H}_A = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\bar{i} + (I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z)\bar{j} + (I_{zz}\omega_z - I_{xz}\omega_x - I_{yz}\omega_y)\bar{k}, \tag{5.28}$$

where

♦ $$I_{xx} = \iiint (y^2+z^2)\, dm, \quad I_{yy} = \iiint (x^2+z^2)\, dm, \quad I_{zz} = \iiint (x^2+y^2)\, dm; \tag{5.29a}$$

♦ $$I_{xy} = \iiint xy\, dm, \quad I_{yz} = \iiint yz\, dm, \quad I_{xz} = \iiint xz\, dm. \tag{5.29b}$$

The terms I_{pp} (repeated subscripts) are *moments of inertia* about the three coordinate axes, and the terms I_{pq} (nonrepeated subscripts) are *products of inertia*. The former are the inertia properties encountered in planar motion, but they are now defined for rotations about three axes. The similarity of any I_{pp} to the parameter for planar motion may be realized by looking down the p axis. Such a view for I_{zz} is shown in Figure 5.5. The distance $R = (x^2+y^2)^{1/2}$ is the perpendicular distance from the z axis to the mass element dm. Thus, I_{zz} is the sum of the $R^2\, dm$ values for all elements. A common way to prescribe a moment of inertia is to give its *radius of gyration* $\kappa_p = \sqrt{I_{pp}/m}$. This is the radius of a thin ring having the same mass as the body and with the same moment of inertia about its axis of symmetry as the body has about axis p. It is clear from the integral definition of I_{pp} that the radius of gyration cannot exceed the largest distance from axis p to a mass point in the body.

The products of inertia describe the symmetry of the mass distribution relative to the coordinate planes. Figure 5.6 shows the cross-section of a body at an arbitrary value of z. The mirror image of the left region $x < 0$ is outlined in the right region

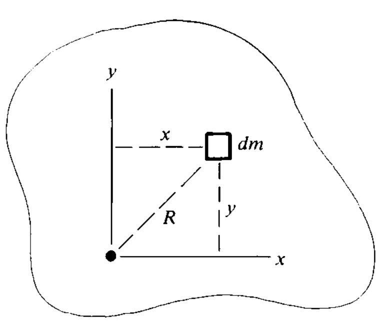


Figure 5.5 Contribution of a mass element to a moment of inertia.

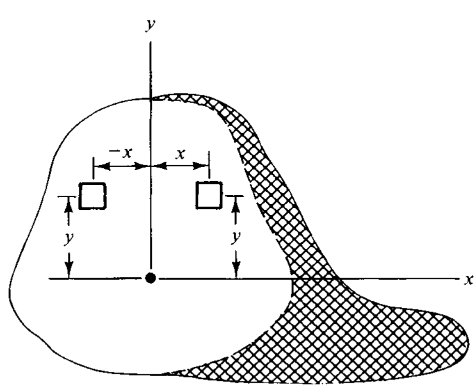


Figure 5.6 Effect of symmetry on a product of inertia.

$x > 0$. If the density is also symmetrical, then every mass element at (x, y, z) within the outlined region in $x > 0$ is matched by a corresponding element at $(-x, y, z)$ to the left. Thus, the $xy\,dm$ values for these two regions cancel, as do the $xz\,dm$ values. All that remains is the contribution of the shaded region outside the outline. The situation in the figure suggests that $I_{xy} < 0$, because more of the shaded area seems to lie in the octants where $x > 0$ and $y < 0$. Increasing the shaded region would increase I_{xy} in magnitude. However, the actual sign of I_{xy} for the situation in Figure 5.6 cannot be judged solely from the drawing, because it depends on the variation of shape and density in the z direction.

A corollary of the foregoing is that if the y–z plane is a plane of symmetry, then $I_{xy} = 0$ and $I_{xz} = 0$. The fact that the x axis is normal to the plane of symmetry leads to the following generalization:

♦ *If two coordinate axes form a plane of symmetry for a body, then all products of inertia involving the coordinate normal to that plane are zero.*

A further corollary is:

♦ *If at least two of the three coordinate planes are planes of symmetry for a body, then all products of inertia are zero.*

Clearly, the last condition is attained for any body of revolution if the axis of symmetry coincides with a coordinate axis. Whenever the coordinate axes correspond to vanishing values of all products of inertia, they are said to be *principal axes.* We shall see that it is possible to identify principal axes for all bodies, not only symmetric ones.

The inertia properties of homogeneous bodies have been tabulated for a variety of common shapes; see the appendix, which follows Chapter 8. Properties for bodies that are not tabulated may be evaluated by treating them as composites of basic shapes or, as a last resort, by carrying out the integrals in Eqs. (5.29).

Once the inertia properties are known, the angular momentum may be evaluated according to Eq. (5.28). For this, the components of $\bar{\omega}$ would be found by the methods for spatial kinematics established in Chapter 3. The vector relation may alternatively be written in matrix form as

♦ $$\{H_A\} = [I]\{\omega\}, \tag{5.30}$$

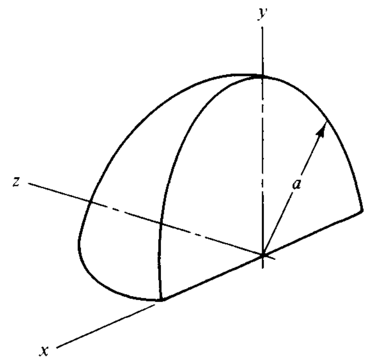
where $\{H_A\}$ and $\{\omega\}$ are formed from the components of $\bar{H}_A$ and $\bar{\omega}$ (respectively), and where $[I]$ is the *inertia matrix,*

♦ $$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}. \tag{5.31}$$

This square array of moments and products of inertia, combined with the mass and the location of the center of mass, fully characterizes the inertia properties of a rigid body.

In some situations the orientation of the desired xyz coordinate system might not match the one appearing in a tabulation. In that case it is necessary to convert the inertia properties. The appropriate transformations shall be developed in the next section.

Example 5.1 Derive the inertia matrix of the quarter-sphere about the xyz axes; then use that result to obtain the inertia matrix for a quarter-spherical shell whose thickness is $\Delta \ll a$. Express each result in terms of the mass m of that body.



Example 5.1

Solution Spherical coordinates are ideal here. Any coordinate axis may be employed as the reference for the polar angle. We select the y axis, so that

$$x = r\sin\phi\cos\theta, \qquad y = r\cos\phi, \qquad z = r\sin\phi\sin\theta.$$

The body occupies the domain $0 \le r \le a$, $0 \le \phi \le \pi/2$, $0 \le \theta \le \pi$, and a differential element of mass is

$$dm = \rho r^2 \sin\phi \, dr \, d\theta \, d\phi.$$

We wish to express the inertia properties in terms of the mass m. We therefore express the density as the ratio of the mass m to the volume of a quarter-sphere,

$$\rho = \frac{m}{V} = \frac{3m}{\pi a^3}.$$

The coordinate axes are such that the body is symmetric with respect to the y-z plane, so

$$I_{xy} = I_{xz} = 0.$$

Also, $I_{zz} = I_{yy}$ by symmetry, so it is necessary to compute only I_{xx}, I_{yy}, and I_{yz}. The integral definitions, Eqs. (5.29), give

$$I_{xx} = \int_0^a \int_0^{\pi/2} \int_0^{\pi} (y^2 + z^2)(\rho r^2 \sin\phi)\, d\theta \, d\phi \, dr$$

$$= \rho \int_0^a r^4 \int_0^{\pi/2} \int_0^{\pi} (\cos^2\phi \sin\phi + \sin^3\phi \sin^2\theta)\, d\theta \, d\phi \, dr = \frac{2}{15}\pi\rho a^5,$$

$$I_{yy} = \int_0^a \int_0^{\pi/2} \int_0^{\pi} (x^2 + z^2)(\rho r^2 \sin\phi)\, d\theta \, d\phi \, dr$$

$$= \rho \int_0^a r^4 \int_0^{\pi/2} \int_0^{\pi} (\sin^3\phi)\, d\theta \, d\phi \, dr = \frac{2}{15}\pi\rho a^5,$$

$$I_{yz} = \int_0^a \int_0^{\pi/2} \int_0^{\pi} yz(\rho r^2 \sin\phi)\, d\theta\, d\phi\, dr$$

$$= \rho \int_0^a r^4 \int_0^{\pi/2} \int_0^{\pi} (\sin^2\phi \cos\phi \sin\theta)\, d\theta\, d\phi\, dr = \frac{2}{15}\rho a^5.$$

In order to express these results in terms of the mass m, we substitute the earlier expression for ρ to find

$$[I] = \frac{2}{5} m a^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/\pi \\ 0 & -1/\pi & 1 \end{bmatrix}.$$

Results for a shell may be obtained by considering a composite body that is formed by removing a quarter sphere of radius $a - \Delta$ from the given body. The mass is then

$$m = \rho \frac{\pi}{3}[a^3 - (a-\Delta)^3] = \rho \frac{\pi}{3}(3a^2\Delta - 3a\Delta^2 + \Delta^3).$$

For a thin shell, $\Delta/a \ll 1$; hence we have

$$\rho = \frac{m}{\pi a^3 \Delta}.$$

The comparable differences for the inertia properties must be formed using the original results, which depended on ρ. Thus

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{15}\pi\rho[a^5 - (a-\Delta)^5] = \frac{2}{3}\pi\rho a^4 \Delta,$$

$$I_{yz} = \frac{2}{15}\rho[a^5 - (a-\Delta)^5] = \frac{2}{3}\rho a^4 \Delta.$$

Eliminating ρ in favor of the mass of the shell yields

$$[I] = \frac{2}{3} m a^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/\pi \\ 0 & -1/\pi & 1 \end{bmatrix}.$$

5.2.2 *Transformation of Inertia Properties*

Suppose that the moments and products of inertia of the body in Figure 5.7 are known relative to the xyz system. The origin O of xyz might not be acceptable for formulating the equation of rotational motion, whereas point O' is acceptable. Then it will be necessary to transfer the inertia properties to an $x'y'z'$ coordinate system having its origin at point O'. Furthermore, it also might be desirable for the $x'y'z'$ axes to be rotated relative to xyz. The general task involves translational and rotational transformations of the inertia properties.

Figure 5.8 depicts a translational transformation. The inertia properties with respect to xyz are known, and the coordinates of the origin O' relative to xyz are denoted $x_{O'}, y_{O'}, z_{O'}$. The coordinate transformation in this case is

$$x' = x - x_{O'}, \qquad y' = y - y_{O'}, \qquad z' = z - z_{O'}. \tag{5.32}$$

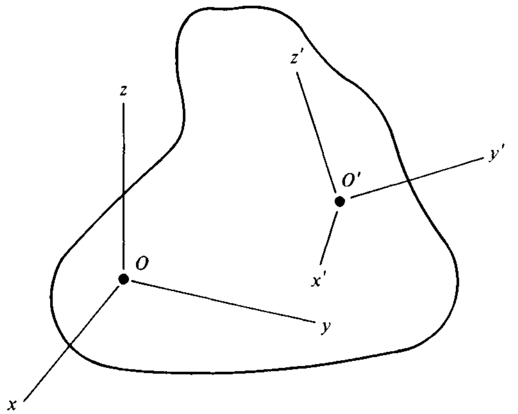


Figure 5.7 Alternate coordinate systems.

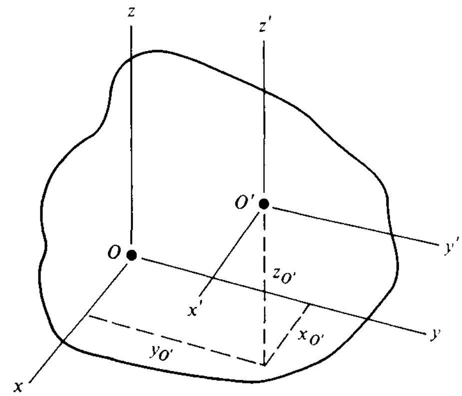


Figure 5.8 Translation transformation.

Consider the definition of a moment of inertia, for example, $I_{x'x'}$. Substitution of Eqs. (5.32) into the first of Eqs. (5.29a) yields

$$\begin{aligned} I_{x'x'} &= \iiint [(y')^2+(z')^2]\,dm \\ &= \iiint [y^2+z^2-2y_{O'}y-2z_{O'}z+(y_{O'})^2+(z_{O'})^2]\,dm \\ &= I_{xx}-2y_{O'}\iiint y\,dm-2z_{O'}\iiint z\,dm+m(y_{O'}^2+z_{O'}^2). \end{aligned} \tag{5.33}$$

The integrals remaining in Eq. (5.33) are first moments of mass with respect to the y and z coordinates. They locate the center of mass G relative to the origin of xyz. Thus

$$I_{x'x'} = I_{xx}-2my_{O'}y_G-2mz_{O'}z_G+m(y_{O'}^2+z_{O'}^2). \tag{5.34}$$

This translation transformation may be simplified if the origin O is restricted to being the center of mass. Then $x_G = y_G = z_G = 0$. Repetition of the derivation for the other terms leads to the *parallel axis theorems for moments of inertia:*

$$\begin{aligned} \blacklozenge \quad & I_{x'x'} = I_{xx}+m(y_{O'}^2+z_{O'}^2), \\ \blacklozenge \quad & I_{y'y'} = I_{yy}+m(x_{O'}^2+z_{O'}^2), \\ \blacklozenge \quad & I_{z'z'} = I_{zz}+m(x_{O'}^2+y_{O'}^2). \end{aligned} \tag{5.35}$$

Note that the sums of squares of the O' coordinates appearing in Eq. (5.35) are actually the square of the perpendicular distance between the parallel axes for the respective moments of inertia. Since the x, y, and z axes are required to intersect the center of mass, it is clear that the moments of inertia for centroidal axes are smaller than those about parallel noncentroidal axes.

The transformation of the products of inertia is obtained in a similar manner. The result is the *parallel axis theorems for products of inertia:*

$$\blacklozenge \quad I_{x'y'} = I_{xy} + m x_{O'} y_{O'},$$
$$\blacklozenge \quad I_{y'z'} = I_{yz} + m y_{O'} z_{O'}, \tag{5.36}$$
$$\blacklozenge \quad I_{x'z'} = I_{xz} + m x_{O'} z_{O'}.$$

It is important to realize that although the signs of the coordinates of point O' are unimportant to the transformation of *moments* of inertia, they must be considered when transforming the *products* of inertia. It is useful in this regard to remember that:

(a) the origin of xyz is the center of mass of the body; and
(b) the values $(x_{O'}, y_{O'}, z_{O'})$ are the coordinates relative to xyz of the origin O' of the translated coordinate system $x'y'z'$.

We developed the parallel axis theorems by directly introducing the coordinate transformation into the integral definitions of the moments and products of inertia. We could follow a comparable line of attack to develop the rotation transformation of inertia properties. However, such a derivation entails many manipulations. A far more elegant approach is to employ the rotational contribution to the kinetic energy of a body. The concept here is to exploit the invariance of the scalar kinetic energy when it is formulated using vector components relative to alternative coordinate systems.

Let the transformation matrix from xyz to $x'y'z'$ in Figure 5.9 be $[R]$, where

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [R] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}. \tag{5.37}$$

Either of Eqs. (5.25) indicates that the kinetic energy associated with rotation about point A is

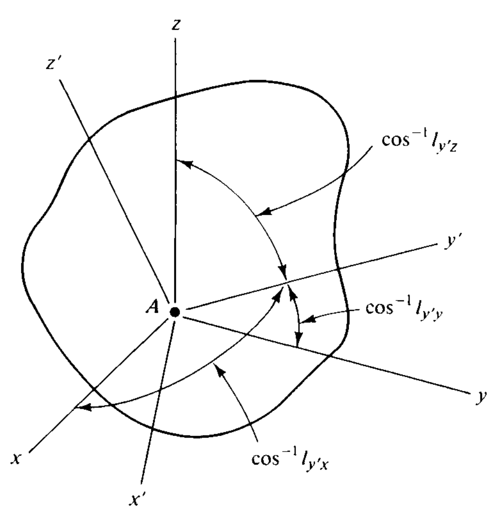


Figure 5.9 Rotation transformation.

$$T_{\text{rot}} = \tfrac{1}{2}\bar{\omega}\cdot\bar{H}_A. \tag{5.38a}$$

In order to employ the transformation matrix $[R]$, we convert this to matrix form by recalling Eq. (5.30), which leads to the equivalent form

♦ $$T_{\text{rot}} = \tfrac{1}{2}\{\omega\}^{\text{T}}[I]\{\omega\}. \tag{5.38b}$$

The scalar nature of kinetic energy requires that the result obtained from this expression be independent of the orientation of the reference frame used to represent $\{\omega\}$ and $[I]$. If the angular velocity and the inertia properties are referred to the $x'y'z'$ axes, then this invariance can be satisfied only if

$$\{\omega'\}^{\text{T}}[I']\{\omega'\} = \{\omega\}^{\text{T}}[I]\{\omega\}, \tag{5.39}$$

where $[I']$ denotes the inertia matrix associated with $x'y'z'$. The rotation transformation and the orthonormal property give

$$\{\omega'\} = [R]\{\omega\}, \quad \{\omega\} = [R]^{\text{T}}\{\omega'\}. \tag{5.40}$$

We substitute the second expression into Eq. (5.39), and require that the equality be satisfied for arbitrary $\{\omega'\}$. This condition can only be satisfied if the inner matrices in the products are identical. Thus, the rotation transformation of inertia properties is

♦ $$[I'] = [R][I][R]^{\text{T}}. \tag{5.41}$$

Any quantity transforming according to Eq. (5.41) is said to be a *tensor of the second rank*. In this viewpoint, vectors - whose components transform according to Eq. (5.40) - are tensors of the first rank.

The transformation in Eq. (5.41) may be resolved into individual inertia values. We write $[R]$ in partition form as a sequence of rows, according to

$$[R] = \begin{bmatrix} l_{x'x} & l_{x'y} & l_{x'z} \\ l_{y'x} & l_{y'y} & l_{y'z} \\ l_{z'x} & l_{z'y} & l_{z'z} \end{bmatrix} = \begin{bmatrix} \{e_{x'}\}^{\text{T}} \\ \{e_{y'}\}^{\text{T}} \\ \{e_{z'}\}^{\text{T}} \end{bmatrix}, \tag{5.42}$$

where the column array $\{e_{p'}\}$ $(p' = x',\ y',\text{ or } z')$ consists of the direction cosines of axis p' relative to xyz or, equivalently, the components of the unit vector $\bar{e}_{p'}$ along the axes of xyz;

$$\{e_{p'}\} = \begin{Bmatrix} l_{p'x} \\ l_{p'y} \\ l_{p'z} \end{Bmatrix}, \quad p' = x',\ y',\text{ or } z'. \tag{5.43}$$

We now employ the partitioned form of $[R]$ to evaluate the inertia transformation in Eq. (5.41). The columns of $[R]^{\text{T}}$ are $\{e_{p'}\}$. Each partition may be treated as a single element when computing a product. This enables us to compute the products in the following manner.

$$[I'] = \begin{bmatrix} \{e_{x'}\}^{\text{T}} \\ \{e_{y'}\}^{\text{T}} \\ \{e_{z'}\}^{\text{T}} \end{bmatrix}[I][\{e_{x'}\}\ \{e_{y'}\}\ \{e_{z'}\}]$$

$$= \begin{bmatrix} \{e_{x'}\}^{\text{T}} \\ \{e_{y'}\}^{\text{T}} \\ \{e_{z'}\}^{\text{T}} \end{bmatrix}[[I]\{e_{x'}\}\ [I]\{e_{y'}\}\ [I]\{e_{z'}\}] =$$

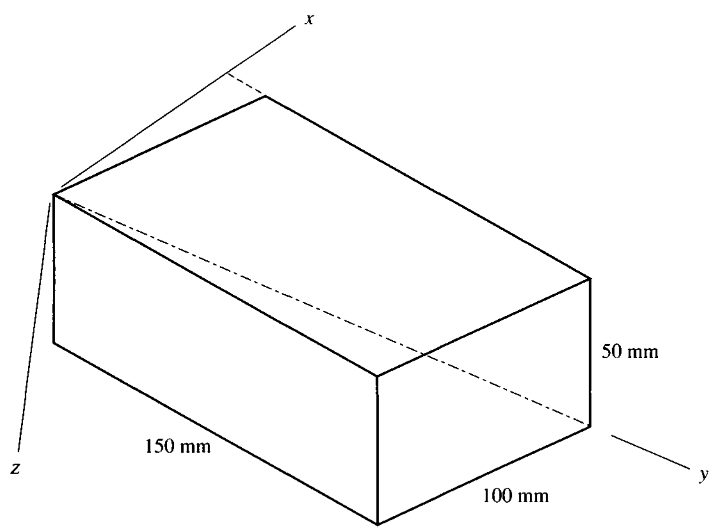
$$= \begin{bmatrix} \{e_{x'}\}^T[I]\{e_{x'}\} & \{e_{x'}\}^T[I]\{e_{y'}\} & \{e_{x'}\}^T[I]\{e_{z'}\} \\ \{e_{y'}\}^T[I]\{e_{x'}\} & \{e_{y'}\}^T[I]\{e_{y'}\} & \{e_{y'}\}^T[I]\{e_{z'}\} \\ \{e_{z'}\}^T[I]\{e_{x'}\} & \{e_{z'}\}^T[I]\{e_{y'}\} & \{e_{z'}\}^T[I]\{e_{z'}\} \end{bmatrix}. \tag{5.44}$$

Each of the products appearing here as an element is a scalar value, so matching like elements on the left and right side yields the following relations for the individual inertia properties:

$$\begin{aligned} &\blacklozenge \quad I_{p'p'} = \{e_{p'}\}^T[I]\{e_{p'}\}, \\ &\blacklozenge \quad I_{p'q'} = -\{e_{p'}\}^T[I]\{e_{q'}\} = -\{e_{q'}\}^T[I]\{e_{p'}\}. \end{aligned} \tag{5.45}$$

The purpose of deriving these relations is to understand how a specific inertia property is altered by a rotation. In most situations we would need all of the transformed inertia properties. In that case, it is much simpler to evaluate Eq. (5.41) directly.

Example 5.2 The x axis lies in the plane of the upper face of the 5-kg homogeneous box, and the y axis is a main diagonal. Determine the inertia matrix of the box relative to xyz.

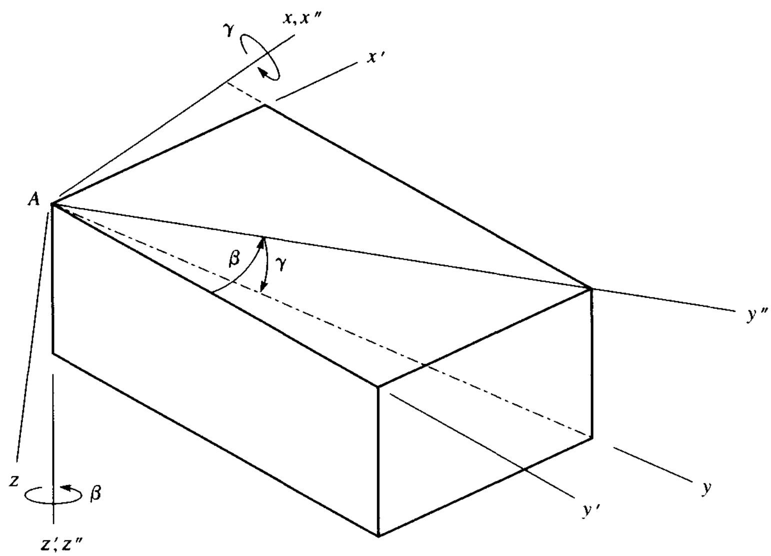


Example 5.2

Solution The appendix gives inertia properties for centroidal axes of a rectangular parallelepiped. These inertia properties may be transferred to a coordinate system whose origin is the designated corner A by means of the parallel axis theorems. For the $x'y'z'$ coordinate axes in the sketch, the coordinates of corner A relative to parallel centroidal axes are $(x_A, y_A, z_A) = (-0.05, -0.075, -0.025)$ meters. Thus we have

$$I_{x'x'} = \frac{1}{12}(5)(0.15^2+0.05^2)+(5)(0.075^2+0.025^2) = 0.04167,$$

$$I_{y'y'} = \frac{1}{12}(5)(0.10^2+0.05^2)+(5)(0.05^2+0.025^2) = 0.02083,$$



Coordinate axes for transforming inertia properties.

$$I_{z'z'} = \frac{1}{12}(5)(0.10^2+0.15^2)+(5)(0.05^2+0.075^2) = 0.05417;$$

$$I_{x'y'} = 0+(5)(-0.050)(-0.075) = 0.01875,$$

$$I_{x'z'} = 0+(5)(-0.050)(-0.025) = 0.00625,$$

$$I_{y'z'} = 0+(5)(-0.075)(-0.025) = 0.009375 \text{ kg-m}^2.$$

We must now evaluate the transformation $[R]$ from $x'y'z'$ to xyz. In Example 3.1, we determined the transformation for a coordinate system whose orientation was specified in a manner similar to the present situation. However, a much more direct solution decomposes the transformation into a sequence of single-axis rotations. We first rotate $x'y'z'$ about the negative z' axis through an angle β, thereby producing an intermediate coordinate system $x''y''z''$ whose y'' axis is the diagonal of the upper face. Then we rotate $x''y''z''$ about the x'' axis through an angle γ in order to bring the y'' axis into coincidence with the x axis. These angles are

$$\beta = \tan^{-1}\left(\frac{100}{150}\right) = 33.690°, \qquad \gamma = \tan^{-1}\left[\frac{50}{(100^2+150^2)^{1/2}}\right] = 15.501°.$$

The successive rotations are about body-fixed axes. We obtain the corresponding transformation by premultiplying the sequence, such that

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R]\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix},$$

where

$$[R] = [R_\gamma][R_\beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8321 & -0.5547 & 0 \\ 0.5345 & 0.8018 & 0.2673 \\ -0.1483 & -0.2224 & 0.9636 \end{bmatrix}.$$

Because $[R]$ transforms $x'y'z'$ to xyz, the inertia transformation is

$$[I] = [R][I'][R]^{\mathrm{T}}.$$

We therefore have

$$[I] = [R]\begin{bmatrix} 0.04167 & -0.01875 & -0.00625 \\ -0.01875 & 0.02083 & -0.009375 \\ -0.00625 & -0.009375 & 0.05417 \end{bmatrix}[R]^{\mathrm{T}}$$

$$= \begin{bmatrix} 0.052569 & 0.002315 & -0.000644 \\ 0.002315 & 0.007292 & 0.001735 \\ -0.000644 & 0.001735 & 0.056810 \end{bmatrix} \text{kg-m}^2.$$

One should note that a common error in using the inertia matrix is to forget that the off-diagonal terms are the negative of the products of inertia. Selecting the appropriate elements from the preceding yields

$$I_{xx} = 0.052569, \quad I_{yy} = 0.007292, \quad I_{zz} = 0.056810,$$

$$I_{xy} = -0.002315, \quad I_{xz} = 0.000644, \quad I_{yz} = -0.001735 \text{ kg-m}^2.$$

5.2.3 *Inertia Ellipsoid*

Comparison of Eq. (5.38b) and Eq. (5.45) leads to an interesting construction that shows the relationship between the moment of inertia about various axes sharing a common origin. (In Chapter 8, we will find this construction to be useful to our understanding of free motion of a rigid body.) Let us represent the instantaneous angular velocity of a body as a rotation at rate ω about axis p', so that the components of the angular velocity are given in matrix form as

$$\{\omega\} = \omega\{e_{p'}\}. \tag{5.46}$$

Substitution of this expression into Eq. (5.38b), followed by application of Eq. (5.45), reveals that

$$2T_{\mathrm{rot}} = \{\omega\}^{\mathrm{T}}[I]\{\omega\} = I_{p'p'}\omega^2. \tag{5.47}$$

Let us consider a variety of rotations about different axes intersecting the origin of xyz. We desire that each rotation have the property that it yields the same value for T_{rot}. How should we adjust the rotation rate about this axis in order to obtain a specified value for the rotational kinetic energy? Equation (5.47) indicates that the appropriate angular speed is

$$\omega = \left(\frac{2T_{\mathrm{rot}}}{I_{p'p'}}\right)^{1/2}. \tag{5.48}$$

Equations (5.46) and (5.48) define the required angular velocity. Let us plot a point P in space that represents the tip of this angular velocity vector when the tail of the

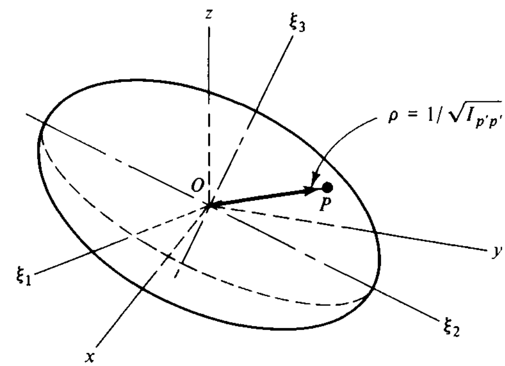


Figure 5.10 Inertia ellipsoid.

vector is placed at the origin. In order to standardize the definition, let us perform the construction for the special case of an angular velocity $\bar{\rho}$ that gives $T_{\text{rot}} = \frac{1}{2}$. The corresponding rotation rate is indicated by Eq. (5.48) to be $\omega = 1/\sqrt{I_{p'p'}}$. A typical point is shown in Figure 5.10. We may represent this vector in either vector or matrix form:

$$\bar{\rho} = \frac{1}{\sqrt{I_{p'p'}}}\bar{e}_{p'}, \qquad \{\rho\} = \frac{1}{\sqrt{I_{p'p'}}}\{e_{p'}\}. \tag{5.49}$$

When we use the second relation to eliminate $\{e_{p'}\}$ from Eq. (5.46), and then substitute that expression for $\bar{\omega}$ into Eq. (5.47), we find that

$$\{\rho\}^{\text{T}}[I]\{\rho\} = 1. \tag{5.50}$$

In view of the way we have defined point P, the (x, y, z) coordinates of the point are identical to the components of $\bar{\rho}$. Hence, expanding Eq. (5.50) leads to

$$I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 - 2I_{xy}xy - 2I_{xz}xz - 2I_{yz}yz = 1. \tag{5.51}$$

This is the equation for an ellipsoidal surface whose centroid coincides with the origin O of the xyz system of axes. This surface is called the *ellipsoid of inertia.* According to Eq. (5.49), the ellipsoid of inertia is the locus of points P for which the distance to the origin is the reciprocal of the square root of the moment of inertia about the axis intersecting point P and origin O. In other words, the distance is inversely proportional to the radius of gyration about the axis. If we alter the rotation rate of a body about various axes having a common origin in order to match this inverse proportionality, in accord with Eq. (5.48), we will find that the rotational kinetic energy is constant for all rotation axes.

The major, minor, and intermediate axes of the ellipsoid of inertia, along which the distance from the origin is an extreme value, do not necessarily coincide with the xyz coordinate system. Suppose that all products of inertia with respect to some coordinate system $\xi_1\xi_2\xi_3$ are zero, which means that they are *principal axes of inertia.* Let us denote the corresponding principal moments of inertia as I_1, I_2, and I_3. The equation for the inertia ellipsoid relative to principal axes is then given by Eq. (5.51) to be

$$I_1(\xi_1)^2 + I_2(\xi_2)^2 + I_3(\xi_3)^2 = 1, \tag{5.52}$$

where (ξ_1, ξ_2, ξ_3) are the coordinates of point P on the inertia ellipsoid relative to the principal axes.

Figure 5.10 shows the inertia ellipsoid relative to principal and nonprincipal axes. Note that Eq. (5.52), which describes the ellipsoid in terms of principal axes, contains only sums of squares. This means that the *major, minor, and intermediate axes of the ellipsoid of inertia coincide with the principal axes.* We could use analytical geometry to evaluate the orientation of the principal axes of inertia. In the next section, we shall instead develop a more direct method for that determination by returning to the fundamental inertia transformation in Eq. (5.41).

5.2.4 *Principal Axes*

Suppose we know the inertia properties relative to a set of axes xyz that are not principal ones, so that $[I]$ has off-diagonal elements. If we can find a transformation matrix $[R]$ for which $[I']$ is diagonal, then the corresponding $\xi_1\xi_2\xi_3$ axes will be principal. Consistent with the use of subscripted numerals to denote principal parameters, let $\{e_j\}$ ($j = 1, 2$, or 3) denote the (unknown) columns of direction cosines of ξ_1, ξ_2, and ξ_3, respectively, relative to xyz. According to Eq. (5.42), the partitioned form of the transformation matrix from the xyz coordinate system to the principal axes is

$$[R] = \begin{bmatrix} \{e_1\}^T \\ \{e_2\}^T \\ \{e_3\}^T \end{bmatrix}. \tag{5.53}$$

Premultiplication of Eq. (5.41) by $[R]^T$ leads to

$$[R]^T[I'] = [I][R]^T. \tag{5.54}$$

Let us now follow steps similar to those leading to Eq. (5.44) and express Eq. (5.54) explicitly in terms of the $\{e_j\}$. This yields

$$\begin{gathered}[\{e_1\}\ \{e_2\}\ \{e_3\}] \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = [I][\{e_1\}\ \{e_2\}\ \{e_3\}], \\ [I_1\{e_1\}\ I_2\{e_2\}\ I_3\{e_3\}] = [[I]\{e_1\}\ [I]\{e_2\}\ [I]\{e_3\}].\end{gathered} \tag{5.55}$$

In order for this equality to hold, corresponding columns must be identical, so that

$$I_i\{e_i\} = [I]\{e_i\}, \quad i = 1, 2, 3. \tag{5.56}$$

In other words, I_i are the eigenvalues λ, and $\{e_i\}$ are the eigenvectors $\{e\}$ of the matrix equation

♦ $$[[I] - \lambda[U]]\{e\} = \{0\}, \tag{5.57}$$

where $[U]$ is the unitary (i.e., identity) matrix.

The solution of matrix eigenvalue problems is a topic in linear algebra, as well as in linear vibration theory. Routines for solving such problems are contained in most mathematical computer software. We shall only highlight the concepts here. Equation (5.57) represents three simultaneous equations for the components of $\{e\}$, which are the direction cosines between a principal axis and the axes associated with inertia matrix $[I]$. If those equations are solvable then the only solution is trivial: $\{e\} = \{0\}$. The equations cannot be solved for a unique value of $\{e\}$ when the coefficient matrix

$[[I]-\lambda[U]]$ cannot be inverted. Hence, nontrivial solutions for $\{e\}$ arise only if the value of λ is selected to satisfy the characteristic equation corresponding to vanishing of the determinant of the coefficients,

$$|[I]-\lambda[U]|=0. \tag{5.58}$$

Evaluation of the determinant with λ as an algebraic parameter leads to a cubic equation for λ. The eigenvalues, which are the three roots of the characteristic equation, are the principal moments of inertia, $\lambda = I_1, I_2, I_3$.

Let us consider first the case where the principal moments of inertia are distinct values. Equating λ to one of these values then reduces the rank of $[A]-\lambda[U]$ to 2, so one of the three simultaneous equations represented by Eq. (5.57) is a linear combination of the others. Because of this loss of an independent equation, any nonzero component of $\{e\}$ may be chosen arbitrarily. The other components may then be found in terms of the arbitrary one by solving the independent equations. The eigenvector associated with each principal moment of inertia I_i is the set of direction cosines $\{e_i\}$ locating that principal axis relative to the original xyz coordinate system.

Although the solution to the eigenvalue problem leaves an element of the eigenvector undetermined, we must remember that $\{e_i\}$ represents the components relative to xyz of a unit vector oriented parallel to the ith principal axis. The condition that such a vector has a unit magnitude is written in matrix form as

$$\{e_i\}^{\mathrm{T}}\{e_i\}=1, \tag{5.59}$$

which provides the additional equation required to uniquely evaluate $\{e_i\}$.

The eigenvectors $\{e_i\}$ form an orthogonal set. To prove this feature, we consider Eq. (5.57) for two different principal values, $i=m$ and $i=n$. Premultiplying each equation by the transpose of the other eigenvector leads to

$$I_m\{e_n\}^{\mathrm{T}}\{e_m\}=\{e_n\}^{\mathrm{T}}[I]\{e_m\}, \qquad I_n\{e_m\}^{\mathrm{T}}\{e_n\}=\{e_m\}^{\mathrm{T}}[I]\{e_n\}. \tag{5.60}$$

Each set of matrix products yields a scalar value, so we may transpose them without altering the result. We perform this operation on the second of Eqs. (5.60), and recall that $[I]$ is symmetric, so that

$$I_n\{e_n\}^{\mathrm{T}}\{e_m\}=\{e_n\}^{\mathrm{T}}[I]\{e_m\}. \tag{5.61}$$

We subtract this equation from the first of Eqs. (5.60). If the moments of inertia are distinct values, $I_n \neq I_m$, differencing these equations leads to the conclusion that

$$\{e_n\}^{\mathrm{T}}\{e_m\}=0, \quad n\neq m, \tag{5.62}$$

which is the matrix form of the dot product $\bar{e}_n\cdot\bar{e}_m=0$. It is interesting to observe that Eqs. (5.61) and (5.62) lead to the condition that

$$\{e_n\}^{\mathrm{T}}[I]\{e_m\}=0, \quad n\neq m. \tag{5.63}$$

In view of Eqs. (5.45), this relation is merely a restatement of the fact that the products of inertia vanish for principal axes.

The case where two of the principal moments of inertia are identical is very much like the one just discussed, except that yet another equation ceases to be independent for each repetition of the principal value. Thus, the number of times a root is repeated is the same as the number of components of the eigenvector $\{e\}$ that are arbitrary. It follows that equating λ in Eq. (5.57) to a principal value and applying Eq. (5.59) is not sufficient to uniquely specify the $\{e_i\}$ in this case.

The method in which to proceed when this ambiguity arises becomes apparent when we consider a body of revolution. Such a body has identical principal moments of inertia for all axes that perpendicularly intersect the axis of symmetry at a common point. The axes we wish to construct should be mutually orthogonal. This suggests that if we make a specific choice for the (several) arbitrary elements of one eigenvector $\{e_i\}$ associated with the repeated principal value, then the other one, $\{e_{i+1}\}$, may be constructed by satisfying the orthogonality condition

$$\{e_i\}^T\{e_{i+1}\} = 0. \tag{5.64}$$

This provides the extra equation required to define the otherwise arbitrary elements of the eigenvector. The greater degree of arbitrariness associated with identical principal moments of inertia arises because the corresponding principal directions are not unique. The case where all three principal values are identical merely means that any set of axes are principal. There is then no need to solve an eigenvalue problem. This feature is exemplified by a homogeneous sphere or cube when the origin is placed at the centroid.

The conversion to principal axes will simplify all equations involving the angular momentum $\bar{H}_A$. The benefits of such simplifications are countered by the need to solve an eigenvalue problem in order to locate the principal axes. Furthermore, the principal axes may not be convenient for the evaluation of $\bar{\omega}$, $\bar{\alpha}$, and $\sum \bar{M}_A$. For this reason, we shall make it a practice when formulating problems to use coordinate axes with the most convenient orientation. Identification of principal axes on the basis of symmetry will be useful, but we usually will not solve the eigenvalue problem. However, we will find it useful to discuss several cases of motion in Chapter 8 in terms of principal axes and moments of inertia. If we wish to apply those results to an arbitrary body, we must be able to locate the principal axes.

Example 5.3 The orthogonal tetrahedron shown on the next page has a mass of 60 kg. For axes having origin at the center of mass, determine the principal moments of inertia and the rotation transformation locating the principal axes.

Solution The appendix gives the inertia properties of an orthogonal tetrahedron with respect to centroidal axes parallel to the orthogonal edges. The values in the present case, where $m = 60$ kg, are

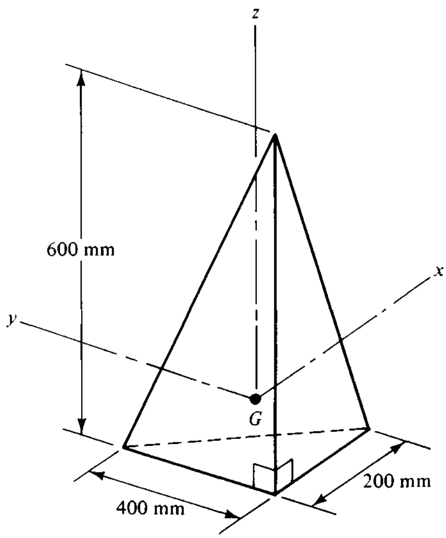
$$[I] = \begin{bmatrix} 1.170 & 0.060 & 0.090 \\ 0.060 & 0.900 & 0.180 \\ 0.090 & 0.180 & 0.450 \end{bmatrix} \text{kg-m}^2.$$

The eigenvector equation for the principal axes is

$$[[I]-\lambda[U]]\{e\} = \{0\}.$$

Expanding the characteristic equation $|[I]-\lambda[U]| = 0$ yields

$$\lambda^3 - 2.520\lambda^2 + 1.9404\lambda - 0.428976 = 0.$$



Example 5.3

The eigenvalues are the principal moments of inertia. In order of decreasing magnitude, they are

$$I_1 = 1.20592, \quad I_2 = 0.93268, \quad I_3 = 0.38140 \text{ kg-m}^2.$$

For the first eigenvalue, the eigenvector equation becomes

$$[[I]-\lambda_1[U]]\{e_1\} = \{0\},$$

$$\begin{bmatrix} -0.03592 & 0.060 & 0.090 \\ 0.060 & -0.30592 & 0.180 \\ 0.090 & 0.180 & -0.75592 \end{bmatrix} \begin{Bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

The solution of the first two equations is

$$e_{11} = 5.1880 e_{13}, \qquad e_{12} = 1.6059 e_{13}.$$

The condition that the values of e_{jk} be components of unit vector $\bar{e}_j$ yields

$$e_{11}^2 + e_{12}^2 + e_{13}^2 = (5.1880^2 + 1.6059^2 + 1) e_{13}^2 = 1.$$

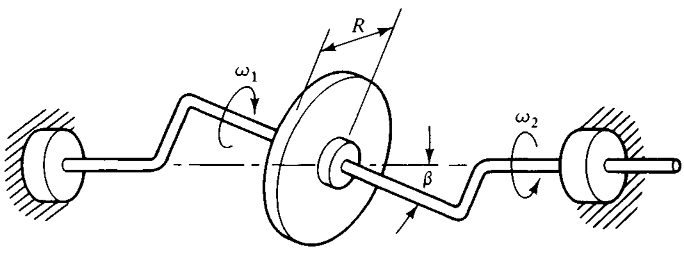
Choosing the positive sign for e_{13} then gives

$$e_{13} = 0.18109, \qquad e_{11} = 0.93948, \qquad e_{12} = 0.29081.$$

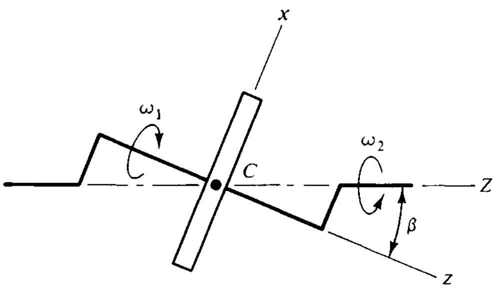
When we carry out the same procedure for $\lambda = \lambda_2$ and $\lambda = \lambda_3$, we obtain $\{e_2\}$ and $\{e_3\}$, respectively. Thus

$$[R] = \begin{bmatrix} \{e_1\}^{\mathrm{T}} \\ \{e_2\}^{\mathrm{T}} \\ \{e_3\}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 0.9395 & 0.2908 & 0.1811 \\ -0.3322 & 0.9023 & 0.2746 \\ -0.0836 & -0.3181 & 0.9444 \end{bmatrix}.$$

Example 5.4 A homogeneous disk of mass m and radius R spins at rate ω_1 about its skewed axis, which rotates about the horizontal at rate ω_2. Derive an expression for the kinetic energy of the disk.



Example 5.4



Coordinate systems.

Solution In order to avoid transforming inertia properties, we align the z axis with the axis of the disk, and place the origin of xyz at the center of mass. Because of the axisymmetry of the disk, we may align the x and y axes in a manner that facilitates the description of $\bar{\omega}$; our choice is shown in the sketch. The inertia properties given in the appendix yield

$$I_{xx} = I_{yy} = \tfrac{1}{4}mR^2, \qquad I_{zz} = \tfrac{1}{2}mR^2, \qquad I_{xy} = I_{yz} = I_{xz} = 0.$$

The angular velocity is

$$\bar{\omega} = \omega_2\bar{K} - \omega_1\bar{k} = (\omega_2 \sin\beta)\bar{i} + (\omega_2\cos\beta - \omega_1)\bar{k}.$$

The corresponding angular momentum relative to the center of mass C is

$$\bar{H}_C = I_{xx}\omega_x\bar{i} + I_{zz}\omega_z\bar{k} = mR^2[\tfrac{1}{4}(\omega_2\sin\beta)\bar{i} + \tfrac{1}{2}(\omega_2\cos\beta - \omega_1)\bar{k}].$$

Point C is fixed, so there is only rotational kinetic energy relative to the center of mass,

$$T = \tfrac{1}{2}\bar{\omega}\cdot\bar{H}_C = \tfrac{1}{8}mR^2[\omega_2^2\sin^2\beta + 2(\omega_2\cos\beta - \omega_1)^2].$$

5.3 Rate of Change of Angular Momentum

The angular momentum is a function of the inertia properties and the angular velocity, as expressed in Eq. (5.28). That form describes the components relative to the xyz reference frame, which is fixed to the body. Thus, the unit vectors associated with the components of $\bar{H}_A$ change with time. The angular velocity components appearing in the components of $\bar{H}_A$ also change with time. However, the fact that xyz is fixed to the body is a substantial benefit, because it yields constant values for all inertia properties. Accordingly, the total derivative of $\bar{H}_A$ is related to the derivative relative to the moving reference frame by

♦ $$\dot{\bar{H}}_A = \frac{\delta \bar{H}_A}{\delta t} + \bar{\omega} \times \bar{H}_A, \tag{5.65}$$

where

♦ $$\frac{\delta \bar{H}_A}{\delta t} = (I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z)\bar{i} + (I_{yy}\dot{\omega}_y - I_{xy}\dot{\omega}_x - I_{yz}\dot{\omega}_z)\bar{j} + (I_{zz}\dot{\omega}_z - I_{xz}\dot{\omega}_x - I_{yz}\dot{\omega}_y)\bar{k}. \tag{5.66}$$

Evaluation of Eq. (5.66) requires determination of the time derivatives of the components of angular velocity. One method for obtaining such terms is to describe the angular velocity in a general functional form that may be differentiated. However, a shortcut is available, because $\bar{\omega}$ (the quantity to be differentiated) is also the angular velocity of the reference frame for the components. It follows that the derivatives of $\bar{\omega}$ relative to the fixed and moving reference frames are identical:

$$\frac{d\bar{\omega}}{dt} = \frac{\delta \bar{\omega}}{\delta t} + \bar{\omega} \times \bar{\omega} \equiv \frac{\delta \bar{\omega}}{\delta t}. \tag{5.67a}$$

By definition, the absolute derivative of $\bar{\omega}$ is the angular acceleration $\bar{\alpha}$. Hence, Eq. (5.67a) may be written in component form as

♦ $$\alpha_x = \dot{\omega}_x, \qquad \alpha_y = \dot{\omega}_y, \qquad \alpha_z = \dot{\omega}_z. \tag{5.67b}$$

The significance of these relations is that they enable evaluation of the kinematical quantities affecting $\bar{H}_A$ by the method developed in Chapter 3. Recall that the procedure begins by writing a general equation for $\bar{\omega}$ that employs unit vectors associated with fixed or moving coordinate systems. We obtain the corresponding expression for $\bar{\alpha}$ by differentiating each unit vector using the angular velocity of its reference frame. Finally, we express $\bar{\omega}$ and $\bar{\alpha}$ in terms of xyz components by resolving the various unit vectors, either by inspection or through the use of rotation transformation matrices.

Example 5.5 Consider the disk in Example 5.4. Determine the rate of change of its centroidal angular momentum. Both ω_1 and ω_2 are constant.

Solution Most of the parameters required to form $\dot{\bar{H}}_A$ were obtained in the solution to Example 5.4. It remains only to form $\bar{\alpha}$. Toward this end, we recall the general expression for $\bar{\omega}$:

$$\bar{\omega} = -\omega_1 \bar{k} + \omega_2 \bar{K}.$$

Both rotation rates are constant, so we have

$$\bar{\alpha} = -\omega_1(\bar{\omega} \times \bar{k}).$$

We resolve $\bar{K}$ into xyz components, which yields

$$\bar{\omega} = (\omega_2 \sin\beta)\bar{i} + (\omega_2 \cos\beta - \omega_1)\bar{k}, \qquad \bar{\alpha} = (\omega_1\omega_2 \sin\beta)\bar{j}.$$

The inertia properties relative to the principal axes xyz were found previously to be

$$I_{xx} = I_{yy} = \tfrac{1}{4}mR^2, \qquad I_{zz} = \tfrac{1}{2}mR^2.$$

Thus, we find

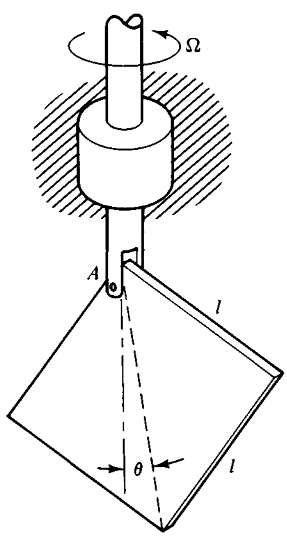
$$\bar{H}_A = I_{xx}\omega_x\bar{i} + I_{zz}\omega_z\bar{k} = mR^2[\tfrac{1}{4}(\omega_2\sin\beta)\bar{i} + \tfrac{1}{2}(\omega_2\cos\beta - \omega_1)\bar{k}],$$

$$\frac{\delta\bar{H}_A}{\delta t} = I_{yy}\alpha_y\bar{j} = mR^2(\tfrac{1}{4}\omega_1\omega_2\sin\beta)\bar{j}.$$

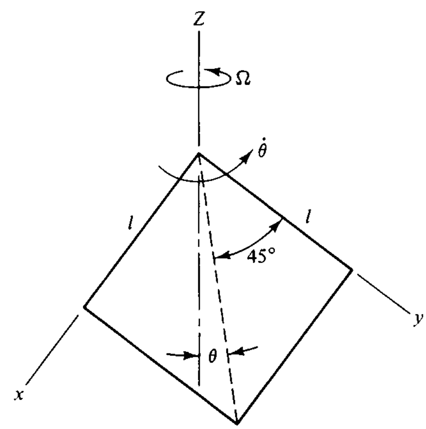
Then

$$\begin{aligned}\dot{\bar{H}}_A &= \frac{\delta\bar{H}_A}{\delta t} + \bar{\omega}\times\bar{H}_A \\ &= mR^2[\tfrac{1}{4}\omega_1\omega_2\sin\beta + (\tfrac{1}{4} - \tfrac{1}{2})(\omega_2\sin\beta)(\omega_2\cos\beta - \omega_1)]\bar{j} \\ &= mR^2(\omega_2\sin\beta)(\tfrac{1}{2}\omega_1 - \tfrac{1}{4}\omega_2\cos\beta)\bar{j}.\end{aligned}$$

Example 5.6 The square plate is pinned at corner A to the vertical shaft, which rotates at the constant angular speed Ω. The angle θ is an arbitrary function of time. Determine $\dot{\bar{H}}_A$ for the plate as a function of θ.



Example 5.6



Coordinate system.

Solution In order to form $\bar{H}_A$, the origin of xyz must coincide with point A. Aligning x and y with the edges of the plate (in accord with the entry shown in the appendix) yields the reference frame shown. The inertia properties obtained from the parallel axis theorems are

$$I_{xx} = I_{yy} = \tfrac{1}{3}ml^2, \qquad I_{zz} = \tfrac{2}{3}ml^2, \qquad I_{xy} = \tfrac{1}{4}ml^2, \qquad I_{xz} = I_{yz} = 0.$$

We next form $\bar{\omega}$ and $\bar{\alpha}$. The general expressions are

$$\bar{\omega} = \Omega\bar{K} + \dot{\theta}\bar{k}, \qquad \bar{\alpha} = \ddot{\theta}\bar{k} + \dot{\theta}(\bar{\omega}\times\bar{k}).$$

It is convenient to resolve these into xyz components using the substitution $\gamma = 45° + \theta$ to describe $\bar{K}$. This yields

$$\bar{\omega} = \Omega[-(\sin\gamma)\bar{i} - (\cos\gamma)\bar{j}] + \dot{\theta}\bar{k},$$
$$\bar{\alpha} = \Omega\dot{\theta}[-(\cos\gamma)\bar{i} + (\sin\gamma)\bar{j}] + \ddot{\theta}\bar{k},$$

from which it follows that

$$\bar{H}_A = (I_{xx}\omega_x - I_{xy}\omega_y)\bar{i} + (I_{yy}\omega_y - I_{xy}\omega_x)\bar{j} + I_{zz}\omega_z\bar{k}$$
$$= ml^2[\Omega(-\tfrac{1}{3}\sin\gamma + \tfrac{1}{4}\cos\gamma)\bar{i} + \Omega(-\tfrac{1}{3}\cos\gamma + \tfrac{1}{4}\sin\gamma)\bar{j} + \tfrac{2}{3}\dot{\theta}\bar{k}].$$
$$\frac{\delta\bar{H}_A}{\delta t} = (I_{xx}\alpha_x - I_{xy}\alpha_y)\bar{i} + (I_{yy}\alpha_y - I_{xy}\alpha_x)\bar{j} + I_{zz}\alpha_z\bar{k}$$
$$= ml^2[\Omega\dot{\theta}(-\tfrac{1}{3}\cos\gamma - \tfrac{1}{4}\sin\gamma)\bar{i} + \Omega\dot{\theta}(\tfrac{1}{3}\sin\gamma + \tfrac{1}{4}\cos\gamma)\bar{j} + \tfrac{2}{3}\ddot{\theta}\bar{k}].$$

The corresponding expression for the rate at which angular momentum changes is

$$\dot{\bar{H}}_A = \frac{\delta H_A}{\delta t} + \bar{\omega} \times \bar{H}_A$$
$$= ml^2\{\Omega\dot{\theta}(-\tfrac{2}{3}\cos\gamma - \tfrac{1}{2}\sin\gamma)\bar{i} + \Omega\dot{\theta}(\tfrac{2}{3}\sin\gamma + \tfrac{1}{2}\cos\gamma)\bar{j}$$
$$+ [\tfrac{2}{3}\ddot{\theta} + \tfrac{1}{4}\Omega^2(\cos^2\gamma - \sin^2\gamma)]\}\bar{k}.$$

Because $\gamma = 45° + \theta$, the trigonometric identities for the sine and cosine of the sum of two angles yield

$$\dot{\bar{H}}_A = \frac{\sqrt{2}}{12} ml^2\Omega\dot{\theta}[(-7\cos\theta + \sin\theta)\bar{i} + (7\cos\theta + \sin\theta)\bar{j}]$$
$$+ ml^2(\tfrac{2}{3}\ddot{\theta} - \tfrac{1}{4}\Omega^2\sin 2\theta)\bar{k}.$$

5.4 Equations of Motion

The developments in the preceding section provide the foundation for synthesizing the relationship between properties of the external force system and motion of the body. Such relations are called *equations of motion.*

We employed Eqs. (5.67b) to evaluate $\delta\bar{H}_A/\delta t$ in Eq. (5.66), and then used that expression to form $\dot{\bar{H}}_A$ in Eq. (5.65). Equating $\dot{\bar{H}}_A$ with $\sum \bar{M}_A$ leads to the equation of rotational motion. In addition, the equation for the motion of the center of mass must be satisfied. For completeness, the expressions needed to formulate the equations of translational and rotational acceleration are summarized here:

♦ $$\sum \bar{F} = m\bar{a}_G, \tag{5.68}$$

♦ $$\sum \bar{M}_A = \frac{\delta\bar{H}_A}{\delta t} + \bar{\omega} \times \bar{H}_A, \tag{5.69}$$

♦ $$\bar{H}_A = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\bar{i} + (I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z)\bar{j} + (I_{zz}\omega_z - I_{xz}\omega_x - I_{yz}\omega_y)\bar{k}, \tag{5.70}$$

♦ $$\frac{\delta\bar{H}_A}{\delta t} = (I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z)\bar{i} + (I_{yy}\alpha_y - I_{xy}\alpha_x - I_{yz}\alpha_z)\bar{j} + (I_{zz}\alpha_z - I_{xz}\alpha_x - I_{yz}\alpha_y)\bar{k}. \tag{5.71}$$

It must be emphasized that the moment equation should be formulated such that point A is selected to be either

(1) the center of mass of the body, or
(2) a fixed point in a body that is in a state of pure rotation.

Matrix notation offers a compact scheme for performing calculations, and several symbolic-mathematics software packages are well-attuned to such notation. The angular momentum was written in this form in Eq. (5.30) in terms of the inertia matrix $[I]$, which was defined in Eq. (5.31). The corresponding forms of the equations of motion are

♦ $$\begin{Bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{Bmatrix} = m \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ a_{Gz} \end{Bmatrix}, \tag{5.72}$$

♦ $$\begin{Bmatrix} \sum M_{Ax} \\ \sum M_{Ay} \\ \sum M_{Az} \end{Bmatrix} = [I] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} [I] \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}. \tag{5.73}$$

In the foregoing, $\sum F_x$, $\sum F_y$, and $\sum F_z$ are the sum of the external forces acting on the body in the three coordinate directions. Similarly, $\sum M_{Ax}$, $\sum M_{Ay}$, and $\sum M_{Az}$ represent the sum of the moments of the external forces about the coordinate axes whose origin is point A. We should note that in some circumstances it might be advantageous to formulate the resultant-force equation in component directions that are different from the body-fixed axes used for the moment equation.

The special case where xyz are principal axes leads to *Euler's equations* of rotational motion, which explicitly express the dependence on the angular velocity and acceleration:

♦ $$\sum M_{Ax} = I_{xx}\alpha_x - (I_{yy} - I_{zz})\omega_y\omega_z,$$
♦ $$\sum M_{Ay} = I_{yy}\alpha_y - (I_{zz} - I_{xx})\omega_x\omega_z, \tag{5.74}$$
♦ $$\sum M_{Az} = I_{zz}\alpha_z - (I_{xx} - I_{yy})\omega_x\omega_y.$$

The repetitive pattern of Euler's equations can be used to help recall the individual components by a mnemonic algorithm based on permutations of the alphabetical order. Euler's equations are particularly useful when it is only necessary to address the moment exerted about one axis.

One aspect of the moment equation of motion that can puzzle a novice is the presence of a moment even when the rotation rates are constant. This effect arises because the orientation of $\bar{\omega}$ is not constant, so that $\bar{\alpha} \neq \bar{0}$. Even if $\bar{\alpha} = \bar{0}$, it is likely in spatial motion that $\bar{H}_A$ is not parallel to $\bar{\omega}$, so that $\bar{\omega} \times \bar{H}_A \neq \bar{0}$. Both effects lead to $d\bar{H}_A/dt \neq 0$. The moment equation merely requires that the force system apply a moment that balances the rate at which the angular momentum changes. (It is irrelevant to the discussion whether the moment is considered to sustain the angular motion, or the angular motion is considered to require the moment.) The portion of $\dot{\bar{H}}_A$ that features products of rotation rates, and therefore is present even if the rotation rates are constant, is often referred to as the *gyroscopic moment.*

Various questions may be investigated using the equations of motion. In the simplest case, the motion of a rigid body is fully specified. This permits complete evaluation of the right side of the translational and rotational equations. The forcing effects, which appear on the left side of the equations, originate from known loads, as well as reactions. The latter are particularly important to characterize. A *free-body diagram,* in which the body is isolated from its surroundings, is essential to the correct description of the reactions.

As an aid in drawing a free-body diagram, recall that reactions are the physical manifestations of kinematical constraints. Thus, if a support prevents a point in the body from moving in a certain direction, then at that point there must be a reaction force exerted on the body in that direction. Similarly, a kinematical constraint on rotation about an axis is imposed by a reaction couple exerted about that axis. The reactions are not known in advance - they are unknown values that will appear in some or all of the equations of motion.

There are only six scalar equations of motion for each body contained in the system (three force sums and three moment sums). It is possible for the number of unknown reactions to exceed the number of available equations. Assuming that this condition does not result from erroneous omission of some characteristic of the supports, it can result from *redundant constraints.* This is the dynamic analog of the condition of *static indeterminacy,* whose resolution requires consideration of deformation effects.

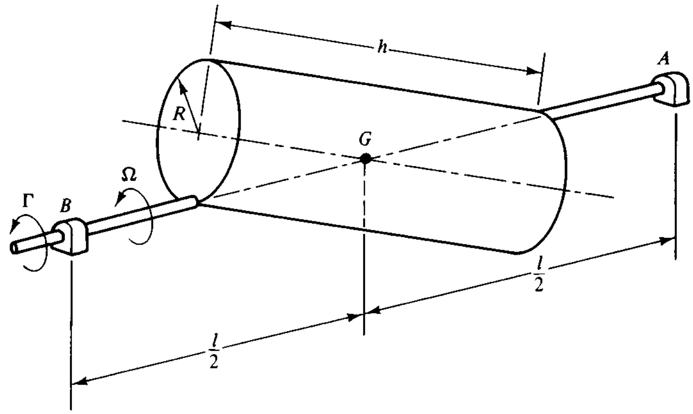
In some situations, the qualitative features of a motion are known except for the value of an angle of orientation or a rotation rate. Such conditions lead to the same type of formulation as that in which the motion is fully specified, except that the list of unknowns also will contain the unspecified kinematical parameters.

A more difficult situation arises when the motion's nature is not known in advance. The orientation of the body may then be described in terms of Eulerian angles (precession, nutation, and spin). The result will be differential equations for the Eulerian angles. Recall that $\bar{\alpha}$ depends on these angles and on their first and second derivatives. Also, the product $\bar{\omega} \times \bar{H}_A$ enters into the evaluation of $\dot{\bar{H}}_A$. As a result, the equations of rotational motion will usually be coupled, nonlinear, second-order differential equations. Analytical solutions of such equations are available in limited situations, but numerical techniques are often necessary. In any case, it is standard practice when the motion is unknown to eliminate the reactions from the equations of motion. The reactions enter into the equations of motion algebraically through the force and moment sums. Hence, their elimination involves, at the worst, a process of simultaneous solution of algebraic equations. (This, of course, assumes that a condition of redundant constraint does not exist.)

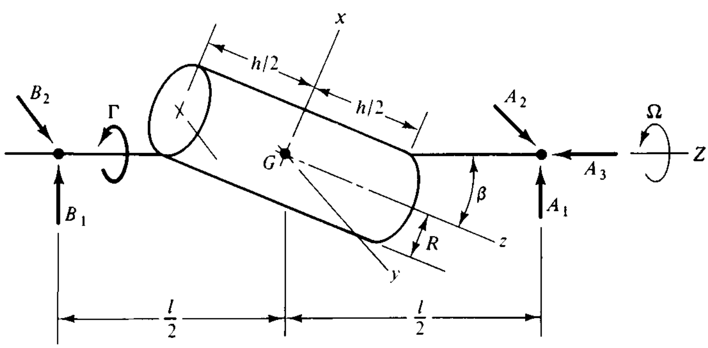
Example 5.7 The cylinder, whose mass is m, is welded to the shaft such that its center is situated on the axis of rotation. The presence of a torque Γ causes the rotation rate Ω to vary. Derive expressions for Γ and the reactions at bearings A and B in terms of Ω and $\dot{\Omega}$.

Solution The first step is to draw a free-body diagram of the cylinder and the shaft. This diagram shows two transverse reactions at each bearing, and a thrust reaction at bearing A. We shall ignore the weight, because it is a static force. The corresponding static reactions divide equally between the bearings, and superpose on the dynamic reactions we shall evaluate.

It is reasonable to assume that the mass of the shaft is negligible in comparison to the mass of the cylinder. The xyz axes we select match those given in the appendix, so the inertia properties for the centroidal principal axes are



Example 5.7



Free-body diagram.

$$I_{xx} = I_{yy} = m\left(\frac{1}{4}R^2 + \frac{1}{12}h^2\right), \qquad I_{zz} = \frac{1}{2}mR^2, \qquad I_{xy} = I_{xz} = I_{yz} = 0.$$

The center of mass is on the axis of rotation, so $\bar{a}_G = \bar{0}$. The angular velocity and angular acceleration are

$$\bar{\omega} = -\Omega\bar{K}, \qquad \bar{\alpha} = -\dot{\Omega}\bar{K},$$

which when resolved into xyz components become

$$\bar{\omega} = -\Omega[(\sin\beta)\bar{i} + (\cos\beta)\bar{k}], \qquad \bar{\alpha} = -\dot{\Omega}[(\sin\beta)\bar{i} + (\cos\beta)\bar{k}].$$

We may employ Euler's equations because xyz are principal axes. To form the moment resultants the applied couple $\bar{\Gamma}$ must be resolved into components. Referring to the free-body diagram for the lever arms of the forces yields

$$\sum M_{Gx} = -\Gamma\sin\beta + (B_2 - A_2)\frac{l}{2}\cos\beta = I_{xx}\alpha_x$$
$$= m\left(\frac{1}{4}R^2 + \frac{1}{12}h^2\right)(-\dot{\Omega}\sin\beta),$$

$$\sum M_{Gy} = (A_1 - B_1)\frac{l}{2} = -(I_{zz} - I_{xx})\omega_x\omega_z$$
$$= -m\left(\frac{1}{4}R^2 - \frac{1}{12}h^2\right)\Omega^2 \sin\beta\cos\beta,$$
$$\sum M_{Gz} = -\Gamma\cos\beta + (A_2 - B_2)\frac{l}{2}\sin\beta = I_{zz}\alpha_z = \frac{1}{2}mR^2(-\dot{\Omega}\cos\beta).$$

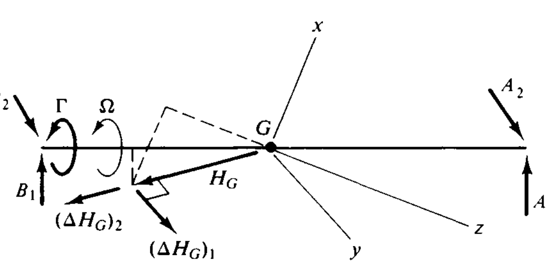
The force sums may be formed in terms of components relative to any set of axes. It is convenient to use the directions of the bearing reactions. Because $\bar{a}_G = \bar{0}$, we have

$$A_3 = 0, \qquad A_1 + B_1 = 0, \qquad A_2 + B_2 = 0.$$

When we consider Ω to be a known function of time, we have three force equations and three moment equations for the six unknowns A_1, A_2, B_1, B_2, A_3, and Γ. The solutions are

$$A_3 = 0, \qquad A_1 = -B_1 = \frac{1}{24}\frac{m\Omega^2}{l}(h^2 - 3R^2)\sin 2\beta,$$
$$A_2 = -B_2 = \frac{1}{24}\frac{m\dot{\Omega}}{l}(h^2 - 3R^2)\sin 2\beta,$$
$$\Gamma = \frac{1}{12}m\dot{\Omega}[6R^2 + (h^2 - 3R^2)\sin^2\beta].$$

Each of these results may be readily explained from the viewpoint of changes in the angular momentum $\bar{H}_G$. Let us suppose for this discussion that $h^2 > 3R^2$, so that $I_{xx} > I_{zz}$. Then at any instant, such as the one depicted in the sketch, the angle from the negative z axis to $\bar{H}_G$ is larger than θ. In this sketch $(\Delta\bar{H}_G)_1$ and $(\Delta\bar{H}_G)_2$ represent increments in $\bar{H}_G$ that would be observed over a small time interval. The effect of the rotation about the horizontal axis is to change the direction of $\bar{H}_G$, such that $(\Delta\bar{H}_G)_1$ is parallel to the y direction. The force system must exert a corresponding net moment about the y axis, which can only be produced by the reactions A_1 and B_1. The increase in Ω due to its time dependence results in an increase in the magnitude of $\bar{H}_G$, which corresponds in the sketch to $(\Delta\bar{H}_G)_2$, parallel to $\bar{H}_G$. The component of this change that is parallel to the horizontal axis of rotation must be matched by the torque Γ, which is the only portion of the force system that exerts a moment about the rotation axis. In addition, $(\Delta\bar{H}_G)_2$ also has a component transverse to the rotation axis in the z-Z plane. The corresponding moment is obtained from the



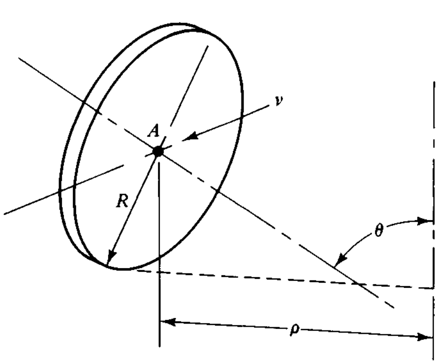
Effect of rotation on the angular momentum.

forces A_2 and B_2. Because the center of mass is on the rotation axis, A_1 and B_1 must form a couple, as must A_2 and B_2. The sense of each couple suggested by the increments in $\bar{H}_G$ appearing in the sketch is consistent with the results of our analysis.

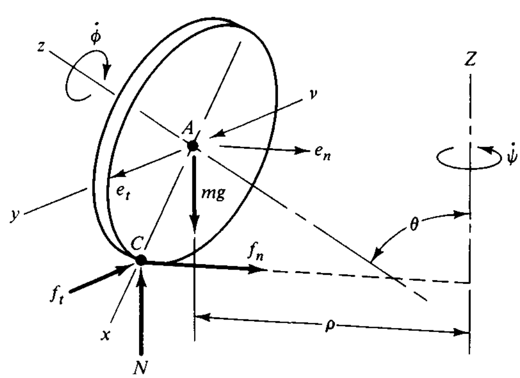
A convenient additional check on the solutions is to ascertain that the results are physically sensible when $\beta = 0°$ and $\beta = 90°$. We observe that $\bar{H}_A$ is then always parallel to the Z axis because in either case the fixed axis of rotation Z coincides with a principal axis. Correspondingly, we expect that the reactions should vanish, and that $M = I_{ZZ}\dot{\Omega}$. Both expectations are borne out by the above expressions. Notice also that the reactions vanish if $h = \sqrt{3}R$, regardless of β. The moments of inertia in this case are equal. This is dynamically similar to letting the rotating body be a sphere. All centroidal axes, including the rotation axis, are principal axes in this case.

The usage of coordinate systems in the preceding example is illustrative of a feature common to the analysis of many systems. It is mandatory for the application of Eqs. (5.69)-(5.74) that the xyz coordinate system be attached to the body of interest. Suppose that the body has axial symmetry and we wish to employ the tabulated inertia properties without recourse to rotation transformations. We then need to select one of the coordinate axes to coincide with, or at least be parallel to, the symmetry axis. Because of the axisymmetry, the inertia properties with respect to the other coordinate axes are invariant. As a result, we may select the instantaneous orientation of those axes in any manner that expedites description of the relevant vectorial quantities. In contrast, if the body is not axisymmetric and we desire to avoid a rotation transformation of the inertia properties, we must select xyz to be parallel to the axes employed in the tabulation. In that case, all vectorial components will need to be resolved into components relative to an xyz system having an arbitrary instantaneous orientation. This aspect of the formulation will be evident in Example 5.10.

Example 5.8 A thin homogeneous disk of mass m rolls without slipping on a horizontal plane such that the center A has a constant speed v as it follows a circular path of radius ρ. The angle of inclination of the axis relative to the vertical is a constant value θ. Derive an expression relating v to the other parameters.



Example 5.8



Free-body diagram.

Solution In addition to the gravitational force, there are reactions at the contact between the disk and the ground. We have depicted in the free-body diagram the frictional forces lying in the horizontal plane as a component F_n toward the vertical axis about which the disk rotates, and a component F_t opposite the velocity of point A. We select a body-fixed coordinate system xyz such that the x and y axes lie in the plane of the disk. At the instant of interest the y axis is aligned horizontal, in the direction of the velocity of point A. The rotation of the disk consists of a precession $\dot{\psi}$ about the vertical axis and a spin $\dot{\phi}$ about the z axis; the speed of the center is related to the precession rate by

$$\dot{\psi} = v/\rho.$$

The angular velocity of the disk is a superposition of the rotations about the two axes,

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\phi}\bar{k}.$$

The corresponding angular acceleration is

$$\bar{\alpha} = \dot{\phi}\bar{\omega} \times \bar{k}.$$

At the instant we have depicted in the free-body diagram, we have

$$\bar{K} = (\cos\theta)\bar{k} - (\sin\theta)\bar{i},$$

which leads to instantaneous expressions for the angular motion of the disk:

$$\bar{\omega} = -(\dot{\psi}\sin\theta)\bar{i} + (\dot{\phi} + \dot{\psi}\cos\theta)\bar{k}, \qquad \bar{\alpha} = (\dot{\psi}\dot{\phi}\sin\theta)\bar{j}.$$

The rotation rates are related to the speed v by the no-slip condition at the contact point C, which requires that $\bar{v}_C = \bar{0}$. We may describe the velocity of the center A by relating it to point C, and by using the fact that it is undergoing circular motion. Thus,

$$\bar{v}_A = \bar{\omega} \times \bar{r}_{A/C} = v\bar{j} = \rho\dot{\psi}\bar{j}.$$

Substituting $\bar{r}_{A/C} = -R\bar{i}$ yields

$$\rho\dot{\psi} = -R(\dot{\phi} + \dot{\psi}\cos\theta) \Rightarrow \dot{\phi} = -\left(\frac{\rho}{R} + \cos\theta\right)\dot{\psi}.$$

We substitute this relation into the angular motion expressions, with the result that

$$\bar{\omega} = \dot{\psi}\left[-(\sin\theta)\bar{i} - \frac{\rho}{R}\bar{k}\right], \qquad \bar{\alpha} = -\dot{\psi}^2\left(\frac{\rho}{R} + \cos\theta\right)(\sin\theta)\bar{j}.$$

The xyz axes are principal, with $I_{xx} = I_{yy} = \frac{1}{4}mR^2$ and $I_{zz} = \frac{1}{2}mR^2$. Combining the inertia properties with the angular rotation components leads to

$$\bar{H}_A = mR^2\dot{\psi}\left[-\frac{1}{4}(\sin\theta)\bar{i} - \frac{1}{2}\frac{\rho}{R}\bar{k}\right],$$

$$\frac{\delta\bar{H}_A}{\delta t} = -\frac{1}{4}mR^2\dot{\psi}^2\left(\frac{\rho}{R} + \cos\theta\right)(\sin\theta)\bar{j}.$$

The moment equation of motion is formed by substituting the foregoing expressions into Eq. (5.69), and matching the result to the moments exerted by the actual force system. The latter quantities must be described as moments about the x, y, and z axes, in order to match them to the components of $\dot{\bar{H}}_A$. The result of this step is

$$\sum \bar{M}_A = (f_n R \sin\theta - NR\cos\theta)\bar{j} - f_t R\bar{k}$$
$$= \frac{\delta \bar{H}_A}{\delta t} + \bar{\omega}\times\bar{H}_A$$
$$= -\frac{1}{4}mR^2\dot{\psi}^2\left(\frac{\rho}{R}+\cos\gamma\right)(\sin\theta)\bar{j} + mR^2\dot{\psi}^2\left(-\frac{1}{4}\frac{\rho}{R}\sin\theta\right)\bar{j}.$$

Because $\dot{\psi} = v/\rho$, the corresponding component equations are

$$f_n R\sin\theta - NR\cos\theta = -\frac{1}{4}mv^2\left(\frac{R}{\rho}\right)^2\left(2\frac{\rho}{R}+\cos\theta\right)\sin\theta, \qquad f_t = 0.$$

We must eliminate the dependence on the reaction forces if we are to obtain the desired relationship for the speed v. Additional equations of motion are those for the resultant force on the body. The center of mass follows a horizontal circular path of radius ρ at constant speed v, so it undergoes a centripetal acceleration. Rather than resolving $\bar{a}_A$ into components relative to the xyz reference frame, a representation in terms of the path variables for that point yields equations having a more convenient form. Thus, we have

$$\bar{a}_A = \frac{v^2}{\rho}\bar{e}_n.$$

The corresponding force equations are

$$\sum F_n = f_n = m\left(\frac{v^2}{\rho}\right), \qquad \sum F_t = f_t = 0, \qquad \sum F_b = N - mg = 0.$$

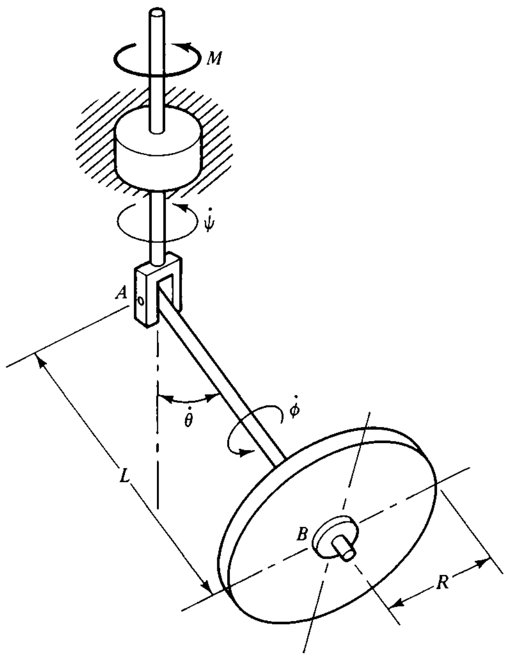
Substituting $f_n = mv^2/\rho$ and $N = mg$ into the moment equation yields

$$v^2 = \frac{4g\rho^2\cot\theta}{6\rho + R\cos\theta}.$$

There is a simple explanation for this steady motion. The gravitational force and normal reaction form a couple about the horizontal diametral line of the disk because the disk is tilted. The frictional reaction required to impart the centripetal acceleration to the center of mass also exerts a moment about this line. The net moment must be matched by a change in the angular momentum. The latter effect is achieved by the precession, which alters the true direction of the angular momentum, even though its components relative to the x and z axes remain constant.

Example 5.9 A servomotor maintains at a constant value the spin rate $\dot{\phi}$ at which the disk rotates relative to the pivoted shaft AB. The precession rate $\dot{\psi}$ about the vertical axis is also held constant by a torque $M(t)$. Derive the differential equation governing the nutation angle θ. Also derive an expression for M.

Solution A free-body diagram of the disk and shaft assembly must account for the reactions. Toward this end, we also draw a free-body diagram of the vertical



Example 5.9

shaft. The pin exerts an arbitrary force $\bar{A}$, which we decompose into components parallel and transverse to shaft AB. The couple $\bar{\Gamma}$ exerted by the pin has no component about the axis of the pin (assuming there is no friction). The vertical shaft carries equal, but opposite, reactions at the pin, as well as transverse forces and couples at its bearing. We assume that both shafts are massless. Then equilibrium of the vertical shaft requires that

$$\sum(M_{\text{vert shaft}})_Z = M - \Gamma_x \sin\theta + \Gamma_z \cos\theta = 0.$$

The xyz axes we selected for this resolution of the forces are parallel to the axes appearing in the appendix for a disk. However, we placed the origin at the fixed point A in order to eliminate the reaction at the pin. This is an allowable point for forming the moment equation because it has a fixed position relative to the disk. Also, because the disk is axisymmetric, we may define the y axis to be instantaneously normal to the plane of the two shafts, without loss of generality. The parallel axis theorems provide

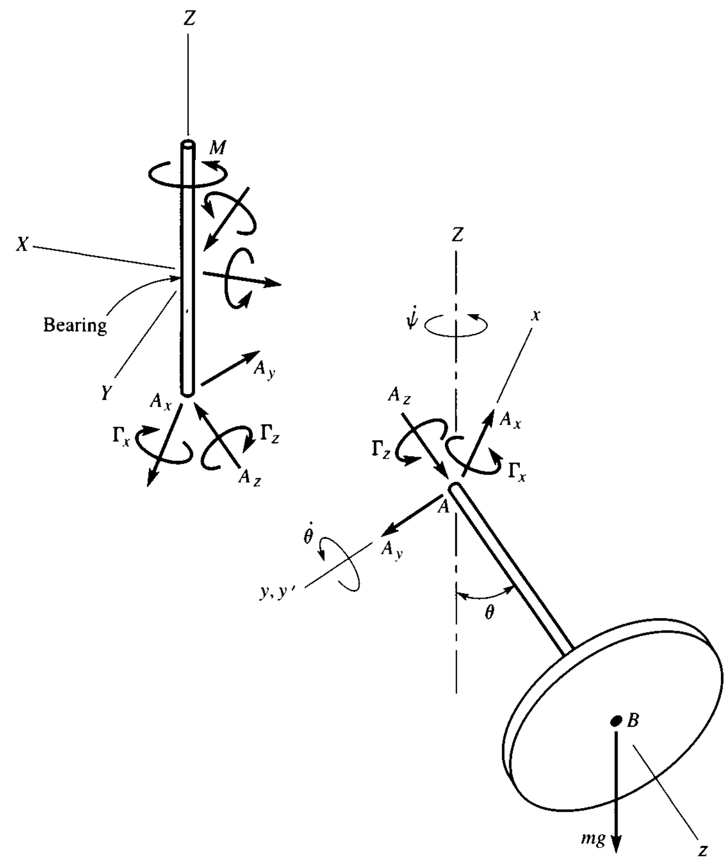
$$I_{xx} = I_{yy} = m(\tfrac{1}{4}R^2 + L^2), \qquad I_{zz} = \tfrac{1}{2}mR^2, \qquad I_{xy} = I_{yz} = I_{xz} = 0.$$

We could employ the Eulerian angles in Chapter 4 to form $\bar{\omega}$ and $\bar{\alpha}$, but it is just as easy to re-derive the results here using the y' axis as the line of nodes for the nutation. Thus, for constant $\dot{\psi}$ and $\dot{\phi}$, we have

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k}, \qquad \bar{\omega}' = \dot{\psi}\bar{K},$$

$$\bar{\alpha} = \ddot{\theta}\bar{j}' + \dot{\theta}(\bar{\omega}' \times \bar{j}') + \dot{\phi}(\bar{\omega} \times \bar{k}).$$

At this instant $\bar{K} = (\sin\theta)\bar{i} - (\cos\theta)\bar{k}$ and $\bar{j}' = \bar{j}$, so



Free-body diagrams.

$$\bar{\omega} = (\dot{\psi}\sin\theta)\bar{i} + \dot{\theta}\bar{j} + (-\dot{\psi}\cos\theta + \dot{\phi})\bar{k}, \qquad \bar{\omega}' = \dot{\psi}[(\sin\theta)\bar{i} - (\cos\theta)\bar{k}],$$

$$\bar{\alpha} = (\dot{\psi}\dot{\theta}\cos\theta + \dot{\theta}\dot{\phi})\bar{i} + (\ddot{\theta} - \dot{\psi}\dot{\phi}\sin\theta)\bar{j} + (\dot{\psi}\dot{\theta}\sin\theta)\bar{k}.$$

Euler's equations are applicable because xyz are principal axes. Using the free-body diagram of the disk and shaft AB to form the moment sums leads to

$$\begin{aligned} \sum M_{Ax} &= \Gamma_x = I_{xx}\alpha_x - (I_{yy} - I_{zz})\omega_y\omega_z \\ &= m[2L^2\dot{\psi}\dot{\theta}\cos\theta + \tfrac{1}{2}R^2\dot{\theta}\dot{\phi}], \end{aligned}$$

$$\begin{aligned} \sum M_{Ay} &= -mgL\sin\theta = I_{yy}\alpha_y - (I_{zz} - I_{xx})\omega_x\omega_z \\ &= m[(L^2 + \tfrac{1}{4}R^2)\ddot{\theta} - (L^2 - \tfrac{1}{4}R^2)\dot{\psi}^2\sin\theta\cos\theta - \tfrac{1}{2}R^2\dot{\psi}\dot{\phi}\sin\theta], \end{aligned}$$

$$\sum M_{Az} = \Gamma_z = I_{zz}\alpha_z - (I_{xx} - I_{yy})\omega_x\omega_y = \tfrac{1}{2}mR^2\dot{\psi}\dot{\phi}\sin\theta.$$

There is no need to form $\sum \bar{F} = m\bar{a}_G$ because those equations would merely lead to relations for the reaction forces A_x, A_y, and A_z. The equation for $\sum M_{Ay}$ yields the differential equation of motion:

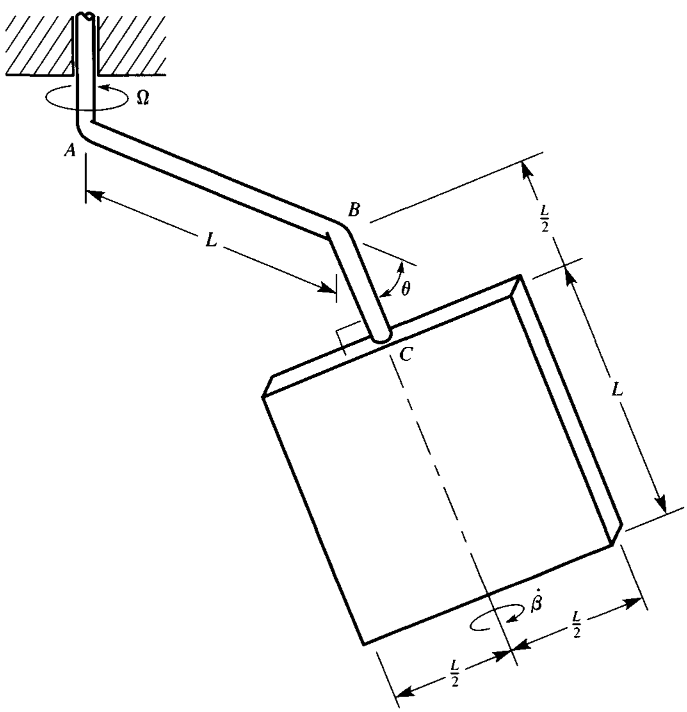
$$(L^2 + \tfrac{1}{4}R^2)\ddot{\theta} - (L^2 - \tfrac{1}{4}R^2)\dot{\psi}^2\sin\theta\cos\theta + (gL - \tfrac{1}{2}R^2\dot{\psi}\dot{\phi})\sin\theta = 0.$$

The equations for $\sum M_{Ay}$ and $\sum M_{Az}$ describe the couple reactions Γ_x and Γ_z. Substituting those expressions into the moment equilibrium equation for the vertical shaft yields the couple M required to sustain the motion:

$$M = \Gamma_x \sin\theta - \Gamma_z \cos\theta = 2m(L^2 - \tfrac{1}{4}R^2)\dot{\psi}\dot{\theta}\sin\theta\cos\theta + \tfrac{1}{2}mR^2\dot{\theta}\dot{\phi}\sin\theta.$$

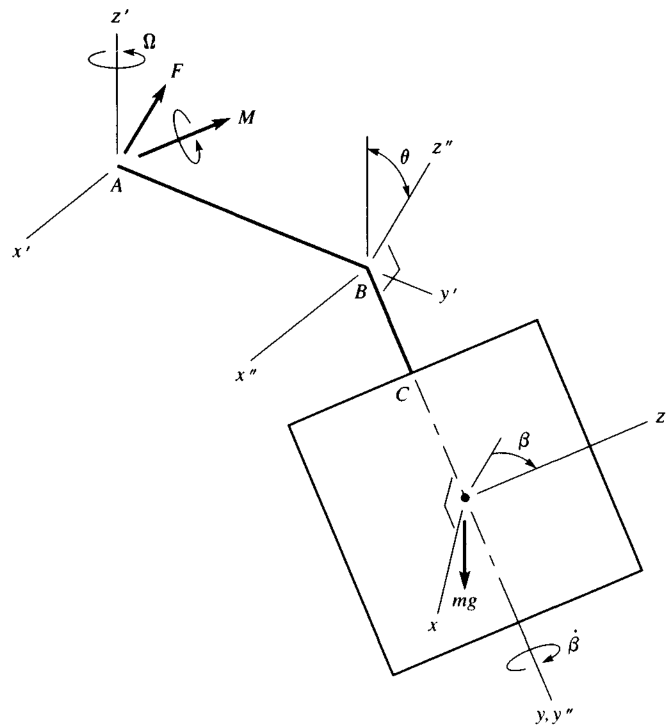
It is interesting to note that couples must be applied about both shafts in order to sustain the precession and spin rates. In particular, if a servomotor was not used to maintain a constant spin rate, that is, if Γ_z was identically zero, the spin angle would be an unknown variable.

Example 5.10 The chemical stirrer consists of a square plate of mass m that spins at a constant rate $\dot{\beta}$ about axis BC of the bent shaft, as that shaft rotates about the vertical axis at the constant rate Ω. The angle of rotation β is defined such that $\beta = 0$ when the plane of the plate is vertical. The mass of the shaft is negligible. As a function of β, determine the internal reactions at joint A required to sustain the motion.



Example 5.10

Solution The free-body diagram cuts the bent shaft at corner A. Because the shaft is rigid, the internal reactions at that location consist of an arbitrary force-couple system, whose components constitute the unknowns we must determine. Because no point on the plate is stationary, we place the origin of the body-fixed xyz axes at the center of mass. We align the axes of this coordinate system with the edges of the plate in order to employ directly the tabulated inertia properties.



Free-body diagram and coordinate systems.

We define two additional coordinate systems in the free-body diagram. The $x'y'z'$ system, which executes the precession, will be useful to describe the force equation of motion. The $x''y''z''$ system also executes only the precession; it will assist the evaluation of the rotation transformation relating $x'y'z'$ and xyz. Each coordinate system differs from the preceding by a single rotation about one of the axes. To go from xyz to $x''y''z''$, we perform a rotation of $+\beta$ about the y axis, while $x'y'z'$ is obtained by rotating $x''y''z''$ by $+\theta$ about the x'' axis. We therefore have

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = [R_\beta] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \quad \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [R] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \quad [R] = [R_\theta][R_\beta],$$

where

$$[R_\beta] = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}, \qquad [R_\theta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix},$$

$$[R] = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ (\sin\theta\sin\beta) & \cos\theta & (\sin\theta\cos\beta) \\ (\cos\theta\sin\beta) & -\sin\theta & (\cos\theta\cos\beta) \end{bmatrix}.$$

We may now form the angular velocity and angular acceleration. Adding the precessional motion about the vertical z' axis and the spin about the y axis leads to

$$\bar{\omega} = \Omega \bar{k}' - \dot{\beta}\bar{j}, \qquad \bar{\alpha} = -\dot{\beta}(\bar{\omega} \times \bar{j}).$$

The direction cosines of $\bar{k}'$ with respect to xyz form the last row of $[R]$. Thus, we have

$$\bar{\omega} = (\Omega \cos\theta \sin\beta)\bar{i} - (\Omega \sin\theta + \dot{\beta})\bar{j} + (\Omega \cos\theta \cos\beta)\bar{k}$$

$$\bar{\alpha} = (\dot{\beta}\Omega \cos\theta \cos\beta)\bar{i} - (\dot{\beta}\Omega \cos\theta \sin\beta)\bar{k}.$$

The inertia properties for a square plate are found from the appendix to be

$$I_{xx} = \frac{1}{6}mL^2, \qquad I_{yy} = I_{zz} = \frac{1}{12}mL^2, \qquad I_{xy} = I_{xz} = I_{yz} = 0.$$

We combine these properties and the components of $\bar{\omega}$ and $\bar{\alpha}$ to form $\dot{\bar{H}}_G$ as follows:

$$\bar{H}_G = I_{xx}\omega_x \bar{i} + I_{yy}\omega_y \bar{j} + I_{zz}\omega_z \bar{k}$$
$$= \frac{1}{12}mL^2[(2\Omega \cos\theta \sin\beta)\bar{i} - (\Omega \sin\theta + \dot{\beta})\bar{j} + (\Omega \cos\theta \cos\beta)\bar{k}];$$

$$\frac{\delta \bar{H}_G}{\delta t} = I_{xx}\alpha_x \bar{i} + I_{yy}\alpha_y \bar{j} + I_{zz}\alpha_z \bar{k}$$
$$= \frac{1}{12}mL^2[(2\dot{\beta}\Omega \cos\theta \cos\beta)\bar{i} - (\dot{\beta}\Omega \cos\theta \sin\beta)\bar{k}];$$

$$\dot{\bar{H}}_G = \frac{\delta \bar{H}_G}{\delta t} + \bar{\omega} \times \bar{H}_G$$
$$= \frac{1}{12}mL^2[(2\dot{\beta}\Omega \cos\theta \cos\beta)\bar{i} + \Omega^2(\cos^2\theta \sin\beta \cos\beta)\bar{j} + \Omega^2(\sin\theta \cos\theta \sin\beta)\bar{k}].$$

We also require an expression for the acceleration of the center of mass. The simplest way of determining this comes from recognizing that the center of the plate precesses about the vertical axis in a circular path of radius $L + L\cos\theta$, so that

$$\bar{a}_G = -(L + L\cos\theta)\Omega^2 \bar{e}_R = -(L + L\cos\theta)\Omega^2 \bar{j}'.$$

The resultant force acting on the system isolated in the free-body diagram is the sum of the reaction force $\bar{F}$ and the weight; the force equation of motion thus gives

$$\bar{F} - mg\bar{k}' = -m(L + L\cos\theta)\Omega^2 \bar{j}'.$$

We add the moment of $\bar{F}$ about point G to the couple reaction in order to form the corresponding moment equation of motion, from which we obtain

$$\bar{M} + \bar{r}_{A/G} \times \bar{F} = \dot{\bar{H}}_G.$$

The force equation yields an expression for $\bar{F}$ in terms of $x'y'z'$ coordinates, $\bar{F} = -m(L + L\cos\theta)\Omega^2 \bar{j}' + mg\bar{k}'$. The position $\bar{r}_{A/G}$ needed to express the moment of $\bar{F}$ about point G is readily found in terms of $x'y'z'$ components, consistent with the expression for $\bar{F}$. We therefore write

$$\bar{r}_{A/G} = -(L + L\cos\theta)\bar{j}' + (L\sin\theta)\bar{k}',$$

$$\bar{r}_{A/G} \times \bar{F} = mL(1 + \cos\theta)(-g + \Omega^2 \sin\theta)\bar{i}'.$$

The moment equation of motion indicates that $\bar{M} = -\bar{r}_{A/G} \times \bar{F} + \dot{\bar{H}}_G$. In order to add the two terms we must express both in terms of a common set of components. Using $x'y'z'$ is more meaningful if one is interested in a stress analysis of joint A. Because the transformation from xyz to $x'y'z'$ is described by $[R]$, we use matrix notation to show the computation:

$$\{M\} = mL(1+\cos\theta)(g-\Omega^2 \sin\theta)\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + [R]\{\dot{H}_G\},$$

where $\{\dot{H}_G\}$ contains the xyz components of $\dot{\bar{H}}_G$. It is a simple matter to carry out these operations for any specified value of the parameters.

A term of particular interest is the $\bar{k}'$ component of $\bar{M}$, which represents the torque that must be applied about the vertical segment of the shaft in order to maintain the precession rate at a constant value. This term is

$$\bar{M} \cdot \bar{k}' = \frac{1}{12} mL^2 \Omega \dot{\beta} (\cos\theta)^2 (\sin 2\beta).$$

The dependence of this moment on the product of two rotation rates indicates that it is a gyroscopic effect. The dependence on $\sin 2\beta$ is not surprising because the system has the same configuration if β is increased by 180°, and $\beta = 0$ corresponds to both rotation axes coinciding with a symmetry plane of the plate. The dependence on $\cos\theta$ is also reasonable, because $\theta = \pm 90°$ corresponds to both rotation axes being vertical, in which case the angular momentum has a constant direction.

5.5 Planar Motion

The principles governing spatial motion provide an interesting perspective for the kinetics of planar motion. Let x and y represent convenient directions in the plane, so that $\bar{\omega}$ is parallel to the z axis. Then

$$\bar{a}_G = a_{Gx}\bar{i} + a_{Gy}\bar{j}, \qquad \bar{\omega} = \omega\bar{k}, \qquad \bar{\alpha} = \dot{\omega}\bar{k}. \tag{5.75}$$

The corresponding angular momentum for a coordinate system whose origin is at an allowable point is

$$\bar{H}_A = -I_{xz}\omega\bar{i} - I_{yz}\omega\bar{j} + I_{zz}\omega\bar{k}. \tag{5.76}$$

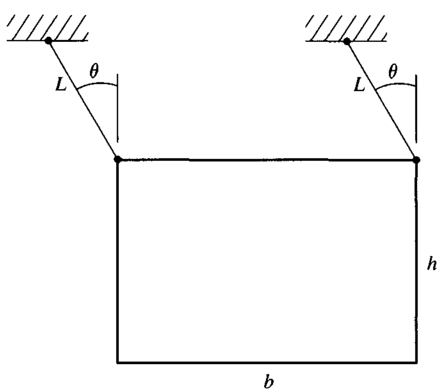
This shows that, if z is not a principal axis, then the angular momentum is not parallel to the angular velocity. In that case, couples in the plane of motion (in the vectorial sense) are required to sustain the planar motion. The equations of motion corresponding to Eqs. (5.68) and (5.69) are

$$\begin{aligned} &\sum F_x = ma_{Gx}, \quad \sum F_y = ma_{Gy}, \quad \sum F_z = 0; \\ &\sum M_{Ax} = -I_{xz}\dot{\omega} + I_{yz}\omega^2, \quad \sum M_{Ay} = -I_{yz}\dot{\omega} + I_{xz}\omega^2, \quad \sum M_{Az} = I_{zz}\dot{\omega}. \end{aligned} \tag{5.77}$$

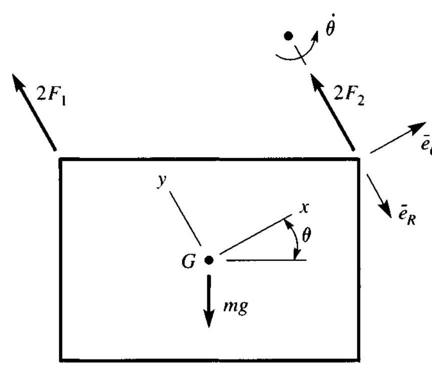
The three force equations and the equation governing the moment about the z axis are the same as those developed in elementary courses. The moments about the x and y axes are gyroscopic moments. They are the result of asymmetrical distributions of mass relative to the x-y plane, corresponding to nonzero values of I_{xz} and

I_{yz}. The dependence of these moments on the square of the rate of rotation means that very large couples must be exerted by the external force system in order to sustain a planar motion at large values of ω. This has serious implications for balancing rotating machinery, because the reactions must be provided by the supports (e.g. bearings). This condition, which is known as *dynamic imbalance,* can occur even though the center of mass is on a fixed axis of rotation ($\bar{a}_G = \bar{0}$), as it would be after the rotating system has been statically balanced. The process of dynamically balancing a rotating part entails making the axis of rotation a principal axis of inertia, so that $I_{xz} = I_{yz} = 0$. Then the angular momentum will always coincide with the axis of rotation. We encountered a dynamically unbalanced system in Example 5.7. We elected there to use principal axes that do not align with the axis of rotation, rather than nonprincipal axes having an axis that coincides with the rotation axis.

Example 5.11 Four cables attached to each corner support the box. (Only two are visible in the side view.) Derive a differential equation governing the angle of inclination θ of the cables, and derive expressions for the tensile force in each cable.



Example 5.11



Free-body diagram and coordinate system.

Solution In the free-body diagram, the tensile forces on the left side are assumed to be different from those on the right, because the arrangement of cables is not symmetrical when $\theta \neq 0$. The cables remain parallel for any θ, so the box undergoes a pure translation. As a result, the acceleration of the center of mass matches that of any of the points at which a cable is attached. We therefore have

$$\bar{\alpha} = \bar{0}, \qquad \bar{a}_G = -L\dot{\theta}^2\bar{e}_R + L\ddot{\theta}\bar{e}_\theta,$$

where $\bar{e}_R$ and $\bar{e}_\theta$ are polar coordinate unit vectors relative to the fixed point of any cable.

Because the box translates, we must sum moments about the center of mass. The body is symmetric about the (vertical) plane of motion intersecting its center of mass, so we need not compute any moments of inertia. Consequently, we may orient the x and y axes in any convenient direction. The directions appearing in the free-body diagram are selected to align with the directions for the acceleration components, $\bar{i} = \bar{e}_\theta$ and $\bar{j} = -\bar{e}_R$. The corresponding equations of motion are

$$\sum F_x = -mg\sin\theta = m(a_G)_x = m(L\ddot{\theta}),$$

$$\sum F_y = 2F_1 + 2F_2 - mg\cos\theta = m(a_G)_y = m(L\dot{\theta}^2),$$

$$\sum M_{Gz} = (2F_1\sin\theta)\frac{h}{2} - (2F_1\cos\theta)\frac{b}{2} + (2F_2\sin\theta)\frac{h}{2} + (2F_2\cos\theta)\frac{b}{2} = 0.$$

The solution of these equations is

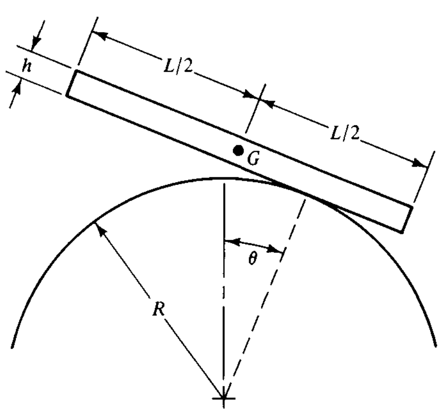
$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0;$$

$$F_1 = \frac{1}{4}\left(1 + \frac{h}{b}\tan\theta\right)(mg\cos\theta + mL\dot{\theta}^2),$$

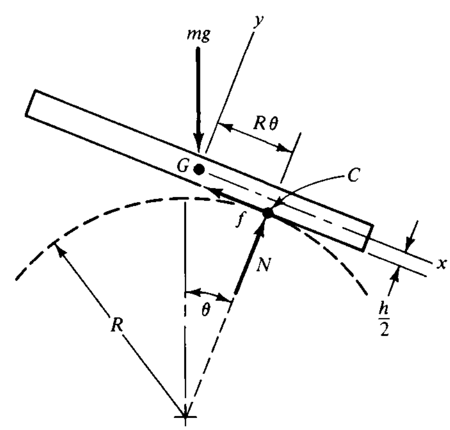
$$F_2 = \frac{1}{4}\left(1 - \frac{h}{b}\tan\theta\right)(mg\cos\theta + mL\dot{\theta}^2).$$

Note that the differential equation for θ is identical to that for a simple pendulum formed by attaching a particle to the end of a cable of length L.

Example 5.12 The bar of mass m is placed horizontally on the semicylinder, such that contact is below the centroid G. Assuming that the bar does not slip, derive the differential equation of motion governing the angle θ.



Example 5.12



Free-body diagram.

Solution The free-body diagram of the bar must show the bar at an arbitrary angle of elevation θ. The friction and normal forces act at the contact point C. Due to the absence of slippage, point C is at a distance $R\theta$ from the center of mass G. The moment equation must be formulated relative to point G because the body is in general motion.

We attach xyz to the bar in order to perform a kinematical analysis. For the assumed sense of the rotation, the angular velocity is

$$\bar{\omega} = -\dot{\theta}\bar{k}, \qquad \bar{\alpha} = -\ddot{\theta}\bar{k}.$$

The velocity of point C is zero, so the velocity of the center of mass is

$$\bar{v}_G = \bar{\omega} \times \bar{r}_{G/C} = (-\dot{\theta}\bar{k}) \times \left(-R\theta\bar{i} + \frac{h}{2}\bar{j}\right) = \frac{h}{2}\dot{\theta}\bar{i} + R\theta\dot{\theta}\bar{j}.$$

This expression is generally valid. It may therefore be differentiated to describe the acceleration of the center of mass,

$$\begin{aligned}\bar{a}_G &= \frac{\delta \bar{v}_G}{\delta t} + \bar{\omega} \times \bar{v}_G \\ &= \frac{h}{2}\ddot{\theta}\bar{i} + R(\theta\ddot{\theta} + \dot{\theta}^2)\bar{j} + (-\dot{\theta}\bar{k}) \times \left(\frac{h}{2}\dot{\theta}\bar{i} + R\theta\dot{\theta}\bar{j}\right) \\ &= \left(R\theta\dot{\theta}^2 + \frac{h}{2}\ddot{\theta}\right)\bar{i} + \left[R\theta\ddot{\theta} + \left(R - \frac{h}{2}\right)\dot{\theta}^2\right]\bar{j}.\end{aligned}$$

For the purpose of evaluating the moment of inertia, we assume that the cross-section of the bar is rectangular. Then

$$I_{zz} = \frac{1}{12}m(L^2 + h^2).$$

The corresponding equations of the motion are

$$\sum M_{Gz} = N(R\theta) - f\frac{h}{2} = \frac{1}{12}m(L^2 + h^2)(-\ddot{\theta}),$$

$$\sum F_x = mg\sin\theta - f = m\left(R\theta\dot{\theta}^2 + \frac{h}{2}\ddot{\theta}\right),$$

$$\sum F_y = N - mg\cos\theta = m\left[R\theta\ddot{\theta} + \left(R - \frac{h}{2}\right)\dot{\theta}^2\right].$$

In order to obtain the desired differential equation, we eliminate the reactions. The force equations give

$$f = m\left(g\sin\theta - R\theta\dot{\theta}^2 - \frac{h}{2}\ddot{\theta}\right),$$

$$N = m\left[g\cos\theta + R\theta\ddot{\theta} + \left(R - \frac{h}{2}\right)\dot{\theta}^2\right],$$

which when substituted into the moment equations yields

$$\left[\frac{1}{12}(L^2 + 4h^2) + R^2\theta^2\right]\ddot{\theta} + R^2\theta\dot{\theta}^2 + gR\theta\cos\theta - g\frac{h}{2}\sin\theta = 0.$$

When θ is small, we may linearize the differential equation by introducing the approximations $\cos\theta \approx 1$, $\sin\theta \approx \theta$, and dropping any terms that have quadratic or higher powers of θ. The resulting equation for small rotations is

$$\frac{1}{12}(L^2 + 4h^2)\ddot{\theta} + g\left(R - \frac{h}{2}\right)\theta = 0.$$

When $h < 2R$, the response obtained from this equation is sinusoidal, corresponding to oscillations about a stable static equilibrium position. In contrast, when $h > 2R$, the solution of the linearized equation of motion is exponential, corresponding to continuous movement away from an unstable static equilibrium position. The transition from stability to instability has a simple explanation. In the case where the bar is slender, $h < 2R$, the center of mass rises as θ increases. Thus, $\theta = 0$ is a position of minimum potential energy. In contrast, when $h > 2R$, the center of mass descends with movement away from the equilibrium position, which means that $\theta = 0$ corresponds to maximum potential energy. Note that the stability transition is independent of the value of the length L. The magnitude of L affects the equivalent moment of inertia of the bar, which is the coefficient of the angular acceleration term in the equation of motion. Thus, when the equilibrium position is stable, the value of L will affect the frequency of the stable oscillation.

In actual practice it may not be possible to satisfy the supposition that the bar does not slip. This would lead to a loss of stability in a different manner. Coulomb's friction laws state that the maximum friction force that can be developed between surfaces that rub against each other is $\mu_s N$, where μ_s is the coefficient of static friction. The friction force f occurring in the present problem is the force required to prevent sliding. If we apply the linearization approximation associated with $\theta \ll 1$ to the solutions for f and N, as well as to the differential equation of motion, we find that

$$f \approx m\left(g\theta - \frac{h}{2}\ddot{\theta}\right), \qquad N \approx mg.$$

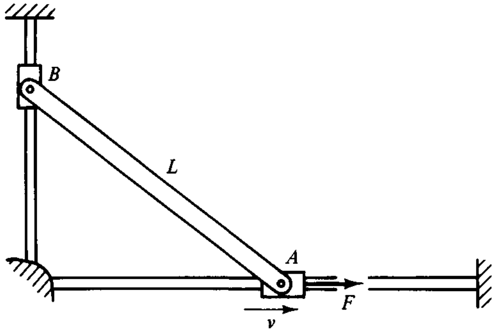
Using the linearized differential equation to eliminate $\ddot{\theta}$ from the foregoing leads to an expression for the minimum coefficient of sliding friction required to prevent slipping at any position:

$$\mu_s = \frac{f}{N} = \frac{L^2 + 6hR + h^2}{L^2 + 4h^2} g\theta.$$

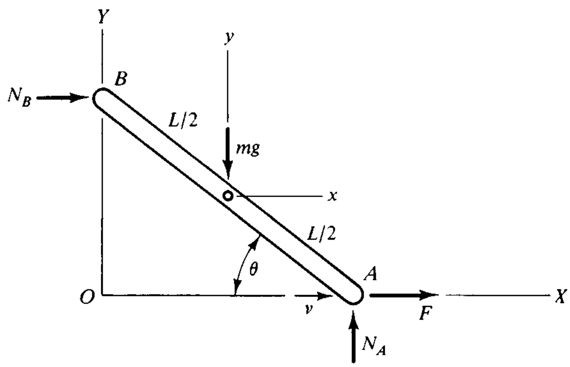
In general, if the friction force f required to prevent surfaces from slipping exceeds $\mu_s N$ at some position, then sliding will be initiated at that position. Coulomb's laws state that the magnitude of the friction force then will be $\mu_k N$, where μ_k is the coefficient of kinetic friction. The sense of the sliding friction force then opposes the sense in which one surface moves relative to the other. This is the same as the sense of the static friction force obtained under the assumption that slippage does not occur. Note that if sliding occurs, the kinematical no-slip relations are lost, which offsets the extra equation relating the friction and normal forces.

Example 5.13 The 4-kg bar lies in the vertical plane. The masses of collars A and B are negligible, and the guide bars are smooth. Collar A is pushed to the right at a constant speed of v by the horizontal force F. Determine the value of F as a function of the angle of elevation θ.

Solution It is convenient to define the body-fixed reference frame such that the x axis is horizontal at the instant under consideration. The kinematical relationship between the motion of collar A, the center of mass G, and the rotation of the bar are readily obtained by differentiating their positions relative to the fixed reference frame XYZ. For the constrained point A, we have



Example 5.13



Free-body diagram.

$$\bar{r}_{A/O} = (L\cos\theta)\bar{I}, \qquad \bar{v}_A = -(L\dot{\theta}\sin\theta)\bar{I}.$$

We also know that $\bar{v}_A = v\bar{I}$. Matching the two descriptions leads to

$$\dot{\theta} = -\frac{v}{L\sin\theta},$$

which, when differentiated again with v held constant, yields

$$\ddot{\theta} = \frac{v\dot{\theta}\cos\theta}{L\sin^2\theta} = -\frac{v^2\cos\theta}{L^2\sin^3\theta}.$$

Then, for point G, we find

$$\bar{r}_{G/O} = \frac{L}{2}[(\cos\theta)\bar{I} + (\sin\theta)\bar{J}];$$

$$\bar{v}_G = \dot{\bar{r}}_{G/O} = \frac{L}{2}\dot{\theta}[-(\sin\theta)\bar{I} + (\cos\theta)\bar{J}] = \frac{1}{2}v[\bar{I} - (\cot\theta)\bar{J}],$$

$$\bar{a}_G = \dot{\bar{v}}_G = \frac{1}{2}v\frac{\dot{\theta}}{\sin^2\theta}\bar{J} = -\frac{1}{2}\frac{v^2}{L\sin^3\theta}\bar{J}.$$

The foregoing gives $\bar{a}_G$ in terms of horizontal and vertical components. The angular acceleration is

$$\bar{\alpha} = -\ddot{\theta}\bar{k}.$$

The corresponding equations of motion are

$$\sum M_{Gz} = N_A\left(\frac{L}{2}\cos\theta\right) + (F - N_B)\left(\frac{L}{2}\sin\theta\right) = \frac{1}{12}mL^2(-\ddot{\theta}) = \frac{mv^2\cos\theta}{12\sin^3\theta},$$

$$\sum F_x = F + N_B = 0, \qquad \sum F_y = N_A - mg = m\left(-\frac{v^2}{2L\sin^3\theta}\right).$$

We solve the force equations for the reactions, and then use those expressions to eliminate the reactions from the moment equation. The resulting expression for F is

$$F = \frac{mv^2\cos\theta}{3L\sin^4\theta} - \frac{1}{2}mg\cot\theta.$$

5.6 Impulse-Momentum and Work-Energy Principles

The force and moment equations discussed thus far govern the linear and angular acceleration of a body. Momentum and energy principles, which represent standard integrals of these equations, may be used to relate the linear and angular velocity of the body at successive instants or locations. The evaluation of the associated impulse and work quantities in some situations requires knowledge of the body's motion, so these integral relations supplement, rather than replace, the basic acceleration equations.

5.6.1 *Momentum Principles*

Equations (5.18) and (5.19) are the time derivative forms of the impulse-momentum relations. Definite integration of them between two instants t_1 and t_2 leads to

$$\blacklozenge \quad \bar{P}_2 = \bar{P}_1 + \int_{t_1}^{t_2} \sum \bar{F}\,dt,$$

$$\blacklozenge \quad (\bar{H}_A)_2 = (\bar{H}_A)_1 + \int_{t_1}^{t_2} \sum \bar{M}_A\,dt. \tag{5.78}$$

Both of these equations state that the final value of the respective momenta exceeds the initial value by the corresponding *impulse,* that is, the time integral of the resultant force or moment. The angular momentum is described by Eq. (5.70). It is clear from this equation that, unless xyz constitutes a set of principle axes, a moment acting about one coordinate axis may lead to rotation about several axes.

Both momentum principles are vector equations, so they each yield three scalar equations obtained from equating like components. If the impulses can be evaluated, then the scalar equations fully define the corresponding change in the linear or angular velocities. The difficulty lies in that evaluation. The resultant force and moment acting on a body are seldom known in advance, because the reactions are unknown. Furthermore, it is not sufficient to know the force or moment in terms of components relative to the body because the corresponding unit vectors are not constant. Carrying out the impulse integrals in this case requires knowledge of the orientation of the unit vectors, and of the components, as functions of time. Such information usually is not available, because it depends on the bodily motion being studied.

One situation where momentum-impulse relations are useful is in treating *impulsive forces,* such as those generated by impacts and explosions. Impulsive forces are defined as imparting very large accelerations to a body over a very short time interval. The velocity change (linear and angular) may be evaluated in this case by considering the time interval to be so short that we may neglect any change in position during the action of the impulsive forces. Further simplifications stem from the fact that only the impulse of the forces, and not their true time dependence, need be known in order to evaluate the velocities. Also, impulsive forces are usually large enough that the influence of nonimpulsive forces is negligible during the brief interval of the impulse. (Note in this regard that reactions are usually impulsive, because they may be as large as necessary to impose the associated motion constraint.)

Either impulse-momentum principle can also prove useful by yielding a conservation equation. It might be that the component of the resultant force in a specific fixed direction, denoted $\bar{e}_F$, is known as a function of time. Similarly, we might know the resultant moment about an axis $\bar{e}_M$ having fixed orientation and intersecting the origin for the moment equation of motion. The corresponding type of impulse in these directions may therefore be evaluated by taking the component of the appropriate impulse-momentum equation in the direction of the known resultant force or moment:

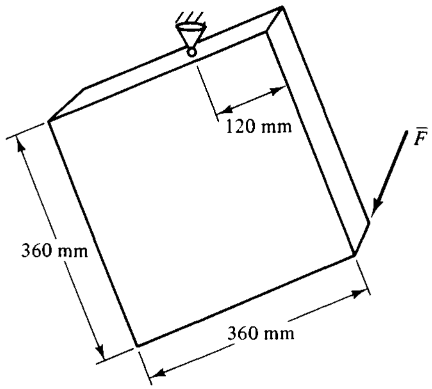
$$\blacklozenge \quad m(\bar{v}_G)_2\cdot\bar{e}_F = m(\bar{v}_G)_1\cdot\bar{e}_F + \int_{t_1}^{t_2} \sum \bar{F}\cdot\bar{e}_F\,dt,$$

$$\blacklozenge \quad (\bar{H}_A)_2\cdot\bar{e}_M = (\bar{H}_A)_1\cdot\bar{e}_M + \int_{t_1}^{t_2} \sum \bar{M}_A\cdot\bar{e}_M\,dt, \qquad (5.79)$$

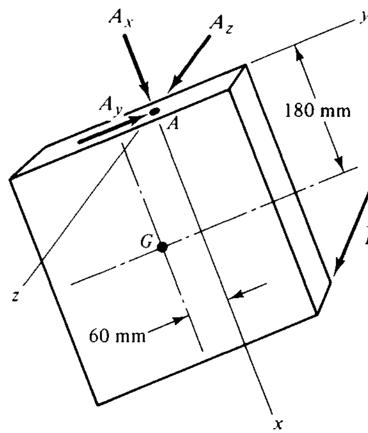
where point A is one of the allowable points for formulating the equations of rotational motion. The most common situation where a force or moment component is known as a function of time is the case where it vanishes. Then Eqs. (5.79) become conservation principles, stating that $\bar{v}_G\cdot\bar{e}_F$ is constant or that $\bar{H}_A\cdot\bar{e}_M$ is constant.

Example 5.14 A 10-kg square plate suspended by a ball-and-socket joint is at rest when it is struck by a hammer. The impulsive force $\bar{F}$ generated by the hammer is normal to the surface of the plate, and its average value during the 4-ms interval that it acts is 5,000 N. Determine the angular velocity of the plate at the instant following the impact, and the average reaction at the support.

Solution The force $\bar{F}$ is much larger than the weight of the plate, so the latter is omitted from the free-body diagram. In contrast, the reaction exerted by the ball-and-socket joint is impulsive, because it must be as large as necessary to prevent movement of point A. We place the origin of xyz at point A in order to eliminate the angular impulse of this reaction. The coordinates of point A relative to parallel centroidal axes are $(-0.18, 0.06, 0)$ meters, so the inertia properties are



Example 5.14



Free-body diagram.

$$I_{xx} = \frac{1}{12}(10)(0.36^2) + 10(0.06^2) = 0.144 \text{ kg-m}^2,$$

$$I_{yy} = \frac{1}{12}(10)(0.36^2) + 10(0.18^2) = 0.432,$$

$$I_{zz} = \frac{1}{12}(10)(0.36^2 + 0.36^2) + (10)(0.18^2 + 0.06^2) = 0.576,$$

$$I_{xy} = 0 + 10(-0.18)(0.06) = -0.108, \qquad I_{xz} = I_{yz} = 0.$$

The angular velocity is initially zero. Let $\bar{\omega}_2 = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}$ denote the angular velocity at the termination of the impulsive action. Then the corresponding velocity of the center of mass is

$$(\bar{v}_G)_2 = \bar{\omega}_2 \times \bar{r}_{G/A} = 0.06\omega_z \bar{i} + 0.18\omega_z \bar{j} - (0.06\omega_x + 0.18\omega_y)\bar{k}.$$

The final angular momentum about pivot A is

$$(\bar{H}_A)_2 = (0.144\omega_x + 0.108\omega_y)\bar{i} + (0.432\omega_y + 0.108\omega_x)\bar{j} + 0.576\omega_z \bar{k}.$$

Applying the angular impulse-momentum principle to the 4-ms interval of the force leads to

$$\begin{aligned}(\bar{H}_A)_2 = \sum \bar{M}_A \, \Delta t &= [(0.36\bar{i} + 0.12\bar{j}) \times 5000\bar{k}](0.004) \\ &= 2.4\bar{i} - 7.2\bar{j} \text{ N-s}.\end{aligned}$$

The result of matching like components of $(\bar{H}_A)_2$ is

$$(\bar{H}_A)_2 \cdot \bar{i} = 0.144\omega_x + 0.108\omega_y = 2.4,$$

$$(\bar{H}_A)_2 \cdot \bar{j} = 0.432\omega_y + 0.108\omega_x = -7.2,$$

$$(\bar{H}_A)_2 \cdot \bar{k} = 0.576\omega_z = 0,$$

from which we obtain

$$\bar{\omega}_2 = 35.90\bar{i} - 25.64\bar{j} \text{ rad/s}.$$

The next step is to form the linear impulse-momentum principle in order to determine the reaction. Using the earlier expression for $(\bar{v}_G)_2$ leads to

$$m(\bar{v}_G)_2 = \sum \bar{F} \Delta t,$$

$$10[0.06\omega_z \bar{i} + 0.18\omega_z \bar{j} - (0.06\omega_x + 0.18\omega_y)\bar{k}] = [A_x \bar{i} + A_y \bar{j} + (A_z + F)\bar{k}]\Delta t.$$

After substitution of the result for the angular velocity $\bar{\omega}_2$, the components of this equation yield

$$A_x = A_y = 0,$$

$$A_z = -F - 10(0.06\omega_x + 0.18\omega_y)/\Delta t = 1{,}153 \text{ N}.$$

These are average values over the 4-ms interval; the maximum values exceed these.

It might surprise you that the reaction is in the same sense as the impulsive force $\bar{F}$. This result indicates that if the ball-and-socket joint were not present then the plate would rotate about its mass center owing to the moment of the force, and point A would move in the negative z direction. It is possible to locate a curve on the plate representing the locus of points at which the force can be applied without generating a dynamic reaction at the joint. Any such point is sometimes referred to as a *center of percussion.*

5.6.2 *Energy Principles*

The derivation of work-energy principles for a rigid body begins from a more fundamental viewpoint than did the momentum principles. We recall the work-energy principle for a single particle, Eq. (1.18), and consider this particle to be one of many in a system. Correspondingly, we categorize the forces acting on particle i according to whether they are associated with interactions that are internal or external to the system. The work-energy equation in this case becomes

$$\frac{1}{2}m_i(v_i^2)_2 = \frac{1}{2}m_i(v_i^2)_1 + \oint_1^2 \left[\bar{F}_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \bar{f}_{ij}\right] \cdot d\bar{r}_i. \tag{5.80}$$

The line integral expresses the work done by the forces acting on particle i as it moves along its path from its initial position 1 to final position 2. Both work and kinetic energy are scalar quantities. When we add Eq. (5.80) for each particle, we obtain

$$T_2 = T_1 + W_{1\to 2}, \tag{5.81}$$

where T is the total kinetic energy of the system. For a rigid body, this quantity is described by either of Eqs. (5.25). Implementation of the work-energy principle for motion of a rigid body will therefore be possible if we can develop an appropriate method of evaluating the total work, $W_{1\to 2}$, done by all of the forces as the body moves from its initial position 1 to final position 2.

Adding Eq. (5.80) for each particle shows the total work to be

$$W_{1\to 2} = \sum_{i=1}^{N} \oint_1^2 \bar{F}_i \cdot d\bar{r}_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \oint_1^2 \bar{f}_{ij} \cdot d\bar{r}_i. \tag{5.82}$$

Note that the line integrals remain inside the summation, reminding us that the equation requires that we follow each point at which a force is applied. However, this expression may be simplified greatly by specializing it to a rigid body. In that case, the infinitesimal displacements $d\bar{r}_i$ are not independent. Let B denote the point for translational motion in Chasle's theorem. (At this juncture, we allow point B to be arbitrarily selected.) Because an infinitesimal displacement is the result of movement over an infinitesimal time interval, we have

$$d\bar{r}_i = \bar{v}_i\,dt = (\bar{v}_B + \bar{\omega}\times\bar{r}_{i/B})\,dt = d\bar{r}_B + \overline{d\theta}\times\bar{r}_{i/B}, \tag{5.83}$$

where $\overline{d\theta}$ is the corresponding infinitesimal rotation of the body. Substituting Eq. (5.83) into Eq. (5.82) yields

$$\begin{aligned} W_{1\to 2} = {} & \oint_1^2 \sum_{i=1}^N \bar{F}_i\cdot d\bar{r}_B + \oint_1^2 \sum_{i=1}^N \bar{F}_i\cdot(\overline{d\theta}\times\bar{r}_{i/B}) \\ & + \oint_1^2 \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \bar{f}_{ij}\cdot d\bar{r}_B + \oint_1^2 \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \bar{f}_{ij}\cdot(\overline{d\theta}\times\bar{r}_{i/B}). \end{aligned} \tag{5.84}$$

The vector identity for re-arranging a scalar triple product indicates that

$$\bar{F}\cdot(\overline{d\theta}\times\bar{r}_{i/B}) \equiv (\bar{r}_{i/B}\times\bar{F})\cdot\overline{d\theta}.$$

Equation (5.84) then becomes

$$\begin{aligned} W_{1\to 2} = {} & \oint_1^2 \left(\sum_{i=1}^N \bar{F}_i\right)\cdot d\bar{r}_B + \oint_1^2 \left(\sum_{i=1}^N \bar{r}_{i/B}\times\bar{F}_i\right)\cdot\overline{d\theta} \\ & + \oint_1^2 \left(\sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \bar{f}_{ij}\right)\cdot d\bar{r}_B + \oint_1^2 \left(\sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \bar{r}_{i/B}\times\bar{f}_{ij}\right)\cdot\overline{d\theta}. \end{aligned} \tag{5.85}$$

The third and fourth terms enclosed in parentheses describe the resultant force and moment about point B exerted by the internal forces. Both terms vanish according to Eqs. (5.1) and (5.2). Similarly, the first and second terms describe the resultant force, $\sum\bar{F}$, and resultant moment about point B, $\sum\bar{M}_B$, exerted by the external forces. Thus, the work may be computed according to

♦ $$W_{1\to 2} = \oint_1^2 \sum\bar{F}\cdot d\bar{r}_B + \oint_1^2 \sum\bar{M}_B\cdot\overline{d\theta}. \tag{5.86}$$

This expression could have been anticipated from Chasle's theorem. It shows that the total work is the sum of (a) the work done by the resultant of the external forces in moving an arbitrary point B and (b) the work done by the moment of the external forces in the rotation about that point. Equation (5.86) provides an alternative to computing the work done by an external force as an integral along the path of its point of application, as described by Eq. (5.82).

Another alternative to direct evaluation of the work done by a force arises when the force is *conservative*. A conservative force is defined to be one that does no net work when the point where it is applied follows an arbitrary closed path. Consider a force that is exerted on a particle that moves over a closed path from position 1 to position 2, and then back to position 1, as in Figure 5.11. By definition, a conservative force does no work in the overall movement, so

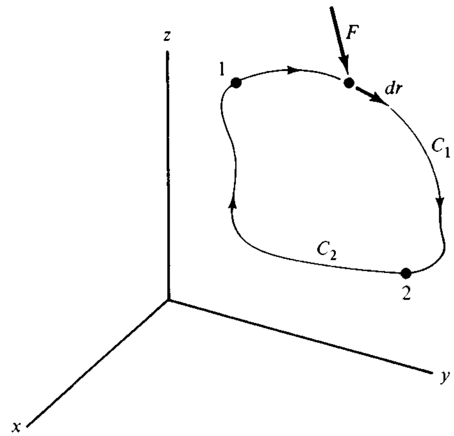


Figure 5.11 Work done by a force.

$$W_{1\to 2}+W_{2\to 1}=0. \tag{5.87}$$

This must be true for all paths between the two positions. Hence, the work done in each phase of the movement can depend only on the initial and final positions. Furthermore, Eq. (5.87) must be valid for any pair of positions. These conditions can only be satisfied if the work done by a conservative force is determined by the change in the value of a function of position.

This function, which is called the *potential energy*, is defined such that

♦ $$W_{1\to 2}=V_1-V_2. \tag{5.88}$$

In other words, the work done by a conservative force equals the amount by which its potential energy is depleted.

For further insight, suppose that position 2 is infinitesimally close to position 1, so that Eq. (5.88) may be written in differential form as

$$dW=-dV=-\frac{\partial V}{\partial X}\,dX-\frac{\partial V}{\partial Y}\,dY-\frac{\partial V}{\partial Z}\,dZ, \tag{5.89a}$$

where the chain rule has been introduced because the potential energy is a function of position. This relation may be rewritten in the notation of vector calculus as

$$dW=-\nabla V\cdot d\bar{r}_P, \tag{5.89b}$$

where $d\bar{r}=dX\bar{I}+dY\bar{J}+dZ\bar{K}$ and ∇V represents the gradient of the potential energy,

$$\nabla V=\frac{\partial V}{\partial X}\bar{I}+\frac{\partial V}{\partial Y}\bar{J}+\frac{\partial V}{\partial Z}\bar{K}. \tag{5.90}$$

However, the infinitesimal work done by a force is

$$dW=\bar{F}\cdot d\bar{r}_P. \tag{5.91}$$

Comparison of Eqs. (5.89b) and (5.91) leads to recognition that a conservative force is the negative of the gradient of the potential energy function,

♦ $$\bar{F}_{\text{conservative}}=-\nabla V. \tag{5.92}$$

The potential energy function may be derived by integrating either of Eqs. (5.89). However, for the common forces, it often is easier to derive an expression for potential energy by evaluating the work done by the force between two arbitrary positions. Comparing the resultant expression with the general expression for the work done by a conservative force, Eq. (5.88), then permits identification of the potential energy function. The few conservative forces that commonly arise in mechanical systems are as follows.

Gravitational Attraction near the Earth's Surface
The distance Z is measured vertically from a reference elevation called the *datum;*

♦ $$\bar{F}_{\text{grav}} = -mg\bar{K} \Leftrightarrow V_{\text{grav}} = mgZ. \qquad (5.93)$$

Gravitational Attraction between the Earth and an Orbiting Object
The distance r is measured between the center of the earth and the object, and m is the mass of the object;

♦ $$\bar{F}_{\text{grav}} = -\frac{GMm}{r^2}\bar{e}_r \Leftrightarrow V = -\frac{GMm}{r}, \qquad (5.94)$$

where GM is the product of the universal gravitational constant and the mass of the earth:

$$GM = 5.990(10^{14})\ \text{m}^3/\text{s}^2.$$

Linear Elastic Spring
The stiffness of the spring is k, Δ is its elongation, L is the length of the spring in the current position, and L_0 is the unstretched length. We have

♦ $$\bar{F}_{\text{spr}} = k\Delta\bar{e}_s \Leftrightarrow V_{\text{spr}} = \tfrac{1}{2}k\Delta^2, \quad \Delta = L - L_0, \qquad (5.95)$$

where the unit vector $\bar{e}_s$ extends from the end where the force $\bar{F}_{\text{spr}}$ is applied to the other end.

The work-energy principle may be reformulated to account explicitly for conservative forces. When the work done by those forces is computed according to Eq. (5.88), the result is

♦ $$T_2 + V_2 = T_1 + V_1 + W_{1\to 2}^{(\text{nc})}, \qquad (5.96)$$

where $W_{1\to 2}^{(\text{nc})}$ represents the work done by all forces that are not included in the potential energy. The quantity $T+V$ is often called the *mechanical energy E.* When $W_{1\to 2}^{(\text{nc})} = 0$, the mechanical energy is conserved in the motion.

Equation (5.96) is a scalar equation relating the motion at two positions. It is adequate by itself only when there is one unknown - for example, a parameter describing a force or a speed. Additional equations relating motion at two positions might be available from conservation of momentum, particularly the angular momentum, or from one of the components of the equations of motion.

The freedom to select point B arbitrarily in Eq. (5.86) can be a significant aid, because constrained points usually follow comparatively simple paths, whereas the motion of an unconstrained point is often quite complicated. This arbitrariness also leads to another relation between the motion at two positions. Suppose that point B is selected as the center of mass G. The equation of motion for the center of mass, $\sum \bar{F} = m\bar{a}_G$, may be integrated to form an energy relation directly. A dot product with the displacement of the center of mass leads to

$$\oint_1^2 \sum \bar{F}\cdot d\bar{r}_G = \oint_1^2 m\bar{a}_G\cdot d\bar{r}_G = \oint_1^2 m\dot{\bar{v}}_G\cdot(\bar{v}_G\, dt)$$
$$= \oint_1^2 \frac{1}{2} m \frac{d}{dt}(\bar{v}_G\cdot\bar{v}_G)\, dt. \qquad (5.97)$$

The last integrand in Eq. (5.97) is a perfect differential, so the relation reduces to

♦ $$\frac{1}{2}v(\bar{v}_G\cdot\bar{v}_G)_2 = \frac{1}{2}m(\bar{v}_G\cdot\bar{v}_G)_1 + \oint_1^2 \sum \bar{F}\cdot d\bar{r}_G. \qquad (5.98)$$

Let us now formulate the basic work-energy principle, Eq. (5.81), using Eqs. (5.25a) and (5.86) with point B set to be point G. Subtracting Eq. (5.98) from that expression yields

♦ $$\frac{1}{2}(\bar{\omega}\cdot\bar{H}_G)_2 = \frac{1}{2}(\bar{\omega}\cdot\bar{H}_G)_1 + \oint_1^2 \sum \bar{M}_G\cdot\overline{d\theta}. \qquad (5.99)$$

The conclusion to be drawn from Eq. (5.98) is that the increase in the *translational kinetic energy* associated with motion of the center of mass, $\frac{1}{2}m\bar{v}_G\cdot\bar{v}_G$, results from the work of the resultant force in moving that point. Similarly, Eq. (5.99) shows that the increase in the *rotational kinetic energy*, $\frac{1}{2}\bar{\omega}\cdot\bar{H}_G$, associated with rotation about the center of mass results from the work done by the moment of the force system about that point.

We obtain another viewpoint when we use Eq. (5.86) to resolve the force system to a force and couple acting at the fixed point O for a body in pure rotation. Substituting Eqs. (5.25b) and (5.86) into the work-energy principle then yields

♦ $$\frac{1}{2}(\bar{\omega}\cdot\bar{H}_O)_2 = \frac{1}{2}(\bar{\omega}\cdot\bar{H}_O)_1 + \oint_1^2 \sum \bar{M}_O\cdot\overline{d\theta}. \qquad (5.100)$$

In this case, the kinetic energy is entirely associated with rotation about the fixed point.

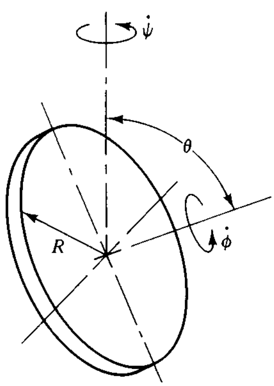
By the foregoing discussion, two independent work-energy equations are available for a rigid body. In the case of a general motion, in which no point in the body is stationary, one can employ any two of Eqs. (5.96) (total kinetic energy), Eq. (5.98) (translational kinetic energy relative to the center of mass), and Eq. (5.99) (rotational kinetic energy about the center of mass). When the body is in pure rotation, we would usually employ Eq. (5.100) (rotational kinetic energy about the fixed point). The second equation in this case may be selected as either Eq. (5.96) or Eq. (5.98). The fact that these alternative formulations treat the kinetic energy in terms of different rotational effects emphasizes that the concept of rotational kinetic energy is meaningful only if the point of reference is specified.

As is true for momentum principles, the work-energy principles have inherent limitations. Most profound of these is the necessity to evaluate the work done by nonconservative forces. Equation (5.86), which replaces any external force by an equivalent force-couple system acting at an arbitrary point, is an aid. Obviously, the motion of such a point must be known in order to evaluate the path integrals. It is equally important to know how the resultant varies as the position of the selected point changes, and how the moment depends on the angle of orientation. If the forces are explicit functions of time or velocity, such dependencies can only be expressed after the motion has been evaluated. Even when it might seem that the work could be evaluated, evaluation of the line integrals may be quite complicated.

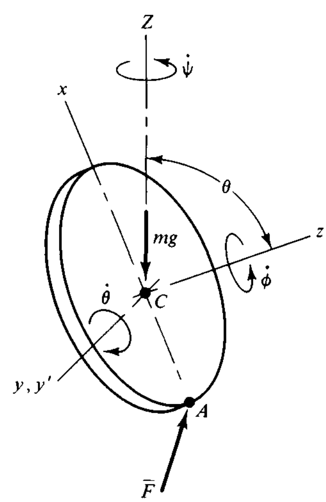
Of course, if a force is conservative then the work it does may be described in terms of its potential energy. However, it is necessary to recognize that the potential energy was derived by following the displacement of the actual point at which the force is applied. It might be necessary to re-derive the energy function if the force is transferred to a different point.

In closing, it must be emphasized that the impulse-momentum and work-energy principles offer possible approaches for avoiding the solution of differential equations of motion. The conditions for usefully employing those principles are quite restrictive. Although it might be more difficult to solve differential equations, formulation and solution of the equations of motion is often the only valid approach.

Example 5.15 The coin is rolling without slipping, but the angle θ at which the plane of the coin is inclined is not constant. Evaluate the kinetic energy of the disk in terms of θ, the precession rate $\dot{\psi}$, and spin rate $\dot{\phi}$. Also, prove that the work done by the friction and normal forces is zero.



Example 5.15



Free-body diagram.

Solution Rotational symmetry allows us to select body-fixed xyz axes whose instantaneous orientation is such that the y axis coincides with the line of nodes, which is the y' axis in the sketch. The inertia properties are

$$I_{xx} = I_{yy} = \tfrac{1}{4}mR^2, \qquad I_{zz} = \tfrac{1}{2}mR^2, \qquad I_{xy} = I_{yz} = I_{xz} = 0.$$

The angular velocity is

$$\bar{\omega} = \dot{\psi}\bar{K} - \dot{\theta}\bar{j}' + \dot{\phi}\bar{k} = (\dot{\psi}\sin\theta)\bar{i} - \dot{\theta}\bar{j} + (\dot{\psi}\cos\theta + \dot{\phi})\bar{k}.$$

We shall form the kinetic energy using the center of mass as the reference point. Since there is no slippage at point A, the velocity of the center of mass is

$$\bar{v}_C = \bar{\omega} \times \bar{r}_{C/A} = R(\dot{\psi}\cos\theta + \dot{\phi})\bar{j} + R\dot{\theta}\bar{k}.$$

The angular momentum about the center of mass is

$$\begin{aligned}\bar{H}_C &= I_{xx}\omega_x\bar{i} + I_{yy}\omega_y\bar{j} + I_{zz}\omega_z\bar{k} \\ &= \tfrac{1}{4}mR^2[(\dot{\psi}\sin\theta)\bar{i} - \dot{\theta}\bar{j} + 2(\dot{\psi}\cos\theta + \dot{\phi})\bar{k}].\end{aligned}$$

Then the kinetic energy is

$$\begin{aligned}T &= \tfrac{1}{2}m\bar{v}_C\cdot\bar{v}_C + \tfrac{1}{2}\bar{\omega}\cdot\bar{H}_C \\ &= \tfrac{1}{2}mR^2[(\dot{\psi}\cos\theta + \dot{\phi})^2 + \dot{\theta}^2] + \tfrac{1}{8}mR^2[\dot{\psi}^2\sin^2\theta + \dot{\theta}^2 + 2(\dot{\psi}\cos\theta + \dot{\phi})^2] \\ &= \tfrac{1}{8}mR^2[\dot{\psi}^2\sin^2\theta + 5\dot{\theta}^2 + 6(\dot{\psi}\cos\theta + \dot{\phi})^2].\end{aligned}$$

For evaluation of the work, we let $\bar{F}$ denote the frictional and normal components of the reaction. Since the contact point A follows a complicated path, we replace these reactions by a force $\bar{F}$ acting at point C and a couple $\bar{M}$ equal to the moment exerted by $\bar{F}$ about point C. Therefore

$$\bar{M} = \bar{r}_{A/C} \times \bar{F}.$$

The infinitesimal work done by the reactions is

$$dW = \bar{F}\cdot d\bar{r}_C + \bar{M}\cdot\overline{d\theta},$$

where

$$\overline{d\theta} = \bar{\omega}\,dt, \qquad d\bar{r}_C = \bar{v}_C\,dt = (\bar{\omega} \times \bar{r}_{C/A})\,dt = \overline{d\theta} \times \bar{r}_{C/A}.$$

Thus

$$dW = \bar{F}\cdot(\overline{d\theta} \times \bar{r}_{C/A}) + (\bar{r}_{A/C} \times \bar{F})\cdot\overline{d\theta}.$$

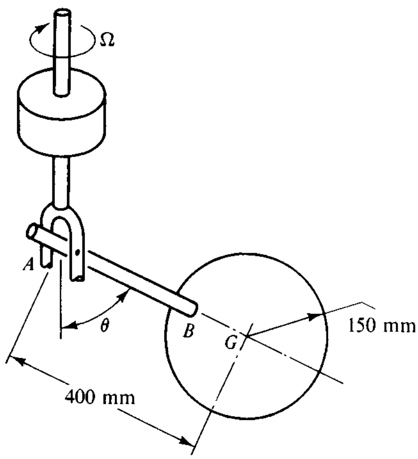
Because $\bar{r}_{A/C} = -\bar{r}_{C/A}$, re-arranging the second scalar triple product leads to

$$dW = \bar{F}\cdot(\overline{d\theta} \times \bar{r}_{C/A}) - \bar{F}\cdot(\overline{d\theta} \times \bar{r}_{C/A}) = 0.$$

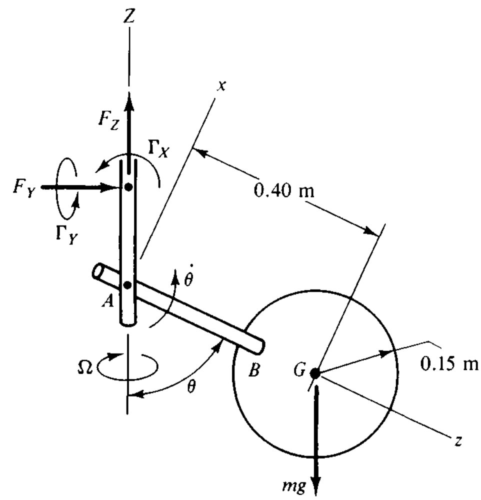
This proves that the friction and normal force acting on a rolling disk never do any work if there is no slippage. Indeed, the same result may be proven for any situation where rigid bodies roll without slipping.

Example 5.16 The sphere, whose mass is m, is fastened to the end of bar AB. The connection at end A is an ideal pin, which allows θ to vary. The vertical shaft rotates freely, and its mass, as well as that of bar AB, are negligible. Initially, θ is held constant at 90° by a string, and the precession rate is $\Omega = 5$ rad/s. Determine the minimum value of θ in the motion following breakage of the string, and the corresponding precession rate. Then determine the values of $\dot{\theta}$ and Ω at the instant when θ is 10° greater than its minimum value.

Solution A free-body diagram of the entire system, including the vertical shaft, shows that there is no moment exerted about the fixed axis of rotation. The force-couple reaction at the bearing, $\bar{F}$ and $\bar{\Gamma}$, does no work, and gravity is conservative. Therefore, both angular momentum about the Z axis and mechanical energy



Example 5.16



Free-body diagram.

are conserved throughout the motion. We shall develop an expression for $\bar{H}_A$ under arbitrary conditions in order to address all aspects of the problem. The inertia properties for the principal body-fixed axes in the sketch are $I_{xx} = I_{yy} = I_1$ and $I_{zz} = I_2$, where

$$I_1 = \tfrac{2}{5}m(0.15^2) + m(0.40^2) = 0.1690m, \qquad I_2 = \tfrac{2}{5}m(0.15^2) = 0.009m.$$

The angular velocity of the sphere is

$$\bar{\omega} = -\Omega\bar{K} + \dot{\theta}\bar{j} = -(\Omega\sin\theta)\bar{i} + \dot{\theta}\bar{j} + (\Omega\cos\theta)\bar{k}.$$

The corresponding angular momentum is

$$\bar{H}_A = -(I_1\Omega\sin\theta)\bar{i} + I_1\dot{\theta}\bar{j} + (I_2\Omega\cos\theta)\bar{k}.$$

The component of $\bar{H}_A$ parallel to the Z axis is

$$\bar{H}_A\cdot\bar{K} = -I_1\Omega\sin^2\theta - I_2\Omega\cos^2\theta,$$

and the kinetic energy is

$$T = \tfrac{1}{2}\bar{\omega}\cdot\bar{H}_A = \tfrac{1}{2}(I_1\Omega^2\sin^2\theta + I_1\dot{\theta}^2 + I_2\Omega^2\cos^2\theta).$$

If the datum for the gravitational potential energy is placed at the elevation of pin A, then

$$V = -mg(0.40\cos\theta)\text{ J}.$$

The initial condition is that $\Omega = 5$ rad/s, $\theta = 90°$, and $\dot{\theta} = 0$. Thus,

$$(\bar{H}_A\cdot\bar{K})_1 = -5I_1 \text{ kg-m-s}, \qquad T_1 = 12.5I_1 \text{ J}, \qquad V_1 = 0.$$

The first question is the minimum value of θ, which means that $\dot{\theta} = 0$ at this position. Conservation of angular momentum then requires that

$$(\bar{H}_A\cdot\bar{K})_2 = (\bar{H}_A\cdot\bar{K})_1 \;\Rightarrow\; \Omega(I_1\sin^2\theta + I_2\cos^2\theta) = 5I_1,$$

while energy is conserved if

$$T_2+V_2=T_1+V_1 \Rightarrow \tfrac{1}{2}(I_1\Omega^2\sin^2\theta+I_2\Omega^2\cos^2\theta)-0.4mg\cos\theta=12.5I_1.$$

Eliminating Ω between the angular momentum and energy equations yields

$$\Omega=\frac{5I_1}{I_1\sin^2\theta+I_2\cos^2\theta}$$

and

$$\tfrac{1}{2}(5I_1)^2=(0.4mg\cos\theta+12.5I_1)(I_1\sin^2\theta+I_2\cos^2\theta).$$

For the present values of I_1 and I_2, the root giving the largest value of $\cos\theta$, which corresponds to the minimum angle, is

$$\theta_{\min}=0.65408 \text{ rad}=37.517°.$$

The corresponding precession rate is obtained from the foregoing equation for Ω as

$$\Omega=12.364 \text{ rad/s}.$$

For the second part, we must find Ω and $\dot{\theta}$ when $\theta_2=\theta_{\min}+10°=0.82861$ rad. The earlier equation for conservation of angular momentum is generally valid. In this case, it gives

$$\Omega_2=\frac{5I_1}{I_1\sin^2\theta_2+I_2\cos^2\theta_2}=8.800 \text{ rad/s}.$$

We now determine the corresponding value of $\dot{\theta}$ from the energy conservation equation, for which we use the general expression for kinetic energy. The result is

$$\tfrac{1}{2}(I_1\Omega_2^2\sin^2\theta_2+I_1\dot{\theta}^2+I_1\Omega_2^2\cos^2\theta_2)-0.4mg\cos\theta_2=12.5I_1,$$

which leads to

$$\dot{\theta}_2=3.515 \text{ rad/s}.$$

5.7 A System of Rigid Bodies

The representation of a collection of particles as a rigid body offers two primary advantages: an enormous reduction in the number of kinematical variables, and a commensurate reduction in the reaction forces that appear in the equations of motion. Both gains result from the recognition that the particles forming a rigid body are mutually constrained. In the same manner, we occasionally find it useful to consider interacting rigid bodies as a unified system. Consider the pair of bodies in Figure 5.12 which are loaded by a set of external forces, as well as by reaction forces associated with their interaction. If each of these bodies is executing a three-dimensional motion, then there are a total of twelve scalar equations of motion: three force-component equations and three moment-component equations for each body. Part of the formulation of these equations of motion involves accounting for Newton's third law when describing the reaction forces exerted between the bodies. Then the solution of the equations must determine, or at least eliminate, the reactions. Treating the two bodies as a single system, so that the interaction forces exerted between them become internal to the system, leads to equations of motion that do not contain these forces.

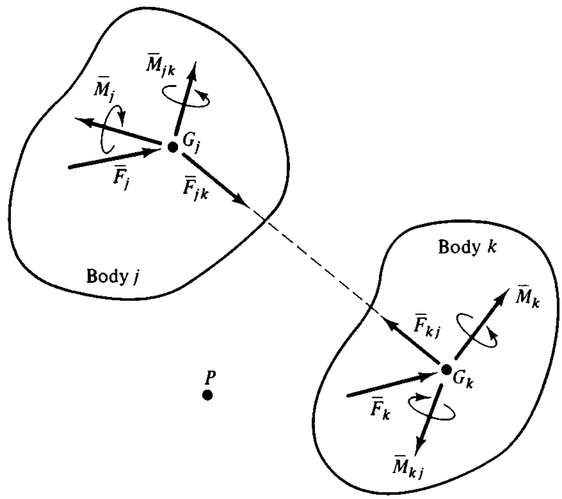


Figure 5.12 Forces acting on interacting bodies.

Without loss of generality, we may resolve the external forces applied to body j into an equivalent force-couple system, $\sum \bar{F}_j$ and $\sum \bar{M}_j$, acting at the center of mass G_j. Similarly, the interaction forces exerted on body j by body k may be represented as a force-couple system, $\sum \bar{F}_{jk}$ and $\sum \bar{M}_{jk}$, acting at point G_j. A comparable representation also applies to the forces acting on body k.

Note that the individual internal force and moment resultants are equivalent to the system of distributed forces exerted by the particles of one body on the particles of the other body. Hence, according to Newton's third law, the internal forces $\sum \bar{F}_{kj}$ and $\sum \bar{F}_{jk}$ cancel in the force sum. For the same reason, the net moment of the interaction forces and moments about an arbitrary point P also cancel. Let $\sum \bar{F}$ denote the resultant of all external forces exerted on the system by bodies not included in the system. Similarly, let $\sum \bar{M}_P$ denote the total moment about point P of all external forces.

Each body in Figure 5.12 must obey its own equation of motion. Hence, the forces acting on body j are equivalent to a force-couple system consisting of the linear inertia vector $m_j \bar{a}_{Gj}$ and the rotational inertia vector $d\bar{H}_{Gj}/dt$. Similarly, the forces acting on body k are equivalent to $m_k \bar{a}_{Gk}$ and $d\bar{H}_{Gk}/dt$. Both equivalencies are illustrated in Figure 5.13. As we did for the actual forces acting on the system of rigid bodies, we may combine the equivalent inertial force-couple systems into a single resultant. Note that the moment sum must account for the moment of the inertial forces $m_j(a_G)_j$ about point P. Thus, we find that the equations of motion for a system of rigid bodies are

$$\blacklozenge \quad \sum \bar{F} = \sum_j m_j(\bar{a}_{Gj}),$$
$$\blacklozenge \quad \sum \bar{M}_P = \sum_j \dot{\bar{H}}_{Gj} + \sum_j [\bar{r}_{Gj/P} \times m_j(\bar{a}_{Gj})]. \tag{5.101}$$

This principle may be interpreted as follows.

♦ *The forces and couples exerted on a system of rigid bodies resolve into a force-couple system acting at an arbitrary point P. The force in the equivalent system is the sum of the linear inertia vectors, $m_j(\bar{a}_{Gj})$, associated with*

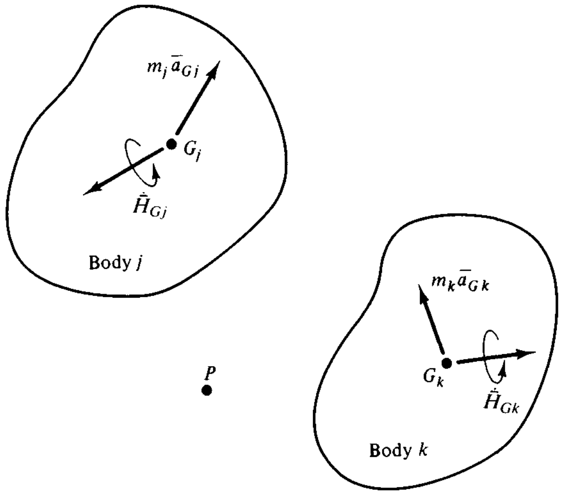


Figure 5.13 Inertial force-couple systems equivalent to the forces acting on interacting rigid bodies.

the motion of the center of mass of each body. The couple in the equivalent system is the sum of the rotational inertia vector for each body, $d\bar{H}_{Gj}/dt$, and the moment of the linear inertia vectors about point P.

The usefulness of Eqs. (5.101) becomes apparent when we recall the techniques for static equilibrium. The main consideration in selecting the point for a moment sum in the static case is the ability to prevent forces (usually reactions) from appearing in the moment equilibrium equations. We have developed here a comparable ability for dynamic systems, because the point P in Eqs. (5.101) was arbitrarily selected.

Treating a set of moving parts as a system leads to a qualitative understanding of how a bicyclist can maintain balance and maneuver without falling. In order to avoid details that would obscure the discussion, we shall employ a simplified model of the steering configuration. Our model, which is shown in top view in Figure 5.14, considers the axis of the steering fork to be perpendicular to the longitudinal axis, and to intersect vertically the axis of the front wheel. Under perfect conditions for following a straight path, the bicycle would be oriented vertically, with the rider's

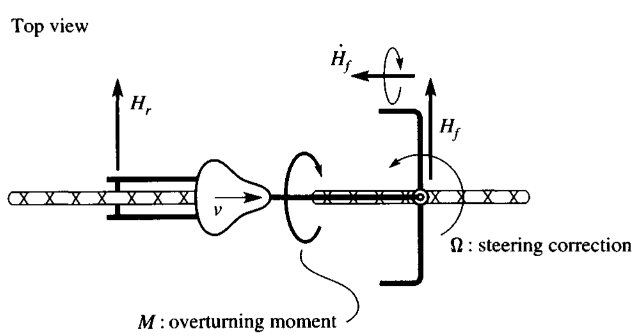


Figure 5.14 Balancing of a bicycle.

center of mass situated directly over the line connecting the centers of the wheels. The angular momenta of the forward and rear wheels, $\bar{H}_f$ and $\bar{H}_r$, are then horizontal, as shown.

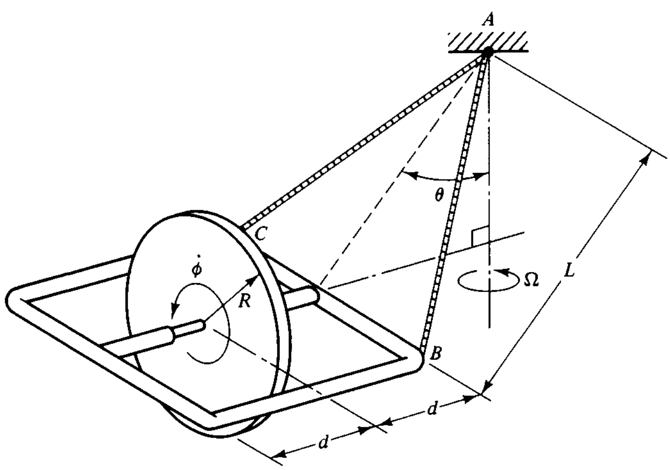
Such ideal conditions cannot be maintained. For example, the rider might lean over or a gust of wind might arise. Such disturbance, in combination with the reaction of the wheels, creates an overturning moment $\bar{M}$ that acts about the longitudinal axis. (The situation in the figure corresponds to the rider leaning to the left.) This moment must be matched by a corresponding change in the angular momentum. The rider achieves this by applying a torque to the handlebars, which causes the steering fork to rotate at some angular speed Ω. This rotation causes the tip of $\bar{H}_f$ to move in the direction of $\bar{\Omega} \times \bar{H}_f$. In order for $d\bar{H}_f/dt$ to equal $\bar{M}$, the sense of $\bar{\Omega}$ must have the effect of turning the bicycle to the side toward which it is tending to lean (left, in the case of the figure). If the rider wishes to return to the direction initially set, then this correctional maneuver must be reversed. As a result, the rider turns the steering wheel back in the opposite sense, thereby changing the sense of $d\bar{H}_f/dt$ (forward in the case of Figure 5.14). In order to generate a force system whose resultant moment about the longitudinal axis matches the required rate of change of the angular momentum, the rider simultaneously leans to the right, thereby shifting the center mass to the other side. Thus, riding in a straight line is actually a sequence of corrective steering maneuvers and shifts of the center of mass. In essence, the bicyclist is both the actuator and controller of a feedback control system. This feature is most evident in children who have just begun to ride a bicycle. For a very experienced bicyclist, the corrective maneuvers are barely perceptible. Also, if the rider's hands are not placed on the handlebars, then the steering wheel turns of its own accord in the manner required to change the angular momentum at a rate that matches the unbalanced moment. In any event, the ability to steer is essential to maintaining one's balance.

The same considerations can be used to explain why a bicyclist must lean to the side toward which the bicycle is turning. In that case, the directions of the angular momenta of the front and rear wheels are both changing as a result of the changing horizontal direction of movement. For a left turn, both $\dot{\bar{H}}_f$ and $\dot{\bar{H}}_r$ will be rearward in Figure 5.14. By leaning into the turn, the rider generates the overturning moment required to change total angular momentum. This situation is essentially the same as that for the rolling disk in Example 5.8.

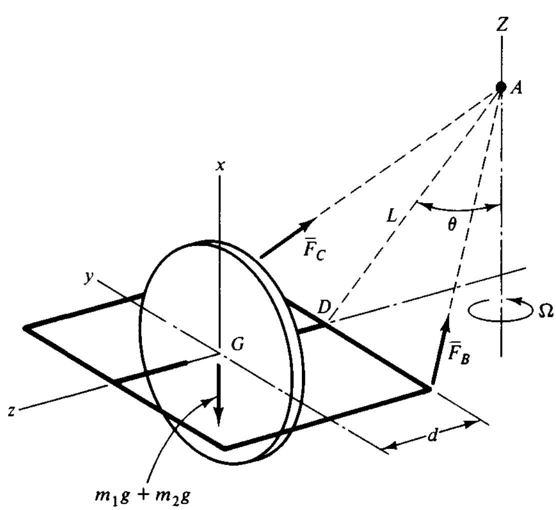
A performer riding a unicycle exploits these same phenomena to maintain left-right balance. Thus, falling to the left is controlled by turning the wheel left, and vice versa. Forward-rear balance requires a different control strategy, which relies on the fact that a falling stick can be kept at a constant angle of tilt if it is given the correct translational acceleration in the horizontal direction. Thus, the unicyclist compensates a tendency to fall forward or back by accelerating in the direction of that tendency. Obviously, riding a unicycle is substantially more difficult than riding a bicycle.

Despite the apparent simplicity of the principles by which a set of rigid bodies may be grouped together, the concept should not be relied on too heavily. One pitfall lies in the need to include the linear inertia vectors in the formation of the moment equations of motion. A comparable operation arose in the formulation of the equations of motion for a single rigid body. We chose there to restrict the point for the moment sum, since any formulation associated with an arbitrary selection of the point would be more prone to error - particularly in cases where the motion is to be determined. Another shortcoming is that Eqs. (5.101) provide only six scalar equations for the entire system, while the individual equations of motion provide six equations of motion for each body.

Example 5.17 The gimbal supporting the flywheel is suspended from pivot A by two cables of equal length. The flywheel, which may be modeled as a thin disk of mass m_1, spins at the constant rate $\dot{\phi}$. The mass of the gimbal is m_2. It is observed that under the appropriate conditions the system will precess about the vertical axis through pivot A at a constant rate Ω, such that the angle of inclination θ is constant and the axis of the flywheel is horizontal. Derive expressions for the required precession and spin rates as a function of the angle θ and the other system parameters.



Example 5.17



Free-body diagram.

Solution In order to avoid considering the bearing forces exerted between the gimbal and the flywheel, we draw a free-body diagram of the assembly. Let body 1 be the flywheel and body 2 be the gimbal. The $x_2y_2z_2$ reference frame is fixed to the gimbal, while $x_1y_1z_1$ is fixed to the flywheel. Letting the instantaneous orientation of $x_1y_1z_1$ coincide with $x_2y_2z_2$ expedites the overall description of the kinematic and kinetic properties.

The center of mass for each body is point G, whose path is circular. Thus

$$\bar{a}_{G1} = \bar{a}_{G2} = -(L\sin\theta + d)\Omega^2\bar{k}.$$

The angular motion of the flywheel is

$$\bar{\omega}_1 = \Omega\bar{K} + \dot{\phi}\bar{k} = \Omega\bar{i} + \dot{\phi}\bar{k},$$

$$\bar{\alpha}_1 = \dot{\phi}(\bar{\omega}\times\bar{k}) = -\Omega\dot{\phi}\bar{j}.$$

The corresponding expression for $\dot{\bar{H}}_G$ may be obtained from Euler's equations because xyz are principal axes. Setting $I_{xx} = I_{yy} = \frac{1}{2}I_{zz} = \frac{1}{4}m_1R^2$ yields

$$(\dot{\bar{H}}_G)_1 = [I_{yy}(\alpha_1)_y - (I_{zz} - I_{xx})(\omega_1)_x(\omega_1)_z]\bar{j} = -\tfrac{1}{2}m_1R^2\Omega\dot{\phi}\bar{j}.$$

The corresponding analysis for the gimbal shows that it precesses about the Z axis only. By symmetry, the x axis, which is parallel to Z, is a principal axis for the gimbal, and the precession rate is constant. Thus, $(\dot{\bar{H}}_G)_2 = \bar{0}$.

We may now form the equations of motion for the system. Point A is convenient for the moment sum, because both cable forces intersect that point. The moment of the actual force system about point A must match the sum of the moments about point A of the $m\bar{a}_G$ vector for each body, and of the couple $\dot{\bar{H}}_G$ for each body. Thus,

$$\begin{aligned}\textstyle\sum\bar{M}_A &= -(m_1g + m_2g)(L\sin\theta + d)\bar{j}\\ &= (\dot{\bar{H}}_G)_1 + (\dot{\bar{H}}_G)_2 + \bar{r}_{G/A}\times[m_1(\bar{a}_G)_1 + m_2(\bar{a}_G)_2]\\ &= [-\tfrac{1}{2}m_1R^2\Omega\dot{\phi} - (m_1+m_2)(L\sin\theta + d)\Omega^2(L\cos\theta)]\bar{j}.\end{aligned}$$

This yields only one algebraic equation. Additional equations may be obtained from the force equations of motion. As an aid in forming those equations, we note that the system is symmetric with respect to the z-Z plane, and neither center of mass has an acceleration transverse to that plane. Therefore, the tensile forces $\bar{F}_B$ and $\bar{F}_C$ have equal magnitude. We may replace them by an equivalent force $\bar{F}$ directed from point D to point A. The force sums are

$$\textstyle\sum F_x = F\cos\theta - (m_1g + m_2g) = 0,$$

$$\begin{aligned}\textstyle\sum F_z = -F\sin\theta &= m_1(a_G)_{1z} + m_2(a_G)_{2z}\\ &= -(m_1+m_2)(L\sin\theta + d)\Omega^2.\end{aligned}$$

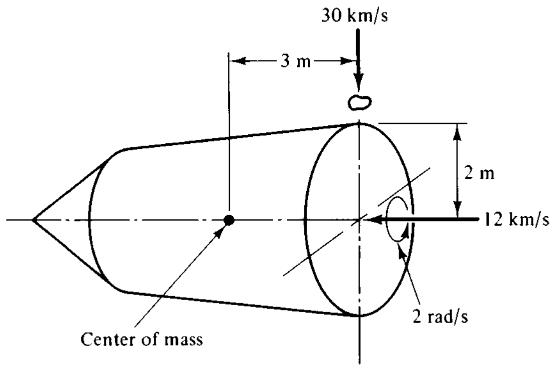
Eliminating F from the force equations leads to a solution for Ω:

$$\Omega = \left(\frac{g\tan\theta}{L\sin\theta + d}\right)^{1/2}.$$

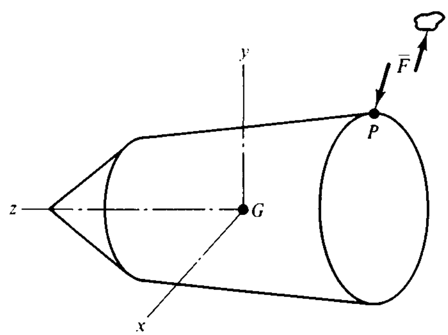
We substitute this expression for Ω into the moment equation and solve for $\dot{\phi}$, which results in

$$\dot{\phi} = \left(1 + \frac{m_2}{m_1}\right)\frac{2d}{R}\left[\frac{g}{R}\left(\frac{L\sin\theta + d}{R\tan\theta}\right)\right]^{1/2}.$$

Example 5.18 An orbiting satellite is spinning about its axis of symmetry at 2 rad/s, and its velocity is 12 km/s parallel to the axis. The mass of the satellite is 2,000 kg, and its radii of gyration for centroidal axes are 1.5 m about the axis of symmetry and 2.5 m transverse to that axis. A 1-kg meteorite traveling at 30 km/s impacts the satellite as shown, and then is embedded in the satellite's wall. Determine the velocity of the center of mass and the angular velocity immediately after the collision.



Example 5.18



Free-body diagrams.

Solution The only force that is significant during the impact is the impulsive interaction force $\bar{F}$, which is shown in the free-body diagrams of the satellite and of the meteorite, where xyz is fixed to the satellite. Furthermore, due to the short duration of the impact, we model the process by considering the positions to not change. Thus, we may also employ xyz as space-fixed axes for the analysis.

Because $\bar{F}$ is an interaction force, we may avoid its appearance in the formulation by considering the satellite and the meteorite as a system. The external force and moment resultants vanish for this system, so both linear and angular momentum are conserved. The appropriate conservation principle may be obtained by multiplying Eqs. (5.101) by dt, then integrating over the interval Δt for the impact. The center of mass G is a convenient reference point for the angular momentum, because the moment of the linear momentum of the satellite about that point is identically zero. Also, we may treat the meteorite as a particle, which means that its angular momentum about its own center of mass is also zero. The conservation principles for the system therefore reduce to

$$m_s(\bar{v}_G)_2 + m_m(\bar{v}_P)_2 = m_s(\bar{v}_G)_1 + m_m(\bar{v}_P)_1,$$

$$(\bar{H}_{Gs})_2 + \bar{r}_{P/G} \times m_m(\bar{v}_P)_2 = (\bar{H}_{Gs})_1 + \bar{r}_{P/G} \times m_m(\bar{v}_P)_1.$$

In these expressions, the subscript s refers to parameters for the satellite, while the subscript m is for the meteorite. Note that we were able to treat the position $\bar{r}_{P/G}$ as a constant for the integration owing to our idealization that changes in position are insignificant during the impact.

It is given that the satellite is axisymmetric, so xyz are principal axes. The inertia properties are found from the given radii of gyration to be

$$I_{xx} = I_{yy} = m_s(2.5^2), \quad I_{zz} = m_s(1.5^2) \text{ kg-m}^2.$$

Let the angular velocity of the satellite after impact be $\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}$. The angular momentum terms are then found to be

$$(\bar{H}_{Gs})_2 = m_s(6.25\omega_x \bar{i} + 6.25\omega_y \bar{j} + 2.25\omega_z \bar{k}),$$

$$(\bar{H}_{Gs})_1 = m_s(2.25)(-2)\bar{k} \text{ kg-m}^2.$$

We have not yet used the fact that the meteorite is embedded after the impact, which means that it has the same velocity as point P on the satellite. Therefore,

$$(\bar{v}_P)_2 = (\bar{v}_G)_2 + \bar{\omega}_2 \times \bar{r}_{P/G}.$$

Next, we substitute this expression and the expressions for $\bar{H}_{Gs}$ into the conservation principles. Dividing each equation by $m_s + m_m$ yields

$$(\bar{v}_G)_2 + \sigma\bar{\omega}_2 \times \bar{r}_{P/G} = -30{,}000\sigma\bar{j} + 12{,}000(1-\sigma)\bar{k},$$

$$(1-\sigma)(6.25\omega_x \bar{i} + 6.25\omega_y \bar{j} + 2.25\omega_z \bar{k}) + \sigma\bar{r}_{P/G} \times (\bar{v}_G)_2 + \sigma\bar{r}_{P/G} \times (\bar{\omega}_2 \times \bar{r}_{P/G})$$
$$= -4.50(1-\sigma)\bar{k} + \bar{r}_{P/G} \times (-30{,}000\sigma\bar{j}),$$

where σ is the ratio of the mass of the meteorite to the total mass,

$$\sigma = \frac{m_m}{m_s + m_m}.$$

We could represent $\bar{v}_G$ in terms of an unknown set of components, as we did for $\bar{\omega}$. Correspondingly, we would obtain six simultaneous equations by matching like components in each conservation equation. However, it is possible to simplify the equations to be solved. We solve the linear momentum equation for $(\bar{v}_G)_2$, and substitute the result into the angular momentum equation. After like terms are collected, this operation gives

$$(\bar{v}_G)_2 = -\sigma\bar{\omega}_2 \times \bar{r}_{P/G} - 30{,}000\sigma\bar{j} + 12{,}000(1-\sigma)\bar{k}$$

and

$$6.25\omega_x \bar{i} + 6.25\omega_y \bar{j} + 2.25\omega_z \bar{k} + \sigma\bar{r}_{P/G} \times (\bar{\omega}_2 \times \bar{r}_{P/G})$$
$$= -4.50\bar{k} + \sigma\bar{r}_{P/G} \times (-30{,}000\bar{j} - 12{,}000\bar{k}).$$

Upon substitution of the component form of $\bar{\omega}$, as well as $\bar{r}_{P/G} = 2\bar{j} - 3\bar{k}$ m, we obtain a vector equation whose only unknowns are ω_x, ω_y, and ω_z. Matching like components then yields

$$(6.25+13\sigma)\omega_x = -114{,}000\sigma,$$

$$(6.25+9\sigma)\omega_y + 6\sigma\omega_z = 0,$$

$$(2.25+4\sigma)\omega_z + 6\sigma\omega_y = -4.5.$$

Since $\sigma = 1/2{,}001$ in the present case, these equations yield

$$\omega_2 = -9.106\bar{i} + 0.00096\bar{j} - 1.9982\bar{k} \text{ rad/s.}$$

We obtain the corresponding velocity of the center of mass by substitution, which yields

$$(\bar{v}_G)_2 = -0.002\bar{i} - 14.979\bar{j} + 11{,}994.0\bar{k} \text{ m/s.}$$

These results have a ready explanation. The velocity in the z direction is decreased by the impact because the mass of the satellite increased, while the y velocity component is attributable to the linear momentum transfer from the initial motion of the meteorite. The small change in the x component of velocity may be traced to the fact that the center of mass of the system shifts as a result of the capture of the meteorite. For the angular velocity, we note that the predominant effect is to induce a rotation about the x axis. The small changes in rotation about the y and z axes are a consequence of the alteration of the mass distribution due to the embedded meteorite, which results in xyz no longer being principal axes.

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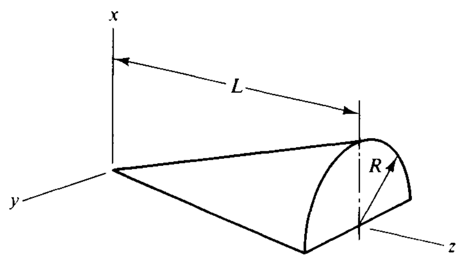
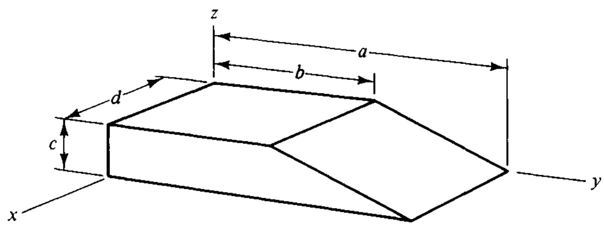
Problems

5.1 Derive the centroidal location and centroidal inertia properties of a homogeneous semicone, as tabulated in the appendix.

5.2 Determine I_{xx} and I_{xy} for the truncated rectangular parallelepiped relative to the xyz coordinate axes shown in the sketch.

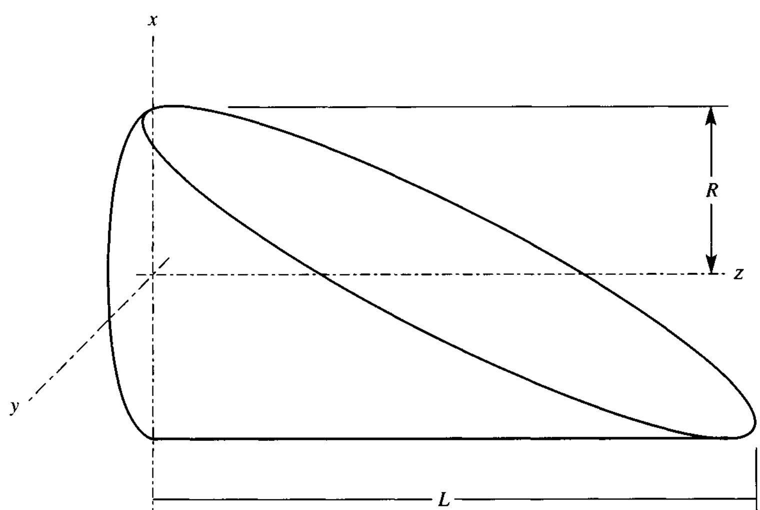


Problem 5.1



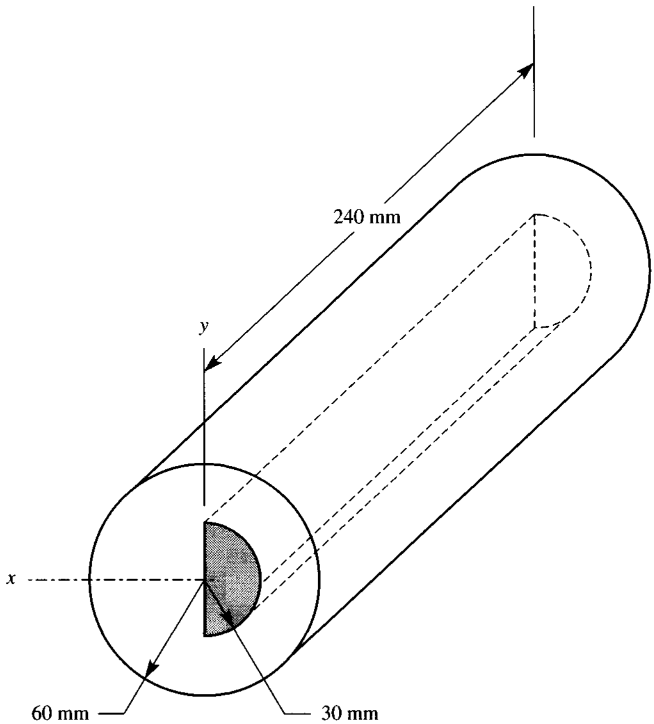
Problem 5.2

5.3 A cylinder is sliced in half along its diagonal. Determine the location of the center of mass and the inertia properties relative to a coordinate system whose z axis coincides with the axis of the cylinder and whose origin is situated at the circular end.



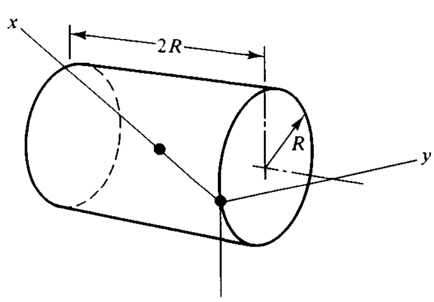
Problem 5.3

5.4 (See figure, next page.) The semicircular cut-out in the steel cylinder is filled with lead. Determine the centroidal location and the inertia properties of this body with respect to the xyz system shown.



Problem 5.4

5.5 The length of the homogeneous cylinder is twice its radius. Consider an xyz coordinate system whose origin is located on the perimeter of one end, whose x–y plane contains the axis of the cylinder, and whose x axis intersects the centroid. Determine the inertia properties of the cylinder relative to xyz.

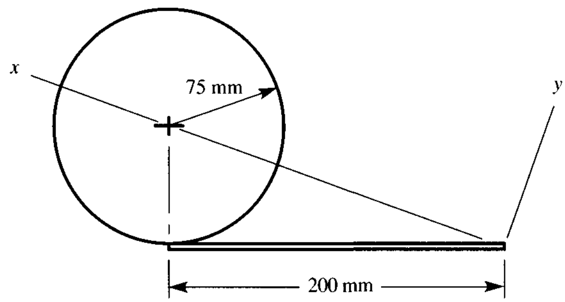


Problem 5.5

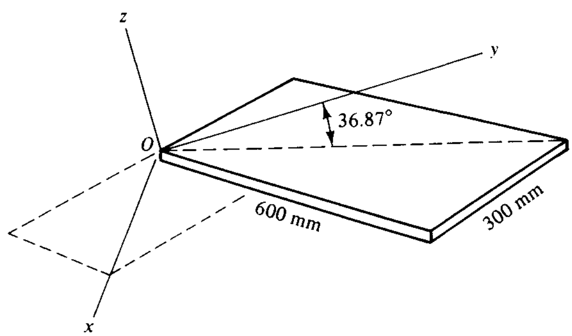
5.6 The 2-kg slender bar is welded to the 10-kg disk. Determine the inertia properties of this system relative to the xyz coordinate system shown in the sketch.

5.7 The x axis lies in the plane of the 10-kg plate, and the y axis is elevated at 36.87° above the diagonal. Determine the inertia matrix of the plate relative to xyz.

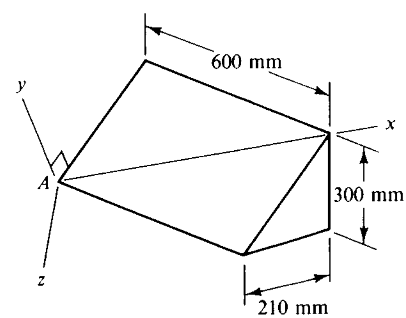
5.8 The mass of the plate is 10 kg. Determine the principal moments of inertia relative to corner O.



Problem 5.6



Problems 5.7 and 5.8

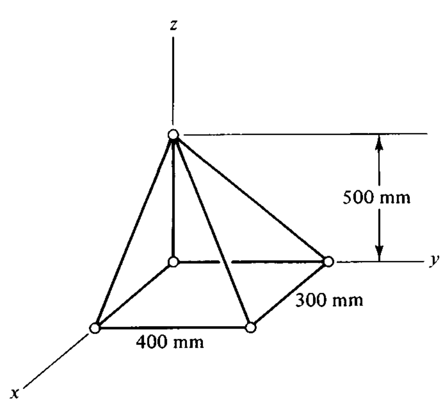


Problems 5.9 and 5.10

5.9 The x axis lies in the inclined face of the 3-kg homogeneous prism, and the y axis is normal to that face. Determine the inertia matrix of the prism relative to this coordinate system.

5.10 The prism's mass is 3 kg. For an origin coincident with corner A, determine the principal moments of inertia. Also determine the rotation transformation for the principal axes relative to coordinate axes aligned with the orthogonal edges.

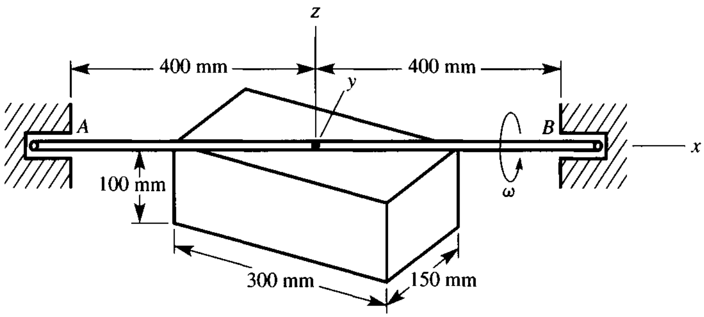
5.11 A rigid body consists of five small spheres of mass m mounted at the corners of a lightweight wire frame in the shape of a pyramid. Determine the principal moments



Problem 5.11

of inertia and the rotation matrix of the principal axes relative to the given xyz coordinate system.

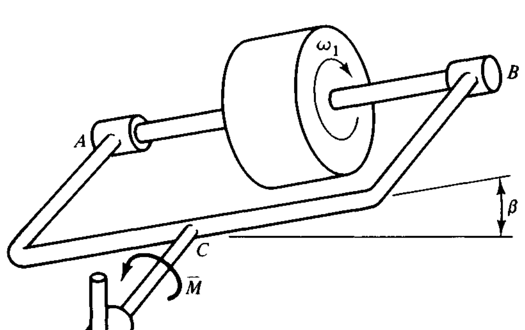
5.12 The 24-kg block is welded to a shaft that rotates about bearings A and B at a constant rate ω. The shaft is collinear with the diagonal to a face of the block, as shown. Determine the inertia properties of the block relative to the xyz coordinate system whose origin is situated at the midpoint of the diagonal. The x axis is aligned with the shaft, and the z axis is normal to the face of the block. Then use these properties to evaluate the dynamic reactions.



Problem 5.12

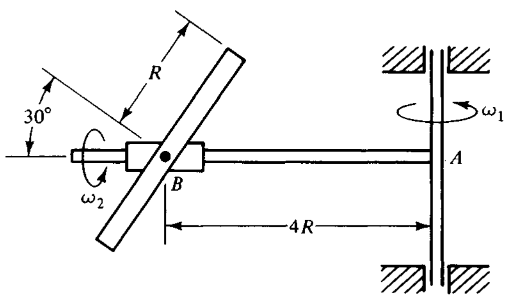
5.13 The gyroscopic turn indicator consists of a 1-kg flywheel whose principal radii of gyration are $k_x = 50$ mm and $k_y = k_z = 40$ mm. The center of mass of the flywheel coincides with the intersection of axes AB and CD. The flywheel spins relative to the gimbal at the constant rate $\omega_1 = 10{,}000$ rev/min. A couple $\bar{M}$ acts about shaft CD, which supports the gimbal, in order to control the angle β between the gimbal and the horizontal. Determine $\bar{M}$ when the rotation rate about the vertical axis is $\Omega = 0.8$ rad/s.

5.14 The disk of mass m spins about the horizontal shaft AB at angular rate ω_2, and the precession rate is ω_1; both rotation rates are constant. The angle between the center



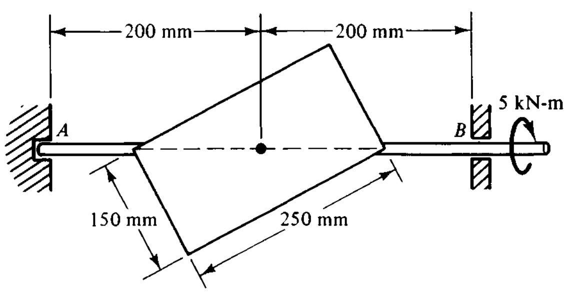
Problem 5.13

line of the disk and the shaft is 30°, so that the illustrated position occurs only once for each rotation of the disk relative to shaft *AB*. Derive an expression for the couple reaction at joint *A* at the instant depicted in the sketch. Then use that answer to explain physically whether or not a moment about the vertical shaft is required to hold ω_1 constant at this instant.



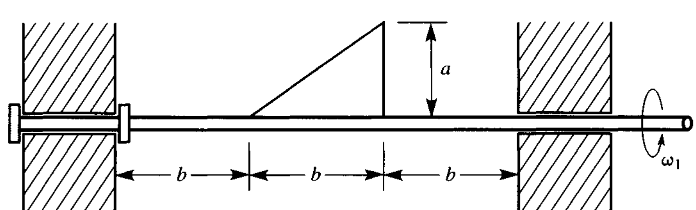
Problem 5.14

5.15 A 50-kg rectangular plate is mounted diagonally on a shaft whose mass is negligible. The system was initially at rest when a constant torque of 5 kN-m is applied to the shaft. Determine the reactions at bearings *A* and *B* four seconds after the application of the torque.



Problem 5.15

5.16 The right triangular plate is welded to the shaft, which rotates at constant speed ω. Determine the dynamic reactions at the bearings.



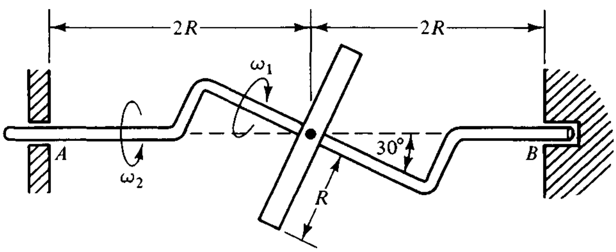
Problem 5.16

5.17 The device shown is a wobble plate, in which the spin rate ω_1 of the disk relative to the shaft, and the precession rate ω_2 of the shaft, are both constant. The mass of shaft AB is negligible. Let λ denote the ratio of the angular speeds such that $\omega_2 = \lambda\omega_1$.

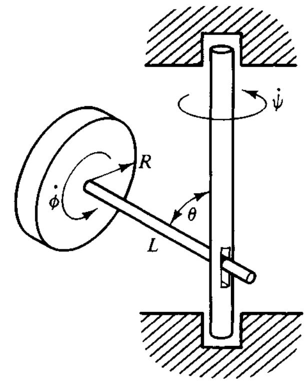
(a) In terms of ω_1 and λ, derive expressions for the angle between the angular velocity $\bar{\omega}$ and shaft AB, the angle between the angular momentum $\bar{H}_G$ and shaft AB, and for the gyroscopic moment $\dot{\bar{H}}_G$.

(b) Evaluate the results in part (a) for the case where $\lambda = 3$, and determine the corresponding reactions at bearings A and B.

(c) Determine whether there is any value of λ for which no dynamic reactions are generated at bearings A and B. Explain your answer.



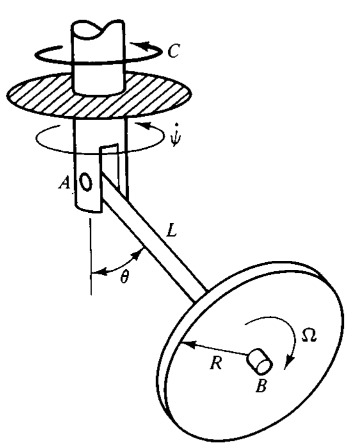
Problem 5.17



Problem 5.18

5.18 The disk spins freely at the constant rate $\dot{\phi}$ relative to its shaft, which is pinned to the vertical shaft. The system precesses freely about the vertical axis, and the mass of both shafts is negligible. The nutation angle θ is initially held constant by a cable, in which condition the precession rate is $\dot{\psi}$. Derive expressions for $\ddot{\phi}$, $\ddot{\theta}$, and $\ddot{\psi}$ at the instant after the cable is cut.

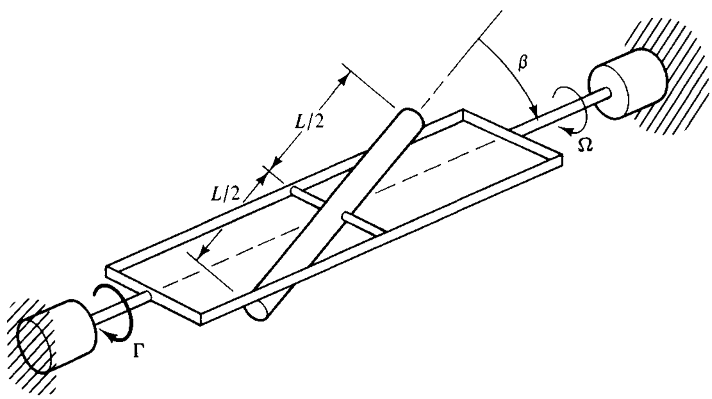
5.19 A flywheel, whose mass is m, is mounted on shaft AB. A servomotor holds the spin rate constant at Ω. A torque $C(t)$ is applied to the vertical shaft, and pivot A has



Problem 5.19

ideal properties. Derive differential equations of motion for the precession angle ψ and the nutation angle θ.

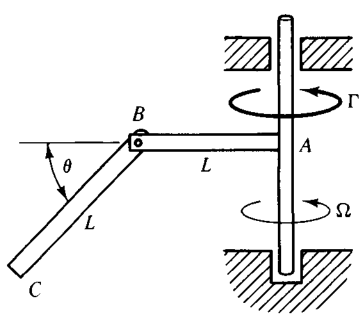
5.20 The slender bar is mounted on a gimbal that rotates about the horizontal axis at constant rate Ω due to torque Γ. Derive the differential equation governing the angle β between the bar and the horizontal axis, and also derive an expression for Γ.



Problem 5.20

5.21 Consider the gyroscopic turn indicator in Problem 5.13 in a situation where the torque $\bar{M}$ is not present, so that β is a variable angle. The precession rate Ω is a specified function of time, and the spin rate ω_1 is held constant by a servomotor. Let I_1 denote the moment of inertia of the flywheel about axis AB, and let I_2 be the centroidal moment of inertia perpendicular to axis AB. Derive the differential equation of motion for β in terms of Ω and ω_1.

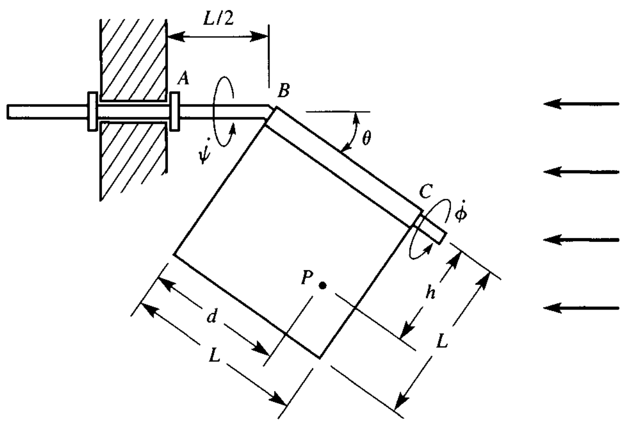
5.22 Bar BC is pivoted from the end of the T-bar. The torque Γ is such that the system rotates about the vertical axis at the constant speed Ω. Derive the differential equation of motion for the angle of elevation θ.



Problem 5.22

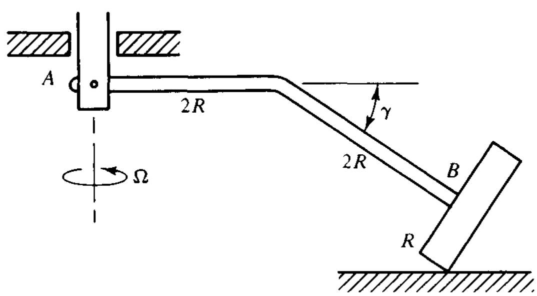
5.23 The system in Problem 5.14 precesses freely about the vertical axis at $\dot{\psi} = \omega_1$, while the spin rate $\dot{\phi} = \omega_2$ is held constant by a servomotor. The spin angle is defined such that $\phi = 0$ in the illustrated position. Derive the differential equation of motion for ψ.

5.24 The square plate is free to spin at rate $\dot{\phi}$ about axis BC of the bent shaft, while the precession rate $\dot{\psi}$ about the horizontal is held constant. The angles are defined such that $\psi = \phi = 0$ when the plate is situated in the vertical plane as shown. This system is situated in a wind tunnel whose flow is horizontal. The resultant of the aerodynamic pressure is a known force $F(t)$ always normal to the plane of the plate acting at the center of pressure P. Derive the differential equation of motion for ϕ. The gravitational force is negligible.



Problem 5.24

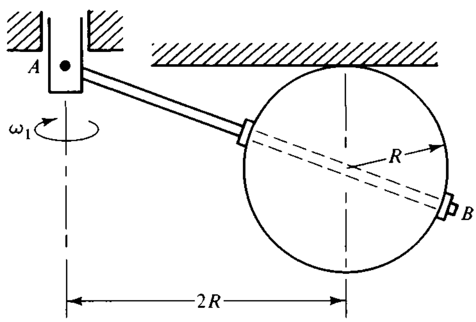
5.25 A thin disk of mass m rolls over the ground without slipping as it rotates freely relative to bent shaft AB. The connection at end A is an ideal pin. The precession rate of the bent shaft about the vertical axis is the constant value Ω. Determine the magnitude of the normal force $\bar{N}$ exerted between the disk and the ground.



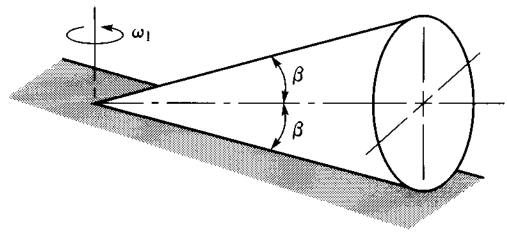
Problem 5.25

5.26 The sphere, whose mass is m, spins freely relative to shaft AB, whose mass is negligible. The system precesses about the vertical axis at the constant rate ω_1, and joint A is an ideal pin. Consider the possibility that the sphere rolls over the horizontal surface without slipping. Determine whether there is a range of values of ω_1 for which such a motion can occur.

5.27 The cone, whose mass is m and apex angle is 2β, rolls without slipping over the horizontal surface. The rolling motion of the cone is such that it is observed to rotate about a fixed vertical axis intersecting its apex at constant angular rate ω_1. Determine the maximum value of ω_1 for which the cone will not tip over its rim in this motion. Also, determine the minimum coefficient of static friction corresponding to that value of ω_1.

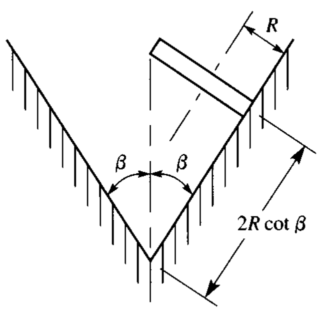


Problem 5.26

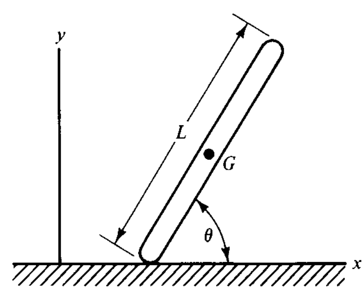


Problem 5.27

5.28 As shown in the cross-sectional view, a disk of radius R rolls without slipping over the interior surface of a cone whose apex angle is 2β. The axis of the cone is vertical. The motion is such that the disk is always normal to the contact plane, and the distance from the apex to the point of contact is always $2R\cot\beta$. Derive expressions for the precession rate of the disk about the cone's axis and the minimum coefficient of static friction required to sustain this motion.



Problem 5.28



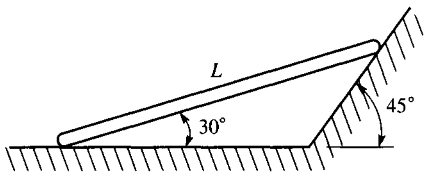
Problem 5.29

5.29 The bar of mass m is falling toward the horizontal surface. Friction is negligible. Derive differential equations of motion for the position coordinates (x_G, y_G) of the center of mass of the bar, and for the angle of inclination θ. Also obtain an expression for the contact force exerted by the ground on the bar in terms of x_G, y_G, θ, and their derivatives.

5.30 (See figure, next page.) The bar, whose mass is m, may slide over the ground and the 45° incline as it moves in the vertical plane. It is released from rest in the position shown. Determine the angular acceleration of the bar at the instant of release. Frictional resistance is negligible.

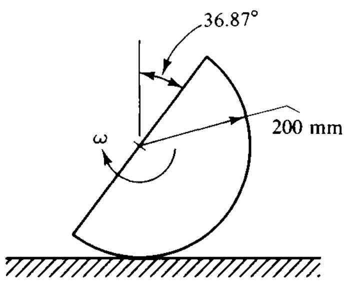
5.31 (See figure, next page.) The bar in Problem 5.30 has a clockwise angular velocity of 5 rad/s in the position shown. The coefficients of sliding friction at the ground and the incline are $\mu = 0.10$, and $L = 360$ mm. Determine the angular acceleration of the bar in this position.

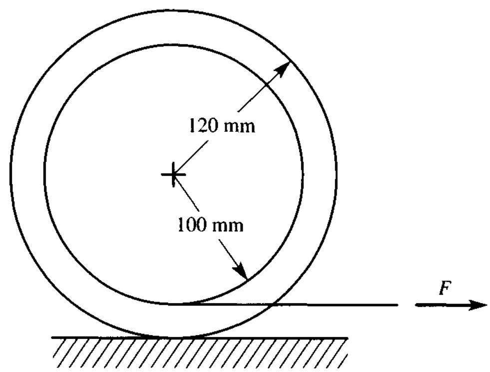
5.32 Consider an automobile whose wheelbase is L and whose center of mass is located at distance b behind the front wheel and distance h above the ground. The coefficient of friction between the tires and the ground is μ. Determine the maximum possible acceleration for cases of front-wheel, rear-wheel, and all-wheel drive.



Problems 5.30 and 5.31

5.33 The 20-kg semicylinder has an angular speed $\omega = 10$ rad/s in the position shown. The coefficient of static friction between the ground and the semicylinder is μ. Determine the minimum value of μ for which slipping between the semicylinder and the ground will not occur in this position. What is the corresponding angular acceleration $\dot{\omega}$ of the semicylinder?





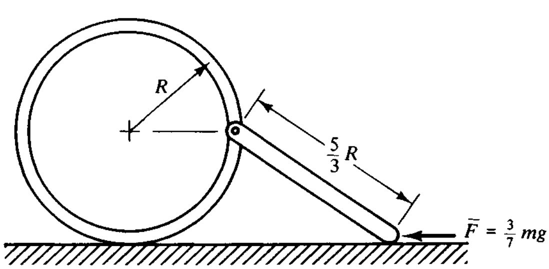
Problem 5.33 **Problem 5.34**

5.34 A cable is wrapped around the drum of the stepped cylinder and held horizontally as a tensile force F is applied to it. The mass of the cylinder is 10 kg, and its centroidal radius of gyration is 90 mm. The coefficients of static and kinetic friction with the ground are both 0.20.

(a) Derive an expression for the maximum value of F for which the cylinder will roll without slipping, and also determine the corresponding acceleration of the center of the cylinder.

(b) If F is 10% greater than the value in part (a), determine the acceleration of the center of the cylinder and the angular acceleration.

5.35 The bar is pinned at its left end to a ring. The system is initially at rest in the position shown, when a horizontal force $\bar{F}$ of magnitude $\frac{3}{7}mg$ is applied to the right end of



Problems 5.35 and 5.36

the bar. The ring does not slip over the ground in the ensuing motion, and friction between the bar and the ground is negligible. Determine the acceleration of the center of the ring at the instant the force is applied. The mass of the ring is negligible.

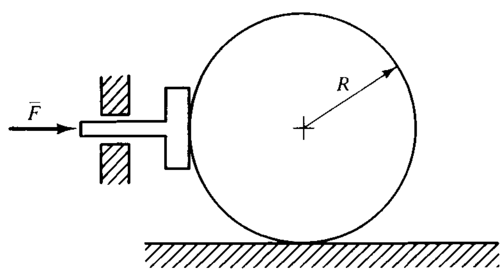
5.36 Solve Problem 5.35 for the case where the ring and the bar have equal masses.

5.37 Horizontal force $\bar{F}$ is applied to the piston, whose mass is small compared to that of the circular cylinder of mass m. The coefficients of friction are μ and ν for static and kinetic friction, respectively.

(a) For the case where there is no slipping relative to the ground, determine the acceleration of the cylinder's center.

(b) Determine the largest value of $\bar{F}$ for which the motion in part (a) is possible.

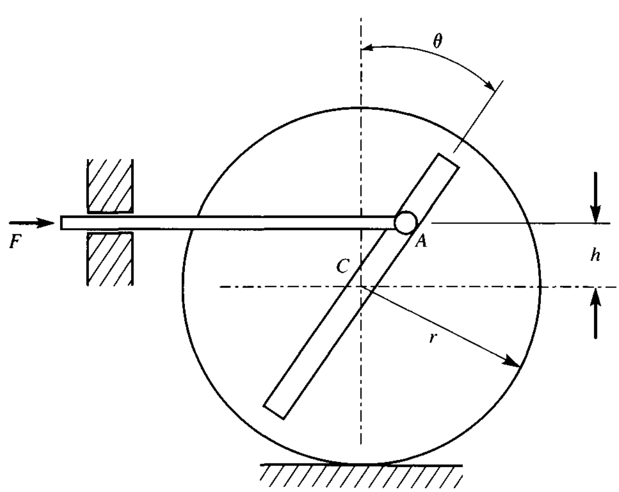
(c) Determine the translational and angular acceleration of the disk when the magnitude of $\bar{F}$ exceeds the value in part (b).



Problem 5.37

5.38 A horizontal force $\bar{F}$ is applied to the actuating rod. This rod is connected to the wheel by pin A, which may slide through the groove. The mass of the wheel is m, the radius of gyration is κ, and μ_s and μ_k are respectively the coefficients of static and kinetic friction between the wheel and the ground. Friction between the pin and the groove is negligible, as is the mass of the rod. Consider the system when the groove is at an arbitrary angle θ from the vertical.

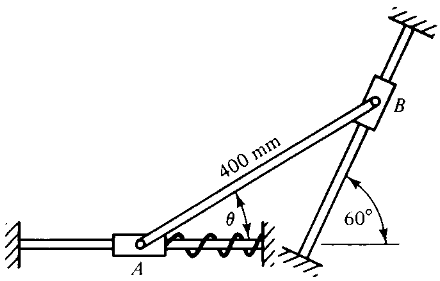
(a) If the wheel rolls without slipping, determine the acceleration of center C and the angular acceleration of the gear.



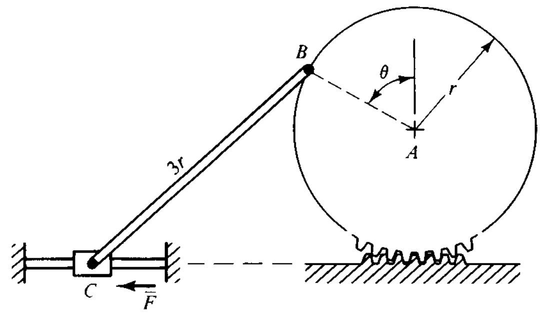
Problem 5.38

(b) Derive an expression for the largest force for which slipping will not occur at a specified θ.
(c) If $\bar{F}$ is larger than the force determined in part (b), determine the acceleration of the center C and the angular acceleration of the gear.

5.39 Bar AB has a mass of 40 kg, and the stiffness of the spring is 20 kN/m. The bar is released from rest at $\theta = 30°$, at which position the spring is compressed by 100 mm. Determine the angular velocity of the bar at $\theta = 0$.



Problem 5.39

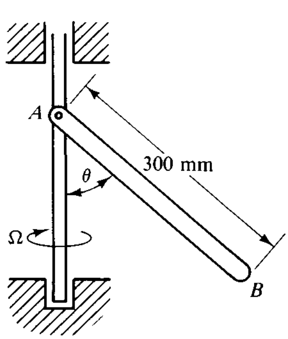


Problem 5.40

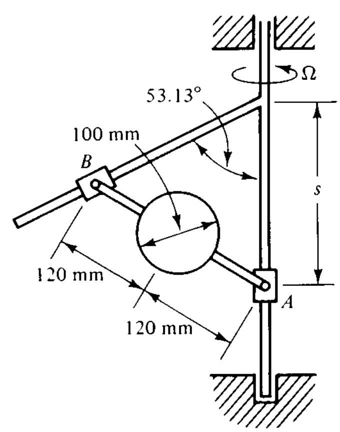
5.40 Gear A has mass m and centroidal radius of gyration κ. It rolls over the horizontal rack due to a constant horizontal force $\bar{F}$ acting on collar C. The connecting bar and collar C have negligible mass. The system was at rest at $\theta = 0$. Derive an expression for the speed v of the center of gear A as a function of θ.

5.41 Bar AB is pinned to the vertical shaft, which rotates freely. When the bar is inclined at $\theta = 10°$ from the vertical, the rotation rate about the vertical axis is $\Omega = 10$ rad/s, and $\dot{\theta} = 4$ rad/s at that instant. Determine the maximum value of θ in the subsequent motion. The mass of the vertical shaft may be neglected.

5.42 Collars A and B are interconnected by a bar on which a 10-kg sphere of 100-mm diameter is mounted. The mass of the collars and the bar is negligible. The system rotates freely about the vertical axis. Initially, $s = 240$ mm, and the collars are not



Problem 5.41

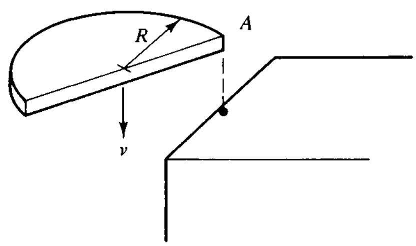


Problems 5.42 and 5.43

moving relative to the guide bars. What minimum value of the initial angular speed Ω about the vertical axis, if any, is required for collar A to reach $s = 180$ mm?

5.43 Initially, the system in Problem 5.42 is at $s = 180$ mm, at which position the rotation rate about the vertical axis is 30 rev/min. Determine whether collar A attains the position $s = 300$ mm in the subsequent motion. If so, what is the angular velocity of the sphere at that position?

5.44 A semicircular plate is falling at speed v with its plane oriented horizontally. It strikes the ledge at corner A, and the impact is perfectly elastic (i.e., the recoil velocity of corner A is v upward). The interval of the collision is Δt. Derive expressions for the velocity of the center of mass and the angular velocity at the instant following the collision. Also, derive an expression for the collision force exerted between the plate and the ledge.

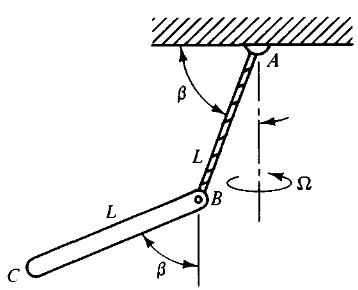


Problem 5.44

5.45 A slender bar of mass m, which is suspended by a cable from pivot A, executes a steady precession about the vertical axis at angular speed Ω as it maintains the orientation shown.

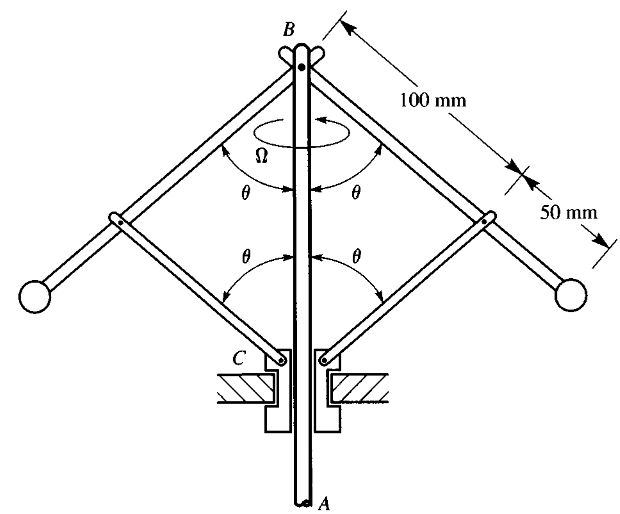
(a) Determine Ω and the angle of inclination β.

(b) An impulsive force at end B, parallel to the initial velocity of that end, acts over a short time interval Δt. Determine the magnitude of $\bar{F}$ for which the angular velocity of the bar at the conclusion of the impulsive action is horizontal. What are the corresponding velocity of the center of mass G and the angular velocity of the bar?



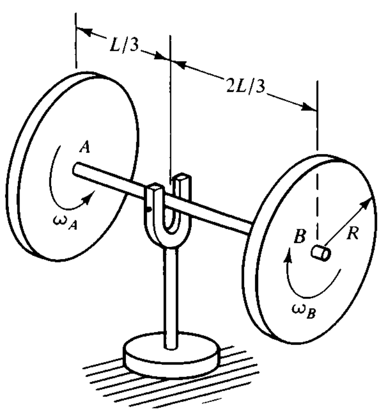
Problem 5.45

5.46 The flyball device consists of two 500-g spheres connected to the vertical shaft by a parallelogram linkage. This shaft, which passes through collar C supporting the linkage, rotates freely. The system is initially rotating steadily at 900 rev/min about the vertical axis, with $\theta = 75°$. A constant upward force $\bar{F}$ is applied to the vertical shaft, causing point B to move upward and θ to decrease. Determine the minimum magnitude of $\bar{F}$ for which the system will reach $\theta = 15°$. The mass of the shaft and the bars in the linkage is negligible.



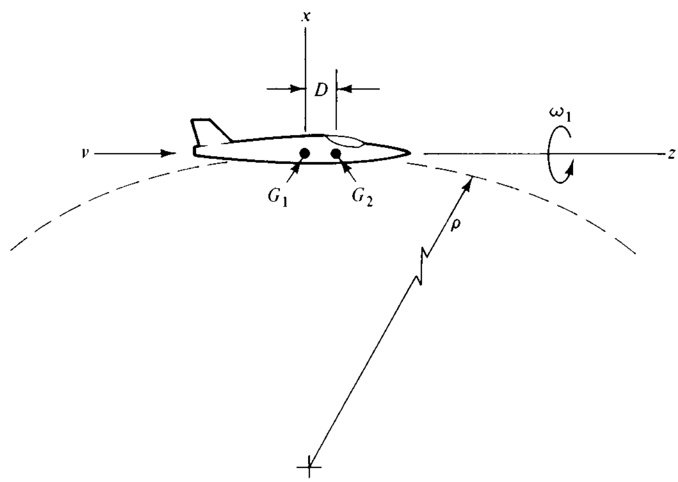
Problem 5.46

5.47 Identical disks A and B are separated by distance L on a massless, rigid shaft, about which they may spin freely. The system is suspended at the $L/3$ position from a pivot on a vertical shaft, as shown. Determine the relationship between the spin rates ω_B and ω_A of the disks for which the system will precess at a steady rate about the vertical axis, with shaft AB always horizontal.



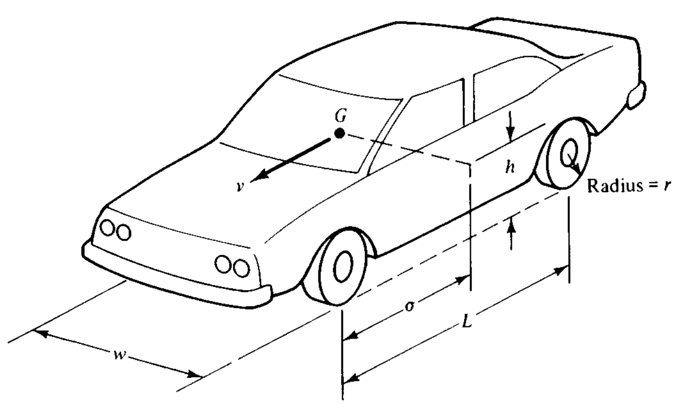
Problem 5.47

5.48 A single-engine turbojet has its minimum speed v at the top of a vertical circle of radius ρ. At that instant, the airplane is executing a roll, clockwise as viewed by the pilot, at angular speed ω_1. The engine turns at angular speed Ω, counterclockwise from the pilot's viewpoint. The rotating parts of the engine have mass m_2, and centroidal moments of inertia J about the rotation axis and J' transverse to the rotation axis. The mass of the airplane, excluding the rotating parts of the engine, is m_1, and the corresponding moments of inertia about centroidal xyz axes are I_x, I_y, and I_z. The spin axis of the engine is collinear with the z axis of the airplane, and the centers of mass G_1 and G_2, associated respectively with m_1 and m_2, both lie on this axis. Derive expressions for the aerodynamic force and moment about the center of mass of the airplane required to execute this maneuver.



Problem 5.48

5.49 The tires of an automobile have mass m and radius of gyration κ about their respective axles. The wheelbase of the automobile is L and the track is w. The center of mass G of the automobile is on the centerline of the automobile at distance σ behind the front axle and height h above the ground. Consider the situation where the automobile executes a right turn of radius ρ (measured to point G) at constant speed v. Determine the change in the normal reaction exerted between each tire and the ground resulting from the rotatory inertia of the wheels.



Problem 5.49

CHAPTER 6

Introduction to Analytical Mechanics

The constraints imposed on the motion of a system enter the Newton-Euler formulation of the equations of motion in two ways. The kinematical relations must account for the restrictions imposed on the motion, while the kinetics principles must account for the reaction force (or moment) associated with each constraint. When the system consists of more than one body, the need to account individually for the constraints associated with each connection substantially enhances the level of difficulty.

The Lagrangian formulation we shall develop in this chapter takes a different view of systems. The principles are based on an overview of the system and its mechanical energy (kinetic and potential). In contrast, the Newtonian equations of motion are time derivatives of momentum principles. Another, and perhaps the most important, difference is that the reactions exerted by supports will usually not appear in the Lagrangian formulation. This is a consequence of the fact that the reactions and the geometrical description of the system are two manifestations of the same physical feature. It is from the kinematical perspective that we shall begin our study.

6.1 Generalized Coordinates and Degrees of Freedom

Suppose the reference location of a system is given. (Such a location might be the starting position or the static equilibrium position.) We must select a set of geometrical parameters whose value uniquely defines a new position of the system relative to the initial position. For example, it should be possible to draw a diagram of the system in its current position by knowing only the fixed dimensions and the position parameters. Geometrical quantities that meet this specification are called *generalized coordinates.* The minimum number of generalized coordinates required to specify the position of the system are the *number of degrees of freedom* of that system.

A simple system consisting of a single rigid bar in planar motion is adequate to develop these concepts. According to Chasle's theorem, the general motion of a rigid body is a superposition of a translation following any specified point and a rotation about that point. In Figure 6.1, the movement of end A is described by its position coordinates, x_A and y_A, and the rotation is described by angle θ measured from the horizontal. We always consider fixed parameters, such as L, to be known, so they are system properties. Thus, the generalized coordinates selected here are (x_A, y_A, θ). We observe that these three parameters may independently be assigned arbitrary values, and that knowledge of those values would enable us to locate any point in the bar. We therefore conclude that this bar has three degrees of freedom.

Generalized coordinates do not form a unique set - other parameters may be equally suitable for describing the motion. In Figure 6.2, the generalized coordinates (x_B, y_B, θ) describe the motion in terms of the position of end B and the rotation. In

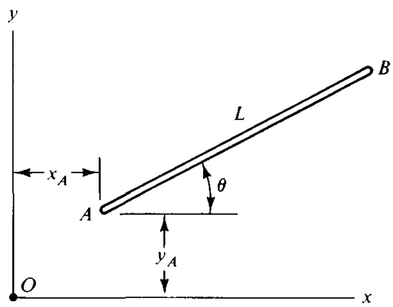


Figure 6.1 Generalized coordinates for a bar in arbitrary planar motion.

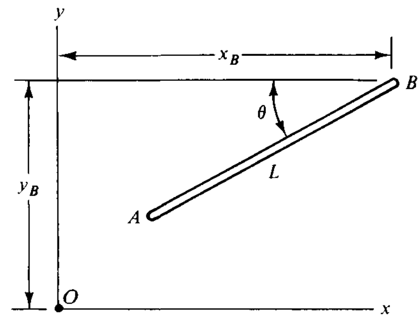


Figure 6.2 Different generalized coordinates for a rigid bar in planar motion.

general, it must be possible to express one set of generalized coordinates in terms of another, for each must be capable of describing the position of all points in the system. The transformation from the set in Figure 6.1 to that in Figure 6.2 is

$$x_B = x_A + L\cos\theta, \qquad y_B = y_A + L\sin\theta, \qquad \theta = \theta.$$

Another choice, shown in Figure 6.3, leads to a difficulty. The three generalized coordinates depicted there are (x_A, y_A, x_B). As shown in the figure, the difficulty is that for given values of (x_A, y_A, x_B), the bar can have one of two orientations. Specifically, since the length L is a fixed parameter, the vertical position of end B is given by $y_B = y_A \pm [L^2 - (x_B - x_A)^2]^{1/2}$. Recall that the generalized coordinates must *uniquely* specify the location. This means that (x_A, y_A, x_B) can serve as generalized coordinates only if the case where $y_B > y_A$ (positive sign in the previous relation for y_B) is to be considered, or alternatively, only $y_B < y_A$ (negative sign). In most cases where the orientation of a body is significant, it is best to select an angle as a generalized coordinate.

The situations in Figures 6.1–6.3 correspond to cases where the number of generalized coordinates equals the number of degrees of freedom. The generalized coordinates in such cases are *unconstrained.* This means that their values may be set independently, without violating any kinematical conditions. (Indeed, the generalized coordinates in this case are sometimes called *independent coordinates.*) When the number of generalized coordinates exceeds the number of degrees of freedom, the generalized coordinates are *constrained,* because they must satisfy additional conditions other than those arising from kinetics principles.

A set of constrained generalized coordinates for the bar in Figures 6.1–6.3 could be the position coordinates of each end and the angle of orientation, $(x_A, y_A, x_B, y_B, \theta)$. Two independent relations exist between these five variables; for example,

$$x_B - x_A = L\cos\theta, \qquad y_B - y_A = L\sin\theta.$$

These relations are consistent with the fact that the system has three degrees of freedom, because the existence of two relations between five variables means that only three variables may be selected independently. The relations between constrained generalized coordinates are called *constraint equations.* The number of degrees of freedom equals the number of generalized coordinates minus the number of constraint

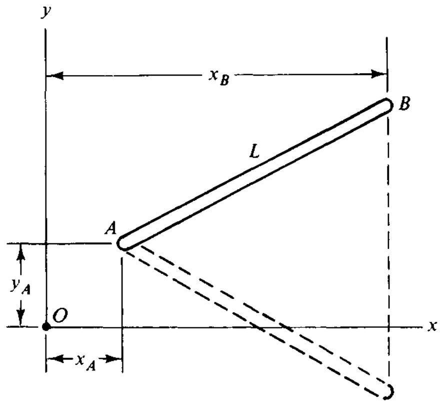


Figure 6.3 Ambiguous generalized coordinates.

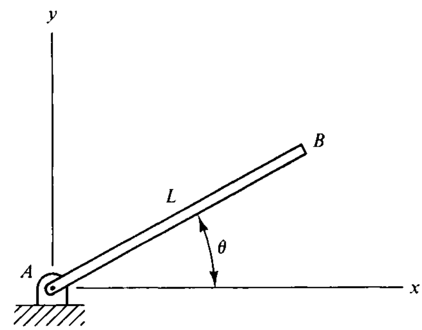


Figure 6.4 Generalized coordinates for a pinned bar.

equations. The question of constrained and unconstrained generalized coordinates will be examined in greater detail in the next section.

Other than the restriction to planar motion, the bar that has been discussed thus far is free to move in space. Any constraint imposed on its motion by supporting it in some manner alters the number of degrees of freedom and, therefore, the selection of unconstrained generalized coordinates. Figure 6.4 shows a common way in which a bar might be supported. The pin at end A prevents movement of that end in both the x and y directions. This reduces the number of degrees of freedom to one, because the position of the bar is now completely specified by the value of θ.

Another way of regarding the bar in Figure 6.4 is to say that the set of generalized coordinates (x_A, y_A, θ) for the bar are now constrained to satisfy $x_A = 0$ and $y_A = 0$. The latter are two constraint equations, which confirms that two of the three generalized coordinates selected to represent this one-degree-of-freedom system cannot be assigned independent values.

A different manner of supporting the bar also leads to a system with one degree of freedom. The bar in Figure 6.5 is constrained by the collars at its ends. A suitable unconstrained generalized coordinate is the angle θ. If this parameter is known, then

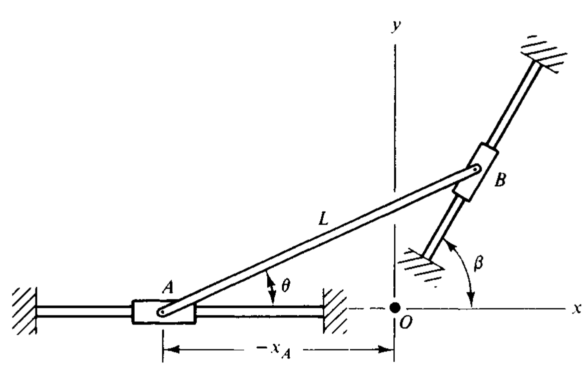


Figure 6.5 Generalized coordinates for a sliding bar.

the (x, y) coordinates of the ends may be evaluated with the aid of the law of sines. (Recall that the constant geometrical parameters, such as L and β in this system, are always considered to be known.)

An aspect of Figure 6.5 that should be noted is the selection of the origin of xyz. Because it is desirable that xyz be fixed, it is useful to place its origin at a point in the system that is stationary. This practice will avoid the possibility of inadvertently writing position coordinates relative to a moving coordinate system when the position relative to a fixed system is what we need.

We could, of course, select constrained generalized coordinates for the system in Figure 6.5. For example, the variables (x_A, y_A, θ) used earlier are related by

$$x_A = -L\frac{\sin(\beta-\theta)}{\sin\beta}, \qquad y_A = 0.$$

These constraint equations are more complicated than those for the previous system. We shall see in later sections that there are situations where it might be necessary or desirable to use constrained generalized coordinates.

The discussion thus far has dealt only with a planar system consisting of a single body. A useful set of generalized coordinates for a body in spatial motion consists of the xyz coordinates of the center of mass, and the Eulerian angles defined relative to convenient sets of axes. In other words, a body in unconstrained spatial motion has six degrees of freedom. This number is decreased by the number of constraints that are imposed.

A common system consisting of a multitude of bodies is a mechanical linkage. A typical one is depicted in Figure 6.6. If nothing is specified regarding the motion, this linkage has two degrees of freedom. (One way of recognizing this number is to ask what motion parameters are required to draw a picture of the system.) For example, in order to specify the position of bar AB, it is necessary to know s_A and θ. This will define the location of end B. Then the orientation of bar BC may be established by seeking the intersection of an arc of length L_2, centered at end B, with the inclined guide bar. Note that two, or possibly no, intersections might occur. This means that (s_A, θ) is not a suitable set of generalized coordinates, unless the range of values is restricted and we stipulate which of the intersections (upper or lower) is of interest.

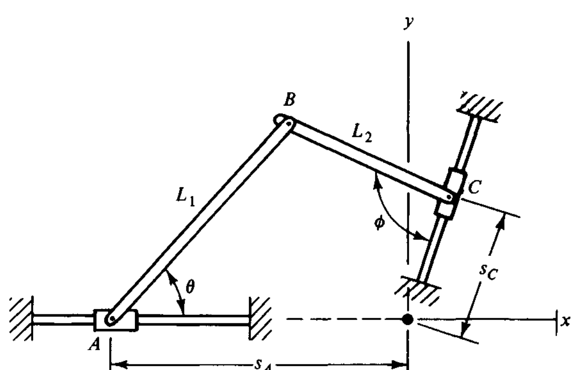


Figure 6.6 Generalized coordinates for a linkage.

A better set of generalized coordinates for this system would have been the distances s_A and s_C locating both ends of the linkage. This would eliminate the question of which intersection should be considered. Nevertheless, the set (s_A, s_C) is still limited in its range of values, because the largest possible distance between ends A and C is L_1+L_2. Indeed, other combinations of variables, such as (θ, ϕ), might be preferable.

We remarked that the system in Figure 6.6 has two degrees of freedom, provided that nothing is specified about the motion. This is not necessarily the case. For example, collar A in Figure 6.6 might be required to move in a given manner along its guide, meaning $s_A = s_A(t)$ is given. This is a constraint on the motion of the system, reducing it to only one degree of freedom.

6.2 Constraints - Holonomic and Nonholonomic

The discussion in the previous section was purposefully qualitative in order to focus on the important concept of generalized coordinates. It is useful now to change our approach. Suppose we select a set of M constrained generalized coordinates to represent a system having N degrees of freedom. Because the coordinates are a constrained set, $M > N$, there must be $M-N$ constraint equations. If the constraint equations are like those arising in the previous section, each may be written in the functional form

$$f_i(q_1, q_2, \ldots, q_M, t) = 0, \tag{6.1}$$

where the subscript i denotes which of the $M-N$ constraints are under consideration. A relation such as Eq. (6.1) is sometimes referred to as a *configuration constraint*. This term stems from the fact that any limitation imposed on the generalized coordinates restricts the overall position that the system can attain. In the most general situation, the value of time must be specified because a motion imposed on one of the physical supports can move the system, even if the generalized coordinates do not vary.

We may equivalently replace a configuration constraint by a *velocity constraint*, which is a restriction on the velocity that a system may have when it is in a specified position. This viewpoint is obtained when we differentiate Eq. (6.1) with respect to time. The chain rule for differentiation must be employed because the generalized coordinates are (unknown) functions of time. The time derivative of constraint equation (6.1) is therefore

$$\dot{f}_i = \sum_{j=1}^{M} \left[\frac{\partial}{\partial q_j} f_i(q_1, q_2, \ldots, q_M, t) \right] \dot{q}_j + \frac{\partial}{\partial t} f_i(q_1, q_2, \ldots, q_M, t) = 0. \tag{6.2}$$

When the values of the generalized coordinates and time are specified, Eq. (6.2) represents one relation among the M rates $\dot{q}_j$, which are called *generalized velocities*. Equations (6.1) and (6.2) are fully equivalent in the restriction they impose, provided that the initial position is specified in conjunction with Eq. (6.2).

A simple example of this dual viewpoint is to consider using x and y as constrained generalized coordinates for a point following the circular path in Figure 6.7. Clearly, one form of the constraint equation is the equation for a circle,

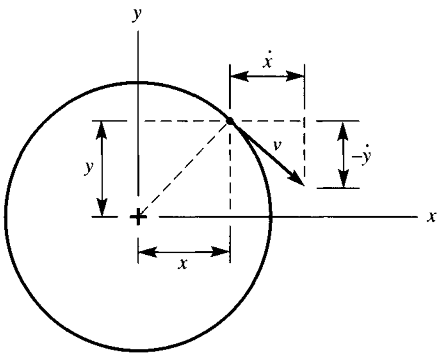


Figure 6.7 Constraint condition for circular motion.

$$x^2+y^2=R^2.$$

Differentiation of this relation yields

$$\dot{y}=-\frac{x}{y}\dot{x},$$

which is merely a statement that the velocity of a point following the circle must be parallel to the tangent to the circle at the instantaneous location of the point. Note that the radius R does not appear in the velocity constraint form. However, R must be known in order to select the x and y values at the instant when the motion is initiated.

A large class of problems involving mechanical systems may be treated by considering a more general form of velocity constraint than Eq. (6.2). Rather than having coefficients that are derivatives of f_i, these coefficients could be arbitrary functions of the generalized coordinates and time. Such constraint equations have the form

♦ $$\sum_{j=1}^{M} a_{ij}(q_1,q_2,\ldots,q_M,t)\dot{q}_j+b_i(q_1,q_2,\ldots,q_M,t)=0. \qquad (6.3)$$

We see from this that configuration constraints represent algebraic/transcendental equations relating the generalized coordinates, whereas velocity constraints are differential equations relating those quantities.

Constraint conditions that match Eq. (6.3) are said to be *linear velocity constraints,* because they depend linearly on the generalized velocities. They are not the most general type of kinematical constraint that can be imposed on the motion of a system. Some types of motion restrictions cannot be treated in any general manner. One such situation arises in treating inequality relationships, such as the limitation that the brakes of a car reaching the top of a hill are functional only if the wheels remain in contact with the road. Also, some systems feature constraints in which the generalized velocities occur nonlinearly. Another generalization is to allow accelerations to be constrained. Both cases are possible in feedback control systems.

The restrictions imposed by Eqs. (6.2) and (6.3) are equivalent if corresponding coefficients of each generalized velocity, and of the velocity-independent term, are identical to within a multiplicative factor. This factor may be a function g_i of the generalized coordinates and time, $g_i = g_i(q_1, q_2, \ldots, q_M, t)$. Hence, we may conclude that a velocity constraint is derivable from a configuration constraint if and only if

$$a_{ij} g_i = \frac{\partial f_i}{\partial q_j}, \qquad b_i g_i = \frac{\partial f_i}{\partial t}. \tag{6.4}$$

The constraint equation(s) relating the generalized velocities are said to be *holonomic* (which may be taken to mean "integrable") if they satisfy Eq. (6.4). If Eq. (6.4) is not satisfied, the constraint is *nonholonomic.* Because the terms $\partial f_i/\partial q_j$ constitute the Jacobian of a set of holonomic constraints, the coefficients a_{ij} are referred to as the *Jacobian constraint matrix,* even when the constraint conditions are not holonomic. We will encounter this matrix in several contexts.

This terminology refers to the *Pfaffian* form of a constraint equation, which is the differential form obtained by multiplying Eq. (6.3) by dt; specifically,

♦ $$\sum_{j=1}^{M} a_{ij}(q_1, q_2, \ldots, q_M, t)\, dq_j + b_i(q_1, q_2, \ldots, q_M, t)\, dt = 0. \tag{6.5}$$

When Eq. (6.4) is true, multiplying Eq. (6.5) by the function g_i converts the Pfaffian form to a perfect differential of the function f_i. In other words:

♦ *If a velocity constraint is holonomic, then there exists an integrating factor g_i for which the Pfaffian form of the constraint equation becomes a perfect differential.*

In that case the constraint may be integrated to obtain the configuration constraint imposed on the generalized coordinates. When Eq. (6.4) is not valid, the kinematical relation between the generalized coordinates can only be established after those parameters have been solved as functions of time, in other words, after the equations of motion have been solved.

It often is quite difficult to determine whether a complicated velocity constraint is holonomic. A test that is occasionally useful comes from the fact that a mixed derivative may be evaluated in any order. Thus, differentiating each of Eqs. (6.4) with respect to an arbitrarily selected generalized coordinate q_k leads to the conclusion that if constraint equation number i is holonomic then

$$\left.\begin{aligned} \frac{\partial}{\partial q_k}(g_i a_{ij}) &= \frac{\partial}{\partial q_j}(g_i a_{ik}), \\ \frac{\partial}{\partial q_k}(g_i b_i) &= \frac{\partial}{\partial t}(g_i a_{ik}), \end{aligned}\right. \qquad j, k = 1, 2, \ldots, M, \;\; j \neq k. \tag{6.6}$$

The difficulty in applying these relations is that the integrating function g_i is not known. If such a function leading to satisfaction of Eqs. (6.6) is determined, it is relatively straightforward to integrate the Pfaffian form and thereby find the corresponding configuration constraint.

We may obtain a different viewpoint for the role played by constraints by introducing the concept of the *configuration space.* The position of a point in real space is associated with the (x, y, z) coordinates it occupies at each time instant. In the same manner, the position of any system may be associated with an M-dimensional Euclidean space. The directions in this space are described by an orthogonal set of

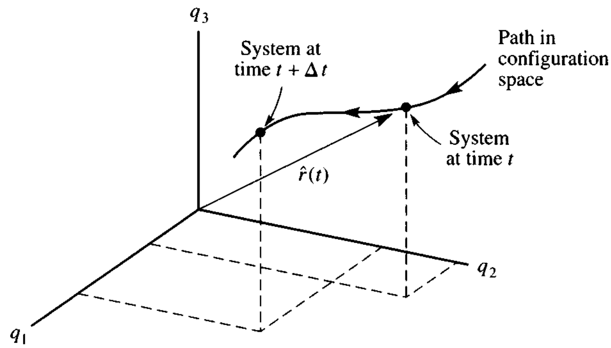


Figure 6.8 Position in the configuration space.

axes, with distance along each defined to represent the value of one of the generalized coordinates $(q_1, q_2, \ldots, q_M)$. The path in the configuration space is the locus of points formed as the motion evolves in time. Figure 6.8 depicts the path in configuration space for a system having three degrees of freedom; the picture may be conceptually extended to systems for which $M > 3$. The position of the system at any instant corresponds to a vector $\hat{r}$, where the caret ("hat") is the notation we shall use to indicate that a vector is associated with the configuration space. Thus, we have

$$\hat{r} = q_1\hat{e}_1 + q_2\hat{e}_2 + \cdots = \sum_{j=1}^{M} q_j\hat{e}_j. \tag{6.7}$$

The path the system follows through the configuration space depends on the forces acting on the system - applying forces at different locations or altering the time dependence of the forces will change the configuration path. If the generalized coordinates are unconstrained then any path through the configuration space is possible, assuming one has the ability to generate the forces required to attain such a motion. In contrast, a constraint equation represents a restriction on the values that the generalized coordinates may have. Thus, if a system is described by a set of constrained generalized coordinates, then there are restrictions on the possible paths the system may follow through the configuration space.

Consider a holonomic constraint that is independent of time, $f_i(q_1, q_2, \ldots, q_M) = 0$. This restricts the position in the configuration space to be somewhere along a surface, the shaded portion of Figure 6.9. At any point along the path in the configuration space, a multitude of displacements are possible. The configuration constraint requires that the next point also be situated on the constraint surface. A displacement in which the new point in the configuration space is on the constraint surface is said to satisfy the constraint condition. It represents a *kinematically admissible movement* of the system.

The corresponding Pfaffian form of the velocity constraint, Eq. (6.5), is merely a statement that the infinitesimal displacement of the system must be along the plane in the configuration space tangent to the constraint surface. Any other type of displacement would move the point in the configuration space off the constraint surface. Should a holonomic constraint be time-dependent, its shape changes as time evolves. The b_i term in the Pfaffian form merely represents an additional term required to compensate for the changing nature of the constraint.

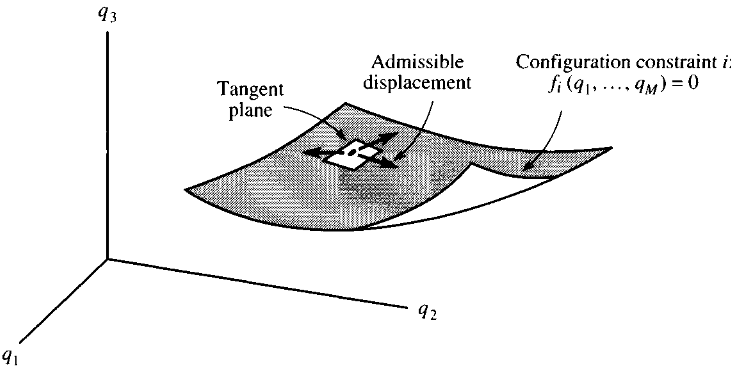


Figure 6.9 Configuration constraint.

When the constraint is nonholonomic, it is not possible to identify a constraint surface. Nevertheless, the effect of the Pfaffian constraint equation is to restrict infinitesimal displacements of the system to lie on a common tangent plane that is dictated by the current state of motion. Such a plane may be considered to be a local manifestation of a constraint surface.

There are two types of holonomic constraint. If the constraint is time-independent, then the holonomic constraint is said to be *scleronomic;* otherwise, it is *rheonomic.* ("Schlero" and "rheo" are Greek phrases meaning "rigid" and "flowing," respectively.) It follows that the integrated form of a scleronomic constraint is $f_i(q_1, q_2, \ldots, q_M) = 0$. A velocity constraint also may be classified according to the presence of the coefficient b_i. If $b_i = 0$, the equation is a *catastatic* constraint, whereas $b_i \neq 0$ means that the generalized coordinates are related by an *acatastatic* constraint equation. It is evident that a scleronomic constraint is catastatic, while a rheonomic constraint is acatastatic. However, it is possible in the case of a nonholonomic constraint for the coefficients a_{ij} to be time-dependent, even though the relation is catastatic.

In the next section we shall return to the concept of a configuration space. First, we need to gain experience in formulating constraint equations. An interesting example of a nonholonomic constraint, which does not usually arise in a course in mechanics, is a pursuit problem. Consider Figure 6.10, which depicts an airplane A

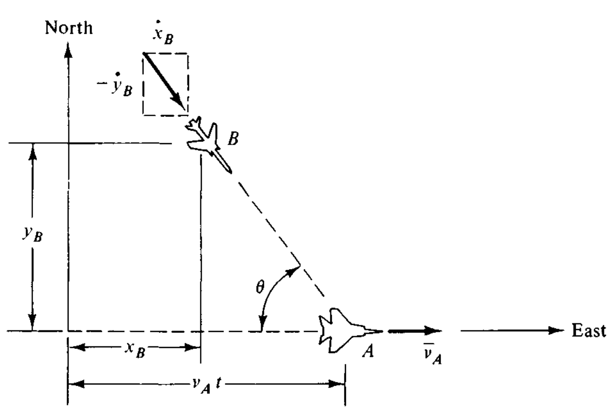


Figure 6.10 Example of a nonholonomic constraint.

that flies eastward at a constant velocity $\bar{v}_A$. Airplane B has a laser mounted parallel to its axis. This airplane must always keep its laser aimed at airplane A. This restriction constrains the path that airplane B follows, but it is only possible to express the restriction on the generalized coordinates (x_B, y_B) as a velocity constraint. We formulate this equation by noting that the angle θ describing the orientation of the velocity vector also may be expressed geometrically in terms of the position coordinates. This yields

$$\tan\theta = -\frac{\dot{y}_B}{\dot{x}_B} = \frac{y_B}{v_A t - x_B},$$

which may be written in the standard form of Eq. (6.3) as

$$y_B \dot{x}_B + (v_A t - x_B)\dot{y}_B = 0.$$

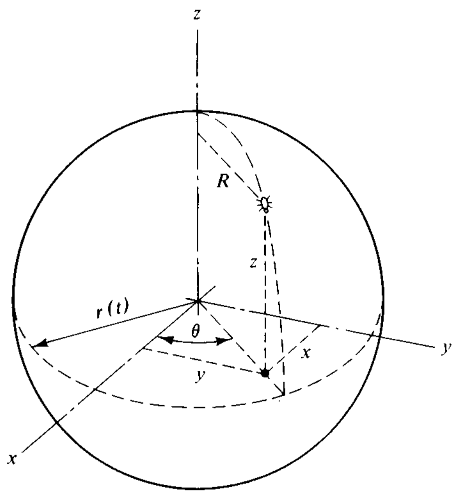
This is a time-dependent nonholonomic constraint; it cannot be integrated with respect to time unless x_B or y_B is given as a function of t.

It is important to know that a constraint is holonomic, because such constraints may be used to eliminate excess generalized coordinates. Suppose that H is the number of holonomic constraint equations relating M generalized coordinates. If these constraints were originally stated in velocity form, they may be integrated to obtain configurational constraints in the form of Eq. (6.1). Because these would represent H equations, we could solve them to express H generalized coordinates in terms of the remaining $M-H$ generalized coordinates.

A *holonomic system* is one in which $H = M - N$, that is, the number of generalized coordinates in excess of the number of degrees of freedom equals the number of holonomic constraints. It is always possible (but not necessarily desirable) to describe a holonomic system by a set of unconstrained generalized coordinates that satisfy all kinematical conditions arising from the physical manner in which the system is supported. In contrast, a *nonholonomic system,* $H < M - N$, must always be described by a set of constrained generalized coordinates. In such cases it is necessary to supplement the equations of motion with explicit statements of the kinematical constraint equations. Nonholonomic constraints often arise in systems having parts that roll.

We use the same terms to describe a system as we use to describe a constraint. Thus, if determining the position of any point in the system requires that we know the value of t, as well as the generalized coordinates, we say that the system is *time-dependent.* Holonomic systems are further classified according to whether they are *scleronomic* (all constraints are independent of time) or *rheonomic* (one or more time-dependent constraints). We shall see that treatment of nonholonomic systems requires modifications of the methods used for holonomic systems, whereas dependence of the kinematical relations on time only influences the details by which such modifications are implemented.

Example 6.1 An insect walks along the surface of a spherical balloon whose radius is a specified function $r(t)$. Describe in terms of both rectangular Cartesian and cylindrical coordinates the constraint imposed on the motion of the insect by the condition that it remain on the surface. Express the result for each set of coordinates as a configuration constraint and as a velocity constraint.



Position coordinates for a point on an expanding sphere.

Solution We may obtain the configuration constraint corresponding to a set of Cartesian coordinates by placing the origin at the center of the sphere, as shown in the sketch. The equation of a sphere describes the distance from a point on the sphere to the center:

$$f(x,y,z,t)=x^2+y^2+z^2-r(t)^2=0.$$

The corresponding velocity constraint results from a differentiation of the configuration constraint. This yields

$$x\dot{x}+y\dot{y}+z\dot{z}-r(t)\frac{d}{dt}[r(t)]=0.$$

For cylindrical coordinates, we note that the radius of the sphere may be expressed in terms of the transverse distance R and axial distance z by the Pythagorean theorem. The configurational constraint therefore is

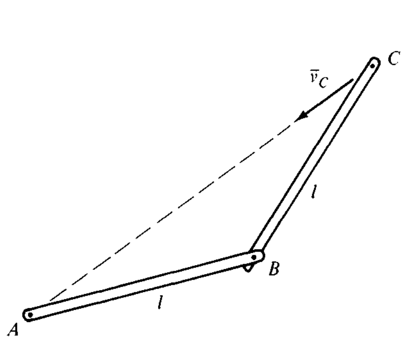
$$R^2+z^2-r(t)^2=0.$$

Differentiation of the configurational constraint leads to the velocity constraint that

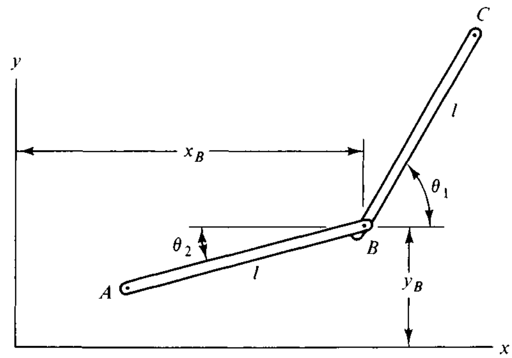
$$R\dot{R}+z\dot{z}-r(t)\frac{d}{dt}[r(t)]=0.$$

Example 6.2 Two bars, pinned at joint B, move in the horizontal plane subject only to the restriction that the velocity of end C must be directed toward end A. Determine the corresponding velocity constraint. Is this constraint holonomic?

Solution The position of each bar is uniquely specified by the coordinates x_B and y_B of the pin connection and the angles of rotation θ_1 and θ_2; these are the generalized coordinates we select. The given condition on the velocity of point C may be written in vector form as



Example 6.2



Generalized coordinates.

$$\bar{v}_C = v_C \bar{e}_{A/C},$$

which leads to the constraint condition

$$\bar{v}_C \times \bar{r}_{A/C} = \bar{0}.$$

We must express this condition in terms of the generalized coordinates. Because points B and C are common to the same body, we have

$$\bar{v}_C = \bar{v}_B + (\dot{\theta}_1 \bar{k} \times \bar{r}_{C/B}) = (\dot{x}_B - l\dot{\theta}_1 \sin\theta_1)\bar{i} + (\dot{y}_B + l\dot{\theta}_1 \cos\theta_1)\bar{j}.$$

Also, the position vector is

$$\bar{r}_{A/C} = -l(\cos\theta_1 + \cos\theta_2)\bar{i} - l(\sin\theta_1 + \sin\theta_2)\bar{j}.$$

Hence, the constraint equation leads to

$$(\bar{v}_C \times \bar{r}_{A/C})\cdot\bar{k} = l(\dot{y}_B + l\dot{\theta}_1\cos\theta_1)(\cos\theta_1 + \cos\theta_2) - l(\dot{x}_B - l\dot{\theta}_1 \sin\theta_1)(\sin\theta_1 + \sin\theta_2) = 0,$$

$$\dot{y}_B(\cos\theta_1 + \cos\theta_2) - \dot{x}_B(\sin\theta_1 + \sin\theta_2) + l\dot{\theta}_1[\cos(\theta_1 - \theta_2) + 1] = 0.$$

The relation has the standard form of a linear velocity constraint,

$$a_{11}\dot{x}_B + a_{12}\dot{y}_B + a_{13}\dot{\theta}_1 + a_{14}\dot{\theta}_2 + b_1 = 0,$$

where

$$a_{11} = -\sin\theta_1 - \sin\theta_2, \qquad a_{12} = \cos\theta_1 + \cos\theta_2,$$

$$a_{13} = l[\cos(\theta_1 - \theta_2) + 1], \qquad a_{14} = b_1 = 0.$$

In order to test whether the constraint is holonomic, we shall first assume that it is and then determine if any contradictions arise. Applying Eqs. (6.4) in the present case yields

$$\frac{\partial f_1}{\partial x_B} = g_1 a_{11}, \qquad \frac{\partial f_1}{\partial y_B} = g_1 a_{12}, \qquad \frac{\partial f_1}{\partial \theta_1} = g_1 a_{13}, \qquad \frac{\partial f_1}{\partial \theta_2} = g_1 a_{14} = 0.$$

Integrating the first with respect to x_B yields

$$f_1 = -(\sin\theta_1 + \sin\theta_2)\int g_1\, dx_B + h_1(y_B, \theta_1, \theta_2),$$

where h_1 is an arbitrary function. In contrast, integrating the last assumed equality leads to

$$f_1 = h_4(x_B, y_B, \theta_1),$$

which states that f_1 must be independent of θ_2. Clearly, this is incompatible with the first form of f_1. We therefore conclude that f_1 does not exist, which means that the constraint is nonholonomic.

6.3 Virtual Displacements

The concept of virtual movement of a system plays a central role in the kinetics principles of analytical mechanics. In a sense, the word "virtual" may be thought to mean "fictitious" or "artificial." It represents an alteration of the system's position that would result if the values of the generalized coordinates were changed. However, before we formally define the concept, it is useful to contemplate the reasons why we should consider altered positions of a system.

Recall that the path a system follows through its configuration space depends on the forces that are applied to the system. Suppose we consider two such paths, as shown in Figure 6.11. It is natural for us to contemplate which aspects cause there to be a difference between the positions on either path at a specific time instant. If we consider alternative paths in the configuration space that are infinitesimally close, we can use differential calculus to evaluate the change in any quantity associated with alternative positions. Thus, if $\hat{r}$ is the position of the system at time t when it follows one path, then the corresponding point on another path will be $\hat{r}+\delta\hat{r}$, where the

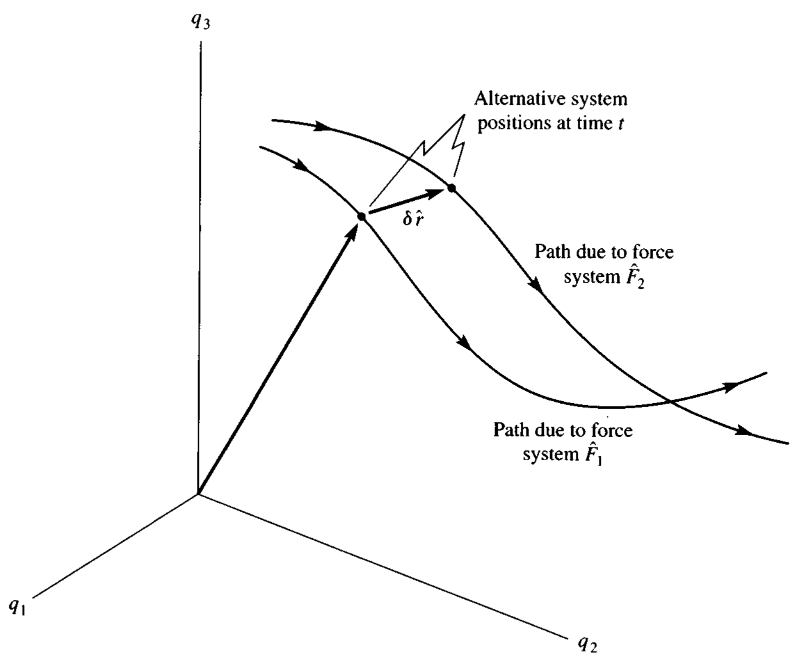


Figure 6.11 Virtual displacement in the configuration space.

symbol δ is our way of indicating that we are considering infinitesimally different states at a specified time. Because the unit vectors $\hat{e}_j$ are mutually orthogonal and the generalized coordinates are the components of $\hat{r}$, we have

$$\delta\hat{r} = \sum_{j=1}^{M} \delta q_j \hat{e}_j. \tag{6.8}$$

The foregoing equation may be stated in words as follows.

♦ *In a virtual movement, the generalized coordinates of the system are considered to be incremented by infinitesimal amounts δq_j from the values they have at an arbitrary instant, with time held constant.*

If this change were actually introduced into the system, physical points in the system would move by an infinitesimal amount. The term *virtual displacement* refers to the movement of the system in the physical three-dimensional space corresponding to a change in the configuration space position by $\delta\hat{r}$. These are not actual movements. Nevertheless, the similarity between virtual and actual differential displacements helps us to evaluate virtual displacements. Such an evaluation is an important aspect of an analytical approach to the laws of mechanics.

6.3.1 *Analytical Method*

In order to characterize a virtual displacement, consider an arbitrary point A in a system at a specified time t. The position vector $\bar{r}_{A/O}$ relative to a fixed origin depends on the generalized coordinates and time, so

$$\bar{r}_{A/O} = \bar{r}_A(q_1, q_2, \ldots, q_M, t). \tag{6.9}$$

In a virtual movement of the system, the generalized coordinate values are incremented by $\delta q_1, \delta q_2, \ldots$. The analytical method for virtual displacements evaluates the change in position resulting from these increments by differentiating an algebraic expression for position, such as Eq. (6.9).

Recall that time is held fixed at arbitrary t in a virtual movement. This is significant for two aspects of Eq. (6.9). First, because t is arbitrary, the generalized coordinates have arbitrary values; this means that the generalized coordinates must be treated as algebraic, rather than numerical, parameters. Also, since time is held constant, the explicit dependence of $\bar{r}_{A/O}$ on t should not be considered.

The change in the position of point A in a virtual movement is the virtual displacement $\delta\bar{r}_A$. Differentiation of Eq. (6.9) with t held constant shows that

♦ $$\delta\bar{r}_A = \sum_{j=1}^{M} \frac{\partial\bar{r}_A}{\partial q_j} \delta q_j. \tag{6.10a}$$

If the position of point A is described in terms of Cartesian coordinates, then the component form of the virtual displacement of the point is

$$\delta\bar{r}_A = \sum_{j=1}^{M} \left[\frac{\partial x_A}{\partial q_j}\bar{i} + \frac{\partial y_A}{\partial q_j}\bar{j} + \frac{\partial z_A}{\partial q_j}\bar{k}\right] \delta q_j. \tag{6.10b}$$

One could alternatively express the position of point A in terms of a set of curvilinear coordinates or path variables, in which case the equivalent of Eq. (6.10b) would need to recognize the variability of the associated unit vectors.

Before we apply these relations we shall develop Eq. (6.10a) by a different method, one whose viewpoint ties into the earlier discussion of alternative paths in the configuration space. Suppose we consider two kinematically admissible displacements of point A at an arbitrary instant t, each of which could be produced by applying a different set of forces to the system. Let superscript (1) or (2) denote variables associated with each set. The chain rule for differentiation indicates that the differential displacement in each case is

$$d\bar{r}_A^{(k)} = \sum_{j=1}^{M} \frac{\partial \bar{r}_{A/O}}{\partial q_j}\, dq_j^{(k)} + \frac{\partial \bar{r}_{A/O}}{\partial t}\, dt, \quad k = 1, 2. \tag{6.11}$$

The difference between the two possible displacements is

$$d\bar{r}_A^{(2)} - d\bar{r}_A^{(1)} = \sum_{j=1}^{M} \frac{\partial \bar{r}_{A/O}}{\partial q_j}(dq_j^{(2)} - dq_j^{(1)}). \tag{6.12}$$

If we define the difference between the differential increments in the generalized coordinates as the virtual increment, that is,

$$\delta q_j = dq_j^{(2)} - dq_j^{(1)}, \tag{6.13}$$

then we recover Eq. (6.10a). This strengthens our earlier observation:

♦ *When we form a virtual displacement, we are studying the differences in the movement of a system that possibly could result from the action of different sets of forces.*

Equations (6.10) are the essence of the analytical approach to evaluating the virtual displacement of a point. They require that the position coordinates of a point be expressed as algebraic functions of the generalized coordinates. (Such dependencies may usually be obtained from the laws of geometry.) As mentioned earlier, the origin O should be selected to be an actual fixed point in the system, in order to assure that the expression for $\bar{r}_{A/O}$ is actually an absolute position.

It will be necessary in many cases to express the change in the angles of orientation of various bodies in the system. This will usually involve using the law of sines and/or cosines to relate such angles to the generalized coordinates. We may then evaluate the virtual rotation by a differentiation process, such as that followed in Eqs. (6.10). Two cases arise, depending on whether the angle of orientation is expressed in explicit or implicit form. Let β be the angle of orientation. In the explicit case, $\beta = f(q_1, q_2, \ldots, q_M, t)$ has been determined. Then

♦ $$\delta\beta = \sum_{j=1}^{M} \frac{\partial f}{\partial q_j}\, \delta q_j. \tag{6.14}$$

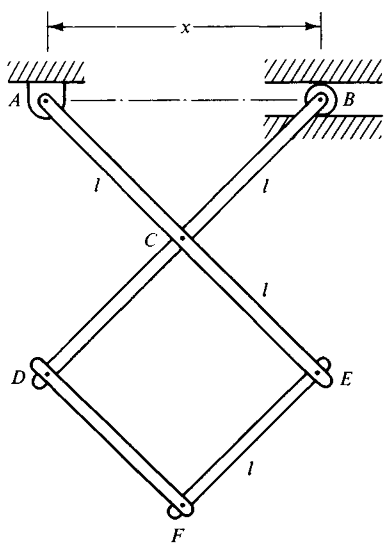
The case where β is known implicitly has the functional form $h(\beta, q_1, q_2, \ldots, q_M, t) = 0$. The virtual change in the function h resulting from incrementing all variables except time is

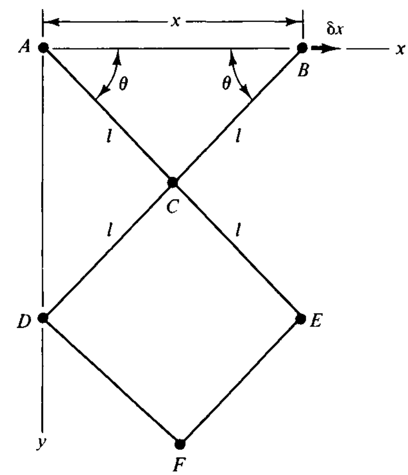
$$\delta h = \frac{\partial h}{\partial \beta}\, \delta\beta + \sum_{j=1}^{M} \frac{\partial h}{\partial q_j}\, \delta q_j = 0,$$

which yields

♦ $$\delta\beta = -\left(\frac{\partial h}{\partial \beta}\right)^{-1} \sum_{j=1}^{M} \frac{\partial h}{\partial q_j}\, \delta q_j. \tag{6.15}$$

Example 6.3 The horizontal distance x between pin A and roller B is selected as the generalized coordinate for the parallelogram linkage. Describe the virtual displacement of pin F and the virtual rotation of bar EF resulting from a virtual increment δx.





Example 6.3 Generalized coordinates and geometry.

Solution Virtual displacements are found in the analytical method by differentiating expressions for the position parameters. We place the origin at pin A because that is the only fixed point in the system. The horizontal and vertical distances between two joints on a diagonal bar are $x/2$ and $(l^2 - x^2/4)^{1/2}$, respectively, so

$$\theta = \cos^{-1}\left(\frac{x}{2l}\right), \qquad \bar{r}_{F/A} = \frac{x}{2}\bar{i} + \frac{3}{2}(4l^2 - x^2)^{1/2}\bar{j}.$$

Then

$$\delta\theta = \frac{d}{dx}\left[\cos^{-1}\left(\frac{x}{2l}\right)\right]\delta x = -\frac{\delta x}{(4l^2 - x^2)^{1/2}},$$

$$\delta\bar{r}_F = \frac{d}{dx}(\bar{r}_{F/A})\,\delta x = \left[\frac{1}{2}\bar{i} - \frac{3x}{2(4l^2 - x^2)^{1/2}}\bar{j}\right]\delta x.$$

6.3.2 *Kinematical Method*

The analytical method for evaluating virtual displacements relies on the laws of geometry for the derivation of differentiable expressions for position. Simple systems, such as those whose parts form isosceles triangles or right angles, are relatively easy to describe geometrically. However, increasing the complexity of the geometry can lead to substantial complication.

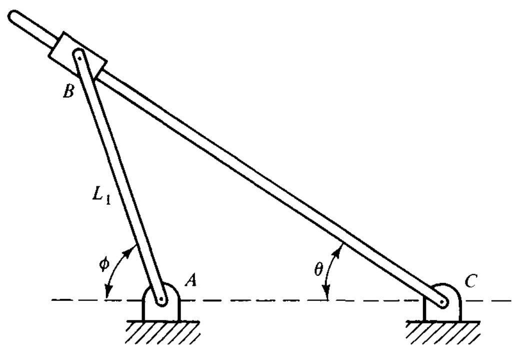


Figure 6.12 Typical linkage.

Consider the common task of evaluating the velocity relationships for a linkage such as the one in Figure 6.12. We could, in principle, write general expressions for the position of the joints, and then differentiate those expressions with respect to time in order to determine the velocity. However, to do so would ignore other kinematical techniques that expedite the analysis. The kinematical method for virtual displacement employs such techniques.

The essence of our approach is to recognize the analogous relationship between virtual and real displacements. Suppose that the position coordinates of some point A in an arbitrary system were known as a function of the generalized coordinates and time. Then the velocity of this point would be

$$\bar{v}_A = \frac{d\bar{r}_A}{dt} = \sum_{j=1}^{M} \frac{\partial \bar{r}_A}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_A}{\partial t}. \tag{6.16}$$

The actual displacement of this point in an infinitesimal time interval dt would be

$$d\bar{r}_A = \bar{v}_A\, dt = \sum_{j=1}^{M} \frac{\partial \bar{r}_A}{\partial q_j}\, dq_j + \frac{\partial \bar{r}_A}{\partial t}\, dt. \tag{6.17}$$

This expression is reminiscent of Eq. (6.10a) for virtual displacements. One difference is that the infinitesimal increments in the generalized coordinates were denoted as δq_j for virtual increments, whereas they are now dq_j for actual increments. Another difference is that, by definition, time is held constant in a virtual change. Hence, Eq. (6.10a) does not contain a time derivative term. However, the time derivative term also is not present in the case of a time-independent system in which all kinematical conditions imposed on the system do not change with elapsed time.

This similarity between $\delta\bar{r}_A$ and $d\bar{r}_A$ can be exploited. Suppose we perform a conventional velocity analysis of $\bar{v}_A$ at an arbitrary position and time. If the system is time-dependent, we retain in that expression only those terms that are proportional to the generalized velocities. The result, which we denote as $\bar{v}_A^c$, represents the velocity resulting from changes in the generalized coordinates with the physical constraints imposed on the system held constant. Hence, the "velocity" will be

♦ $$\bar{v}_A^c = \sum_{j=1}^{M} \frac{\partial \bar{r}_A}{\partial q_j}\dot{q}_j. \tag{6.18}$$

This expression for $\bar{v}_A^c$ is identical to Eq. (6.10a), aside from the presence of generalized velocities instead of virtual increments. Thus:

♦ *An expression for a virtual displacement may be obtained directly from a velocity relation by replacing generalized velocities $\dot{q}_j$ with virtual increments δq_j. The only restriction on this approach is that, when a system is time-dependent, all terms in the velocity that do not contain a generalized velocity should be dropped. Alternatively, the velocity analysis for a time-dependent system may be performed by holding constant those features that change with time in a specified manner.*

As an example, suppose the generalized coordinate for a particle as it moves along a specified path is the arclength s. Then the path-variable formula for velocity leads to

$$\delta\bar{r}_A = \delta s\bar{e}_t.$$

In the same way, the velocity formulas in cylindrical and spherical coordinates may be adapted to virtual displacement, as follows.

(a) Cylindrical coordinates:

$$\delta\bar{r}_A = \delta R\bar{e}_R + R\,\delta\theta\bar{e}_\theta + \delta z\bar{k};$$

see Figure 6.13(a).

(b) Spherical coordinates:

$$\delta\bar{r}_A = \delta r\bar{e}_r + r\,\delta\phi\bar{e}_\phi + (r\sin\phi)\,\delta\theta\bar{e}_\theta;$$

see Figure 6.13(b).

Similarly, we may relate the virtual displacements of two points in a rigid body by modifying their velocity relationship. Let $\overline{\delta\theta}$ represent the infinitesimal rotation (a vector quantity, according to the right-hand rule) in a virtual movement. In the velocity equation for rigid-body motion, we replace velocities of points by virtual displacements, and angular velocity by virtual rotation. The result is that

$$\delta\bar{r}_B = \delta\bar{r}_A + \overline{\delta\theta}\times\bar{r}_{B/A}. \tag{6.19}$$

The only subtlety involved in using this concept is the question of eliminating the effect of time-dependence of the constraints. The velocity analog must have been obtained for the time-dependent version of the constraint condition. For example, if the transverse distance R in cylindrical coordinates is constrained to be a specified function of time, then the velocity equation for a time-independent constraint would hold R constant at an arbitrary value, very much like a partial derivative. The result would be

$$\bar{v}_A^c = R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{k} \;\Rightarrow\; \delta\bar{r}_A = R\,\delta\theta\bar{e}_\theta + \delta z\bar{k}.$$

The primary utility of Eq. (6.18) is that it leads to techniques for virtual displacements that parallel those for velocity analysis. Notable among these is the method of instant centers, which is particularly useful for planar situations involving linkages and rolling without slipping. As an illustration of this concept, consider the linkage in Figure 6.14. This system was considered earlier in the discussion of generalized

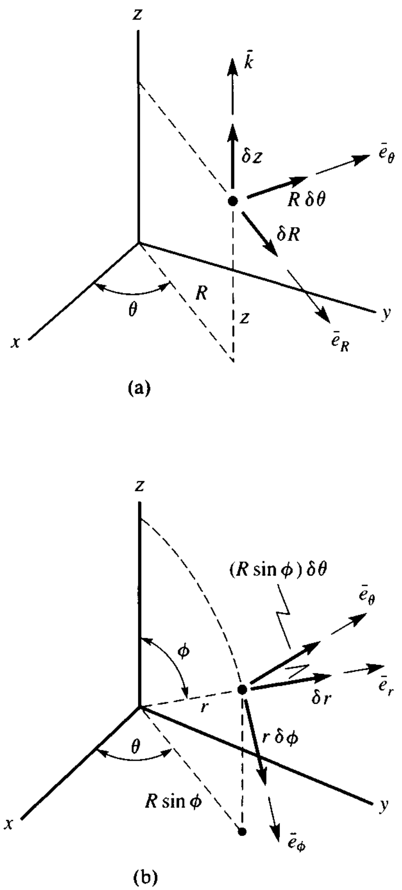


Figure 6.13 Virtual displacement in curvilinear coordinates.
(a) Cylindrical coordinates. (b) Spherical coordinates.

coordinates. Let us first consider the case where the movement of end C is specified, so that $s_C(t)$ is known. Then the linkage has only one degree of freedom. We select s_A as the generalized coordinate. Because the movement of end C is specified, we hold that end fixed in a virtual displacement, $\delta s_A = 0$.

In order to exploit the analogy between virtual displacements and velocities, we perform an instant center analysis based on end C not moving from its current position. The corresponding instant center for the virtual movement is at point D for bar AB and at point C for bar BC. (Note that the virtual rotation depicted in the figure corresponds to an increase in the generalized coordinate s_A. We make it a standard practice to depict positive increments of the generalized coordinates in order to avoid sign errors.)

The laws of trigonometry yield the distances l_A and l_B in terms of s_A, s_C, and the system parameters β, L_1, and L_2. Then the desired virtual displacements may be constructed by considering the virtual movement of bar AB to be a rotation about point D. This yields

$$\delta s_A = -l_A\,\delta\theta, \qquad |\delta\bar{r}_B| = l_B\,\delta\theta = L_2\,\delta\phi.$$

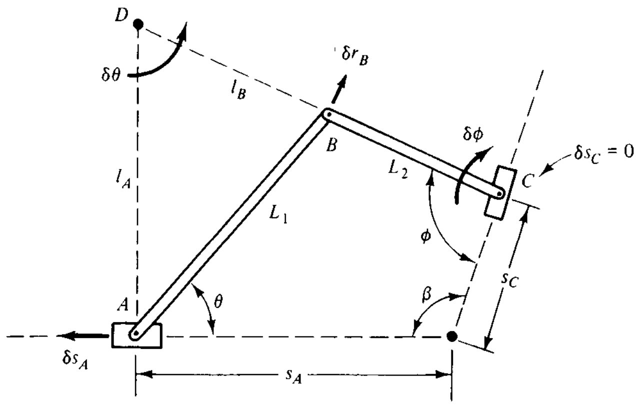


Figure 6.14 Instant center analysis of virtual displacements.

These relations may be solved for the virtual rotations:

$$\delta\theta = -\frac{1}{l_A}\,\delta s_A, \qquad \delta\phi = -\frac{l_B}{l_A L_2}\,\delta s_A.$$

If we were to use these expressions in the context of an overall formulation of the equations of motion, we would need to employ the expression for l_A and l_B in terms of s_A and s_C. The derivation of such expressions is not a trivial task, but at least the kinematical method avoids the need to differentiate the corresponding position vectors.

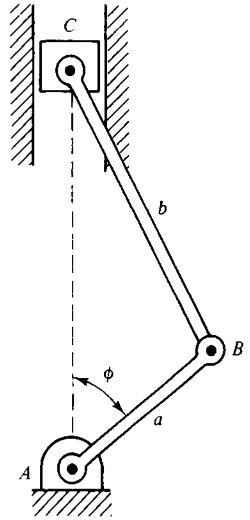
A significant aid to the kinematical method is the principle of superposition, which permits the effects of changing each generalized coordinate to be considered individually. Indeed, the basic relation, Eq. (6.10a), represents a superposition of virtual displacements, since it may be rewritten as

$$\delta\bar{r}_A = \sum_{j=1}^{M} (\delta\bar{r}_A)_j, \qquad (\delta\bar{r}_A)_j = \frac{\partial \bar{r}_{A/O}}{\partial q_j}\,\delta q_j. \tag{6.20}$$

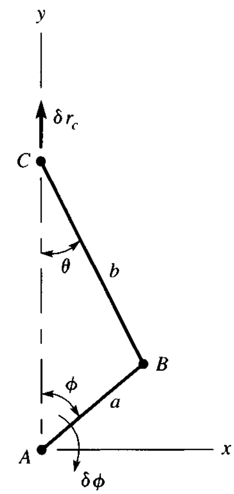
The jth contribution, $(\delta\bar{r}_A)_j$, is the virtual displacement obtained when only the corresponding q_j is incremented. This converts the kinematical analysis to investigations of a sequence of one-degree-of-freedom systems associated with each generalized coordinate. The overall virtual displacements would then be obtained by a *vectorial* superposition of the individual contributions.

The linkage in Figure 6.14 can serve also to illustrate the superposition principle. Suppose now that the collars at both ends of the linkage slide without constraint over their guide bars. The total virtual movement of the system is the superposition of the effects of incrementing s_A and s_C. The first increment was treated previously. A comparable analysis could be performed to increment s_C with s_A held fixed. The overall virtual displacements and rotations would then be the vector sum of the individual effects.

Example 6.4 The crankshaft AB is given a virtual rotation $\delta\phi$ when it is at an arbitrary orientation ϕ. Determine the corresponding virtual displacement of the piston.



Example 6.4



Kinematical parameters for virtual displacement analysis.

Solution This linkage has one degree of freedom because the other angles may be evaluated when ϕ is known. Specifically, the law of sines gives

$$\sin\theta = (a/b)\sin\phi.$$

We could use the method of instant centers to evaluate the virtual displacement. For variety, let us use a vectorial kinematical analysis. We employ Eq. (6.19) to relate the virtual displacement of each constrained point in the linkage. For the coordinate system shown in the sketch, we have

$$\delta\bar{r}_B = \delta\phi(-\bar{k})\times\bar{r}_{B/A} = \delta\bar{r}_C + \delta\theta(\bar{k})\times\bar{r}_{B/C}.$$

Because the piston is only free to move up and down, we have $\delta\bar{r}_C = \delta r_C\bar{j}$. The last step before evaluating the constraint equation is to express the position vectors solely in terms of the generalized coordinate ϕ. Using the Pythagorean theorem to determine the vertical component of $\bar{r}_{B/C}$ yields

$$\bar{r}_{B/A} = (a\sin\phi)\bar{i} + (a\cos\phi)\bar{j}, \qquad \bar{r}_{B/C} = (a\sin\phi)\bar{i} - (b^2 - a^2\sin^2\phi)^{1/2}\bar{j}.$$

Substitution of each term into the linkage equation leads to

$$\delta\phi[(a\cos\phi)\bar{i} - (a\sin\phi)\bar{j}] = \delta r_C\bar{j} + \delta\theta[(b^2 - a^2\sin^2\phi)^{1/2}\bar{i} + (a\sin\phi)\bar{j}].$$

Equating like components leads us to two scalar equations:

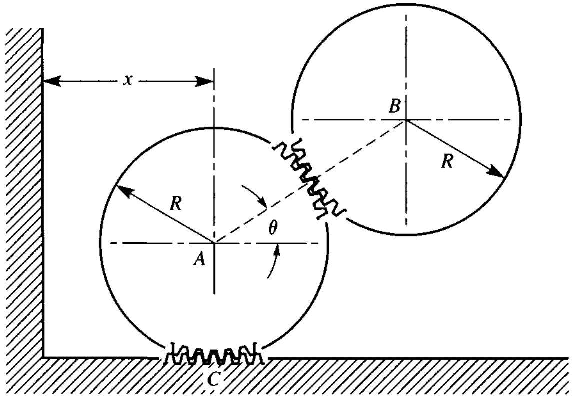
$$\delta\phi(a\cos\phi) = \delta\theta(b^2 - a^2\sin^2\phi)^{1/2},$$

$$\delta\phi(-a\sin\phi) = \delta r_C + \delta\theta(a\sin\phi).$$

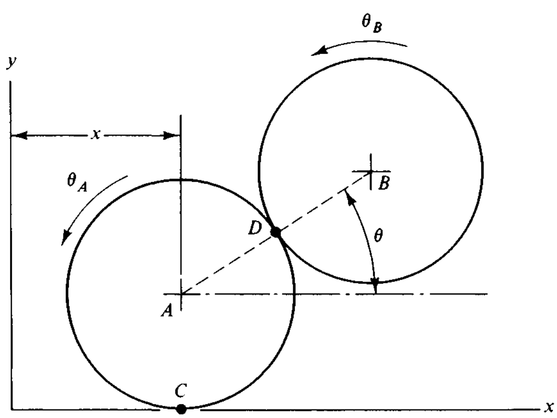
We solve the second equation for δr_C and use the first equation to eliminate $\delta\theta$, with the result that

$$\delta\bar{r}_C = -(a\sin\phi)\left[1 + \frac{a\cos\phi}{(b^2 - a^2\sin^2\phi)^{1/2}}\right]\delta\phi\bar{j}.$$

Example 6.5 In the position shown, gear B is falling as it rolls over gear A, which is rolling over rack C. Generalized coordinates are the horizontal distance x to gear A and the angle of elevation θ for the line connecting the centers. Determine the virtual displacement of the center of each gear and the virtual rotation of each gear resulting from virtual increments in the generalized coordinates.



Example 6.5



Kinematical parameters.

Solution The analogy with the relative velocity equation is useful for evaluating virtual displacements of rolling bodies. We shall use the vector equations to satisfy the constraint that the virtual displacements of contacting points must match because the gear teeth prevent slippage. We therefore write

$$\delta\bar{r}_C = \bar{0}, \qquad \delta\bar{r}_A = \delta x\,\bar{i} = \delta\theta_A\bar{k}\times\bar{r}_{A/C}, \qquad \delta\bar{r}_B = \delta\bar{r}_A + \delta\theta\bar{k}\times\bar{r}_{B/A},$$

$$\delta\bar{r}_D = \delta\bar{r}_A + \delta\theta_A\bar{k}\times\bar{r}_{D/A} = \delta\bar{r}_B + \delta\theta_B\bar{k}\times\bar{r}_{D/B}.$$

Note that the angles of rotation of the gears, θ_A and θ_B, are different variables from θ.

We describe the position vectors in terms of θ and x:

$$\bar{r}_{A/C} = R\bar{j}, \qquad \bar{r}_{D/A} = -\bar{r}_{D/B} = \tfrac{1}{2}\bar{r}_{B/A} = (R\cos\theta)\bar{i} + (R\sin\theta)\bar{j}.$$

Then, matching the two expressions for $\delta\bar{r}_A$ leads to

$$\delta\theta_A = -\frac{\delta x}{R}.$$

Using $\bar{r}_{B/A}$ to construct the virtual displacement of the center of gear B yields

$$\delta\bar{r}_B = [\delta x - (2R\sin\theta)\,\delta\theta]\bar{i} + (2R\cos\theta)\,\delta\theta\bar{j},$$

which, when substituted into the two descriptions of the virtual displacement of the contact point D, yields

$$\begin{aligned}\delta\bar{r}_D &= [\delta x - (R\sin\theta)\,\delta\theta_A]\bar{i} + (R\cos\theta)\,\delta\theta_A\bar{j}\\ &= [\delta x - (2R\sin\theta)\,\delta\theta]\bar{i} + (2R\cos\theta)\,\delta\theta\bar{j} + [(R\sin\theta)\bar{i} - (R\cos\theta)\bar{j}]\,\delta\theta_B.\end{aligned}$$

Matching either set of components in the foregoing leads to

$$-\delta\theta_A = -2\,\delta\theta + \delta\theta_B \;\Rightarrow\; \delta\theta_B = 2\,\delta\theta - \delta\theta_A = 2\,\delta\theta + (1/R)\,\delta x.$$

6.4 Generalized Forces

The selection of a set of generalized coordinates, and the evaluation of the virtual displacements in terms of those quantities, are primary aspects of the Lagrangian approach to the derivation of a system's equations of motion. It is necessary to recognize which parameters are appropriate to the kinematical description. By doing so, we create the model upon which the rest of the analysis will be based. The kinematics phase of the formulation is essentially complete when the physical velocities and virtual displacements have been related to the generalized coordinates. The kinetics principles, which we shall derive in the following sections, tend to be much more straightforward. The first task is to represent the effect of the forces exerted on a system.

6.4.1 *Virtual Work*

Consider the particle in Figure 6.15 that is subjected to a variety of forces. When this particle is given a virtual displacement $\delta\bar{r}$, the forces acting on the particle do *virtual work,* denoted δW. Because the virtual displacement is infinitesimal, the virtual work is also infinitesimal. Also, because the change is virtual, time is held constant at an arbitrary value. This means that the force is constant throughout the virtual displacement.

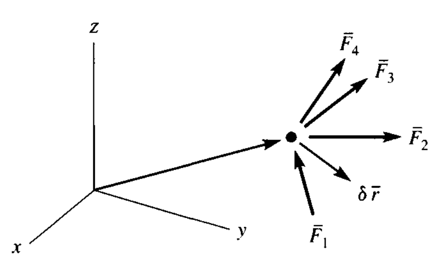


Figure 6.15 Virtual work of forces on a particle.

The virtual work of the force system in Figure 6.15 may be evaluated by taking the dot product of $\delta\bar{r}$ with each individual force. Alternatively, the resultant of the force system, $\sum \bar{F}$, may be formed first. Thus,

$$\delta W = \sum_i (\bar{F}_i \cdot \delta\bar{r}_i) = (\sum \bar{F}) \cdot \delta\bar{r}.$$

Equation (6.10a) for virtual displacement then yields

$$\delta W = \sum \bar{F} \cdot \sum_{j=1}^{M} \frac{\partial \bar{r}}{\partial q_j} \delta q_j. \tag{6.21}$$

The resultant force may be brought inside the sum over the generalized coordinates, with the result that

$$\delta W = \sum_{j=1}^{M} \left(\sum \bar{F} \cdot \frac{\partial \bar{r}}{\partial q_j} \right) \delta q_j. \tag{6.22}$$

Equation (6.22) shows that virtual work is a sum of force terms multiplying the virtual change in each generalized coordinate. A *generalized force*, denoted Q_j, is defined as the coefficient of the corresponding increment δq_j in the expression for virtual work. Thus

♦ $$\delta W = \sum_{j=1}^{M} Q_j \, \delta q_j. \tag{6.23}$$

A comparison of Eqs. (6.22) and (6.23) shows that, for a single particle,

♦ $$Q_j = \sum \bar{F} \cdot \frac{\partial \bar{r}}{\partial q_j}. \tag{6.24}$$

One reason for calling Q_j a generalized force may be recognized from Eq. (6.24). Suppose that Cartesian coordinates (x, y, z) are selected as the generalized coordinates. Then

$$\frac{\partial \bar{r}}{\partial x} = \bar{i}, \qquad \frac{\partial \bar{r}}{\partial y} = \bar{j}, \qquad \frac{\partial \bar{r}}{\partial z} = \bar{k},$$

so that

$$Q_1 = \sum \bar{F} \cdot \bar{i} = \sum F_x, \qquad Q_2 = \sum \bar{F} \cdot \bar{j} = \sum F_y, \qquad Q_3 = \sum \bar{F} \cdot \bar{k} = \sum F_z.$$

In other words, the generalized forces in this case are the force components in the respective directions. Equation (6.24) is an extension to the case where the generalized coordinates are any type of geometrical variable. For example, if q_j is an angle of rotation, then Q_j will be a moment. (This is recognizable from the fact that δW in Eq. (6.24) must have units of work.)

A different reason to call the coefficient of δq_j in Eq. (6.24) a generalized force may be seen by returning to the configuration space. Let $\hat{F}$ denote a vector in that space that is defined to represent the effect of forces acting on the system. Because $\delta\hat{r}$ describes the virtual movement of the system, it is reasonable to define this vector so that the work it does in the virtual movement is the same as the virtual work done by the actual forces in the physical space. Thus, we define $\hat{F}$ to satisfy

$$\delta W = \hat{F} \cdot \delta\hat{r}. \tag{6.25}$$

The components of $\delta\hat{r}$ are the virtual increments δq_j, so Eq. (6.25) for δW will match Eq. (6.24) if the components of $\hat{F}$ are the generalized forces, that is, if

$$\hat{F} = Q_1\hat{e}_1 + Q_2\hat{e}_2 + \cdots = \sum_{j=1}^{M} Q_j\hat{e}_j. \tag{6.26}$$

In general, the evaluation of the generalized forces corresponding to the physical forces applied to a system rests on the definition in Eq. (6.23). However, shortcuts are available to handle two special classes of forces: reactions and conservative forces. These are the subjects of the next sections.

6.4.2 *Relation between Reactions and Constraint Conditions*

The terms "reactions" and "constraint forces" are synonyms we employ to describe the forces required to enforce constraint equations. We will see that it is possible to describe the influence of such a force based solely on knowledge of the Jacobian constraint matrix.

We begin by considering the case where a particle is constrained to move along a specified curve. As shown in Figure 6.16, this situation leaves only a single degree of freedom, for which the arclength s is a convenient generalized coordinate. The resultant force may be resolved into tangent, normal, and binormal components. The latter two are constraint forces because they prevent the particle from moving perpendicular to the path. In a virtual movement that increases s by δs, the particle moves in the tangential direction by that amount, so $\delta\bar{r} = \delta s\bar{e}_t$. The corresponding virtual work is $\delta W = F_t\,\delta s$, because the virtual displacement is perpendicular to the normal and binormal force components.

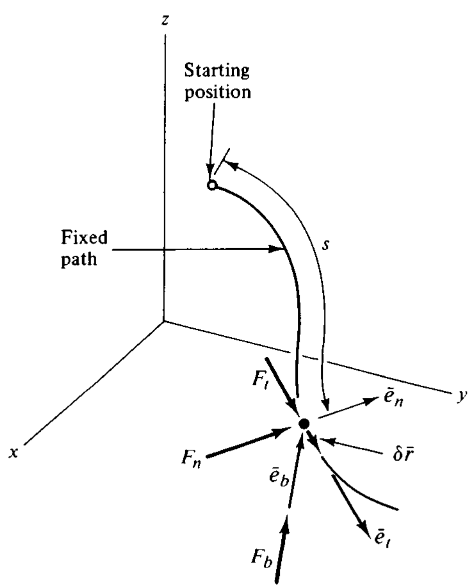


Figure 6.16 Virtual work in movement along a constrained path.

This result, that the reactions do no virtual work, is not a chance occurrence. In fact, we consider it to be the hallmark of a reaction.

♦ *A force acting on a system is a reaction associated with a constraint condition if, and only if, it does no work in a virtual movement of the system that does not violate the constraint condition.*

An important corollary comes from the observation that unconstrained generalized coordinates satisfy all constraint equations. Thus:

♦ *The virtual work done by the reactions is always zero in a holonomic system, provided that the motion of the system is described by unconstrained generalized coordinates.*

In order to establish how reactions enter into the generalized forces when the generalized coordinates form a constrained set, we return to the configuration space. Let $\hat{R}^{(i)}$ be the configuration space vector representing the generalized forces attributable to constraint condition i. The component $R_j^{(i)}$ of this vector is the contribution of the reaction associated with this constraint to the jth generalized force. Because a reaction force does no work in a virtual movement that is consistent with the corresponding kinematical restriction, we conclude that $\hat{R}$ must be perpendicular to *any* $\delta\hat{r}$ that satisfies the ith constraint equation. Thus, if $\delta\hat{r}$ satisfies constraint equation i, we have

$$\hat{R}^{(i)}\cdot\delta\hat{r} = \sum_{j=1}^{M} R_j^{(i)}\,\delta q_j = 0. \tag{6.27}$$

We now consider which condition $\delta\hat{r}$ must satisfy in order to satisfy a constraint equation. Equation (6.5) is the Pfaffian form of a general velocity constraint. Time is held constant in a virtual displacement, so we set $dt = 0$ and replace all differential increments by virtual increments. Thus, Eq. (6.5) requires that the components of $\delta\hat{r}$ in the configuration space satisfy

$$\sum_{j=1}^{M} a_{ij}(q_1, q_2, \ldots, q_M, t)\,\delta q_j = 0. \tag{6.28}$$

This has the form of a dot product in the configuration space. Let $\hat{a}^{(i)}$ be a vector whose components are the coefficients a_{ij}, with i fixed:

$$\hat{a}^{(i)} = a_{i1}\hat{e}_1 + a_{i2}\hat{e}_2 + \cdots = \sum_{j=1}^{M} a_{ij}\hat{e}_j. \tag{6.29}$$

Then any virtual displacement that is consistent with the constraint condition must satisfy

$$\hat{a}^{(i)}\cdot\delta\hat{r} = 0. \tag{6.30}$$

When the constraint is holonomic, we have $a_{ij} = \partial f_i/\partial q_j$, where $f_i = 0$ is the configuration constraint. Because this represents the gradient of the constraint equation, we see that $\hat{a}^{(i)}$ is normal to the constraint surface. It follows that Eq. (6.30) is merely a restatement that $\delta\hat{r}$ must be tangent to the constraint surface. When the constraint is nonholonomic, there is no constraint surface, so we interpret $\hat{a}^{(i)}$ as the normal to the local manifestation of a constraint surface.

The crucial aspect of Eqs. (6.27) and (6.30) is that both must be satisfied for *any* $\delta\hat{r}$ that is consistent with the constraint. This will only be true if $\hat{R}^{(i)}$ and $\hat{a}^{(i)}$ are parallel. We define the ratio of their magnitudes to be the *Lagrange multiplier* λ_i, so that

$$\hat{R}^{(i)} = \lambda_i \hat{a}^{(i)}. \tag{6.31a}$$

The components of this relation provide expressions for the contributions of the physical constraint force to the generalized forces,

♦ $$R_j^{(i)} = \lambda_i a_{ij}, \quad j = 1, 2, \ldots, M. \tag{6.31b}$$

Equation (6.31b) enables us to describe the contributions of reactions solely by knowing the Jacobian constraint matrix, a_{ij}, without actually evaluating the virtual work done by these forces.

Let us examine a few situations that illustrate when constraint forces may be avoided in the formulation. First, consider Figure 6.17(a), where two small spheres, which may be modeled as particles, are constrained to move in the x-y plane. The spheres are connected by a massless rigid bar. Because the bar is rigid, the unconstrained generalized coordinates are (x_A, y_A, ϕ). The free-body diagram for each sphere, Figure 6.17(b), shows that there is an axial force $\bar{F}$ exerted by the rigid bar on each sphere. The position vectors for the particles are

$$\bar{r}_A = x_A \bar{i} + y_A \bar{j}, \qquad \bar{r}_B = \bar{r}_A + L\bar{e}_R.$$

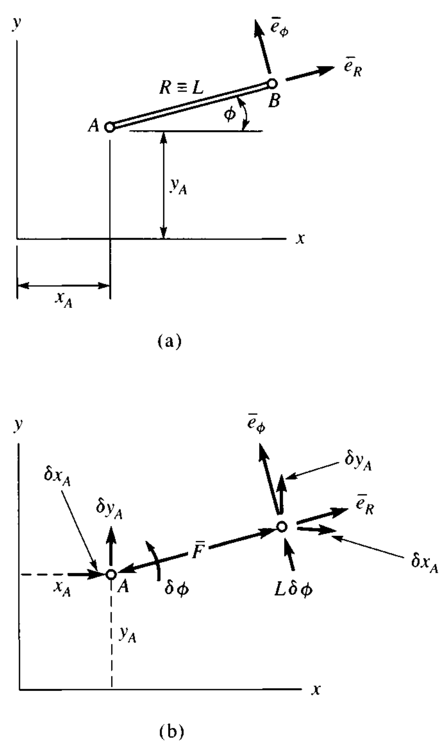


Figure 6.17 Virtual work for connected particles. (a) Generalized coordinates. (b) Virtual displacement.

Because L is constant, the virtual displacement $\delta\bar{r}_B$ differs from $\delta\bar{r}_A$ due only to the change in ϕ. Thus,

$$\delta\bar{r}_A = \delta x_A \bar{i} + \delta y_A \bar{j}, \qquad \delta\bar{r}_B = \delta\bar{r}_A + L\,\delta\phi\bar{e}_\phi.$$

The virtual work done by the axial force is therefore

$$\delta W = (-F\bar{e}_R)\cdot\delta\bar{r}_A + (F\bar{e}_R)\cdot\delta\bar{r}_B = FL\,\delta\phi\bar{e}_R\cdot\bar{e}_\phi = 0.$$

The axial force does no work in this situation because it is the constraint force required to keep L constant. If we wish to violate this constraint, we must employ a set of constrained generalized coordinates. Suppose that (x_A, y_A, R, ϕ) are used as generalized coordinates subject to the constraint condition $R = L$. Then the virtual displacements would satisfy

$$\delta\bar{r}_B = \delta\bar{r}_A + \delta R\bar{e}_R + L\,\delta\phi\bar{e}_\phi,$$

so that

$$\delta W = F\,\delta R.$$

If $\delta R \neq 0$ – that is, if the constraint condition is violated – then $\bar{F}$ does virtual work.

The case where the spheres in Figure 6.17(a) are connected by a spring offers an instructive contrast. Instead of being an unknown reaction that restricts motion, a spring force is known in terms of position. There would be no constraints on the motion of the spheres in this case, so the system would have four degrees of freedom. The set (x_A, y_A, R, ϕ) would then constitute unconstrained generalized coordinates. When the spring force is written as $F = k\Delta = k(R - R_0)$, the virtual work done by the spring force is $\delta W = F\,\delta R = k(R - R_0)\,\delta R$.

This system provides an important analogy for the general task of modeling. Let the spheres represent two adjacent particles in a body. Correspondingly, the force $\bar{F}$ represents the stress resultant exerted between them. When a body is considered to be rigid, the internal stress resultants are equivalent to reactions that maintain the particles at fixed relative distances. These forces do no virtual work. In a deformable-body model, the internal-stress resultants are equivalent to spring forces. An analysis of such a system requires consideration of the deformation associated with internal stresses.

Connections between bodies are an important element in most dynamic systems. In the absence of friction (i.e., if the connection is ideal) then the reactions associated with the connections will do no virtual work. In Figure 6.18, two bars are connected

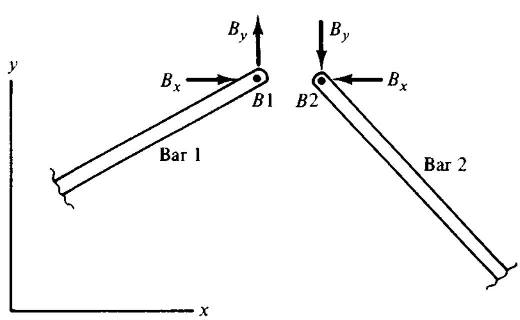


Figure 6.18 Reactions for a pin constraint.

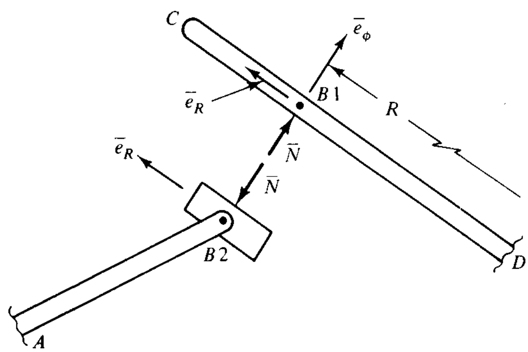


Figure 6.19 Reactions for a sliding-collar constraint.

by a pin. Let B_1 and B_2 denote the connection point on the respective bars. The corresponding virtual work is

$$\delta W = (B_x\bar{i} + B_y\bar{j})\cdot\delta\bar{r}_{B1} + (-B_x\bar{i} - B_y\bar{j})\cdot\delta\bar{r}_{B2}.$$

The pin connection constrains the points to displace by the same amount. If that constraint is not violated, then $\delta\bar{r}_{B2} = \delta\bar{r}_{B1}$, which leads to $\delta W = 0$. Thus, if the system is described by a set of unconstrained generalized coordinates, there is no need to explicitly consider the pin reactions.

Another connection that is commonly encountered is a sliding collar, such as the one in Figure 6.19. The collar can only move inward or outward relative to bar *CD*, so the virtual displacements of point B_1 on bar *CD*, and of point B_2 on the collar, satisfy

$$\delta\bar{r}_{B2} = \delta\bar{r}_{B1} + \delta R\bar{e}_R.$$

If the connection is ideal then the only force that is developed at the collar is the normal reaction $\bar{N}$. The virtual work is therefore

$$\delta W = (N\bar{e}_\phi)\cdot\delta\bar{r}_{B1} + (-N\bar{e}_\phi)\cdot\delta\bar{r}_{B2} = -(N\bar{e}_\phi)\cdot(\delta R\bar{e}_R) = 0.$$

The reaction $\bar{N}$ is the constraint force enforcing the kinematical requirement that the collar and bar *CD* execute the same movement in the direction normal to bar *CD*. This condition was satisfied by the virtual displacement, so the reaction $\bar{N}$ did no virtual work.

Suppose that friction is present. Then a friction force $\bar{f}$ parallel to bar *CD* would act in opposition to the sliding motion. The situation appearing in Figure 6.20 is based on the collar sliding outward, $\dot{R} > 0$. The virtual work in this case would be

$$\delta W = (N\bar{e}_\phi + f\bar{e}_R)\cdot\delta\bar{r}_{B1} + (-N\bar{e}_\phi - f\bar{e}_R)\cdot\delta\bar{r}_{B2} = -f\,\delta R.$$

Hence, $\bar{f}$ does work in a virtual displacement that is consistent with the constraint imposed by the collar. Note that friction does not represent a constraint force, because it does not prevent sliding motion. (The exception is static Coulomb friction. However, in that case the connection acts like a pin, because the collar does not move relative to the bar.)

Many other types of constraint forces could be considered at this juncture – for example, the reaction forces associated with rolling motion. The normal force is a constraint force that prevents interpenetration of the contacting surfaces. In the case

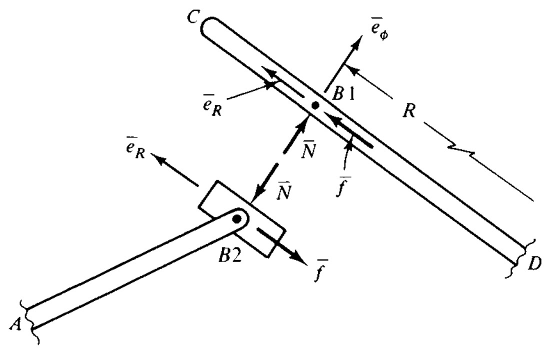
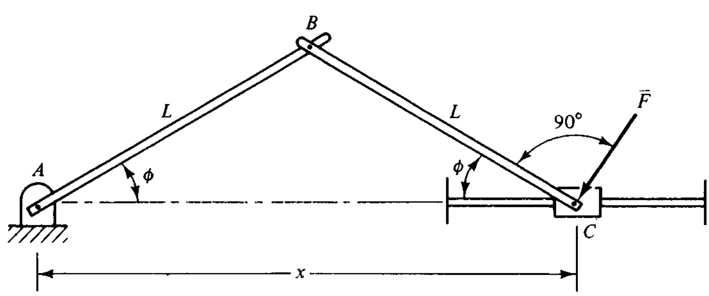


Figure 6.20 Effect of friction in a sliding collar.

of no slipping, the tangential force developed by friction or gear teeth makes the points of contact on the two bodies move by the same amount. Hence, it is also a constraint force. It follows that the forces exerted between rolling bodies will do no virtual work if there is no slippage. Conversely, the tangential force will do virtual work when there is slippage, because it is then not constraining the motion.

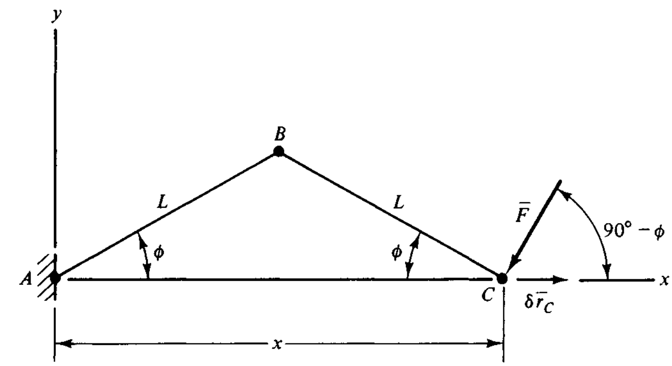
These examples emphasize the fact that a reaction does no work in a virtual displacement that is consistent with the constraint condition imposed by the force. We also see that some internal forces exerted between parts of a system do virtual work (friction and elastic forces in the examples). Thus, before using the theorem regarding virtual work done by reactions, it is important to characterize the various forces. Free-body diagrams are valuable aids for this task.

Example 6.6 A force $\bar{F}$ having constant magnitude is applied to end C of the linkage such that it always is perpendicular to link BC. Consider the alternative choice for the generalized coordinate as either the angle ϕ or the distance x. Determine the corresponding generalized force.



Example 6.6

Solution The virtual work done by the external force at end C is $\bar{F}\cdot\delta\bar{r}_C$. Our selection of the generalized coordinate affects the description of both $\bar{F}$ and $\delta\bar{r}_C$, although the latter is directed along the x axis in any case. When ϕ is the generalized coordinate, the analytical method for virtual displacement gives



Kinematical parameters.

$$\bar{F} = -(F\sin\phi)\bar{i} - (F\cos\phi)\bar{j}, \qquad \bar{r}_{C/A} = (2L\cos\phi)\bar{i}$$

$$\delta\bar{r}_C = \frac{\partial \bar{r}_{C/A}}{\partial\phi}\,\delta\phi = -(2L\sin\phi)\,\delta\phi\bar{i}.$$

The corresponding virtual work is

$$\delta W = [-(F\sin\phi)\bar{i} - (F\cos\phi)\bar{j}]\cdot(-2L\sin\phi\,\delta\phi\bar{i}) = 2FL\sin^2\phi\,\delta\phi.$$

Matching this to the standard form $\delta W = Q_\phi\,\delta\phi$ then yields

$$Q_\phi = 2FL\sin^2\phi.$$

When x is used as the generalized coordinate, the virtual displacement of point C is

$$\delta\bar{r}_C = \delta x\bar{i}.$$

We must describe the components of $\bar{F}$ in terms of the generalized coordinate. The angle ϕ is related to this generalized coordinate by

$$\cos\phi = \frac{x}{2L}, \qquad \sin\phi = \frac{1}{L}\left(L^2 - \frac{x^2}{4}\right)^{1/2},$$

so the force is

$$\bar{F} = -\frac{F}{L}\left[\left(L^2 - \frac{x^2}{4}\right)^{1/2}\bar{i} + \frac{x}{2}\bar{j}\right].$$

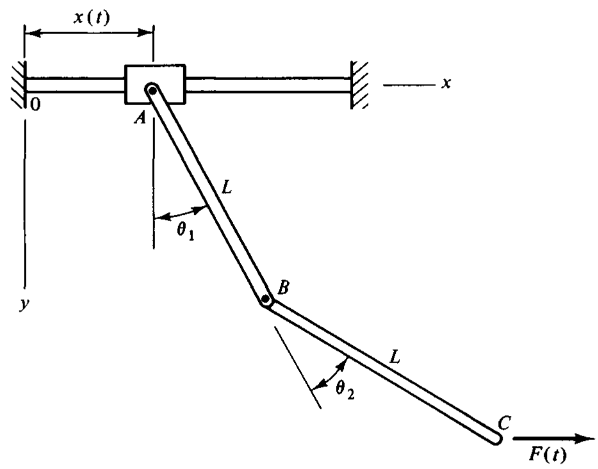
The virtual work in this case is

$$\delta W = \bar{F}\cdot\delta\bar{r}_C = -\frac{F}{L}\left(L^2 - \frac{x^2}{4}\right)^{1/2}\delta x = Q_x\,\delta x.$$

Matching the two expressions for δW yields

$$Q_x = -\frac{F}{L}\left(L^2 - \frac{x^2}{4}\right)^{1/2}.$$

Example 6.7 A horizontal force $F(t)$ is applied to the end of the compound pendulum whose pivot is given a specified horizontal displacement $x(t)$. Generalized



Example 6.7

coordinates are the absolute angle of rotation θ_1 for the upper bar and the relative angle θ_2 for the lower arm. Determine the corresponding generalized forces.

Solution The virtual work is

$$\delta W = \bar{F} \cdot \delta \bar{r}_C.$$

We shall employ the analytical method to determine the virtual displacement. The position of point C with respect to the fixed support is

$$\bar{r}_{C/O} = [x + L\sin\theta_1 + L\sin(\theta_1+\theta_2)]\bar{i} + [L\cos\theta_1 + L\cos(\theta_1+\theta_2)]\bar{j}.$$

It is important to the evaluation of virtual displacement that time be held constant. Hence, in this evaluation collar A is held stationary, $\delta x = 0$. The chain rule for differentiation then yields the displacement resulting from virtual increments in θ_1 and θ_2, according to

$$\begin{aligned}\delta \bar{r}_C &= \frac{\partial \bar{r}_{C/O}}{\partial \theta_1}\delta\theta_1 + \frac{\partial \bar{r}_{C/O}}{\partial \theta_2}\delta\theta_2 \\ &= L\{[\cos\theta_1 + \cos(\theta_1+\theta_2)]\bar{i} - [\sin\theta_1 + \sin(\theta_1+\theta_2)]\bar{j}\}\,\delta\theta_1 \\ &\quad + L[\cos(\theta_1+\theta_2)\bar{i} - \sin(\theta_1+\theta_2)\bar{j}]\,\delta\theta_2.\end{aligned}$$

Because the force $\bar{F}$ is horizontal, $\bar{F} = F\bar{i}$, only the horizontal component of $\delta\bar{r}_C$ affects the virtual work:

$$\delta W = FL[\cos\theta_1 + \cos(\theta_1+\theta_2)]\,\delta\theta_1 + FL\cos(\theta_1+\theta_2)\,\delta\theta_2.$$

We identify the generalized forces by matching these coefficients of $\delta\theta_1$ and $\delta\theta_2$ to the definition,

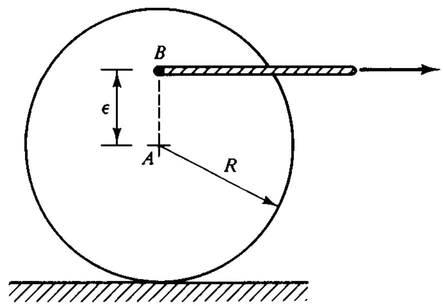
$$\delta W = Q_1\,\delta\theta_1 + Q_2\,\delta\theta_2.$$

This yields

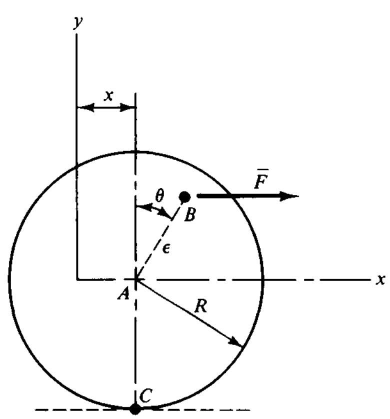
$$Q_1 = FL[\cos\theta_1 + \cos(\theta_1+\theta_2)], \qquad Q_2 = FL\cos(\theta_1+\theta_2).$$

These expressions have a simple interpretation. Note that $L\cos\theta_1 + L\cos(\theta_1+\theta_2)$ is the lever arm of $\bar{F}$ about collar A, whereas $L\cos(\theta_1+\theta_2)$ is the lever arm of the force about pin B. Correspondingly, Q_1 and Q_2 are the moments about the respective points.

Example 6.8 A cable is tied to pin B on pinion gear A. A tensile force $\bar{F}$ is applied to the free end of the cable so that the cable remains horizontal. Determine the generalized force corresponding to the choice of rotation of gear A as the generalized coordinate. The gear rolls without slipping.



Example 6.8



Kinematical parameters.

Solution It is important to remember that evaluation of generalized forces must be performed on the basis of the system's arbitrary position. Thus, we draw a sketch with the radial line to the pin rotated by an angle θ away from the vertical. The kinematical method is appropriate for the description of rolling motion. The analogy with a velocity analysis, combined with the observation that contact point C does not slip when its constraint condition is satisfied, leads to

$$\delta\bar{r}_C = \bar{0}, \qquad \delta\bar{r}_A = \delta x_A \bar{i} = \delta\theta(-\bar{k})\times\bar{r}_{A/C}, \qquad \bar{r}_{A/C} = R\bar{j},$$

$$\delta\bar{r}_B = \delta\bar{r}_A + \delta\theta(-\bar{k})\times\bar{r}_{B/A}, \qquad \bar{r}_{B/A} = \epsilon[(\sin\theta)\bar{i} + (\cos\theta)\bar{j}].$$

Substitution of the position vectors yields

$$\delta x_A = R\,\delta\theta, \qquad \delta\bar{r}_B = [(R+\epsilon\cos\theta)\bar{i} - (\epsilon\sin\theta)\bar{j}]\,\delta\theta.$$

The cable is maintained at a horizontal orientation, so the corresponding virtual work is

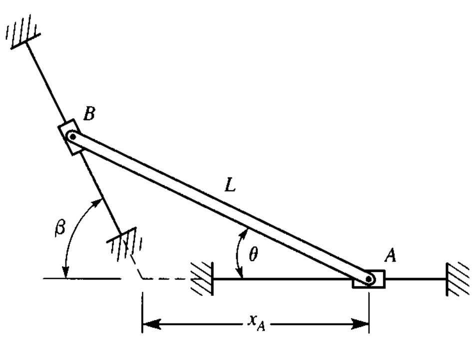
$$\delta W = F\bar{i}\cdot\delta\bar{r}_B = F(R+\epsilon\cos\theta)\,\delta\theta.$$

Matching this expression to the standard form $\delta W = Q_1\,\delta\theta$ leads to

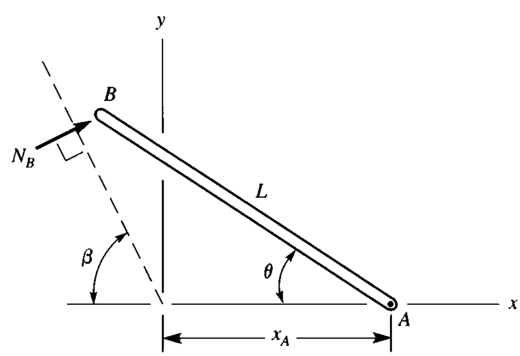
$$Q_1 = F(R+\epsilon\cos\theta).$$

This term is recognizable as the moment of $\bar{F}$ about contact point C. In fact, an easier approach to solving this problem is to replace the cable force by force $\bar{F}$ and clockwise couple $F\epsilon\cos\theta$ acting at point A. Then, the virtual work is the sum of work done by the moment in a virtual rotation $\delta\theta$ and work done by the force F in a virtual displacement of point A by $R\,\delta\theta$.

Example 6.9 Consider the bar, which is constrained by collars to follow the guide bars. Use x_A and θ as a pair of generalized coordinates that are subject to the constraint condition that collar B must follow its guide. Derive the contribution of the reaction at collar B to the generalized forces by using a Lagrange multiplier. Then compare that analysis to a formulation based on assessing the virtual work done by the reaction force.



Example 6.9



Configuration for arbitrary values of constrained generalized coordinates.

Solution We place end A on the horizontal guide using x_A, and then use θ to determine the orientation of the bar. As shown in the sketch, end B then does not contact the inclined guide bar because the generalized coordinates have arbitrary values. We know that the reaction associated with a sliding collar is perpendicular to the guide, and we show this force as $\bar{N}_B$ acting at end B.

To use a Lagrange multiplier to describe the contribution of this reaction, we need the coefficients a_{ij}, so we describe the constraint condition in velocity form. Accordingly, we have

$$\bar{v}_A = \dot{x}_A\bar{i}, \qquad \bar{v}_B = v_B[-(\cos\beta)\bar{i} + (\sin\beta)\bar{j}], \qquad \bar{v}_B = \bar{v}_A + (-\dot{\theta}\bar{k})\times\bar{r}_{B/A}.$$

For arbitrary values of x_A and θ, the position vector is $\bar{r}_{B/A} = -(L\cos\theta)\bar{i} + (L\sin\theta)\bar{j}$, so the components of the velocity equations yield

$$-v_B\cos\beta = \dot{x}_A + L\dot{\theta}\sin\theta, \qquad v_B\sin\beta = L\dot{\theta}\cos\theta.$$

Because v_B is not one of the generalized velocities, we eliminate it from these two equations, with the result that

$$\dot{x}_A\sin\beta + L\dot{\theta}(\sin\theta\sin\beta - \cos\theta\cos\beta) = 0.$$

In terms of the standard form of a velocity constraint in Eq. (6.3), we find that

$$a_{11} = \sin\beta, \qquad a_{12} = L(\sin\theta\sin\beta - \cos\theta\cos\beta) \equiv -L\cos(\theta+\beta).$$

It follows from Eq. (6.33b) that the contributions of $\bar{N}_B$ to the generalized forces are

$$R_1^{(1)} = \lambda_1\sin\beta, \qquad R_2^{(1)} = -\lambda_1 L\cos(\theta+\beta).$$

The corresponding analysis considering the virtual work done by $\bar{N}_B$ begins by describing the virtual displacement of end B. The position corresponding to arbitrary values of x_A and θ is

$$\bar{r}_{B/O} = (x_A - L\cos\theta)\bar{i} + (L\sin\theta)\bar{j},$$

from which it follows that

$$\delta\bar{r}_B = \frac{\partial\bar{r}_{B/O}}{\partial x_A}\,\delta x_A + \frac{\partial\bar{r}_{B/O}}{\partial\theta}\,\delta\theta = \delta x_A\bar{i} + [(L\sin\theta)\bar{i} + (L\cos\theta)\bar{j}]\,\delta\theta.$$

The component representation of the force is

$$\bar{N}_B = N_B[(\sin\beta)\bar{i} + (\cos\beta)\bar{j}],$$

so the virtual work done by this force is

$$\delta W = \bar{N}_B\cdot\delta\bar{r}_B = N_B\sin\beta(\delta x_A + L\sin\theta\,\delta\theta) + N_B\cos\beta(L\cos\theta)\,\delta\theta.$$

Collecting the coefficients of δx_A and $\delta\theta$ according to the definition of generalized forces, Eq. (6.23), yields

$$R_1^{(1)} = N_B, \qquad R_2^{(1)} = N_B L(\sin\beta\sin\theta + \cos\beta\cos\theta) \equiv -N_B L\cos(\beta+\theta).$$

A comparison of these generalized forces to the corresponding Lagrange multiplier forms reveals that $\lambda_1 = N_B$.

It is instructive also to consider this holonomic system when it is described by a single generalized coordinate, for which θ would be suitable. Of course, the reaction $\bar{N}_B$ would not enter the formulation because the constraint it imposes would be satisfied. However, suppose we consider some excitation force applied to the bar. Evaluating the virtual work of such a force in this case would require that we describe the virtual displacement of the point at which the force is applied. This, in turn, would require that we invoke the laws of trigonometry to express x_A in terms of θ. In contrast, using constrained generalized coordinates (as we did here) would enable us to avoid such an analysis. In general, one reason for employing constrained generalized coordinates to describe a holonomic system is that doing so might simplify the geometrical analysis. This consideration is addressed in greater detail in Section 7.1.

6.4.3 *Conservative Forces*

The ability to avoid the virtual work done by constraint forces in a compatible virtual displacement substantially simplifies evaluation of the generalized forces. Another simplification stems from the potential energy function associated with a conservative force. The work done by a conservative force is expressible as the difference of the potential energy at two positions, $W_{1\to 2} = V_1 - V_2$. The potential energy V depends only on the position of the system, so it is an explicit function of the generalized coordinates. In a system whose constraints are time-dependent, the position is also an explicit function of time, so that $V = V(q_1, q_2, \ldots, q_m, t)$.

A virtual movement is obtained by incrementing each of the generalized coordinates. The corresponding virtual work is found to be

$$\begin{aligned}\delta W &= V(q_1, q_2, \ldots, q_M, t) - V(q_1 + \delta q_1, q_2 + \delta q_2, \ldots, t) \\ &= -\delta V = -\sum_{j=1}^{M} \frac{\partial V}{\partial q_j}\,\delta q_j. \end{aligned} \tag{6.32}$$

The coefficient in δW of each virtual increment δq_j is the generalized force, so the contribution of conservative forces to the generalized forces must be

$$Q_j^{(\text{conservative})} = -\frac{\partial V}{\partial q_j}. \tag{6.33}$$

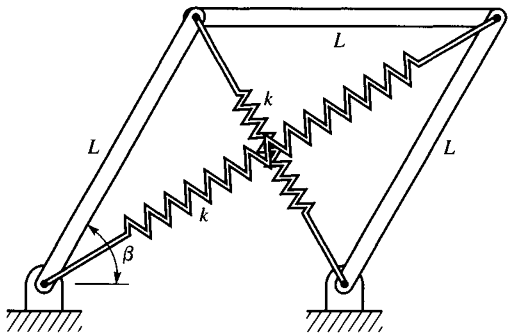
Of course, not all forces are conservative. The virtual work in a general situation may be apportioned between conservative and nonconservative effects. The corresponding expression for the generalized forces may be written as

♦ $$Q_j^{(\text{total})} = -\frac{\partial V}{\partial q_j} + Q_j. \tag{6.34}$$

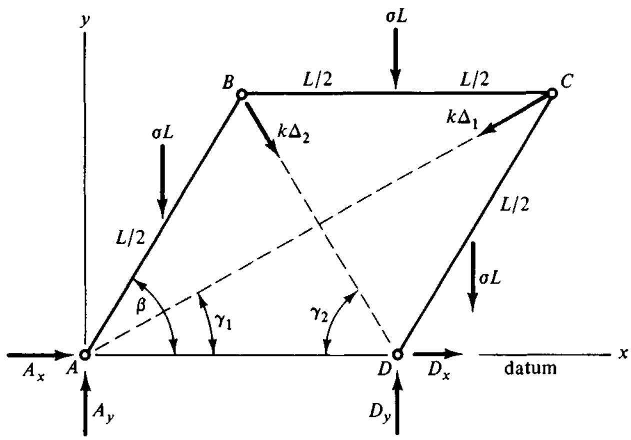
Here, and in all future developments, the symbol Q_j without a superscript denotes generalized forces associated with forces that are not described by the potential energy. This provides a degree of flexibility. It is not necessary to formulate the potential energy of a conservative force. If the nature of a force is uncertain, or if it is straightforward to evaluate the virtual work of a force that is known to be conservative, then that force may be considered to be nonconservative. Hence, the generalized forces Q_j describe all forces, conservative and nonconservative, whose effect is derived from an analysis of the virtual work. Obviously, it would not be correct to include the conservative force both in the potential energy and in the generalized force Q_j.

Example 6.10 The parallelogram linkage, which is situated in the vertical plane, is steadied by identical springs that are fastened across the diagonals. The stiffness of the springs is k and the springs are unstretched when the angle of elevation $\beta = 90°$. The mass per unit length of each bar is σ. Determine the potential energy of this system as a function of β.

Solution The free-body diagram shows the conservative forces exerted by the springs and gravity. We designate the elongation of the springs (from their unstretched length) as Δ_1 and Δ_2. The mass of each bar is σL, and the individual weight



Example 6.10



Conservative forces and kinematical parameters.

forces act at the center of mass of the respective bars. The elevation of the fixed pins A and D provides a useful datum for the potential energy of gravity. The total potential energy is the sum of the individual effects, so

$$V = \frac{1}{2}k\Delta_1^2 + \frac{1}{2}k\Delta_2^2 + 2(\sigma Lg)\left(\frac{L}{2}\sin\beta\right) + (\sigma Lg)(L\sin\beta).$$

It still remains to express the elongations in terms of β. For this, we note that because $ABCD$ is an equilateral parallelogram, the diagonals intersect perpendicularly and bisect the interior angles. Thus, $\gamma_1 = \beta/2$ and $\gamma_2 = 90° - \beta/2$. We form the elongation at each spring by subtracting the unstretched length L_0 from the respective diagonal length, with the result that

$$\Delta_1 = 2L\sin\gamma_2 - L_0 = 2L\cos(\beta/2) - L_0,$$

$$\Delta_2 = 2L\sin\gamma_1 - L_0 = 2L\sin(\beta/2) - L_0.$$

The value of L_0 was not given explicitly. Instead, we were told that $\Delta_1 = \Delta_2 = 0$ when $\beta = 90°$, from which we find that

$$L_0 = 2L\sin 45° = \sqrt{2}L.$$

When we substitute the expressions for Δ_1 and Δ_2 into V, we obtain

$$V = \frac{1}{2}kL^2\left[2\cos\frac{\beta}{2} - \sqrt{2}\right]^2 + \frac{1}{2}kL^2\left[2\sin\frac{\beta}{2} - \sqrt{2}\right]^2 + 2\sigma L^2 g\sin\beta$$
$$= \frac{1}{2}kL^2\left[8 - 4\sqrt{2}\left(\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\right)\right] + 2\sigma L^2 g\sin\beta.$$

6.5 Hamilton's Principle

We will soon see that the equations of motion for a system of rigid bodies may be obtained by evaluating a standard set of relations known as Lagrange's equations. It is possible to derive the latter directly from Newton's laws, which is the approach that Joseph-Louis Lagrange followed. Instead, we shall deviate from chronological order and first derive Hamilton's principle, which was formulated by Sir William Rowan Hamilton. This principle has much wider applicability in that it is also valid for deformable continuous media.

We begin by considering a single particle. Newton's second law was reformulated by d'Alembert† as

$$\bar{F} - m\bar{a} = \bar{0}. \tag{6.35}$$

The essence of this transposition is that it converts a dynamic system into a static one, with $-m\bar{a}$ considered to represent an inertial force. The virtual work done by this force system in a virtual displacement of the particle is

$$(\bar{F} - m\bar{a})\cdot\delta\bar{r} = \delta W - (m\bar{a}\cdot\delta\bar{r}) = 0, \tag{6.36}$$

where δW is the virtual work done by the force acting on the particle. This expression is reminiscent of an intermediate step in the derivation of the work-energy principle. In a similar manner, we eliminate acceleration by introducing the rule for differentiating a product; this leads to

$$\delta W - \frac{d}{dt}(m\bar{v}\cdot\delta\bar{r}) + m\bar{v}\cdot\frac{d}{dt}(\delta\bar{r}) = 0. \tag{6.37}$$

The last term involves two types of derivatives: a virtual increment in which time is held constant, and a true time derivative. The order in which these derivatives are taken does not matter. To prove this fundamental property, we consider the particle's position to depend on M generalized coordinates and time. In that case we have

$$\frac{d}{dt}(\delta\bar{r}) = \frac{d}{dt}\sum_{i=1}^{M}\frac{\partial\bar{r}}{\partial q_i}\,\delta q_i$$
$$= \sum_{j=1}^{M}\frac{\partial}{\partial q_j}\left(\sum_{i=1}^{M}\frac{\partial\bar{r}}{\partial q_i}\,\delta q_i\right)\dot{q}_j + \frac{\partial}{\partial t}\left(\sum_{i=1}^{M}\frac{\partial\bar{r}}{\partial q_i}\,\delta q_i\right).$$

† As explained by Rosenberg (1977), this is actually an oversimplification of d'Alembert's principle. Essentially, d'Alembert categorized forces as to whether they are the given forces inducing the motion or constraint forces. He grouped only the given forces with the $(-m\bar{a})$ terms, and then employed the principle of virtual work for static systems. Because constraint forces do no virtual work, this procedure enabled him to formulate equations of motion in which only the given forces and inertial effects appear.

We may take partial derivatives of the position vector in any order. Re-arranging them such that the derivatives for the virtual increment are taken last leads to

$$\frac{d}{dt}(\delta\bar{r}) = \sum_{i=1}^{M} \frac{\partial}{\partial q_i}\left(\sum_{j=1}^{M} \frac{\partial \bar{r}}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}}{\partial t}\right)\delta q_i = \delta\left(\frac{d\bar{r}}{dt}\right) = \delta\bar{v}. \tag{6.38}$$

In other words, it does not matter whether the virtual increment or the time differentiation is done first. With the aid of the foregoing, Eq. (6.37) may be rewritten as

$$\delta W - \frac{d}{dt}(m\bar{v}\cdot\delta\bar{r}) + m\bar{v}\cdot(\delta\bar{v}) = 0. \tag{6.39}$$

Aside from holding t constant, the rules for a virtual increment are like those for a differential. In particular, we may write that $\bar{v}\cdot(\delta\bar{v}) \equiv \frac{1}{2}\delta(\bar{v}\cdot\bar{v})$, from which it follows that

$$\delta W + \delta T - m\frac{d}{dt}(\bar{v}\cdot\delta\bar{r}) = 0, \tag{6.40}$$

where T is the kinetic energy of the particle.

We treat a system of particles by modifying Eq. (6.40). Let i denote the particle number in the system. Then, addition of Eq. (6.40) for each particle leads to redefinition of T as the kinetic energy of all particles in the system, and δW as the virtual work done by all forces. The latter was decomposed by Eq. (6.34) into two parts. The virtual work done by the conservative forces is the negative of the virtual change in the potential energy. We henceforth restrict the symbol δW to denote the virtual work done by all forces that have not been described by a potential-energy function. In that case, addition of Eq. (6.40) for each particle in the system yields

$$\delta T - \delta V + \delta W - \sum_i m_i \frac{d}{dt}(\bar{v}_i\cdot\delta\bar{r}_i) = 0. \tag{6.41}$$

The last step is to integrate over the time interval $t_1 \le t \le t_2$, where t_1 and t_2 are arbitrary. Because the last term in Eq. (6.41) is a sum of perfect differentials, this step leads to

$$\int_{t_1}^{t_2}(\delta T - \delta V + \delta W)\,dt - \sum_i m_i\bar{v}_i\cdot\delta\bar{r}_i\Big|_{t=t_1}^{t=t_2} = 0. \tag{6.42}$$

This relation is best understood by considering the motion of the system through the configuration space. In Figure 6.21, points P_1 and P_2 represent the values of the generalized coordinates at the initial and final instants, t_1 and t_2, respectively. The solid curve C represents the actual evolution of the generalized coordinates as time elapses. (The equations of motion have not been solved, so curve C is not yet known.) Curve C' represents a kinematically admissible motion that would be obtained if the set of external forces were infinitesimally different from their given values. Common terminology states that curve C' is the *variational path,* because it is obtained from an infinitesimal variation of the generalized coordinates away from their values along the true path C.

The virtual displacement imparted to each particle at each instant is arbitrary, with two exceptions. Any change in the initial condition described by point P_1 would produce a different state of motion. In other words, if the initial conditions are specified, then $\delta\bar{r}_i = \bar{0}$ for each particle at time t_1. Also, the virtual displacement must be such that the alternative curve C' leads to the actual final position represented by

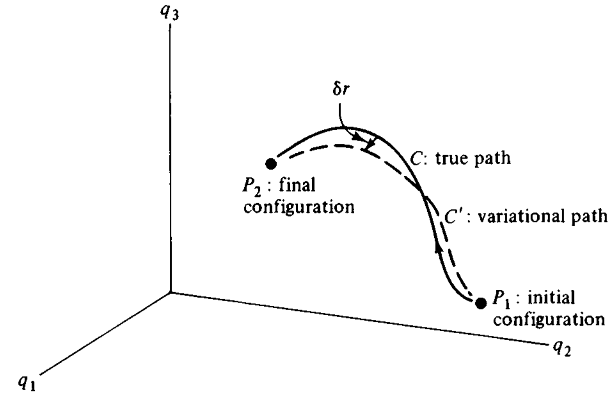


Figure 6.21 Variational path.

point P_2. In order to arrive at the true final position P_2, the virtual displacement must be such that $\delta\bar{r}_i = \bar{0}$ when $t = t_2$. Because $\delta\bar{r}_i = \bar{0}$ for each particle at the initial and final positions, Eq. (6.42) reduces to

$$\blacklozenge \qquad \int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt = 0. \tag{6.43}$$

This is *Hamilton's principle,* according to which the true path is distinguishable from all possible variational paths by the fact that the time dependence of the generalized coordinates and forces yields a zero mean value for $\delta T - \delta V + \delta W$.

Note that δT and δV represent virtual increments in the values of the corresponding quantities at an arbitrary position, whereas there is no "work" quantity W from which the virtual work of a nonconservative force may be derived. Suppose we consider the restricted group of systems for which $\delta W = 0$. Clearly, this can only be the case if a system is conservative, but we must impose the further restriction that the system be holonomic. Otherwise, the constraint forces *will* do work when the generalized coordinates are given arbitrary virtual increments. Thus, in the case of a conservative holonomic system, we obtain a more enlightening view of Hamilton's principle. Specifically, among all variational paths connecting the initial and final positions, the true path is the one for which the *action integral*

$$I = \int_{t_1}^{t_2} (T - V)\, dt \tag{6.44}$$

is *stationary,* which means that it has an extreme value (maximum, minimum, or inflection point).

It is logical at this juncture to question the significance of these results, since Hamilton's principle seems to represent only one relation. For example, the work-energy principle $\Delta T + \Delta V = W_{1\to 2}^{(\text{nc})}$ is not adequate by itself to solve problems involving several generalized coordinates. The difference is that Eq. (6.43) leads to many relations, because the virtual movement is arbitrary except at the initial and final instants. An infinite number of variational curves C' can be constructed, and Hamilton's principle must be satisfied for each. For example, we can construct different variational paths by holding constant all generalized coordinates except one, and imparting any virtual change to the remaining generalized coordinate.

In the next section, we will derive Lagrange's equations for the generalized coordinates in a system having a finite number of degrees of freedom. Systems containing deformable bodies may be conceptualized as having an infinite number of degrees of freedom. The equations of motion for such systems, which are partial differential equations in space and time for the displacements, may be obtained by applying the calculus of variations in conjunction with Hamilton's principle. (Weinstock 1974 gives a good introductory treatment of the calculus of variations and its applications.) Also, for static systems, the kinetic energy vanishes and the system is independent of time. Hamilton's principle then reduces to

$$\delta V = \delta W, \tag{6.45}$$

which is the *principle of virtual work and stationary potential energy*. It is particularly useful for the analysis of statically indeterminate structures and machines.

6.6 Lagrange's Equations

Hamilton's principle may be specialized to a system having a discrete number of degrees of freedom by accounting for the functional form of the energies and virtual work. The dependence of the kinetic energy may be recognized by considering an arbitrary particle k. The position $\bar{r}_k$ in a general situation is a function of the generalized coordinates and time, $\bar{r}_k = \bar{r}_k(q_1, q_2, \ldots, q_M, t)$. We have seen on several occasions that the corresponding velocity expression is a function of the generalized velocities, as well as the generalized coordinates and time. We therefore know that the kinetic energy may be an explicit function of the generalized coordinates q_j, generalized velocities $\dot{q}_j$, and time t. Hence, the virtual change in the kinetic energy resulting from virtual changes in the q_j and $\dot{q}_j$ is given by

$$\delta T = \sum_{j=1}^{M} \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right). \tag{6.46}$$

We seek to express δT in terms of virtual increments in the generalized coordinates only. We obtain such a form by using the interchangeability of a virtual increment and a time derivative described by Eq. (6.38), which indicates that $\delta \dot{q}_i = (d/dt)\,\delta q_i$. Then, manipulating the derivative of a product leads to

$$\begin{aligned} \delta T &= \sum_{j=1}^{M} \left[\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt} (\delta q_j) \right] \\ &= \sum_{j=1}^{M} \left[\frac{\partial T}{\partial q_j} \delta q_j + \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \delta q_j \right) - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j \right]. \end{aligned} \tag{6.47}$$

The virtual change in the potential energy and the virtual work done by nonconservative forces were related previously to the generalized coordinates; specifically,

$$\delta V = \sum_{j=1}^{M} \frac{\partial V}{\partial q_j} \delta q_j, \qquad \delta W = \sum_{j=1}^{M} Q_j \, \delta q_j.$$

When these expressions are substituted into Hamilton's principle, the coefficients of δq_j may be collected, with the result that

$$\int_{t_1}^{t_2} \sum_{j=1}^{M} \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial V}{\partial q_j} + Q_j \right] \delta q_j \, dt + \int_{t_1}^{t_2} \sum_{j=1}^{M} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \delta q_j \right) dt = 0. \tag{6.48}$$

The second integrand is a perfect differential; its integration yields a sum of terms $(\partial T/\partial \dot{q}_j)\,\delta q_j$ evaluated at $t = t_1$ and $t = t_2$. According to the derivation of Hamilton's principle, the variational path must be such that the virtual displacement at the initial and final instants is zero, so $\delta q_j = 0$ at $t = t_1$ and $t = t_2$. Consequently, the second integral vanishes, which reduces Eq. (6.48) to

$$\int_{t_1}^{t_2} \sum_{j=1}^{M} \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial V}{\partial q_j} + Q_j \right] \delta q_j \, dt = 0. \tag{6.49}$$

Recall that the virtual increments assigned to each generalized coordinate in an unconstrained set are arbitrary. In terms of the configuration space in Figure 6.21, each variational path is obtained by assigning, at each time instant, a different set of δq_j to the generalized coordinates for the true path. Even if the set is constrained ($M > N$), the increments assigned to any unconstrained subset of N generalized coordinates is arbitrary, and the choice for generalized coordinates to form that set is also arbitrary. The only way Hamilton's principle can be satisfied under these conditions is if the bracketed term in Eq. (6.49) is zero for each generalized coordinate. This term forms *Lagrange's equations of motion:*

◆ $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j, \quad j = 1, 2, \ldots, M. \tag{6.50}$$

Note that the Lagrange equations yield M equations (usually differential) for the M generalized coordinates. Suppose that a system is holonomic, so that a set of unconstrained generalized coordinates can be selected. In that case, the number of generalized coordinates matches the number of degrees of freedom, $M = N$. Also, a virtual movement of the system associated with arbitrarily selected values for δq_j satisfies all kinematical constraint conditions, so none of the (unknown) reactions appear in the generalized forces. In this situation, Lagrange's equations fully define the motion of the system. In contrast, Lagrange's equations must be supplemented by other relations when the generalized coordinates form a constrained set. Constrained generalized coordinates must be employed when a system is nonholonomic. Also, as we have seen in Example 6.9, they may be desirable for holonomic systems. Consideration of these matters will be deferred until the next chapter.

An alternative form of Lagrange's equations, preferred by some practitioners, features the *Lagrangian function* $\mathcal{L}$, which is defined to be

◆ $$\mathcal{L} = T - V. \tag{6.51}$$

The potential energy can depend only on position, so it is independent of the generalized velocities. Thus

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}, \qquad \frac{\partial \mathcal{L}}{\partial q_j} = \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j},$$

from which Lagrange's equations (6.50) convert to

◆ $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j, \quad j = 1, 2, \ldots, M. \tag{6.52}$$

Although Eq. (6.52) has one fewer term than Eq. (6.50), both require equivalent mathematical evaluations. The primary reason for introducing the Lagrangian function is its utility for further development of principles.

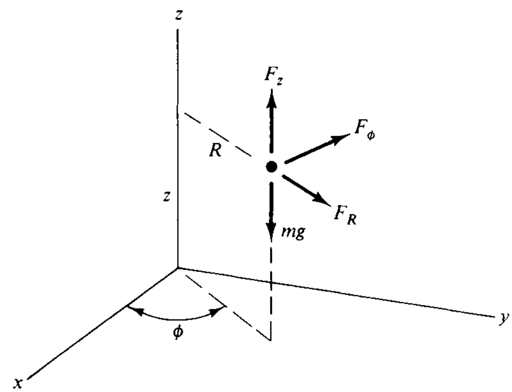


Figure 6.22 Force acting on a particle described by cylindrical coordinates.

The actual evaluation of either form of Lagrange's equations for a specific system is straightforward, provided that one is cognizant of the difference between partial and total derivatives. For the partial derivatives, the generalized coordinates q_j and generalized velocities $\dot{q}_j$ are treated like independent variables. In the derivative with respect to time, all quantities that are time-dependent must be differentiated.

A simple example that demonstrates the equivalence of Lagrange's equations and Newton's second law is a particle in spatial motion, under the influence of gravity and a force $\bar{F}$ that is described in terms of its cylindrical components (F_R, F_ϕ, F_z). Let the cylindrical coordinates (R, ϕ, z) in Figure 6.22 be the generalized coordinates. Because $\bar{v} = \dot{R}\bar{e}_R + R\dot{\phi}\bar{e}_\phi + \dot{z}\bar{k}$, the kinetic energy is

$$T = \tfrac{1}{2}m(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2).$$

The partial derivatives are

$$\frac{\partial T}{\partial \dot{R}} = m\dot{R}, \qquad \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi}, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z};$$

$$\frac{\partial T}{\partial R} = mR\dot{\phi}^2, \qquad \frac{\partial T}{\partial \phi} = 0, \qquad \frac{\partial T}{\partial z} = 0.$$

The time derivatives of the first group of terms are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}}\right) = m\ddot{R}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = 2mR\dot{R}\dot{\phi} + mR^2\ddot{\phi}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = m\ddot{z}.$$

Let the datum of the conservative gravitational force be the x-y plane, so

$$V = mgz.$$

The virtual work done by the force $\bar{F}$ is

$$\begin{aligned}\delta W &= (F_R\bar{e}_R + F_\phi\bar{e}_\phi + F_z\bar{k}) \cdot (\delta R\bar{e}_R + R\,\delta\phi\bar{e}_\phi + \delta z\bar{k}) \\ &= F_R\,\delta R + F_\phi R\,\delta\phi + F_z\,\delta z.\end{aligned}$$

The generalized forces are the coefficients of the virtual increments of the generalized coordinates in δW. Thus,

$$Q_1 = F_R, \qquad Q_2 = RF_\phi, \qquad Q_3 = F_z.$$

The corresponding set of Lagrange equations is

$$q_1 = R\colon\ m\ddot{R} - mR\dot{\phi}^2 = F_R;$$

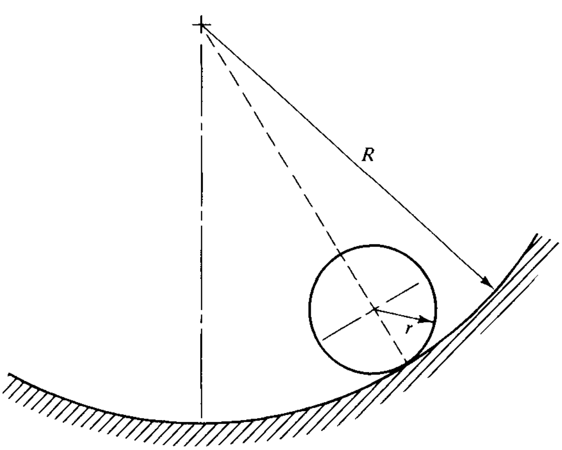
$$q_2 = \phi\colon\ 2mR\dot{R}\dot{\phi} + mR^2\ddot{\phi} = RF_\phi;$$

$$q_3 = z\colon\ m\ddot{z} + mg = F_z.$$

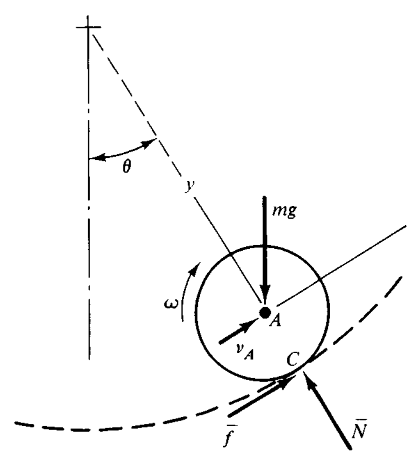
Each of these is merely Newton's second law in terms of polar coordinates, with the exception of the equation for ϕ, which has an additional factor R. That form of the equation results because $Q_2 = RF_R$ represents the moment of the external force system about the z axis. Correspondingly, the left side of the second of Lagrange's equations is the derivative of the angular momentum, $mR^2\dot{\phi}$, about that axis.

The steps we followed in this simple example parallel those for all systems for which unconstrained generalized coordinates have been selected. The bulk of the analysis usually lies in the kinematical analysis of the virtual displacements and kinetic energy. Then, after the potential energy and generalized forces have been determined, Lagrange's equations directly yield the equations of motion.

Example 6.11 Determine the equations of motion for the homogeneous sphere of radius r that rolls without slipping along the interior of the semicircular cavity. The sphere is constrained to remain in the vertical plane shown.



Example 6.11



Free-body diagram.

Solution A useful generalized coordinate for this system is the angle θ to the center of the sphere. It was specified that the sphere rolls without slipping; this constrains the absolute angle through which the sphere rotates. We impart to the sphere a virtual movement that satisfies this constraint. The normal force $\bar{N}$ and friction force $\bar{f}$ impose the constraint. Hence, they do no virtual work and will not appear in the formulation. The only other force acting on the sphere is gravity. We place the datum for the potential energy of gravity at the center of the cavity, because that is a fixed point of geometrical significance. We therefore have

$$\delta W = 0, \qquad V = mg[-(R-r)\cos\theta],$$

where the negative sign in V arises because the center of mass is below the datum.

The kinetic energy is

$$T = \tfrac{1}{2} m v_A^2 + \tfrac{1}{2} I_{zz} \omega^2,$$

where $I_{zz} = \frac{2}{5} m r^2$ for a sphere. We must express v_A in terms of the generalized coordinate. Because the center of the wheel follows a circular path of radius $R - r$, we write $v_A = (R - r)\dot{\theta}$. We obtain an expression for ω in terms of the generalized coordinate by noting that the point of contact is the instant center. Hence, $\omega = v_A / r = (R - r)\dot{\theta}/r$. The resulting kinetic energy expression is

$$T = \frac{1}{2}\left[m(R-r)^2\dot{\theta}^2 + \left(\frac{2}{5}mr^2\right)\left(\frac{R-r}{r}\dot{\theta}\right)^2\right] = \frac{7}{10}m(R-r)^2\dot{\theta}^2.$$

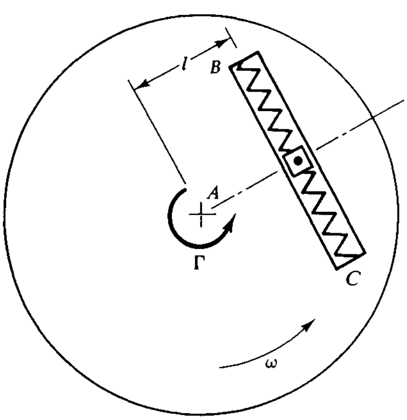
The kinetic energy is independent of θ, and the generalized force vanishes because $\delta W = 0$. Therefore, Lagrange's equation reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) + \frac{\partial V}{\partial \theta} = 0 \;\Rightarrow\; \frac{7}{5}m(R-r)^2\ddot{\theta} + mg(R-r)\sin\theta = 0,$$

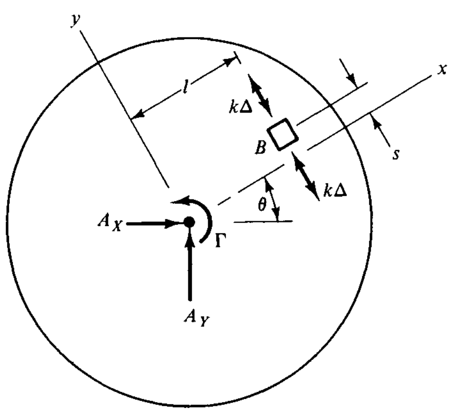
$$\frac{7}{5}\ddot{\theta} + \frac{g}{R-r}\sin\theta = 0.$$

The last form is recognizable as being comparable to the equation of motion for a simple pendulum.

Example 6.12 The table rotates in a horizontal plane about bearing A due to a torque $\Gamma(t)$. The mass of the table is M and its radius of gyration about its center is κ. The slider, whose mass is m, moves within groove BC under the restraint of a pair of springs that are unstretched in the position shown. Derive the equations of motion for this system.



Example 6.12



Free-body diagram.

Solution Our selection of generalized coordinates for this system is based on observing that the table is in pure rotation, while the slider executes a rectilinear motion relative to the table. Correspondingly, the generalized coordinates we select are the angle of rotation, $q_1 = \theta$, and the displacement of the slider relative to the unstretched position of the springs, $q_2 = s$. The only nonconservative force that does work when the generalized coordinates are given virtual increments is the torque load $\Gamma(t)$. Hence,

$$\delta W = \Gamma\,\delta\theta \;\Rightarrow\; Q_1 = \Gamma(t), \qquad Q_2 = 0.$$

The kinetic energy is the sum of the values for the table and for the slider:

$$T = \tfrac{1}{2}(I_{zz})_{\text{table}}\dot{\theta}^2 + \tfrac{1}{2}m v_s^2,$$

where $(I_{zz})_{\text{table}} = M\kappa^2$. We relate $\bar{v}_s$ to the generalized coordinates by using the relative motion equation, based on a moving reference frame xyz attached to the table. Then

$$\begin{aligned}\bar{v}_s &= \bar{v}_A + (\bar{v}_s)_{xyz} + \bar{\omega}\times\bar{r}_{s/A} = \bar{0} + \dot{s}\bar{j} + \dot{\theta}\bar{k}\times(l\bar{i} + s\bar{j})\\ &= -s\dot{\theta}\bar{i} + (\dot{s} + l\dot{\theta})\bar{j}.\end{aligned}$$

The kinetic energy therefore becomes

$$\begin{aligned}T &= \tfrac{1}{2}M\kappa^2\dot{\theta}^2 + \tfrac{1}{2}m[(s\dot{\theta})^2 + (\dot{s} + l\dot{\theta})^2]\\ &= \tfrac{1}{2}(M\kappa^2 + ms^2 + ml^2)\dot{\theta}^2 + \tfrac{1}{2}m\dot{s}^2 + ml\dot{s}\dot{\theta}.\end{aligned}$$

It is a simple matter to derive the potential energy because the displacement s is, by definition, the deformation of each spring. Therefore

$$V = 2[\tfrac{1}{2}k\Delta^2] = ks^2.$$

We must evaluate Lagrange's equations corresponding to θ and s. When we evaluate the various derivatives of T, we must always be cognizant of the difference between partial and total derivatives. Thus, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) &= \frac{d}{dt}[(M\kappa^2 + ms^2 + ml^2)\dot{\theta} + ml\dot{s}]\\ &= (M\kappa^2 + ms^2 + ml^2)\ddot{\theta} + 2ms\dot{s}\dot{\theta} + ml\ddot{s},\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) = \frac{d}{dt}[m\dot{s} + ml\dot{\theta}] = m\ddot{s} + ml\ddot{\theta};$$

$$\frac{\partial T}{\partial \theta} = \frac{\partial V}{\partial \theta} = 0, \qquad \frac{\partial T}{\partial s} = ms\dot{\theta}^2, \qquad \frac{\partial V}{\partial s} = 2ks.$$

The corresponding Lagrange equations are

$$(M\kappa^2 + ms^2 + ml^2)\ddot{\theta} + 2ms\dot{s}\dot{\theta} + ml\ddot{s} = \Gamma,$$

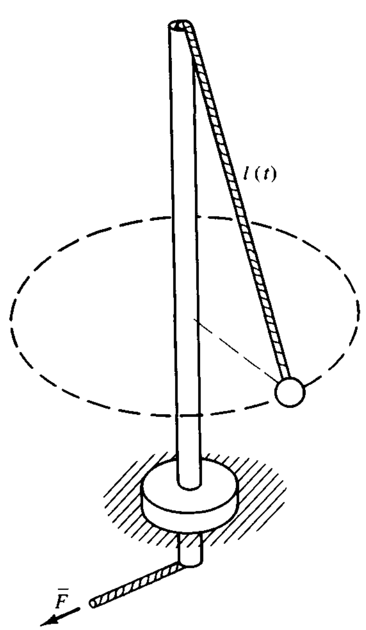
$$m\ddot{s} + ml\ddot{\theta} - ms\dot{\theta}^2 + 2ks = 0.$$

Some of the terms in these equations may be understood intuitively. For example, $M\kappa^2 + m(l^2 + s^2)$ represents the total moment of inertia of the system about pivot A. The significance of other terms may be understood by resorting to Newton's laws. Specifically, when we isolate the slider from the table, we see that there is a normal force exerted by the walls of the groove. This force is related to the acceleration of the slider in the x direction through $\bar{F} = m\bar{a}$. In turn, the reaction to this normal force exerts a moment about point A that must be considered in the equation of motion for the table. These observations demonstrate that although Lagrange's equations lead to the appropriate set of equations of motion, they often will not provide the physical insight that we obtain from Newtonian methods.

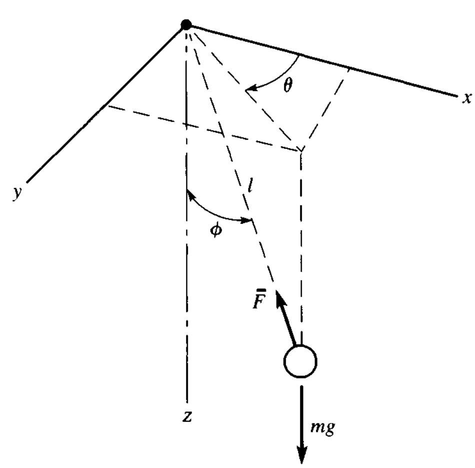
In the preceding problem, the torque Γ excited a generalized coordinate whose equation of motion we wished to determine. When the excitation causes some part of a system to move in a specified manner, the excitation is actually a reaction that imposes a time-dependent constraint condition. In that case, Lagrange's equations may be used to obtain an equation for the reaction. To do so, we merely delay using the fact that the motion is constrained until after Lagrange's equations have been formed. Thus, in the preceding example, if Γ were stipulated to be the torque required to produce a constant rotation rate ω, we would follow the same procedure as used there to determine the equations of motion. Substituting $\dot{\theta} = \omega$ into those equations would lead to an equation for Γ.

If the system of interest contains a reaction that imposes a specific motion, *and* we do not wish to obtain an equation for that reaction, we may simplify the analysis. We do so by selecting a set of generalized coordinates that will satisfy the constraint condition imposed by the reaction. Consider the previous example with Γ specified to produce a specified $\theta(t)$, so that the system has only one degree of freedom. If we select s as the only generalized coordinate then the torque will not arise in the formulation. This aspect of the selection of generalized coordinates is featured in the next example.

Example 6.13 A small sphere of mass m is suspended from the top of a hollow pole through which the cable passes. The cable's free end is pulled inward by the tensile force $\bar{F}$, whose magnitude is a function of time, such that the length of the



Example 6.13



Free-body diagram.

cable is a specified function $l(t)$. The sphere is given an initial velocity that causes it to rotate about the pole, as well as to swing outward from the pole. Determine the equations of motion for the sphere.

Solution The position of the sphere relative to the pivot may be conveniently defined in terms of spherical coordinates, which are depicted in the free-body diagram. However, the radial distance l is a specified function of time. Because the problem statement did not request evaluation of the tension force causing this distance to change, we do not select l as a generalized coordinate. We therefore have $q_1 = \phi$ and $q_2 = \theta$.

The only nonconservative force acting on the sphere is tension $\bar{F}$. This force is the constraint force that imposes the restriction that $l(t)$ is specified. Therefore, $\bar{F}$ does no virtual work in a virtual movement that is consistent with the constraint, and

$$\delta W = 0 \;\Rightarrow\; Q_1 = Q_2 = 0.$$

This same result may also be obtained in another way. We know that l is a specified function of time, and that time is constant in a virtual movement. When we employ the kinematical method, we find that the displacement of the sphere resulting from virtual increments in the generalized coordinates ϕ and θ is

$$\delta\bar{r}_s = l\,\delta\phi\,\bar{e}_\phi + l\sin\phi\,\delta\theta\,\bar{e}_\theta.$$

Because $\bar{F}$ acts in the radial direction, we have $\bar{F} = -F\bar{e}_r$. It follows that

$$\delta W = \bar{f}\cdot\delta\bar{r}_s = 0.$$

That the length l is not constant affects the kinetic energy. The velocity in terms of spherical coordinates is

$$\bar{v}_s = \dot{l}\bar{e}_r + l\dot{\phi}\bar{e}_\phi + (l\dot{\theta}\sin\phi)\bar{e}_\theta.$$

The corresponding kinetic energy is

$$T = \tfrac{1}{2}m(\bar{v}_s\cdot\bar{v}_s) = \tfrac{1}{2}m(\dot{l}^2 + l^2\dot{\phi}^2 + l^2\dot{\theta}^2\sin^2\phi).$$

The elevation of the pivot O serves as a convenient datum for gravitational potential energy,

$$V = mg(-l\cos\phi).$$

The derivatives for Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) = \frac{d}{dt}(ml^2\dot{\phi}) = m(l^2\ddot{\phi} + 2l\dot{l}\dot{\phi}),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) = \frac{d}{dt}(ml^2\dot{\theta}\sin^2\phi)$$
$$= m(l^2\ddot{\theta}\sin^2\phi + 2l\dot{l}\dot{\theta}\sin^2\phi + 2l^2\dot{\theta}\dot{\phi}\sin\phi\cos\phi);$$

$$\frac{\partial T}{\partial\phi} = ml^2\dot{\theta}^2\sin\phi\cos\phi, \qquad \frac{\partial V}{\partial\phi} = mgl\sin\phi, \qquad \frac{\partial T}{\partial\theta} = \frac{\partial V}{\partial\theta} = 0.$$

Note that in these derivatives, l is held constant in the partial differentiations, whereas the variability of l must be recognized when total derivatives are evaluated with respect to t. The corresponding Lagrange equations are

$$l^2\ddot{\phi}+2l\dot{l}\dot{\phi}-l^2\dot{\theta}^2\sin\phi\cos\phi+gl\sin\phi=0,$$
$$l^2\ddot{\theta}\sin^2\phi+2l\dot{l}\dot{\theta}\sin^2\phi+2l^2\dot{\theta}\dot{\phi}\sin\phi\cos\phi=0.$$

We may verify that these equations are correct by recalling the relations for acceleration in terms of spherical coordinates. The first of the equations just displayed is merely $\sum F_\phi = ma_\phi$, multiplied by a factor l. Similarly, the second equation is $\sum F_\theta = ma_\theta$, multiplied by a factor $l\sin\phi$.

It is possible to remove one equation of motion by a procedure that anticipates the treatment of ignorable generalized coordinates in Section 7.3.2. We note that T and V do not explicitly depend on θ, and that the generalized force $Q_\theta = 0$. This enables us to integrate the θ differential equation with respect to time, with the result that

$$\frac{\partial T}{\partial\dot{\theta}} = ml^2\dot{\theta}\sin^2\phi = m\beta,$$

where β is a constant. The values of the generalized coordinates and generalized velocities at the initial instant may be deduced from initial conditions, which must be specified if the response is to be uniquely defined. We substitute these values into the foregoing relation, from which we determine the value of β for the motion. This, in turn, allows us to solve the expression for the precession rate corresponding to any θ:

$$\dot{\theta} = \frac{\beta}{l^2\sin^2\phi}.$$

Substitution of $\dot{\theta}$ into the Lagrange equation for ϕ yields

$$l\ddot{\phi}+2\dot{l}\dot{\phi}-\frac{\beta^2\cos\phi}{l^3\sin^3\phi}+g\sin\phi=0.$$

It is easier to solve this equation than the two Lagrange equations we obtained originally.

Example 6.14 The linkage, which lies in the vertical plane, is loaded by a force $\bar{F}(t)$ that is always parallel to bar BC. The torsional spring, whose stiffness is k, is undeformed when $\theta = 60°$. Determine the equations of motion for the system.

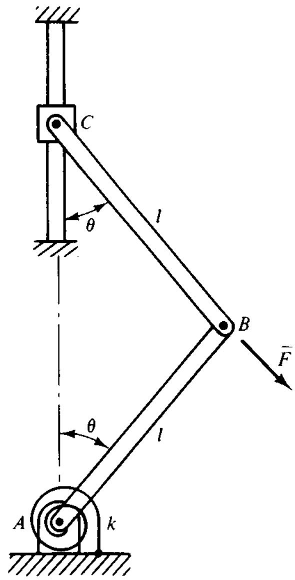
Solution The angle θ defines the position of both links, so it is suitable as the generalized coordinate. The reaction forces at pin A and collar C, which are shown in the free-body diagram, do no work in a virtual movement that increments θ. The weight of each bar, which we assume to be mg for each, is conservative. Therefore, the virtual work is

$$\delta W = \bar{F}\cdot\delta\bar{r}_B.$$

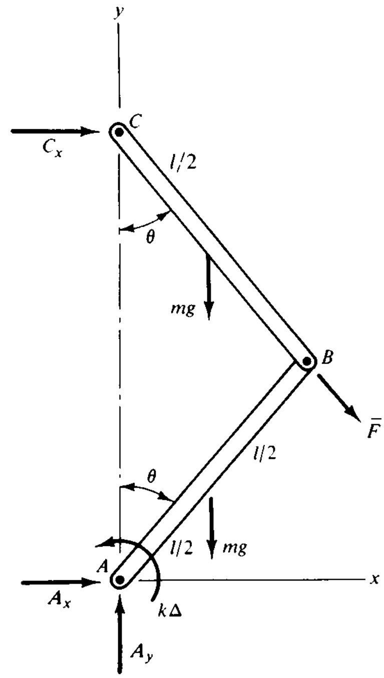
Because the linkage forms an isosceles triangle, we develop the virtual displacement and velocity relations by the geometrical method. For $\delta\bar{r}_B$, we have

$$\bar{r}_{B/A} = (l\sin\theta)\bar{i}+(l\cos\theta)\bar{j},$$

$$\delta\bar{r}_B = \frac{\partial\bar{r}_{B/A}}{\partial\theta}\,\delta\theta = [(l\cos\theta)\bar{i}-(l\sin\theta)\bar{j}]\,\delta\theta.$$



Example 6.14



Free-body diagram.

The corresponding virtual work is

$$\begin{aligned}\delta W &= [(F\sin\theta)\bar{i}-(F\cos\theta)\bar{j}]\cdot[(l\cos\theta)\bar{i}-(l\sin\theta)\bar{j}]\,\delta\theta \\ &= 2Fl\sin\theta\cos\theta\,\delta\theta \equiv Fl\sin 2\theta\,\delta\theta = Q_1\,\delta\theta,\end{aligned}$$

so the generalized force is

$$Q_1 = Fl\sin 2\theta.$$

Bar AB is in pure rotation about pin A, and bar BC is in general motion. The kinetic energy is therefore

$$T=\tfrac{1}{2}(I_A)_{AB}\omega_{AB}^2+\tfrac{1}{2}m(v_G)_{BC}^2+\tfrac{1}{2}(I_G)_{BC}\omega_{BC}^2,$$

where $(I_A)_{AB}$ and $(I_G)_{BC}$ are (respectively) the moments of inertia of bar AB about end A and of bar BC about its center of mass, both about axes perpendicular to the plane of motion. Thus,

$$(I_A)_{AB}=\frac{1}{3}ml^2, \qquad (I_G)_{BC}=\frac{1}{12}ml^2.$$

The angle θ defines the orientation of each bar, so $\omega_{AB}=\omega_{BC}=\dot{\theta}$. For the velocity of the center of mass of bar BC, we write

$$(\bar{v}_G)_{BC}=\frac{d}{dt}\bar{r}_{G/A}=\frac{d}{dt}\left[\left(\frac{l}{2}\sin\theta\right)\bar{i}+\left(\frac{3l}{2}\cos\theta\right)\bar{j}\right]=\frac{l}{2}\dot{\theta}[(\cos\theta)\bar{i}-(3\sin\theta)\bar{j}].$$

Thus the kinetic energy is

$$T = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 + \frac{1}{2}m\left(\frac{l}{2}\dot{\theta}\right)^2(\cos^2\theta + 9\sin^2\theta) + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}^2$$

$$= \frac{1}{2}ml^2\dot{\theta}^2\left(\frac{2}{3} + 2\sin^2\theta\right).$$

The last step is to evaluate the potential energy. For this we observe that the torsional spring is undeformed when $\theta = 60° = \pi/3$. (We assume that the spring constant is expressed as moment units per radian.) Thus, the rotational deformation is $\Delta = \theta - \pi/3$. We place the datum for gravity at the elevation of the fixed pin, so that

$$V = \frac{1}{2}k\Delta^2 + mg\left(\frac{l}{2}\cos\theta\right) + mg\left(\frac{3l}{2}\cos\theta\right) = \frac{1}{2}k\left(\theta - \frac{\pi}{3}\right)^2 + 2mgl\cos\theta.$$

The derivatives of T for Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[ml^2\dot{\theta}\left(\frac{2}{3} + 2\sin^2\theta\right)\right]$$

$$= ml^2\left[\ddot{\theta}\left(\frac{2}{3} + 2\sin^2\theta\right) + 4\dot{\theta}^2\sin\theta\cos\theta\right],$$

$$\frac{\partial T}{\partial \theta} = 2ml^2\dot{\theta}^2\sin\theta\cos\theta.$$

The corresponding equation of motion is

$$ml^2\left[\ddot{\theta}\left(\frac{2}{3} + 2\sin^2\theta\right) + \dot{\theta}^2\sin 2\theta\right] + k\left(\theta - \frac{\pi}{3}\right) - 2mgl\sin\theta = Fl\sin 2\theta.$$

Example 6.15 The disk spins about shaft AB at the constant rate ω_1, whereas the vertical shaft, to which shaft AB is pinned, precesses freely. The masses are m_1 for the disk and m_2 for shaft AB. Derive the equations of motion.

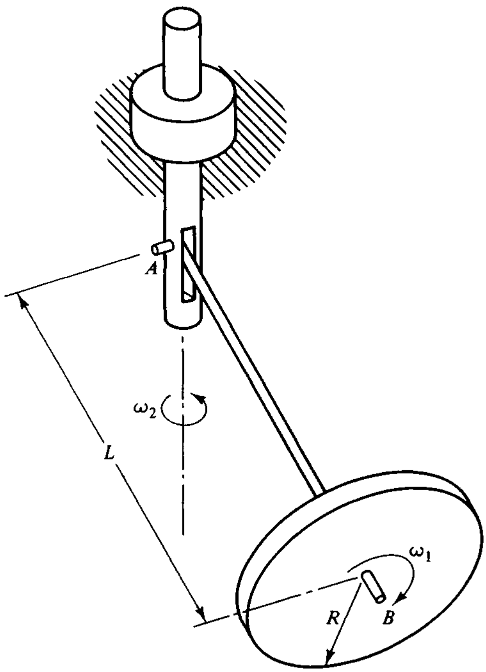
Solution Specifying ω_1 means that the spin angle ϕ is constrained, because $\dot{\phi} = \omega_1$. Hence, locating the position of the system at any instant requires that we determine the precession angle ψ and the nutation angle θ. These are the parameters we select as generalized coordinates for this two-degree-of-freedom system.

We have isolated the system in a free-body diagram, which is shown in side view. If we neglect the inertia for the vertical shaft, there is no couple affecting the precession. Consequently, no virtual work is done in a virtual movement that results from increments in ψ and θ, $\delta W = 0$, which leads to $Q_1 = Q_2 = 0$.

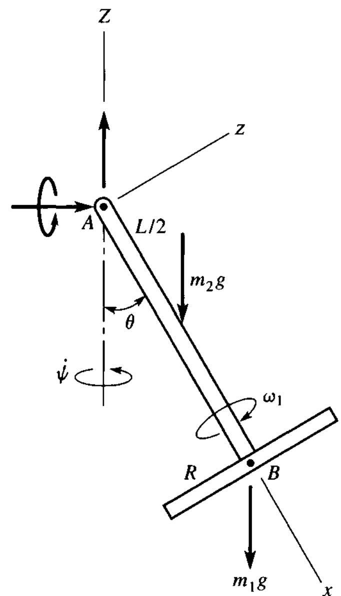
Because pin A has a fixed position relative to the disk, we may formulate the kinetic energy of each body relative to that point:

$$T = \tfrac{1}{2}(\bar{H}_A\cdot\bar{\omega})_{\text{disk}} + \tfrac{1}{2}(\bar{H}_A\cdot\bar{\omega})_{\text{shaft}}.$$

We shall use the xyz coordinate system shown in the free-body diagram to formulate the kinetic energy at the disk. (It is imperative to recognize that this coordinate system is acceptable only because the disk is axisymmetric. Otherwise, we would need to develop expressions based on the z axis being at an arbitrary spin angle relative to



Example 6.15



Free-body diagram.

the vertical plane.) Because xyz are principal axes for both bodies, the kinetic energy reduces to

$$T = \tfrac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2)_{\text{disk}} + \tfrac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2)_{\text{shaft}}.$$

The angular velocity of the disk is the vector sum of the precession rate $\dot\psi$, the nutation rate $\dot\theta$, and the spin rate ω_1, all of which must be resolved into xyz components:

$$\bar\omega_{\text{disk}} = \dot\psi \bar K + \dot\theta(-\bar j) + \omega_1(-\bar i) = -[(\dot\psi\cos\theta + \omega_1)\bar i - \dot\theta \bar j + (\dot\psi \sin\theta)\bar k].$$

The shaft is not spinning, so

$$\bar\omega_{\text{shaft}} = -(\dot\psi\cos\theta)\bar i - \dot\theta \bar j + (\dot\psi\sin\theta)\bar k.$$

We obtain the respective moments of inertia from the tabulated properties and the parallel axis theorems. The result is

$$T = \tfrac{1}{2}(\tfrac{1}{2}m_1R^2)(\dot\psi\cos\theta + \omega_1)^2 + \tfrac{1}{2}(\tfrac{1}{4}m_1R^2 + m_1L^2)[\dot\theta^2 + (\dot\psi\sin\theta)^2]$$
$$+ \tfrac{1}{2}(\tfrac{1}{3}m_2L^2)[\dot\theta^2 + (\dot\psi\sin\theta)^2],$$

where we have considered $I_{xx} = 0$ for the slender shaft.

When we place the datum for gravitational potential energy at the elevation of pin A, we find that

$$V = m_1 g(-L\cos\theta) + m_2 g\left(-\frac{L}{2}\cos\theta\right) = -\left(m_1 + \frac{1}{2}m_2\right)gL\cos\theta.$$

For brevity, let us define the following moment of inertia parameters:

$$I_1 = \tfrac{1}{2}m_1R^2, \qquad I_2 = \tfrac{1}{4}m_1R^2 + (m_1 + \tfrac{1}{3}m_2)L^2.$$

Then, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) = \frac{d}{dt}[I_1(\dot\psi\cos\theta+\omega_1)\cos\theta + I_2\dot\psi\sin^2\theta]$$

$$= I_1(\ddot\psi\cos^2\theta - 2\dot\psi\dot\theta\cos\theta\sin\theta - \omega_1\dot\theta\sin\theta)$$

$$+ I_2\ddot\psi\sin^2\theta + 2I_2\dot\psi\dot\theta\sin\theta\cos\theta,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = \frac{d}{dt}(I_2\dot\theta) = I_2\ddot\theta, \qquad \frac{\partial T}{\partial \psi} = \frac{\partial V}{\partial \psi} = 0,$$

$$\frac{\partial T}{\partial \theta} = I_1(\dot\psi\cos\theta+\omega_1)(-\dot\psi\sin\theta) + I_2(\dot\psi^2\sin\theta\cos\theta),$$

$$\frac{\partial V}{\partial \theta} = \left(m_1 + \frac{1}{2}m_2\right)gL\sin\theta.$$

The corresponding Lagrange equations are

$$(I_1\cos^2\theta + I_2\sin^2\theta)\ddot\psi - 2(I_1 - I_2)\dot\psi\dot\theta\sin\theta\cos\theta - I_1\omega_1\dot\theta\sin\theta = 0,$$

$$I_2\ddot\theta + (I_1 - I_2)\dot\psi^2\sin\theta\cos\theta + I_1\omega_1\dot\psi\sin\theta + (m_1 + \tfrac{1}{2}m_2)gL\sin\theta = 0.$$

In order to recognize the physical significance of these equations of motion, suppose that a nonconservative force were present. The generalized forces in that case would be the moments of the force about the precession (i.e. vertical) axis and about the nutation (i.e. y) axis, corresponding to the virtual work done when ψ and θ, respectively, are incremented. We therefore conclude that the equations of motion are the Z and y components of $\sum \bar M_A = \dot{\bar H}_A$. Only the latter is identical to what we would have obtained from Euler's equations for a rigid body, because those equations always express the moment equations of motion in terms of body-fixed axes.

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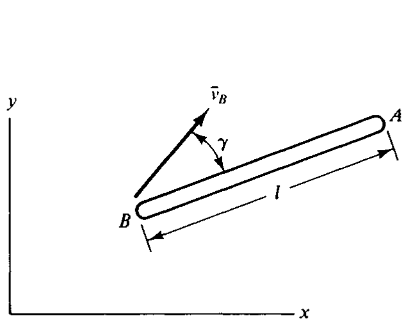
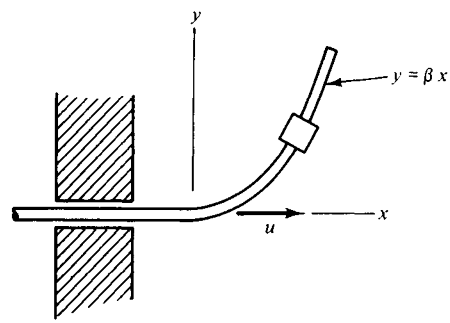
Problems

6.1 The bar is made to slide along the horizontal plane such that the velocity of end B is always directed at a constant angle γ relative to the bar. Select three generalized



Problem 6.1

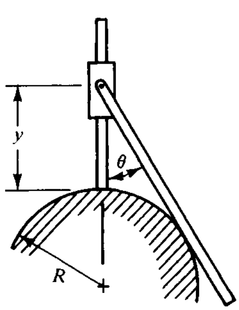


Problem 6.2

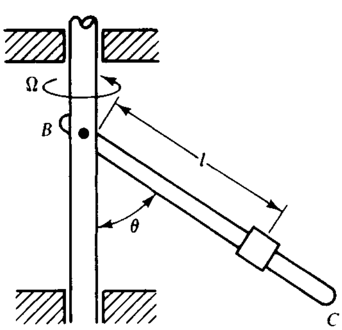
coordinates and describe the corresponding velocity constraint. Then determine whether the constraint is holonomic.

6.2 The slider descends along a curved guide in the shape of a parabola, $y = \beta x^2$, as the guide translates to the right at the constant speed u. Describe the constraint equations on the (absolute) Cartesian coordinates of the slider as a velocity constraint, and show that the constraint is holonomic. Derive the corresponding configurational constraint by integration of the velocity constraint, and also by geometrical analysis of the position.

6.3 The bar remains in contact with the semicylinder of radius R as the collar slides over the vertical guide. Determine the velocity constraint condition between the distance y locating the collar and the angle θ. Prove that this constraint condition is holonomic by integrating the velocity constraint. Then derive the same configuration constraint from a geometrical analysis.



Problem 6.3

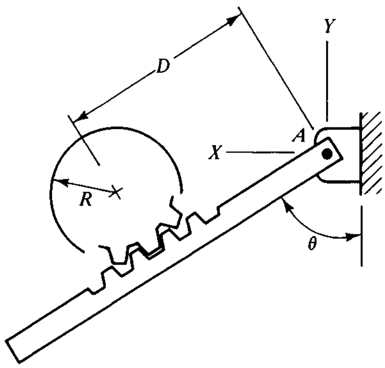


Problem 6.4

6.4 The system is forced to precess about the vertical axis at the constant rate Ω, but the nutation angle θ for bar BC and the distance l locating the collar are unknown. Consider using l, θ, and the cylindrical coordinates of the collar as a set of constrained generalized coordinates. Perform a velocity analysis to derive the corresponding constraint equations. Are they holonomic? If so, determine the corresponding configuration constraints.

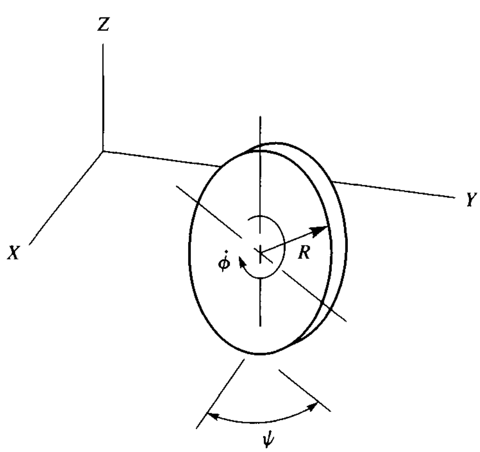
6.5 The Cartesian coordinate (x, y, z) of a particle relative to a fixed reference frame are related by $(z/x - 2\alpha y)\dot{x} - \alpha x\dot{y} + \dot{z} = \dot{\alpha}xy + \dot{\beta}/x$, where α and β are specified functions of time. Prove that this constraint is holonomic, and derive the corresponding configuration constraint.

6.6 The gear rolls without slipping over the rack, which pivots about pin A. Generalized coordinates for this system are selected to be the angle of rotation θ of the rack, the distance D from the pivot to the center of the gear, and the (X, Y) coordinates of the center of the gear. Determine the velocity constraints relating these generalized coordinates. Are these constraints holonomic? How many degrees of freedom does this system have?



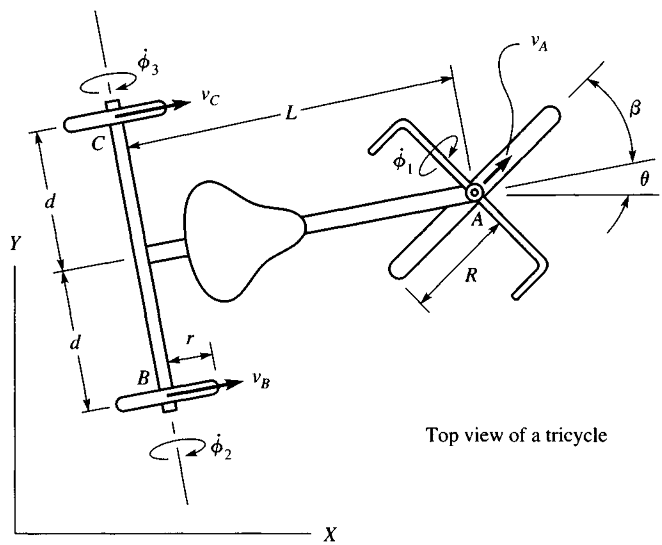
Problem 6.6

6.7 The figure shows a disk whose plane remains vertical as it rolls without slipping on a horizontal plane. Let the position coordinates X, Y of the geometric center, the heading angle ψ, and the spin angle ϕ be generalized coordinates. Describe the velocity constraints between these generalized coordinates. From those results, determine the number of degrees of freedom, and whether the system is holonomic.



Problem 6.7

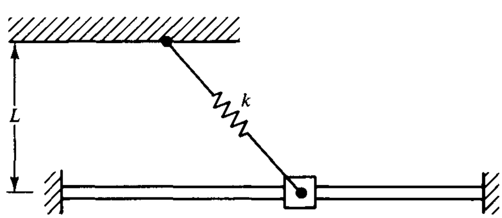
6.8 The figure shows a child's tricycle as viewed from above. When the wheels do not slip over the ground, the velocity of each wheel's center must be perpendicular to the wheel's shaft in the horizontal plane, as shown. Consider a set of generalized coordinates consisting of the position coordinates X_A and Y_A of the steering joint, the angle of orientation θ of the frame, the steering angle β, and the spin angles ϕ_1, ϕ_2, ϕ_3 of the wheels. Derive the velocity constraints among these seven generalized coordinates. From that result, determine the number of degrees of freedom.



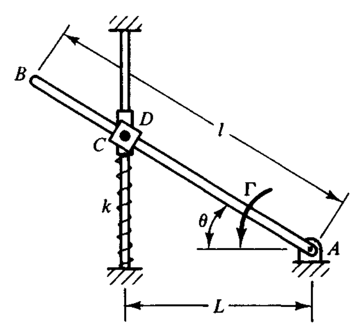
Problem 6.8

6.9 The slider in Problem 6.2 has mass m. Determine its equations of motion. Friction has negligible effect.

6.10 The collar of mass m slides over the smooth horizontal guide under the restraint of a spring whose stiffness is k. The unstretched length of the spring is $0.8L$. Determine the equations of motion for the system.

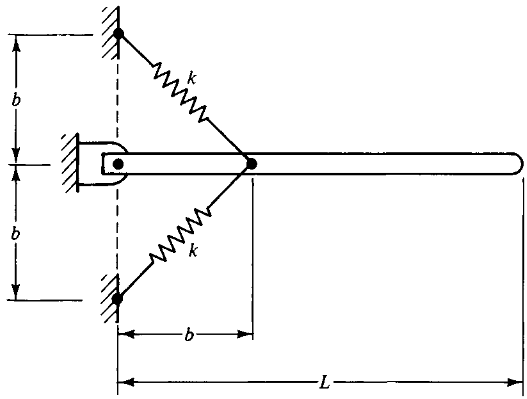


Problem 6.10



Problem 6.11

6.11 A torque $\Gamma(t)$ is applied to rod AB, whose mass is m. Collar C is pinned to collar D. The mass of each collar is $m/4$. The system lies in the vertical plane, and the spring is unstretched in the position where $\theta = 20°$. Determine the equations of motion for the system.

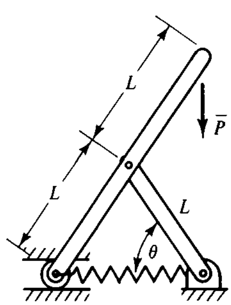


Problem 6.12

6.12 The bar is supported by two springs whose stiffness is k. The springs are unstretched when the bar is horizontal. Determine the equations of motion.

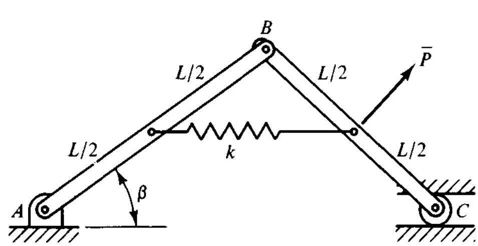
6.13 Determine the equations of motion of the compound pendulum in Example 6.7.

6.14 The linkage is braced by a spring of stiffness k in order to support the vertical force $\bar{P}$. The system lies in the vertical plane, and the spring is unstretched when $\theta = 45°$. Derive the equations of motion.

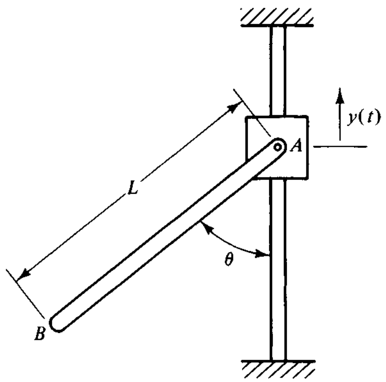


Problem 6.14

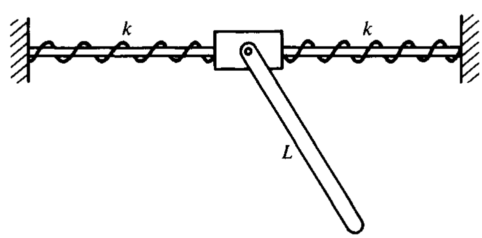
6.15 The force $\bar{P}(t)$ acting on link BC is always perpendicular to that link. The linear spring is unstretched in the position where $\beta = 53.13°$, and the identical bars each have mass m. The spring can sustain both compressive and tensile forces. Derive the equations of motion. The system is situated in the vertical plane.



Problem 6.15



Problem 6.16

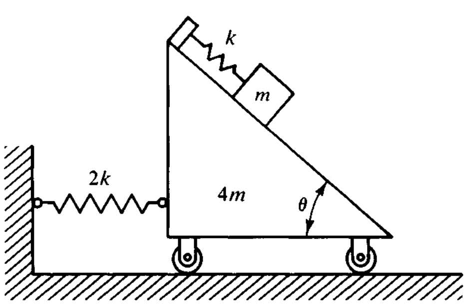


Problem 6.17

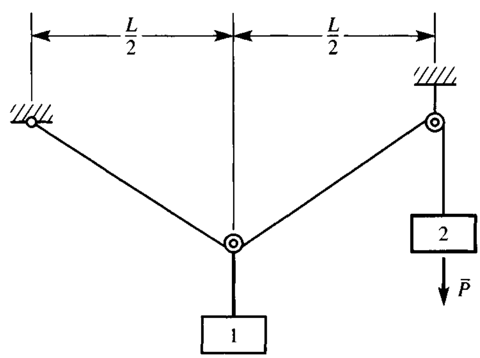
6.16 The collar supporting bar AB is given a specified displacement $y(t)$. The collar and the bar have equal mass m. Derive the equation of motion for the angle of rotation θ.

6.17 The collar, whose mass is m_1, supports a bar whose mass is m_2. The springs restraining the collar each have stiffness k. Determine the equations of motion for this system.

6.18 The horizontal spring has stiffness $2k$, while the spring holding the small block has stiffness k, where k is a basic unit of stiffness. The masses are $4m$ and m for the cart and the block, respectively. Determine the equations of motion for the system.



Problem 6.18

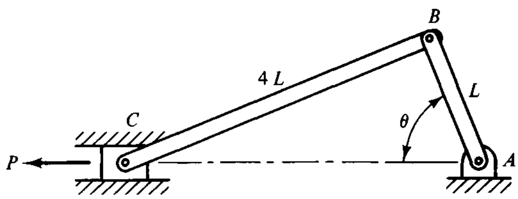


Problem 6.19

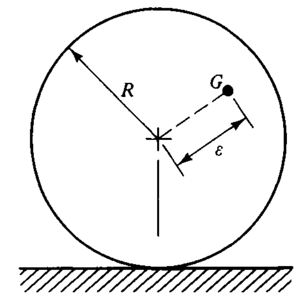
6.19 A downward force $\bar{P}(t)$ is applied to block 2. The masses are $m_1 = 2m$ and $m_2 = 3m$. Derive the equations of motion for the system.

6.20 A force $P(t)$ acting on the piston causes crankshaft AB to rotate. The mass per unit length of each bar is σ, and the mass of piston C is σL. The system lies in the horizontal plane. Derive the equations governing the generalized coordinate θ.

6.21 The cylinder is unbalanced such that its center of mass G is situated at an eccentricity ϵ from the geometric center C. The centroidal moment of inertia of the cylinder is I_C. Determine the equation of motion for arbitrarily large movements.

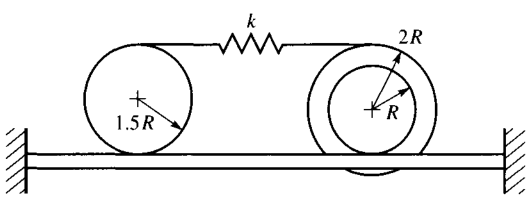


Problem 6.20

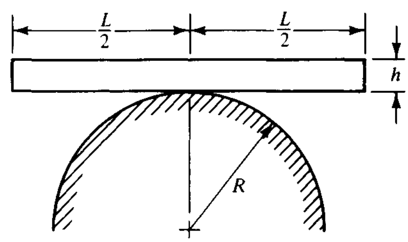


Problem 6.21

6.22 The pulleys roll over the rack without slipping. The masses and centroidal radii of gyration are m_i and σ_i ($i = 1$ for the left pulley and $i = 2$ for the right). The spring, whose stiffness is k, is capable of sustaining both compressive and tensile forces. Determine the equations of motion.



Problem 6.22

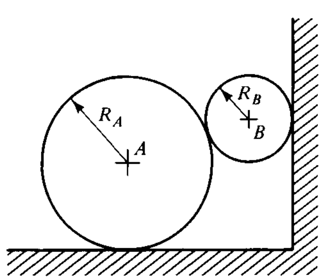


Problem 6.23

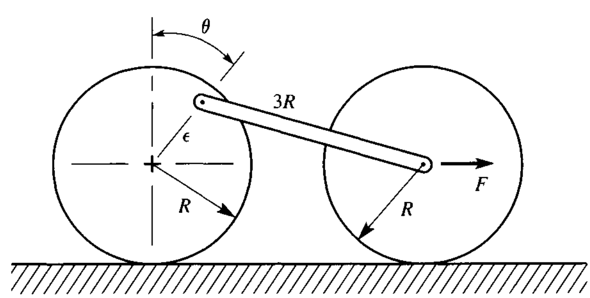
6.23 The bar is initially positioned horizontally at the top of the stationary semicylinder, as shown. It is then given a small push. There is no slipping between the bar and the semicylinder. Determine the equations of motion.

6.24 The cylinders roll in the vertical plane such that there is no slipping between them, nor between cylinder A and the ground. The vertical surface is smooth. The mass of each cylinder is m. Derive the equations of motion for the system.

6.25 Two cylinders, each having mass m, are linked by a connecting rod whose mass is negligible. A horizontal force $F(t)$ is applied to the right cylinder, and neither cylinder slips in its rolling motion. In the initial position, the angle θ locating the connecting pin is zero. Derive the equation of motion for this angle.

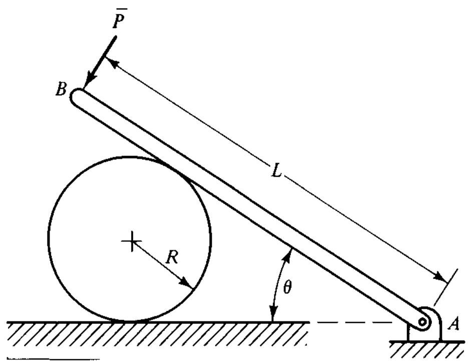


Problem 6.24



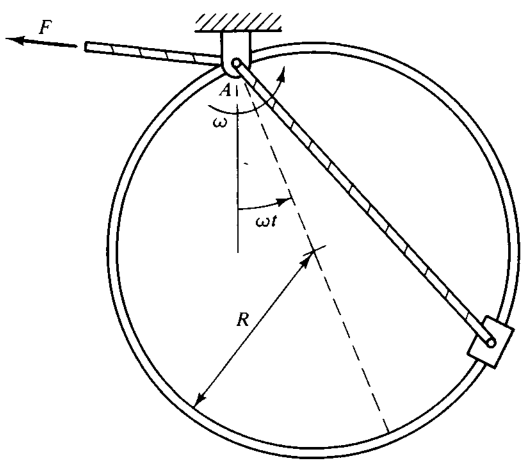
Problem 6.25

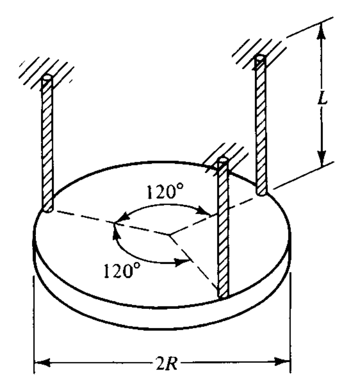
6.26 Force $\bar{P}$ acts normal to bar AB, whose mass is m. This causes the disk, whose mass is $2m$, to move to the left. The disk does not slip relative to the bar, and friction between the disk and the ground is negligible. Derive the equations of motion for the system.



Problem 6.26

6.27 The collar of mass m slides over the circular guide bar that rotates about its pivot at the constant angular speed ω. The force applied to the free end of the cable after it passes through pivot A is known to be $F(t)$. Derive the equation of motion for this system, which lies in the vertical plane.

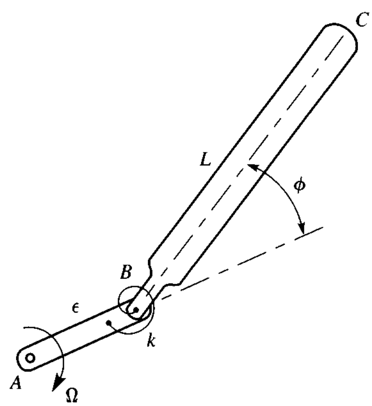




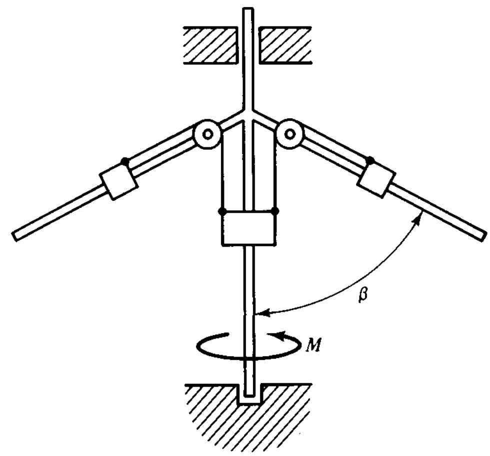
Problem 6.27 **Problem 6.28**

6.28 A circular disk of mass m is suspended in the horizontal plane by three cables of equal length L. The cables are vertical when the system is at its equilibrium position. Derive the equation of motion for the angle θ by which the disk rotates about its axis. Assume that all cables remain taut.

6.29 A simplified model of one blade of a helicopter is shown in the sketch. The short segment AB is driven at a constant rotational speed Ω. The blade BC is connected to AB by a pin and a torsional spring of stiffness k. The spring is unstressed when the



Problem 6.29

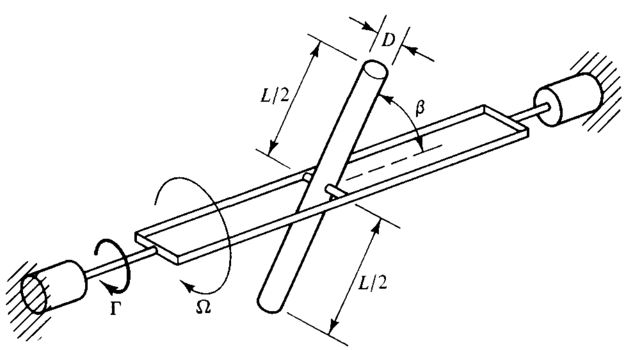


Problem 6.30

lag angle ϕ is zero. Derive the equation of motion for ϕ. Blade BC may be considered to be a homogeneous bar whose cross-section is uniform.

6.30 Each of the collars has mass m. The bar assembly on which they ride has negligible mass and rotates about the vertical axis due to a torsional load $M(t)$. Derive the equations of motion for the system.

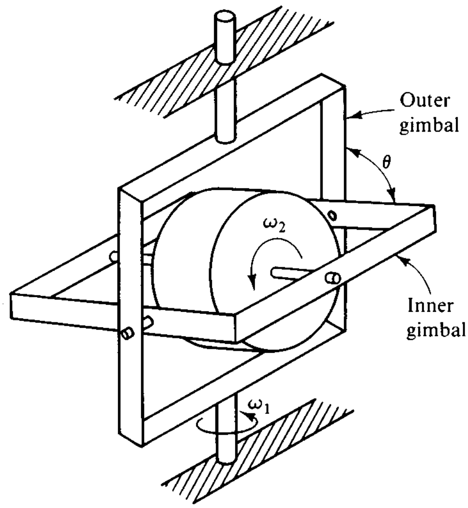
6.31 The orientation of the homogeneous cylinder relative to the gimbal is described by the angle β. The torque Γ is such that the rotation rate Ω of the gimbal about the horizontal axis is constant. The gimbals have negligible mass. Derive the equation of motion for β.



Problem 6.31

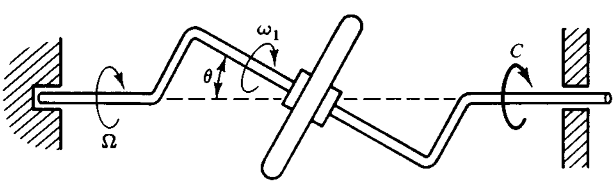
6.32 (See figure, next page.) Servomotors make the flywheel spin at a constant rate ω_2, and also impose a precession rate ω_1 that is a function of time. The center of mass of the flywheel is situated on the precession axis, and the centroidal moments of inertia are I_1 about the spin axis and I_2 transverse to that axis. Derive the equations of motion for the system.

6.33 (See figure, next page.) Solve Problem 6.32 for the case where a known torque $Q(t)$ acts about the vertical axis, so that the precession rate is unknown.



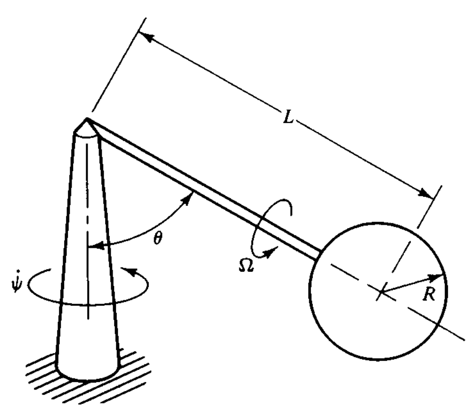
Problems 6.32 and 6.33

6.34 A servomotor makes the disk spin at the variable angular speed ω_1. The couple $C(t)$ induces rotation at rate Ω of the system about the horizontal shaft. Derive the differential equation for Ω.



Problem 6.34

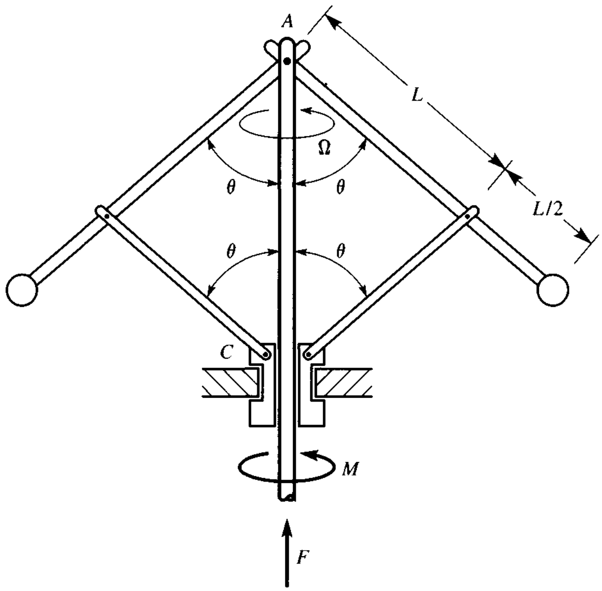
6.35 A servomotor makes the sphere spin at constant angular speed Ω relative to its shaft, which is connected to the vertical post by a ball-and-socket joint. Derive the equations of motion for the precession ψ and nutation θ of the system. Then establish the conditions for which the nutation angle is constant.



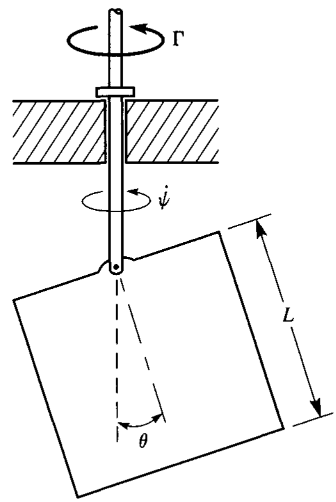
Problem 6.35

6.36 Use Lagrange's equations to solve Problem 5.24.

6.37 The elevation of pin A is controlled by a force $\bar{F}$ applied to the vertical control rod in the flyball speed governor. The system is made to precess at a constant rate Ω about the vertical axis by a torque $\bar{M}$. Determine the equation of motion governing θ. The mass of each sphere is m, and the links have negligible mass.



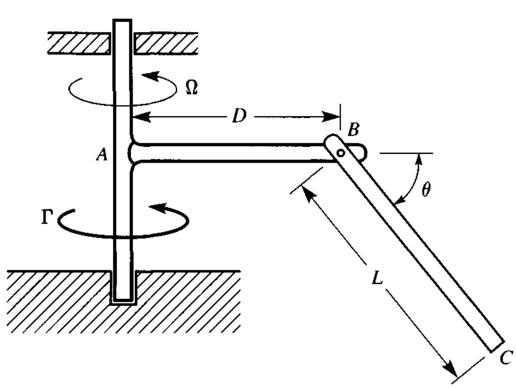
Problem 6.37



Problem 6.38

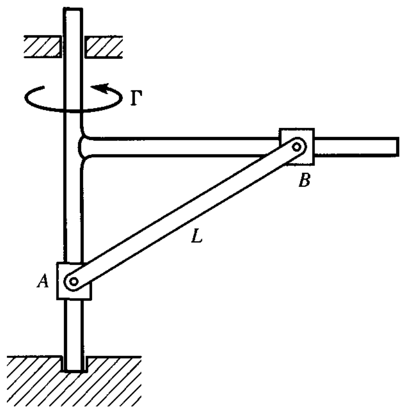
6.38 The square plate is pinned to the vertical shaft, which is made to rotate by a known torque $\Gamma(t)$. Derive differential equations of motion for the precession angle ψ and nutation angle θ.

6.39 Bar BC is pivoted from the end of the T-bar, which rotates about the vertical axis owing to a constant torque Γ. Derive the differential equations of motion for the angle of elevation θ and the precession rate Ω.



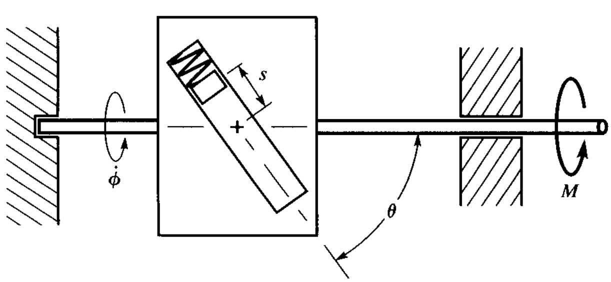
Problem 6.39

6.40 A known couple $\Gamma(t)$ induces rotation of the system about the vertical axis. Collars A and B, each of whose mass is m, are interconnected by a rigid bar whose mass is $4m$. The moment of inertia of the T-bar about the vertical axis is I. Derive the equations of motion for this system.

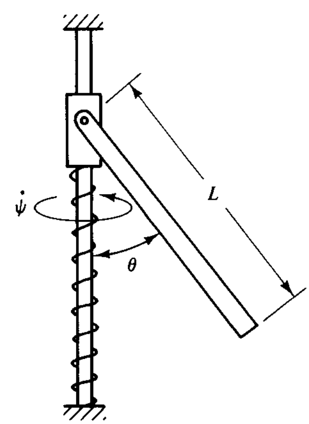


Problem 6.40

6.41 The slider, whose mass is m_1, oscillates within the groove in the housing. The moment of inertia of the housing about the axis of rotation is I. The spring restraining the slider is unstretched when $s = 0$. Derive differential equations for the distance s and spin angle ϕ resulting from application of a torque $M(t)$ to the shaft.



Problem 6.41



Problem 6.42

6.42 The bar, whose mass is m, is pinned to a collar that permits precessional rotation ψ about the vertical guide, as well as nutational rotation θ. The collar is fastened to a spring whose extensional stiffness is k_e and whose torsional stiffness for precessional rotation is k_t. Derive the equations of motion for this system.

CHAPTER 7

Further Concepts in Analytical Mechanics

The basic principles in the preceding chapters provide a sufficient foundation to treat the majority of modeling tasks that arise in engineering practice. Our goal in this chapter is to expand these capabilities. The first priority is to be able to apply Lagrange's equations in situations where constrained generalized coordinates have been selected. We will find that such a description might be desirable, even if the system is holonomic, especially if friction is present in the system.

We will also develop alternative, and sometimes simpler, forms of the equations of motion. Those developments are partially intended to assist the phase of a dynamics study in which the equations of motion are solved. However, they also will enhance our understanding of the basic concepts of analytical mechanics, and their relationship to the principles of Newtonian mechanics.

7.1 Constrained Generalized Coordinates

Whenever we formulate equations of motion using more generalized coordinates than the number of degrees of freedom, we must deal with constrained generalized coordinates. This is unavoidable in the case of a nonholonomic system. The key feature of such formulations is the need to account for constraint forces in the equations of motion. Also, because the number M of generalized coordinates exceeds the number N of degrees of freedom, there are $M-N$ constraint conditions that must be explicitly satisfied. Such conditions may be written as velocity constraints having the form of Eq. (6.3), even if a constraint is holonomic. Specifically,

♦ $$\sum_{k=1}^{M} a_{ik}\dot{q}_k + b_i = 0, \quad i = 1, 2, \dots, M-N. \tag{7.1}$$

Each constraint condition that must be explicitly stated would be violated if the generalized coordinates were chosen arbitrarily. The constraint force associated with each is an unknown reaction that does virtual work, and therefore occurs as an unknown in the generalized force. Lagrange's equations may be employed in this case by defining $Q_j^{(a)}$ to be the contribution of the given applied forces to the jth generalized force, and R_j to be the contribution of all reactions. Then

♦ $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j^{(a)} + R_j, \quad j = 1, 2, \dots, M. \tag{7.2}$$

Each reaction force appears in some, or all, of the terms R_j. Thus, $M-N$ reactions and M unknown generalized coordinates appear in the Lagrange equations. The combination of the $M-N$ constraint equations (7.1) and the M Lagrange equations (7.2) yields the required number of equations of motion.

If there is no need to determine the actual reaction forces, then it is possible to form the equations of motion without actually evaluating the virtual work done by those forces. Instead, we may account for constraint forces indirectly through Lagrange multipliers. These factors were introduced in Eq. (6.31b), where constraint forces were related to their corresponding constraint conditions. We let $R_j^{(i)}$ denote the portion of the jth generalized force attributable to the reaction enforcing constraint condition i. We found that $R_j^{(i)} = \lambda_i a_{ij}$, where λ_i is the Lagrange multiplier for the constraint. The combined contribution resulting from each of the $M-N$ constraints that are not satisfied is the sum of the individual contributions, so that

$$R_j = \sum_{i=1}^{M-N} \lambda_i a_{ij}. \tag{7.3}$$

Because we determine the Jacobian coefficients a_{ij} in the course of the derivation of the constraint equations for the generalized coordinates, there is very little extra effort required to describe reaction forces in this manner. It follows immediately from the foregoing that Eq. (7.2) becomes

$$\blacklozenge \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = \sum_{i=1}^{M-N} \lambda_i a_{ij} + Q_j^{(a)}, \quad j = 1, 2, \ldots, M. \tag{7.4}$$

The unknown quantities appearing in this form of Lagrange's equations are the M generalized coordinates and the $M-N$ Lagrange multipliers. The M Lagrange equations are supplemented by $M-N$ constraint equations, Eqs. (7.1), with the result that the number of system equations again balances the number of unknowns.

We generally employ the constraint force form, Eq. (7.2), rather than the Lagrange multiplier formulation, Eq. (7.4), whenever we wish to study the reactions. In either approach, the constraint equations are auxiliary conditions that must be satisfied in addition to Lagrange's equations. Although such equations may always be written as velocity constraints in the form of Eq. (7.1), there is an advantage in describing a holonomic constraint configurationally as $f_i(q_1, q_2, \ldots, q_M, t) = 0$. This relation may be solved for one of the generalized coordinates. Substitution of that result into the equations of motion and the other constraint equations will remove the selected generalized coordinate from the formulation. The result will be a reduction in the number of system equations to be solved.

There are several reasons why we might choose to formulate the equations of motion of a holonomic system in terms of constrained generalized coordinates. Most common is the situation where it is necessary to evaluate a reaction, which was discussed briefly in the previous chapter. If the desired reaction is to appear in the equations of motion, a set of generalized coordinates that do not satisfy the corresponding constraint equation must be employed. Simultaneous solution of the Lagrange equations and the constraint equation would yield the reactions and the response at the same stage of the solution process. The alternative approach, in which unconstrained generalized coordinates are employed, would require a separate analysis using the Newton-Euler equations of motion (developed in Chapter 5) after the response has been evaluated.

For example, consider the pendulum in Figure 7.1, whose length l is a specified function of time, $l = L(t)$, due to the application of the tensile force $\bar{F}$. One could consider the pendulum to be a rheonomic, one-degree-of-freedom system that is described by the angle ϕ. In that approach $\bar{F}$ does no virtual work, because l does not change in a virtual displacement, which holds time constant. Lagrange's equations

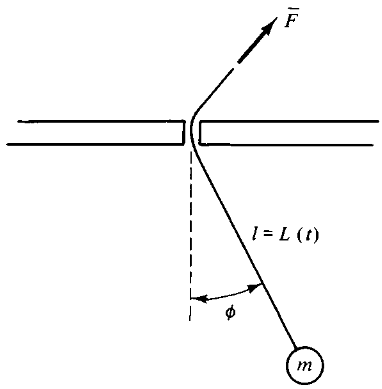


Figure 7.1 Rheonomic system versus constrained generalized coordinates.

yield a single differential equation for ϕ. An expression for the tensile force $\bar{F}$ required to obtain the motion can be obtained from the radial component of Newton's second law.

An alternative approach to this problem employs ϕ and l as constrained generalized coordinates that must satisfy the condition $l = L(t)$. The length l increases by δl in a virtual movement, so the force $\bar{F}$ does virtual work. Hence, it contributes to the generalized forces. The system equations in this formulation are the two Lagrange equations and the constraint equation; the corresponding unknowns are the two generalized forces and the magnitude of the axial force. Note that we would formulate the equations of motion using Eq. (7.2), which employs the constraint forces, because we have a specific interest in the reaction force $\bar{F}$.

The need to evaluate a reaction force is often a discretionary matter that depends on the application. However, in situations involving Coulomb sliding friction, the evaluation of the normal force is an intrinsic part of the solution process because the magnitude of the tangential (i.e. friction) force depends on the normal force. The sliding friction force does not prevent motion. Therefore, it acts like an applied force that does virtual work. It follows that the magnitude of the normal force will always occur in some of the generalized forces, even though the force itself is a reaction.

A system illustrating this aspect is the linkage in Figure 7.2, which because of symmetry must satisfy the configurational constraint equation $\phi = \theta$. However, if the equations of motion for this one-degree-of-freedom system were to be formulated

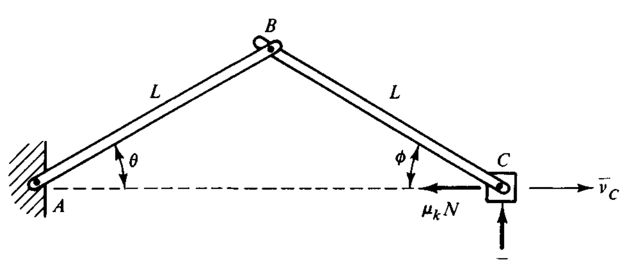


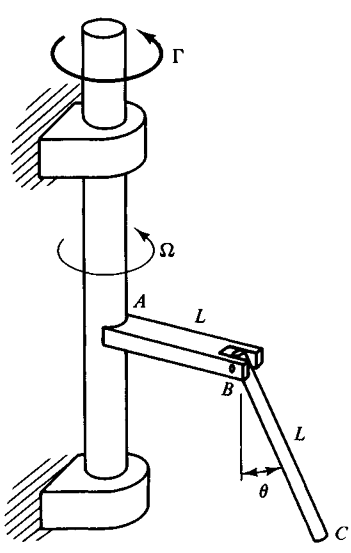
Figure 7.2 Use of constrained coordinates to account for friction.

using either angle as the unconstrained generalized coordinate, we would find that the friction force $\mu_k N$ does work. (Strictly speaking, we should describe the friction force as $-\mu_k|\bar{N}|\bar{v}/|\bar{v}|$. Doing so would accommodate the possibility that we have not correctly guessed the sense of the normal force or the sense of the velocity when we drew the free-body diagram.) In a formulation using unconstrained generalized coordinates, the only system equation is the one derived from Lagrange's equations. There is no direct way in which to obtain an equation featuring the normal force.

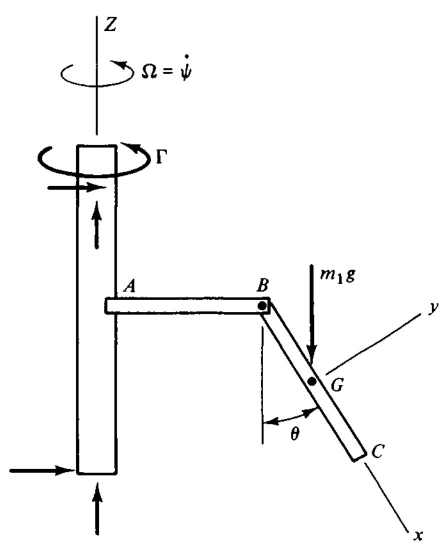
An analysis using constrained generalized coordinates can treat this system effectively. Let θ and ϕ be the generalized coordinates. A virtual movement corresponding to arbitrary increments of both generalized coordinates produces a displacement of collar C perpendicular, as well as parallel, to the horizontal guide bar. The virtual work done by $\bar{N}$ in this displacement leads to terms R_1 and R_2 representing the contributions of $\bar{N}$ to the generalized forces. The contributions of the friction force $\mu_k N$ would appear in the applied generalized forces, $Q_1^{(a)}$ and $Q_2^{(a)}$. The system of equations governing the two generalized coordinates and N will consist of two Lagrange equations and the constraint equation. This technique will be employed in Example 7.2.

Another reason for using constrained generalized coordinates to describe a holonomic system is to simplify the analysis. This feature rests on the fact that, aside from the constraint equations, all generalized coordinates are considered to be independent variables. Thus, there is no need in such a formulation to eliminate the excess generalized coordinates in the expressions for kinetic and potential energy and in the virtual work. This feature will also be prominent in Example 7.2.

Example 7.1 A torque Γ applied to the vertical shaft of the T-bar causes the rotation rate Ω about the vertical axis to increase in proportion to the angle θ by which bar BC swings outward, that is, $\Omega = c\theta$. The mass of bar BC is m_1 and the moment of inertia of the T-bar about its axis of rotation is I_2. Determine the equations of motion for the system, and for the torque Γ.



Example 7.1



Free-body diagram.

Solution The location of the bar is fully specified by the precession angle and the nutation angle. Since $\Omega = \dot\psi$, the given constraint on the motion is $\dot\psi = c\theta$, which is nonholonomic. In addition, we wish to obtain an equation for Γ, which imposes the constraint. Hence, we employ both angles as generalized coordinates, $q_1 = \psi$ and $q_2 = \theta$, even though the system has only one degree of freedom.

The kinetic energy of the T-bar is $\frac{1}{2}I_2\dot\psi^2$, to which we must add the kinetic energy of bar BC. This bar is in general motion, so

$$T_{BC} = \tfrac{1}{2}m_1(\bar v_G\cdot\bar v_G) + \tfrac{1}{2}\bar\omega_{BC}\cdot\bar H_G,$$

where the velocity parameters are

$$\bar\omega_{BC} = \dot\psi\bar K + \dot\theta\bar k = -(\dot\psi\cos\theta)\bar i + (\dot\psi\sin\theta)\bar j + \dot\theta\bar k,$$

$$\bar v_G = \bar v_B + \bar\omega_{BC}\times\bar r_{G/B} = \dot\psi L(-\bar k) + \bar\omega_{BC}\times\left(\frac{L}{2}\bar i\right)$$

$$= \frac{L}{2}\dot\theta\bar j - L\dot\psi\left(1+\frac{1}{2}\sin\theta\right)\bar k.$$

Considering bar BC to be slender leads to

$$I_{yy} = I_{zz} = \frac{1}{12}m_1L^2, \qquad I_{xx} = I_{xy} = I_{yz} = I_{xz} = 0,$$

so its angular momentum is

$$\bar H_G = I_{yy}\omega_y\bar j + I_{zz}\omega_z\bar k = \frac{1}{12}m_1L^2[(\dot\psi\sin\theta)\bar j + \dot\theta\bar k].$$

The corresponding kinetic energy of the system is

$$T = \frac{1}{2}\left\{m_1\left[\frac{L^2}{4}\dot\theta^2 + L^2\dot\psi^2\left(1+\frac{1}{2}\sin\theta\right)^2\right] + \left(\frac{1}{12}m_1L^2\right)(\dot\psi^2\sin^2\theta + \dot\theta^2) + I_2\dot\psi^2\right\}$$

$$= \frac{1}{2}\left[\left(m_1L^2 + I_2 + m_1L^2\sin\theta + \frac{1}{3}m_1L^2\sin^2\theta\right)\dot\psi^2 + \frac{1}{3}m_1L^2\dot\theta^2\right].$$

We select the elevation of pin B as the datum for gravitational potential energy, so

$$V = -\tfrac{1}{2}m_1gL\cos\theta.$$

In order to evaluate the torque Γ, we explicitly account for reactions in the virtual work, rather than using Lagrange multipliers. Arbitrary increments $\delta\psi$ and $\delta\theta$ violate only the constraint imposed by Γ on $\dot\psi$. Therefore, Γ is the only nonconservative force that does work,

$$\delta W = \Gamma\,\delta\theta = Q_1\,\delta\psi + Q_2\,\delta\theta \;\Rightarrow\; Q_1 = \Gamma,\ Q_2 = 0.$$

Both T and V are independent of ψ, so the first Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\psi}\right) = Q_1,$$

$$\left(m_1L^2 + I_2 + m_1L^2\sin\theta + \frac{1}{3}m_1L^2\sin^2\theta\right)\ddot\psi$$

$$+ m_1L^2\left(1+\frac{2}{3}\sin\theta\right)(\cos\theta)\dot\psi\dot\theta = \Gamma. \quad (1)$$

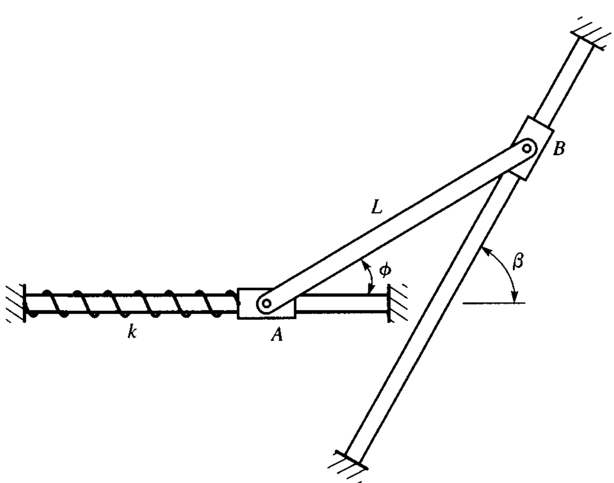
For the second Lagrange equation, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}+\frac{\partial V}{\partial \theta}=0,$$

$$\frac{1}{3}\ddot{\theta}-\left(\frac{1}{2}+\frac{1}{3}\sin\theta\right)(\cos\theta)\dot{\psi}^2+\frac{g}{2L}\sin\theta=0. \quad (2)$$

These two Lagrange equations, in combination with the constraint equation $\dot{\psi}=c\theta$, govern the three unknowns ψ, θ, and Γ. Substituting the constraint equation into eq. (2) yields an ordinary differential equation for θ. After the response $\theta(t)$ has been obtained, the value of $\Gamma(t)$ may be found by substituting $\theta(t)$ and the constraint equation into eq. (1).

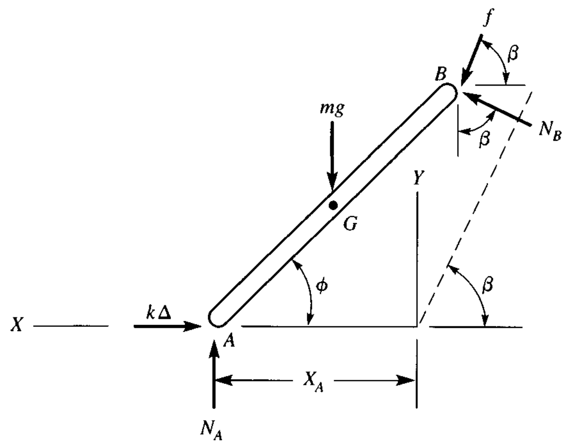
Example 7.2 The coefficient of sliding friction between collar B and its guide is μ, but friction between collar A and the horizontal guide bar is negligible. The spring, whose stiffness is k, is unstretched when $\phi=0$, and the mass of the bar is m. Determine the equations of motion of the system.



Example 7.2

Solution The position of the bar is fully specified by the angle ϕ, so this is a holonomic system with one degree of freedom. However, two features suggest that we should employ two generalized coordinates. The first results from the observation that a virtual movement of the bar in which only ϕ is incremented will not violate the constraint that end B cannot move transversely to the incline. Hence, if we employ only $q_1=\phi$, the friction force, but not the reaction, at end B will appear in the generalized force. Because the magnitude of the friction force is $\mu|\bar{N}_B|$, we would find that the single equation of motion would contain two unknowns: ϕ and $|\bar{N}_B|$. We will see that selecting two generalized coordinates will lead to a solvable set of equations.

The second reason for employing two generalized coordinates is relevant to the frictionless case also. Formulating Lagrange's equations using only $q_1=\phi$ would



Free-body diagram and constrained generalized coordinates.

require that we express the velocity of the center of mass solely in terms of ϕ and $\dot{\phi}$. Obtaining such an expression is complicated by the fact that the guidebars are not mutually orthogonal. It is substantially simpler to carry out the kinematical analysis of this system by using two position variables whose relationship is described by an additional constraint equation.

For both reasons, we select as generalized coordinates the angle of orientation and the absolute position of collar A along its guide, $q_1 = \phi$, $q_2 = X_A$. In order to derive the constraint equation, we first express the velocity of end B in terms of ϕ and X_A. Thus,

$$\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A} = \dot{X}_A \bar{I} + (-\dot{\phi}\bar{K}) \times [-(L\cos\phi)\bar{I} + (L\sin\phi)\bar{J}]$$
$$= (\dot{X}_A + L\dot{\phi}\sin\phi)\bar{I} + (L\dot{\phi}\cos\phi)\bar{J}.$$

The requirement that $\bar{v}_B$ be parallel to the incline corresponds to the condition $\bar{v}_B \cdot \bar{e}_n = 0$, where $\bar{e}_n$ is the normal to the incline,

$$\bar{e}_n = (\sin\beta)\bar{I} + (\cos\beta)\bar{J}.$$

Substitution of these expressions for $\bar{v}_B$ and $\bar{e}_n$ leads to the following constraint equation:

$$L\dot{\phi}\cos(\beta - \phi) + \dot{X}_A \sin\beta = 0. \tag{1}$$

In the present situation, where the friction force depends on the normal force, we require equations in which the normal force occurs explicitly. For this reason, we include the normal force in the evaluation of the virtual work, rather than using a Lagrange multiplier to account for it. Note that the expression for $\bar{v}_B$ is independent of time. Hence, the virtual displacement of end B may be described by forming $\bar{v}_B\,dt$, and then replacing differentials by virtual increments. This yields

$$\delta\bar{r}_B = (\delta X_A + L\,\delta\phi\sin\phi)\bar{I} + (L\,\delta\phi\cos\phi)\bar{J}.$$

The spring and gravity forces are conservative, and therefore are not included in the virtual work. Also, the constraint on the motion of collar A is satisfied regardless of the values of ϕ and X_A. As a result, the virtual work is

$$\delta W = [(f\cos\beta + N_B\sin\beta)\bar{I} + (-f\sin\beta + N_B\cos\beta)\bar{J}]\cdot\delta\bar{r}_B.$$

We substitute for $\delta\bar{r}_B$ and collect the coefficients of $\delta\phi$ and δX_A, which are the corresponding generalized forces:

$$Q_1 = -fL\sin(\beta-\phi) + N_B L\cos(\beta-\phi),$$
$$Q_2 = f\cos\beta + N_B\sin\beta.$$

According to Coulomb's law for sliding friction, $\bar{f} = \mu|\bar{N}_B|$ in the direction opposite the velocity of end B. (Note that we use $|\bar{N}_B|$ to describe the magnitude of the normal force, in order to emphasize that friction force would have the same value if the reaction were opposite to the sense assumed in the free-body diagram.) To describe the sense of the friction force we note that, when the constraint conditions are satisfied, counterclockwise rotation of the bar, $\dot{\phi} > 0$, produces an upward movement of collar B. Such movement corresponds to $\bar{f}$ being down and to the left, as was assumed in the free-body diagram. Thus, we set

$$f = \mu|\bar{N}_B|\,\mathrm{sgn}(\dot{\phi}),$$

where $\mathrm{sgn}(\dot{\phi})$ is the signum function: $\mathrm{sgn}(\dot{\phi}) = \dot{\phi}/|\dot{\phi}|$ if $\dot{\phi} \neq 0$.

We must express the kinetic energy for arbitrary ϕ and X_A. The velocity of the center of mass is

$$\bar{v}_G = \bar{v}_A + \bar{\omega}\times\bar{r}_{G/A} = \left(\dot{X}_A + \frac{L}{2}\dot{\phi}\sin\phi\right)\bar{I} + \left(\frac{L}{2}\dot{\phi}\cos\phi\right)\bar{J},$$

from which we obtain

$$T = \tfrac{1}{2}m(\bar{v}_G\cdot\bar{v}_G) + \tfrac{1}{2}I_G\dot{\phi}^2 = \tfrac{1}{2}m(\tfrac{1}{3}L^2\dot{\phi}^2 + L\dot{\phi}\dot{X}_A\sin\phi + \dot{X}_A^2).$$

The spring and gravity contribute to the potential energy. We let the elevation of end A be the gravitational datum. The elongation of the spring is

$$\Delta = X_A|_{\phi=0} - X_A = L - X_A,$$

so that

$$V = \tfrac{1}{2}k(L-X_A)^2 + \tfrac{1}{2}mgL\sin\phi.$$

We now form Lagrange's equations, using the earlier expressions for Q_1, Q_2, and f. For the latter we use $|\bar{N}_B| = N_B\,\mathrm{sgn}(N_B)$ in order to account for the possibility that the reaction is not in the assumed sense. The resulting equation for $q_1 = \phi$ is

$$\begin{aligned}&\tfrac{1}{3}mL^2\ddot{\phi} + \tfrac{1}{2}mL\ddot{X}_A\sin\phi + \tfrac{1}{2}mgL\cos\phi \\ &\quad = N_B L[\cos(\beta-\phi) - \mu\,\mathrm{sgn}(N_B\dot{\phi})\sin(\beta-\phi)],\end{aligned} \tag{2}$$

while the equation for $q_2 = X_A$ is

$$\begin{aligned}&m\ddot{X}_A + \tfrac{1}{2}mL\ddot{\phi}\sin\phi + \tfrac{1}{2}mL\dot{\phi}^2\cos\phi - k(L-X_A) \\ &\quad = N_B[\sin\beta + \mu\,\mathrm{sgn}(N_B\dot{\phi})\cos\beta].\end{aligned} \tag{3}$$

There are three unknowns: ϕ, X_A, and N_B. The third equation is the constraint condition, eq. (1).

It is possible without too much effort to reduce the number of equations in the present problem. Because the system is holonomic, the generalized coordinates in excess of the number of degrees of freedom may be eliminated. Toward that end we write eq. (1) in Pfaffian form:

$$L\,d\phi\cos(\beta-\phi)+dX_A\sin\beta=0.$$

Each term is a perfect differential, so the corresponding configuration constraint is

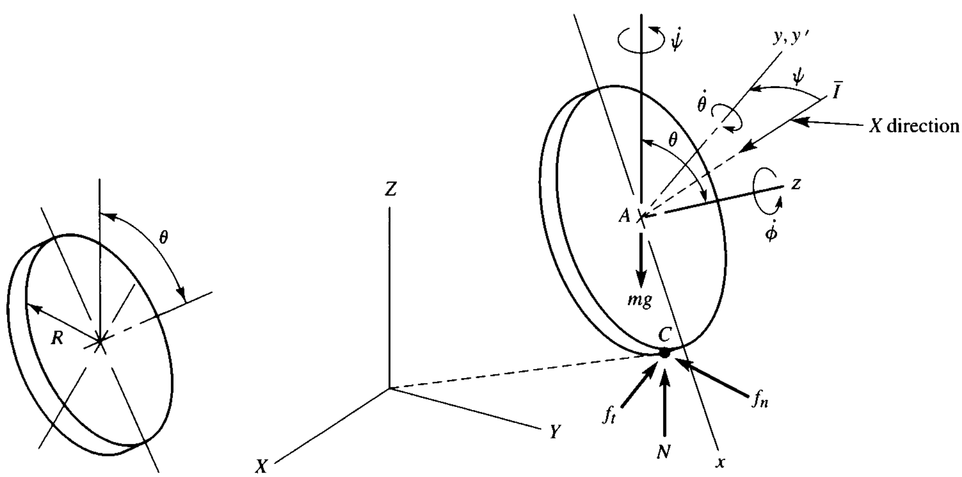
$$-L\sin(\beta-\phi)+X_A\sin\beta=C.$$

The value of the constant of integration C must be such that $X_A=L$ when $\phi=0$, which yields $C=0$, and

$$X_A=L\frac{\sin(\beta-\phi)}{\sin\beta}. \tag{4}$$

As a check, we note that eq. (4) is identical to the expression for X_A given by the law of sines. If we were to substitute equation (4) into the Lagrange equations (2) and (3), we would remove X_A. Then, forming the ratio of eqs. (2) and (3) would eliminate N_B. The resulting differential equation, which we shall not detail, is second-order and highly nonlinear.

Example 7.3 A thin disk wobbles as it rolls without slipping along the ground. Consequently, the plane of the disk is inclined at an unsteady angle θ. Derive the equations of motion for the system. Then specialize the result to the steady precession case, in which θ is constant and the center A follows a circular path. The radius of gyration of the disk about its axis of symmetry is κ.



Example 7.3 Free-body diagram and kinematical parameters.

Solution The position of any rigid body may always be described in terms of three position coordinates for any point, such as the center of mass, and three Eulerian angles. The circular shape of the disk and the absence of slipping constrain some of these variables, so it is not apparent at the outset which of the variables are independent. In order to identify the appropriate choice, we recall from Section 4.4 the kinematical analysis of a disk that wobbles as it rolls without slipping.

Let (X, Y, Z) be the Cartesian coordinates of the center of mass. The Eulerian angles (ψ, θ, ϕ) are the precession, nutation, and spin angles, respectively; they are defined by letting the vertical direction define the precession axis. Note that the y' axis is always horizontal (it is the line of nodes), whereas the y axis is a body-fixed axis. The angle between the y' axis and the negative X axis is ψ. The angular velocity of the disk is

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k} = -(\dot{\psi}\sin\theta)\bar{i} + \dot{\theta}\bar{j} + (\dot{\psi}\cos\theta + \dot{\phi})\bar{k}.$$

The velocity of the center obtained from the no-slip condition is

$$\bar{v}_A = \bar{\omega} \times \bar{r}_{A/C} = -R(\dot{\psi}\cos\theta + \dot{\phi})\bar{j} + R\dot{\theta}\bar{k}.$$

The velocity of the center may also be described in terms of the Cartesian coordinates,

$$\bar{v}_A = \dot{X}\bar{I} + \dot{Y}\bar{J} + \dot{Z}\bar{K}.$$

We match these two descriptions by resolving the unit vectors of one set of axes onto the other set of axes, as follows:

$$\bar{i} = -(\sin\psi\cos\theta)\bar{I} + (\cos\psi\cos\theta)\bar{J} - (\sin\theta)\bar{K},$$

$$\bar{j} = -(\cos\psi)\bar{I} - (\sin\psi)\bar{J},$$

$$\bar{k} = -(\sin\psi\sin\theta)\bar{I} + (\cos\psi\sin\theta)\bar{J} + (\cos\theta)\bar{K}.$$

We substitute these expressions into the first equation for $\bar{v}_A$, and compare the result to the second equation. We find from these operations that

$$\dot{X} = R(\dot{\psi}\cos\theta + \dot{\phi})\cos\psi - R\dot{\theta}\sin\psi\sin\theta,$$

$$\dot{Y} = R(\dot{\psi}\cos\theta + \dot{\phi})\sin\psi + R\dot{\theta}\cos\psi\sin\theta,$$

$$\dot{Z} = R\dot{\theta}\cos\theta.$$

These relations are three velocity constraints that the six position variables must satisfy, so the disk only has three degrees of freedom. The constraints on $\dot{X}$ and $\dot{Y}$ are nonholonomic. However, the one governing $\dot{Z}$ may be integrated. Multiplying each rate variable in the last equation by dt shows that both sides are perfect differentials. Setting $Z = 0$ when $\theta = 0$ leads to

$$Z = R\sin\theta.$$

This position constraint permits us to eliminate Z from the formulation. Hence, we shall employ five generalized coordinates in the sequence: X, Y, ψ, θ, ϕ.

The generalized coordinates are a constrained set that must satisfy the velocity constraints on $\dot{X}$ and $\dot{Y}$. We are not specifically interested in the reactions at the ground, which enforce these constraints. Therefore, we employ the Lagrange multiplier formulation. In order to identify the coefficients that correspond to each multiplier, we adapt the standard form of a velocity constraint to the present system,

$$\sum_{k=1}^{5} a_{jk}\dot{q}_k + b_j = 0, \quad j = 1, 2. \tag{1, 2}$$

Comparing this form to the actual constraint equations shows that

$$a_{11}=1, \quad a_{12}=0, \quad a_{13}=-R\cos\psi\cos\theta,$$
$$a_{14}=R\sin\psi\sin\theta, \quad a_{15}=-R\cos\psi, \quad b_1=0;$$
$$a_{21}=0, \quad a_{22}=1, \quad a_{23}=-R\sin\psi\cos\theta,$$
$$a_{24}=-R\cos\psi\sin\theta, \quad a_{25}=-R\sin\psi, \quad b_2=0.$$

We now proceed to formulate the mechanical energies. We use the fact that the disk is thin to set $I_{xx}=I_{yy}=\frac{1}{2}I$. Then, adding the translational kinetic energy associated with the center of mass to the rotational kinetic energy yields

$$T=\tfrac{1}{2}m(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)+\tfrac{1}{2}(\tfrac{1}{2}I)(\omega_x^2+\omega_y^2)+\tfrac{1}{2}I\omega_z^2$$
$$=\tfrac{1}{2}m[\dot{X}^2+\dot{Y}^2+R^2\dot{\theta}^2\cos^2\theta+\tfrac{1}{2}\kappa^2\dot{\psi}^2\sin^2\theta+\tfrac{1}{2}\kappa^2\dot{\theta}^2+\kappa^2(\dot{\psi}\cos\theta+\dot{\phi})^2],$$
$$V=mgZ=mgR\sin\theta.$$

Note that we have not used the velocity constraint to remove the dependence on $\dot{X}$ and $\dot{Y}$, because Lagrange's equations must describe the effect of variations in each of the generalized coordinates.

The only nonconservative forces acting on the disk are the reactions at the ground, whose effect we shall describe by Lagrange multipliers. Hence, we have $\delta W=0$, which leads to $Q_j=0$ for each generalized coordinate.

Applying the constrained Lagrange equations (7.4) to the present system leads to a set of five differential equations:

$$m\ddot{X}=\lambda_1, \qquad m\ddot{Y}=\lambda_2, \tag{3, 4}$$

$$m\kappa^2[\tfrac{1}{2}\ddot{\psi}(1+\cos^2\theta)-\dot{\psi}\dot{\theta}\sin\theta\cos\theta+\ddot{\phi}\cos\theta-\dot{\theta}\dot{\phi}\sin\theta]$$
$$=-\lambda_1 R\cos\psi\cos\theta-\lambda_2 R\sin\psi\cos\theta, \tag{5}$$

$$m[\ddot{\theta}(\tfrac{1}{2}\kappa^2+R^2\cos^2\theta)+(\tfrac{1}{2}\kappa^2\dot{\psi}^2-R^2\dot{\theta}^2)\sin\theta\cos\theta+\kappa^2\dot{\psi}\dot{\phi}\sin\theta+gR\cos\theta]$$
$$=\lambda_1 R\sin\psi\sin\theta-\lambda_2 R\cos\psi\sin\theta, \tag{6}$$

$$m\kappa^2(\ddot{\psi}\cos\theta+\ddot{\phi}-\dot{\psi}\dot{\theta}\sin\theta)=-\lambda_1 R\cos\psi-\lambda_2 R\sin\psi. \tag{7}$$

There are seven unknowns in this formulation: the five generalized coordinates and the two Lagrange multipliers. These variables must satisfy the constraint equations (1) and (2), and the Lagrange equations (3)-(7).

When $\dot{\theta}\equiv 0$, the center A of the disk follows a circular path. Let ρ be the radius of curvature of that path, and let the center of the path be situated on the Z axis. We form a trial solution for X and Y by assuming that the precession rate is also the rotation rate for the radial line. The position of the center A is then given by

$$X=-\rho\sin\psi, \qquad Y=\rho\cos\psi.$$

These relations satisfy constraint equations (1) and (2), with $\dot{\theta}=0$, when

$$\dot{\phi}=-\left(\frac{\rho}{R}+\cos\theta\right)\dot{\psi}.$$

Next, we substitute these expressions for X, Y, and $\dot{\phi}$, as well as $\dot{\theta}=0$, into the Lagrange equations (3)-(7). Equations (3) and (4) yield expressions for the Lagrange multipliers,

$$\lambda_1 = m\rho(-\ddot{\psi}\cos\psi + \dot{\psi}^2\sin\psi), \tag{3'}$$

$$\lambda_2 = m\rho(-\ddot{\psi}\sin\psi - \dot{\psi}^2\cos\psi). \tag{4'}$$

We use eqs. (3′) and (4′) to eliminate the Lagrange multipliers from eqs. (5)–(7), with the result that

$$\kappa^2\left(\frac{1}{2}\sin^2\theta - \frac{\rho}{R}\right)\ddot{\psi} = \rho\ddot{\psi}\cos\theta, \tag{5'}$$

$$\kappa^2\left(-\frac{1}{2}\sin\theta\cos\theta - \frac{\rho}{R}\sin\theta\right)\dot{\psi}^2 + gR\cos\theta = \rho\dot{\psi}^2 R\sin\theta, \tag{6'}$$

$$\kappa^2\left(-\frac{\rho}{R}\right)\ddot{\psi} = \rho\ddot{\psi}. \tag{7'}$$

Equation (7′) requires that $\ddot{\psi} = 0$, which also satisfies eq. (5′). Notice that $\ddot{\psi} = 0$ corresponds to a constant speed $\rho\dot{\psi}$ for the center A. The remaining Lagrange equation (6′) yields an expression for the value of $\dot{\psi}$ required to maintain a specified constant nutation angle θ:

$$\dot{\psi}^2 = \frac{2gR^2\cot\theta}{\kappa^2 R\cos\theta + 2\rho(R^2+\kappa^2)}.$$

If the disk is homogeneous, the radius of gyration is $\kappa = R/\sqrt{2}$, which leads to

$$\dot{\psi}^2 = \frac{4g\cot\theta}{R\cos\theta + 6\rho}.$$

This solution is in complete agreement with the result derived in Example 5.8 by using the Newton–Euler formulation. The earlier methods provide greater physical insight. However, they would have been much more difficult to use in deriving the equations of motion for the general case treated here, where the nutation angle is not constant.

7.2 Computational Methods in the State Space

When the motion of a system is known, the equations of motion may be solved algebraically for the forces (such as applied loads) required to sustain that motion. A more interesting situation arises when some aspect of the motion is unknown. In that case, the generalized coordinates are governed by differential equations of motion. When these equations are derived by direct application of Lagrange's equations, they have a standard form. The highest-order derivatives are generalized accelerations $\ddot{q}_i$; such derivatives occur linearly.

To demonstrate this, we consider the general form of the kinetic energy. The position $\bar{r}_k$ of particle k in a system may be a function of the generalized coordinates and time, $\bar{r}_k = \bar{r}_k(q_1, q_2, \ldots, q_M, t)$. The corresponding velocity expression is

$$\bar{v}_k = \frac{d\bar{r}_k}{dt} = \sum_{j=1}^{M} \frac{\partial \bar{r}_k}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_k}{\partial t}. \tag{7.5}$$

From this, the kinetic energy of this particle may be expressed in terms of the generalized coordinates as

$$T_k = \frac{1}{2} m_k \left[\sum_{j=1}^{M} \frac{\partial \bar{r}_k}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_k}{\partial t} \right] \cdot \left[\sum_{j=1}^{M} \frac{\partial \bar{r}_k}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_k}{\partial t} \right]$$
$$= \frac{1}{2} m_k \sum_{i=1}^{M} \sum_{j=1}^{M} \left[\frac{\partial \bar{r}_k}{\partial q_i} \cdot \frac{\partial \bar{r}_k}{\partial q_j} \right] \dot{q}_i \dot{q}_j + m_k \sum_{j=1}^{M} \left[\frac{\partial \bar{r}_k}{\partial t} \cdot \frac{\partial \bar{r}_k}{\partial q_j} \right] \dot{q}_j$$
$$+ \frac{1}{2} m_k \left[\frac{\partial \bar{r}_k}{\partial t} \cdot \frac{\partial \bar{r}_k}{\partial t} \right]. \tag{7.6}$$

This expression indicates that the terms forming the kinetic energy of a particle fall into one of three categories: they either contain the generalized velocities as quadratic products or as linear terms, or they are independent of the generalized velocities; no term in the kinetic energy contains the $\dot{q}_j$ in any other manner.

The total kinetic energy of the system is obtained by adding the contribution of each particle. (For a rigid body, such a sum involves an integration over the differential mass elements.) The basic terms being accumulated all depend on the generalized velocities in the same manner as Eq. (7.6). It follows that the kinetic energy of a system consists of three groups of terms: T_2 is quadratic in the generalized velocities, T_1 is linear in the generalized velocities, and T_0 is independent of the generalized velocities. The general form is

♦ $$T = T_2 + T_1 + T_0, \tag{7.7a}$$

where

♦ $$T_2 = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} M_{ij} \dot{q}_i \dot{q}_j, \qquad T_1 = \sum_{j=1}^{M} N_j \dot{q}_j. \tag{7.7b}$$

The coefficients M_{ij} and N_j, as well as T_0, might depend on the generalized coordinates and time, because that is the dependence of the partial derivatives in Eq. (7.6). A useful property obeyed by the coefficients of the quadratic terms M_{ij} is *symmetry:*

$$M_{ij} = M_{ji}, \quad i, j = 1, 2, \ldots, M, \tag{7.8}$$

which is a consequence of the fact that the order in which the product $\dot{q}_i \dot{q}_j$ is formed is unimportant.

It will be crucial for some later developments to recognize situations where all terms in the kinetic energy are quadratic in the generalized velocities, so that $T = T_2$. To identify such a condition, we note that the terms contributing to T_1 and T_0 originated from $\partial \bar{r}_k / \partial t$ in Eq. (7.5). This term vanishes if the constraint conditions for the system are independent of t, so that the relations for position in terms of the generalized coordinates are invariant in time. The condition where all constraints are catastatic is an alternative leading to the same result. In either case, T_0 and all N_j are identically zero.

Let us consider the result of using Eqs. (7.7) to form the first term of Lagrange's equations, $d(\partial T/\partial \dot{q}_n)/dt$, where q_n is arbitrarily selected. We begin with the quadratic terms, T_2. Because the generalized velocities are independent quantities for the partial derivative, we have $\partial \dot{q}_i / \partial \dot{q}_n = 1$ or 0 according to whether or not $i = n$.

Furthermore, the M_{ij} coefficients do not depend on the generalized velocities, so they are constant in the partial differentiation. It follows that

$$\frac{\partial T_2}{\partial \dot{q}_n} = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} M_{ij}\frac{\partial}{\partial \dot{q}_n}(\dot{q}_i\dot{q}_j) = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} M_{ij}\left(\frac{\partial \dot{q}_i}{\partial \dot{q}_n}\dot{q}_j + \dot{q}_i\frac{\partial \dot{q}_j}{\partial \dot{q}_n}\right)$$
$$= \frac{1}{2}\sum_{j=1}^{M} M_{nj}\dot{q}_j + \frac{1}{2}\sum_{i=1}^{M} M_{in}\dot{q}_i = \sum_{j=1}^{M} M_{nj}\dot{q}_j, \tag{7.9a}$$

where the last step is a consequence of the symmetry of the coefficients M_{nj}. The corresponding terms obtained from T_1 and T_0 are

$$\frac{\partial T_1}{\partial \dot{q}_n} = N_n, \qquad \frac{\partial T_0}{\partial \dot{q}_n} = 0. \tag{7.9b}$$

Differentiation of the sum of Eqs. (7.9) with respect to time yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) = \sum_{j=1}^{M}(M_{nj}\ddot{q}_j + \dot{M}_{nj}\dot{q}_j) + \sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial M_{nj}}{\partial q_i}\dot{q}_i\dot{q}_j$$
$$+ \sum_{j=1}^{M}\frac{\partial N_n}{\partial q_j}\dot{q}_j + \dot{N}_n. \tag{7.10}$$

A similar analysis of $\partial T/\partial q_n$ based on Eqs. (7.7) leads to

$$\frac{\partial T}{\partial q_n} = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial M_{ij}}{\partial q_n}\dot{q}_i\dot{q}_j + \sum_{j=1}^{M}\frac{\partial N_j}{\partial q_n}\dot{q}_j + \frac{\partial T_0}{\partial q_n}. \tag{7.11}$$

The Lagrange equations corresponding to the foregoing expressions are

$$\sum_{j=1}^{M}\left[M_{nj}\ddot{q}_j + \left(\dot{M}_{nj} + \frac{\partial N_n}{\partial q_j} - \frac{\partial N_j}{\partial q_n}\right)\dot{q}_j\right] + \sum_{i=1}^{M}\sum_{j=1}^{M}\left(\frac{\partial M_{nj}}{\partial q_i} - \frac{1}{2}\frac{\partial M_{ij}}{\partial q_n}\right)\dot{q}_i\dot{q}_j$$
$$+ \dot{N}_n - \frac{\partial T_0}{\partial q_n} + \frac{\partial V}{\partial q_n} = Q_n, \quad n = 1, 2, \ldots, M. \tag{7.12}$$

Let $F_n(\dot{q}_j, q_j, t)$ denote all the terms in Eq. (7.12) that are independent of the genereralized accelerations $\ddot{q}_n$. Correspondingly, we may write Eq. (7.12) as a set of second-order differential equations for the unknown q_n having the general form

$$\sum_{j=1}^{M} M_{nj}\ddot{q}_j = F_n, \quad n = 1, 2, \ldots, M. \tag{7.13a}$$

This expression may be written equivalently in matrix form as

$$[M]\{\ddot{q}\} = \{F\}, \tag{7.13b}$$

where the elements of the column array $\{\ddot{q}\}$ are the second derivatives of the sequence of generalized coordinates. It is significant to the developments that follow that the elements M_{ij} of the array $[M]$ depend only on the generalized coordinates and time. In contrast, the elements of $\{F\}$ might be functions of the generalized velocities, as well as the generalized coordinates and time.

As we have seen in several examples, the coefficients M_{ij} may depend nonlinearly on the generalized coordinates, and they may be time-dependent as well. The functions F_n may be nonlinear in the generalized velocities and generalized coordinates. It is likely that, unless we introduce approximations, we will be unable to use analytical methods to solve the differential equations of motion. Numerous numerical methods and associated standard computerized routines have been developed to assist us in solving sets of coupled differential equations. Usually, such techniques require that we express the differential equations of motion in first-order form – that is, as equations that contain only first derivatives of the unknown variables,

$$\frac{d}{dt}\{z\} = \{G(z_j, t)\}. \tag{7.14}$$

A discussion of numerical methods by which a set of equations in this form may be solved is beyond the scope of this book. Assistance in that task may be obtained from a variety of texts, FORTRAN subroutine libraries, computational software packages, and symbolic-mathematics programs. A good starting point to learn about possible techniques is the text by Press et al. (1992). From this juncture onward, we shall assume that a reliable technique, capable of solving a system of differential equations in the form of Eq. (7.14), is available to us. Thus, the task we must address is how we can convert the Lagrange equations of motion, as well as any additional constraint equations, into a form that is compatible with Eq. (7.14). The simplest case is that of a holonomic system decribed by unconstrained generalized coordinates. That is where our development begins.

7.2.1 *State-Space Transformation for Holonomic Systems*

We do not usually consider a derivative of an unknown variable to be a new unknown. However, doing so leads to a simple transformation that converts Eq. (7.13b) into a system of first-order differential equations. We define a set of $2M$ variables x_i such that the first group of M variables are the generalized coordinates while the second group are the generalized velocities. This may be described in matrix form as upper and lower partitions of a column:

♦ $$\{x\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix}. \tag{7.15}$$

Note that second derivatives, $\{\ddot{q}\}$, are not considered to be new variables because their values are specified by the Lagrange equations, Eq. (7.13b).

The derivative of $\{q\}$ is obviously $\{\dot{q}\}$. This identity for the derivative may be written in partitioned form by recognizing that $\{q\}$ is the upper partition of $\{x\}$ and $\{\dot{q}\}$ is the lower partition. Recall that when the partitioning of a matrix equation is consistent, a product may be formed by treating the partitions as though they were individual elements. Thus, we have

$$[[U]\ \ [0]]\frac{d}{dt}\begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix} = [[0]\ \ [U]]\begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix}, \tag{7.16}$$

where $[U]$ is the identity matrix. In view of the definition of $\{x\}$, the foregoing is equivalent to

$$[[U]\ \ [0]]\frac{d}{dt}\{x\} = [[0]\ \ [U]]\{x\}. \tag{7.17}$$

The partitioned form of $\{x\}$ converts the equation of motion Eq. (7.13b) to

$$[[0]\ [M]]\frac{d}{dt}\{x\} = \{F\}. \tag{7.18}$$

Equations (7.17) and (7.18) may be combined as

$$\begin{bmatrix} [U] & [0] \\ [0] & [M] \end{bmatrix} \frac{d}{dt}\{x\} = \begin{Bmatrix} [[0]\ [U]]\{x\} \\ \{F\} \end{Bmatrix}. \tag{7.19}$$

The foregoing represents a set of $2M$ first-order differential equations for the $2M$ elements of $\{x\}$. When the generalized coordinates are unconstrained, this is the full set of equations governing the motion of the system. Note that the dependencies of $[M]$ and $\{F\}$ on the generalized coordinates and velocities must be expressed in terms of the appropriate elements of $\{x\}$, consistent with the overall change of variables.

The set of variables x_i constitute the *state space* for a system, just as the generalized coordinates q_i form the configuration space. We must specify the initial value of the state-space vector, $\{x\}$ at $t = 0$, which is populated with the initial values of the generalized coordinates and velocities. These values, combined with $\{F\}$ at $t = 0$, define the initial value of $d\{x\}/dt$ according to Eq. (7.19). In turn, this defines the value of $\{x\}$ at an infinitesimal time-instant later. Indeed, this crude view of how the system's state evolves is equivalent to the Euler integration algorithm for differential equations in the form of Eq. (7.14).

We presumed earlier to have available a numerical integration scheme capable of solving differential equations in the standard form of Eq. (7.14). To convert Eq. (7.19) to such a form we need only solve the second partition for the generalized accelerations. Let us denote with an asterisk any quantity that is known at a specified time t. As a result of numerical integration up to that time, we have determined $\{x^*\}$, from which we know $\{q^*\}$ and $\{\dot{q}^*\}$. This in turn enables us to determine $[M^*]$ and $\{F^*\}$. Solving Eq. (7.13b) by Gauss elimination or LU decomposition yields the corresponding generalized accelerations $\{\ddot{q}^*\}$, so the form of the state-space equations we would implement in conjunction with Eq. (7.14) is

$$\frac{d}{dt}\{x\} = \begin{Bmatrix} \{\dot{q}^*\} \\ \{\ddot{q}^*\} \end{Bmatrix} = \begin{Bmatrix} \{\dot{q}^*\} \\ [M^*]^{-1}\{F^*\} \end{Bmatrix}. \tag{7.20}$$

Note that although $[M^*]^{-1}$ appears in Eq. (7.20), there is no need to actually compute an inverse.

7.2.2 *Approaches for Constrained Generalized Coordinates*

When the generalized coordinates form a constrained set, the reactions associated with the constraint conditions enter into the Lagrange equations. If we use Lagrange multipliers to account for these reactions, then Eq. (7.4) describes the basic equations of motion. In order to track the effect of these forces on the generalized accelerations in Eq. (7.13b), we add a suitable term to the array $\{F\}$. If $\{\lambda\}$ is a vector whose elements are the Lagrange multipliers, then the required term is $[a]^{\mathrm{T}}\{\lambda\}$, where $[a]$ is the Jacobian constraint matrix. Thus, the M Lagrange equations may be written as

$$[M]\{\ddot{q}\} = \{F\} + [a]^{\mathrm{T}}\{\lambda\}. \tag{7.21}$$

Also, the constraint equations must be enforced. When we express them in velocity form, Eqs. (7.1) represent a set of $M-N$ additional differential equations to be satisfied. In the matrix notation used here, these additional equations are

$$[a]\{\dot{q}\} = -\{b\}. \tag{7.22}$$

The combination of Eqs. (7.21) and (7.22) constitutes $2M-N$ differential equations governing the constrained set of M generalized coordinates and the $M-N$ Lagrange multipliers. We have not assembled these equations into a single matrix equation for a basic reason: the unknown Lagrange multipliers only appear algebraically, while the state-space variables appear as derivatives. Without adjustment, such a form would not suit the computational technique that solves Eq. (7.14). It might seem as though the procedure used in Example 7.3 to eliminate the Lagrange multipliers should be applicable. However, that approach is effective only when we wish to employ analytical techniques to fit a trial solution to the differential equations, as we did in that example.

To identify the source of the difficulty, suppose we knew the values $\{q\}^*$ and $\{\dot{q}\}^*$ at some instant t^*. We wish to obtain a set of equations in the form of Eq. (7.20). The basic equations to be satisfied are the state-space transformation identity, Eq. (7.16), the equations of motion, Eq. (7.21), and the constraint equations, Eq. (7.22). We bring the Lagrange multipliers to the left side, because they are unknowns that we must eliminate to place the system of equations into the desired form. Thus, we have

$$\begin{bmatrix} [U] & [0] & [0] \\ [0] & [M^*] & -[a^*]^{\mathrm{T}} \\ [a^*] & [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\dot{q}\}^* \\ \{\ddot{q}\}^* \\ \{\lambda\}^* \end{Bmatrix} = \begin{Bmatrix} \{\dot{q}\}^* \\ \{F\}^* \\ -\{b\}^* \end{Bmatrix}. \tag{7.23}$$

Unfortunately, we cannot solve these equations for $\{\ddot{q}^*\}$, and thereby eliminate $\{\lambda^*\}$, because the first and third rows of the coefficient matrix merely differ by a factor $[a]$. Consequently, this matrix is not full rank, and the equations are not solvable. Several strategies have been developed to convert the coupled differential algebraic equations of Eqs. (7.21) and (7.22) to a form suiting the differential equation solver associated with Eq. (7.14). We shall survey a few of them here.

A widely employed technique for solving constrained equations of motion is the *augmented* method. The basic philosophy here is consistent with the intent of Eq. (7.23), in that the method algebraically eliminates the instantaneous values of the Lagrange multipliers. To generate a set of independent equations, we convert the constraint equations to acceleration form by differentiating with respect to time,†

$$[a]\{\ddot{q}\} = -\{\dot{b}\} - [\dot{a}]\{\dot{q}\}. \tag{7.24}$$

We consider this and the Lagrange equations (7.21) to be a set of $2M-N$ algebraic equations for the values of $\{\ddot{q}\}$ and $\{\lambda\}$, whose assembled form we write as

$$\begin{bmatrix} [M] & -[a]^{\mathrm{T}} \\ -[a] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ [\dot{a}]\{\dot{q}\} - \{\dot{b}\} \end{Bmatrix}. \tag{7.25}$$

† The procedures developed here are equally valid if Eq. (7.24) is further generalized. Replacing the right side by a vector $\{d\}$ whose elements may depend arbitrarily on the q_j, the $\dot{q}_j$, and t leads to a linear acceleration constraint that is appropriate to some problems in active control using feedback.

Note that the signs in the second row have been reversed in the process of assembling Eq. (7.25), because doing so makes the coefficient matrix symmetric.

The values of the generalized coordinates at any instant are $\{q\}^*$ and $\{\dot{q}\}^*$, so the corresponding $[M]^*$, $[a]^*$, $[\dot{a}]^*$, and $\{b\}^*$ may be determined. Then the simultaneous equations represented by Eq. (7.25) may be solved by any convenient method. We thereby obtain the values $\{\ddot{q}\}^*$ representing the generalized accelerations at that instant. (Although the Lagrange multipliers are also obtained as part of this solution, their values are discarded unless we wish to retain them to evaluate the constraint forces.) The generalized accelerations found in this manner form the right side of Eq. (7.20), which is the form required of the differential equation solver.

Solution of this set of differential equations requires the initial value of $\{x\}$, which would be formed from the initial values of the M generalized coordinates and M generalized velocities. The latter cannot be selected independently, because they must be compatible with the velocity constraint equations at $t=0$. Specifically, let $\{q_0\}$ denote the initial values of the generalized coordinates. If any of the velocity constraint equations are holonomic, these initial values must satisfy those conditions. Otherwise, they may be selected arbitrarily. Because Eq. (7.22) represents $M-N$ equations relating the generalized velocities, we may assign to a set of N generalized velocities any initial values. The other $M-N$ initial velocities must be determined by solving Eq. (7.22). This matter will be featured in Example 7.4, where we solve the equations of motion for a wobbling disk.

The augmented method seems to be straightforward to implement, but it does have a potential difficulty. The velocity constraint equations are not satisfied directly at every instant. Rather, they occur only in differentiated form as acceleration constraint equations. Consequently, the method can lead to a solution for the response whose error relative to the velocity constraint equations accumulates with time. Hence, it would be wise to implement an auxiliary step to monitor this error. The associated operations are simple. The generalized coordinate and velocity values obtained numerically at each time step may be substituted into each of the velocity constraint equations. We employ an asterisk to denote the numerical values at any time instant, so the error in satisfying the constraint equation may be written as

$$\{\epsilon\}^* = [a]^*\{\dot{q}\}^* + \{b\}^*. \tag{7.26}$$

If the norm of $\{\epsilon\}^*$ becomes significant in comparison to the norm of the terms on the right side, that is, if $\|\{\epsilon\}^*\|$ is a large fraction of $\|[a]^*\|\,\|\{\dot{q}\}^*\|$, one would be wise to halt the numerical solution.

In fact, regardless of the numerical algorithm selected to solve differential equations of motion, one should verify at regular time intervals that any relevant auxiliary conditions are satisfied. Such conditions might be configuration constraints, $f(q_j, t) = 0$, that have been converted to velocity form in order to formulate the differential equations of motion. An important general class of auxiliary conditions that might be available to monitor errors are conservation principles that apply to the system. These could be conservation of mechanical energy, a linear or angular momentum component, or the Hamiltonian, which is discussed in Section 7.3.3.

The possible accumulation of error is sometimes addressed by using Baumgarte's (1972) *constraint stabilization method,* in which the acceleration constraint, Eq. (7.24), is modified by adding to it the velocity constraint, Eq. (7.22), multiplied by some constant. This modified form is

$$[a]\{\ddot{q}\} = -\{\dot{b}\} - [\dot{a}]\{\dot{q}\} - 2\alpha\{[a]\{\dot{q}\} + \{b\}\}. \tag{7.27}$$

The coefficient α is in some respect like an artificial viscosity term used to solve partial differential equations by finite differences. Clearly, the added term would have no effect if the velocity constraint equations had actually been satisfied. Nevertheless, this method does have limitations, as discussed by Haug (1989).

Another algorithm for solving differential equations governing constrained generalized coordinates uses the *orthogonal complement* of the Jacobian constraint matrix. This is a matrix $[C]$, with N rows and M columns, satisfying the condition that

$$[C][a]^{\mathrm{T}} = [0]. \tag{7.28}$$

Note that $[C]$ is not unique, as one can recognize by considering the foregoing to be a set of simultaneous equations obtained on an element-by-element basis. There are NM elements of $[C]$, but the product yields only $N(M-N)$ elemental equations because $[a]^{\mathrm{T}}$ has M rows and $M-N$ columns. Let us assume for the moment that we can find a suitable $[C]$.

When we multiply the constrained Lagrange equations (7.21) by $[C]$, we find that

$$[C][M]\{\ddot{q}\} = [C]\{F\} + [C][a]^{\mathrm{T}}\{\lambda\} \equiv [C]\{F\}. \tag{7.29}$$

Hence, this operation removes the Lagrange multipliers from the equations to be solved. Furthermore, because $[C]$ has N rows, the number of Lagrange equations to be solved is reduced to the number of degrees of freedom, N.

It is a simple matter to form the state-space equations corresponding to the orthogonal complement method. Because Eq. (7.29) represents N equations for the M generalized accelerations at any instant, we augment Eq. (7.29) with the $M-N$ acceleration constraint equations, Eq. (7.24). The result is M equations having the partitioned form

$$\begin{bmatrix} [C][M] \\ [a] \end{bmatrix} \{\ddot{q}\} = \begin{Bmatrix} [C]\{f\} \\ -[\dot{a}]\{\dot{q}\} - \{\dot{b}\} \end{Bmatrix}. \tag{7.30}$$

At any time-instant in a numerical integration process, the generalized coordinates $\{q\}^*$ have been determined and so all matrices except $\{\ddot{q}\}$ are known. Correspondingly, $\{\ddot{q}\}^*$ are the generalized acceleration values obtained by algebraically solving this equation. The state-space differential equations to be solved are then as given by Eq. (7.20). Once again, the initial values of the generalized velocities must be consistent with the constraint equations.

It is obvious that the crucial step in implementing the orthogonal complement method is determining the orthogonal complement! Several techniques are discussed by Amirouche (1992) under the category of coordinate reductions. One method not discussed there employs singular value decomposition, which is described by Press et al. (1992). We shall briefly summarize the method here.

A theorem in linear algebra states that if $[A]$ is a known $I \times J$ array with $I \geq J$, then $[A]$ may be broken down into the form

$$[A]_{I\times J} = [L]_{I\times J}[w]_{J\times J}[R]^{\mathrm{T}}_{J\times J}. \tag{7.31}$$

In this representation, $[w]$ is a diagonal array that holds the singular values of $[A]$, and $[L]$ and $[R]$ consist of columns that are orthonormal,

$$[L]^{\mathrm{T}}[L] = [U]_{J\times J}, \qquad [R]^{\mathrm{T}}[R] = [U]_{J\times J}. \tag{7.32}$$

It is important for our development that the singular values are real and nonnegative, $w_j \geq 0$. Equations (7.31) and (7.32) are the *singular value decomposition* of $[A]$. Press et al. (1992) give a reliable subroutine for carrying out the process, and comparable routines are contained in some of the popular mathematical software. It is important to note that if one compares the singular value decomposition resulting from different algorithms, only the singular values $[w]$ are unique.

Next we consider the case where $[A]$ is square, $J \times J$, and we are confronted with the task of solving

$$[A]\{y\} = \{0\}. \tag{7.33}$$

Any $\{y\} \neq \{0\}$ satisfying this equation is said to be in the *null space* of $[A]$. The number K of independent solutions lying in the null space of $[A]$ is the *nullity* of $[A]$. Thus, the rank of $[A]$ is $J-K$. One reason for performing a singular value decomposition is that the nullity of $[A]$ is the same as the number of zero singular values. (In general, the ratio of the largest to the smallest singular value is the *condition number* of $[A]$.)

The relevance of the foregoing to the tasks of implementing Eq. (7.30) is the following theorem: If $w_j = 0$, then column j of $[R]$ is an independent vector in the null space of $[A]$. Let us arrange the arrays in Eq. (7.31) such that the values of w_j occur in ascending order, which leads to $w_1 = \cdots = w_K = 0$. Then the first K columns of $[R]$ form an orthonormal basis for any vector $\{y\}$ in the null space of $[A]$; that is,

$$\{y\} = c_1\{R_1\} + c_2\{R_2\} + \cdots + c_K\{R_K\} \;\Rightarrow\; [A]\{y\} = \{0\}. \tag{7.34}$$

Stated in a different way, we have

$$[A][\{R_1\} \; \{R_2\} \; \ldots \; \{R_K\}] = \{0\}. \tag{7.35}$$

In order to employ this result to form the orthogonal complement $[C]$, we take the transpose of Eq. (7.28),

$$[a][C]^{\mathrm{T}} = [0]. \tag{7.36}$$

We form a square array $[A]$ from $[a]$, which has $M-N$ rows and M columns, by augmenting $[a]$ with N rows of zeros,

$$[A]_{M\times M} = \begin{bmatrix} [a] \\ [0] \end{bmatrix}. \tag{7.37}$$

Aside from degenerate conditions that might occur at some instant, the $M-N$ constraint conditions are independent, so the rank of $[a]$ is $M-N$. It follows that the nullity of $[A]$ as formed here is N. Hence, if the singular values are arranged in the prescribed manner, the first N columns of $[R]$ obtained from Eq. (7.31) will be an orthonormal basis of solutions in the null space of $[A]$. We define the rows of $[C]$ to be the transpose of these columns of $[R]$, so that

$$[C] = \begin{bmatrix} \{R_1\}^{\mathrm{T}} \\ \vdots \\ \{R_N\}^{\mathrm{T}} \end{bmatrix}. \tag{7.38}$$

When we use the definition of $[A]$ in Eq. (7.37) in conjunction with this representation of $[C]$, it follows from Eq. (7.35) that $[C]$ is the orthogonal complement of $[a]$.

Although using singular value decomposition may not be the most efficient method for finding the orthogonal complement, it is reliable. Example 7.5 will illustrate the singular value decomposition method for a typical step in the solution of a set of constrained equations of motion by the orthogonal complement method.

An astute observer might note that the orthogonal complement method and the augmented method are very much alike. Both employ the acceleration form of the constraint equations, and therefore share the same potential problem regarding numerical error. Furthermore, both yield a set of values for the M generalized accelerations at any instant. The methods differ only in the manner in which the equations for those quantities are formed. The augmented method entails solving $2M-N$ equations, while the orthogonal complement method requires solution of only N equations. Balancing this is the increased number of operations required to evaluate the orthogonal complement of the Jacobian constraint matrix. If the number of constraint equations is not too large, so that $M \approx N$, it is likely that the orthogonal complement method will be computationally less efficient.

A fundamentally different algorithm that has received some advocacy involves selecting a set of N generalized coordinates as an independent set, on which the remaining $M-N$ generalized coordinates depend. The approach uses the constraint equations to algebraically eliminate the dependent constrained generalized velocities and accelerations. Amirouche (1992) refers to this as the *embedding method,* because the constraint equations become intertwined with the Lagrange equations of motion.

The method begins by sequencing the generalized coordinates such that the first N elements of $\{q\}$ form partition $\{q\}_u$, which contains the quantities one regards as independent, or unconstrained. The remaining generalized coordinates form partition $\{q\}_c$, so we have

$$\{q\} = \begin{Bmatrix} \{q\}_u \\ \{q\}_c \end{Bmatrix}. \tag{7.39}$$

We partition the acceleration constraint equation, Eq. (7.24), in the same manner:

$$[[a]_u \ [a]_c]\begin{Bmatrix} \{\ddot{q}\}_u \\ \{\ddot{q}\}_c \end{Bmatrix} = \{G\}, \tag{7.40}$$

where

$$\{G\} = -[\dot{a}]\{\dot{q}\} - \{\dot{b}\}. \tag{7.41}$$

It is important for the present discussion to recognize that $\{G\}$ may depend on all of the generalized coordinates, not just the unconstrained ones. Solving Eq. (7.40) yields

$$\{\ddot{q}\}_c = -[a_c]^{-1}\{[a]_u\{\ddot{q}\}_u - \{G\}\}. \tag{7.42}$$

Without going into the details of the derivation, one finds that using this expression to eliminate $\{\ddot{q}\}_c$ from the Lagrange equations leads to a set of N differential equations that are linear in the N elements of $\{\ddot{q}\}_u$. The important aspect of the derivation is the fact that the substitution also eliminates the Lagrange multipliers. Thus, it would appear that the result is a solvable set of differential equations, as Amirouche (1992) implies. That view is deceptive, because of the possible dependence of $\{G\}$ on the constrained generalized coordinates. Thus, one has not necessarily eliminated those variables from the equations to be solved. It might therefore be necessary to use additional differential equations to track the constrained generalized coordinates.

The discussion has not fully explored the embedding method because it suffers from a potential difficulty. Specifically, one might find at some instant that they cannot form the solution in Eq. (7.42) because $[a]_c$ is rank-deficient, so that $[a]_c^{-1}$ does not exist. A similar difficulty would arise if $[a]_c$ should become ill-conditioned. The remedy in either case is to include in the computational algorithm a process that identifies rank-deficient and ill-conditioned cases. If such a situation is identified, it is necessary to consider a different set of generalized coordinates to be the unconstrained set. This entails re-arranging the sequence in which the elements of $\{q\}$ are defined. The sequence in which the equations of motion and constraint equations are arranged would need to be adjusted correspondingly. Both the augmented and orthogonal complement methods are more robust, in the sense that they do not suffer from problems of solvability. (This assumes that the Jacobian constraint matrix itself never becomes rank-deficient.)

Example 7.4 The disk in Example 7.3, which rolls without slipping, is set into motion with an initial precession rate $\dot{\psi}_0 = \dot{\psi}_s$ and spin rate $\dot{\phi}_0$ that match the values required to make the center follow a circular path having radius $\rho = 5R$ with a constant nutation angle $\theta = \pi/3$. However, the initial nutation angle is $\theta = \pi/4$ and the initial nutation rate is $\dot{\theta} = \dot{\psi}_s/2$. Other initial conditions are $X = 0$, $Y = 5R$, $\psi = 0$, and $\phi = 0$, which are consistent with the nominal steady-precession solution. The disk is homogeneous with a radius $R = 0.25$ m. Use the augmented method to evaluate the subsequent motion of the disk, and compare that result to the case of nominal steady precession.

Solution The equations of motion for a rolling, wobbling disk were numbered (1) to (7) in the solution to Example 7.3. The unknowns appearing in those equations are the two Cartesian coordinates X and Y, the three Eulerian angles ψ, ϕ, and θ, and the two Lagrange multipliers. In order to make direct use of the previous development, we define the column array $\{q\}$ that contains the generalized coordinates in the same sequence as before. There are two Lagrange multipliers, so we define

$$\{q\} = [X \;\; Y \;\; \psi \;\; \theta \;\; \phi]^{\mathrm{T}}, \qquad \{\lambda\} = [\lambda_1 \;\; \lambda_2]^{\mathrm{T}}.$$

The velocity constraint equations were denoted as (1) and (2) in the previous solution. For the vector $\{q\}$ just defined, writing these terms in the form $[a]\{\dot{q}\} = -\{b\}$ corresponds to

$$[a] = \begin{bmatrix} 1 & 0 & -R\cos\psi\cos\theta & R\sin\psi\sin\theta & -R\cos\psi \\ 0 & 1 & -R\sin\psi\cos\theta & -R\cos\psi\sin\theta & -R\sin\psi \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & -R\cos q_3\cos q_4 & R\sin q_3\sin q_4 & -R\cos q_3 \\ 0 & 1 & -R\sin q_3\cos q_4 & -R\cos q_3\sin q_4 & -R\sin q_3 \end{bmatrix},$$
$$\{b\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

We obtain $[\dot{a}]$ for the augmented method by differentiating each term, which yields

$$[\dot{a}] = R\begin{bmatrix} 0 & 0 & (\dot{q}_3 \sin q_3 \cos q_4 + \dot{q}_4 \cos q_3 \sin q_4) \\ 0 & 0 & (-\dot{q}_3 \cos q_3 \cos q_4 + \dot{q}_4 \sin q_3 \sin q_4) \end{bmatrix}$$

$$\cdots \begin{bmatrix} (\dot{q}_3 \cos q_3 \sin q_4 + \dot{q}_4 \sin q_3 \cos q_4) & (\dot{q}_3 \sin q_3) \\ (\dot{q}_3 \sin q_3 \sin q_4 - \dot{q}_4 \cos q_3 \cos q_4) & (-\dot{q}_3 \cos q_3) \end{bmatrix}.$$

To write the Lagrange equations (3)–(7) obtained previously in the standard form $[M]\{\ddot{q}\} = \{F\} + [a]^{\mathrm{T}}\{\lambda\}$, we form the inertia matrix $[M]$ by identifying the coefficients of the terms in T that are quadratic in the generalized velocities. We bring all remaining terms that do not contain the Lagrange multipliers to the right side of the equations, and place them in the respective elements of $\{F\}$. After doing so, we replace the generalized coordinates and velocities with the corresponding elements of $\{q\}$ and $\{\dot{q}\}$. This yields

$$[M] = m\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \kappa^2(1+\cos^2 q_4)/2 & 0 & \kappa^2 \cos q_4 \\ 0 & 0 & 0 & (\kappa^2/2 + R^2 \cos^2 q_4) & 0 \\ 0 & 0 & \kappa^2 \cos q_4 & 0 & \kappa^2 \end{bmatrix},$$

$$\{F\} = m\begin{Bmatrix} 0 \\ 0 \\ \kappa^2(\dot{q}_3 \dot{q}_4 \sin q_4 \cos q_4 + \dot{q}_4 \dot{q}_5 \sin q_4) \\ (R^2 \dot{q}_4^2 - \kappa^2 \dot{q}_3^2/2) \sin q_4 \cos q_4 - \kappa^2 \dot{q}_3 \dot{q}_5 \sin q_4 - gR \cos q_4 \\ \kappa^2 \dot{q}_3 \dot{q}_4 \sin q_4 \end{Bmatrix}.$$

In order to start the differential equation solver, we need to provide the initial conditions. The initial value of $\{q\}$ is specified by the problem statement,

$$\{q\}_0 = [0 \;\; 5R \;\; 0 \;\; \pi/4 \;\; 0]^{\mathrm{T}}.$$

Only three of the five initial generalized velocities were specified. We obtain the other two velocities by satisfying the two nonholonomic constraint equations corresponding to the given initial conditions. Because the disk is homogeneous, we compute the initial precession rate from the last equation in the solution to Example 7.3, with $\theta = \theta_s = \pi/3$ and $\rho = 5R$:

$$\dot{\psi}_0 = \dot{\psi}_s = \left(\frac{4g \cot \theta_s}{R \cos \theta_s + 6\rho}\right)^{1/2} = 1.723446 \text{ rad/s}.$$

We obtain the initial spin rate from the relation previously determined for the no-slip condition at a constant nutation angle,

$$\dot{\phi}_0 = -\left(\frac{\rho}{R} + \cos \theta_s\right)\dot{\psi}_s = -9.478951 \text{ rad/s},$$

and the initial nutation rate is specified to be

$$\dot{\theta}_0 = 0.5\dot{\psi}_s = 0.861723 \text{ rad/s}.$$

In general, determining initial values of the dependent generalized velocities would involve simultaneous solution of the constraint equations, but the values of $\dot{X}$ and $\dot{Y}$ appear separately in the constraint equations for the rolling disk. Thus, we compute

$$\dot{X}_0 = R(\dot{\psi}_0 \cos\theta_0 + \dot{\phi}_0)\cos\psi_0 - R\dot{\theta}_0 \sin\psi_0 \sin\theta_0 = -2.674403 \text{ m/s},$$

$$\dot{Y}_0 = R(\dot{\psi}_0 \cos\theta_0 + \dot{\phi}_0)\sin\psi_0 + R\dot{\theta}_0 \cos\psi_0 \sin\theta_0 = 0.152333 \text{ m/s}.$$

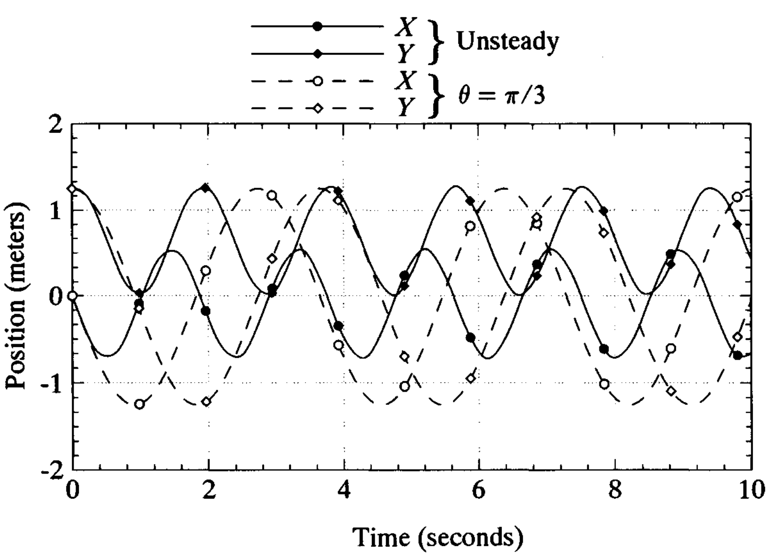
Correspondingly, the initial generalized velocity is

$$\{\dot{q}\}_0 = [\dot{X}_0 \;\; \dot{Y}_0 \;\; \dot{\psi}_0 \;\; \dot{\theta}_0 \;\; \dot{\phi}_0]^{\mathrm{T}}.$$

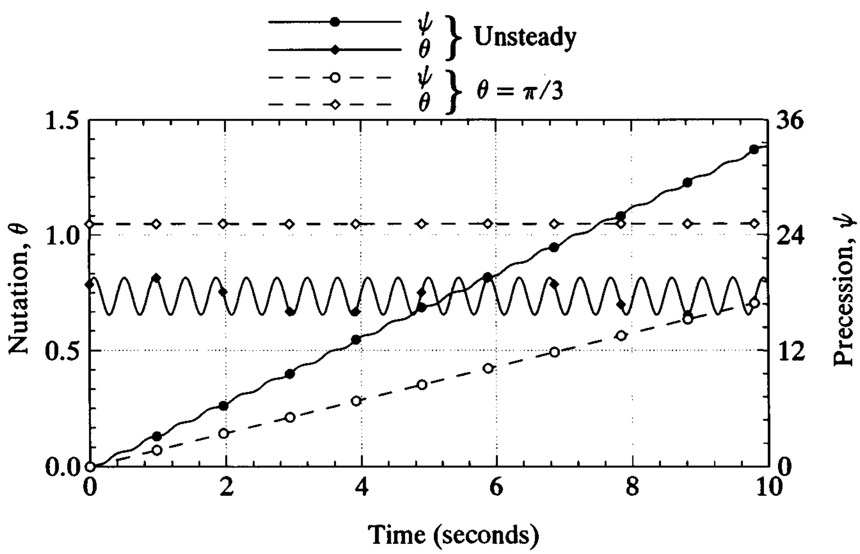
The solution of the differential equations presented here was obtained by MATLAB® using the ODE23 subroutine, which employs second- and third-order Runge-Kutta formulas. The solver addresses the standard form of Eq. (7.20), where the generalized accelerations are obtained by solving the augmented equations, Eq. (7.25). Note that the mass m appears as a common factor of all terms except the Lagrange multipliers, so its value may be set to unity for the computation.

One check of the formulation is to confirm that the case of steady precession demonstrated in Example 7.3 is obtained when the initial values are set as here, except that $\theta_0 = \theta_s = \pi/3$ and $\dot{\theta}_0 = 0$. This solution is shown in the figures that follow. Another check is to monitor the mechanical energy $E = T + V$. Because the disk rolls without slipping, friction does no work. Consequently, this quantity should be constant. The computed value of E was found to remain at the initial value $E_0 = 4.9727$ throughout the time interval in the figures.

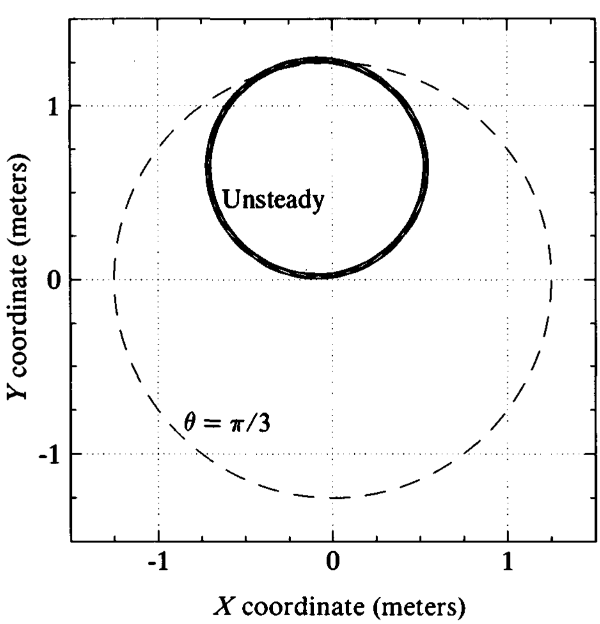
The computed responses are displayed as plots of X, Y, ψ, and θ as functions of time. Another plot shows Y as a function of X, with t serving as the parameter. Each figure exhibits the strong deviation of the motion from the nominal steady precession. The time traces suggest that the precession angle increases nearly linearly with time. The oscillation of the nutation angle, which represents the wobble, appears to be periodic, as do the oscillations of X and Y. Aside from θ, each time dependence has the character of the steady-precession case that was intended, although the details



Position of the center of the disk as a function of time.



Nutation and precession angles as a function of time.



Path of the center of the disk.

of the various responses differ drastically from the nominal case. This suggests that the motion is close to a *different* steady precession. This is most vividly depicted in the plots of Y versus X. We see there that the center of the disk appears to follow a circular path, although this is precisely correct only for the case of nominal steady precession. This graph clearly shows the much smaller circle resulting from the given initial conditions.

One might be surprised that a nearly steady precession is obtained, despite the difference between the given initial conditions and those for the intended motion. This is a manifestation of the overall stability of the rolling motion associated with the gyroscopic moment, which induces a rotation about an axis perpendicular to the axis of an applied moment. Of course, an actual disk would not continue in motion because of rolling friction. Energy is conserved in the present idealized model, so the only way the motion can end in our analysis is if it predicts $\theta < 0$ or $\theta > \pi$, both of which correspond to the center falling to the ground. Furthermore, the solution presented here assumes that the coefficient of static friction is sufficiently large to develop the required frictional forces. Note in this regard that the Lagrange multipliers represent the components of the friction force in the X and Y directions, as is apparent from the Lagrange equations of motion corresponding to the X and Y generalized coordinates.

An interesting response that highlights the shortcomings of a formulation that ignores friction losses is encountered when a slight disturbance is imparted to a rolling motion in which the disk is upright. If the forward speed is too low, $v < (gR/3)^{1/2}$, such a motion is unstable (see Meirovitch 1970, p. 164). In the present case, initial conditions of $X = Y = \psi = \phi = 0$, $\theta = \pi/2 - 0.001$, $\dot{\psi} = \dot{\theta} = 0$, and $R\dot{\phi} = 0.2(gR/3)^{1/2}$ result in an unexpected response. The disk begins by rolling almost upright in a fairly constant direction. This is followed by a short interval in which the disk loops around and nearly falls to the ground, but then recovers, after which it resumes upright rolling in a different direction. Continued integration of the equations of motion indicates that this motion continues periodically and that the overall path of the center of the disk is nearly circular.

Solving the present problem by the orthogonal complement method leads to an interesting observation. The same MATLAB® formulation was used, with the orthogonal complement obtained from the NULL function, which implements singular value decomposition. Because the orthogonal complement eliminates the Lagrange multipliers, the instantaneous generalized accelerations are obtained by solving five simultaneous equations. In contrast, the augmented method used here entails solving seven equations. Nevertheless, the orthogonal complement method required 27% more floating point operations, due to the computational overhead required to perform a singular value decomposition.

Example 7.5 At a particular instant, the generalized coordinates and generalized velocities for the rolling disk in Example 7.4 are

$$\{q\} = [-0.194103 \;\; 1.422270 \;\; -0.528468 \;\; 1.410973 \;\; -0.716483]^{\mathrm{T}},$$

$$\{\dot{q}\} = [-1.761367 \;\; 1.646433 \;\; -9.839692 \;\; 2.162469 \;\; -7.839004]^{\mathrm{T}}.$$

For this state, use the orthogonal complement method to determine the corresponding generalized accelerations.

Solution For brevity, we shall not repeat the definitions of the matrices in the solution to Example 7.4. The first step in the present approach is to compute the

Jacobian constraint matrix $[a]$ and its time derivative, the inertia matrix $[M]$, and the excitation matrix $\{F\}$ corresponding to the given values of $\{q\}$ and $\{\dot{q}\}$. We assign the elements of the given $\{q\}$ to the first partition of $\{x\}$, while the elements $\{\dot{q}\}$ form the second partition. Substituting the appropriate values into the earlier definitions yields

$$[a] = \begin{bmatrix} 1 & 0 & -0.034358 & -0.124446 & -0.215895 \\ 0 & 1 & 0.020060 & -0.213144 & 0.126053 \end{bmatrix},$$

$$[\dot{a}] = \begin{bmatrix} 0 & 0 & 0.658306 & -2.140648 & 1.240319 \\ 0 & 0 & 0.068964 & 1.150213 & 2.124342 \end{bmatrix},$$

$$[m] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.016021 & 0 & 0.004973 \\ 0 & 0 & 0 & 0.017208 & 0 \\ 0 & 0 & 0.004973 & 0 & 0.031250 \end{bmatrix},$$

$$\{F\} = [0 \;\; 0 \;\; -0.627458 \;\; -2.961644 \;\; -0.105821]^{\mathrm{T}}.$$

To perform the singular value decomposition, we define $[A]$ such that $[a]$ forms its first two rows and the last three rows are zeros. The decomposition results described below were obtained from the SVD function contained within the MATLAB® software. The SVD subroutine returns the singular values in descending order. These values are

$$w_1 = 1.031544, \qquad w_2 = 1.030008, \qquad w_3 = 0, \qquad w_4 = 0, \qquad w_5 = 0,$$

and the corresponding left and right unitary matrices are

$$[L] = \begin{bmatrix} 0.863581 & -0.504211 & 0 & 0 & 0 \\ -0.504211 & -0.863581 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[R] = \begin{bmatrix} 0.837173 & -0.489521 & 0.243948 & 0 & 0 \\ -0.488792 & -0.838421 & -0.005006 & 0.190742 & -0.147393 \\ -0.038569 & 0.000000 & 0.132361 & -0.663563 & -0.735307 \\ 0.000000 & 0.239623 & 0.480842 & 0.667389 & -0.515717 \\ -0.242355 & 0.000000 & 0.831706 & -0.279094 & 0.414289 \end{bmatrix}.$$

Because $w_3 = w_4 = w_5 = 0$, we use columns 3, 4, and 5 of $[R]$ to form the rows of the orthogonal complement of $[a]$,

$$[C] = \begin{bmatrix} 0.243948 & -0.005006 & 0.132361 & 0.480842 & 0.831706 \\ 0 & 0.190742 & -0.663563 & 0.667389 & -0.279094 \\ 0 & -0.147393 & -0.735307 & -0.515717 & 0.414289 \end{bmatrix}.$$

We next use this value of $[C]$ to form Eq. (7.30), which yields

$$\begin{bmatrix} 0.243948 & -0.005006 & 0.006257 & 0.008274 & 0.026649 \\ 0 & 0.190742 & -0.012019 & 0.011484 & -0.012022 \\ 0 & -0.147393 & -0.009720 & -0.008874 & 0.009289 \\ -1 & 0 & 0.034358 & 0.124446 & 0.215895 \\ 0 & -1 & -0.020060 & 0.213144 & -0.126053 \end{bmatrix} \{\ddot{q}\}$$
$$= [-1.595147 \ \ -1.530675 \ \ 1.944903 \ \ -20.829479 \ \ -14.844011]^{\mathrm{T}}.$$

The solution of this equation is

$$\{\ddot{q}\} = [0.6522 \ \ -5.2671 \ \ -40.0950 \ \ -111.5867 \ \ -22.7570]^{\mathrm{T}}.$$

These values of $\{\ddot{q}\}$ and the given values of $\{\dot{q}\}$, which were obtained from the previous integration step, would be used to form the right side of Eq. (7.20). The latter are the values, corresponding to the given instant, that should be sent to an integration solver.

7.3 Hamiltonian Mechanics and Further Conservation Principles

W. R. Hamilton developed a standard (i.e. canonical) set of first-order equations of motion that are quite different from the state-space formulation in the preceding section. These equations will lead us to recognize that additional conservation principles may be applicable to a system. As we have seen, such principles can be useful in formulating and solving equations of motion.

7.3.1 *Hamilton's Canonical Equations*

Hamilton altered the appearance of Lagrange's equations of motion by relating momentum and kinetic energy. The motivation for this formulation may be recognized by referring to Eq. (7.6), which describes the kinetic energy of particle k in a system. Let us evaluate $\partial T_k/\partial \dot{q}_n$ by the procedure we used to obtain Eqs. (7.9). The result is

$$\frac{\partial T_k}{\partial \dot{q}_n} = \sum_{i=1}^{M}\left(m_k \frac{\partial \bar{r}_k}{\partial q_i} \cdot \frac{\partial \bar{r}_k}{\partial q_n}\right)\dot{q}_i + m_k \frac{\partial \bar{r}_k}{\partial t} \cdot \frac{\partial \bar{r}_k}{\partial q_n}. \tag{7.43}$$

A simple re-arrangement of terms leads to the observation that

$$\frac{\partial T_k}{\partial \dot{q}_n} = m_k\left(\sum_{i=1}^{M} \frac{\partial \bar{r}_k}{\partial q_i}\dot{q}_i + \frac{\partial \bar{r}_k}{\partial t}\right) \cdot \frac{\partial \bar{r}_k}{\partial q_n} = m_k \bar{v}_k \cdot \frac{\partial \bar{r}_k}{\partial q_n}. \tag{7.44}$$

The term $m_k \bar{v}_k$ is the momentum of this particle. The partial derivative $\partial \bar{r}_k/\partial q_n$ serves to generalize the derivative to fit the type of parameter associated with q_n, for example, linear or angular motion.

For this reason, the derivative $\partial T/\partial \dot{q}_n$ for any system is called the *generalized momentum* p_n corresponding to q_n. Since the potential energy is independent of the generalized velocities, the generalized momentum may also be defined in terms of the Lagrangian function $\mathcal{L} = T - V$. Thus

♦ $$p_n = \frac{\partial T}{\partial \dot{q}_n} = \frac{\partial \mathcal{L}}{\partial \dot{q}_n}. \tag{7.45}$$

We now use the generalized momenta to form the *Hamiltonian function* $\mathcal{H}$ according to

♦ $$\mathcal{H} = \sum_{i=1}^{M} p_i \dot{q}_i - \mathcal{L}. \tag{7.46}$$

In this formulation, the state variables are considered to be the generalized coordinates and momenta, so the generalized velocities must be removed from all relationships. Such a change of variables may be achieved in a general situation by the following sequence of operations.

(1) Form the Lagrangian $\mathcal{L} = T - V$ in the usual manner, as a function of the generalized coordinates q_i, generalized velocities $\dot{q}_i$, and time t.
(2) Derive expressions for the generalized momenta p_i as functions of the q_i, $\dot{q}_i$, and t according to Eq. (7.45).
(3) Solve the equations found in the preceding step for the $\dot{q}_i$ in terms of the q_i, p_i, and t.
(4) Substitute the expressions for the $\dot{q}_i$ into Eq. (7.46), thereby obtaining the functional form $\mathcal{H} = \mathcal{H}(q_1, \ldots, q_M, p_1, \ldots, p_M, t)$. (The operations entailed in this step may be reduced by referring to Eq. (7.63), which is a relation between $\mathcal{H}$ and the potential and kinetic energies we shall derive later.)

Let us consider the time derivative of the Hamiltonian function. Because $\mathcal{H}$ depends on the generalized coordinates and momenta, we have

$$\dot{\mathcal{H}} = \sum_{i=1}^{M} \left(\frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i \right) + \frac{\partial \mathcal{H}}{\partial t}. \tag{7.47}$$

We can also use the definition of $\mathcal{H}$, Eq. (7.46), to form the derivative. This yields

$$\begin{aligned} \dot{\mathcal{H}} &= \sum_{i=1}^{M} \left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right) - \frac{\partial \mathcal{L}}{\partial t} \\ &= \sum_{i=1}^{M} \left(\dot{p}_i \dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i \right) - \frac{\partial \mathcal{L}}{\partial t}, \end{aligned} \tag{7.48}$$

where the simplified form results from substitution of Eq. (7.45). The two descriptions of $\dot{\mathcal{H}}$ must match for any set of values of generalized velocities and momenta. Thus, a comparison of like terms in Eqs. (7.47) and (7.48) reveals that

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i}, \qquad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{7.49}$$

These identities and the definition of p_i make it possible to express Lagrange's equations in terms of $\mathcal{H}$, rather than $\mathcal{L}$. This yields

$$\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, M. \tag{7.50}$$

The combination of the first of Eqs. (7.49) and (7.50) form a set of first-order differential equations, which are called *Hamilton's canonical equations:*

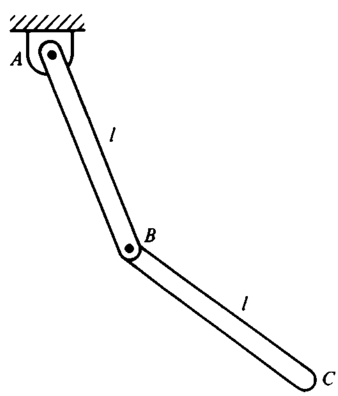
♦ $$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad i = 1, 2, \ldots, M;$$

♦ $$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + Q_i, \quad i = 1, 2, \ldots, M. \tag{7.51}$$

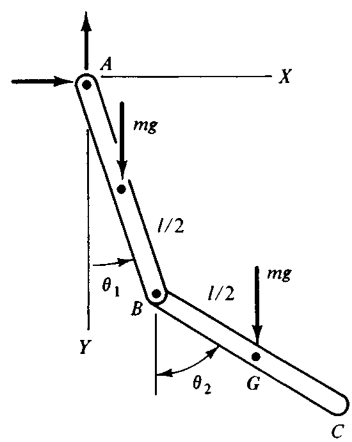
These constitute $2M$ coupled, first-order differential equations for the M values of q_i and the M values of p_i. If the generalized coordinates are constrained, these equations must be supplemented by the constraint equations. The constraint forces would then appear in the generalized forces, or, alternatively, the Lagrange multiplier terms may be added to the generalized force array.

Once formulated, Hamilton's canonical equations are slightly easier than the state-space form to implement for numerical solution. Equations (7.51) give the derivative of each state variable explicitly at each instant. In contrast, the state-space equations of motion, Eq. (7.19) or (7.25), require solution of the coupled equations for the generalized accelerations in order to obtain the corresponding result. The avoidance of this operation is a substantial benefit of Hamilton's canonical equations, particularly when the inertia matrix or the Jacobian constraint matrix is not constant. This gain is balanced by the fact that evaluation of the Hamiltonian function requires the intricate change of variables just described, in order to remove the generalized velocities in favor of the generalized momenta.

Example 7.6 The double pendulum consists of identical bars of mass m connected at the ideal pin B. Derive the Hamiltonian equations of motion for the system.



Example 7.6



Free-body diagram.

Solution The angles $q_1 = \theta_1$ and $q_2 = \theta_2$, which describe the orientation of each bar relative to the vertical, are convenient generalized coordinates for this holonomic, two-degree-of-freedom system. The first step in forming $\mathcal{H}$ is to express the Lagrangian in terms of the generalized coordinates and velocities. Bar AB is in pure rotation about end A, but bar BC is in general motion. We shall obtain the velocity of the center of mass G of bar BC by differentiating its position. Thus,

$$\bar{r}_{G/A} = (l\sin\theta_1 + \tfrac{1}{2}l\sin\theta_2)\bar{I} + (l\cos\theta_1 + \tfrac{1}{2}l\cos\theta_2)\bar{J},$$

$$\bar{v}_G = l(\dot{\theta}_1\cos\theta_1 + \tfrac{1}{2}\dot{\theta}_2\cos\theta_2)\bar{I} - l(\dot{\theta}_1\sin\theta_1 + \tfrac{1}{2}\dot{\theta}_2\sin\theta_2)\bar{J}.$$

The kinetic energy is the sum of the rotational energy of bar AB relative to end A, and of the translational and rotational energy of bar BC relative to point G. Thus,

$$T = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}_1^2 + \frac{1}{2}m\bar{v}_G\cdot\bar{v}_G + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}_2^2$$
$$= \frac{1}{2}ml^2\left[\frac{4}{3}\dot{\theta}_1^2 + \frac{1}{3}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right].$$

The elevation of pin A is a convenient datum for gravitational potential energy. The corresponding Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}ml^2\left[\frac{4}{3}\dot{\theta}_1^2 + \frac{1}{3}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right]$$
$$+ mg\frac{l}{2}\cos\theta_1 + mgl\left(\cos\theta_1 + \frac{1}{2}\cos\theta_2\right).$$

The generalized momenta are

$$p_1 = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = ml^2\left[\frac{4}{3}\dot{\theta}_1 + \frac{1}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)\right],$$
$$p_2 = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = ml^2\left[\frac{1}{3}\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_1\cos(\theta_2-\theta_1)\right].$$

We solve these relations for $\dot{\theta}_1$ and $\dot{\theta}_2$, which yields

$$\dot{\theta}_1 = \frac{12p_1 - 18p_2c}{ml^2K}, \qquad \dot{\theta}_2 = \frac{-18p_1c + 48p_2}{ml^2K},$$

where c and K depend only on $\theta_2 - \theta_1$,

$$c = \cos(\theta_2-\theta_1), \qquad K = [16 - 9\cos^2(\theta_2-\theta_1)].$$

We may now form the Hamiltonian. By definition,

$$\mathcal{H} = p_1\dot{\theta}_1 + p_2\dot{\theta}_2 - \mathcal{L},$$

from which we eliminate $\dot{\theta}_1$ and $\dot{\theta}_2$ by substituting the previous expressions. Thus,

$$\mathcal{H} = \frac{12p_1^2 - 36p_1p_2c + 48p_2^2}{ml^2K}$$
$$- \frac{1}{2ml^2K^2}\left[\frac{4}{3}(12p_1 - 18p_2c)^2 + \frac{1}{3}(-18p_1c + 48p_2)^2\right.$$
$$\left. + (12p_1 - 18p_2c)(-18p_1c + 48p_2c)\right]$$
$$- mgl\left(\frac{3}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2\right).$$

Collecting like coefficients enables us to rewrite $\mathcal{H}$ in a much simpler form as

$$\mathcal{H} = \frac{1}{2ml^2K}[12p_1^2 - 36cp_1p_2 + 48p_2^2] - mgl\left(\frac{3}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2\right).$$

The last step before forming Hamilton's canonical equations is to identify that the generalized forces vanish, $Q_1 = Q_2 = 0$, because the constraints imposed by pins A and B are satisfied.

Differentiating $\mathcal{H}$ with respect to p_1 and p_2, in accord with the first of Eqs. (7.51), yields

$$\dot{\theta}_1 = \frac{1}{ml^2K}(12p_1 - 18cp_2), \qquad \dot{\theta}_2 = \frac{1}{ml^2K}(-18cp_1 + 48p_2).$$

Note that these expressions for $\dot{\theta}_1$ and $\dot{\theta}_2$ are the same as those obtained from the equations defining p_1 and p_2. The second part of Hamilton's equations (7.51) requires differentiation of $\mathcal{H}$ with respect to θ_1 and θ_2. However, K and c depend only on $\theta_2 - \theta_1$, so their derivatives with respect to θ_2 are the negative of those with respect to θ_1. The resulting equations are

$$\dot{p}_1 = -\frac{18}{ml^2K^2}\sin(\theta_2-\theta_1)[6cp_1^2 - (16+9c^2)p_1p_2 + 24cp_2^2] - \frac{3}{2}mgl\sin\theta_1,$$

$$\dot{p}_2 = \frac{18}{ml^2K^2}\sin(\theta_2-\theta_1)[6cp_1^2 - (16+9c^2)p_1p_2 + 24cp_2^2] - \frac{1}{2}mgl\sin\theta_2.$$

It is clear from this example that the operations required to form Hamilton's canonical equations are far more tedious than those required for Lagrange's equations. Of course, some of the complicated work can be performed by a symbolic computer language.

7.3.2 *Ignorable Coordinates and Routh's Method*

The concept of generalized momenta leads to further understanding of the laws of mechanics in some special circumstances. The first arises when the Lagrangian function depends on a particular generalized velocity $\dot{q}_n$, but not on the generalized coordinate q_n itself. Because $\partial\mathcal{L}/\partial q_n = 0$ in this case, Lagrange's equation for this generalized coordinate is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_n}\right) = Q_n. \tag{7.52}$$

By virtue of the definition of p_n, this equation may be integrated to obtain

$$p_n|_{t_2} = p_n|_{t_1} + \int_{t_1}^{t_2} Q_n(\tau)\,d\tau. \tag{7.53}$$

This relation is a generalization of the impulse-momentum principles. It describes both linear momentum and angular momentum, depending on the type of geometric quantity associated with q_n. It is interesting to note that, if Q_n depends on any of the generalized coordinates, then we cannot employ this momentum principle to relate the states at t_1 and t_2. Such a situation corresponds to the Newtonian formulation of position-dependent forces, where momentum principles are not used because the impulse cannot be evaluated.

Consider the more restrictive situation, in which $\mathcal{L}$ does not depend on q_n *and* the corresponding generalized force Q_n vanishes, $Q_n = 0$. We find from the foregoing that p_n is constant, which corresponds to *conservation of generalized momentum.* When such a situation occurs, the corresponding generalized coordinate q_n is said to be *ignorable.* This term arises because the relation

$$p_n = \frac{\partial T}{\partial \dot{q}_n} = \frac{\partial \mathcal{L}}{\partial \dot{q}_n} = \text{constant} \tag{7.54}$$

may be solved for $\dot{q}_n$ in terms of the other generalized coordinates and velocities. That solution may be substituted into each of the remaining Lagrange equations in order to obtain equations of motion in which neither q_n nor $\dot{q}_n$ appears.†

We used this procedure to simplify the equations of motion in Example 6.12. The procedure may be extended to treat a system in which several generalized coordinates are ignorable. However, it is important to recognize that all of the Lagrange equations must be formed prior to substitution for the ignorable coordinates. (It is incorrect to treat the conserved momenta as constants in the expression for T or $\mathcal{L}$ when Lagrange's equations are formulated, because doing so does not allow for a description of the full effect of a variation of each generalized coordinate.)

It is possible, as an alternative, to use the relations for the ignorable coordinates to derive the Lagrangian for an equivalent system having a reduced number of degrees of freedom. The method by which this reduction may be achieved is *Routh's method for the ignoration of coordinates.* Suppose that, from the original set of M generalized coordinates, there are $M-J$ ignorable coordinates, which we designate as $q_{J+1}, \ldots, q_M$. Then Eqs. (7.54) apply for $n = J+1, \ldots, M$; that is, $p_{J+1}, \ldots, p_M$ are constants. Solving Eqs. (7.54) simultaneously allows us to evaluate the generalized velocities $\dot{q}_{J+1}, \ldots, \dot{q}_M$ as functions of the J generalized coordinates and velocities that are not ignorable, and, possibly, time. The *Routhian function* is defined, in terms of the Lagrangian for the system, as

♦ $$\mathcal{R} = \mathcal{L} - \sum_{n=J+1}^{M} \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\dot{q}_n = \mathcal{L} - \sum_{n=J+1}^{M} p_n \dot{q}_n. \tag{7.55}$$

After substitution of the relations for the generalized velocities of the ignorable coordinates, the dependence of the Routhian is $\mathcal{R} = \mathcal{R}(q_1, \ldots, q_J, \dot{q}_1, \ldots, \dot{q}_J, p_{J+1}, \ldots, p_M, t)$.

We consider first the variation of $\mathcal{R}$ based on its functional dependence. Even though $p_{J+1}, \ldots, p_M$ are constants in the actual motion, they must be varied when deriving the equations of motion. Thus,

$$\delta\mathcal{R} = \sum_{n=1}^{J}\left(\frac{\partial \mathcal{R}}{\partial q_n}\,\delta q_n + \frac{\partial \mathcal{R}}{\partial \dot{q}_n}\,\delta \dot{q}_n\right) + \sum_{n=J+1}^{M} \frac{\partial \mathcal{R}}{\partial p_n}\,\delta p_n. \tag{7.56a}$$

We may also form the variation of $\mathcal{R}$ based on its definition, Eq. (7.55), as follows:

$$\begin{aligned}\delta\mathcal{R} &= \sum_{n=1}^{J} \frac{\partial \mathcal{L}}{\partial q_n}\,\delta q_n + \sum_{n=1}^{M} \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\,\delta \dot{q}_n - \sum_{n=J+1}^{M} (p_n\,\delta \dot{q}_n + \delta p_n \dot{q}_n) \\ &= \sum_{n=1}^{J}\left(\frac{\partial \mathcal{L}}{\partial q_n}\,\delta q_n + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\,\delta \dot{q}_n\right) - \sum_{n=J+1}^{M} \delta p_n \dot{q}_n.\end{aligned} \tag{7.56b}$$

The alternative forms of $\delta\mathcal{R}$ must be valid for arbitrary virtual increments, which means that like coefficients must match. Hence, we find from Eqs. (7.56) that

$$\frac{\partial \mathcal{R}}{\partial q_n} = \frac{\partial \mathcal{L}}{\partial q_n} \quad \text{and} \quad \frac{\partial \mathcal{R}}{\partial \dot{q}_n} = \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \quad \text{for } n = 1, \ldots, J, \tag{7.57a}$$

† Ignorable coordinates are sometimes called *cyclic coordinates.* This name stems from the observation that many cases where a generalized coordinate is ignorable involve rotation about an axis. They are also known as *kinosthenic coordinates.*

$$\frac{\partial \mathcal{R}}{\partial p_n} = -\dot{q}_n \quad \text{for } n = J+1, \ldots, M. \tag{7.57b}$$

According to Eqs. (7.57a), the original Lagrange equations for the nonignorable generalized coordinates may be replaced by equations of the same form that feature the Routhian:

♦ $$\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{q}_n}\right) - \frac{\partial \mathcal{R}}{\partial q_n} = Q_n, \quad n = 1, \ldots, J. \tag{7.58}$$

In other words, the Routhian may be considered to be the Lagrangian for a system described by J generalized coordinates. If the generalized coordinates are an unconstrained set, then the Routhian represents an equivalent system having J degrees of freedom.

Example 7.7 A particle slides on the interior of a smooth surface of revolution whose shape is defined in cylindrical coordinates as $r = f(z)$, where r is the transverse distance from the axis and z is the vertical distance along the axis. Derive the differential equation of motion whose solution gives z as a function of time.

Solution The azimuthal angle θ and elevation z are useful generalized coordinates, so we set $q_1 = z$ and $q_2 = \theta$. The distance r is subject to a configuration constraint, $r = f(z)$, so the radial velocity is

$$\dot{r} = f'\dot{z},$$

where a prime denotes differentiation with respect to z. The kinetic energy for a particle of mass m is therefore

$$T = \tfrac{1}{2}m[(f')^2+1]\dot{z}^2 + \tfrac{1}{2}mf^2\dot{\theta}^2.$$

We let $z = 0$ be the gravitational datum. The corresponding Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}m[(f')^2+1]\dot{z}^2 + \tfrac{1}{2}mf^2\dot{\theta}^2 - mgz.$$

We note that θ does not appear explicitly in $\mathcal{L}$, and that the generalized force $Q_\theta = 0$ because the system is conservative. These are the conditions for which θ is ignorable. The corresponding generalized momentum p_θ is constant, where

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mf^2\dot{\theta}.$$

We solve this expression for the generalized velocity,

$$\dot{\theta} = \frac{p_\theta}{mf^2},$$

and use this relation to replace $\dot{\theta}$ wherever it occurs in the Routhian. The result is

$$\mathcal{R} = \mathcal{L} - p_\theta\dot{\theta} = \frac{1}{2}m[(f')^2+1]\dot{z}^2 + \frac{1}{2}mf^2\left(\frac{p_\theta}{mf^2}\right)^2 - mgz - p_\theta\left(\frac{p_\theta}{mf^2}\right)$$

$$= \frac{1}{2}m[(f')^2+1]\dot{z}^2 - \frac{p_\theta^2}{2mf^2} - mgz.$$

The Routhian $\mathcal{R}$ depends only on z and $\dot{z}$, so it represents an equivalent system with one degree of freedom. The first term in $\mathcal{R}$, which contains the generalized velocity, is the equivalent kinetic energy, while the equivalent potential energy is the negative of the sum of the remaining terms.

The equation of motion is Lagrange's equation for $q_1 = z$, with $\mathcal{R}$ used instead of the Lagrangian. It is necessary to recognize that f and f' are functions of z, which is, in turn, a function of time. Thus, the required derivatives of $\mathcal{R}$ are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{z}}\right) = m\frac{d}{dt}\{[(f')^2+1]\dot{z}\} = m[(f')^2+1]\ddot{z}+2mf'f''\dot{z}^2,$$

$$\frac{\partial \mathcal{R}}{\partial z} = mf'f''\dot{z}^2+\frac{p_\theta^2}{mf^3}f'-mg.$$

Setting $Q_1 = 0$ leads to

$$[(f')^2+1]\ddot{z}+f'f''\dot{z}^2-\frac{p_\theta^2}{m^2f^3}f' = -g.$$

7.3.3 *Conservation Theorems*

The previous section derived an extended principle for conservation of momentum. Here we shall develop concepts that are related to conservation of energy. In the course of deriving Hamilton's canonical equations, we found in Eq. (7.48) that the time derivative of the Hamiltonian function $\mathcal{H}$ is given by

$$\dot{\mathcal{H}} = \sum_{i=1}^{M}\left(\dot{p}_i\dot{q}_i-\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i\right)-\frac{\partial \mathcal{L}}{\partial t}. \tag{7.59}$$

Let us remove p_i from this relation by factoring out $\dot{q}_i$ and then recalling the definition of p_i and Lagrange's equations. This yields

$$\dot{\mathcal{H}} = \sum_{i=1}^{M}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}\right)-\frac{\partial \mathcal{L}}{\partial q_n}\right]\dot{q}_i-\frac{\partial \mathcal{L}}{\partial t};$$

♦ $$\dot{\mathcal{H}} = \sum_{i=1}^{M} Q_i\dot{q}_i-\frac{\partial \mathcal{L}}{\partial t}. \tag{7.60}$$

This relation has a useful corollary in the case of a conservative system, for which $Q_i = 0$, when that system's Lagrangian is independent of time, so that $\partial\mathcal{L}/\partial t = 0$. One finds in this special case that $\dot{\mathcal{H}} = 0$, so that $\mathcal{H}$ is a constant whose value is conserved from the initial motion. This is *Jacobi's integral.*

In order to understand the significance of Eq. (7.60), and its associated conservation principle, we recall that the virtual work is written as

$$\delta W = \sum_{i=1}^{M} Q_i\,\delta q_i. \tag{7.61}$$

If the virtual increments in δW were instead time derivatives, then the summation would represent the power input from the generalized forces. If this were so, then it would appear from Eq. (7.60) that $\dot{\mathcal{H}}$ is related to the rate of change of the mechanical energy. To cast $\mathcal{H}$ into the appropriate form, we eliminate the generalized momenta. From Eqs. (7.9) and (7.45), we have

$$p_n = \frac{\partial T}{\partial \dot{q}_n} = \sum_{j=1}^{M} M_{nj}\dot{q}_j + N_n. \tag{7.62}$$

Substitution of this expression into the definition of $\mathcal{H}$, Eq. (7.46) leads to

$$\mathcal{H} = \sum_{n=1}^{M}\sum_{j=1}^{M} M_{nj}\dot{q}_j\dot{q}_n + \sum_{n=1}^{M} N_n\dot{q}_n - \mathcal{L} = 2T_2 + T_1 - (T_2 + T_1 + T_0 - V);$$

♦ $$\mathcal{H} = T_2 - T_0 + V. \tag{7.63}$$

(This expression, compared to the definition in Eq. (7.46), usually proves to be an easier means to forming $\mathcal{H}$.)

It is clear from the preceding form of $\mathcal{H}$ that Eq. (7.60) is like an energy-rate principle. Whether or not this relation actually provides a new perspective depends on the nature of the constraints imposed on the system's motion. We consider first a system that is time-independent, so that the position of any point depends only on the generalized coordinates, with no explicit dependence on time. In such systems, $\partial\mathcal{L}/\partial t \equiv 0$ and $T_0 \equiv 0$. The latter converts Eq. (7.63) to

$$\mathcal{H} = T + V = E, \tag{7.64}$$

where E is the total mechanical energy of the system. Now let us recall from Chapter 6 the development of the kinematical method for virtual displacement. We found that if a system is time-independent, then the velocity and virtual displacement of a point have analogous forms,

$$\bar{v} = \sum_{j=1}^{M} \frac{\partial \bar{r}}{\partial q_j}\dot{q}_j \Leftrightarrow \delta\bar{r} = \sum_{j=1}^{M} \frac{\partial \bar{r}}{\partial q_j}\,\delta q_j. \tag{7.65}$$

As a consequence, virtual increments in Eq. (7.61) may be replaced by time derivatives, which leads to the following expression for the instantaneous power input to a system by the generalized forces:

$$\dot{W} = \sum_{i=1}^{M} Q_i\dot{q}_i. \tag{7.66}$$

When these observations concerning a time-independent system are combined, we find that, for such systems, the Hamiltonian rate principle, Eq. (7.60), reduces to the power form of the work–energy principle:

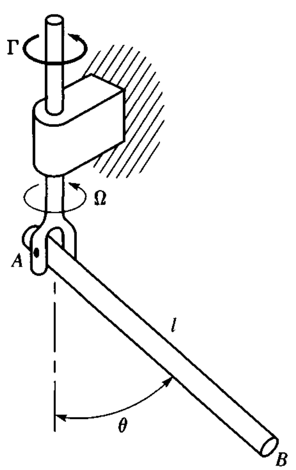
$$\text{power} = \dot{W} = \dot{E}. \tag{7.67}$$

We therefore conclude that the Hamiltonian rate equation provides no additional insight to the motion of a system if the kinematics of that system do not depend explicitly on time. Clearly, the important case of a scleronomic system fits this specification.

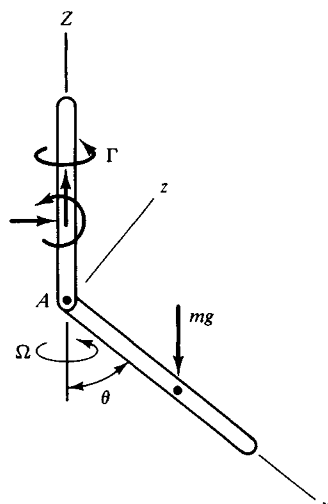
A less restrictive situation is one where the system's constraints are catastatic. This means that, although the constraint equations depend explicitly on time, all velocity variables and constraint conditions are homogeneous in the $\dot{q}_j$. In such a system we still have $T = T_2$, so that $\mathcal{H} = E$. Although Eqs. (7.65) are applicable for a catastatic system, time does appear explicitly in the Lagrangian, so that $\partial\mathcal{L}/\partial t \neq 0$. Obviously, the latter also applies to more general situations, in which the kinematical features of a system depend arbitrarily on time.

It follows that Eq. (7.60) for $\dot{\mathcal{H}}$ and Eq. (7.67) for $\dot{E}$ are independent principles governing any time-dependent system. The work-energy principle, Eq. (7.67), is generally valid. However, the power input to such systems comes from reactions that impose the time-dependent motion, as well as from the applied forces. Such forces do not appear in the generalized forces of a system that is described by unconstrained generalized coordinates. It is possible for the Hamiltonian of a system to be constant (Jacobi's integral), even if the mechanical energy is not conserved. When $\mathcal{H}$ is conserved, its constant value is determined from the initial conditions, which leads to a relation between the generalized velocities at different positions.

Example 7.8 Bar AB, whose mass is m, is pinned to the vertical shaft. The assembly precesses about the vertical axis at the constant rate Ω due to the torque Γ. Consider the angle of nutation θ as the sole generalized coordinate. Compare the rate of change of the Hamiltonian in such a formulation to the rate of change of the mechanical energy of the system. The inertia of the vertical shaft is negligible.



Example 7.8



Free-body diagram.

Solution We may select $q_1 = \theta$ as the sole generalized coordinate, because the restriction that Ω be constant serves to define the precession angle ψ. We must express the kinetic energy in terms of θ in order to form both $\mathcal{H}$ and E. The angular velocity of the bar is

$$\bar{\omega} = \Omega\bar{K} - \dot{\theta}\bar{j} = -(\Omega\cos\theta)\bar{i} + \dot{\theta}\bar{j} + (\Omega\sin\theta)\bar{k}.$$

The kinetic energy may be considered to be purely rotational relative to the stationary end A. The bar is slender, and xyz are principal axes, so we have

$$T = \tfrac{1}{2}(I_{yy}\omega_y^2 + I_{zz}\omega_z^2) = \tfrac{1}{6}ml^2[\dot{\theta}^2 + \Omega^2\sin^2\theta].$$

For a gravitational datum at the elevation of end A, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{6}ml^2[\dot{\theta}^2 + \Omega^2 \sin^2\theta] + \tfrac{1}{2}mgl\cos\theta.$$

The generalized momentum corresponding to the only generalized coordinate is

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{3}ml^2\dot{\theta}.$$

We solve this expression for $\dot{\theta}$, and use the result to eliminate $\dot{\theta}$ from the Hamiltonian. Thus,

$$\mathcal{H} = p_1\dot{\theta} - \mathcal{L} = \frac{3}{2ml^2}p_1^2 - \frac{1}{6}ml^2\Omega^2\sin^2\theta - \frac{1}{2}mgl\cos\theta.$$

No constraints are violated in a virtual movement resulting from incrementing θ by $\delta\theta$. Therefore, $Q_1 = 0$. Furthermore, $\mathcal{L}$ does not explicitly depend on time, so $\partial\mathcal{L}/\partial t = 0$. (This would not be true if Ω were a given time-dependent function.) According to Eq. (7.60), these conditions correspond to $\dot{\mathcal{H}} = 0$, from which it follows that the Hamiltonian is conserved,

$$\frac{3}{2ml^2}p_1^2 - \frac{1}{6}ml^2\Omega^2\sin^2\theta - \frac{1}{2}mgl\cos\theta = \mathcal{H}_0,$$

where $\mathcal{H}_0$ is the value when the motion was initiated. This conservation equation may be expressed in terms of $\dot{\theta}$ by substituting $p_1 = ml^2\dot{\theta}/3$, which yields

$$\tfrac{1}{6}ml^2[\dot{\theta}^2 - \Omega^2\sin^2\theta] - \tfrac{1}{2}mgl\cos\theta = \mathcal{H}_0.$$

The mechanical energy is

$$E = T + V = \frac{1}{6}ml^2[\dot{\theta}^2 + \Omega^2\sin^2\theta] - \frac{L}{2}mgl\cos\theta.$$

The only nonconservative force that does work is the couple Γ, so the power input is

$$\dot{W} = \Gamma\Omega.$$

The work-energy principle, $\dot{E} = \dot{W}$, then yields

$$\dot{E} = \tfrac{1}{3}ml^2[\dot{\theta}\ddot{\theta} + \Omega^2\dot{\theta}\sin\theta\cos\theta] + \tfrac{1}{2}mgl\sin\theta = \Gamma\Omega.$$

For comparison, we now differentiate the equation for the constant Hamiltonian;

$$\dot{\mathcal{H}} = \tfrac{1}{3}ml^2[\dot{\theta}\ddot{\theta} - \Omega^2\dot{\theta}\sin\theta\cos\theta] + \tfrac{1}{2}mgl\dot{\theta}\sin\theta = 0.$$

Aside from the presence of the common factor $\dot{\theta}$, this is the same equation of motion as we would obtain from the application of Lagrange's equations for $q_1 = \theta$. We also obtain an expression for Γ by forming $\dot{E} - \dot{\mathcal{H}}$, which yields

$$\Gamma = \tfrac{2}{3}ml^2\Omega\dot{\theta}\sin\theta\cos\theta.$$

This relation could have been obtained from the Lagrangian formulation by using constrained generalized coordinates $q_1 = \theta$, $q_2 = \psi$, which must satisfy the constraint $\dot{\psi} = \Omega$.

7.4 Gibbs-Appell Equations for Quasicoordinates

The methods of the preceding sections, by which dynamic system equations were reduced to first-order form, are closely linked with the kinetic energy of the system. The formulation of first-order equations we shall develop here is out of this mainstream, in that it is not founded on kinetic energy. It has other assets beyond leading to a first-order form, the primary one being that it can yield equations for nonholonomic systems in which constraint forces and/or Lagrange multipliers do not occur. The method is not used as widely as Lagrange's equations, probably because its merits only become important for nonholonomic systems. One can derive Kane's (1985) equations, which is a formulation favored by some individuals, as a special case of the developments that follow. We begin by extending the concept of generalized coordinates. Then we will derive equations of motion by considering a system of particles. The last step in the derivation will be specializing the equations to the case of rigid-body systems.

7.4.1 *Quasicoordinates and Generalized Forces*

By definition, we may uniquely describe the instantaneous position of any point in a system in terms of M generalized coordinates q_j, where $M \geq N$ and N is the number of degrees of freedom of the system. We can correspondingly describe the instantaneous velocity of that point in terms of M generalized velocities $\dot{q}_j$. However, it might be more desirable to use a different set of parameters, called *quasicoordinates* γ_j. We use the prefix "quasi" because we require only that the time derivative of these quantities have physical meaning. It is not necessary that there be an actual position associated with a quasicoordinate, because the equations of motion will only involve $\dot{\gamma}_j$. However, it is acceptable to employ a position variable as a quasicoordinate. A familiar example of quasicoordinates arises in Euler's equations of motion for a rigid body, which are expressed in terms of the angular velocity components. In that case, there is no angular orientation vector that may be used to form a corresponding set of generalized coordinates.

Consider a system of P particles. Let x_i, $i = 1, \ldots, 3P$, denote the set of (absolute) Cartesian coordinates for all particles in the system. The position of the system is known in terms of the generalized coordinates, but the value of time t must also be specified if the physical constraints imposed on the system are time-dependent. We therefore have $x_i = x_i(q_j, t)$, where functional dependence on a set of variables is indicated by a generic variable in that set. Differentiation of the position leads to expressions for the velocity components of each particle,

$$\dot{x}_i = \sum_{j=1}^{M} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t}, \tag{7.68}$$

where all partial derivatives may be functions of the generalized coordinates and time. Clearly, $\dot{x}_i \, dt$ is an exact differential that may be integrated to return to the functional dependence of the position coordinates.

Let us generalize Eq. (7.68) such that the rate variable it defines is no longer a perfect differential. Specifically, we replace the physical velocity components $\dot{x}_i$ by parameters $\dot{\gamma}_i$, and also replace the partial derivatives by arbitrary functions. Thus, the *quasicoordinate velocities* $\dot{\gamma}_i$ (or, more briefly, the *quasivelocities*) are related to the generalized coordinates by

$$\dot{\gamma}_i = \sum_{j=1}^{M} u_{ij}(q_k, t)\dot{q}_j + g_i(q_k, t), \quad i = 1, \ldots, M, \tag{7.69}$$

where the functions u_{ij} and g_i depend on the definitions of the quasivelocities and generalized coordinates. Note that we have defined only M quasicoordinates, because that is the number of generalized velocities $\dot{q}_i$.

The differential displacement in a time interval dt corresponding to Eq. (7.69) is

$$d\gamma_i = \dot{\gamma}_i\, dt = \sum_{j=1}^{M} u_{ij}(q_k, t)\, dq_j + g_i(q_k, t)\, dt, \quad i = 1, \ldots, M. \tag{7.70}$$

Recall that the coefficients u_{ij} and g_i are arbitrary, depending on how we define the quasivelocities. Therefore, integrating $d\gamma_i$ in order to obtain a value of the quasicoordinate γ_i is only possible if we express the time dependence of the generalized coordinates. However, we do not know such dependencies until we have solved the equations of motion, so the quasicoordinates are not useful for specifying position.

The overall philosophy in the developments that follow is to use generalized coordinates to describe any position-related effects, whereas quasicoordinates are used to represent the various aspects of movement, such as velocity and virtual displacement. We may replace any dependence on the generalized velocities $\dot{q}_i$ by a comparable dependence on the quasivelocities $\dot{\gamma}_i$. Equations (7.69) represent a set of M linear equations expressing the values $\dot{\gamma}_i$ in terms of the values of $\dot{q}_i$. We may solve these equations to find $\dot{q}_i$ in terms of $\dot{\gamma}_i$; the form of that solution is

♦ $$\dot{q}_i = \sum_{j=1}^{M} v_{ij}(q_k, t)\dot{\gamma}_j + h_i(q_k, t). \tag{7.71}$$

Formulation of the Gibbs–Appell equations of motion for a specific system will begin by defining the generalized coordinates and quasivelocities. For the system of interest, we will need to derive expressions like Eq. (7.71) for each generalized velocity in terms of the quasivelocities. This is a purely kinematical issue, for which any of the techniques in Chapters 2–4 might be useful. Consequently, we consider the coefficients v_{ij} and h_i to be known. In contrast, we will have no need for inverse relations in the form of Eq. (7.70).

The linear form of the transformation between the $\dot{q}_i$ and the $\dot{\gamma}_i$ has an important implication for constraint equations. Consider the substitution of Eqs. (7.71) into Eqs. (7.1). The result is a set of constraints on the quasivelocities, whose form is similar to Eq. (7.1):

♦ $$\sum_{j=1}^{M} A_{ij}(q_k, t)\dot{\gamma}_j + B_i(q_k, t) = 0, \quad i = 1, \ldots, M-N. \tag{7.72}$$

The coefficients A_{ij} form the Jacobian constraint matrix for the quasivelocities. Once again, although we could relate the coefficients A_{ij} and B_i appearing here to the coefficients in Eq. (7.1), there will seldom be any reason to do so. Instead, we will formulate the constraint conditions for a system directly in terms of the quasivelocities.

In order to treat virtual displacements, we multiply Eq. (7.71) by dt, which leads to an expression relating the increments of the quasi- and generalized coordinates in an infinitesimal time interval,

$$dq_i = \sum_{j=1}^{M} v_{ij}(q_k, t)\, d\gamma_j + h_i(q_k, t)\, dt. \tag{7.73}$$

Time is held constant in a virtual movement, so the virtual increments imparted to the generalized coordinates are related to the corresponding increments in the quasicoordinates by

$$\delta q_i = \sum_{j=1}^{M} v_{ij}(q_k, t)\, \delta\gamma_j, \quad i = 1, \ldots, M. \tag{7.74}$$

As a consequence of Eq. (7.74), we may form a generalized force Γ_j corresponding to each quasicoordinate γ_j. The definition of the generalized forces Q_i associated with a set of generalized coordinates is that the virtual work has the form $\delta W = \sum Q_i\, \delta q_i$. Substitution of Eq. (7.74) into this definition yields an analogous form for the virtual work in terms of the Γ_j; specifically,

♦ $$\delta W = \sum_{j=1}^{M} \Gamma_j\, \delta\gamma_j, \quad \Gamma_j = \sum_{j=1}^{M} Q_i v_{ij} = \sum_{i=1}^{M} Q_i \frac{\partial \dot{q}_i}{\partial \dot{\gamma}_j}. \tag{7.75}$$

Note that here we have invoked Eq. (7.71) to replace the coefficient v_{ij} by the partial derivative $\partial\dot{q}_i/\partial\dot{\gamma}_j$, because this form has greater meaning in the context of a problem solution.

The second of Eqs. (7.75) allows us to convert the generalized forces associated with the q_i parameters for a system. Such an approach is particularly useful for conservative forces. We merely use $Q_i = -\partial V/\partial q_i$, where $V(q_k, t)$ is the potential energy of the conservative forces. In contrast, the first of Eqs. (7.75) shows how we usually proceed with nonconservative forces. The approach is quite similar to the kinematical method for virtual displacement. Let point P be the location where a nonconservative force $\bar{F}$ is applied. By definition, the various $\dot{\gamma}_i$ are variables we have selected to describe the motion of the system. Hence, a kinematical analysis of the velocity of point P would yield an expression of the form

$$\bar{v}_P = \sum_{j=1}^{M} \bar{c}_j(q_k, t)\dot{\gamma}_j + \bar{d}(q_k, t). \tag{7.76}$$

(If the quasicoordinates were the same as the generalized coordinates, then $\bar{c}_j = \partial\bar{r}_P/\partial q_j$ and $\bar{d} = \partial\bar{r}_P/\partial t$.) To obtain the corresponding virtual displacement, we convert the foregoing to a differential and then set $dt = 0$. From this we may form the virtual work done by $\bar{F}$,

$$\delta\bar{r}_P = \sum_{j=1}^{M} \bar{c}_j\, \delta\gamma_j \;\Rightarrow\; \delta W^F = \bar{F}\cdot\delta\bar{r}_P = \sum_{j=1}^{M} \bar{F}\cdot\bar{c}_j\, \delta\gamma_j. \tag{7.77}$$

When we match δW^F to the first of Eqs. (7.75), we find that the contribution of $\bar{F}$ to the generalized forces is

$$\Gamma_j^F = \bar{F}\cdot\bar{c}_j = \bar{F}\cdot\frac{\partial\bar{v}_P}{\partial\dot{\gamma}_j}. \tag{7.78}$$

The similarity of virtual increments in generalized and quasicoordinates leads to an explicit statement of the manner in which the constraint forces affect the generalized forces. Because the constraint equations (7.72) have the same form as Eq. (7.1) for constrained generalized coordinates, the contribution of the reactions to the generalized forces must be as indicated in Eq. (7.3). Let λ_i be the Lagrange multiplier for the ith constraint on the quasicoordinates. Then the generalized force Γ_j is a superposition of $\Gamma_j^{(a)}$ due to active forces applied to the system, and the Lagrange multiplier contributes $\lambda_i A_{ij}$ due to each constraint. The generalized forces are therefore given by

$$\Gamma_j = \Gamma_j^{(a)} + \sum_{i=1}^{M-N} \lambda_i A_{ij}. \tag{7.79}$$

As we found in Section 7.1, using Lagrange multipliers allows us to account for the reactions associated with constraints, without actually formulating the virtual work they do. Even better, it is possible to employ a set of quasicoordinates that entirely avoid the appearance of reaction forces in the equations of motion. Toward that end we select a subset of N quasicoordinates $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_N$. We correspondingly rewrite Eqs. (7.72) as

$$\sum_{j=1}^{M-N} K_{ij}(q_k, t)\dot{\gamma}_{(j+N)} = -\sum_{j=1}^{N} A_{ij}(q_k, t)\dot{\tilde{\gamma}}_j - B_i(q_k, t), \quad i = 1, \ldots, M-N, \tag{7.80a}$$

where $[K]$ is the $(M-N)\times(M-N)$ submatrix of $[A]$ associated with the remaining quasicoordinates,

$$K_{ij} = A_{i(j+N)}, \quad i, j = 1, \ldots, M-N. \tag{7.80b}$$

If the square array of coefficients K_{ij} is not singular (i.e., if $|[K]| \neq 0$), then the constraint equations may be solved for the remaining quasicoordinates in terms of the N parameters $\tilde{\gamma}_i$. This means that any set of values assigned to the N quasivelocities $\dot{\tilde{\gamma}}_i$ will represent a kinematically admissible motion, provided that the remaining $M-N$ quasivelocities are selected to satisfy the constraint equations. Hence, the $\tilde{\gamma}_i$ are *unconstrained quasicoordinates*. Note that the case where $|[K]| = 0$ corresponds to selecting a set of N quasicoordinates that are not kinematically independent. Because we require the constraint equations to be distinct, there must be some submatrix $[K]$ that is not singular, from which it follows that a set of unconstrained quasicoordinates can be defined for any system.

As a corollary of Eq. (7.80a), the generalized velocities are uniquely related to the unconstrained quasivelocities. Such relations have the same linear form as Eq. (7.71), but the coefficient functions are altered. The new expressions are

$$\dot{q}_i = \sum_{j=1}^{M} \tilde{v}_{ij}(q_k, t)\dot{\gamma}_j + \tilde{h}_i(q_k, t), \quad i = 1, \ldots, M. \tag{7.81}$$

Note that we will obtain the coefficients $\tilde{v}_{ij}$ appropriate to a specific system directly from a kinematical analysis, rather than by using Eq. (7.80a) to transform the coefficients v_{ij} associated with the constrained quasicoordinates.

Because the values of the $\dot{\tilde{\gamma}}_i$ are not constrained, an arbitrary set of values may be assigned to the virtual increments $\delta\tilde{\gamma}_i$ without violating any of the constraint conditions. Thus, the generalized forces $\tilde{\Gamma}_i$ corresponding to a set of unconstrained quasicoordinates $\tilde{\gamma}_i$ will not contain any terms associated with reactions; that is, $\tilde{\Gamma}_i = \tilde{\Gamma}_i^{(a)}$. We may obtain these generalized forces according to either of Eqs. (7.75), which become

$$\delta W = \sum_{j=1}^{M} \tilde{\Gamma}_j^{(a)}\,\delta\tilde{\gamma}_j, \quad \tilde{\Gamma}_j^{(a)} = \sum_{i=1}^{M} Q_i^{(a)} \frac{\partial \dot{q}_i}{\partial \dot{\tilde{\gamma}}_j}. \tag{7.82}$$

By eliminating the reactions from the generalized force, unconstrained quasicoordinates simplify nonholonomic systems in the same way that unconstrained generalized coordinates simplify holonomic systems.

7.4.2 *Gibbs-Appell Equations*

It is possible to modify Lagrange's equations such that the term containing a derivative with respect to generalized velocities $\dot{q}_j$ is replaced by terms that depend on the quasivelocities $\dot{\gamma}_j$. However, the transformation may be conveniently carried out only for a scleronomic system. There is little to be gained from such a derivation, especially when an alternative principle that is more widely applicable is available.

We return to a system of P particles, in a three-dimensional space, whose Cartesian coordinates are x_i $(i = 1, \ldots, 3P)$. Suppose we have selected M generalized coordinates to describe the position of all particles, so that $x_i = x_i(q_k, t)$. We also define M quasicoordinates γ_i. When we invoke Eq. (7.76) in component form, we find that the Cartesian velocity components of the particles have the form

$$\dot{x}_i = \sum_{j=1}^{M} c_{ij}(q_k, t)\dot{\gamma}_j + d_i(q_k, t). \tag{7.83}$$

Correspondingly, the virtual displacement components are

$$\delta x_i = \sum_{j=1}^{M} c_{ij}(q_k, t)\,\delta\gamma_j. \tag{7.84}$$

In view of the dependencies of the coefficients c_{ij} and d_i, the acceleration components obtained by differentiating Eq. (7.83) are

$$\ddot{x}_i = \sum_{j=1}^{M}\left[c_{ij}\ddot{\gamma}_j + \left(\sum_{k=1}^{M} \frac{\partial c_{ij}}{\partial q_k}\dot{q}_k + \frac{\partial c_{ij}}{\partial t}\right)\dot{\gamma}_j\right] + \sum_{k=1}^{M} \frac{\partial d_i}{\partial q_k}\dot{q}_k + \frac{\partial d_i}{\partial t}. \tag{7.85}$$

We now implement d'Alembert's principle of virtual work. First, we equate the Cartesian components f_i of the resultant force acting on each particle to the corresponding inertial effect $m_i\ddot{x}_i$. When we use Eq. (7.84) to describe the virtual displacements associated with virtual increments of the quasicoordinates, we find that the virtual work done by all forces is

$$\begin{gathered} \delta W = \sum_{i=1}^{3P} f_i\,\delta x_i = \sum_{i=1}^{3P} m_i\ddot{x}_i\,\delta x_i, \\ \sum_{i=1}^{3P}\sum_{j=1}^{M} f_i c_{ij}\,\delta\gamma_j = \sum_{i=1}^{3P}\sum_{j=1}^{M} m_i\ddot{x}_i c_{ij}\,\delta\gamma_j. \end{gathered} \tag{7.86}$$

Upon interchange of the order in which the summations are performed, we see that the coefficients of $\delta\gamma_j$ to the left of the equality sign are the generalized forces associated with the quasicoordinates, so we have

$$\sum_{j=1}^{M} \Gamma_j \,\delta\gamma_j = \sum_{j=1}^{M} \sum_{i=1}^{3P} m_i \ddot{x}_i c_{ij} \,\delta\gamma_j. \tag{7.87}$$

If the number of degrees of freedom N is the same as M, so that the quasicoordinates form an unconstrained set, then the $\delta\gamma_j$ may be assigned arbitrary values. Even if $M > N$, any subset formed from N of the $\delta\gamma_j$ parameters may be assigned arbitrary values. Furthermore, how we select that subset is also arbitrary. (This assumes that the subset we select does not lead to the degenerate condition $|[K]| = 0$ in Eq. (7.80a).) Given the arbitrary nature of the virtual increments, the coefficients of like $\delta\gamma_j$ in either side of Eq. (7.87) must match, so that

$$\sum_{i=1}^{3P} m_i \ddot{x}_i c_{ij} = \Gamma_j. \tag{7.88}$$

The last step in the derivation comes from recognizing that the coefficients c_{ij} are also the coefficients of the quasiaccelerations in Eq. (7.85), so that

$$\frac{\partial \ddot{x}_i}{\partial \ddot{\gamma}_j} = c_{ij} \;\Rightarrow\; \ddot{x}_i c_{ij} = \ddot{x}_i \frac{\partial \ddot{x}_i}{\partial \ddot{\gamma}_j} \equiv \frac{\partial}{\partial \ddot{\gamma}_j}\left[\frac{1}{2}(\ddot{x}_i)^2\right]. \tag{7.89}$$

We substitute this expression into Eq. (7.88), and recall Eq. (7.79) in order to make the role of the constraint forces explicit. The result is the *Gibbs-Appell equations,* which are named for the researchers who pioneered their development (see the references at the end of this chapter):

$$\blacklozenge \qquad \frac{\partial S}{\partial \ddot{\gamma}_j} = \Gamma_j + \sum_{i=1}^{M-N} \lambda_i A_{ij}. \tag{7.90}$$

The quantity S is the *Gibbs-Appell function,*

$$\blacklozenge \qquad S = \frac{1}{2} \sum_{i=1}^{3P} m_i (\ddot{x}_i)^2. \tag{7.91}$$

In some presentations S is referred to as the energy of acceleration, but doing so adds little to one's understanding. We see from the derivation that although the Gibbs-Appell equations differ drastically from Lagrange's equations, these differences are a consequence solely of the way virtual displacements are described in the two approaches.

The unknowns in the formulation here are the M quasivelocities $\dot{\gamma}_j$, the M generalized coordinates q_j, and the $M-N$ Lagrange multipliers λ_i. The Gibbs-Appell equations represent M first-order differential equations for the $\dot{\gamma}_j$. In addition, the motion must satisfy Eqs. (7.71), which are M first-order differential equations for the q_j derived from a kinematics analysis. Additional $M-N$ first-order differential equations for the $\dot{\gamma}_j$ are obtained from the constraint conditions in Eq. (7.72). Thus, the number of equations balance the number of unknowns.

That the Lagrange multipliers arise algebraically complicates numerical solutions of the foregoing set of equations, just as it does for Lagrange's equations. (We discussed this matter in Section 7.2.2.) The Gibbs-Appell formulation offers a simple remedy, because we can avoid the difficulty completely by using unconstrained quasicoordinates. Those parameters lead to virtual displacements that always satisfy all constraint conditions, so there is no need to employ Lagrange multipliers. The Gibbs-Appell equations in this case reduce to

$$\frac{\partial S}{\partial \ddot{\gamma}_j} = \tilde{\Gamma}_j^{(a)}, \quad j = 1, \ldots, N. \tag{7.92}$$

Also, there are no constraint equations to be satisfied. Consequently, the full set of equations of motion consists of the N Gibbs-Appell equations (7.92) and the M kinematical equations (7.81). These are first-order differential equations for the N quasivelocities and the M generalized coordinates, respectively. The first-order nature of these differential equations is important, because it expedites application of numerical techniques to determine the response, as discussed in Section 7.2.

In some situations, especially those involving Coulomb friction forces, it is necessary for a constraint force to appear in the equations of motion. We address such cases by employing a set of quasicoordinates that violate only the constraint of interest, but not the others. Without loss of generality, we may consider this constraint to be numbered 1. Because we wish to violate only one constraint, we employ $N+1$ quasicoordinates. We form this set from the N unconstrained parameters we would otherwise use, supplemented by a quasicoordinate γ_{N+1} that is associated with movement in violation of the constraint. When we specialize the standard constraint equations (7.72) to the present situation, we find that

$$A_{1(N+1)}\dot{\gamma}_{N+1} + \sum_{j=1}^{N} A_{1j}(q_k, t)\dot{\gamma}_j + B_1(q_k, t) = 0. \tag{7.93}$$

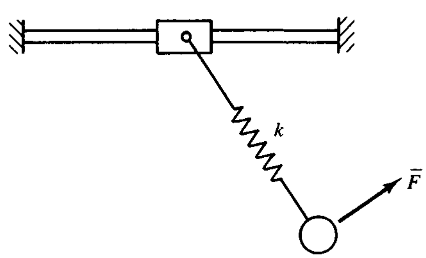
(We assume that $A_{1(N+1)}$ does not equal zero. If it did, then our choice for $\dot{\gamma}_{N+1}$ would not be suitable for the extra quasivelocity, because it would not represent a movement that violates the constraint.) The Gibbs-Appell equations in this case consist of the set of N equations (7.92), and the one for the constrained quasicoordinate,

$$\frac{\partial S}{\partial \ddot{\gamma}_{N+1}} = \Gamma_{N+1}. \tag{7.94}$$

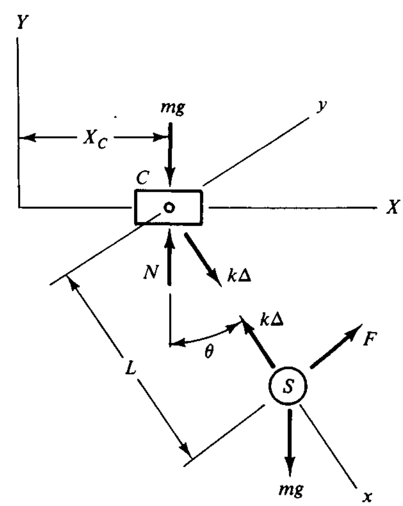
Note that the reaction associated with constraint number 1 appears only in Γ_{N+1}, but not in the generalized forces $\tilde{\Gamma}_j^{(a)}$ for $j = 1, \ldots, N$. Thus, we gain the reaction and $\dot{\gamma}_{(N+1)}$ as unknowns, which are balanced by the addition of Eqs. (7.93) and (7.94). Obviously, this approach can be extended to treat systems featuring several frictional constraints.

As a closure to the development, we shall summarize the procedure required to formulate the Gibbs-Appell equations of motion when reaction forces are not of interest. The first step is to define a set of M generalized coordinates q_i that satisfy any constraints we have identified as holonomic, as well as a set of N unconstrained quasivelocities $\dot{\gamma}_i$, where N is the number of degrees of freedom and $M-N$ is the number of nonholonomic constraints. The kinematical portion of the analysis requires derivation of Eqs. (7.81) relating the $\dot{q}_i$ to the $\dot{\gamma}_i$. The kinematical analysis must also describe, in terms of the q_i, $\dot{\gamma}_i$, and $\ddot{\gamma}_i$, all acceleration parameters required to form S. The kinetics analysis requires formation of the expression for S, and also evaluation of the generalized forces $\tilde{\Gamma}_i^{(a)}$. The latter may be determined either directly from an analysis of the virtual work, or by converting generalized forces associated with the generalized coordinates. These are the two approaches described by Eq. (7.82). In any case, all reactions may be omitted in the evaluation of generalized forces. The complete set of equations consists of the N Gibbs-Appell equations (7.92) and the M kinematical relations in Eqs. (7.81). (Advocates of the Gibbs-Appell equations, as well as the related Kane's equations, often claim that those approaches lead to a much smaller set of equations to solve. However, they fail to mention that there is an associated set of kinematical equations that also must be satisfied.) Finally, in order to simplify notation, we shall retain the tilde identifying unconstrained quasi-coordinates only when there is a possibility of confusion with constrained parameters.

Example 7.9 A small sphere is suspended by a spring of stiffness k from the collar. The collar and the sphere each have mass m, and the horizontal bar guiding the collar is smooth. A force $\bar{F}$ acting transversely to the spring is applied to the sphere, with the result that the velocity of the sphere is always parallel to the spring. Derive the Gibbs-Appell equations of motion for the system.



Example 7.9



Free-body diagrams and kinematical parameters.

Solution The position of the system may be described by the length L and angle of orientation θ of the spring, and the horizontal position X_C of the collar, so $q_1 = L$, $q_2 = \theta$, and $q_3 = X_C$. A constraint is imposed on the direction of the velocity of the sphere, so the system has two degrees of freedom. Correspondingly, two quasivelocities are unconstrained. A variety of definitions are possible; we shall employ

$$\dot{L} = \dot{\gamma}_1, \qquad \dot{\theta} = \dot{\gamma}_2. \tag{1,2}$$

Note that these definitions are system equations relating two of the generalized coordinates to the quasivelocities. The relation for $\dot{X}_C$ requires a kinematical analysis.

Let xyz be a moving reference frame whose origin coincides with collar C, and whose x axis is always collinear with the spring. Then

$$\bar{v}_C = \dot{X}_C \bar{I} = \dot{X}_C[(\sin\theta)\bar{i} + (\cos\theta)\bar{j}], \qquad \bar{\omega} = \dot{\theta}\bar{k},$$

$$\bar{r}_{S/C} = L\bar{i}, \qquad (\bar{v}_S)_{xyz} = \dot{L}\bar{i}, \qquad \bar{v}_S = \bar{v}_C + (\bar{v}_S)_{xyz} + \bar{\omega}\times\bar{r}_{S/C}.$$

Using the definitions of the quasivelocities, eqs. (1, 2), leads to

$$\bar{v}_S = (\dot{X}_C \sin\theta + \dot{\gamma}_1)\bar{i} + (\dot{X}_C\cos\theta + \dot{\gamma}_2 L)\bar{j}.$$

The y axis is normal to the spring in the plane of motion, so the constraint condition that $\bar{v}_s$ be parallel to the x axis gives

$$\bar{v}_S\cdot\bar{j} = \dot{X}_C\cos\theta + \dot{\gamma}_2 L = 0.$$

Thus, the relation between the third generalized coordinate and the quasivelocities is

$$\dot{X}_C = -\dot{\gamma}_2 \frac{L}{\cos\theta}, \tag{3}$$

from which it follows that

$$\bar{v}_S = (\dot{\gamma}_1 - \dot{\gamma}_2 L\tan\theta)\bar{i}.$$

We will require an expression for the acceleration of each body in order to form the function S. Differentiating the expression for $\dot{X}_C$ yields

$$\ddot{X}_C = -\ddot{\gamma}_2\frac{L}{\cos\theta} - \dot{\gamma}_1\dot{\gamma}_2\frac{1}{\cos\theta} - \dot{\gamma}_2^2 L\frac{\sin\theta}{\cos^2\theta}.$$

We could employ the standard relative acceleration equation to describe $\bar{a}_S$. A simpler alternative is to differentiate the expression for $\bar{v}_S$ just derived:

$$\begin{aligned}\bar{a}_S &= \frac{d}{dt}(\bar{v}_S)\\ &= \left(\ddot{\gamma}_1 - (\ddot{\gamma}_2 L + \dot{\gamma}_2\dot{L})\tan\theta - \dot{\gamma}_2 L\dot{\theta}\frac{1}{\cos^2\theta}\right)\bar{i} + (\dot{\gamma}_1 - \dot{\gamma}_2 L\tan\theta)(\bar{\omega}\times\bar{i})\\ &= \left(\ddot{\gamma}_1 - (\ddot{\gamma}_2 L + \dot{\gamma}_1\dot{\gamma}_2)\tan\theta - \dot{\gamma}_2^2 L\frac{1}{\cos^2\theta}\right)\bar{i} + (\dot{\gamma}_1\dot{\gamma}_2 - \dot{\gamma}_2^2 L\tan\theta)\bar{j}.\end{aligned}$$

The Gibbs-Appell function is

$$\begin{aligned}S &= \frac{1}{2}m\bar{a}_S\cdot\bar{a}_S + \frac{1}{2}m\bar{a}_C\cdot\bar{a}_C = \frac{1}{2}m\bar{a}_S\cdot\bar{a}_S + \frac{1}{2}m(\ddot{X}_C)^2\\ &= \frac{1}{2}m\left(\ddot{\gamma}_1 - (\ddot{\gamma}_2 L + \dot{\gamma}_1\dot{\gamma}_2)\tan\theta - \dot{\gamma}_2^2 L\frac{1}{\cos^2\theta}\right)^2\\ &\quad + \frac{1}{2}m\left(\ddot{\gamma}_2\frac{L}{\cos\theta} + \dot{\gamma}_1\dot{\gamma}_2\frac{1}{\cos\theta} + \dot{\gamma}_2^2 L\frac{\sin\theta}{\cos^2\theta}\right)^2 + \text{irrelevant terms.}\end{aligned}$$

The terms not listed in the preceding expression do not contain quasiaccelerations $\ddot{\gamma}_j$, so they cannot contribute to the equations of motion. Also, note that there is no reason to collect like terms in S before evaluating its derivatives.

We begin the evaluation of the Γ_j by considering the force $\bar{F}$. The virtual displacement of the sphere may be obtained by using the expression following eq. (3) to form $\bar{v}_s\,dt$. Replacing differentials by virtual increments yields

$$\delta\bar{r}_S = (\delta\gamma_1 - \delta\gamma_2 L\tan\theta)\bar{i}.$$

Because $\delta\bar{r}_S$ is in the $\bar{i}$ direction and $\bar{F} = F\bar{j}$, we find that $\bar{F}\cdot\delta\bar{r}_S = 0$. We should note that the same result could have been obtained much more simply: The displacements resulting from virtual increments of unconstrained quasicoordinates do not violate any constraint conditions. Therefore, $\bar{F}$, which is the reaction required to prevent the sphere from moving in the transverse direction, does no virtual work.

The forces doing virtual work are the spring and gravity, which are conservative. Let L_0 be the unstretched length of the spring, and let the datum for gravity be the elevation of the collar. Then

$$V = \tfrac{1}{2}k(L-L_0)^2 - mgL\cos\theta.$$

Because the potential energy is independent of X_C, the virtual work done by these forces is

$$\delta W = -\frac{\partial V}{\partial L}\,\delta L - \frac{\partial V}{\partial \theta}\,\delta\theta.$$

Equations (1) and (2) developed earlier correspond to Eqs. (7.81) relating the generalized and quasicoordinates. Those relations also characterize the relation between the virtual increments of both types of quantities, so we have

$$\dot{L} = \dot{\gamma}_1 \;\Rightarrow\; \delta L = \delta\gamma_1, \qquad \dot{\theta} = \dot{\gamma}_2 \;\Rightarrow\; \delta\theta = \delta\gamma_2.$$

Thus, matching the δW displayed previously to the standard form in the first of Eqs. (7.82) leads to

$$\Gamma_1 = -\frac{\partial V}{\partial L} = -k(L-L_0) + mg\cos\theta, \qquad \Gamma_2 = -\frac{\partial V}{\partial \theta} = -mgL\sin\theta.$$

Next, we form the derivatives for the Gibbs-Appell equations:

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = m\left(\ddot{\gamma}_1 - \ddot{\gamma}_2 L\tan\theta - \dot{\gamma}_1\dot{\gamma}_2\tan\theta - \dot{\gamma}_2^2 L\,\frac{1}{\cos^2\theta}\right),$$

$$\frac{\partial S}{\partial \ddot{\gamma}_2} = m\left(\ddot{\gamma}_1 - \ddot{\gamma}_2 L\tan\theta - \dot{\gamma}_1\dot{\gamma}_2\tan\theta - \dot{\gamma}_2^2 L\,\frac{1}{\cos^2\theta}\right)(-L\tan\theta)$$
$$+ m\left(\ddot{\gamma}_2\,\frac{L}{\cos\theta} + \dot{\gamma}_1\dot{\gamma}_2\,\frac{1}{\cos\theta} + \dot{\gamma}_2^2 L\,\frac{\sin\theta}{\cos^2\theta}\right)\frac{L}{\cos\theta}.$$

Equating each derivative to the corresponding Γ_j, followed by clearing $\cos\theta$ from the denominators, yields

$$\ddot{\gamma}_1\cos^2\theta - (\ddot{\gamma}_2 L + \dot{\gamma}_1\dot{\gamma}_2)\sin\theta\cos\theta - \dot{\gamma}_2^2 L + \frac{k}{m}(L-L_0)\cos^2\theta - g\cos^3\theta = 0, \quad (4)$$

$$-\ddot{\gamma}_1 L\sin\theta\cos^2\theta + (\ddot{\gamma}_2 + \dot{\gamma}_1\dot{\gamma}_2)(L\cos\theta)(1+\sin^2\theta) + 2\dot{\gamma}_2^2 L^2\sin\theta + gL\sin\theta\cos^3\theta = 0. \quad (5)$$

The full set of equations of motion comprises the first-order differential equations (1)-(5), in which the five dependent variables are L, θ, X_c, $\dot{\gamma}_1$, and $\dot{\gamma}_2$.

7.4.3 *Gibbs-Appell Function for a Rigid Body*

The definition of S, Eq. (7.91), is similar in form to that for kinetic energy, so the derivation of an expression for S in rigid-body motion will also follow a comparable approach. Indeed, we will find that S may be expressed in terms of the angular motion and angular momentum. As a preliminary to this development, it is appropriate to consider a basic aspect of the manner in which S is employed.

Because derivatives of S are taken only with respect to the quasicoordinate accelerations, any terms in S that are independent of the $\ddot{\gamma}_j$ cannot be relevant to the equations of motion. However, the quasicoordinate accelerations occur only in the physical acceleration components, as evidenced by a comparison of Eqs. (7.83) and (7.85). Therefore, any term in S that we find to depend solely on velocity parameters may be ignored.

We describe the acceleration of a point P in a rigid body in terms of the translational effect following a reference point A in that body and a rotational effect about point A. If point P locates an element of mass dm, then the infinitesimal contribution of this element to S is given by

$$\begin{aligned} dS &= \tfrac{1}{2}[\bar{a}_A+\bar{\alpha}\times\bar{r}+\bar{\omega}\times(\bar{\omega}\times\bar{r})]\cdot[\bar{a}_A+\bar{\alpha}\times\bar{r}+\bar{\omega}\times(\bar{\omega}\times\bar{r})]\,dm \\ &= \tfrac{1}{2}\{\bar{a}_A\cdot\bar{a}_A+(\bar{\alpha}\times\bar{r})\cdot(\bar{\alpha}\times\bar{r})+2\bar{a}_A\cdot(\bar{\alpha}\times\bar{r})+2\bar{a}_A\cdot[\bar{\omega}\times(\bar{\omega}\times\bar{r})] \\ &\quad +2(\bar{\alpha}\times\bar{r})\cdot[\bar{\omega}\times(\bar{\omega}\times\bar{r})]\}\,dm+\text{irrelevant terms,} \end{aligned} \tag{7.95}$$

where $\bar{\omega}$ and $\bar{\alpha}$ are the angular velocity and angular acceleration of the body, and $\bar{r}$ is the position of dm relative to reference point A. The terms marked as irrelevant do not contain derivatives of the quasivelocities.

As we did for angular momentum, we now restrict point A to be either the center of mass or the fixed point when the body is in pure rotation. In the first case, the first moment of mass (which is the integral of $\bar{r}\,dm$) vanishes, while the second case gives $\bar{a}_A=\bar{0}$. In either case, the third and fourth terms in Eq. (7.95) will make no contribution to S. The first term requires no modification. The dependence of the other terms on the angular momentum may be displayed by using the following identities for the scalar and vector triple products:

$$\bar{a}\cdot(\bar{b}\times\bar{c})\equiv\bar{b}\cdot(\bar{c}\times\bar{a})\equiv\bar{c}\cdot(\bar{a}\times\bar{b}), \tag{a}$$

$$\bar{a}\times(\bar{b}\times\bar{c})\equiv\bar{b}(\bar{a}\cdot\bar{c})-\bar{c}(\bar{a}\cdot\bar{b}). \tag{b}$$

Applying identity (a) to the second term in Eq. (7.95) leads to

$$(\bar{\alpha}\times\bar{r})\cdot(\bar{\alpha}\times\bar{r})=\bar{\alpha}\cdot[\bar{r}\times(\bar{\alpha}\times\bar{r})], \tag{7.96}$$

while the same step changes the fifth term to

$$(\bar{\alpha}\times\bar{r})\cdot[\bar{\omega}\times(\bar{\omega}\times\bar{r})]=\bar{\alpha}\cdot\{\bar{r}\times[\bar{\omega}\times(\bar{\omega}\times\bar{r})]\}.$$

To simplify this we invoke identity (b), which shows that

$$\bar{r}\times[\bar{\omega}\times(\bar{\omega}\times\bar{r})]\equiv-(\bar{\omega}\times\bar{r})(\bar{r}\cdot\bar{\omega})$$

because $\bar{r}\cdot(\bar{\omega}\times\bar{r})=0$. Let us now consider a term that resembles the left side of the foregoing, the only difference being the order in which the individual vectors occur. Applying identity (b) shows that

$$\bar{\omega}\times[\bar{r}\times(\bar{\omega}\times\bar{r})]\equiv-(\bar{\omega}\times\bar{r})(\bar{\omega}\cdot\bar{r}),$$

where the simplified form stems from the fact that $\bar{\omega}\cdot(\bar{\omega}\times\bar{r})\equiv 0$. We see that the re-arranged product yields the same result as the original, from which it follows that

$$(\bar{\alpha}\times\bar{r})\cdot[\bar{\omega}\times(\bar{\omega}\times\bar{r})]=\bar{\alpha}\cdot\{\bar{\omega}\times[\bar{r}\times(\bar{\omega}\times\bar{r})]\}. \tag{7.97}$$

The next step is to substitute Eqs. (7.96) and (7.97) into Eq. (7.95), and then to integrate over the entire body. Factoring out of the integrals those terms that are solely functions of time yields

$$S=\frac{1}{2}\bar{a}_A\cdot\bar{a}_A\iiint dm+\frac{1}{2}\bar{\alpha}\cdot\iiint\bar{r}\times(\bar{\alpha}\times\bar{r})\,dm$$
$$+\bar{\alpha}\cdot\bar{\omega}\times\iiint\bar{r}\times(\bar{\omega}\times\bar{r})\,dm. \tag{7.98}$$

The first integral is the mass of the body, while the third integral is the angular momentum $\bar{H}_A$. Furthermore, the second integral is the same as the third, except that $\bar{\alpha}$ replaces $\bar{\omega}$. Hence, the second integral reduces to the rate of change of the components of the angular momentum, which we write as $\delta\bar{H}_A/\delta t$. The final form of S is therefore

♦ $$S=\frac{1}{2}m\bar{a}_A\cdot\bar{a}_A+\frac{1}{2}\bar{\alpha}\cdot\frac{\delta\bar{H}_A}{\delta t}+\bar{\alpha}\cdot(\bar{\omega}\times\bar{H}_A), \tag{7.99}$$

where point A is the center of mass of the body, or a fixed point in a purely rotating body. A feature of primary significance in this result is that the Gibbs–Appell function for a rigid body involves the same parameters (acceleration of the center of mass, angular velocity, angular acceleration, and the inertia properties) as those arising in the Newton–Euler equations of motion.

Given the apparent simplicity of the Gibbs–Appell equations of motion, it might seem that there is no reason to use Lagrange's equations. However, each approach has certain merits. For a holonomic system, Lagrange's equations are easier to formulate. They do not require evaluation of accelerations, and the expression for kinetic energy is simpler to form than the Gibbs–Appell function. Also, the potential energy may be employed directly in Lagrange's equations. However, if the system is subject to nonholonomic constraints, solving the Lagrange equations is complicated by the algebraic manner in which the Lagrange multipliers arise in the differential equations of motion. In that case, the Gibbs–Appell approach, which avoids the appearance of constraint forces and Lagrange multipliers, becomes more attractive. A last feature is that the Gibbs–Appell differential equations often have a somewhat simpler appearance than Lagrange's equations. This improvement is attributable to the freedom to select quasicoordinates that match the kinematical aspects of a system, independent of the choice of generalized coordinates used to describe the configuration of the system.

Example 7.10 Use the Gibbs–Appell equations to derive Euler's equations of motion for a rigid body.

Solution Because we are concerned with rotational motion, we define the quasivelocities to be the components of angular velocity relative to a set of body-fixed axes: $\dot{\gamma}_1 = \omega_x$, $\dot{\gamma}_2 = \omega_y$, $\dot{\gamma}_3 = \omega_z$. It is sufficient to demonstrate the equivalence between a Gibbs-Appell equation for one γ_j and the corresponding Euler equation, because the components of $\bar{\omega}$ and $\bar{H}_A$ are symbolic permutations. Hence, we shall evaluate the equation for $\partial S/\partial \ddot{\gamma}_1$.

We begin by expressing $\bar{\omega}$ and $\bar{\alpha}$ in terms of the $\dot{\gamma}_j$:

$$\bar{\omega} = \dot{\gamma}_1 \bar{i} + \dot{\gamma}_2 \bar{j} + \dot{\gamma}_3 \bar{k}, \qquad \bar{\alpha} = \ddot{\gamma}_1 \bar{i} + \ddot{\gamma}_2 \bar{j} + \ddot{\gamma}_3 \bar{k}.$$

Note that the expression for $\bar{\alpha}$ results from the identity $\bar{\alpha} \equiv \delta\bar{\omega}/\delta t$. Euler's equations are based on xyz being principal axes, so we have

$$\bar{H}_A = I_{xx}\dot{\gamma}_1 \bar{i} + I_{yy}\dot{\gamma}_2 \bar{j} + I_{zz}\dot{\gamma}_3 \bar{k},$$

$$\frac{\delta \bar{H}_A}{\delta t} = I_{xx}\ddot{\gamma}_1 + I_{yy}\ddot{\gamma}_2 \bar{j} + I_{zz}\ddot{\gamma}_3 \bar{k}.$$

Substitution of the foregoing into Eq. (7.99) yields S in the required functional form.

A virtual rotation is the vector sum of the infinitesimal rotations $\delta\gamma_1$, $\delta\gamma_2$, and $\delta\gamma_3$ about the respective axes. Let $\bar{M}$ denote the resultant moment acting on the body. Then the virtual work is

$$\delta W = \bar{M}\cdot\overline{\delta\theta} = (\bar{M}\cdot\bar{i})\,\delta\gamma_1 + (\bar{M}\cdot\bar{j})\,\delta\gamma_2 + (\bar{M}\cdot\bar{k})\,\delta q_3,$$

which corresponds to generalized forces that are the moments about the body-fixed axes:

$$\Gamma_1 = \bar{M}\cdot\bar{i}, \qquad \Gamma_2 = \bar{M}\cdot\bar{j}, \qquad \Gamma_3 = \bar{M}\cdot\bar{k}.$$

The next step is to evaluate $\partial S/\partial \ddot{\gamma}_1$. We assume that $\bar{a}_A$ is independent of $\dot{\gamma}_1$, $\dot{\gamma}_2$, and $\dot{\gamma}_3$, in which case differentiating Eq. (7.99) yields

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = \frac{1}{2}\frac{\partial \bar{\alpha}}{\partial \ddot{\gamma}_1}\cdot\frac{\delta \bar{H}_A}{\delta t} + \frac{1}{2}\bar{\alpha}\cdot\left[\frac{\partial}{\partial \ddot{\gamma}_1}\left(\frac{\delta \bar{H}_A}{\delta t}\right)\right] + \frac{\partial \bar{\alpha}}{\partial \ddot{\gamma}_1}\cdot(\bar{\omega}\times\bar{H}_A).$$

In view of the earlier expressions, we have

$$\frac{\partial \bar{\alpha}}{\partial \ddot{\gamma}_1} = \bar{i}, \qquad \frac{\partial}{\partial \ddot{\gamma}_1}\left(\frac{\delta \bar{H}_A}{\delta t}\right) = I_{xx}\bar{i}.$$

Thus,

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = \frac{1}{2}\bar{i}\cdot\frac{\delta \bar{H}_A}{\delta t} + \frac{1}{2}\bar{\alpha}\cdot(I_{xx}\bar{i}) + \bar{i}\cdot(\bar{\omega}\times\bar{H}_A) = \Gamma_1,$$

$$I_{xx}\ddot{\gamma}_1 + (I_{zz} - I_{yy})\dot{\gamma}_2\dot{\gamma}_3 = \bar{M}\cdot\bar{i}.$$

Permutations replacing the various $\dot{\gamma}_j$ by the appropriate term ω_x, ω_y, or ω_z demonstrates the equivalence.

Example 7.11 Consider the bar in Example 7.8 under the conditions stated there. Use the Gibbs-Appell equations to derive the differential equation of motion for the angle θ, and an equation for the torque Γ.

Solution In the Gibbs-Appell formulation, generalized coordinates and quasivelocities are defined individually. Because the precession rate is constant, the precession angle at any instant is $\psi = \Omega t$. Thus, the position of the system is fully specified by the nutation angle θ, which we select as the single generalized coordinate for this one-degree-of-freedom system. A useful quasivelocity is

$$\dot{\theta} = \dot{\gamma}_1. \qquad (1)$$

Also, in order to obtain an equation for Γ, we must violate the constraint that Ω be constant. Hence, we let $\dot{\gamma}_1 = \dot{\psi}$ be the precession rate, which is subject to the constraint equation

$$\dot{\gamma}_2 = \Omega. \qquad (2)$$

Note that deriving the Gibbs-Appell equation associated with γ_2 requires that we consider $\ddot{\gamma}_2 \neq 0$ in the derivation.

We may formulate the angular momentum of the bar relative to its fixed point A. This enables us to avoid formulating the acceleration of the center of mass. The angular motion (for nonconstant $\dot{\gamma}_2$) is

$$\begin{aligned}\bar{\omega} &= \Omega \bar{K} + \dot{\theta}(-\bar{j}) = \dot{\gamma}_2 \bar{K} - \dot{\gamma}_1 \bar{j} \\ &= -(\dot{\gamma}_2 \cos\theta)\bar{i} - \dot{\gamma}_1 \bar{j} + (\dot{\gamma}_2 \sin\theta)\bar{k}, \\ \bar{\alpha} &= \ddot{\gamma}_2 \bar{K} - \ddot{\gamma}_1 \bar{j} - \dot{\gamma}_1(\bar{\omega} \times \bar{j}) \\ &= (-\ddot{\gamma}_2 \cos\theta + \dot{\gamma}_1 \dot{\gamma}_2 \sin\theta)\bar{i} - \ddot{\gamma}_1 \bar{j} + (\ddot{\gamma}_2 \sin\theta + \dot{\gamma}_1 \dot{\gamma}_2 \cos\theta)\bar{k}.\end{aligned}$$

Setting $I_{xx} = 0$ for a slender bar then gives

$$\begin{aligned}\bar{H}_A &= I_{yy}\omega_y \bar{j} + I_{zz}\omega_z \bar{k} = \frac{1}{3}ml^2[-\dot{\gamma}_1 \bar{j} + (\dot{\gamma}_2 \sin\theta)\bar{k}], \\ \frac{\delta \bar{H}_A}{\delta t} &= I_{yy}\alpha_y \bar{j} + I_{zz}\alpha_z \bar{k} \\ &= \frac{1}{3}ml^2[-\ddot{\gamma}_1 \bar{j} + (\ddot{\gamma}_2 \sin\theta + \dot{\gamma}_1 \dot{\gamma}_2 \cos\theta)\bar{k}].\end{aligned}$$

From these expressions, we obtain

$$\begin{aligned}S &= \frac{1}{2}\bar{\alpha} \cdot \frac{\delta \bar{H}_A}{\delta t} + \bar{\alpha} \cdot (\bar{\omega} \times \bar{H}_A) \\ &= \frac{1}{2}\left(\frac{1}{3}ml^2\right)[\ddot{\gamma}_1^2 + (\ddot{\gamma}_2 \sin\theta + \dot{\gamma}_1 \dot{\gamma}_2 \cos\theta)^2] \\ &\quad + \left(\frac{1}{3}ml^2\right)[-\ddot{\gamma}_1 \dot{\gamma}_2^2 \sin\theta\cos\theta + (\ddot{\gamma}_2 \sin\theta)(\dot{\gamma}_1 \dot{\gamma}_2 \cos\theta)] + \text{irrelevant terms.}\end{aligned}$$

The constraints imposed by the bearing force $\bar{F}$ and couple $\bar{M}$ are not violated, so virtual work is done only by the applied torque and gravity. For a datum at the elevation of pin A, the potential energy as a function of the generalized coordinate is

$$V = -\tfrac{1}{2}mgl\cos\theta.$$

Hence, the virtual work is

$$\delta W = \Gamma\,\delta\gamma_2 - \frac{\partial V}{\partial \theta}\frac{\partial \dot{\theta}}{\partial \dot{\gamma}_1}\,\delta\gamma_1 = \Gamma\,\delta\gamma_2 - \frac{1}{2}mgl\sin\theta\,\delta\gamma_1.$$

The corresponding generalized forces are

$$\Gamma_1 = -\tfrac{1}{2}mgl\sin\theta, \qquad \Gamma_2 = \Gamma.$$

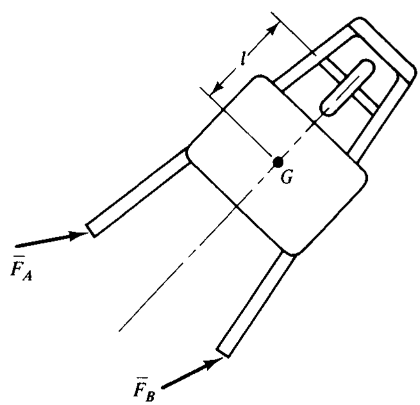
The Gibbs-Appell equations are

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = \frac{1}{3}ml^2(\ddot{\gamma}_1 - \dot{\gamma}_2^2 \sin\theta\cos\theta) = -\frac{1}{2}mgl\sin\theta, \tag{3}$$

$$\frac{\partial S}{\partial \ddot{\gamma}_2} = \frac{1}{3}ml^2(\ddot{\gamma}_2 \sin\theta + 2\dot{\gamma}_1\dot{\gamma}_2 \sin\theta\cos\theta) = \Gamma. \tag{4}$$

In combination with eqs. (1) and (2), these equations are identical to the results in Example 7.8.

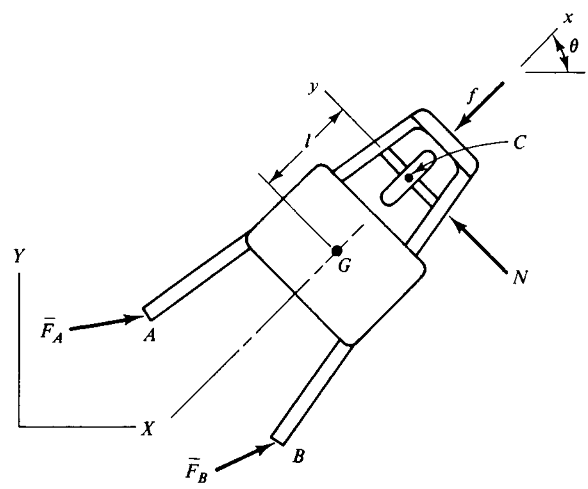
Example 7.12 The wheelbarrow is pushed in the horizontal plane by forces $\bar{F}_A$ and $\bar{F}_B$ acting at each handle. The body of the wheelbarrow has mass m_1 with its center of mass at point G; the corresponding centroidal moment of inertia about a vertical axis is I. The wheel, which rolls without slipping, has mass m_2, moment of inertia J about its axle, and radius r. Derive the Gibbs-Appell equations of motion.



Example 7.12

Solution We assume that the wheelbarrow remains upright. Then, because the wheel rolls without slipping, we know that the angular velocity component of the wheel about its bearing axis is $\omega_1 = v/r$, where v is the speed of the center C of the axle. Also, because there is no slippage, the velocity of point C must be along the longitudinal axis, which we define as x. Convenient generalized coordinates for the wheelbarrow are the absolute position coordinates of point C, $q_1 = X_C$ and $q_2 = Y_C$, and the angle θ locating the x axis. For the wheel, we let $q_4 = \phi$, the angle of spin about the axle.

We let v be a quasivelocity, $\dot{\gamma}_1 = v$, because that expedites the description of the constraint on the movement of point C. The other quasivelocity we define is $\dot{\gamma}_2 = \dot{\theta}$. Note that although four generalized coordinates have been defined, there are two constraint conditions due to rolling without slipping. Therefore, the system has only two degrees of freedom, and only two quasivelocities are unconstrained.



Free-body diagram.

We relate the $\dot{q}_i$ to the $\dot{\gamma}_i$ by describing the constraint conditions. By definition,

$$\dot{\theta} = \dot{\gamma}_2. \tag{1}$$

The constraint on the velocity of point C is

$$\bar{v}_C = \dot{X}_C \bar{I} + \dot{Y}_C \bar{J} = v\bar{i} = v[(\cos\theta)\bar{I} + (\sin\theta)\bar{J}],$$

so

$$\dot{X}_C = \dot{\gamma}_1 \cos\theta, \qquad \dot{Y}_C = \dot{\gamma}_1 \sin\theta. \tag{2, 3}$$

Also, for rolling without slipping, $v = r\dot{\phi}$, which yields

$$\dot{\phi} = \dot{\gamma}_1 / r. \tag{4}$$

The next task is to describe the acceleration variables required to form S. For the center of mass G, we use

$$\bar{a}_G = \bar{a}_C + \bar{\alpha} \times \bar{r}_{G/C} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{G/C}),$$

where $\bar{\omega}$ and $\bar{\alpha}$ pertain to the xyz reference frame, which is fixed to the wheelbarrow. For the set of variables just defined, we have $\bar{\omega} = \dot{\theta}\bar{k} = \dot{\gamma}_2 \bar{k}$ and $\bar{\alpha} = \ddot{\gamma}_2 \bar{k}$. The most direct method of characterizing $\bar{a}_C$ in terms of the quasicoordinates is to recall the expression $\bar{v}_C = v\bar{i} = \dot{\gamma}_1 \bar{i}$. This is true at any time-instant, and therefore may be differentiated:

$$\bar{a}_C = \frac{d}{dt}(\dot{\gamma}_1 \bar{i}) = \ddot{\gamma}_1 \bar{i} + \dot{\gamma}_1 (\bar{\omega} \times \bar{i}) = \ddot{\gamma}_1 \bar{i} + \dot{\gamma}_1 \dot{\gamma}_2 \bar{j}.$$

We substitute these expressions and $\bar{r}_{G/C} = -l\bar{i}$ into the foregoing equation for $\bar{a}_G$, which yields

$$\bar{a}_G = (\ddot{\gamma}_1 + \dot{\gamma}_2^2 l)\bar{i} + (-\ddot{\gamma}_2 l + \dot{\gamma}_1 \dot{\gamma}_2)\bar{j}.$$

In order to express the contribution of the wheel to S, we must describe its angular motion. If the wheelbarrow does not tilt, then the wheel precesses at $\dot{\theta}$ about the vertical axis while it spins about its axle at $\dot{\phi}$. Thus

$$\bar{\omega}_w = \frac{1}{r}\dot{\gamma}_1\bar{j} + \dot{\gamma}_2\bar{K} = \frac{1}{r}\dot{\gamma}_1\bar{j} + \dot{\gamma}_2\bar{k},$$

$$\bar{\alpha}_w = \frac{1}{r}[\ddot{\gamma}_1\bar{j} + \dot{\gamma}_1(\bar{\omega}_w \times \bar{j})] + \ddot{\gamma}_2\bar{k} = -\frac{1}{r}\dot{\gamma}_1\dot{\gamma}_2\bar{i} + \frac{1}{r}\ddot{\gamma}_1\bar{j} + \ddot{\gamma}_2\bar{k}.$$

Now that the kinematical parameters have been characterized, we form the angular momentum properties. For the wheelbarrow, we have

$$\bar{H}_G = I\dot{\gamma}_2\bar{k}, \qquad \frac{\delta\bar{H}_G}{\delta t} = I\ddot{\gamma}_2\bar{k}.$$

We consider the wheel to be very thin, so its moment of inertia about a centroidal in-plane axis is $\frac{1}{2}J$. Thus

$$\bar{H}_C = \frac{J}{r}\dot{\gamma}_1\bar{j} + \frac{J}{2}\dot{\gamma}_2\bar{k}, \qquad \frac{\delta\bar{H}_C}{\delta t} = -\frac{J}{2r}\dot{\gamma}_1\dot{\gamma}_2\bar{i} + \frac{J}{r}\ddot{\gamma}_1\bar{j} + \frac{J}{2}\ddot{\gamma}_2\bar{k}.$$

The Gibbs-Appell function for the system is the sum of the contributions of each body:

$$\begin{aligned} S &= \frac{1}{2}m\bar{a}_G\cdot\bar{a}_G + \frac{1}{2}\bar{\alpha}\cdot\frac{\delta\bar{H}_G}{\delta t} + \bar{\alpha}\cdot(\bar{\omega}\times\bar{H}_G) \\ &\quad + \frac{1}{2}m\bar{a}_C\cdot\bar{a}_C + \frac{1}{2}\bar{\alpha}_w\cdot\frac{\delta\bar{H}_C}{\delta t} + \bar{\alpha}_w\cdot(\bar{\omega}_w\times\bar{H}_C) \\ &= \frac{1}{2}m_1[(\ddot{\gamma}_1 + \dot{\gamma}_2^2 l)^2 + (-\ddot{\gamma}_2 l + \dot{\gamma}_1\dot{\gamma}_2)^2] + \frac{1}{2}I\ddot{\gamma}_2^2 + \frac{1}{2}m_2\ddot{\gamma}_1^2 \\ &\quad + \frac{1}{2}J\left(\frac{1}{r^2}\ddot{\gamma}_1^2 + \frac{1}{2}\ddot{\gamma}_2^2\right) + \text{irrelevant terms.} \end{aligned}$$

Note that the terms associated with a gyroscopic moment for the wheel do not appear in S because the system has been constrained to move in the horizontal plane. Momentum effects tending to cause the wheelbarrow to tilt are balanced by reactions, which are the out-of-plane components of the forces $\bar{F}_A$ and $\bar{F}_B$.

The next step is to evaluate the virtual work. The reaction forces $\bar{f}$ and $\bar{N}$ exerted on the wheel by the ground do no virtual work. Their role is to prevent the wheel from slipping and from penetrating the ground, and those constraints are not violated. To characterize the effect of the applied forces $\bar{F}_A$ and $\bar{F}_B$, we recall that we have already described the velocity of point C. We therefore replace $\bar{F}_A$ and $\bar{F}_B$ by a force-couple system, $\bar{R}$ and $\bar{M}$, acting at point C:

$$\bar{R} = \bar{F}_A + \bar{F}_B, \qquad \bar{M} = \bar{r}_{A/C}\times\bar{F}_A + \bar{r}_{B/C}\times\bar{F}_B.$$

The analogy between differential and virtual displacements leads to

$$d\bar{r}_C = \bar{v}_C\,dt = d\gamma_1\bar{i} \;\Rightarrow\; \delta\bar{r}_C = \delta\gamma_1\bar{i},$$

$$\overline{d\theta} = \dot{\theta}\,dt\,\bar{k} = d\gamma_2\bar{k} \;\Rightarrow\; \overline{\delta\theta} = \delta\gamma_2\bar{k},$$

so the virtual work associated with the forces applied to the handles is

$$\delta W = \bar{R}\cdot\delta\bar{r}_C + \bar{M} + \overline{\delta\theta} = \Gamma_1\,\delta\gamma_1 + \Gamma_2\,\delta\gamma_2.$$

Matching the two descriptions of δW yields the generalized forces,

$$\Gamma_1 = \bar{R}\cdot\bar{i}, \qquad \Gamma_2 = \bar{M}\cdot\bar{k}.$$

The Gibbs–Appell equations resulting from the expression for S are

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = \left(m_1 + m_2 + \frac{J}{r^2}\right)\ddot{\gamma}_1 + m_1 l \dot{\gamma}_2^2 = \bar{R}\cdot\bar{i}, \tag{5}$$

$$\frac{\partial S}{\partial \ddot{\gamma}_2} = \left(m_1 l^2 + I + \frac{1}{2}J\right)\ddot{\gamma}_2 - m_1 l \dot{\gamma}_1 \dot{\gamma}_2 = \bar{M}\cdot\bar{k}. \tag{6}$$

In combination with eqs. (1)–(4), which relate the generalized and quasivelocities, we have derived six first-order differential equations for the unknowns X_C, Y_C, θ, ϕ, $\dot{\gamma}_1 = v$, and $\dot{\gamma}_2 = \dot{\theta}$.

7.5 Linearization

Once we have obtained the differential equations of motion, our next task is to solve them for the dynamic response. We are likely to encounter two sources of difficulty in that effort. Clearly, if the system contains many independently moving pieces, then the number of equations will be rather large. In addition, the equations might be complicated, particularly if the geometrical details of the system are intricate. The combination of these features often makes it necessary to devote a major effort to the process of solving the equations of motion.

Of course, one could employ numerical techniques, as discussed in Section 7.2. However, little is known about the response when a system is studied for the first time. It might then be adequate to simplify the equations of motion. A broad class of systems feature vibratory responses, in which displacements oscillate about a reference state. We focus our attention on situations where such displacements are a small fraction of the overall dimensions of the system. In that case, we may simplify the equations of motion through the process of linearization.

For a system whose physical constraints are independent of time, a suitable reference state would be the static equilibrium position. When the system's constraints are time-dependent, two cases where linearization might be useful commonly arise. First, suppose we are able to identify a steady-state solution of the nonlinear equations of motion. It is possible that such a solution cannot actually occur because it represents a dynamically unstable response. A common way of examining the stability of a steady-state response is to use that response as the reference state for a linearization of the equations of motion. Another case where linearization is useful occurs when the time-dependent aspects of a constraint condition correspond to a small imposed motion of a support, such as the displacement of a pin. Then we could consider the reference state to be the static equilibrium position that the system would occupy if the support were fixed.

Let us denote as q_i^* the generalized coordinates evaluated at the reference state. For the sake of simplicity, we shall address only holonomic systems that are described by unconstrained generalized coordinates. Applying Lagrange's equations to the reference state then yields

◆ $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)^* - \left(\frac{\partial T}{\partial q_i}\right)^* + \left(\frac{\partial V}{\partial q_i}\right)^* = Q_i^*, \quad i = 1, \ldots, M, \tag{7.100}$$

where an asterisk here denotes terms that are evaluated at the reference state, at which $q_i = q_i^*$. Note that the q_i^* will be functions of time if the kinematical features of the system depend explicitly on time. Determination of the q_i^* requires simultaneous solution of Eq. (7.100) for all i.

In a linearized analysis, we restrict our attention to those cases where the displacements relative to the reference state are very small. We measure such displacements by a set of *relative generalized coordinates* ξ_i, such that

$$\xi_i = q_i - q_i^* \;\Rightarrow\; q_i = \xi_i + q_i^* \;\&\; \dot{q}_i = \dot{\xi}_i + \dot{q}_i^*. \tag{7.101}$$

There are essentially two approaches whereby linear equations of motion for the ξ_i may be obtained. The first, and most reliable, operates directly on the equations of motion. Let F be a term in an equation of motion that is an arbitrary function of q_i, $\dot{q}_i$, $\ddot{q}_i$, and t. We employ Eqs. (7.101) to replace this dependence by an equivalent dependence on the ξ_i. Assuming that the function F is analytic, a Taylor series yields

$$\begin{aligned} F(q_i, \dot{q}_i, \ddot{q}_i, t) &= F(q_i^* + \xi_i, \dot{q}_i^* + \dot{\xi}_i, \ddot{q}_i^* + \ddot{\xi}_i, t) \\ &= F^* + \sum_{j=1}^{M}\left[\left(\frac{\partial F}{\partial q_j}\right)^* \xi_j + \left(\frac{\partial F}{\partial \dot{q}_j}\right)^* \dot{\xi}_j + \left(\frac{\partial F}{\partial \ddot{q}_j}\right)^* \ddot{\xi}_j\right] + \cdots. \end{aligned} \tag{7.102}$$

As before, an asterisk indicates quantities to be evaluated at the reference state, where $q_i = q_i^*$. Thus:

◆ *One method by which the equations of motion may be linearized is to substitute $q_i = q_i^* + \xi_i$ into those equations, and then to truncate at linear terms a Taylor series of each term in which ξ_i appears.*

Truncation of a series in this manner implies that quadratic products of the ξ_i are negligible (in a nondimensional sense) in comparison to the variables. Technically, this is only true if all ξ_j are infinitesimal. Indeed, linearized equations of motion are often said to constitute an *infinitesimal displacement theory.* There are numerous situations, however, where linearized equations of motion accurately describe the system response for very substantial displacements.

In order to illustrate this process, consider the pendulum in Figure 7.3, whose cable is elastic with a linear stiffness k and an unstretched length L_0. The nonlinear equations of motion for the system are

$$m\ddot{R} - mR\dot{\theta}^2 + k(R - L_0) - mg\cos\theta = 0,$$

$$mR\ddot{\theta} + 2m\dot{R}\dot{\theta} + mg\sin\theta = 0.$$

The equilibrium positions of the system are obtained by solving these equations with all time derivatives set to zero. The stable position is

$$R^* = L_0 + mg/k, \qquad \theta^* = 0.$$

For the linearization, we let $R = R^* + \xi_1$ and $\theta = \xi_2$. Because $\xi_2 \ll 1$, the series for the trigonometric functions may be truncated as

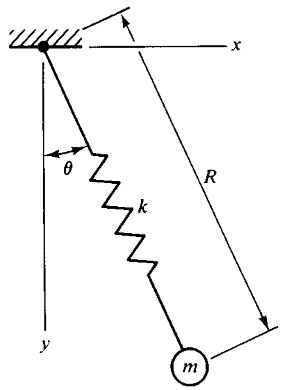


Figure 7.3 Elastically supported pendulum.

$$\sin\theta = \sin\xi_2 \approx \xi_2, \qquad \cos\theta = \cos\xi_2 \approx 1 - \tfrac{1}{2}\xi_2^2 \approx 1.$$

In addition, nonlinear terms arise in the accelerations in each equation of motion. When all terms that contain products of the ξ_i are dropped, the result is

$$m\ddot{\xi}_1 + k\xi_1 = 0, \qquad R^*\ddot{\xi}_2 + g\xi_2 = 0.$$

It is clear that the linearized equations of motion for this system are substantially easier to solve than the original nonlinear equations.

The process of directly linearizing the equations of motion is straightforward. However, it introduces the simplifications of linearization subsequent to the application of Lagrange's equations. Consequently, this approach requires derivation of energy expressions that are more accurate than necessary. Let us therefore consider the alternative of deriving simplified energy expressions based on a kinematical restriction to small displacements. We shall explore this method only for the important case of scleronomic systems. Time-dependent systems are best linearized after the equations of motion have been derived. The kinetic energy of such systems has a much more complicated form, which enhances the likelihood of inadvertently omitting an important effect. Also, if one seeks to identify a steady reference state in a time-dependent system, it will be necessary in any event to formulate the full set of equations of motion.

When the system is scleronomic, the reference state of interest will be a static equilibrium position, so that the q_k^* represent constant values. Correspondingly, we may identify the reference state by solving Eq. (7.100) with $T \equiv 0$. (Equation (7.100) in this case reduces to the static principle of virtual work and potential energy.) Thus, linearization of a scleronomic system entails the substitutions $q_k = q_k^* + \xi_k$, $\dot{q}_k = \dot{\xi}_k$, and $\ddot{q}_k = \ddot{\xi}_k$.

Our analysis of the mechanical energies appropriate to a linearization of the equations of motion rests on a single observation. The kinetic and potential energies appear in Lagrange's equations as first derivatives with respect to the generalized coordinates. Consequently, quadratic and higher terms in the equations of motion may be avoided by neglecting cubic and higher terms in the energies. In other words:

♦ *A linearized set of equations of motion may be obtained by retaining in the kinetic and potential energy expressions only those terms that are either* *quadratic or linear in generalized coordinates measured relative to the static equilibrium configuration.*

The potential energy of a scleronomic system is independent of time. A Taylor series expansion relative to the static equilibrium configuration leads to

$$V(q_k^* + \xi_k) = V^* + \sum_{i=1}^{M} \left(\frac{\partial V}{\partial q_i}\right)^* \xi_i + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} K_{ij} \xi_i \xi_j, \tag{7.103}$$

where the constants K_{ij} are the second derivatives evaluated at the reference state,

$$K_{ij} = \left(\frac{\partial^2 V}{\partial q_i \, \partial q_j}\right)^*, \quad K_{ij} = K_{ji}, \; i, j = 1, \ldots, M. \tag{7.104}$$

These constants are commonly known as the *stiffness coefficients.* The reason for this term is obvious when one considers the analogy between the quadratic terms in Eq. (7.103) and the potential energy for a spring, $V = \frac{1}{2} k \Delta^2$.

Our treatment of kinetic energy begins by recalling from Eqs. (7.7) that the kinetic energy of a scleronomic system is quadratic in the generalized velocities. Because $\dot{q}_k = \dot{\xi}_k$ when the reference state is a static equilibrium position, we have

$$T = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} M_{ij} \dot{\xi}_i \dot{\xi}_j. \tag{7.105}$$

In a general situation, the coefficients M_{ij} can depend on the generalized coordinates. However, the preceding expression for T is already quadratic in the $\dot{\xi}_k$. If we allow these coefficients to be variables, their Taylor series expansions in powers of the ξ_i would cause cubic and higher-order terms to appear in T. Consequently, we should evaluate the coefficients M_{ij} at the equilibrium position. We therefore have

$$T = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} M_{ij}^* \dot{\xi}_i \dot{\xi}_j. \tag{7.106}$$

The constants M_{ij}^* are known as *inertia coefficients,* analogous to the terminology for the K_{ij}.

As a corollary of evaluating the coefficients M_{ij}^* at the reference position, we observe that:

- ♦ *When a system is scleronomic, the kinematical relationship between the linear and angular velocities and the generalized velocities may be evaluated using the geometrical configuration at the static equilibrium position.*

The ability to perform a kinematical analysis based on the geometry of the static equilibrium position can substantially simplify the determination of linearized equations. For example, relations between the velocities of points may be formulated using the relative position vector at the equilibrium position, rather than at a general position.

Linearization procedures for the potential energy usually do not admit the kinematical simplifications available for the kinetic energy of a scleronomic system. Let us refer back to Eq. (7.103) in order to identify the reason for this difference. The parameter V^* is the potential energy at the equilibrium position. That quantity is unimportant, because only derivatives of V occur in Lagrange's equations. Also, the term $(\partial V / \partial q_i)^*$ must have been evaluated already if the reference state is known; see

Eq. (7.100). Therefore, we need only evaluate the stiffness coefficients K_{ij} in order to characterize the potential energy function for a linearized analysis. The difficulty arises in the description of the geometric parameters required to form these coefficients, as may be demonstrated by considering a one-degree-of-freedom system that contains a linear spring. Suppose that the spring is stretched by an amount Δ^* when the system is at its static equilibrium position. Then, expanding the spring length in a Taylor series in the relative displacement ξ leads to a power series for the total elongation Δ,

$$\Delta \approx \Delta^* + c_1\xi + c_2\xi^2,$$

where c_1 and c_2 depend on the details of the system's configuration. The potential energy corresponding to this expression is

$$V = \tfrac{1}{2}k(c_1^2 + 2\Delta^* c_2)\xi^2 + \text{nonquadratic terms.}$$

Note that c_2 is the coefficient of the quadratic term in the series expansion for Δ. This simple example leads us to the following general conclusion.

♦ *For the derivation of linearized equations of motion, the elongation Δ of each spring should be expanded in a Taylor series that retains quadratic terms in the relative displacements ξ_i. The elongation Δ may be truncated with certainty at linear terms in the ξ_i only if all springs are unstretched at the static equilibrium position.*

We should mention that the foregoing is a subtlety that is often forgotten, because there are many systems in which Δ may be linearized even though the various Δ^* are nonzero. Example 7.13 will treat a system where the higher-order terms are required.

It is equally important to retain higher-order terms when we treat other types of conservative forces. For example, consider the potential energy of the gravitational attraction on the bar in Figure 7.4. Let θ^* denote the angle of inclination of the bar at the static equilibrium position. (A nonzero value of θ^* may be obtained by attaching springs to the bar, or by letting the bar be part of a linkage.) For a datum at the elevation of the pivot, the potential energy is

$$V = -\tfrac{1}{2}mgL\cos(\theta^* + \xi) = -\tfrac{1}{2}mgL(\cos\theta^*\cos\xi - \sin\theta^*\sin\xi).$$

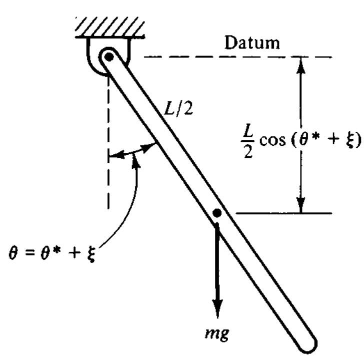


Figure 7.4 Linearization parameters.

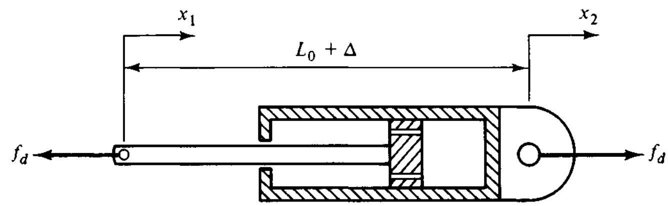


Figure 7.5 Dashpot.

We expand the cosine and sine terms in powers of ξ, and drop nonquadratic terms, with the result that

$$V = \tfrac{1}{4} mgL(\cos\theta^*)\xi^2 + \text{nonquadratic terms.}$$

In this case, it is the height of the center of mass G relative to the datum that must be expanded in a Taylor series that retains quadratic terms. Only when the bar is horizontal in the static equilibrium position, $\theta^* = 90°$, does gravity have a purely static effect.

A different aid is available when it is necessary to account for the effects of energy dissipation in a linearized analysis. Dissipation mechanisms may be modeled by *dashpots,* such as the one depicted schematically in Figure 7.5. The most common use of this device is as a shock absorber in an automobile. The magnitude of the force exerted by a dashpot depends on the rate of change of its elongation, as well as on the elongation itself in some designs; $f_d = f_d(\Delta, \dot{\Delta})$. The direction of the forces exerted at each end are tensile if the elongation is increasing, that is, if $\dot{\Delta} > 0$.

When the dashpot force is linearized, the force it exerts is taken to be proportional to its elongation rate,

$$f_d = \mu\dot{\Delta}, \tag{7.107}$$

where the coefficient μ is the *dashpot constant,* whose dimensions are FT/L. This relation has obvious similarities to the relation between elongation and force for a linear spring. The only difference is the proportionality to the elongation rate, rather than the elongation itself.

We exploit the analogy between the forces exerted by a spring and a dashpot by defining the *Rayleigh dissipation function D.* Just as the potential energy in a spring is $\frac{1}{2}k\Delta^2$, we define

$$D = \tfrac{1}{2}\mu\dot{\Delta}^2. \tag{7.108}$$

Because the relations for a linear dashpot differ from those for a linear spring only by the presence of a time derivative of the elongation, the generalized force associated with the dashpot is obtained by a differentiation with respect to the generalized velocity,

$$(Q_d)_i = -\frac{\partial D}{\partial \dot{q}_i}. \tag{7.109}$$

When the system contains several dashpots, the dissipation function is the sum of the effects associated with each,

♦ $$D = \frac{1}{2}\sum_i \mu_i \dot{\Delta}_i^2. \tag{7.110}$$

The dissipation function is related to the instantaneous power absorbed by the dashpot. For the case depicted in Figure 7.5, the elongation is $\Delta = x_2 - x_1$. The dashpot force is tensile, corresponding to an elongation that is increasing, $\dot{x}_2 > \dot{x}_1$. Thus, power is input to the dashpot at the right end at the rate $f_d \dot{x}_2$, and is expended at the left at the rate $f_d \dot{x}_1$. The net power input is

$$P_d = f_d \dot{x}_2 - f_d \dot{x}_1 = \mu(\dot{x}_2 - \dot{x}_1)^2 = 2D. \tag{7.111}$$

The power input to the dashpot accumulates as energy that is lost from the system. Thus, we may conclude that the instantaneous rate at which mechanical energy is lost from a system due to linear dashpots is twice the Rayleigh dissipation function.

The Rayleigh dissipation function may be used to write Lagrange's equations in a modified form. Let $Q_i^{(a)}$ now denote the generalized force associated with given applied forces that are not included in the potential energy or dissipation function. It follows from Eq. (7.109) that the equations of motion may be written as

♦ $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i^{(a)}, \quad i = 1, 2, \ldots, M. \tag{7.112}$$

It must be emphasized that *the Rayleigh dissipation function is valid solely for linear dashpot models.* In systems where the dashpot force does not depend linearly on the rate of elongation, the generalized forces associated with the dashpot must be evaluated from the virtual work, in the usual manner; Eq. (7.112) is not valid in that case.

When a system is scleronomic, the dissipation function will be a quadratic sum of the generalized velocities,

$$D = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} D_{ij} \dot{\xi}_i \dot{\xi}_j, \quad D_{ij} = D_{ji}, \tag{7.113}$$

where the coefficients D_{ij} are *damping constants.* This has the same form as the quadratic terms in Eq. (7.103) for the potential energy, and in Eq. (7.106) for the kinetic energy. Let us examine the corresponding form of the linearized equations of motion. Because the ξ_i differ from the original generalized coordinates only by the constant q_i^*, it must be that $\delta q_i = \delta \xi_i$. The consequence of the equivalence of the virtual increments of the two sets of generalized coordinates is that the generalized forces are the same: $Q_i^{(a)}$ for either set.

The representation of T, V, and D as quadratic sums leads to a standard form for the linearized equations of motion. Let ξ_n be a specified generalized coordinate. The details of the derivative of a quadratic sum were treated previously in Eq. (7.9a). We apply the same procedure to Eqs. (7.103), (7.106), and (7.113) to find

$$\frac{\partial T}{\partial \dot{\xi}_n} = \sum_{j=1}^{M} M_{nj}^* \dot{\xi}_j, \quad \frac{\partial D}{\partial \dot{\xi}_n} = \sum_{j=1}^{M} D_{nj} \dot{\xi}_j, \quad \frac{\partial V}{\partial \xi_n} = \sum_{j=1}^{M} K_{nj} \xi_j + \left(\frac{\partial V}{\partial q_n}\right)^*. \tag{7.114}$$

Also, because the kinetic energy of a scleronomic system after linearization does not depend on the relative generalized coordinates, we have $\partial T / \partial \xi_n = 0$. When we substitute the derivatives in Eqs. (7.114) into the modified Lagrange equations, Eq. (7.112), we obtain

$$\sum_{j=1}^{M} M_{nj}^* \ddot{\xi}_j + \sum_{j=1}^{M} D_{nj} \dot{\xi}_j + \sum_{j=1}^{M} K_{nj} \xi_j + \left(\frac{\partial V}{\partial q_n}\right)^* = Q_n^{(a)}, \quad n = 1, \ldots, M. \tag{7.115}$$

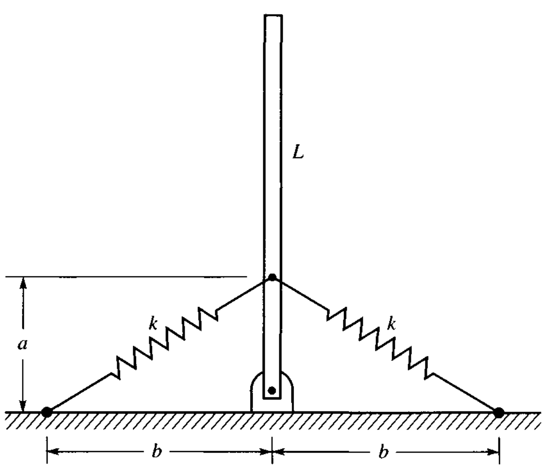
These equations may be written more compactly in matrix form. Let ξ_i be the ith element of $\{\xi\}$, and let M_{ij}^*, D_{ij}, and K_{ij} be the elements of (respectively) the mass, dissipation, and stiffness arrays, $[M]$, $[D]$, and $[K]$. Then Eqs. (7.115) are equivalent to

$$\blacklozenge \quad [M]\{\ddot{\xi}\}+[D]\{\dot{\xi}\}+[K]\{\xi\}=\{Q^{(a)}\}-\left\{\frac{\partial V}{\partial q}\right\}^*. \tag{7.116}$$

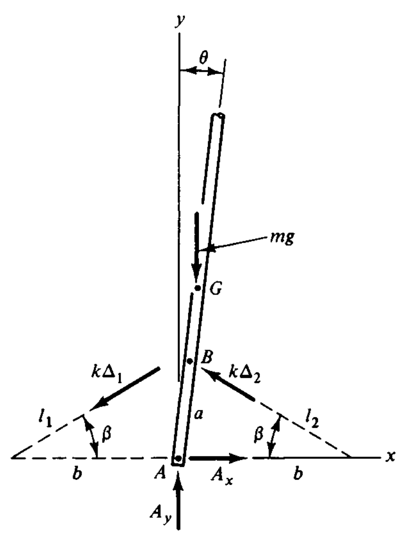
In this expression, $\{Q^{(a)}\}$ is the set of active generalized forces, excluding the conservative and dissipative forces whose effects appear in the potential energy and dissipation function, respectively. Also, $\{\partial V/\partial q\}^*$ is the set of static external forces that establish the equilibrium position. Thus, the right side of Eq. (7.116) represents the set of nondissipative generalized forces that tend to move the system away from its static equilibrium position, at which $\{\xi\}=\{0\}$.

Many methods are available for solving these standard equations of motion for a scleronomic system, as they have constant coefficients. Study of the responses they describe forms a major portion of texts on linear vibrations. It must be emphasized, however, that Eq. (7.116) is valid only for analysis of a scleronomic system. If the physical constraints of a system are time-dependent, then there will be additional terms due to the more general form of the kinetic energy. As mentioned earlier, it is preferable with such systems to derive equations of motion without approximations, and then to linearize those equations. Also, if there are nonholonomic constraint conditions to be satisfied, then the differential equations of motion must be modified to include either the constraint forces or the Lagrange multipliers. The system of equations for such systems must be supplemented by constraint equations, which also may be linearized.

Example 7.13 The bar is in static equilibrium in the upright position shown. The unstretched length of the identical springs is $l_0 < (a^2+b^2)^{1/2}$, in order that the springs remain taut throughout any small displacement away from this position. Derive the corresponding equation of motion.



Example 7.13



Free-body diagram.

Solution We select as the generalized coordinate the angle θ relative to the vertical. Because $\theta = 0$ at the static equilibrium position, this variable serves directly as a relative generalized coordinate. The bar is in pure rotation about end A, so the kinetic energy is

$$T = \tfrac{1}{2}(\tfrac{1}{3}mL^2)\dot{\theta}^2.$$

This expression needs no simplification, because it is quadratic in $\dot{\theta}$.

We select the elevation of the pin as the datum for the gravitational potential energy. The springs also are conservative, so

$$V = \tfrac{1}{2}k\Delta_1^2 + \tfrac{1}{2}k\Delta_2^2 + \tfrac{1}{2}mgL\cos\theta.$$

In order to obtain some general properties regarding the potential energy of a spring, we shall consider the springs individually. We use the law of cosines to describe the length of each spring at an arbitrary position in terms of the generalized coordinate:

$$l_1 = [a^2+b^2-2ab\cos(90°+\theta)]^{1/2} = [a^2+b^2+2ab\sin\theta]^{1/2},$$
$$l_2 = [a^2+b^2-2ab\cos(90°-\theta)]^{1/2} = [a^2+b^2-2ab\sin\theta]^{1/2}.$$

We need terms only up to quadratic in the elongation, so we drop cubic and higher-degree terms in the expansion of the sine, as well as in the binomial expansion of the square root. The resulting elongation is

$$\begin{aligned}\Delta_1 &= l_1 - l_0\\ &\approx [(a^2+b^2)^{1/2} - l_0] + ab\theta(a^2+b^2)^{-1/2} - \tfrac{1}{2}(ab\theta)^2(a^2+b^2)^{-3/2},\\ \Delta_2 &\approx [(a^2+b^2)^{1/2} - l_0] - ab\theta(a^2+b^2)^{-1/2} - \tfrac{1}{2}(ab\theta)^2(a^2+b^2)^{-3/2}.\end{aligned}$$

Some terms cancel when we form Δ_1^2 and Δ_2^2, with the result that the corresponding potential energy of each spring is

$$V_1 = V_1^* + kab\theta\left[1 - \frac{l_0}{(a^2+b^2)^{1/2}}\right] + \frac{1}{2}\frac{ka^2b^2l_0}{(a^2+b^2)^{3/2}}\theta^2,$$
$$V_2 = V_2^* - kab\theta\left[1 - \frac{l_0}{(a^2+b^2)^{1/2}}\right] + \frac{1}{2}\frac{ka^2b^2l_0}{(a^2+b^2)^{3/2}}\theta^2,$$

where we have dropped the higher-order terms. When we add the potential energy in each spring, we find that the terms that are linear in θ cancel. This is a consequence of the fact that the linear terms characterize the static forces affecting the equilibrium position, $\theta = 0$, because only they give rise to a nonzero value of $(\partial V/\partial\theta)^*$. The static forces in each spring cancel. We may also consider V_1^* and V_2^* to be zero. An interesting form of the total potential energy of the springs results from expressing the horizontal distance b in terms of the angle of elevation β, from which we find that

$$V_1 + V_2 = \frac{kl_0(a\cos\beta)^2}{(a^2+b^2)^{1/2}}\theta^2.$$

In the small-displacement approximation, $a\theta$ is the horizontal movement of point B. Then $a\theta\cos\beta$ is the component of this displacement parallel to either spring, which is the same as the linearized change in the length of either spring relative to its length at the equilibrium position. This observation is illustrative of the following general property.

♦ *Suppose a spring is stretched very little at the static equilibrium position ($l_0 \approx (a^2+b^2)^{1/2}$ in the present case). Then the quadratic terms in the potential energy of that spring may be obtained with reasonable accuracy by constructing linearized displacement components parallel to the spring at both of its ends.*† *Note that, if the stiffness k is large, this condition can be obtained even if the spring force at the equilibrium position is large.*

We now return to the derivation of the equation of motion. For the gravitational potential energy, we expand $\cos\theta$ in a series for small θ, and truncate that series at the second degree, which yields $\cos\theta \approx 1-\theta^2/2$. The total potential energy that results is

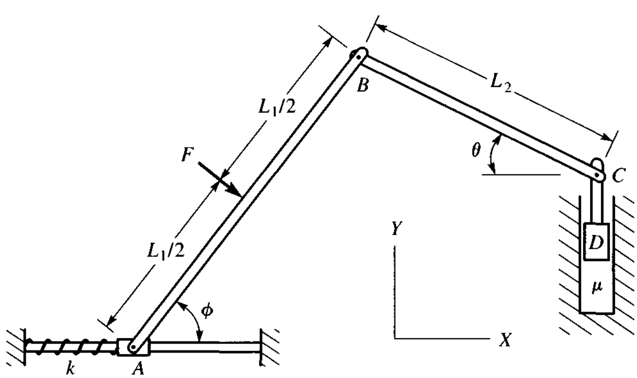
$$V=\left[\frac{kl_0(a\cos\beta)^2}{(a^2+b^2)^{1/2}}-\frac{1}{4}mgL\right]\theta^2+\text{higher-order terms.}$$

It is interesting to note that the appearance of l_0 in this expression is surprising for some who have learned linearization concepts as ad hoc procedures.

The only forces remaining are the reactions, whose constraints are not violated. Therefore, $Q_1 = 0$. Substitution of T and V into Lagrange's equations therefore leads to

$$\frac{1}{3}mL^2\ddot{\theta}+\left[\frac{2kl_0(a\cos\beta)^2}{(a^2+b^2)^{1/2}}-\frac{1}{2}mgL\right]\theta=0.$$

Example 7.14 The linkage lies in the horizontal plane. The dashpot constant for the piston is μ. The bars have mass m_1 and m_2, and the mass of collar A and of piston D is negligible. The force $\bar{F}(t)$ remains perpendicular to bar AB. The spring, whose stiffnesses is k, is unstretched in the position where $\phi = 45°$ and $\theta = 30°$. Derive the linearized equations of motion relative to this position.



Example 7.14

Solution If we know both ϕ and θ, we can locate the position of this system, from which we conclude that the system has two degrees of freedom and is scleronomic. The relative generalized coordinates we employ are

† This analysis of the spring's potential energy is not valid when $\beta = 0$. However, in such a case it is not difficult to show that the static elongation is never significant to the quadratic terms in potential energy.

$$\xi_1 = \phi - \phi^* = \phi - \pi/4, \qquad \xi_2 = \theta - \theta^* = \theta - \pi/6.$$

Because the system is scleronomic, we may use the approach in which the mechanical energies are reduced to quadratic forms. We begin by evaluating the linearized kinetic energy. Both bars execute general motion. The angular velocities of the bars are $\bar{\omega}_{AB} = \dot{\phi}\bar{k} = \dot{\xi}_1\bar{k}$ and $\bar{\omega}_{BC} = -\dot{\theta}\bar{k} = -\dot{\xi}_2\bar{k}$. We must describe the velocity of each center of mass in terms of the generalized coordinates. However, the centers of mass are not constrained points, so we first perform a kinematical analysis of the linkage using

$$\bar{v}_B = \bar{v}_A + \bar{\omega}_{AB} \times \bar{r}_{B/A} = \bar{v}_C + \bar{\omega}_{BC} \times \bar{r}_{B/C}.$$

For a linearized analysis, we may describe the position vectors by their values in the reference position, so we have

$$\begin{aligned}\bar{v}_B &= v_A\bar{i} + \dot{\xi}_1\bar{k} \times L_1(0.7071\bar{i} + 0.7071\bar{j}) \\ &= v_C\bar{j} + (-\dot{\xi}_2\bar{k}) \times L_2(-0.8660\bar{i} + 0.50\bar{j}).\end{aligned}$$

Matching like components in the foregoing yields

$$v_A = 0.7071L_1\dot{\xi}_1 + 0.50L_2\dot{\xi}_2, \qquad v_C = 0.7071L_1\dot{\xi}_1 - 0.8660L_2\dot{\xi}_2.$$

We may now relate the velocities of the centers of mass to the generalized coordinates by writing

$$\begin{aligned}\bar{v}_{G1} &= v_A\bar{i} + \dot{\xi}_1\bar{k} \times (\tfrac{1}{2}L_1)(0.7071\bar{i} + 0.7071\bar{j}) \\ &= (0.3536L_1\dot{\xi}_1 + 0.50L_2\dot{\xi}_2)\bar{i} + 0.3536L_1\dot{\xi}_1\bar{j}, \\ \bar{v}_{G2} &= v_C\bar{j} + (-\dot{\xi}_2\bar{k}) \times (\tfrac{1}{2}L_2)(-0.8660\bar{i} + 0.50\bar{j}) \\ &= 0.250L_2\dot{\xi}_2\bar{i} + (0.7071L_1\dot{\xi}_1 - 0.4330L_2\dot{\xi}_2)\bar{j}.\end{aligned}$$

From these expressions we find the kinetic energy to be

$$\begin{aligned}T &= \frac{1}{2}m_1\bar{v}_{G1}\cdot\bar{v}_{G1} + \frac{1}{2}\left(\frac{1}{12}m_1L_1^2\right)\dot{\xi}_1^2 + \frac{1}{2}m_2\bar{v}_{G2}\cdot\bar{v}_{G2} + \frac{1}{2}\left(\frac{1}{12}m_2L_2^2\right)\dot{\xi}_2^2 \\ &= \frac{1}{2}m_1\left[(0.3536L_1\dot{\xi}_1 + 0.50L_2\dot{\xi}_2)^2 + (0.3536L_1\dot{\xi}_1)^2 + \frac{1}{12}L_1^2\dot{\xi}_1^2\right] \\ &\quad + \frac{1}{2}m_2\left[(0.250L_2\dot{\xi}_2)^2 + (0.7071L_1\dot{\xi}_1 - 0.4330L_2\dot{\xi}_2)^2 + \frac{1}{12}L_2^2\dot{\xi}_2^2\right].\end{aligned}$$

To identify the inertia coefficients, we collect like coefficients of the generalized velocities, which yields

$$\begin{aligned}T = \tfrac{1}{2}[&(0.3333m_1 + 0.5m_2)L_1^2\dot{\xi}_1^2 + (0.3536m_1 - 0.6124m_2)L_1L_2\dot{\xi}_1\dot{\xi}_2 \\ &+ (0.250m_1 + 0.3333m_2)L_2^2\dot{\xi}_2^2].\end{aligned}$$

The next step is to identify the M_{ij}^* values by matching this expression to the standard form in Eq. (7.106), which leads to

$$M_{11}^* = (0.3333m_1 + 0.50m_2)L_1^2, \quad M_{12}^* = M_{21}^* = (0.1768m_1 - 0.3062m_2)L_1L_2,$$
$$M_{22}^* = (0.250m_1 + 0.3333m_2)L_2^2.$$

We should note that this step often leads to an error, because it is easy to forget that the quadratic sum has two terms associated with the mixed product $\dot{\xi}_1\dot{\xi}_2$.

Because the system lies in the horizontal plane, the only conservative force is that of the spring. We must relate its deformation to the generalized coordinates. Since the spring is unstretched in the reference position, we have $\Delta = s_A$. We are performing a linearized analysis, so we may approximate $s_A \approx v_A \Delta t$ with $\xi_j \approx \dot{\xi}_j \Delta t$. The earlier expression for v_A then leads to

$$\Delta = (0.7071 L_1 \dot{\xi}_1 + 0.50 L_2 \dot{\xi}_2)\Delta t = 0.7071 L_1 \xi_1 + 0.50 L_2 \xi_2.$$

The potential energy therefore is

$$V = \tfrac{1}{2}k\Delta^2 = \tfrac{1}{2}k(0.50 L_1^2 \xi_1^2 + 0.7071 L_1 L_2 \xi_1 \xi_2 + 0.25 L_2^2 \xi_2^2).$$

We match this to the standard form of V, Eq. (7.103), which yields

$$k_{11} = 0.50 k L_1^2, \qquad k_{12} = k_{21} = 0.3536 k L_1 L_2, \qquad k_{22} = 0.250 k L_2^2.$$

We account for the dashpot by using the Rayleigh dissipation function. To form D we use the earlier expression for v_C, from which we find

$$D = \tfrac{1}{2}\mu v_C^2 = \tfrac{1}{2}\mu(0.50 L_1^2 \dot{\xi}_1^2 - 1.2427 L_1 L_2 \dot{\xi}_1 \dot{\xi}_2 + 0.75 L_2^2 \dot{\xi}_2^2).$$

Matching this to the standard form of D, Eq. (7.113), leads to

$$D_{11} = 0.50 \mu L_1^2, \qquad D_{12} = D_{21} = -0.6214 \mu L_1 L_2, \qquad D_{22} = 0.75 \mu L_2^2.$$

All that remains now is to identify the generalized forces. The only force not accounted for is $\bar{F}$, which acts at the center of mass of bar AB. Because we have established an expression for the velocity of this point, we invoke the kinematical method to form the corresponding virtual displacement,

$$d\bar{r}_{G1} = \bar{v}_{G1}\,dt \;\Rightarrow\; \delta\bar{r}_{G1} = (0.3536 L_1\,\delta\xi_1 + 0.50 L_2\,\delta\xi_2)\bar{i} + (0.3536 L_1\,\delta\xi_1)\bar{j}.$$

We now form the virtual work, for which we use the reference state to describe the components of the force. This yields

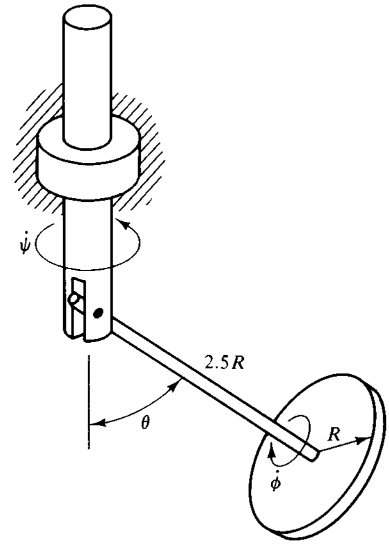
$$\delta W = F(0.7071\bar{i} - 0.7071\bar{j})\cdot\delta\bar{r}_{G1} = F(0.3536 L_2\,\delta\xi_2);$$

$$Q_1 = 0, \qquad Q_2 = 0.3536 L_2 F.$$

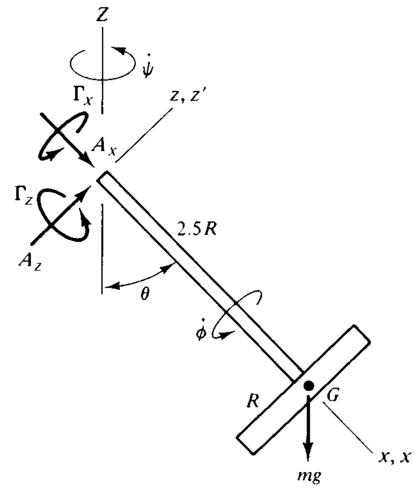
The inertia, stiffness, and dissipation coefficients that we have found, along with the above generalized forces, enable us to form the matrix equations of motion, Eq. (7.116). Because the springs are unstressed in the static equilibrium position, we set $\{\partial V/\partial q\}^* = \{0\}$ in those equations.

Example 7.15 Servomotors maintain the spin rate $\dot{\phi}$ and the precession rate $\dot{\psi}$ at constant values. One possible motion is a steady precession with $\theta = 0$. However, when $\dot{\psi}$ exceeds a minimum value, another steady precessional motion is possible in which θ is a constant nonzero value. Determine this minimum value of $\dot{\psi}$. Then evaluate the stability of both steady precessions as a function of $\dot{\psi}$.

Solution The reference states we wish to study are steady-state precessions. Studying the stability of such a response will require that we consider deviations from the steady motion. In order to derive equations of motion that can be used for both phases of the analysis, we begin by considering the nutation angle θ to be



Example 7.15



Free-body diagram.

arbitrary, $q_1 = \theta$. The precession and spin rates are constrained to be constant, so the system is rheonomic with one degree of freedom. As a way of emphasizing that the linearization concepts developed in this section do not require the use of Lagrange's equations, we shall derive the general equations of motion by applying Euler's angular momentum equations.

The flywheel is in pure rotation about pin A. To describe the angular velocity of the disk, we fix xyz to the disk, and let $x'y'z'$ execute the precession only. Selecting the instantaneous orientation of both axes as shown in the sketch yields

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}(-\bar{j}') + \dot{\phi}\bar{i} = (\dot{\phi} - \dot{\psi}\cos\theta)\bar{i} - \dot{\theta}\bar{j} + (\dot{\psi}\sin\theta)\bar{k},$$

$$\bar{\omega}' = \dot{\psi}\bar{K} = -(\dot{\psi}\cos\theta)\bar{i} + (\dot{\psi}\sin\theta)\bar{k},$$

$$\begin{aligned}\bar{\alpha} &= -\ddot{\theta}\bar{j}' - \dot{\theta}(\bar{\omega}' \times \bar{j}') + \dot{\phi}(\bar{\omega} \times \bar{i}) \\ &= (\dot{\psi}\dot{\theta}\sin\theta)\bar{i} + (-\ddot{\theta} + \dot{\psi}\dot{\phi}\sin\theta)\bar{j} + (\dot{\psi}\dot{\theta}\cos\theta - \dot{\theta}\dot{\phi})\bar{k}.\end{aligned}$$

We assume that the mass of the flywheel is much greater than that of either shaft. Considering the flywheel to be a thin disk leads to $I_{xx} = 0.5mR^2$ and $I_{yy} = I_{zz} = 6.50mR^2$, with xyz the principal axes. The pin exerts no moment about its own axis, so Euler's equation for moment about the pin's axis yields a differential equation for θ,

$$\begin{aligned}\sum M_{Ay} &= mg(2.5R\sin\theta) = I_{yy}\alpha_y - (I_{zz} - I_{xx})\omega_x\omega_z \\ &= mR^2[6.5(-\ddot{\theta} + \dot{\psi}\dot{\phi}\sin\theta) - 6.0(\dot{\phi} - \dot{\psi}\cos\theta)(\dot{\psi}\sin\theta)];\end{aligned}$$

$$13\ddot{\theta} + \left(\frac{5g}{R} - \dot{\psi}\dot{\phi}\right)\sin\theta - 12\dot{\psi}^2\sin\theta\cos\theta = 0. \qquad (1)$$

For steady precession, we set $\dot{\theta} = \ddot{\theta} = 0$. In that case, eq. (1) yields

$$\sin\theta^* = 0 \quad \text{or} \quad \cos\theta^* = \frac{5g - R\dot{\psi}\dot{\phi}}{12R\dot{\psi}^2}. \qquad (2)$$

The first case is the vertical position. The other solution is possible only if $|\cos\theta^*| < 1$, which leads to

$$-12R\dot{\psi}^2 < 5g - R\dot{\psi}\dot{\phi} < 12R\dot{\psi}^2.$$

This condition may be rewritten as

$$\dot{\phi}_{\min} < \dot{\phi} < \dot{\phi}_{\max}, \quad \dot{\phi}_{\min} = \frac{5g}{R\dot{\psi}} - 12\dot{\psi}, \quad \dot{\phi}_{\max} = \frac{5g}{R\dot{\psi}} + 12\dot{\psi}. \tag{3}$$

When the spin rate is outside the range shown, only the vertical position $\sin\theta^* = 0$ is possible. This could be satisfied by either $\theta^* = \pi$ or $\theta^* = 0$, although the upright position is not possible because of interference with the vertical shaft. In both positions, gravity exerts no moment about the pin, and the angular momentum is constant, aligned along the precession axis. When the spin rate is within the range described in (3), steady precession with $\theta^* \neq 0$ becomes a possibility. Within this range, the angle θ^* is that for which the moment of gravity about the pin equals the rate at which the angular momentum $\bar{H}_A$ changes.

In order to study the effect of small changes in θ away from a steady-state position, we introduce the relative generalized coordinate ξ such that

$$\theta = \theta^* + \xi, \quad |\xi| \ll 1.$$

Linearization of equations of motion entails dropping quadratic and higher-order terms in ξ in those equations. Thus, we write

$$\sin\theta = \sin(\theta^* + \xi) = \sin\theta^*\cos\xi + \cos\theta^*\sin\xi \approx \sin\theta^* + \xi\cos\theta^*,$$

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta \approx \tfrac{1}{2}\sin 2\theta^* + \xi\cos 2\theta^*.$$

Substitution of these expressions into the nonlinear equation of motion (1) gives

$$13\ddot{\xi} + \left[\left(\frac{5g}{R} - \dot{\psi}\dot{\phi}\right)\sin\theta^* - 6\dot{\psi}^2\sin 2\theta^*\right] + \left[\left(\frac{5g}{R} - \dot{\psi}\dot{\phi}\right)\cos\theta^* - 12\dot{\psi}^2\cos 2\theta^*\right]\xi = 0.$$

The first bracketed term vanishes identically, by virtue of eq. (2) for θ^*, so the stability of a steady precession is governed by

$$13\ddot{\xi} + \left[\left(\frac{5g}{R} - \dot{\psi}\dot{\phi}\right)\cos\theta^* - 12\dot{\psi}^2\cos 2\theta^*\right]\xi = 0.$$

This equation has some interesting ramifications for dynamic stability, all of which result from the following observation: The sign of the coefficient of ξ determines whether $\xi(t)$ is bounded or grows without limit. We define β to be this coefficient,

$$\beta = \left(\frac{5g}{R} - \dot{\psi}\dot{\phi}\right)\cos\theta^* - 12\dot{\psi}^2\cos 2\theta^*, \qquad 13\ddot{\xi} + \beta\xi = 0. \tag{4}$$

If $\beta > 0$ then $\xi(t)$ is a sinusoidal function, which means that ξ will remain small. Thus, $\beta > 0$ means that $\theta = \theta^*$ is a stable steady precession. In contrast, if $\beta < 0$ then one of the general solutions for $\xi(t)$ grows exponentially. We consider this to indicate that the steady precession is unstable. (The differential equation for ξ was derived under the assumption that ξ is small, so one cannot infer that θ will actually grow without bound in the unstable case. Instability here merely means that the motion will differ strongly from the steady-precession solution.)

Consider first the case $\theta^* = 0$, which is always a possibility. We then have

$$\beta = \frac{5g}{R} - \dot{\psi}\dot{\phi} - 12\dot{\psi}^2. \tag{5}$$

The steady precession at $\theta^* = 0$ is stable when $\beta > 0$, which is satisfied if

$$\dot{\phi} < 5g/R\dot{\psi} - 12\dot{\psi},$$

that is, if $\dot{\phi} < \dot{\phi}_{\min}$. When $\dot{\phi} < \dot{\phi}_{\min}$, the dynamic tendency to swing outward is inadequate to overcome the tendency of gravity to return the system to the upright position. This situation is reversed when $\dot{\phi} > \dot{\phi}_{\min}$.

Suppose now that $\dot{\phi}$ satisfies inequality (3), so that a steady precession with $\theta^* > 0$ is possible. We substitute equation (2) and $\cos 2\theta^* = 2\cos^2\theta^* - 1$ into the definition of β. After simplification, the result is

$$\beta = \frac{(12R\dot{\psi}^2)^2 - (5g - R\dot{\psi}\dot{\phi})^2}{12R^2\dot{\psi}^2}.$$

In view of inequality (3), $\beta > 0$ is satisfied by the same conditions required for $\theta^* > 0$. We conclude that if the rotation rates are such that a steady precession at $\theta^* > 0$ is possible, then such a motion is stable.

In summary, the flywheel remains below the pivot at $\theta^* = 0$ if the spin rate is less than

$$\dot{\phi}_{\min} < \frac{5g}{R\dot{\psi}^2} - 12\dot{\psi}.$$

This steady motion becomes unstable if $\dot{\phi} > \dot{\phi}_{\min}$. In that case, a stable steady precession may be established with

$$\theta^* = \cos^{-1}\left(\frac{5g - R\dot{\psi}\dot{\phi}}{12\dot{\psi}^2}\right).$$

Such a precessional motion ceases to be possible if the spin rate exceeds the upper limit

$$\dot{\phi}_{\max} > \frac{5g}{R\dot{\psi}^2} + 12\dot{\psi}.$$

If the upright position $\theta^* = \pi$ were physically admissible, we could prove by examining β that it is unstable if $\dot{\phi} < \dot{\phi}_{\max}$ and stable if $\dot{\phi}$ exceeds that value.

References

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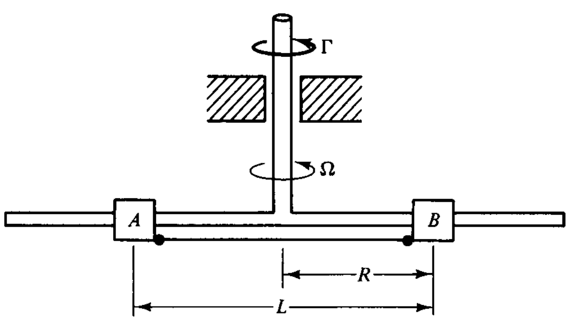
Problems

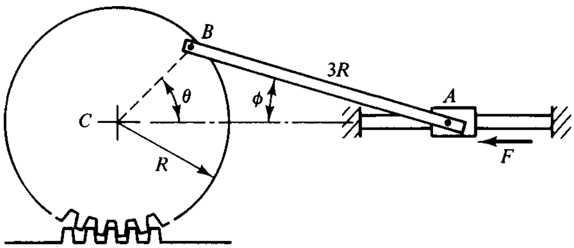
7.1 The torque Γ acting about the vertical shaft is such that the rotation rate Ω is constant. The sliders, having masses m_A and m_B, are tied together by an inextensible cable. Derive the equation of motion for the radial distance R, and also obtain an expression for Γ.



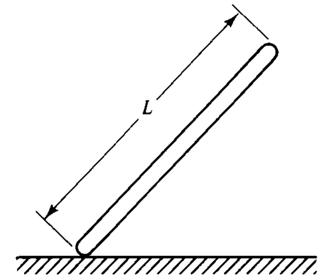
Problem 7.1

7.2 Derive the Lagrange equations of motion for the system in Example 7.8, and also obtain a relation for Γ. Compare the results to those obtained in the solution to the example.

7.3 (See figure, next page.) Force $\bar{F}(t)$ pushes piston A, whose mass is m, to the left. This causes the gear to roll over the horizontal rack. The mass of the gear is $2m$ and its radius of gyration about center point C is κ; the mass of bar AB is m. Use the angles θ and ϕ as constrained generalized coordinates to derive the equations of motion for the system.



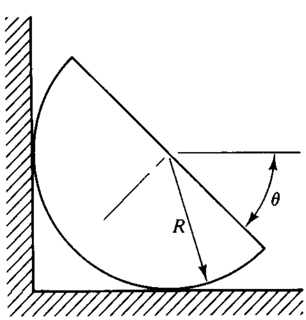
Problem 7.3



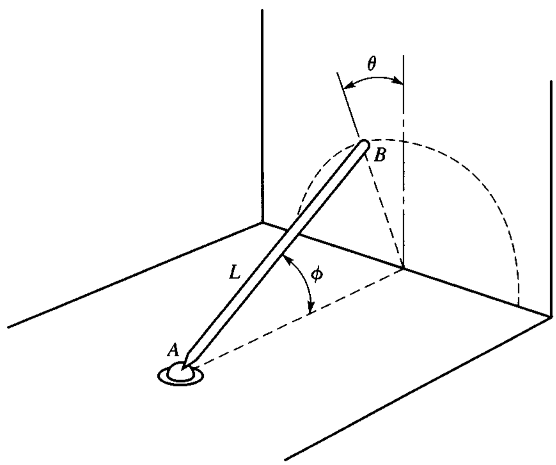
Problem 7.4

7.4 The bar is slipping relative to the ground as it falls. The coefficient of kinetic friction is μ. Use Lagrange's equations to derive the equations of motion for the bar.

7.5 The semicylinder, whose mass is m, is released from rest at an initial orientation $\theta > 0$. The floor is smooth, and the coefficient of kinetic friction μ between the cylinder and the wall is not adequate to prevent sliding. Use Lagrange's equations to derive the equations of motion for the bar.



Problem 7.5

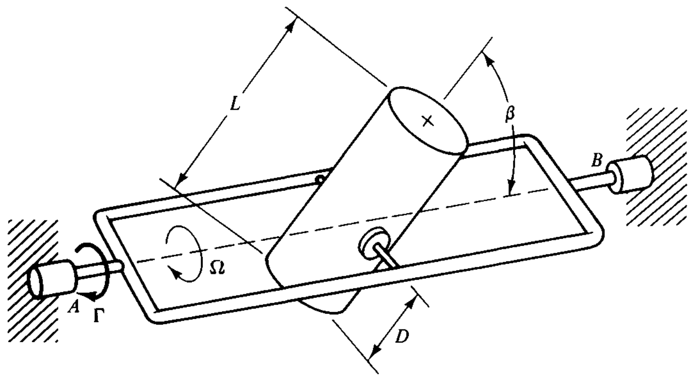


Problem 7.6

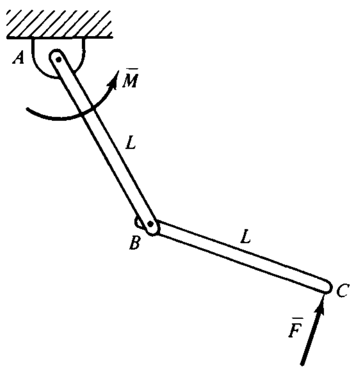
7.6 The bar is supported by a ball-and-socket joint at end A and the rough wall at end B; the coefficient of sliding friction is μ. Use Lagrange's equations to derive the equation of motion governing the angle of inclination θ.

7.7 The cylinder of mass m is free to rotate by angle β relative to the gimbal, which rotates about the horizontal axis. The precessional rate Ω is held constant by varying the torque Γ. Use Lagrange's equations to derive the equation of motion governing β, as well as an expression for Γ.

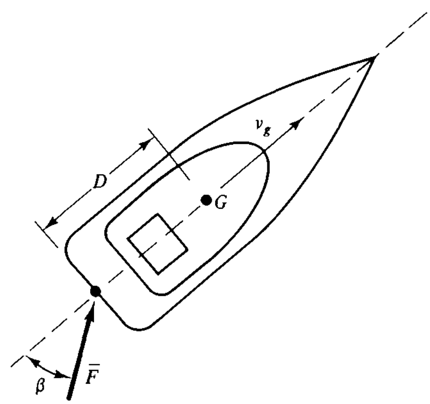
7.8 A known couple $\bar{M}(t)$ is applied to the upper bar. Force $\bar{F}$, which is applied perpendicularly to the lower bar, acts to make the velocity of end C always collinear with the line from joint B to end C. The bars have equal mass m, and the system lies in the horizontal plane. Use the method of Lagrange multipliers to derive the equations of motion.



Problem 7.7



Problem 7.8



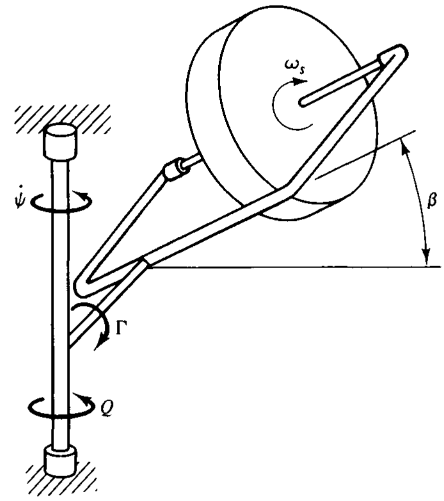
Problem 7.9

7.9 The thrust of an outboard motor on a boat may be represented as a force $\bar{F}$ acting at an angle β relative to the axis of the boat. The hydrodynamic properties of the boat are such that the velocity of the center of mass G is constrained to be parallel to the longitudinal axis of the boat. The component of the hydrodynamic force parallel to the axis of the boat is the drag f_d. Derive the equations of motion for the boat using Lagrange multipliers. The mass of the boat is m, and the centroidal moment of inertia is I.

7.10 Use Lagrange's equations to derive the equations of motion of the spring-mass system in Example 7.9.

7.11 Use Lagrange's equations to derive the equations of motion of the wheelbarrow in Example 7.12.

7.12 (See figure, next page.) The torque Γ causing the gimbal rotation β is a specified function of time. Moment Q about the vertical axis causes the gyroscope to rotate such that the precession angle $\psi = c\dot{\beta}$, where c is a constant. The spin rate ω_s is maintained at a constant value by a servomotor. The mass of this motor and the gimbal are negligible. The mass of the flywheel is m and its principal radii of gyration for centroidal axes are κ_1 about its spin axis and κ_2 normal to that axis. Also, derive an expression for Q.

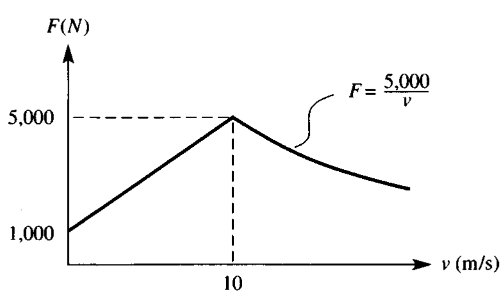


Problem 7.12

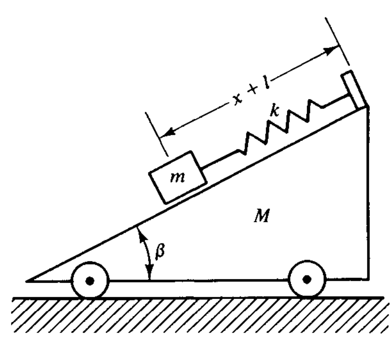
7.13 Consider the system in Example 7.1 for the case where the torque increases linearly according to $\Gamma = 5t$ N-m, where t is measured in seconds and the precession rate is unknown. The system was at rest at $t = 0$ with $\theta = \dot{\psi} = 0$. Use numerical methods to solve the state-space equations for this system. From that solution, determine the maximum value of θ attained in the response, and the corresponding value of $\dot{\psi}$. Parameters for the system are $m_1 = 2$ kg, $L = 500$ mm, and $I_2 = 0.1$ kg-m^2.

7.14 Consider the bar in Example 7.2, for which $k/m = 5$ (rad/s)^2, $g/L = 20$ (rad/s)^2, $\beta = 60°$, and $\mu = 0.5$. Initially, the bar is at rest in the position where $\phi = 30°$. Use numerical methods to determine the elapsed time until $\phi = -90°$. From that result determine the mechanical energy, initially stored in the system, that is dissipated during the descent.

7.15 Consider the motorboat in Problem 7.9. The thrust at full throttle, which is shown in the graph, increases linearly with increasing speed v, until the maximum power output is attained at 10 m/s. Beyond that speed, the power output remains constant, so the thrust decreases inversely with v. The hydrodynamic drag is given by $f_d = kv$, where k is such that the maximum speed of the boat along a straight path is 15 m/s. The mass of the boat is 500 kg and its centroidal radius of gyration about the vertical



Problem 7.15



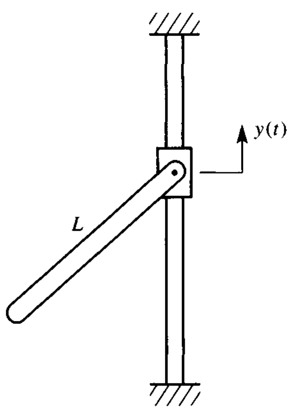
Problems 7.16 and 7.17

axis is 1.20 m. Also, the distance $D = 1.50$ m. Suppose the boat is at rest pointing northward when the motor is set and held at full throttle. The steering angle is increased linearly from $\beta = 0$ to $\beta = 2°$ during an interval of 1 min. Use numerical methods to determine the path of the center G during this 1-min interval, and also determine the speed and heading of the boat at the end of that interval.

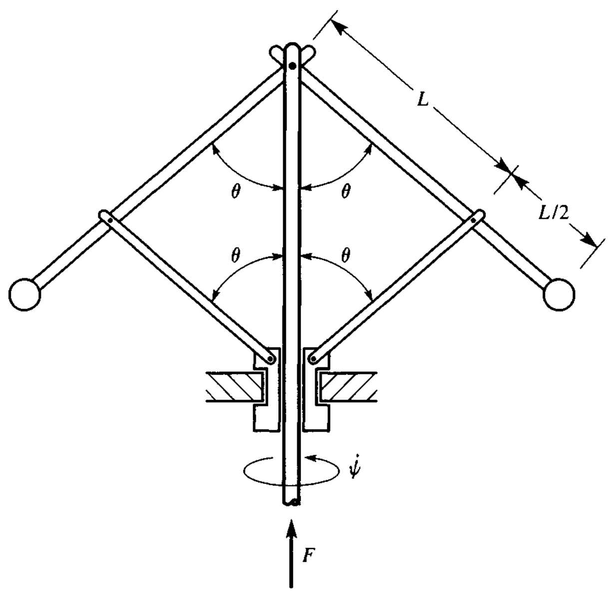
7.16 The small mass m is supported by a spring as it moves along the smooth incline on the cart, whose mass is M. The spring has stiffness k and its unstretched length is l. Derive Hamilton's canonical equations for the system.

7.17 Consider the system in Problem 7.16. Derive a single differential equation of motion for the relative distance x using Routh's method for the ignoration of coordinates.

7.18 The collar supporting bar AB is given a specified displacement $y(t)$. The collar and the bar have equal mass m. Derive Hamilton's canonical equations for this system.



Problem 7.18



Problem 7.19

7.19 Angle θ for the flyball governor is controlled by applying force $\bar{F}$, which moves the vertical shaft up or down. The system rotates freely about the vertical axis. The mass of each sphere is m and the mass of the linkage is negligible. Use Routh's method for the ignoration of coordinates to derive a single differential equation governing the nutation angle θ.

7.20 Consider the system in Problem 7.1. Use conservation of the Hamiltonian and the work-energy principle to derive the differential equation for the angle θ, and to obtain an expression for the couple Γ.

7.21 Consider the system in Problem 7.7 when the torque $\Gamma = 0$, in which case the rotation rate Ω about the horizontal axis is unknown. Let $D = L/2$, so that the center of mass is coincident with the horizontal axis. Use Routh's method for the ignoration of coordinates to derive a single differential equation governing the nutation angle β. Can such a formulation be used when $D \neq L/2$? Explain your answer.

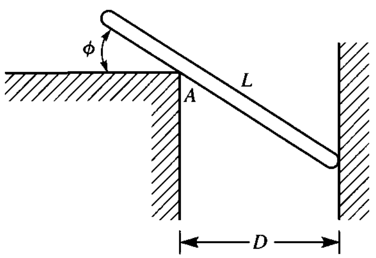
7.22 Use conservation of the Hamiltonian and the work-energy principle to solve Problem 7.7.

7.23 The absolute velocity of a particle may be represented by its components v_x, v_y, v_z along the axes of a moving reference system xyz. Suppose that the angular velocity $\bar{\omega}$ of xyz and the velocity $\bar{v}_O$ of the origin of xyz are known as functions of time. Derive the Gibbs-Appell equations of motion relating the quasivelocities $\dot{\gamma}_1 = v_x$, $\dot{\gamma}_2 = v_y$, and $\dot{\gamma}_3 = v_z$ to the components of the resultant force acting on the particle.

7.24 Derive the Gibbs-Appell equations of motion for the flyball governor in Problem 7.19.

7.25 Derive the Gibbs-Appell equations of motion for the boat in Problem 7.9.

7.26 Friction between the rod and the surfaces it contacts is negligible. Determine the Gibbs-Appell equations of motion for the system. Assume that the rod remains in contact with the wall.

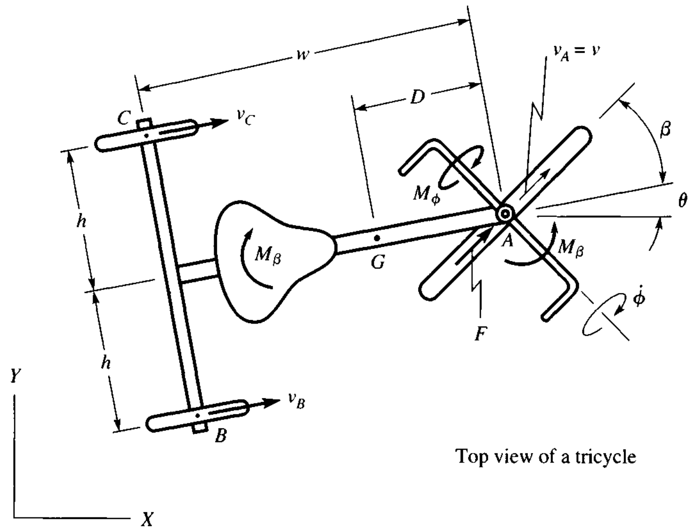


Problems 7.26 and 7.27

7.27 The coefficient of kinetic friction between the rod and corner A is μ, while frictional resistance at the wall is negligible. Use the Gibbs-Appell formulation to derive equations of motion for the system. Assume that the rod remains in contact with the wall.

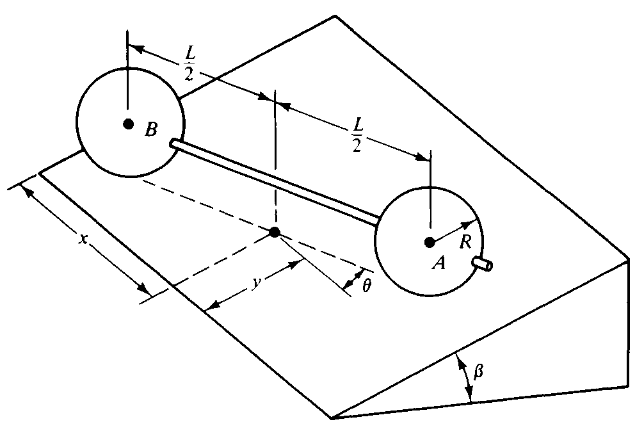
7.28 Derive Gibbs-Appell equations of motion for the system in Example 7.1.

7.29 The sketch shows a child's tricycle. All wheels roll without slipping, so the velocity of each wheel's center must be perpendicular to that wheel's axle, as shown. The rear wheels are small enough that we may neglect their inertia. A suitable set of generalized coordinates for the tricycle are therefore the position coordinates (X_A, Y_A) of the steering pivot A, the heading angle θ, the steering angle β, and the spin angle ϕ of the front wheel. Some of the forces applied by the rider have no external resultant, or else are balanced by reactions at the wheels. Under the assumption that the rider is stationary with respect to the bicycle, the effective force system causing the tricycle to move may be reduced to the traction force $\bar{F}$ exerted between the front wheel and the ground, a couple $\bar{M}_\phi$ about the axis of the front wheel representing the propulsive torque exerted by the rider, a couple $\bar{M}_\beta$ representing the steering effort applied by the rider to the handle bars, and a corresponding reaction couple $-\bar{M}_\beta$ exerted on the seat by the rider. (Note that $\bar{F}$ is a constraint force that imposes the no-slip condition at the front wheel.) In this model, the combined mass and centroidal moment of inertia of the tricycle frame and rider may be taken to be (respectively) m_1 and I, based on a center of mass G at distance D relative to the front axle, while the mass and moment of inertia of the front wheel about its axle are (respectively) m_2 and J. The inertia of the rear wheels is negligible. Use $\dot{\gamma}_1 = v$ and $\dot{\gamma}_2 = \dot{\beta}$ as unconstrained quasivelocities to formulate the Gibbs-Appell equations of motion for this system.



Problem 7.29

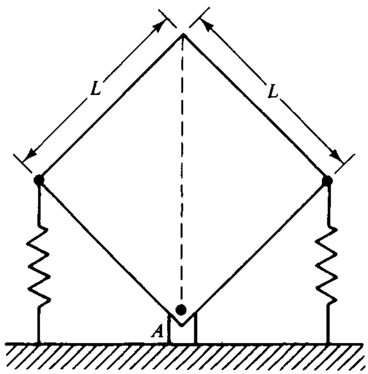
7.30 The identical spheres of mass m spin freely relative to the massless shaft, such that their centers are at constant distance L. Derive equations of motion for position coordinates x and y and angle θ of the shaft relative to the incline.



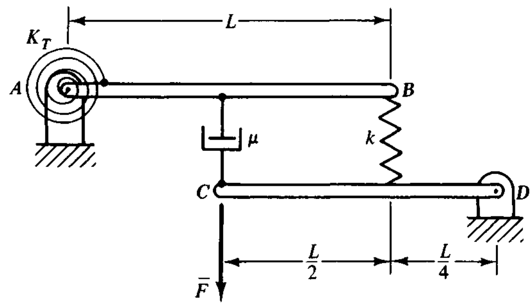
Problem 7.30

7.31 (See figure, next page.) A square plate pivoted about corner A is supported by two springs of stiffness k, such that the inverted position shown is a static equilibrium position. Derive the equation of motion for small rotation away from this position. From that equation, determine the minimum allowable value of k for which the equilibrium position is stable.

7.32 (See figure, next page.) Bars AB and CD, each of whose mass per unit length is m/L, are connected by a spring whose stiffness is k and a dashpot whose constant is μ. In addition, a torsional spring of stiffness K_T restrains rotation at pivot A. The system is



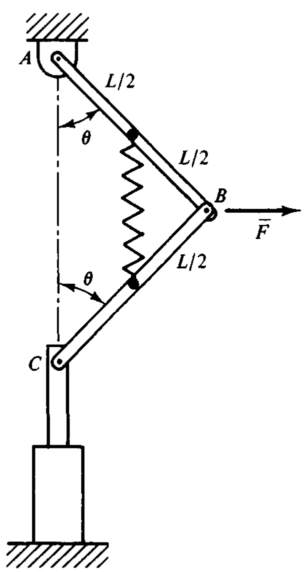
Problem 7.31



Problem 7.32

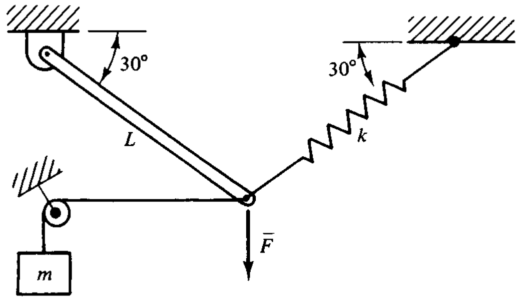
in static equilibrium in the horizontal position shown when the force $\bar{F}$ is not present. Determine the equations of motion for small displacements relative to this position.

7.33 The linkage, which lies in the vertical plane, is loaded by a horizontal force $\bar{F}(t)$. The mass of each bar is m, and the spring, whose unstretched length is $L/2$, has stiffness k. The hydraulic cylinder at end C, which permits only vertical movement, acts like a dashpot whose constant is μ. The static equilibrium position of the system is $\theta = 36.87°$ when $F = 0$. Derive an expression for k. Then derive the equation of motion for small displacements away from the static equilibrium position.



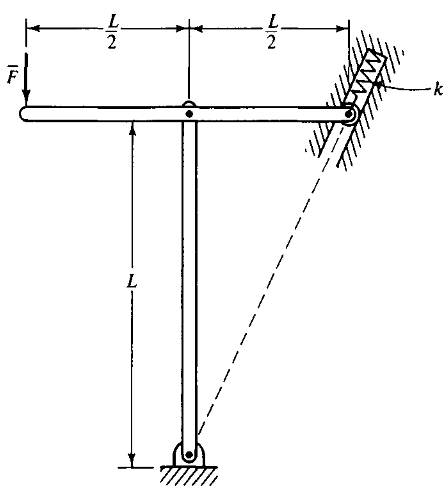
Problem 7.33

7.34 When the vertical force $\bar{F}$ is not present, the system is in static equilibrium in the position shown. The masses of the bar and of the block, which is attached to the bar by the inextensible cable, are each m. Determine the equation of motion for small rotations of the bar away from this position.



Problem 7.34

7.35 When the vertical force $\bar{F}(t)$ is not present, the linkage is in equilibrium in the position shown. The mass of each bar is m and the stiffness of the spring is k. Derive the linearized equation of motion.

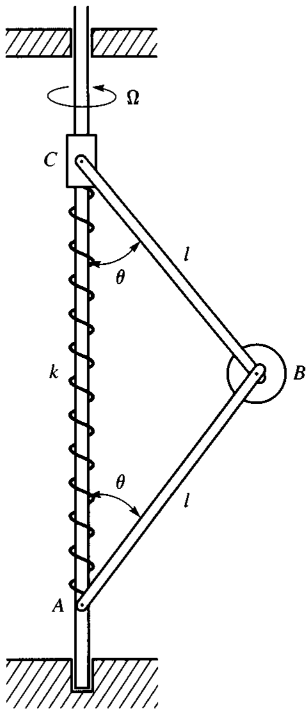


Problem 7.35

7.36 Consider the T-bar assembly in Example 7.1 in the case where the torque Γ induces a constant precession rate $\dot{\psi} = 100$ rev/min and $L = 500$ mm. Determine the constant values of the nutation angle $-\pi \le \theta^* < \pi$ for which steady precession is possible. Then evaluate the stability of each position.

7.37 Consider the system in Problem 7.1 in the case where $m_A = m$ and $m_B = m/2$, where m is a basic mass unit. It is possible for the system to rotate such that the radial distance R to collar A is constant. Derive an expression for this constant distance. Then evaluate its stability by considering a small displacement away from the steady position.

7.38 The linkage precesses about the vertical axis at the constant rate Ω. The small disk B and slider C each have mass m, and the mass of each link is negligible. The spring has stiffness k and its unstretched length is $2l$. Identify the two possible constant values of θ in the physically meaningful range $0 \le \theta^* < \pi/2$ corresponding to steady precession with $\dot{\theta} = 0$. Prove that one of these possibilities is always unstable, while the other is a stable position that exists only if k is sufficiently large.



Problem 7.38

CHAPTER 8

Gyroscopic Effects

Our emphasis thus far has been on development of basic principles for treating the kinematics and kinetics of rigid-body motion. Regardless of whether we employed a Newtonian or Lagrangian formulation, it was usually necessary to account for constraints associated with the way in which the system is supported. Sometimes the goal was to characterize the force system required to produce a specified motion, as when the reactions must be evaluated. Other situations required the determination of conditions that are satisfied during the motion, as typified by the task of deriving differential equations of motion. In this chapter we will formulate and solve the equations of motion governing the rotational motion of a rigid body. Because the angular momentum in spatial motion is usually not aligned with the instantaneous axis of rotation, a portion of the rotational effect does not coincide with that axis. Such phenomena are exploited in gyroscopes, whose theory will be introduced here. However, we may learn much about the nature of dynamical responses by beginning with studies of simpler, yet more common, systems that display comparable effects.

8.1 Free Motion

One of the first types of spatial motion treated in basic physics and engineering courses on mechanics is projectile motion, whose study is devoted to the determination of the motion of the center of mass. In contrast, the manner in which the body rotates about its center of mass is seldom discussed in a fundamental course. Our study of this *free motion* will be based on the assumption that gravitational attraction is the only external force acting at the center of mass. In that case, the rotational motion is uncoupled from that of the mass center. In reality, aerodynamic forces acting on a body usually may be represented as a force–couple system acting at the center of pressure, which does not necessarily coincide with the mass center. Such forces depend on the orientation (angle of attack), as well as the overall velocity. Hence, accurate models of the motion of objects through the air might require consideration of coupling between the translational and rotational motions.

8.1.1 *Arbitrary Bodies*

As a direct consequence of assuming that the force system acts through the center of mass, the angular momentum $\bar{H}_G$ about the center of mass is constant. This simple fact provides the foundation for our entire development. The constant magnitude of $\bar{H}_G$ provides a constant that relates the rates of rotation. The direction of $\bar{H}_G$ is also invariable, so it provides a fixed direction in space that may be employed as a reference.

Eulerian angles are useful for the kinematical formulation of free motion. Suppose that we know the orientation of the body at the instant it is released, and that

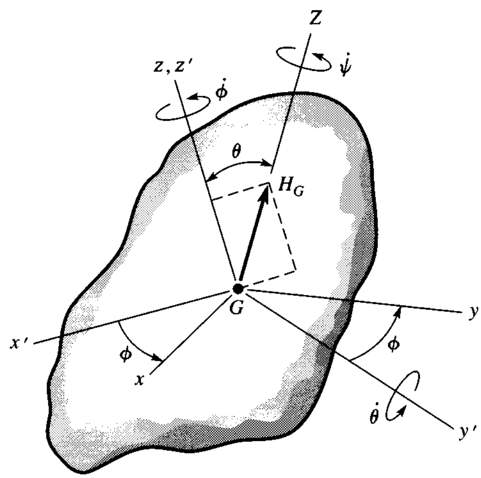


Figure 8.1 Eulerian angles and angular momentum for free motion.

the angular velocity at that instant is also known. It is a simple matter to use such information to form $\bar{H}_G$. Let us *define the precession axis to coincide with the direction of the angular momentum.* It is convenient to use as the body-fixed reference frame a set of *xyz* axes that are principal, about which the principal moments of inertia are I_1, I_2, and I_3, respectively. (Evaluation of the orientation of these axes for an arbitrary body is discussed in Section 5.2.3.) Definition of the Eulerian angles associated with these axes is depicted in Figure 8.1. The precession rate $\dot\psi$ is about the Z axis, which is parallel to $\bar{H}_G$. The spin rate $\dot\phi$ is about the z axis. The angle between the precession and spin axes is the nutation angle θ. The nutation rate $\dot\theta$ is about the line of nodes, which is the y' axis perpendicular to the plane formed by the Z and z axes.

The angular velocity at any instant is

$$\bar\omega = (-\dot\psi\sin\theta\cos\phi + \dot\theta\sin\phi)\bar{i} + (\dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi)\bar{j} + (\dot\psi\cos\theta + \dot\phi)\bar{k}. \tag{8.1}$$

An expression for the angular momentum may be obtained by combining these angular velocity components with the respective moments of inertia:

$$\bar{H}_G = I_1\omega_x\bar{i} + I_2\omega_y\bar{j} + I_3\omega_z\bar{k}. \tag{8.2}$$

However, projecting the $\bar{H}_G$ vector in Figure 8.1 onto the respective axes leads to a different expression:

$$\bar{H}_G = -(H_G\sin\theta\cos\phi)\bar{i} + (H_G\sin\theta\sin\phi)\bar{j} + (H_G\cos\theta)\bar{k}. \tag{8.3}$$

Substitution of the angular velocity components of Eq. (8.1) into Eq. (8.2), followed by matching like components of that expression to Eq. (8.3), yields

$$\begin{aligned} &\blacklozenge \quad I_1(\dot\psi\sin\theta\cos\phi - \dot\theta\sin\phi) = H_G\sin\theta\cos\phi, \\ &\blacklozenge \quad I_2(\dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi) = H_G\sin\theta\sin\phi, \\ &\blacklozenge \quad I_3(\dot\psi\cos\theta + \dot\phi) = H_G\cos\theta. \end{aligned} \tag{8.4}$$

The value of H_G is known from the motion at the instant the body was released, so these relations constitute a set of first-order, coupled differential equations for the Eulerian angles. Their initial conditions are the values of the Eulerian angles at the instant of release.

It is interesting to note that we could have used Lagrange's equations to obtain an alternative set of differential equations of motion. Such equations would have been second-order. In essence, the constancy of $\bar{H}_G$ led us in Eqs. (8.4) to first integrals of the Lagrangian equations.

These differential equations are highly nonlinear. If one is interested in developing solutions for specified initial conditions, integration schemes are easier to implement if the equations are not coupled in the derivatives. We may obtain equations for the precession and nutation rates from the first two of Eqs. (8.4), after which we find an expression for the spin rate from the last equation. The result is

$$\blacklozenge \quad \dot{\psi} = H_G\left(\frac{\cos^2\phi}{I_1} + \frac{\sin^2\phi}{I_2}\right),$$

$$\blacklozenge \quad \dot{\theta} = H_G\left(\frac{1}{I_2} - \frac{1}{I_1}\right)\sin\theta\sin\phi\cos\phi, \tag{8.5}$$

$$\blacklozenge \quad \dot{\phi} = H_G\left(\frac{1}{I_3} - \frac{\cos^2\phi}{I_1} - \frac{\sin^2\phi}{I_2}\right)\cos\theta.$$

We shall not pursue solutions of these differential equations, although analytical solutions in the form of elliptic functions are possible (see Synge and Griffith 1959). Also, numerical solutions may be readily implemented. We will study a graphical way of understanding the rotation in a later section on the Poinsot construction, after we treat the special case of an axisymmetric body in Section 8.1.2. Before continuing to those topics, there are a few general observations that we can make.

First, note that $\dot{\psi}$ is always positive, which means that a body in free motion never changes the direction in which it precesses. However, the signs of $\dot{\theta}$ and $\dot{\phi}$ depend on the relative magnitudes of the moments of inertia, and on the current quadrant in which θ and ϕ reside. The latter observation suggests that there might be free motions in which the nutation and spin rates, and therefore the corresponding angles, oscillate. This, in turn, leads us to be concerned with the stability of an established rotational motion. We may obtain specific results regarding stability in the important case where we attempt to make a body rotate at rate Ω about a principal axis. Without loss of generality, we consider such a motion to consist solely of a spin, with the precession axis aligned with the spin axis; in this case, both the precession and nutation rates are zero.

The question of stability for this motion arises from the recognition that we are not likely to impart an initial rotation to a body in which the axis of rotation is *exactly* aligned with one of the principal axes. A more realistic expectation is that, because of small error, the initial motion will feature nutation and precession rates that are much smaller than Ω, and that the nutation angle will be small. If an evaluation of the response confirms that these initial *perturbations* remain small for the overall reponse, then we may conclude that the rotation is stable.

Accordingly, let

$$\dot{\phi} = \Omega + \Delta\dot{\phi}, \qquad \theta = \Delta\theta, \qquad \dot{\psi} = \Delta\dot{\psi}, \tag{8.6}$$

where $\Delta\psi$, $\Delta\theta$, and $\Delta\phi$ represent small deviations from the ideal. We may derive solvable equations for the perturbations in cases where they are sufficiently small to enable linearizing the equations of motion. It is easier to investigate the stability by returning to the original set of equations of motion, Eqs. (8.4). The linearized differential equations resulting from substitution of Eqs. (8.6) are

$$\begin{aligned}
&-I_1(\Delta\dot{\theta}\sin\phi) = H_G(\Delta\theta\cos\phi),\\
&I_2(\Delta\dot{\theta}\cos\phi) = H_G(\Delta\theta\sin\phi),\\
&I_3\Omega = H_G.
\end{aligned} \tag{8.7}$$

In order to solve these equations, let us consider $\overline{\Delta\theta}$ to be a vector aligned along the nodes in the right-hand sense. Then $\Delta\theta\sin\phi$ and $\Delta\theta\cos\phi$ are the projections of the small nutation angle onto the x and y axes. Let u and v denote these components,

$$u = \Delta\theta\sin\phi, \qquad v = \Delta\theta\cos\phi. \tag{8.8}$$

Then, because $\dot{\phi} \approx \Omega$ when higher-order perturbation terms are neglected, differentiation of Eqs. (8.8) produces

$$\begin{aligned}
\dot{u} &= \Delta\dot{\theta}\sin\phi + \Omega(\Delta\theta\cos\phi) = \Delta\dot{\theta}\sin\phi + \Omega v,\\
\dot{v} &= \Delta\dot{\theta}\cos\phi - \Omega(\Delta\theta\sin\phi) = \Delta\dot{\theta}\cos\phi - \Omega u.
\end{aligned} \tag{8.9}$$

Substitution of Eqs. (8.8) and (8.9) into the differential equations (8.7) yields

$$I_1(\dot{u} - \Omega v) = -I_3\Omega v, \qquad I_2(\dot{v} + \Omega u) = I_3\Omega u. \tag{8.10}$$

These are a pair of coupled, homogeneous, linear differential equations with constant coefficients. Their solution must be exponential in time,

$$u = A\exp(\lambda t), \qquad v = B\exp(\lambda t). \tag{8.11}$$

Substitution of these forms into Eq. (8.10) leads to

$$\begin{bmatrix} I_1\lambda & (I_3 - I_1)\Omega \\ (I_2 - I_3)\Omega & I_2\lambda \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{8.12}$$

The foregoing is an eigenvalue problem. In order for there to be a nontrivial solution, the determinant of this set of equations must vanish, which leads to the following characteristic equation:

$$I_1 I_2\lambda^2 + (I_3 - I_1)(I_3 - I_2)\Omega^2 = 0. \tag{8.13}$$

Because the moments of inertia are positive values, the roots of this quadratic equation occur either as a pair of conjugate imaginary values, or as real values with alternating sign, depending on the sign of the last term. A positive value of λ corresponds to exponential growth of u and v, and therefore of $\Delta\theta$. In this case, a small perturbation from the ideal condition grows with time. Hence, the free rotation is unstable if $(I_3 - I_1)(I_3 - I_2) < 0$. In contrast, the case where the values of λ are imaginary, $(I_3 - I_1)(I_3 - I_2) > 0$, corresponds to an oscillation. The perturbation in the nutation angle $\overline{\Delta\theta}$ in that case never exceeds a bounded value that is determined by the initial conditions. This means that the spin axis will remain close to the precession axis, which corresponds to stability of the initial rotation.

The stability condition is obtained if I_3 is either the largest moment of inertia, $I_3 > I_2$ and $I_3 > I_1$, or the smallest, $I_3 < I_2$ and $I_3 < I_1$. In other words, if a body is released with an initial angular velocity that is essentially a spin about the principal axis for which the moment of inertia is either the largest or smallest value, then it will continue with that type of rotation. An initial spin about the principal axis for which the moment of inertia is the intermediate value will show a growth in the nutation angle, such that the eventual rotation does not resemble the attempted initial state. Note in this regard that Eqs. (8.10) merely describe the onset of instability. They cannot be employed to study the unstable response, because the assumption of a small nutation angle would not be valid in such a case.

If you wish, you may test these stability properties by throwing a homogeneous rectangular object, such as a wooden block or a board eraser. Try to impart to it an initial spin about an axis parallel to one of its edges. It is fairly easy to obtain a motion in which the object spins about an axis parallel to the shortest or longest edge. However, a comparable attempt for rotation about the intermediate edge does not produce the desired steady spin.

8.1.2 *Axisymmetric Bodies*

In the preceding stability analysis of an initial spinning rotation, we assumed that the principal moments of inertia about the center of mass are three distinct values. An axisymmetric body - that is, any body whose mass is distributed symmetrically about an axis - has identical moments of inertia for all centroidal axes that perpendicularly intersect the axis of symmetry. Also, any set of axes containing the axis of symmetry are principal axes. Without loss of generality, we select the z axis to be the axis of symmetry, and correspondingly let $I_3 = I$ and $I_1 = I_2 = I'$.†

Equations (8.5) remain valid for an axisymmetric body. Substitution for the moments of inertia converts those relations to

$$\dot{\psi} = \frac{H_G}{I'}, \qquad \dot{\theta} = 0, \qquad \dot{\phi} = H_G\left(\frac{1}{I} - \frac{1}{I'}\right)\cos\theta. \tag{8.14}$$

This shows that the free rotation of an axisymmetric body is characterized by a steady spinning rotation about the axis of symmetry, accompanied by a steady precession about an axis that is parallel to the angular momentum; the nutation angle between these axes is constant.

Although Eqs. (8.14) fully characterize the motion, further examination will greatly enhance our qualitative understanding of free motion. First, we evaluate the angular velocity by substituting Eqs. (8.14) into Eq. (8.1). This yields

$$\bar{\omega} = -\left(\frac{H_G}{I'}\sin\theta\cos\phi\right)\bar{i} + \left(\frac{H_G}{I'}\sin\theta\sin\phi\right)\bar{j} + \left(\frac{H_G}{I}\cos\theta\right)\bar{k}. \tag{8.15}$$

Recall that one of the reference frames used in Chapter 4 to define the Eulerian angles was $x'y'z'$, which undergoes only the precessional and nutational rotations. As

† Any body having two equal principal moments of inertia behaves as though it were axisymmetric. The present analysis is valid for the free motion of such an object, provided the z axis is aligned with the axis that has the distinct moment of inertia.

shown in Figure 8.1, the y' axis is the line of nodes, whereas the z' axis coincides with the z axis. The unit vector along the x' axis is

$$\bar{i}' = (\cos\phi)\bar{i} - (\sin\phi)\bar{j}, \tag{8.16}$$

which means that the angular velocity in Eq. (8.15) may be written as

$$\bar{\omega} = -\left(\frac{H_G}{I'}\sin\theta\right)\bar{i}' + \left(\frac{H_G}{I}\cos\theta\right)\bar{k}. \tag{8.17}$$

One check on the correctness of this expression comes from using its components to reconstitute the angular momentum,

$$\bar{H}_G = I'\omega_{x'}\bar{i}' + I\omega_z\bar{k} = -(H_G\sin\theta)\bar{i}' + (H_G\cos\theta)\bar{k} = H_G\bar{K}. \tag{8.18}$$

This confirms that the angular momentum is aligned along the precession axis.

Because H_G and θ are constant, we find from Eq. (8.17) that the angular velocity is formed from two orthogonal components having constant magnitude. One component is parallel to the axis of symmetry, and the other lies in the plane formed by the axis of symmetry and the fixed direction characterized by the angular momentum. This representation of $\bar{\omega}$ is shown in Figure 8.2, as is the representation obtained by vectorially adding the precession and spin rates.

Suppose that the initial motion of the body is specified, which is equivalent to specifying the initial value of $\bar{\omega}$ and the initial orientation of the body. Such conditions mean that we know the initial angular speed $\omega = |\bar{\omega}|$, as well as the angle β between $\bar{\omega}$ and the axis of symmetry at the instant of release. (In order to avoid ambiguity, and without loss of generality, we consider the angle to be acute, $\beta < 90°$.) We could use the relations already established to express the other parameters in terms of these initial conditions. Instead, we shall develop the appropriate relations by referring to Figure 8.2, which displays three methods for constructing the components of $\bar{\omega}$. Specifically,

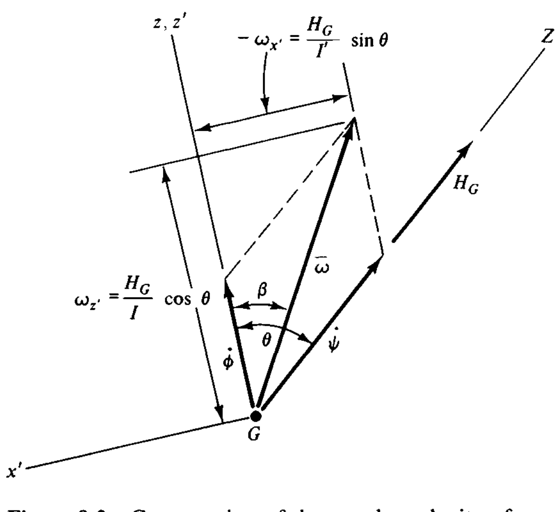


Figure 8.2 Construction of the angular velocity of an axisymmetric body in free motion.

$$\omega_{z'} = \omega\cos\beta = \frac{H_G}{I}\cos\theta = \dot\phi + \dot\psi\cos\theta,$$
$$-\omega_{x'} = \omega\sin\beta = \frac{H_G}{I'}\sin\theta = \dot\psi\sin\theta. \tag{8.19}$$

Eliminating from these relations all kinematical parameters except ω and β yields

$$\blacklozenge \quad \tan\theta = \frac{I'}{I}\tan\beta,$$
$$\blacklozenge \quad H_G = (I'^2\sin^2\beta + I^2\cos^2\beta)^{1/2}\omega,$$
$$\blacklozenge \quad \dot\psi = \left[\sin^2\beta + \left(\frac{I}{I'}\right)^2\cos^2\beta\right]^{1/2}\omega, \tag{8.20}$$
$$\blacklozenge \quad \dot\phi = \left(1 - \frac{I}{I'}\right)\omega\cos\beta.$$

The picture provided by Figure 8.2 may be considered as general, but we must remember that the entire system precesses at a constant rate about the Z axis. As the motion evolves, the angular velocity sweeps out a cone in space whose axis is parallel to $\bar{H}_G$, and whose vertex half-angle is $\theta - \beta$. Similarly, the axis of symmetry precesses such that it is coplanar with the $\bar\omega$ and $\bar{H}_G$ vectors, at a constant angle θ relative to the precession axis.

Such a motion may be represented by a conceptual model formed from (right circular) cones. The *body cone,* which is fixed to the body, rolls without slipping over the stationary *space cone.* The angular velocity of the body cone must be parallel to the line of contact between the cones, because the instantaneous axis of rotation is the locus of points in the body having zero velocity. Hence, $\bar{H}_G$ defines the axis of the space cone whose semivertex angle is $\beta - \theta$, whereas the axis of symmetry is the axis of the body cone whose semivertex angle is β. The sides of the space and body cones contact at $\bar\omega$.

Two types of rotation are recognizable from this model. *Regular precession,* which is shown in Figure 8.3, corresponds to rotations in which $\beta < \theta$. The exterior of the body cone in this case rolls over the exterior of the space cone. Note from the first of Eqs. (8.20) that regular precession is obtained if $I' > I$, which is characteristic of a slender body such as a football. We see from the last of Eqs. (8.20) that whenever $I' > I$, the spin rate is positive; the third equation shows that the precession rate is always positive. Therefore, the angle between the precession and spin-rate vectors is acute in a regular precession.

Figure 8.4 depicts a *retrograde precession,* corresponding to $\beta > \theta$. In this case, the interior of the body cone rolls over the exterior of the space cone. Such a rotation arises when $I' < I$, which corresponds to a squat body such as a disk. Here the spin rate is negative, so the spin-rate vector is oriented along the negative z axis. Because the precession rate is always positive, the angle between the two rotation rate vectors is now $180° - \theta$, so we perceive the precession to be generally opposite to the sense of the spin. This counterrotation is the source of the term "retrograde."

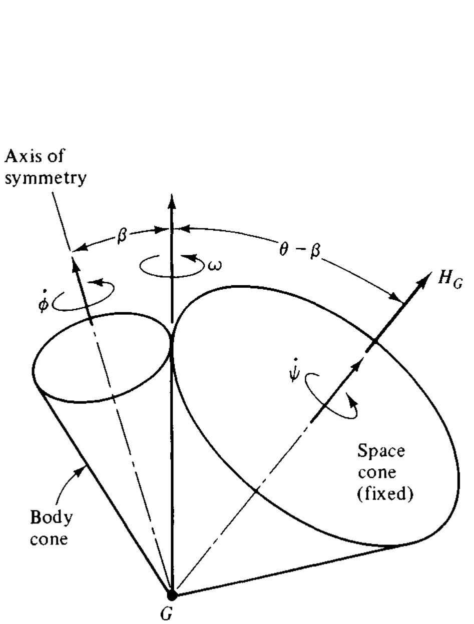


Figure 8.3 Body and space cones for regular precession.

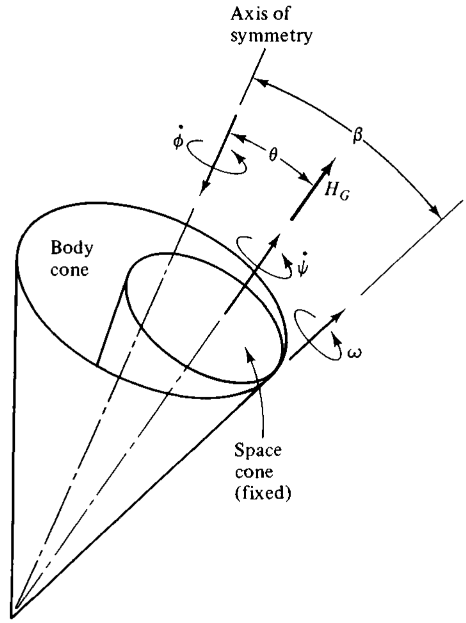


Figure 8.4 Body and space cones for retrograde precession.

Example 8.1 A football has an instantaneous velocity of 25 m/s parallel to its longitudinal axis z, and is spinning about that axis at 5 rev/sec. At that instant, the ball is deflected by a transverse force $\bar{F}$ at the forward tip. As a result of the action of $\bar{F}$, whose duration is very short, the ensuing motion relative to the center of mass is such that the longitudinal axis always lies on the surface of a cone whose apex angle is 60°. The radii of gyration about centroidal axes are 40 mm and 70 mm along and transverse (respectively) to the longitudinal axis. Determine:

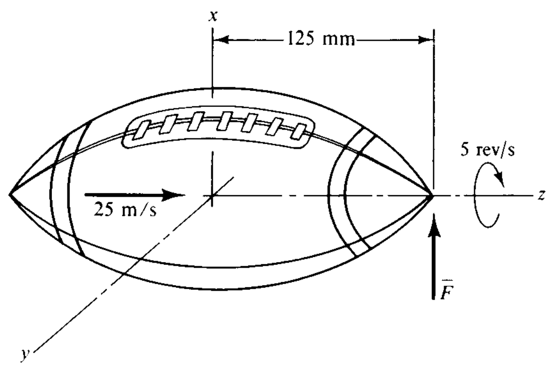
(a) the angular velocity and the velocity of the center of mass immediately after the application of $\bar{F}$;
(b) the orientation of the precession axis for the subsequent rotation relative to the orientation of the longitudinal axis prior to the application of $\bar{F}$; and
(c) the precession and spin rates for the rotational motion.

Solution The force $\bar{F}$ fits the impulsive model because it induces a substantial change in the motion over a short time interval. We neglect position changes during this interval, which means that the orientation of the body-fixed xyz reference frame changes negligibly during the interval of the impulse. The moments of inertia are

$$I = I_{zz} = m(0.040^2) = 0.0016m \text{ kg-m}^2,$$

$$I' = I_{xx} = I_{yy} = m(0.070^2) = 0.0049m \text{ kg-m}^2,$$

where m is the mass (in units of kg).



Example 8.1

The initial linear and angular momenta are

$$\bar{P}_1 = m(\bar{v}_G)_1 = 25m\bar{k} \text{ kg-m/s},$$

$$(\bar{H}_G)_1 = I\omega_z\bar{k} = I(-5)(2\pi)\bar{k} = -0.05027m\bar{k} \text{ kg-m}^2\text{/s}.$$

Because the change in the position of any point on the football is negligible during the impulse interval, the point of application of $\bar{F}$ is essentially constant at $\bar{r}_{P/G} = 0.125\bar{k}$ m. The corresponding impulse-momentum principles are

$$m(\bar{v}_G)_2 = 25m\bar{k} + (F\Delta t)\bar{i},$$

$$(\bar{H}_G)_2 = (\bar{H}_G)_1 + (\bar{r}_{P/G} \times F\Delta t)\bar{i} = -0.05027m\bar{k} + (0.125F\Delta t)\bar{j}.$$

Because $I' > I$, the free rotation of the football is a regular precession. According to Figure 8.3, the given information that the z axis sweeps out a 60° cone in the subsequent rotation means that the nutation angle is $\theta = 30°$, with the precession axis coincident with the axis of that cone. We find the corresponding angle β between the angular velocity and the axis of symmetry from Eqs. (8.20) as follows:

$$\beta = \tan^{-1}\left(\frac{I}{I'}\tan\theta\right) = 10.076°.$$

The preceding expression for $(\bar{H}_G)_2$ indicates that $\omega_x = 0$ at the instant when $\bar{F}$ terminates, while $\omega_y > 0$ and $\omega_z < 0$ at that instant. Hence, the angular velocity at that instant must be

$$\bar{\omega}_2 = -(\omega\cos\beta)\bar{k} + (\omega\sin\beta)\bar{j} = \omega(-0.9027\bar{k} + 0.18526\bar{j}) = \omega_y\bar{j} + \omega_z\bar{k}.$$

The corresponding angular momentum is

$$(\bar{H}_G)_2 = I\omega_z\bar{k} + I'\omega_y\bar{j} = m\omega(-1.5723\bar{k} + 0.9078\bar{j})(10^{-3}).$$

Matching this to the first expression for $\bar{H}_G$ yields

$$1.5723(10^{-3})m\omega = 0.0527m, \qquad 0.9078(10^{-3})m\omega = 0.125F\Delta t,$$

$$\omega = 31.97 \text{ rad/s}, \qquad \frac{F\Delta t}{m} = 0.2322 \text{ m/s}.$$

The corresponding motion parameters are

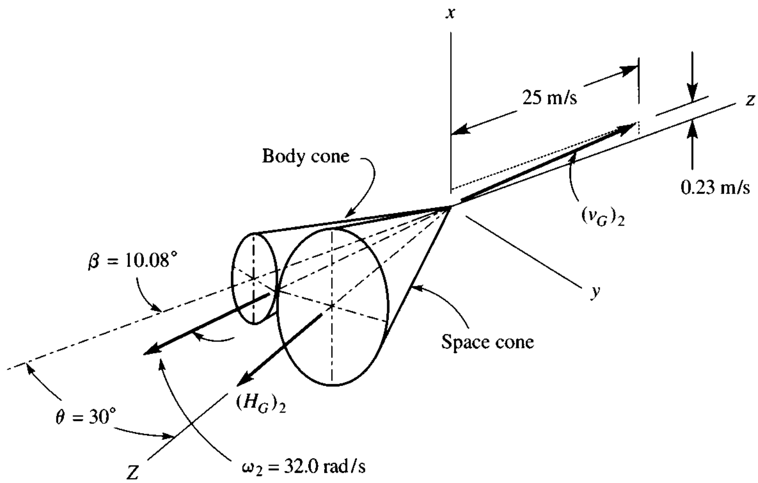
$$\bar{\omega}_2 = -31.42\bar{k} + 5.923\bar{j} \text{ rad/s},$$

$$(\bar{v}_G)_2 = 25\bar{k} + 0.2322\bar{j} \text{ m/s}.$$

The precession axis is parallel to $(\bar{H}_G)_2$, which is the angular momentum for the subsequent free-motion axis. Thus

$$\bar{K} = \frac{(\bar{H}_G)_2}{|(\bar{H}_G)_2|} = \frac{-1.5723\bar{k} + 0.9078\bar{j}}{(1.5723^2 + 0.9078^2)^{1/2}} = -0.8660\bar{k} + 0.500\bar{j}.$$

Note that the angle between $\bar{K}$ and $\bar{k}$ - that is, between the symmetry and precession axes - is $\cos^{-1}(-0.8660) = 150°$, in agreement with the stated conditions.



Motion at the end of the impulse.

From these results we may draw a sketch of the position of the body cone relative to the space cone at the initiation of the free motion. We also show $(\bar{v}_G)_2$ in that sketch. The corresponding precession and spin rates are given by Eqs. (8.20),

$$\dot{\psi} = 11.85 \text{ rad/s}, \qquad \dot{\phi} = 21.16 \text{ rad/s}.$$

8.1.3 *Poinsot's Construction for Arbitrary Bodies*

When the principal moments of inertia are unequal, the nutation angle will generally not be constant. As mentioned earlier, one approach in this case is to seek analytical or numerical solutions of the first-order equations of motion, Eqs. (8.5). Here, we shall develop a pictorial representation of the motion that considerably enhances our qualitative understanding of free rotation. The framework for this development is the ellipsoid of inertia, which was described in Section 5.2.3.

We begin by noting that the general expression for rotational kinetic energy, which employs angular velocity components and inertia properties $[I]$ relative to xyz, is more simply represented in terms of the moment of inertia I about the instantaneous axis of rotation. Specifically,

$$T_{\text{rot}} = \tfrac{1}{2}\bar{\omega}\cdot\bar{H}_G = \tfrac{1}{2}\{\omega\}^{\text{T}}[I]\{\omega\} = \tfrac{1}{2}I\omega^2. \tag{8.21}$$

The ellipsoid of inertia is a fictitious body that moves in unison with the actual body. The major, minor, and intermediate axes of the ellipsoid coincide with the principal axes of the body, which are xyz in the current situation. The distance from the center of mass G to any point P on the inertia ellipsoid is defined to be the rate at which the body should rotate about axis GP in order for the rotational kinetic energy to be one half. Setting $T_{rot} = 1/2$ in Eq. (8.21) shows that the required rotation rate is $1/\sqrt{I}$, where I is the moment of inertia about axis GP.

If (x, y, z) are the coordinates of point P relative to the body-fixed reference frame whose origin is the center of mass, then the position of this point may be written in vector and matrix notation as

$$\bar{\rho} = x\bar{i} + y\bar{j} + z\bar{k}, \qquad \{\rho\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}. \tag{8.22}$$

By definition, we have $\rho = 1/\sqrt{I}$. This does not represent an explicit relation among the (x, y, z) values because I depends on the location of point P. We obtain such a relation in a different manner. By definition, $\bar{\rho}$ represents the angular velocity required for $T_{rot} = 1/2$ when the axis of rotation is parallel to line GP. Substitution of $\bar{\omega} = \bar{\rho}$ and $\omega = 1/\sqrt{I}$ into Eq. (8.21) yields

$$\{\rho\}^T [I] \{\rho\} = 1. \tag{8.23a}$$

This product may be expanded for arbitrary $[I]$. However, we defined x, y, and z to be principal axes, with I_1, I_2, and I_3 as the respective principal values. Thus, the expanded form of Eq. (8.23a) is

♦ $$I_1 x^2 + I_2 y^2 + I_3 z^2 = 1. \tag{8.23b}$$

In order to describe the rotation of an inertia ellipsoid in free motion, we draw a line through the center of mass parallel to the angular velocity $\bar{\omega}$ at an arbitrary instant; this construction appears in Figure 8.5. The point P we shall follow is the

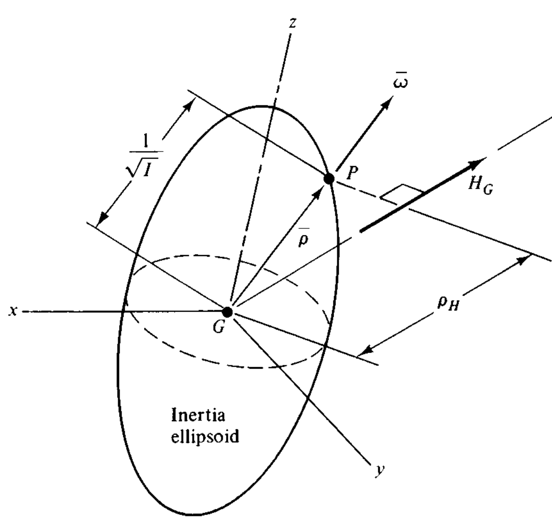


Figure 8.5 Inertia ellipsoid of an arbitrary body in free motion.

intersection of this line with the surface of the ellipsoid. If we know $\bar{\omega}$, the corresponding $\bar{\rho}$ is a vector parallel to $\bar{\omega}$ with magnitude $1/\sqrt{I}$; that is

$$\bar{\rho} = \frac{1}{\sqrt{I}} \frac{\bar{\omega}}{\omega}. \tag{8.24a}$$

Thus, the coordinates of point P are

$$x = \frac{\omega_x}{\sqrt{I}\,\omega}, \qquad y = \frac{\omega_y}{\sqrt{I}\,\omega}, \qquad z = \frac{\omega_z}{\sqrt{I}\,\omega}. \tag{8.24b}$$

Recall that the angular momentum of a body is constant in a free motion. Since we may construct this vector from the rotational motion at the instant the body was released, $\bar{H}_G$ defines a convenient reference direction. The first significant aspect of the inertia ellipsoid's motion comes from evaluating the component of $\bar{\rho}$ that is parallel to the angular momentum. As shown in Figure 8.5, this component, denoted ρ_H, may be obtained from a dot product of $\bar{\rho}$ with a unit vector parallel to $\bar{H}_G$. In view of Eqs. (8.21) and (8.24a), we obtain

$$\rho_H = \bar{\rho} \cdot \frac{\bar{H}_G}{H_G} = \frac{\bar{\omega}}{\sqrt{I}\,\omega} \cdot \frac{\bar{H}_G}{H_G} = \frac{(2T_{\text{rot}})^{1/2}}{H_G}. \tag{8.25}$$

Now note that, because the forces acting on a body in free motion exert no moment about the center of mass G, no work is done in the rotation. Consequently, the rotational kinetic energy is constant. Because both T_{rot} and H_G are constant, the distance ρ_H remains constant as the body and its ellipsoid of inertia rotate.

One definition of a plane states that it is the locus of points whose distance to a specified point, measured parallel to a fixed direction, is constant. This direction is the *normal* to the plane. It follows that point P always lies on a plane that is at a constant distance ρ_H from the center of mass, with $\bar{H}_G$ being the normal to the plane. If we ignore the movement of the center of mass, this plane appears to be stationary; it is the *invariable plane.*

Knowledge of the invariable plane does not fully prescribe the motion of the ellipsoid of inertia. We have not established how point P moves along the plane, nor do we know how the ellipsoid is oriented relative to the plane. In order to address these questions, we shall derive another property of the ellipsoid of inertia.

Let us define a family of concurrent ellipsoids having the same proportions as the inertia ellipsoid. Let C be a constant that scales the magnitude of $\bar{\rho}$. In other words, if point P' on a different ellipsoid is collinear with the line from the center of mass G to point P on the inertia ellipsoid, then the distance from point G to point P' is $C/\sqrt{I}$. It follows from Eq. (8.23b) that the coordinates of point P' satisfy

$$F(x, y, z, T_{\text{rot}}) = I_1 x^2 + I_2 y^2 + I_3 z^2 = C^2, \tag{8.26}$$

where $C = 1$ corresponds to the ellipsoid of inertia.

The gradient operator applied to Eq. (8.26) indicates the direction in which the value of C changes most rapidly when going from one surface to another. Therefore, the gradient of F, which is

$$\nabla F = 2I_1 x\bar{i} + 2I_2 y\bar{j} + 2I_2 z\bar{k}, \tag{8.27a}$$

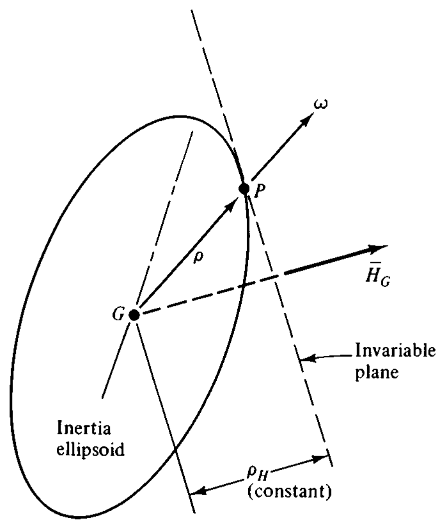


Figure 8.6 Poinsot's construction of the inertia ellipsoid in free motion.

defines the normal to the ellipsoid that contains the point (x, y, z). The coordinates of point P, at which the rotation axis intersects the inertia ellipsoid, are given by Eqs. (8.24b), which when substituted into Eq. (8.27a) yield

$$\nabla F = \frac{2}{\sqrt{I}\,\omega}(I_1\omega_x\bar{i} + I_2\omega_y\bar{j} + I_3\omega_z\bar{k}) = \frac{2}{\sqrt{I}\,\omega}\bar{H}_G. \tag{8.27b}$$

We see now that the normal to the inertia ellipsoid at point P is parallel to the angular momentum. However, the normal to the invariable plane is also parallel to $\bar{H}_G$. As shown in Figure 8.6, these two statements mean that the ellipsoid of inertia is always tangent at point P to the invariable plane. Furthermore, the velocity of point P is zero, because it is on the instantaneous axis of rotation. These observations lead us to the Poinsot construction, which states that

♦ *The inertia ellipsoid of a body in free motion rotates about the center of mass such that it rolls without slipping over the invariable plane. The normal to the invariable plane is parallel to the (constant) angular momentum of the body. The line extending from the center of mass to the point where the ellipsoid tangentially contacts the invariable plane is parallel to the instantaneous axis of rotation. The rolling motion is such that the perpendicular distance from the center of mass to the invariable plane is constant, at a value that depends on the angular momentum and (constant) rotational kinetic energy relative to the center of mass.*

The initial conditions at the instant the body was released define $\bar{H}_G$ and T_{rot}, which, in turn, define the invariable plane and the distance therefrom to the center of mass. At each instant, a different point on the inertia ellipsoid contacts the invariable plane. The locus of contact points on the inertia ellipsoid is a curve called the *polhode,* while the locus on the invariable plane forms the *herpolhode.* The herpolhode is generally an open curve, which means that the rotation does not repeat, but the polhode is a closed curve. In the special case where the body is axisymmetric with respect to the centerline appearing in Figure 8.6, the Poinsot construction reduces to the space- and body-cone representations derived from Figure 8.2. In that case, the herpolhode is closed.

The closure of the polhode, as well as the overall nature of these curves, may be established by noting that the inertia ellipsoid represents the constancy of kinetic energy in rotational motion. However, the angular momentum is also constant, which means that

$$H_G^2 = \bar{H}_G \cdot \bar{H}_G = (I_1 \omega_x)^2 + (I_2 \omega_y)^2 + (I_3 \omega_z)^2 = \text{constant}. \tag{8.28}$$

Let us use Eqs. (8.24b) to express this relation in terms of the coordinates of point P at which the inertia ellipsoid contacts the invariable plane. Thus,

$$H_G^2 = I\omega^2[(I_1 x)^2 + (I_2 y)^2 + (I_3 z)^2]. \tag{8.29}$$

However, at each instant the body is rotating at rate ω about the instantaneous axis through origin G and point P, and I is the moment of inertia about that axis. Thus, $I\omega^2 = 2T_{\text{rot}}$, so the foregoing becomes

♦ $$I_1^2 x^2 + I_2^2 y^2 + I_3^2 z^2 = \frac{H_G^2}{2T_{\text{rot}}} = \frac{1}{\rho_H^2} \equiv D. \tag{8.30}$$

This relation characterizes another ellipsoid that is fixed to the body. The intersection of the ellipsoid given by Eq. (8.30) with the ellipsoid of inertia, Eq. (8.23b), is the polhode. The closure of the polhode is a direct consequence of the fact that both ellipsoids, and therefore their intersection, rotate with the body.

The value of the constant D is determined by the initial motion. In view of Eqs. (8.24b) and (8.30), initial rotations about each of the principal axes correspond to $D = I_1, I_2, I_3$, respectively. Without loss of generality, we now specify the labeling of the xyz axes to be such that I_1 is the smallest value and I_3 the largest. Then, $I_1 \le D \le I_3$. The polhode for a specified value of D may be constructed by picking a value of one coordinate and then solving Eqs. (8.23b) and (8.30) simultaneously for the other two. Alternatively, the projection of a polhode curve onto any of the principal coordinate planes may be derived by eliminating the coordinate normal to that plane from the two relations. These projection equations are

$$\begin{aligned} x\text{-}y \text{ plane:} \quad & I_1(I_3 - I_1)x^2 + I_2(I_3 - I_2)y^2 = I_3 - D; \\ y\text{-}z \text{ plane:} \quad & I_2(I_2 - I_1)y^2 + I_3(I_3 - I_1)z^2 = D - I_1; \\ x\text{-}z \text{ plane:} \quad & I_1(I_2 - I_1)x^2 - I_3(I_3 - I_2)z^2 = I_2 - D. \end{aligned} \tag{8.31}$$

We wrote each of these equations such that the coefficients are positive for the assigned sequence $I_1 < I_2 < I_3$. For this ordering, the projections onto the x-y and y-z planes are ellipses, and the projections onto the x-z plane are hyperbolas. These projections, and the outline of the inertia ellipsoid, are illustrated in Figure 8.7 for the positive quadrants. The curve corresponding to $D = I_2$ is the separatrix between the hyperbolas in the x-z plane, but it appears as an ellipse in the other coordinate planes. The corresponding polhode curves on the ellipsoid of inertia are shown in Figure 8.8.

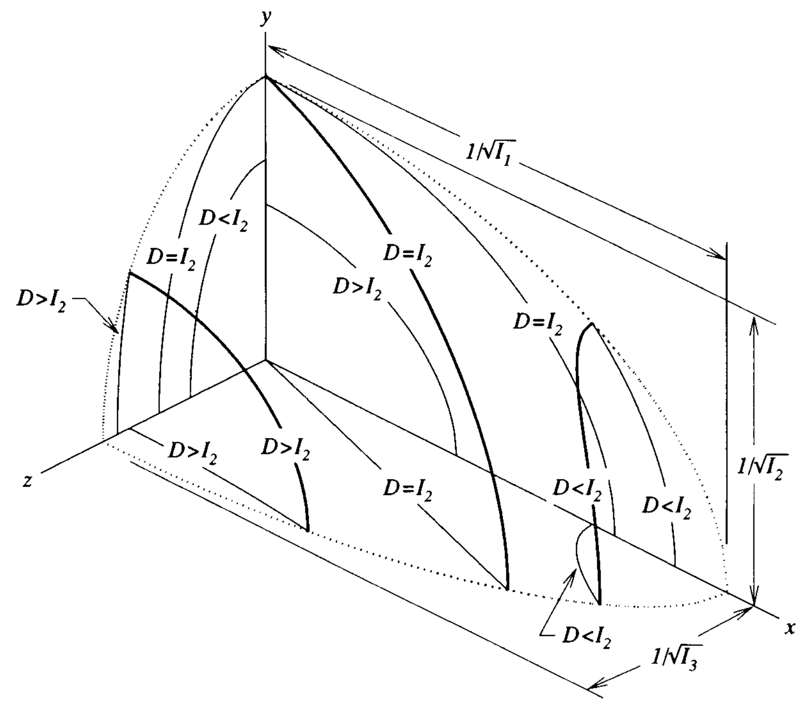


Figure 8.7 Typical polhode curves in the first octant and their projections onto the principal-axis planes for $I_1 = 1$, $I_2 = 4$, $I_3 = 8$.

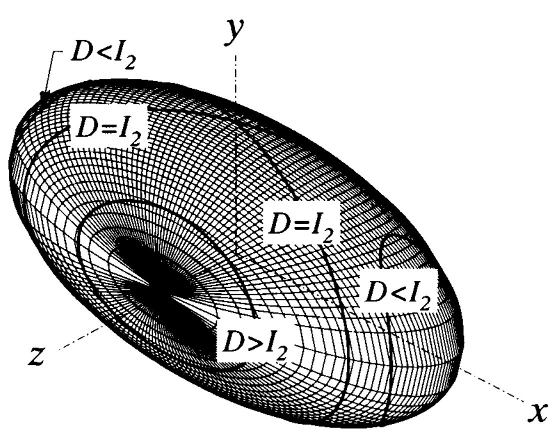
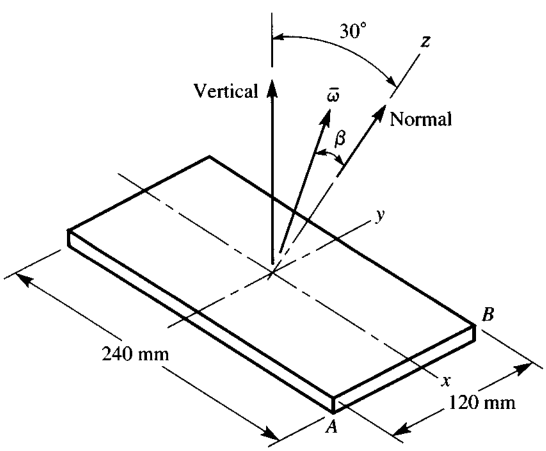


Figure 8.8 Typical polhode curves and the inertia ellipsoid for $I_1 = 1$, $I_2 = 4$, $I_3 = 8$.

We concluded in Section 8.1.1 that an attempt to impart a rotation about the principal axis of smallest or largest moment of inertia would produce a stable rotation. This is further demonstrated here. Recall that the instantaneous angular velocity $\bar{\omega}$ is parallel to a line from the origin (i.e., the center of mass) to the point where the polhode curve contacts the invariable plane. If the initial rotation is approximately about the z axis, then D is slightly smaller than I_3. In that case, the projection of the polhode curve onto the x-y plane is a small ellipse, corresponding to an angular velocity that always is nearly parallel to the z axis. Similarly, an initial rotation approximately about the x axis, which gives a value of D slightly larger than I_1, leads to a polhode curve projection on the y-z plane that is a small ellipse. This corresponds to an angular velocity that is always nearly parallel to the x axis. In either case, the axis of rotation remains close to the respective principal axis. In contrast, if the initial motion is approximately about the y axis, then $D \approx I_2$. Then the polhode curves are close to the separatrices. Depending on whether D is greater than or less than I_2, the closed polhode curve is centered about either the z axis or the x axis, respectively. In either case, the angle between $\bar{\omega}$ and any of the coordinate axes varies greatly in the motion. This explains why the rotation of an arbitrary body is often difficult to observe.

Example 8.2 At the instant the 10-kg rectangular plate is released, edge AB is horizontal. The angle between the plate's normal and the vertical direction is 30°. The angular velocity at that instant lies in the vertical plane containing the normal, with β defined as the angle between $\bar{\omega}$ and the normal.

(a) Determine the value of β for which the precession axis is vertical.
(b) Determine the maximum value of β for which the angle between the normal and the precession axis will not exceed 90° in the rotation after release.
(c) For the case where β is one half the critical value in part (b), determine the minimum and maximum angles between the plate's normal and the precession axis during the rotation. Evaluate the corresponding angular velocity at these limits.



Example 8.2

Solution A centroidal coordinate system whose axes are aligned with the edges of the plate is principal. In accordance with the derivation, the axes in the sketch are labeled such that $I_{xx} = I_1$ is the smallest principal value and $I_{zz} = I_3$ is the largest. These values are

$$I_1 = \frac{1}{12}10(0.12^2) = 0.012, \quad I_2 = \frac{1}{12}10(0.24^2) = 0.048,$$

$$I_3 = \frac{1}{12}10(0.12^2 + 0.24^2) = 0.060 \text{ kg-m}^2.$$

The orientation of the initial angular velocity relative to xyz is described by the angle β, so we have

$$\bar{\omega} = \omega[-(\sin\beta)\bar{i} + (\cos\beta)\bar{k}].$$

The corresponding constant angular momentum is

$$\bar{H}_G = I_1\omega_x\bar{i} + I_2\omega_y\bar{j} + I_3\omega_z\bar{k} = 0.012\omega[-(\sin\beta)\bar{i} + (5\cos\beta)\bar{k}].$$

To answer the first question, we require that the angular momentum be parallel to the vertical axis. Resolving this vector into xyz components gives

$$\bar{H}_G = H_G[-(\sin 30°)\bar{i} + (\cos 30°)\bar{k}].$$

Matching the two descriptions of $\bar{H}_G$ leads to

$$0.012\omega \sin\beta = 0.5H_G, \qquad 0.060\omega\cos\beta = 0.8660H_G,$$

so that

$$0.20\tan\beta = 0.5774 \Rightarrow \beta = 70.89°.$$

In order to address the second question, we examine Figure 8.8, and recall that the normal to the tangent plane at a point represents the precession axis. Consider the polhode curve corresponding to $D > I_2$, which surrounds the z axis. At every point on this curve, the tangent plane has a normal that forms an acute angle with the z axis. The limiting case is $D = I_2$, which defines the separatrices, because the tangent plane's normal at $x = z = 0$ is parallel to the y axis, and therefore perpendicular to the z axis. Thus, the maximum value of β satisfying the specification in part (b) is that which gives $D = I_2$. The earlier expressions for $\bar{\omega}$ and $\bar{H}_G$ in terms of β give

$$2T_{\text{rot}} = \bar{\omega}\cdot\bar{H}_G = 0.012\omega^2(\sin^2\beta + 5\cos^2\beta).$$

Hence, the critical condition is

$$D = \frac{H_G^2}{2T_{\text{rot}}} = 0.012\frac{\sin^2\beta + 25\cos^2\beta}{\sin^2\beta + 5\cos^2\beta} = I_2 = 0.048,$$

which becomes

$$\sin^2\beta + 25\cos^2\beta = 4(\sin^2\beta + 5\cos^2\beta) \Rightarrow \tan^2\beta = 5/3;$$

$$\text{critical } \beta = 52.239°.$$

For part (c), we set $\beta = 26.119°$. The corresponding value of D is readily obtained from the preceding general formula as follows:

$$D = 0.012\frac{\sin^2\beta + 25\cos^2\beta}{\sin^2\beta + 5\cos^2\beta} = 0.057798.$$

Because $D > I_2$, the polhode curve surrounds the z axis. Now examine the polhode curve for this case in Figure 8.8. The minimum and maximum angles between the z

axis and the normal to the tangent plane, which is the fixed precession axis, occur when the polhode curve intersects the x–z plane and y–z plane, respectively. Thus, the task of identifying the minimum and maximum angle conditions reduces to establishing the conditions for which the angular velocity has either a zero y or x component, respectively.

This observation leads us directly to conclude that the initial motion must represent the minimum angle condition, because it is specified that $\omega_y = 0$ at that instant. Thus, we determine the minimum angle between the z axis and the precession axis by forming the angular momentum corresponding to $\beta = 26.119°$:

$$\bar{H}_G = 0.012\omega[-(\sin\beta)\bar{i} + (5\cos\beta)\bar{k}] = \omega(0.005283\bar{i} + 0.053873\bar{k}),$$

$$\theta_{\min} = \cos^{-1}\left(\frac{\bar{H}_G \cdot \bar{k}}{|\bar{H}_G|}\right) = 5.6007°.$$

To determine the maximum angle, we seek the solution of the polhode equations that corresponds to the foregoing value of D and $x = 0$. The polhode curves correspond to values of (x, y, z) that simultaneously satisfy Eqs. (8.23b) (the inertia ellipsoid equation) and (8.30) (constancy of T_{rot}). For the present values of the parameters, these equations are

$$0.012(x^2 + 4y^2 + 5z^2) = 1,$$

$$0.012^2(x^2 + 16y^2 + 25z^2) = 0.057798.$$

The polhode of interest corresponds to $z > 0$, because ω_z was initially positive. We therefore seek the root of these equations for which $x = 0$ and $z > 0$. The roots are

$$y = \pm 1.9553, \qquad z = 3.6889.$$

We shall use the positive value of y; either sign will yield the same angle. According to Eqs. (8.24), the corresponding angular velocity is

$$\bar{\omega}_i = \omega_i \sqrt{I_i}(y\bar{j} + z\bar{k}) = \omega_i \sqrt{I_i}(1.9553\bar{j} + 3.6889\bar{k}),$$

where the subscript i identifies the quantities as instantaneous values. To find the moment of inertia about the instantaneous axis of rotation, we use this relation to form $|\bar{\omega}|$, which yields

$$I_i = \frac{1}{1.9553^2 + 3.6889^2} = 0.057368.$$

Substituting I_i into the previous expression for $\bar{\omega}_i$ leads to

$$\bar{\omega}_i = \omega_i(0.46831\bar{j} + 0.88356\bar{k}).$$

The angular momentum corresponding to this expression for $\bar{\omega}_i$ is

$$\bar{H}_G = I_2\omega_y\bar{j} + I_3\omega_z\bar{k} = 0.048(0.46831\omega_i)\bar{j} + 0.060(0.88536\omega_i)\bar{k}$$
$$= \omega_i(0.022479\bar{j} + 0.053014\bar{k}).$$

We may evaluate ω_i by equating constant $|\bar{H}_G|$ with the value corresponding to the initial instant, at which $\beta = 26.119°$:

$$H_G^2 = \omega_i^2(0.022479^2 + 0.053014^2)$$
$$= (0.012\omega)^2(\sin^2\beta + 25\cos^2\beta) = 0.0029302\omega^2;$$

$$\omega_i = 0.9401\omega.$$

Substituting this value into the expression for $\bar{\omega}_i$ yields

$$\bar{\omega}_i = \omega(0.44024\bar{j} + 0.83060\bar{k}).$$

The corresponding angle between the z axis and $\bar{H}_G$ may be found from a dot product,

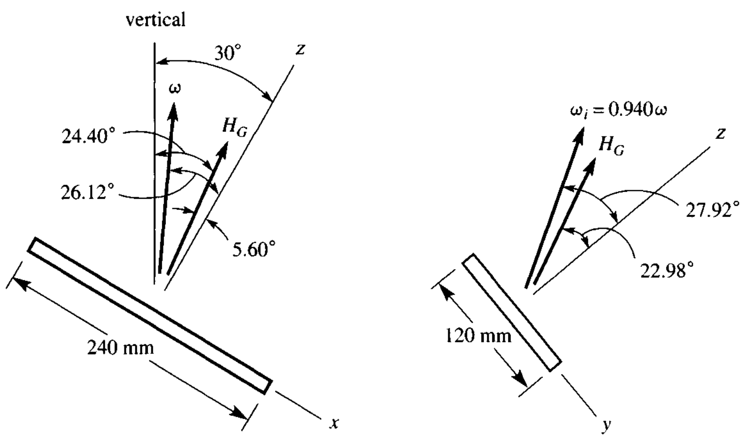
$$\cos\theta_i = \frac{\bar{H}_G \cdot \bar{k}}{|H_G|} = \frac{0.053014}{(0.022479^2 + 0.053014^2)^{1/2}};$$

$$\theta_i = \theta_{\max} = 22.978°.$$

The angle between the angular velocity and the z axis is

$$\beta_i = \cos^{-1}\left(\frac{\bar{\omega}_i \cdot \bar{k}}{\omega_i}\right) = 27.924°.$$

The orientation at this instant of the principal axes and the angular velocity are depicted in the accompanying sketch. Also shown is the initial condition, which we



Extreme conditions of the free motion.

found corresponds to the minimum angle. Note that the planes for each sketch are not the same; to locate the plane for the second sketch, we would need to know the precession angle. Each situation in the sketch is mirrored by another, not shown, in which everything is rotated about the precession axis by 180°. At an arbitrary instant, the angles between the z axis and the precession axis, and between the angular velocity and the precession axis, will be intermediate to the illustrated conditions and its mirror image.

8.2 Spinning Top

The toy known as a spinning top consists of an axially symmetric body that executes a pure rotation about an apex situated on the axis of symmetry. (We shall not worry here about the drift that occurs when the apex is not anchored, primarily because such effects are complicated by minor irregularities in the surface over which the apex would move.) The study of a spinning top leads to many insights regarding the

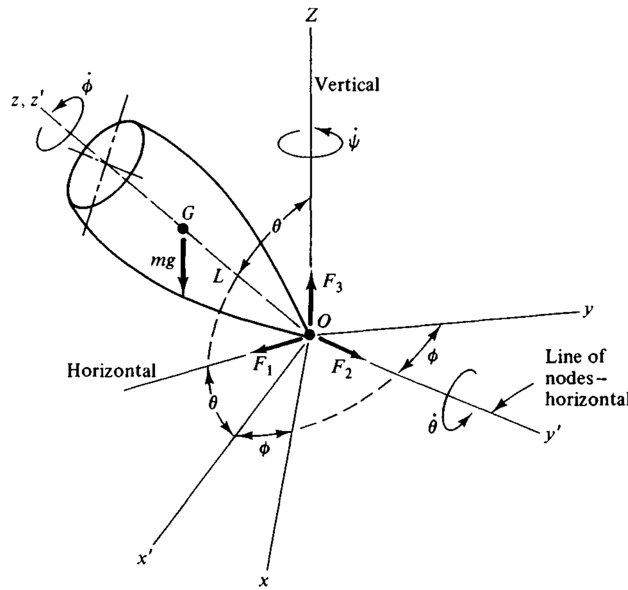


Figure 8.9 Free-body diagram for a spinning top.

interplay between rotation, angular momentum, and the moment exerted by forces. The results for its motion may be extended to other bodies that rotate about a reference point owing to the moment of the gravitational force; such systems include certain types of gyroscopes.

In Figure 8.9, point O is considered to be stationary owing to a reaction force having three components F_j in the horizontal and vertical directions. The gravity force acts through the center of mass G. Its moment about point O is $mgL\sin\theta$ in the direction of the horizontal axis through point O and perpendicular to the axis of symmetry. Because such an axis is the line of nodes (nutation axis) for a set of Eulerian angles, it is natural to formulate the equations of motion in terms of those parameters. Note that the reactions exert no moments about the precession, spin, and nutation axes, so the generalized nonconservative force associated with each angle is identically zero. Thus, the principal difference between a spinning top and an axisymmetric body in free motion is the presence of a moment about the reference point for the rotation. This moment must be balanced by an angular momentum that varies with time.

We shall employ Lagrange's equations to formulate the equations of motion. Let I be the moment of inertia about the axis of symmetry and let I' be the moment of inertia about any axis perpendicular to the axis of symmetry and intersecting point O. When resolved into components relative to the $x'y'z'$ axes for the Eulerian angles, the angular velocity of the body is

$$\bar{\omega} = -(\dot{\psi}\sin\theta)\bar{i}' + \dot{\theta}\bar{j}' + (\dot{\psi}\cos\theta + \dot{\phi})\bar{k}'. \tag{8.32}$$

The moment of inertia is the same for any axis through the apex O and perpendicular to the axis of symmetry, so the kinetic energy corresponding to this expression for $\bar{\omega}$ is

$$T = \tfrac{1}{2}I(\dot{\psi}\cos\theta + \dot{\phi})^2 + \tfrac{1}{2}I'(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2). \tag{8.33}$$

The elevation of the apex is a convenient reference for the gravitational potential energy, so

$$V = mgL\cos\theta. \tag{8.34}$$

We noted earlier that the generalized forces are all zero in our idealized model. Furthermore, in this case the Lagrangian $\mathcal{L} = T - V$ does not depend explicitly on either the precession or spin angles. As a result, the precession and spin angles are ignorable coordinates, corresponding to conservation of the generalized momenta associated with these variables. These momenta are

$$\begin{aligned} p_\psi &= \frac{\partial T}{\partial \dot{\psi}} = I(\dot{\psi}\cos\theta + \dot{\phi})\cos\theta + I'\dot{\psi}\sin^2\theta = I'\beta_\psi, \\ p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = I(\dot{\psi}\cos\theta + \dot{\phi}) = I'\beta_\phi, \end{aligned} \tag{8.35}$$

where β_ψ and β_ϕ are constants having the units of angular speed. The values of β_ψ and β_ϕ are specified by the initial conditions, so Eqs. (8.35) yield the following first-order differential equations for the precession and spin angles:

$$\begin{aligned} \blacklozenge \quad \dot{\psi} &= \frac{\beta_\psi - \beta_\phi\cos\theta}{\sin^2\theta}, \\ \blacklozenge \quad \dot{\phi} &= \frac{\beta_\phi(I'\sin^2\theta + I\cos^2\theta) - \beta_\psi I\cos\theta}{I\sin^2\theta}. \end{aligned} \tag{8.36}$$

In both differential equations, the still-undetermined nutation is the excitation. A constant value of β_ϕ corresponds to constancy of the total rotation rate about the axis of symmetry, $\omega_z = (\dot{\psi}\cos\theta + \dot{\phi})$. The foregoing expressions reveal that the precession and spin rates are individually constant only when the nutation angle is constant.

Constant values of p_ψ and p_ϕ satisfy the Lagrange equations associated with ψ and ϕ. In the derivation of the third Lagrange equation, which governs θ, we must evaluate the derivatives of the energy expressions before we use the conserved momenta to eliminate the ignorable coordinates. Carrying out the appropriate derivatives of Eq. (8.33) yields

$$I'\ddot{\theta} - (\dot{\psi}\sin\theta)[I'\dot{\psi}\cos\theta - I(\dot{\psi}\cos\theta + \dot{\phi})] - mgL\sin\theta = 0. \tag{8.37}$$

We substitute Eqs. (8.36) into this expression in order to remove the precession and spin rates, which yields

$$\ddot{\theta} + \frac{1}{\sin^3\theta}(\beta_\psi - \beta_\phi\cos\theta)(\beta_\phi - \beta_\psi\cos\theta) - \frac{mgL}{I'}\sin\theta = 0. \tag{8.38}$$

We shall employ this equation of motion later. A first integral of Eq. (8.38) could be obtained by separating variables using the chain-rule identity

$$\ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\dot{\theta} = \frac{1}{2}\frac{d}{d\theta}(\dot{\theta}^2).$$

However, it is much simpler to observe that mechanical energy, $E = T + V$, is conserved. Expressions for the kinetic and potential energy are given by Eqs. (8.33) and (8.34). Eliminating the precession and spin rates with the aid of Eqs. (8.36) yields

$$E = \frac{1}{2} I' \dot{\theta}^2 + \frac{1}{2} I' \frac{(\beta_\psi - \beta_\phi \cos\theta)^2}{\sin^2\theta} + \frac{(I')^2}{2I} \beta_\phi^2 + mgL\cos\theta. \tag{8.39}$$

The value of the energy E, just like the generalized momenta, is known from the initial conditions.

When we multiply this equation by $\sin^2\theta$, we see that the derivative of θ appears in the combination $\dot{\theta}\sin\theta$, whereas the terms that do not contain a derivative depend on $\cos\theta$, because $\sin^2\theta = 1 - \cos^2\theta$. This suggests that it would be useful to define a new variable such that

$$u = \cos\theta, \qquad \dot{u} = -\dot{\theta}\sin\theta. \tag{8.40}$$

Also, it is convenient to define the following combination of parameters:

$$\epsilon = \frac{2E}{I'} - \frac{I'}{I}\beta_\phi^2, \qquad \gamma = \frac{2mgL}{I'}. \tag{8.41}$$

Substitution of Eqs. (8.40) and (8.41) converts the energy expression in Eq. (8.39) to

♦ $$\dot{u}^2 = (\epsilon - \gamma u)(1-u^2) - (\beta_\psi - \beta_\phi u^2). \tag{8.42}$$

It is possible to separate variables in this differential equation, which would lead to an expression for the time t required to attain a certain value of θ. Such a relation would have the form of an elliptic integral. Numerical methods provide another approach by which the differential equation, Eq. (8.42), may be solved for a relation between θ and t. However, we can determine much qualitative information about the motion merely by studying the roots of the cubic polynomial in the right side of Eq. (8.42). These roots describe the conditions for which $\dot{\theta}$ is zero, corresponding to either an extreme or a constant value of the nutation angle.

The polynomial in question is

$$f(u) = (\epsilon - \gamma u)(1-u^2) - (\beta_\psi - \beta_\phi u)^2. \tag{8.43}$$

In view of the definition of u by Eqs. (8.40), the physically meaningful values of u must lie in the range $-1 \le u \le 1$, subject to the requirement that $f(u) \ge 0$ in order that $\dot{\theta}$ be real. (For an actual toy top on the ground, $\theta > 0$ is the only realistic case, but $\theta < 0$ is possible by placing the apex O on an elevated pivot.)

Let us investigate the nature of the roots of $f(u)$. When u is very large, we find that $f(u) \approx \gamma u^3 > 0$ because γ is a positive parameter. Furthermore, $f(1) < 0$ because the first term vanishes. It follows that one root of $f(u)$ must be in the range $u > 1$, so it is of no interest. A comparable evaluation for large negative values of θ shows that $f(u)$ is asymptotically negative, and that $f(-1) < 0$. There must be some range of u over which there is a real value of the nutation rate, so we may conclude that $f(u)$ must have two roots in the range $-1 \le u \le 1$. One possible situation for the significant roots u_1 and u_2 is shown in Figure 8.10, although it might be that both roots are positive or negative.

The variable u may be interpreted geometrically as being the elevation above the apex of a point P on the z axis of symmetry at a unit distance from the apex. In this interpretation, the precession angle ψ and the nutation angle θ are spherical coordinates for point P, whose path lies on a sphere of unit radius. Because $f(u) = 0$ corresponds to $\dot{\theta} = 0$, the highest elevation of point P corresponds to the largest value,

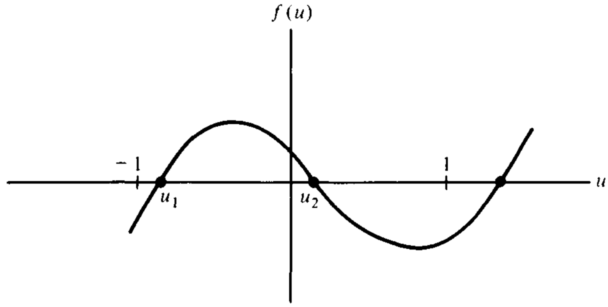


Figure 8.10 Roots of $f(u) = 0$ for a spinning top.

$u = u_2$, for which the nutation angle is the smallest. Similarly, the lowest elevation attained in the motion is $u = u_1$, corresponding to the largest nutation angle. Hence, the nutational motion is such that the symmetry axis oscillates between high and low positions, $u_1 \le u \le u_2$. In the exceptional situation where the roots are repeated, $u = u_1 = u_2$ throughout the motion, corresponding to a constant nutation angle. This is an important possibility, because we saw in Eqs. (8.38) that the precession and spin rate are constant when θ is constant. Thus, the case of repeated roots corresponds to *steady precession,* which we shall treat later.

The values of the parameters β_ψ, β_ϕ, ϵ, and γ are set by the initial conditions. The relation between the elevation u and the precession rate for specified initial conditions is found from Eqs. (8.38) and (8.40) to be

$$\dot{\psi} = \frac{\beta_\phi(u_0 - u)}{1 - u^2}, \quad u_0 = \frac{\beta_\psi}{\beta_\phi}. \tag{8.44}$$

Because $|u| \le 1$, we observe from this relation that the sense of the precession, which is defined by the sign of $\dot{\psi}$, is determined by the parametric combination $u - u_0$. Indeed, $\dot{\psi}$ vanishes at $u = u_0$. Whether $\dot{\psi}$ actually goes to zero in a motion depends on whether the value of u_0 lies in the range $u_1 \le u \le u_2$.

There are three ways in which the value of u_0 may be situated relative to u_1 and u_2. Understanding each requires recognition of the interplay between the alteration in the rotational motions necessary to conserve angular momentum and energy. The second of Eqs. (8.35) shows that the total rate of rotation about the axis of symmetry, $\omega_z = \dot{\phi} + \dot{\psi}\cos\theta$, remains constant in order to conserve momentum about that axis. Thus, a decrease in the precession rate or an increase in the nutation angle must be compensated by an increase in the spin rate. The effect of the nutation on the precession rate may be seen from the first of Eqs. (8.35). The precessional momentum originates from two sources: the projection of the spin momentum onto the precession axis, and the angular momentum associated with the precession itself. The equivalent moment of inertia for the latter effect is $I'\sin^2\theta$. Increasing the nutation angle increases this moment of inertia, while it simultaneously decreases the projection of p_ϕ. Hence, an increase in the nutation angle has competing effects on the precession rate, depending on the value of β_ϕ relative to β_ψ.

In regard to energy, Eq. (8.39) indicates that the portion of mechanical energy E attributable to the precession and spin might increase or decrease when θ increases, depending on the values of β_ψ and β_ϕ. This is accompanied by a decrease in the

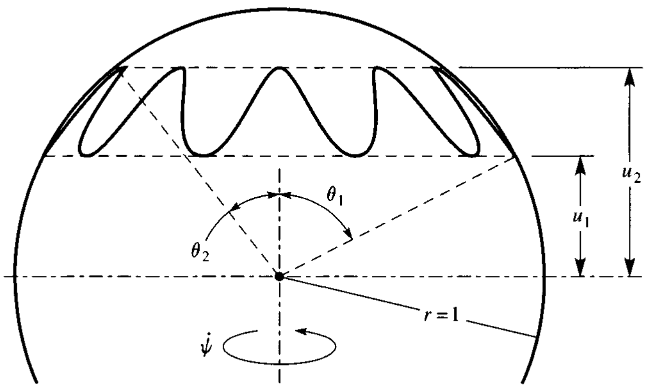


Figure 8.11 Path of the spin axis of a top in unidirectional precession.

potential energy with increasing θ. The nutational portion of the mechanical energy must maintain the balance between kinetic and potential energy. At the extremes of the nutational motion, the change in potential energy is exactly compensated by the change in the precession and spin kinetic energy, so the nutational energy vanishes at those locations.

Unidirectional Precession: $u_0 < u_1$ *or* $u_0 > u_2$
In this situation, we conclude from the first of Eqs. (8.44) that the precession rate is never zero. Whatever sense it has at the initial instant is retained throughout the motion. The nutation angle has its maximum and minimum values at $u = u_1$ and u_2, respectively, but the precession continues at those locations. As shown in Figure 8.11, the path of point P at its highest and lowest elevations is tangent to horizontal circles on the unit sphere. One way of initiating a unidirectional precession is to release the top at the highest elevation of point P, $u = u_2$, with the appropriate angular velocity. The initial nutation rate $\dot{\theta}$ at this location must be zero, corresponding to $\dot{u} = 0$ and $\theta \neq 0$; the initial precession rate should be relatively large, sufficient to make $u_0 = \beta_\psi / \beta_\phi$ exceed u_2.

Looping Precession: $u_1 < u_0 < u_2$
In this case, the first of Eqs. (8.44) indicates that the precession rate is zero at the elevation u_0, which is intermediate to the extreme values u_1 and u_2 that mark the limits of the nutation. This null corresponds to a change in the sense of the precession as the elevation rises and falls. In contrast, the nutation rate vanishes at the lowest and highest elevations. At those locations, point P moves tangent to circles of maximum and minimum elevation in opposite senses, as shown in Figure 8.12. The vertical tangencies in the loops correspond to position where $u = u_0$, so that $\dot{\psi} = 0$.

A looping precession may be attained by releasing a top at the highest elevation, $u = u_2$, with a comparatively small precession rate. The nutation rate at release must be zero in order for u_2 to be the maximum elevation. As the top falls, the portion of the precession associated with β_ψ is eventually overwhelmed by the counter effect associated with β_ϕ. Thus, the overall precession comes to rest at elevation u_0, and then proceeds opposite to the initial sense down to u_1. The process repeats with the

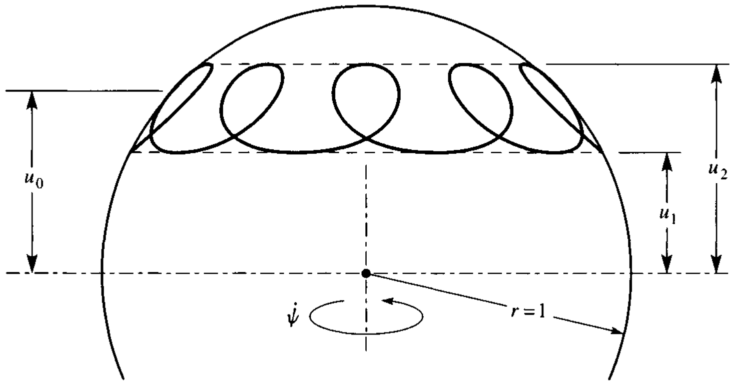


Figure 8.12 Path of the spin axis of a top in looping precession.

return to elevation u_2. As shown in Figure 8.12, the overall precessional motion matches the sense of the precession at the minimum elevations, $u = u_1$, even though the precession rate oscillates.

Cuspidial Motion: $u_0 = u_2$
This case is a transition between the unidirectional and looping precessions discussed previously. Here, the precession comes to rest at the highest elevation, $u = u_2$. Point P approaches the circle of highest elevation perpendicularly, which results in the appearance of cusps in the path of point P at these locations. As shown in Figure 8.13, the path of point P resembles a cycloidal path that is wrapped around the unit sphere.

Cuspidial motion may be attained by releasing the top at the highest elevation, $u = u_2$, with no initial precessional or nutational motion. The precessional motion that arises as the top falls is therefore attributable only to the spin momentum β_ϕ. As the top falls, it gains kinetic energy and loses potential energy, until the changes in the precession and spin rates result in an increase in the kinetic energy that equals the decrease in the potential energy. Incidentally, we may prove by this reasoning that the cusps cannot arise at the largest nutation angle, where $u = u_1$. Such a motion

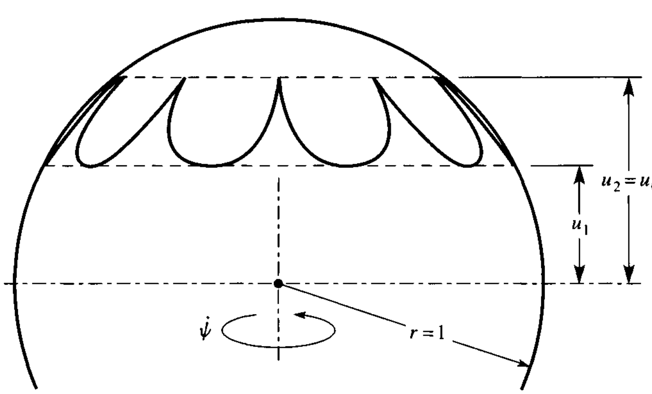


Figure 8.13 Path of the spin axis of a top in cuspidial motion.

would lead to kinetic and potential energies which are both maximum values at $u = u_1$, in violation of energy conservation.

Cuspidial motion shares many characteristics with unidirectional and looping precessions. The coincidence of the values of u_0 and u_2 in this case enables us to derive approximate expressions for the Eulerian angles. We have seen that suitable initial conditions leading to cuspidial motion are $\dot{\psi} = \dot{\theta} = 0$ when $u_0 = u_2 = \cos\theta_2$, with $\dot{\phi}_2$ nonzero. The corresponding momentum parameters are given by Eqs. (8.35) to be

$$\beta_\phi = \frac{I}{I'}\dot{\phi}_2, \qquad \beta_\psi = \beta_\phi u_0. \tag{8.45}$$

The energy-level parameters obtained from Eqs. (8.39) and (8.41) in this case are

$$\gamma = \frac{2mgL}{I'}, \qquad \epsilon = \gamma u_0. \tag{8.46}$$

Upon substitution of these parameters, the energy function $f(u)$ defined in Eq. (8.43) factorizes as

$$f(u) = (u_0 - u)[\gamma(1-u^2) - \beta_\phi^2(u_0 - u)]. \tag{8.47}$$

The roots of $f(u)$ for cuspidial motion are readily found to be

$$\begin{aligned} u_1 &= U - (U^2 - 2u_0 U + 1)^{1/2}, \\ u_2 &= u_0, \\ u_3 &= U + [U^2 - 2u_0 U + 1]^{1/2}, \end{aligned} \tag{8.48}$$

where

$$U = \frac{\beta_\phi^2}{2\gamma} = \frac{I^2\dot{\phi}_2^2}{4I'mgL}. \tag{8.49}$$

Because $|u_0| < 1$ when the top is released away from the vertical, we have

$$U^2 - 2u_0 U + 1 > (U-1)^2.$$

It follows that $u_3 > 1$ (the meaningless root), while $-1 < u_1 < 1$. This, of course, agrees with our earlier assessment of the nature of the roots of $f(u)$ in the general case.

The limits of the nutation in cuspidial motion are given explicitly by Eqs. (8.48) in terms of the initial spin rate defining U in Eq. (8.49). Further simplifications are possible when we consider the typical situation of a *fast top,* in which the spin rate imparted in the initial motion is large. We quantify this restriction by specifying that $U \gg 1$. The corresponding minimum elevation obtained from the leading terms in a series expansion of the first of Eqs. (8.48) is

$$u_1 = u_0 - \frac{1-u_0^2}{2U}. \tag{8.50}$$

In view of Eq. (8.49), we may conclude from this expression that the difference between the maximum and minimum elevations for a fast top decreases as the inverse square of the initial spin rate.

The smallness of the cuspidial motion at high spin rates allows us to evaluate the precessional and nutational rotations as explicit functions of time. The technique for such an investigation is *perturbation analysis.* The value of u_1 in Eq. (8.50) suggests that, in general, the elevation u may be expressed in a series as

$$u = u_0 - \frac{1}{U}v_1(t) - \frac{1}{U^2}v_2(t) + \cdots, \tag{8.51}$$

where the $v_j(t)$ are unknown functions of time that are independent of the parameter U. Many terms would be required to make the series converge when the value of U is arbitrary. In contrast, the error that arises from truncating the series becomes smaller and smaller as the value of U increases. We say that Eq. (8.51) is an *asymptotic series* for the variable u in terms of the perturbation parameter $1/U \ll 1$.

We obtain differential equations for the unknown functions v_j by requiring that the asymptotic series satisfy the equation of motion at each level of approximation, associated with increasing powers of $1/U$. The equation of motion we shall employ is the energy conservation relation, Eq. (8.42), with the function $f(u)$ for cuspidial motion given by Eq. (8.47). The parameter γ may be removed from the expression by applying the definition of U, Eq. (8.49), which leads to

$$\left(\frac{1}{U}\dot{v}_1 + \frac{1}{U^2}\dot{v}_2 + \cdots\right)^2 = \left(\frac{1}{U}v_1 + \frac{1}{U^2}v_2 + \cdots\right)$$
$$\times\left\{\frac{\beta_\beta^2}{2U}\left[1 - \left(u_0 - \frac{1}{U}v_1 + \cdots\right)^2\right] - \beta_\phi^2\left(\frac{1}{U}v_1 + \frac{1}{U^2}v_2 + \cdots\right)\right\}. \tag{8.52}$$

Although we truncated the asymptotic series for u at two terms beyond the initial approximiation $u = u_0$, we shall consider only the first approximation here. In other words, we shall evaluate only v_1. Matching the coefficients of $1/U^2$ on each side of Eq. (8.52) yields

$$\dot{v}_1^2 = \beta_\phi^2[\tfrac{1}{2}(1-u_0^2)v_1 - v_1^2]. \tag{8.53}$$

Taking the square root of this nonlinear, first-order differential equation for v_1, in order to form an equation whose variables may be separated, leads to an ambiguity in sign that can only be resolved by addressing the initial conditions. A simpler technique is to convert the equation to a second-order differential equation by differentiating it once with respect to time. This operation leads to a common factor of $\dot{v}_1$, which may be canceled. Thus,

$$\ddot{v}_1 + \beta_\phi^2 v_1 = \tfrac{1}{4}\beta_\phi^2(1-u_0^2). \tag{8.54}$$

The solution of this differential equation must satisfy the initial conditions for cuspidial motion, which we have taken to be that $u = u_0$ and $\dot{u} = 0$ at the instant of release. The leading term in Eq. (8.51) satisfies these conditions, so the next order of approximation must satisfy rest conditions; that is,

$$v_1 = \dot{v}_1 = 0 \quad \text{when } t = 0.$$

The sum of the complementary and particular solutions satisfying the initial conditions is

$$\begin{aligned} v_1 &= \tfrac{1}{4}(1-u_0^2)(1-\cos\beta_\phi t); \\ u &\approx u_0 - \frac{1}{U}v_1 = u_0 - \frac{\gamma}{2\beta_\phi^2}(1-u_0^2)(1-\cos\beta_\phi t). \end{aligned} \tag{8.55}$$

The rate of change of the elevation thus obtained leads to an expression for the nutation rate. Differentiating Eq. (8.55) gives

$$\dot{u} = -\frac{\gamma}{2\beta_\phi}(1-u_0^2)\sin\beta_\phi t. \tag{8.56}$$

The expression for $\dot{u}$ in Eq. (8.40) may be simplified for the present case because the value of θ remains close to the initial value θ_2, so

$$\begin{aligned} &\sin\theta \approx \sin\theta_2 = (1-u_0^2)^{1/2}; \\ &\dot{u} = -(\sin\theta)\dot{\theta} \approx -(1-u_0^2)^{1/2}\dot{\theta}. \end{aligned} \tag{8.57}$$

The result of equating Eqs. (8.56) and (8.57) is

$$\dot{\theta} \approx \frac{\gamma}{2\beta_\phi}\sin\theta_2 \sin\beta_\phi t. \tag{8.58}$$

We find an expression for the precession rate by using $u \approx u_0$ to simplify the denominator of Eq. (8.44). Substitution of Eqs. (8.45) and (8.55) then yields

$$\dot{\psi} \approx \frac{\gamma}{2\beta_\phi}(1-\cos\beta_\phi t). \tag{8.59}$$

The interpretation of these results is that the average precession rate of a fast top varies harmonically about the mean value $\gamma/2\beta_\phi$, with an amplitude equal to the mean value. When the precession rate is zero ($\cos\beta_\phi t = 1$), the nutation rate is zero and the top is at its highest elevation. At the instant when the precession rate is maximum ($\cos\beta_\phi t = -1$), the nutation rate is also zero, corresponding to the lowest elevation.

Steady Precession

If the appropriate initial motion is imparted to the top, it is possible to obtain a rotation in which the nutation angle is constant. The corresponding spin and precession rates in that case will not vary from their initial values. The most direct approach leading to this response employs the equation of motion for the nutation angle, Eq. (8.38). This equation shows that if the nutation angle is constant then

$$(\beta_\phi - \beta_\psi\cos\theta)(\beta_\psi - \beta_\phi\cos\theta) - \frac{mgL}{I'}\sin^4\theta = 0. \tag{8.60}$$

We could consider this relation as governing the nutation angle for specified values of the momentum parameters. However, it is more meaningful to use Eq. (8.60) to derive an expression for the precession rate corresponding to a specified nutation angle. The definitions of the momentum parameters in Eqs. (8.35) are

$$\beta_\psi = \dot{\psi}\sin^2\theta + \beta_\phi\cos\theta, \qquad \beta_\phi = \frac{I}{I'}(\dot{\psi}\cos\theta + \dot{\phi}). \tag{8.61}$$

Because the spin momentum β_ϕ is proportional to the component of angular velocity parallel to the axis of symmetry, we shall retain β_ϕ rather than the spin rate. We therefore substitute only the first of Eqs. (8.61) into Eq. (8.60), and cancel a common factor of $\sin^4\theta$, which leads to

$$(\beta_\phi - \dot{\psi}\cos\theta)\dot{\psi} - \gamma/2 = 0. \tag{8.62}$$

The solution of this quadratic equation is

$$\dot{\psi} = \frac{\beta_\phi \pm (\beta_\phi^2 - 2\gamma\cos\theta)^{1/2}}{2\cos\theta}. \tag{8.63}$$

An interesting corollary of this result is that, for a specified nutation angle, there is a minimum spin momentum for which steady precession is possible; specifically,

$$(\beta_\phi)_{\min} = (2\gamma\cos\theta)^{1/2}. \tag{8.64}$$

Equation (8.63) seems to be fairly straightforward. However, a complication arises if one desires to determine the steady-precession rate for a specified spin rate, because the spin momentum depends on the value of $\dot{\psi}$ according to the second of Eqs. (8.61). Example 8.3 describes an accurate evaluation of the relation between $\dot{\phi}$ and $\dot{\psi}$. Here, we shall derive simple formulas for the case of a fast top, where $\beta_\phi^2 \ll 2\gamma$. It is permissible in this case to truncate a binomial expansion of the square root in Eq. (8.63) at the first two terms. The corresponding roots are

$$\dot{\psi}_1 = \frac{\gamma}{2\beta_\phi}, \qquad \dot{\psi}_2 = \frac{\beta_\phi}{\cos\theta}. \tag{8.65}$$

The first value is comparatively small, because it varies inversely with β_ϕ; similarly, the second value is large. It follows that we may neglect the contribution of the precession rate to β_ϕ in the first case, but not in the second. Specifically, we find from the second of Eqs. (8.61) and Eq. (8.65) that

$$\begin{aligned} (\beta_\phi)_1 &= \frac{I}{I'}\dot{\phi} \;\Rightarrow\; \dot{\psi}_1 = \frac{I'\gamma}{2I\dot{\phi}} = \frac{mgL}{I\dot{\phi}}, \\ (\beta_\phi)_2 &= \frac{I}{I'}[(\beta_\phi)_2 + \dot{\phi}] \;\Rightarrow\; \dot{\psi}_2 = \frac{I\dot{\phi}}{(I'-I)\cos\theta}. \end{aligned} \tag{8.66}$$

The fast precession rate $\dot{\psi}_2$ matches the value obtained from Eqs. (8.20) for a symmetric body in free motion. In essence, the spin and precession rates in the fast case are so high that the gravitational moment is negligible in comparison to the moments required to alter the angular momentum of the top. Steady precession of a top usually occurs at the slow precession rate, because the kinetic energy required to attain $\dot{\psi}_2$ is prohibitive.

A special case of steady rotation is the *sleeping top,* which is the term used when the axis of symmetry of the top is vertical, $\cos\theta = 1$. The precession and spin are indistinguishable in a sleeping top, because both rotations are about concurrent axes. (The name "sleeping top" stems from the merger of spin and precession, which makes a polished, unmarked, body of revolution appear to be stationary.) Because of the similarity of precession and spin in such a rotation, some of the relations for steady precession become trivial. For example, because $\beta_\psi = \beta_\phi$ when $\theta = 90°$, Eq. (8.60) is satisfied identically. However, all relations for steady precession remain valid in the limit as $\theta \to 0$. We shall treat this degenerate case by noting that the angular velocity of a sleeping top is merely

$$\omega = \dot{\phi} + \dot{\psi},$$

so the definition of the spin momentum in Eq. (8.61) reduces to

$$\beta_\phi = \frac{I}{I'}\omega.$$

Hence, we find from Eq. (8.64) that the minimum rotation rate required for a top to "sleep" is

$$\omega_{cr} = \frac{I'}{I}(2\gamma)^{1/2} = \left(\frac{4mgLI'}{I^2}\right)^{1/2}. \tag{8.67}$$

Our analysis suggests that the axis of symmetry cannot remain vertical if $\omega < \omega_{cr}$. This is not precisely correct, because the vertical position, $\theta = 0$, is a solution of the equations of motion for any ω. The angular momentum in that case is vertical and therefore constant, and the moment of the gravity force about the pivot point vanishes. In essence, by obtaining the sleeping top as a special case, we have demonstrated that $\theta = 0$ is unstable if $\omega < \omega_{cr}$. In actuality, the effect of friction at the apex O is to slow the rate of rotation. When the value of ω for a sleeping top falls below ω_{cr}, the top begins to nutate. Because the nutational velocity is zero at the instant when the rotation rate falls below critical, the ensuing motion is a cuspidial precession. If the spin rate decreases slowly, the amplitude of the nutation will slowly increase until the top hits the ground or falls from its support.

Example 8.3 A 2-kg top is in a state of steady slow precession at a spin rate of 500 rev/min with its axis at $\theta = 120°$. A vertical impulsive force acting through the axis of symmetry suddenly induces an upward nutation, such that the ensuing motion is observed to be cuspidial. The radii of gyration of the top about its pivot are 360 mm and 480 mm parallel and transverse, respectively, to the axis of symmetry, and the distance from the center of mass to the pivot is 200 mm. Determine:

(a) the nutation rate induced by the impulsive force;
(b) the largest and smallest values of the nutation angle in the cuspidial precession;
(c) the number of cusps in the path of the axis of symmetry for one revolution of the top about the vertical axis; and
(d) the maximum, minimum, and average precession rates in the cuspidial motion.

Solution We begin by evaluating the steady precession prior to the application of the impulse force. We could employ Eqs. (8.66) for this purpose, provided that we verify the condition $\beta_\phi^2 \gg 2\gamma$ for a fast top. However, an alternative relation for the steady precession rate, one that does not require preconditions, is available. In the present case we know the spin rate, which is only one contribution to β_ϕ. Therefore, we substitute the second of Eqs. (8.61) into Eq. (8.62) in order to obtain a relation between the precession and spin rates, with the result that

$$\left[\left(\frac{I}{I'} - 1\right)\cos\theta\right]\dot{\psi}^2 + \frac{I}{I'}\dot{\phi}\dot{\psi} - \frac{\gamma}{2} = 0.$$

The roots of this quadratic equation are the fast and slow precession rates,†

† Setting the discriminant of this equation to zero shows that the minimum spin rate for which steady precession is possible is

$$\dot{\phi}_{min} = \left[2\left(\frac{I'}{I} - 1\right)\frac{I'}{I}\gamma\cos\theta\right]^{1/2}.$$

This value can be shown to be smaller than the spin rate corresponding to Eq. (8.64), but the value of β_ϕ associated with $\dot{\phi}_{min}$ is higher than Eq. (8.64).

$$\dot{\psi} = \frac{I\dot{\phi} \pm [I^2\dot{\phi}^2 - 2(I'-I)I'\gamma\cos\theta]^{1/2}}{2(I'-I)\cos\theta}.$$

The parameters for the present system are

$$m = 2 \text{ kg}, \qquad I = m\kappa^2 = 0.2592 \text{ kg-m}^2, \qquad I' = m(\kappa')^2 = 0.4608 \text{ kg-m}^2,$$

$$L = 0.2 \text{ m}, \qquad \gamma = \frac{2mgL}{I'} = 17.026 \text{ rad}^2/\text{s}^2, \qquad \dot{\phi} = 52.36 \text{ rad/s},$$

which leads to the two roots

$$\dot{\psi}_1 = 0.28843 \text{ rad/s}, \qquad \dot{\psi}_2 = -134.93 \text{ rad/s}.$$

Both values are extremely close to the approximations in Eqs. (8.66). Because we know that the initial precession was slow, we use $\dot{\psi}_1$ as the initial rate.

The impulsive force induces an unknown nutation rate $\dot{\theta}$, because it exerts a moment about the horizontal axis through the pivot. However, the spin and precession rates are not altered during the impulse interval. We find $\dot{\theta}$ from the fact that the subsequent precession is cuspidial. We need the values of the precession and spin momentum parameters to evaluate cuspidial motion. Equations (8.61) for the slow precession rate just described and the given spin rate yield

$$\beta_\phi = 29.371 \text{ rad/s}, \qquad \beta_\psi = -14.469 \text{ rad/s}.$$

Then the highest elevation u_0 is

$$u_0 = \beta_\psi/\beta_\phi = -0.49264,$$

which corresponds to the position where the nutation angle is a minimum:

$$\theta_{\min} = \theta_2 = \cos^{-1} u_0 = 119.514°.$$

Before we may employ results of the perturbation analysis of cuspidial motion, we must check the value of the parameter U in Eq. (8.49). We calculate

$$U = \frac{\beta_\phi^2}{2\gamma} = 25.33,$$

which is sufficiently large. Because $u = \cos\theta$, Eq. (8.55) provides an expression for the time dependence of the nutation angle. We find that

$$\theta = \cos^{-1}\left[\cos\theta_2 - \frac{1}{4U}\sin^2\theta_2(1-\cos\beta_\phi t)\right]$$
$$= \cos^{-1}\{-0.49264 - 0.0074733[1-\cos(29.371t)]\} \text{ rad}.$$

We obtain the corresponding nutation and precession rates by direct substitution into Eqs. (8.58) and (8.59), which yield

$$\dot{\theta} \approx 0.2522\sin(29.371t) \text{ rad/s},$$

$$\dot{\psi} \approx 0.2898[1-\cos(29.371t)] \text{ rad/s}.$$

With these relations, we have fully evaluated the cuspidial response. However, we do not yet know the time t_0 when the impulse occurred. (Note that $t = 0$ corresponds to the instant at which a cusp occurs, $\theta = \theta_2$.) To find t_0, from which we may determine the initial nutation rate, we use the fact that $\theta = 2\pi/3$ initially. We therefore have

$$-0.49264 - 0.0074733[1 - \cos(29.371 t_0)] = -0.50, \quad t_0 = 52.99 \text{ ms}.$$

We find the initial value of the nutation rate by evaluating the expression for $\dot{\phi}$ at $t = t_0$,

$$\dot{\theta} = 0.02522 \text{ rad/s}.$$

The average of a sinusoidal function is zero, so the expression for the precession rate indicates that

$$\dot{\psi}_{avg} = 0.2898, \quad \dot{\psi}_{min} = 0, \quad \dot{\psi}_{max} = 0.5796 \text{ rad/s}.$$

Finally, we note that cusps occur when $\theta = \theta_2$, corresponding to the minimum precession angle. This condition occurs whenever $\cos(29.371t) = 1$. Therefore, the time interval between two adjacent cusps is the period of the oscillation,

$$\Delta t = 2\pi/29.371 = 0.2139 \text{ s}.$$

At the average precession rate, the time interval for one revolution about the vertical axis is

$$T = \frac{2\pi}{\dot{\psi}_{avg}} = 21.68 \text{ s}.$$

The ratio $T/\Delta t$ is the number of cusps per precessional revolution. Thus,

$$\frac{T}{\Delta t} = 101.34 \Rightarrow N = 101 \text{ or } 102 \text{ cusps}.$$

8.3 Gyroscopes for Inertial Guidance

We know from our studies thus far that the moment required to change the orientation of a body's rotation axes is directly correlated to the change in the state of motion. We shall explore here a number of ways this effect has been employed in devices that direct moving vehicles without using the frame of reference provided by the earth. These devices are called *inertial guidance systems* because they provide an inertial reference system that moves with the vehicle. Our conceptual pictures will be quite crude. In practice, the various pieces of equipment are manufactured with exceptionally high accuracy and with the finest bearings, in order to match as closely as possible the ideal conditions that we shall treat.

8.3.1 *Free Gyroscopes*

The gyroscope appearing in Figure 8.14 is said to be *free* because the rotation of the rotor is unconstrained. The outer gimbal permits precessional rotation, the inner gimbal permits nutation, and the rotor shaft permits spin. For our introductory study we shall ignore the effect of the vehicle's motion supporting the outer gimbal. In that case the center point O is stationary, because the three rotation axes are concurrent at the center O.

When the center of mass G of the rotor does not coincide with the fixed point O, the gimbals must rotate. The excitation is the moment of the gravity force about the line of nodes, which is the axis about which the inner gimbal rotates relative to the

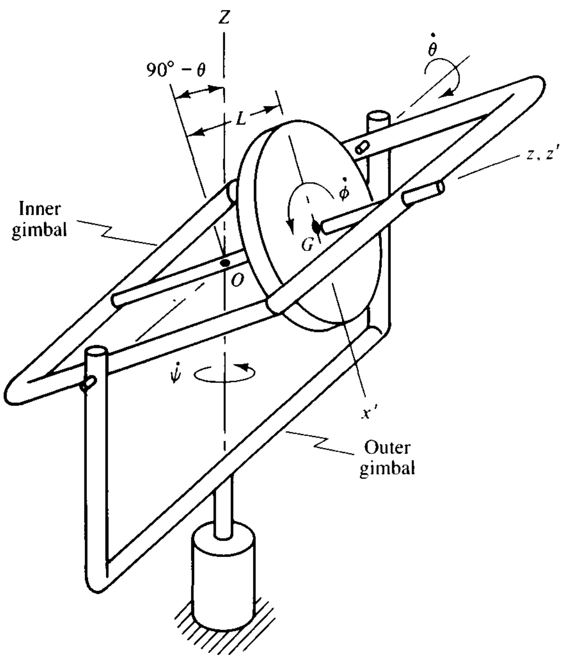


Figure 8.14 Free gyroscope.

outer gimbal. Despite the differences in appearance, the configuration of the system, as represented by the position of the center of mass relative to the fixed point, is identical to that of a spinning top. It follows that the two systems behave in the same manner. In the special case where the center of mass coincides with the fixed point O, the free gyroscope behaves like a body in free motion (Section 8.1) because there are no external moments. We shall employ the results of previous sections, as necessary.

Suppose a steady precession, in which the nutation angle is constant, has been established. The relation between the precession rate, the spin rate, and the nutation angle is given by Eq. (8.63). In order for there to be a steady precession, the value of β_ϕ for the gyroscope must exceed the minimum rotation rate about the symmetry axis, given by Eq. (8.64). Note that if the center of mass coincides with the fixed point O, then $\gamma = 0$. The steady slow precession rate in that case is zero, which means that the axis of symmetry has a constant orientation.

An important question that must be addressed is whether the steady precession is a stable response. If it is not, then such motion would not be observed in reality. One technique by which *dynamic stability* may be studied is to perturb the nutation angle from the steady value it has when a steady precession has been established. Thus, let

$$\theta = \theta^* + \Delta\theta, \tag{8.68}$$

where θ^* denotes the constant value for steady precession, and $\Delta\theta$ is a small disturbance that may vary with time.

A linearized equation of motion governing $\Delta\theta$ may be obtained from the general equation, Eq. (8.38), which we multiply by $\sin^3\theta$. We substitute Eq. (8.68), and then expand in a Taylor series that we truncate at terms that contain quadratic and higher powers of $\Delta\theta$. For example,

$$\begin{aligned} \cos\theta &\approx \cos\theta^* - \Delta\theta\sin\theta^*, \\ (\sin\theta)^4 &\approx (\sin\theta^*)^4 + (4\Delta\theta)(\sin\theta^*)^3\cos\theta^*. \end{aligned} \tag{8.69}$$

By definition, θ^* is a solution of Eq. (8.60). Consequently, the zero-order terms (i.e., those that are independent of $\Delta\theta$) cancel. The first-order equation that results from the foregoing procedure is

$$\sin^3\theta^*\Delta\ddot{\theta} + [\beta_\phi(\beta_\phi - \beta_\psi\cos\theta^*) + \beta_\psi(\beta_\psi - \beta_\phi\cos\theta^*)]\,\Delta\theta\sin\theta^* - 2\gamma\,\Delta\theta\sin^3\theta^*\cos\theta^* = 0. \tag{8.70}$$

We may further simplify this expression by eliminating β_ψ with the aid of Eq. (8.61). This yields

$$\Delta\ddot{\theta} + \omega^2\Delta\theta = 0, \tag{8.71}$$

where

$$\omega^2 = \beta_\phi^2 + \dot{\psi}^2\sin^2\theta^* - 2\gamma\cos\theta^*. \tag{8.72}$$

The steady precession is stable to small disturbances if the value of Δ remains bounded. Such a condition is obtained if $\omega^2 > 0$, which corresponds to oscillatory solutions of Eq. (8.71). However, Eq. (8.64) states that a steady precession can exist only if the spin momentum is sufficiently large, $\beta_\phi^2 > 2\gamma\cos\theta$. It follows that $\omega^2 > 0$ for any free precession. In other words, if the spin momentum is sufficiently large to establish a steady precession at nutation angle θ^*, then an attempt to change the nutation angle by a small amount will result in an oscillatory nutational motion whose mean value is θ^*.

The balanced free gyroscope, for which $\gamma = 0$ ($L = 0$), is stable regardless of the spin momentum. Its primary application is in inertial navigation systems that track vehicle motion in aircraft and missiles. The concept is remarkably simple. The invariability of the direction of the balanced gyroscope provides a translating reference frame. Measurements of the vehicle's rotation relative to this reference frame are used to drive servomotors, which maintain a platform in a horizontal orientation relative to the earth's surface. Accelerometers are mounted on this platform. Because the platform remains horizontal, the accelerometers measure the acceleration of the platform relative to the earth's surface. The displacement relative to the earth may then be determined by electronically integrating the accelerations twice in time.

Example 8.4 In order to overcome the effects of friction, a servomotor applies a torque about the spin axis of an unbalanced gyroscope, with the result that the spin rate is constant. The initial conditions are such that the initial precession rate $\dot{\psi}^*$ and nutation angle θ^* correspond to steady precession. Determine whether the action of the servomotor can cause the gyroscope to be unstable to small disturbances.

Solution The primary difference between the present system and a free gyroscope is that there are only two degrees of freedom, because the spin rate is constrained to be constant. Thus $\dot{\phi}$, rather than the spin momentum β_ϕ, is constant. We commence to derive the equations of motion for the servogyroscope by using the energies in Eqs. (8.33) and (8.34):

$$T = \tfrac{1}{2}I(\dot\psi\cos\theta+\dot\phi)^2+\tfrac{1}{2}I'(\dot\psi^2\sin^2\theta+\dot\theta^2), \qquad V = mgL\cos\theta.$$

The precession angle is an ignorable generalized coordinate, because only its derivative appears in T. The corresponding conservation of momentum equation is identical to the first of Eqs. (8.35),

$$\beta_\psi = \dot\psi\left(\frac{I}{I'}\cos^2\theta+\sin^2\theta\right)+\frac{I}{I'}\dot\phi\cos\theta. \tag{1}$$

The Lagrange equation for θ may be written as

$$\ddot\theta+g(\dot\psi,\theta)=0, \tag{2}$$

where the function g is found from Eq. (8.37) to be

$$g(\dot\psi,\theta)=\left(\frac{I}{I'}-1\right)\dot\psi^2\sin\theta\cos\theta+\frac{I}{I'}\dot\psi\dot\phi\sin\theta-\frac{\gamma}{2}\sin\theta. \tag{3}$$

Because the initial conditions are those appropriate to a steady precession, we have

$$g(\dot\psi^*,\theta^*)=0. \tag{4a}$$

We consider $\sin\theta\neq 0$ for steady precession in the nonvertical position, so eq. (4a) is satisfied when

$$\left[\left(\frac{I}{I'}-1\right)\cos\theta^*\right](\dot\psi^*)^2+\frac{I}{I'}\dot\phi\dot\psi^*-\frac{\gamma}{2}=0, \tag{4b}$$

which matches the expression established in Example 8.3 for the free gyroscope.

Although the angular motion in steady precession is identical to that of a free gyroscope, the stability situation is different. Constancy of β_ψ now requires that any fluctuation in the nutation angle will be compensated solely by a change in the precession rate. We consider very small changes $\Delta\theta$ for a stability analysis, so the corresponding increment in the precession rate, $\Delta\dot\psi$, is also small. We therefore substitute $\theta=\theta^*+\Delta\theta$ and $\dot\psi=\dot\psi^*+\Delta\dot\psi$ into eqs. (1) and (2), which are the basic equations of motion for this system. The zero-order terms in Taylor series expansions of these equations combine to form the value of β_ψ in eq. (1), and they also satisfy eq. (4a). Hence, the first-order terms obtained from eq. (1) require

$$\Delta\beta_\psi = 0 = \Delta\dot\psi\left(\frac{I}{I'}\cos^2\theta^*+\sin^2\theta^*\right) - \left[\dot\psi^*\left(\frac{I}{I'}-1\right)\sin 2\theta^*+\frac{I}{I'}\dot\phi\sin\theta^*\right]\Delta\theta, \tag{5}$$

while the first-order terms obtained from eq. (2) are

$$\Delta\ddot\theta+\left(\frac{\partial g}{\partial\dot\psi}\right)^*\Delta\dot\psi+\left(\frac{\partial g}{\partial\theta}\right)^*\Delta\theta=0; \tag{6}$$

the derivatives are marked by an asterisk to signify their evaluation at the steady-precession state. We use eq. (5) to solve for $\Delta\dot\psi$ in terms of $\Delta\theta$, and substitute that expression into eq. (6). The result is

$$\Delta\ddot{\theta}+\omega^2\Delta\theta=0, \tag{7}$$

where expanding the partial derivatives in eq. (6) yields

$$\omega^2=\left[\left(\frac{I}{I'}-1\right)(\dot{\psi}^*)^2\cos 2\theta^*+\left(\frac{I}{I'}\dot{\phi}\dot{\psi}^*-\frac{\gamma}{2}\right)\cos\theta^*\right] \\ +\frac{I'\sin^2\theta^*}{I\cos^2\theta^*+I'\sin^2\theta^*}\left[2\left(\frac{I}{I'}-1\right)\dot{\psi}^*\cos\theta^*+\frac{I}{I'}\dot{\phi}\right]^2.$$

To simplify this further, we employ eq. (4b) to eliminate $\dot{\phi}$. The result after many manipulations is

$$\omega^2=\frac{\sin^2\theta^*}{\dot{\psi}^2[(I/I')\cos^2\theta^*+\sin^2\theta^*]} \\ \times\left\{\frac{1}{4}\gamma^2+\left(\frac{I}{I'}-1\right)(\dot{\psi}^*)^2[\gamma\cos\theta^*-(\dot{\psi}^*)^2]\right\}. \tag{8}$$

As with the free gyroscope, whose stability was described by Eqs. (8.71) and (8.72), $\omega^2<0$ indicates situations where the servo-driven gyroscope is unstable to small disturbances of the nutation angle. However, it is not a trivial matter to identify the sign of ω^2 obtained from eq. (8). Suppose we are interested in evaluating whether a specific design, corresponding to specified values of $\dot{\phi}$, γ, and I/I', is unstable in some range of θ^*. To address the question we recognize that eq. (4b) is a quadratic equation. We use the smaller root, corresponding to slow precession, to describe $\dot{\psi}^*$ as a function of θ^*. Substituting this function into eq. (8) yields ω^2 as a function of θ^*. By scanning the range $0\le\theta^*\le\pi$, we may determine whether $\omega^2<0$ in some range of θ^*. As an example of such a computation, consider $I/I'=1.5$ and $\gamma=20$ rad/s^2. Setting $\dot{\phi}>3.366$ rad/s leads to stability for any θ^*, while $\dot{\phi}=2$ rad/s leads to stability only for $0<\theta^*<97.09°$.

The occurrence of instability could have been anticipated on the basis of physical arguments. The free gyroscope is a conservative system. In contrast, the servo-driven gyroscope is not, because the servomotor does work to hold the spin rate constant. The energy provided to the system from this source can drive the nutational motion away from steady precession. However, most situations of practical interest are like the numerical example above, in that the spin rate below which the gyroscope would lose stability is sufficiently low to be of no concern.

8.3.2 *Gyrocompass*

A fundamental requirement for earthbound navigation is knowledge of the orientation of true north. The balanced free gyroscope maintains a fixed orientation as the earth rotates; an observer on the earth perceives the gyroscope to be rotating. The gyrocompass has the feature that its steady precession always matches the earth's rotation, so an observer on the earth perceives the rotor axis always to point in a constant direction.

The gyrocompass bears much similarity to an unbalanced free gyroscope, except for the placement of the mass causing the imbalance. As shown in Figure 8.15, the

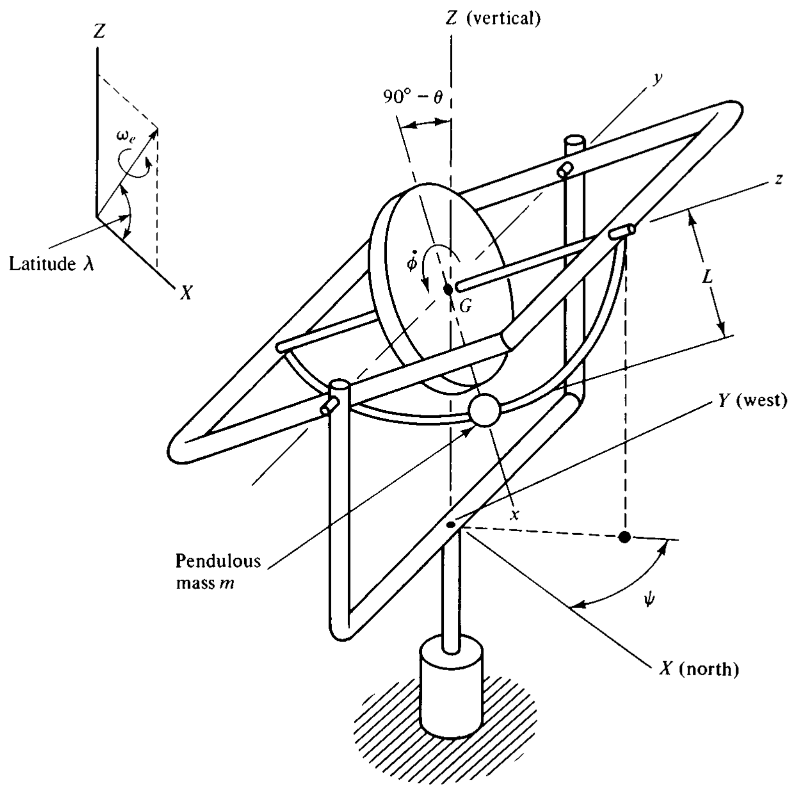


Figure 8.15 Gyrocompass.

center of mass of the rotor is situated on the intersection of the precession and spin axes, but a small additional mass m is attached to the inner gimbal. This arrangement is selected so that the gravitational moment will be like that for a pendulum. We must include the rotation of the earth in the analysis, for which the earth-based reference frame defined in Chapter 3 is useful. Thus, in Figure 8.15 the Z axis is oriented in the direction perceived as vertical to an observer on the earth and the X axis is situated in the northerly direction. The angular velocity of the Earth is therefore

$$\bar{\omega}_e = \omega_e[(\cos\lambda)\bar{I} + (\sin\lambda)\bar{K}], \tag{8.73}$$

where $\omega_e = 7.292(10^{-5})$ rad/s $\approx 2\pi$ rad/(24 h) is the rotation rate.

The rotation of the gyrocompass's rotor is unconstrained. We use the Eulerian angles to describe the orientation of the rotor *relative to the earth-fixed reference frame XYZ*. Toward that end, we introduce an intermediate reference frame xyz that is fixed to the inner gimbal. The z axis is aligned with the axis of symmetry of the rotor, and the y axis is the line of nodes formed by the bearings of the inner gimbal. Our goal here is to determine whether there is any set of precession and nutation angles for which the axis of the rotor remains stationary relative to the earth. For this reason we consider the values of ψ and θ to be constant, and assume also that $\dot{\phi}$ remains constant. The corresponding angular velocity of the rotor relative to XYZ is

$\dot{\phi}\bar{k}$. In order to combine this term with the rotation of the earth, we transform the unit vectors according to

$$\begin{aligned}
\bar{I} &= (\cos\psi)[(\cos\theta)\bar{i}+(\sin\theta)\bar{k}]-(\sin\psi)\bar{j},\\
\bar{J} &= (\sin\psi)[(\cos\theta)\bar{i}+(\sin\theta)\bar{k}]+(\cos\psi)\bar{j},\\
\bar{K} &= -(\sin\theta)\bar{i}+(\cos\theta)\bar{k}.
\end{aligned} \tag{8.74}$$

Adding the earth's rotation to the rotor spin then leads to the following absolute angular velocity of the rotor:

$$\begin{aligned}
\bar{\omega} &= \omega_e(\cos\lambda\cos\psi\cos\theta-\sin\lambda\sin\theta)\bar{i}-\omega_e(\cos\lambda\sin\psi)\bar{j}\\
&\quad+(\omega_e\cos\lambda\cos\psi\sin\theta+\omega_e\sin\lambda\cos\theta+\dot{\phi})\bar{k}.
\end{aligned} \tag{8.75}$$

Terms containing ω_e have a very small value, so we may simplify the kinetic energy of the system by neglecting effects that are of the order of ω_e^2. The corresponding kinetic energy for the system is

$$\begin{aligned}
T &= \tfrac{1}{2}(I'\omega_x^2+I'\omega_y^2+I\omega_z^2)\\
&= \tfrac{1}{2}I[\dot{\phi}^2+2\omega_e\dot{\phi}(\cos\lambda\cos\psi\sin\theta+\sin\lambda\cos\theta)].
\end{aligned} \tag{8.76}$$

The corresponding potential energy is associated with the unbalanced mass on the inner gimbal. When the datum is set at the fixed point G, we find that

$$V = mgL\cos(\pi/2+\theta) = -mgL\sin\theta. \tag{8.77}$$

There are no nonconservative forces in this ideal model, so the Lagrange equations for the generalized coordinates ψ, θ, and ϕ in this case of steady precession are

$$\begin{aligned}
&\omega_e\dot{\phi}\cos\lambda\sin\theta = 0,\\
&I\omega_e\dot{\phi}(-\cos\lambda\cos\psi\cos\theta+\sin\lambda\sin\theta)-mgL\cos\theta = 0,\\
&\dot{\phi}+\omega_e(\cos\lambda\cos\psi\sin\theta+\sin\lambda\cos\theta) = \beta_\phi,
\end{aligned} \tag{8.78}$$

where $\beta_\phi = p_\phi/I$ is the spin momentum parameter associated with the ignorable coordinate ϕ.

Because of the smallness of ω_e, the last of Eqs. (8.78) indicates that the spin momentum is primarily associated with the spin itself. The first equation is satisfied when $\sin\theta = 0$ or $\sin\psi = 0$. The first possibility is not useful, because then the rotor does not provide any directional information. The second case is the one we seek, because it is satisfied when $\psi = 0$ or π, so that the spin axis is aligned along the north-south meridian. Setting $\cos\psi = \pm 1$ in the second of Eqs. (8.78) yields

$$(mgL\pm I\omega_e\dot{\phi}\cos\lambda)\cos\theta = I\omega_e\dot{\phi}\sin\lambda\sin\theta. \tag{8.79}$$

We observe that the smallness of ω_e means that the value of $\tan\theta$ obtained from the foregoing is much larger than unity, which corresponds to $\theta\approx\pi/2$. Thus, we set

$$\theta = \pi/2-\Delta\theta, \quad \Delta\theta \ll 1.$$

Furthermore, we may neglect $I\omega_e\dot{\phi}$ in comparison to mgL. We therefore find from Eq. (8.79) that

$$\Delta\theta = \frac{I\omega_e\dot{\phi}\sin\lambda}{mgL}, \qquad \theta = \frac{\pi}{2}-\Delta\theta, \qquad \psi = 0 \text{ or } \pi. \tag{8.80}$$

As a summary of these results, recall that the analysis treated a balanced gyroscope having a pendulous mass mg attached to the inner gimbal. We have established that if the rotor is released with its spin axis tilted at an angle $\Delta\theta$ above the north–south horizontal, then the precession of the gyroscope will match the component of the earth's angular velocity in the direction of the local vertical. Thus, the plane containing the rotor and the bearings of the outer gimbal will indicate the northward direction.

Our analysis of the gyrocompass has established the conditions for dynamic equilibrium at a specified latitude λ. We will show in the next example that the gyrocompass is stable to small disturbances. The primary limitation on its use is loss of accuracy due to rapid movement of the vehicle in which it is mounted. To learn why such an effect arises, we first observe that the angular velocity of the earth entered into the derivation of Eq. (8.80) because it represented the rotation of the base. Linear motion relative to the earth is actually motion along a great circle. Such an effect adds to the angular velocity of the base.

Consider the situation in Figure 8.16, where the velocity $\bar{v}$ of the base of the gyrocompass is oriented at angle β west of north. This velocity may be considered as produced by rotating the base at $\bar{\omega}_{\rm rel}$ relative to the earth, with the center for the relative motion situated at the earth's center; that is, $\bar{v} = \bar{\omega}_{\rm rel} \times \bar{r}_{O'/O}$, where $\bar{r}_{O'/O} = R_e\bar{K}$. For the velocity appearing in Figure 8.16, we have

$$\bar{\omega}_{\rm rel} = -\left(\frac{v}{R_e}\sin\beta\right)\bar{I} + \left(\frac{v}{R_e}\cos\beta\right)\bar{J}. \tag{8.81}$$

Then the total angular velocity of the base of the gyroscope is the sum of the earth's rotation, Eq. (8.73), and the foregoing rotation of the base relative to the earth,

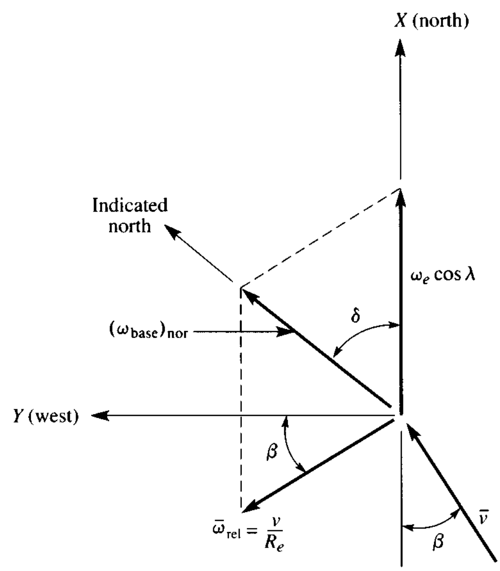


Figure 8.16 Directional error in a gyrocompass due to movement of the vehicle in a great circle.

$$\bar{\omega}_{\text{base}} = \left(\omega_e \cos\lambda - \frac{v}{R_e}\sin\beta\right)\bar{I} + \left(\frac{v}{R_e}\cos\beta\right)\bar{J} + (\omega_e \sin\lambda)\bar{K}. \tag{8.82}$$

The component of this angular velocity parallel to the earth's surface is deviated from the true northerly direction by angle δ, as shown in Figure 8.16.

When Eq. (8.82) applies, $\bar{\omega}_{\text{base}}$ plays the same role for the gyrocompass as did $\bar{\omega}_e$ in the derivation of Eq. (8.80). Hence, the axis of the gyrocompass will align with the horizontal component of $\bar{\omega}_{\text{base}}$, even though this rotation is not solely due to the earth's rotation. The angle δ in Figure 8.16 represents the error in the northerly direction indicated by the moving gyrocompass. This error is found from either Figure 8.16 or Eq. (8.82) to be

$$\delta = \tan^{-1}\left(\frac{v\cos\beta}{\omega_e R_e \cos\lambda - v\sin\beta}\right). \tag{8.83}$$

If $v \ll \omega_e R_e \cos\lambda$, this error is quite small. However, if the gyrocompass is mounted on a moving vehicle near either the North or South Poles, $|\lambda| = \pm\pi/2$, the error will be quite substantial, even if v is quite low. In practice, it is possible to use the fact that δ is known from Eq. (8.83) to compensate readings for this error. However, this does not entirely remove the difficulty near the Poles, because the manner in which a gyrocompass responds to disturbances at the Poles introduces additional errors, as discussed in Example 8.5.

Another source of error arises from acceleration of the pendulous mass, which introduces inertial forces in addition to the weight of that body. In effect, this alters the apparent magnitude and direction of the gravitational force. For all of these reasons, the gyrocompass is used primarily as a navigational aid for slowly moving vehicles, such as ships.

Example 8.5 A gyrocompass tracking the northerly direction in a steady precession is given a small initial nutational disturbance $\Delta\dot{\theta}$, causing it to deviate from its proper direction. Determine the response to this initial disturbance. Then, from that result, assess the stability of the gyrocompass.

Solution The precessional and nutational motions resulting from the disturbance are time-dependent, so Eqs. (8.78) are not adequate for the stability analysis. In order to derive the equations of motion for this case, we form the angular velocity of the flywheel as a superposition of the rotor's spin and the rotation of the earth as given by Eq. (8.75), the nutational motion $\dot{\theta}$, and the precession rate $\dot{\psi}$ relative to the earth. Thus,

$$\begin{aligned}\bar{\omega} = {} & [-\dot{\psi}\sin\theta + \omega_e(\cos\lambda\cos\psi\cos\theta - \sin\lambda\sin\theta)]\bar{i} + (\dot{\theta} - \omega_e\cos\lambda\sin\psi)\bar{j} \\ & + (\dot{\psi}\cos\theta + \omega_e\cos\lambda\cos\psi\sin\theta + \omega_e\sin\lambda\cos\theta + \dot{\phi})\bar{k}.\end{aligned}$$

Because the disturbance is small, the nutation angle θ will remain close to $\pi/2$, provided that the system is stable. We therefore define a new variable such that

$$\eta = \pi/2 - \theta, \qquad \dot{\eta} = -\dot{\theta}.$$

The corresponding mechanical energies are

$$T=\tfrac{1}{2}I'[-(\dot\psi+\omega_e\sin\lambda)\cos\eta+\omega_e\cos\lambda\cos\psi\sin\eta]^2+\tfrac{1}{2}I'(\dot\eta+\omega_e\cos\lambda\sin\psi)^2$$
$$+\tfrac{1}{2}I[\dot\phi+(\dot\psi+\omega_e\sin\lambda)\sin\eta+\omega_e\cos\lambda\cos\psi\cos\eta]^2,$$
$$V=-mgL\cos\eta.$$

The spin angle ϕ is ignorable, as it was in the case of steady precession. The corresponding generalized momentum parameter β_ϕ is constant, where now

$$\beta_\phi=\dot\phi+(\dot\psi+\omega_e\sin\lambda)\sin\eta+\omega_e\cos\lambda\cos\psi\cos\eta.$$

This expression may be substituted into the Lagrange equations for η and ψ, after the derivatives with respect to the generalized coordinates and velocities have been evaluated. The result is

$$\begin{aligned}&I'(\ddot\eta+\dot\psi\omega_e\cos\lambda\cos\psi)\\&\quad-[I\beta_\phi-I'\dot\psi\sin\eta-I'\omega_e(\sin\lambda\sin\eta+\cos\lambda\cos\psi\cos\eta)]\\&\qquad\times[(\dot\psi+\omega_e\sin\lambda)\cos\eta-\omega_e\cos\lambda\cos\psi\sin\eta]+mgL\sin\eta=0,\end{aligned}\tag{1}$$

$$\begin{aligned}&I'[\ddot\psi\cos^2\eta-(\dot\psi+\omega_e\sin\lambda)(\dot\eta+\tfrac{1}{2}\omega_e\cos\lambda\sin\psi)\sin 2\eta\\&\qquad-2\dot\eta\omega_e\cos\lambda\cos\psi\cos^2\eta-\tfrac{1}{2}\omega_e^2\cos^2\lambda\sin 2\psi\cos^2\eta]\\&\quad+I\beta_\phi(\dot\eta+\omega_e\cos\lambda\sin\psi)\cos\eta=0.\end{aligned}\tag{2}$$

In the case of steady precession, $\psi=0$ and η is the constant value $\Delta\theta$ given by Eq. (8.80), which is a very small value. If the disturbance of that state does not destabilize the system, then $\psi(t)$ and $\eta(t)$ will remain small. Hence, we linearize equations of motion (1) and (2). In this process we also ignore terms that are quadratic in ω_e, and use $mgL\gg I\beta_\phi\omega_e$, $\beta_\phi\gg\dot\psi$, $\beta_\phi\gg\omega_e$. The linearized equations of motion simplify to

$$I'\ddot\eta-I\beta_\phi\dot\psi+mgL\eta=I\beta_\phi\omega_e\sin\lambda,\tag{3}$$

$$I'\ddot\psi+(I\beta_\phi-2I'\omega_e\cos\lambda)\dot\eta+(I\beta_\phi\omega_e\cos\lambda)\psi=0.\tag{4}$$

We form the solution of these differential equations by adding complementary and particular solutions. The latter are the values for steady precession,

$$\psi_s=0,\qquad \eta_s=\frac{I\beta_\phi\omega_e\sin\lambda}{mgL}.$$

Because $\beta_\phi\approx\dot\phi$, the latter equation is equivalent to $\eta_s=\Delta\theta$.

We have assumed that the system is stable. We thus anticipate that η and θ vary sinusoidally. Note that the homogeneous terms in eqs. (3) and (4) relate a generalized coordinate and its second derivative to the first derivative of the other generalized coordinate. Consequently, one generalized coordinate must be 90° out of phase relative to the other. A suitable trial form for the complementary solution is therefore

$$\eta_c=A\sin(\sigma t-\nu),\qquad \psi_c=B\cos(\sigma t-\nu),\tag{5}$$

where A, B, σ, and ν are constants. Requiring that these expressions be solutions of the homogeneous portions of eqs. (3) and (4) leads to

$$\begin{aligned}&(mgL-I'\sigma^2)A+(I\beta_\phi\sigma)B=0,\\&(I\beta_\phi-2I'\omega_e\cos\lambda)\sigma A+(I\beta_\phi\omega_e\cos\lambda-I'\sigma^2)B=0.\end{aligned}\tag{6}$$

In order for there to be a nontrivial solution, the determinant of the coefficients of A and B must vanish, which leads to the characteristic equation

$$(I')^2\sigma^4 + (I'I\beta_\phi\omega_e\cos\lambda - I^2\beta_\phi^2 - I'mgL)\sigma^2 + mgLI\beta_\phi\omega_e\cos\lambda = 0. \tag{7}$$

For practical applications, the spin rate is sufficiently large that $\beta_\phi^2 \gg mgL/I$. Then the two values of $\sigma \geq 0$ obtained from this quadratic equation are well approximated as

$$\sigma_1 \approx \left(\frac{mgL\omega_e\cos\lambda}{I\beta_\phi}\right)^{1/2}, \qquad \sigma_2 \approx \frac{I\beta_\phi}{I'}. \tag{8}$$

For each frequency σ_j, there is a corresponding ratio B/A. The first of eqs. (6) indicates that

$$B_j = \mu_j A_j, \quad \mu_j = \frac{I'\sigma_j^2 - mgL}{I\beta_\phi\sigma_j}. \tag{9}$$

For the assumed orders of magnitude of β_ϕ, mgL/I, and ω_e, substitution of each of eqs. (8) leads to

$$\mu_1 \approx -\left(\frac{mgL}{I\beta_\phi\omega_e\cos\lambda}\right)^{1/2}, \qquad \mu_2 \approx 1.$$

We conclude from the foregoing that the complementary solution, which is a free vibration, occurs as either of two modes. The first is a low-frequency mode at σ_1, in which the amplitude of the nutation is much smaller than that of the precession ($\mu_1 \gg 1$), whereas the second is a high-frequency mode at σ_2, in which the amplitudes of the nutation and the precession are approximately equal.

The most general solution is a sum of the two modes, and of the particular solution. Thus, we find that the response to the disturbance is

$$\eta = \Delta\theta + A_1\sin(\sigma_1 t - \nu_1) + A_2\sin(\sigma_2 t - \nu_2),$$

$$\psi = \mu_1 A_1\cos(\sigma_1 t - \nu_1) + \mu_2 A_2\cos(\sigma_2 t - \nu_2).$$

The actual values of the amplitudes A_j and phase angles ν_j depend on the initial conditions, which are not stated specifically. In most actual situations, dissipation effects damp the high-frequency mode much more than the low-frequency mode, in which case the oscillation at frequency σ_1 is most likely to be observed.

In regard to the question of stability, we note that the values of σ_1 and σ_2 are always real. Hence, disturbing the gyrocompass always results in bounded oscillations, corresponding to a stable steady motion. However, the value of σ_1 becomes very small if $\cos\lambda \approx 0$, corresponding to locations near the North or South Poles. Hence, at those locations the gyrocompass executes very slow oscillations when disturbed, which makes it difficult to obtain accurate readings.

8.3.3 *Single-Axis Gyroscope*

An important aspect to the operation of vehicles, particularly airplanes, is measurement of the vehicle's angular velocity. The single-axis gyroscope, which has only an inner gimbal, provides such information because its nutation is essentially proportional to the precession rate. A conceptual model of a single-axis gyro appears

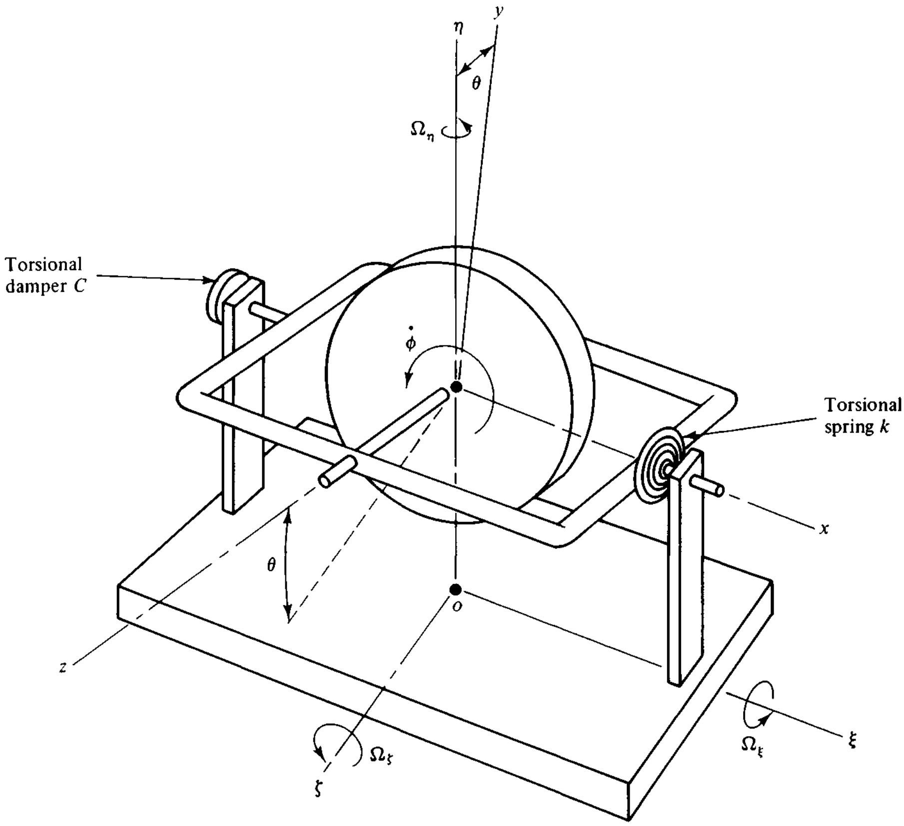


Figure 8.17 Single-axis gyroscope.

in Figure 8.17, where the platform is assumed to undergo arbitrary rotations about the axes $\xi\eta\zeta$. These axes are defined such that the ξ-ζ plane is parallel to the platform, with ξ aligned parallel to the bearings of the gimbal. The xyz reference frame is attached to the gimbal. The gimbal is mounted on the platform by a shaft that is loaded by a linear torsional spring of stiffness K and a torsional damper whose constant is C. We require that the spring be mounted such that, in the absence of movement of the platform, the rotor axis would align parallel to the platform.

If we assume that the rotor spins freely about its axis, then supporting it with one gimbal gives it two degrees of freedom relative to the platform. Several alternative definitions of the Eulerian angles are possible in this case. We shall consider the x axis to be the nutation axis and set $\psi = 0$. The nutation angle θ is the angle from the ξ axis to the rotor axis, and the spin angle ϕ is the rotation of the rotor about its axis. According to this definition, $\theta = 0$ represents the undeformed position of the spring.

We describe the rotation of the platform in terms of the rotation rates $\Omega_\xi, \Omega_\eta, \Omega_\zeta$ about the axes fixed to the platform. We employ the Eulerian angles to describe the angular velocity of the rotor relative to the platform. Adding this relative quantity to the angular velocity of the platform yields the absolute angular velocity of the rotor,

$$\bar{\omega} = \Omega_\xi \bar{e}_\xi + \Omega_\eta \bar{e}_\eta + \Omega_\zeta \bar{e}_\zeta - \dot{\theta}\bar{i} + \dot{\phi}\bar{k}. \tag{8.84}$$

Let I denote the moment of inertia of the rotor about the z axis. Due to the axial symmetry, the moments of inertia of the rotor about the x and y axes are both I', regardless of the angle of spin of the rotor. Correspondingly, we express $\bar{\omega}$ in terms of components relative to xyz in order to form the kinetic energy. The result is

$$\bar{\omega} = (\Omega_\xi - \dot{\theta})\bar{i} + (\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta)\bar{j} + (\Omega_\eta \sin\theta + \Omega_\zeta \cos\theta + \dot{\phi})\bar{k}. \tag{8.85}$$

The corresponding general expression for the kinetic energy is

$$T = \tfrac{1}{2}I'[(\Omega_\xi - \dot{\theta})^2 + (\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta)^2] + \tfrac{1}{2}I(\Omega_\eta \sin\theta + \Omega_\zeta \cos\theta + \dot{\phi})^2. \tag{8.86}$$

The position where the nutation angle is zero corresponds to the unstretched position of the torsional spring, so the potential energy is

$$V = \tfrac{1}{2}K\theta^2. \tag{8.87}$$

We shall describe the effect of the linear torsional dashpot by employing the Rayleigh dissipation function, which treats damping forces analogously to a linear spring. The dashpot constant is C, so we have

$$D = \tfrac{1}{2}C\dot{\theta}^2. \tag{8.88}$$

It is important to recognize that the base rotations are specified, so they are not generalized coordinates. Also, the spin angle appears in the formulation only as a time derivative. Hence, ϕ is an ignorable coordinate; the corresponding Lagrange equation reduces to $\partial T/\partial\dot{\phi} = I\beta_\phi$, where the spin momentum parameter β_ϕ is a constant. For the kinetic energy in Eq. (8.86), this reduces to

$$\beta_\phi = \dot{\phi} + \Omega_\eta \sin\theta + \Omega_\zeta \cos\theta. \tag{8.89}$$

The equation of motion for the nutation angle is the full Lagrange equation, including the term $\partial D/\partial\dot{\theta}$ for the dashpot. The result is

$$I'(\ddot{\theta} - \dot{\Omega}_\xi) + I'(\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta)(\Omega_\eta \sin\theta + \Omega_\zeta \cos\theta) - I(\Omega_\eta \sin\theta + \Omega_\zeta \cos\theta + \dot{\phi})(\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta) + C\dot{\theta} + K\theta = 0. \tag{8.90}$$

We remove the spin rate from this relation by substituting the spin momentum given by Eq. (8.89). When all terms containing the rotation rates of the platform are moved to the right side, the result is

$$I'\ddot{\theta} + C\dot{\theta} + K\theta = -I'[\tfrac{1}{2}(\Omega_\eta^2 - \Omega_\zeta^2)\sin 2\theta + \Omega_\zeta\Omega_\eta \cos 2\theta] + I\beta_\phi(\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta) + I'\dot{\Omega}_\zeta. \tag{8.91}$$

The rotor in a practical single-axis gyroscope is made to spin much more rapidly than the highest anticipated rate of rotation of the platform. Also, the stiffness and damping parameters are usually selected to restrict the nutation angle to a small magnitude. Under these assumptions, the right side is dominated by two terms. The main effect of the rotation rates is contained in $I\beta_\phi\Omega_\eta \cos\theta$, which may be linearized by setting $\cos\theta \approx 1$, while the effect of unsteadiness in the rates appears only in $I'\dot{\Omega}_\zeta$. If β_ϕ is sufficiently high, the latter term may also be neglected. The equation of motion then reduces to

$$I'\ddot{\theta} + C\dot{\theta} + K\theta \approx I\beta_\phi\Omega_\eta \quad \text{if } \beta_\phi \gg \Omega_\eta, \Omega_\zeta, \dot{\Omega}_\zeta/\Omega_\eta. \tag{8.92}$$

This is the equation of motion for a damped linear oscillator with one degree of freedom. Its natural frequency ω and ratio of critical damping σ are

$$\omega = \left(\frac{K}{I'}\right)^{1/2}, \qquad \sigma = \frac{C}{2\omega I'} = \frac{C}{2(KI')^{1/2}}. \tag{8.93}$$

Let us begin by evaluating the nutation when Ω_η is a constant, nonzero value. The corresponding response may be obtained by adding the complementary and particular solutions. In the absence of rotations of the platform, the gimbal will be at rest at its equilibrium position $\theta = 0$, so we set $\theta = \dot{\theta} = 0$ when $t = 0$ as initial conditions. If the damping is light, $\sigma < 1$, the corresponding response is

$$\theta = \frac{I\beta_\phi \Omega_\eta}{K}\left\{1 - \exp(-\sigma\omega t)\left[\cos\omega_d t + \frac{\sigma}{(1-\sigma)^{1/2}}\sin\omega_d t\right]\right\}, \tag{8.94}$$

where $\omega_d = \omega(1-\sigma^2)^{1/2}$ is the damped natural frequency. Equation (8.94) indicates that the steady-state response, which is obtained as $t \to \infty$, is a constant nutation angle that is proportional to the platform's rotation rate about the η axis,

$$\theta_s = \frac{I\beta_\phi}{K}\Omega_\eta. \tag{8.95}$$

Thus, the nutation angle may be measured and compared to a scale that is calibrated in units of the rotation rate Ω_η. We should note that the foregoing steady-state response would also be obtained if Ω_η were time-dependent, provided that the free vibration response decays in a much smaller time than the interval required to observe substantial changes in Ω_η. This condition may be achieved by designing the system to have a high natural frequency and to be highly damped, subject to $\sigma < 1$. A single-axis gyro that is constructed with a spring and a dashpot that are both stiff is called a *rate gyroscope*. Because a rate gyroscope indicates the rotation about only one axis, inertial guidance systems for aerospace applications employ three such gyros, mounted about orthogonal axes.

There is an alternative configuration for a single-axis gyro that is employed frequently. Suppose the torsional spring is not present. Setting $K = 0$ in Eq. (8.92) leads to

$$I'\ddot{\theta} + C\dot{\theta} = I\beta_\phi\Omega_\eta. \tag{8.96}$$

It is possible to obtain a solution valid for arbitrary Ω_η, not necessarily constant. Such a result consists of a convolution integral that may be derived either from a Laplace transform, or from a Duhamel integral using the impulse response of a second-order linear oscillator that has no spring. The result is

$$\theta = \frac{I\beta_\phi}{C}\int_0^t \Omega_\eta(\tau)\left\{1 - \exp\left[-\frac{C}{I'}(t-\tau)\right]\right\} d\tau. \tag{8.97}$$

It is desirable that the damping rate be large, in order to make the exponential term in the integrand decay quickly. Then, after an initial start-up interval, the nutation angle will be well approximated by

$$\theta = \frac{I\beta_\phi}{C}\int_0^t \Omega_\eta(\tau)\, d\tau. \tag{8.98}$$

We see that the nutation angle in this case is proportional to the integral of the rotation rate about the η axis, which represents the cumulative rotation. For this reason, a well-damped single-axis gyro that is not restrained by a spring is called an *integrating gyroscope.* As with rate gyroscopes, a complete guidance system would require three integrating gyroscopes each of whose nutation axis is aligned with mutually orthogonal axes.

We must note in closing that our discussions of inertial guidance systems have been drastically simplified, both through the models we created and the assumptions used to obtain responses. For example, we generally idealized systems by neglecting the inertia of the gimbals. In some cases this merely affects oscillation frequencies. However, the additional inertial resistance can lead to qualitative differences. Such is the case for a free gyroscope that is subjected to a small disturbance. The inertia of the outer gimbal can cause a precession that drifts away from the initial orientation, rather than oscillating about it. In regard to the analysis of responses, linearization often avoids some important questions, such as loss of dynamic stability through nonlinear mechanisms. Practical use of the gyroscope as a tool for navigation over long ranges requires more sophisticated analyses, accounting for gimbal inertia and bearing friction, than those we have presented here. However, the features of such investigations would show many similarities to the steps we have pursued.

Example 8.6 An airplane initially in level flight executes a body-fixed rotation about an axis that lies in the ζ–η plane in Figure 8.17, at angle γ from the η axis. The rotation rate Ω about this axis is a sinusoidal pulse over a time interval τ.

$$\Omega = \begin{cases} \Omega_0 \sin(\pi t/\tau) & \text{for } 0 \le t \le \tau, \\ 0 & \text{for } t \ge \tau. \end{cases}$$

The rotor was spinning in its reference position, $\theta = 0$, when the aircraft began its rotation. Determine the nutational response $\theta(t)$ of the rate gyroscope for the case where damping is less than critical. From that solution, determine the conditions for which the value of $K\theta/I\beta_\phi$ closely matches the nominal response in Eq. (8.95).

Solution Because the rotation is about a body-fixed axis, the components of the angular velocity $\bar{\Omega}$ of the aircraft relative to the (body-fixed) $\xi\eta\zeta$ axes are constant at

$$\Omega_\xi = 0, \qquad \Omega_\zeta = \Omega \sin\gamma, \qquad \Omega_\eta = \Omega \cos\gamma.$$

The response we seek is the solution to Eq. (8.92) for the specified rotation of the base, subject to the initial conditions that $\theta = \dot{\theta} = 0$ when $t = 0$. Several methods are suitable for determining this response. We shall exploit the similarity of the problem to that encountered in conventional transient vibrations.

For the given angular velocity components, the conditions $\beta_\phi \gg \Omega_\eta, \Omega_\zeta, \dot{\Omega}_\zeta/\Omega_\eta$ required to employ Eq. (8.92) are satisfied if $\beta_\phi \gg \Omega_0$ and $\beta_\phi \gg (\pi/\tau)\tan\gamma$. (When $\lambda \approx \pm\pi/2$, the rotation is essentially about the ζ axis; such a rotation would presumably be measured by a rate gyroscope arranged orthogonally to the one under consideration.) We assume that β_ϕ meets these conditions. Substitution of the given functional form of Ω for $t < \tau$ then leads to

$$I'\ddot{\theta} + C\dot{\theta} + K\theta = I\beta_\phi\Omega_0 \cos\gamma \sin\left(\frac{\pi t}{\tau}\right).$$

This represents a one-degree-of-freedom system, with natural frequency $\omega = (K/I')^{1/2}$ and ratio of critical damping $\sigma = C/2(I'K)^{1/2}$, that is being subjected to a sinusoidal excitation at frequency π/τ. We may construct the particular solution, known in vibration theory as the steady-state response, by multiplying the quasistatic response ($\tau \gg \pi/\omega$) by a dynamic magnification factor F. Also, the steady-state response is delayed by a phase lag δ. Hence, we construct the particular solution for the present response as

$$\theta_p = F\frac{I}{I'}\frac{\Omega_0}{\omega}\frac{\beta_\phi}{\omega}\cos\gamma\sin\left(\frac{\pi t}{\tau}-\delta\right),$$

where

$$F = \left\{\left[1-\left(\frac{\pi}{\omega\tau}\right)^2\right]^2+\left(\frac{2\pi\sigma}{\omega\tau}\right)^2\right\}^{-1/2},$$

$$\delta = \tan^{-1}\left[\frac{2\sigma\pi/\omega\tau}{1-(\pi/\omega\tau)^2}\right], \quad 0\le\delta\le\pi.$$

The initial conditions must be satisfied by the combination of the particular and complementary solutions. We recall for the latter that damping is less than critical, $\sigma < 1$. The complementary solution, which is equivalent to the free-vibration response, therefore consists of an oscillatory response that decays exponentially in time according to

$$\theta_c = \exp(-\sigma\omega t)[D\sin(\omega_d t)+E\cos(\omega_d t)], \quad \omega_d = \omega(1-\sigma^2)^{1/2}.$$

Setting $\theta_p+\theta_c = 0$ at $t = 0$ yields

$$E = F\frac{I}{I'}\frac{\Omega_0}{\omega}\frac{\beta_\phi}{\omega}\cos\gamma\sin\delta.$$

Similarly, the condition that $\dot{\theta}_p+\dot{\theta}_c = 0$ at $t = 0$ requires that

$$\omega_d D-\sigma\omega E = -F\frac{\pi}{\tau}\frac{I}{I'}\frac{\Omega_0}{\omega}\frac{\beta_\phi}{\omega}\cos\gamma\cos\delta.$$

When we substitute the preceding expression for E into this relation, we obtain the value of D. The total response then consists of

$$\theta = F\frac{I}{I'}\frac{\Omega_0}{\omega}\frac{\beta_\phi}{\omega}\cos\gamma\sin\left(\frac{\pi t}{\tau}-\delta\right)$$
$$+\exp(-\sigma\omega t)[D\sin(\omega_d t)+E\cos(\omega_d t)], \quad t<\tau. \quad (1)$$

As we have noted, this remains true only so long as the body-fixed rotation is active. At $t = \tau$, the airplane's rotation ceases, so the response for $t > \tau$ consists of a free vibration. However, the values of θ and $\dot{\theta}$ must be continuous at $t = \tau$. Let θ^* and $\dot{\theta}^*$ denote these values, which we find by evaluating the response in the interval $t < \tau$. The complementary solution that we found earlier may be used to form the free-vibration response. The task of satisfying the continuity conditions at $t = \tau$ is expedited by using the retarded time $t-\tau$. Hence, we let

$$\theta = \exp[-\sigma\omega(t-\tau)]\{D^*\sin[\omega_d(t-\tau)]+E^*\cos[\omega_d(t-\tau)]\}, \quad t>\tau, \quad (2)$$

where satisfying the continuity conditions yields

$$E^* = \theta^*, \qquad \omega_d D^* - \sigma\omega D^* = \dot{\theta}^*.$$

Given values of the system parameters, it would be a simple matter to use eqs. (1) and (2) to determine the value of θ at any instant. For comparison, the nominal response given by Eq. (8.95) would be

$$\theta = \frac{I\beta_\phi}{I'\omega^2}\Omega_\eta,$$

which for the given rotation becomes

$$\theta = \frac{\Omega_0}{\omega}\frac{I}{I'}\frac{\beta_\phi}{\omega}\cos\gamma \sin\left(\frac{\pi t}{\tau}\right) \quad \text{if } t < \tau, \tag{3}$$

$$\theta = 0 \quad \text{if } t > \tau. \tag{4}$$

We want the expression in eqs. (1) and (2) to match (respectively) eqs. (3) and (4) closely. We observe that $I'/I \geq 0.5$ for any rotor, with the lower bound corresponding to a thin disk. Also, the values of the coefficients D and E are comparable to the amplitudes of the sinusoidal term. Therefore, if the expressions are to match, the following conditions must apply:

(a) the exponential factor, representing the decay of the complementary solution, must become very small in a time interval much shorter than τ;
(b) the magnification factor F must be close to unity; and
(c) the phase lag δ must be close to zero.

These conditions, as well as the conditions required to neglect the effect of acceleration of the base, are satisfied when

$$\sigma\omega\tau \gg 1, \qquad \sigma < 1, \qquad \pi/\omega\tau \ll 1, \qquad \beta_\phi \gg \Omega_0, \qquad \beta_\phi \gg (\pi/\tau)\tan\gamma.$$

In general, the spin rate will be much larger than the rate at which the aircraft rotates, so $\beta_\phi \approx \dot{\phi}$. The conditions shown then lead to the following requirements: the natural period of free vibration, $2\pi/\omega$, should be much smaller than the time interval τ over which the pulse occurs; the damping should be reasonably close to critical; the spin rate should be much larger than the peak rotation rate of the base, as well as the frequency at which the airplane's rotation fluctuates; and the angle between the angular velocity of the base and the η axis should not be close to 90°.

These requirements are not difficult to meet, because the spin, roll, and yaw motions of even a very high-performance aircraft are moderate from a mechanical standpoint. For example, a very violent maneuver might consist of several rolls in a few seconds, for which τ might be of the order of 2 s. In contrast, a natural frequency of 1,000 rad/s and a spin rate of 20,000 rev/min are readily attainable.

When all these criteria are met, differences between the response in eqs. (1) and (3), and between eqs. (2) and (4), are significant only for early elapsed times. For example, if $\sigma\omega t$ (in the initial phase) or $\sigma\omega(t-\tau)$ (after cessation of rotation) equals unity, the complementary solution will have decayed to 37% of its initial magnitude. Clearly, damping designed to be as close as possible to critical, $\sigma = 1$, substantially helps reduce the duration over which discrepancies between the measured and nominal responses are significant.

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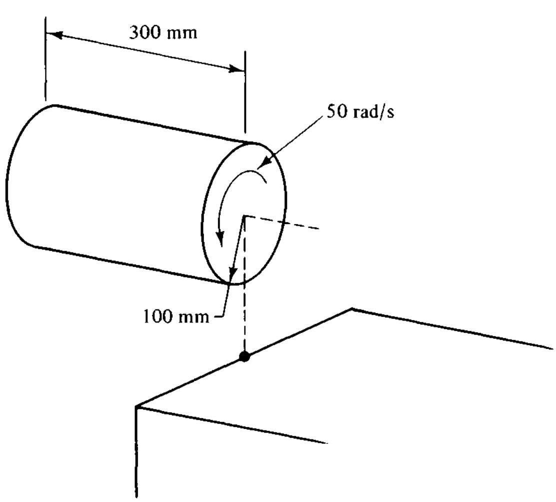
Problems

8.1 Prove that the polhode description of free motion for an arbitrary body reduces to the space- and body-cone analogy when the body is axisymmetric.

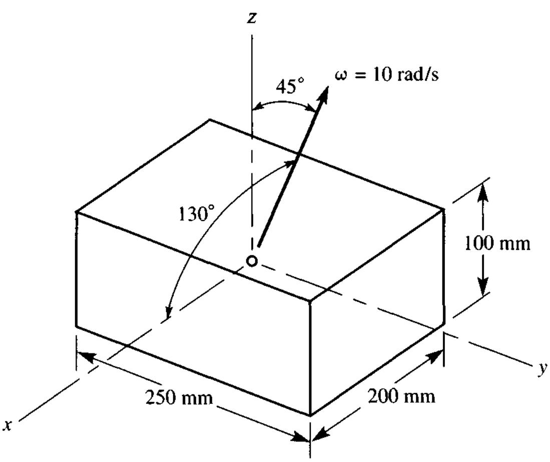
8.2 An axially symmetric earth satellite, whose ratio of principal moments of inertia is $I/I' = 1.6$, precesses about its axis once every 2 s. The spin rate in this state is 0.1 rad/s. Determine the overall rate of rotation and the angle from the axis of symmetry to the precession axis. Then determine the minimum angular impulse that a set of control rockets fastened to the satellite must exert in order to bring the precession axis into coincidence with the axis of symmetry. What is the rotation rate of the satellite at the conclusion of such a maneuver? (Assume that the rockets act impulsively.)

8.3 The cylinder, whose mass is 2 kg, translates downward such that its axis of symmetry remains horizontal. The spin rate about that axis is 50 rad/s. The cylinder has a speed of 40 m/s when it collides with the ledge. Immediately after impact, the center of mass of the cylinder has a downward velocity of 10 m/s. Describe the rotational motion of the cylinder after impact.

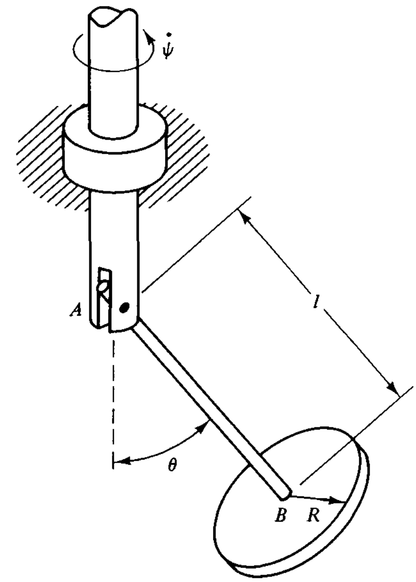


Problem 8.3

8.4 The angular velocity of a wooden block at the instant it is released is as shown. Which body-fixed axis is surrounded by the polhode curve for the free rotation? What are the maximum and minimum angles between this axis and the constant direction of the angular momentum? What are the angular velocities of the block at these maximum and minimum conditions?



Problem 8.4



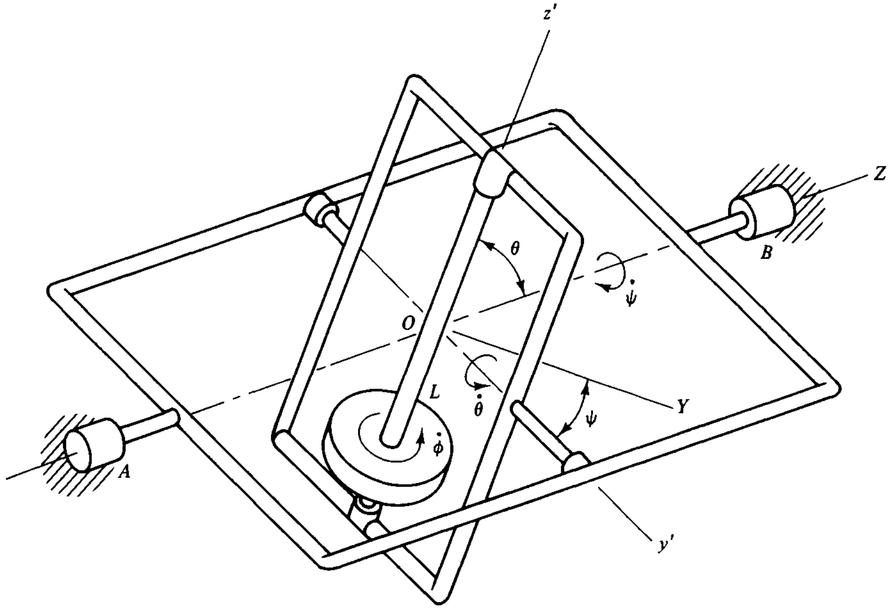
Problem 8.5

8.5 The thin disk of mass m is welded to bar AB, which is fastened to the vertical shaft by a pin. The rotation rate of this shaft is $\dot{\psi}$. The mass of bar AB is negligible. Evaluate the stability of a steady precession about the vertical orientation of bar AB, $\theta = 0$, as a function of the precession rate $\dot{\psi}$ and the length ratio R/l.

8.6 A free gyroscope is in a state of slow, steady precession at a nutation angle of 53.13°, with a spin rate of 10,000 rev/min. The rotor's mass is 5 kg, its radii of gyration about its pivot are $\kappa = 100$ mm and $\kappa' = 180$ mm, and its center of mass is 120 mm from the pivot. A person accidentally touches the outer gimbal, causing the precession rate to decrease suddenly by 0.6 rad/s. Determine whether the ensuing motion is unidirectional, looping, cuspidial, or steady precession. What are the maximum and minimum nutation angles in that motion?

8.7 A free symmetric gyroscope, initially in a state of steady slow precession, is subjected to a small disturbing torque $\epsilon mgL \sin \Omega t$ acting about the fixed vertical shaft supporting the outer gimbal. Use a perturbation analysis for $\epsilon \ll 1$ to determine the frequency, if any, at which the system resonates.

8.8 The device shown is a *gyropendulum*, a system used in some inertial guidance applications to locate the vertical direction. The spin rate $\dot{\phi}$ is held constant by a servomotor. Let m be the mass, and let I and I' be the centroidal principal moments of inertia of the flywheel parallel and transverse to the spin axis; ignore the inertia of the gimbals. Evaluate the nutation response $\theta(t)$ and the precessional response $\dot{\psi}(t)$ of the flywheel to a disturbance that causes it to rotate by very small angles away from the vertical reference position, at which $\theta = \pi/2$. Compare the frequency of these responses to that of a simple pendulum, and use that result to discuss an advantage of the gyropendulum.



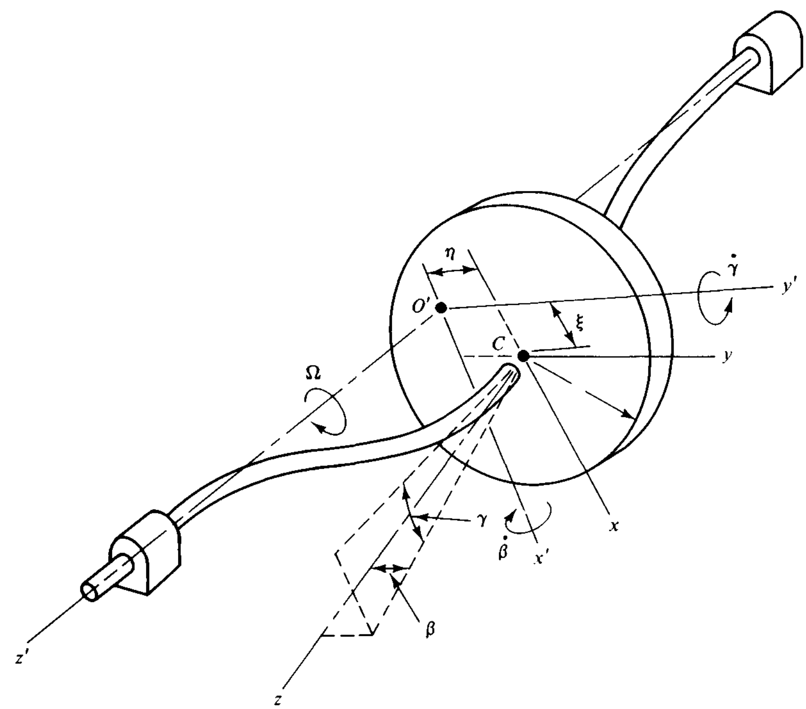
Problem 8.8

8.9 Consider the gyropendulum in Problem 8.8. Because of movement of the vehicle, the center point O has a constant acceleration $\dot{v}$ directed parallel to the axis of the outer gimbal (i.e., $\bar{a}_0 = \dot{v}\bar{e}_{B/A}$). Let this acceleration be directed at angle β north of east. Derive equations of motion for the Eulerian angles including the effect of the earth's rotation, and of the movement of the vehicle in a great circle at angular speed v/R_e.

8.10 The platform of an integrating gyroscope rotates about the η axis in a time-dependent manner. Consider an angular speed that consists of an average value Ω_0, over which is superposed a harmonic fluctuation at amplitude Ω_1 and frequency λ, such that $\Omega_\eta = \Omega_0 + \Omega_1 \sin(\lambda t)$. What conditions must be true if the nutational response $\theta(t)$ following the initial treatment phase is to be proportional to the mean rotation $\Omega_0 t$?

8.11 A top is initially in a state of steady precession at a precession rate $\dot{\psi}$ and a nutation angle θ^*. The precession rate is suddenly increased by the amount $\Delta\dot{\psi}$, due to the application of an impulsive force. Determine the precessional and nutational responses after cessation of the impulsive force. Use a perturbation analysis of the basic equations for a top, Eqs. (8.35)–(8.38), in which the small parameter is $\epsilon = \Delta\dot{\psi}/\dot{\psi}^*$.

8.12 *Whirling* is a phenomenon in turbomachinery in which a rotating shaft undergoes deformation as a beam. In order to study this effect, consider the shaft supporting the disk of mass m and radius R to be flexible in bending, rigid in extension and torsion, and massless. The disk is welded to the shaft, such that its center C is on the centerline of the shaft. The rotation rate of the shaft about its bearing is constant at Ω. Let $x'y'z'$ be a reference frame that rotates at this rate, with its z' axis concurrent with the line connecting the bearings. Let the deflected position of the center of the disk relative to $x'y'z'$ be $\bar{r}_{C/O'} = \xi\bar{i}' + \eta\bar{j}'$. Furthermore, let $x'y'z'$ be principal centroidal axes for the disk, and let β and γ be very small rotations about the x' and y' axes, respectively, of xyz relative to $x'y'z'$. For a shaft of symmetric cross-section, each deformation is resisted solely by a corresponding proportional elastic force or moment; the elastic constants are k_ξ, k_η, k_β, and k_γ. Derive the corresponding linearized equations of motion.



Problem 8.12

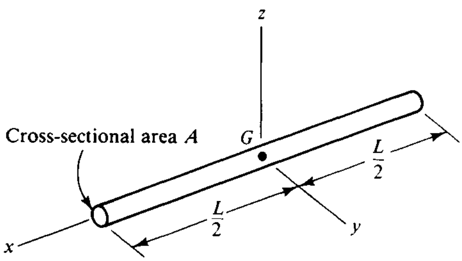
8.13 Consider the effects of the inertia of the gimbals in a balanced free gyroscope ($L = 0$). Let A, B, and C denote the (principal) moments of inertia of the inner gimbal about the xyz axes, where y is the line of nodes and z is the spin axis of the rotor. Also, let A' denote the moment of inertia of the outer gimbal about the precession axis. Derive the equations of motion for the gyroscope in this case.

8.14 Suppose that the gyroscope in Problem 8.13 is initially spinning at $\dot{\phi}$ and that the nutation angle is constant at θ_0; there is no precession in this initial motion. At $t = 0$, a nutational velocity $\epsilon \ll \dot{\phi}$ is imparted to the inner gimbal. Use a perturbation analysis in which $\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2$ and $\dot{\psi} = \epsilon\dot{\psi}_1 + \epsilon^2\dot{\psi}_2$ to determine the nutational and precessional fluctuations induced by the disturbance. Show that, because of gimbal inertia, the response exhibits *gimbal walk*, in which there is a nonzero average precessional rotation rate, even though the gyroscope is balanced.

APPENDIX

Centroidal Inertia Properties

Slender Bar

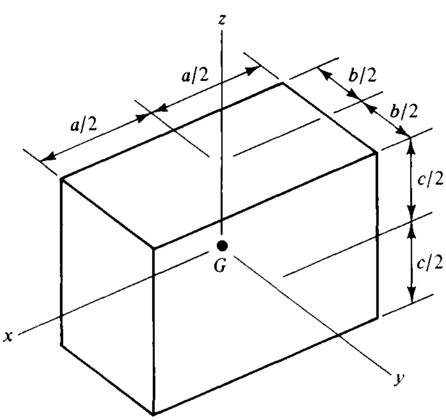


$$V = LA$$

$$I_{xx} = 0$$

$$I_{yy} = I_{zz} = \frac{1}{12} mL^2$$

Rectangular Prism



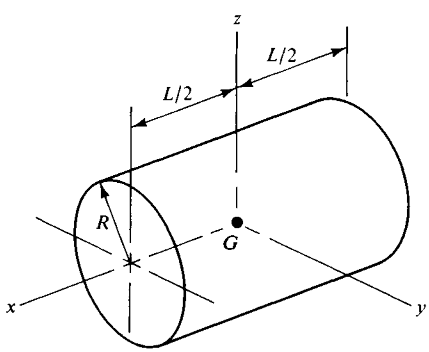
$$V = abc$$

$$I_{xx} = \frac{1}{12} m(b^2 + c^2)$$

$$I_{yy} = \frac{1}{12} m(a^2 + c^2)$$

$$I_{zz} = \frac{1}{12} m(a^2 + b^2)$$

Cylinder

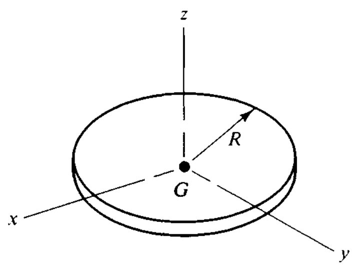


$$V = \pi R^2 L$$

$$I_{xx} = \frac{1}{2} mR^2$$

$$I_{yy} = I_{zz} = \frac{1}{12} m(3R^2 + L^2)$$

Thin Disk

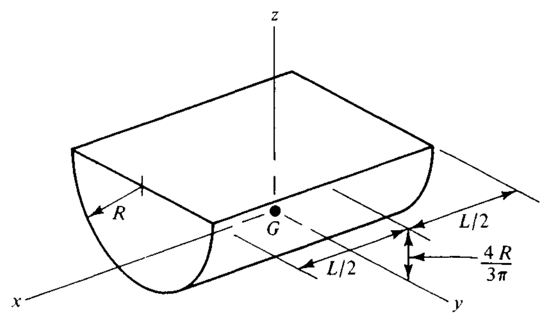


$$V = \pi R^2 h$$

$$I_{xx} = I_{yy} = \frac{1}{4} m R^2$$

$$I_{zz} = \frac{1}{2} m R^2$$

Semicylinder



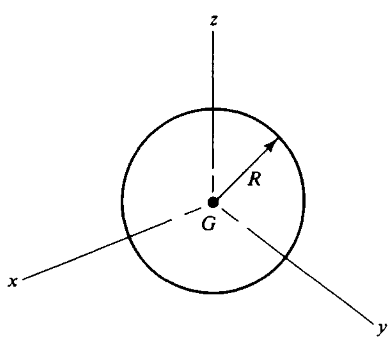
$$V = \frac{1}{2} \pi R^2 L$$

$$I_{xx} = \frac{9\pi^2 - 32}{18\pi^2} m R^2$$

$$I_{yy} = \frac{9\pi^2 - 64}{36\pi^2} m R^2 + \frac{1}{12} m L^2$$

$$I_{zz} = \frac{1}{12} m (3R^2 + L^2)$$

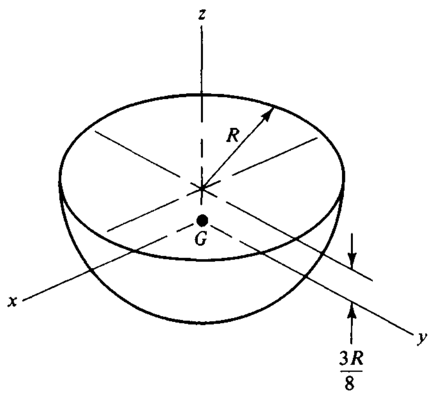
Sphere



$$V = \frac{4}{3} \pi R^3$$

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} m R^2$$

Hemisphere

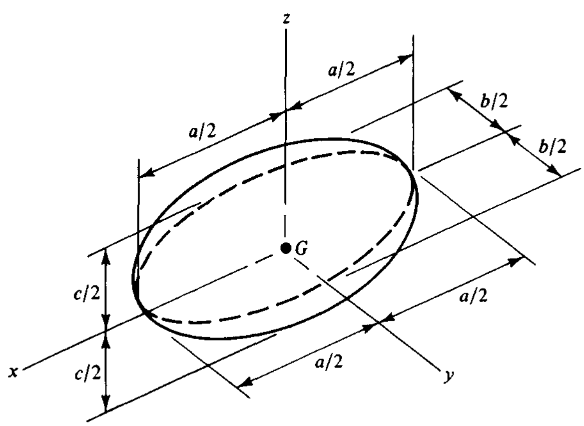


$$V = \frac{2}{3}\pi R^3$$

$$I_{xx} = I_{yy} = \frac{83}{320} mR^2$$

$$I_{zz} = \frac{2}{5} mR^2$$

Ellipsoid



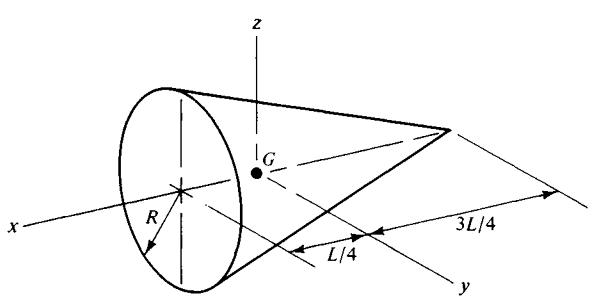
$$V = \frac{4}{3}\pi abc$$

$$I_{xx} = \frac{1}{5} m(b^2 + c^2)$$

$$I_{yy} = \frac{1}{5} m(a^2 + c^2)$$

$$I_{zz} = \frac{1}{5} m(a^2 + b^2)$$

Cone

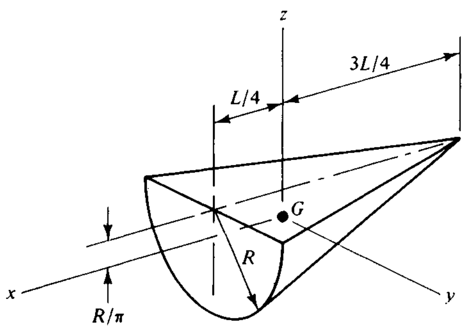


$$V = \frac{1}{3}\pi R^2 L$$

$$I_{xx} = \frac{3}{10} mR^2$$

$$I_{yy} = I_{zz} = \frac{3}{80} m(4R^2 + L^2)$$

Semicone



$$V = \frac{1}{6}\pi R^2 L$$

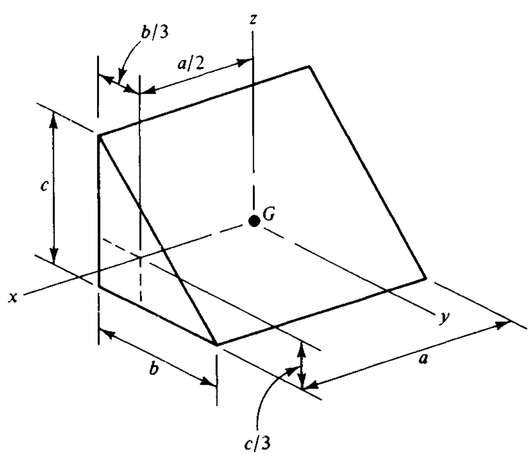
$$I_{xx} = \left(\frac{3}{10} - \frac{1}{\pi^2}\right) mR^2$$

$$I_{yy} = \left(\frac{3}{20} - \frac{1}{\pi^2}\right) mR^2 + \frac{3}{80} mL^2$$

$$I_{zz} = \frac{3}{80} m(4R^2 + L^2)$$

$$I_{xz} = -\frac{1}{20\pi} mRL \qquad I_{xy} = I_{yz} = 0$$

Right Triangular Prism



$$V = \frac{1}{2} abL$$

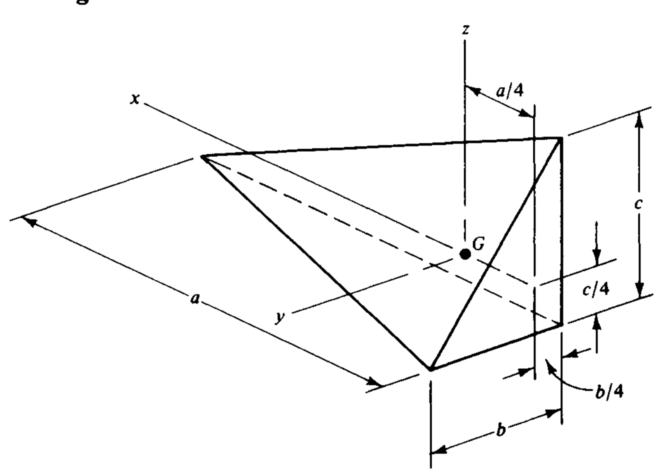
$$I_{xx} = \frac{1}{18} m(b^2 + c^2)$$

$$I_{yy} = \frac{1}{36} m(3a^2 + 2c^2)$$

$$I_{zz} = \frac{1}{36} m(3a^2 + 2b^2)$$

$$I_{yz} = -\frac{1}{36} mbc \qquad I_{xy} = I_{xz} = 0$$

Orthogonal Tetrahedron



$$V = \frac{1}{6} abc$$

$$I_{xx} = \frac{3}{80} m(b^2 + c^2)$$

$$I_{yy} = \frac{3}{80} m(a^2 + c^2)$$

$$I_{zz} = \frac{3}{80} m(a^2 + b^2)$$

$$I_{xy} = -\frac{1}{80} mab$$

$$I_{xz} = -\frac{1}{80} mac$$

$$I_{yz} = -\frac{1}{80} mbc$$

Answers to Even-Numbered Problems

2.2 $\bar{v} = \sqrt{2gR\sin(s/R)}\,\bar{e}_t, \quad \bar{a} = g\sin(s/R)\bar{e}_t + 2g\sin(s/R)\bar{e}_n.$

2.4 $\bar{v} = v_0(1-k\sinh x)\bar{e}_t,$

$$\bar{a} = v_0^2\left[-k(1-k\sinh x)\bar{e}_t + \left(\frac{1-k\sinh x}{\cosh x}\right)^2\bar{e}_n\right].$$

2.6 $\bar{v} = \alpha k(-1.028\bar{i} + 4.830\bar{j} + 0.783\bar{k}),$

$\bar{a} = \alpha k^2(1.259\bar{i} - 1.694\bar{j} - 0.671\bar{k}).$

2.8 $\bar{v} = 2k\omega t[(\sin\beta + \beta\cos\beta)\bar{i} + 2\beta\bar{j} + 2\beta(\sin^2\beta + \beta\sin\beta\cos\beta + 2\beta^2)\bar{k}], \quad \beta = \omega t^2,$

$$\bar{a} = 2k\omega\{[(1-2\beta^2)\sin\beta + 5\beta\cos\beta]\bar{i} + 6\beta\bar{j} + 2\beta[(3-4\beta^2)\sin^2\beta + 9\beta\sin\beta\cos\beta + 16\beta^2]\bar{k}\}.$$

2.10 $D = \dfrac{2u^2\sin(\theta+\beta)\cos\beta}{g\cos^2\theta}, \quad H = \dfrac{u^2\sin^2(\theta+\beta)}{2g\cos\theta}.$

2.12 $\bar{r} = 1.4051\bar{i} + 0.4\bar{j} + 1.2759\bar{k}$ m.

2.14 $\bar{r} = \left[\dfrac{\dot{y}_0}{\mu}(1-\cos(\mu t)) + \dfrac{\dot{x}_0}{\mu}\sin(\mu t)\right]\bar{i} + \left[\dfrac{\dot{y}_0}{\mu}\sin(\mu t) - \dfrac{\dot{x}_0}{\mu}(1-\cos(\mu t))\right]\bar{\partial}, \quad \mu = \dfrac{B\beta}{m}.$

2.16 $\bar{v} = 1.773\omega\sqrt{bc}\,\bar{e}_R + 33.41\omega\sqrt{bc}\,\bar{e}_\theta + 10.883\omega b\bar{k},$

$\bar{a} = -366.7\omega^2\sqrt{bc}\,\bar{e}_R - 12.566\omega^2 b\bar{k}.$

2.18 $\bar{a} = -\left(\dfrac{k}{ab}\right)^2 R\bar{e}_R.$

2.20 $\bar{v} = \dot{\theta}(R'\bar{e}_R + R\bar{e}_\theta), \quad \bar{a} = [(R''-R)\bar{e}_R + 2R'\bar{e}_\theta]\dot{\theta}^2,$

$$\rho = \frac{[(R')^2 + R^2]^{3/2}}{|R''R - 2(R')^2 - R^2|}.$$

2.22 $\bar{a}_P = \left(-\dfrac{3\dot{z}^2}{8L\sin^3\theta} - \dfrac{3}{2}L\omega^2\sin\theta\right)\bar{e}_R - \dfrac{3}{2}(\dot{z}\omega\cot\theta)\bar{e}_\theta.$

2.24 $\bar{v} = (\dot{z}\tan\beta)\bar{e}_R + (z\alpha t\tan\beta)\bar{e}_\theta + \dot{z}\bar{k}$

$= (\dot{z}\sec\beta)\bar{e}_r + (z\alpha t\tan\beta)\bar{e}_\theta,$

$\bar{a} = (\ddot{z} - z\alpha^2 t^2)(\tan\beta)\bar{e}_R + (\alpha z + 2\dot{z}\alpha t)(\tan\beta)\bar{e}_\theta + \ddot{z}\bar{k}$

$= \left(\dfrac{\ddot{z} - z\alpha^2 t^2\sin^2\beta}{\cos\beta}\right)\bar{e}_r - (z\alpha^2 t^2\sin\beta)\bar{e}_\phi + (z\alpha + 2\dot{z}\alpha t)(\tan\beta)\bar{e}_\theta.$

2.26 $\bar{e}_\rho = (\cos\psi\cos\theta)\bar{i} + (\cos\psi\sin\theta)\bar{j} + (\sin\psi)\bar{k},$

$\bar{e}_\theta = -(\sin\theta)\bar{i} + (\cos\theta)\bar{j},$

$\bar{e}_\psi = -(\sin\psi\cos\theta)\bar{i} - (\sin\psi\sin\theta)\bar{j} + (\cos\psi)\bar{k};$

$$\frac{\partial \bar{e}_\rho}{\partial \rho} = \bar{0}, \quad \frac{\partial \bar{e}_\rho}{\partial \theta} = (\cos\psi)\bar{e}_\theta, \quad \frac{\partial \bar{e}_\rho}{\partial \psi} = \bar{e}_\psi;$$

$$\frac{\partial \bar{e}_\theta}{\partial \rho} = \bar{0}, \quad \frac{\partial \bar{e}_\theta}{\partial \theta} = -(\cos\psi)\bar{e}_\rho + (\sin\psi)\bar{e}_\psi, \quad \frac{\partial \bar{e}_\theta}{\partial \psi} = \bar{0};$$

$$\frac{\partial \bar{e}_\psi}{\partial \rho} = \bar{0}, \quad \frac{\partial \bar{e}_\psi}{\partial \theta} = -(\sin\psi)\bar{e}_\theta, \quad \frac{\partial \bar{e}_\psi}{\partial \psi} = -\bar{e}_\rho.$$

2.28 $v = 12.329$ m/s, $\dot{v} = -8.922$ m/s^2, $\rho = 3.719$ m.

2.30 $\dot{\phi} = -0.6154$ rad/s, $\ddot{\phi} = -0.7040$ rad/s^2.

2.32 $\bar{a} = -3.01\bar{i} - 38.04\bar{j}$ m/s^2.

2.34 $v = \sqrt{2}rc(1-\cos\xi)^{1/2}$, $\dot{v} = rc^2 \dfrac{\sin\xi}{\sqrt{2}(1-\cos\xi)^{1/2}}$,

$\rho = 2\sqrt{2}r(1-\cos\xi)^{1/2}$.

3.2 $\Delta\bar{r} = -52.23\bar{I} - 10.19\bar{J} - 131.37\bar{K}$ mm.

3.4 $\beta = 128.68°$, $l_{z'x} = -0.2774$, $l_{z'y} = 0.4804$, $l_{z'z} = 0.8321$.

3.6 $\bar{r}_C = -3.11\bar{I} + 51.86\bar{J} - 5.05\bar{K}$ m.

3.8
$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 0.9143 & 0.3786 & -0.1443 \\ 0.0643 & 0.2161 & 0.9743 \\ 0.4000 & -0.9000 & 0.1732 \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}.$$

3.10 $\Delta\bar{r}_C = -914.3\bar{I} - 535.7\bar{J} - 800\bar{K}$ mm.

3.12 $\Delta\bar{r}_A = -9.65\bar{I} - 759.7\bar{J} - 458.4\bar{K}$ mm.

3.14 $\bar{\omega} = 9.66\bar{i} + 630.9\bar{k}$ rad/s,

$\bar{\alpha} = 96.6\bar{i} - 6{,}069\bar{j} + 25.9\bar{k}$ rad/s^2; $\bar{k}$ = centerline of disk.

3.16 $\bar{\omega} = 1.257\bar{i} + 123.99\bar{k}$ rad/s,

$\bar{\alpha} = -157.91\bar{j}$ rad/s^2; $\bar{k}$ = centerline of disk.

3.18 $\bar{v}_D = (123.85R - 3.908L)\bar{j}$; $\bar{k}$ = centerline of disk,

$\bar{a}_D = (-15{,}339R - 7.09L)\bar{i} + (261.3R - 4.09L)\bar{k}$.

3.20 $\bar{a}_C = -24{,}655L\bar{i} + 1{,}250L\bar{j}$, $\dot{v} = -1{,}923.9L$; $\bar{i} = \bar{e}_{C/B}$.

3.22 $\bar{v}_B = (\omega_2 L\sin\theta)\bar{i} + (\omega_1 L\cos\theta)\bar{j} - (\omega_1 L\sin\theta)\bar{k}$,

$\bar{a}_B = (2\omega_1\omega_2 L\cos\theta)\bar{i} - [(\omega_1^2 + \omega_2^2)L\sin\theta]\bar{j} - (\omega_1^2 L\cos\theta)\bar{k}$.

3.24 $\bar{v}_A = 2.451\bar{i} - 3.371\bar{j} - 10.828\bar{k}$ m/s; $\bar{i} = \bar{e}_{C/B}$,

$\bar{a}_A = 193.82\bar{i} - 158.22\bar{j} + 113.37\bar{k}$ m/s^2.

3.26 $\bar{v} = \dot{\rho}\bar{e}_\rho + (R + \rho\cos\phi)\dot{\theta}\bar{e}_\theta + \rho\dot{\phi}e_\phi$,

$\bar{a} = [\ddot{\rho} - (R+\rho\cos\phi)\dot{\theta}^2\cos\phi - \rho\dot{\phi}^2]\bar{e}_\rho + [(R+\rho\cos\phi)\ddot{\theta} + 2\dot{\rho}\dot{\theta}\cos\phi - 2\rho\dot{\theta}\dot{\phi}\sin\phi]\bar{e}_\theta$
$+ [\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} + (R+\rho\cos\phi)\dot{\theta}^2\sin\phi]\bar{e}_\phi$.

3.28 $\dot{u} = (50 - \sin\theta)g + \dot{\theta}^2 L + \Omega^2 L\cos^2\theta$;

$N_{\text{horiz}} = 2m\Omega(u\cos\theta - \dot{\theta}L\sin\theta)$,

$N_{\text{vert}} = m(g\cos\theta + \Omega^2 L\sin\theta + 2\dot{\theta}u)$.

3.30 $\bar{v} = -41.13\bar{I} + 46.84\bar{J}$ m/s, $\bar{a} = -98{,}116\bar{I} + 56{,}647\bar{J}$ m/s^2.

3.32 $\bar{v}_D = -(\omega_2 R \sin\beta \sin\theta)\bar{i} + [(\omega_1 + \omega_2 \cos\beta) R \sin\theta]\bar{j}$
$-[(\omega_1 + \omega_2 \cos\beta) R \cos\theta]\bar{k}$; $\bar{i}$ opposite $\bar{\omega}_1$;

$\bar{a}_D = [\omega_2(2\omega_1 + \omega_2 \cos\beta) R \sin\beta \cos\theta]\bar{i} + [\dot{\omega}_1 R \sin\theta - (\omega_1 + \omega_2 \cos\beta)^2 R \cos\theta]\bar{j}$
$-[\dot{\omega}_1 R \cos\theta + (\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos\beta) R \sin\theta]\bar{k}$.

3.34 $\bar{\omega} = -(0.9397\omega_1 \cos\theta)\bar{i} + (\dot{\theta} + 0.3420\omega_1)\bar{j} - (0.9397\omega_1 \sin\theta)\bar{k}$,

$\bar{\alpha} = (0.9397\omega_1 \dot{\theta} \sin\theta)\bar{i} + \ddot{\theta}\bar{j} - (0.9397\omega_1 \dot{\theta} \cos\theta)\bar{k}$;

$\bar{v}_D = (1.9397\omega_1 L \sin\theta)\bar{i} - (0.9397\omega_1 L \sin\theta)\bar{j}$
$+(-1.9397\omega_1 L \cos\theta - 0.3420\omega_1 L - L\dot{\theta})\bar{k}$,

$\bar{a}_D = L[(-1 - 0.6634\cos\theta + 0.8830\cos^2\theta)\omega_1^2 - \dot{\theta}^2 - 0.6840\omega_1\dot{\theta}]\bar{i}$
$+L[-(1.8227 + 0.3214\cos\theta)\omega_1^2 - 1.8794\omega_1\dot{\theta}\cos\theta]\bar{j}$
$+L[-\ddot{\theta} + (0.8830\cos\theta - 0.6634)\omega_1^2 \sin\theta]\bar{k}$.

3.36 $\bar{v} = -2.87L\bar{i} + 17.84L\bar{j} - 0.03L\bar{k}$,

$\bar{a} = -165.09L\bar{i} - 24.91L\bar{j} + 16.13L\bar{k}$; $\bar{i} = \bar{e}_{B/A}$, $\bar{k}$ vertical.

3.38 $\bar{v}_{P/G} = 2.50\bar{i} + 0.60\bar{j} + 0.10\bar{k}$ m/s.

$\bar{a}_{P/G} = -1.15\bar{i} - 44.98\bar{j} - 1.37\bar{k}$ m/s^2.

3.40 $(\bar{a}_C)_{xyz} = -(\omega_1^2 s \sin\theta)\bar{j} - (2\omega_1 v_C \sin\theta)\bar{k}$; $\bar{i}$ = axis of cylinder.

3.42 $d^2 + \left(s - \dfrac{u}{2\omega_e \sin\lambda}\right)^2 = \left(\dfrac{u}{2\omega_e \sin\lambda}\right)^2$.

3.44 (a) $x = 0$, $y = -\omega_e\left(\dfrac{8H^3}{9g}\right)^{1/2} \cos\lambda$ (east of O');

(b) $x = 0$, $y = \omega_e\left(\dfrac{2H^3}{9g}\right)^{1/2}\left(1 + \dfrac{3R_e}{H}\right)\cos\lambda$ (west of O').

4.2 $[R] = \begin{bmatrix} -0.5797 & 0.4731 & -0.6634 \\ -0.0400 & -0.8297 & -0.5567 \\ -0.8138 & -0.2962 & 0.50 \end{bmatrix}$.

4.4 $v = 2.188$ m/s, $\dot{v} = 1{,}561.3$ m/s^2.

4.6 $\omega_{AB} = 0.866$ rad/s (cw), $\omega_{CD} = 0.6928$ rad/s (ccw);

$\alpha_{AB} = 21.22$ rad/s^2 (ccw), $\alpha_{CD} = 23.00$ rad/s^2 (cw).

4.8 $\bar{\alpha}_{AB} = -9\bar{I} - 3\bar{J} - 2\bar{K}$ rad/s^2; $\bar{J}$ parallel to $\bar{v}_B$, $\bar{K}$ vertical.

4.10 $\bar{\omega}_{AB} = -\omega_1\bar{j} - \dfrac{v_A}{L}\bar{k}$, $\bar{\alpha}_{AB} = \dfrac{\omega_1 v_A}{L}\bar{i} - 0.75\left(\dfrac{v_A}{L}\right)^2\bar{k}$; $\bar{j}$ vertical.

4.12 $\bar{\omega}_{BC} = -84.21\bar{I} - 181.96\bar{J} + 62.51\bar{K}$ rad/s; $\bar{J}$ parallel to $\bar{\omega}_{AB}$,

$\bar{\alpha}_{BC} = -9{,}651\bar{I} - 11{,}028\bar{J} + 7{,}161\bar{K}$ rad/s^2.

4.14 $\bar{v}_C = -0.3695\bar{I} + 0.2665\bar{J} + 0.2770\bar{K}$ m/s; $\bar{I} = \bar{e}_{D/A}$,

$\bar{a}_C = -8.600\bar{I} - 2.002\bar{J} + 6.450\bar{K}$ m/s^2.

4.16 $\bar{v}_D = [v(1+\cos\beta) - \Omega R\cos\beta]\bar{i} - [(v-\Omega R)\sin\beta]\bar{j}; \quad \bar{j} = \bar{e}_{C/A},$

$$\bar{a}_D = \left[\dot{v}(1+\cos\beta) - \frac{(v-\Omega R)^2}{r}\sin\beta\right]\bar{i} - \left[\dot{v}\sin\beta + \frac{v^2}{r+R} + \frac{(v-\Omega R)^2}{r}\cos\beta\right]\bar{j}.$$

4.18 $\bar{\omega}_A = 0.6\dfrac{v}{R}$ (cw), $\quad \bar{\alpha}_A = 0.008\dfrac{v^2}{R}$ (cw).

4.20 $\bar{v}_C = \frac{1}{2}[(\omega_2 r_2 + \omega_1 r_1) + (\omega_2 r_2 - \omega_1 r_1)\cos\theta]\bar{i} - [(\omega_2 r_2 - \omega_1 r_1)\sin\theta]\bar{j}; \quad \bar{j} = \bar{e}_{B/A},$

$$\bar{a}_C = \frac{(\omega_2 r_2 + \omega_1 r_1)^2}{2(r_1+r_2)}\bar{j} - \frac{(\omega_2 r_2 - \omega_1 r_1)^2}{2(r_1-r_2)}[(\sin\theta)\bar{i} + (\cos\theta)\bar{j}].$$

4.22 $\bar{\omega} = \omega_1[\sin(\beta+\gamma)]\bar{i} - \omega_1\left[\cos(\beta+\gamma) + \dfrac{\sin\gamma}{\sin\beta}\right]\bar{k},$

$\bar{\alpha} = \omega_1^2\left[\dfrac{\sin\gamma}{\sin\beta}\sin(\beta+\gamma)\right]\bar{j}; \quad \bar{k}$ parallel to shaft A.

4.24 $\bar{\omega} = \dfrac{v}{R}[-\bar{i} + (\cos\theta)\bar{j}]; \quad \bar{i}$ parallel to conical surface,

$\bar{\alpha} = \left(\dfrac{v}{R}\right)^2(1+\cos\theta)(\sin\theta)\bar{k}.$

4.26 $\bar{\omega} = -13.898\bar{i} + 16.562\bar{k}$ rad/s,

$\bar{\alpha} = 53.20\bar{j}$ rad/s^2; $\bar{k}$ parallel to axis of body cone.

4.28 $\dot{\psi} = -16.40$ rad/s and $\dot{\theta} = -5.282$ rad/s, or

$\dot{\psi} = -5.90$ rad/s and $\dot{\theta} = -14.680$ rad/s.

4.30 $\bar{\omega} = (\omega_1\cos\beta)\bar{i} - \dfrac{u}{R}\dfrac{1}{\sin\beta - 2\cos\beta}\bar{j} - 2\omega_1(1+\cos\beta)\bar{k},$

$$\bar{\alpha} = \frac{2u\omega_1}{R}\frac{1+\cos\beta-\sin\beta}{\sin\beta-2\cos\beta}\bar{i} + \left[\frac{u^2}{R^2}\frac{\cos\beta+2\sin\beta}{(\sin\beta-2\cos\beta)^3} + \omega_1^2(2+2\cos\beta-\sin\beta)\cos\beta\right]\bar{j}$$
$$+\frac{2u\omega_1}{R}\frac{\sin\beta}{\sin\beta-2\cos\beta}\bar{k};$$

$\bar{k} = \bar{e}_{C/B}.$

5.2 $I_{xx} = \dfrac{1}{6}m\left[a^2+b^2+\dfrac{a+3b}{a+b}c^2\right], \quad I_{xy} = \dfrac{1}{6}md\dfrac{a^2+ab+b^2}{a+b}.$

5.4 $r_{G/O} = -0.6748\bar{i} + 120\bar{k}$ mm; $\quad I_{xy} = I_{yz} = 0,$

$I_{xx} = I_{yy} = 0.4509, \; I_{zz} = 0.03884, \; I_{xz} = -0.00182$ kg-m^2.

5.6 $I_{xx} = 0.01735, \; I_{yy} = 0.49369, \; I_{zz} = 0.51104$ kg-m^2,

$I_{xy} = -0.008767$ kg-m^2, $\; I_{xz} = I_{yz} = 0.$

5.8 $I_1 = 0.1136, \; I_2 = 1.3864, \; I_3 = 1.5$ kg-m^2.

5.10 $I_1 = 0.4647, \; I_2 = 0.4393, \; I_3 = 0.0383$ kg-m^2,

$$[R] = \begin{bmatrix} 0.0778 & 0.6951 & 0.7147 \\ 0.4186 & -0.6734 & 0.6093 \\ 0.9048 & 0.2518 & -0.3433 \end{bmatrix}.$$

5.12 $[I] = \begin{bmatrix} 0.1520 & 0.0540 & 0 \\ 0.0540 & 0.2330 & 0 \\ 0 & 0 & 0.225 \end{bmatrix}$ kg-m^2;

$\bar{A} = \omega^2(-0.0675\bar{j}+0.60\bar{k})$ N, $\bar{B} = \omega^2(0.0675\bar{j}+0.60\bar{k})$ N.

5.14 $\bar{M}_A = \frac{1}{4}mR^2\left[-\frac{3}{2}\omega_1\omega_2+\frac{\sqrt{3}}{4}(\omega_1^2-\omega_2^2)\right]\bar{i}$; $\bar{i}$ is outward.

5.16 $\bar{A} = -\frac{5}{36}m\omega^2 a\bar{j}$, $\bar{B} = -\frac{7}{36}m\omega^2 a\bar{j}$; $\bar{j}$ parallel to side a.

5.18 $\ddot{\theta} = \frac{4gL\sin\theta}{4L^2+R^2}+\frac{4L^2-R^2}{4L^2+R^2}\dot{\psi}^2\sin\theta\cos\theta+\frac{2R^2}{4L^2+R^2}\dot{\psi}\dot{\phi}\sin\theta$,

$\ddot{\psi} = \ddot{\phi} = 0$.

5.20 $\ddot{\beta}-\frac{1}{2}\Omega^2\sin 2\beta = 0$, $\Gamma = \frac{1}{12}mL^2\Omega\dot{\beta}\sin 2\beta$.

5.22 $\frac{1}{3}L^2\ddot{\theta}+(L\Omega^2\sin\theta)(\frac{1}{2}+\frac{1}{3}\cos\theta) = \frac{1}{2}gL\cos\theta$.

5.24 $\ddot{\phi}-\dot{\psi}^2\sin^2\theta\sin\phi\cos\phi = \frac{3Fh}{mL^2}$.

5.26 $\omega_1^2 > \frac{5}{7}\frac{g}{R}$.

5.28 $\dot{\psi}^2 = \frac{4g}{7R\sin\beta}$, $\mu = \frac{7\cos\beta-4}{7\sin\beta+4\cot\beta}$.

5.30 $\dot{\omega} = -0.2856\frac{g}{L}$.

5.32 front: $\dot{v} = \mu g\frac{L-b}{L+\mu h}$; rear: $\dot{v} = \mu g\frac{b}{L-\mu h}$; all: $\dot{v} = \mu g$.

5.34 (a) $F = 21.96$ N, $\dot{v} = 0.2342$ m/s^2;

(b) $\dot{v} = 0.4540$ m/s^2, $\dot{\omega} = 0.7619$ rad/s^2 (ccw).

5.36 $\dot{v} = 0.0585g$.

5.38 (a) $\dot{v} = \dot{\omega}r = \frac{F(h+r\cos^2\theta)r}{m(\kappa^2+r^2)\cos^2\theta}$;

(b) $F_{\max} = \frac{(\kappa^2+r^2)\cos^2\theta}{\kappa^2\cos^2\theta-(\kappa^2+r^2)\mu_s\sin\theta\cos\theta-hr}\mu_s mg$;

(c) $\dot{v} = \frac{F}{m}(1-\mu_k\tan\theta)$, $\dot{\omega} = \frac{F}{m\kappa^2}\left(\frac{h}{\cos^2\theta}+\mu_k r\tan\theta\right)+\frac{\mu_k g}{\kappa^2}$.

5.40 $v^2 = \frac{2Fr^3}{m(r^2+\kappa^2)}[\theta+\sin\theta+(8-2\cos\theta-\cos^2\theta)^{1/2}-\sqrt{5}]$.

5.42 $\Omega_1 = 22.24$ rad/s.

5.44 $\bar{v}_G = -0.7361v\bar{k}$, $\bar{\omega} = -1.0577\frac{v}{R}\bar{i}+1.6032\frac{v}{R}\bar{j}$; $\bar{k}$ upward.

5.46 $F = 8{,}523$ N.

5.48 $\bar{F} = (m_1+m_2)\left(g-\frac{v^2}{\rho}\right)\bar{i},$

$\bar{M} = \frac{v}{\rho}[J\Omega-(I_x-I_y+I_z+J)\omega_1]\bar{i}+m_2D\left(g-\frac{v^2}{\rho}\right)\bar{j}.$

6.2 $\dot{X}=\dot{x}+u, \quad \dot{Y}=2\beta x\dot{x}.$

6.4 $\dot{R}=\dot{l}\sin\theta+z\dot{\theta}, \quad \dot{\phi}=-\Omega, \quad \dot{z}=\dot{l}\cos\theta-R\dot{\theta}.$

6.6 $\dot{X}=\dot{D}\sin\theta+\dot{\theta}(D\cos\theta-R\sin\theta),$

$\dot{Y}=-\dot{D}\cos\theta+\dot{\theta}(\dot{D}\sin\theta+R\cos\theta).$

6.8 $2R\dot{\phi}_1\cos\beta-r\dot{\phi}_2-r\dot{\phi}_3=0, \quad 2d\dot{\theta}-r(\dot{\phi}_2-\dot{\phi}_3)=0,$

$L\dot{\theta}-R\dot{\phi}_1\sin\beta=0, \quad N=4.$

6.10 $m\ddot{x}+kx\left[1-\frac{0.8L}{(L^2+x^2)^{1/2}}\right]=0.$

6.12 $\frac{1}{3}mL^2\ddot{\theta}+2kb^2\sin\frac{\theta}{2}-mg\frac{L}{2}\cos\theta=0.$

6.14 $3mL^2\dot{\theta}+2kL^2(\sqrt{2}\sin\theta-\sin 2\theta)+\left(\frac{5}{2}mg+2P\right)L\cos\theta=0.$

6.16 $\frac{1}{3}L\ddot{\theta}+\frac{1}{2}(\ddot{y}+g)\sin\theta=0.$

6.18 $5\ddot{s}_1+\ddot{s}_2\cos\theta+\frac{2k}{m}(s_1-\hat{s}_1)=0.$

$\ddot{s}_2+\ddot{s}_1\cos\theta+\frac{k}{m}(s_2-\hat{s}_2)=g\sin\theta, \quad s_j=$ spring elongations.

6.20 $\left[\left(\frac{4}{3}+\frac{16}{3}f^2\right)+(4+6f+2f^2)\sin^2\theta\right]\ddot{\theta}$

$+\left[\frac{16}{3}ff'+(3+2f)f'\sin^2\theta+(4+6f+2f^2)\sin\theta\cos\theta\right]\dot{\theta}^2$

$+\frac{5g}{2L}\cos\theta=-\frac{P}{\sigma L^2}(1+f),$

where $f=\frac{\cos\theta}{(16-\sin^2\theta)^{1/2}}$ and $f'=-\frac{15\sin\theta}{(16-\sin^2\theta)^{3/2}}.$

6.22 $m_1\left(1+\frac{4\sigma_1^2}{9R^2}\right)\ddot{x}_1+2k(2x_1-3x_2)=0,$

$m_2\left(1+\frac{\sigma_2^2}{R^2}\right)\ddot{x}_2+3k(3x_2-2x_1)=0.$

6.24 $\frac{1}{2}\ddot{\theta}[3m_A\sin^2\theta+m_B(3+2\sin\theta-\sin^2\theta)]$

$+\frac{1}{2}\dot{\theta}^2[3m_A\sin\theta+m_B(1-\sin\theta)]\cos\theta+\frac{m_Bg}{R_A+R_B}\cos\theta=0,$

where θ is the angle of elevation of the line of centers.

6.26 $m\ddot{\theta}\left[\frac{1}{3}L^2+R^2\frac{2+\cos\theta}{(1-\cos\theta)^2}\right]-mR^2\dot{\theta}^2\frac{(\sin\theta)(2+\cos\theta)}{(1-\cos\theta)^3}+\frac{1}{2}mgL\cos\theta=-PL.$

6.28 $(\frac{1}{2}h^4+4R^2h^2\sin^2\theta)\ddot{\theta}+4R^2(h^2\sin\theta\cos\theta+R^2\sin^3\theta)\dot{\theta}^2+gh^3\sin\theta=0,$
where $h=[L^2-2R^2(1-\cos\theta)]^{1/2}$.

6.30 $3\ddot{l}-2l\dot{\psi}^2\sin^2\beta+g(1-2\cos\beta)=0, \quad 2ml(l\ddot{\psi}+2\dot{l}\dot{\psi})(\sin^2\beta)\dot{\psi}=M,$
where l is the distance to a block on the inclined arm.

6.32 $I_2\ddot{\theta}+\frac{1}{2}(I_1-I_2)\omega_1^2\sin 2\theta+I_1\omega_1\omega_2\sin\theta=0.$

6.34 $\frac{1}{4}mR^2[(1+\cos^2\theta)\dot{\Omega}+2\dot{\omega}_1\cos\theta]=C.$

6.36 $\ddot{\phi}-\dot{\psi}^2\sin^2\theta\sin\phi\cos\phi=\dfrac{3Fh}{mL^2}.$

6.38 $\ddot{\psi}(1+3\sin^2\theta)+6\dot{\psi}\dot{\theta}\sin\theta\cos\theta=0,$

$$\frac{1}{12}mL^2(5\ddot{\theta}+3\dot{\psi}^2\sin\theta\cos\theta)+mg\frac{L}{2}\sin\theta=0.$$

6.40 $\dfrac{7}{3}[\ddot{\theta}-\dot{\psi}^2\sin\theta\cos\theta]+\dfrac{3g}{L}\sin\theta=0,$

$$\left[\frac{7}{3}mL^2\sin^2\theta+I\right]\ddot{\psi}+\frac{14}{3}mL^2\dot{\psi}\dot{\theta}\sin\theta\cos\theta=\Gamma.$$

where θ is the angle of bar AB from the vertical.

7.2 $\dfrac{1}{3}(\ddot{\theta}-\Omega^2\sin\theta\cos\theta)+\dfrac{g}{2L}\sin\theta=0, \quad \dfrac{2}{3}mL^2\Omega\dot{\theta}\sin\theta\cos\theta=\Gamma.$

7.4 $\ddot{x}_G+\mu\left(g+\dfrac{L}{2}\ddot{\theta}\cos\theta-\dfrac{L}{2}\dot{\theta}^2\sin\theta\right)\operatorname{sgn}\left(\dot{x}_G+\dfrac{L}{2}\dot{\theta}\sin\theta\right)=0,$

$$\frac{1}{6}L\ddot{\theta}+\left(g+\frac{L}{2}\ddot{\theta}\cos\theta-\frac{L}{2}\dot{\theta}^2\sin\theta\right)\left[\cos\theta+\mu\sin\theta\operatorname{sgn}\left(\dot{x}_G+\frac{L}{2}\dot{\theta}\sin\theta\right)\right]=0.$$

7.6 $\frac{1}{3}L\ddot{\theta}\sin\phi-\frac{1}{2}g\sin\theta+\mu[\frac{1}{2}g\cot\phi\cos\theta-\frac{1}{3}L\dot{\theta}^2\cos\phi]\operatorname{sgn}\dot{\theta}=0.$

7.8 $mL^2[\frac{4}{3}\ddot{\theta}_1+\frac{1}{2}\ddot{\theta}_2\cos(\theta_1-\theta_2)+\frac{1}{2}\dot{\theta}_2^2\sin(\theta_1-\theta_2)]+\frac{3}{2}mgL\sin\theta_1=\lambda_1\cos(\theta_1-\theta_2)+M,$
$mL^2[\frac{1}{3}\ddot{\theta}_2+\frac{1}{2}\ddot{\theta}_1\cos(\theta_1-\theta_2)-\frac{1}{2}\dot{\theta}_1^2\sin(\theta_1-\theta_2)]+\frac{1}{2}mgL\sin\theta_2=\lambda_1,$
$\dot{\theta}_1\cos(\theta_1-\theta_2)+\dot{\theta}_2=0,$ where θ_j are measured from the vertical.

7.10 $\ddot{X}_C\sin\theta+\ddot{L}-L\dot{\theta}^2+\dfrac{k}{m}(L-L_0)-mg\cos\theta=0,$

$m(\ddot{X}_C\cos\theta+L\ddot{\theta}+2\dot{L}\dot{\theta}+g\sin\theta)=\lambda_1,$

$m(2\ddot{X}_C+\ddot{L}\sin\theta+2\dot{L}\dot{\theta}\cos\theta-L\dot{\theta}^2\sin\theta)=\lambda_1\cos\theta.$

7.12 Eqs. (7.20) and (7.25), with $\{q\}=[\psi \;\; \beta]^T$, $\{\lambda\}=\{\lambda_1\}$, and

$$[M]=\begin{bmatrix}(mL^2+I_1\sin^2\beta+I_2\cos^2\beta) & 0\\ 0 & I_2\end{bmatrix}, \quad I_1=m\kappa_1^2,\; I_2=m\kappa_2^2,$$

$$\{F\}=\begin{Bmatrix}-\frac{1}{2}(I_1-I_2)\dot{\beta}\dot{\psi}\sin 2\beta+I_1\dot{\beta}\omega_s\cos\beta\\ \frac{1}{2}(I_1-I_2)\dot{\psi}^2\sin 2\beta-I_1\omega_s\dot{\psi}\cos\beta-\Gamma\end{Bmatrix},$$

$[a]=[0\;\;1], \quad [\dot{a}]=[0\;\;0], \quad \{\dot{b}\}=\{0\}.$

7.14 $t=0.623$ s, $E_{dis}=11.527mL^2$.

7.16 $D\dot{s} = m(p_1 + p_2 \cos\beta)$, $D\dot{x} = mp_1 \cos\beta + (M+m)p_2$,
$\dot{p}_1 = 0$, $\dot{p}_2 = mg\sin\beta - kx$,
where s is the displacement of the cart and $D = m(M + m\sin^2\beta)$.

7.18 $\dot{\theta} = \dfrac{3p_1}{2ML} - \dfrac{3\dot{y}}{2L}\sin\theta$, $\dot{p}_1 = \dfrac{3p_1\dot{y}}{2L}\cos\theta + mg\dfrac{L}{2}\sin\theta$.

7.20 $(m_A + m_B)\ddot{R} - [(m_A + m_B)R - m_B L]\Omega^2 = 0$,
$\Gamma = 2[(m_A + m_B)R - m_B L]\dot{R}\Omega$.

7.22 $I_2\ddot{\beta} + \dfrac{1}{2}(I_1 - I_2)\Omega^2 \sin 2\beta + mg\left(\dfrac{L}{2} - D\right)\cos\beta\cos\Omega t = 0$,
$\Gamma = (I_2 - I_1)\Omega\dot{\beta}\sin 2\beta - mg\left(\dfrac{L}{2} - D\right)\sin\beta\sin\Omega t$,
where $I_1 = \dfrac{1}{2}mR^2$, $I_2 = m\left(\dfrac{1}{4}R^2 + \dfrac{1}{3}L^2 - LD + D^2\right)$.

7.24 $\dot{\gamma}_1 = \dot{\psi}$, $\dot{\gamma}_2 = \dot{\theta}$, $\ddot{\gamma}_1 \sin^2\theta + \dot{\gamma}_1\dot{\gamma}_2 \sin 2\theta = 0$,
$\ddot{\gamma}_2(1 + 8\cos^2\theta) - \left(\dfrac{9}{2}\dot{\gamma}_1^2 + 4\dot{\gamma}_2^2\right)\sin 2\theta = \dfrac{2mg - 4F}{mL}\sin\theta$.

7.26 $(\frac{1}{3}L^2\cos^4\theta - DL\cos^3\theta + D^2)\ddot{\gamma}_1 + (2D^2 - \frac{1}{2}DL\cos^3\theta)(\tan\theta)\dot{\gamma}_1^2$
$-g(D - \frac{1}{2}L\cos^3\theta)\cos^2\theta = 0$, $\dot{\gamma}_1 = \dot{\theta}$.

7.28 $\dot{\gamma}_1 = \dot{\theta}$, $\dot{\gamma}_2 = \dot{\psi} = c\theta$,
$\dfrac{1}{3}\ddot{\gamma}_1 - \dot{\gamma}_2^2(\cos\theta)\left(\dfrac{1}{2} + \dfrac{1}{3}\sin\theta\right) + \dfrac{g}{2L}\sin\theta = 0$,
$\left[\left(1 + \sin\theta + \dfrac{1}{3}\sin^2\theta\right) + I_2\right]\ddot{\gamma}_2 + \left(1 + \dfrac{2}{3}\sin\theta\right)(\cos\theta)\dot{\gamma}_1\dot{\gamma}_2 = \dfrac{\Gamma}{mL^2}$.

7.30 $\{q\} = [X_G \; Y_G \; \theta \; \phi_A \; \phi_B]^T$, where ϕ_i are spin angles;
$\dot{\gamma}_1 = |\bar{v}_G|$, $\dot{\gamma}_2 = \dot{\theta}$, $\dot{X}_G = -\dot{\gamma}_1\sin\theta$, $\dot{Y}_G = \dot{\gamma}_1\cos\theta$,
$\dot{\phi}_A = \dfrac{\dot{\gamma}_1}{R} + \dfrac{L}{2R}\dot{\gamma}_2$, $\dot{\phi}_B = \dfrac{\dot{\gamma}_1}{R} - \dfrac{L}{2R}\dot{\gamma}_2$,
$\dfrac{14}{5}\ddot{\gamma}_1 = -2g\sin\beta\cos\theta$, $\ddot{\gamma}_2 = 0$.

7.32 $q_1 = \theta_{AB}$, $q_2 = \theta_{CD}$, $[M]\{\ddot{q}\} + [D]\{\dot{q}\} + [K]\{q\} = \{F\}$;
$[M] = mL^2\begin{bmatrix} 1/3 & 0 \\ 0 & 9/64 \end{bmatrix}$, $[D] = \mu L^2\begin{bmatrix} 1/4 & 3/8 \\ 3/8 & 9/16 \end{bmatrix}$,
$[K] = \begin{bmatrix} kL^2 + k_T & kL^2/4 \\ kL^2/4 & kL^2/16 \end{bmatrix}$, $\{F\} = \begin{Bmatrix} 0 \\ 3/4 \end{Bmatrix} FL$.

7.34 $0.5833mL^2\ddot{\xi} + (0.75kL^2 + 1.0580mgL + 0.5FL)\xi = 0.866FL$.

7.36 Stable at $\theta^* = 83.851°$, unstable at $\theta^* = -114.325°$.

7.38 $\theta^* = 0$ is always unstable;
$\theta^* = \cos^{-1}\left(\dfrac{4kl - 3mg}{kl + ml\Omega^2}\right)$ exists and is stable if $k > 0.75\dfrac{mg}{l}$.

8.2 $\omega = 3.135$ rad/s at $\beta = 93.05°$ and $\theta = 94.87°$,

$\Delta \bar{H}_G = -3.130 I'\bar{i}$, $\omega_2 = -0.1667$ rad/s.

8.4 Minimum angle to z axis: $\theta = 27.160°$, $\bar{\omega} = 7.981\bar{i} + 7.588\bar{k}$ rad/s;

Maximum angle to z axis: $\theta = 30.371°$, $\bar{\omega} = 6.096\bar{j} + 7.358\bar{k}$ rad/s.

8.6 Looping precession: $53.13° \le \theta \le 53.30°$.

8.8 $\theta = \frac{\pi}{2} + C_1 \sin\left(\frac{\omega^2}{\sigma} t + \nu_1\right) - C_2 \sin(\sigma t + \nu_2)$,

$\psi = C_1 \cos\left(\frac{\omega^2}{\sigma} t + \nu_1\right) + C_2 \cos(\sigma t + \nu_2)$,

where $\omega^2 = \frac{mgL}{I' + mL^2}$, $\sigma = \frac{\dot{\phi} I}{I' + mL^2}$.

8.10 $\frac{I'\lambda}{c} \gg 1$, $\frac{ct}{I'} \gg \frac{\Omega_1}{\Omega_0}$, $\frac{I'}{I}$ not large.

8.12 $\ddot{\xi} + 2\Omega\dot{\eta} + \left(\frac{k_\xi}{m} - \Omega^2\right)\xi = g \cos \Omega t$,

$\ddot{\eta} - 2\Omega\dot{\xi} + \left(\frac{k_\eta}{m} - \Omega^2\right)\eta = g \sin \Omega t$,

$\ddot{\beta} + 2\Omega\dot{\gamma} + \left(\frac{k_\beta}{I_2} - \Omega^2\right)\beta = 0$,

$\ddot{\gamma} - 2\Omega\dot{\beta} + \left(\frac{k_\gamma}{I_2} - \Omega^2\right)\gamma = 0$.

8.14 $\theta = \theta_0 + \frac{\epsilon}{\lambda} \sin \lambda t + \frac{\epsilon^2}{8\lambda^2}\left[2\frac{(I' + A - C)}{J} - \frac{1}{\sin^2\theta_0}\right](3 - 4\cos\lambda t + \cos 2\lambda t)$,

$$\frac{\dot{\psi} J}{p_\phi} = \frac{\epsilon}{\lambda} \sin\theta_0 \sin\lambda t + \frac{\epsilon^2}{2\lambda^2}(\cos\theta_0)\left[1 - 4\frac{I' + A - C}{J}\sin^2\theta_0\right]\sin^2\lambda t$$
$$+ \frac{\epsilon^2}{4\lambda^2}\left[\frac{(I' + A - C)}{J}\sin 2\theta_0 - \cot\theta_0\right](\sin\theta_0)(3 - 4\cos\lambda t + \cos 2\lambda t),$$

where $J = A' + (I' + A)\sin^2\theta_0 + C\cos^2\theta_0$ and $\lambda^2 = \frac{p_\phi^2 \sin^2\theta_0}{J(I' + B)}$.

Index